Lecture 1: Introduction & Gaussian Elimination

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Importance of Linear Algebra

- Most mathematicians define *Linear Algebra* **a**s that branch of mathematics that deals with the study of vectors, vector spaces and linear equations.

- Modern mathematics also relies upon linear transformations and systems of vector matrix. Analytic geometry utilizes the techniques learned during a study of linear algebra, for analytically computing complex geometrical shapes.

- In addition to science, engineering and mathematics, linear algebra has extensive applications in the natural as well as the social sciences.

- Linear algebra today has been extended to consider *n*-dimnsional space. Although it is very difficult to visualize vectors in *n*-space, such n-dimensional vectors are extremely useful in representing data.

- One can easily summarize and manipulate data efficiently in this framework, when data are *ordered* as a list of *n* components.

Usage and Applications

❖ Geometry of ambient space

The modeling of ambient space is based on geometry. Sciences concerned with this space use geometry widely. This is the case with mechanics and robotics, for describing rigid body dynamics; geodesy for describing Earth shape; perspectivity, computer vision, and computer graphics, for describing the relationship between a scene and its plane representation; and many other scientific domains.

In all these applications, synthetic geometry is often used for general descriptions and a qualitative approach, but for the study of explicit situations, one must compute with coordinates. This requires the heavy use of linear algebra.

❖ Functional analysis

Functional analysis studies function spaces. These are vector spaces with additional structure, such as Hilbert spaces. Linear algebra is thus a fundamental part of functional analysis and its applications, which include, in particular, quantum mechanics (wave functions).

❖ Study of complex systems

Most physical phenomena are modeled by partial differential equations. To solve them, one usually decomposes the space in which the solutions are searched into small, mutually interacting cells. For linear systems this interaction involves linear functions. For nonlinear systems, this interaction is often approximated by linear functions. In both cases, very large matrices are generally involved. Weather forecasting is a typical example, where the whole Earth atmosphere is divided in cells of, say, 100 km of width and 100 m of height.

❖ Scientific computation

Nearly all scientific computations involve linear algebra. Consequently, linear algebra algorithms have been highly optimized. BLAS and LAPACK are the best known implementations. For improving efficiency, some of them configure the algorithms automatically, at run time, for adapting them to the specificities of the computer (cache size, number of available cores, ...).

Some Applications in…

- In Data Science
  - Linear Algebra in Machine Learning
    - Loss functions
    - Regularization
    - Covariance Matrix
    - Support Vector Machine Classification
  - Linear Algebra in Dimensionality Reduction
    - Principal Component Analysis (PCA)
    - Singular Value Decomposition (SVD)
  - Linear Algebra in Natural Language Processing
    - Word Embeddings
    - Latent Semantic Analysis
  - Linear Algebra in Computer Vision
    - Image Representation as Tensors
    - Convolution and Image Processing

- In Robotics
  - Basic kinematic equations (modeling)
    - Interpreting time derivatives
    - Tangent vectors and co-tangent vectors
  - A geometric perspective on linear algebra
    - Linear maps are bijections between fundamental spaces
    - A natural generalized inverse
    - Using the SVD to solve reduced rank linear equations
  - Quadratic forms and their manipulation

Why Study Linear Algebra?

- Why should you spend time learning Linear Algebra when you can simply import a package in Python and build your model?

I consider Linear Algebra as one of the foundational blocks of Data Science. You cannot build a skyscraper without a strong foundation, can you? Think of this scenario:

You want to reduce the dimensions of your data using Principal Component Analysis (PCA). How would you decide how many Principal Components to preserve if you did not know how it would affect your data? Clearly, you need to know the mechanics of the algorithm to make this decision.

You would also be able to code algorithms from scratch and make your own variations to them as well. The ability to experiment and play around with our models? Consider linear algebra as the key to unlock a whole new world

Introduction and Recall

This course begins with the central problem of linear algebra: *solving linear equations.*

We start from some definitions and Recalls.

