How Inductive Bias in Machine Learning Aligns with Optimality in Economic Dynamics

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Motivation

In the long run, we are all dead—*J.M. Keynes, A Tract on Monetary Reform (1923)*

- Numerical solutions to dynamical systems are central to many quantitative fields in economics.
- Dynamical systems in economics are **boundary value** problems:
  1. The boundary is at **infinity**.
  2. The values at the boundary are potentially **unknown**.
- Resulting from **forward looking** behavior of agents.
- Examples include the transversality and the no-bubble condition.
- Without them, the problems are **ill-posed** and have infinitely many solutions:
  - These forward-looking boundary conditions are a key limitation on increasing dimensionality.

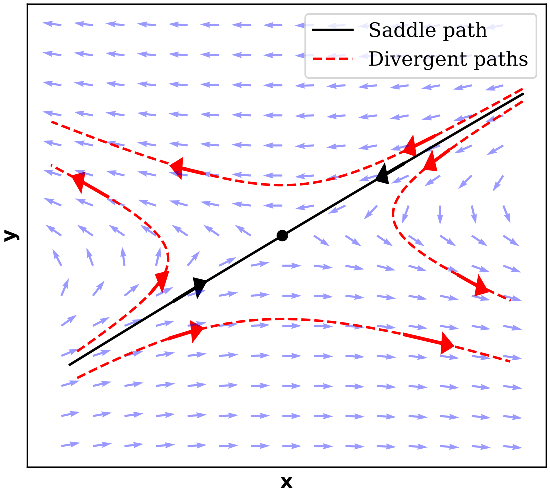
Contribution

1. **Inductive bias alignment**:
   - The minimum-norm implicit bias of modern ML models automatically satisfies economic boundary conditions at infinity.
2. **Learning the right set of steady-states**:
   - Deep neural networks and kernel machines learn the boundary values, thereby extrapolating very accurately.
3. **Robustness and speed**:
   - Competitive in speed and more stable than traditional methods.
4. **Consistency of ML estimates**.

Intuition

- **Minimum-norm implicit bias**:
  - Over-parameterized models (e.g., large neural networks) have more parameters than data points and potentially interpolate the data.
  - They are biased towards interpolating functions with smallest norm.

- **Violation of economic boundary conditions**:
  - Sub-optimal solutions diverge (explode) over time.
  - They have large or explosive norms.
  - This is due to the **saddle-path** nature of econ problems.



The Problem

The class of problems

A differential-algebraic system of equations, coming from an economic optimization problem:

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), \mathbf{y}(t)) \tag{1}$$

$$\dot{\mathbf{y}}(t) = \mathbf{G}(\mathbf{x}(t), \mathbf{y}(t)) \tag{2}$$

$$\mathbf{0} = \mathbf{H}(\mathbf{x}(t), \mathbf{y}(t)) \tag{3}$$

$\mathbf{x} \in \mathbb{R}^{N_x}$: state variables, $\mathbf{y} \in \mathbb{R}^{N_y}$: jump variables. Initial value $\mathbf{x}(0) = \mathbf{x}_0$ and boundary conditions (at infinity)

$$\mathbf{0} = \lim_{t \to \infty} \mathbf{B}(t, \mathbf{x}(t), \mathbf{y}(t)) \tag{4}$$

Goal: finding an approximation for $\mathbf{x}(t)$ and $\mathbf{y}(t)$.

What is the problem?

- $\mathbf{y}_0$ is unknown.
- The optimal solutions is a **saddle-path**: unstable nature

