HW 3 Solution

2. a) From the full form of the Friedmann equation:

$$H(t)^2 = \frac{8\pi G}{3c^2}\,\varepsilon(t) - \frac{\kappa c^2}{R_0^2\, a(t)}$$

Solving for $\varepsilon(t)$:

$$\varepsilon(t) = \frac{3c^2}{8\pi G} H(t)^2 + \left(\frac{3c^2}{8\pi G}\,\frac{c^2}{R_0^2\, a(t)}\right)\kappa$$

using $\rho = \varepsilon / c^2$:

$$\rho(t) = \frac{3}{8\pi G} H(t)^2 + \left(\frac{3}{8\pi G}\,\frac{c^2}{R_0^2\, a(t)}\right)\kappa$$

defining the current $\rho_{crit} = \rho(t=0, \kappa=0)$:

$$\rho_c = \frac{3}{8\pi G} H_0^2 \Rightarrow \boxed{\rho = \rho_c + \left(\frac{3c^2}{8\pi G R_0^2 a_0}\right)\kappa}$$

$$\boxed{\begin{array}{lll} \text{So if } \kappa = -1, & \rho < \rho_c & \text{(negative curvature)} \\ \text{if } \kappa = 0, & \rho = \rho_c & \text{(flat)} \\ \text{if } \kappa = +1, & \rho > \rho_c & \text{(positive curvature)} \end{array}}$$

~~b)~~ last: $\rho_c = \frac{3}{8\pi \cdot 6.67\times10^{-11} \frac{m^3}{kg\, s^2}} \left(70 \frac{km\, s^{-1}}{Mpc}\right)^2 \times \left(\frac{3.24\times10^{-20}\, Mpc}{1\, km}\right)^2 = \boxed{9.2\times10^{-27}\, kg/m^3}$

b) $9.2\times10^{-27} \frac{kg}{m^3} \times \frac{1\, H\ atom}{1.67\times10^{-27}\, kg} \times \left(\frac{1\, m}{100\, cm}\right)^3 = \boxed{5.5\times10^{-6}\, atoms/cm^3}$

Ryder 4.2

In general: $\frac{\ddot{a}}{a} = \frac{-4\pi G}{3c^2}(\varepsilon + 3P)$

where ε & P have matter + radiation contributions.

In Einstein's static universe Λ components are added.

$$\frac{\ddot{a}}{a} = \frac{-4\pi G}{3c^2}\left(\varepsilon_m + \varepsilon_r + \varepsilon_\Lambda + 3(P_m + P_r + P_\Lambda)\right)$$

we know $P = \omega\varepsilon$, where $\omega = 0$ for matter (non-relativistic)

$\omega = 1/3$ for radiation

$\omega = -1$ for Λ

$$\Rightarrow \frac{\ddot{a}}{a} = \frac{-4\pi G}{3c^2}\left(\varepsilon_m + \varepsilon_r + \varepsilon_\Lambda + 3\left(0 + \frac{\varepsilon_r}{3} - \varepsilon_\Lambda\right)\right)$$

$$= \frac{-4\pi G}{3c^2}\left(\varepsilon_m + 2\varepsilon_r - 2\varepsilon_\Lambda\right)$$

we know $\varepsilon_\Lambda = \frac{c^2\Lambda}{8\pi G} = \frac{c^2}{8\pi G}\cdot 4\pi G\rho = \frac{\rho c^2}{2}$

and $\varepsilon_m = \frac{mc^2}{V} = \rho c^2$

$$\Rightarrow \frac{\ddot{a}}{a} = \frac{-4\pi G}{3c^2}\left(\rho c^2 + 2\varepsilon_r - 2\left(\frac{\rho c^2}{2}\right)\right) = \boxed{-\frac{8\pi G}{3c^2}\varepsilon_r}$$

since $\frac{\ddot{a}}{a}$ is negative, the Universe is decelerating.

Ryder 4.3

$$\rho = 3\cdot 10^{-27}\,\mathrm{kg/m^3}$$

$$R_0 = \frac{c}{\sqrt{4\pi G\rho}} = \frac{2.998\times10^{8}\,\mathrm{m/s}}{\left(4\pi\ 6.67\times10^{-11}\frac{\mathrm{m^3}}{\mathrm{kg\,s^2}}\cdot 3\times10^{-27}\,\mathrm{kg/m^3}\right)^{1/2}} = \boxed{1.89\times10^{26}\,\mathrm{m}}$$

b) $$t = \frac{d}{v} = \frac{2\pi R_0}{v} = \frac{2\pi(1.89\times10^{26}\,\mathrm{m})}{3\times10^{8}\,\mathrm{m/s}} \times \frac{1\,\mathrm{yr}}{\pi\times10^{7}\,\mathrm{s}} = \boxed{126\ \mathrm{Gyr}}$$