第5周作业

1、

(1)

$$\text{取出红球的概率}P_1 = \frac{3}{3+4+5} = \frac{3}{12}$$
$$\text{取出白球的概率}P_2 = \frac{4}{3+4+5} = \frac{4}{12}$$
$$\text{取出黑球的概率}P_3 = \frac{5}{3+4+5} = \frac{5}{12}$$

$P(2\text{红}3\text{白}1\text{黑}) = \binom{6}{2\ 3\ 1} P_1^2 P_2^3 P_3 = \frac{25}{432}$

(2)

$X+Y=3$，分别有$(X,Y) = (0,3),(1,2),(2,1),(3,0)$

$$P(X=0,Y=3) = \frac{C_4^3}{C_{12}^3} = \frac{1}{55}$$
$$P(X=1,Y=2) = \frac{C_3^1C_4^2}{C_{12}^3} = \frac{9}{110}$$
$$\text{(把所有无序3元组看成基本事件)}$$
$$P(X=2,Y=1) = \frac{C_3^2C_4^1}{C_{12}^3} = \frac{3}{55}$$
$$P(X=3,Y=0) = \frac{C_3^3}{C_{12}^3} = \frac{1}{220}$$

事件	$X=0,Y=3$	$X=1,Y=2$	$X=2,Y=1$	$X=3,Y=0$
P	$\frac{1}{55}$	$\frac{9}{110}$	$\frac{3}{55}$	$\frac{1}{220}$

(3)

$$P(X=1) = \frac{C_3^1C_9^2}{C_{12}^3} = \frac{27}{55}$$

2、

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(s,t)dtds$$

$$\begin{aligned}
LHS &= F(b,d) - F(a,d) - F(b,c) + F(a,c) \\
&= (F(b,d) - F(b,c)) - (F(a,d) - F(a,c)) \\
&= (\int_{-\infty}^{b}\int_{-\infty}^{d} f(s,t)dtds - \int_{-\infty}^{b}\int_{-\infty}^{c} f(s,t)dtds) - (\int_{-\infty}^{a}\int_{-\infty}^{d} f(s,t)dtds - \int_{-\infty}^{a}\int_{-\infty}^{c} f(s,t)dtds) \\
&= \int_{-\infty}^{b}\int_{c}^{d} f(s,t)dtds - \int_{-\infty}^{a}\int_{c}^{d} f(s,t)dtds \\
&= \int_{a}^{b}\int_{c}^{d} f(s,t)dtds \\
&= P(a < X \leq b, c < Y \leq d)
\end{aligned}$$

3、

(1)

$$
\begin{aligned}
S &= \pi \\
\iint_S p(x,y) &= 1 \\
p &= \frac{1}{\pi}
\end{aligned}
$$

$$
f(X,Y) = \begin{cases} \frac{1}{\pi}, x^2+y^2 \le 1 \\ 0, x^2+y^2 > 1 \end{cases}
$$

(2)

$$
X\text{的}pdf: f_x(X) = \int_{-\infty}^{\infty} f(X,Y)dY = \frac{2}{\pi}\sqrt{4-X^2}
$$
$$
\text{由对称性，}Y\text{的}pdf: f_y(Y) = \frac{2}{\pi}\sqrt{4-Y^2}
$$

(3)

记以r为半径的圆为s

$$
P(R \le r) = \iint_S f(x,y)dxdy = \frac{\pi r^2}{\pi} = r^2
$$

(4)

R的cdf为$F(r)=r^2, pdf$为$f(r)=\frac{d}{dr}F(r)=2r$

$$
\begin{aligned}
E(R) &= \int_0^1 rf(r)dr \\
&= \int_0^1 r \times 2rdr \\
&= \frac{2}{3}
\end{aligned}
$$

