7MR10040: Medical Robotics: Theory and Applications
Semester 1

Assignment 3

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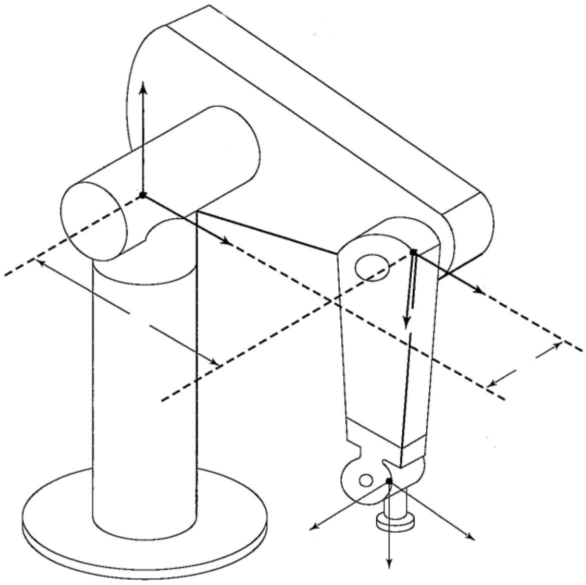
Question 1



a) Give the homogeneous transformation $T_t^0(0)$, which relates Frame t to Frame 0 when θ = 0, as it is shown in the figure.

With frame 0 at the base:

We rotate 90 degrees on the y-axis and translate -1 on x and 1 on y, giving us:

$$T_t^0(0) = \begin{pmatrix} R_y(90) & T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) & -1 \\ 0 & 1 & 0 & 1 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

With frame 0 at the tip:

There is no rotation, so this is a pure translation. And since frame 0 is the tip, there is also no translation, giving us:

$$T_t^0(0) = \begin{pmatrix} I & T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

b) Give the screw parameters that geometrically describe the screw motions of the joints with respect to Frame 0.

The screw parameters are given by:

$$\text{Pitch: } h = \frac{\omega^{\mathrm{T}} v}{||\omega||}, \text{since } \omega \neq 0$$

$$\text{Axis: } l = \frac{\omega \mathrm{x} \mathrm{v}}{||\omega||^2}, \; since\ \omega \neq 0$$

$$Magnitude\text{: } M = ||\omega||$$

With frame 0 at the base:

$$\omega_1 = (1{,}0{,}0), \omega_2 = (1{,}0{,}0)$$

$$q_1 = (0{,}0{,}1), \mathrm{q}_2 = (0{,}1{,}1)$$

Therefore for $\boldsymbol{\omega_1} = (\mathbf{1}, \mathbf{0}, \mathbf{0})$ and $\boldsymbol{q_1} = (\mathbf{0}, \mathbf{0}, \mathbf{1})$:

$$v_1 = -\omega x q = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

$$\text{Pitch: } h_1 = \frac{\omega^{\mathrm{T}} v}{||\omega||} = \frac{0}{1} = 0$$

$$\text{Axis: } \mathrm{l}_1 = \frac{\omega_1 \mathrm{x} \mathrm{v}_1}{||\omega_1||^2} + \lambda\omega_1 = q_1 + \lambda\omega_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \; since\ \omega \neq 0$$

$$Magnitude\text{: } M = ||\,\omega|| = 1\ since\ M = ||\omega|| = 1\ since\ \omega \neq 0$$

And for $\boldsymbol{\omega_2} = (\mathbf{1}, \mathbf{0}, \mathbf{0})$ and $\mathbf{q_2} = (\mathbf{0}, \mathbf{1}, \mathbf{1})$:

$$v_2 = -\omega x q = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$Pitch\text{: } h_2 = \frac{\omega^{\mathrm{T}} v}{||\omega||} = \frac{0}{1} = 0$$

$$\text{Axis: } \mathrm{l}_2 = \frac{\omega_2 \mathrm{x} \mathrm{v}_2}{||\omega_2||^2} + \lambda\omega_2 = q_2 + \lambda\omega_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \; since\ \omega \neq 0$$

$$Magnitude\text{: } M = ||\,\omega|| = 1\ since\ M = ||\omega|| = 1\ since\ \omega \neq 0$$

With frame 0 at the tip:

$\omega_1 = (0,0,1), \omega_2 = (0,0,1)$

$q_1 = (-1,-1,0), q_2 = (-1,0,0)$

Therefore for $\boldsymbol{\omega_1} = (\mathbf{0}, \mathbf{0}, \mathbf{1})$ and $\boldsymbol{q_1} = (-\mathbf{1}, -\mathbf{1}, \mathbf{0})$:

$$v_1 = -\omega x q = -\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Pitch: $h_1 = \frac{\omega^T v}{||\omega||} = \frac{0}{1} = 0$