Method

Method

- Pick a set of points $\mathcal{D} \equiv \{t_1, \cdots, t_N\}$ for some fixed interval $[0, T]$
- Large machine learning models to learn $\hat{\mathbf{x}}(t)$ and $\hat{\mathbf{y}}(t)$

$$\min_{\hat{\mathbf{x}},\hat{\mathbf{y}}} \sum_{t_i \in \mathcal{D}} \Bigg[\eta_1 \underbrace{\left\| \dot{\hat{\mathbf{x}}}(t_i) - \mathbf{F}(\hat{\mathbf{x}}(t_i), \hat{\mathbf{y}}(t_i)(t_i)) \right\|_2^2}_{\text{Residuals}^2\text{: state variables}} + \eta_2 \underbrace{\left\| \dot{\hat{\mathbf{y}}}(t_i) - \mathbf{G}(\hat{\mathbf{x}}(t_i), \hat{\mathbf{y}}(t_i)) \right\|_2^2}_{\text{Residuals}^2\text{: jump variables}}$$

$$+ \eta_3 \underbrace{\left\| \mathbf{H}(\hat{\mathbf{x}}(t_i), \hat{\mathbf{y}}(t_i)) \right\|_2^2}_{\text{Residuals}^2\text{: algebraic constraint}} \Bigg] + \eta_4 \underbrace{\left\| \hat{\mathbf{x}}(0) - \mathbf{x}_0 \right\|_2^2}_{\text{Residuals}^2\text{: initial conditions}} .$$

- This optimization **ignores** the boundary conditions.
- The implicit bias automatically satisfy the boundary conditions.
- Recent works suggest the implicit bias is toward smallest Sobolev semi-norms.

Ridgeless kernel regression

$$\hat{\dot{\mathbf{x}}}(t) = \sum_{j=1}^{N} \boldsymbol{\alpha}_j^x K(t, t_j), \quad \hat{\dot{\mathbf{y}}}(t) = \sum_{j=1}^{N} \boldsymbol{\alpha}_j^y K(t, t_j)$$

$$\hat{\mathbf{x}}(t) = \mathbf{x}_0 + \int_0^t \hat{\dot{\mathbf{x}}}(\tau) d\tau, \quad \hat{\mathbf{y}}(t) = \hat{\mathbf{y}}_0 + \int_0^t \hat{\dot{\mathbf{y}}}(\tau) d\tau$$

- $\mathbf{x}_0$ is given.
- $\hat{\mathbf{y}}_0$, $\boldsymbol{\alpha}_j^x$, and $\boldsymbol{\alpha}_j^y$ are learnable parameters.
- $K(\cdot, \cdot)$: Matérn Kernel, with smoothness parameter ν and length scale ℓ.

Ridgeless kernel regression: minimum Sobolev seminorm solutions

We also solve the ridgeless kernel regression

$$\lim_{\lambda\to 0}\min_{\hat{\mathbf{x}},\hat{\mathbf{y}}}\sum_{t_i\in\mathcal{D}}\Bigg[\eta_1\left\|\dot{\hat{\mathbf{x}}}(t_i)-\mathbf{F}(\hat{\mathbf{x}}(t_i),\hat{\mathbf{y}}(t_i)(t_i))\right\|_2^2+\eta_2\left\|\dot{\hat{\mathbf{y}}}(t_i)-\mathbf{G}(\hat{\mathbf{x}}(t_i),\hat{\mathbf{y}}(t_i))\right\|_2^2$$

$$+\eta_3\left\|\mathbf{H}(\hat{\mathbf{x}}(t_i),\hat{\mathbf{y}}(t_i))\right\|_2^2\Bigg]+\eta_4\left\|\hat{\mathbf{x}}(0)-\hat{\mathbf{x}}_0\right\|_2^2+\lambda\underbrace{\left[\sum_{m=1}^{N_x}\|\hat{\mathbf{x}}^{(m)}\|_{\mathcal{H}}^2+\sum_{m=1}^{N_y}\|\hat{\mathbf{y}}^{(m)}\|_{\mathcal{H}}^2\right]}_{\text{The Sobolev semi-norm}}$$

- Targeting Sobolev semi-norm.
- This choice is very natural: it solves the instability issues of the classical algorithm.

