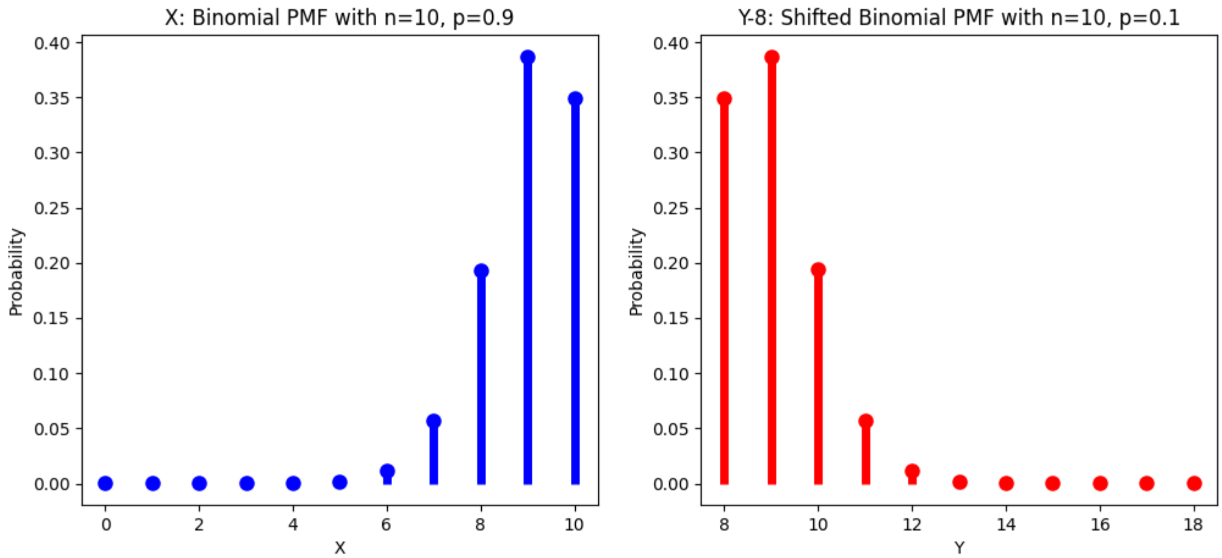
第7次作业

1、

(1)



(2)

$$
\begin{gathered}
u(X)=np=10\times 0.9=9\\
\sigma^2(X)=np(1-p)=10\times 0.9\times 0.1=0.9\\
mean(X)=9\\
mode(X)=9\\
u(Y-8)=np=10\times 0.1=1\\
u(Y)=9\\
\sigma^2(Y)=np(1-p)=0.9\\
mean(Y)=9\\
mode(Y)=9
\end{gathered}
$$

(3)

$$
\begin{aligned}
Skew(X)&=\frac{E[(X-\mu)^3]}{\sigma^3}\\
&=\frac{E(X^3)-3\mu E(X^2)+3\mu^2E(X)-\mu^3}{\sigma^3}\\
&=\frac{E(X^3)-3\mu(\sigma^2+\mu^2)+3\mu^3-\mu^3}{\sigma^3}\\
&=\frac{E(X^3)-3\mu\sigma^2-\mu^3}{\sigma^3}
\end{aligned}
$$

$$
\begin{aligned}
E(X^3) &= 1^3C_n^1p^1(1-p)^{n-1}+\cdots+n^3C_n^np^n(1-p)^0\\
&= np\sum_{k=0}^{n-1}C_{n-1}^k(k+1)^2p^k(1-p)^{n-1-k}\\
&= np\sum_{k=0}^{n-1}C_{n-1}^k(k^2+2k+1)p^k(1-p)^{n-1-k}\\
&= np[\sum_{k=0}^{n-1}C_{n-1}^k k^2p^k(1-p)^{n-1-k}+2\sum_{k=0}^{n-1}C_{n-1}^k kp^k(1-p)^{n-1-k}+\sum_{k=0}^{n-1}C_{n-1}^k p^k(1-p)^{n-1-k}]\\
&= np[E(X'^2)+2E(X')+1], X'\sim B(9,0.9)\\
&= np[\sigma^2(X')+E^2(X')+2E(X')+1]\\
&= np[\sigma^2(X')+(E(X')+1)^2]\\
&= 10\times 0.9[9\times 0.9\times 0.1+(9\times 0.9+1)^2]\\
&= 752.58\\
E(X^2) &= \sigma^2(X)+E^2(X)\\
&= np(1-p)+(np)^2\\
&= 0.9+9^2\\
&= 81.9\\
Skew(X) &= \frac{752.58-3\times 9\times 0.9-9^3}{0.9^{1.5}}\approx -0.843\\
\text{由对称性}, Skew(Y) &= 0.843
\end{aligned}
$$

