Chapter 1

VECTORS

Perhaps the single most important mathematical device used in computer graphics and many engineering and physics applications is the *vector*. A vector is a geometric object of a sort, because, as we will soon see, it fits our notion of a displacement or motion. (We can think of displacement as a change in position. If we move a book from a shelf to a table, we have displaced it a specific distance and direction.) Vector methods offer a distinct advantage over traditional analytic geometry by minimizing our computational dependence on a specific coordinate system until the later stages of solving a problem. Vectors are direct descendants of complex numbers and are generalizations of hypercomplex numbers. Interpreting these numbers as directed line segments makes it easier for us to understand their properties and to apply them to practical problems.

Length and direction are the most important vector properties. Scalar multiplication of a vector (that is, multiplication by a constant), vector addition, and scalar and vector products of two vectors reveal more geometric subtleties. Representing straight lines and planes using vector equations adds to our understanding of these elements and gives us powerful tools for solving many geometry problems. Linear vector spaces and basis vectors provide rigor and a deeper insight into the subject of vectors. The history of vectors and vector geometry tells us much about how mathematics develops, as well as how gifted mathematicians established a new discipline.

1.1 Introduction

In the 19th century, mathematicians developed a new mathematical object—a new kind of number. They were motivated in part by an important observation in physics: Physicists had long known that while some phenomena can be described by a single number—the temperature of a beaker of water (6°C), the mass of a sample of iron (17.5g), or the length of a rod (31.736 cm)—other phenomena require something more.

A ball strikes the side rail of a billiard table at a certain speed and angle (Figure 1.1); we cannot describe its rebound by a single number. A pilot steers north with an air speed of 800 kph in a cross wind of 120 kph from the west (Figure 1.2); we cannot describe the airplane's true motion relative to the ground by a single number. Two spherical bodies collide; if we know the momentum (mass $\times$ velocity) of each body before impact, then we can determine their speed and direction after impact (Figure 1.3). The

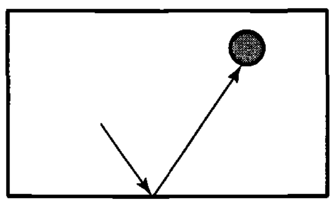

Figure 1.1 *Billiard ball rebound.*

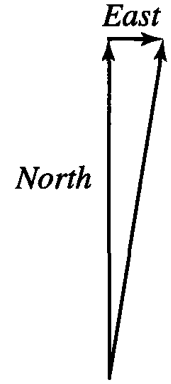

Figure 1.2 *Effect of a cross wind on an airplane's course.*

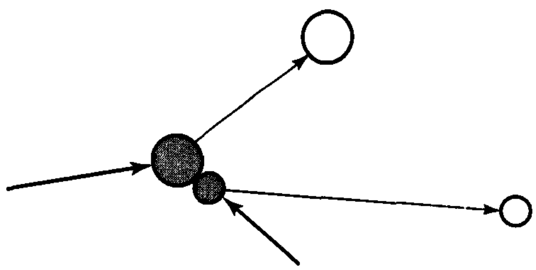

Figure 1.3 *Collision of two spherical bodies.*

description of the momentum of each body seems to require more information than a single number can convey.

Billiard balls, airplanes, and colliding bodies need a number that describes both the speed and the direction of their motion. Mathematicians found that they could do this by using a *super number* made up of two or more normal numbers, called *components*. When the component member numbers of this super number are combined according to certain rules, the results define magnitude and direction.

For the example of the airplane flying in a crosswind, its magnitude is given in kilometers per hour, and its direction is given by a compass reading. For any specified elapsed time t and velocity v, the problem immediately reduces to one of directed distance or displacement d because we know that $d = vt$.

When a super number is associated with a distance and direction, it is called a *vector*, and we can use vectors directly to solve problems of the kind just described. Vectors are derived from a special class of numbers called *hypercomplex numbers*. We will use a simple version of a hyper-complex number and call it a *hypernumber*. In the next section we look at hypernumbers in a way that lets us interpret them geometrically.

1.2 Hypernumbers

Hypernumbers are a generalization of complex numbers. Recall that a complex number such as $a + bi$ consists of a real part a and an imaginary part bi, $i = \sqrt{-1}$. If we eliminate the coefficient i of the imaginary part and retain a and b as an ordered pair of real numbers (a, b), we create a two-dimensional hypernumber. Ordered triples, quadruples, or n-tuples, for that matter, of real numbers comprise and define higher-dimension hypernumbers.

Order is very important, as we shall soon see, and we will often need to use subscripts to indicate that order. For example, suppose we have the ordered triple of real numbers (a, b, c). None of the letters a, b, or c alone tells us about its position in the sequence. However, if we make the substitutions $a_1 = a$, $a_2 = b$, and $a_3 = c$, the hypernumber becomes (a_1, a_2, a_3). This notation not only lets us know the position of each number in the sequence, but it also frees b and c for other duties.

It is tedious to write out the sequence of numbers for a hypernumber. Therefore, whenever we can, we will use an italicized uppercase letter to represent a hypernumber. Thus, we let $A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$, and so on. We generalize this to any number of dimensions so that, for example, A might represent the n-dimensional hypernumber $(a_1, a_2, \ldots, a_n)$, where $a_1, a_2, \ldots, a_n$ are the *components* of A.

Hypernumbers have their own special arithmetic and algebra. One of the most basic questions we can ask about two hypernumbers is: Are they equal? Two hypernumbers A and B are equal, that is $A = B$, if and only if their corresponding components are equal. This means that if $A = B$, then $a_1 = b_1, a_2 = b_2, \ldots a_n = b_n$. This also means, of course, that A and B must have the same number of components. We say that they must have the same *dimension*. Is the hypernumber $A = (5, 2)$ equal to the hypernumber $B = (2, 5)$? No, because $a_1 = 5$, $b_1 = 2$, and $a_1 \neq b_1$. What about $C = (-1, 6, 2)$ and $D = (-1, 6, 2)$? Clearly, here $C = D$.

We can add or subtract two hypernumbers of the same dimension by adding (or subtracting) their corresponding components. Thus,

$$A + B = (a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) \tag{1.1}$$

or

$$A + B = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) \tag{1.2}$$

We find that the sum of two hypernumbers is another hypernumber; for example, $A + B = C$, where $c_1 = a_1 + b_1$, and $c_2 = a_2 + b_2, \ldots, c_n = a_n + b_n$. For the two-dimensional hypernumbers, $A = (14, -5)$, $B = (0, 8)$, and $A + B = (14, 3)$.

The simplest kind of multiplication involving a hypernumber is scalar multiplication, where we multiply each component by a common factor k. We write this as follows:

$$kA = k(a_1, a_2, a_3) = (ka_1, ka_2, ka_3) \tag{1.3}$$

We find that k has the effect of scaling each component equally. In fact, k is called a *scalar* to distinguish it from the hypernumber. The two hypernumbers $(3, 5)$ and $(9, 15)$ differ by a scale factor of $k = 3$. And, as it turns out, division is not defined for hypernumbers.

When we multiply two hypernumbers, we expect to find a product such as $(a_1, a_2, a_3) \times (b_1, b_2, b_3)$. In Section 1.7 we will see that there are actually two kinds of multiplication, one of which is not commutative (i.e., $AB \neq BA$).

1.3 Geometric Interpretation

To see the full power of hypernumbers, we must give them a geometric interpretation. Distance and direction are certainly important geometric properties, and we will now see how to derive them from hypernumbers.

Imagine that an ordered pair of numbers, a hypernumber, is really just a set of instructions for moving about on a flat two-dimensional surface. For the moment, let's agree that the first number of the pair represents a displacement (how we are to move) east if plus (+), or west if minus (−), and that the second number represents a displacement north (+) or south (−). Then we can interpret the two-dimensional hypernumber, (16.3, −10.2) for example, as a displacement of 16.3 units of length (feet, meters, light-years, or whatever) to the east, followed by a displacement of 10.2 units of length (same as the east-west units) to the south (Figure 1.4). Note something interesting here: By applying the Pythagorean theorem we find that this is equivalent to a total (resultant) displacement of $\sqrt{(16.3)^2 + (-10.2)^2} = 19.23$ units of length in a southeasterly direction. We can, of course, be more precise about the direction. The compass heading from our initial position would be

$$90^\circ + \arctan \frac{10.2}{16.3} = 122^\circ \text{ SE} \tag{1.4}$$

We can apply this interpretation to any ordered pair of numbers (a_1, a_2) so that each pair produces both a magnitude (the total displacement) and a direction. Think of this interpretation as an algorithm, or mathematical machine, that uses as input an ordered pair of real numbers and produces as output two real numbers that we interpret as magnitude and direction. In fact, this interpretation is so different from that of an ordinary hypernumber that we are justified in giving the set $A = (a_1, a_2)$ a new name and its own notation. W. R. Hamilton (1805–1865) was the first to use the term *vector* (from the Latin word *vectus*, to carry over) to describe this new mathematical object, and that is what we will do. We will use boldface lowercase letters to represent vectors, and list the components inside brackets without separating commas, so that $\mathbf{a} = [a_1 \quad a_2]$, for example.

