Other Perspectives To Prove Benkoski's Theorem.

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1 Introduction.

One branch of analytic number theory concerns itself with the distribution of integers with certain special properties. The prototypical result in this field is Dirichlet's theorem for primes lying in an arithmetic progression of 1837: For any two positive relatively prime integers a and d, there are infinitely many primes of the form a modulo d. In 1896, Hadamard and de la Vallé Poussin proved the Prime Number Theorem: The number of primes less than or equal to n is asymptotically equal to $n/\ln n$.

Unlike these well-known results, the following two fascinating facts about the integers are not as notorious: (1) The probability that an integer is squarefree is $6/\pi^2$, and (2) The probability that two integers are relatively prime is $6/\pi^2$. These results predate the Prime Number Theorem. Gegenbauer proved the first in 1885, while Mertens proved the second in 1874.

Gegenbauer actually proved the more general result that the probability that an integer is not divisible by an rth power, with $r \geq 2$, is $1/\zeta(r)$. (Note that $1/\zeta(2) = 6/\pi^2$.) In 1900, Lehmer extended Mertens' result to show that the probability that k integers are relatively prime is $1/\zeta(k)$.

Given the short period of time that it took to establish the probabilities for the relative primality of k integers and for an integer to be rth power free, it may be surprising that not until 1976 did Benkoski prove a result that combines both lines of thought by showing that the probability that k integers do not have a common rth power is $1/\zeta(rk)$. While his proof uses a generalisation of Jordan totient functions to prove this result, I will outline two more accessible proofs of this result. The first proof uses a discrete version of Lebesgue's Dominated Convergence Theorem. While one does not obtain growth estimates this way (and is therefore rendered useless for most Analytic Number Theorists), it gains in its brevity and transparency. The second proof follows the methods of Nymann in deriving the classic growth estimates without having to make a detour to the aforementioned totient functions, while still being quite direct and accessible to any motivated undergraduate student.

Related probabilistic questions have been subsequently considered in other PID's. In particular, Morrison and Dong proved an analogue of Benkoski's statement for the ring $\mathbb{F}_q[x]$ in 2004. For the case of the algebraic integers, all that has been published is the case of relative primality. We will remedy this situation by proving a full generalisation of Benkoski's result for a ring of algebraic integers.

2 More concise manners to prove Benkoski's Theorem.

2.1 Some background arithmetic material.

As a reminder, the Riemann Zeta function ζ is defined as follows:

$$\zeta(s) = \sum_{j=1}^{\infty} \frac{1}{j^s}.$$

This series converges for all $s > 1$. Without going too far on a tangent, its importance to Number Theory can be duly noted through its infinite product expansion

$$\zeta(s) = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}.$$

Next, we define the Möbius function $\mu : \mathbb{Z} \to \{0, \pm 1\}$ as follows:

$$\mu(n) = \begin{cases} 1 & \text{if } n = \pm 1 \\ 0 & \text{if } n \text{ is not squarefree} \\ (-1)^r & \text{if } n = p_1 p_2 ... p_r. \end{cases}$$

Then, one may check that

$$\sum_{j=1}^{\infty} \frac{\mu(j)}{j^s} = \prod_{p \text{ prime}} (1 - p^{-s}) = \frac{1}{\zeta(s)}.$$

This should be more than sufficient for the purposes of the rest of this article. Onto the proofs!

2.2 Benkoski's Theorem.

For a fixed integer $r \geq 1$, we say that the integers $m_1, m_2, ..., m_k$ are **relatively r-prime** if they have no common factor of the form n^r for any integer $n > 1$. When $r = 1$, this is the definition of being relatively prime.

Theorem 2.1 ***(Benkoski)***
Fix $k, r \in \mathbb{N}$ not both equal to 1, and let $q(n)$ denote the number of ordered k-tuples of positive integers less than or equal to n that are relatively r-prime. Then,

$$\lim_{n\to\infty} \frac{q(n)}{n^k} = \frac{1}{\zeta(rk)}.$$

Proof:
Fix $n \in \mathbb{N}$. Since an ordered k-tuple of integers is relatively r-prime if and only if there exists no prime p such that p^r divides all k integers, the Inclusion-Exclusion Principle shows that $q(n)$ may be written as

$$q(n) = n^k - \sum_{p_1} \Big\lfloor \frac{n}{p_1^r} \Big\rfloor^k + \sum_{p_1<p_2} \Big\lfloor \frac{n}{(p_1p_2)^r} \Big\rfloor^k - \sum_{p_1<p_2<p_3} \Big\lfloor \frac{n}{(p_1p_2p_3)^r} \Big\rfloor^k + ...,$$

where p_1, p_2, ... denote distinct primes less than or equal to n. (Note that these sums terminate, since n is fixed.) By using the Möbius function μ, we may rewrite this sum more compactly as

$$q(n) = \sum_{j=1}^{\infty} \mu(j) \Big\lfloor \frac{n}{j^r} \Big\rfloor^k.$$

However, we do not need the infinity at the upper end of the summation. In fact, note that the floor functions inside the sums annihilate terms with indices $j > \lfloor \sqrt[r]{n} \rfloor$.

So, we may rewrite the sum as $q(n) = \displaystyle\sum_{j=1}^{\lfloor \sqrt[r]{n} \rfloor} \mu(j) \Big\lfloor \frac{n}{j^r} \Big\rfloor^k$.

Since

$$\Big\lfloor \frac{n}{j^r} \Big\rfloor^k - \Big(\frac{n}{j^r}\Big)^k = \Big(\Big\lfloor \frac{n}{j^r} \Big\rfloor - \frac{n}{j^r}\Big)\Big(\Big\lfloor \frac{n}{j^r} \Big\rfloor^{k-1} + \Big\lfloor \frac{n}{j^r} \Big\rfloor^{k-2}\Big(\frac{n}{j^r}\Big) + ... + \Big(\frac{n}{j^r}\Big)^{k-1}\Big)$$

and $0 \leq x - \lfloor x \rfloor \leq 1$ for all $x \in \mathbb{R}$, we obtain

$$\Big\lfloor \frac{n}{j^r} \Big\rfloor^k - \Big(\frac{n}{j^r}\Big)^k = O\Big(\Big(\frac{n}{j^r}\Big)^{k-1}\Big).$$

Applying this growth estimate to $q(n)$ yields

$$q(n) = \sum_{j=1}^{\lfloor \sqrt[r]{n} \rfloor} \mu(j) \Big(\frac{n}{j^r}\Big)^k + O\Big(\sum_{j=1}^{\lfloor \sqrt[r]{n} \rfloor} |\mu(j)| \Big(\frac{n}{j^r}\Big)^{k-1}\Big).$$

Now, we must estimate how fast each sum in $q(n)$ grows. For the first sum, after rewriting it as

$$\sum_{j=1}^{\lfloor \sqrt[r]{n} \rfloor} \frac{\mu(j)}{j^{rk}} = \frac{1}{\zeta(rk)} - \sum_{j=\lfloor \sqrt[r]{n} \rfloor + 1}^{\infty} \frac{\mu(j)}{j^{rk}},$$

we find that its tail is bounded above by the integral $\displaystyle\int_{\sqrt[r]{n}}^{\infty} \frac{dx}{x^{rk}} = O(n^{1/r-k})$.

