

# Derivation of State-Space Representation for a Quarter Car Model

PS

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## 1 Introduction

This document provides the derivation of the state-space representation from Newton's second law for a quarter car model. The quarter car model consists of a sprung mass  $m_s$ , representing the car body, and an unsprung mass  $m_u$ , representing the wheel assembly. The system also includes a damper and a spring connecting the two masses, as well as an actuator force  $F_a$  and a sensor for feedback, see figure 1. We aim to derive the equations of motion for both masses and represent the system in state-space form, focusing on the velocity and position of both masses as the outputs.

## 2 Equations of Motion

To derive the equations of motion, we apply Newton's second law  $F = ma$ , considering the forces due to the spring, damper, actuator, and the external road profile.

### 2.1 Sprung Mass

For the sprung mass  $m_s$ , the forces include the spring force  $k_s(z_u - z_s)$ , the damping force  $b_s(\dot{z}_u - \dot{z}_s)$ , and the actuator force  $F_a$ . The equation of motion is:

$$m_s \ddot{z}_s = -k_s(z_s - z_u) - b_s(\dot{z}_s - \dot{z}_u) + F_a \quad (1)$$

### 2.2 Unsprung Mass

For the unsprung mass  $m_u$ , the forces include the tire stiffness  $k_t(z_r - z_u)$  and the forces from the spring and damper. The equation of motion is:

$$m_u \ddot{z}_u = -k_t(z_u - z_r) + k_s(z_s - z_u) + b_s(\dot{z}_s - \dot{z}_u) \quad (2)$$

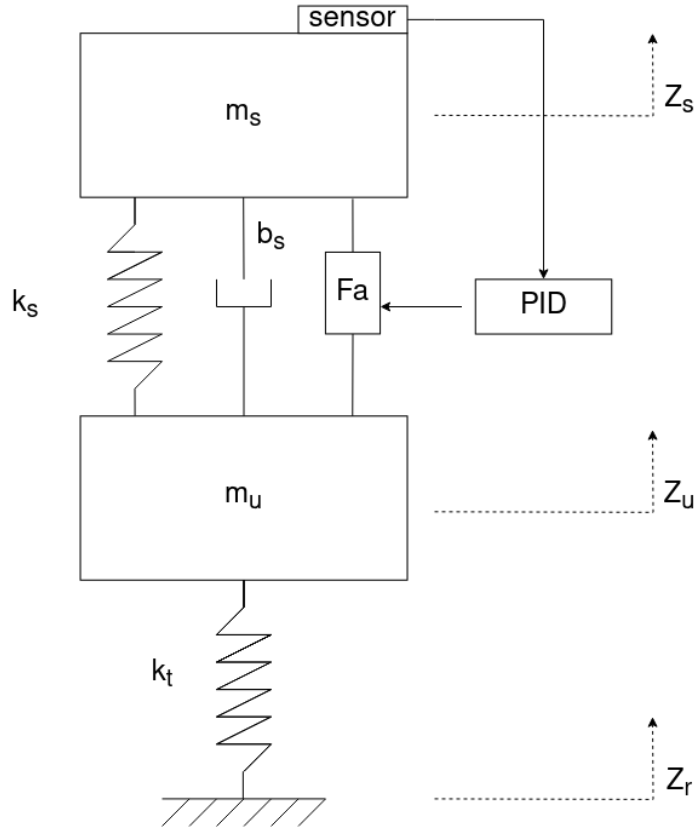


Figure 1: Quarter Car Model

### 3 State-Space Representation

To form the state-space representation, we define the state vector  $\mathbf{x}$ , the input vector  $\mathbf{u}$ , and the output vector  $\mathbf{y}$ .

The state vector is chosen as  $\mathbf{x} = [z_s, \dot{z}_s, z_u, \dot{z}_u]^T$ , where:

- $z_s$  is the displacement of the sprung mass,
- $\dot{z}_s$  is the velocity of the sprung mass,
- $z_u$  is the displacement of the unsprung mass,
- $\dot{z}_u$  is the velocity of the unsprung mass.

The input vector  $\mathbf{u}$  is composed of the road displacement  $z_r$  and the actuator force  $F_a$ :

$$\mathbf{u} = \begin{bmatrix} z_r \\ F_a \end{bmatrix} \quad (3)$$

The output vector  $\mathbf{y}$  is defined to include the velocity and displacement of both the sprung and unsprung masses:

$$\mathbf{y} = \begin{bmatrix} z_s \\ \dot{z}_s \\ z_u \\ \dot{z}_u \end{bmatrix} \quad (4)$$

The state-space equations can then be written as:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (5)$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \quad (6)$$

where  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$ , and  $\mathbf{D}$  are matrices that define the system dynamics and output relationships.