We treat ordered triples of numbers (a_1, a_2, a_3) in the same way, by creating a third dimension to complement the compass headings of the two-dimensional plane. This is easy enough: Adding up (+) and down (−) is all that we need to do. We have,

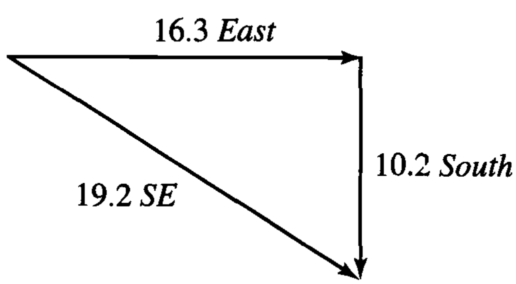

Figure 1.4 *Displacement as a vector.*

then, for an ordered triple, an agreement that the first number represents a displacement east (+) or west (−), the second number is a displacement north (+) or south (−), and the third number is a displacement up (+) or down (−) (Figure 1.5). Later we will see how an ordered triple yields a distance and direction.

Before doing this, let's see how all of this relates to adding hypernumbers. We will consider only ordered pairs, with the understanding that the treatment easily extends to ordered triples in three dimensions and n-tuples in n dimensions. Given two ordered pairs A and B, we saw that their sum, $A + B$, is simply $(a_1 + b_1, a_2 + b_2)$, or $C = A + B$ where $c_1 = a_1 + b_1$ and $c_2 = a_2 + b_2$. Nothing we discussed earlier prevents us from interpreting $c_1 = a_1 + b_1$ as the total east-west displacement and $c_2 = a_2 + b_2$ as the total north-south displacement. This means that the resulting grand total displacement is $\sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2}$ or $\sqrt{c_1^2 + c_2^2}$, with the compass direction given by $90° - \arctan[(a_2 + b_2)/(a_1 + b_1)]$. Thus, the distance-and-direction interpretation holds true for addition, as well.

The distance-and-direction interpretation suggests a powerful way for us to visualize a vector, and that is as a directed line segment or arrow (Figure 1.6). The length of the arrow (at some predetermined scale) represents the magnitude of the vector, and the orientation of the segment and placement of the arrowhead (at one end of the segment or the other) represent its direction. The figure shows several examples lying in the plane of the paper. Two vectors are equal if they have the same length and direction, so that $\mathbf{a} = \mathbf{b}$. Although $\mathbf{c}$ is the same length as $\mathbf{a}$, it is in the opposite direction, so $\mathbf{a} \neq \mathbf{c}$. Clearly, neither $\mathbf{d}$ nor $\mathbf{e}$ is the equivalent of $\mathbf{a}$. Another way of stating this is that if any vector $\mathbf{a}$ can be transported, remaining parallel to its initial orientation, into coincidence with another vector $\mathbf{b}$, then $\mathbf{a} = \mathbf{b}$.

We can use this idea of parallel transport of directed line segments (arrows) to add two vectors $\mathbf{a}$ and $\mathbf{b}$ as follows: Transport $\mathbf{b}$ until its tail is coincident with the head

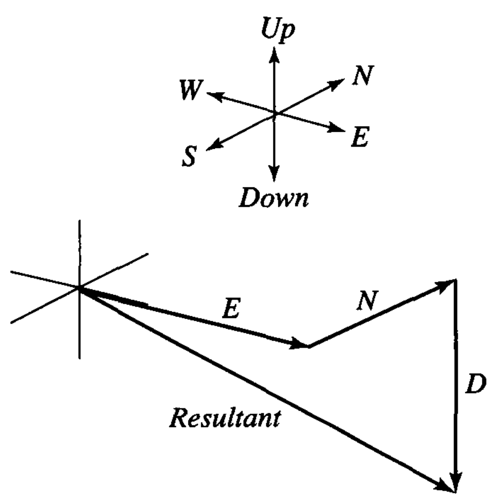

Figure 1.5 *A three-dimensional displacement.*

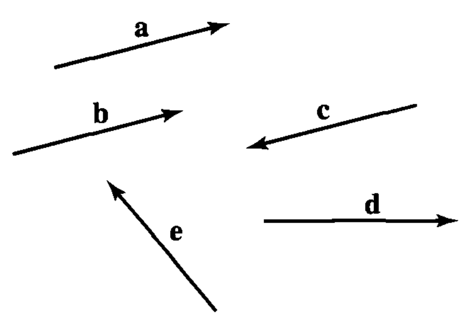

Figure 1.6 *Vectors as directed line segments.*

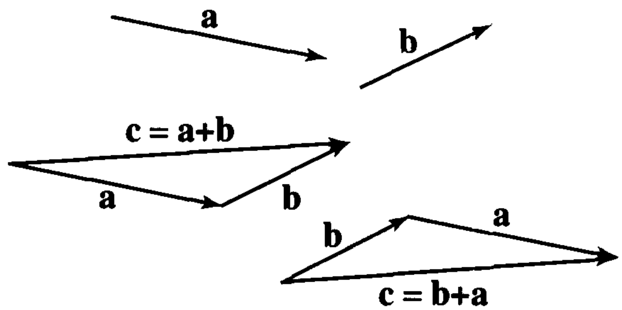

Figure 1.7 *Using parallel transport to add two vectors.*

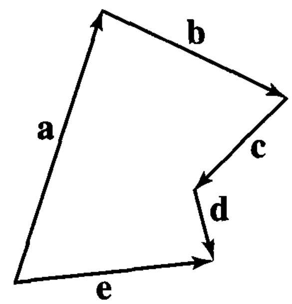

Figure 1.8 *Head-to-tail chain of vectors.*

of **a** (Figure 1.7). Then their sum, $\mathbf{a} + \mathbf{b}$, is the directed line segment beginning at the tail of **a** and ending at the head of **b**. Note that transporting **a** so that its tail coincides with the head of **b** produces the same result. Thus, $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$. We extend this method of adding two vectors to adding many vectors by simply transporting each vector so as to form a head-to-tail chain (Figure 1.8). Connecting the tail of the first vector to the head of the last vector in the chain produces the resultant vector. In the figure we see that this construction yields $\mathbf{e} = \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}$.

This process leads directly to the *parallelogram law* for adding two vectors, where we transport **a** and/or **b** so that their tails are coincident and then complete the construction of the suggested parallelogram. The diagonal **c** represents their sum. (Figure 1.9). The parallelogram law of addition suggests a way to find the components of a vector **a** along any two directions L and M (Figure 1.10). We construct lines L and M through the tail of **a** and their parallel images L' and M' through the head of **a**. This construction produces a parallelogram whose adjacent sides $\mathbf{a}_L$ and $\mathbf{a}_M$ are the components of **a** along L and M, respectively. We see, of course, that $\mathbf{a} = \mathbf{a}_L + \mathbf{a}_M$ are not unique! We could just as easily construct other lines, say PQ and $P'Q'$, to find $\mathbf{a}_P$ and $\mathbf{a}_Q$. Obviously, $\mathbf{a} = \mathbf{a}_P + \mathbf{a}_Q$ and, in general, $\mathbf{a}_P \neq \mathbf{a}_L, \mathbf{a}_M$ or $\mathbf{a}_Q \neq \mathbf{a}_L, \mathbf{a}_M$. This property of the non-uniqueness of the vector's components is a very powerful feature, which we will see often in the sections to follow.

So far we have not constrained vectors to any particular location, so we call them *free vectors*. We have moved them around and preserved their properties of length and orientation. This is possible only if we always move them parallel to themselves. It

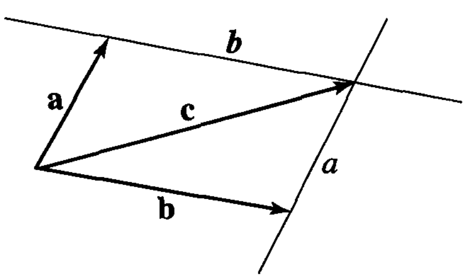

Figure 1.9 *The parallelogram law of vector addition.*

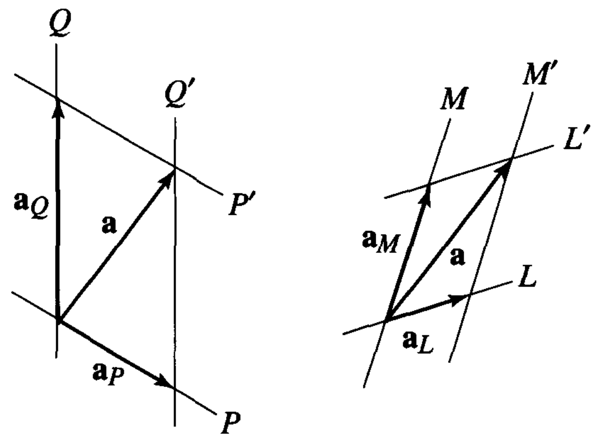

Figure 1.10 *Using parallelogram construction and vector components.*

is also true in three dimensions (Figure 1.11). The components must not be coplanar, and in the figure we use them to construct a rectangular parallelepiped.

When we use vectors simply to represent pure displacements, then we use unrestricted parallel transport, and the solution is in terms of free vectors. Many problems, however, require a *bound* or *fixed vector*, which is often the case in physics, computer graphics, and geometric modeling.

Fixed vectors always begin at a common point, usually (but not necessarily) the origin of a coordinate system. If we use the origin of a rectangular Cartesian coordinate system, then the components of a fixed vector lie along the principal axes and correspond to the coordinates of the point at the tip of the arrowhead (Figure 1.12). Thus, the vector $\mathbf{p}$ in the figure has vector components $\mathbf{p}_x$ and $\mathbf{p}_y$, also known as the x and y component. This means that $\mathbf{p} = \mathbf{p}_x + \mathbf{p}_y$, and in three dimensions $\mathbf{p} = \mathbf{p}_x + \mathbf{p}_y + \mathbf{p}_z$ (Figure 1.13).

