Paul Wilmott Introduces Quantitative Finance

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Paul Wilmott Introduces Quantitative Finance

In praise of Paul Wilmott

(the unique spelling of his admirers has been retained)

I'm a junior derivatives trader in Mexico City. I've seen your book and I have only one comment: SEXY!......... After reading the book I'd like to follow one of your courses, but they are way too expensive..... ***Purchased both Quant Finance and Derivatives a couple of days ago. Will not be able to afford steak or wine for weeks as a result*****.....** Loved your book, which is a breath of fresh air, amongst all those arid derivatives books!! It really is in a class of its own. I have wasted so much money on stupid derivative books which too elementary or way too complicated........... **BTW, I want to congratulate you for the *best* book in Financial Engineering I've read in the last years** .... I found it to be easily the best book that I have read/worked through on the subject..... ***Congratulation to your book Derivatives!!! The way you describe, present, and deliver Derivative knowledge is unique! One can feel your passion on the topic. It's a pleasure to read, study, re-read*****.....** Congratulations on a great new book — 'PW on Quant Finance'. I bought the DERIVATIVES one but cannot afford this one!!........... **I am fanatical follower of your book ''Derivatives''. You are best and this not flattery. Sorry from my English!** ..... Congratulations for your brilliant book........... **I had a course on derivatives and your book was not suggested by the teacher (a stupid teacher)** .... ***I'd like to say that this is a great book but you already know that!*****.....** What I like about it is that it has this no-nonsense kind of approach that you'd expect in a physics text and it spells out the ''stuff between the equations''........... **We use it for the part of our Banking and Risk Management course and it's much more comprehensive than the books that we have recommended in our study guide**..... Your book ''Derivatives: the Theory and Practice of Financial Engineering'' is the best in the market so far........... ***I shall waste no more precious words but to say that I am very sympathetic to your humor and irony*****.....** ***what most don't always seem to understand: irony is one of the GREAT filters to access knowledge in this world and an elegant one for that matter***........... your book rocks........... Congratulations to the success of your book (I got my copy of it for Christmas)..... **I thought it might amuse you to know that I think your book got me a job!**..... I would like to thank you for writing Derivatives........... ***Derivatives is the Greatest! Thank you, thank you, thank you! Just read the first 7 chapters of Derivatives, and it speaks to me.*****.....** complete, brilliant and amusing, stimulating for some original ideas and examples, didactically ready to be used by Students; it employs mathematical tools as tools only, not as a target; it is the last but the best book on derivatives in my library...

Love you for ever, baby. xxx

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To a rising Star

Contents

Preface

In this book I present classical quantitative finance. The book is suitable for students on advanced undergraduate finance and derivatives courses, MBA courses, and graduate courses that are mainly taught. The text is quite self-contained, with, I hope helpful, 'Time Out' sections covering the more mathematical aspects of the subject for those who feel a little bit uncomfortable. Little prior knowledge is assumed, other than basic calculus, even *stochastic* calculus is explained here in a simple, accessible way.

By the end of the book you should know enough quantitative finance to understand most derivative contracts, to converse knowledgeably about the subject at dinner parties, to land a job on Wall Street, and to pass your exams.

The structure of the book is quite logical. Markets are introduced, followed by the necessary math, and then the two are melded together. The technical complexity is never that great, nor need it be. The last couple of chapters are on the numerical methods you will need for pricing. In the more advanced subjects, such as credit risk, the mathematics is kept to a minimum. Also, plenty of the chapters can be read without reference to the mathematics at all. The structure, mathematical content, intuition, etc. are based on many years teaching at universities and training bank personnel at all levels.

The accompanying CD contains spreadsheets and Visual Basic programs implementing many of the techniques described in the text. The CD icon will be seen throughout the book, indicating material to be found on the CD, naturally. There is also a full list of its contents at the end of the book.

This book is a shortened version of *Paul Wilmott on Quantitative Finance*. It's also more affordable than the 'full' version. However, I hope that you'll eventually upgrade, perhaps when you go on to more advanced, research-based studies, or take that job on The Street.

PWoQF and its predecessor *Derivatives* are, I am told, standard texts within the banking industry. Students brought up on other popular texts quickly convert once reality bites.

In *Paul Wilmott Introduces Quantitative Finance* I have specifically the university student in mind.

The differences between the university and the full versions are outlined at the end of the book. And to help you make the leap, we've included a form for you to upgrade, giving you a nice discount. Roughly speaking, the full version includes a great deal of nonclassical, more modern approaches to quantitative finance, including several nonprobabilistic models. There are more mathematical techniques for valuing exotic options and more markets are covered, including energy. The numerical methods are described in more detail. All in all, well worth the extra pennies.

If you have any problems understanding anything in the book, find errors, or just want a chat, email me at `paul@paulwilmott.com`. I'll do my very best to respond as quickly as possible.

ACKNOWLEDGEMENTS

I would like to thank Mark 'woolly sweater, woolly brain' Davis, Dominic 'the welsher' Welsh, Peter Fondleme, Sir Pppeter 'Piper picked a peck of pickled pepper' Nnnorth, Hans Foolmer, Elizabeth Llewd-Snatch and Peter Sniffturd. Collectively these people changed my life enormously for the better. Who said committees never achieved anything?[1]

ABOUT THE AUTHOR

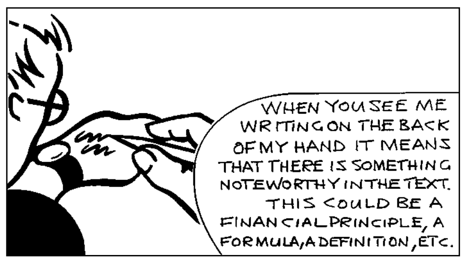
Paul Wilmott needs no introduction to those studying and working in quantitative finance. However, for completeness ... he is a researcher, trainer and consultant working in quantitative finance. He has written several best-selling finance textbooks and is a world-renowned debunker of bad mathematical models,[2] as well as being highly innovative in his own work. He is the proprietor of a financial magazine; see `www.wilmott.com`.

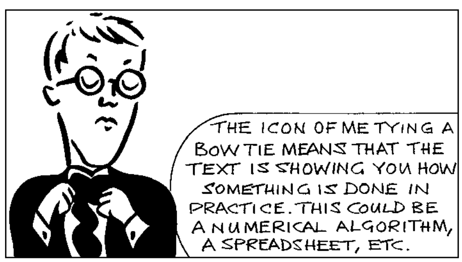
Paul lives in Bayswater, London, with 'La Estrellita.' He has two children, Oscar and Zachary, from a previous entanglement.

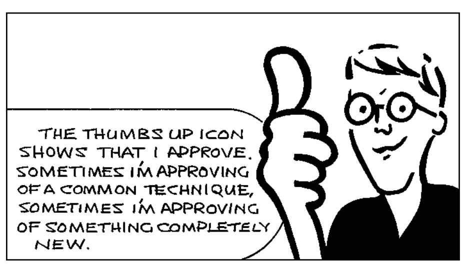
[1] Akshuly, thes are awl swots and snekes hem-hem and uterly wet and weedy. Down with swots!

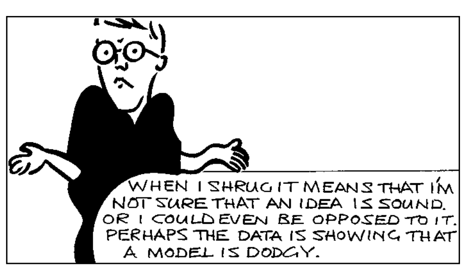
[2] And the scourge of pomposity everywhere.

You will see this icon whenever a method is implemented on the CD.









More info about the particular meaning of an icon is contained in its ‘speech box.’

Lecturer Support

Paul Wilmott has lectured on finance for many years and to many people. He has taught in the public sector, at universities, and the private sector, to financial institutions. His audiences have been every size, from the lone student to packed auditoria. His students have had every conceivable technical background, some with no math at all and others with multiple doctorates.

Within classes he has experienced the broadest range of skills...imagine trying to educate a room full of people ranging from a temp. secretary to an astrophysics Ph.D. How is it possible to keep the attention of such an extreme audience and have them all leave satisfied?

To help instructors cope with these typical demands, Paul Wilmott has put together a special information pack. This pack is available free of charge to instructors who adopt this book at **`www.wiley.co.uk/wilmott`**

- 25 lectures in portable document format (pdf) following the contents of this book. These pages can be used directly from a PC or laptop, or printed onto overhead transparencies. Each lecture is approximately one hour long. (Adobe Acrobat Reader is supplied.)
- Detailed notes on each lecture explaining how to address each topic, common pitfalls for students and instructors, which parts to emphasise and suggestions for further discussion and projects: What to say, what to ask and what to write up on a board.
- Option classification tables for testing students and for encouraging interaction.
- Term sheets for many derivative contracts with notes discussing the pricing and hedging. To be used for discussion or for examination purposes.
- Spreadsheets that can be implemented during computer classes, with notes on how to build these up from scratch.

Paul Wilmott uses this material himself for training students.

Adopting instructors should contact **`www.wiley.co.uk/wilmott`** for their free copy.

CHAPTER 1

products and markets: equities, commodities, exchange rates, forwards and futures

The aim of this Chapter...

... is to describe some of the basic financial market products and conventions, to slowly introduce some mathematics, to hint at how stocks might be modeled using mathematics, and to explain the important financial concept of 'no free lunch.' By the end of the chapter you will be eager to get to grips with more complex products and to start doing some proper modeling.

In this Chapter...

- an introduction to equities, commodities, currencies and indices
- the time value of money
- fixed and floating interest rates
- futures and forwards
- no arbitrage, one of the main building blocks of finance theory

1.1 INTRODUCTION

This first chapter is a very gentle introduction to the subject of finance, and is mainly just a collection of definitions and specifications concerning the financial markets in general. There is little technical material here, and the one technical issue, the 'time value of money,' is extremely simple. I will give the first example of 'no arbitrage.' This is important, being one part of the foundation of derivatives theory. Whether you read this chapter thoroughly or just skim it will depend on your background.

1.2 EQUITIES

The most basic of financial instruments is the **equity**, **stock** or **share**. This is the ownership of a small piece of a company. If you have a bright idea for a new product or service then you could raise capital to realize this idea by selling off future profits in the form of a stake in your new company. The investors may be friends, your Aunt Joan, a bank, or a venture capitalist. The investor in the company gives you some cash, and in return you give him a contract stating how much of the company he owns. The **shareholders** who own the company between them then have some say in the running of the business, and technically the directors of the company are meant to act in the best interests of the shareholders. Once your business is up and running, you could raise further capital for expansion by issuing new shares.

This is how small businesses begin. Once the small business has become a large business, your Aunt Joan may not have enough money hidden under the mattress to invest in the next expansion. At this point shares in the company may be sold to a wider audience or even the general public. The investors in the business may have no link with the founders. The final point in the growth of the company is with the quotation of shares on a regulated stock exchange so that shares can be bought and sold freely, and capital can be raised efficiently and at the lowest cost.

Figures 1.1 and 1.2 show screens from Bloomberg giving details of Microsoft stock, including price, high and low, names of key personnel, weighting in various indices etc. There is much, much more info available on Bloomberg for this and all other stocks. We'll be seeing many Bloomberg screens throughout this book.

In Figure 1.3 I show an excerpt from *The Wall Street Journal Europe* of 5th January 2000. This shows a small selection of the many stocks traded on the New York Stock Exchange. The listed information includes, from left to right, highest stock price in previous 52 weeks, lowest price in previous 52 weeks, stock name, dividend payment, dividend as percentage of stock price, PE ratio, volume traded (in thousands), highs and lows for the day, closing price and change in price since the previous day's close. The **PE** or **price-to-earnings ratio** is the ratio of the stock price to the earnings of the company per share. High PE ratio means that investors believe that the company has good growth prospects. At least, that's the theory.

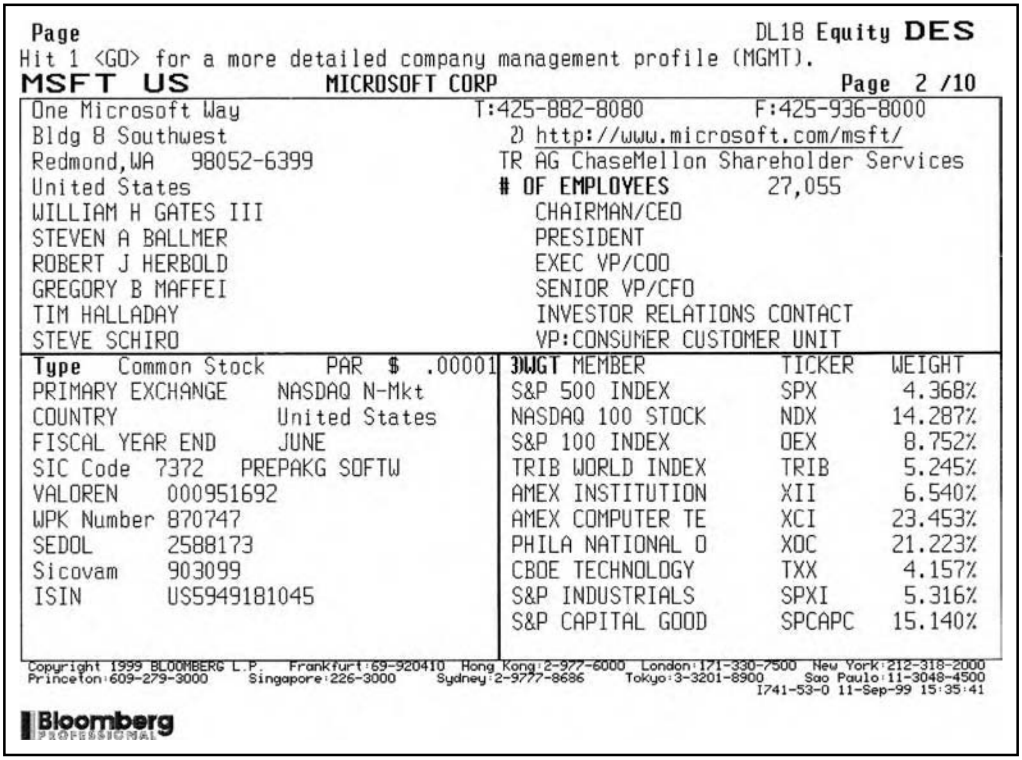
The behavior of the quoted prices of stocks is far from being predictable. In Figure 1.4 I show the Dow Jones Industrial Average over the period August 1964 to February 1999. In Figure 1.5 is a time series of the Glaxo–Wellcome share price, as produced by Bloomberg.

If we could predict the behavior of stock prices in the future then we could become very rich. Although many people have claimed to be able to predict prices with varying degrees

Figure 1.1 Details of Microsoft stock. Source: Bloomberg L.P.

of accuracy, no one has yet made a completely convincing case. In this book I am going to take the point of view that prices have a large element of randomness. This does *not* mean that we cannot model stock prices, but it does mean that the modeling must be done in a probabilistic sense. No doubt the reality of the situation lies somewhere between complete predictability and perfect randomness, not least because there have been many cases of market manipulation where large trades have moved stock prices in a direction that was favorable to the person doing the moving. Having said that, I will digress slightly in Chapter 3 where I describe some of the popular methods for supposedly predicting future stock prices.

To whet your appetite for the mathematical modeling later, I want to show you a simple way to simulate a random walk that looks something like a stock price. One of the simplest random processes is the tossing of a coin. I am going to use ideas related to coin tossing as a model for the behavior of a stock price. As a simple experiment start with the number 100 which you should think of as the price of your stock, and toss a coin. If you throw a head multiply the number by 1.01, if you throw a tail multiply by 0.99. After one toss your number will be either 99 or 101. Toss again. If you get a head multiply your *new* number by 1.01 or by 0.99 if you throw a tail. You will now have either $1.01^2 \times 100$, $1.01 \times 0.99 \times 100 = 0.99 \times 1.01 \times 100$ or $0.99^2 \times 100$. Continue this process and plot your value on a graph each time you throw the coin. Results of one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MSFT US MICROSOFT CORP Page 2 /10

One Microsoft Way	T:425-882-8080	F:425-936-8000	
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Redmond,WA 98052-6399	TR AG ChaseMellon Shareholder Services		
United States	# OF EMPLOYEES	27,055	
WILLIAM H GATES III	CHAIRMAN/CEO		
STEVEN A BALLMER	PRESIDENT		
ROBERT J HERBOLD	EXEC VP/COO		
GREGORY B MAFFEI	SENIOR VP/CFO		
TIM HALLADAY	INVESTOR RELATIONS CONTACT		
STEVE SCHIRO	VP:CONSUMER CUSTOMER UNIT		

Type	Common Stock PAR $ .00001	3)WGT MEMBER	TICKER	WEIGHT
PRIMARY EXCHANGE	NASDAQ N-Mkt	S&P 500 INDEX	SPX	4.368%
COUNTRY	United States	NASDAQ 100 STOCK	NDX	14.287%
FISCAL YEAR END	JUNE	S&P 100 INDEX	OEX	8.752%
SIC Code	7372 PREPAKG SOFTW	TRIB WORLD INDEX	TRIB	5.245%
VALOREN	000951692	AMEX INSTITUTION	XII	6.540%
WPK Number	870747	AMEX COMPUTER TE	XCI	23.453%
SEDOL	2588173	PHILA NATIONAL O	XOC	21.223%
Sicovam	903099	CBOE TECHNOLOGY	TXX	4.157%
ISIN	US5949181045	S&P INDUSTRIALS	SPXI	5.316%
		S&P CAPITAL GOOD	SPCAPC	15.140%

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Princeton:609-279-3000 Singapore:226-3000 Sydney:2-9777-8686 Tokyo:3-3201-8900 Sao Paulo:11-3048-4500
1741-53-0 11-Sep-99 15:35:41

Bloomberg PROFESSIONAL

Figure 1.2 Details of Microsoft stock continued. Source: Bloomberg L.P.

particular experiment are shown in Figure 1.6. Instead of physically tossing a coin, the series used in this plot was generated on a spreadsheet like that in Figure 1.7. This uses the Excel spreadsheet function `RAND()` to generate a uniformly distributed random number between 0 and 1. If this number is greater than one half it counts as a 'head' otherwise a 'tail.'

Time Out...

More about coin tossing

Notice how in the above experiment I've chosen to *multiply* each 'asset price' by a factor, either 1.01 or 0.99. Why didn't I simply *add* a fixed amount, 1 or -1, say? This is a very important point in the modeling of asset prices; as the asset price gets larger so do the changes from one day to the next. It seems reasonable to model the asset price changes as being proportional to the current level of the asset, since they are still random but the magnitude of the randomness depends on the level of the asset. This will be made more precise in later chapters, where we'll see how it is important to model the return on the asset, its percentage change, rather than its absolute value.

If we use the multiplicative rule we get an approximation to what is called a **lognormal random walk**, also **geometric random walk**. If we use the additive rule we get an approximation to a **Normal** or **arithmetic random walk**.

As an experiment, using Excel try to simulate both the arithmetic and geometric random walks, and also play around with the probability of a rise in asset price; it doesn't have to be one half. What happens if you have an arithmetic random walk with a probability of rising being less than one half?

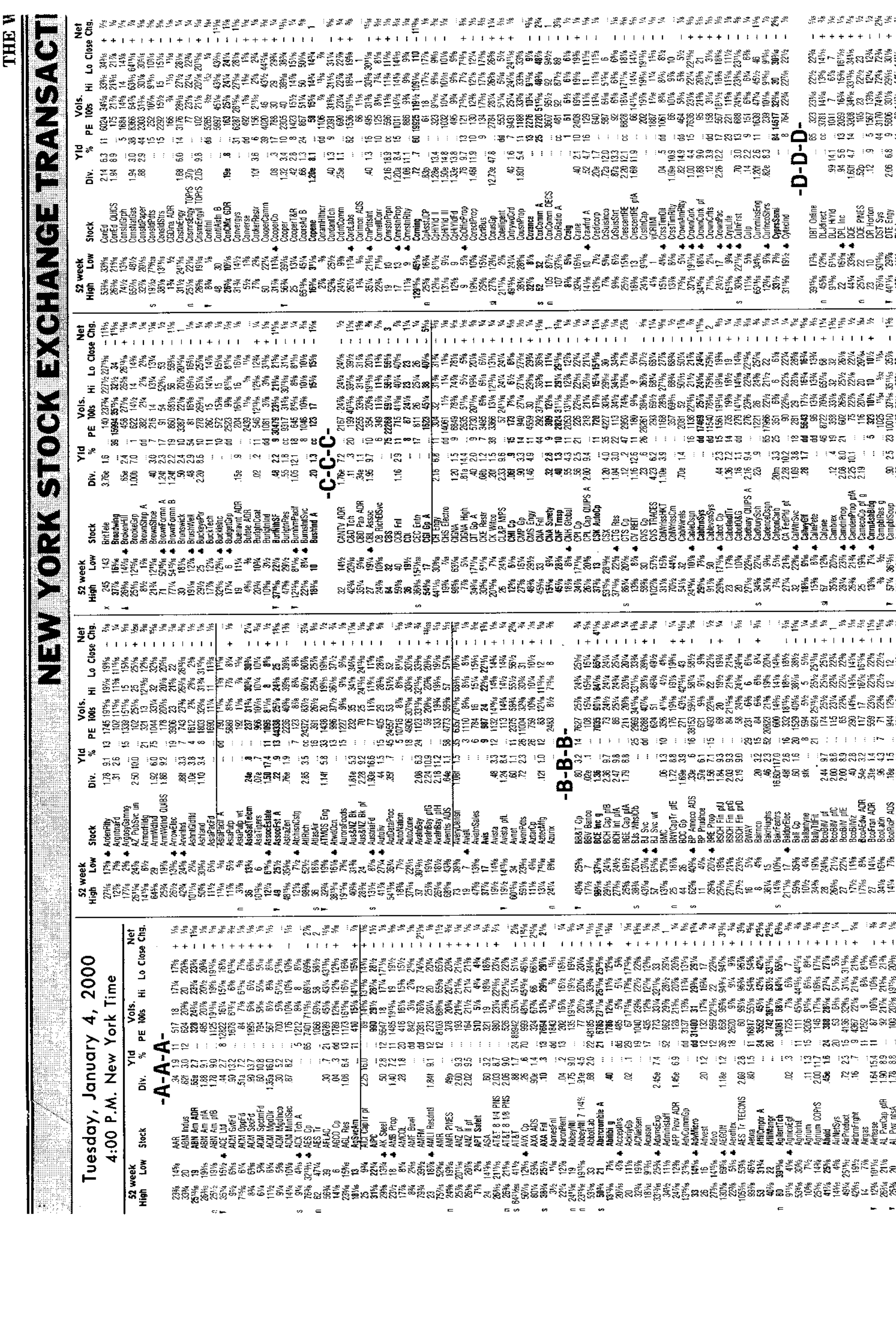
1.2.1 Dividends

The owner of the stock theoretically owns a piece of the company. This ownership can only be turned into cash if he owns so many of the stock that he can take over the company and keep all the profits for himself. This is unrealistic for most of us. To the average investor the value in holding the stock comes from the **dividends** and any growth in the stock's value. Dividends are lump sum payments, paid out every quarter or every six months, to the holder of the stock.

The amount of the dividend varies from year to year depending on the profitability of the company. As a general rule companies like to try to keep the level of dividends about the same each time. The amount of the dividend is decided by the board of directors of the company and is usually set a month or so before the dividend is actually paid.

When the stock is bought it either comes with its entitlement to the next dividend (**cum**) or not (**ex**). There is a date at around the time of the dividend payment when the stock goes from cum to ex. The original holder of the stock gets the dividend but the person who buys it obviously does not. All things being equal a stock that is cum dividend is better than one that is ex dividend. Thus at the time that the dividend is paid and the stock goes ex dividend there will be a drop in the value of the stock. The size of this drop in stock value offsets the disadvantage of not getting the dividend.

This jump in stock price is in practice more complex than I have just made out. Often capital gains due to the rise in a stock price are taxed differently from a dividend, which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Tuesday, January 4, 2000

4:00 P.M. New York Time

-A-A-A-

-B-B-B-

-C-C-C-

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Figure 1.3 The *Wall Street Journal Europe* of 5th January 2000. Reproduced by permission of Dow Jones & Company, Inc.

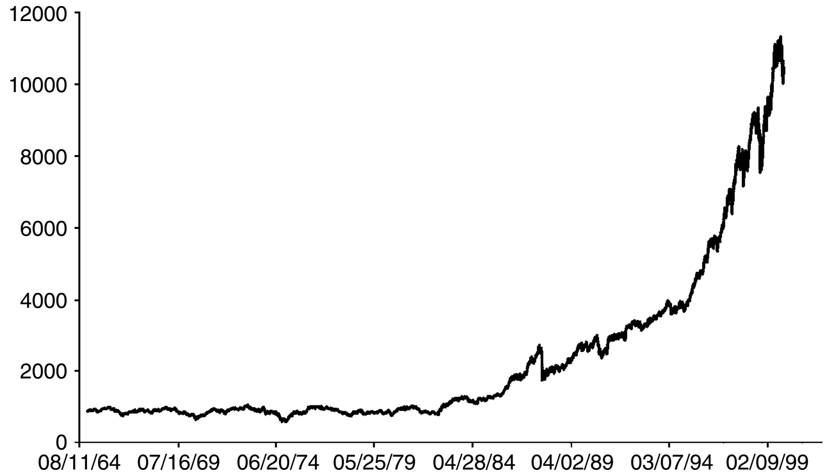


Figure 1.4 A time series of the Dow Jones Industrial Average from August 1964 to February 1999.

Figure 1.5 Glaxo–Wellcome share price (volume below). Source: Bloomberg L.P.

