The Structure of White Dwarf Stars

Katie Chynoweth

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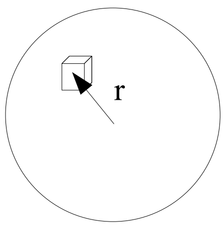
A white dwarf star is the end stage in the life of a low to medium mass star, such as our Sun. When the star runs out of helium to burn, it is not heavy enough to fuse carbon. The star blows off its outer layers in a planetary nebula, and hot core of carbon remains. Over time, the star cools. Its structure depends on the balance of gravity and electron degeneracy pressure.

Gravity and Pressure

Consider a cube of matter in the white dwarf, of volume δV:



The force on the cube is

$$F = \frac{-Gm(r)}{r^2}\rho(r)\delta V \tag{1}$$

where

G = gravitational constant
$\rho(r)$ = density at r
m(r) = mass contained with r

The force per unit volume is due to the change in pressure with radius:

$$\frac{dP}{dr} = \frac{-Gm(r)}{r^2}\rho(r) \tag{2}$$

We can write density as a function of radius using:

$$\frac{dP}{dr} = \frac{d\rho}{dr}\frac{dP}{d\rho} \tag{3}$$

Combining (2) and (3):

$$\frac{d\rho}{dr} = -\left(\frac{dP}{d\rho}\right)^{-1}\frac{Gm(r)}{r^2}\rho \tag{4}$$

Also, we can write the mass as a function of radius:

$$\frac{dm}{dr}=4\pi r^2 \rho(r) \tag{5}$$

We now have two coupled differential equations.

The boundary conditions at r = 0 are:

$$\begin{aligned}\rho&=\rho_c\\ m&=0\end{aligned} \tag{6}$$

Equation of state

In order to solve the differential equations, we must find pressure as a function of density, also known as the equation of state of the white dwarf. When the temperature of the star is low enough, the electrons fill the lowest possible quantum levels. Since the nuclei are very heavy compared to the electrons, we can assume that most of the mass of the star comes from the nuclei. The electrons are moving much faster than the nuclei, so most of the pressure comes from the electrons. The density in the star is so high that the electrons are not bound to individual nuclei, but rather move freely about the star. Therefore, a good model for the white dwarf star is a free Fermi gas of relativistic electrons at zero temperature.

Number density of electrons:

$$n=\frac{N}{V}=Y_e\frac{\rho}{m_N} \tag{7}$$

where

Y_e = electrons per nucleon
m_N = nucleon mass

The nucleon mass from now on will be treated as the mass of a proton, since protons and neutrons have nearly the same mass.

Fermi momentum:

There are two spin states per level, and

$$\frac{4\pi p^2 dp}{h^3} \tag{8}$$

levels per unit volume; this is the density of states. Integrating to the Fermi momentum:

$$n=\frac{2\circ 4\pi}{h^3}\int_0^{p_f} p^2 dp=\frac{8\pi}{3\mathrm{h}^3}p_f^3 \tag{9}$$

Combining (7) and (9),

$$p_f = h\left(\frac{3Y_e \rho}{8\pi m_p}\right)^{\frac{1}{3}} \tag{10}$$

Energy

Relativistic kinetic energy of the electrons:

$$\epsilon = m_e c^2 \{(1+(\frac{p}{m_e c})^{\frac{1}{2}} - 1)\} \tag{11}$$

Speed:

$$u = \frac{d\epsilon}{dp} = \frac{(p/m_e)}{(1+(\frac{p}{m_e c})^2)^{\frac{1}{2}}} \tag{12}$$

Pressure:

$$P = \frac{1}{3}\frac{N}{V}\langle pu\rangle = \frac{8\pi}{3h^3}\int_0^{p_f} \frac{(p^2/m_e)}{(1+(\frac{p}{m_e c})^2)^{\frac{1}{2}}} p^2\, dp \tag{13}$$

Now introduce a new variable:

$$p = m_e c \sinh(\vartheta) \tag{14}$$
$$u = c \tanh(\vartheta) \tag{15}$$

Substitute these into (9):

$$n = \frac{8\pi m_e^3 c^3}{3h^3} x^3 \tag{16}$$

where

$$x = \sinh(\vartheta_f) = \frac{p_f}{m_e c} = \left(\frac{3n}{8\pi}\right)^{\frac{1}{3}}\left(\frac{h}{m_e c}\right) \tag{17}$$