The distinction between free and fixed vectors is often blurred by the nature of the problem and because most arithmetic and algebraic operations are identical for both kinds of vectors. The distinction may be as important for visualization and intuition as for any other reason.

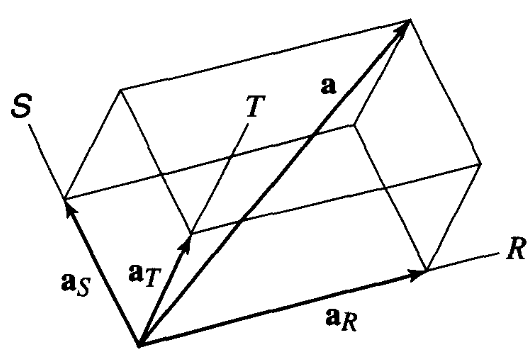

Figure 1.11 *Free vectors and parallel transport in three dimensions.*

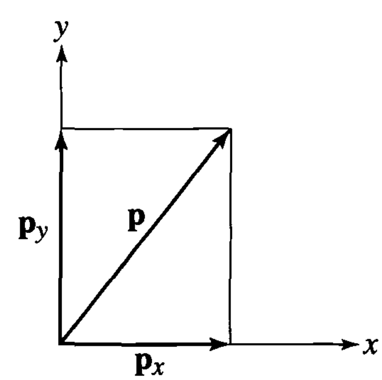

Figure 1.12 *Fixed vector in two dimensions.*

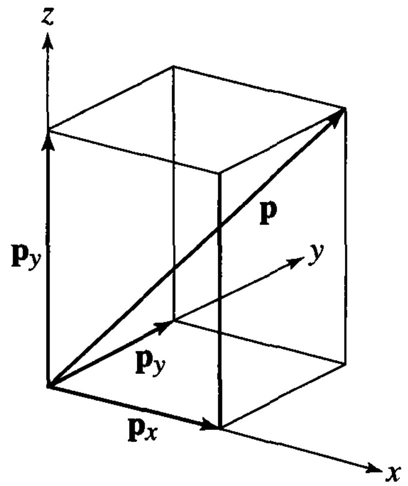

Figure 1.13 *Fixed vector in three dimensions.*

1.4 Vector Properties

It is time to introduce some special vectors, denoted **i**, **j**, **k**, each of which has a length equal to one. The vector **i** lies along the x axis, **j** lies along the y axis, and **k** lies along the z axis (Figure 1.14), where

$$\begin{aligned} \mathbf{i} &= [1 \quad 0 \quad 0] \\ \mathbf{j} &= [0 \quad 1 \quad 0] \\ \mathbf{k} &= [0 \quad 0 \quad 1] \end{aligned} \tag{1.5}$$

Because we can multiply a vector by some constant that changes its magnitude but not its direction, we can express any fixed vector **a** as

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \tag{1.6}$$

This follows from $\mathbf{a} = \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z$, where $\mathbf{a}_x = a_x\mathbf{i}$, $\mathbf{a}_y = a_y\mathbf{j}$, and $\mathbf{a}_z = a_z\mathbf{k}$ (Figure 1.15).

We can describe the vector **a** more simply as an ordered triple and enclose it in brackets. Thus, $\mathbf{a} = [a_x \quad a_y \quad a_z]$, where a_x, a_y, and a_z are the components of **a**. The

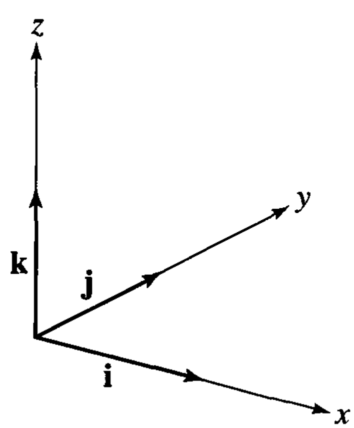

Figure 1.14 *Unit vectors along the coordinate axes.*

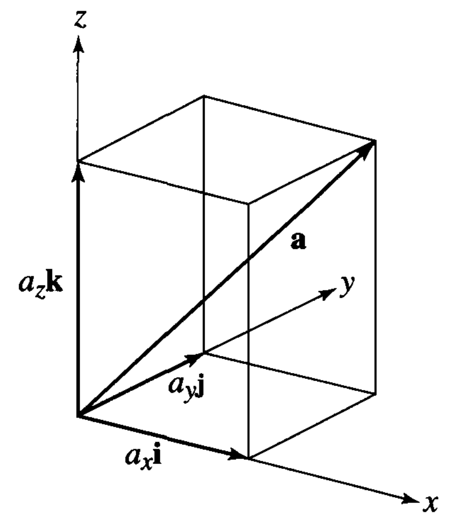

Figure 1.15 *Fixed-vectors components along the coordinate axes.*

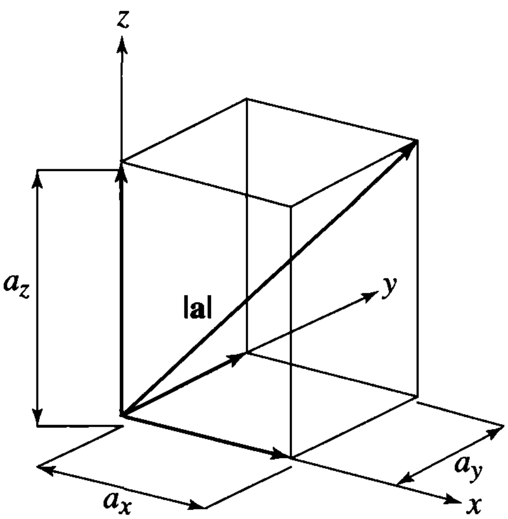

Figure 1.16 *Determining the magnitude of a vector.*

components may be negative, depending on the direction of the vector. We reverse the direction of any vector by multiplying each of its components by -1. Thus, the reverse of $\mathbf{a} = [3 \quad 2 \quad -7]$ is $-\mathbf{a}$, or $-\mathbf{a} = [-3 \quad -2 \quad 7]$.

Magnitude (length) and direction are the most important properties of a vector. Because a vector's magnitude is best described as a length, it is always positive. The length of $\mathbf{a}$ is a scalar, denoted as $|\mathbf{a}|$ and given by

$$|\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \tag{1.7}$$

which is a simple application of the Pythagorean theorem for finding the length of the diagonal of a rectangular solid (Figure 1.16).

We define a unit vector as any vector whose length or magnitude is equal to one, independent of its direction, of course. As we saw, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are special cases, with specific directions assigned to them. A *unit vector* in the direction of $\mathbf{a}$ is denoted as $\hat{\mathbf{a}}$, where

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} \tag{1.8}$$

and its components are

$$\hat{\mathbf{a}} = \begin{bmatrix} \dfrac{a_x}{|\mathbf{a}|} & \dfrac{a_y}{|\mathbf{a}|} & \dfrac{a_z}{|\mathbf{a}|} \end{bmatrix} \tag{1.9}$$

We can make this more concise with the following substitutions:

$$\hat{\mathbf{a}}_x = \frac{a_x}{|\mathbf{a}|}, \quad \hat{\mathbf{a}}_y = \frac{a_y}{|\mathbf{a}|}, \quad \hat{\mathbf{a}}_z = \frac{a_z}{|\mathbf{a}|} \tag{1.10}$$

so that

$$\hat{\mathbf{a}} = [\hat{\mathbf{a}}_x \quad \hat{\mathbf{a}}_y \quad \hat{\mathbf{a}}_z] \tag{1.11}$$

Note that if α, β, and γ are the angles between $\mathbf{a}$ and the x, y, and z axes, respectively, then

$$\hat{\mathbf{a}}_x = \frac{a_x}{|\mathbf{a}|} = \cos\alpha, \quad \hat{\mathbf{a}}_y = \frac{a_y}{|\mathbf{a}|} = \cos\beta, \quad \hat{\mathbf{a}}_z = \frac{\mathbf{a}_z}{|\mathbf{a}|} = \cos\gamma \tag{1.12}$$

This means that $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, and $\hat{\mathbf{a}}_z$ are also the *direction cosines* of $\mathbf{a}$.