Linear Equations in n Variables

A linear equation in n variables $x_1, x_2, \dots, x_n$ has the form:

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

where

Coefficients: $a_1, a_2, \dots, a_n$

Constant term: b

Examples

Linear Equations:	Nonlinear Equations:
$3x + 2y = 7,$ $\frac{1}{2}x + y - \pi z = \sqrt{2},$ $(\sin \frac{\pi}{2})x_1 - 4x_2 = e^2.$ $x_1 - 2x_2 + 10x_3 + x_4 = 0,$	$xy + z = 2,$ $e^x - 2y = 4,$ $\sin x_1 + 2x_2 - 3x_3 = 0,$ $\frac{1}{x} + \frac{1}{y} = 4.$

Parametric Representation of a Solution Set

- Solve the linear equation $x_1 + 2x_2 = 4$.

Solution

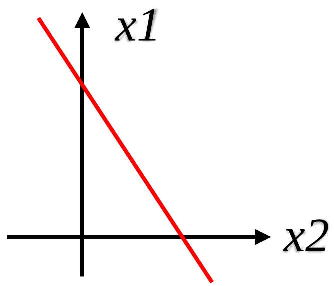
$$x_1 = 4 - 2x_2$$

Variable x_2 is <u>**free**</u> (it can take on *any* real value).

Variable x_1 is <u>**not free**</u> (its value *depends on* the value of x_2).

By letting $x_2 = p$ (p: the third variable, ***parameter***),

you can represent the solution set as



$$\begin{cases} x_1 = 4 - 2p, \\ x_2 = p, \end{cases} \qquad p \in R$$

Infinite number of solutions

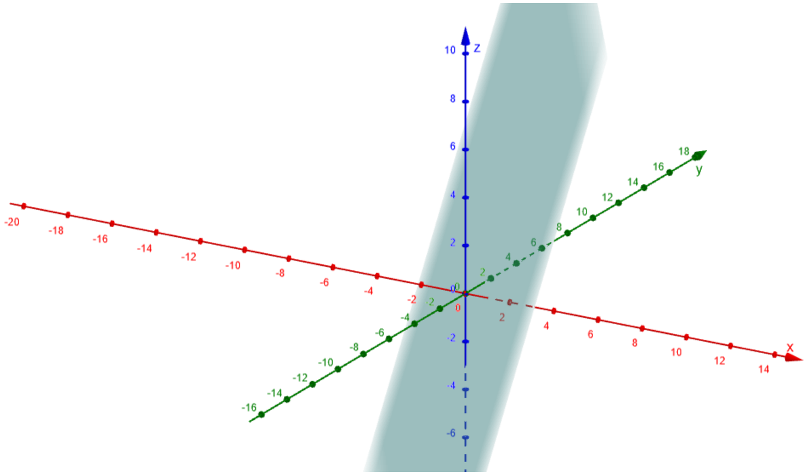
- Solve the linear equation $3x + 2y - z = 3$.

Solution

Choosing y and z to be the free variables

$$x = 1 - \tfrac{2}{3}y + \tfrac{1}{3}z.$$

Letting $y = p$ and $z = t$, you obtain the parametric representation



$$\begin{cases} x = 1 - \tfrac{2}{3}p + \tfrac{1}{3}t, \\ y = p, \\ z = t. \end{cases} \qquad p, t \in R$$

$\boxed{\text{Infinite number of solutions}}$

Systems of Linear Equations

- A system of m linear equations in n variables is a set of m equations, each of which is linear in the same n variables:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

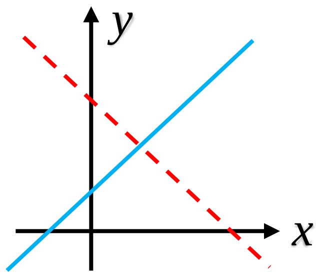
- The double-subscript notation indicates that a_{ij} is the coefficient of x_j in the i th equation.
- A system of linear equations has
  - exactly one solution,
  - or an infinite number of solutions,
  - or no solution.
- A system of linear equations is called
  - **consistent** if it has <u>at least one solution</u>,
  - **inconsistent** if it has <u>no solution</u>.