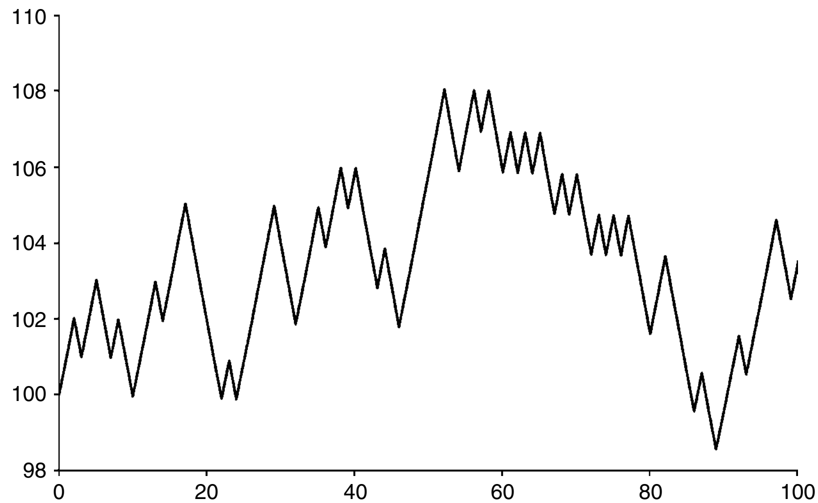


Figure 1.6 A simulation of an asset price path?

	A	B	C	D	E
1	**Initial stock price**	**100**		**Stock**	
2	**Up move**	**1.01**		100	
3	**Down move**	**0.99**		99	
4	**Probability of up**	**0.5**		98.01	
5				97.0299	
6				96.0596	
7				97.0202	
8				97.9904	
9				[illegible]	
10				97.9806	
11				98.96041	
12				99.95001	
13				98.95051	
14				97.961	
15				98.94061	
16				99.93002	
17				100.9293	
18				99.92003	
19				100.9192	
20				101.9284	
21				100.9091	
22				99.90004	
23				98.90104	
24				99.89005	
25				100.889	
26				99.88007	
27				98.88127	
28				97.89245	
29				96.91353	
30				95.94439	
31				96.90384	

=B1

=D6*IF(RAND()>1-$B$4,$B$2,$B$3)

Figure 1.7 Simple spreadsheet to simulate the coin-tossing experiment.

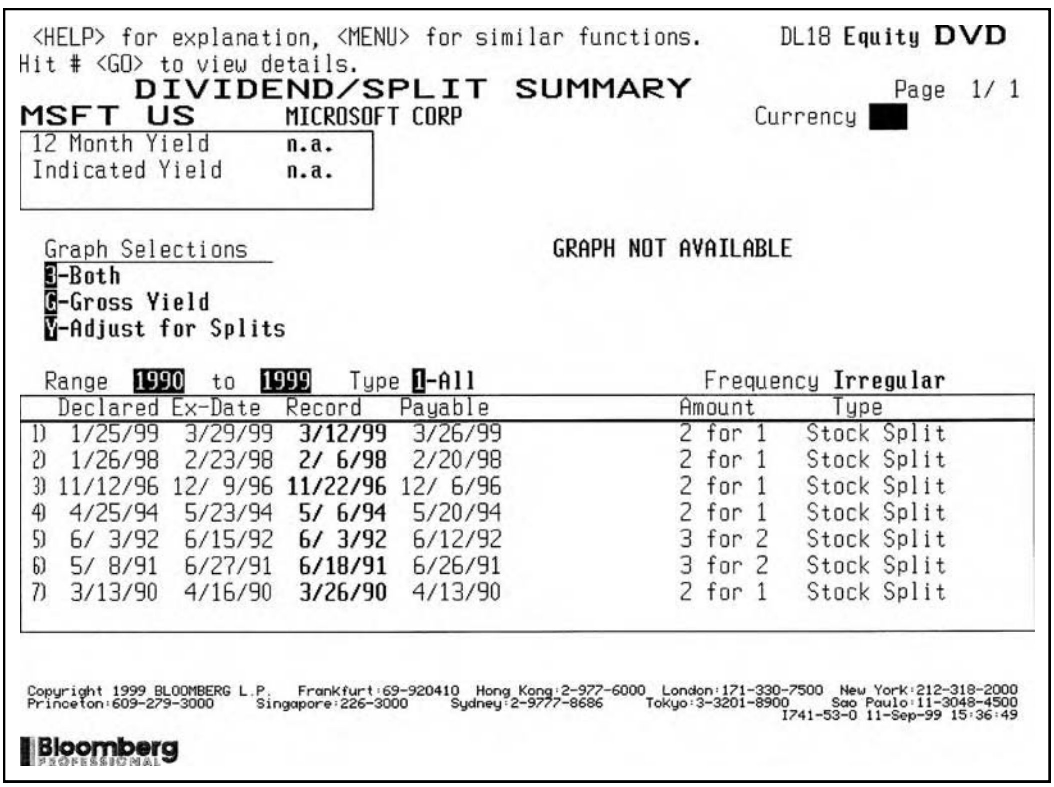
is often treated as income. Some people can make a lot of risk-free money by exploiting tax 'inconsistencies.'

1.2.2 Stock splits

Stock prices in the US are usually of the order of magnitude of $100. In the UK they are typically around £1. There is no real reason for the popularity of the number of digits, after all, if I buy a stock I want to know what percentage growth I will get, the absolute level of the stock is irrelevant to me, it just determines whether I have to buy tens or thousands of the stock to invest a given amount. Nevertheless there is some psychological element to the stock size. Every now and then a company will announce a **stock split** (Figure 1.8). For example, the company with a stock price of $900 announces a three-for-one stock split. This simply means that instead of holding one stock valued at $900, I hold three valued at $300 each.[1]

1.3 COMMODITIES

Commodities are usually raw products such as precious metals, oil, food products etc. The prices of these products are unpredictable but often show seasonal effects. Scarcity



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DIVIDEND/SPLIT SUMMARY Page 1/ 1

MSFT US MICROSOFT CORP Currency

12 Month Yield n.a.
Indicated Yield n.a.

Graph Selections
B-Both
G-Gross Yield
Y-Adjust for Splits

GRAPH NOT AVAILABLE

Range 1990 to 1999 Type 1-All Frequency Irregular

	Declared	Ex-Date	Record	Payable	Amount	Type
1)	1/25/99	3/29/99	3/12/99	3/26/99	2 for 1	Stock Split
2)	1/26/98	2/23/98	2/ 6/98	2/20/98	2 for 1	Stock Split
3)	11/12/96	12/ 9/96	11/22/96	12/ 6/96	2 for 1	Stock Split
4)	4/25/94	5/23/94	5/ 6/94	5/20/94	2 for 1	Stock Split
5)	6/ 3/92	6/15/92	6/ 3/92	6/12/92	3 for 2	Stock Split
6)	5/ 8/91	6/27/91	6/18/91	6/26/91	3 for 2	Stock Split
7)	3/13/90	4/16/90	3/26/90	4/13/90	2 for 1	Stock Split

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Figure 1.8 Stock split info for Microsoft. Source: Bloomberg L.P.

[1] In the UK this would be called a two-for-one split.

of the product results in higher prices. Commodities are usually traded by people who have no need of the raw material. For example they may just be speculating on the direction of gold without wanting to stockpile it or make jewelry. Most trading is done on the futures market, making deals to buy or sell the commodity at some time in the future. The deal is then closed out before the commodity is due to be delivered. Futures contracts are discussed below.

Figure 1.9 shows a time series of the price of pulp, used in paper manufacture.

1.4 CURRENCIES

Another financial quantity we shall discuss is the **exchange rate**, the rate at which one currency can be exchanged for another. This is the world of **foreign exchange**, or **Forex** or **FX** for short. Some currencies are pegged to one another, and others are allowed to float freely. Whatever the exchange rates from one currency to another, there must be consistency throughout. If it is possible to exchange dollars for pounds and then the pounds for yen, this implies a relationship between the dollar/pound, pound/yen and dollar/yen exchange rates. If this relationship moves out of line it is possible to make **arbitrage profits** by exploiting the mispricing.

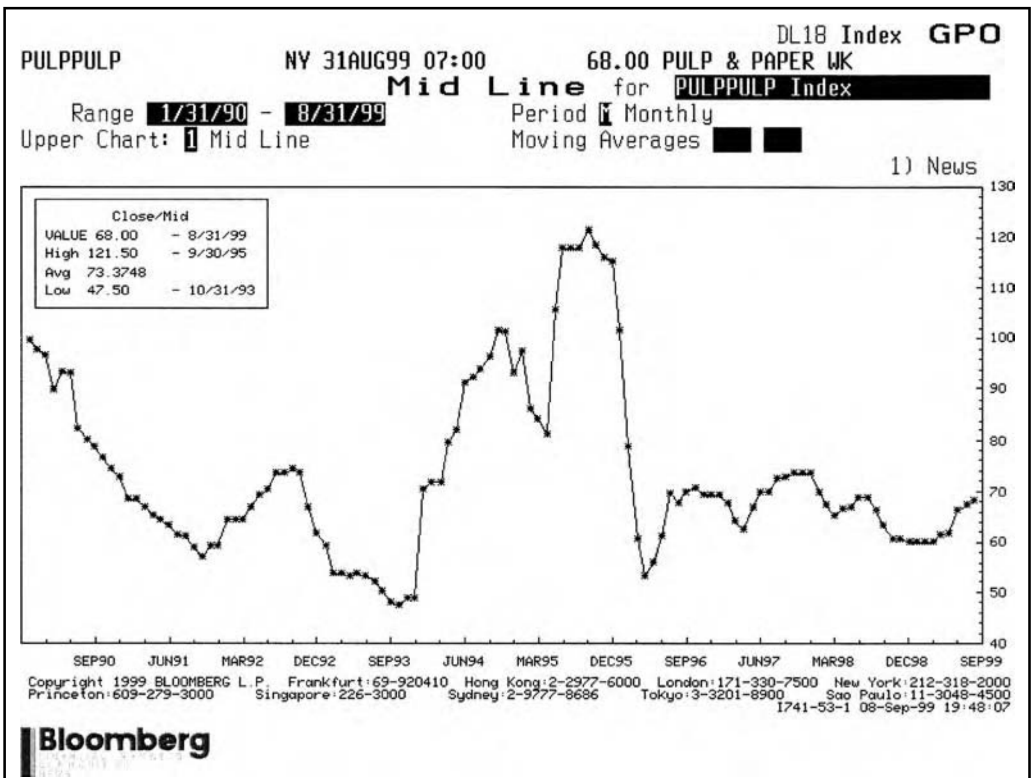


Figure 1.9 Pulp price. Source: Bloomberg L.P.

CURRENCY TRADING

Tuesday, January 4, 2000

DOLLAR EXCHANGE RATES

The New York foreign exchange mid-range rates below apply to trading among banks in amounts of $1 million and more, as quoted at 4 p.m. Eastern time by Reuters and other sources. Retail transactions provide fewer units of foreign currency per dollar. Rates for the 11 Euro currency countries are derived from the latest dollar-euro rate using the exchange ratios set 1/1/99.

	U.S. $ equiv.		Currency per U.S. $	
Country	**Tue**	**Mon**	**Tue**	**Mon**
Argentina (Peso)	1.0001	1.0002	.9999	.9998
Australia (Dollar)	.6551	.6582	1.5266	1.5192
Austria (Schilling)	.07491	.07462	13.349	13.402
Bahrain (Dinar)	2.6525	2.6525	.3770	.3770
Belgium (Franc)	.0256	.0255	39.1325	39.2885
Brazil (Real)	.5408	.5495	1.8490	1.8200
Britain (Pound)	1.6370	1.6371	.6109	.6108
1-month forward	1.6372	1.6373	.6108	.6108
3-months forward	1.6371	1.6373	.6108	.6108
6-months forward	1.6365	1.6368	.6111	.6109
Canada (Dollar)	.6887	.6913	1.4521	1.4465
1-month forward	.6892	.6919	1.4510	1.4453
3-months forward	.6902	.6929	1.4489	1.4433
6-months forward	.6915	.6943	1.4462	1.4404
Chile (Peso) (d)	.001890	.001894	529.05	527.85
China (Renminbi)	.1208	.1208	8.2799	8.2798
Colombia (Peso)	.0005238	.0005306	1909.00	1884.50
Czech. Rep. (Koruna)				
Commercial rate	.02839	.02814	35.226	35.541
Denmark (Krone)	.1385	.1379	7.2186	7.2491
Ecuador (Sucre)				
Floating rate	.00004202	.00004598	23800.00	21750.00
Finland (Markka)	.1734	.1727	5.7678	5.7908
France (Franc)	.1572	.1565	6.3633	6.3887
1-month forward	.1575	.1569	6.3488	6.3736
3-months forward	.1582	.1576	6.3205	6.3460
6-months forward	.1592	.1586	6.2800	6.3054
Germany (Mark)	.5273	.5250	1.8966	1.9049
1-month forward	.5285	.5262	1.8922	1.9004
3-months forward	.5308	.5285	1.8838	1.8921
6-months forward	.5343	.5319	1.8717	1.8800
Greece (Drachma)	.003120	.003113	320.52	321.26
Hong Kong (Dollar)	.1286	.1286	7.7773	7.7763
Hungary (Forint)	.004044	.004034	247.30	247.90
India (Rupee)	.02300	.02302	43.480	43.435
Indonesia (Rupiah)	.0001396	.0001417	7165.00	7055.00
Ireland (Punt)	1.3094	1.3040	.7637	.7669
Israel (Shekel)	.2413	.2436	4.1436	4.1053
Italy (Lira)	.0005324	.0005303	1878.33	1885.83
Japan (Yen)	.009681	.009843	103.29	101.60
1-month forward	.009730	.009893	102.77	101.08
3-months forward	.009825	.009989	101.79	100.11
6-months forward	.009974	.010141	100.26	98.61
Jordan (Dinar)	1.4085	1.4085	.7100	.7100
Kuwait (Dinar)	3.2927	3.2895	.3037	.3040
Lebanon (Pound)	.0006634	.0006634	1507.50	1507.50
Malaysia (Ringgit)	.2632	.2632	3.8001	3.8000
Malta (Lira)	2.4691	2.4643	.4050	.4058
Mexico (Peso)				
Floating rate	.1045	.1063	9.5700	9.4050
Netherland (Guilder)	.4678	.4659	2.1378	2.1463
New Zealand (Dollar)	.5209	.5255	1.9198	1.9029
Norway (Krone)	.1260	.1266	7.9388	7.8969
Pakistan (Rupee)	.01929	.01927	51.850	51.890
Peru (new Sol)	.2841	.2840	3.5200	3.5215
Philippines (Peso)	.02509	.02503	39.850	39.950
Poland (Zloty)	.2427	.2418	4.1195	4.1350
Portugal (Escudo)	.005142	.005121	194.49	195.26
Russia (Ruble) (a)	.03632	.03643	27.530	27.450
Saudi Arabia (Riyal)	.2666	.2666	3.7510	3.7508
Singapore (Dollar)	.6039	.6038	1.6560	1.6562
Slovak Rep. (Koruna)	.02434	.02427	41.077	41.202
South Africa (Rand)	.1643	.1632	6.0875	6.1288
South Korea (Won)	.0008909	.0008869	1122.50	1127.50
Spain (Peseta)	.006196	.006171	161.41	162.05
Sweden (Krona)	.1195	.1195	8.3671	8.3668
Switzerland (Franc)	.6429	.6391	1.5555	1.5647
1-month forward	.6454	.6416	1.5494	1.5585
3-months forward	.6498	.6459	1.5389	1.5483
6-months forward	.6567	.6527	1.5228	1.5321
Taiwan (Dollar)	.03276	.03188	30.525	31.365
Thailand (Baht)	.02683	.02702	37.265	37.015
Turkey (Lira)	.00000186	.00000185	536260.00	539255.00
United Arab (Dirham)	.2722	.2723	3.6731	3.6729
Uruguay (New Peso)				
Financial	.08607	.08608	11.619	11.618
Venezuela (Bolivar)	.001538	.001540	650.00	649.25
SDR	1.3820	1.3761	.7236	.7267
Euro	1.0313	1.0268	.9696	.9739

Special Drawing Rights (SDR) are based on exchange rates for the U.S., German, British, French , and Japanese currencies. Source: International Monetary Fund.
a-Russian Central Bank rate. Trading band lowered on 8/17/98. b-Government rate. d-Floating rate; trading band suspended on 9/2/99.
The 3-month and 6-month forward rates for France, Germany, Japan and Switzerland appearing in the Foreign Exchange column were incorrectly calculated for the period beginning with August 13 and ending with October 7. Corrected data is available from Readers' Reference Service (413) 592-3600.

PHILADELPHIA EXCHANGE OPTIONS

Euro					102.33
62,500 Euro-cents per unit.					
102	Jan	2	1.70	...	...
102	Feb	31	2.42	...	0.01
102	Mar	69	2.92	...	...
104	Feb	10	1.41	...	...
104	Mar	153	1.95	...	0.01
106	Jan	5	0.12	...	...
106	Feb	10	0.80	...	0.01
106	Mar	2	1.15	...	0.01
110	Mar	29	0.40	...	0.01
Australian Dollar					65.97
50,000 Australian Dollars-European Style.					
66	Mar	...	0.01	20	1.11
British Pound					163.46
31,250 Brit. Pounds-European Style.					
162	Feb	...	0.01	20	0.83
166	Feb	20	1.00	...	0.01
Euro					102.33
62,500 Euro-European style					
102	Jan	20	1.72	40	1.13
62,500 Euro-European style.					
102	Jan	10	1.26	...	...
102	Mar	...	0.01	12	1.41
106	Jan	8	0.25	...	0.01
Japanese Yen					98.52
6,250,000J.Yen-100ths of a cent per unit.					
95	Jan	...	0.01	3	0.13
96	Jan	...	0.01	2	0.38
98	Mar	...	0.01	2	2.26
Swiss Franc					63.72
62,500 Swiss Francs-European Style.					
64	Jan	16	0.86		
65	Mar	...	0.01	16	1.42
62,500 Swiss Francs-cents per unit.					
62	Jan	5	2.31	...	0.01
62	Mar	...	0.01	5	0.38
63	Feb	5	1.95	5	0.60
63	Mar	10	2.40	...	...
64	Feb	...	0.01	1	0.82
64	Mar	124	1.67	...	...
66	Feb	20	0.55	2	1.98

Call Vol ...... 1,586 Open Int ... 12,719
Put Vol .... 277 Open Int ... 9,240

r - None of these particular options were traded on this day;
s - No option in this particular month at this strike price

OTHER EURO VALUES

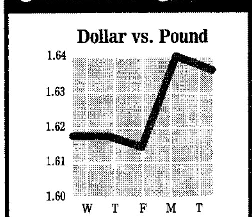
Currency	Tue.	Prev.	Currency	Tue.	Prev.
Dan Kr.	7.4429	7.4404	Swed Kr.	8.6215	8.5520
Nor Kr.	8.1500	8.0620	Slov Tolar	198.80	198.89
Aust Dollar	1.5677	1.5346	Nz Dollar	1.9745	1.9331
Czech Crown	36.270	36.063	Est Kroon	15.647	15.647
Gr Drachma	330.40	329.85	Hun Forint	254.52	254.53
Cypr Pound	0.5775	0.5767			

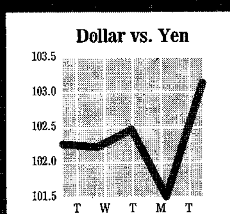
TRADE WEIGHTED VALUES

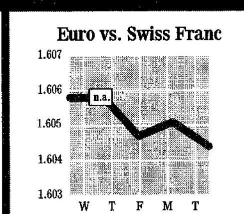
J.P. Morgan calculates the daily exchange rate indices based upon a 1990 average equalling 100. Depreciation or appreciation of each currency is against a basket of 18 other currencies, weighted by each country's 1990 bilateral manufactures trade pattern.

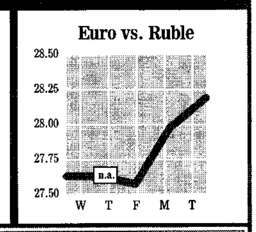
Currency	Tue.	Prev.	Currency	Tue.	Prev.
Dollar	105.4	105.4	Sterling	106.1	106.4
Sw Franc	103.6	103.1	Can Dlr	80.2	80.6
Yen	155.1	158.2	Nor Krone	92.9	93.6

CURRENCY CROSSRATES









	Dollar	Pound	SF	Peseta	Ruble	Escudo	Zloty	DFL	Yen	Lira	Punt	DM	FF	Markka	Euro	CDN	BF	ATS
Austria	13.364	21.871	8.5769	.08270	.48813	.06864	3.2454	6.2442	.12957	.00711	17.472	7.0355	2.0977	2.3143	13.760	9.2073	.34111	---
Belgium	39.178	64.118	25.144	.24245	1.4310	.20121	9.5141	18.305	.37985	.02083	51.221	20.625	6.1498	6.7847	40.340	26.992	---	2.9316
Canada	1.4515	2.3754	0.9315	.00898	.05302	.00745	.35248	.67817	.01407	.00077	1.8976	.76413	.22784	.25136	1.4945	---	.03705	.10861
Euro	.97120	1.5894	.62331	.00601	.03547	.00499	.23585	.45378	.00942	.00052	1.2697	.51129	.15245	.16819	---	0.6691	0.0248	0.0727
Finland	5.7745	9.4504	3.7060	.03573	.21092	.02966	1.4023	2.6981	.05599	.00307	7.5495	3.0400	.90642	---	5.9457	3.9784	.14739	.43209
France	6.3707	10.4261	4.0886	.03942	.23269	.03272	1.5471	2.9766	.06177	.00339	8.3289	3.3539	---	1.1032	6.5596	4.3891	.16261	.47670
Germany	1.8995	3.1087	1.2191	.01175	.06938	.00976	.46128	.88752	.01842	.00101	2.4834	---	.29816	.32895	1.9558	1.3087	.04848	.14214
Ireland	.76489	1.2518	.49089	.00473	.02794	.00393	.18575	.35738	.00742	.00041	---	.40268	.12006	.13246	.78756	.52697	.01952	.05723
Italy	1880.5	3077.6	1206.9	11.637	68.686	9.6581	456.67	878.64	18.232	---	2458.6	990.00	295.18	325.66	1936.3	1295.6	47.999	140.71
Japan	103.14	168.80	66.195	.63827	3.7673	.52972	25.047	48.191	---	.05485	134.85	54.299	16.190	17.862	106.20	71.061	2.6326	7.7179
Netherlands	2.1403	3.5027	1.3736	.01324	.07817	.01099	.51974	---	.02075	.00114	2.7981	1.1267	.33595	.37064	2.2037	1.4745	.05463	.16015
Poland	4.1179	6.7393	2.6428	.02548	.15041	.02115	---	1.9240	.03992	.00219	5.3837	2.1679	.64638	.71312	4.2400	2.8371	.10511	.30813
Portugal	194.71	318.66	124.96	1.2049	7.1118	---	47.283	90.975	1.8878	.10354	254.56	102.50	30.563	33.719	200.48	134.15	4.9698	14.570
Russia	27.378	44.806	17.571	.16943	---	.14061	6.6486	12.792	.26544	.01456	35.794	14.413	4.2975	4.7412	28.190	18.862	.69881	2.0486
Spain	161.59	264.46	103.71	---	5.9023	.82993	39.242	75.503	1.5667	.08593	211.27	85.072	25.365	27.984	166.39	111.33	4.1246	12.092
Switzerland	1.5582	2.5500	---	.00964	.05691	.00800	.37838	.72802	.01511	.00083	2.0371	.82029	.24458	.26983	1.6044	1.07350	.03977	.11659
U.K.	.61103	---	.39215	.00378	.02232	.00314	.14838	.28550	.00592	.00032	.79886	.32168	.09591	.10582	.62915	.42098	.01560	.04572
U.S.	---	1.6366	.64179	.00619	.03663	.00514	.24284	.46723	.00970	.00053	1.3074	.52645	.15697	.17317	1.0297	.68896	.02552	.07483

Figure 1.10 The *Wall Street Journal Europe* of 5th January 2000, currency exchange rates. Reproduced by permission of Dow Jones & Company, Inc.

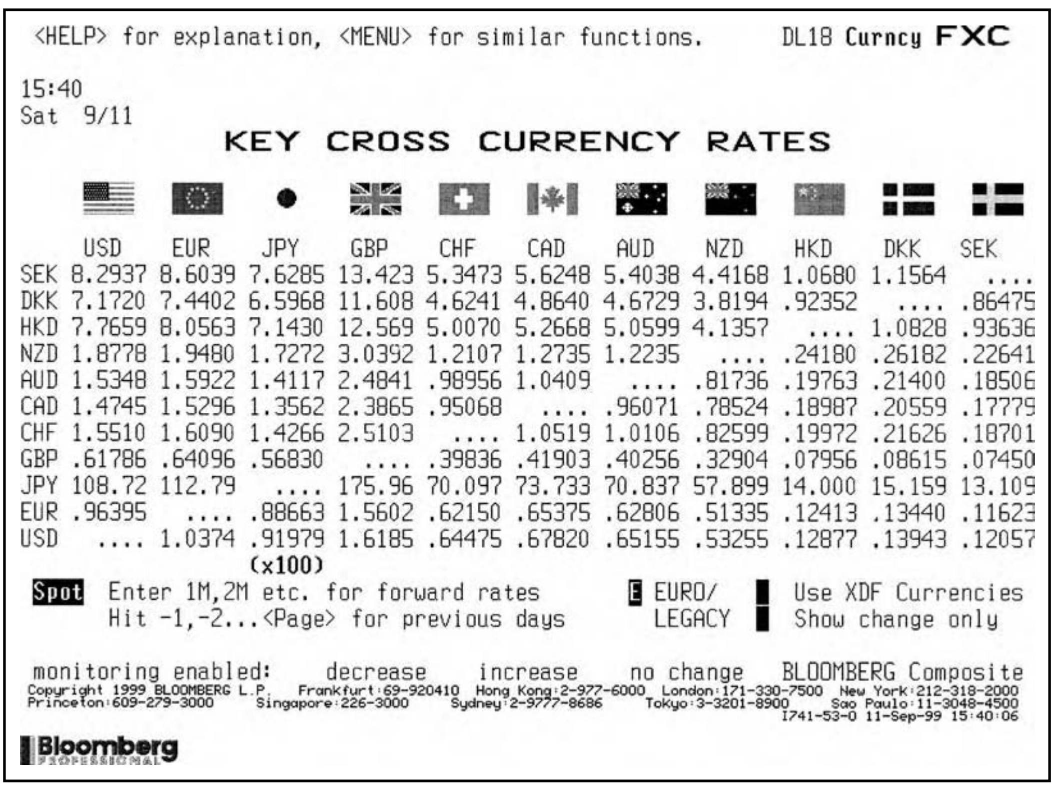
Figure 1.10 is an excerpt from *The Wall Street Journal Europe* of 5th January 2000. At the bottom of this excerpt is a matrix of exchange rates. A similar matrix is shown in Figure 1.11 from Bloomberg.

