Orginasatorisch ...

→ TUM-Moodle
→ Zulip
→ TUM-Live
→ TUM-Online
→ Artemis

Logik

a) Aussagenlogik

Junktoren

- UND: ∧
- ODER: ∨
- NICHT: ¬
- Genau DANN WENN: ⟺
- FALLS, DANN: ⟹
- XOR: ⊙

unär

binär

b) Prädikantenlogik

Quantoren

Für alle: ∀
Es existiert: ∃
} Spoiler Alert !!!

Atomare Aussagen A, B ....
kann man mit Junktoren verbinden

→ wahr 1
→ falsch 0

Bsp: Aussage A: Es regnet
B: Die Sonne scheint

mit Junktoren: Es regnet und die Sonne scheint

genauer: A und B

A	¬A
wahr	falsch
falsch	wahr

2^1

$2^2 = 4$

A	B	A UND B
falsch 0	falsch 0	falsch
falsch 0	wahr 1	falsch
wahr 1	falsch 0	falsch
wahr 1	wahr 1	wahr

A	B	A ODER B
falsch	falsch	falsch
falsch	wahr	wahr
wahr	falsch	wahr
wahr	wahr	wahr

IF THEN

A	B	A IMP B
falsch	falsch	wahr
falsch	wahr	wahr
wahr	falsch	falsch
wahr	wahr	wahr

A	B	A GDW B
falsch	falsch	wahr
falsch	wahr	falsch
wahr	falsch	falsch
wahr	wahr	wahr

A	B	A ⊕ B
falsch	falsch	wahr
falsch	wahr	falsch
wahr	falsch	falsch
wahr	wahr	wahr

Präzedenzordnung

→ man soll wissen, auf welche Teilaussagen die Junktoren genau beziehen.

() bindet stärker als alles
NICHT " " " UND
UND " " " ODER
ODER " " " IMP

!

Semantische Äquivalenzen

De Morgan'sche Gesetze:

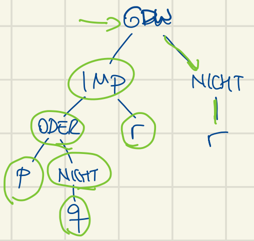
NICHT (A UND B) = NICHT A ODER NICHT B

NICHT (A ODER B) = NICHT A UND NICHT B

Syntaxbaum:

⤳ von innen nach außen lösen (Wahrheitstabelle) | von außen nach innen Syntaxbaum

⤳ Ausblick: ((P ODER NICHT q) IMP r) GDW NICHT r



Reihenfolge wichtig !!!

1.1 Truth tables and syntax trees

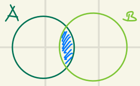
Given are the following formalized propositions: →KV-Diagrammen–VENN Diagrammen

$$F_1 = ((A \text{ AND}(\text{IF } B \text{ THEN } C)) \text{ OR}(C \text{ AND NOT } A))$$
$$F_2 = ((A \text{ AND NOT } B) \text{ OR}((C \text{ OR } B) \text{ AND}(C \text{ OR NOT } B)))$$
$$F_3 = (((\text{NOT } A) \text{ IFF } B) \text{ AND}(B \text{ AND}(A \text{ OR } C)))$$
$$F_4 = (((\text{NOT } A) \text{ AND}(\text{IF } C \text{ THEN } B)) \text{ AND}(A \text{ OR } C))$$

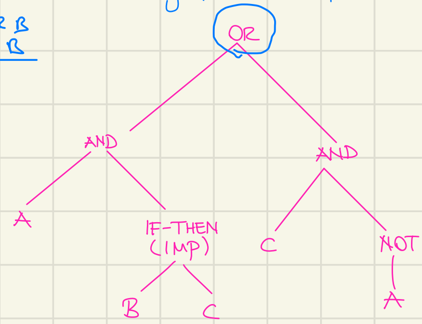
A, B and C stand for arbitrary propositions. Analogously to arithmetic expressions, brackets indicate in which order to evaluate the connectives.
Klammern

A UND B
$A \wedge B$

$A \cap B$ geschnitten
$A \cup B$ vereinigt



Syntaxbaum



F1: 2 3

A	B	A IMP B
falsch	falsch	wahr
falsch	wahr	wahr
wahr	falsch	falsch
wahr	wahr	wahr

innen → außen

A	B	A ODER B $A \vee B$
0	0	0
0	1	1
1	0	1
1	1	1

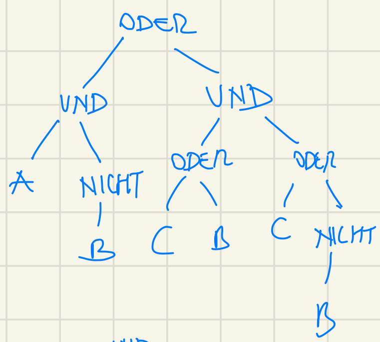
$B \Rightarrow C$

A		B	C			((A AND(IF B THEN C))			OR		(C AND NOT A))	
FALSE	0	FALSE	FALSE	0	0	FALSE	1	0	FALSE	0	FALSE	1
FALSE	0	FALSE	TRUE	0	0	FALSE	1	1	TRUE	1	TRUE	1
FALSE	1	TRUE	FALSE	0	0	FALSE	0	0	FALSE	0	FALSE	1
FALSE	1	TRUE	TRUE	0	0	FALSE	1	1	TRUE	1	TRUE	1
TRUE	0	FALSE	FALSE	1	1	TRUE	1	0	TRUE	0	FALSE	0
TRUE	0	FALSE	TRUE	1	1	TRUE	1	1	TRUE	1	FALSE	0
TRUE	1	TRUE	FALSE	1	0	FALSE	0	0	FALSE	0	FALSE	0
TRUE	1	TRUE	TRUE	1	1	TRUE	1	1	TRUE	1	FALSE	0

markierte Ergebnisse können in VENN-Diagrammen gezeigt werden

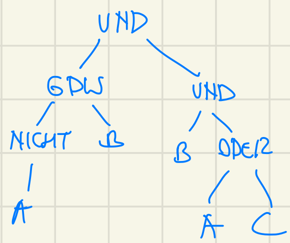
F2:

A	B	C	((A AND NOT B) OR((C OR B)	AND	(C OR NOT B)))
FALSE	FALSE	FALSE	FALSE	FALSE	TRUE
FALSE	FALSE	TRUE	TRUE	TRUE	TRUE
FALSE	TRUE	FALSE	TRUE	FALSE	FALSE
FALSE	TRUE	TRUE	TRUE	TRUE	TRUE
TRUE	FALSE	FALSE	TRUE	TRUE	TRUE
TRUE	FALSE	TRUE	TRUE	TRUE	TRUE
TRUE	TRUE	FALSE	TRUE	FALSE	FALSE
TRUE	TRUE	TRUE	TRUE	TRUE	TRUE



F3:

A	B	C	(((NOT A) IFF B)	AND	(B AND(A OR C))))
FALSE	FALSE	FALSE	FALSE	FALSE	FALSE
FALSE	FALSE	TRUE	FALSE	FALSE	FALSE
FALSE	TRUE	FALSE	TRUE	FALSE	FALSE
FALSE	TRUE	TRUE	TRUE	TRUE	TRUE
TRUE	FALSE	FALSE	TRUE	FALSE	FALSE
TRUE	FALSE	TRUE	TRUE	FALSE	FALSE
TRUE	TRUE	FALSE	FALSE	FALSE	TRUE
TRUE	TRUE	TRUE	FALSE	FALSE	TRUE

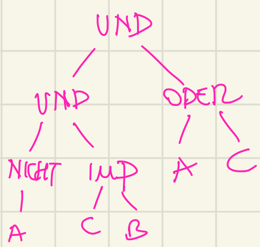


iff ≡ GDW ≡ $\Leftrightarrow$

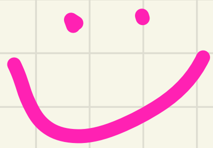
F4:

$C \Rightarrow B$

A	B	C	(((NOT A) AND(IF C THEN B))	AND	(A OR C))
FALSE	FALSE	FALSE	TRUE	FALSE	FALSE
FALSE	FALSE	TRUE	FALSE	FALSE	TRUE
FALSE	TRUE	FALSE	TRUE	FALSE	FALSE
~~FALSE~~	~~TRUE~~	~~TRUE~~	~~TRUE~~	~~TRUE~~	~~TRUE~~
TRUE	FALSE	FALSE	FALSE	FALSE	TRUE
TRUE	FALSE	TRUE	FALSE	FALSE	TRUE
TRUE	TRUE	FALSE	FALSE	FALSE	TRUE
TRUE	TRUE	TRUE	FALSE	FALSE	TRUE



Ihr seid dran



Hier war
Roman

1.2 Brackets and ambiguity

(a) In formalized propositions (formulas, expressions, terms) brackets are used to resolve ambiguities: For arithemtic expressions we usually have some sort of "operator precedence" e.g. "evaluate multiplication before addition":

Präsidenz Ordnung

$$(5 \cdot 3) + 3 = 5 \cdot 3 + 3 \neq 5 \cdot (3 + 3)$$

Aussagenlogik

Similar precedence rules exist for propositional logic, but these often depend on the author/lecture/book/programming language.

Consider the following formalized proposition:

$$\boxed{\text{NOT}\, A \,\text{AND}\, \text{NOT}\, B \,\text{OR}\, C}$$

Without precedence rules, we could read it as

$$\text{NOT}(A\,\text{AND}((\text{NOT}\,B)\,\text{OR}\,C))$$

but also as

$$((\text{NOT}\,A)\,\text{AND}(\text{NOT}(B\,\text{OR}\,C)))$$

Enumerate all possible ways of how to structure

$$\text{NOT}\,A\,\text{AND}\,\text{NOT}\,B\,\text{OR}\,C$$

using brackets; for each proposition you obtain, determine the syntax tree and truth table.

1. (NICHT A) UND (NICHT B ODER C)
2. ((NICHT A UND NICHT B) ODER C)
3. NICHT (A UND B) ODER C
4. NICHT (A UND (NICHT B ODER C)
5. (NICHT (A UND NICHT B)) ODER C)
6. (NICHT A) UND (NICHT (B ODER C))
7. (NICHT (A UND NICHT B ODER C))

Syntax tree ?

(b) In *discrete structures* we will only make use of the single precedence rule:

"Negation (NOT) always binds the strongest"

What is now the answer to (a)?

(((NICHT A) UND (NICHT B)) ODER C)

() bindet stärker als alles
NICHT " " " UND
UND " " " ODER
ODER " " " IMP

!

((NOT A) AND (NOT B)) OR (C)

1.3 NOR and NAND

One can show that for propositional logic the connectives "AND", "OR" and "NOT" suffice; any other connective can be expressed using these three standard connectives. e.g. "IF A THEN B" is synonymous with (semantically equivalent to) "B OR NOT A" (*independent of the concrete choice of* A, B).

A IMP B ≡ NICHT A ODER B

Every conventional processor is ultimately a physical realization of propositional logic; but processors are often not built using the three standard connectives, instead they exclusively use either "NAND" or "NOR" gates as defined by the following truth table:

A	B	(A NAND B)	(A NOR B)
FALSE	FALSE	TRUE Falsch	TRUE
FALSE	TRUE	TRUE	FALSE
TRUE	FALSE	TRUE	FALSE
TRUE	TRUE	FALSE	FALSE

In words: "A NAND B" is a shorthand for "NOT(A AND B)", while "A NOR B" is one for "NOT(A OR B)".

Express "AND", "OR" and "NOT" using only "NAND" resp. only using "NOR".

(E.g. "NOT A" is "(A NAND A)" resp. "(A NOR A)")

A	B	A UND B
falsch	falsch	falsch
falsch	wahr	falsch
wahr	falsch	falsch
wahr	wahr	wahr

$2^2 = 4$

NAND

A UND B ≡ (A NUND B) NUND (A NUND B) ≡ NICHT (A NOR B)

A	UND	B	(A NUND B)	NUND	(A NUND B)
0	0	0	1	0	1
0	0	1	1	0	1
1	0	0	1	0	1
1	1	1	0	1	0

A ODER B ≡ (A NUND A) NUND (B NUND B) ≡ (NICHT A) nor (NICHT B)

NICHT A ≡ A NUND A ≡ A nor A

1.4 XOR

In formal logic, mathematics and programming languages "OR" (disjunction) has the meaning of "at least one". But in natural language "or" is often used in the sense of "either-or", i.e. "exactly one out of two". The latter is typically called *exclusive-or* in computer science and often abbreviated to "XOR".

For comparison:

A	B	A OR B	A XOR B	A GDW B
FALSE	FALSE	FALSE	FALSE	1
FALSE	TRUE	TRUE	TRUE	0
TRUE	FALSE	TRUE	TRUE	0
TRUE	TRUE	TRUE	FALSE	1

(So, the two connectives differ only a situation where both A and B are true.)

Express "XOR" using only the standard connectives "NOT", "AND" and "OR".

A xor B ≡ (A UND (NICHT B)) ODER ((NICHT A) UND B) !!!

A GDW B = (A → B) UND (B → A)

NICHT A ODER B

A	B	A XOR B	$\overbrace{(A \text{ ODER } B)}^{q}$	(NOT A)	$\overbrace{\text{ODER}}^{m}$	(NOT B)	q UND m
0	0	0	0	1	1	1	0
0	1	1	1	1	1	0	1
1	0	1	1	0	1	1	1
1	1	0	1	0	0	0	0

1.6 Syntax versus semantics

Consider the two propositions

- NOT(A AND B)
- (NOT A) OR (NOT B)

semantische Äquivalenzen

"Syntactically" they are different (their syntax trees are different, they differ in the connectives that occur in them, they are simply different texts), but "semantically" they are the "same" (they are synonymous, their meaning is the same in every possible situation. their truth tables coincide).

(a) Check that the two statements are semantically equivalent by determining the truth tables.

A	B	NOT (A UND B)		(NICHT A) ODER (NICHT B)		
0	0	1	0	1	1	1
0	1	1	0	1	1	0
1	0	1	0	0	1	1
1	1	0	1	0	0	0

De Morgan'sche Gesetze

De Morgan'sche Gesetze:

NICHT (A UND B) = NICHT A ODER NICHT B

NICHT (A ODER B) = NICHT A UND NICHT B

(b) Also check that

- NOT(A OR B)
- (NOT A) AND (NOT B)

are semantically equivalent.

Auch mit Wahrheitstabelle zu zeigen

Äquivalenzumwandlung

NICHT (A ODER B) = NICHT A UND NICHT B

2. Mengen

$\{1,1,2,3\} \overset{?}{=} \{1,2,3\}$

$\Leftrightarrow$ → theoretische Informatik

$\leftrightarrow$ → DS

Unter einer Menge verstehen wir

verstehen wir jede Zusammenfassung

M von bestimmten wohlunterschiedenen

Objekten m

Bsp: Mengen, Zahlen ....

* Per Definition entweder $m \in M$ oder $m \notin M$

Russel'sche Antinomie: Gibt es eine Menge aller Mengen, die sich nicht selbst enthalten. } Paradox Alarm!!!