4、

$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right)$

$$
\begin{aligned}
f_x(x) &= \int_{-\infty}^{+\infty} \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right) dy \\
&= \int_{-\infty}^{+\infty} \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[(\frac{y-\mu_y}{\sigma_y} - \frac{x-\mu_x}{\sigma_x})^2 + (1-\rho^2)\frac{(x-\mu_x)^2}{\sigma_x^2}\right]\right) dy \\
&= \int_{-\infty}^{+\infty} \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[(\frac{y-\mu_y}{\sigma_y} - \frac{x-\mu_x}{\sigma_x})^2\right] - \frac{1}{2}\frac{(x-\mu_x)^2}{\sigma_x^2}\right) dy \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2}\pi\sigma_x} \exp\left(-\frac{1}{2(1-\rho^2)}\left[(\frac{y-\mu_y}{\sigma_y} - \frac{x-\mu_x}{\sigma_x})^2\right] - \frac{1}{2}\frac{(x-\mu_x)^2}{\sigma_x^2}\right) d(\frac{y-\mu_y}{\sqrt{2(1-\rho^2)}\sigma_y}) \\
&= \frac{1}{\sqrt{2}\pi\sigma_x}\sqrt{\pi} exp(-\frac{1}{2}\frac{(x-\mu_x)^2}{\sigma_x^2}) \\
&= \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}}
\end{aligned}
$$

5、

$$
\begin{aligned}
f(x|y) &= \frac{f(x,y)}{f_Y(y)} \\
&= \frac{\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2}+\frac{(y-\mu_y)^2}{\sigma_y^2}-\frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right)}{\frac{1}{\sqrt{2\pi}\sigma_y}e^{-\frac{(y-\mu_y)^2}{2\sigma_y^2}}} \\
&= \frac{1}{\sqrt{2\pi}\sigma_x\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[-(1-\rho^2)\frac{(y-\mu_y)^2}{\sigma_y^2}+\frac{(x-\mu_x)^2}{\sigma_x^2}+\frac{(y-\mu_y)^2}{\sigma_y^2}-\frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right) \\
&= \frac{1}{\sqrt{2\pi}\sigma_x\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\rho^2\frac{(y-\mu_y)^2}{\sigma_y^2}+\frac{(x-\mu_x)^2}{\sigma_x^2}-\frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right) \\
&= \frac{1}{\sqrt{2\pi}\sigma_x\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[(\frac{x-\mu_x}{\sigma_x}-\rho\frac{y-\mu_y}{\sigma_y})^2\right]\right) \\
&= \frac{1}{\sqrt{2\pi}\sigma_x\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2\sigma_x^2)}(x-\mu_x-\rho\frac{\sigma_x}{\sigma_y}(y-\mu_y))^2\right)
\end{aligned}
$$

6、

(1)

$$S = \frac{1}{2}$$

$$P(X,Y) = \begin{cases} 2, X \ge 0\text{且}Y \ge 0\text{且}X+Y \le 1 \\ 0, \text{其他} \end{cases}$$

(2)

当$Y<0$或$Y>1$，$P(X,Y)=0, f_Y(y)=0$

当$0 \le Y \le 1, P(X,Y)=2, X\in[0,1-y], f_Y(y)=\int_0^{1-y}2dx=2(1-y)$

(3)

当$Y<0$或$Y>1$，$f_Y(y)=0$,无条件密度函数

当$0 \le Y \le 1, f_Y(y)=2(1-y)$

$$f(x|y) = \frac{f(x,y)}{f_Y(y)} = \begin{cases} \frac{1}{1-y}, y \le 1 \\ 0, y>1 \end{cases}$$

7、

(1)

$$\begin{aligned}P(X_1=k|X_1+X_2=n)&=\frac{P(X_1=k,X_1+X_2=n)}{P(X_1+X_2=n)}\\&=\frac{P(X_1=k,X_2=n-k)}{P(X_1+X_2=n)}\\ \text{由独立性}&=\frac{P(X_1=k)P(X_2=k)}{P(X_1+X_2=n)}\\&=\frac{\frac{\lambda_1^k}{k!}e^{-\lambda_1}\frac{\lambda_2^{n-k}}{(n-k)!}e^{-\lambda_2}}{\frac{(\lambda_1+\lambda_2)^n}{n!}e^{-(\lambda_1+\lambda_2)}}\\&=C_n^k(\frac{\lambda_1}{\lambda_1+\lambda_2})^k(\frac{\lambda_2}{\lambda_1+\lambda_2})^{n-k}\end{aligned}$$

(2)
泊松分布可近似理解为二项分布，分别做k次实验一和$n-k$次实验二，得到的就是类似的二项分布

8、

(1)
考虑$0-60$分钟,$P(X=k)=P(Y=k)=\frac{1}{60},k\in[0,60]$
因为X,Y独立，$P(X=k,Y=m)=(\frac{1}{60})^2=\frac{1}{3600}$