Although the fluctuation in exchange rates is unpredictable, there is a link between exchange rates and the interest rates in the two countries. If the interest rate on dollars is raised while the interest rate on pounds sterling stays fixed we would expect to see sterling depreciating against the dollar for a while. Central banks can use interest rates as a tool for manipulating exchange rates, but only to a degree.

At the start of 1999 Euroland currencies were fixed at the rates shown in Figure 1.12.

1.5 INDICES

For measuring how the stock market/economy is doing as a whole, there have been developed the stock market **indices**. A typical index is made up from the weighted sum of a selection or **basket** of representative stocks. The selection may be designed to represent the whole market, such as the Standard & Poor's 500 (S&P500) in the US or the Financial Times Stock Exchange index (FTSE100) in the UK, or a very special part of a market. In Figure 1.4 we saw the DJIA, representing major US stocks. In Figure 1.13 is shown JP Morgan's Emerging Market Bond Index. The EMBI+ is an index of emerging market debt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KEY CROSS CURRENCY RATES

	USD	EUR	JPY	GBP	CHF	CAD	AUD	NZD	HKD	DKK	SEK
SEK	8.2937	8.6039	7.6285	13.423	5.3473	5.6248	5.4038	4.4168	1.0680	1.1564	....
DKK	7.1720	7.4402	6.5968	11.608	4.6241	4.8640	4.6729	3.8194	.92352	....	.86475
HKD	7.7659	8.0563	7.1430	12.569	5.0070	5.2668	5.0599	4.1357	....	1.0828	.93636
NZD	1.8778	1.9480	1.7272	3.0392	1.2107	1.2735	1.2235	....	.24180	.26182	.22641
AUD	1.5348	1.5922	1.4117	2.4841	.98956	1.0409	....	.81736	.19763	.21400	.18506
CAD	1.4745	1.5296	1.3562	2.3865	.95068	....	.96071	.78524	.18987	.20559	.17779
CHF	1.5510	1.6090	1.4266	2.5103	....	1.0519	1.0106	.82599	.19972	.21626	.18701
GBP	.61786	.64096	.56830	....	.39836	.41903	.40256	.32904	.07956	.08615	.07450
JPY	108.72	112.79	....	175.96	70.097	73.733	70.837	57.899	14.000	15.159	13.109
EUR	.96395	....	.88663	1.5602	.62150	.65375	.62806	.51335	.12413	.13440	.11623
USD	....	1.0374	.91979	1.6185	.64475	.67820	.65155	.53255	.12877	.13943	.12057
			(x100)								

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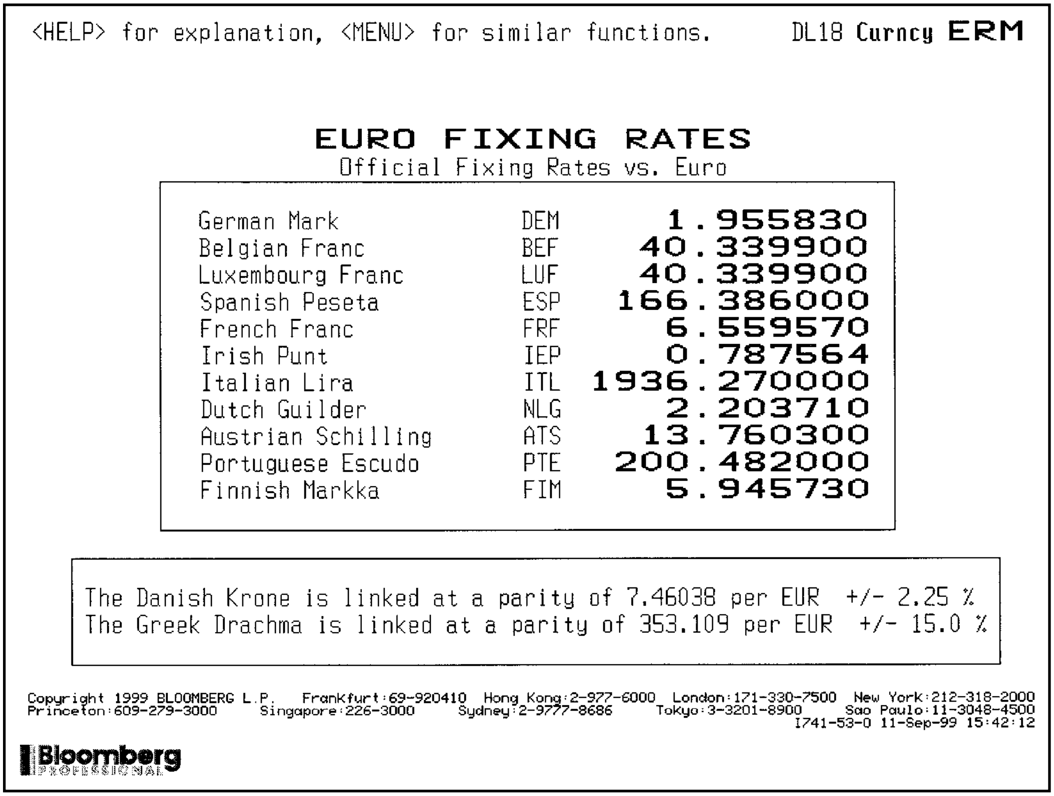
Figure 1.11 Key cross currency rates. Source: Bloomberg L.P.



<HELP> for explanation, <MENU> for similar functions. DL18 Curncy ERM

EURO FIXING RATES

Official Fixing Rates vs. Euro

German Mark	DEM	1.955830
Belgian Franc	BEF	40.339900
Luxembourg Franc	LUF	40.339900
Spanish Peseta	ESP	166.386000
French Franc	FRF	6.559570
Irish Punt	IEP	0.787564
Italian Lira	ITL	1936.270000
Dutch Guilder	NLG	2.203710
Austrian Schilling	ATS	13.760300
Portuguese Escudo	PTE	200.482000
Finnish Markka	FIM	5.945730

The Danish Krone is linked at a parity of 7.46038 per EUR +/- 2.25 %
The Greek Drachma is linked at a parity of 353.109 per EUR +/- 15.0 %

Figure 1.12 Euro fixing rates. Source: Bloomberg L.P.

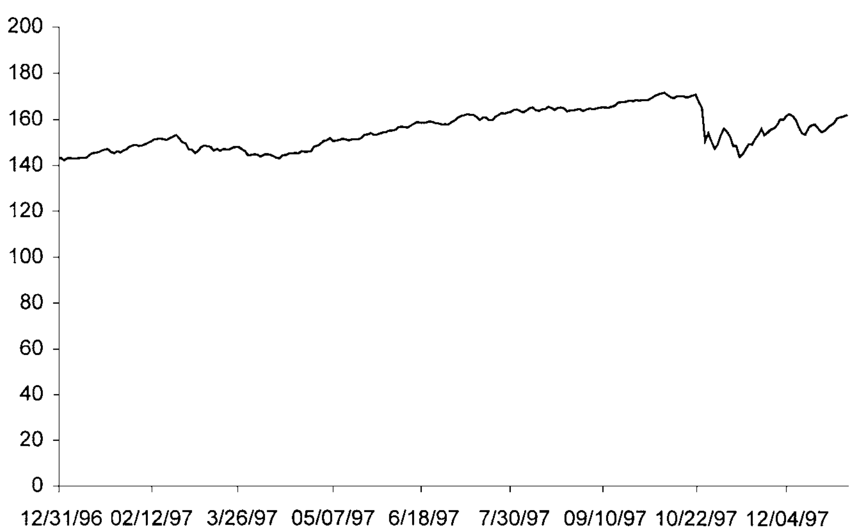


Figure 1.13 JP Morgan's EMBI+.

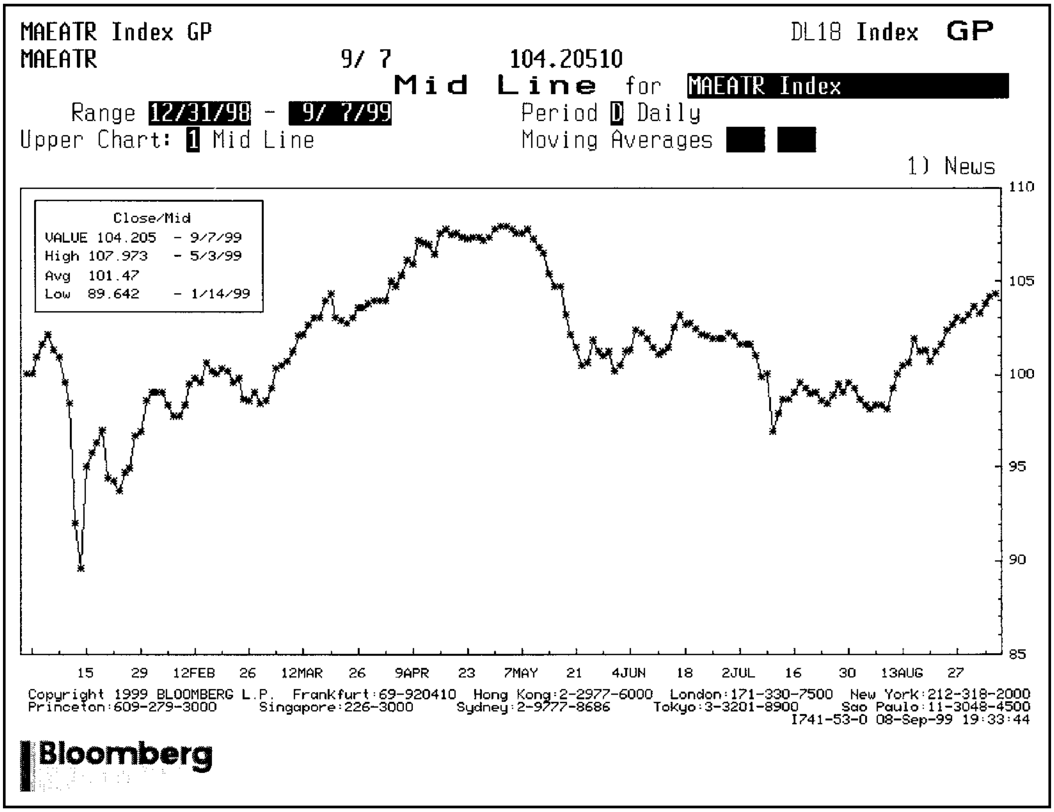
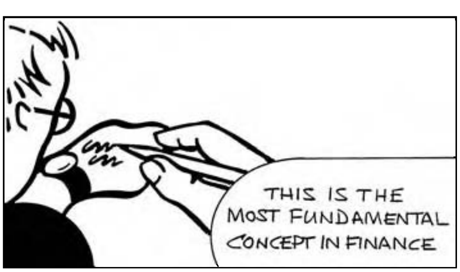


Figure 1.14 A time series of the MAE All Bond Index. Source: Bloomberg L.P.

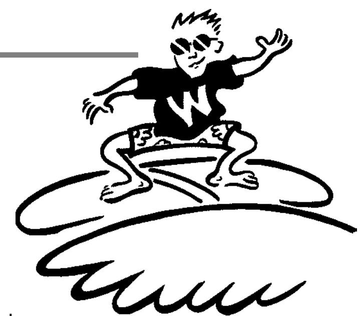
instruments, including external-currency-denominated Brady bonds, Eurobonds and US dollar local markets instruments. The main components of the index are the three major Latin American countries, Argentina, Brazil and Mexico. Bulgaria, Morocco, Nigeria, the Philippines, Poland, Russia and South Africa are also represented.

Figure 1.14 shows a time series of the MAE All Bond Index which includes peso and US dollar denominated bonds sold by the Argentine Government.

1.6 THE TIME VALUE OF MONEY



The simplest concept in finance is that of the **time value of money**; $1 today is worth more than $1 in a year's time. This is because of all the things we can do with $1 over the next year. At the very least, we can put it under the mattress and take it out in one year. But instead of putting it under the mattress we could invest it in a gold mine, or a new company. If those are too risky, then lend the money to someone who is willing to take the risks and will give you back the dollar with a little bit extra, the **interest**. That is what banks do, they borrow your money and invest it in various risky ways, but by spreading their risk over many investments they reduce their overall risk. And by borrowing money from many people they can invest in ways that the average individual cannot. The banks compete for your money by offering high interest rates. Free markets and the ability to quickly and cheaply change banks ensure that interest rates are fairly consistent from one bank to another.



Time Out...

Symbols

It had to happen sooner or later, and the first chapter is as good as anywhere. Our first mathematical symbol is nigh. Please don't be put off by the use of symbols if you feel more comfortable with numbers and concrete examples. I know that math is the one academic subject that can terrify adults, just because of poor teaching in schools. If you fall into this category, just go with the flow, concentrate on the words, the examples and the Time Outs, and before you know it...

I am going to denote interest rates by r. Although rates vary with time I am going to assume for the moment that they are constant. We can talk about several types of interest. First of all there is **simple** and **compound interest**. Simple interest is when the interest you receive is based only on the amount you initially invest, whereas compound interest is when you also get interest on your interest. Compound interest is the only case of relevance. And compound interest comes in two forms, **discretely compounded** and **continuously compounded**. Let me illustrate how they each work.

Suppose I invest \$1 in a bank at a discrete interest rate of r paid once *per annum*. At the end of one year my bank account will contain

$$1 \times (1 + r).$$

If the interest rate is 10% I will have one dollar and ten cents. After two years I will have

$$1 \times (1 + r) \times (1 + r) = (1 + r)^2,$$

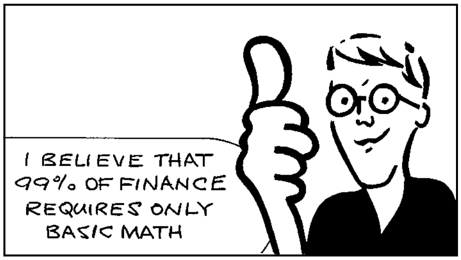
or one dollar and twenty-one cents. After n years I will have $(1 + r)^n$. That is an example of discrete compounding.

Now suppose I receive m interest payments at a rate of r/m *per annum*. After one year I will have

$$\left(1 + \frac{r}{m}\right)^m. \tag{1.1}$$

Now I am going to imagine that these interest payments come at increasingly frequent intervals, but at an increasingly smaller interest rate: I am going to take the limit $m \to \infty$. This will lead to a rate of interest that is paid continuously. Expression (1.1) becomes[2]

$$\left(1+\frac{r}{m}\right)^m = e^{m\log\left(1+\frac{r}{m}\right)} \sim e^r.$$



That is how much money I will have in the bank after one year if the interest is continuously compounded. And similarly, after a time t I will have an amount

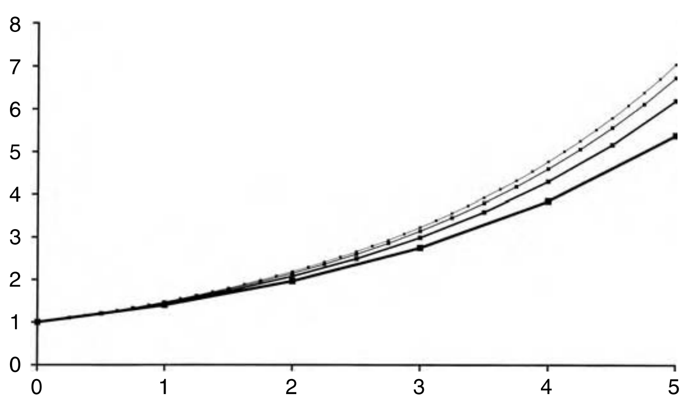
$$\left(1+\frac{r}{m}\right)^{mt} \sim e^{rt} \tag{1.2}$$

in the bank. Almost everything in this book assumes that interest is compounded continuously.

Time Out...

The math so far

Let's see m getting larger and larger in an example. I produced the next figure in Excel.



As m gets larger and larger, so the curve seems to get smoother and smoother, eventually becoming the exponential function. We'll be seeing this function a lot. In Excel the exponential function e^x (also written $\exp(x)$) is EXP( ).

[2] The symbol $\sim$, called 'tilde' is like 'approximately equal to,' but with a slightly more technical, in a math sense, meaning. The symbol $\to$ means 'tends to.'

What mathematics have we seen so far? To get to (1.2) all we needed to know about are two functions, the **exponential function** e (or exp) and the **logarithm** log, and Taylor series. Believe it or not, you can appreciate almost all finance theory by knowing these three things together with 'expectations.' I'm going to build up to the basic Black–Scholes and derivatives theory assuming that you know all four of these. Don't worry if you don't know about these things yet, in Chapter 4 I review these requisites.

En passant, what would the above figures look like if interest were simple rather than compound? Which would you prefer to receive?

Another way of deriving the result (1.2) is via a differential equation. Suppose I have an amount $M(t)$ in the bank at time t, how much does this increase in value from one day to the next? If I look at my bank account at time t and then again a short while later, time $t + dt$, the amount will have increased by

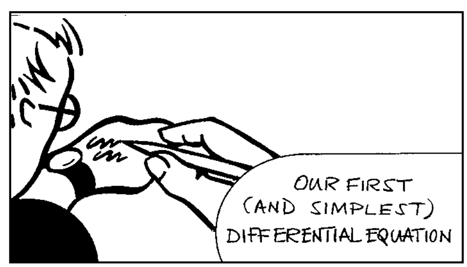
$$M(t + dt) - M(t) \approx \frac{dM}{dt} dt + \cdots,$$

where the right-hand side comes from a Taylor series expansion. But I also know that the interest I receive must be proportional to the amount I have, M, the interest rate, r, and the timestep, dt. Thus

$$\frac{dM}{dt} dt = rM(t)\, dt.$$

Dividing by dt gives the ordinary differential equation

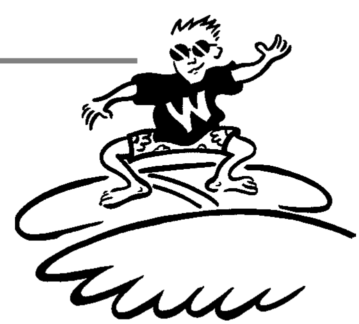
$$\boxed{\frac{dM}{dt} = rM(t)}$$



the solution of which is

$$M(t) = M(0)\, e^{rt}.$$

If the initial amount at $t = 0$ was \$1 then I get (1.2) again.



Time Out...

Differential equations

Our first **differential equation**, hang on in there, it'll become second nature soon. Whenever you see d something over d something else you know you're looking at a slope, or gradient, also known as rate of change or sensitivity. So here we've got the rate of change of money with time, i.e. rate of growth of money in the bank. You don't need to know how I solved this differential equation really. In Chapter 4 I explain all about slope, sensitivities and differential equations.

This first differential equation is an example of an **ordinary differential equation**, there is only one **independent variable** t. M is the **dependent variable**, its value depends on t. We'll also be seeing **partial differential equations** where there is more than one independent variable. And we'll also see quite a few **stochastic differential equations**. These are equations with a random term in them, used for modeling the randomness in the financial world.

For the next few chapters there will be no more mention of differential equations. Whew.

This equation relates the value of the money I have now to the value in the future. Conversely, if I know I will get one dollar at time T in the future, its value at an earlier time t is simply

$$\frac{1}{e^{r(T-t)}} = e^{-r(T-t)}.$$

I can relate cashflows in the future to their **present value** by multiplying by this factor. As an example, suppose that r is 5% i.e. $r = 0.05$, then the present value of $1,000,000 to be received in two years is

$$\$1,000,000 \times e^{-0.05\times 2} = \$904,837.$$

The present value is clearly less than the future value.

Interest rates are a very important factor determining the present value of future cashflows. For the moment I will only talk about one interest rate, and that will be constant. In later chapters I will generalize.

1.7 FIXED-INCOME SECURITIES

In lending money to a bank you may get to choose for how long you tie your money up and what kind of interest rate you receive. If you decide on a fixed-term deposit the bank will offer to lock in a fixed rate of interest for the period of the deposit, a month, six months, a year, say. The rate of interest will not necessarily be the same for each period, and generally the longer the time that the money is tied up the higher the rate of interest, although this is not always the case. Often, if you want to have immediate access to your money then you will be exposed to interest rates that will change from time to time, since interest rates are not constant.

These two types of interest payments, **fixed** and **floating**, are seen in many financial instruments. **Coupon-bearing bonds** pay out a known amount every six months or year etc. This is the **coupon** and would often be a fixed rate of interest. At the end of your fixed term you get a final coupon and the return of the **principal**, the amount on which the interest was calculated. **Interest rate swaps** are an exchange of a fixed rate of interest for a floating rate of interest. Governments and companies issue bonds as a form of borrowing. The less creditworthy the issuer, the higher the interest that they will have to pay out. Bonds are actively traded, with prices that continually fluctuate.

1.8 INFLATION-PROOF BONDS

A very recent addition to the list of bonds issued by the US government is the **index-linked bond**. These have been around in the UK since 1981, and have provided a very successful way of ensuring that income is not eroded by inflation.

In the UK inflation is measured by the **Retail Price Index** or **RPI**. This index is a measure of year-on-year inflation, using a 'basket' of goods and services including mortgage interest payments. The index is published monthly. The coupons and principal of the index-linked bonds are related to the level of the RPI. Roughly speaking, the amounts of the coupon and principal are scaled with the increase in the RPI over the period from the issue of the bond to the time of the payment. There is one slight complication in that the actual RPI level used in these calculations is set back *eight months*. Thus the base measurement is eight months before issue and the scaling of any coupon is with respect to the increase in the RPI from this base measurement to the level of the RPI eight months before the coupon is paid. One of the reasons for this complexity is that the initial estimate of the RPI is usually corrected at a later date.

Figure 1.15 shows the UK gilts prices published in *The Financial Times* of 11th January 2000. The index-linked bonds are on the right. The figures in parentheses give the base for the index, the RPI eight months prior to the issue of the gilt.

In the US the inflation index is the **Consumer Price Index** (**CPI**). A time series of this index is shown in Figure 1.16.