Therefore, the first sum is $\dfrac{n^k}{\zeta(rk)} + O(n^{1/r})$.

As for the second sum, we first note that

$$\sum_{j=1}^{\lfloor \sqrt[r]{n} \rfloor} |\mu(j)| \Big(\frac{n}{j^r}\Big)^{k-1} = O\Big(n^{k-1} \sum_{j=1}^{\lfloor \sqrt[r]{n} \rfloor} \frac{1}{j^{r(k-1)}}\Big).$$

Since

$$\sum_{j=1}^{\lfloor \sqrt[r]{n} \rfloor} \frac{1}{j^{r(k-1)}} \leq 1 + \int_1^{\sqrt[r]{n}} \frac{dx}{x^{r(k-1)}} = \begin{cases} O(\ln n) & \text{if } r = 1 \text{ and } k = 2 \\ O(n^{1/r}) & \text{if } r \geq 2 \text{ and } k = 1 \\ O(1) & \text{otherwise,} \end{cases}$$

the second term is $\begin{cases} O(n \ln n) & \text{if } r = 1 \text{ and } k = 2 \\ O(n^{1/r}) & \text{if } r \geq 2 \text{ and } k = 1 \\ O(n^{k-1}) & \text{otherwise.} \end{cases}$

Hence, we find that

$$q(n) = \frac{n^k}{\zeta(rk)} + \begin{cases} O(n \ln n) & \text{if } r = 1 \text{ and } k = 2 \\ O(n^{1/r}) & \text{if } r \geq 2 \text{ and } k = 1 \\ O(n^{k-1}) & \text{otherwise.} \end{cases}$$

Finally, with the application of L'Hôpital's Rule, we conclude that

$$\lim_{n\to\infty} \frac{q(n)}{n^k} = \frac{1}{\zeta(rk)}. \blacksquare$$

3 Extensions of Benkoski's Theorem.

3.1 A Hasty Review of Basic Algebraic Number Theory.

One natural question to ask is whether Benkoski's Theorem extends to the set of algebraic integers. In order to formulate this question precisely, we need to review some basic algebraic number theory.

An **algebraic number** (over $\mathbb{Q}$) is a complex number that satisfies a polynomial equation with rational coefficients. Note that the set of all algebraic numbers forms a field. Subfields of this field are known as algebraic number fields. Examples of these include $\mathbb{Q}$ and $\mathbb{Q}[i]$. Next, an **algebraic integer** (over $\mathbb{Z}$) is any number that satisfies a monic polynomial equation with integer coefficients. For example, i is an algebraic integer, as it satisfies the monic quadratic equation $x^2 + 1 = 0$. The set of all algebraic integers forms a ring (and not a field), which we will denote as $\mathbb{A}$. Finally, corresponding to any given algebraic number field K is an **algebraic number ring** (which we will denote by $\mathcal{O}$) defined by $\mathcal{O} = K \cap \mathbb{A}$.

One classic example is the algebraic number ring corresponding to $\mathbb{Q}[i]$, which is the set of Gaussian integers $\mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}\}$. Like $\mathbb{Z}$, $\mathbb{Z}[i]$ also possesses concepts such as divisibility, greatest common divisors, and prime numbers. However, numbers that are prime in $\mathbb{Z}$ may no longer be prime in $\mathbb{Z}[i]$. For example, 2 is not prime in $\mathbb{Z}[i]$, since $2 = -i(1 + i)^2$. In fact, here is how our primes fare in $\mathbb{Z}[i]$:

- $2 = -i(1 + i)^2$ (and so $1 + i$ is a prime in $\mathbb{Z}[i]$).
- If $p \equiv 1 \mod 4$, then $p = \alpha\overline{\alpha}$ for some $\alpha \in \mathbb{Z}[i]$ (and so α and $\overline{\alpha}$ are primes in $\mathbb{Z}[i]$).
- If $p \equiv 3 \mod 4$, then p remains prime in $\mathbb{Z}[i]$.

With our readjusted list of primes, it turns out that $\mathbb{Z}[i]$, like $\mathbb{Z}$ itself, enjoys unique factorisation into prime numbers. Unfortunately, there exist many algebraic number rings that do not enjoy unique factorisation. The simplest example is to note that in $\mathbb{Z}[\sqrt{-5}]$, $6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$, and yet all of the numbers involved in these factorisations are have no nontrivial factors. Can we work around such difficulties?

In order to recapture unique factorisation to some extent, we must pass to *ideals*, sets that are closed under addition and multiplication by (ordinary) integers. Any algebraic number with no nontrivial factorisation over a number ring is usually said to be **irreducible**. (Note that I did not use the use the word prime!) In order to bring primes back into the forefront, recall that an integer p is prime iff whenever $a, b \in \mathbb{Z}$ such that $p|ab$, then $p|a$ or $p|b$. (Over rings possessing unique factorisation, primes and irreducibles are equivalent.) Passing to ideals, we say that an ideal $\mathfrak{p}$ is **prime** iff whenever $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals such that $\mathfrak{ab} \subseteq \mathfrak{p}$, either $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$. With the notion of a prime ideal, we can say (though this takes some effort to prove) that ideals in an algebraic number ring have unique factorisation into ideals. (In the case algebraic rings with unique factorisation of *numbers*, all ideals are principal.) In order to illustrate this, let us return to our example from $\mathbb{Z}[\sqrt{-5}]$, it can be shown that that $\mathfrak{p}_1 = \langle 2, 1+\sqrt{-5}\rangle$, $\mathfrak{p}_2 = \langle 3, 1+\sqrt{-5}\rangle$ and $\mathfrak{p}_3 = \langle 3, 1-\sqrt{-5}\rangle$ are prime ideals. Then, $\langle 2\rangle\langle 3\rangle = (\mathfrak{p}_1^2)(\mathfrak{p}_2\mathfrak{p}_3)$, while $\langle 1+\sqrt{-5}\rangle\langle 1-\sqrt{-5}\rangle = (\mathfrak{p}_1\mathfrak{p}_2)(\mathfrak{p}_1\mathfrak{p}_3)$. Either way, the prime factorisation of $\langle 6\rangle$ is $\mathfrak{p}_1^2\mathfrak{p}_2\mathfrak{p}_3$.

