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The probability that random algebraic integers are relatively r-prime

Brian D. Sittinger

CSU Channel Islands, Department of Mathematics, 1 University Drive, Camarillo, CA 93012, United States

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ABSTRACT

Benkoski (1976) [1] proved that the probability that k randomly chosen integers do not have a nontrivial common rth power is $1/\zeta(rk)$. We first give a more concise proof of this result before proceeding to establish its analogue in the ring of algebraic integers.

1. Introduction

One branch of analytic number theory concerns itself with the distribution of integers with certain special properties. The prototypical result in this field is Dirichlet's Theorem for primes in an arithmetic progression: For any two positive relatively prime integers a and d, there are infinitely many primes of the form a mod d. In 1896, Hadamard and de la Vallée Poussin proved the Prime Number Theorem: The probability of randomly picking a positive integer less than or equal to n that is prime is asymptotically equal to $1/\ln n$.

Unlike these well-known results, the following two fascinating facts about the integers are not as notorious: (1) The probability that an integer is squarefree is $6/\pi^2$, and (2) the probability that two integers are relatively prime is $6/\pi^2$. These results predate the Prime Number Theorem. Gegenbauer proved the first in 1885, while Mertens proved the second in 1874.

These last two facts have natural generalisations that were also investigated around the same time. In 1885, Gegenbauer proved the more general result that the probability that an integer is not divisible by an rth power, with $r \geqslant 2$, is $1/\zeta(r)$. (Note that $1/\zeta(2) = 6/\pi^2$.) In 1900, Lehmer extended Mertens' result to show that the probability that k integers are relatively prime is $1/\zeta(k)$.

Given the short period of time that it took to establish the probabilities for the relative primality of k integers and for an integer to be rth power free, it may be surprising that not until 1976 did Benkoski prove a result that combines both lines of thought by showing that the probability that

E-mail address: brian.sittinger@csuci.edu.

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k integers do not have a common rth power is $1/\zeta(rk)$. While his proof uses a generalisation of Jordan totient functions, we will establish this result more concisely by following the methods of Nymann [6].

Related probabilistic questions have been subsequently considered in other PID's. In particular, Morrison and Dong [4] proved an analogue of Benkoski's statement for the ring $\mathbb{F}_q[x]$ in 2004. For the case of the algebraic integers, all that has been published is the case of relative primality [2]. We will remedy this situation by proving a full generalisation of Benkoski's result for a ring of algebraic integers.

2. Benkoski's Theorem

For a fixed integer $r \geqslant 1$, we say that the integers $m_1, m_2, \dots, m_k$ are **relatively r-prime** if they have no common factor of the form n^r for any integer $n > 1$. When $r = 1$, this is the definition of being relatively prime.

Theorem 2.1. *Fix $k, r \in \mathbb{N}$ not both equal to 1, and let $q(n)$ denote the number of ordered k-tuples of positive integers less than or equal to n that are relatively r-prime. Then,*

$$\lim_{n\to\infty} \frac{q(n)}{n^k} = \frac{1}{\zeta(rk)}.$$

Proof. Fix $n \in \mathbb{N}$. Since an ordered k-tuple of integers is relatively r-prime if and only if there exists no prime p such that p^r divides all k integers, the Inclusion–Exclusion Principle shows that $q(n)$ may be written as

$$q(n) = n^k - \sum_{p_1} \left\lfloor \frac{n}{p_1^r} \right\rfloor^k + \sum_{p_1<p_2} \left\lfloor \frac{n}{(p_1 p_2)^r} \right\rfloor^k - \sum_{p_1<p_2<p_3} \left\lfloor \frac{n}{(p_1 p_2 p_3)^r} \right\rfloor^k + \cdots,$$

where $p_1, p_2, \dots$ denote distinct primes less than or equal to n. (Note that these sums terminate, since n is fixed.) By using the Möbius function μ, we may rewrite this sum more compactly as

$$q(n) = \sum_{j=1}^{\infty} \mu(j) \left\lfloor \frac{n}{j^r} \right\rfloor^k.$$