Axis: $l_1 = \frac{\omega_1 x v_1}{\left||\omega_1|\right|^2} + \lambda\omega_1 = q_1 + \lambda\omega_1 = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $since\ \omega \neq 0$

$Magnitude: M = ||\,\omega|| = 1\ since\ M = \left||\omega|\right| = 1\ since\ \omega \neq 0$

And for $\boldsymbol{\omega_2} = (\mathbf{0}, \mathbf{0}, \mathbf{1})$ and $\mathbf{q_2} = (-\mathbf{1}, \mathbf{0}, \mathbf{0})$:

$$v_2 = -\omega x q = -\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$Pitch: h_2 = \frac{\omega^T v}{||\omega||} = \frac{0}{1} = 0$

Axis: $l_2 = \frac{\omega_2 x v_2}{\left||\omega_2|\right|^2} + \lambda\omega_2 = q_2 + \lambda\omega_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $since\ \omega \neq 0$

$Magnitude: M = ||\,\omega|| = 1\ since\ M = \left||\omega|\right| = 1\ since\ \omega \neq 0$

c) Express the geometric “screws” in their abstract mathematical “twist” notations.

Using:

$$\xi = \begin{pmatrix} -\omega x q \\ \omega \end{pmatrix} = \begin{pmatrix} v \\ \omega \end{pmatrix}$$

With frame 0 at the base:

$$v_1 = -\omega x q = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

$$v_2 = -\omega x q = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Therefore:

$$\xi_1 = \begin{pmatrix} -\omega x q \\ \omega \end{pmatrix} = \begin{pmatrix} v \\ \omega \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\xi_2 = \begin{pmatrix} -\omega x q \\ \omega \end{pmatrix} = \begin{pmatrix} v \\ \omega \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

With frame 0 at the tip:

$$v_1 = -\omega x q = -\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$v_2 = -\omega x q = -\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Therefore:

$$\xi_1 = \begin{pmatrix} -\omega x q \\ \omega \end{pmatrix} = \begin{pmatrix} v \\ \omega \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\xi_2 = \begin{pmatrix} -\omega x q \\ \omega \end{pmatrix} = \begin{pmatrix} v \\ \omega \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

d) Give the homogeneous transformation $\boldsymbol{T}_0^b$ which relates Frame 0 to the robot base frame.

With frame 0 at the base:

There is no rotation, so this is a pure translation. And since frame 0 is the tip, there is also no translation, giving us:

$$\boldsymbol{T}_0^b = \begin{pmatrix} I & T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

With frame 0 at the tip:

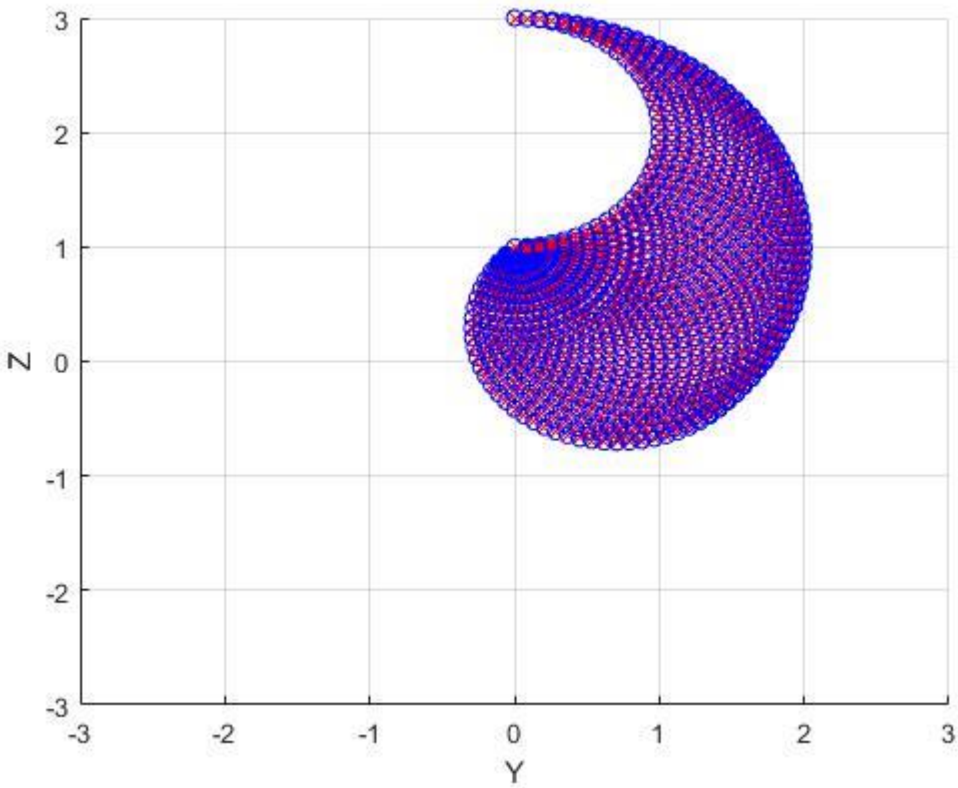
We rotate 90 degrees on the y-axis and translate -1 on x and 1 on y, giving us:

$$\boldsymbol{T}_0^b = \begin{pmatrix} R_y(90) & T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos(90) & 0 & \sin(90) & -1 \\ 0 & 1 & 0 & 1 \\ -\sin(90) & 0 & \cos(90) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

e) Write a Matlab function that takes in the two joint angles and returns the homogeneous transformation T_t^b, using the product of exponential

```
% Multiplies two exponential of xi theta
function transformation = HomogeneousTransformation(q1, w1, th1, q2, w2, th2, g0)


    exp_xi_th1 = expi_xi_th_parameter(q1,w1,th1);
    exp_xi_th2 = expi_xi_th_parameter(q2,w2,th2);


    transformation = subs(exp_xi_th1 * exp_xi_th2 * g0);

end


% Calculates the exponential of xi theta
function exp_xi_th = expi_xi_th_parameter(q,w,th)
    v = cross(-w, q);
    xi = [v ; w];
    xi_ = twist(xi);
    exp_xi_th = expm(xi_ * th);
end




% Example:
syms l th1 th2 real
l=1
q1 = [0; 0; l];
w1 = [1; 0 ; 0];
q2 = [-l; l; l];
w2 = [1; 0 ; 0];
translation = [-1; 1; 0;];


g0 = [roty(pi/2) , translation; zeros(1,3) , 1];


HomogeneousTransformation(q1,w1,th1,q2,w2,th2,g0)
```

f) The robot joints can move within the following ranges: $\theta_)$= [$-\pi/4$, $\pi/2$] and θ_+ = [$-\pi/2$,$\pi/2$]. For every combination of the two joints at 5-degree increments, plot the origin of the tool frame with respect to the base frame (in the y-z plane). For one case, use “o”s, and for the other use “x”s. If you have done the forward kinematics correctly, the “x”s and “o”s should line up on your plot. Turn in this plot. Make sure that the axes are drawn with equal magnitude so that the workspace of the manipulator is not distorted. The relative density of the points in the workspace, using this method of equally spaced joint angles, gives information about the manipulability of the manipulator (which we will discuss more in the future). In locations where the points are densely packed, the robot is better conditioned.



This graph was plotted by looping over the script defined in the previous exercise for all the combinations of the angles and by transforming the second case to the same frame as the first

```
function [] = plotpuma(q11, w11, q12, w12, g10, q21, w21, q22, w22, g20,trans-
lation2)

    clear figure;
    figure(1);
    xlabel('Y');
    ylabel('Z');
    hold on;
    grid on;
    axis([-3 3 -3 3]);
    step = 5*pi/180;
    %Transformation matrix to bring the second case in the same frame as
    %the first
    tm = [roty(pi/2) , translation2; zeros(1,3) , 1];
    for i=-pi/4:step:pi/2
        for j=-pi/2:step:pi/2
            g1 = HomogeneousTransformation(q11, w11, i, q12, w12, j, g10);
            g2 = tm * HomogeneousTransformation(q21, w21, i, q22, w22,j, g20);
            y1 = g1(2,4);
            z1 = g1(3,4);
            y2 = g2(2,4);
            z2 = g2(3,4);
            scatter(y1, z1,'r','x');
            scatter(y2, z2,'b','o');
        end
     end

end

% Example
% Define parameters for the first case
l = 1;
q11 = [0; 0; l];
w11 = [1; 0; 0];
q12 = [0; l; l];
w12 = [1; 0; 0];
translation1 = [-1; 1; 0;];


g10 = [roty(pi/2), translation1; zeros(1,3), 1];


% Define parameters for the second case
q21 = [-l; -l; l];
w21 = [0; 0; 1];
q22 = [-l; 0; 0];
w22 = [0; 0; 1];
translation2 = [0; 0; 0;];
g20 = [eye(3) , translation2; zeros(1,3) , 1];


% Plot the graph
plotpuma(q11, w11, q12, w12, g10, q21, w21, q22, w22, g20, translation1)
```