第15周作业

1、

(1)

$$B(t) \sim N(0,t)$$
$$P(B(2)>0)=0.5$$

(2)

$$\begin{aligned} &B(t-s) \sim N(0,t-s), 2B(s) \sim N(0,4s) \\ &B(t)+B(s)=B(t-s)+2B(s), \text{且}B(t-s)\text{和}B(s)\text{独立} \\ &=N(0,t+3s) \end{aligned}$$

(3)

$$\begin{aligned} P(B(1)>0)&=0.5 \\ P(B(1)>0,B(2)>0)&=P(B(1)>0,B(2)-B(1)>-B(1)) \\ \text{设}X_1=B(1),X_2=B(2)-B(1),\text{则有}LHS&=P(X_1>0,X_2>-X_1), \\ \text{且}X_{1/2}&\sim N(0,1) \\ P(X_1>0,X_2>-X_1)&=\int_0^{+\infty}\int_{-x_1}^{+\infty}f(x_1)f(x_2)dx_1dx_2 \\ &=\int_0^{+\infty}f(x_1)(1-\Phi(-x_1))dx_1 \end{aligned}$$

上面的积分比较困难，考虑$P(X_2<-X_1|X_1>0)=0.5(X_2\text{是负数})\times 0.5(X_2\text{绝对值更大})=0.25$

$$P(B(1)>0,B(2)>0)=P(X_1>0)P(X_2>-X_1|X_1>0)=0.5\times(1-0.25)=0.375$$
$$P(B(2)>0,B(1)>0)\neq P(B(1)>0)\times P(B(2)>0)\text{，不独立}$$

(4)

$$\text{设}X_1=B(1),X_2=B(2)-B(1),X_3=B(3)-B(2)$$
$$\text{考虑3维空间}x<0,x+y<0,x+y+z<0\text{围成的面积占比为}\frac{1}{3}$$

3、

$$\text{设}T_a\text{为}a\text{的首中时刻}P(T_a\le t)=2P(B(t)\ge a)=2(1-\Phi(\frac{a}{\sqrt{t}}))\to 1, t\to+\infty$$

4、

(1)

$$E(Z(t))=E(X(t))-tE(X(1))=0$$

5、

(1)

由$Brown$的$Markov$性质，在s时刻之后的事件仅与s时刻的状态有关
由对称性，s时刻后状态的均值为s时刻的值$Y(s)$

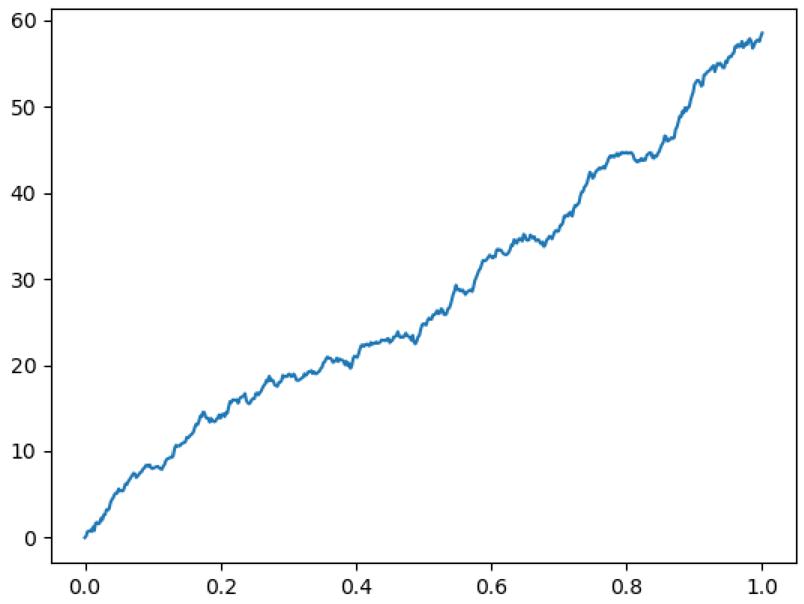
(2)

$$\begin{aligned} B(t)&=B(s)+X, X\sim N(0,t-s) \\ E(Y(t)|Y(u),0\le u\le s)&=E((B(s)+X)^2)-t \\ &=B(s)^2+E(2B(s)X)+X^2-t \\ &=B(s)^2+t-s-t \\ &=B(s)^2-s=Y(s) \end{aligned}$$

(3)

$$
\begin{aligned}
E(Y(t)) &= E(e^{-\frac{1}{2}\sigma^2 t+\sigma B(t)}) \\
&= Y(s)E(e^{-\frac{1}{2}\sigma^2(t-s)+\sigma X}), X \sim B(0, t-s) \\
&= Y(s)\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\sigma^2(t-s)+\sigma X} f(X)dX \\
&= Y(s)\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\sigma^2(t-s)+\sigma X} \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{X^2}{2\sigma^2}} dX \\
&= Y(s)\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2-2\sigma(t-s)+\sigma^2(t-s)^2}{2(t-s)}} dX \\
&= Y(s)\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-(t-s))^2}{2(t-s)}} dX \\
&= Y(s)
\end{aligned}
$$

8、




```python
import numpy as np

def generate_drift_brownian_motion(T, N, mu, sigma, n):
    """
    Generate n paths of a drift brownian motion.

    Parameters:
    T: total time
    N: number of time steps
    n: number of paths
    mu: drift term
    sigma: standard deviation of the random term

    Returns:
    B: a (n, N+1) array of n paths of the brownian motion
    """
    dt = T/N
    # Initialize the brownian motion
    B = np.zeros((n, N+1))
    # Generate the brownian motion
    for i in range(n):
        B[i, 1:] = mu * dt + sigma * np.sqrt(dt) * np.random.standard_normal(size=N)
    return np.cumsum(B, axis=1)

# Test the function
B = generate_drift_brownian_motion(T=50, N=500, mu=1, sigma=1, n=1)
```

```python
# Plot the result
import matplotlib.pyplot as plt
for i in range(B.shape[0]):
    plt.plot(np.linspace(0, 1, 501), B[i])
plt.savefig('8.png')
plt.show()
```