- Solve the following systems of linear equations, and graph each system as a pair of straight lines.

$$(a)\begin{cases} x+y=3 \\ x-y=-1 \end{cases} \Rightarrow x=1, y=2. \qquad (b)\begin{cases} x+y=3 \\ 2x+2y=6 \end{cases} \Rightarrow x=3-p, y=p, \qquad p\in R. \qquad (c)\begin{cases} x+y=3 \\ x+y=1 \end{cases} \Rightarrow \text{ no solution}$$

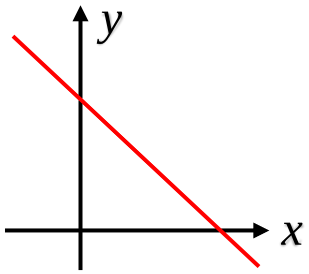
Solution

Finding the solution set of a system of two linear equations in two variables is easy because it amounts to finding the intersection of two lines.



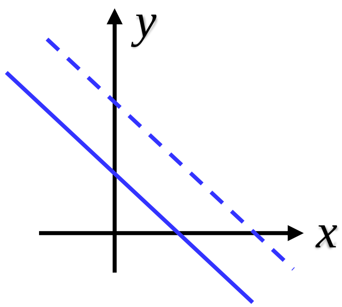
Two intersecting lines

(one solution)



Two coincident lines

(many solutions)



Two parallel lines

(no solution)

Equivalent Systems

- Two systems of linear equations are called equivalent if they have precisely the same solution set.
- Each of the following operations on a system of linear equations produces an equivalent system.
  1) Interchange two equations.
  2) Multiply an equation by a nonzero constant.
  3) Add a multiple of an equation to another equation.

Elementary Row Operations

- Interchange two rows.
- Multiply a row by a nonzero constant.
- Add a multiple of a row to another row.

- Two matrices are said to be *row-equivalent* if one can be obtained from the other by a finite sequence of elementary row operations.

Augmented/Coefficient Matrix

- The matrix derived from the *coefficients* and *constant terms* of a system of linear equations is called the **augmented matrix** of the system.

- The matrix containing ***only*** the coefficients of the system is called the **coefficient matrix** of the system.

System

$$\begin{aligned} x - 4y + 3z &= 5 \\ -x + 3y - z &= -3 \\ 2x \qquad - 4z &= 6 \end{aligned}$$

Augmented Matrix

$$\begin{array}{cccc} x & y & z & const. \end{array}$$
$$\left[\begin{array}{ccc:c} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & -4 & 6 \end{array}\right]$$

Coefficient Matrix

$$\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix}$$

The Geometry of Linear Equations

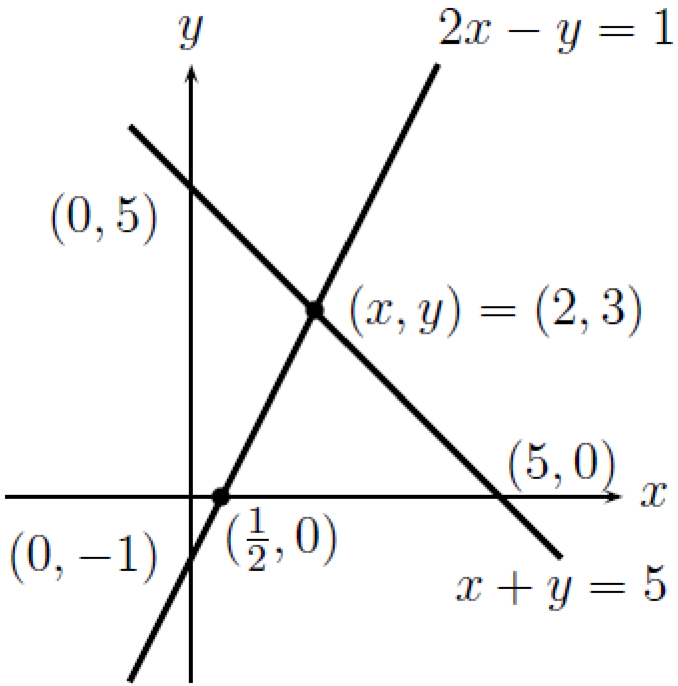
The Geometry of Linear Equations (2D)

The way to understand this subject is by example. We begin with two extremely humble equations

$$2x - y = 1$$
$$x + y = 5$$

We can look at that system by rows or by columns. We want to see them both.