Next, we define the **norm** of an ideal $\mathfrak{a}$ to be $\mathfrak{N}(\mathfrak{a}) := |\mathcal{O}/\mathfrak{a}|$. It can be shown that this is always finite. For example in $\mathbb{Z}[i]$, $\mathfrak{N}(\langle a+bi\rangle) = a^2+b^2$, which is geometrically the square of the distance of the generator to the origin.

Now, define the **Dedekind zeta function** of an algebraic number ring $\mathcal{O}$ as follows:

$$\zeta_{\mathcal{O}}(s) = \sum_{\mathfrak{a}\subseteq\mathcal{O}} \frac{1}{\mathfrak{N}(\mathfrak{a})^s},$$

where the sum is over all nonzero ideals of $\mathcal{O}$. Observe that if $\mathcal{O} = \mathbb{Z}$, then this definition reduces to the Riemann zeta function, with the norm of an ideal (n) (which is always principal in $\mathbb{Z}$) being the familiar absolute value of n. Another way to express the Dedekind zeta function, which will be important to us, is as follows:

$$\zeta_{\mathcal{O}}(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s},$$

where c_n represents the number of ideals with norm n.

As with the Riemann zeta function, the Dedekind zeta function converges for all $s > 1$ and possesses the product expansion

$$\zeta_{\mathcal{O}}(s) = \prod_{\mathfrak{p}\text{ prime}} (1-\mathfrak{N}(\mathfrak{p})^{-s})^{-1}.$$

Define the Möbius function $\mu : \mathcal{O} \to \{0, \pm 1\}$ as

$$\mu(\mathfrak{a}) = \begin{cases} 1 & \text{if } \mathfrak{N}(\mathfrak{a}) = 1 \\ 0 & \text{if } \mathfrak{a} \subseteq \mathfrak{p}^2 \text{ for some prime } \mathfrak{p} \\ (-1)^r & \text{if } \mathfrak{a} = \mathfrak{p}_1\mathfrak{p}_2...\mathfrak{p}_r \text{ for distinct primes } \mathfrak{p}_1, \mathfrak{p}_2, ..., \mathfrak{p}_r, \end{cases}$$

then

$$\sum_{\mathfrak{a}\subseteq\mathcal{O}} \frac{\mu(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^s} = \prod_{\mathfrak{p}\text{ prime}} (1-\mathfrak{N}(\mathfrak{p})^{-s}) = \frac{1}{\zeta_{\mathcal{O}}(s)}.$$

For the details for the proofs of these results, one can see Neukirch or Marcus.

4 Extending Benkoski's Result to the Algebraic Integers

The definition of relative r-primality in $\mathbb{Z}$ naturally extends to that in $\mathcal{O}$ as follows: For a fixed integer $r \geq 1$, we say that the ideals $\mathfrak{a}_1, \mathfrak{a}_2, ..., \mathfrak{a}_k \subseteq \mathcal{O}$ are **relatively r-prime** if $\mathfrak{a}_1, \mathfrak{a}_2, ..., \mathfrak{a}_k \nsubseteq \mathfrak{b}^r$ for any nonzero proper ideal $\mathfrak{b}$.

Theorem 4.1 *Let K be an algebraic number field with $\mathcal{O}$ its algebraic number ring. Fix $k, r \in \mathbb{N}$ not both equal to 1. Let $H(n)$ denote the number of ideals in $\mathcal{O}$ with norm less than or equal to n, and $Q(n)$ denote the number of ordered k-tuples of ideals in $\mathcal{O}$ with norm less than or equal to n that are relatively r-prime. Then,*

$$\lim_{n\to\infty} \frac{Q(n)}{H(n)^k} = \frac{1}{\zeta_{\mathcal{O}}(rk)}.$$

Proof:
Fix $n \in \mathbb{N}$. An ordered k-tuple of ideals $(\mathfrak{a}_1, \mathfrak{a}_2, ..., \mathfrak{a}_k)$ is relatively r-prime if and only if there exists no prime ideal $\mathfrak{p}$ such that $\mathfrak{a}_1, \mathfrak{a}_2, ..., \mathfrak{a}_k \subseteq \mathfrak{p}^r$, the Inclusion-Exclusion Principle shows that $Q(n)$ may be written as

$$Q(n) = H(n)^k - \sum_{\mathfrak{p}_1} H\Big(\frac{n}{\mathfrak{N}(\mathfrak{p}_1^r)}\Big)^k + \sum_{\mathfrak{p}_1,\mathfrak{p}_2} H\Big(\frac{n}{\mathfrak{N}((\mathfrak{p}_1\mathfrak{p}_2)^r)}\Big)^k - ...$$

where $\mathfrak{p}_1$, $\mathfrak{p}_2$, ... denote distinct prime ideals with norm less than or equal to n. Moreover, the second sum is on pairs of distinct prime ideals not counting repetitions, et cetera. Using the Möbius function μ permits us to rewrite this sum more compactly as

$$Q(n) = \sum_{\mathfrak{a}} \mu(\mathfrak{a})\, H\Big(\frac{n}{\mathfrak{N}(\mathfrak{a}^r)}\Big)^k.$$

This sum actually ranges over all nonzero ideals $\mathfrak{a}$ such that $\mathfrak{N}(\mathfrak{a}) \leq \lfloor \sqrt[r]{n} \rfloor$ since the summands corresponding to the other indices are annihilated by H.

In order to estimate H, we use the fact that there exists a positive constant c such that $H(n) = cn + O(n^{1-\epsilon})$ where $\epsilon = [K : \mathbb{Q}]^{-1}$ (see Marcus). If $K = \mathbb{Q}$, then H reduces to the floor function, and the above estimate follows suit.