$\{\ \} :=$ Mengen ~~Reihenfolge~~, ~~Vielfachheit~~

$(\ ) :=$ Tupel ✓Reihenfolge, ✓Vielfachheit $(1,2,3) \overset{?}{=} (1,2,3,3)$

Bsp: Un/endliche Mengen

$[3] := \{0,1,2\}$

$\mathbb{N} := \{n \mid n$ ist eine positive natürliche Zahl$\}$ implizite Darstellung → Anforderung

$:= \{1,2,3\}$ explizite Darstellung

$\{x \in M \mid x$ erfüllt Anforderung $\varphi\}$

die Menge, die genau die Elemente aus M enthält, welche die Anforderung φ erfüllen (bzw. die die Eigenschaft φ besitzen)

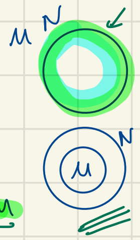
* Leere Menge

$\{\} \equiv \emptyset$ $\qquad \{\emptyset\} \neq \emptyset$ oder $\{\{\emptyset\}, \emptyset\}$ $\qquad \forall m\ m \notin \{\}$

→ $\emptyset$ ist Teilmenge jeder Menge

siehe: Potenzmenge

* Teilmengenbeziehung



Eine Menge M ist Teilmenge einer Menge N

Bsp: $\emptyset \subseteq \{\} \subset \{1,2,3,1\} \subseteq \{3,1,2\} \subsetneq \{1,2,4,3\}$

* Falls jedes Element von M auch Element von N ist → $M \subseteq N$ $\qquad m \in N$ und $m \in M$

* echte Teilmenge $\subsetneq$ $\qquad M \subset N \Leftrightarrow N \subset M$ bzw.

$\subsetneq$

* Gleichheit der Mengen

$M = N$

$m \in M$ bzw. $m \in N$

$\{1,2,3,1\} = \{3,1,2\}$

× Mehrfachheit

× Reihenfolge

* Vereinigung | Schnitt

$\bigcup \{\{a,b\},\{1,2\},\{1,a\}\} = \{a,b,1,2\}$

$\boxed{\{a,b\} \cup \{1,2\}} = \{a,b,1,2\}$

$\{a,b\} \cup \{1,2\} \cup \{1,a\}$

$\boxed{\bigcap \emptyset \text{ undefiniert}}$

M eine nicht leere Menge von Mengen

$\bigcap \{\{1,2,3,5\},\{3,2,4,1\},\{4,3,1\}\} = \{1,3\}$

$\{1,2,3,5\} \cap \{3,2,4,1\} \cap \{4,3,1\}$

$\{1,3,2\} \cap \{4,3,1\} = \{1,3\}$

* wichtige Notationen:

$$\bigcup_{i=1}^{k} M_i = M_1 \cup \ldots \cup M_k = \bigcup \{M_1, \ldots, M_k\}$$

$$\bigcap_{i=1}^{k} M_i = M_1 \cap \ldots \cap M_k = \bigcap \{M_1, \ldots, M_k\}$$

* Potenzmenge

M sei eine Menge, dann gibt es auch eine Menge, die als Elemente genau die Teilmengen von M enthält. $P(M) :\equiv 2^M$

Definition: $2^M := \{N \mid N \subseteq M\}$

$M = \{1,2\}$, $2^M = \{\emptyset, \{1\}, \{2\}, \overset{id}{\{1,2\}}\}$

$\{\emptyset\}$ $\emptyset$ $\emptyset$

$|M| = 2$ Kardinalität der Mengen

$2^2 = 4$ $\{\ \} = \emptyset$

* Rechenregel

$A \cap B = B \cap A$
$A \cup B = B \cup A$ } kommutativ

$(A \cap B) \cap C = A \cap (B \cap C)$ } assoziativ

$\bigcap \{A, B, C\}$

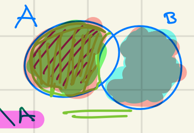
analog für $\bigcup$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ } Distributivität

$a \cdot (b+c) = (a \cdot b) + (a \cdot c)$

$A \setminus B$

$A \Delta B$



$A \setminus B$ gdw $B \setminus A$

De Morgan'sche Gesetze (analog)

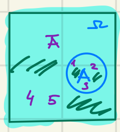
$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

$\Omega :=$ Universum, Grundmenge

Bsp: $\Omega = \{1,2,3,4,5\}$

$A = \{1,2,3\}$

$\bar{A} = \{4,5\}$ $\bar{A} := \Omega \setminus A$

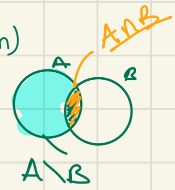


$\overline{A \cup B} = \bar{A} \cap \bar{B}$

$\overline{A \cap B} = \bar{A} \cup \bar{B}$ (De Morgan)

$A \setminus B = A \cap \bar{B}$

$\Omega = A \cup \bar{A}$

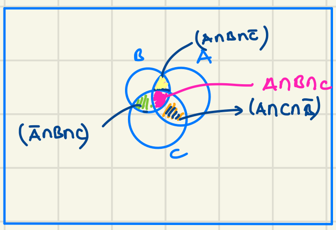


* Tupel, Sequenzen, Folge

↓

Reihenfolge
Häufigkeit

$(a,b,c) \neq (a,c,b)$



$[k] = \{1,3,\ldots,k\}$

$[0] = \emptyset$

$x_i :=$ i-ter Eintrag

Folgen: $m \in M$

$(m_i)_{i \in \mathbb{N}_0}$ bzw. $(m_i)_{i \in \mathbb{N}}$

Bsp: Für jedes $i \in \mathbb{N}_0$ sei $a_i = i$
$(a_i)_{i \in \mathbb{N}_0} = (i)_{i \in \mathbb{N}_0} = (0,1,2,3,4)$

$n \in \mathbb{N}_0$ $a_n = n^2$
$(a_n)_{n \in \mathbb{N}_0} = (n^2)_{n \in \mathbb{N}_0} = (0,1,4,9,16,\ldots)$

Häufig definiert man unendliche Folgen, indem man eine Konstruktionsvorschrift angibt, wie man a_i aus den bereits konstruierten Werten $a_0, \dots, a_{i-1}$

2 First-order Logic

2.1 Syllogisms

Decide whether the following syllogisms are valid/correct/always true, i.e. that in every situation (independent of the precise meaning of the predicates), in which both statements above the dividing line (premisses/assumptions/hypotheses) are true/satisfied, the statement underneath the dividing line (conclusion/consequence) is also *necessarily* true.

Visualize the relationship between the types of objects graphically as discussed in the lecture (see the Bandersnatch example). For each example also provide the first-order logic structure of each statement ("FOR ALL x with ...", "NOT FOR ALL x ...", "EXISTS x ...", "NOT EXISTS x ...") as shown here:

All bandersnatches are borogoves
All borogoves are slithy

All bandersnatches are slithy

has the structure

FOR ALL x: IF $P(x)$ THEN $Q(x)$
FOR ALL x: IF $Q(x)$ THEN $R(x)$

FOR ALL x: IF $P(x)$ THEN $R(x)$

where

$P(x) := x$ is a bandersnatch
$Q(x) := x$ is a borogove
$R(x) := x$ is slithy

rectangles
squares
quadrilaterals

Then group the syllogisms by their respective first-order logic structure (ignoring the order of the statements above the dividing line).

All rectangles are quadrilaterals
All squares are rectangles

All squares are quadrilaterals

No rectangle is a circle
All squares are rectangles

No square is a circle

Some rhombi are rectangles
All squares are rectangles

Some rhombi are squares

Some rhombi are squares
All squares are rectangles

Some rhombi are rectangles

All squares are rectangles
Some rhombi are squares

Some rhombi are rectangles

All rectangles are quadrilaterals
All squares are rectangles

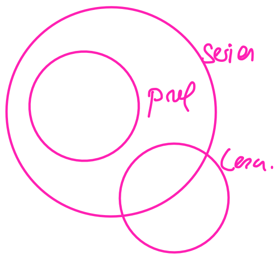
Some squares are quadrilaterals

No rectangle is a circle
All squares are rectangles

Some squares are not circles

$\forall x\,(\text{Professor}(x) \rightarrow \text{Serious}(x))$
All professors are serious
$\exists x\,(\text{Lecturer}(x) \wedge \neg \text{Serious}(x))$
Some lecturers are not serious

Some lecturers are not professors
$\exists\,(\text{Lecturer}(x) \wedge \neg \text{Professor}(x))$



No mammal breathes through gills
All fish breathe through gills

No fish is a mammal

No mammal breathes through gills
All fish breathe through gills

No mammal is a fish

No animal breathing through gills is a mammal
Some marine animals are mammals

Some marine animals don't breath through gills

Some fruits are apples
All fruits are parts of plants

Some parts of plants are apples

For all x, if x is a professor, then x is Serious

32

3.1

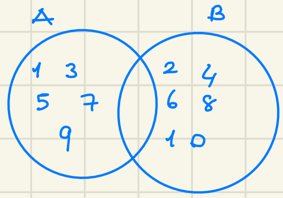
$2^{|A|} = 2^{5} = 32$ $\quad P(A) = \{\emptyset, \{1\}, \{3\}, \{5\}, \{7\}, \dots\}$

3.1 Operations on sets

Let the sets $A = \{1,3,5,7,9\}$, $B = \{2,4,6,8,10\}$ and $D = \{5,6,7,8,9,10\}$.

Explicitly write down the sets described by the following expressions:

a) $A \cup B$,	b) $A \cap B$,	c) $A \setminus B$,
d) $A \setminus D$,	e) $B \setminus D$,	f) $D \setminus A$,
g) $D \setminus B$,	h) $D \setminus (A \cup B)$,	i) $D \setminus (A \cap B)$.



$\emptyset$

Lösung:

a) $A \cup B = \{1, \dots, 10\}$

b) $A \cap B = \emptyset$ $\quad \{\emptyset\}$

c) $A \setminus B = \{1,3,5,7,9\}$

d) $A \setminus D = \{1,3\}$

e) $B \setminus D = \{2,4\}$

f) $D \setminus A = \{6,8,10\}$

g) $D \setminus B = \{5,7,9\}$

h) $D \setminus (A \cup B) = \emptyset$

$\to (D \setminus A) \cap (D \setminus B)$

i) $D \setminus (A \cap B) = \{5,6,7,8,9,10\}$

$\to (D \setminus A) \cup (D \setminus B)$

$\emptyset$

Made with Goodnotes

3.2)

3.2 Cardinality of finite sets

Determine the number of elements of the following sets:

a) $\{1,4,6\}$, b) $\varnothing$, c) $\{\varnothing\}$, d) $\{\varnothing,\{1,2,3\}\}$.

Solution:

a) $|\{1,4,6\}| = 3$, b) $|\varnothing| = 0$, c) $|\{\varnothing\}| = 1$, d) $|\{\varnothing,\{1,2,3\}\}| = 2$.

3.3)

3.3 Intersection, union, and power sets

Let $M = \{\{1,2,3,5\},\{3,2,4,1\},\{4,3,1\}\}$.

1. Determine $\bigcap M$ and $\bigcup M$.
2. Determine $N = \{x \in 2^{\bigcup M} \mid \text{for every } m \in M \text{ we have } x \subseteq m\}$.
3. Determine $\bigcup N$.

$\forall m \in M$ → Teilmenge

$x \subseteq m$

$m = \{1,2,3,5\}$, $\{3,2,4,1\}$, $\{4,3,1\}$

Solution:

1. $\bigcap M = \{1,3\}$ ✓
   $\boxed{\bigcup M = \{1,2,3,4,5\}}$ ✓
2. $N = \{\{\},\{1\},\{3\},\{1,3\}\}$
3. $\bigcup N = \{1,3\}$

$x \in 2^5 \Rightarrow 32$ Teilmengen $= \{\varnothing, \dots, \{1,2,3,4,5\}\}$

$2^2 = 4$

$\{\} \cup \{1\} \cup \{3\} \cup \{1,3\}$

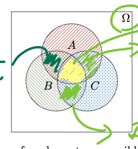
$\{1\} \cup \{3\} = \{1,3\} \cup \{1,3\} = \{1,3\}$

3.4)

3.4 Venn-Diagrams

Relations between sets can be visualized using Venn diagrams.

For this the sets are drawn as circles or ellipses, that overlap so that every possibility for an object to be an element of the given sets is represented. For three sets $A, B, C \subseteq \Omega$ the diagram looks as follows:



Determine three sets A, B, C with as few elements as possible using Venn diagrams so all of the following constraints are satisfied at once.

(1) $A \cap B \cap C = \varnothing$ (2) $A \cap B \neq \varnothing$ → $e \in A$ und $e \in B$ $= \{e\}$
(3) $B \cap C \neq \varnothing$ (4) $A \cap C \neq \varnothing$ → $m \in A$ und $m \in C$ $= \{m\}$

$A \neq \varnothing$

$k \in B$ und $k \in C$ $= \{k\}$

mindestens 1 Element

Lösung: $A = \{1,2\}$, $B = \{1,3\}$, $C = \{2,3\}$

$A \setminus B = A \cap \overline{B}$

$A \setminus (A \cap B \cap C)$

$(A \setminus A) \cup (A \setminus B) \cup (A \setminus C)$

$\{\}$

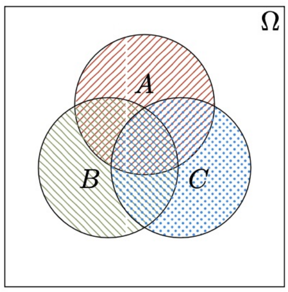
$B \cap C \cap \overline{A} \cup$

3.5 Simplifying set expeessions

20 min

We fix a set Ω as our universe. Let A, B, C be subsets of Ω.

The general Venn diagram is then



Each of the eight regions of the Venn diagram corresponds to an intersection of the given sets A, B, C or their complements $\overline{A}, \overline{B}, \overline{C}$ (wrt. the fixed Ω).

Hence, no matter how complicated a set expression built from A, B, C might be, it can be simplified to a union of those 8 basic intersections.

E.g. the expression $(A \setminus (B \cup C)) \cup ((B \cap C) \setminus A)$ describes the same set as $(A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap C)$.

Similar express the following set expressions as union of those eight basic intersection that correspond to the eight regions of the Venn diagram.

1. $((A \cup (B \cup C)) \cap (C \setminus A))$.
2. $\overline{(A \cap \overline{B} \cap C)} \setminus (A \setminus B)$.