✓ The first approach concentrates on the separate equations (the **rows**).



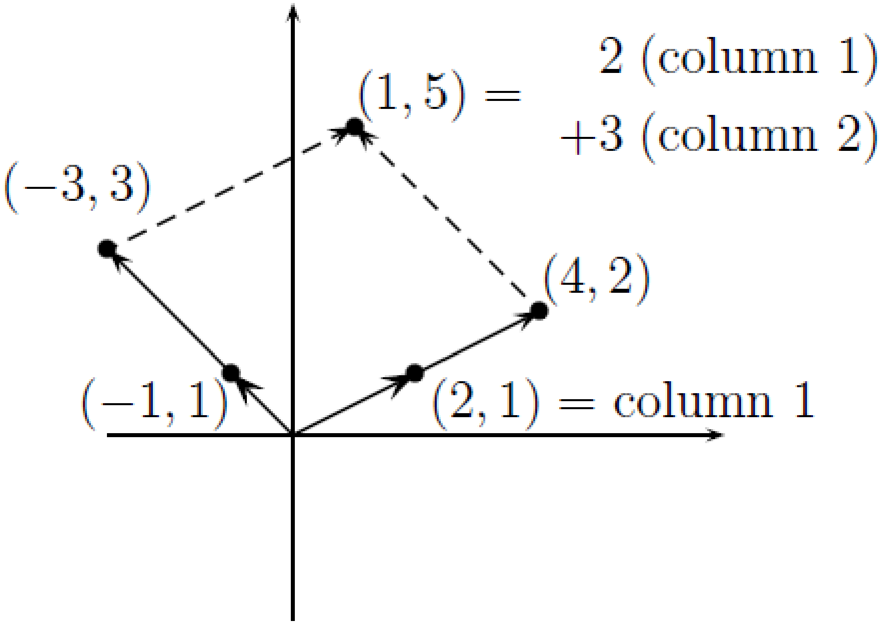
$\boxed{\text{Lines meet at } x = 2,\ y = 3}$

✓ The second approach looks at the columns of the linear system. The two separate equations are really one vector equation:

$$\begin{cases} 2x - y = 1 \\ x + y = 5 \end{cases} \rightarrow \text{Column form} \quad x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

The problem is to find the combination of the column vectors on the left side that produces the vector on the right side.

The unknowns are the numbers *x* and *y* that multiply the column vectors.



Columns combine with 2 and 3

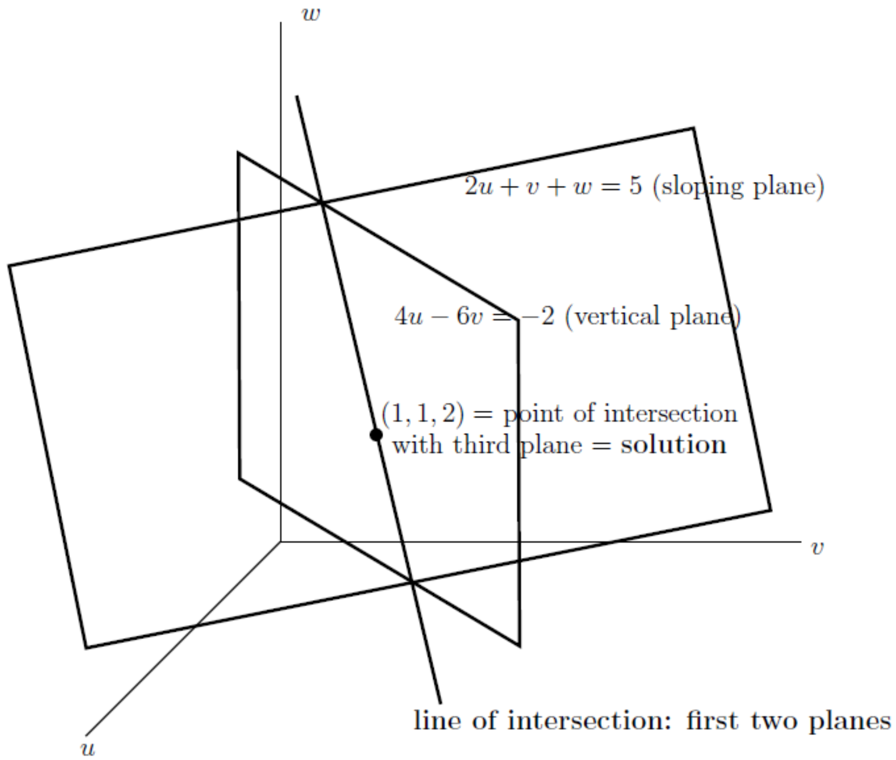
We would rather move forward to n = 3 (Three equations are still manageable).