UK GILTS PRICES

	Notes	Yield Int	Yield Red	Price £	+ or –	52 week High	52 week Low
Shorts" (Lives up to Five Years)							
Treas $8\frac{1}{2}$pc 2000	✠	8.49	5.54	100.13	....	103.51	100.13
Conv 9pc 2000		8.96	5.56	100.47	–.01	104.36	100.40
Treas 13pc 2000		12.57	6.03	103.44xd	–.01	111.61	103.44
Treas 8pc 2000		7.89	6.35	101.42	+.01	105.82	101.41
Treas Fltg Rate 2001		–	0.24	100.16	....	100.81	100.11
Treas 10pc 2001		9.63	6.43	103.80	+.02	110.36	103.77
Conv $9\frac{1}{2}$pc 2001	✠	9.11	6.43	104.33xd	+.03	110.88	104.28
Conv $9\frac{3}{4}$pc 2001	✠	9.29	6.43	104.91	+.03	111.79	104.85
Treas 7pc 2001		6.94	6.44	100.93	+.08	106.51	100.75
Conv 10pc 2002	✠	9.34	6.57	107.04	+.11	116.41	106.82
Treas 7pc 2002		6.92	6.45	101.19	+.11	108.13	100.95
Conv $9\frac{1}{2}$pc 2002	✠	8.92	6.57	106.45	+.12	115.64	106.21
Treas $9\frac{3}{4}$pc 2002		9.07	6.57	107.54	+.13	117.29	107.28
Exch 9pc 2002	✠	8.47	6.57	106.21	+.14	115.62	105.92
Conv $9\frac{3}{4}$pc 2003	✠	8.92	6.57	109.33	+.17	120.19	108.99
Treas 8pc 2003		7.64	6.42	104.78	+.17	114.74	104.45
Treas 10pc 2003		8.98	6.46	111.34	+.19	123.52	110.99
Treas $13\frac{3}{4}$pc 2000–3	✠	13.22	6.03	104.02xd	–.01	112.93	104.02
Treas $6\frac{1}{2}$pc 2003		6.47	6.34	100.53	+.19	110.21	100.15
Treas $11\frac{1}{2}$pc 2001–4		10.89	6.49	105.62	+.04	113.56	105.58
Treas 10pc 2004	✠	8.83	6.46	113.22	+.23	126.55	112.80
Five to Fifteen Years							
Treas 5pc 2004		5.25	6.24	95.28	+.23	98.80	94.53
Funding $3\frac{1}{2}$pc 1999–4		3.89	6.06	90.03xd	+.19	98.95	89.66
Conv $9\frac{1}{2}$pc 2004	✠	8.40	6.28	113.12	+.26	126.43	112.66
Treas $6\frac{3}{4}$pc 2004		6.61	6.22	102.19	+.25	113.28	101.39
Conv $9\frac{1}{2}$pc 2005		8.30	6.23	114.46	+.27	128.16	113.68
Exch $10\frac{1}{2}$pc 2005	✠	8.74	6.23	120.16	+.30	135.46	119.38
Treas $12\frac{1}{2}$pc 2003–5	✠	10.39	6.46	120.32	+.21	135.55	119.95
Treas $8\frac{1}{2}$pc 2005		7.63	6.16	111.41	+.32	125.31	110.35
Conv $9\frac{3}{4}$pc 2006	✠	8.10	6.06	120.41	+.43	136.21	119.10
Treas $7\frac{3}{4}$pc 2006		7.13	6.14	108.69	+.38	122.40	107.42
Treas 8pc 2002–6		7.75	6.66	103.29	+.12	111.73	103.05
Treas $7\frac{1}{2}$pc 2006		6.95	6.08	107.89	+.40	121.62	106.44
Treas $11\frac{3}{4}$pc 2003–7	✠	10.31	6.57	114.02xd	+.16	126.29	113.72
Treas $8\frac{1}{2}$pc 2007		7.42	6.06	114.56xd	+.45	129.86	112.98
Treas $7\frac{1}{4}$pc 2007		6.72	5.98	107.93	+.46	122.55	106.17
Treas $13\frac{1}{2}$pc 2004–8	✠	10.67	6.22	126.56	+.24	142.61	126.15
Treas 9pc 2008		7.46	5.94	120.68	+.55	138.52	118.77
Treas 8pc 2009	✠	6.86	5.73	116.70	+.55	132.91	114.73
Treas $5\frac{3}{4}$pc 2009		5.70	5.63	100.93	+.58	114.67	99.10
Treas $6\frac{1}{4}$pc 2010		5.92	5.57	105.51	+.59	118.76	103.35
Conv 9pc Ln 2011		6.99	5.58	128.75	+.69	145.31	126.38
Treas 9pc 2012		6.85	5.50	131.48	+.75	147.74	128.45
Treas $5\frac{1}{2}$pc 2008–12		5.63	5.76	97.70	+.39	112.26	95.68
Treas 8pc 2013		6.34	5.29	126.13	+.60	139.64	122.43
Treas $7\frac{3}{4}$pc 2012–15		6.54	5.61	118.59	+.64	133.56	116.18
Over Fifteen Years							
Treas 8pc 2015		6.05	5.04	132.13	+.81	144.12	127.70
Treas $8\frac{3}{4}$pc 2017		6.03	4.90	145.04	+1.00	156.08	139.82
Exch 12pc 2013–17	✠	7.44	5.60	161.31	+.96	183.80	158.27
Treas 8pc 2021		5.57	4.73	143.69	+1.45	153.21	137.64
Treas 6pc 2028		4.84	4.50	124.06	+1.55	132.62	116.63
Undated							
Consols 4pc	✠	5.03	–	79.50	+1.30	87.19	73.95
War Loan $3\frac{1}{2}$pc		4.82	–	72.66	+1.17	79.83	67.14
Conv $3\frac{1}{2}$pc '61 Aft.	✠	4.13	–	84.66	+1.17	95.68	79.14
Treas 3pc '66 Aft.	✠	5.38	–	55.72	+.80	61.93	51.78
Consols $2\frac{1}{2}$pc	✠	4.93	–	50.67	+.79	56.93	46.78
Treas. $2\frac{1}{2}$pc		4.96	–	50.41	+.51	55.17	46.70

Index–Linked (b)	Notes	Yield (1)	Yield (2)	Price £	+ or –	52 week High	52 week Low
$2\frac{1}{2}$pc '01	(78.3)	3.72	4.35	203.28	+.09	206.02	201.90
$2\frac{1}{2}$pc '03	(78.8)	3.17	3.48	201.98	+.18	207.83	201.24
$4\frac{3}{8}$pc '04	(135.6)	2.70	2.93	128.72	+.13	134.77	127.26
2pc '06	(69.5)	1.85	2.02	235.96xd	+.29	239.80	230.16
$2\frac{1}{2}$pc '09	(78.8)	1.85	1.97	217.78	+.39	221.45	208.87
$2\frac{1}{2}$pc '11	(74.6)	1.96	2.06	229.97	+.62	235.62	219.11
$2\frac{1}{2}$pc '13	(89.2)	1.94	2.03	194.34	+.66	199.07	183.57
$2\frac{1}{2}$pc '16	(81.6)	1.91	1.99	215.37	+.67	221.10	203.03
$2\frac{1}{2}$pc '20	(83.0)	1.83	1.89	218.00	+.88	221.68	203.17
$2\frac{1}{2}$pc '24	(97.7)	1.72	1.78	192.00xd	+.88	195.05	176.32
$4\frac{1}{8}$pc '30	(135.1)	1.65	1.70	191.40xd	+.78	194.33	174.35

Prospective real redemption rate on projected inflation of (1) 5% and (2) 3%. (b) Figures in parentheses show RPI base for indexing (ie 8 months prior to issue) and have been adjusted to reflect rebasing of RPI to 100 in February 1987. Conversion factor 3.945. RPI for April 1999: 165.2 and for November1999: 166.7.

Other Fixed Interest

	Notes	Yield Int	Yield Red	Price £	+ or –	52 week High	52 week Low
Asian Dev $10\frac{1}{4}$pc 2009		8.23	6.60	$124\frac{1}{2}$	....	$140\frac{23}{32}$	$124\frac{1}{2}$
B'ham $11\frac{1}{2}$pc 2012		8.04	6.40	143	....	$158\frac{3}{4}$	142
Leeds $13\frac{1}{2}$pc 2006		9.93	6.80	136	....	152	136
Liverpool $3\frac{1}{2}$pc Irred.		5.38	5.40	65	....	73	55
LCC 3pc '20 Aft.		5.33	5.30	$56\frac{1}{4}$	....	64	50
Manchester $11\frac{1}{2}$pc 2007		8.97	6.70	$128\frac{1}{4}$	....	$151\frac{3}{4}$	$128\frac{1}{4}$
Met. Wtr. 3pc 'B'		3.33	6.40	90	....	$93\frac{1}{8}$	85
N'wide Anglia $3\frac{7}{8}$pc IL 2021		–	2.90	$196\frac{3}{4}$	....	$200\frac{1}{2}$	$174\frac{21}{32}$
$4\frac{1}{4}$pc IL 2024		–	2.90	$192\frac{1}{2}$	....	$196\frac{3}{4}$	169

● 'Tap' stock. All UK Gilts are tax-free to non-residents on application. E Auction basis. xd Ex dividend. Closing mid-prices are shown in pounds per £100 nominal of stock. Prospective real Index-Linked redemption yields are calculated by HSBC Bank plc from Gemma closing prices. ✠ Indicative price.

Figure 1.15 UK gilts prices from *The Financial Times* of 11th January 2000. Reproduced by permission of *The Financial Times*.

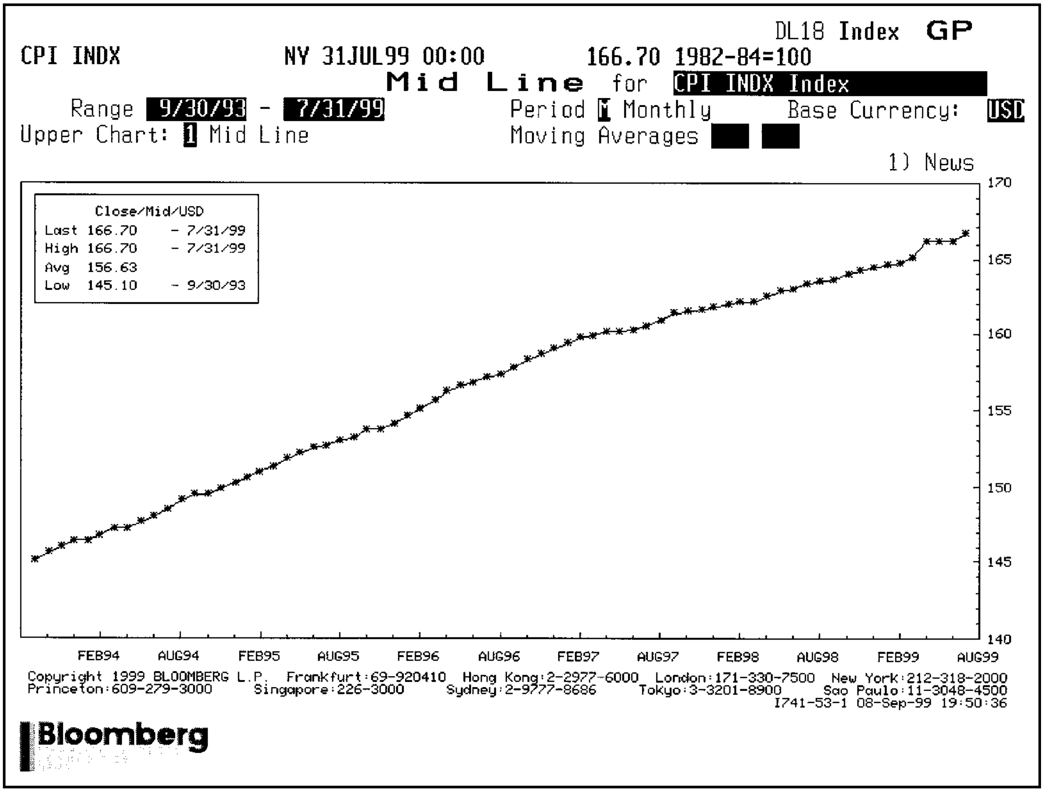
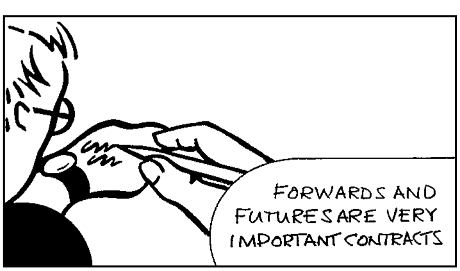


Figure 1.16 The CPI index. Source: Bloomberg L.P.

I will not pursue the modeling of inflation or index-linked bonds in this book. I would just like to say that the dynamics of the relationship between inflation and short-term interest rates is particularly interesting. Clearly the level of interest rates will affect the rate of inflation directly through mortgage repayments, but also interest rates are often used by central banks as a tool for keeping inflation down.

1.9 FORWARDS AND FUTURES



A **forward contract** is an agreement where one party promises to buy an asset from another party at some specified time in the future and at some specified price. No money changes hands until the **delivery date** or **maturity** of the contract. The terms of the contract make it an obligation to buy the asset at the delivery date, there is no choice in the matter. The asset could be a stock, a commodity or a currency.

The amount that is paid for the asset at the delivery date is called the **delivery price**. This price is set at the time that the forward contract is entered into, at an amount that gives the forward contract a value of zero initially. As we approach maturity the value of *this particular forward contract* that we hold will change in value, from initially zero to, at maturity, the difference between the underlying asset and the delivery price.

In the newspapers we will also see quoted the **forward price** for different maturities. These prices are the delivery prices for forward contracts of the quoted maturities, should we enter into such a contract *now*.

Try and distinguish between the value of a particular contract during its life, and the specification of the delivery price at initiation of the contract. It's all very subtle. You might think that the forward price is the market's view on the asset value at maturity, but this is not quite true as we'll see shortly. In theory, the market's expectation about the value of the asset at maturity of the contract is irrelevant.

A **futures contract** is very similar to a forward contract. Futures contracts are usually traded through an exchange, which standardizes the terms of the contracts. The profit or loss from the futures position is calculated every day and the change in this value is paid from one party to the other. Thus with futures contracts there is a gradual payment of funds from initiation until maturity.

Because you settle the change in value on a daily basis, the value of a futures contract at any time during its life is zero. The futures price varies from day to day, but must at maturity be the same as the asset that you are buying.

I'll show later that, provided interest rates are known in advance, forward prices and futures prices of the same maturity must be identical.

Forwards and futures have two main uses, in speculation and in hedging. If you believe that the market will rise you can benefit from this by entering into a forward or futures contract. If your market view is right then a lot of money will change hands (at maturity or every day) in your favor. That is speculation and is very risky. Hedging is the opposite, it is avoidance of risk. For example, if you are expecting to get paid in yen in six months' time, but you live in America and your expenses are all in dollars, then you could enter into a futures contract to lock in a guaranteed exchange rate for the amount of your yen income. Once this exchange rate is locked in you are no longer exposed to fluctuations in the dollar/yen exchange rate. But then you won't benefit if the yen appreciates.

1.9.1 A first example of no arbitrage

Although I won't be discussing futures and forwards very much they do provide us with our first example of the **no-arbitrage** principle. I am going to introduce some more mathematical notation now, it will be fairly consistent throughout the book. Consider a forward contract that obliges us to hand over an amount \$$F$ at time T to receive the underlying asset. Today's date is t and the price of the asset is currently \$$S(t)$, this is the **spot price**, the amount for which we could get immediate delivery of the asset. When we get to maturity we will hand over the amount \$$F$ and receive the asset, then worth \$$S(T)$. How much profit we make cannot be known until we know the value \$$S(T)$, and we can't know this until time T. From now on I am going to drop the '\$' sign from in front of monetary amounts.

We know all of F, $S(t)$, t and T, is there any relationship between them? You might think not, since the forward contract entitles us to receive an amount $S(T) - F$ at expiry and this is unknown. However, by entering into a special portfolio of trades *now* we can eliminate all randomness in the future. This is done as follows.

Enter into the forward contract. This costs us nothing up front but exposes us to the uncertainty in the value of the asset at maturity. Simultaneously sell the asset. It is called **going short** when you sell something you don't own. This is possible in many markets, but with some timing restrictions. We now have an amount $S(t)$ in cash due to the sale of the asset, a forward contract, and a short asset position. But our net position is zero. Put the cash in the bank, to receive interest.

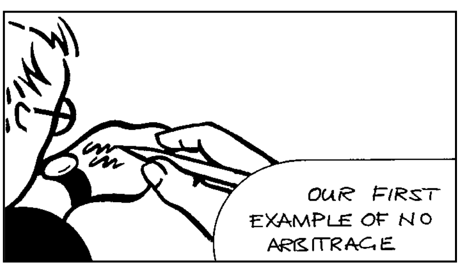
When we get to maturity we hand over the amount F and receive the asset. This cancels our short asset position regardless of the value of $S(T)$. At maturity we are left with a guaranteed $-F$ in cash as well as the bank account. The word 'guaranteed' is important because it emphasizes that it is independent of the value of the asset. The bank account contains the initial investment of an amount $S(t)$ with added interest, this has a value at maturity of

$$S(t)e^{r(T-t)}.$$

Our net position at maturity is therefore

$$S(t)e^{r(T-t)} - F.$$

Since we began with a portfolio worth zero and we end up with a predictable amount, that predictable amount should also be zero. We can conclude that



$$\boxed{F = S(t)e^{r(T-t)}. \quad (1.3)}$$

This is the relationship between the spot price and the forward price. It is a linear relationship, the forward price is proportional to the spot price.

The cashflows in this special hedged portfolio are shown in Table 1.1.

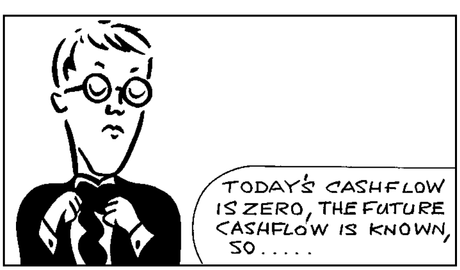
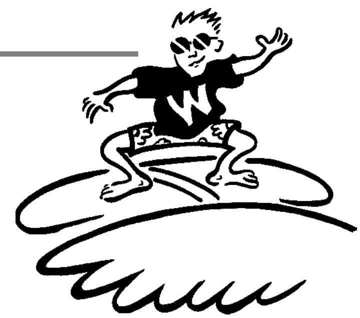


Table 1.1 Cashflows in a hedged portfolio of asset and forward.

Holding	Worth today (t)	Worth at maturity (T)
Forward	0	$S(T) - F$
−Stock	$-S(t)$	$-S(T)$
Cash	$S(t)$	$S(t)e^{r(T-t)}$
Total	0	$S(t)e^{r(T-t)} - F$



Time Out.

No arbitrage again

Example: The spot asset price S is 28.75, the one-year forward price F is 30.20 and the one-year interest rate is 4.92%. Are these numbers consistent with no arbitrage?

$$F - Se^{r(T-t)} = 30.20 - 28.75e^{0.0492\times 1} = 0.0001.$$

This is effectively zero to the number of decimal places quoted.

If we know any three out of S, F, r and $T - t$ we can find the fourth, assuming there are no arbitrage possibilities. Note that the forward price in no way depends on what the asset price is expected to do, whether it is expected to increase or decrease in value.

In Figure 1.17 is a path taken by the spot asset price and its forward price. As long as interest rates are constant, these two are related by (1.3).

If this relationship is violated then there will be an arbitrage opportunity. To see what is meant by this, imagine that F is less than $S(t)e^{r(T-t)}$. To exploit this and make a riskless arbitrage profit, enter into the deals as explained above. At maturity you will have $S(t)e^{r(T-t)}$ in the bank, a short asset and a long forward. The asset position cancels when

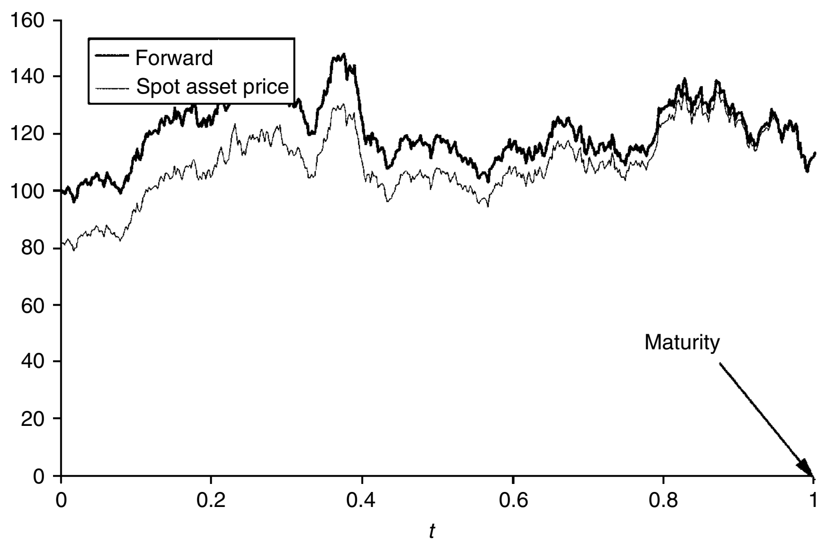


Figure 1.17 A time series of a spot asset price and its forward price.

you hand over the amount F, leaving you with a profit of $S(t)e^{r(T-t)} - F$. If F is greater than that given by (1.3) then you enter into the opposite position, going short the forward. Again you make a riskless profit. The standard economic argument then says that investors will act quickly to exploit the opportunity, and in the process prices will adjust to eliminate it.

1.10 MORE ABOUT FUTURES

Futures are usually traded through an exchange. This means that they are very liquid instruments and have lots of rules and regulations surrounding them. Here are a few observations on the nature of futures contracts.

Available assets A futures contract will specify the asset which is being invested in. This is particularly interesting when the asset is a natural commodity because of nonuniformity in the type and quality of the asset to be delivered. Most commodities come in a variety of grades. Oil, sugar, orange juice, wheat etc. futures contracts lay down rules for precisely what grade of oil, sugar, etc. may be delivered. This idea even applies in some financial futures contracts. For example, bond futures may allow a range of bonds to be delivered. Since the holder of the short position gets to choose which bond to deliver he naturally chooses the cheapest.

The contract also specifies how many of each asset must be delivered. The quantity will depend on the market.

Delivery and settlement The futures contract will specify when the asset is to be delivered. There may be some leeway in the precise delivery date. Most futures contracts are closed out before delivery, with the trader taking the opposite position before maturity. But if the position is not closed then delivery of the asset is made. When the asset is another financial contract, settlement is usually made in cash.

Margin I said above that changes in the value of futures contracts are settled each day. This is called **marking to market**. To reduce the likelihood of one party defaulting, being unable or unwilling to pay up, the exchanges insist on traders depositing a sum of money to cover changes in the value of their positions. This money is deposited in a **margin account**. As the position is marked to market daily, money is deposited or withdrawn from this margin account.

Margin comes in two forms, the **initial margin** and the **maintenance margin**. The initial margin is the amount deposited at the initiation of the contract. The total amount held as margin must stay above a prescribed maintenance margin. If it ever falls below this level then more money (or equivalent in bonds, stocks etc.) must be deposited. The levels of these margins vary from market to market.

Margin has been much neglected in the academic literature. But a poor understanding of the subject has led to a number of famous financial disasters, most notably Metallgesellschaft and Long-Term Capital Management. We'll discuss the details of these cases in Chapter 24, and we'll also be seeing how to model margin and how to margin hedge.

1.10.1 Commodity futures

Futures on commodities don't necessarily obey the no-arbitrage law that led to the asset/future price relationship explained above. This is because of the messy topic of storage. Sometimes we can only reliably find an upper bound for the futures price. Will the futures price be higher or lower than the theoretical no-storage-cost amount? Higher. The holder of the futures contract must compensate the holder of the commodity for his storage costs. This can be expressed in percentage terms by an adjustment s to the risk-free rate of interest.

But things are not quite so simple. Most people actually holding the commodity are benefiting from it in some way. If it is something consumable, such as oil, then the holder can benefit from it immediately in whatever production process they are engaged in. They are naturally reluctant to part with it on the basis of some dodgy theoretical financial calculation. This brings the futures price back down. The benefit from holding the commodity is commonly measured in terms of the **convenience yield** c:

$$F = S(t)e^{(r+s-c)(T-t)}.$$

Observe how the storage cost and the convenience yield act in opposite directions on the price. Whenever

$$F < S(t)e^{r(T-t)}$$

the market is said to be in **backwardation**. Whenever

$$F > S(t)e^{r(T-t)}$$

the market is in **contango**.

1.10.2 FX futures

There are no problems associated with storage when the asset is a currency. We need to modify the no-arb. result to allow for interest received on the foreign currency r_f. The result is

$$F = S(t)e^{(r-r_f)(T-t)}.$$

The confirmation of this is an easy exercise.

1.10.3 Index futures

Futures contracts on stock indices are settled in cash. Again, there are no storage problems, but now we have dividends to contend with. Dividends play a role similar to that of a foreign interest rate on FX futures. So

$$F = S(t)e^{(r-q)(T-t)}.$$

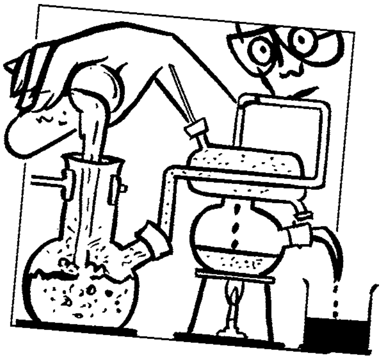
Here q is the dividend yield. This is clearly an approximation. Each stock in an index receives a dividend at discrete intervals, but can these all be approximated by one continuous dividend yield?

1.11 SUMMARY

The above descriptions of financial markets are enough for this introductory chapter. Perhaps the most important point to take away with you is the idea of no arbitrage. In the example here, relating spot prices to futures prices, we saw how we could set up a very simple portfolio which completely eliminated any dependence on the future value of the stock. When we come to value derivatives, in the way we just valued a forward, we will see that the same principle can be applied albeit in a far more sophisticated way.

FURTHER READING

- For general financial news visit www.bloomberg.com and www.reuters.com. CNN has online financial news at www.cnnfn.com. There are also online editions of *The Wall Street Journal*, www.wsj.com, *The Financial Times*, www.ft.com and *Futures and Options World*, www.fow.com.
- For more information about futures see the Chicago Board of Trade website www.cbot.com.
- Many, many financial links can be found at Wahoo!, www.io.com/~gibbonsb/wahoo.html.
- See Bloch (1995) for an empirical analysis of inflation data and a theoretical discussion of pricing index-linked bonds.
- In the main, we'll be assuming that markets are random. For insight about alternative hypotheses see Schwager (1990, 1992).
- See Brooks (1967) for how the raising of capital for a business might work in practice.
- Cox, Ingersoll & Ross (1981) discuss the relationship between forward and future prices.



CHAPTER 2
derivatives

The aim of this Chapter...

... is to describe the basic forms of option contracts, make the reader comfortable with the jargon, explain the relevant pages of financial newspapers, give a basic understanding of the purpose of options, and to expand on the 'no free lunch,' or no-arbitrage, idea. By the end of the chapter you will be familiar with the most common forms of derivatives.

In this Chapter...

- the definitions of basic derivative instruments
- option jargon
- no arbitrage and put-call parity
- how to draw payoff diagrams
- simple option strategies

2.1 INTRODUCTION

The previous chapter dealt with some of the basics of financial markets. I didn't go into any detail, just giving the barest outline and setting the scene for this chapter. Here I introduce the theme that is central to the book, the subject of options, a.k.a. derivatives or contingent claims. This chapter is nontechnical, being a description of some of the most common option contracts, and an explanation of the market-standard jargon. It is in later chapters that I start to get technical.

Options have been around for many years, but it was only on 26th April 1973 that they were first traded on an exchange. It was then that The Chicago Board Options Exchange (CBOE) first created standardized, listed options. Initially there were just calls on 16 stocks. Puts weren't even introduced until 1977. In the US options are traded on CBOE, the American Stock Exchange, the Pacific Stock Exchange and the Philadelphia Stock Exchange. Worldwide, there are over 50 exchanges on which options are traded.

2.2 OPTIONS

If you are reading the book in a linear fashion, from start to finish, then the last topics you read about will have been futures and forwards. The holder of future or forward contracts is *obliged* to trade at the maturity of the contract. Unless the position is closed before maturity the holder must take possession of the commodity, currency or whatever is the subject of the contract, regardless of whether the asset has risen or fallen. Wouldn't it be nice if we only had to take possession of the asset if it had risen?