1) $((A \cup (B \cup C)) \cap (C \setminus A))$

$(A \cap (C \setminus A)) \cup ((B \cup C) \cap (C \setminus A))$

$((A \cap (C \cap \bar{A})) \cup ((B \cup C) \cap (C \cap \bar{A}))$ lassoziativität

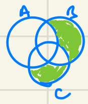
$A \cap \bar{A} \cap C$

$\emptyset$

$(\emptyset)$ $\cup$ $(((B \cup C) \cap C) \cap ((B \cup C) \cap \bar{A}))$ $\cap$ $= \emptyset$

$((B \cap C) \cup (C \cap C)) \cap$

C



$((B \cap C) \cup C)$

$C \quad \cap \bar{A} = (\bar{A} \cap C)$

$C \setminus A = C \cap \bar{A}$

$\overline{B \cap C} = \bar{B} \cup \bar{C}$

2) $(A \cap \overline{B \cap C}) \setminus (A \setminus B)$ $\quad (\overline{m \cap n}) = \bar{m} \cup \bar{n}$

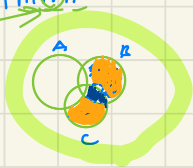
$= (\bar{A} \cup \overline{B \cap C}) \setminus (A \cap \bar{B})$ $\quad \overline{B \cap C}$

$(\bar{A} \cup (B \cap C))$ $\quad \bar{m} = \bar{\bar{A}} = A$

$m \setminus n$ $\quad \overline{\overline{B \cap C}} \} \; B \cap C$

$= ((\bar{A} \cup B) \cap (\bar{A} \cup C)) \setminus (A \cap \bar{B})$ $\quad | m \cap \bar{n}$

$m \quad n$



$= ((\bar{A} \cup B) \cap (\bar{A} \cup C)) \cap \overline{(A \cap \bar{B})}$

$= (B \cap C \cap \bar{A}) \quad \cap (\bar{A} \cup \bar{\bar{B}})$

$= (B \cap C) \quad \cap (\bar{A} \cup B)$

$((B \cap C) \cap \bar{A}) \cup ((B \cap C) \cap B)$

$((B \cap C) \cap \bar{A}) \cup (B \cap C)$

$(B \cap C) \cup (B \cap C)$

$(B \cap C) \cup \bar{A}$

Induktion:

man benutzt Induktionsbeweis, wenn man für alle gültige Werte die gegebene mathematische Formel beweisen möchte.

für alle $x \in \mathbb{N}_0 : P(x)$

1) erstes Element
was kann man für x als erstes einsetzen?

2) Gilt das für n+1?

3) Die Gleichheit bei beiden Seiten überprüfen r.s ≡ l.s

Struktur ist wichtig !!!

Induktionsbasis (I.B.) : P(0)

Induktionsschritt (I.S.): Fixiere ein beliebiges $n \in \mathbb{N}_0$, über das bis jetzt noch keine Annahmen getroffen worden sind

Induktionsannahme → I.A : P(n) gilt für das fixierte n

Induktionsbehauptung → I.B : P(n+1) gilt für das fixierte n

→ Schließlich beweise P(n+1) unter den bereits getroffenen Annahmen unter zusätzlichen Annahmen P(n) für das fixierte n.

4.1)

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)\cdot(2n+1)}{6}$$

Induktionsbasis: n = 0

$$\sum_{i=0}^{0} i = \frac{0\cdot(0+1)}{2}$$

LHS: $\sum_{i=n}^{0} i \overset{n=0}{=} 0$

RHS: $\frac{n(n+1)}{2} \overset{n=0}{=} \frac{0\cdot 1}{2} = 0$

I.S: n≥0 beliebig, aber fixiert

I.A: $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

I.B: $\sum_{i=0}^{n+1} i = \frac{(n+1)\cdot(n+2)}{2}$

$$\sum_{i=0}^{n+1} i = 0+1+2+\dots+n+(n+1)$$

$$= (n+1) + \sum_{i=0}^{n} i \quad \left\{\frac{n\cdot(n+1)}{2}\right\}$$

$$\overset{IA}{=} (n+1) + \frac{n\cdot(n+1)}{2}$$

$$= \frac{2(n+1)}{2} + \frac{n\cdot(n+1)}{2}$$

$$= \frac{2(n+1)+n\cdot(n+1)}{2}$$

$$= \frac{(2+n)\cdot(n+1)}{2}$$

$$= \boxed{\frac{(n+1)\cdot(n+2)}{2}}$$

□ q.e.d

2) $\sum_{i=0}^{n} i^2 = \frac{n \cdot (n+1) \cdot (2n+1)}{6}$

I.B: LHS: $\sum_{i=n}^{0} i^2 \overset{n=0}{=} 0$

gehalten ✓

RHS: $\frac{n \cdot (n+1) \cdot (2n+1)}{6} \overset{n=0}{=} \frac{0 \cdot 1 \cdot 1}{6} = 0$

I.S: n sei $n \geq 0$ aber fixiert

I.A: $\sum_{i=0}^{n} i^2 = \frac{n \cdot (n+1) \cdot (2n+1)}{6}$

I.B: $\sum_{i=0}^{n+1} i^2 = \frac{(n+1) \cdot (n+2) \cdot (2n+3)}{6}$

$$\sum_{i=0}^{n+1} i^2 = 0^2 + 1^2 + \dots + n^2 + (n+1)^2$$

$$= (n+1)^2 + (0^2 + 1^2 + 2^2 + \dots + n^2)$$

$$= (n+1)^2 + \sum_{i=0}^{n} n^2$$

$$\overset{IA}{=} (n+1)^2 + \frac{n \cdot (n+1) \cdot (2n+1)}{6}$$

$$= \frac{6(n+1) \cdot (n+1)}{6} + \frac{(n \cdot (2n+1)) \cdot (n+1)}{6}$$

$$\vdots$$

$$= \frac{(n+1) \cdot (n+2) \cdot (2(n+1)+1)}{6}$$

Relationen

$R \subseteq A \times B$

$a R b$ $\quad$ $a \not R b$

Typische binäre Relationen auf $\mathbb{R}$ $(\mathbb{N}_0, \mathbb{Z}, \mathbb{Q})$

$\leq, <, =, >, \geq$

$\neq$

$3 \leq_{\mathbb{N}} 5$ statt $(3,5) \in \leq_{\mathbb{N}}$

$3 <_{\mathbb{N}} 5$

Transponierte
Inverse Relation:

$R^T := R^{-1} = \{(b,a) \mid (a,b) \in R\}$

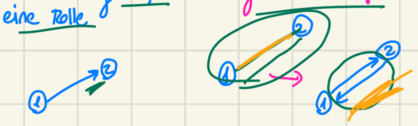
R: $a\ R\ b$
R^{-1}: $b\ R\ a$

Falls $A \times B$ endlich ist, kann R als gerichteter Graph dargestellt werden.

$(a,b) \in R$ => Kante
$x \in A \cup B$ => Punkt
$a = b$ => Schleife

die Richtung spielt eine Rolle

ungerichteter Graph



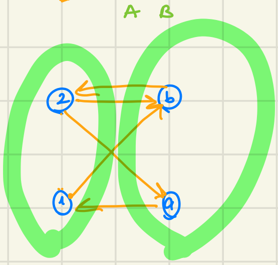
mit Pfeilen

jeder ungerichteter Graph ist ein gerichteter Graph

Bsp: $E = \{(1,b), (2,b), (2,a), (1,2), (b,b), (b,2), (a,1)\}$

$V = \{1, 2, a, b\}$

A $\quad$ B



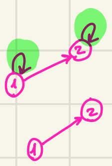
A $\quad$ B

Bipartit: falls $V = A \cup B$ mit $A \cap B = \emptyset$

mit $E \subseteq A \times B \cup B \times A$

es gibt nur Kanten zwischen A und B

Eigenschaften von Relationen



jeder Knoten hat eine Schleife

- reflexiv, falls: für alle $a \in A : (a,a) \in R$ $\quad (id_A \subseteq R)$
- irreflexiv, falls: für alle $a \in A : (a,a) \notin R$ $\quad (id_A \cap R = \emptyset)$
- symmetrisch, falls: „für alle $(s,t) \in R : (t,s) \in R$" $\quad (R = R^T)$
- antisymmetrisch, falls „für alle $(s,t) \in R$: falls $(t,s) \in R$, dann $s = t$

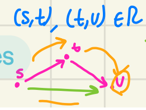
$R \cap R^T \subseteq id_A$

zwischen verschieden Knoten existiert höchstens eine Kante

- asymmetrisch, falls $\forall (s,t) \in R$: falls $(t,s) \notin R$, dann $(R \cap R^T = \emptyset)$

keine Schleifen zwischen je zwei verschieden Knoten höchstens eine Kante

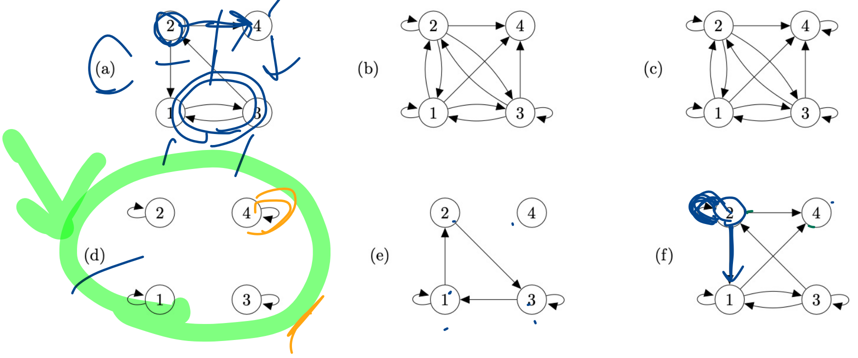
- transitiv, falls: „für alle $(s,t), (t,v) \in R : (s,v) \in R$" $(R R \subseteq R)$



in zwei Schritten

5.1 Properties of relations

For every of the depicted relations on $\{1, 2, 3, 4\}$ decide whether it is reflexive, irreflexive, symmetric, antisymmetric, asymmetric, and transitive, respectively?



transitive: zwei Schritten

antysymmetrie erlaubt schleifen

asymmetrie erlaubt keine Schleifen

Solution:

(a) irreflexive

(b) transitive

(c) reflexive, transitive

(d) reflexive, symmetric, anti-symmetric, transitive

(e) anti-symmetric

(f) none

	a	b	c	d	e	f
reflexiv	✗	✗	✓	✓	✗	✗
irreflexiv	✓	✗	✗	✗	✗	✗
symmetrisch	✗	✗	✗	✓	✗	✗
antisymmetrisch	✗	✗	✗	✓	✗	✗
asymmetrisch	✗	✗	✗	✗	✗	✗
transitiv	✗	✓	✓	✓	✗	✗

Äquivalenzrelationen:

$=_{\mathbb{Z}}$, $\equiv_k$ (Kongruenzmodulo k)

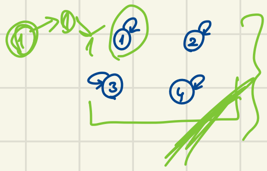
Gemeinsamkeiten: reflexiv, symmetrisch, transitiv

— $a =_{\mathbb{Z}} b$ „a und b dieselbe Zahl"

— $a \equiv_m b$ „a und b derselbe Rest bei Division durch m"

— Äquivalenzrelation

* R reflexiv, symmetrisch, transitiv

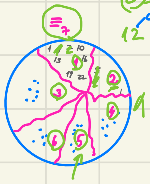


"Partitionierung"

5 –

$5 \equiv_7 5$

$12 \equiv_7 5$



$\frac{a-b}{n} = \frac{7}{7} = 1$

0, 1, 2, 3, 4, 5, 6

— Ordnungsrelation

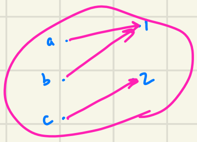
reflexiv, antisymmetrisch, transitiv

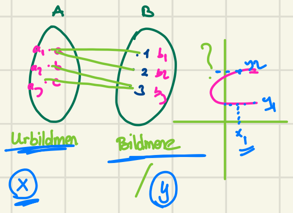
$\longrightarrow$ $f = \{(a,1),(b,1),(c,2)\} \subseteq A \times B$ für $A = \{a,b,c\}$ und $B = [3]$

— Funktionen

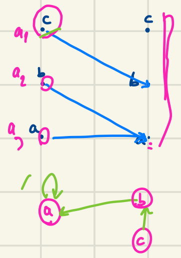
$R \subseteq A \times B$ ist eine Funktion, falls:

Für jedes $a \in A$ gibt es genau ein $b \in B$ mit $(a,b) \in R$

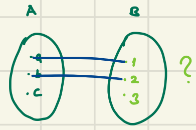




Als Funktion:



Als Relation:



— Funktionkomposition (Nacheinanderausführung) $(g \circ f)(a) := g(f(a)) :=$ „g nach f"

$f: A \to B$

$g: B \to C$

* Komposition ist assoziativ, aber nicht kommutativ

$(g \circ f): A \to C$

$(h \circ (g \circ f))(a) = h((g \circ f)(a))$

$= h(g(f(a)))$

$= (h \circ g)(f(a))$

$= ((h \circ g) \circ f)(a)$

$f(x)$

$\boxed{f(h(x))}$

$A \to B \to C$

$g(f(x)) \neq f(g(x))$

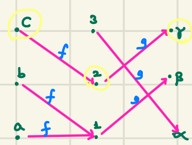
$f: x+1$

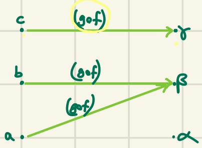
$f: x+2$

Relationales Produkt

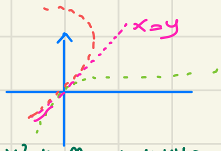
Komposition

$fg = \{(a, g(f(a))) \mid a \in A\} = (g \circ f)$





Eigenschaften von Funktionen



— Erinnerung: Für $f: A \to B$ und $Y \subseteq B$ bezeichnet

$f^{-1}(Y) = \{a \in A \mid f(a) \in Y\}$ die Menge der Urbilder von Y.

— $f: A \to B$

* injektiv $f(a) = f(a')$ stets $\Rightarrow$ $a = a'$
  $|f^{-1}(\{b\})| \leq 1$
* surjektiv falls für jedes $b \in B$ ein $a \in A$ mit $f(a) = b$ gibt

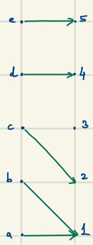
  Äquivalent: $f(A) = B$, $\operatorname{im}(f) = B$

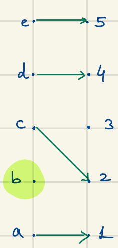
  $|f^{-1}(\{b\})| \geq 1$ für jedes $b \in B$

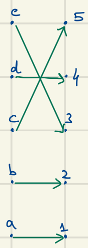
* bijektiv: injektiv + surjektiv

  Äquivalent: $|f^{-1}(\{b\})| = 1$ für jedes $b \in B$

Eine bijektive Funktion $f: A \to A$ nennt man Permutation







5.2 Properties of relations (2)

Consider the set

$$M = \{x, y, z\}$$

For each subproblem, define a binary relation on M satisfying the stated properties:

(a) reflexive, but not symmetric

(b) neither symmetric nor antisymmetric

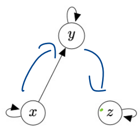
(c) antisymmetric, but not asymmetric

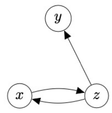
(d) not transitiv, but each pair of elements is in relation with each other in at least one of the two possible ways

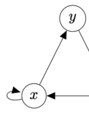
(e) symmetric and each pair of elements is in relation with each other in at least one of the two possible ways

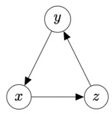
(f) reflexive and asymmetric

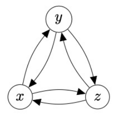
Solution: Other solutions are possible. These are only examples.