However, we do not need the infinity at the upper end of the summation. In fact, note that the floor functions in the summands annihilate terms with indices $j > \lfloor \sqrt[r]{n} \rfloor$. So, we may rewrite the sum as

$$q(n) = \sum_{j=1}^{\lfloor \sqrt[r]{n} \rfloor} \mu(j) \left\lfloor \frac{n}{j^r} \right\rfloor^k.$$

Since

$$\left\lfloor \frac{n}{j^r} \right\rfloor^k - \left(\frac{n}{j^r}\right)^k = \left(\left\lfloor \frac{n}{j^r} \right\rfloor - \frac{n}{j^r}\right)\left(\left\lfloor \frac{n}{j^r} \right\rfloor^{k-1} + \left\lfloor \frac{n}{j^r} \right\rfloor^{k-2}\left(\frac{n}{j^r}\right) + \cdots + \left(\frac{n}{j^r}\right)^{k-1}\right)$$

and $0 \leqslant x - \lfloor x \rfloor \leqslant 1$ for all $x \in \mathbb{R}$, we obtain

$$\left\lfloor \frac{n}{j^r} \right\rfloor^k - \left(\frac{n}{j^r}\right)^k = O\left(\left(\frac{n}{j^r}\right)^{k-1}\right).$$

Applying this growth estimate to $q(n)$ yields

$$q(n)=\sum_{j=1}^{\lfloor\sqrt[r]{n}\rfloor}\mu(j)\left(\frac{n}{j^r}\right)^k+O\left(\sum_{j=1}^{\lfloor\sqrt[r]{n}\rfloor}\left|\mu(j)\right|\left(\frac{n}{j^r}\right)^{k-1}\right).$$

Now, we must estimate how fast each sum in $q(n)$ grows. For the first sum, after rewriting it as

$$\sum_{j=1}^{\lfloor\sqrt[r]{n}\rfloor}\frac{\mu(j)}{j^{rk}}=\frac{1}{\zeta(rk)}-\sum_{j=\lfloor\sqrt[r]{n}\rfloor+1}^{\infty}\frac{\mu(j)}{j^{rk}},$$

we find that its tail is bounded above by the integral $\int_{\sqrt[r]{n}}^{\infty}\frac{dx}{x^{rk}}=O(n^{1/r-k})$.

Therefore, the first sum is $\frac{n^k}{\zeta(rk)}+O(n^{1/r})$.

As for the second sum, we first note that

$$\sum_{j=1}^{\lfloor\sqrt[r]{n}\rfloor}\left|\mu(j)\right|\left(\frac{n}{j^r}\right)^{k-1}=O\left(n^{k-1}\sum_{j=1}^{\lfloor\sqrt[r]{n}\rfloor}\frac{1}{j^{r(k-1)}}\right).$$

Since

$$\sum_{j=1}^{\lfloor\sqrt[r]{n}\rfloor}\frac{1}{j^{r(k-1)}}\leqslant 1+\int_{1}^{\sqrt[r]{n}}\frac{dx}{x^{r(k-1)}}=\begin{cases}O(\ln n) & \text{if } r=1 \text{ and } k=2,\\ O(n^{1/r}) & \text{if } r\geqslant 2 \text{ and } k=1,\\ O(1) & \text{otherwise,}\end{cases}$$

the second term is

$$\begin{cases}O(n\ln n) & \text{if } r=1 \text{ and } k=2,\\ O\left(n^{1/r}\right) & \text{if } r\geqslant 2 \text{ and } k=1,\\ O\left(n^{k-1}\right) & \text{otherwise.}\end{cases}$$

Hence, we find that

$$q(n)=\frac{n^k}{\zeta(rk)}+\begin{cases}O(n\ln n) & \text{if } r=1 \text{ and } k=2,\\ O(n^{1/r}) & \text{if } r\geqslant 2 \text{ and } k=1,\\ O(n^{k-1}) & \text{otherwise.}\end{cases}$$

Finally, with the application of L'Hôpital's Rule, we conclude that

$$\lim_{n\to\infty}\frac{q(n)}{n^k}=\frac{1}{\zeta(rk)}. \qquad \square$$

3. The Dedekind zeta function

In order to state the extension of Benkoski's result to algebraic numbers, we need to introduce the Dedekind zeta function.