1.5 Scalar Multiplication

Multiplying any vector $\mathbf{a}$ by a scalar k produces a vector $k\mathbf{a}$ or, in component form,

$$k\mathbf{a} = [k\mathbf{a}_x \quad k\mathbf{a}_y \quad k\mathbf{a}_z] \tag{1.13}$$

If k is positive, then $\mathbf{a}$ and $k\mathbf{a}$ are in the same direction. If k is negative, then $\mathbf{a}$ and $k\mathbf{a}$ are in opposite directions. The magnitude (length) of $k\mathbf{a}$ is

$$|k\mathbf{a}| = \sqrt{k^2a_x^2 + k^2a_y^2 + k^2a_z^2} \tag{1.14}$$

so that

$$|k\mathbf{a}| = k|\mathbf{a}| \tag{1.15}$$

We can see that scalar multiplication is well named, because it changes the scale of the vector. Here are the possible effects of a scalar multiplier k:

$k > 1$	Increases length
$k = 1$	No change
$0 < k < 1$	Decreases length
$k = 0$	Null vector (0 length, direction undefined)
$-1 < k < 0$	Decreases length and reverses direction
$k = -1$	Reverses direction only
$k < -1$	Reverses direction and increases length

1.6 Vector Addition

Vector addition (or subtraction) in terms of components is perhaps the simplest of all vector operations (except for multiplication of a vector by a scalar). Given $\mathbf{a} = [a_x \quad a_y \quad a_z]$ and $\mathbf{b} = [b_x \quad b_y \quad b_z]$, then

$$\mathbf{a} + \mathbf{b} = [a_x + b_x \quad a_y + b_y \quad a_z + b_z] \tag{1.16}$$

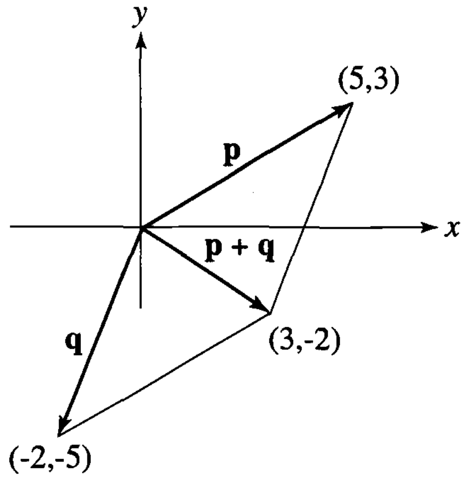

Figure 1.17 *Confirming the parallelogram law of vector addition.*

For example, given the two-dimensional vectors $\mathbf{p} = [5 \quad 3]$ and $\mathbf{q} = [-2 \quad -5]$, we readily obtain their sum

$$\mathbf{p} + \mathbf{q} = [5 + (-2) \quad 3 + (-5)] = [3 \quad -2] \tag{1.17}$$

In Figure 1.17 we see that the parallelogram law of addition is satisfied.

Given vectors **a**, **b**, and **c** and scalars k and l, then vector addition and scalar multiplication have the following properties:

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $k(l\mathbf{a}) = kl\mathbf{a}$
4. $(k + l)\mathbf{a} = k\mathbf{a} + l\mathbf{a}$
5. $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$

1.7 Scalar and Vector Products

We multiply two vectors **a** and **b** in two very different ways. One way produces a single real number, or scalar, identified as the *scalar product*. The other way produces a vector, identified as the *vector product*. Both kinds of multiplication require that **a** and **b** have the same dimension. To avoid problems beyond the scope of this textbook, we will work only with three-dimensional vectors.

The scalar product of two vectors **a** and **b** is the sum of the products of their corresponding components:

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z \tag{1.18}$$

which is a scalar, not another vector. Occasionally you will see or hear it referred to as the *dot product*, particularly in older texts. It is easy to show that the scalar product is commutative, that is, $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.

It is interesting to note that the scalar product of a vector with itself produces the square of the vector's length $\mathbf{a}\cdot\mathbf{a} = a_x^2 + a_y^2 + a_z^2$, so that

$$\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2 \tag{1.19}$$

Using the law of cosines we can demonstrate that the angle θ between two vectors $\mathbf{a}$ and $\mathbf{b}$ satisfies the equation

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta \tag{1.20}$$

Solving this equation for θ yields

$$\theta = \cos^{-1}\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} \tag{1.21}$$

This means that if $\mathbf{a}\cdot\mathbf{b} = 0$, then $\mathbf{a}$ and $\mathbf{b}$ are perpendicular. If $\theta = 0$, then they are parallel.

The scalar product has the following properties:

1. $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$
2. $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$
3. $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$, commutative property
4. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$, distributive property
5. $(k\mathbf{a})\cdot\mathbf{b} = \mathbf{a}\cdot(k\mathbf{b}) = k(\mathbf{a}\cdot\mathbf{b})$, associative property
6. If $\mathbf{a}$ is perpendicular to $\mathbf{b}$, then $\mathbf{a}\cdot\mathbf{b} = 0$

The vector product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a}\times\mathbf{b} = (a_y b_z - a_z b_y)\mathbf{i} - (a_x b_z - a_z b_x)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k} \tag{1.22}$$

In component form this becomes

$$\mathbf{a}\times\mathbf{b} = [(a_y b_z - a_z b_y) \quad -(a_x b_z - a_z b_x) \quad (a_x b_y - a_y b_x)] \tag{1.23}$$

You might wonder how such a collection of terms arises. Although the detailed derivation of this expression is too long and complex to include here (W. R. Hamilton spent years investigating the problem of multiplying hyper-complex numbers), we can find meaningful patterns. For example, each possible permutation of the product $a_i b_j$ (where $i, j = x, y, z$) appears only once. There are no a_x or b_x terms in the first, or x, component of the product, no a_y or b_y terms in the y component, and no a_z or b_z terms in the z component. One way to remember this multi-term expression is as the expansion of the following determinant:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \tag{1.24}$$

If $\mathbf{c} = \mathbf{a}\times\mathbf{b}$, then $\mathbf{c}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ and, thus, it is also perpendicular to the plane defined by $\mathbf{a}$ and $\mathbf{b}$. We can prove this assertion by computing

$\mathbf{a} \cdot \mathbf{c}$ and $\mathbf{b} \cdot \mathbf{c}$:

$$\begin{aligned}\mathbf{a} \cdot \mathbf{c} &= a_x(a_y b_z - a_z b_y) - a_y(a_x b_z - a_z b_x) + a_z(a_x b_y - a_y b_x) \\ &= a_x a_y b_z - a_x a_z b_y - a_x a_y b_z + a_y a_z b_x + a_x a_z b_y - a_y a_z b_x \\ &= 0\end{aligned} \tag{1.25}$$

Because $\mathbf{a} \cdot \mathbf{c} = 0$, we know that **a** and **c** are perpendicular. We can also show that $\mathbf{b} \cdot \mathbf{c} = 0$.

If two vectors **a** and **b** are parallel, then $\mathbf{a} \times \mathbf{b} = 0$. To prove this, we let $\mathbf{b} = k\mathbf{a}$; this guarantees that **a** and **b** are parallel. Then we compute $\mathbf{a} \times k\mathbf{a}$:

$$\mathbf{a} \times k\mathbf{a} = [(ka_y a_z - ka_z a_y) \quad -(ka_x a_z - ka_z a_x) \quad (ka_x a_y - ka_y a_x)] \tag{1.26}$$

This reduces to

$$\mathbf{a} \times k\mathbf{a} = [0 \quad 0 \quad 0] \tag{1.27}$$

and $[0 \quad 0 \quad 0]$ is the so-called *null vector*, or **0**. Obviously, this means that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$, because we have put no restrictions on k.

The vector product is not commutative; in fact, $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$. Thus, reversing the order of the two vectors reverses the direction of their vector product. This is easy to verify.

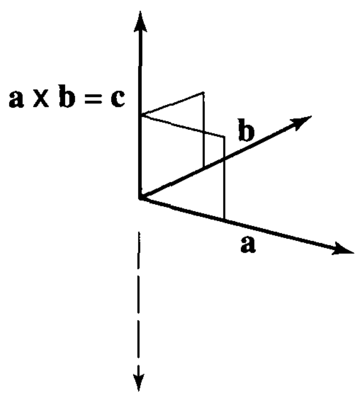

Figure 1.18 *Vector product.*

To say that if $\mathbf{c} = \mathbf{a} \times \mathbf{b}$, then **c** is perpendicular to both **a** and **b** only gives the line of action of the vector **c**. In which of two directions does **c** point? (Figure 1.18) The direction is, of course, inherent in the components of the vector product. However, an intuitive rule applies: Imagine rotating **a** into **b** through the smallest of the angles formed by their lines of action, curling the fingers of your right hand in this angular sense. The extended thumb of your right hand points in the direction of **c**, where $\mathbf{c} = \mathbf{a} \times \mathbf{b}$. For $\mathbf{d} = \mathbf{b} \times \mathbf{a}$, **d** points in the opposite direction, that is, $\mathbf{d} = -\mathbf{c}$.

The vector product has the following properties:

1. $\mathbf{a} \times \mathbf{b} = \mathbf{c}$, where **c** is perpendicular to both **a** and **b**
2. $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$, the expansion of the determinant

3. $\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\hat{\mathbf{n}} \sin\theta$, where $\hat{\mathbf{n}}$ is the unit vector perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ and θ is the angle between them
4. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
5. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
6. $(k\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (k\mathbf{b}) = k(\mathbf{a} \times \mathbf{b})$
7. $\mathbf{i} \times \mathbf{j} = \mathbf{k},\ \mathbf{j} \times \mathbf{k} = \mathbf{i},\ \mathbf{k} \times \mathbf{i} = \mathbf{j}$
8. If $\mathbf{a}$ is parallel to $\mathbf{b}$, then $\mathbf{a} \times \mathbf{b} = 0$
9. $\mathbf{a} \times \mathbf{a} = \mathbf{0}$

1.8 Elements of Vector Geometry

We can use vector equations to describe many geometric objects, from points, lines, and curves to very complex surfaces. We do this by writing a vector equation in terms of one or more variables. We will be exploring only straight lines and planes here.

Lines

The vector equation of a line through some point $\mathbf{p}_0$ and parallel to another vector $\mathbf{t}$ is

$$\mathbf{p}(u) = \mathbf{p}_0 + u\mathbf{t} \tag{1.28}$$

where u is a scalar variable multiplying $\mathbf{t}$ (Figure 1.19). We see that as u takes on successive numerical values, the equation generates points on a straight line. The components of $\mathbf{p}$ are the coordinates of a point on this line. In other words, because $\mathbf{p}_0$ and $\mathbf{t}$ are constant for any specific line, any real value of u generates a point on that line.