Directly applying this estimate to $Q(n)$ and applying the Binomial Theorem, we see that

$$Q(n) = (cn)^k \sum_{\mathfrak{a}} \frac{\mu(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^{rk}} + \sum_{\mathfrak{a}} \mu(\mathfrak{a}) \Big(\frac{cn}{\mathfrak{N}(\mathfrak{a})^r}\Big)^{k-1} O\Big(\frac{n}{\mathfrak{N}(\mathfrak{a})^r}\Big)^{1-\epsilon}.$$

Now, we must estimate how fast each sum of $Q(n)$ grows. For the first sum of $Q(n)$, note that

$$\sum_{\substack{\mathfrak{a}\subseteq\mathcal{O}\\ \mathfrak{N}(\mathfrak{a})\le\lfloor\sqrt[r]{n}\rfloor}} \frac{\mu(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^{rk}} = \frac{1}{\zeta_{\mathcal{O}}(rk)} - \sum_{\substack{\mathfrak{a}\subseteq\mathcal{O}\\ \mathfrak{N}(\mathfrak{a})>\lfloor\sqrt[r]{n}\rfloor}} \frac{\mu(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^{rk}}.$$

In order to estimate the growth of the second sum, we use the fact that the number of ideals with a fixed norm n is given by $H(n) - H(n-1) = O(n^{1-\epsilon})$. Applying this fact yields

$$\sum_{\substack{\mathfrak{a}\subseteq\mathcal{O}\\ \mathfrak{N}(\mathfrak{a})>\lfloor\sqrt[r]{n}\rfloor}} \frac{\mu(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^{rk}} \le \int_{\sqrt[r]{n}}^{\infty} \frac{cx^{1-\epsilon}}{x^{rk}}\,dx = O(n^{(2-\epsilon-rk)/r}).$$

Hence, the first sum of $Q(n)$ is $\dfrac{(cn)^k}{\zeta_{\mathcal{O}}(rk)} + O(n^{(2-\epsilon)/r})$.

As for the second sum of $Q(n)$, we need to estimate the sum $\displaystyle\sum_{\mathfrak{a}} O\Big(\frac{n^{k-\epsilon}}{\mathfrak{N}(\mathfrak{a})^{r(k-\epsilon)}}\Big)$.

Again by using our norm estimate, we see that

$$\sum_{\mathfrak{a}} \frac{n^{k-\epsilon}}{\mathfrak{N}(\mathfrak{a})^{r(k-\epsilon)}} \le n^{k-\epsilon} \sum_{j=1}^{\lfloor\sqrt[r]{n}\rfloor} \frac{cj^{1-\epsilon}}{j^{r(k-\epsilon)}} \le n^{k-\epsilon}\Big(1 + \int_1^{\sqrt[r]{n}} \frac{dx}{x^{r(k-1)}}\Big).$$

Estimating this integral proceeds as in the integer case, and we find (after leaving the tedious details to the reader) that the second sum of $Q(n)$ is

$$\begin{cases} O(n^{k-\epsilon}) & \text{if } k>2, \text{ or } k=2 \text{ and } r\ge 2\\ O(n^{2-\epsilon}\ln n) & \text{if } k=2 \text{ and } r=1\\ O(n^{1-\epsilon}\ln n) & \text{if } k=1 \text{ and } \epsilon=\frac{r-2}{r-1}\\ O(n^{1-\epsilon}) & \text{if } k=1 \text{ and } \epsilon<\frac{r-2}{r-1}\\ O(n^{(2-r-\epsilon+r\epsilon)/r}) & \text{if } k=1 \text{ and } \epsilon>\frac{r-2}{r-1} \end{cases}$$

Hence, we find that

$$Q(n) = \frac{(cn)^k}{\zeta_{\mathcal{O}}(rk)} + \begin{cases} O(n^{k-\epsilon}) & \text{if } k>2, \text{ or } k=2 \text{ and } r\ge 2\\ O(n^{2-\epsilon}\ln n) & \text{if } k=2 \text{ and } r=1\\ O(n^{1-\epsilon}\ln n) & \text{if } k=1 \text{ and } \epsilon=\frac{r-2}{r-1}\\ O(n^{1-\epsilon}) & \text{if } k=1 \text{ and } \epsilon<\frac{r-2}{r-1}\\ O(n^{(2-\epsilon)/r}) & \text{if } k=1 \text{ and } \epsilon>\frac{r-2}{r-1} \end{cases}$$

Finally with this growth estimate, we may conclude that

$$\lim_{n\to\infty}\frac{Q(n)}{H(n)^k} = \lim_{n\to\infty}\frac{Q(n)/n^k}{(H(n)/n)^k} = \frac{c^k\zeta_{\mathcal{O}}(rk)^{-1}+0}{c^k} = \frac{1}{\zeta_{\mathcal{O}}(rk)}. \blacksquare$$

Remark: Note that this theorem reduces to Benkoski's Theorem for ordinary integers if we let $K=\mathbb{Q}$ right down to the growth estimates.

5 Proving Benkoski's Result using the Dominated Convergence Theorem.

We now prove the previous Benkoski results without deriving the error terms. In order to do this, we use the following standard theorem (without proof).

Theorem 5.1 ***(Dominated Convergence Theorem for Sums)***
Suppose that the following conditions are true:

1. *The family of series* $s_n := \sum_{j=1}^{\infty} a_j^{(n)}$ *is dominated by a convergent series of nonnegative terms* $\sum_{j=1}^{\infty} b_j$

2. $a_j := \lim_{n\to\infty} a_j^{(n)}$ *exists for each* j.

Then, $s := \lim_{n\to\infty} s_n$ *exists, and* $\lim_{n\to\infty} \sum_{j=1}^{\infty} a_j^{(n)} = \sum_{j=1}^{\infty} a_j = s$.

Proof of Benkoski's Theorem:
From our previous discussion, we know that $q(n) = \sum_{j=1}^{\infty} \mu(j) \left\lfloor \frac{n}{j^r} \right\rfloor^k$.
Defining $P(n)$ as the probability that an ordered k-tuple of (positive) integers with each entry being less than or equal to n does not have a common rth power, we see that

$$P(n) = \frac{q(n)}{n^k} = \sum_{j=1}^{\infty} \frac{\mu(j)}{n^k} \left\lfloor \frac{n}{j^r} \right\rfloor^k.$$