(a)


(b)


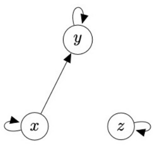
(c)


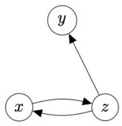
(d)


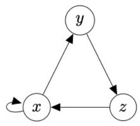
(e)


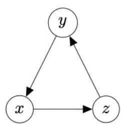
(f) Recall: "asymmetric" is synonymous with "antisymmetric and irreflexiv", thus "reflexiv and asymmetric" contains "reflexive and irreflexive" which is not possible for a nonempty set.

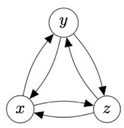
Solution: Other solutions are possible. These are only examples.

(a)


(b)


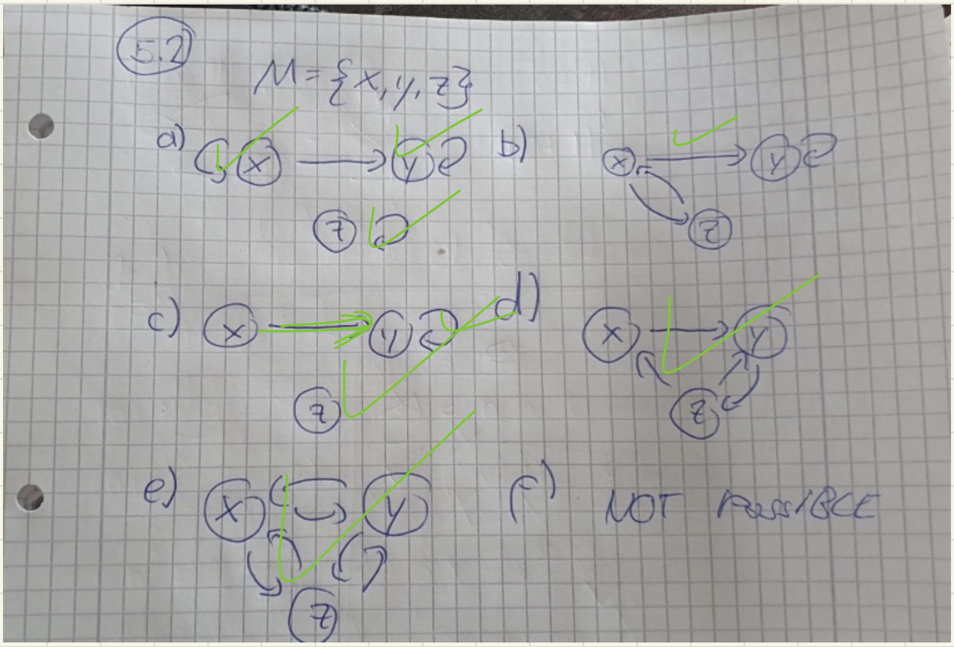
(c)


(d)


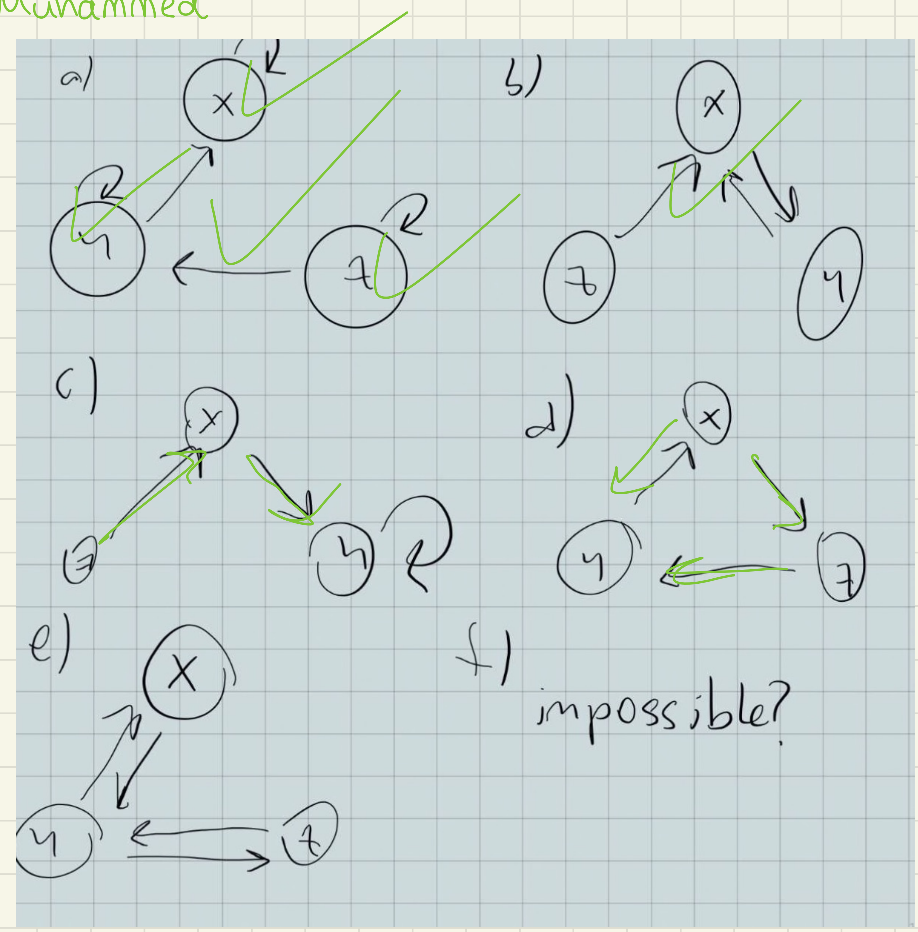
(e)


(f) Recall: "asymmetric" is synonymous with "antisymmetric and irreflexiv", thus "reflexiv and asymmetric" contains "reflexive and irreflexive" which is not possible for a nonempty set.

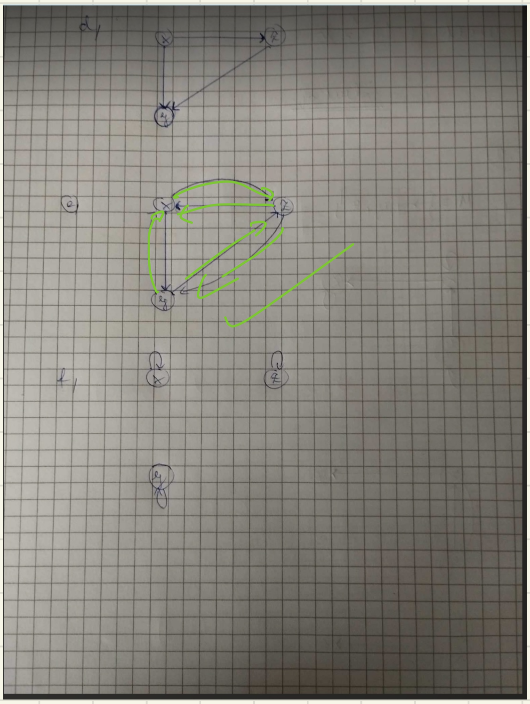
Leon:

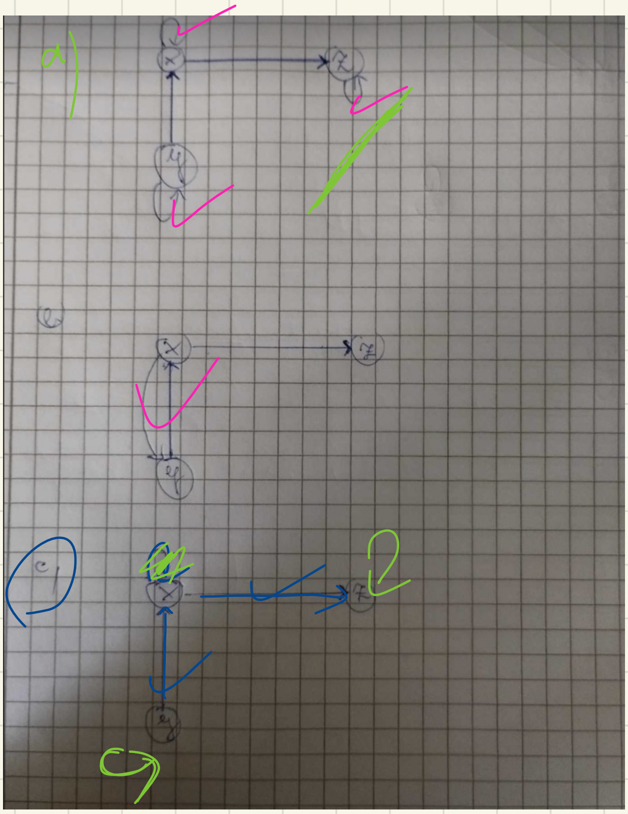


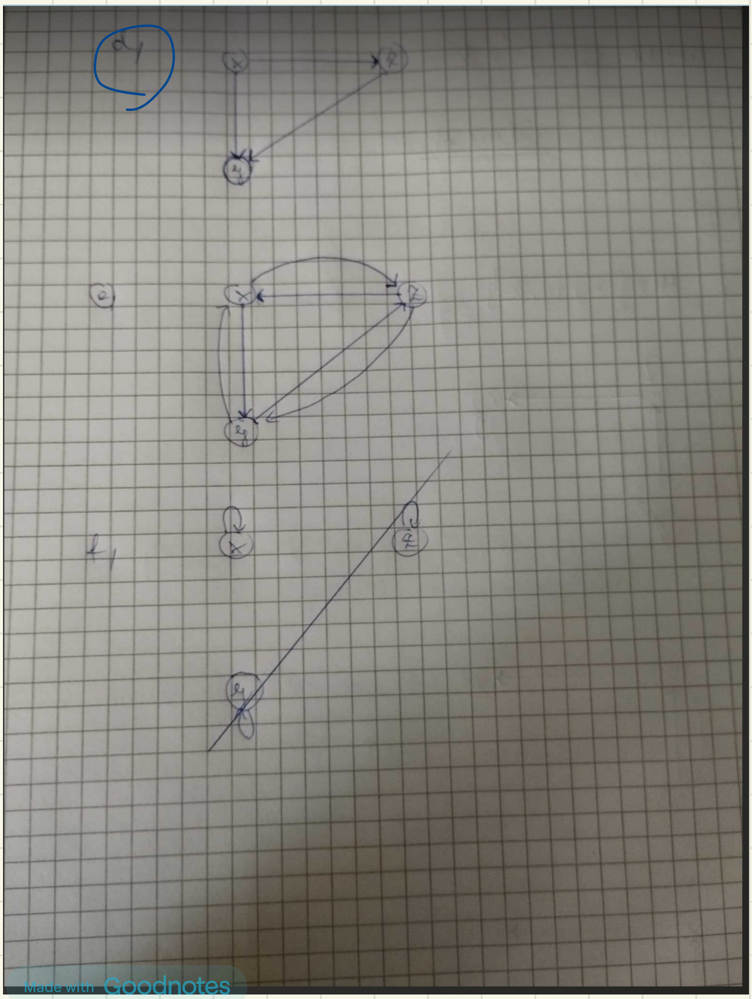
Muhammed



Levent :



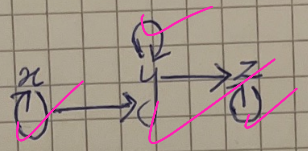


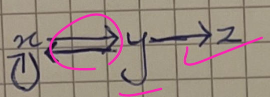


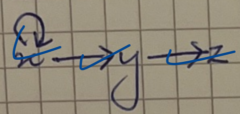
Ayush

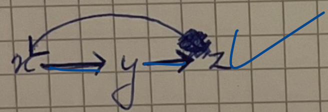
5.2]. Properties of relations-

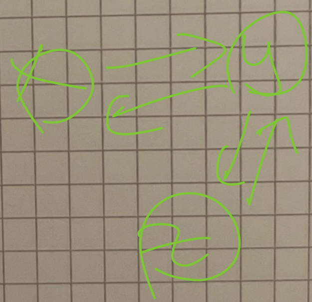
$M = \{x, y, z\}$

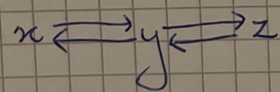
a] 

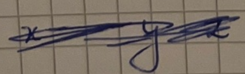
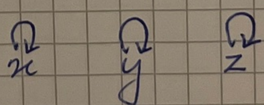
b] 

c] 

d] 

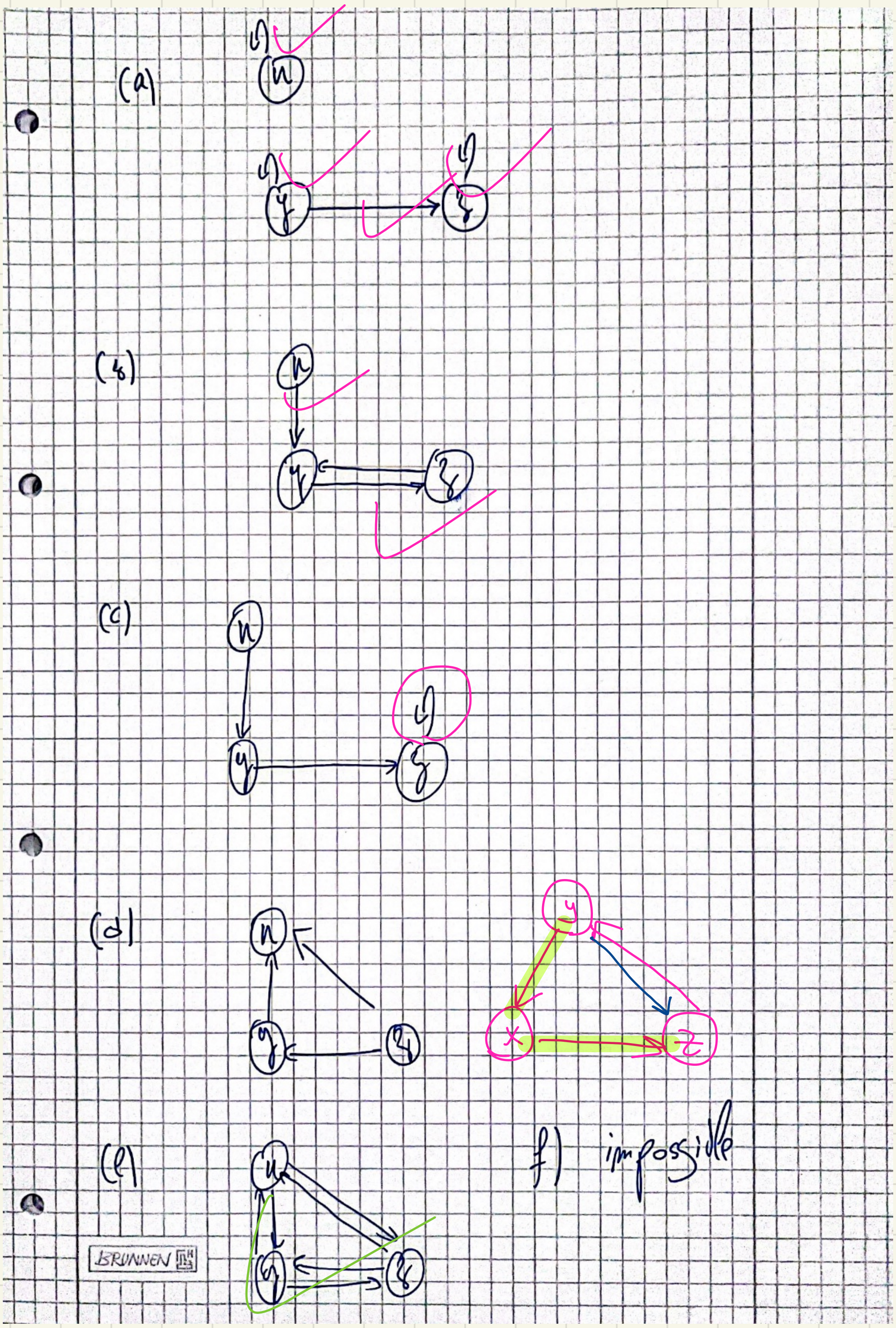


e] 

f]  

asymmetrie dann nicht möglich.