The simplest **option** gives the holder the *right* to trade in the future at a previously agreed price but takes away the obligation. So if the stock falls, we don't have to buy it after all.

> A **call option** is the right to buy a particular asset for an agreed amount at a specified time in the future

As an example, consider the following call option on Iomega stock. It gives the holder the right to buy one of Iomega stock for an amount $25 in one month's time. Today's stock price is $24.5. The amount '25' which we can pay for the stock is called the **exercise price** or **strike price**. The date on which we must **exercise** our option, if we decide to, is called the **expiry** or **expiration date**. The stock on which the option is based is known as the **underlying asset**.

Let's consider what may happen over the next month, up until expiry. Suppose that nothing happens, that the stock price remains at $24.5. What do we do at expiry? We could exercise the option, handing over $25 to receive the stock. Would that be sensible? No, because the stock is only worth $24.5, either we wouldn't exercise the option or if we really wanted the stock we would buy it in the stock market for the $24.5. But what if the stock price rises to $29? Then we'd be laughing, we would exercise the option, paying $25 for a stock that's worth $29, a profit of $4.

We would exercise the option at expiry if the stock is above the strike and not if it is below. If we use S to mean the stock price and E the strike then at expiry the option is worth

$$\max(S - E, 0).$$

This function of the underlying asset is called the **payoff function**. The 'max' function represents the optionality.

Why would we buy such an option? Clearly, if you own a call option you want the stock to rise as much as possible. The higher the stock price the greater will be your profit. I will discuss this below, but our decision whether to buy it will depend on how much it costs; the option is valuable, there is no downside to it unlike a future. In our example the option was valued at \$1.875. Where did this number come from? The valuation of options is one of the subjects of this book, and I'll be showing you how to find this value later on.

What if you believe that the stock is going to fall, is there a contract that you can buy to benefit from the fall in a stock price?

A **put option** is the right to *sell* a particular asset for an agreed amount at a specified time in the future

The holder of a put option wants the stock price to fall so that he can sell the asset for more than it is worth. The payoff function for a put option is

$$\max(E - S, 0).$$

Now the option is only exercised if the stock falls below the strike price.

Figure 2.1 is an excerpt from *The Wall Street Journal Europe* of 5th January 2000 showing options on various stocks. The table lists closing prices of the underlying stocks and the last traded prices of the options on the stocks. To understand how to read this let us examine the prices of options on Gateway. Go to 'Gateway' in the list. The closing price on 4th January was \$65.5, and is written beneath 'Gateway' several times. Calls and puts are quoted here with strikes of \$60 and \$65, others may exist but are not mentioned in the newspaper for want of space. The available expiries are January and March. Part of the information included here is the volume of the transactions in each series, we won't worry about that but some people use option volume as a trading indicator. From the data, we can see that the January calls with a strike of \$60 were worth \$6.875. The puts with same strike and expiry were worth \$2. The March calls with a strike of \$60 were worth \$10.5 and the puts with same strike and expiry were worth \$6. Note that the higher the strike, the lower the value of the calls but the higher the value of the puts. This makes sense when you remember that the call allows you to buy the underlying for the strike, so that the lower the strike price the more this right is worth to you. The opposite is true for a put since it allows you to sell the underlying for the strike price.

There are more strikes and expiries available for options on indices, so let's now look at the Index Options section of *The Wall Street Journal Europe* 5th January 2000, this is shown in Figure 2.2.

U.S. LISTED OPTIONS QUOTATIONS

Tuesday, January 4, 2000

Volume and close for actively traded equity options with results for corresponding put or call contract as of 3 p.m. Volume figures are unofficial. Open interest is total outstanding for all exchanges and reflects previous trading day. Close when possible is shown for the underlying stock on primary market. **CB**-Chicago Board Options Exchange. **AM**-American Stock Exchange. **PB**-Philadelphia Stock Exchange. **PC**-Pacific Stock Exchange. **NY**-New York Stock Exchange. **XC**-Composite. **c**-Call. **p**-Put.

MOST ACTIVE CONTRACTS

Option	Strike		Vol.	Exch.	Last	Net Chg	3 pm Close	Open Int.
Micsft	Jan	100	p 13,675	XC	9/16	+ 1/16	115 3/8	109,572
Disney	Jan	27 1/2	13,388	XC	3 7/8	+ 1 5/16	31 11/16	95,848
AmOnline	Jan	80	11,888	XC	5 1/8	− 2 1/4	78 1/2	211,536
Micsft	Jan	90	p 10,448	XC	3/16	...	115 3/8	140,712
Intel	Jan	70	p 9,805	XC	7/16	+ 1/16	85 3/8	158,564
DellCptr	Jan	45	p 8,784	XC	1 1/8	+ 1/2	48	81,720
Disney	Feb	30	6,982	XC	2 9/16	+ 13/16	31 11/16	4,152
Intel	Jan	90	6,457	XC	2 3/16	− 7/16	85 3/8	148,840
Cisco	Jan	90	p 6,344	XC	3/4	+ 1/4	104 3/16	53,396
Bk of Am	Jan	47 1/2	6,196	XC	1 3/8	− 1 5/16	45 5/16	16,968
Compaq	Jan	30	6,161	XC	1 3/8	− 11/16	28 13/16	256,144
Intel	Jan	80	6,053	XC	7 1/2	− 1 1/8	85 3/8	169,968
Qualcom	Jan	77 1/2	p 6,049	XC	1/16	...	162 3/8	31,856
AmOnline	Jan	90	6,039	XC	2 1/8	− 1 1/4	78 1/2	211,516
Yahoo	Jan 02	135	p 6,002	XC	6 1/8	− 1	481	160
Yahoo	Jan	450	5,857	XC	66	+ 2 1/8	481	27,424
Disney	Jan	30	p 5,619	XC	9/16	− 9/16	31 11/16	32,984
AmOnline	Jan	100	5,383	XC	1 1/16	− 7/16	78 1/2	261,988
Micsft	Jan	125	5,052	XC	1 1/16	− 3/16	115 3/8	65,676
CBS Cp	Feb	60	5,010	XC	2 5/16	− 1 13/16	57	2,211

Option	Strike		Vol.	Exch.	Last	Net Chg	3 pm Close	Open Int.
DellCptr	Feb	45	p 4,772	XC	2 1/2	+ 3/4	48	46,540
Intel	Jan	85	4,571	XC	4 1/2	− 5/8	85 3/8	148,724
CMGI Inc	Jan	320	4,504	XC	28	− 11	307	16,305
Atl R	Feb	85	4,500	XC	4 3/4	− 2 1/4	80 5/8	740
DellCptr	Jan	50	4,456	XC	1 1/2	− 1 3/8	48	187,664
Cmpuwr	Jan	30	4,385	XC	6 7/8	− 1/8	36 9/16	15,858
MCI Wrld	Jan	46 5/8	p 4,332	XC	1	+ 1/4	80	49,432
Compaq	Apr	20	4,227	XC	10	− 3/4	28 13/16	52,660
Caterp	Aug	50	4,154	XC	6	+ 1	48 7/8	240
Disney	Jan	30	4,004	XC	1 13/16	+ 13/16	31 11/16	100,672
Citigrp	Jan	55	3,907	XC	1/2	− 9/16	50 11/16	87,340
LoralSp	Feb	22 1/2	3,811	XC	2 3/8	+ 3/16	22	777
Cendant	Feb	25	3,797	XC	1 7/16	− 7/16	23 3/4	64,845
Intel	Jan	95	3,706	XC	1	− 3/16	85 3/8	70,272
GMagic	Feb	5	3,687	XC	1 5/8	+ 1	5	156,147
Compaq	Feb	30	3,647	XC	2 3/8	− 5/8	28 13/16	34,532
DellCptr	Jan	55	3,610	XC	7/16	− 5/8	48	133,160
SunMicro	Apr	45	p 3,572	XC	1	+ 1/16	73	92,732
ETradeGr	Jan	30	3,564	XC	2	− 1/16	28 5/16	78,180
MerrLyn	Jan	80	3,534	XC	2 5/8	− 1 3/8	77 1/8	48,848

Option	Strike	Exp.	-Call- Vol.	3 pm	-Put- Vol.	3pm
ACTV	35	Jan	186	5 3/8	2503	2 3/16
38 5/8	45	Jan	92	1 1/4	2380	8 1/2
AT&T	45	Jan	1144	7	95	1/4
51 13/16	50	Jan	147	2 7/8	596	1 1/16
51 13/16	55	Jan	665	13/16	185	4 1/8
Abbt L	35	May	1725	3	3	3 1/4
A M D	15	Jan	14	14 3/4	500	1/16
29 5/8	25	Jan	1868	5 1/2	143	15/16
29 5/8	25	Feb	1720	6 3/4	10	1 5/8
29 5/8	30	Jan	1557	2 11/16	356	2 13/16
AdvRdio	22 1/2	Feb	485	1 11/16	...	...
Alcatl	35	Jan	...	...	600	3/8
AlterraHl	5	Feb	...	...	500	5/8
6	7 1/2	Feb	350	5/16	500	2 1/4
Amazon	65	Jan	66	20 3/4	785	1 3/4
86	80	Jan	385	10 1/2	626	6 3/8
86	85	Jan	1203	8 1/8	295	8 1/4
86	90	Jan	1331	6	316	12
86	95	Jan	685	4 1/2	10	12
86	95	Jul	21	20 1/2	500	26 1/8
86	100	Jan	1043	3	131	18 5/8
AmOnline	57 1/2	Jan	5	21 5/8	580	1/2
78 1/2	65	Jan	118	14 3/4	702	1 1/4
78 1/2	70	Jan	847	11	1810	2 5/16
78 1/2	75	Jan	2564	7 1/2	1113	4
78 1/2	75	Feb	808	11 1/8	208	7

Option	Strike	Exp.	-Call- Vol.	3 pm	-Put- Vol.	3pm
108 1/16	100	Jan	1023	10 3/8	415	2 1/2
108 1/16	105	Jan	520	6 7/8	224	4 1/4
Enron	40	Feb	1006	3 1/2	...	...
41 7/16	45	Apr	533	2 3/4	...	...
Equant	115	Jan	715	5 1/4	...	...
EricTel	60	Jan	573	6 1/4	1315	1 1/2
65 1/8	65	Feb	517	5 1/2	605	5 5/8
65 1/8	65	Apr	525	8 3/4	527	7 1/2
eToys	25	Jan	200	3 7/8	578	2 11/16
26 3/8	35	Jan	883	15/16	20	9 5/8
26 3/8	40	Jan	528	1/2	2	13 1/2
ExodsCm	90	Jan	504	8 3/8	64	9 3/4
87 5/16	95	Feb	558	10 5/8	...	...
Exxon	70	Apr	500	9 3/8	10	1 1/2
FEMSA	40	Jul	8	8 7/8	1000	4 5/8
F N M	50	Jan	837	7 3/8	225	3/8
FUnion	30	Feb	1819	2 1/2	193	1 1/2
30 11/16	35	Feb	1749	3/4	22	4 5/8
Firstar	20	Jan	2460	15/16	20	1 1/16
Gateway	50	Mar	...	...	520	2 1/2
65 1/2	60	Jan	46	6 7/8	573	2
65 1/2	60	Mar	25	10 1/2	533	6
65 1/2	65	Jan	2354	3 5/8	516	4 1/4
Gen El	135	Jan	62	11 7/8	1137	1 1/8
145 3/4	140	Jan	503	8 1/8	873	2 1/4
145 3/4	145	Jan	884	4 5/8	478	4

Option	Strike	Exp.	-Call- Vol.	3 pm	-Put- Vol.	3pm
92 1/2	95	Jan	...	...	700	5
OceanEgy	7 1/2	Feb	1010	3/4	...	...
Oracle o	30	Mar	1510	87	...	...
Oracle	70	Jan	58	39 1/2	565	1/8
108	75	Jan	53	33 1/2	1254	5/16
108	115	Mar	563	12 7/8	127	17 7/8
108	120	Jan	2574	4	81	14 1/2
108	120	Feb	504	8	7	18 1/8
PRI Auto	65	Feb	500	7 1/8	...	...
ParmTc	20	Jan	1011	1 7/8	701	1 3/4
19 3/8	22 1/2	Jan	2147	13/16	29	3 7/8
19 3/8	25	Feb	689	1 1/4	206	5 1/8
19 3/8	35	Feb	2140	5/16	...	...
PepsiCo	32 1/2	Apr	...	...	507	7/8
36 5/16	37 1/2	Jan	697	1/2	85	7/16
PetrlGeo	20	Feb	...	...	550	3 5/8
Pfizer	30	Jan	438	1 3/4	546	3/4
31	30	Feb	161	2 1/2	818	1 3/8
31	30	Jun	63	4	1238	2 9/16
31	35	Jan	733	1/4	92	4 1/4
31	35	Feb	498	5/8	6	4 3/8
31	35	Mar	497	1 1/16	1037	4 7/8
Ph Mor	25	Jan	717	9/16	1307	1 3/4
23 7/8	25	Feb	1051	1 7/16	110	2 9/16
23 7/8	25	Mar	275	2 1/16	1130	3 1/4

Figure 2.1 The *Wall Street Journal Europe* of 5th January 2000, Stock Options. Reproduced by permission of Dow Jones & Company, Inc.

INDEX OPTIONS TRADING

Tuesday, January 4, 2000

Volume, close, net change and open interest for all contracts. Volume figures are unofficial. Open interest reflects previous trading day. p-Put. c-Call. The totals for call and put volume and open interest are midday figures.

CHICAGO

CB MEXICO INDEX(MEX)

Strike			Vol.	3 pm Close	Net. Chg.	Open Int.
Jun	90	c	5	19 3/8	- 1 3/8	13
Mar	110	c	10	4 1/4	- 1 1/8	46
Call Vol.			**15**	**Open Int.**		**116**
Put Vol.			**0**	**Open Int.**		**338**

CB TECHNOLOGY(TXX)

Strike			Vol.	3 pm Close	Net. Chg.	Open Int.
Jan	650	p	10	1/8	- 6 1/8	26
Feb	820	p	10	8 1/8	+ 1 3/8	10
Feb	900	p	60	17 1/4	+ 3 1/2	30
Call Vol.			**0**	**Open Int.**		**319**
Put Vol.			**80**	**Open Int.**		**381**

DJ INDUS AVG(DJX)

Strike			Vol.	3 pm Close	Net. Chg.	Open Int.
Jan	90	p	540	1/16	...	9,881
Jan	92	p	180	1/16	...	1,546
Feb	92	p	150	3/16	...	150
Mar	92	c	1	21 1/2	- 1	1
Jun	92	p	5	1 1/2	+ 5/16	9,117
Jan	96	p	13	1/8	+ 1/16	23,136
Feb	96	p	23	9/16	+ 1/4	1,340
Mar	96	p	50	1	+ 3/16	10,585
Jun	96	p	795	1 7/8	+ 1/4	6,974
Jan	100	p	35	5/16	+ 3/16	9,841
Feb	100	p	112	1 1/16	+ 1/4	2,983
Mar	100	c	82	13 1/4	- 3 3/4	258
Mar	100	p	15	1 3/8	+ 7/16	9,170
Jun	100	p	5	2 3/4	+ 3/4	5,023
Jan	102	p	1,088	5/16	+ 1/8	4,358
Feb	102	p	365	1 1/16	+ 1/4	1,750
Mar	102	p	30	1 13/16	+ 3/8	3,147
Jan	104	c	111	9 1/8	- 7/8	2,654
Jan	104	p	449	1/2	+ 1/4	3,671
Feb	104	p	4	1 3/16	...	227
Mar	104	c	2	10 1/4	- 1 5/8	1,051
Mar	104	p	8	1 5/8	+ 3/8	2,367
Jun	104	p	5	3 1/2	+ 3/4	1,632
Jan	105	p	1,281	3/4	+ 3/8	959
Jan	106	c	10	7 1/8	- 1	377
Jan	106	p	28	3/4	+ 3/8	1,999
Mar	106	c	8	9 3/8	- 2 1/4	783
Mar	106	p	1	2 5/16	+ 5/16	757
Jun	106	c	2	12 1/2	+ 1/8	3
Jan	108	c	23	4 1/2	- 3 5/8	618
Jan	108	p	159	1 3/8	+ 9/16	2,157
Feb	108	p	25	2 1/2	+ 5/8	272
Mar	108	c	8	7 3/8	- 3	3,594
Mar	108	p	8	3 1/8	+ 9/16	4,769
Jun	108	c	12	10 1/4	- 2 3/4	311
Jan	109	c	6	3 7/8	- 3 3/8	30
Jan	109	p	43	1 1/2	+ 11/16	689
Jan	110	c	143	2 3/4	- 3 1/8	1,519
Jan	110	p	213	1 13/16	+ 13/16	5,035
Feb	110	c	13	4 5/8	- 2	63
Feb	110	p	58	3	+ 15/16	514
Mar	110	c	34	5 7/8	- 2 5/8	7,029

Strike			Vol.	3 pm Close	Net. Chg.	Open Int.
Feb	470	c	2	27 1/8	- 4	8
Feb	470	p	1	8	+ 3/8	615
Jan	480	c	15	11	- 14 3/4	19
Mar	480	p	1	15 1/2	- 1/8	2,364
Jan	485	c	2	10	- 6 7/8	9
Jan	490	c	4	9 3/8	- 1 3/4	75
Feb	490	c	86	12 3/8	- 6 3/8	219
Mar	490	c	3	23	- 2	100
Mar	490	p	3	18 1/2	+ 1/2	100
Feb	500	c	400	8 1/4	- 5	186
Feb	500	p	100	20 3/4	+ 1 1/4	195
Jan	510	c	5	2	- 2 1/4	5
Feb	510	c	10	5 3/4	- 3 1/4	21
Mar	520	c	3	6 1/2	- 3 3/4	5
Call Vol.			**533**	**Open Int.**		**8,978**
Put Vol.			**266**	**Open Int.**		**14,765**

S & P 100 INDEX(OEX)

Strike			Vol.	3 pm Close	Net. Chg.	Open Int.
Mar	540	p	33	1 1/8	+ 1/4	486
Jan	550	p	796	1/16	- 1/16	6,551
Jan	560	p	60	3/16	+ 1/16	1,498
Feb	560	p	10	5/8	...	272
Jan	580	p	226	1/4	+ 1/8	1,195
Feb	580	p	32	3/4	+ 1/8	231
Jan	600	p	400	3/16	...	3,531
Feb	600	c	1	187	- 17	1
Feb	600	p	14	1	+ 1/8	1,088
Jan	610	c	1	172	- 18	25
Jan	610	p	251	5/16	+ 1/16	1,200
Jan	620	p	95	3/8	+ 1/8	2,490
Feb	620	c	1	165	- 19	10
Feb	620	p	43	2	...	496
Mar	620	p	81	4 3/8	+ 1 1/8	712
Jan	630	p	29	3/8	+ 1/16	1,623
Jan	640	c	3	144	- 12 1/2	199
Jan	640	p	50	7/16	- 1/16	2,022
Feb	640	c	1	146	- 19	31
Feb	640	p	60	3 1/8	+ 3/4	838
Mar	640	p	41	5 1/8	+ 1 3/4	315
Apr	640	p	6	6 1/4	+ 1/8	32
Jan	650	p	36	9/16	- 3/16	4,315
Jan	660	p	90	1	+ 1/2	3,318
Feb	660	c	1	125	- 7 1/4	5
Feb	660	p	42	4 1/8	+ 1	1,083
Mar	660	c	4	132	- 18	59
Mar	660	p	90	7	+ 1	789
Apr	660	p	2	8 7/8	+ 2 1/8	4
Jan	670	c	18	99 1/2	- 31 1/2	137
Jan	670	p	413	1 1/4	+ 1/4	2,349
Feb	670	p	44	4 7/8	+ 7/8	125
Jan	680	c	706	102	- 7 3/8	1,309
Jan	680	p	244	1 9/16	+ 7/16	3,464
Feb	680	p	92	6	+ 2 1/8	223
Mar	680	p	76	10 1/4	+ 3 1/2	1,657
Jan	690	p	291	1 3/4	+ 3/8	4,583
Feb	690	c	4	91	- 14 1/4	11
Feb	690	p	171	6 1/2	+ 2 1/4	1,977
Jan	695	p	283	2 3/8	+ 7/8	1,415
Jan	700	c	124	73	- 18 7/8	6,023
Jan	700	p	968	2 1/4	+ 5/8	12,174
Feb	700	p	278	7 1/2	+ 2	1,305
Mar	700	p	7	10	+ 1/2	2,731
Apr	700	p	3	18 1/2	+ 5 1/2	98
Jan	705	p	245	2 5/8	+ 5/8	1,569
Jan	710	c	74	61 1/2	- 19 1/2	2,386
Jan	710	p	726	3 1/4	+ 1 3/8	7,712
Feb	710	p	6	9 3/8	+ 2 3/4	580
Jan	715	p	237	3 1/4	+ 1 1/4	3,720

Strike			Vol.	3 pm Close	Net. Chg.	Open Int.
Feb	1500	c	669	12	- 12 1/4	5,484
Feb	1500	p	55	78 1/2	+ 9	116
Mar	1500	c	1,001	32 1/2	- 4 1/2	13,524
Mar	1500	p	17	83	+ 8	740
Jan	1525	c	541	3/4	- 3 1/4	5,628
Jan	1525	p	203	97	+ 16	278
Feb	1525	c	181	7 3/4	- 8 1/4	1,861
Mar	1525	c	1,606	15	- 25 1/2	7,549
Mar	1525	p	500	114	+ 28	218
Jan	1550	c	789	3/8	- 5/8	13,957
Feb	1550	c	221	3 5/8	- 4 7/8	3,059
Feb	1550	p	2	111	+ 25	77
Mar	1550	c	1,397	11	- 5 1/2	9,956
Jan	1575	c	10	3/8	+ 1/8	2,102
Feb	1575	c	6	2	- 3 5/8	1,779
Mar	1575	c	209	6 3/4	- 6	619
Mar	1575	p	10	152 1/2	+ 38 1/2	11
Feb	1600	c	155	1	- 1 7/16	1,925
Mar	1600	c	502	3 3/4	- 3 1/4	9,199
Mar	1650	c	55	1 1/4	- 1 3/4	2,052
Jan	1700	p	1	267	+ 15	373
Mar	1700	c	100	1	- 1/8	1,416
Mar	1700	p	3	264	+ 22	32
Call Vol.			**36,803**	**Open Int.**		**820,987**
Put Vol.			**45,134**	**Open Int.**		**975,013**

AMERICAN

COMP TECH(XCI)

Strike			Vol.	3 pm Close	Net. Chg.	Open Int.
Jan	860	p	3	16 1/4	...	...
Jan	1120	c	22	259 3/4	+ 160 1/8	22
Jan	1170	p	3	7	+ 2 7/8	3
Call Vol.			**22**	**Open Int.**		**71**
Put Vol.			**6**	**Open Int.**		**60**

JAPAN INDEX(JPN)

Strike			Vol.	3 pm Close	Net. Chg.	Open Int.
Mar	170	c	2	25 1/4	+ 3 3/4	651
Mar	175	c	9	21	- 1	39
Jan	180	p	30	11/16	- 1/8	125
Mar	180	c	8	17 1/2	+ 3 1/2	3,390
Jan	185	c	10	9 5/8	+ 1/8	10
Jan	185	p	10	11/16	+ 1/4	91
Feb	185	p	10	3	- 3/4	10
Mar	185	c	1	14 3/8	- 1 1/4	99
Jan	190	p	35	2 1/8	+ 1/16	145
Mar	190	c	156	10 3/4	- 1 1/2	252
Jan	195	c	1	3 5/8	- 7/8	42
Feb	195	p	35	5 3/4	- 4 1/8	39
Mar	195	c	4	8 1/4	- 1/8	101
Call Vol.			**556**	**Open Int.**		**23,806**
Put Vol.			**120**	**Open Int.**		**7,561**

MS CYCLICAL(CYC)

Strike			Vol.	3 pm Close	Net. Chg.	Open Int.
Jan	520	p	125	3 1/8	...	125
Feb	520	p	2,000	7 1/2	- 1/2	5,000
Feb	560	c	2,000	22	- 4	5,000
Jan	590	c	20	4 1/2	- 7 1/2	400
Call Vol.			**2,020**	**Open Int.**		**12,676**
Put Vol.			**2,125**	**Open Int.**		**11,224**

MS HITECH 35(MSH)

Strike			Vol.	3 pm Close	Net. Chg.	Open Int.
Jan	1560	c	50	310 1/4	+ 10 1/4	39
Jan	1570	c	850	303 3/4	- 36 1/4	865
Jan	1580	c	50	291 1/4	+ 23 1/2	14
Feb	1600	p	275	45	+ 10	260
Jan	1610	p	60	14 3/4	- 8 1/8	21
Jan	1630	c	4	218 3/8	- 11 7/8	3

Figure 2.2 The *Wall Street Journal Europe* of 5th January 2000, Index Options. Reproduced by permission of Dow Jones & Company, Inc.

In Figure 2.3 are the quoted prices of the March and June DJIA calls against the strike price. Also plotted is the payoff function *if the underlying were to finish at its current value at expiry*, the current closing price of the DJIA was 10997.93.

This plot reinforces the fact that the higher the strike the lower the value of a call option. It also appears that the longer time to maturity the higher the value of the call. Is it obvious that this should be so? As the time to expiry decreases what would we see happen? As there is less and less time for the underlying to move, so the option value must converge to the payoff function.

Time Out...

Plotting

When plotting using Excel you'll find it best to use the 'XY Scatter' option. This allows you to get the correct scale on the horizontal axis without any hassle. Also, don't use the smoothing option as it can give spurious wiggles in the plots.