Three planes $\begin{cases} 2u + v + w = 5 \\ 4u - 6v = -2 \\ -2u + 7v + 2w = 9 \end{cases}$

We start with the rows. Each equation describes a *plane* in three dimensions.

The intersection of the second plane with the first is a line.

Finally the third plane intersects this line in a point at u = 1, v = 1, w = 2..



The Geometry of Linear Equations (nD)

I will be in trouble if that example from relativity goes any further. The point is that linear algebra can operate with any number of equations. The first equation produces an $(n-1)$-dimensional plane in n dimensions, The second plane intersects it (we hope) in a smaller set of "dimension $n-2$." Assuming all goes well, every new plane (every new equation) reduces the dimension by one. At the end, when all n planes are accounted for, the intersection has dimension zero. *It is a point*, it lies on all the planes, and its coordinates satisfy all n equations. <u>It is the solution</u>!

Three planes $\begin{cases} 2u + v + w = 5 \\ 4u - 6v = -2 \\ -2u + 7v + 2w = 9 \end{cases}$

We turn to the columns. This time the vector equation is

Column form $$u\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + v\begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} + w\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = b$$

Those are three-dimensional column vectors. The vector b is identified with the point whose coordinates are $5, -2, 9$. Every point in three-dimensional space is matched to a vector, and vice versa. That was the idea of Descartes, who turned geometry into algebra by working with the coordinates of the point.

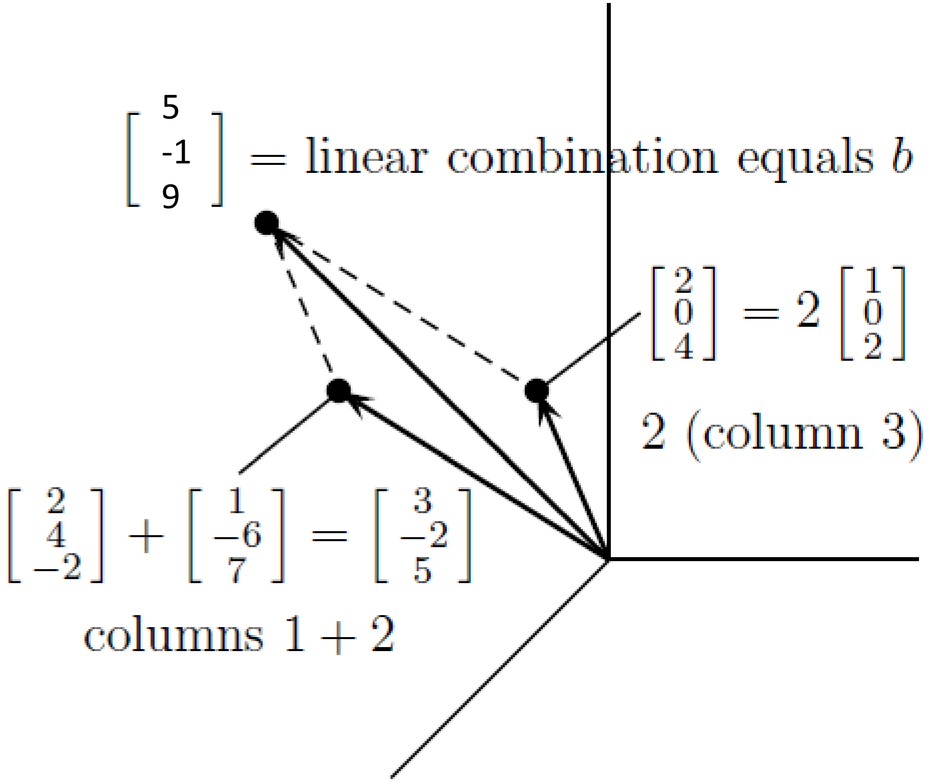
Recall:
We use parentheses and commas when the components are listed horizontally, and square brackets (with no commas) when a column vector is printed vertically.