In order to apply the Dominated Convergence Theorem, first observe that for each n, $P(n)$ is bounded above by the convergence series $\sum_{j=1}^{\infty} \frac{1}{n^k} \left(\frac{n}{j^r}\right)^k = \zeta(rk)$.
Moreover, since $\frac{n}{j^r} - 1 \leq \left\lfloor \frac{n}{j^r} \right\rfloor \leq \frac{n}{j^r}$, applying the Squeeze Law of Limits permits us to deduce that $\lim_{n\to\infty} \frac{1}{n^k} \left\lfloor \frac{n}{j^r} \right\rfloor^k = \frac{1}{j^{rk}}$.
Hence, the Dominated Convergence Theorem for Sums applies, and we obtain

$$\lim_{n\to\infty} P(n) = \sum_{j=1}^{\infty} \frac{\mu(j)}{j^{rk}} = \frac{1}{\zeta(rk)}$$

which completes the proof. ■

Proof of Benkoski's result extended to $\mathcal{O}$:
From our previous discussion, we know that $Q(n) = \sum_{\mathfrak{a}} \mu(\mathfrak{a})\, H\Big(\frac{n}{\mathfrak{N}(\mathfrak{a}^r)}\Big)^k$.
Defining $P(n)$ as the probability that an ordered k-tuple of ideals does not have a common rth power, with the norm of each entry being less than or equal to n, we see that

$$P(n) = \frac{Q(n)}{H(n)^k} = \sum_{\mathfrak{a}} \frac{\mu(\mathfrak{a})}{H(n)^k}\, H\Big(\frac{n}{\mathfrak{N}(\mathfrak{a}^r)}\Big)^k.$$

In order to apply the Dominated Convergence Theorem, first observe that for each n, $P(n)$ is bounded above by the convergence series $A\zeta_{\mathcal{O}}(rk)$ for some positive constant A. To show this, we use the estimate $H(n) = cn + O(n^{1-\epsilon})$ for some positive constants c and ϵ. From this, we find that there exists $D > 0$ such that for all $n \in \mathbb{N}$

$$\frac{H(\frac{n}{\mathfrak{N}(\mathfrak{a})^r})}{H(n)} \leq \frac{D}{\mathfrak{N}(\mathfrak{a})^r}.$$

(By the estimate for $H(n)$, this is true for sufficiently large n. Then, we rescale D as needed to make this inequality true for all $n \in \mathbb{N}$.)
Therefore, we have

$$P(n) \leq \sum_{\mathfrak{a}} \frac{1}{H(n)^k}\, H\Big(\frac{n}{\mathfrak{N}(\mathfrak{a}^r)}\Big)^k \leq D\zeta_{\mathcal{O}}(rk),$$

where M is a positive constant determined from the upper bound for the growth estimate of $H(n)$.

Next, fix $b \in \mathbb{N}$. Since $H(n) = cn + O(n^{1-\epsilon})$, there exists $K > 0$ such that

$$\Big| H\Big(\frac{n}{b}\Big) - \frac{cn}{b}\Big| \leq Kn^{1-\epsilon}.$$

Dividing both sides by n and applying the Squeeze Law of Limits as $n \to \infty$ yields $\lim_{n\to\infty} \frac{H(\frac{n}{b})}{n} = \frac{c}{b}$.

Hence, the Dominated Convergence Theorem for Sums applies, and we obtain

$$\lim_{n\to\infty} P(n) = \sum_{\mathfrak{a}} \frac{\mu(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^{rk}} = \frac{1}{\zeta_{\mathcal{O}}(rk)}$$

which completes the proof. ■

6 Further Results

Benkoski also investigated probabilities that a randomly chosen k-tuple of integers is allowed to have a common rth power m^r where m is an integer whose factors arise from a specific list of prime numbers S (or its complement $\overline{S}$) as long as S (or $\overline{S}$) is finite. Not only do we show that the finiteness condition is unnecessary, we will prove this in the case of the ring of algebraic integers $\mathcal{O}$.

Before stating the result, let's fix some notation. S will denote a fixed subset of distinct prime ideals, and $\overline{S}$ will denote the set of prime ideals not in S. Moreover, $\langle\overline{S}\rangle$ represents all ideals which arise as products of (not necessarily distinct) elements from $\mathcal{O} \cup \overline{S}$.

Theorem 6.1 *Fix $k, r \in \mathbb{N}$ not both equal to 1 and a set of prime ideals S. Let $Q(n, S)$ denote the number of ordered k-tuples of ideals in $\mathcal{O}$ with norm less than or equal to n that is have a common rth power of a prime ideal $\mathfrak{p} \in S$ Then,*

$$\lim_{n\to\infty} \frac{Q(n,S)}{H(n)^k} = \sum_{\mathfrak{a}\subseteq\langle\overline{S}\rangle} \frac{\mu(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^{rk}}.$$

Proof:
Remarkably proving this is no more difficult than the result from the previous section. I will only highlight the essential details to prove this result below.

Fix $n \in \mathbb{N}$, and let $\mathfrak{p}_1$, $\mathfrak{p}_2$, ... denote distinct prime ideals with norm less than or equal to n in $\overline{S}$. Then, applying the Inclusion-Exclusion Principle as before, we find that $Q(n, S)$ can be written (using the Möbius function μ) as

$$Q(n,S) = \sum_{\mathfrak{a}\subseteq\langle\overline{S}\rangle} \mu(\mathfrak{a})\, H\Big(\frac{n}{\mathfrak{N}(\mathfrak{a}^r)}\Big)^k.$$

Proving this using growth estimates proceeds as before, as the previous integral estimates still apply since $\langle\overline{S}\rangle$ is a subset of $\mathcal{O}$. We thus obtain

$$Q(n,S) = (cn)^k \cdot \sum_{\mathfrak{a}\subseteq\langle\overline{S}\rangle} \frac{\mu(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^{rk}} + \begin{cases} O(n^{k-\epsilon}) & \text{if } k>2, \text{ or } k=2 \text{ and } r\geq 2 \\ O(n^{2-\epsilon}\ln n) & \text{if } k=2 \text{ and } r=1 \\ O(n^{1-\epsilon}\ln n) & \text{if } k=1 \text{ and } \epsilon=\frac{r-2}{r-1} \\ O(n^{1-\epsilon}) & \text{if } k=1 \text{ and } \epsilon<\frac{r-2}{r-1} \\ O(n^{(2-\epsilon)/r}) & \text{if } k=1 \text{ and } \epsilon>\frac{r-2}{r-1} \end{cases}$$

Finally with this growth estimate, we may conclude that $\lim_{n\to\infty} \dfrac{Q(n,S)}{H(n)^k} = \displaystyle\sum_{\mathfrak{a}\subseteq\langle\overline{S}\rangle} \frac{\mu(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^{rk}}$,

as desired. ■

To make an analogy with the classic zeta functions, let us define

$$\zeta_{\mathcal{O},S}(s) = \sum_{\mathfrak{a} \subseteq \langle \overline{S} \rangle} \frac{1}{\mathfrak{N}(\mathfrak{a}^s)}.$$