One of the most interesting features of calls and puts is that they have a nonlinear dependence on the underlying asset. This contrasts with futures which have a linear dependence on the underlying. This nonlinearity is very important in the pricing of options, the randomness in the underlying asset and the curvature of the option value with respect to the asset are intimately related.

Calls and puts are the two simplest forms of option. For this reason they are often referred to as **vanilla** because of the ubiquity of that flavor. There are many, many more

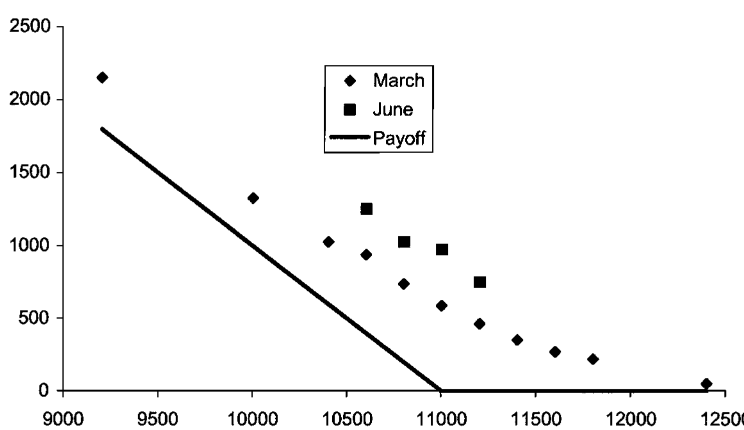
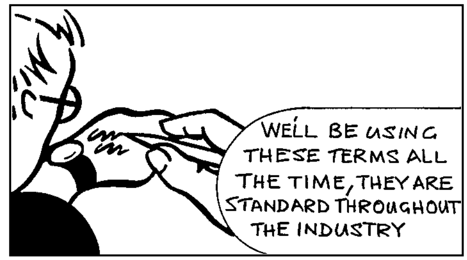


Figure 2.3 Option prices versus strike, March and June series of DJIA.

kinds of options, some of which will be described and examined later on. Other terms used to describe contracts with some dependence on a more fundamental asset are **derivatives** or **contingent claims**.

Figure 2.4 shows the prices of call options on Glaxo–Wellcome for a variety of strikes. All these options are expiring in October. The table shows many other quantities that we will be seeing later on.

2.3 DEFINITION OF COMMON TERMS



The subjects of mathematical finance and derivatives theory are filled with jargon. The jargon comes from both the mathematical world and the financial world. Generally speaking the jargon from finance is aimed at simplifying communication, and to put everyone on the same footing.[1] Here are a few loose definitions to be going on with, some you have already seen and there will be many more throughout the book.

GLXO LN GBp ↑ 1688 −13 L 5s L 1686/1689 L Trd Equity OCM
At 12:50 Vol 854,194 Op 1694 L Hi 1703 L Lo 1686 L Prev 1701
OPTION MONITOR 3 COMP Center: 1687 1 <GO> to Edit Spreadsheet

	BID m	ASK m	LAST m	1CHG m	IVBD	IVAS	BEST	DEBS	GABS	VEBS	THEO	7DEC
GLXO LN CALLS	Bid Price	Ask Price	Last Trade	1 Day Net Change	Imp Volat Bid	Imp Volat Ask	Best Price	Delta Best Price	Gamma Best Price	Vega Best Price	Theo. Value	7 Day Decay
GLXOCT99	1686.0	1689.0	1688.0	-13.0			1687					
1) 1200	489.50	504.50	509.50	unch	N.A.	69.97	504.50	.942	.0003	.674	494.09	4.6870
2) 1250	440.00	455.00	460.00	unch	N.A.	63.58	455.00	.936	.0003	.689	444.92	4.6853
3) 1300	390.50	405.50	410.50	unch	N.A.	57.36	405.50	.928	.0004	.837	396.33	4.6828
4) 1350	342.00	357.00	362.00	unch	N.A.	52.29	357.00	.915	.0005	.853	348.72	4.8888
5) 1400	294.50	309.50	314.50	unch	N.A.	48.07	309.50	.895	.0007	1.018	302.62	5.2385
6) 1450	249.00	264.00	268.50	unch	29.45	45.11	264.00	.864	.0008	1.194	258.66	5.8316
7) 1500	203.00	218.00	224.00	unch	30.67	42.27	220.00	.823	.0011	1.538	217.53	6.3538
8) 1600	125.00	137.50	136.00	-6.00	29.86	37.59	136.00	.706	.0017	2.013	146.02	7.0423
9) 1700	69.00	76.00	80.00	unch	30.95	34.02	76.00	.516	.0020	2.280	90.956	7.4785
10) 1800	32.00	38.00	40.00	unch	30.62	33.12	37.00	.319	.0019	2.005	52.713	6.2390
11) 1900	16.00	20.00	21.50	unch	32.84	35.47	20.00	.190	.0013	1.552	28.386	4.8611
12) 2000	6.00	9.00	9.00	unch	32.53	35.83	9.00	.099	.0008	1.041	14.273	2.9660
13) 2100	2.00	4.00	3.50	unch	32.32	36.52	3.50	.044	.0005	.581	6.728	1.4568
14) 2200		2.00	1.00	unch	N.A.	38.08	1.00	.015	.0002	.232	2.968	.5272
15) 2300		1.50	.50	unch	N.A.	41.58	.50	.008	.0001	.132	1.262	.2929
16) 2400		1.00	.50	unch	N.A.	43.98	.50	.007	.0001	.126	.502	.2977
17) 2500		1.00	.50	unch	N.A.	48.40	.50	.007	.0001	.101	.195	.3010

Bloomberg PROFESSIONAL

Figure 2.4 Prices for Glaxo–Wellcome calls expiring in October. Source: Bloomberg L.P.

[1] I have serious doubts about the purpose of most of the math jargon.

- **Premium**: The amount paid for the contract initially. How to find this value is the subject of much of this book.
- **Underlying** (**asset**): The financial instrument on which the option value depends. Stocks, commodities, currencies and indices are going to be denoted by S. The option payoff is defined as some function of the underlying asset at expiry.
- **Strike** (**price**) or **exercise price**: The amount for which the underlying can be bought (call) or sold (put). This will be denoted by E. This definition only really applies to the simple calls and puts. We will see more complicated contracts in later chapters and the definition of strike or exercise price will be extended.
- **Expiration** (**date**) or **expiry** (**date**): Date on which the option can be exercised or date on which the option ceases to exist or give the holder any rights. This will be denoted by T.
- **Intrinsic value**: The payoff that would be received if the underlying is at its current level when the option expires.
- **Time value**: Any value that the option has above its intrinsic value. The uncertainty surrounding the future value of the underlying asset means that the option value is generally different from the intrinsic value.
- **In the money**: An option with positive intrinsic value. A call option when the asset price is above the strike, a put option when the asset price is below the strike.
- **Out of the money**: An option with no intrinsic value, only time value. A call option when the asset price is below the strike, a put option when the asset price is above the strike.
- **At the money**: A call or put with a strike that is close to the current asset level.
- **Long position**: A positive amount of a quantity, or a positive exposure to a quantity.
- **Short position**: A negative amount of a quantity, or a negative exposure to a quantity. Many assets can be sold short, with some constraints on the length of time before they must be bought back.

2.4 PAYOFF DIAGRAMS

The understanding of options is helped by the visual interpretation of an option's value at expiry. We can plot the value of an option at expiry as a function of the underlying in what is known as a **payoff diagram**. At expiry the option is worth a known amount. In the case of a call option the contract is worth $\max(S - E, 0)$. This function is the bold line in Figure 2.5.

Figure 2.6 shows Bloomberg's standard option valuation screen and Figure 2.7 shows the value against the underlying and the payoff.

The payoff for a put option is $\max(E - S, 0)$, this is the bold line plotted in Figure 2.8.

Figure 2.9 shows Bloomberg's option valuation screen and Figure 2.10 shows the value against the underlying and the payoff.

These payoff diagrams are useful since they simplify the analysis of complex strategies involving more than one option.

Make a mental note of the thin lines in all of these figures. The meaning of these will be explained very shortly.

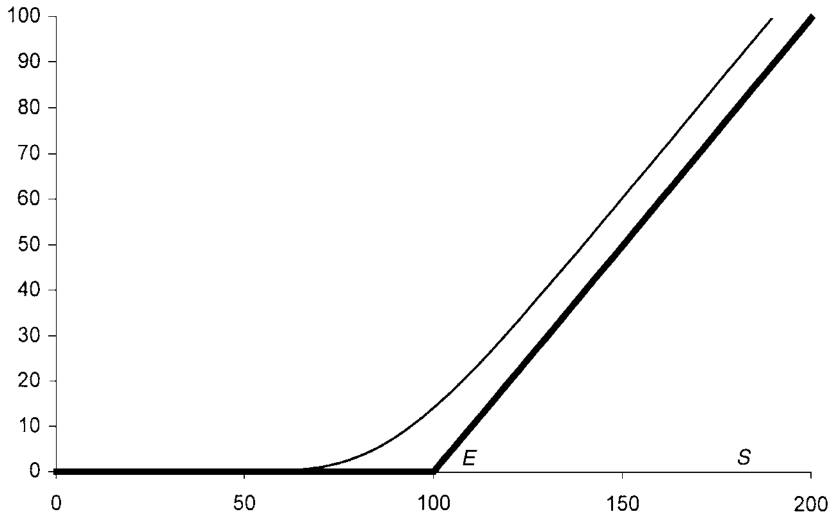


Figure 2.5 Payoff diagram for a call option.

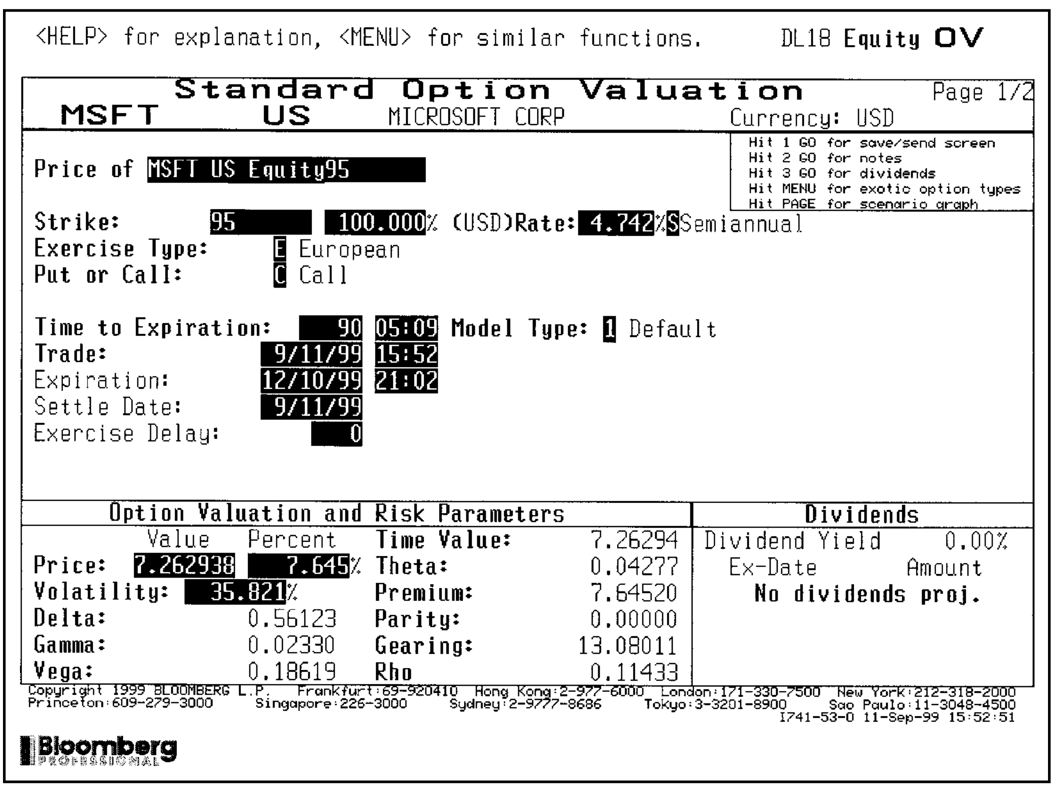


Figure 2.6 Bloomberg option valuation screen, call. Source: Bloomberg L.P.

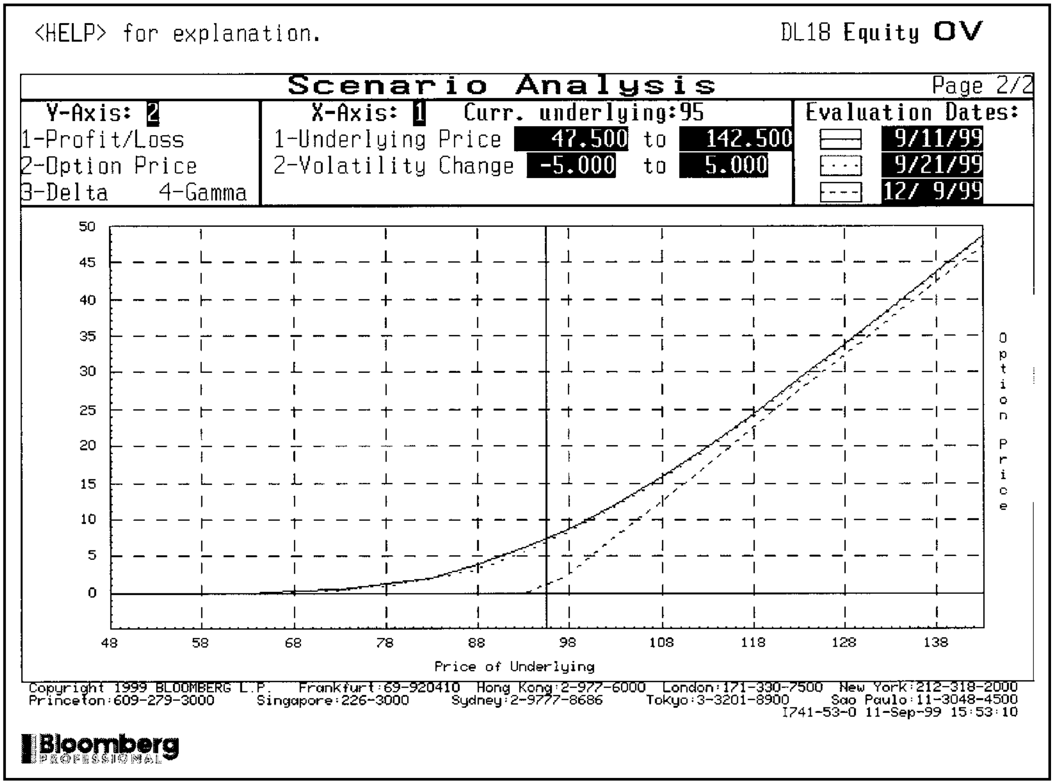


Figure 2.7 Bloomberg scenario analysis, call. Source: Bloomberg L.P.

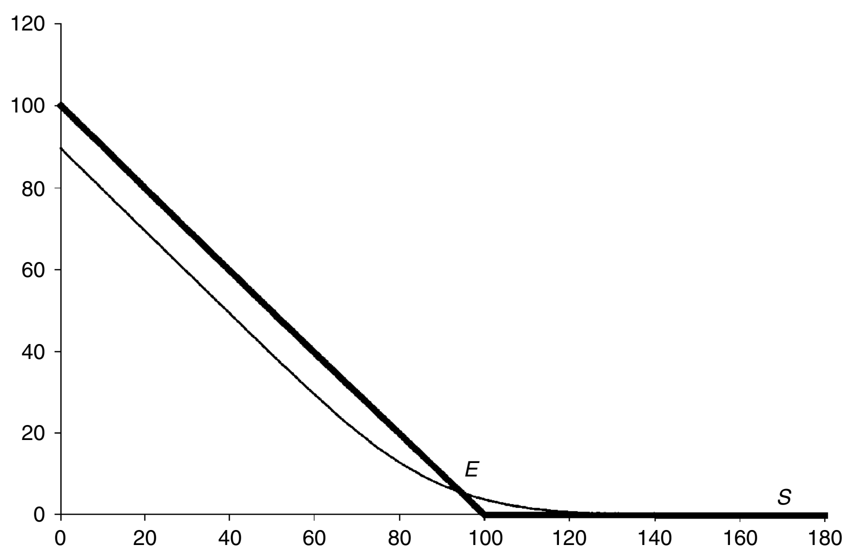


Figure 2.8 Payoff diagram for a put option.

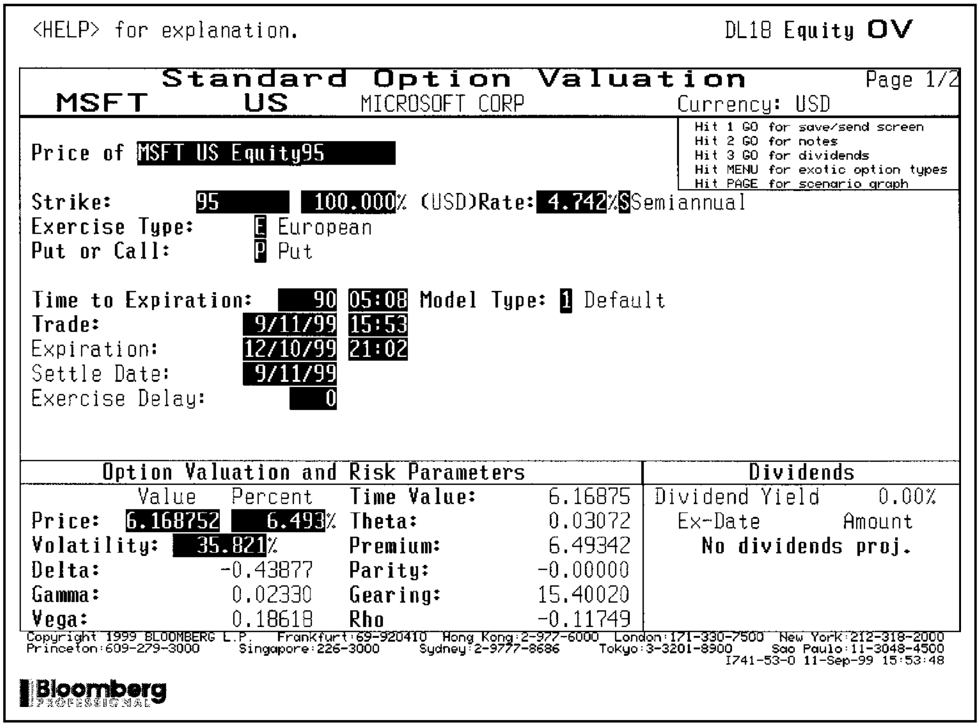


Figure 2.9 Bloomberg option valuation screen, put. Source: Bloomberg L.P.

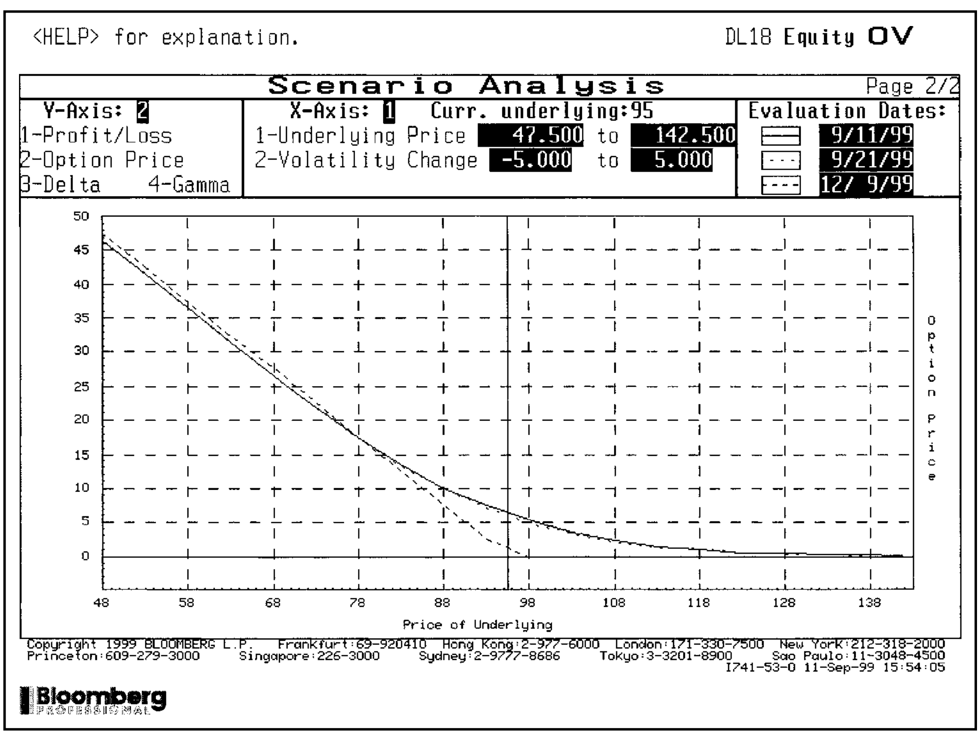


Figure 2.10 Bloomberg scenario analysis, put. Source: Bloomberg L.P.

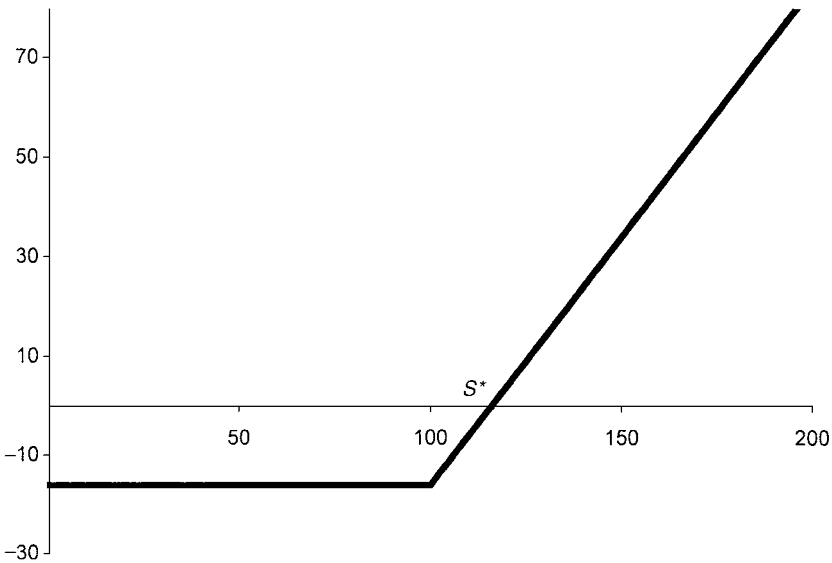


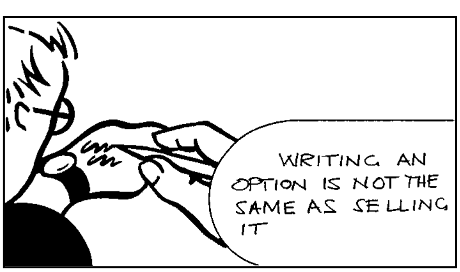
Figure 2.11 Profit diagram for a call option.

2.4.1 Other representations of value

The payoff diagrams shown above only tell you about what happens at expiry, how much money your option contract is worth at that time. It makes no allowance for how much premium you had to pay for the option. To adjust for the original cost of the option, sometimes one plots a diagram such as that shown in Figure 2.11. In this **profit diagram** for a call option I have subtracted off from the payoff the premium originally paid for the call option. This figure is helpful because it shows how far into the money the asset must be at expiry before the option becomes profitable. The asset value marked S^* is the point which divides profit from loss; if the asset at expiry is above this value then the contract has made a profit, if below the contract has made a loss.

As it stands, this profit diagram takes no account of the time value of money. The premium is paid up front but the payoff, if any, is only received at expiry. To be consistent one should either discount the payoff by multiplying by $e^{-r(T-t)}$ to value everything at the present, or multiply the premium by $e^{r(T-t)}$ to value all cashflows at expiry.

Figure 2.12 shows Bloomberg's call option profit diagram. Note that the profit today is zero; if we buy the option and immediately sell it we make neither a profit nor a loss (this is subject to issues of transaction costs).



2.5 WRITING OPTIONS

I have talked above about the rights of the purchaser of the option. But for every option that is sold, someone somewhere must be liable if the option is exercised. If I hold a call option entitling me to buy a stock some time in the future, who do I buy this stock from? Ultimately, the stock must be delivered by the person who **wrote** the option. The **writer** of an option is the person who promises

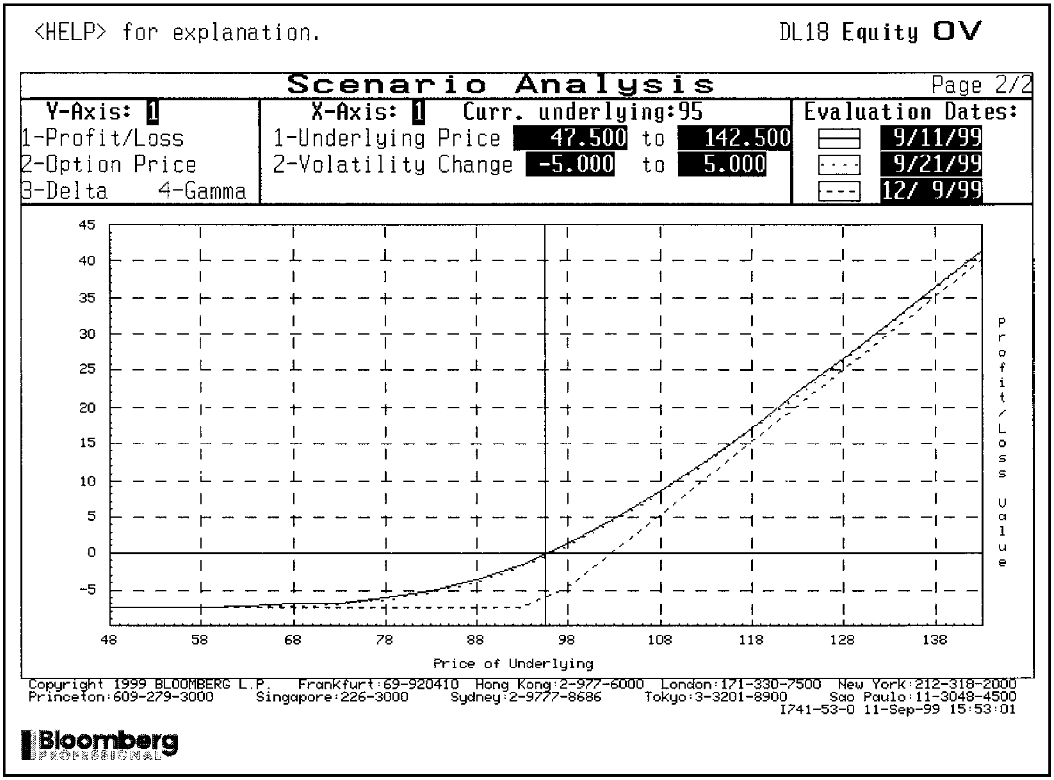


Figure 2.12 Profit diagram for a call. Source: Bloomberg L.P.

to deliver the underlying asset, if the option is a call, or buy it, if the option is a put. The writer is the person who receives the premium.