(2)
如果甲先到,$P=\int_0^{50}\int_{x+10}^{60}\frac{1}{3600}dydx=\frac{25}{72}$
对称可得乙先到等甲超过十分钟的概率为$\frac{25}{72}$
$P_{tot}=\frac{25}{36}$

9、

(1)

$$\begin{aligned}H(x,y)&=\int_{-\infty}^{x}\int_{-\infty}^{y}f(s,t)dtds\\ \frac{\partial}{\partial x}H(x,y)&=\int_{-\infty}^{y}f(x,t)dt\\ lim_{y\to+\infty}\frac{\partial}{\partial x}H(x,y)&=\int_{-\infty}^{+\infty}f(x,t)dt=f_x(x)\end{aligned}$$

$$lim_{y\to+\infty}\frac{\partial}{\partial x}H(x,y)=G(y)(F'(x)[1+\alpha(1-F(x))(1-G(y))]+F(x)[-\alpha F'(x)(1-G(y))])$$
$$\text{代入}G(y)=1,LHS=F'(x)$$

由对称性，Y的边际分布函数$f_y(y)=G'(y)$

(2)
令$F(x)=x,x\in[0,1]$，则X的边际分布函数$F'(x)=x'=1$，类似定义$G(y)=y$,取$\alpha=-1$和1

10、

令$U=F(x),V=G(y)$,则有

$$\begin{aligned}C(u,v)&=P(U\le u,V\le v)\\&=P(F_x(X)\le F_x(x),F_y(Y)\le F_y(y))\\ \text{由累计分布函数单调性}&=P(X\le x,Y\le y)\end{aligned}$$

11、

(1)X连续，Y连续

$$全概率公式：\int_{-\infty}^{+\infty} f_{X|Y}(x|y)f_Y(y)dy$$
$$Bayes公式：f_{Y|X}(y|x)=\frac{f(x,y)}{f_X(x)}=\frac{f_{X|Y}(x|y)f_Y(y)}{\int_{-\infty}^{+\infty} f_{X|Y}(x|y)f_Y(y)dy}$$

(2)X连续，Y离散

$$全概率公式：\sum_i f_{X|Y}(x|y)P_{Yi}(y)$$
$$Bayes公式：P_{Y|X}(y|x)=\frac{P(x,y)}{f_X(x)}=\frac{f_{X|Y}(x|y)P_Y(y)}{\sum_i f_{X|Y}(x|y)P_{Yi}(y)}$$

(3)X离散，Y连续

$$全概率公式：\int_{-\infty}^{+\infty} P(x|y)f_Y(y)dy$$
$$Bayes公式：f_{Y|X}(y|x)=\frac{P(x,y)}{f_X(x)}=\frac{P_{X|Y}(x|y)f_Y(y)}{\int_{-\infty}^{+\infty} P_{X|Y}(x|y)f_Y(y)}$$

(4)X离散，Y离散

$$全概率公式：\sum_i P(x|y_i)P(y_i)$$
$$Bayes公式：P(y|x)=\frac{P(x|y)P(y)}{\sum_i P(x|y_i)P(y_i)}$$

12、

(1)

$$f(x,y)=\frac{c}{1+x^2+y^2},x^2+y^2\le 1$$

$$\begin{aligned}\iint_{x^2+y^2\le 1}\frac{c}{1+x^2+y^2}dxdy&=\iint_{0\le r\le 1,0\le\theta\le 2\pi}\frac{c}{1+r^2}rdrd\theta\\&=\int_{0\le r\le 1}\frac{2\pi rc}{1+r^2}dr\\&=\int_{0\le r\le 1}\frac{\pi c}{1+r^2}dr^2\\&=\pi cln(1+r^2)\big|_{r=0}^{1}\\&=\pi c(ln(2)-ln(1))\\&=\pi cln2\end{aligned}$$
$$\pi cln2=1$$
$$c=\frac{1}{\pi ln2}$$

(2)