Note that if we let $S = \varnothing$, then we $\zeta_{\mathcal{O},S}(s) = \zeta_{\mathcal{O}}(s)$. It is easy to see that this series converges for all $s > 1$. Since $\langle \overline{S} \rangle$ is multiplicative, we see that

$$\zeta_{\mathcal{O},S}(s) = \prod_{\mathfrak{p} \subseteq \overline{S}} (1 - \mathfrak{N}(\mathfrak{p})^{-s})^{-1}.$$

Thus,

$$\sum_{\mathfrak{a} \subseteq \langle \overline{S} \rangle} \frac{\mu(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^{rk}} = \prod_{\mathfrak{p} \subseteq \overline{S}} (1 - \mathfrak{N}(\mathfrak{p})^{-s}) = \frac{1}{\zeta_{\mathcal{O},S}(s)}.$$

Consequently, we may restate the conclusion of the above theorem as

$$\lim_{n \to \infty} \frac{Q(n,S)}{H(n)^k} = \frac{1}{\zeta_{\mathcal{O},S}(s)}.$$

7 The Pairwise Relatively Prime case

In this section, we now consider the problem of finding the probability that k randomly chosen integers are pairwise relatively prime. To do this, we alter our approach and directly establish the product representation of the probability. We will first reprove Benkoski's result in this manner.

Theorem 7.1 *Fix $t \in \mathbb{N}$. The probability that an ordered k-tuple of positive integers which are relatively r-prime for all primes $p \leq t$ equals* $\prod_{p \leq t} \left(1 - \frac{1}{p^{rk}}\right)$.

Proof: First, we fix a prime $p \leq t$ and a positive integer n. Then, the probability that an ordered k-tuple of positive integers with each entry at most n has no common factor p^r equals $\frac{n^k - \lfloor \frac{n}{p^r} \rfloor^k}{n^k} = 1 - \frac{1}{n^k} \cdot \left\lfloor \frac{n}{p^r} \right\rfloor^k$. Letting $n \to \infty$, the Squeeze Theorem implies that the probability that an ordered k-tuple of positive integers has no common factor p^r equals $1 - \frac{1}{p^{rk}}$. The claim now follows from the Chinese Remainder Theorem. ■

Proof of Benkoski's Theorem (again): By the previous theorem, it suffices to bound the error of the fraction of k-tuples of positive integers with each entry at most n is divisible by p^r for some prime $p > t$. More precisely, we need to show that

$$\lim_{t \to \infty} \lim_{n \to \infty} \frac{\#\{(a_1, ..., a_k) \in (\mathbb{N} \cap [1, n])^k : p^r \mid a_1, ..., a_k \text{ for some } p > t\}}{n^k} = 0.$$

To this end, note that
$\#\{(a_1, ..., a_k) \in (\mathbb{N} \cap [1, n])^k : p^r \mid a_1, ..., a_k \text{ for some } p > t\}$

$$\begin{aligned}
&\leq \sum_{p > t} \#\{(a_1, ..., a_k) \in (\mathbb{N} \cap [1, n])^k : p^r \mid a_1, ..., a_k\}, \text{ by Inclusion-Exclusion} \\
&\leq \sum_{p > t} \left\lfloor \frac{n}{p^r} \right\rfloor^k \\
&\leq \sum_{m > t} \left(\frac{n}{m^r}\right)^k \\
&\leq n^k \cdot \int_t^\infty \frac{1}{x^{rk}} \, dx, \text{ via Riemann sums} \\
&= \frac{n^k}{(rk - 1) t^{rk-1}}.
\end{aligned}$$

So, $\frac{\#\{(a_1, ..., a_k) \in (\mathbb{N} \cap [1, n])^k : p^r \mid a_1, ..., a_k \text{ for some } p > t\}}{n^k} = \frac{1}{(rk - 1) t^{rk-1}}$.
Now, letting $n \to \infty$ followed by $t \to \infty$ shows that the error term goes to 0. Hence, the desired probability equals $\lim_{t \to \infty} \prod_{p \leq t} \left(1 - \frac{1}{p^{rk}}\right) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^{rk}}\right)$. ■

Now, we consider the probability that k randomly chosen positive integers are pairwise relatively r-prime. We will actually prove a bit more.

For a fixed integer $r \geq 1$, we say that the integers $m_1, m_2, ..., m_k$ are **pairwise relatively r-prime** if any pair of these integers have no common factor of the form n^r for any integer $n > 1$. When $r = 1$, this is the definition of being pairwise relatively prime.

Theorem 7.2 *Fix $k, r \in \mathbb{N}$ not both equal to 1, and let $q(n)$ denote the number of ordered k-tuples of positive integers less than or equal to n that are pairwise relatively r-prime. Then,*

$$\lim_{n\to\infty} \frac{q(n)}{n^k} = \prod_{p\ prime} \Big(1 - \frac{1}{p^r}\Big)^{k-1} \cdot \Big(1 + \frac{k-1}{p^r}\Big).$$

Proof: First, we fix positive integers t and n, and consider a prime $p \leq t$. Then, the probability that an ordered k-tuple of positive integers in which each pair of entries at most n has no common factor p^r equals

$$\frac{\Big(n - \lfloor\frac{n}{p^r}\rfloor\Big)^k + k\lfloor\frac{n}{p^r}\rfloor\Big(n - \lfloor\frac{n}{p^r}\rfloor\Big)^{k-1}}{n^k} = \Big(1-\frac{1}{n}\Big\lfloor\frac{n}{p^r}\Big\rfloor\Big)^k + k\cdot\Big(\frac{1}{n}\cdot\Big\lfloor\frac{n}{p^r}\Big\rfloor\Big)\cdot\Big(1-\frac{1}{n}\Big\lfloor\frac{n}{p^r}\Big\rfloor\Big)^{k-1}.$$

Observe that the first term deals with the case that no entry is divisible by p^r, while the second term deals with the case that exactly one entry is divisible by p^r.

Letting $n \to \infty$, the Squeeze Theorem implies that the probability that an ordered k-tuple of positive integers in which each pair of entries has no common factor p^r equals $\Big(1 - \frac{1}{p^r}\Big)^{k-1} \cdot \Big(1 + \frac{k-1}{p^r}\Big)$.

By the Chinese Remainder Theorem, the probability that an ordered k-tuple of positive integers in which each pair of entries at most n has no common factor p^r for some prime $p \leq t$ equals $\displaystyle\prod_{p\leq t} \Big(1 - \frac{1}{p^r}\Big)^{k-1} \cdot \Big(1 + \frac{k-1}{p^r}\Big)$.