### 3.1 State-Space Matrices

Using the equations of motion, we derive the state-space matrices:

System matrix  $\mathbf{A}$ :

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_s}{m_s} & -\frac{b_s}{m_s} & \frac{k_s}{m_s} & \frac{b_s}{m_s} \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{m_u} & \frac{b_s}{m_u} & -\frac{k_t+k_s}{m_u} & -\frac{b_s}{m_u} \end{bmatrix} \quad (7)$$

Input matrix  $\mathbf{B}$ :

$$\mathbf{B} = \begin{bmatrix} 0 & 0 \\ \frac{1}{m_s} & 0 \\ 0 & 0 \\ 0 & \frac{k_t}{m_u} \end{bmatrix} \quad (8)$$

Output matrix  $\mathbf{C}$ :

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (9)$$

Feedthrough matrix  $\mathbf{D}$ :

$$\mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (10)$$

## 4 Modified Output Matrix for Acceleration and Displacement

The output vector  $\mathbf{y}$  is defined to include the acceleration and displacement of both the sprung and unsprung masses:

$$\mathbf{y} = \begin{bmatrix} z_s \\ \ddot{z}_s \\ z_u \\ \ddot{z}_u \end{bmatrix} \quad (11)$$

System matrix  $\mathbf{A}$ :

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_s}{m_s} & -\frac{b_s}{m_s} & \frac{k_s}{m_s} & \frac{b_s}{m_s} \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{m_u} & \frac{b_s}{m_u} & -\frac{k_t+k_s}{m_u} & -\frac{b_s}{m_u} \end{bmatrix} \quad (12)$$

Input matrix  $\mathbf{B}$ :

$$\mathbf{B} = \begin{bmatrix} 0 & 0 \\ \frac{1}{m_s} & 0 \\ 0 & 0 \\ 0 & \frac{k_t}{m_u} \end{bmatrix} \quad (13)$$

The modified output matrix  $\mathbf{C}$  to extract the acceleration of the sprung mass  $\ddot{z}_s$ , the displacement of the sprung mass  $z_s$ , the acceleration of the unsprung mass  $\ddot{z}_u$ , and the displacement of the unsprung mass  $z_u$  is given by:

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{k_s}{m_s} & -\frac{b_s}{m_s} & \frac{k_s}{m_s} & \frac{b_s}{m_s} \\ 0 & 0 & 1 & 0 \\ \frac{k_s}{m_u} & \frac{b_s}{m_u} & -\frac{k_t+k_s}{m_u} & -\frac{b_s}{m_u} \end{bmatrix} \quad (14)$$

Note that the feedthrough matrix  $\mathbf{D}$  remains a zero matrix unless there is a direct relationship between the inputs and the outputs:

$$\mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (15)$$

## 5 Transfer Functions

The state-space model can be represented as a series of transfer functions from each input to each output. The transfer functions derived from the state-space representation are as follows:

## 5.1 From Input 1 to Outputs

1. For Output 1:

$$G_{1,1}(s) = \frac{-0.016s^3 - 0.2976s^2 - 62.72s - 1167}{s^4 + 24s^3 + 4366s^2 + 1.568 \times 10^4s + 2.916 \times 10^5} \quad (16)$$

2. For Output 2:

$$G_{2,1}(s) = \frac{0.004s^2 + 0.08s + 17.17}{s^4 + 24s^3 + 4366s^2 + 1.568 \times 10^4s + 2.916 \times 10^5} \quad (17)$$

3. For Output 3:

$$G_{3,1}(s) = \frac{0.08s^3 + 1.488s^2 + 1.921 \times 10^{-13}s + 3.359 \times 10^{-12}}{s^4 + 24s^3 + 4366s^2 + 1.568 \times 10^4s + 2.916 \times 10^5} \quad (18)$$

4. For Output 4:

$$G_{4,1}(s) = \frac{0.08s + 1.488}{s^4 + 24s^3 + 4366s^2 + 1.568 \times 10^4s + 2.916 \times 10^5} \quad (19)$$

## 5.2 From Input 2 to Outputs

1. For Output 1:

$$G_{1,2}(s) = \frac{15680s^3 + 2.916 \times 10^5s^2}{s^4 + 24s^3 + 4366s^2 + 1.568 \times 10^4s + 2.916 \times 10^5} \quad (20)$$

2. For Output 2:

$$G_{2,2}(s) = \frac{1.568 \times 10^4s + 2.916 \times 10^5}{s^4 + 24s^3 + 4366s^2 + 1.568 \times 10^4s + 2.916 \times 10^5} \quad (21)$$

3. For Output 3:

$$G_{3,2}(s) = \frac{-7.84 \times 10^4s^3 - 1.682 \times 10^7s^2 - 6.147 \times 10^7s - 1.143 \times 10^9}{s^4 + 24s^3 + 4366s^2 + 1.568 \times 10^4s + 2.916 \times 10^5} \quad (22)$$

4. For Output 4:

$$G_{4,2}(s) = \frac{3920s^2 + 1.568 \times 10^4s + 2.916 \times 10^5}{s^4 + 24s^3 + 4366s^2 + 1.568 \times 10^4s + 2.916 \times 10^5} \quad (23)$$