We can expand this equation by writing it in its component form. This time we will list the components in a vertical or column array instead of the horizontal or row array we have used. (The row and column forms are mathematically equivalent, demanding only that we do not mix the two and that we use some simple bookkeeping

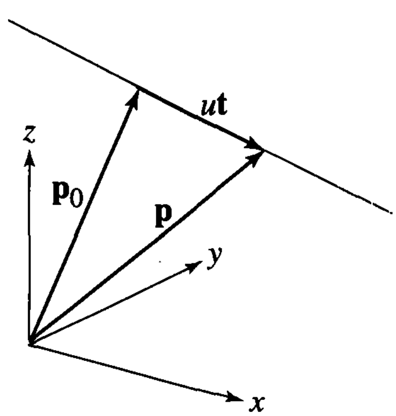

Figure 1.19 *Vector equation of a line.*

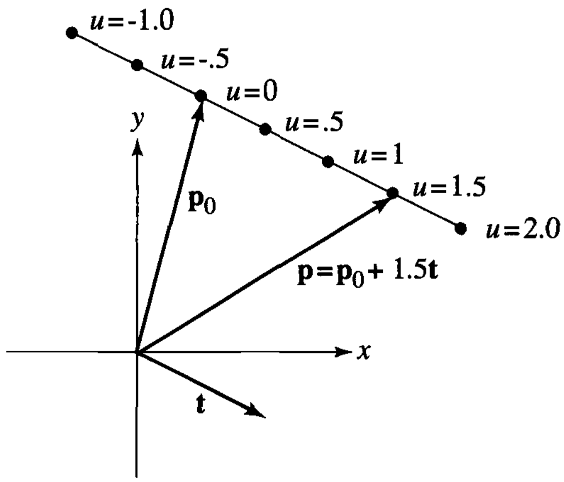

Figure 1.20 *Example of a vector equation of a straight line.*

techniques when doing algebra on them.) This produces

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + u \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} \tag{1.29}$$

In ordinary algebraic form we have

$$\begin{aligned} x &= x_0 + ut_x \\ y &= y_0 + ut_y \\ z &= z_0 + ut_z \end{aligned} \tag{1.30}$$

where u is the independent variable; x, y, and z are dependent variables; and x_0, y_0, z_0, t_x, t_y, and t_z are constants. Mathematicians call this set of equations the *parametric equations* of a straight line: (x, y, z) are the coordinates of any point on the line, (x_0, y_0, z_0) are the coordinates of a given point on the line, and (t_x, t_y, t_z) are the components of a vector (in point form) parallel to the line.

Here is a simple example in two dimensions (Figure 1.20): Let's find the vector equation for a straight line that passes through the point described by the vector $\mathbf{p}_0 = [1 \quad 4]$ and parallel to the vector $\mathbf{t} = [2 \quad -1]$. In vector component form, we have

$$\mathbf{p} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} + u \begin{bmatrix} 2 \\ -1 \end{bmatrix} \tag{1.31}$$

or, in algebraic form,

$$\begin{aligned} x &= 1 + 2u \\ y &= 4 - u \end{aligned} \tag{1.32}$$

Now we can compute the coordinates of points on this line for a series of values of u and tabulate the results (Table 1.1).

Of course, we can easily expand this tabulation in a variety of ways. We can, for example, use values of u closer together or farther apart, as well as values beyond (in

Table 1.1 Points on a line

u	x	y
−1.0	−1.0	5.0
−0.5	0	4.5
0	1.0	4.0
0.5	2.0	3.5
1.0	3.0	3.0
1.5	4.0	2.5
2.0	5.0	2.0

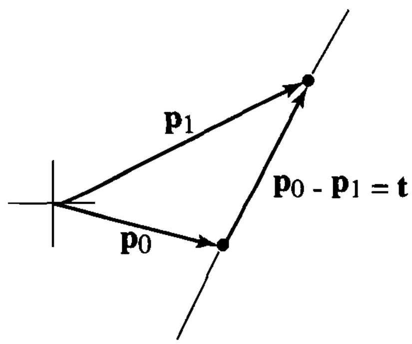

Figure 1.21 *Vector equation of a line through two points.*

either direction) those tabulated. Using equations like this as part of a computer-graphics program, a computer scientist can construct and display straight lines.

An interesting variation of this vector geometry of a straight line will help us find the vector equation of a line through two given points, say, $\mathbf{p}_0$ and $\mathbf{p}_1$ (Figure 1.21). In this figure we see that we can define t as $\mathbf{p}_1 - \mathbf{p}_0$. Making the appropriate substitution into our original vector equation for a straight line, we obtain

$$\mathbf{p} = \mathbf{p}_0 + u(\mathbf{p}_1 - \mathbf{p}_0) \tag{1.33}$$

If we limit the allowable value of u to the interval $0 \leq u \leq 1$, then this equation defines a line segment extending from $\mathbf{p}_0$ to $\mathbf{p}_1$. In algebraic form, this vector equation expands to

$$\begin{aligned} x &= x_0 + u(x_1 - x_0) \\ y &= y_0 + u(y_1 - y_0) \\ z &= z_0 + u(z_1 - z_0) \end{aligned} \tag{1.34}$$

(Note that we have arbitrarily used both fixed and free vector interpretations. Compare Figures 1.20 and 1.21.)

Planes

There are four ways to define a plane in three dimensions using vector equations. One way is by the vector equation of a plane through $\mathbf{p}_0$ and parallel to two independent vectors $\mathbf{s}$ and $\mathbf{t}$:

$$\mathbf{p} = \mathbf{p}_0 + u\mathbf{s} + w\mathbf{t} \tag{1.35}$$

where $\mathbf{s} \neq k\mathbf{t}$ and where u and w are scalar independent variables multiplying $\mathbf{s}$ and $\mathbf{t}$, respectively (Figure 1.22). The vector $\mathbf{p}$ represents the set of points defining a plane as the parameters u and w vary independently. In terms of the vector's components, the dependent variables x, y, and z, we have the three equations

$$\begin{aligned} x &= x_0 + us_x + wt_x \\ y &= y_0 + us_y + wt_y \\ z &= z_0 + us_z + wt_z \end{aligned} \tag{1.36}$$

We can also write Equation 1.36 as the matrix equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + u \begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix} + w \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} \tag{1.37}$$

A second way: three points $\mathbf{p}_0$, $\mathbf{p}_1$, and $\mathbf{p}_2$ are sufficient to define a plane in space if they are not collinear (Figure 1.23). We can rewrite Equation 1.35 in terms of these points:

$$\mathbf{p} = \mathbf{p}_0 + u(\mathbf{p}_1 - \mathbf{p}_0) + w(\mathbf{p}_2 - \mathbf{p}_1) \tag{1.38}$$

Any vector perpendicular to a plane is called a *normal vector* to that plane. We usually denote it as $\mathbf{n}$. Thus far we have two ways to compute it:

$$\mathbf{n} = \mathbf{s} \times \mathbf{t} \tag{1.39}$$

or

$$\mathbf{n} = (\mathbf{p}_1 - \mathbf{p}_0) \times (\mathbf{p}_2 - \mathbf{p}_1) \tag{1.40}$$

Note that we can construct a normal at any point on the plane and that, of course, all normals to the plane are parallel to one another. If the magnitude of $\mathbf{n}$ is not of interest,

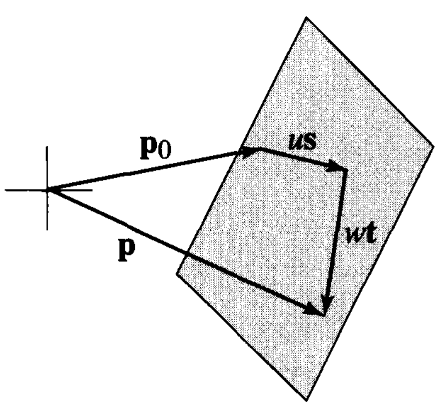

Figure 1.22 *Vector equation of a plane.*

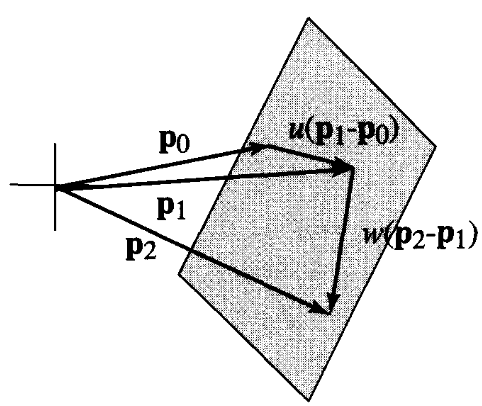

Figure 1.23 *Three points defining a plane.*

we can work with the unit normal $\hat{\mathbf{n}}$, where

$$\hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|} \tag{1.41}$$

A third way to define a plane is by using a point it passes through and the normal vector to the plane. This means that any point $\mathbf{p}$ lies on the plane if and only if $\mathbf{p} - \mathbf{p}_0$ is perpendicular to $\hat{\mathbf{n}}$, because $\hat{\mathbf{n}}$ is perpendicular to all lines in the plane. In terms of a vector equation, this statement becomes

$$(\mathbf{p} - \mathbf{p}_0) \cdot \hat{\mathbf{n}} = 0 \tag{1.42}$$

(Remember: The scalar product of two mutually perpendicular vectors is zero.) Performing the indicated scalar product yields

$$(x - x_0)\hat{n}_x + (y - y_0)\hat{n}_y + (z - z_0)\hat{n}_z = 0 \tag{1.43}$$

where $\hat{n}_x$, $\hat{n}_y$, and $\hat{n}_z$ are the components of $\hat{\mathbf{n}}$.