We first fix some notation. Let K be an extension field of $\mathbb{Q}$, and let $\mathcal{O}$ denote the corresponding ring of algebraic integers in K. Since $\mathcal{O}$ generally does not enjoy unique factorisation into prime *algebraic integers*, we pass from the algebraic integers to *ideals*. When we do this, we have unique factorisation into prime ideals. Next, we define the **norm** of an ideal $\mathfrak{a} \subseteq \mathcal{O}$ as $\mathfrak{N}(\mathfrak{a}) := |\mathcal{O}/\mathfrak{a}|$. It can be shown that this is always finite.

Now, define the **Dedekind zeta function** of an algebraic number ring $\mathcal{O}$ as follows:

$$\zeta_{\mathcal{O}}(s) = \sum_{\mathfrak{a} \subseteq \mathcal{O}} \frac{1}{\mathfrak{N}(\mathfrak{a})^s},$$

where the sum is over all nonzero ideals of $\mathcal{O}$. Observe that if $\mathcal{O} = \mathbb{Z}$, then this definition reduces to the Riemann zeta function, with the norm of an ideal (n) (which is always principal in $\mathbb{Z}$) being the familiar absolute value of n. Another way to express the Dedekind zeta function, which will be important to us, is as follows:

$$\zeta_{\mathcal{O}}(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s},$$

where c_n represents the number of ideals with norm n.

As with the Riemann zeta function, the Dedekind zeta function converges for all $s > 1$ and possesses the product expansion

$$\zeta_{\mathcal{O}}(s) = \prod_{\mathfrak{p} \text{ prime}} \left(1 - \mathfrak{N}(\mathfrak{p})^{-s}\right)^{-1}.$$

Define the Möbius function $\mu : \mathcal{O} \to \{0, \pm 1\}$ as

$$\mu(\mathfrak{a}) = \begin{cases} 1 & \text{if } \mathfrak{N}(\mathfrak{a}) = 1, \\ 0 & \text{if } \mathfrak{a} \not\subseteq \mathfrak{p}^2 \text{ for some prime } \mathfrak{p}, \\ (-1)^r & \text{if } \mathfrak{a} = \mathfrak{p}_1\mathfrak{p}_2 \cdots \mathfrak{p}_r \text{ for distinct primes } \mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r, \end{cases}$$

then

$$\sum_{\mathfrak{a} \subseteq \mathcal{O}} \frac{\mu(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^s} = \prod_{\mathfrak{p} \text{ prime}} \left(1 - \mathfrak{N}(\mathfrak{p})^{-s}\right) = \frac{1}{\zeta_{\mathcal{O}}(s)}.$$

For the details for the proofs of these results, one can see Neukirch [5] or Marcus [3].

4. Extending Benkoski's result to the algebraic integers

The definition of relative r-primality in $\mathbb{Z}$ naturally extends to that in $\mathcal{O}$ as follows: For a fixed integer $r \geqslant 1$, we say that the ideals $\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_k \subseteq \mathcal{O}$ are **relatively** r**-prime** if $\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_k \not\subseteq \mathfrak{b}^r$ for any nonzero proper ideal $\mathfrak{b}$.

Theorem 4.1. *Let K be an algebraic number field with $\mathcal{O}$ its algebraic number ring. Fix $k, r \in \mathbb{N}$ not both equal to 1. Let $H(n)$ denote the number of ideals in $\mathcal{O}$ with norm less than or equal to n, and $Q(n)$ denote the number of ordered k-tuples of ideals in $\mathcal{O}$ with norm less than or equal to n that are relatively r-prime. Then,*

$$\lim_{n\to\infty} \frac{Q(n)}{H(n)^k} = \frac{1}{\zeta_{\mathcal{O}}(rk)}.$$

Proof. Fix $n \in \mathbb{N}$. An ordered k-tuple of ideals $(\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_k)$ is relatively r-prime if and only if there exists no prime ideal $\mathfrak{p}$ such that $\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_k \subseteq \mathfrak{p}^r$, the Inclusion–Exclusion Principle shows that $Q(n)$ may be written as