Applications

Linear asset pricing

$$\dot{\mathbf{x}}(t) = c + g\mathbf{x}(t) \tag{5}$$
$$\dot{\mathbf{y}}(t) = r\mathbf{y}(t) - \mathbf{x}(t) \tag{6}$$
$$0 = \lim_{t\to\infty} e^{-rt}\mathbf{y}(t) \tag{7}$$

- $\mathbf{x}(t) \in \mathbb{R}$: dividends, $\mathbf{y}(t) \in \mathbb{R}$: prices, and $\mathbf{x}_0$ given.
- Equation (5): how the dividends evolve in time.
- Equation (6): how the prices evolve in time.
- Equation (7): "no-bubble" condition, the boundary condition at infinity.

Why do we need the boundary condition?

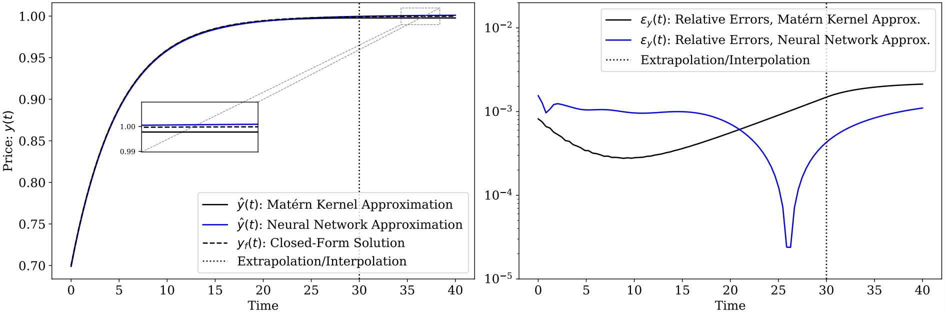
$$\dot{\mathbf{x}}(t) = c + g\mathbf{x}(t)$$
$$\dot{\mathbf{y}}(t) = r\mathbf{y}(t) - \mathbf{x}(t)$$

- The solutions:

$$\mathbf{y}(t) = \mathbf{y}_f(t) + \zeta e^{rt}$$

- $\mathbf{y}_f(t) = \int_0^\infty e^{-r\tau}\mathbf{x}(t+s)ds$: price based on the fundamentals.
- ζe^{rt}: explosive bubble terms, it has to be **ruled out** by the boundary condition.
- Triangle inequality: $\|\mathbf{y}_f\| < \|\mathbf{y}\|$.
- The price based on the fundamentals has the **lowest norm**.

Results



$$\mathcal{D} = \{0, 1, \cdots, 30\}$$

- The explosive solutions are ruled out without directly imposing the boundary condition.
- Very accurate approximations, both in the short- and medium-run.
- Learns the steady-state.

Neoclassical growth model: the agent's problem

$$\max_{\mathbf{y}(t)} \int_0^\infty e^{-rt} \ln(\mathbf{y}(t))dt$$
$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = f(\mathbf{x}(t)) - \mathbf{y}(t) - \delta \mathbf{x}(t)$$

for a given $\mathbf{x}_0$.

- $\mathbf{x}(t) \in \mathbb{R}$: capital, $\mathbf{y}(t) \in \mathbb{R}$: consumption, and a concave production function $f(x) = x^a$.

Constructing the Hamiltonian ...

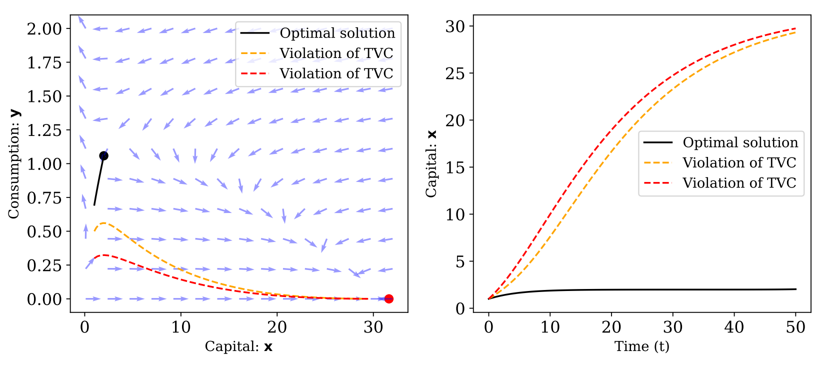
$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t)) - \mathbf{y}(t) - \delta \mathbf{x}(t) \tag{8}$$
$$\dot{\mathbf{y}}(t) = \mathbf{y}(t)\left[f'(\mathbf{x}(t)) - \delta - r\right] \tag{9}$$
$$0 = \lim_{t\to\infty} e^{-rt} \frac{\mathbf{x}(t)}{\mathbf{y}(t)} \tag{10}$$

