第6周作业

1、

$$
\begin{aligned}
P(Y=k) &= P(X_1+X_2=k) \\
&= \sum_i P(X_1=k_i, X_2=k-k_i) \\
&= \sum_i P(X_1=k_i)P(X_2=k-k_i) \\
&= \sum_i \frac{\lambda_1^{k_i}}{k_i!}e^{-\lambda_1}\frac{\lambda_2^{k-k_i}}{(k-k_i)!}e^{-\lambda_2} \\
&= \sum_i \frac{\lambda_1^{k_i}\lambda_2^{k-k_i}}{k_i!(k-k_i)!}e^{-(\lambda_1+\lambda_2)} \\
&= \sum_i C_k^{k_i}\lambda_1^{k_i}\lambda_2^{k-k_i}\frac{e^{-(\lambda_1+\lambda_2)}}{k!} \\
&= \frac{(\lambda_1+\lambda_2)^k}{k!}e^{-(\lambda_1+\lambda_2)}
\end{aligned}
$$

可以看出$Y \sim P(\lambda_1+\lambda_2)$，泊松分布是二项分布$n \to \infty$的形式，因此期望具有可加性

2、

$$
\begin{cases}
Z = \dfrac{Y}{X} \\
T = X
\end{cases}
$$

(1)

$$
\begin{aligned}
f'(T,Z) &= f(X,Y)|J| \\
&= |T|f(T,TZ) \\
f'(Z) &= \int_{-\infty}^{+\infty} f(T,TZ)|T|dT \\
&= 2\int_0^{+\infty} f(T,TZ)TdT \\
&= \int_0^{+\infty} \frac{1}{\pi}e^{-\frac{T^2}{2}(1+Z^2)}TdT \\
&= \frac{1}{\pi(1+Z^2)}
\end{aligned}
$$

(2)

$$
\begin{aligned}
g(R,\theta) &= f(x,y)|J| \\
&= f(R\cos\theta, R\sin\theta)R \\
&= \frac{1}{2\pi}e^{-\frac{R^2}{2}}R \\
g(R) &= \int_0^{2\pi} g(R,\theta)d\theta \\
&= e^{-\frac{R^2}{2}}R \\
g(\theta) &= \int_0^{+\infty} g(R,\theta)dR \\
&= \frac{1}{2\pi} \\
g(R,\theta) &= g(R)g(\theta)\text{，所以}R,\theta\text{独立}
\end{aligned}
$$

(3)

$$
\begin{aligned}
g(U,V) &= f(x,y)|J| \\
&= \frac{1}{2}f(\frac{U+V}{2},\frac{U-V}{2}) \\
&= \frac{1}{4\pi}e^{-\frac{U^2+V^2}{4}} \\
g(U) &= \int_{-\infty}^{+\infty} g(U,V)dV \\
&= \int_{-\infty}^{+\infty} \frac{1}{4\pi}e^{-\frac{U^2+V^2}{4}}dV \\
&= \frac{1}{2\sqrt{\pi}}e^{-\frac{V^2}{4}} \\
\text{对称得} g(V) &= \frac{1}{2\sqrt{\pi}}e^{-\frac{U^2}{4}} \\
g(U,V) &= g(U)g(V), U,V\text{独立}
\end{aligned}
$$

3、

设Y的分布函数为G, Z的分布函数为H

$$
\begin{gathered}
P(Y \le y) = P(X_1 \le y)P(X_2 \le y)\cdots P(X_n \le y) \\
G(y) = F(y)^n \\
P(Z \ge z) = P(X_1 \ge z)P(X_2 \ge z)\cdots P(X_n \ge z) \\
1 - H(z) = (1 - F(z))^n \\
H(z) = 1 - (1 - F(z))^n
\end{gathered}
$$

4、

卡方分布

设 $X_1, X_2, \ldots, X_n$ 相互独立，且都服从 标准正态分布 $\mathrm{N}(0,1)$，则称随机变量

$$\chi^2 = X_1^2 + X_2^2 + \cdots + X_n^2$$

服从自由度为 n 的 χ^2 分布，记为 $\chi^2 \sim \chi^2(n)$

t分布

设 $X \sim N(0,1), Y \sim \chi^2(n)$ 且 X, Y 相互独立, 令

$$t = \frac{X}{\sqrt{Y/n}}$$

称 t 服从的分布为自由度为 n 的 t-分布，记为 $t \sim t(n)$.