In order to prove the theorem, it suffices by the previous assertion to bound the error of the fraction of k-tuples of positive integers in which each pair of entries has no common factor p^r for some prime $p > t$. More precisely, we need to show that

$$\lim_{t\to\infty}\lim_{n\to\infty} \frac{\#\{(a_1, ..., a_k) \in (\mathbb{N} \cap [1, n])^k : p^r \mid a_i, a_j \text{ for some } p > t, 1 \leq i < j \leq k\}}{n^k} = 0.$$

To this end, note that
$\#\{(a_1, ..., a_k) \in (\mathbb{N} \cap [1, n])^k : p^r \mid a_i, a_j \text{ for some } p > t, 1 \leq i < j \leq k\}$

$$\leq \sum_{p>t} \#\{(a_1, ..., a_k) \in (\mathbb{N} \cap [1, n])^k : p^r \mid a_i, a_j \text{ for some } 1 \leq i < j \leq k\}$$

$$\leq \sum_{p>t} \binom{k}{2} \left\lfloor \frac{n}{p^r} \right\rfloor^2 \cdot n^{k-2}$$
$$\leq \sum_{m>t} \binom{k}{2} \left(\frac{n}{m^r}\right)^2 \cdot n^{k-2}$$
$$\leq \tbinom{k}{2} n^k \cdot \int_t^\infty \frac{1}{x^{2r}}\, dx, \text{ via Riemann sums}$$
$$= \frac{\binom{k}{2} n^k}{(2r-1)t^{2r-1}}.$$

So, $\dfrac{\#\{(a_1, ..., a_k) \in (\mathbb{N} \cap [1,n])^k : p^r \mid a_i, a_j \text{ for some } p > t, 1 \leq i < j \leq k\}}{n^k} = \dfrac{\binom{k}{2}}{(2r-1)t^{2r-1}}$.
Now, letting $n \to \infty$ followed by $t \to \infty$ shows that the error term goes to 0. Hence, the desired probability equals

$$\lim_{t\to\infty} \prod_{p \leq t} \left(1 - \frac{1}{p^r}\right)^{k-1} \cdot \left(1 + \frac{k-1}{p^r}\right) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^r}\right)^{k-1} \cdot \left(1 + \frac{k-1}{p^r}\right). \blacksquare$$

Next, we will extend this result to an algebraic ring of integers. The definition of pairwise relative r-primality in $\mathbb{Z}$ naturally extends to that in $\mathcal{O}$ as follows: For a fixed integer $r \geq 1$, we say that the ideals $\mathfrak{a}_1, \mathfrak{a}_2, ..., \mathfrak{a}_k \subseteq \mathcal{O}$ are **pairwise relatively r-prime** if for any distinct $i, j \in \{1, 2, ..., k\}$, we have $\mathfrak{a}_i, \mathfrak{a}_j \nsubseteq \mathfrak{b}^r$ for any nonzero proper ideal $\mathfrak{b}$.

Theorem 7.3 *Let K be an algebraic number field with $\mathcal{O}$ its algebraic number ring. Fix $k, r \in \mathbb{N}$ not both equal to 1. Let $H(n)$ denote the number of ideals in $\mathcal{O}$ with norm less than or equal to n, and $Q(n)$ denote the number of ordered k-tuples of ideals in $\mathcal{O}$ with norm less than or equal to n that are pairwise relatively r-prime. Then,*

$$\lim_{n\to\infty} \frac{Q(n)}{H(n)^k} = \prod_{\mathfrak{p} \text{ prime}} \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})^r}\right)^{k-1} \cdot \left(1 + \frac{k-1}{\mathfrak{N}(\mathfrak{p})^r}\right).$$

Proof: First, we fix positive integers t and n, and consider a prime ideal $\mathfrak{p}$ such that $\mathfrak{N}(\mathfrak{p}) \leq t$. Then, the probability that an ordered k-tuple of ideals in which each pair of entries with norm at most n has no common factor $\mathfrak{p}^r$ equals

$$\frac{\left(H(n) - H(\frac{n}{\mathfrak{N}(\mathfrak{p}^r)})\right)^k + kH(\frac{n}{\mathfrak{N}(\mathfrak{p}^r)})\left(H(n) - H(\frac{n}{\mathfrak{N}(\mathfrak{p}^r)})\right)^{k-1}}{H(n)^k}$$

$$= \left(1 - \frac{1}{H(n)} H(\frac{n}{\mathfrak{N}(\mathfrak{p})^r})\right)^k + k \cdot \left(\frac{1}{H(n)} \cdot H(\frac{n}{\mathfrak{N}(\mathfrak{p})^r})\right) \cdot \left(1 - \frac{1}{H(n)} H(\frac{n}{\mathfrak{N}(\mathfrak{p})^r})\right)^{k-1}.$$

Observe that the first term deals with the case that no entry is divisible by $\mathfrak{p}^r$, while the second term deals with the case that exactly one entry is divisible by $\mathfrak{p}^r$.

However, since $H(n) = cn+O(n^{1-\epsilon})$, there exists a constant $M > 0$ such that for sufficiently large n and any constant $b > 0$, we have $|H(\frac{n}{b})-\frac{cn}{b}| \leq Mn^{1-\epsilon}$. Then, the Squeeze Theorem implies that $\lim\limits_{n\to\infty} \dfrac{H(\frac{n}{b})}{n} = \dfrac{c}{b}$. Therefore, the probability that an ordered k-tuple of ideals in which each pair of entries has no common factor $\mathfrak{p}^r$ equals

$$\Big(1-\frac{1}{\mathfrak{N}(\mathfrak{p})^r}\Big)^k + k\cdot\frac{1}{\mathfrak{N}(\mathfrak{p})^r}\cdot\Big(1-\frac{1}{\mathfrak{N}(\mathfrak{p})^r}\Big)^{k-1} = \Big(1-\frac{1}{\mathfrak{N}(\mathfrak{p})^r}\Big)^{k-1}\cdot\Big(1+\frac{k-1}{\mathfrak{N}(\mathfrak{p})^r}\Big).$$

By the Chinese Remainder Theorem, the probability that an ordered k-tuple of ideals in which each pair of entries at most n has no common factor $\mathfrak{p}^r$ where $\mathfrak{N}(\mathfrak{p}) \leq t$ equals $\displaystyle\prod_{\mathfrak{N}(\mathfrak{p})\leq t}\Big(1-\frac{1}{\mathfrak{N}(\mathfrak{p})^r}\Big)^{k-1}\cdot\Big(1+\frac{k-1}{\mathfrak{N}(\mathfrak{p})^r}\Big)$.