- Equation (10) : transversality condition (TVC)

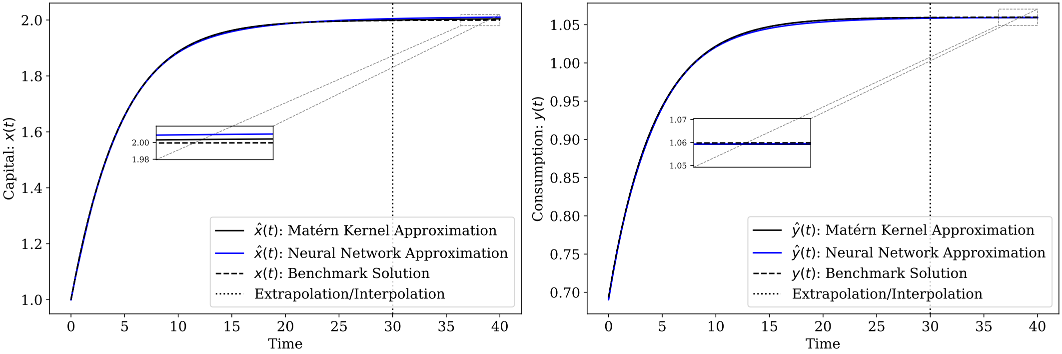
Why do we need the boundary condition?

Ignoring the transversality condition:

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t)) - \mathbf{y}(t) - \delta\mathbf{x}(t)$$
$$\dot{\mathbf{y}}(t) = \mathbf{y}(t)\left[f'(\mathbf{x}(t)) - \delta - r\right]$$
$$\mathbf{x}(0) = \mathbf{x}_0 \text{ given.}$$



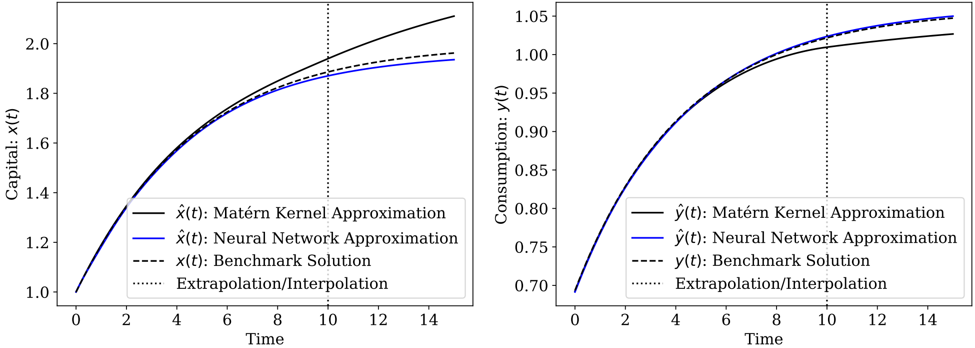
Results



$$\mathcal{D} = \{0, 1, \cdots, 30\}$$

- The explosive solutions are ruled out without directly imposing the boundary condition.
- Very accurate approximations, both in the short- and medium-run.
- Learns the **right steady-state**. Relative errors

Short-run planning: “In the long run, we are all dead”



$$\mathcal{D} = \{0, 1, \cdots, 10\}$$

- The explosive solutions are ruled out without directly imposing the boundary condition.
- Provides a very accurate approximation in the short-run.