Column form $u\begin{bmatrix}2\\4\\-2\end{bmatrix}+v\begin{bmatrix}1\\-6\\7\end{bmatrix}+w\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}5\\-2\\9\end{bmatrix}=b$ (*)

The results shown in figure uses both of the basic operations; vectors are multiplied by numbers and then added. The result is called a linear combination, and this combination solves our equation:

Linear combination $1\begin{bmatrix}2\\4\\-2\end{bmatrix}+1\begin{bmatrix}1\\-6\\7\end{bmatrix}+2\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}5\\-2\\9\end{bmatrix}$

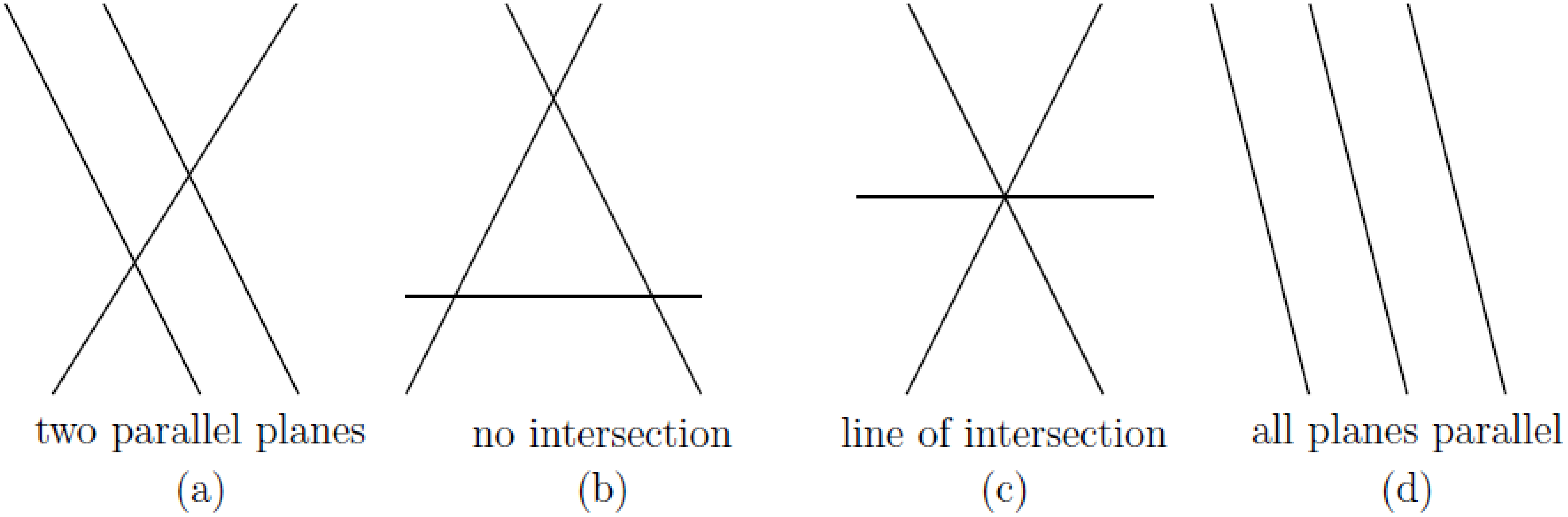
Equation (*) asked for multipliers u, v, w that produce the right side b. Those numbers are $u = 1, v = 1, w = 2$. They give the correct combination of the columns. They also gave the point (1, 1, 2) in the row picture (where the three planes intersect).