F分布

设X服从自由度为n_1的卡方分布, Y服从自由度为n_2的卡方分布，且X，Y独立，则称随机变量$F = \frac{X/n1}{Y/n2}$服从自由度为$(n1，n2)$的F分布，记为$F \sim F(n1，n2)$。

5、

(1)
$1000 \times \frac{5}{100} = 50, 1000 - 50 = 950$
如果A赢了,赔率$\eta_1 = \frac{950-500}{500} = 0.9$
如果B赢了，赔率$\eta_2 = \frac{950-300}{300} = \frac{13}{6} \approx 2.17$
如果C赢了，赔率$\eta_3 = \frac{950-200}{200} = 3.75$

(2)
若赔率为x,则获胜概率为$\frac{1}{1+x}$
A的获胜概率为$\frac{1}{1+0.9} = \frac{10}{19} \approx 0.526$
B的获胜概率为$\frac{1}{1+13/6} = \frac{6}{19} \approx 0.316$
C的获胜概率为$\frac{1}{1+3.75} = \frac{4}{19} \approx 0.211$

(3)
$0.526+0.316+0.211>1$，说明投注站更倾向高估每匹马的胜率，从而降低赔率

6、

(1)
错误

设X,Y服从$p=0.5$的$0-1$分布，则$Var(X)=Var(Y)=\frac{1}{4},Var(XY)=\frac{3}{16},Var(XY)\neq Var(X)Var(Y)$

(2)
错误

当X为1维变量，$E(X)=\bar{x}$,对数列0，1，3，$\bar{x}=1$显然不成立

7、

$$
\begin{gathered}
\text{要证}E\left((X-c)^2\right)\geq \operatorname{Var}(X)\\
\text{只要证}E(X^2-2cX+c^2)\geq \operatorname{Var}(X)\\
\text{只要证}E(X^2)-2cE(X)+c^2\geq E(X^2)-E^2(X)\\
\text{只要证}E^2(X)-2cE(x)+c^2\geq 0\\
(E(X)-c)\geq 0\text{显然成立，当且仅当}E(X)=c\text{取等}
\end{gathered}
$$

8、

不妨设$c>m$

$$
\begin{aligned}
E(|X-c|)&=\int_{-\infty}^{c}(c-x)f(x)dx+\int_{c}^{+\infty}(x-c)f(x)dx\\
&=\int_{-\infty}^{m}(c-x)f(x)dx+\int_{m}^{c}(c-x)f(x)dx+\int_{c}^{+\infty}(x-c)f(x)dx
\end{aligned}
$$

$$
\begin{aligned}
E(|X-m|)&=\int_{-\infty}^{m}(m-x)f(x)dx+\int_{m}^{+\infty}(x-m)f(x)dx\\
&=\int_{-\infty}^{m}(m-x)f(x)dx+\int_{m}^{c}(x-m)f(x)dx+\int_{c}^{+\infty}(x-m)f(x)dx
\end{aligned}
$$

$$
\begin{aligned}
E(|X-c|)-E(|X-m|)&=\int_{-\infty}^{m}(c-m)f(x)+\int_{m}^{c}(c+m-2x)f(x)dx+\int_{c}^{+\infty}(m-c)f(x)dx\\
&=0.5(c-m)+\int_{m}^{c}(c+m-2x)f(x)dx+(0.5(m-c)-\int_{m}^{c}(m-c)f(x)dx)\\
&=\int_{m}^{c}(2c-2x)f(x)dx\geq 0
\end{aligned}
$$

9、

$$
\text{对数正态分布的}pdf\text{为：}f(y)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(lny-\mu)^2}{2\sigma^2}}\times\frac{1}{y}
$$

$$
\begin{aligned}
E(Y) &= \int_0^{+\infty} yf(y)dy \\
&= \int_0^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(lny-\mu)^2}{2\sigma^2}} dy
\end{aligned}
$$

$$
令t = \frac{lny-\mu}{\sigma}，有y = e^{\sigma t+\mu}
$$

$$
\begin{aligned}
&= \int_0^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2}} d(e^{\sigma t+\mu}) \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}+\sigma t+\mu} \\
&= e^{\mu} + \frac{\sigma^2}{2} \\
Var(Y) &= \int_{-\infty}^{+\infty} (y-\mu)^2 f(y)dy \\
&= (e^{\sigma^2}-1)e^{2\mu+\sigma^2}
\end{aligned}
$$

11、

$(1) \iff (2) \iff (3) \iff (4)$
1、2由定义知等价
$Cov(X,Y) = E(XY) - E(X)E(Y)$
$Var(X,Y) = Var(X) + Var(Y) + 2Cov(X,Y)$

12、

$$
\begin{aligned}
Corr(X,Y) &= E(\frac{X-\mu_1}{\sigma_1}\frac{Y-\mu_2}{\sigma_2}) \\
&= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \iint_\Omega \frac{X-\mu_1}{\sigma_1}\frac{Y-\mu_2}{\sigma_2} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)}{\sigma_2^2}\right.\right.
\end{aligned}
$$

$$
令m = \frac{X-\mu_1}{\sigma_1}, n = \frac{Y-\mu_2}{\sigma_2}
$$

$$
\begin{aligned}
&= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \iint_\Omega mn\exp(-\frac{m^2-2\rho mn+n^2}{2(1-\rho^2)})\sigma_1\sigma_2 dmdn \\
&= \frac{1}{2\pi\sqrt{1-\rho^2}} \iint_\Omega mn\exp(-\frac{(m-\rho n)^2+(1-\rho^2)n^2}{2(1-\rho^2)})dmdn \\
&= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{+\infty} ne^{-\frac{n^2}{2}}dn \int_{-\infty}^{+\infty} me^{-\frac{(m-\rho n)^2}{2(1-\rho^2)}}dm
\end{aligned}
$$

