第15次作业

1、

$$
\begin{aligned}
&\text{分划} 0 = t_0 < t1 < \cdots < t_n = t, \lambda_n = \max\{t_1 - t_0, \cdots, t_k - t_{k-1}\} \\
\int_0^t s dB(s) &= \sum t_{k-1}(B(t_k) - B(t_{k-1})) \\
&= \sum [(t_k B(t_k) - t_{k-1} B(t_{k-1})) - B(t_k)(t_k - t_{k-1})] \\
&= tB(t) - 0 \times B(0) - \sum B(t_k)(t_k - t_{k-1}) \\
&= tB(t) - \sum B(t_{k-1})(t_k - t_{k-1}) + (B(t_k) - B(t_{k-1}))(t_k - t_{k-1}) \\
&= tB(t) - \int_0^t B(s) ds + \sum (B(t_k) - B(t_{k-1}))(t_k - t_{k-1}) \\
&= tB(t) - \int_0^t B(s) ds + E(\sum (B(t_k) - B(t_{k-1}))(t_k - t_{k-1})) \\
&= tB(t) - \int_0^t B(s) ds + E(\sum 0 \times (t_k - t_{k-1})) \\
&= tB(t) - \int_0^t B(s) ds
\end{aligned}
$$

2、

由$It\hat{o}$微分公式：

$$
\begin{aligned}
dX(t) &= (\frac{\partial X}{\partial t} + \frac{1}{2}\frac{\partial^2 X}{\partial B(t)^2})dt + \frac{\partial X}{\partial B} dB(t) \\
&= (-\frac{1}{2} e^{B(t)} e^{-\frac{t}{2}} + \frac{1}{2} e^{B(t)} e^{-\frac{t}{2}})dt + e^{B(t)} e^{-\frac{t}{2}} dB(t) \\
&= e^{B(t)} e^{-\frac{t}{2}} dB(t) \\
&= X(t) dB(t)
\end{aligned}
$$

3、

$$
d(\frac{1}{3}B^3) = (0 + \frac{1}{2} \times 2B)dt + B^2 dB
$$

$$
B^2 dB = d(\frac{1}{3}B^3) - Bdt
$$

由$It\hat{o}$微分定义，$\int_0^t B(s)^2 dB(s) = \frac{1}{3}B(s)^3 - \int_0^t B(s) ds$

4、

$$\begin{aligned}
\text{考虑} d(B(t)e^{at}) &= (aB(t)e^{at} + \frac{1}{2} \times 0)dt + e^{at}dB(t) \\
&= aB(t)e^{at}dt + e^{at}dB(t) \\
\text{则有} \int_0^t e^{as}dB(s) &= B(t)e^{at} - \int_0^t aB(s)e^{as}ds \\
\text{可得} \frac{\partial}{\partial t}(\int_0^t e^{as}dB(s)) &= aB(t)e^{at} - aB(t)e^{at} = 0 \\
\frac{\partial}{\partial B(t)}(\int_0^t e^{as}dB(s)) &= e^{at} \\
\frac{\partial^2}{\partial B(t)^2}(\int_0^t e^{as}dB(s)) &= \frac{\partial}{\partial B(t)}(e^{at}) = 0 \\
Y(t) &= \sigma e^{-at}\int_0^t e^{as}dB(s) \\
dY(t) &= (\frac{\partial Y}{\partial t} + \frac{\partial^2 Y}{\partial B^2})dt + \frac{\partial Y}{\partial B}dB \\
&= ((-aY(t) + \frac{\partial}{\partial t}(\int_0^t e^{as}dB(s)))\sigma e^{-at} + 0)dt + \sigma e^{-at}e^{at}dB(t) \\
&= -aY(t)dt + \sigma dB(t)
\end{aligned}$$