2、

(1)(2)(3)

$Exp(1)$的矩母函数：

$$
\begin{aligned}
M_x(t) &= \int_0^{+\infty}e^{tx}e^{-x}dx\\
&= \int_0^{+\infty}e^{(t-1)x}dx\\
&= \frac{e^{(t-1)x}}{t-1}\Big|_0^{+\infty}\\
&= \frac{1}{1-t}, t\leq 1\\
M_x^{(1)} &= (t-1)^{-2}, M_x^{(1)}(0)=1\\
M_x^{(2)} &= -2(t-1)^{-3}, M_x^{(2)}(0)=2\\
M_x^{(3)} &= 6(t-1)^{-4}, M_x^{(3)}(0)=6\\
M_x^{(4)} &= -24(t-1)^{-5}, M_x^{(4)}(0)=24\\
\mu &= 1/1=1, \sigma^2=1, \sigma=1\\
Skew(X) &= E((X-1)^3)\\
&= E(X^3)-3E(X^2)+3E(X)-1\\
&= 6-3\times 2+3-1\\
&= 2\\
Kurt(X) &= E((X-1)^4)\\
&= E(X^4)-4E(X^3)+6E(X^2)-4E(X)+1\\
&= 24-4\times 6+6\times 2-4+1\\
&= 9
\end{aligned}
$$

$P(4)$的矩母函数：

$$
\begin{aligned}
M_x(t) &= E(e^{tx}) \\
&= \sum_k e^{tk}\frac{4^k}{k!}e^{-4} \\
&= e^{-4}\sum_k \frac{(4e^t)^k}{k!} \\
&= e^{-4}e^{4e^t} \\
&= e^{4e^t-4} \\
M_x^{(1)} &= 4e^te^{4e^t-4} = 4e^{4e^t+t-4} \\
M_x^{(2)} &= (4e^t+1)4e^{4e^t+t-4} = 16e^{4e^t+2t-4} + M_x^{(1)} \\
M_x^{(3)} &= 16(4e^t+2)e^{4e^t+2t-4} + M_x^{(2)} \\
&= 64e^{4e^t+3t-4} + 2(M_x^{(2)} - M_x^{(1)}) + M_x^{(2)} \\
M_x^{(4)} &= 64(4e^t+3)e^{4e^t+3t-4} + 3M_x^{(3)} - 2M_x^{(2)} \\
&= 256e^{4e^t+4t-4} + 192e^{4e^t+3t-4} + 3M_x^{(3)} - 2M_x^{(2)} \\
M_x^{(1)}(0) &= 4 \\
M_x^{(2)}(0) &= 16+4 = 20 \\
M_x^{(3)}(0) &= 64 + 2\times 16 + 20 = 116 \\
M_x^{(4)}(0) &= 256 + 192 + 3\times 116 - 2\times 20 = 756 \\
Skew(X) &= E(\frac{X-4}{\sigma})^3 = (E(X^3) - 3E(X^2) + 3E(X) - 64)/2^3 = 0.5 \\
Kurt(X) &= E(\frac{X-4}{2})^4 = (E(X^4) - 16E(X^3) + 96E(X^2) - 256E(X) + 256)/16 = 3.25
\end{aligned}
$$

$U(0,1)$的矩母函数：

$$
\begin{aligned}
M_x(t) &= \int_0^1 e^{tx}dx \\
&= \frac{e^{tx}}{t}\Big|_0^1 \\
&= \frac{e^t-1}{t} \\
M_x^{(n)}(0)&\text{没有意义} \\
E(X^n) &= \frac{1}{n+1} \\
\mu = 0.5, \sigma^2 &= \frac{1}{12} \\
Skew(X) &= E[(\frac{X-0.5}{\sigma})^3] = 0(\text{对称性}) \\
Kurt(X) &= E[(\frac{X-0.5}{\sigma})^4] = 3.36
\end{aligned}
$$

