Derivation of State-Space Representation for a Quarter Car Model

PS

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1 Introduction

This document provides the derivation of the state-space representation from Newton's second law for a quarter car model. The quarter car model consists of a sprung mass m_s, representing the car body, and an unsprung mass m_u, representing the wheel assembly. The system also includes a damper and a spring connecting the two masses, as well as an actuator force F_a and a sensor for feedback, see figure 1. We aim to derive the equations of motion for both masses and represent the system in state-space form, focusing on the velocity and position of both masses as the outputs.

2 Equations of Motion

To derive the equations of motion, we apply Newton's second law $F = ma$, considering the forces due to the spring, damper, actuator, and the external road profile.

2.1 Sprung Mass

For the sprung mass m_s, the forces include the spring force $k_s(z_u - z_s)$, the damping force $b_s(\dot{z}_u - \dot{z}_s)$, and the actuator force F_a. The equation of motion is:

$$m_s\ddot{z}_s = -k_s(z_s - z_u) - b_s(\dot{z}_s - \dot{z}_u) + F_a \tag{1}$$

2.2 Unsprung Mass

For the unsprung mass m_u, the forces include the tire stiffness $k_t(z_r - z_u)$ and the forces from the spring and damper. The equation of motion is:

$$m_u\ddot{z}_u = -k_t(z_u - z_r) + k_s(z_s - z_u) + b_s(\dot{z}_s - \dot{z}_u) \tag{2}$$

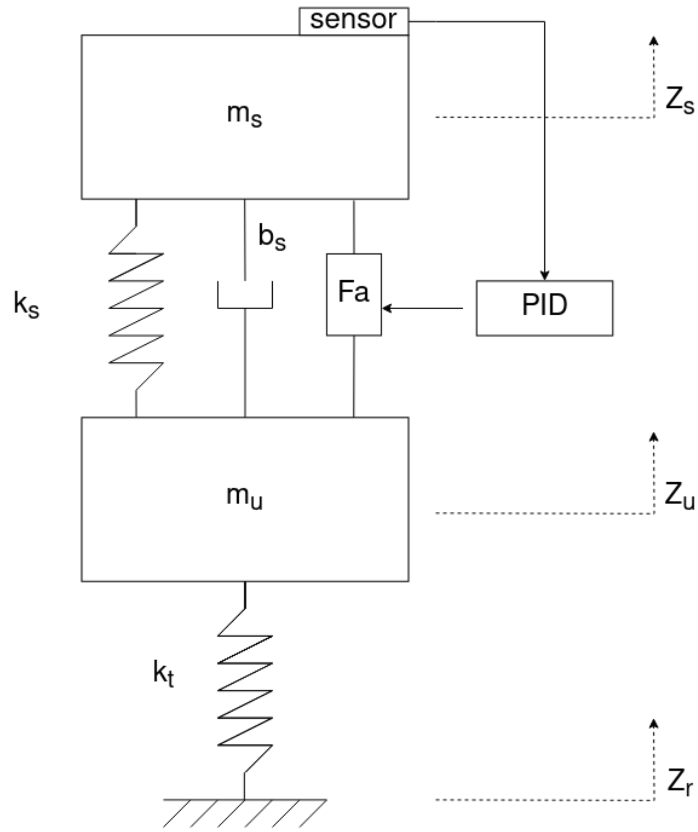

Figure 1: Quarter Car Model

3 State-Space Representation

To form the state-space representation, we define the state vector $\mathbf{x}$, the input vector $\mathbf{u}$, and the output vector $\mathbf{y}$. The state vector is chosen as $\mathbf{x} = [z_s, \dot{z}_s, z_u, \dot{z}_u]^T$, where z_s and z_u are the displacements of the sprung and unsprung masses, respectively, and $\dot{z}_s$ and $\dot{z}_u$ are their velocities.

The input vector $\mathbf{u}$ is composed of the road displacement Z_r and the actuator force F_a:

$$\mathbf{u} = \begin{bmatrix} z_r \\ F_a \end{bmatrix} \tag{3}$$

The state-space equations can then be written as:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{4}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{5}$$

where **A**, **B**, **C**, and **D** are matrices that define the system dynamics and output relationships.

3.1 State-Space Matrices

Using the equations of motion, we derive the state-space matrices:

System matrix **A**:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_s}{m_s} & -\frac{b_s}{m_s} & \frac{k_s}{m_s} & \frac{b_s}{m_s} \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{m_u} & \frac{b_s}{m_u} & -\frac{k_t+k_s}{m_u} & -\frac{b_s}{m_u} \end{bmatrix} \tag{6}$$

Input matrix **B**:

$$\mathbf{B} = \begin{bmatrix} 0 & 0 \\ \frac{1}{m_s} & 0 \\ 0 & 0 \\ 0 & \frac{k_t}{m_u} \end{bmatrix} \tag{7}$$

Output matrix **C**:

$$\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{8}$$

Feedthrough matrix **D**:

$$\mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{9}$$

4 Modified Output Matrix for Acceleration and Displacement

The modified output matrix **C** to extract the acceleration of the sprung mass $\ddot{z}_s$, the displacement of the sprung mass z_s, the acceleration of the unsprung mass $\ddot{z}_u$, and the displacement of the unsprung mass z_u is given by:

$$\mathbf{C} = \begin{bmatrix} -\frac{k_s}{m_s} & -\frac{b_s}{m_s} & \frac{k_s}{m_s} & \frac{b_s}{m_s} \\ 1 & 0 & 0 & 0 \\ \frac{k_s}{m_u} & \frac{b_s}{m_u} & -\frac{k_t+k_s}{m_u} & -\frac{b_s}{m_u} \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{10}$$

Note that the feedthrough matrix **D** remains a zero matrix unless there is a direct relationship between the inputs and the outputs:

$$\mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{11}$$