In practice, most simple option contracts are handled through an exchange so that the purchaser of an option does not know who the writer is. The holder of the option can even sell the option on to someone else via the exchange to close his position. However, regardless of who holds the option, or who has handled it, the writer is the person who has the obligation to deliver or buy the underlying.

The asymmetry between owning and writing options is now clear. The purchaser of the option hands over a premium in return for special rights, and an uncertain outcome. The writer receives a guaranteed payment up front, but then has obligations in the future.

2.6 MARGIN

Writing options is very risky. The downside of buying an option is just the initial premium, the upside may be unlimited. The upside of writing an option is limited, but the downside could be huge. For this reason, to cover the risk of default in the event of an unfavorable outcome, the **clearing houses** that register and settle options insist on the deposit of a margin by the writers of options. Clearing houses act as counterparty to each transaction. Margin was described in Chapter 1.

2.7 MARKET CONVENTIONS

Most of the simpler options contracts are bought and sold through exchanges. These exchanges make it simpler and more efficient to match buyers with sellers. Part of this simplification involves the conventions about such features of the contracts as the available strikes and expiries. For example, simple calls and puts come in **series**. This refers to the strike and expiry dates. Typically a stock has three choices of expiries trading at any time. Having standardized contracts traded through an exchange promotes liquidity of the instruments.

Some options are an agreement between two parties, often brought together by an intermediary. These agreements can be very flexible and the contract details do not need to satisfy any conventions. Such contracts are known as **over the counter** or **OTC** contracts. I give an example at the end of this chapter.

2.8 THE VALUE OF THE OPTION BEFORE EXPIRY

We have seen how much calls and puts are worth at expiry, and drawn these values in payoff diagrams. The question that we can ask, and the question that is central to this book, is ‘How much is the contract worth *now*, before expiry?’ How much would you pay for a contract, a piece of paper, giving you rights in the future? You may have no idea what the stock price will do between now and expiry in six months, say, but clearly the contract has value. At the very least you know that there is no downside to owning the option, the contract gives you specific rights but no *obligations*. Two things are clear about the contract value before expiry: the value will depend on how high the asset price is today and how long there is before expiry.

The higher the underlying asset today, the higher we might expect the asset to be at expiry of the option and therefore the more valuable we might expect a call option to be. On the other hand a put option might be cheaper by the same reasoning.

The dependence on time to expiry is more subtle. The longer the time to expiry, the more time there is for the asset to rise or fall. Is that good or bad if we own a call option? Furthermore, the longer we have to wait until we get any payoff, the less valuable will that payoff be simply because of the time value of money.

I will ask you to suspend disbelief for the moment (it won’t be the first time in the book) and trust me that we will be finding a ‘fair value’ for these options contracts. The aspect of finding the ‘fair value’ that I want to focus on now is the dependence on the asset price and time. I am going to use V to mean the value of the option, and it will be a function of the value of the underlying asset S at time t. Thus we can write $V(S,t)$ for the value of the contract.

We know the value of the contract *at expiry*. If I use T to denote the expiry date then at $t=T$ the function V is known, it is just the payoff function. For example if we have a call option then

$$V(S,T)=\max(S-E,0).$$

This is the function of S that I plotted in the earlier payoff diagrams. Now I can tell you what the fine lines are in Figures 2.5 and 2.8, they are the values of the contracts $V(S, t)$ *at some time before expiry*, plotted against S. I have not specified how long before expiry, since the plot is for explanatory purposes only.

Time Out...

Functions of two variables

The option value is a function of two variables, asset price S and time t. If it helps, think of V as being the height of a mountain with the two variables being distances in the northerly and westerly directions. Later we're going to be looking at the slope of this mountain in each of the two directions... these will be sensitivities of the option price to changes in the asset and in time. These slopes or gradients are what you experience in your car when you see a sign such as '1-in-10 gradient.' That is precisely the same as a slope of 0.1.

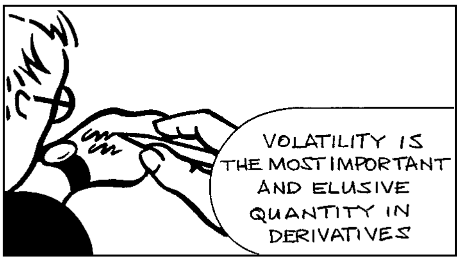
2.9 FACTORS AFFECTING DERIVATIVE PRICES

The two most important factors affecting the prices of options are the value of the underlying asset S and the time to expiry t. These quantities are **variables** meaning that they inevitably change during the life of the contract; if the underlying did not change then the pricing would be trivial. This contrasts with the **parameters** that affect the price of options.

Examples of parameters are the interest rate and strike price. The interest rate will have an effect on the option value via the time value of money since the payoff is received in the future. The interest rate also plays another role which we will see later. Clearly the strike price is important, the higher the strike in a call, the lower the value of the call.

If we have an equity option then its value will depend on any dividends that are paid on the asset during the option's life. If we have an FX option then its value will depend on the interest rate received by the foreign currency.

There is one important parameter that I have not mentioned, and which has a major impact on the option value. That parameter is the **volatility**. Volatility is a measure of the amount of fluctuation in the asset price, a measure of the randomness. Figure 2.13 shows two asset price paths, the more jagged of the two has the higher volatility. The technical definition of volatility is the 'annualized standard deviation of the asset returns.' I will show how to measure this parameter in Chapter 6.



Volatility is a particularly interesting parameter because it is so hard to estimate. And having estimated it, one finds that it never stays constant and is unpredictable.

The distinction between parameters and variables is very important. I shall be deriving equations for the value of options, partial differential equations. These equations will involve differentiation with respect to the variables, but the parameters, as their name suggests, remain as parameters in the equations.

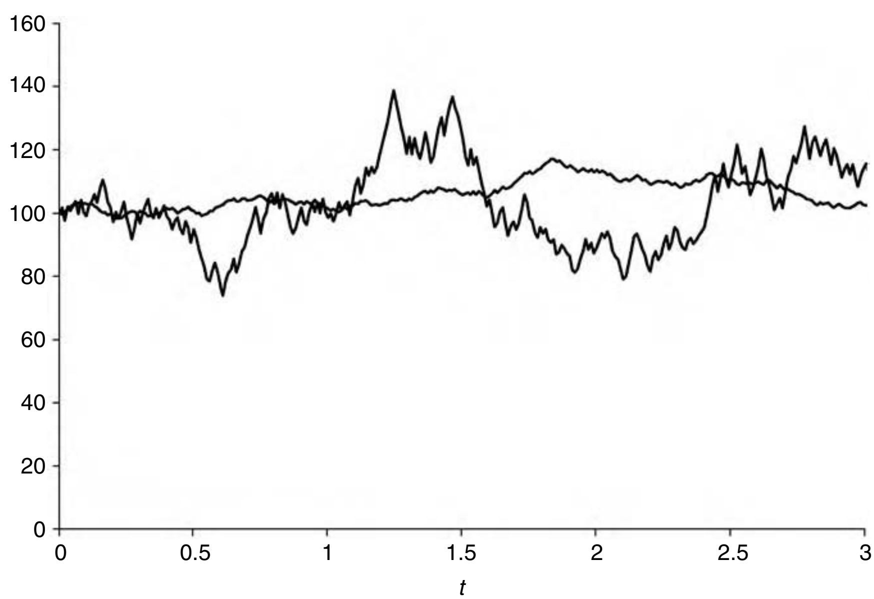
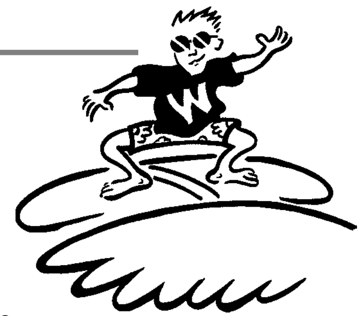


Figure 2.13 Two asset price paths, one is much more volatile than the other.



Time Out

Volatility

Remember our first coin tossing experiment back in Chapter 1? Try this again, but instead of multiplying by a factor of 1.01 or 0.99, use factors of 1.02 and 0.98. Now plot the time series. This is an example of a more volatile path. If you're feeling strong, try the following experiment.

Play around with different scale factors, 1.01 and 0.99, 1.02 and 0.98, 1.05 and 0.95, keeping them symmetric about 1 to start with. Now try different 'timescales,' i.e. toss the coin only once every two units of time, then once every four units. Would you call a path with large but infrequent moves as volatile as one with smaller but more frequent moves?

2.10 SPECULATION AND GEARING

If you buy a far out-of-the-money option it may not cost very much, especially if there is not very long until expiry. If the option expires worthless, then you also haven't lost very much. However, if there is a dramatic move in the underlying, so that the option expires in the money, you may make a large profit relative to the amount of the investment. Let me give an example.

Example Today's date is 14th April and the price of Wilmott Inc. stock is \$666. The cost of a 680 call option with expiry 22nd August is \$39. I expect the stock to rise significantly between now and August, how can I profit it I am right?

Buy the stock Suppose I buy the stock for \$666. And suppose that by the middle of August the stock has risen to \$730. I will have made a profit of \$64 per stock. More importantly my investment will have risen by

$$\frac{730-666}{666}\times 100 = 9.6\%.$$

Buy the call If I buy the call option for \$39, then at expiry I can exercise the call, paying \$680 to receive something worth \$730. I have paid \$39 and I get back \$50. This is a profit of \$11 per option, but in percentage terms I have made

$$\frac{\text{value of asset at expiry} - \text{strike} - \text{cost of call}}{\text{cost of call}}\times 100 = \frac{730-680-39}{39}\times 100 = 28\%.$$

This is an example of **gearing** or **leverage**. The out-of-the-money option has a high gearing, a possible high payoff for a small investment. The downside of this leverage is that the call option is more likely than not to expire completely worthless and you will lose all of your investment. If Wilmott Inc. remains at \$666 then the stock investment has the same value but the call option experiences a 100% loss.

Highly leveraged contracts are very risky for the writer of the option. The buyer is only risking a small amount; although he is very likely to lose, his downside is limited to his initial premium. But the writer is risking a large loss in order to make a probable small profit. The writer is likely to think twice about such a deal unless he can offset his risk by buying other contracts. This offsetting of risk by buying other related contracts is called **hedging**.

Gearing explains one of the reasons for buying options. If you have a strong view about the direction of the market then you can exploit derivatives to make a better return, if you are right, than buying or selling the underlying.

2.11 EARLY EXERCISE

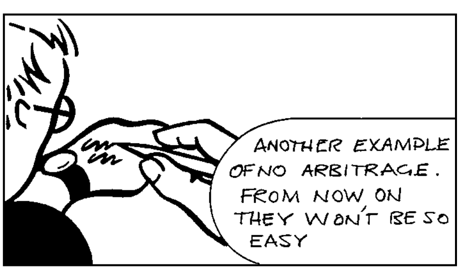
The simple options described above are examples of **European options** because exercise is only permitted *at expiry*. Some contracts allow the holder to exercise *at any time* before expiry, and these are called **American options**. American options give the holder more rights than their European equivalent and can therefore be more valuable, and they can never be less valuable. The main point of interest with American-style contracts is deciding *when* to exercise. In Chapter 5 I will discuss American options, and show how to determine when it is *optimal* to exercise, so as to give the contract the highest value.

Note that the terms 'European' and 'American' do not in any way refer to the continents on which the contracts are traded.

Finally, there are **Bermudan options**. These allow exercise on specified dates, or in specified periods. In a sense they are halfway between European and American since exercise is allowed on some days and not on others.

2.12 PUT-CALL PARITY

Imagine that you buy one European call option with a strike of E and an expiry of T and that you write a European put option with the same strike and expiry. Today's date is t. The payoff you receive at T for the call will look like the line in the first plot of Figure 2.14. The payoff for the put is the line in the second plot in the figure. Note that the sign of the payoff is negative, you *wrote* the option and are liable for the payoff. The payoff for the portfolio of the two options is the sum of the individual payoffs, shown in the third plot. The payoff for this portfolio of options is



$$\max(S(T) - E, 0) - \max(E - S(T), 0) = S(T) - E,$$

where $S(T)$ is the value of the underlying asset at time T.

The right-hand side of this expression consists of two parts, the asset and a fixed sum E. Is there another way to get exactly this payoff? If I buy the asset today it will cost me $S(t)$ and be worth $S(T)$ at expiry. I don't know what the value $S(T)$ will be but I do know how to guarantee to get that amount, and that is to buy the asset. What about the E

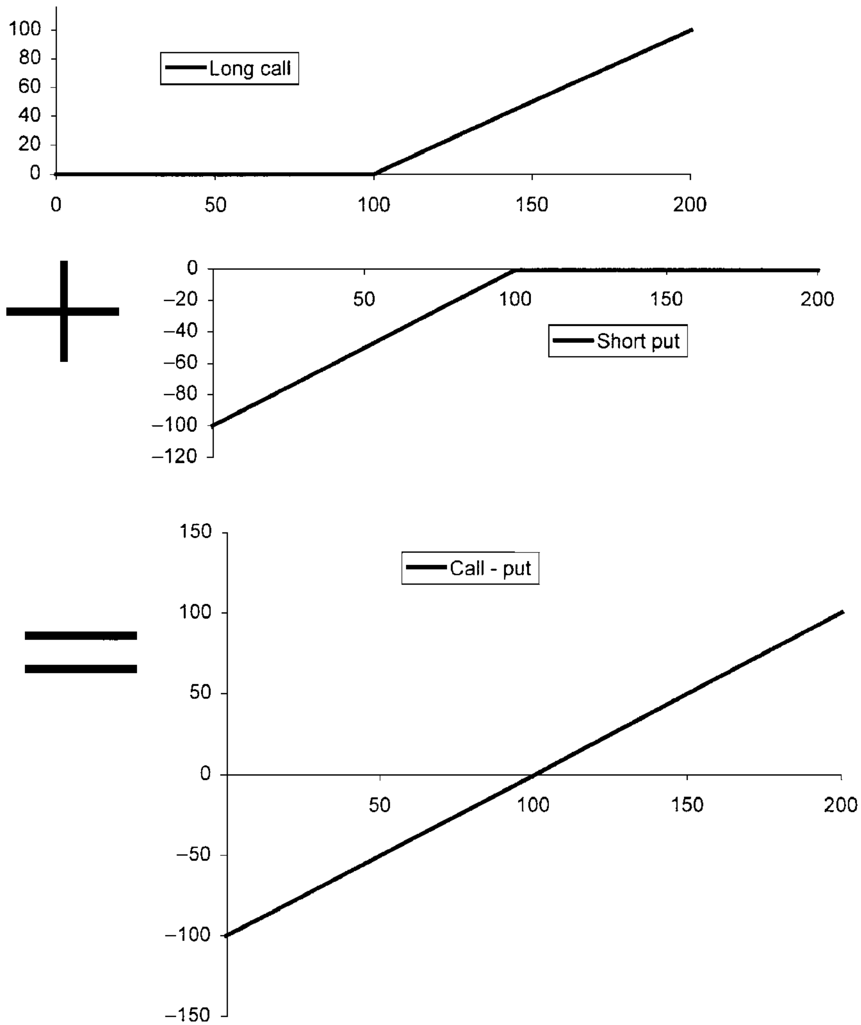


Figure 2.14 Schematic diagram showing put-call parity.

term? To lock in a payment of E at time T involves a cash flow of $Ee^{-r(T-t)}$ at time t. The conclusion is that the portfolio of a long call and a short put gives me exactly the same payoff as a long asset, short cash position. The equality of these cashflows is independent of the future behavior of the stock and is model independent:

$$C - P = S - Ee^{-r(T-t)},$$

where C and P are today's values of the call and the put respectively. This relationship holds at any time up to expiry and is known as **put-call parity**. If this relationship did not hold then there would be riskless arbitrage opportunities.

In Table 2.1 I show the cashflows in the perfectly hedged portfolio. In this table I have set up the cashflows to have a guaranteed value of zero at expiry.

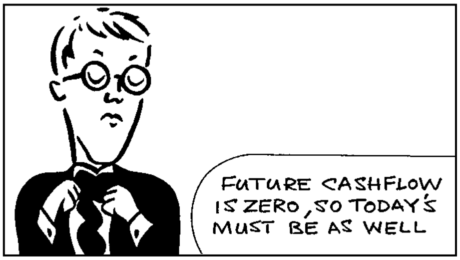
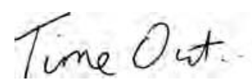


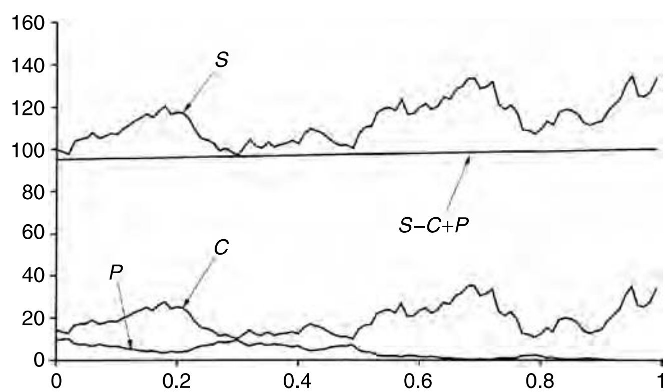
Table 2.1 Cashflows in a hedged portfolio of options and asset.

Holding	Worth today (t)	Worth at expiry (T)
Call	C	$\max(S(T)-E,0)$
$-$Put	$-P$	$-\max(E-S(T),0)$
$-$Stock	$-S(t)$	$-S(T)$
Cash	$Ee^{-r(T-t)}$	E
Total	$C-P-S(t)+Ee^{-r(T-t)}$	0



A simulation of put-call parity

Below are four plots, all with time along the horizontal axis. The first is of some asset price. The second is the value of a call option on that asset. You don't need to know details of the contract, such as strike and expiry. Nor do you need to know how I calculated the value. The third plot is of a put option (same strike and expiry as the call, whatever they were). The fourth plot is stock value minus call value plus put value. Observe how it grows exponentially, just like cash in the bank. This is a graphical illustration of put-call parity.



2.13 BINARIES OR DIGITALS

The original and still most common contracts are the vanilla calls and puts. Increasingly important are the **binary** or **digital options**. These contracts have a payoff at expiry that is discontinuous in the underlying asset price. An example of the payoff diagram for one of these options, a **binary call**, is shown in Figure 2.15. This contract pays $1 at expiry, time T, if the asset price is then greater than the exercise price E. Again, and as with the rest of the figures in this chapter, the bold line is the payoff and the fine line is the contract value some time before expiry.

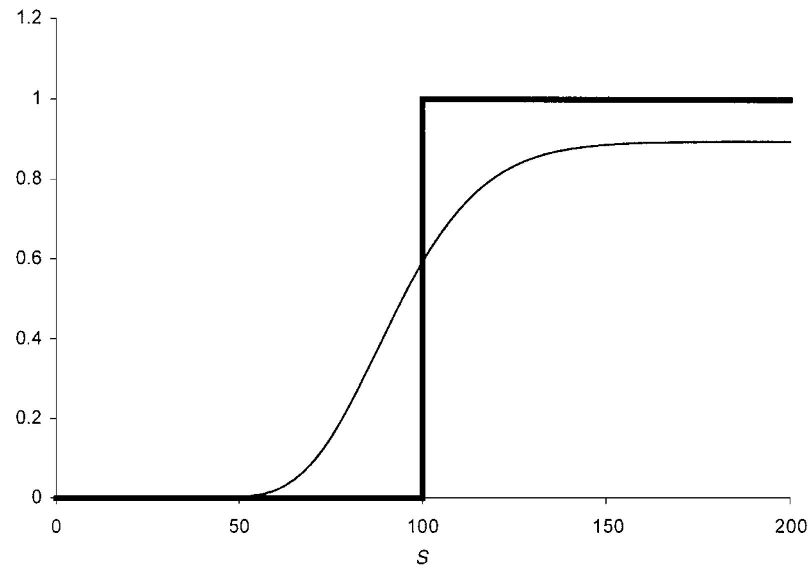


Figure 2.15 Payoff diagram for a binary call option.

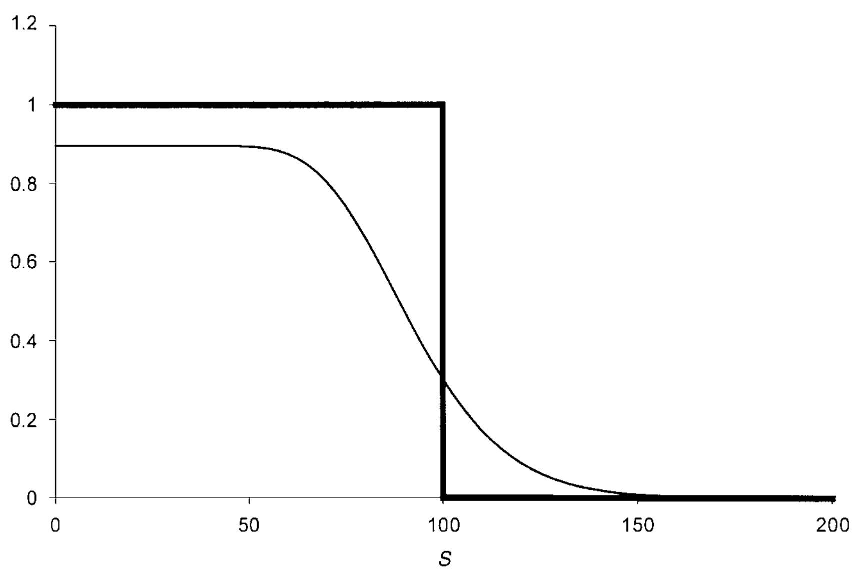


Figure 2.16 Payoff diagram for a binary put option.

Why would you invest in a binary call? If you think that the asset price will rise by expiry, to finish above the strike price then you might choose to buy either a vanilla call or a binary call. The vanilla call has the best upside potential, growing linearly with S beyond the strike. The binary call, however, can never pay off more than the \$1. If you expect the underlying to rise dramatically then it may be best to buy the vanilla call. If you believe that the asset rise will be less dramatic then buy the binary call. The gearing of the vanilla call is greater than that for a binary call if the move in the underlying is large.

Figure 2.16 shows the payoff diagram for a **binary put**, the holder of which receives \$1 if the asset is *below E* at expiry. The binary put would be bought by someone expecting a modest fall in the asset price.

There is a particularly simple binary put-call parity relationship. What do you get at expiry if you hold both a binary call and a binary put with the same strikes and expiries? The answer is that you will always get \$1 regardless of the level of the underlying at expiry. Thus

$$\text{Binary call} + \text{Binary put} = e^{-r(T-t)}.$$

What would the table of cashflows look like for the perfectly hedged digital portfolio?

2.14 BULL AND BEAR SPREADS

A payoff that is similar to a binary option can be made up with vanilla calls. This is our first example of a **portfolio of options** or an **option strategy**.

Suppose I buy one call option with a strike of 100 and write another with a strike of 120 and with the same expiration as the first then my resulting portfolio has a payoff that is shown in Figure 2.17. This payoff is zero below 100, 20 above 120 and linear in between. The payoff is continuous, unlike the binary call, but has a payoff that is superficially similar. This strategy is called a **bull spread** because it benefits from a bull, i.e. rising, market.

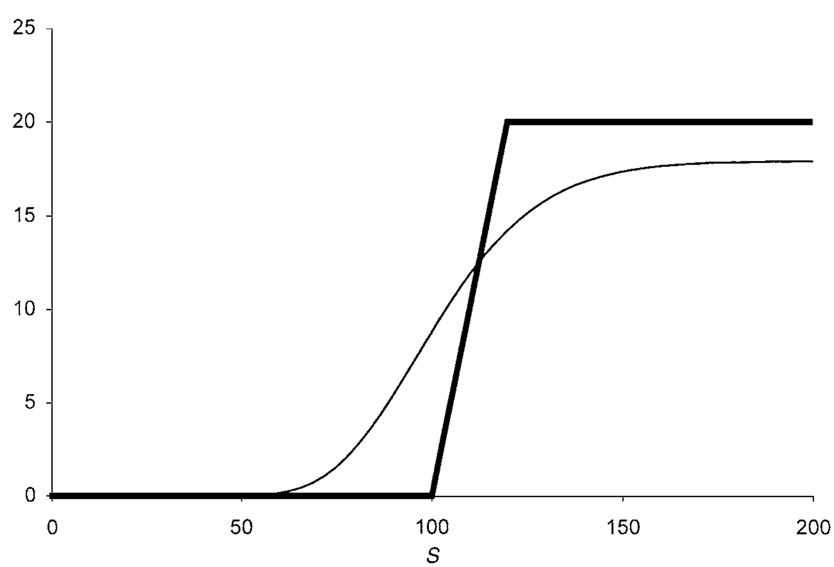


Figure 2.17 Payoff diagram for a bull spread.