$$Q(n) = H(n)^k - \sum_{\mathfrak{p}_1} H\left(\frac{n}{\mathfrak{N}(\mathfrak{p}_1^r)}\right)^k + \sum_{\mathfrak{p}_1,\mathfrak{p}_2} H\left(\frac{n}{\mathfrak{N}((\mathfrak{p}_1\mathfrak{p}_2)^r)}\right)^k - \cdots,$$

where $\mathfrak{p}_1, \mathfrak{p}_2, \dots$ denote distinct prime ideals with norm less than or equal to n. Moreover, the second sum is on pairs of distinct prime ideals not counting repetitions, et cetera. Using the Möbius function μ permits us to rewrite this sum more compactly as

$$Q(n) = \sum_{\mathfrak{a}} \mu(\mathfrak{a}) H\left(\frac{n}{\mathfrak{N}(\mathfrak{a}^r)}\right)^k.$$

This sum actually ranges over all nonzero ideals $\mathfrak{a}$ such that $\mathfrak{N}(\mathfrak{a}) \leqslant \lfloor \sqrt[r]{n} \rfloor$ since the summands corresponding to the other indices are annihilated by H.

In order to estimate H, we use the fact that there exists a positive constant c such that $H(n) = cn + O(n^{1-\epsilon})$ where $\epsilon = [K:\mathbb{Q}]^{-1}$ (see [3]). If $K = \mathbb{Q}$, then H reduces to the floor function, and the above estimate follows suit.

Directly applying this estimate to $Q(n)$ and applying the Binomial Theorem, we see that

$$Q(n) = (cn)^k \sum_{\mathfrak{a}} \frac{\mu(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^{rk}} + \sum_{\mathfrak{a}} \mu(\mathfrak{a}) \left(\frac{cn}{\mathfrak{N}(\mathfrak{a})^r}\right)^{k-1} O\left(\frac{n}{\mathfrak{N}(\mathfrak{a})^r}\right)^{1-\epsilon}.$$

Now, we must estimate how fast each sum of $Q(n)$ grows. For the first sum of $Q(n)$, note that

$$\sum_{\substack{\mathfrak{a}\subseteq\mathcal{O} \\ \mathfrak{N}(\mathfrak{a})\leqslant\lfloor\sqrt[r]{n}\rfloor}} \frac{\mu(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^{rk}} = \frac{1}{\zeta_{\mathcal{O}}(rk)} - \sum_{\substack{\mathfrak{a}\subseteq\mathcal{O} \\ \mathfrak{N}(\mathfrak{a})>\lfloor\sqrt[r]{n}\rfloor}} \frac{\mu(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^{rk}}.$$

In order to estimate the growth of the second sum, we use the fact that the number of ideals with a fixed norm n is given by $H(n) - H(n-1) = O(n^{1-\epsilon})$. Applying this fact yields

$$\sum_{\substack{\mathfrak{a}\subseteq\mathcal{O} \\ \mathfrak{N}(\mathfrak{a})>\lfloor\sqrt[r]{n}\rfloor}} \frac{\mu(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^{rk}} \leqslant \int_{\sqrt[r]{n}}^{\infty} \frac{cx^{1-\epsilon}}{x^{rk}}\,dx = O\left(n^{(2-\epsilon-rk)/r}\right).$$

Hence, the first sum of $Q(n)$ is $\frac{(cn)^k}{\zeta_{\mathcal{O}}(rk)} + O(n^{(2-\epsilon)/r})$.

As for the second sum of $Q(n)$, we need to estimate the sum $\sum_{\mathfrak{a}} O(\frac{n^{k-\epsilon}}{\mathfrak{N}(\mathfrak{a})^{r(k-\epsilon)}})$.