So, pressure is now written as

$$P=\frac{8\pi m_e^4 c^5}{3h^3}\int_0^{\vartheta_f}\sinh(\vartheta)^4 d\vartheta=\frac{\pi m_e^4 c^5}{3h^3}A(x) \tag{18}$$

where

$$A(x)=x(x^2+1)^{\frac{1}{2}}(2x^2-3)+3\sinh^{-1}(x) \tag{19}$$

Find x for our model:

$$x=\frac{p_f}{m_e c}=\left(\frac{n}{n_0}\right)^{\frac{1}{3}}=\left(\frac{\rho}{\rho_0}\right)^{\frac{1}{3}} \tag{20}$$

where

$$n_0=\frac{8\pi}{3}\frac{m_e^3 c^3}{h^3} \tag{21}$$

$$\rho_0=\frac{m_p n_0}{Y_e} \tag{22}$$

Now, the equation for pressure is:

$$P=\frac{Y_e m_e c^2}{8 m_p}A(x) \tag{23}$$

And the equation of state is:

$$\frac{dP}{d\rho}=\frac{Y_e c^2 m_e}{m_p}\gamma(x) \tag{24}$$

where

$$\gamma(x)=\frac{x^2}{3(1+x^2)^{\frac{1}{2}}} \tag{25}$$

Final differential equations

Mass:

$$\frac{dm}{dr} = 4\pi r^2 \rho(r) \tag{26}$$

Density:

$$\frac{d\rho}{dr} = \frac{-Gm(r)\rho(r)}{r^2} \frac{m_p}{m_e} \frac{1}{Y_e c^2} \frac{1}{\gamma(x)} \tag{27}$$

These can be solved numerically for the structure of a white dwarf, given the boundary conditions, for different values of the central density.

The program

The differential equations were solved using the program EULER. In particular, the differential equation solver "runge" was used, which employs the Runge-Kutta method. A sample white dwarf code follows with outputs (Figure 1). The resulting plots are also included (Figure 2).

The Chandrasekhar limit

As the central density increases, the radius of the star decreases and the mass increases. Therefore, there must be a maximum mass at which the radius of the star goes to zero. Physically, this is the point at which electron degeneracy pressure can no longer support the star against gravity. The theoretical limit, calculated by Chandrasekhar, is 1.44 solar masses for a carbon white dwarf. A plot of the mass-radius relationship of a carbon white dwarf is included (Figure 3); it is apparent that the limiting mass is indeed 1.44 solar masses.

If a white dwarf star becomes more massive than the Chandrasekhar limit by accretion of matter from a nearby star, the outer layers of the star collapse and "bounce" off the core, creating a type Ia supernova.