The fourth way we can define a plane is a variation of the third way: Given the vector $\mathbf{d}$ to a point on the plane and where $\mathbf{d}$ is itself perpendicular to the plane, then any point $\mathbf{p}$ on the plane must satisfy

$$(\mathbf{p} - \mathbf{d}) \cdot \mathbf{d} = 0 \tag{1.44}$$

Point of Intersection between a Plane and a Straight Line

We can use vector equations to solve many kinds of problems in geometry. For example, given the plane $\mathbf{p}_P = \mathbf{a} + u\mathbf{b} + w\mathbf{c}$ and the straight line $\mathbf{p}_L = \mathbf{d} + t\mathbf{e}$, at their point of intersection it must be true that $\mathbf{p}_P = \mathbf{p}_L$, so that

$$\mathbf{a} + u\mathbf{b} + w\mathbf{c} = \mathbf{d} + t\mathbf{e} \tag{1.45}$$

This represents, of course, a system of three linear equations in three unknowns, u, w, and t. In algebraic form this equation becomes

$$\begin{aligned} a_x + ub_x + wc_x &= d_x + te_x \\ a_y + ub_y + wc_y &= d_y + te_y \\ a_z + ub_z + wc_z &= d_z + te_z \end{aligned} \tag{1.46}$$

However, we can use the properties of vectors to solve Equation 1.46 directly. We solve for u, w, and t by isolating each in turn. To solve for t, we take the scalar product of both sides of the equation with the vector product $(\mathbf{b} \times \mathbf{c})$ as follows:

$$(\mathbf{b} \times \mathbf{c}) \cdot (\mathbf{a} + u\mathbf{b} + w\mathbf{c}) = (\mathbf{b} \times \mathbf{c}) \cdot (\mathbf{d} + t\mathbf{e}) \tag{1.47}$$

Because $(\mathbf{b} \times \mathbf{c})$ is perpendicular to both $\mathbf{b}$ and $\mathbf{c}$, Equation 1.47 reduces to

$$(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{d} + t(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{e} \tag{1.48}$$

Solving Equation 1.48 for t yields

$$t = \frac{(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} - (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{d}}{(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{e}} \tag{1.49}$$

We continue in the same way to solve for u and w:

$$u = \frac{(\mathbf{c} \times \mathbf{e}) \cdot \mathbf{d} - (\mathbf{c} \times \mathbf{e}) \cdot \mathbf{a}}{(\mathbf{c} \times \mathbf{e}) \cdot \mathbf{b}} \tag{1.50}$$

$$w = \frac{(\mathbf{b} \times \mathbf{e}) \cdot \mathbf{d} - (\mathbf{b} \times \mathbf{e}) \cdot \mathbf{a}}{(\mathbf{b} \times \mathbf{e}) \cdot \mathbf{c}} \tag{1.51}$$

1.9 Linear Vector Spaces

If we algebraically treat vectors as if they originated at a common point, then we work in a *linear vector space*. Not only do all vectors have a common origin, but any vector combines with any other vector according to the parallelogram law of addition. Vectors must be subjected to the following two operations to qualify as members of a linear vector space:

1. Addition of any two vectors must produce a third vector, identified as their sum: $\mathbf{a} + \mathbf{b} = \mathbf{c}$.
2. Multiplication of a vector $\mathbf{a}$ by a scalar k must produce another vector $k\mathbf{a}$ as the product.

The set of all vectors is *closed* with respect to these two operations, which means that both the sum of two vectors and the product of a vector and a scalar are themselves vectors. These two operations have the following properties, some of which we have seen before:

1. Commutativity: $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. Associativity: $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$
3. Identity element: $\mathbf{a} + \mathbf{0} = \mathbf{a}$
4. Inverse: $\mathbf{a} - \mathbf{a} = \mathbf{0}$
5. Identity under scalar multiplication: $k\mathbf{a} = \mathbf{a}$, when $k = 1$
6. $c(d\mathbf{a}) = (cd)\mathbf{a}$
7. $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
8. $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$

A set of vectors that can be subjected to the two operations with these eight properties forms a linear vector space. The other vector operations we have discussed, such as the scalar and vector products, are not pertinent to this definition of a linear vector space. This brings us back to hypernumbers. It is easy to show that the set of all vectors of the form $\mathbf{r} = [r_1 \quad r_2 \quad \dots \quad r_n]$, where $r_1, r_2, \dots, r_n$ are real numbers, constitutes a linear vector space.

The set of all linear combinations of a given set of vectors (none of which is a scalar multiple of any other in the set) forms a vector space. For example, if we let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be any n vectors, then $a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_n\mathbf{x}_n$ (where $a_1, a_2, \ldots, a_n$ are scalars) is a linear combination of the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$. But that is not all we can do. If we let

$$\begin{aligned} \mathbf{s} &= a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_n\mathbf{x}_n \\ \mathbf{t} &= b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + \cdots + b_n\mathbf{x}_n \end{aligned} \tag{1.52}$$

so that the vectors s and t are linear combinations of the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, then

$$\mathbf{s} + \mathbf{t} = (a_1 + b_1)\mathbf{x}_1 + (a_2 + b_2)\mathbf{x}_2 + \cdots + (a_n + b_n)\mathbf{x}_n \tag{1.53}$$

We also have

$$k\mathbf{s} = (ka_1)\mathbf{x}_1 + (ka_2)\mathbf{x}_2 + \cdots + (ka_n)\mathbf{x}_n \tag{1.54}$$

which is also a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$. Mathematicians point out that the space of all linear combinations of a given set of vectors is the space generated by that set.

Given a single vector $\mathbf{x}$, the space generated by all scalar multiples of $\mathbf{x}$ is a straight line collinear with $\mathbf{x}$ (for $\mathbf{x} \neq \mathbf{0}$). Given two vectors $\mathbf{s}$ and $\mathbf{t}$, where $\mathbf{t}$ is not a scalar multiple of $\mathbf{s}$, then the space generated by their linear combinations is the plane containing $\mathbf{s}$ and $\mathbf{t}$. For example, if we let $\mathrm{r} = a\mathbf{s} + b\mathbf{t}$, then from the parallelogram law of addition we know that vectors $\mathbf{r}$, $\mathbf{s}$, and $\mathbf{t}$ are coplanar. Of course, we could continue this process, generating spaces of three and more dimensions simply by increasing the number of vectors in the generating set. To do this, we must impose certain conditions, as in the previous example where we did not allow $\mathbf{s}$ and $\mathbf{t}$ to be scalar multiples of each other. This leads us to the concepts of linear independence and dependence.

1.10 Linear Independence and Dependence

Vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are *linearly dependent* if and only if there are real numbers $a_1, a_2, \ldots, a_n$ not all equal to zero, such that

$$a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_n\mathbf{x}_n = 0 \tag{1.55}$$

If this equation is true only if $a_1, a_2, \ldots, a_n$ are all zero, then $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are *linearly independent*.

If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are linearly dependent, then we can express any one of them as a linear combination of the others. On the other hand, if one of the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is a linear combination of the others, then the vectors are linearly dependent. Another way of saying this is that vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are linearly dependent if and only if one of them belongs to the space generated by the remaining $n - 1$ vectors. We can now define the dimension of a linear space as equal to the maximum number of linearly independent vectors that it can contain. This fact underlies our study of basis vectors.

We observe that any two vectors are dependent if and only if they are parallel (or collinear); three vectors are dependent if and only if they are coplanar; four vectors are dependent in a space of three dimensions; n vectors are dependent in a space of $n-1$ dimensions. Finally, in a space of three dimensions, a set of three vectors $\mathbf{r}$, $\mathbf{s}$, and $\mathbf{t}$ is linearly dependent if and only if the following determinant is equal to zero:

$$\begin{vmatrix} r_x & r_y & r_z \\ s_x & s_y & s_z \\ t_x & t_y & t_z \end{vmatrix} = 0 \tag{1.56}$$

If the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are linearly independent, it is impossible to represent any one of them as a linear combination of the other $n-1$ vectors.

1.11 Basis Vectors and Coordinate Systems

Fixed vectors always relate to some frame of reference. All vectors have components, but only the components of fixed vectors are also coordinates. In our discussion of basis vectors we will use Cartesian systems. The familiar rectangular Cartesian coordinates are a special case of the more general Cartesian systems. Associated with each dimension, or principal direction, of a general Cartesian system is a family of parallel straight lines, with a uniform scale or metric. All principal directions may have the same scale, or each may be different. The principal directions need not be mutually orthogonal, but one point must serve as the origin (Figure 1.24).

We create a set of linearly independent vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ to form the *basis* of a Cartesian space S_n of n dimensions, expressing any point vector $\mathbf{r}$ in S_n as a linear combination of these basis vectors. In a space of three dimensions, the set of basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ originating at a common point O defines three families of parallel lines and forms a Cartesian system. The three lines X_1, X_2, and X_3 concurrent at O and collinear with $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, respectively, define the principal coordinate axes (Figure 1.25). This system is a right-handed one, but it could just as well be left-handed. The basis vectors are analogous to the set of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ that we introduced previously for a rectangular Cartesian coordinate system, although $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are not necessarily unit vectors or mutually perpendicular.