To conclude the proof of this theorem, it suffices by the last assertion to bound the error of the fraction of k-tuples of ideals in which each pair of entries has no common factor $\mathfrak{p}^r$ such that $\mathfrak{N}(\mathfrak{p}) > t$. More precisely, we need to show that

$$\lim_{t\to\infty}\lim_{n\to\infty}\frac{\#\{(\mathfrak{a}_1,...,\mathfrak{a}_k)\in\mathcal{O}^k : \mathfrak{N}(\mathfrak{a}_i)\leq n, \mathfrak{p}^r \mid a_i, a_j \text{ for some } \mathfrak{p}, \mathfrak{N}(\mathfrak{p}) > t, 1\leq i<j\leq k\}}{H(n)^k} = 0.$$

To this end, note that for sufficiently large n, t, we have

$$\begin{aligned}
&\frac{\#\{(\mathfrak{a}_1,...,\mathfrak{a}_k)\in\mathcal{O}^k : \mathfrak{N}(\mathfrak{a}_i)\leq n, \mathfrak{p}^r \mid a_i, a_j \text{ for some } \mathfrak{p}, \mathfrak{N}(\mathfrak{p}) > t, 1\leq i<j\leq k\}}{H(n)^k}\\
&\leq \frac{1}{H(n)^k}\cdot\sum_{\mathfrak{N}(\mathfrak{p})>t}\#\{(\mathfrak{a}_1,...,\mathfrak{a}_k)\in\mathcal{O}^k : \mathfrak{N}(\mathfrak{a}_i)\leq n, \mathfrak{p}^r \mid a_i, a_j\ , 1\leq i<j\leq k\}\\
&\leq \frac{1}{H(n)^k}\cdot\sum_{\mathfrak{N}(\mathfrak{p})>t}\binom{k}{2}H\Big(\frac{n}{\mathfrak{N}(\mathfrak{p})^r}\Big)^2\cdot H(n)^{k-2}\\
&\leq \sum_{\mathfrak{N}(\mathfrak{b})>t}\binom{k}{2}\frac{1}{H(n)^2}\cdot H\Big(\frac{n}{\mathfrak{N}(\mathfrak{b})^r}\Big)^2 \text{ for any nonzero } \mathfrak{b}\subseteq\mathcal{O}.
\end{aligned}$$

However, since $H(n) = cn + O(n^{1-\epsilon})$ for some positive constants c and ϵ, there exists $A > 0$ such that for sufficiently large n we have $|H(n) - cn| \leq An^{1-\epsilon}$. Therefore, for sufficiently large n, t there exists a constant $D > 0$ such that

$$\frac{H(\frac{n}{\mathfrak{N}(\mathfrak{b})^r})}{H(n)} \leq \frac{\frac{cn}{\mathfrak{N}(\mathfrak{b})^r} + A(\frac{n}{\mathfrak{N}(\mathfrak{b})^r})^{1-\epsilon}}{cn - An^{1-\epsilon}} \leq \frac{D}{\mathfrak{N}(\mathfrak{b})^r}.$$

(For the last inequality, we used the fact that the numerator and denominator are both $\Theta(n)$).) So, the error is bounded above by

$$\sum_{\mathfrak{N}(\mathfrak{b})>t}\binom{k}{2}D^2\cdot\frac{1}{\mathfrak{N}(\mathfrak{b})^{2r}} \leq \binom{k}{2}D^2\cdot\int_t^\infty\frac{1}{x^{2r}}\,dx = \frac{\binom{k}{2}D^2}{(2r-1)t^{2r-1}}.$$

Now, letting $n \to \infty$ followed by $t \to \infty$ shows that the error term goes to 0. Hence, the desired probability equals

$$\lim_{t\to\infty} \prod_{\mathfrak{N}(\mathfrak{p})\leq t} \Big(1-\frac{1}{\mathfrak{N}(\mathfrak{p})^r}\Big)^{k-1} \cdot \Big(1+\frac{k-1}{\mathfrak{N}(\mathfrak{p})^r}\Big) = \prod_{\mathfrak{p}\text{ prime}} \Big(1-\frac{1}{\mathfrak{N}(\mathfrak{p})^r}\Big)^{k-1} \cdot \Big(1+\frac{k-1}{\mathfrak{N}(\mathfrak{p})^r}\Big). \blacksquare$$

As before, if we set $\mathcal{O} = \mathbb{Z}$, then this reduces to the result about the probability that k integers being pairwise relatively prime. This can be readily generalised in a couple of ways. One way is to consider a set of prime ideals S as we did with Benkoski's result. Another generalisation of this result is to consider a k-tuple being j-wise relatively r-prime (that is, now any j of the k entries are relatively r-prime; note that the case of being pairwise relatively r-prime is when $j = 2$). Here are the results without proof (as the above proof can be readily adapted as necessary).

Theorem 7.4 *Let K be an algebraic number field with $\mathcal{O}$ its algebraic number ring. Fix $k, r \in \mathbb{N}$ not both equal to 1 and a set of prime ideals S. Let $H(n)$ denote the number of ideals in $\mathcal{O}$ with norm less than or equal to n, and $Q(n, S)$ denote the number of ordered k-tuples of ideals in $\mathcal{O}$ with norm less than or equal to n that are pairwise relatively r-prime. Then,*

$$\lim_{n\to\infty} \frac{Q(n,S)}{H(n)^k} = \prod_{\mathfrak{p}\subseteq \overline{S}} \Big(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})^r}\Big)^{k-1} \cdot \Big(1 + \frac{k-1}{\mathfrak{N}(\mathfrak{p})^r}\Big).$$

Theorem 7.5 *Let K be an algebraic number field with $\mathcal{O}$ its algebraic number ring. Fix $j, k, r \in \mathbb{N}$, where k and r are not both equal to 1 and $j < k$. Let $H(n)$ denote the number of ideals in $\mathcal{O}$ with norm less than or equal to n, and $Q(n)$ denote the number of ordered k-tuples of ideals in $\mathcal{O}$ with norm less than or equal to n that are j-wise relatively r-prime. Then,*

$$\lim_{n\to\infty} \frac{Q(n)}{H(n)^k} = \prod_{\mathfrak{p}\ prime} \Big[\sum_{m=0}^{j-1} \binom{k}{m} \Big(\frac{1}{\mathfrak{N}(\mathfrak{p}^r)}\Big)^m \Big(1 - \frac{1}{\mathfrak{N}(\mathfrak{p}^r)}\Big)^{k-m}\Big].$$