Extensions

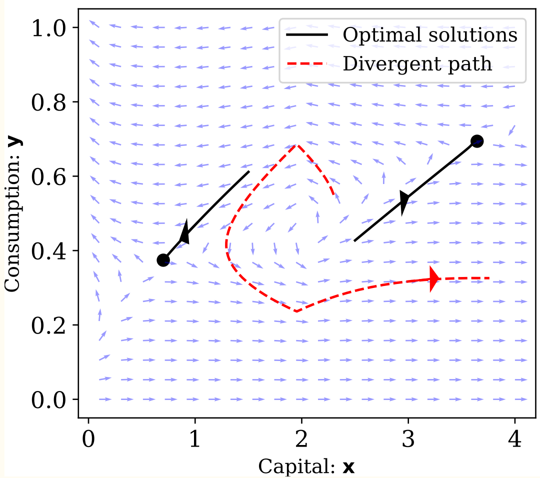
Neoclassical Growth Model: Concave-Convex Production Function

- So far we have had a **unique** saddle-path converging to a unique **saddle** steady state.
- What if we have **two** saddle steady states, very close to each other (equilibrium multiplicity)?
- Neoclassical growth model with a concave-convex production function:

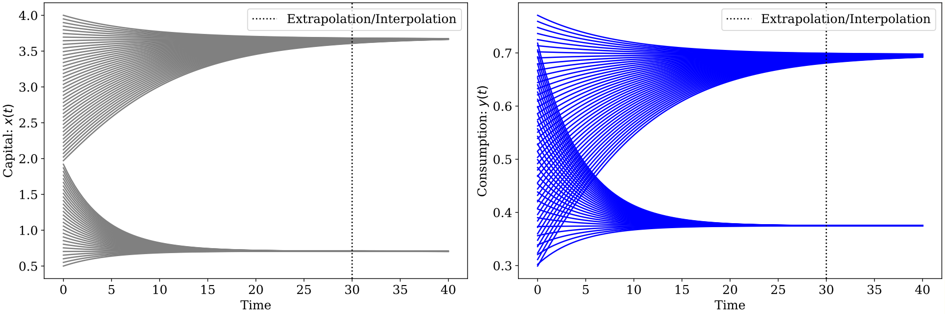
$$f(x) = A \max\{x^a, b_1 x^a - b_2\}$$

Concave-convex production function: vector field

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t)) - \mathbf{y}(t) - \delta \mathbf{x}(t)$$
$$\dot{\mathbf{y}}(t) = \mathbf{y}(t)\left[f'(\mathbf{x}(t)) - \delta - r\right]$$
$$\mathbf{x}(0) = \mathbf{x}_0 \text{ given.}$$



Results



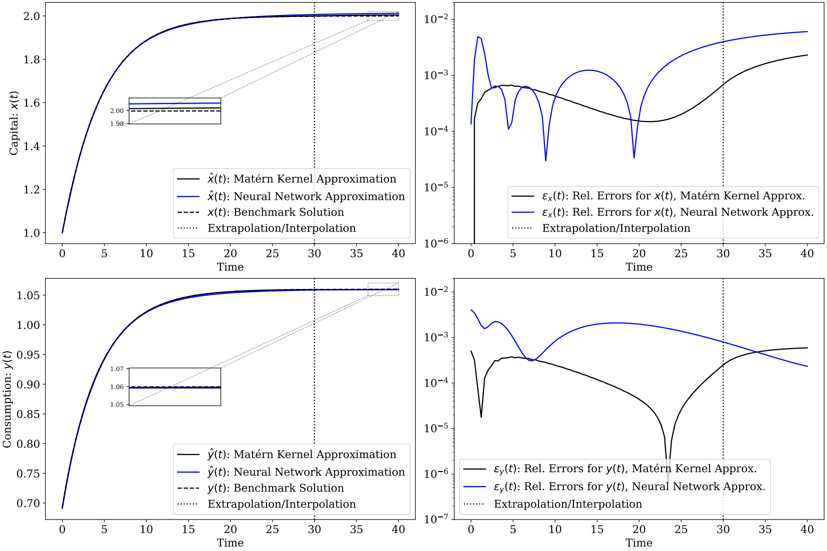
- The approximate solutions approach the right steady states.
- The transversality conditions are satisfied without being directly imposed.
- The steady states are learned. Full DAE

Conclusion

- Long-run (**global**) conditions can be replaced with appropriate regularization (**local**) to achieve the optimal solution.
- The minimum-norm implicit bias of large ML models aligns with optimality in economic dynamic models.
- Both kernel and neural network approximations accurately learn the right steady state(s).
- Proceeding with **caution**: can regularization be thought of as an equilibrium selection device?