3、

$f(x) = \frac{1}{3}\times 2e^{-2x} + \frac{2}{3}\times 3e^{-3x}$

4、

(1)

$$
\begin{aligned}
M_x(t) &= \int_0^{+\infty} e^{tx}f(x)dx \\
lim_{x\to+\infty}e^{tx}f(x) &= +\infty\text{，积分不收敛}
\end{aligned}
$$

(2)

$$
\begin{aligned}
E(Y^n) &= E((e^X)^n) \\
&= E(e^{nX}) \\
&= M_x(n) \\
&= e^{\frac{\sigma^2 n^2}{2}+\mu n}
\end{aligned}
$$

5、

$$
\begin{gathered}
\text{对}X \sim P(\lambda), M_X(t) = e^{\lambda(e^t-1)} \\
M_Y(t) = M_{X_1}(t)M_{X_2}(t) \\
= e^{(\lambda_1+\lambda_2)(e^t-1)} \\
\text{所以}Y \sim P(\lambda_1+\lambda_2)
\end{gathered}
$$

6、

$X \sim U(0,1), Y \sim U(X,1)$

$$
\begin{aligned}
E(Y) &= E[E(Y|X)] \\
&= \int_0^1 \frac{x+1}{2}dx \\
&= \frac{3}{4}
\end{aligned}
$$

$$
\text{余下长度为}1-\frac{3}{4}=\frac{1}{4}
$$

7、

$$
\begin{gathered}
t = \frac{1}{3}\times 2 + \frac{1}{3}\times(t+3) + \frac{1}{3}\times(t+1) \\
t = 6
\end{gathered}
$$

8、

$$
\begin{aligned}
E(E(Y|X)) = E(Y) = E(X) \\
Cov(X,Y) &= E(X-E(X)(Y-E(Y))) \\
&= E(X-E(X)(Y-X+X-E(X))) \\
&= E(X-E(X)(Y-X)) + Var(X) \\
&= \sum_x (x-E(X))E(Y-x|x) + Var(X) \\
&= \sum_x (x-E(X))E(x-x|x) + Var(X) \\
&= Var(X)
\end{aligned}
$$

9、

(1)

$$
\begin{aligned}
E(Y|x) &= \int_{-\infty}^{+\infty} y f_{Y|X}(y|x)dy \\
&= \int_{-\infty}^{+\infty} y f(y)dy \\
&= E(Y)
\end{aligned}
$$

(2)
不成立

	A	B	C	D
Y	0	3	1	2
X	0	0	1	1

$E(Y|X)=1.5$恒成立，但Y与X不独立

10、

$$\begin{aligned}
Var(\hat{Y}) &= E((E(Y|X)-E(E(Y|X)))^2) \\
&= E((E(Y|X)-E(Y))^2) \\
&= E(E^2(Y|X))-2E(E(Y|X))E(Y)+E^2(Y) \\
&= E(E^2(Y|X))-E^2(Y) \\
Var(\tilde{Y}) &= E((\tilde{Y}-E(\tilde{Y}))^2) \\
\text{因为}E(\tilde{Y})=0, &= E(\tilde{Y}^2) \\
&= E((Y-E(Y|X))^2) \\
&= E(Y^2)-E(E^2(Y|X)) \\
Var(\hat{Y})+Var(\tilde{Y}) &= E(Y^2)-E(Y^2)=Var(Y)
\end{aligned}$$

11、

(1)

$$\begin{aligned}
Var(X|Y) &= E[Y^2-2YE(Y|X)+E^2(Y|X)|X] \\
&= E(Y^2|X)-2E^2(Y|X)+E^2(Y|X) \\
&= E(Y^2|X)-E^2(Y|X)
\end{aligned}$$

(2)

$$\begin{aligned}
Var[E(Y|X)] &= E((E(Y|X)-E(E(Y|X)))^2) \\
&= E((E(Y|X)-E(Y))^2) \\
&= E(E^2(Y|X)-2E(Y|X)E(Y)+E^2(Y)) \\
&= E(E^2(Y|X))-E^2(Y) \\
E[Var(X|Y)+Var[E(Y|X)]] &= E(Y^2)-E(E^2(Y|X))+E(E^2(Y|X))-E^2(Y) \\
&= E(Y^2)-E^2(Y) \\
&= Var(Y)
\end{aligned}$$