We find the components (coordinates) of any point $\mathbf{r}$ by constructing a parallelepiped, with the origin, O, and one vertex $\mathbf{r}$ as a body diagonal, and concurrent edges

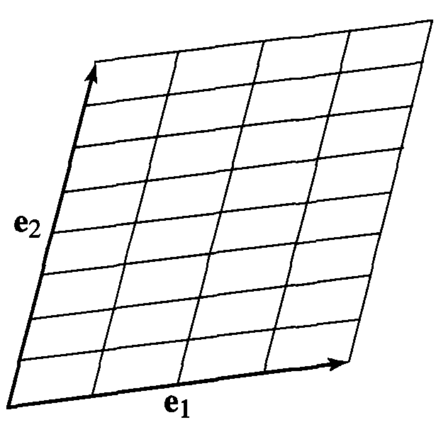

Figure 1.24 *Principal directions of basis vectors.*

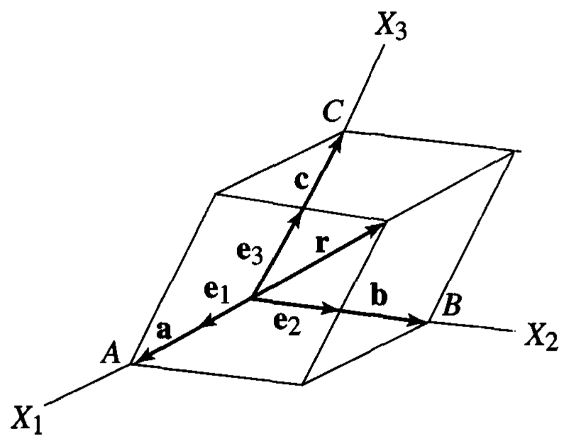

Figure 1.25 *Basis vectors and principal coordinate axes.*

collinear with the basis vectors. The three directed line segments corresponding to edges OA, OB, and OC define the vector components of $\mathbf{r}$ in this basis system. We denote them as $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. Note that this construction technique guarantees that the parallelogram law of vector addition applies, so that

$$\mathbf{r} = \mathbf{a} + \mathbf{b} + \mathbf{c} \tag{1.57}$$

The coordinates of r with respect to this basis are:

$$r_1 = \frac{|\mathbf{a}|}{|\mathbf{e}_1|}, \quad r_2 = \frac{|\mathbf{b}|}{|\mathbf{e}_2|}, \quad r_3 = \frac{|\mathbf{c}|}{|\mathbf{e}_3|} \tag{1.58}$$

Using these coordinates, we rewrite Equation 1.57 to obtain

$$\mathbf{r} = r_1\mathbf{e}_1 + r_2\mathbf{e}_2 + r_3\mathbf{e}_3 \tag{1.59}$$

where r_1, r_2, and r_3 are the coordinates or components of $\mathbf{r}$ with respect to the coordinate system defined by $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. Mathematicians call these terms the parallel coordinates of the point $\mathbf{r}$. These coordinates coincide with the coordinates of an affine three-dimensional space if the basis vectors are unit vectors.

1.12 A Short History

The complete history of vectors and vector geometry is far too complex to cover completely here. Many textbook-length works do an excellent job of it. The development of vectors could serve as a model for the development of the most important subdisciplines of mathematics. The subject of vectors has an ill-defined beginning and, as an important branch of applied mathematics, it is still sprouting new surprises and directions of inquiry.

Vectors are taught in courses on linear algebra, structural mechanics, introductory physics, and many other courses in engineering and the physical sciences. They are an indispensable tool in computer graphics, geometric modeling, and computer-aided design and manufacturing.

The story of vectors goes back at least to the sixteenth century, when Rafael Bombelli (1526?–1572?), an Italian mathematician, first treated $\sqrt{-1}$ as a number, albeit an *imaginary number*, and defined arithmetic operations on imaginary numbers. He combined real and imaginary numbers to form *complex numbers* and defined arithmetic operations on them.

Caspar Wessel (1745–1818), a Norwegian surveyor, gave complex numbers a geometric interpretation. Wessel's work, and in 1806 the work of Jean Argand (1768–1822)—the Swiss-French mathematician—led to the association of complex numbers with points on a plane. A complex number $a + ib$ is associated with the real number pair (a, b), which is then interpreted as the coordinates of a point in the plane (Figure 1.26). We can represent this complex number as a directed line segment, with its tail, or initial point, at the origin and its terminal point (or arrowhead) at the point (a, b). An example of the "geometry" of complex numbers is that each successive multiplication of a complex number by i rotates its equivalent directed line segment by 90° counterclockwise, so that $i(a + ib) = -b + ia$ (Figure 1.27).

William Rowan Hamilton (1805–1865), the great Irish mathematician and scientist, was a child prodigy. He had mastered over a dozen languages before becoming a teenager and was Ireland's preeminent mathematician before he was 20 years old. Hamilton treated complex numbers as merely an ordered pair of real numbers and generalized this concept to create hyper-complex numbers. For 15 years he struggled to develop an arithmetic for hyper-complex numbers, looking for operations on them that would correspond to operations on ordinary numbers and that would obey the customary laws of arithmetic (closure, commutativity, associativity, distributivity). This, he finally found, could not be done. So Hamilton created a new, logically consistent arithmetic, dropping the commutative law of multiplication. He focused on quadruples of real numbers which, with their special properties, became known as *quaternions*. These mathematical objects contain what we now recognize to be a scalar part and a vector part.

Hermann Grassman (1809–1877), a German geometer, soon formulated a more general algebra of hyper-complex numbers. The British mathematician Arthur Cayley (1821–1895) also generalized some of Hamilton's ideas and in the process developed

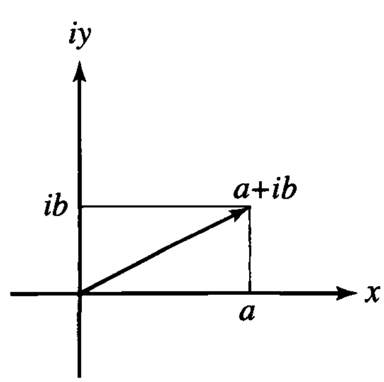

Figure 1.26 *The complex plane.*

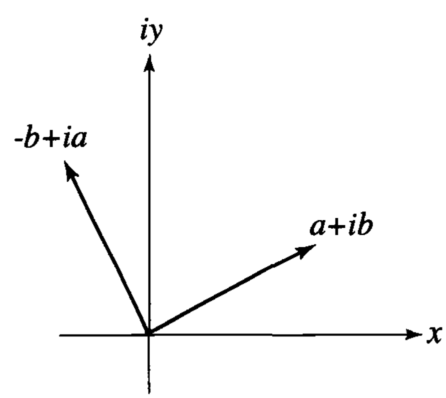

Figure 1.27 *Rotation in the complex plane.*

matrix algebra. However, it was Josiah Willard Gibbs (1839–1903), a Yale professor of mathematical physics, who refined vectors and vector analysis into the discipline as we know it today.

Exercises

1.1 Perform the indicated operations on the following complex numbers:
  a. $(3+2i)+(1-4i)$
  b. $(7-i)+(1+3i)$
  c. $4i+(-3+2i)$
  d. $(6-5i)\times i$
  e. $(a+bi)(a-bi)$

1.2 Perform the indicated addition or subtraction on the following hypernumbers:
  a. $(6,4,1)+(1,-3,0)$
  b. $(9,-8)-(2,-2)$
  c. $(a,b)+(c,d)$
  d. $(5,9,1,12)+(5,-3,6,-1)$
  e. $(0,0,8)-(4,1,8)$

1.3 Given the five vectors shown in Figure 1.28, write them in component form.

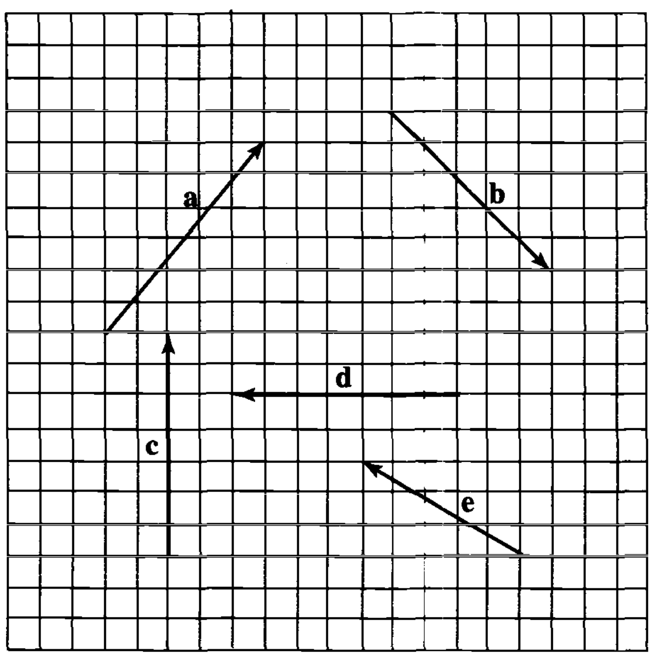

Figure 1.28 *Exercise 1.3.*

1.4 Compute the magnitudes of the vectors given in Exercise 1.3.

1.5 Compute the direction cosines of the vectors given in Exercise 1.3.

1.6 Given **a**, **b**, and **c** are three-component vectors, express the compact vector equation $\mathbf{a}x+\mathbf{b}y=\mathbf{c}$
  a. in expanded vector form
  b. in ordinary algebraic form

1.7 Compute the direction cosines of the following vectors:
   a. $\mathbf{a} = [5 \quad 6]$
   b. $\mathbf{b} = [5 \quad -5]$
   c. $\mathbf{c} = [0 \quad 7]$
   d. $\mathbf{d} = [-7 \quad 0]$
   e. $\mathbf{e} = [-5 \quad 3]$

1.8 Given $\mathbf{a} = [-2 \quad 0 \quad 7]$ and $\mathbf{b} = [4 \quad 1 \quad 3]$, compute
   a. $\hat{\mathbf{a}}$
   b. $\hat{\mathbf{b}}$
   c. $\mathbf{c} = \mathbf{a} - 2\mathbf{b}$
   d. $\mathbf{c} = 3\mathbf{a}$
   e. $\mathbf{c} = \mathbf{a} + \mathbf{b}$

1.9 Given $\mathbf{a} = [6 \quad 2 \quad -5]$ and $\mathbf{b} = 2\mathbf{a}$, compare the unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$.