Adem:



Bitte nur a und b

(a) Graphically depict all equivalence relations on the set $\{1, 2, 3\}$.

(b) The *equivalence class* of an element a wrt. a given equivalence relation $\sim \subseteq A \times A$ is the set of all elements that are equivalent to a. Often, this set is denoted by $[a]_\sim := \{a' \in A \mid a \sim a'\}$.

Graphically this means that $[a]_\sim$ is simply the set of all elements that are somehow connected to a (in graph theory this is also called the "connected component of a").

For each of the equivalence relations of (a), determine all equivalence classes. (Note that there is at least one, but at most $|A|$ many equivalence classes.)

(c) (*) In (a) you have already seen that wrt. $n = 3$ elements, there are exactly 3 equivalence relations which have exactly two equivalence classes. $[2] = \{ \ldots$

Determine the number of equivalence relations on $\{1, 2, 3, 4\}$ with exactly 2 equivalence classes.

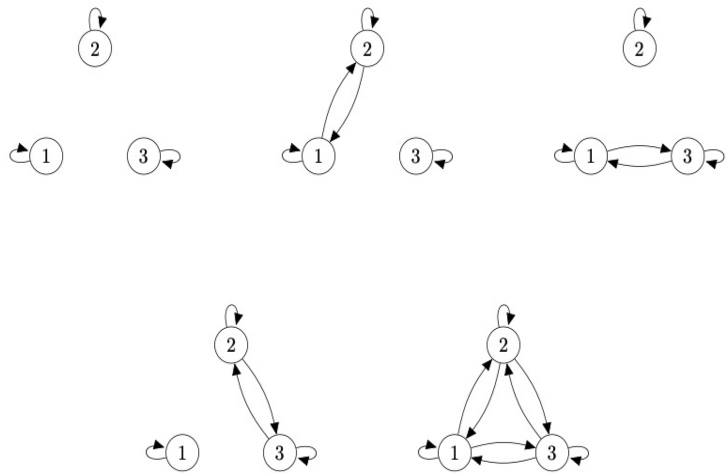
Try to understand how the problem can be reduced to counting the equivalence relations on $\{1, 2, 3\}$ which have either 1 or 2 equivalence classes.

Try to also determine the number of equivalence relations on $\{1, 2, 3, 4, 5\}$ with exactly 2 equivalence classes.

Can you derive a formula in n to determine the number of equivalence relations on $\{1, 2, \ldots, n\}$ that have exactly two equivalence classes? (These numbers are usually denoted by $S_{n,2}$.)

Solution:

(a) The following equivalence relations exist:



(b) From left to right, from top to bottom:

Every element is its own equivalence class, short: $\{\{1\}, \{2\}, \{3\}\}$.

1 and 2 are equivalent and distinct from 3, so $\{\{1, 2\}, \{3\}\}$

1 and 3 are equivalent and distinct from 2, so $\{\{1, 3\}, \{2\}\}$

2 and 3 are equivalent and distinct from 1, so $\{\{1\}, \{2, 3\}\}$

All elements are considered equivalent, so $\{\{1, 2, 3\}\}$.

5.5 Modulo

$5 \equiv_7 5$ $\quad a \equiv_n b \quad \frac{a-b}{n} \quad \frac{5-5}{7} = 0 \equiv 0$

For a fixed $n \in \mathbb{N}$ the equivalence relation $\equiv_n$ is defined on $\mathbb{Z}$ by:

We write $a \equiv_n b$ IF $\frac{a-b}{n} \in \mathbb{Z}$ and otherwise $a \not\equiv_n b$.

$a \equiv_n b$ can also be read as "a and b have the same remainder ("residue") when divided by n".

The equivalence class of a number $a \in \mathbb{Z}$ is denoted by $[a]_n = \{b \in \mathbb{Z} \mid a \equiv_n b\}$.

(The relation $\equiv_n$ was studied already by Gauss in his disquisitiones arithmeticae; it is often refered to also as "congruence (relation)" and plays an important role in number theory with applications in cryptography.)

(a) Determine the equivalences classes of $\equiv_n$ for $n \in \{3, 5, 15\}$.
E.g. first consider the set $\{0, 1, 2 \ldots, 44\}$ and determine which numbers are equivalent.

(b) (*) Show: Let $m, n \in \mathbb{N}$ be any two positive integers.

- If m divides n and $a \equiv_n b$, then also $a \equiv_m b$; hence: $[a]_n \subseteq [a]_m$.
  Recall that "m divides n (without remainder)" is usually denoted by "$m|n$" and "$|$" is a partial order on $\mathbb{N}$.
- $[a]_m \cap [a]_n = [a]_k$ for k the least common multiple of m and n.
- $[a]_m \cap [a]_n = [a]_{mn}$ IFF the greatest common divisor of m and n is 1.

$7 \equiv_3 1$

$6 \equiv 0$

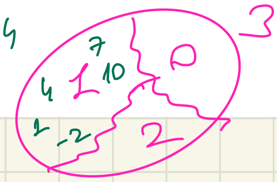
$0 \quad 3 \equiv_3 0$

Solution:

$3, 5, 15$

(a) In general: $[a]_n = a + n\mathbb{Z} = \{a + in \mid i \in \mathbb{Z}\}$ (simply the set of all numbers that have the same remainder as a modulo n).

- $n = 3$: We only have the remainders $0, 1, 2$ giving rise to the classes $3\mathbb{Z} = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$ (all multiples of 3), $\boxed{1 + 3\mathbb{Z} =} \{\ldots, -5, -2, 1, 4, 7, \ldots\}$ (all numbers with remainder 1 modulo 3), and $2 + 3\mathbb{Z} = \{\ldots, -4, -1, 2, 5, 8, \ldots\}$
- $n = 5$: analogously $5\mathbb{Z}$, $1 + 5\mathbb{Z}$, $2 + 5\mathbb{Z}$, $3 + 5\mathbb{Z}$, $4 + 5\mathbb{Z}$ — 0, 1, 2, 3, 4
- $n = 15$: analogously $15\mathbb{Z}$, $1 + 15\mathbb{Z}$, $2 + 15\mathbb{Z}$, ..., $\boxed{14 + 15\mathbb{Z}.}$



→ 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

10, 11, 12, 13, 14 $\qquad$ 0 . . . . . n−1

6.1 Neither Injective nor Surjective

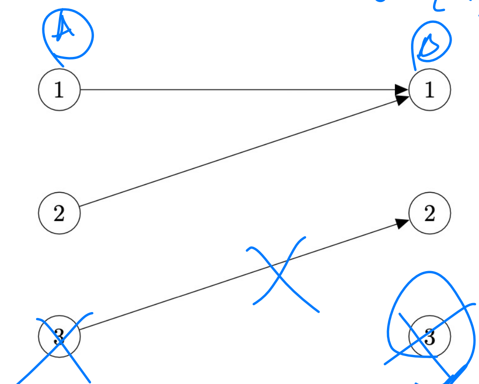
$A = \{ \dots \}$ $\quad B = \{ \dots \}$

$f = \{(1,1),(2,1),(3,2)\}$

Define a function that is neither injective nor surjective.

Annahme:

Solution:



Since both 1 and 2 map to 1 ($f(1) = f(2) = 1$) f is not injective.
Since no element maps to 3, f is not surjective.

6.2 Injectivity and surjectivity

Check, whether the following functions are injective and/or surjective:

(a) $f\colon \mathbb{Z} \to \mathbb{N}_0,\ x \mapsto x^2$

(b) $f\colon \mathbb{N}_0 \to \mathbb{N}_0,\ x \mapsto x^2$

(c) $f\colon \mathbb{N} \to \mathbb{N},\ x \mapsto \begin{cases} 1 & \text{if } x = 1 \\ x - 1 & \text{else} \end{cases}$

(d) $f\colon \mathbb{Z} \to \mathbb{Z},\ x \mapsto x - 1$

$\mathbb{N}_0 = \{0, 1, 2, 3, 4, 5 \dots \quad \infty$

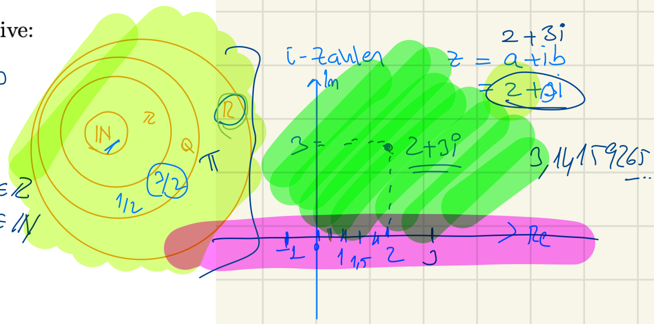
$\mathbb{Z} = \{-1, 2, 3, 4, 1000\}$

$\mathbb{Q} = \{3/2, 1/4, 1/5 \dots\}$

$\mathbb{R} = \{\pi, e, \sqrt{2}, \dots$

$a \to \in \mathbb{Z}$

$b \to \in \mathbb{N}$



Urbildmenge

Bildmenge

Solution:

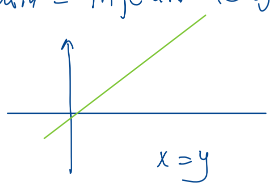
(a) neither injective nor surjective

(b) only injective

(c) only surjective

(d) both injective and surjective = bijektiv

Bijektiv = injektiv + surjektiv



$\mathbb{Z} \to \mathbb{Z}$

$\mathbb{N}_0 \to \mathbb{N}_0$

$\mathbb{R} \to \mathbb{R}$

Laws:	$\neg\neg A \equiv A$	double negation
	$A \wedge A \equiv A$	idempotence
	$A \vee A \equiv A$	
	$\neg(A \vee B) \equiv \neg A \wedge \neg B$	De Morgan
	$\neg(A \wedge B) \equiv \neg A \vee \neg B$	
	$A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$	distributivity
	$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$	
	$A \vee (B \wedge A) \equiv A$	absorption
	$A \wedge (B \vee A) \equiv A$	

!!!

6.3 Graphical representation of functions

Let $A = \{1, 2, 3\}$.   3 = 27

1. How many different functions $f : A \to A$ are there? ~~How many of them are bijective?~~
2. Consider the two functions $f, g\colon A \to A$ defined as follows:

$$f(1) = 2, \quad f(2) = 3, \quad f(3) = 1 \qquad g(1) = 2, \quad g(2) = 1, \quad g(3) = 3$$

   (a) Depict f and g graphically:

   Both as a bipartite graph with separate nodes for the pre-images and images, and also as relation on A with only one node for each element of A.

   (b) Graphically determine

$$f \circ g, \quad g \circ f, \quad f \circ f, \quad \cancel{f \circ f \circ f}, \quad g \circ g, \quad \cancel{g \circ g \circ g}$$

   (c) (*) Show: for every bijective self-map (permutation) $h\colon A \to A$ we always have that for every $a \in A$:

$$(h \circ h \circ h \circ h \circ h \circ h)(a) = a$$

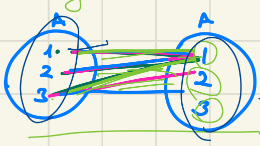
1. Wie viele Funktionen? _ _ _ mehr dazu in Kombinatorik

$f : A \to A$

in DS
Stirling-Zahlen 1. Art

$|A|^{|A|} = 3^3 = 27$ verschiedene Funktionen

$|A|! = 3! = 6$ bijektive Funktionen von A zu A.
→ surjektiv + injektiv



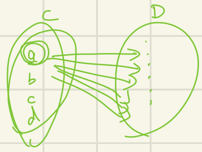
→ 6

Sei $f : [n] \xrightarrow{\text{bij.}} [n]$ eine Permutation von $[n]$

3^3 $\underline{3}$ $\underline{3}$ $\underline{3} = 3^3$
      1 2 3

Bonus: wie viele Möglichkeiten gibt es, um eine Funktion von [4] nach [23] zu konstruieren?

23^4 $\underline{23}$ $\underline{23}$ $\underline{23}$ $\underline{23} = 23^4$
      a b c d



Determine all solutions of the following inequality (over $\mathbb{R}$):

$$\frac{x-8}{x+1} < \frac{1-x}{x+3}$$

Remark: $2x^2 - 5x - 25 = (x-5)(2x+5)$. **Solution:**

$$\frac{x-8}{x+1} - \frac{1-x}{x+3} < 0$$

$$\frac{(x-8)(x+3) - (1-x)(1+x)}{(x+1)(x+3)} < 0$$

$$\frac{2x^2 - 5x - 25}{(x+1)(x+3)} < 0$$

$$\frac{(x-5)(2x+5)}{(x+1)(x+3)} < 0$$

$$\frac{(x-5)(x+2.5)}{(x+1)(x+3)} < 0$$

$x + 2.5 < 0$

$| x < -2,5$

The enumerator is positive IF $x < -2.5$ or $x > 5$, and negative IF $-2.5 < x < 5$.

The denominator is positive IF $x < -3$ or $x > -1$, and negative IF $-3 < x < -1$.

For the fraction to be negative, enumerator and denominator have to have opposite sign.

- "($x < -2.5$ or $x > 5$) and $-3 < x < -1$" is syn. with "$-3 < x < -2.5$".
- "($x < -3$ or $x > -1$) and $-2.5 < x < 5$" is syn. with "$-1 < x < 5$".

So, "$-3 < x < -2.5$ or $-1 < x < 5$".

7 Primes and divisibility

7.1 Prime factorization

Let the following numbers be given:

$$110, \quad 111, \quad 112, \quad 113, \quad 114, \quad 115$$

1. Determine the prime factorization of these numbers.
2. Determine for each pair of numbers the least common multiple (LCM) as well as the greatest common divisor (GCD). How can you use the previous sub-task for this?