12、

$E(Y|0.5)=\frac{\sqrt{3}}{4}$

13、

(1)当$\rho>0, a=\frac{\sigma_2}{\sigma_1}, b=\mu_2-\frac{\sigma_2}{\sigma_1}\mu_1$；当$\rho<0$，用$-\sigma_2$替换$\sigma_2$

14、

(1)

$$E(Y)=E(E(Y|X))=\sum_n n\mu p_n=a\mu$$

(2)
$M_N(t)=\sum_n e^{tn}p_n$

15、

考虑一个随机变量N，它是一个离散型随机变量，表示要相加的正态随机变量的个数。假设N服从某个离散分布，例如泊松分布

考虑一系列独立的标准正态随机变量$X_1,X_2,X_3,...$，它们都服从$N(0,1)$。

定义随机变量S，它表示前N个正态随机变量的和，即$S=X_1+X_2+...+X_N$。在这种情况下，S的分布取决于N的取值，即实际相加的正态随机变量的个数。

因为N是随机的，所以S的分布实际上是所有可能的N值下正态随机变量之和的混合分布。也即S的分布是多个正态分布的加权和，

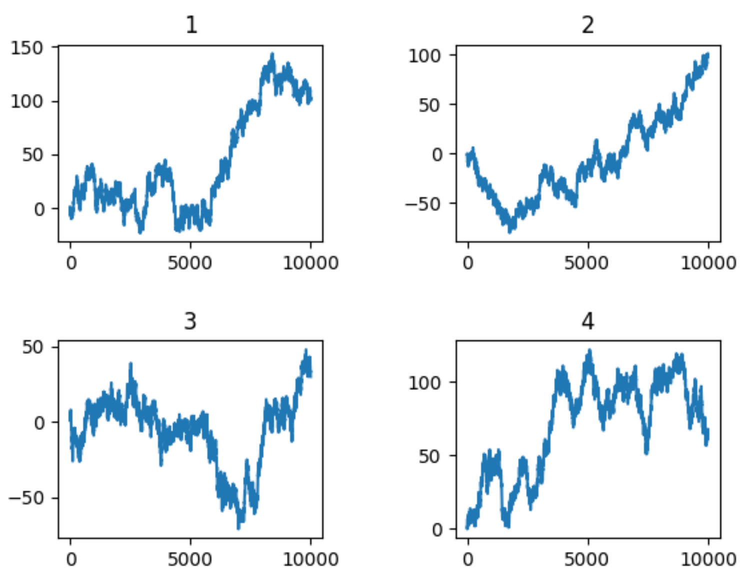
16、

1、

$$
\begin{gathered}
M_x(t)=\sum_i e^{tx_i}p_i=\frac{1}{2}(e^t+e^{-t})\\
M_x(t)^n=\frac{C_n^0e^{tn}+C_n^1e^{t(n-2)}+\cdots+C_n^ne^{-tn}}{2^n}\\
P(X_n=n)=\frac{C_n^0}{2^n}\\
P(X_n=n-2)=\frac{C_n^1}{2^n}\\
\cdots\\
\text{由对称性}, E(X_n)=0
\end{gathered}
$$

$$
\begin{aligned}
Var(X_n)&=\frac{1}{2^n}(n^2C_n^0+(n-2)^2C_n^1+\cdots+(-n)^2C_n^n)\\
&=\sum_{k=0}^{n}(n-2k)^2C_n^kp^k(1-p)^{n-k},\text{其中}p=0.5\\
&=\sum_{k=0}^{n}(n^2-4kn+4k^2)C_n^kp^k(1-p)^{n-k}\\
&=n^2-4nE(X)+4E(X^2),X\sim B(n,0.5)\\
&=n^2-4n\times0.5n+4(Var(X)+E^2(X))\\
&=n^2-2n^2+4(n\times0.5\times0.5+n^2\times0.25)\\
&=n
\end{aligned}
$$

2、



图像会多次在0左右震荡，最大偏离在100左右，与$\sigma^2=10000,\sigma=100$一致