1.10 Given $\mathbf{a} = [2 \quad 3 \quad 5]$ and $\mathbf{b} = [6 \quad -1 \quad 3]$, compute
   a. $|\mathbf{a}|$
   b. $|\mathbf{b}|$
   c. $\mathbf{c} = \mathbf{a} + \mathbf{b}$
   d. $\mathbf{c} = \mathbf{a} - \mathbf{b}$
   e. $\mathbf{c} = 2\mathbf{a} + 3\mathbf{b}$

1.11 Compute the following scalar products:
   a. $\mathbf{i} \cdot \mathbf{i}$
   b. $\mathbf{i} \cdot \mathbf{j}$
   c. $\mathbf{i} \cdot \mathbf{k}$
   d. $\mathbf{j} \cdot \mathbf{j}$
   e. $\mathbf{j} \cdot \mathbf{k}$
   f. $\mathbf{k} \cdot \mathbf{k}$

1.12 Given $\mathbf{a} = [2 \quad 0 \quad 5]$, $\mathbf{b} = [-1 \quad 3 \quad 1]$, and $\mathbf{c} = [6 \quad -2 \quad -4]$, compute
   a. $\mathbf{a} \cdot \mathbf{a}$
   b. $\mathbf{a} \cdot \mathbf{b}$
   c. $\mathbf{a} \cdot \mathbf{c}$
   d. $\mathbf{b} \cdot \mathbf{b}$
   e. $\mathbf{b} \cdot \mathbf{c}$
   f. $\mathbf{c} \cdot \mathbf{c}$
   g. $\mathbf{b} \cdot \mathbf{a}$
   h. $\mathbf{c} \cdot \mathbf{b}$
   i. $\hat{\mathbf{a}} \cdot \hat{\mathbf{a}}$
   j. $\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}$

1.13 Compute the magnitude and direction cosines for each of the following vectors:
   a. $\mathbf{a} = [3 \quad 4]$
   b. $\mathbf{b} = [0 \quad -2]$
   c. $\mathbf{c} = [-3 \quad -5 \quad 0]$
   d. $\mathbf{d} = [1 \quad 4 \quad -3]$
   e. $\mathbf{e} = [x \quad y \quad z]$

1.14 Compute the scalar product of the following pairs of vectors:
   a. $\mathbf{a} = [0 \quad -2]$, $\mathbf{b} = [1 \quad 3]$
   b. $\mathbf{a} = [4 \quad -1]$, $\mathbf{b} = [2 \quad 1]$
   c. $\mathbf{a} = [1 \quad 0]$, $\mathbf{b} = [0 \quad 4]$
   d. $\mathbf{a} = [3 \quad 0 \quad -2]$, $\mathbf{b} = [0 \quad -1 \quad -3]$
   e. $\mathbf{a} = [5 \quad 1 \quad 7]$, $\mathbf{b} = [-2 \quad 4 \quad 1]$

1.15 Compute the following vector products:
   a. $\mathbf{i} \times \mathbf{i}$
   b. $\mathbf{j} \times \mathbf{j}$
   c. $\mathbf{k} \times \mathbf{k}$
   d. $\mathbf{i} \times \mathbf{j}$
   e. $\mathbf{j} \times \mathbf{k}$
   f. $\mathbf{k} \times \mathbf{i}$
   g. $\mathbf{j} \times \mathbf{i}$
   h. $\mathbf{k} \times \mathbf{j}$
   i. $\mathbf{i} \times \mathbf{k}$

1.16 Show that $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$.

1.17 Given $\mathbf{a} = [1 \quad 0 \quad -2]$, $\mathbf{b} = [3 \quad 1 \quad 4]$, and $\mathbf{c} = [-1 \quad 6 \quad 2]$, compute

a. $\mathbf{a} \times \mathbf{a}$
b. $\mathbf{a} \times \mathbf{b}$
c. $\mathbf{b} \times \mathbf{a}$
d. $\mathbf{b} \times \mathbf{c}$
e. $\mathbf{c} \times \mathbf{a}$

1.18 Given $\mathbf{a} = [4 \quad -1]$, $\mathbf{b} = [2 \quad 8]$, $\mathbf{c} = [-4 \quad 1]$, and $\mathbf{d} = [3 \quad 2]$, compute the angle between

a. **a** and **b**
b. **a** and **c**
c. **c** and **b**
d. **a** and **d**
e. **c** and **d**

1.19 Compute the angle between the pairs of vectors given in Exercise 1.14.

1.20 Compute the vector product for each of the following pairs of vectors:

a. $\mathbf{a} = [3 \quad -1 \quad 2]$, $\mathbf{b} = [2 \quad 0 \quad 2]$
b. $\mathbf{a} = [4 \quad 1 \quad -5]$, $\mathbf{b} = [3 \quad 6 \quad 2]$
c. $\mathbf{a} = [2 \quad -1 \quad 3]$, $\mathbf{b} = [-4 \quad 2 \quad -6]$
d. $\mathbf{a} = [0 \quad 1 \quad 0]$, $\mathbf{b} = [1 \quad 0 \quad 0]$
e. $\mathbf{a} = [0 \quad 0 \quad 1]$, $\mathbf{b} = [1 \quad 0 \quad 0]$

1.21 Show that the vectors $\mathbf{a} = [-\frac{1}{3} \quad \frac{2}{3} \quad \frac{2}{3}]$, $\mathbf{b} = [\frac{2}{3} \quad -\frac{1}{3} \quad \frac{2}{3}]$, and $\mathbf{c} = [-\frac{2}{3} \quad -\frac{2}{3} \quad \frac{1}{3}]$ are mutually perpendicular.

1.22 Show that the line joining the midpoints of two sides of a triangle is parallel to the third side and has one half its magnitude.

1.23 Find the midpoint of the line segment between $\mathbf{p}_0 = [3 \quad 5 \quad 1]$ and $\mathbf{p}_1 = [-2 \quad 6 \quad 4]$.

1.24 Given $\mathbf{a} = [6 \quad -1 \quad -2]$, $\mathbf{b} = [3 \quad 2 \quad 4]$, and $\mathbf{c} = [7 \quad 0 \quad 2]$, write the vector equation of a line

a. through **a** and parallel to **b**
b. through **b** and parallel to **c**
c. through **c** and parallel to **a**
d. through **a** and parallel to **a**
e. through **b** and parallel to the **a** axis

1.25 Find the equations of the x, y, and z vector components for the line segments given by the following pairs of endpoints:

a. $\mathbf{p}_0 = [0 \quad 0 \quad 0]$, $\mathbf{p}_1 = [1 \quad 1 \quad 1]$
b. $\mathbf{p}_0 = [-3 \quad 1 \quad 6]$, $\mathbf{p}_1 = [2 \quad 0 \quad 7]$
c. $\mathbf{p}_0 = [1 \quad 1 \quad -4]$, $\mathbf{p}_1 = [5 \quad -3 \quad 9]$
d. $\mathbf{p}_0 = [6 \quad 8 \quad 8]$, $\mathbf{p}_1 = [-10 \quad 0 \quad -3]$
e. $\mathbf{p}_0 = [0 \quad 0 \quad 1]$, $\mathbf{p}_1 = [0 \quad 0 \quad -1]$

1.26 Given $x = 3 + 2u$, $y = -6 + u$, and $z = 4$, find $\mathbf{p}_0$ and $\mathbf{p}_1$.

1.27 What is the difference between the following two line segments: for line 1, $\mathbf{p}_0 = [2 \quad 1 \quad -2]$ and $\mathbf{p}_1 = [3 \quad -3 \quad 1]$; for line 2, $\mathbf{p}_0 = [3 \quad -3 \quad 1]$ and $\mathbf{p}_1 = [2 \quad 1 \quad -2]$.

1.28 Write the vector equation of the plane passing through **a** and parallel to **b** and **c**.

1.29 Write the vector equation of a plane that passes through the origin and is perpendicular to the y axis.

1.30 Is the set of all vectors lying in the first quadrant of the x, y plane a linear space? Why?

1.31 Determine nontrivial linear relations for the following sets of vectors:

a. $\mathbf{p} = [1 \quad 0 \quad -2]$, $\mathbf{q} = [3 \quad -1 \quad 3]$, $\mathbf{r} = [5 \quad -2 \quad 8]$

b. $\mathbf{p} = [2 \quad 0 \quad 1]$, $\mathbf{q} = [0 \quad 5 \quad 1]$, $\mathbf{r} = [6 \quad -5 \quad 4]$

c. $\mathbf{p} = [3 \quad 0]$, $\mathbf{q} = [1 \quad 4]$, $\mathbf{r} = [2 \quad -1]$

1.32 Are the vectors $\mathbf{r} = [1 \quad -1 \quad 0]$, $\mathbf{s} = [0 \quad 2 \quad -1]$, and $\mathbf{t} = [2 \quad 0 \quad -1]$ linearly dependent? Why?