$\boxed{\text{Add columns } 1+2+(3+3)}$

The Singular Case

Suppose we are again in three dimensions, and the three planes in the row picture do not intersect. What can go wrong?



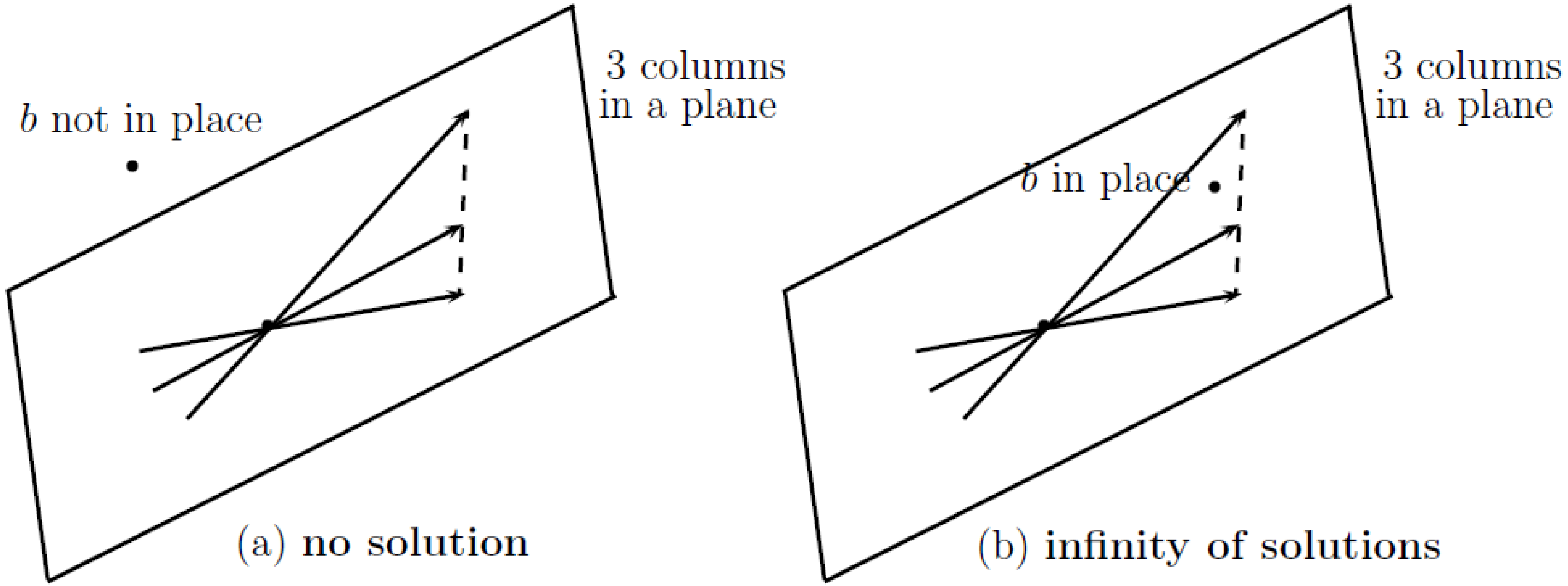
Singular cases: no solution for (a), (b), or (d), an infinity of solutions for (c).

The Singular Case - (cntd.)

What happens to the *column picture* when the system is singular?

Singular case: Column picture
Three columns in the same plane
Solvable only for b in that plane

$$u\begin{bmatrix}1\\2\\3\end{bmatrix}+v\begin{bmatrix}1\\0\\1\end{bmatrix}+w\begin{bmatrix}1\\3\\4\end{bmatrix}=b.$$



Singular cases: b outside or inside the plane with all three columns.