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{+\infty} f(x,y)dy \\
&= \int_{-\infty}^{+\infty} \frac{c}{1+x^2+y^2}dy \\
\text{令}\alpha^2 = 1+x^2, &= \int_{-\infty}^{+\infty} \frac{c}{\alpha^2+y^2}dy \\
&= \int_{-\infty}^{+\infty} \frac{c}{\alpha(1+(\frac{y}{\alpha})^2)}d\frac{y}{\alpha} \\
&= arctan(\frac{y}{\alpha})|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\frac{c}{\alpha}
\end{aligned}
$$

$$
\begin{gathered}
\text{类似可定义}\beta^2 = 1+y^2, f_Y(y) = arctan(\frac{x}{\beta})|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\frac{c}{\beta} \\
\text{显然}f(x) \neq f_X(x)f_Y(y)
\end{gathered}
$$

13、

(1)

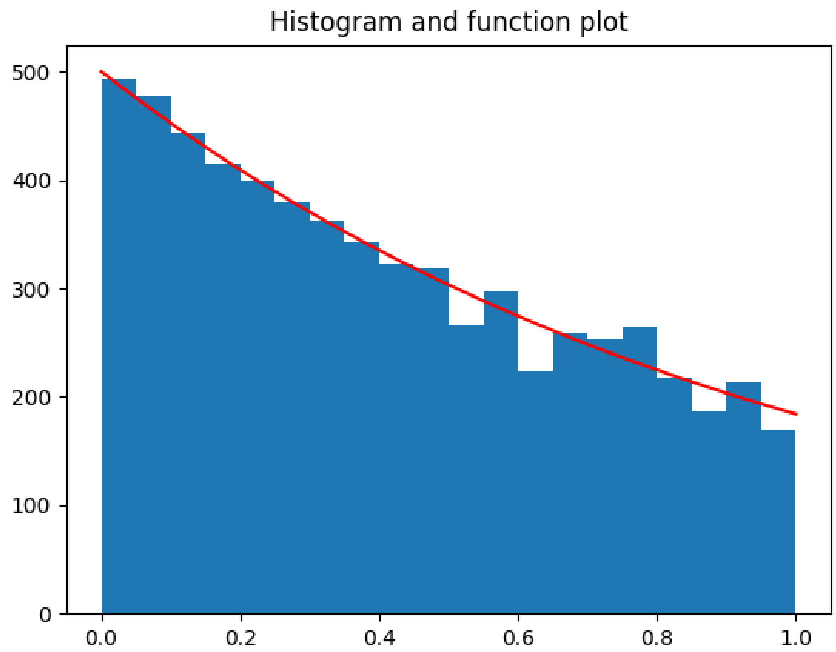
$$
\begin{aligned}
f(x,y) &= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2}+\frac{(y-\mu_y)^2}{\sigma_y^2}-\frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right) \\
&= \frac{1}{2\pi\sqrt{1-\rho^2}}\exp(-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}) \\
\text{由}X\text{，}Y\text{独立，}\rho = 0 &= \frac{1}{2\pi}\exp(-\frac{x^2+y^2}{2}) \\
&\geq \frac{1}{2\pi}e^{-1} \approx 0.059
\end{aligned}
$$

在$x^2+y^2 \leq 1$时，$f(x,y)+\frac{xy}{100} \geq 0$，且$\iint_{x^2+y^2\leq 1}\frac{xy}{100}dxdy = 0$满足$\iint_\Omega f(x,y) = 1$，所以$g(x,y)$是二维密度函数

(2)

$$
\begin{gathered}
f(x,y) = \frac{1}{2\pi}\exp(-\frac{x^2+y^2}{2}) \\
g(x,y) = \frac{1}{2\pi}\exp(-\frac{x^2+y^2}{2})+\frac{xy}{100} \\
g_X(x) = \int g(x,y)dy = \frac{1}{2\pi}e^{-\frac{x^2}{2}} \\
\text{类似得}g_Y(y) = \frac{1}{2\pi}e^{-\frac{y^2}{2}} \\
\text{且}(U,V)\text{显然不符合二元正态分布}
\end{gathered}
$$

14、




```python
import random
import math
import matplotlib.pyplot as plt
import numpy as np

# exp = np.random.exponential(scale=1, size=10000)
y = []
x = []
for i in range(10000):
    y.append(random.uniform(0, 1))
    x.append(-math.log(1 - y[i]))
plt.hist(x, bins=20, range=(0,1))

x_func = np.linspace(0, 1, 1000)
y_func = np.exp(-x_func) * 500
plt.title('Histogram and function plot')
plt.plot(x_func, y_func, 'r', label='500 * pdf function')
plt.savefig('2.png')
plt.show()
```