1. a) $\dot{\varepsilon} + 3\frac{\dot{a}}{a}(1+\omega)\varepsilon = 0$

$$\frac{d\varepsilon}{\cancel{dt}} = -\frac{3}{a}\frac{da}{\cancel{dt}}(1+\omega)\varepsilon$$

$$\int_{\varepsilon_0}^{\varepsilon}\frac{d\varepsilon}{\varepsilon} = -3(1+\omega)\int_{a_0}^{a}\frac{da}{a}$$

$$\ln\left(\frac{\varepsilon}{\varepsilon_0}\right) = -3(1+\omega)\ln\left(\frac{a}{a_0}\right) \qquad e^{ab} = (e^a)^b = (e^b)^a$$

$e\,\hat{}\qquad\qquad e\,\hat{}$

$$\frac{\varepsilon}{\varepsilon_0} = \left(e^{\ln\left(\frac{a}{a_0}\right)}\right)^{-3(1+\omega)} = \left(\frac{a}{a_0}\right)^{-3(1+\omega)} = \left(\frac{a_0}{a}\right)^{3(1+\omega)}$$

$$\boxed{\varepsilon = \varepsilon_0\left(\frac{a_0}{a}\right)^{3(1+\omega)}}$$

b) $\omega_m = 0$ $\qquad \varepsilon_m \sim \frac{1}{a^{3(1+\omega)}} = \frac{1}{a^3}$

$\omega_r = \frac{1}{3}$ $\qquad \varepsilon_r \sim \frac{1}{a^{3(1+\omega)}} = \frac{1}{a^4}$

$\rightarrow$ as the Universe expands, a increases, so ε_m + ε_r both decrease.

c) $P = \omega\varepsilon = -\varepsilon_\Lambda \Rightarrow \boxed{\omega_\Lambda = -1}$

$\Rightarrow \frac{\varepsilon_\Lambda}{\varepsilon_0} = \frac{1}{a^0} = 1 \rightarrow \boxed{\varepsilon_\Lambda(t) = \varepsilon_{0,\Lambda}}$

$\rightarrow$ ε_Λ does not change with scale factor; nor does it change with time.

d) $\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}} = \frac{\varepsilon_{\Lambda,0}}{\varepsilon_{m,0}}$ $\qquad \frac{\varepsilon_\Lambda(t)}{\varepsilon_m(t)} = 1 = \frac{a^3\varepsilon_{\Lambda,0}}{\varepsilon_{m,0}} = \frac{a^3\Omega_{\Lambda,0}}{\Omega_{m,0}} \rightarrow a^3 = \frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}$

$\varepsilon_\Lambda(t) \sim \varepsilon_{\Lambda,0}$

$\varepsilon_m(t) \sim \frac{\varepsilon_{m,0}}{a^3}$

$$\boxed{a = \sqrt[3]{\frac{\Omega_m}{\Omega_\Lambda}} = \sqrt[3]{\frac{.2}{.7}} \sim .65}$$

2. a) isotropy and homogeneity

b) $\dot{a} \rightarrow$ rate of change of scale factor

$a \rightarrow$ scale factor; describes how the expansion of universe changes with time

$\varepsilon \rightarrow$ energy density of universe, usually described in terms of matter, radiation, and dark energy

$K \rightarrow$ curvature constant (0 = flat, -1 = negative, +1 = positive)

$R_0 \rightarrow$ radius of curvature of the Universe now

c) $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon - \frac{kc^2}{R_0^2 a^2}$; $\dot{\varepsilon} + 3\frac{\dot{a}}{a}(1+w)\varepsilon = 0$

$$\frac{d}{dt}(\dot{a}\cdot\dot{a}) = \frac{d}{dt}\left(\frac{8\pi G}{3c^2}\varepsilon a^2 - \frac{kc^2}{R_0^2}\right)$$

product rule x 2: $f = \varepsilon$, $g = a^2$; $f' = \dot{\varepsilon}$, $g' = 2a\dot{a}$

$\frac{d}{dt}(\varepsilon a^2) = 2a\dot{a}\varepsilon + \dot{\varepsilon}a^2$

product rule:

$f = \dot{a}$, $g = \dot{a}$; $f' = \ddot{a}$, $g' = \ddot{a}$ $\rightarrow \frac{d}{dt}(\dot{a}\cdot\dot{a}) = 2\dot{a}\ddot{a}$

$$\frac{2\ddot{a}\dot{a}}{2\dot{a}a} = \frac{\frac{8\pi G}{3c^2}\left(\dot{\varepsilon}a^2 + 2a\dot{a}\varepsilon\right)}{2\dot{a}a}$$