Gaussian Elimination

The way to understand elimination is by example. We begin in three dimensions:

Original system
$$\begin{array}{rcrcrcr} 2u & + & v & + & w & = & 5 \\ 4u & - & 6v & & & = & -2 \\ -2u & + & 7v & + & 2w & = & 9. \end{array}$$

The method starts by subtracting multiples of the first equation from the other equations. The goal is to eliminate u from the last two equations. This requires that we
(a) subtract 2 times the first equation from the second
(b) subtract -1 times the first equation from the third.

Equivalent system
$$\begin{array}{rcrcrcr} \mathbf{2}u & + & v & + & w & = & 5 \\ & - & \mathbf{8}v & - & 2w & = & -12 \\ & & 8v & + & 3w & = & 14. \end{array}$$

The coefficient 2 is the first pivot. Elimination is constantly dividing the pivot into the numbers underneath it, to find out the right multipliers.
The pivot for the second stage of elimination is¡8. We now ignore the first equation. A multiple of the second equation will be subtracted from the remaining equations (in this case there is only the third one) so as to eliminate v.

Gaussian Elimination (2/2)

We add the second equation to the third or, in other words, we
(c) subtract -1 times the second equation from the third.

The elimination process is now complete (This process is called ***forward eliminations***.):

Triangular system

$$\begin{array}{rcrcrcr} \mathbf{2}u & + & v & + & w & = & 5 \\ & - & \mathbf{8}v & - & 2w & = & -12 \\ & & & & \mathbf{1}w & = & 2. \end{array}$$

This system is solved backward, bottom to top. The last equation gives $w = 2$. Substituting into the second equation, we find $v = 1$. Then the first equation gives $u = 1$. This process is called ***back-substitution***.
Remark. One good way to write down the forward elimination steps is to include the right-hand side as an extra column. There is no need to copy *u* and *v* and *w* and "=" at every step, so we are left with the bare minimum:

$$\left[\begin{array}{rrr|r} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

At the end is the triangular system, ready for back-substitution.
By definition**, pivots cannot be zero**. We need to divide by them.

The Breakdown of Elimination (1/2)

Under what circumstances could the process break down?

The answer is: With a full set of n pivots, there is only one solution. The system is non singular, and it is solved by forward elimination and back-substitution. But if a zero ***appears*** in a pivot position, elimination has to stop—either temporarily or permanently. The system might or might not be singular.

Roughly speaking, ***we do not know whether a zero will appear until we try***, by actually going through the elimination process.

In many cases this problem can be cured, and elimination can proceed. Such a system still counts as nonsingular.

The Breakdown of Elimination (2/2)

Example 1. Nonsingular (cured by exchanging equations 2 and 3)

$$\begin{array}{rcrcrcl} u & + & v & + & w & = & \_\_ \\ 2u & + & 2v & + & 5w & = & \_\_ \\ 4u & + & 6v & + & 8w & = & \_\_ \end{array} \rightarrow \begin{array}{rcrcrcl} u & + & v & + & w & = & \_\_ \\ & & & & 3w & = & \_\_ \\ & & 2v & + & 4w & = & \_\_ \end{array} \rightarrow \begin{array}{rcrcrcl} u & + & v & + & w & = & \_\_ \\ & & 2v & + & 4w & = & \_\_ \\ & & & & 3w & = & \_\_ \end{array}$$

The system is now triangular, and back-substitution will solve it.

Example 2. Singular (incurable)

$$\begin{array}{rcrcrcl} u & + & v & + & w & = & \_\_\_\_ \\ 2u & + & 2v & + & 5w & = & \_\_\_\_ \\ 4u & + & 4v & + & 8w & = & \_\_\_\_ \end{array} \longrightarrow \begin{array}{rcrcrcl} u & + & v & + & w & = & \_\_\_\_ \\ & & & & 3w & = & \_\_\_\_ \\ & & & & 4w & = & \_\_\_\_ \end{array}$$

There is no exchange of equations that can avoid zero in the second pivot position.

Next Week Topics

- Matrix operations, including inverses.
- Triangular Factors (LU and LDU factorization) and Row Exchanges.