$$
考虑\mu = \rho n, \sigma^2 = 1-\rho^2的一元正态分布, \mu = \int_{-\infty}^{+\infty} x\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx
$$

$$
因此\int_{-\infty}^{+\infty} me^{-\frac{(m-\rho n)^2}{2(1-\rho^2)}}dm = \sqrt{2\pi}\mu\sigma = \sqrt{2\pi}\rho n\sqrt{1-\rho^2}
$$

$$
\begin{aligned}
LHS &= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{+\infty} ne^{-\frac{n^2}{2}}dn\sqrt{2\pi}\rho n\sqrt{1-\rho^2} \\
&= \frac{\rho}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} n^2 e^{-\frac{n^2}{2}}dn
\end{aligned}
$$

$$
注意到\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} n^2e^{-\frac{n^2}{2}}dn是\mu = 0, \sigma = 1的一元正态分布的方差，值为1
$$

$$
LHS = \rho
$$

14、

(1)

$$
E((U-tV)^2) \geq 0, \forall t \in R
$$

$$
\begin{aligned}
E((U-tV)^2) &= E(U^2+t^2V^2-2tUV) \\
&= E(U^2)+t^2E(V^2)-2tE(UV) \geq 0
\end{aligned}
$$

$$E(U^2)+t^2E(V^2) \geq 2tE(UV)$$

$$\frac{1}{2}(\frac{E(U^2)}{t}+tE(V^2)) \geq E(UV)$$

$$\frac{1}{4}[(\frac{E(U^2)}{t}-tE(V^2))^2+4E(U^2)E(V^2)] \geq E^2(UV)$$

$$E^2(UV) \leq E(U^2)E(V^2)+\frac{1}{4}(\frac{E(U^2)}{t}-tE(V^2))^2$$

$$\text{取}t^2=\frac{E(U^2)}{E(V^2)}\text{，则有}E^2(UV) \leq E(U^2)E(V^2)$$

$$\text{等号成立当且仅当}U=tV$$

(2)

$$Corr(X,Y)=E(\frac{X-\mu_1}{\sigma_1}\frac{Y-\mu_2}{\sigma_2})$$

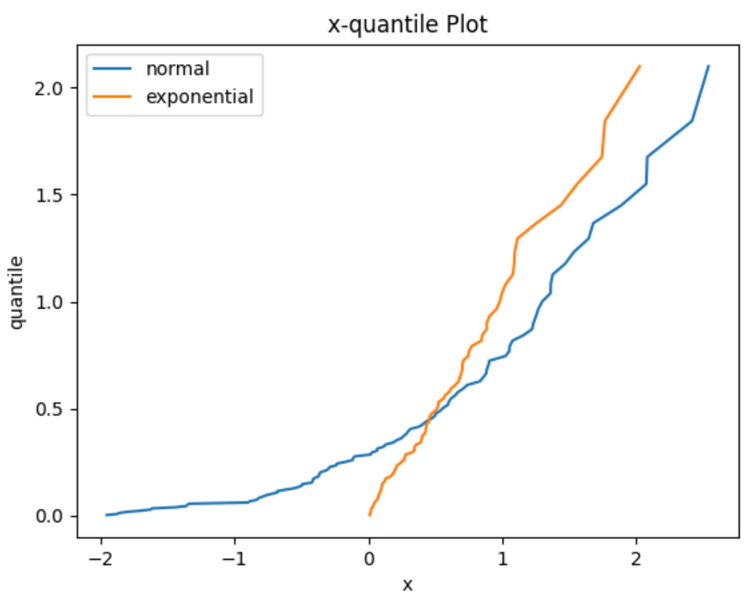
$$|Corr(X,Y)|^2=|E(\frac{X-\mu_1}{\sigma_1}\frac{Y-\mu_2}{\sigma_2})|^2 \leq E((\frac{X-\mu_1}{\sigma_1})^2)E((\frac{Y-\mu_2}{\sigma_2})^2)=1$$

$$\text{等号成立当且仅当}\frac{X-\mu_1}{\sigma_1}=t\frac{Y-\mu_2}{\sigma_2}$$

$$\text{即}Y=\frac{\sigma_2}{t\sigma_1}(X-\mu_1)+\mu_2$$

16、

(1)(2)



(3)
可以，设Φ^{-1}是假设的分布cdf的反函数，将数据排序并考察x_i与$\Phi^{-1}(\frac{i-1}{n})$的线性关系，可以推断是否服从假设的分布

(4)
如果数据量足够多，可以拟合出它们的cdf，从而求出反函数并绘制$Q-Q$图