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3c^2}\left(\frac{\dot{\varepsilon}a}{\dot{a}} + 2\varepsilon\right)$$

$\dot{\varepsilon} = -\frac{3\dot{a}}{a}(1+w)\varepsilon$

$$= \frac{4\pi G}{3c^2}\left(-3(1+w)\varepsilon + 2\varepsilon\right)$$

$-3\varepsilon - 3w\varepsilon + 2\varepsilon = -3w\varepsilon - \varepsilon = -(3P + \varepsilon)$

$$\rightarrow \boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P)}$$

d) $\ddot{a} \sim -(\varepsilon + 3P)$

Matter: $P = 0 \rightarrow \ddot{a} \sim -\varepsilon \rightarrow \ddot{a} < 0$

Rad. $P = \frac{1}{3}\varepsilon \rightarrow \ddot{a} \sim -(\varepsilon + \varepsilon) \sim -2\varepsilon \rightarrow \ddot{a} < 0$

The universe will slow its acceleration in either case, but the pressure from radiation slows it faster than matter.

e) $P = -1\varepsilon \rightarrow \ddot{a} \sim -(\varepsilon - 3\varepsilon) \sim -2\varepsilon \rightarrow \ddot{a} > 0$

$\Rightarrow$ the rate of acceleration will increase

3(a) The cosmological principle states that there is nothing special about the location from which we find ourselves observing the universe - in other words the universe should be the same everywhere (ie homogeneous) and in all directions (isotropic)

1 point

The cosmological principle does not apply to our local universe, we clearly see structures (galaxies, groups etc) and also a special direction (~~everything~~ many galaxies seems to move towards the great attractor!)

1 point

Our universe does not appear homogeneous and isotropic until we consider size scales > 100 Mpc. Galaxy superclusters appear to be distributed in a filamentary structure with voids in between → these filaments and voids are distributed homogeneously and isotropically as best as we can observe.

1 point

observations of large scale galaxy cluster structures from galaxy counts, and the near-constant temperature of the CMB across the sky support the cosmological principle

1 point

(b) The FLWR metric describes a homogeneous, isotropic universe because
- the speed of light is universal
- curvature is global and so describes the whole universe
- a, the scale factor, depends only on time but is the same everywhere in space.

1 point

(c) The horizon distance is ~~given by~~ the farthest distance a particle moving at the speed of light could have traveled since the start of the Universe. 1 point

Particles that travel at the speed of light travel on null geodesics ie $ds^2 = 0$ 1 point

The also travel on un-twisting paths so $d\Omega = 0$

This means

$$c^2 dt^2 = a(t)^2 dr^2$$

$$dt = \frac{a(t)}{c} dr$$

$$c \int_0^{t_0} \frac{dt}{a(t)} = \int_0^{d_h} dr$$

ie $d_{hor}(t_0) = c \int_0^{t_0} \frac{dt}{a(t)}$ where t_0 is age of universe.

2 points

d) $d_{hor}(t_0) = c\int_0^{t_0} \frac{dt}{a(t)}$

$a(t) = a_0 \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}$. $\qquad a_0 = 1.$

so $d_{hor}(t_0) = c\int_0^{t_0} \left(\frac{t_0}{t}\right)^{\frac{2}{3(1+w)}} dt.$ $\qquad$ 1 point

$= c\, t_0^{\frac{2}{3(1+w)}} \; t_0^{\left(\frac{-2}{3(1+w)}+1\right)} , \left(\frac{-2}{3(1+w)}+1\right).$

$\frac{-2}{3(1+w)} + \frac{3(1+w)}{3(1+w)}$

$= \frac{1+3w}{3(1+w)}$

$= c\, t_0^{\frac{2}{3(1+w)}} \; t_0^{\frac{1+3w}{3(1+w)}} \Big/ \frac{1+3w}{3(1+w)}$

$= c t_0 \left(\frac{1+3w}{3+3w}\right)^{-1}.$ $\qquad \boxed{H_0 = \frac{2}{3(1+w)} \frac{1}{t_0}}$

$= \frac{c}{H_0} \frac{2}{1+3w}$ $\qquad H_0 = \frac{\dot{a}}{a}$

matter $\rightarrow w = 0$.

$\boxed{\text{so } d_{p}(t_0) = \quad .} \; \boxed{3ct_0 = \frac{2c}{H_0}}$

for radiation, $w = \frac{1}{3}$

$\boxed{d_{p}(t_0) = \quad . = 2ct_0 = \frac{c}{H_0}}$

so horizon distance in radiation dominated universe is closer than in matter dominated.

1 point