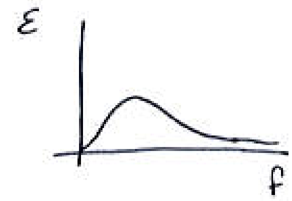
HW 2 Solution

1) $$\varepsilon(f)\,df = \frac{8\pi h}{c^3}\frac{f^3\,df}{e^{hf/kT}-1}$$



a) show $hf_{pk} \sim 2.82kT$

- peak occurs @ $\frac{d\varepsilon}{df} = 0$ because calc
- we know from plot that there's only a maximum, so no need to check that $\frac{d^2\varepsilon}{df^2} > 0$

rewriting: $$\varepsilon(f) = \frac{8\pi h}{c^3}\frac{h^2}{h^2}\frac{f^3}{e^{hf/kT}-1}\left(\frac{kT}{kT}\right)^3 = \frac{8\pi(kT)^3}{h^2c^3}\frac{\left(\frac{hf}{kT}\right)^3}{e^{hf/kT}-1}$$

let $x \equiv hf/kT$

$$\frac{d}{dx}\left(\frac{x^3}{e^x-1}\right) = 0$$

$f = x^3$, $f' = 3x^2$

$g = (e^x-1)^{-1}$, $g' = \frac{-e^x}{(e^x-1)^2}$

$$\rightarrow \frac{3\cancel{x^2}}{\cancel{e^x-1}} - \frac{x^{\cancel{3}}e^x}{(e^x-1)^{\cancel{2}}} = 0$$

$$\rightarrow 3(e^x-1) = xe^x$$

$$3e^x - 3 = xe^x$$

$$(3-x)e^x = 3 \xrightarrow{\text{Mathematica}} x \sim 2.82 = \frac{hf_{pk}}{kT} \rightarrow \boxed{hf_{pk} = 2.82kT}$$

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b) $$\varepsilon\,df = \frac{8\pi(kT)^3}{h^2c^3}\frac{x^3}{e^x-1}\,df$$

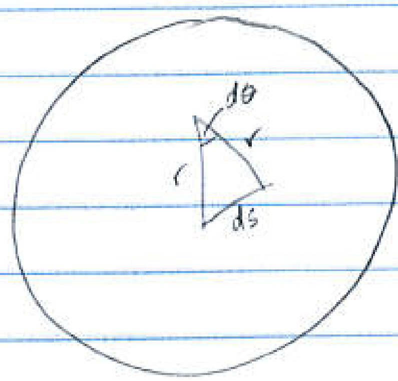
$x = \frac{h}{kT}f$

$dx = \frac{h}{kT}df \rightarrow df = \frac{kT}{h}dx$

$$\rightarrow \varepsilon\,df = \frac{8\pi(kT)^4}{(hc)^3}\frac{x^3}{e^x-1}dx$$

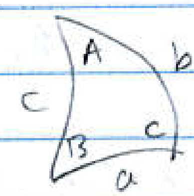
$$\varepsilon_{TOT} = \frac{8\pi k^4T^4}{(hc)^3}\int_0^\infty \frac{x^3}{e^x-1}dx = \underbrace{\frac{8\pi k^4}{(hc)^3}\int_0^\infty\frac{x^3}{e^x-1}dx}_{\alpha}\,T^4 = \boxed{\alpha T^4}$$

(a)



An object of length ds forms a triangle with the observer as shown. Since this is all on the surface of a sphere, we can use spherical trigonometry to determine $d\theta$.

Cosine rule For generic triangle



$\cos a = \cos b \cos c + \sin b \sin c \cos A$

our triangle:

$$\cos(ds) = \cos r \cos r + \sin r \sin r \cos(d\theta)$$
$$= \cos^2 r + \sin^2 r \cos(d\theta)$$