Figure 1: Sample Euler code

```
Euler [ /home/katiemae/wd_18.en ]
File   Edit   Misc

>pi=3.14159;
>g=6.67e-11;
>mp=1.67e-27;
>me=9.11e-31;
>ye=.5;
>c=3e8;
>h=6.626e-34;
>no=8*pi*me**3*c**3/(3*h**3);
>rho0=mp*no/ye;
>function gamma(rho)
$useglobal;
$x=(rho/rho0)**(1/3);
$return x**2/(3*sqrt(1+x**2));
$endfunction
>function dmrhodr(r,mrho)
$useglobal;
$return [4*pi*r**2*mrho[2],-(g*mrho[1]*mrho[2]*mp)/(r**2*me*ye*c**2*gamma(mrho[2]))] ;
$endfunction
>r=.001:100:4.5343e6;
>4.5343e6/6378e3
      0.710928
>mrho=runge("dmrhodr",r,[.001,8e10]);
>max(mrho[1])
  2.26888e+30
>2.26888e30/1.98892e30
       1.14076
>min(mrho[2])
       323.319
>xplot(r,mrho[1,:]);
>text("Radius = 0.71 R_earth", [300,250]);
>text("Total mass = 1.14 M_sun", [300,200]);
>text("Central density = 8*10^10 kg/m^3", [300,150]);
>title("Mass Profile");
>vutext("Mass in kg",[100,600]);
>text("Radius in m",[500,960]);
>xplot(r,mrho[2,:]);
>title("Density Profile");
>vutext("Density in kg/m^3",[100,600]);
>text("Radius in km",[500,960]);
>text("Radius = 0.71 R_earth", [300,250]);
>text("Total mass = 1.14 M_sun", [300,200]);
>text("Central density = 8*10^10 kg/m^3", [300,150]);
>

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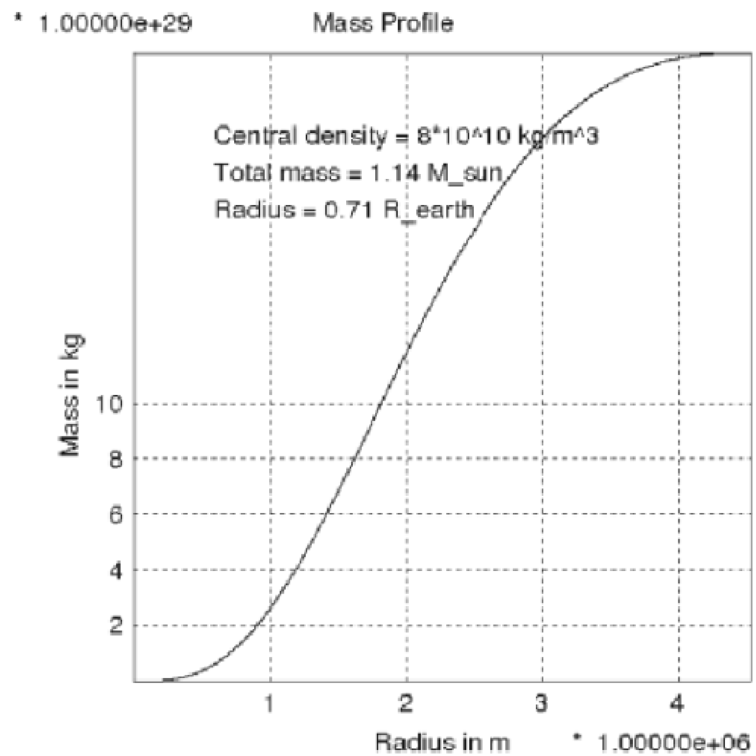
Figure 2: Sample white dwarf mass and density profiles



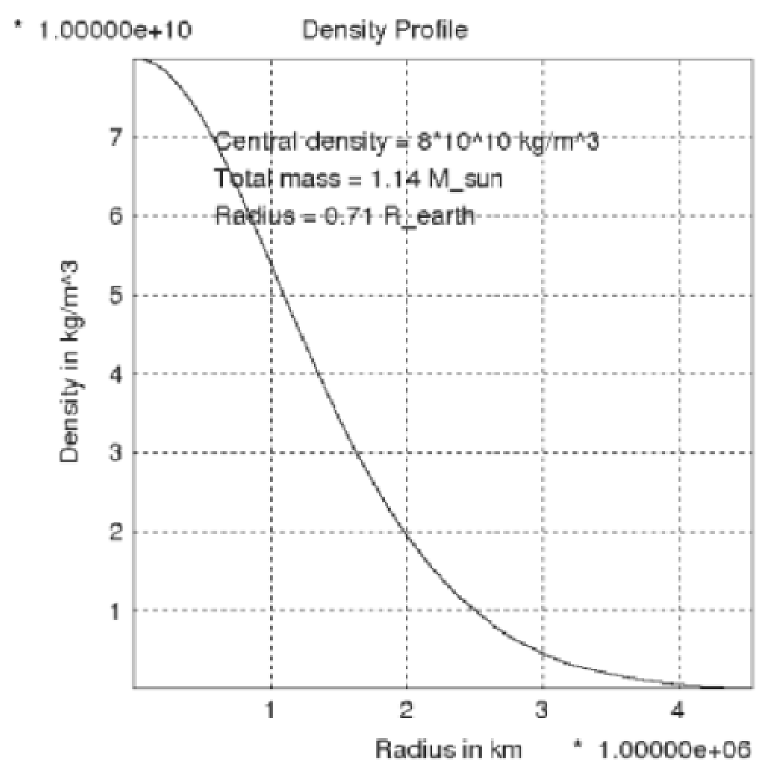


Figure 3: Mass-radius relationship for a carbon white dwarf