Again by using our norm estimate, we see that

$$\sum_{\mathfrak{a}} \frac{n^{k-\epsilon}}{\mathfrak{N}(\mathfrak{a})^{r(k-\epsilon)}} \leqslant n^{k-\epsilon} \sum_{j=1}^{\lfloor \sqrt[r]{n} \rfloor} \frac{cj^{1-\epsilon}}{j^{r(k-\epsilon)}} \leqslant n^{k-\epsilon}\left(1 + \int_1^{\sqrt[r]{n}} \frac{dx}{x^{r(k-1)}}\right).$$

Estimating this integral proceeds as in the integer case, and we find (after leaving the tedious details to the reader) that the second sum of $Q(n)$ is

$$\begin{cases} O\left(n^{k-\epsilon}\right) & \text{if } k > 2,\ \text{or } k = 2 \text{ and } r \geqslant 2, \\ O\left(n^{2-\epsilon} \ln n\right) & \text{if } k = 2 \text{ and } r = 1, \\ O\left(n^{1-\epsilon} \ln n\right) & \text{if } k = 1 \text{ and } \epsilon = \dfrac{r-2}{r-1}, \\ O\left(n^{1-\epsilon}\right) & \text{if } k = 1 \text{ and } \epsilon < \dfrac{r-2}{r-1}, \\ O\left(n^{(2-r-\epsilon+r\epsilon)/r}\right) & \text{if } k = 1 \text{ and } \epsilon > \dfrac{r-2}{r-1}. \end{cases}$$

Hence, we find that

$$Q(n) = \frac{(cn)^k}{\zeta_{\mathcal{O}}(rk)} + \begin{cases} O(n^{k-\epsilon}) & \text{if } k > 2,\ \text{or } k = 2 \text{ and } r \geqslant 2, \\ O(n^{2-\epsilon} \ln n) & \text{if } k = 2 \text{ and } r = 1, \\ O(n^{1-\epsilon} \ln n) & \text{if } k = 1 \text{ and } \epsilon = \frac{r-2}{r-1}, \\ O(n^{1-\epsilon}) & \text{if } k = 1 \text{ and } \epsilon < \frac{r-2}{r-1}, \\ O(n^{(2-\epsilon)/r}) & \text{if } k = 1 \text{ and } \epsilon > \frac{r-2}{r-1}. \end{cases}$$

Finally with this growth estimate, we may conclude that

$$\lim_{n\to\infty} \frac{Q(n)}{H(n)^k} = \lim_{n\to\infty} \frac{Q(n)/n^k}{(H(n)/n)^k} = \frac{c^k \zeta_{\mathcal{O}}(rk)^{-1} + 0}{c^k} = \frac{1}{\zeta_{\mathcal{O}}(rk)}. \qquad \square$$

Remark. Note that this theorem reduces to Benkoski's Theorem for ordinary integers if we let $K = \mathbb{Q}$ right down to the growth estimates.

5. Further results

In [1], Benkoski also investigates probabilities that a randomly chosen k-tuple of integers is allowed to have a common rth power m^r where m is an integer whose factors arise from a specific list of prime numbers S (or its complement $\overline{S}$) as long as S (or $\overline{S}$) is finite. Not only do we show that the finiteness condition is unnecessary, we will prove this in the case of the ring of algebraic integers $\mathcal{O}$.

Before stating the result, let's fix some notation. S will denote a fixed subset of distinct prime ideals, and $\overline{S}$ will denote the set of prime ideals not in S. Moreover, $\langle \overline{S} \rangle$ represents all ideals which arise as products of (not necessarily distinct) elements from $\mathcal{O} \cup \overline{S}$.

Theorem 5.1. *Fix $k, r \in \mathbb{N}$ not both equal to 1 and a set of prime ideals S. Let $Q(n, S)$ denote the number of ordered k-tuples of ideals in $\mathcal{O}$ with norm less than or equal to n that is have a common rth power of a prime* *ideal* $\mathfrak{p} \in S$ *Then,*

$$\lim_{n\to\infty} \frac{Q(n,S)}{H(n)^k} = \sum_{\mathfrak{a}\subseteq\langle\bar{S}\rangle} \frac{\mu(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^{rk}}.$$

Proof. Remarkably proving this is no more difficult than the result from the previous section. I will only highlight the essential details to prove this result below.