$\cos^2\theta + \sin^2\theta = 1$

$$\rightarrow \quad = 1 - \sin^2 r + \sin^2 r \cos(d\theta)$$

$$\cos ds = 1 + \sin^2 r\left(\cos(d\theta) - 1\right)$$

$$\frac{\cos(ds) - 1}{\cos(d\theta) - 1} = \sin^2 r$$

Maclaurin expansion of $\cos x$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

ignore higher order terms (ok since ds, $d\theta$ small)

$$\frac{ds^2}{d\theta^2} = \sin^2 r$$

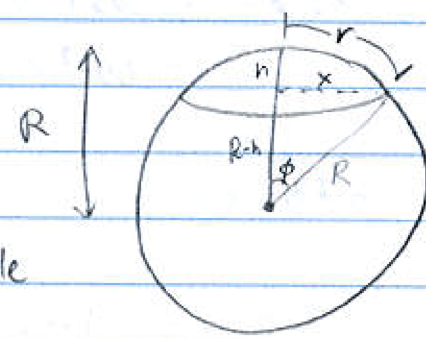
$$\boxed{d\theta = \frac{ds}{\sin r}}$$

as $r \to \pi R$, then object is on opposite side of sphere

$\sin r \to 0$

so $\underline{\underline{d\theta \to \infty}}$

(b) To calculate the circumference of a circle on the surface of a sphere, we can relate r, the distance 'down' from the north pole to ϕ, the latitude

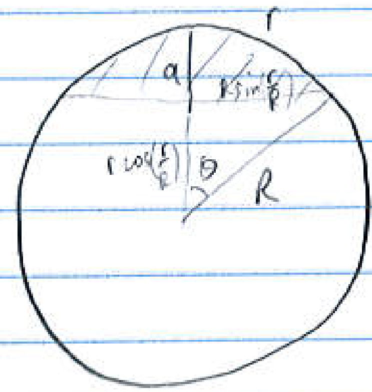


$$\frac{r}{2\pi R} = \frac{\phi}{2\pi} \quad \text{so} \quad \boxed{\phi = \frac{r}{R}}$$

circumference of circle located at latitude ϕ $C = 2\pi x$

$$x = R\sin\phi$$
$$= R\sin\left(\frac{r}{R}\right)$$

So $C = 2\pi R\sin\left(\frac{r}{R}\right)$, as required

(c) 

Again, imagine being located at the north pole — we want to know the surface area enclosed by the circle at r

surface area of spherical 'cap' $= 2\pi R a$

$a = R - R\cos\left(\frac{r}{R}\right)$ from triangle above

so $\text{Area} = 2\pi R\left(R - R\cos\left(\frac{r}{R}\right)\right)$

$= 2\pi R^2\left[1 - \cos\left(\frac{r}{R}\right)\right]$ as required.

$N \text{ stars} = \text{area} \times n$

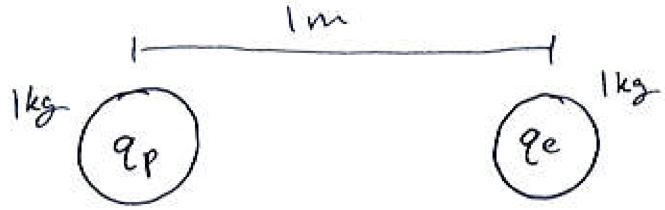
$= 2\pi n R^2\left[1 - \cos\left(\frac{r}{R}\right)\right]$.

if $r << R$, then $\cos\left(\frac{r}{R}\right) \to 1 - \frac{1}{2}\frac{r^2}{R^2}$

so $N \to 2\pi n R^2\left[1 - 1 + \frac{1}{2}\frac{r^2}{R^2}\right]$

$\to \pi n r^2$ as expected in flat space.

3)



if we assume the rocks are ~1/2 protons and ~1/2 neutrons by mass:

$$N_{protons} = \frac{\frac{1}{2}M_{rock}}{m_p} = \frac{.5kg}{1.673\times10^{-27}kg} \approx 3\times10^{26} \text{ protons}$$

since 1,000,000,001 e = 1,000,000,000 p:

$$N_{electrons} = N_{protons} \times \frac{1,000,000,001 e}{1,000,000,000 p} = 3.000000003\times10^{26} \text{ electrons}$$

The net charge on each rock is:

$$q = (N_{electrons} - N_{protons})e^- = 3\times10^{17}e^- = .048C$$

$$\rightarrow F = \frac{kq_1q_2}{r^2} = \frac{9\times10^9\frac{Nm^2}{C^2}(.048C)^2}{1m^2} \approx \boxed{2\times10^7 N}$$

b) It's not hard to push 2 rocks together.