Appendix

Neoclassical growth: relative errors



Back

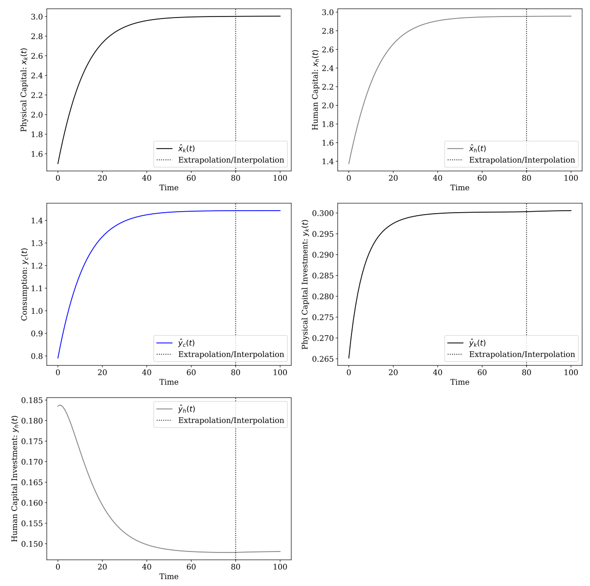
Human capital and growth

$$
\begin{aligned}
\dot{\mathbf{x}}_k(t) &= \mathbf{y}_k(t) - \delta_k \mathbf{x}_k(t),\\
\dot{\mathbf{x}}_h(t) &= \mathbf{y}_h(t) - \delta_k \mathbf{x}_h(t)\\
\dot{\mathbf{y}}_c(t) &= \mathbf{y}_c(t)\left[f_1\left(\mathbf{x}_k(t), \mathbf{x}_h(t)\right) - \delta_k - r\right],\\
0 &= f\left(\mathbf{x}_k(t), \mathbf{x}_h(t)\right) - \mathbf{y}_c(t) - \mathbf{y}_k(t) - \mathbf{y}_h(t),\\
0 &= f_2\left(\mathbf{x}_k(t), \mathbf{x}_h(t)\right) - f_1\left(\mathbf{x}_k(t), \mathbf{x}_h(t)\right) + \delta_k - \delta_h.\\
0 &= \lim_{t\to\infty} e^{-rt}\frac{\mathbf{x}_k(t)}{\mathbf{y}_c(t)},\ 0 = \lim_{t\to\infty} e^{-rt}\frac{\mathbf{x}_h(t)}{\mathbf{y}_c(t)}.
\end{aligned}
$$

- $\mathbf{x}_k$: physical capital, $\mathbf{x}_h$: human capital, $\mathbf{y}_c$: consumption, $\mathbf{y}_k$: investment in physical capital, $\mathbf{y}_h$: investment in human capital
- $f\left(x_k, x_h\right) = x_k^{a_k} x_h^{a_h}$

Back

Results



Physical Capital: $x_k(t)$
$\hat{x}_k(t)$
Extrapolation/Interpolation
Time
Human Capital: $x_h(t)$
$\hat{x}_h(t)$
Extrapolation/Interpolation
Time
Consumption: $y_c(t)$
$\hat{y}_c(t)$
Extrapolation/Interpolation
Time
Physical Capital Investment: $y_k(t)$
$\hat{y}_k(t)$
Extrapolation/Interpolation
Time
Human Capital Investment: $y_h(t)$
$\hat{y}_h(t)$
Extrapolation/Interpolation
Time