Fix $n \in \mathbb{N}$, and let $\mathfrak{p}_1, \mathfrak{p}_2, \ldots$ denote distinct prime ideals with norm less than or equal to n in $\bar{S}$. Then, applying the Inclusion–Exclusion Principle as before, we find that $Q(n,S)$ can be written (using the Möbius function μ) as

$$Q(n,S) = \sum_{\mathfrak{a}\subseteq\langle\bar{S}\rangle} \mu(\mathfrak{a}) H\left(\frac{n}{\mathfrak{N}(\mathfrak{a}^r)}\right)^k.$$

Proving this using growth estimates proceeds as before, as the previous integral estimates still apply since $\langle\bar{S}\rangle$ is a subset of $\mathcal{O}$. We thus obtain

$$Q(n,S) = (cn)^k \cdot \sum_{\mathfrak{a}\subseteq\langle\bar{S}\rangle} \frac{\mu(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^{rk}} + \begin{cases} O(n^{k-\epsilon}) & \text{if } k>2, \text{ or } k=2 \text{ and } r \geqslant 2, \\ O(n^{2-\epsilon}\ln n) & \text{if } k=2 \text{ and } r=1, \\ O(n^{1-\epsilon}\ln n) & \text{if } k=1 \text{ and } \epsilon = \frac{r-2}{r-1}, \\ O(n^{1-\epsilon}) & \text{if } k=1 \text{ and } \epsilon < \frac{r-2}{r-1}, \\ O(n^{(2-\epsilon)/r}) & \text{if } k=1 \text{ and } \epsilon > \frac{r-2}{r-1}. \end{cases}$$

Finally with this growth estimate, we may conclude that $\lim_{n\to\infty} \frac{Q(n,S)}{H(n)^k} = \sum_{\mathfrak{a}\subseteq\langle\bar{S}\rangle} \frac{\mu(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^{rk}}$, as desired. □

To make an analogy with the classic zeta functions, let us define

$$\zeta_{\mathcal{O},S}(s) = \sum_{\mathfrak{a}\subseteq\langle\bar{S}\rangle} \frac{1}{\mathfrak{N}(\mathfrak{a}^s)}.$$

Note that if we let $S = \emptyset$, then we get $\zeta_{\mathcal{O},S}(s) = \zeta_{\mathcal{O}}(s)$. It is easy to see that this series converges for all $s > 1$. Since $\langle\bar{S}\rangle$ is multiplicative, we see that

$$\zeta_{\mathcal{O},S}(s) = \prod_{\mathfrak{p}\subseteq\bar{S}} \left(1 - \mathfrak{N}(\mathfrak{p})^{-s}\right)^{-1}.$$

Thus,

$$\sum_{\mathfrak{a}\subseteq\langle\bar{S}\rangle} \frac{\mu(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^{rk}} = \prod_{\mathfrak{p}\subseteq\bar{S}} \left(1 - \mathfrak{N}(\mathfrak{p})^{-s}\right) = \frac{1}{\zeta_{\mathcal{O},S}(s)}.$$

Consequently, we may restate the conclusion of the above theorem as

$$\lim_{n\to\infty} \frac{Q(n,S)}{H(n)^k} = \frac{1}{\zeta_{\mathcal{O},S}(s)}.$$

References

[1] S.J. Benkoski, The probability that k positive integers are relatively r-prime, J. Number Theory 8 (1976) 218–223.
[2] G.E. Collins, J.R. Johnson, The probability of relative primality of gaussian integers, in: Proc. 1988 International Symposium Symbolic and Algebraic Computation, Springer-Verlag, 1989, pp. 252–258.
[3] D.A. Marcus, Number Fields, second ed., Springer-Verlag, 1995.
[4] K. Morrison, Z. Dong, The probability that random polynomials are relatively r-prime, 2004. Available at http://www.calpoly.edu/~kmorriso/Research/RPFF04-2.pdf.
[5] J. Neukirch, Algebraic Number Theory, Comprehensive Stud. Math., vol. 322, Springer-Verlag, 1999.
[6] J.E. Nymann, On the probability that k positive integers are relatively prime, J. Number Theory 4 (1972) 469–473.