Solution:

(a)

$$110 = 2 \cdot 5 \cdot 11$$
$$111 = 3 \cdot 37$$
$$112 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 7$$
$$113 = 113$$
$$114 = 2 \cdot 3 \cdot 19$$
$$115 = 5 \cdot 23$$

110	2
55	5
11	11
1	

111	3
37	37
1	

112	2
56	2
28	2
14	2
7	7
1	

114	2
57	3
19	19
1	

115	5
23	23
1	

Teilbarkeit, Primzahlen

$d \in \mathbb{Z} \setminus \{0\}$ $\quad \frac{x}{0} = \infty$ X

Divisor

$\frac{n}{m}$ $\to \in \mathbb{Z}$ (n), $\to \in \mathbb{N}$ (m)

$d|z$ → ohne Rest $z = k d$
→ mit $\boxed{z = kd + r}$ →

$n \in \mathbb{N}$ mit $n > 1$ ist prim falls $k | n$
$\{1, 2, \ldots, \infty\}$ $k = 1$ oder $k = n$

Teilerfremd

$\mathbb{P} = \{2, 3, 5, 7\}$

ggT, euklidischer Algorithmus....

Teilerfremd $\Leftrightarrow$ ggT $(x, y) = 1$

$$\frac{x-8}{x+1} < \frac{1-x}{x+3}$$

$$\frac{(x-8)(x+3) - (1-x)(1+x)}{(x+1)(x+3)} < 0$$

$$\frac{2x^2 - 5x - 25}{(x+1)(x+3)} < 0$$

$$\frac{(x-5)(2x+5)}{(x+1)(x+3)} \overset{|:2}{<} 0$$

⊕ ⊖ $$\frac{(x-5)(x+2.5)}{(x+1)(x+3)} < 0$$

$(x-5) > 0$

$x + 2.5 > 0$

$|x| > -2.5$

Betrag

< 5

$|x| < -2{,}5$

$x > 2{,}5$

a	b	k	s	t
935	1491	1	716	−449
556	935	1	−449	267
379	556	1	267	−182
177	379	2	−182	85
25	177	7	85	−12
2	25	12	−12	1
1	2	–	1	0

inverse

1491 − 935 · 1

$$1 = \mathrm{ggT}(a, n) = \alpha \cdot a + \beta \cdot n \text{ für } \alpha = 716 \text{ und } \beta = -449.$$

9.5 Base 2 Representation

Represent the following numbers in base 2 (most significant digit first/left)

1. 2
2. 31
3. 5
4. 0.5
5. 0.31
6. 2.5
7. 13.31

7.) 64
8.) 38
9.) 26

1.25 → 8 Bit darstellen

2 = 10₂ → 10_b

2	2	0
1	2	1

31	2	1
15	2	1
7	2	1
3	2	1
1	2	1

38	2	0
19	2	1
9	2	1
4	2	0
2	2	0
1	2	1

100110_b

Solution:

Conversion to binary. For 0.31, the graph can be drawn as in the "Representation: non-integer part slide".

1. 10
2. 11111
3. 101
4. 0.1
5. 0.0100111101011100001010
6. 10.1
7. 1101. fraction of 0.31

26	2	0
13	2	1
6	2	0
3	2	1
1	2	1

11010_b

$\boxed{\text{LGS}} \to \boxed{\text{Der Gaußalgorithmus}}$

Ziel: Konkreter Algorithmus
für das Lösen von linearen Gleichungssystemen
$\longleftrightarrow$ Gaußalgorithmus

Geg: Lineares Gleichungssystem

Bsp: Z2.3 (a):

$$\begin{aligned} 2\cdot x_1 \quad\quad\quad + x_3 &= 3 \\ 4\cdot x_1 + 2\cdot x_2 + x_3 &= 3 \\ -2\cdot x_1 + 8\cdot x_2 + 2\cdot x_3 &= -8 \end{aligned}$$

$$\longleftrightarrow \underbrace{\begin{pmatrix} 2 & 0 & 1 \\ 4 & 2 & 1 \\ -2 & 8 & 2 \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 3 \\ 3 \\ -8 \end{pmatrix}}_{b} \longleftrightarrow \left(\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 4 & 2 & 1 & 3 \\ -2 & 8 & 2 & -8 \end{array}\right)$$

Ges: $\text{Lös}_{A,b}$

Gaaaanz einfaches Beispiel

$$\to \begin{aligned} 2x_1 \quad\quad &= 2 \\ 2x_2 \quad &= -2 \\ 1\cdot x_3 &= 1 \end{aligned} \longleftrightarrow \left(\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$ Diagonal-matrix

$$\Rightarrow \begin{aligned} x_1 &= 1 \\ x_2 &= -1 \\ x_3 &= 1 \end{aligned} \longleftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$ Einheits-matrix links

$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Einfaches Beispiel:

$$\to \boxed{\begin{aligned} 2x_1 + 0\cdot x_2 + x_3 &= 3 \\ 2x_2 - x_3 &= -3 \\ 7x_3 &= 7 \end{aligned}}$$

$\Rightarrow x_1 = 1$
$\Rightarrow x_2 = -1$
$\Rightarrow x_3 = 1$

Lösen durch Rückwärts-einsetzen

0 0 Dreiecks-form

$\updownarrow$

$2x_2 - x_3 = -3 \Rightarrow 2x_2 = -3 + 1$ $\quad 2x_2 = -2$ $\quad x_2 = -1$

$$\left(\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 0 & 2 & -1 & -3 \\ 0 & 0 & 7 & 7 \end{array}\right)$$

obere Dreiecksmatrix

Zeilenstufenform

$7x_3 = 7 \Rightarrow x_3 = 1$

$\Rightarrow$ wie erhält man aus einer beliebigen (erweiterten) Matrix $(A|b)$ eine obere Dreiecksmatrix $(A^*|b^*)$ mit derselben Lösungsmenge?

Satz: Elementare Zeilenumformungen

Die nachfolgenden Zeilenumformungen ändern nichts an der Lösungsgesamtheit / Lösungsmenge eines LGS:

(i) Vertauschen zweier Zeilen,

(ii) Multiplikation einer Zeile mit $\lambda \neq 0$, $\quad \lambda \in \mathbb{R} \setminus \{0\}$

d.h. ersetze Zeile $i \longleftrightarrow \lambda \cdot$ Zeile i

(iii) Addition des Vielfachen einer anderen Zeile zu einer Zeile,

d.h. Zeile $i \longleftrightarrow$ Zeile $i + \lambda\cdot$ Zeile j, $\lambda \in \mathbb{R}$

[Nicht vergessen: Zeile j bleibt !!]

Achtung: Nicht zu viele Schritte gleichzeitig.

Bsp: Wird aus dem LGS

$$\left(\begin{array}{c|c} \cdots & \vdots \\ \cdots & \vdots \\ \cdots & \vdots \end{array}\right) \begin{array}{l} \text{I} \\ \text{II} \\ \text{III} \end{array} \rightsquigarrow \left(\begin{array}{c|c} \cdots & \vdots \\ \cdots & \vdots \\ \cdots & \vdots \end{array}\right) \begin{array}{l} \text{I} \\ \text{II}-\text{I} \\ \text{III}-\text{II} \end{array}$$

so geht Information verloren, denn

$\text{I}+\text{II}+\text{III}$

$\text{I} + (\text{II}-\text{I}) + (\text{III}-\text{II}) = \text{III} \neq \text{I}+\text{II}+\text{III}$

Sicheres Vorgehen:

$$\left(\begin{array}{c|c} \cdots & \vdots \\ \cdots & \vdots \\ \cdots & \vdots \end{array}\right) \begin{array}{l} \text{I} \\ \text{II} \\ \text{III} \end{array} \rightsquigarrow \left(\begin{array}{c|c} \cdots & \vdots \\ \cdots & \vdots \\ \cdots & \vdots \end{array}\right) \begin{array}{l} \text{I} \\ \text{II}+\lambda\text{I} \\ \text{III}+\mu\text{I} \end{array}$$

also 1 Zeile fix lassen
und nur mit dieser bei den anderen Zeilen arbeiten

$$\left(\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 4 & 2 & 1 & 3 \\ -2 & 8 & 2 & -8 \end{array}\right) \begin{array}{l} \text{I} \\ \text{II} \\ \text{III} \end{array}$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 0 & 2 & -1 & -3 \\ 0 & 8 & 3 & -5 \end{array}\right) \begin{array}{l} \text{I} \\ \text{II}-2\text{I} \\ \text{III}+\text{I} \end{array} \quad \begin{array}{l} \text{I} \\ \text{II} \\ \text{III} \end{array}$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 0 & 2 & -1 & -3 \\ 0 & 0 & 7 & 7 \end{array}\right) \begin{array}{l} \text{I} \\ \text{II} \\ \text{III}-4\cdot\text{II} \end{array}$$

Es ist möglich mit (i), (ii), (iii) gezielt Variablen zu "eliminieren" und das LGS auf obere Dreiecksstufenform zu bringen!

$\Rightarrow$ durch Rückwärtseinsetzen
(siehe einfaches Beispiel :))

$x_3 = 1, \; x_2 = -1, \; x_1 = 1$

$\mathbb{L} = \text{Lös}_{A,b} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$

genau eine (eindeutige) Lösung

8.3 Systems of Linear Equations

There are different algorithms for solving systems of linear equations that work for any field, e.g. "substitution" and "Gaussian elimination".

(a) *By Substitution:*
Solve the following system of linear equations over the real numbers step-by-step by solving one equation for one variable and then substituting it into another equation as shown in the lecture slides:

$$x - 2y = 4, \quad -y - z = -1, \quad -x + 3z + y = -1$$

(b) *By Gaussian elimination (elementary transformations):*
Recall: if we (i) multiple a single equation by a non-zero number or (ii) add a multiple of one equation to another (distinct) equation, then the set of possible solutions does not change (as we can reverse these transformations).

a) $x - 2y = 4 \quad x = 4 + 2y$

$-y - z = -1 \quad \Rightarrow -z = -1 + y \;|\cdot(-1) \quad z = 1 - y$

$-x + 3z + y = -1$

$-(4 + 2y) + 3 - 3y + y = -1$

$-4 - 2y + 3 \cancel{- 3y + y} = -1$

$-4y - 1 = -1$

$-4y = 0$

$y = 0$

$x = 4$

$z = 1$

b) $-x + y + 3z = -1$

$x - 2y = 4$

$-y - z = -1$

$$\left(\begin{array}{ccc|c} -1 & 1 & 3 & -1 \\ 1 & -2 & 0 & 4 \\ 0 & -1 & -1 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} -1 & 1 & 3 & -1 \\ 0 & -1 & 3 & 3 \\ 0 & -1 & -1 & -1 \end{array}\right) -\mathrm{II} + \mathrm{III}$$

$$\to \left(\begin{array}{ccc|c} -1 & 1 & 3 & -1 \\ 0 & -1 & 3 & 3 \\ 0 & 0 & -4 & -4 \end{array}\right)$$

$z = 1$

$-y + 3z = 3$

$-y = 0$

$y = 0$

$x = 4$

Eine Folge $(a_n)_{n\in\mathbb{N}_0}$ komplexer Zahlen $\rightarrow z = a+ib$

ist eine Abbildung $\mathbb{N}_0 \rightarrow \mathbb{C}, n \rightarrow a_n$

man interessiert sich für das asymptotische Verhalten einer Folge

Sei $q \in \mathbb{R} \Rightarrow a_0 = 1$

$a_{n+1} = q \cdot a_n$ für $n \in \mathbb{N}_0$

$(a_n)_{n\in\mathbb{N}_0}$ $\quad a_n = q^n, n \in \mathbb{N}_0$

$a_0 = q^0 = 1 \quad a_3 = q^3$

$a_1 = q^1 \quad a_4 = q^4$

$a_2 = q^2 \quad a_5 = q^5$

$q = 2 \quad a_n = 2^n$ strebt nach ∞

$q = 1/2 \quad a_n = \frac{1}{2}^n$ " 0

$q = -1 \quad a_n = (-1)^n \quad [-1; 1]$

Eine komplexe Folge $(a_n)_{n\in\mathbb{N}_0}$ konvergiert gegen $a \in \mathbb{C}$, falls für jede

Genauigkeit $\varepsilon > 0$ ein $n_0 \in \mathbb{N}$ existiert, sodass für alle $n \geq n_0$ gilt

$\rightarrow |a_n - a| < \varepsilon \quad \forall n \in \mathbb{N}_0, \geq n_0$

gegen a konvergiert

2

1,999999999

$\forall \varepsilon > 0 \; \exists n_0 \in \mathbb{N} \; \forall n \geq n_0 \; |a_n - a| < \varepsilon$

$\lim_{n\to\infty} a_n = a$ oder $a_n \xrightarrow{n\to\infty} a$

Rechenregel für Grenzwerte

Falls $\lim_{n\to\infty} a_n = a \in \mathbb{R}$ und $\lim_{n\to\infty} b_n = b \in \mathbb{R}$

(a) $\lim_{n\to\infty} (a_n + b_n) = a+b \Rightarrow \underbrace{\lim_{n\to\infty} a_n}_{a} + \underbrace{\lim_{n\to\infty} b_n}_{b} \Rightarrow a+b$

(b) $\lim_{n\to\infty} c\,a_n = ca$ für alle $c \in \mathbb{R}$

(c) $\lim_{n\to\infty} a_n b_n = ab$

$\lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n = a \cdot b$

(d) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$ falls $b \neq 0$

$\Rightarrow \infty$

Bsp: $\lim_{n\to\infty} \frac{4n^2+3n-7}{2n^2+1}$ oder $\frac{\lim a}{\lim b} = \frac{\lim_{n\to\infty} a}{\lim_{n\to\infty} b}$

Zähler $\approx 4n^2$

Nenner $\approx 2n^2$

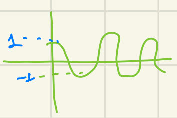
Bruch $\approx \frac{4n^2}{2n^2} = 2$

$\frac{\lim_{n\to\infty} a}{\lim_{n\to\infty} b} = \frac{\lim_{n\to\infty} 4n^2 + 3n - 7}{\lim_{n\to\infty} 2n^2 + \underbrace{\lim_{n\to\infty} 1}_{1}}$

Die Folge $(a_n)_{n\in\mathbb{N}}$ divergiert gegen $+\infty$

$-\infty$

Nicht jede divergente Folge konvergiert uneigentlich, z.B. $(-1)^n$ $\quad -1 \quad 1$



9 Limits, sequences, and series

9.1 Limits of sequences

For each of the following sequences determine whether it converges and the limit:

a) $a_n = \frac{3n+2(-1)^n}{n}$

b) $b_n = \frac{nx^n}{nx^n+1}$ for a fixed, but real $x \in (1, \infty)$ $= 1$

c) $c_n = 2n - \frac{6n^3}{3n^2-2n+1}$ $= \frac{2n(3n^2-2n+1) - 6n^3}{3n^2-2n+1} = \frac{\cancel{6n^3} - 4n^2 + 2n - \cancel{6n^3}}{3n^2-2n+1} = \frac{-4n^2+2n}{3n^2-2n+1}$

d) $d_n = n(\sqrt{9n^2+6n+5} - (3n-1))$ for $n > 0$

You can verify your result e.g. via Wolfram Alpha.

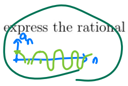
Syntax e.g. `limit (3n+2(-1)^n)/n as n->infinity` **Solution:** Using the rules

for calculating limits of convergent sequences, one can try to reduce the current sequence to a sequence whose convergence is known..

A frequently used approach herefore is by calculating the limit by means of fractions and cancelling by the highest power of n.

a) As it is okay to discard finitely many elements of the sequence w.r.t. convergence, we can assume $n > 0$

With this assumption we can express the rational expression as a sum.



$a_n = \frac{3n+2(-1)^n}{n} = 3 + \frac{2(-1)^n}{n}$

We have $\lim_{n\to\infty} 3 = 3$.

Furthermore $|(-1)^n/n| \leq 1/n \xrightarrow{n\to\infty} 0$ (compare with limit example from slides $2 - 1/(1+i)$).

$1 \; -1$

As such $\lim_{n\to\infty} 2(-1)^n/n = 2\lim_{n\to\infty} \boxed{(-1)^n}/n = 2 \cdot 0 = 0$.

Overall:

[redacted]

b) $b_n = \frac{nx^n}{nx^n+1} = \frac{nx^n}{nx^n+1} \cdot \frac{nx^n}{nx^n} = \frac{\frac{nx^n}{nx^n}}{\frac{nx^n}{nx^n}+\frac{1}{nx^n}} = \frac{1}{1+\frac{1}{nx^n}}$

$\lim_{n\to\infty} \frac{1}{1+\frac{1}{nx^n}} = \frac{1}{1+0} = 1$

($\frac{1}{nx^n} \to 0$, da $nx^n \to \infty$, reminiscent of a))

c) $c_n = 2n - \frac{6n^3}{3n^2-2n+1} = \frac{6n^3-4n^2+2n}{3n^2-2n+1} - \frac{6n^3}{3n^2-2n+1} = \frac{-4n^2+2n}{3n^2-2n+1}$

$\lim_{n\to\infty} \frac{-4n^2+2n}{3n^2-2n+1} = \lim_{n\to\infty} \frac{-4-\frac{2}{n}}{3-\frac{-2}{n}+\frac{1}{n^2}} = -\frac{4}{3}$

d) $d_n = n(\sqrt{9n^2+6n+5}-(3n-1)) = n(\frac{9n^2+6n+5-(3n-1)^2}{\sqrt{9n^2+6n+5}+(3n-1)})$ (as $a-b = \frac{a^2-b^2}{a+b}$) $= n(\frac{9n^2+6n+5-(9n^2-6n+1)}{\sqrt{9n^2+6n+5}+(3n-1)})$

$\lim_{n\to\infty} n(\frac{9n^2+6n+5-(9n^2-6n+1)}{\sqrt{9n^2+6n+5}+(3n-1)}) = \lim_{n\to\infty} n(\frac{12+\frac{4}{n}}{\sqrt{9+\frac{6}{n}+\frac{5}{n^2}}+(3-\frac{1}{n})}) = \lim_{n\to\infty} n\frac{12}{6} = \lim_{n\to\infty} 2n = \infty$

Please don't use L'Hospital. Basic rules for limits should be practiced.

e) $a_n = \sqrt{4n+1} - \sqrt{4n-3}$ konvergiert? ¨

7 min

wende hier die Definition an, zeige Konvergenz, für ein festes $\varepsilon > 0$

$$\left\{ \begin{array}{l} |a_n - 0| = \frac{1}{\sqrt{n}} \leq \varepsilon \\ |a_n - *| = \varepsilon, \; \forall n \geq N \end{array} \right\}$$

$$\frac{\sqrt{4n+1} - \sqrt{4n-3}}{1} \quad \Big| \cdot \sqrt{4n+1} + \sqrt{4n-3} \quad \} \; a^2 - b^2$$

$$\frac{4n+1-(4n-3)}{\sqrt{4n+1}+\sqrt{4n-3}} \Rightarrow \frac{\cancel{4n}+1-\cancel{4n}+3}{\sqrt{4n+1}+\sqrt{4n-3}} = \frac{4}{4}$$

4

$$\Rightarrow \frac{4}{\sqrt{4n}} = \frac{4}{2\sqrt{n}}$$

$$\lim_{n\to\infty} \frac{4}{2\sqrt{n}} = 0$$

$\sqrt{4n+1}$ ist gleich $\sqrt{4n}$ weil 1 im zu $4n$ vernachlässigbar ist

$\sqrt{4n-3}$ gleich $\sqrt{4n}$ weil -3 im Vergleich zu ebenfalls vernachlässigbar ist.

Falls $\lim_{n\to\infty} a_n = a \in \mathbb{R}$ und $\lim_{n\to\infty} b_n = b \in \mathbb{R}$, so gilt auch:

(a) $\lim_{n\to\infty} (a_n + b_n) = a + b$,

(b) $\lim_{n\to\infty} ca_n = ca$, $\forall c \in \mathbb{R}$,

(c) $\lim_{n\to\infty} a_n b_n = ab$,

(d) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$, falls $b \neq 0$.

Einschließungsregel: 3 Folgen

$$a_n \leq b_n \leq c_n$$

für alle bis auf endlich viele n. Falls $\alpha \in \mathbb{R}$ existiert mit

$$\lim_{n\to\infty} a_n = \alpha = \lim_{n\to\infty} c_n,$$

dann gilt auch

$$\lim_{n\to\infty} b_n = \alpha.$$

9.2 To infinity and sometimes beyond

For each subtask provide sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ $\boxed{\text{tending to } \pm\infty}$ (of course, "either or") such that:

a) $\lim_{n\to\infty}(a_n - b_n) = 0$

b) $\lim_{n\to\infty}(a_n - b_n) = +\infty$

c) $\lim_{n\to\infty}(a_n - b_n) = -\infty$

d) $\lim_{n\to\infty}(\frac{a_n}{b_n}) = 0$ $\quad \frac{n}{n^2}$

e) $\lim_{n\to\infty}(\frac{a_n}{b_n}) = 1$ $\quad \frac{n}{n}$

$a_n = \frac{1}{n}$

$b_n = \frac{1}{n^2}$

$n^2 \quad n^2$

$a_n = n^2$

$b_n = n$

$\frac{n}{n^2}$

Solution:

a) $a_n = n$, $b_n = n$

b) $a_n = n$, $b_n = -n$; $a_n = n^2$, $b_n = n$

c) $a_n = -n$, $b_n = n$; $a_n = -n^2$, $b_n = -n$

d) $a_n = n$, $b_n = n^2$

e) $a_n = n$, $b_n = n$

Tutoriumsaufgabe 2. Geben Sie, falls möglich, Folgen $(a_n)_{n\in\mathbb{N}}$ und $(b_n)_{n\in\mathbb{N}}$ reeller Zahlen an mit:

$1/n$ n

(a) $(a_n)_{n\in\mathbb{N}}$ konvergiert gegen 0, $(b_n)_{n\in\mathbb{N}}$ konvergiert nicht und $(a_nb_n)_{n\in\mathbb{N}}$ konvergiert.

$1/n$ n^2

(b) $(a_n)_{n\in\mathbb{N}}$ konvergiert gegen 0, $(b_n)_{n\in\mathbb{N}}$ konvergiert nicht und $(a_nb_n)_{n\in\mathbb{N}}$ konvergiert nicht.

$\frac{1}{n} \cdot n^2 \to n$

$\lim_{n\to\infty} n = \infty$

Lösung:

(a) $a_n := \frac{1}{n^2}$ und $b_n := n$, dann $a_nb_n \to 0$.

Oder $a_n := \frac{1}{n}$ und $b_n := n$, dann $a_nb_n \to 1$.

Oder $a_n := 0$ und $b_n := n$, dann $a_nb_n \to 0$.

(b) $a_n := \frac{1}{n}$ und $b_n := n^2$, dann konvergiert $(a_nb_n)_{n\in\mathbb{N}}$ nicht.

9.6 Binomial theorem

Verify the generalisation of the first binomial formula.

$$\boxed{(x+y)^m = \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} x^k y^{m-k}}$$

for $m = 3, 4, 5$ via explicit expansion (you can also try to prove the general formula using induction on $m \in \mathbb{N}_0$).

Reminder: The factorial $n!$ is defined inductively for $n \in \mathbb{N}_0$ by $0! := 1$, if $n = 0$, and $(n+1)! := (n!) \cdot (n+1)$, if $n > 0$; further $\binom{m}{k} := \frac{m!}{k!(m-k)!}$ is also called binomial coefficient.

Solution:

- m = 3:

  $(x+y)^3 = (x^2+2xy+y^2)(x+y) = x^3+2x^2y+y^2x+x^2y+2xy^2+y^3 = x^3+3x^2y+3xy^2+y^3$

  $(x+y)^3 = \sum_{k=0}^{3} \frac{3!}{k!(3-k)!} x^k y^{3-k} =$

  $\frac{3!}{0!(3-0)!} x^0 y^{3-0} + \frac{3!}{1!(3-1)!} x^1 y^{3-1} + \frac{3!}{2!(3-2)!} x^2 y^{3-2} + \frac{3!}{3!(3-3)!} x^3 y^{3-3} =$

  $\frac{3!}{3!} y^3 + \frac{3!}{2!} xy^2 + \frac{3!}{2!} x^2 y + \frac{3!}{3!} x^3 =$

  $y^3 + 3xy^2 + 3x^2y + x^3$

- m = 4:

  $(x+y)^4 = (x^3+3x^2y+3xy^2+y^3)(x+y) = x^4+4x^3y+6x^2y^2+4xy^3+y^4$

  $(x+y)^4 = \sum_{k=0}^{4} \frac{4!}{k!(4-k)!} x^k y^{4-k} =$

  $\frac{4!}{4!} y^4 + \frac{4!}{3!} xy^3 + \frac{4!}{2!\cdot 2!} x^2y^2 + \frac{4!}{3!} x^3 y + \frac{4!}{4!} x^4 =$

  $y^4 + 4xy^3 + 6x^2y^2 + 4x^3y + y$

- m = 5:

  $(x+y)^5 = (x^4+4x^3y+6x^2y^2+4xy^3+y^4)(x+y) = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$

  $(x+y)^5 = \sum_{k=0}^{5} \frac{5!}{k!(5-k)!} x^k y^{5-k} =$

  $\frac{5!}{5!} y^5 + \frac{5!}{4!} xy^4 + \frac{5!}{2!\cdot 3!} x^2y^3 + \frac{5!}{3!\cdot 2!} x^3y^2 + \frac{5!}{4!} x^4 y + \frac{5!}{5!} x^5 =$

  $y^5 + 5xy^4 + 10x^2y^3 + 10x^3y^2 + 5x^4y + x^5$

- (*) General case: induction on $m \in \mathbb{N}_0$:
  - I.B.: $m = 0$

    $\forall\, x, y \in \mathbb{R}$

    RHS: $(x+y)^m = (x+y)^0 = 1$ for all $x, y \in \mathbb{R}$.

    LHS: $\sum_{k=0}^{m} \frac{m!}{k!(m-k)!} x^k y^{m-k} \overset{m=0}{=} \frac{0!}{0!0!} x^0 y^0 = 1$ for all $x, y \in \mathbb{R}$.

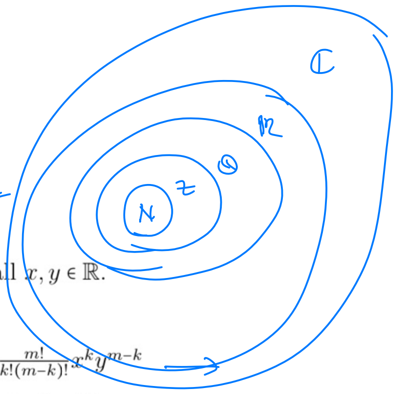
    $\frac{1}{1}$

  - I.S.: Fix any $m \in \mathbb{N}_0$.

    $0! = 1$

    I.A.: For all $x, y \in \mathbb{R}$ we have $(x+y)^m = \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} x^k y^{m-k}$

    I.C.: For all $x, y \in \mathbb{R}$ we have $(x+y)^{m+1} = \sum_{k=0}^{m+1} \frac{(m+1)!}{k!(m+1-k)!} x^k y^{m+1-k}$



Proof:

$$(x+y)^{m+1} = (x+y)(x+y)^{m} \overset{I.A.}{=} (x+y)\sum_{k=0}^{m}\frac{m!}{k!(m-k)!}x^{k}y^{m-k}$$

Expanding the right-most term:

$$\ldots = \sum_{k=0}^{m}\frac{m!}{k!(m-k)!}x^{k+1}y^{m-k} + \sum_{k=0}^{m}\frac{m!}{k!(m-k)!}x^{k}y^{m+1-k}$$

Shifting the index in the first sum so that it starts at 1:

$$\ldots = \sum_{k=1}^{m+1}\frac{m!}{(k-1)!(m-(k-1))!}x^{k}y^{m-(k-1)} + \sum_{k=0}^{m}\frac{m!}{k!(m-k)!}x^{k}y^{m+1-k}$$

Simplifying the index in the first sum:

$$\ldots = \sum_{k=1}^{m+1}\frac{m!}{(k-1)!(m+1-k)!}x^{k}y^{m+1-k} + \sum_{k=0}^{m}\frac{m!}{k!(m-k)!}x^{k}y^{m+1-k}$$

$$\ldots = \frac{m!}{m!0!}x^{m+1}y^{0} + \sum_{k=1}^{m}\left(\frac{m!}{(k-1)!(m+1-k)!}x^{k}y^{m+1-k} + \frac{m!}{k!(m-k)!}x^{k}y^{m+1-k}\right) + \frac{m!}{0!m!}x^{0}y^{m+1}$$

$$\ldots = \frac{m!}{m!0!}x^{m+1}y^{0} + \sum_{k=1}^{m}\left(\frac{m!}{k!(m+1-k)!}x^{k}y^{m+1-k}\cdot(k+m+1-k)\right) + \frac{m!}{0!m!}x^{0}y^{m+1}$$

$$\ldots = \frac{(m+1)!}{(m+1)!0!}x^{m+1}y^{0} + \sum_{k=1}^{m}\left(\frac{(m+1)!}{k!(m+1-k)!}x^{k}y^{m+1-k}\right) + \frac{(m+1)!}{0!(m+1)!}x^{0}y^{m+1}$$

$$\ldots = \sum_{k=0}^{m+1}\frac{(m+1)!}{k!(m+1-k)!}x^{k}y^{m+1-k}$$

Elementare Funktionen

Potenzen:

$x^0 := 1 \qquad x^{n+1} = x^n \cdot x$

$\to x^{-n} := (x^{-1})^n = (x^n)^{-1} = \left(\frac{1}{x}\right)^n$

$\to \sqrt{x}^n := (x^{1/2})^n = x^{n/2} = (x^n)^{1/2} \to \sqrt{x^n}$

$\to \sqrt[3]{x} := (x^{1/3})$

$\vdots$

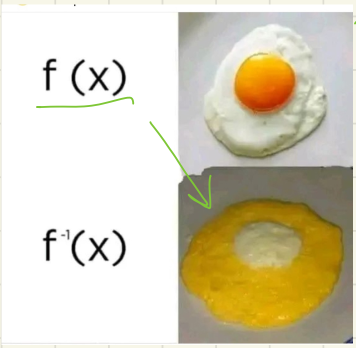
$x^m \cdot x^n = x^{m+n} \qquad (x^m)^n = x^{mn}$

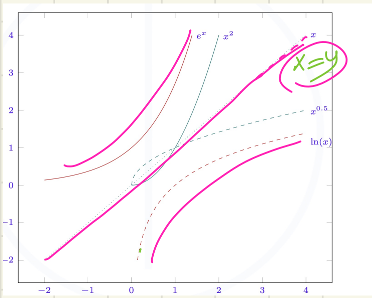
$\sqrt[n]{y^m} = (y^m)^{1/n}$

$(\sqrt[5]{3})^2 = (3^{1/5})^2 = 3^{2/5} = \sqrt[5]{3^2}$

$(3^2)^{1/5}$

Umkehrfunktion: streng monoton, stetig $\Rightarrow$ keine Lücken





Rechenregel

$\to \log_b (xy) = \log_b(x) + \log_b(y) \qquad \log_b \frac{x}{y} = \log_b x - \log_b y$

$\log_b x^y = y \log_b x$

$\log x^2 = 2 \log x$

$\to \log_b (x \cdot y) = \log_b x + \log_b y \qquad \log_b \frac{x}{y} = \log_b x - \log_b y$

$\log_b x^y = y \log_b x \qquad \frac{\log x}{\log b} \Rightarrow \log_b x$

$\to$ natürlicher Logarithmus

$\ln(x) := \log(x) := \log_e(x) \qquad \ln(e) = 1$

$x = \ln(e^x)$ für $x \in \mathbb{R}$ und $y = e^{\ln(y)}$ für $y \in \mathbb{R}_{>0}$

Es gilt: $\ln(z) = \ln\left(b^{\log_b(z)}\right) = \ln(b) \cdot \log_b(z) \to \log_b(z)\ \ln(b)$

d.h $\boxed{\log_b(z) = \frac{\ln(z)}{\ln(b)}} = \boxed{\frac{\log_e z}{\log_e b}} \Rightarrow \log_b z$

10 Elementary functions on $\mathbb{R}$

10.1 Exponentiation and logarithms

1. Sketch the graph of the function $x \mapsto \log_2 x$ for $x = 2^{-10} \dots 10$.
   Use GeoGebra for example.
2. For an arbitrary positive $b \neq 1$ determine the following values: $\log_b 1$ and $\log_b b$.
3. Using the rules for exponentiation and logarithms, simplify $\log_b \sqrt[n]{x}$ so that you can compute it using a standard calculator.
4. Simplify the following expressions (b, c, x, y are positive and $b, c \neq 1$):

$$(i)\ b^{x+\log_b y} \qquad (ii)\ \left(\sqrt{b}\right)^{\log_b x} \qquad (iii)\ \log_c\left(x^{\frac{1}{\log_c b}}\right).$$

5. Assume $x > y > 0$. Then simplify

$$\ln(x^2 - y^2) - \ln(x - y).$$

$a^2 - b^2$

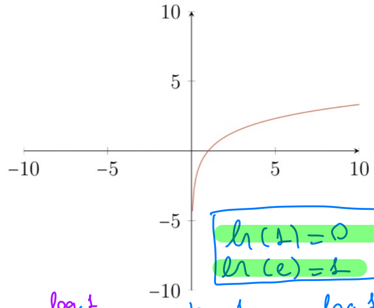
$\ln((x+y)\cdot(x-y)) - \ln(x-y) = \ln\left(\frac{(x+y)(x-y)}{(x-y)}\right)$

6. Evaluate the following expressions (e.g. using a calculator)

$$(2^3)^5 \quad 2^{(3^5)} \quad (2^3)^{-5} \quad 2^{(3^{-5})} \quad (2^{-3})^5 \quad 2^{((-3)^5)}$$

7. Show tbat for all $x \in \mathbb{R}_{>0}$ it holds that: $\ln(x)^{\ln(x)} = x^{\ln(\ln(x))}$

Solution:
log and ln both denote the logarithm wrt. e.



$\ln(1) = 0$
$\ln(e) = 1$

1.

2. $\log_b 1 = \frac{\ln 1}{\ln b} = \frac{0}{\ln b} = 0$ $\quad \log_b 1 \Rightarrow \frac{\log 1}{\log b}$

$\log_b b = \frac{\ln b}{\ln b} = 1$ $\quad \frac{\log b}{\log b} = \frac{2}{2} = 1$

($b \neq 1$, as $\ln 1 = 0$, which would mean dividing by 0)

$\left(b^{\log_b x}\right)^{\frac{1}{2}}$ $\quad 10^{\log_{10} 2} = 2$ $\quad e^{\ln(x)} \Rightarrow x$ $\quad x^{1/2} \Rightarrow \sqrt{x}$

3. $\log_b \sqrt[n]{x} = \log_b x^{1/n} = \frac{1}{n}\log_b x = \frac{\log x}{n \log b}$

4.
   - $b^{x+\log_b y} = b^x \cdot b^{\log_b y} = b^x \cdot y$
   - $\left(\sqrt{b}\right)^{\log_b x} = (b^{1/2})^{\log_b x} = b^{\left(\frac{1}{2}\cdot\log_b x\right)} = b^{\left(\log_b x^{1/2}\right)} = x^{\frac{1}{2}} = \sqrt{x}$ $\quad \log_c b = \frac{\ln b}{\ln c}$
   - $\log_c\left(x^{\frac{1}{\log_c b}}\right) = \boxed{\frac{1}{\log_c b} \log_c x} = \frac{\ln c}{\ln b}\frac{\ln x}{\ln c} = \frac{\ln x}{\ln b} = \log_b x$

5. $\ln(x^2 - y^2) - \ln(x - y) = \ln((x-y)(x+y)) - \ln(x-y) = \ln(x-y) + \ln(x+y) - \ln(x-y) = \ln(x+y)$

$a^2 - b^2$

6.
   - $(2^3)^5 = 2^{3\cdot 5} = 2^{15} = 32768$
   - $2^{(3^5)} = 2^{243} = 1.4 \cdot 10^{73}$
   - $(2^3)^{-5} = \frac{1}{32768}$
   - $2^{(3^{-5})} = 1.002\dots$
   - $(2^{-3})^5 = \frac{1}{32768}$
   - $2^{((-3)^5)} = \frac{1}{1.4\cdot 10^{73}}$

$\ln(x^2) - \ln(y^2)$
$2\ln(x) - 2\ln(y)$
$2(\ln(x) - \ln(y))$

7. $\ln(x)^{\ln(x)} = e^{\ln(\ln(x)^{\ln(x)})} = e^{\ln(x)\cdot\ln(\ln(x))} = e^{\ln(x^{\ln(\ln(x))})} = x^{\ln(\ln(x))}$

L' Hopital Regel :

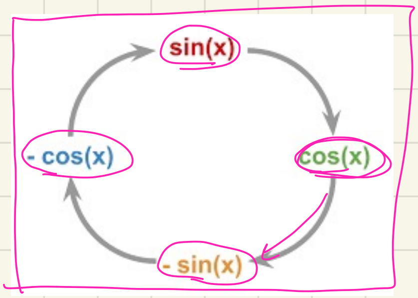
1) Bilde Ableitung von Zähler
Nenner
Separat

$\lim_{x \to 0} \left\{ \frac{\sin(x)}{x} \right\}$

$f'(x) = \cos(x)$ und $g'(x) = 1$

$\lim_{x \to 0} \frac{\cos(x)}{1} = \cos(0) = 1 = 1$

$\cos(0)$



$\frac{f(x)}{g(x)}$

12.6 L'Hospital (*)

Use L'Hospital's rule to determine the following limits:

(a) $\lim_{x\to 0} \frac{\sin(x)}{x}$

(b) $\lim_{x\to 0} \frac{e^x-1}{x}$

(c) $\lim_{x\to 0} \frac{\ln(x+1)}{x}$

(d) $\lim_{x\to \infty} \frac{\ln(x+1)}{x}$

(e) $\lim_{x\to \infty} \frac{\ln(x+1)}{x^r}$ for arbitrary, but fixed $r \in \mathbb{R}^+$

(f) $\lim_{x\to \infty} \frac{e^{rx}}{x}$ for arbitrary, but fixed $r \in \mathbb{R}^+$

(g) $\lim_{0<x\to 0} x^2 \ln(x)$

(h) $\lim_{0<x\to 0} x \ln(x)^2$

(i) $\lim_{x\to -\infty} xe^x$

(j) $\lim_{x\to \infty} x^{\frac{1}{x}}$

(k) $\lim_{x\to \infty} x^{(e^{-x})}$

Verify your results once more with GeoGebra. To this extent transform $x \mapsto x^{-1}$ for limits that go to $\pm\infty$ (such that the limit goes to 0 instead).

Solution:

(a) $\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{\cos(x)}{1} = 1$

(b) $\lim_{x\to 0} \frac{e^x-1}{x} = \lim_{x\to 0} \frac{e^x}{1} = 1$

(c) $\lim_{x\to 0} \frac{\ln(x+1)}{x} = \lim_{x\to 0} \frac{\frac{1}{x+1}}{1} = \lim_{x\to 0} \frac{1}{x+1} = 1$

(d) $\lim_{x\to \infty} \frac{\ln(x+1)}{x} = \lim_{x\to \infty} \frac{1}{x+1} = 0$

(e) $\lim_{x\to \infty} \frac{\ln(x+1)}{x^r} = \lim_{x\to \infty} \frac{1}{x+1} \cdot \frac{-r}{x^{r+1}} = \lim_{x\to \infty} -\frac{r}{x^{r+2}+x^{r+1}} = 0$

(f) $\lim_{x\to \infty} \frac{e^{rx}}{x} = \lim_{x\to \infty} \frac{re^{rx}}{1} = \infty$

(g) Reformulation as $\lim_{0<x\to 0} \frac{x^2}{\frac{1}{\ln(x)}} = \lim_{0<x\to 0} \frac{2x}{-x\ln(x)^2} = \lim_{0<x\to 0} \frac{2}{-\ln(x)^2} = 0$

(h) Reformulation as $\lim_{0<x\to 0} \frac{x}{\frac{1}{\ln(x)^2}} = \lim_{0<x\to 0} \frac{1}{\frac{-x\ln(x)^3}{2}} = \lim_{0<x\to 0} \frac{-2}{x\ln(x)^3} = 0$

(i) Reformulation as $\lim_{x\to -\infty} \frac{x}{\frac{1}{e^x}} = \lim_{x\to -\infty} \frac{1}{-\frac{1}{e^x}} = \lim_{x\to -\infty} -e^x = 0$

(j) Reformulation as $\lim_{x\to \infty} e^{\frac{\ln(x)}{x}}$. Da $\lim_{x\to \infty} \frac{\ln(x)}{x} = \lim_{x\to \infty} \frac{\frac{1}{x}}{1} = \lim_{x\to \infty} \frac{1}{x} = 0$ gilt, ist $\lim_{x\to \infty} e^{\frac{\ln(x)}{x}} = \lim_{x\to \infty} e^0 = 1$

(k) Reformulation as $\lim_{x\to \infty} e^{\frac{\ln(x)}{e^x}}$. Da $\lim_{x\to \infty} \frac{\ln(x)}{e^x} = \lim_{x\to \infty} \frac{\frac{1}{x}}{e^x} = \lim_{x\to \infty} \frac{1}{xe^x} = 0$ gilt, ist $\lim_{x\to \infty} e^{\frac{\ln(x)}{e^x}} = \lim_{x\to \infty} e^0 = 1$

12.5 Little-o notation and comparison of runtime bounds

$f \in o(g)$ ("little-o") is commonly used as an abbreviation for $\lim_{x\to\infty} \left|\frac{f(x)}{g(x)}\right| = 0$.

This essentially means f is asymptotically negligible compared to g, which can prove to be useful when it comes to describing runtimes of algorithms.

The following functions are frequently used for classifying algorithms by their run time. Verify the statements below using L'Hopital's rule for example.

(a) $\ln(\ln(x)) \in o(\ln(x))$

(b) $\ln(x) \in o(\sqrt{x})$

(c) $\sqrt{x} \in o(x)$

(d) $x \in o(x\ln(x))$

(e) $x\ln(x) \in o(x^2)$

(f) $x^k \in o(e^{\sqrt{x}})$ for every fixed k

(g) $e^{\sqrt{x}} \in o(e^x)$

Solution:

We verify the sufficient condition each time: $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$ implies $f \in o(g)$. To this end we use L'Hospital's rule.

(a) $\lim_{x\to\infty} \frac{\ln(\ln(x))}{\ln(x)} = \lim_{x\to\infty} \frac{\frac{1}{x\ln(x)}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{1}{\ln(x)} = 0$

(b) $\lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$

(c) $\lim_{x\to\infty} \frac{\sqrt{x}}{x} = \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{1} = \lim_{x\to\infty} \frac{1}{2\sqrt{x}} = 0$

(d) $\lim_{x\to\infty} \frac{x}{x\ln(x)} = \lim_{x\to\infty} \frac{1}{\ln(x)+1} = 0$

(e) $\lim_{x\to\infty} \frac{x\ln(x)}{x^2} = \lim_{x\to\infty} \frac{\ln(x)+1}{2x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{2} = \lim_{x\to\infty} \frac{1}{2x} = 0$

(f) It holds that $\lim_{x\to\infty} \frac{x^k}{e^{\sqrt{x}}} = \lim_{x\to\infty} \frac{e^{\ln(x)k}}{e^{\sqrt{x}}} = \lim_{x\to\infty} e^{k\ln(x)-\sqrt{x}} = 0$ if and only if. $\lim_{x\to\infty} k\ln(x) - \sqrt{x} = -\infty$. In other words: $k\ln(x)$ has to grow negligibly slower than $\sqrt{x}$, i.e. $k\ln(x) \in o(\sqrt{x})$. Thereby we can reduce the problem to (b): $\lim_{x\to\infty} \frac{k\ln(x)}{\sqrt{x}} = \lim_{x\to\infty} \frac{2k}{\sqrt{x}} = 0$

(g) It holds that $\lim_{x\to\infty} \frac{e^{\sqrt{x}}}{e^x} = \lim_{x\to\infty} e^{\sqrt{x}-x} = 0$ gdw. $\lim_{x\to\infty} \sqrt{x} - x = -\infty$. In other words: $\sqrt{x}$ has to grow negligibly slower than x, i.e. $\sqrt{x} \in o(x)$. This has already been shown in (c).