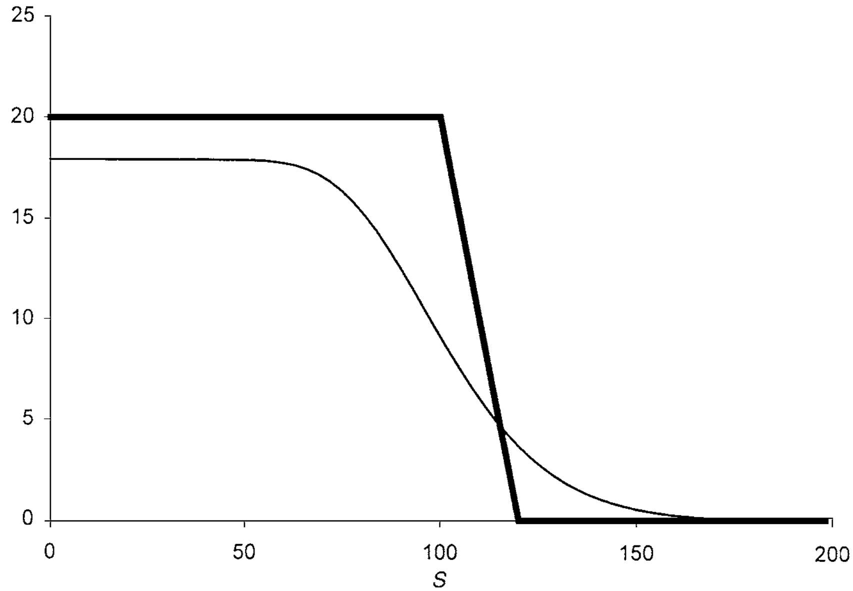


Figure 2.18 Payoff diagram for a bear spread.

The payoff for a general bull spread, made up of calls with strikes E_1 and E_2, is given by

$$\frac{1}{E_2 - E_1}(\max(S - E_1, 0) - \max(S - E_2, 0)),$$

where $E_2 > E_1$. Here I have bought/sold $(E_2 - E_1)^{-1}$ of each of the options so that the maximum payoff is scaled to 1.

If I write a put option with strike 100 and buy a put with strike 120 I get the payoff shown in Figure 2.18. This is called a **bear spread**, benefitting from a bear, i.e. falling, market. Again, it is very similar to a binary put except that the payoff is continuous.

Because of put-call parity it is possible to build up these payoffs using other contracts.

A strategy involving options of the same type (i.e. calls or puts) is called a **spread**.

2.15 STRADDLES AND STRANGLES

If you have a precise view on the behavior of the underlying asset, you may want to be precise in your choice of option; simple calls, puts, and binaries may be too crude.

The **straddle** consists of a call and a put with the same strike. The payoff diagram is shown in Figure 2.19. Such a position is usually bought at the money by someone who expects the underlying to either rise or fall, but not to remain at the same level. For example, just before an anticipated major news item stocks often show a 'calm before the storm.' On the announcement the stocks suddenly move either up or down depending on whether or not the news was favorable to the company. They may also be bought by technical traders who see the stock at a key support or resistance level and expect the stock to either break through dramatically or bounce back.

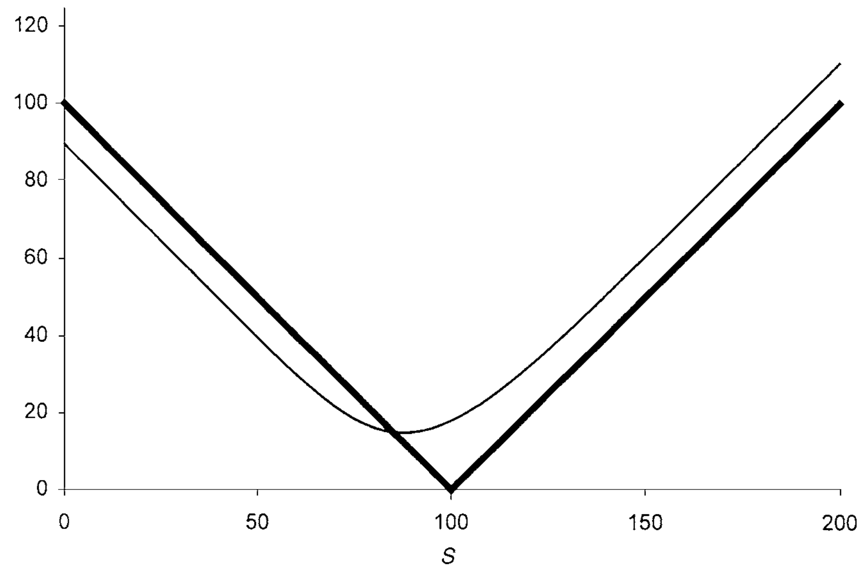


Figure 2.19 Payoff diagram for a straddle.

The straddle would be sold by someone with the opposite view, someone who expects the underlying price to remain stable.

Figure 2.20 shows the Bloomberg screen for setting up a straddle. Figure 2.21 shows the profit and loss for this position at various times before expiry. The profit/loss is the option value less the upfront premium.

The **strangle** is similar to the straddle except that the strikes of the put and the call are different. The contract can be either an **out-of-the-money strangle** or **an in-the-money strangle**. The payoff for an out-of-the money strangle is shown in Figure 2.22. The motivation behind the purchase of this position is similar to that for the purchase of a straddle. The difference is that the buyer expects an even larger move in the underlying one way or the other. The contract is usually bought when the asset is around the middle of the two strikes and is cheaper than a straddle. This cheapness means that the gearing for the out-of-the-money strangle is higher than that for the straddle. The downside is that there is a much greater range over which the strangle has no payoff at expiry, for the straddle there is only the one point at which there is no payoff.

There is another reason for a straddle or strangle trade that does not involve a view on the direction of the underlying. These contracts are bought or sold by those with a view on the direction of volatility, they are one of the simplest **volatility trades**. Because of the relationship between the price of an option and the volatility of the asset one can speculate on the direction of volatility. Do you expect the volatility to rise? If so, how can you benefit from this? Until we know more about this relationship, we cannot go into this in more detail.

Straddles and strangles are rarely held until expiry.

A strategy involving options of different types (i.e. both calls and puts) is called a **combination**.

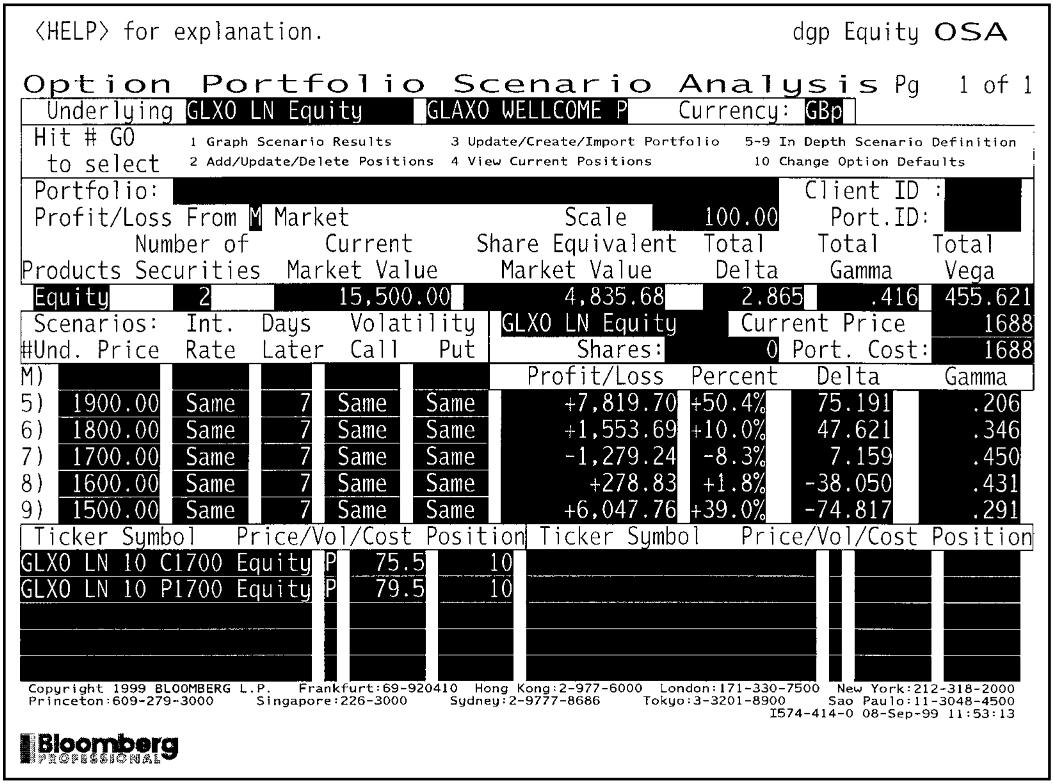


Figure 2.20 A portfolio of two options making up a straddle. Source: Bloomberg L.P.

2.16 RISK REVERSAL

The **risk reversal** is a combination of a long call, with strike above the current spot, and a short put with a strike below the current spot. Both have the same expiry. The payoff is shown in Figure 2.23.

The risk reversal is a very special contract, popular with practitioners. Its value is usually quite small and related to the market's expectations of the behavior of volatility.

2.17 BUTTERFLIES AND CONDORS

A more complicated strategy involving the purchase and sale of options with *three* different strikes is a **butterfly spread**. Buying a call with a strike of 90, writing two calls struck at 100 and buying a 110 call gives the payoff in Figure 2.24. This is the kind of position you might enter if you believe that the asset is not going anywhere, either up or down. Because it has no large upside potential (in this case the maximum payoff is 10) the position will be relatively cheap. With options, cheap is good.

The **condor** is like a butterfly except that four strikes, and four call options, are used. The payoff is shown in Figure 2.25.

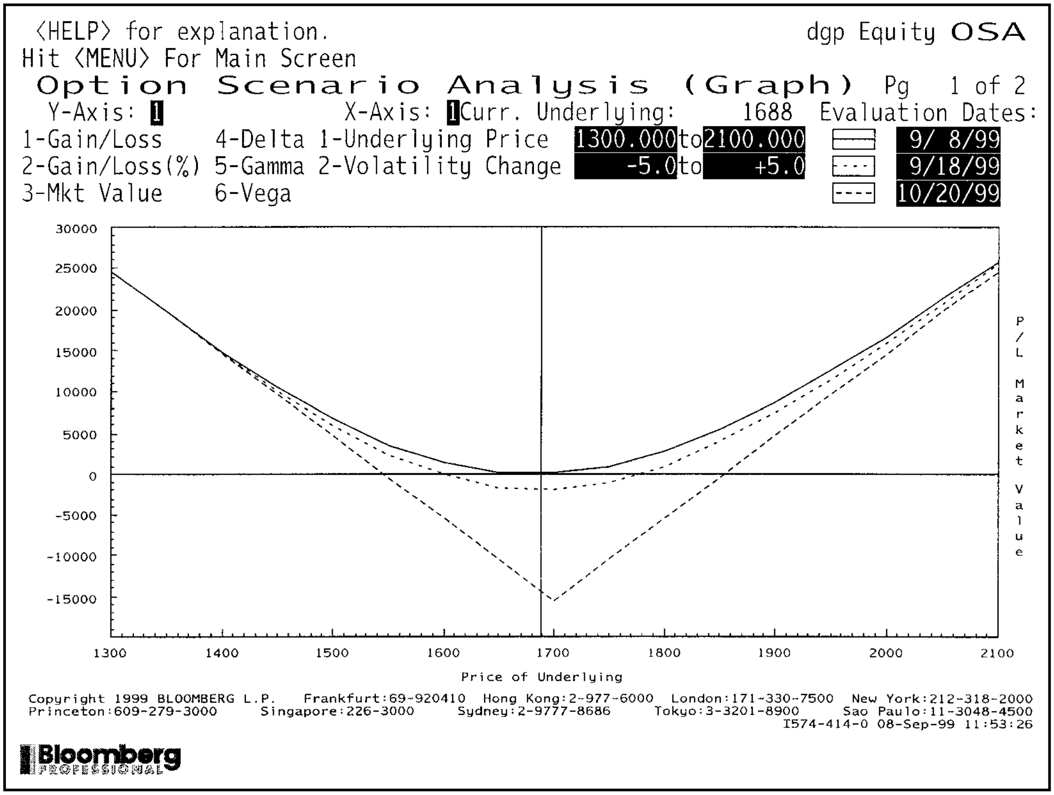


Figure 2.21 Profit/loss for the straddle at several times before expiry. Source: Bloomberg L.P.

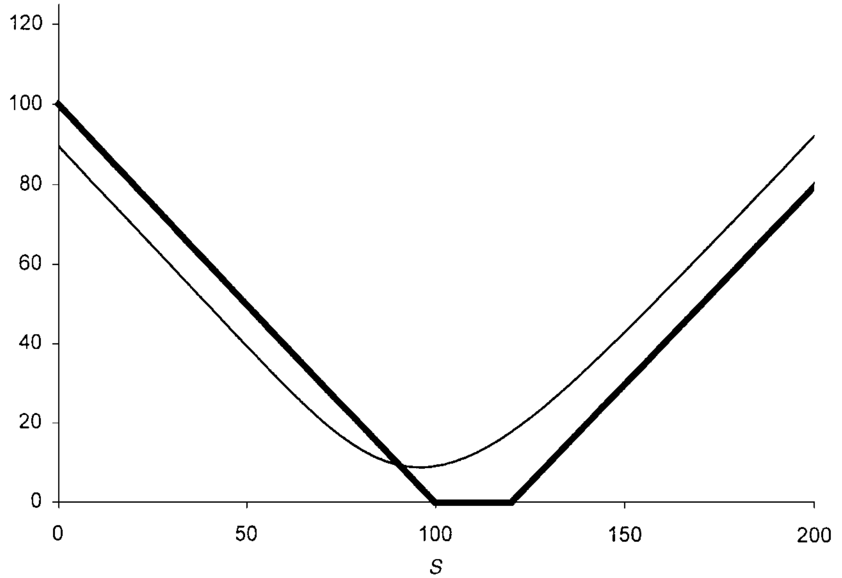


Figure 2.22 Payoff diagram for a strangle.

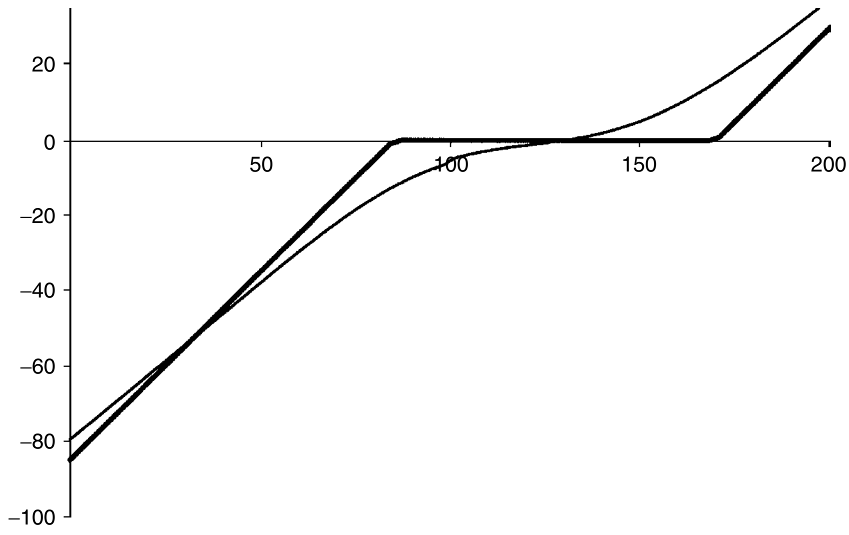


Figure 2.23 Payoff diagram for a risk reversal.

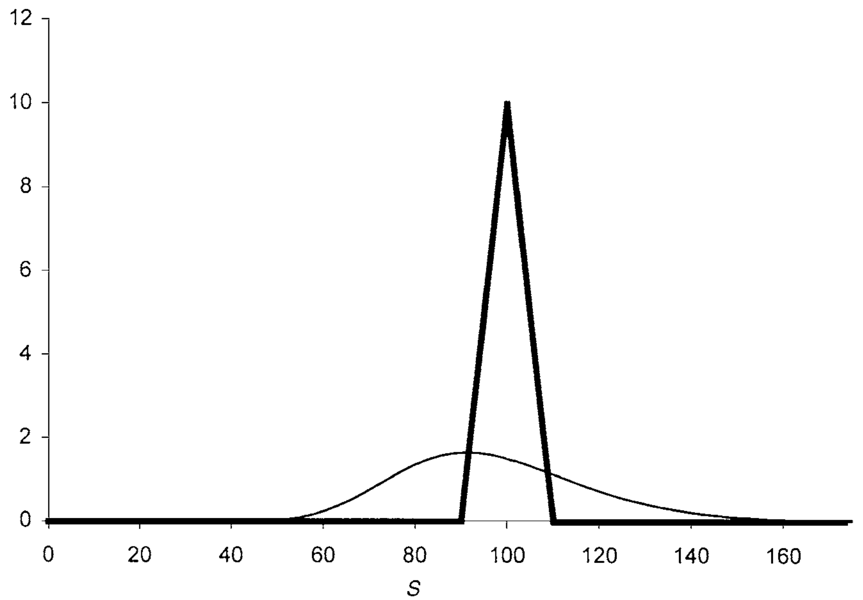


Figure 2.24 Payoff diagram for a butterfly spread.

2.18 CALENDAR SPREADS

All of the strategies I have described above have involved buying or writing calls and puts with different strikes *but all with the same expiration*. A strategy involving options with different expiry dates is called a **calendar spread**. You may enter into such a position if you have a precise view on the timing of a market move as well as the direction of the

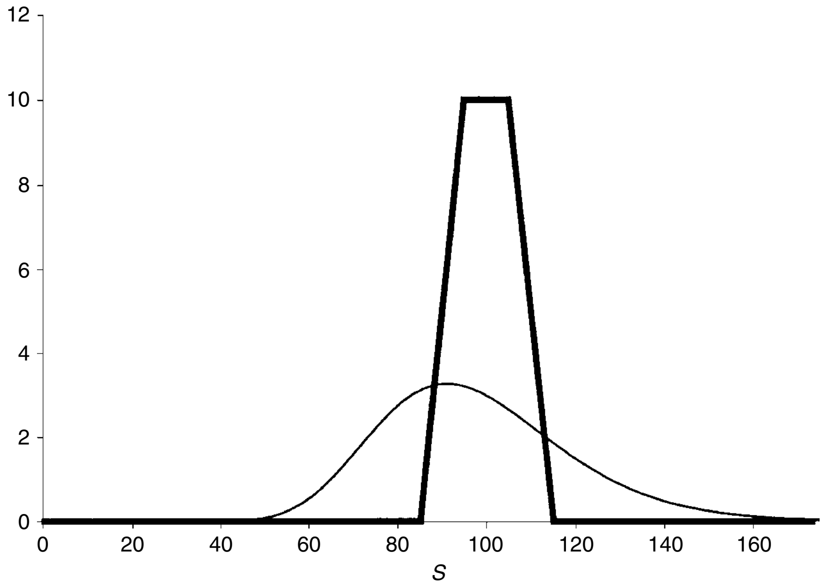
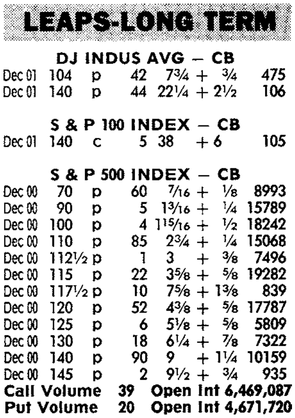


Figure 2.25 Payoff diagram for a condor.

move. As always the motive behind such a strategy is to reduce the payoff at asset values and times which you believe are irrelevant, while increasing the payoff where you think it will matter. Any reduction in payoff will reduce the overall value of the option position.

2.19 LEAPS AND FLEX

LEAPS or **long-term equity anticipation securities** are longer-dated exchange-traded calls and puts. They began trading on the CBOE in the late 1980s. They are standardized so that they expire in January each year and are available with expiries up to three years.



LEAPS-LONG TERM

		DJ INDUS AVG – CB				
Dec 01	104	p	42	7¾	+ ¾	475
Dec 01	140	p	44	22¼	+ 2½	106
		S & P 100 INDEX – CB				
Dec 01	140	c	5	38	+ 6	105
		S & P 500 INDEX – CB				
Dec 00	70	p	60	7/16	+ ⅛	8993
Dec 00	90	p	5	1 3/16	+ ¼	15789
Dec 00	100	p	4	1 15/16	+ ½	18242
Dec 00	110	p	85	2¾	+ ¼	15068
Dec 00	112½	p	1	3	+ ⅜	7496
Dec 00	115	p	22	3⅝	+ ⅝	19282
Dec 00	117½	p	10	7⅝	+ 1⅜	839
Dec 00	120	p	52	4⅜	+ ⅝	17787
Dec 00	125	p	6	5⅛	+ ⅝	5809
Dec 00	130	p	18	6¼	+ ⅞	7322
Dec 00	140	p	90	9	+ 1¼	10159
Dec 00	145	p	2	9½	+ ¾	935
Call Volume			**39**	**Open Int 6,469,087**		
Put Volume			**20**	**Open Int 4,671,720**		

Figure 2.26 *The Wall Street Journal Europe* of 5th January 2000, LEAPS. Reproduced by permission of Dow Jones & Company, Inc.

They come with three strikes, corresponding to at the money and 20% in and out of the money with respect to the underlying asset price when issued.

Figure 2.26 shows LEAPS quoted in *The Wall Street Journal Europe*.

In 1993 the CBOE created **FLEX** or **FLexible EXchange-traded options** on several indices. These allow a degree of customization, in the expiry date (up to five years), the strike price and the exercise style.

2.20 WARRANTS

A contract that is very similar to an option is a **warrant**. Warrants are call options issued by a company on its own equity. The main differences between traded options and warrants are the timescales involved, warrants usually have a longer lifespan, and on exercise the company issues new stock to the warrant holder. On exercise, the holder of a *traded* option receives stock that has already been issued. Exercise is usually allowed any time before expiry, but after an initial waiting period.

The typical lifespan of a warrant is five or more years. Occasionally **perpetual warrants** are issued, these have no maturity.

2.21 CONVERTIBLE BONDS

Convertible bonds or **CB**s have features of both bonds and warrants. They pay a stream of coupons with a final repayment of principal at maturity, but they can be converted into the underlying stock before expiry. On conversion rights to future coupons are lost. If the stock price is low then there is little incentive to convert to the stock, the coupon stream is more valuable. In this case the CB behaves like a bond. If the stock price is high then conversion is likely and the CB responds to the movement in the asset. Because the CB can be converted into the asset, its value has to be at least the value of the asset. This makes CBs similar to American options; early exercise and conversion are mathematically the same.

2.22 OVER THE COUNTER OPTIONS

Not all options are traded on an exchange. Some, known as over the counter or OTC options are sold privately from one counterparty to another. In Figure 2.27 is the term sheet for an OTC put option, having some special features. A **term sheet** specifies the precise details of an OTC contract. In this OTC put the holder gets a put option on S&P500, but more cheaply than a vanilla put option. This contract is cheap because part of the premium does not have to be paid until and unless the underlying index trades above a specified level. Each time that a new level is reached an extra payment is triggered. This feature means that the contract is not vanilla, and makes the pricing more complicated. We will be discussing special features like the ones in this contract in later chapters. Quantities in square brackets will be set at the time that the deal is struck.

Over-the-counter Option linked to the S&P500 Index

Option Type	European put option, with contingent premium feature
Option Seller	XXXX
Option Buyer	[dealing name to be advised]
Notional Amount	USD 20MM
Trade Date	[]
Expiration Date	[]
Underlying Index	S&P500
Settlement	Cash settlement
Cash Settlement Date	5 business days after the Expiration Date
Cash Settlement Amount	Calculated as per the following formula: #Contracts * max[0, S&Pstrike – S&Pfinal] where #Contracts = Notional Amount / S&Pinitial This is the same as a conventional put option: **S&Pstrike** will be equal **to 95% of the closing price on the Trade Date** **S&Pfinal** will be the level of the Underlying Index at the valuation time on the Expiration Date **S&Pinitial** is the level of the Underlying Index at the time of execution
Initial Premium Amount	[2%] of Notional Amount
Initial Premium Payment Date	5 business days after Trade Date
Additional Premium Amounts	[1.43%] of Notional Amount per Trigger Level
Additional Premium Payment Dates	The Additional Premium Amounts shall be due only if the Underlying Index at any time from and including the Trade Date and to and including the Expiration Date is equal to or greater than any of the Trigger Levels.
Trigger Levels	103%, 106% and 109% of **S&P500initial**
Documentation	ISDA
Governing Law	New York

This indicative termsheet is neither an offer to buy or sell securities or an OTC derivative product which includes options, swaps, forwards and structured notes having similar features to OTC derivative transactions, nor a solicitation to buy or sell securities or an OTC derivative product. The proposal contained in the foregoing is not a complete description of the terms of a particular transaction and is subject to change without limitation.

Figure 2.27 Term sheet for an OTC 'Put'.

2.23 SUMMARY

We now know the basics of options and markets, and a few of the simplest trading strategies. We know some of the jargon and the reasons why people might want to buy an option. We've also seen another example of no arbitrage in put-call parity. This is just the beginning. We don't know how much these instruments are worth, how they are affected by the price of the underlying, how much risk is involved in the buying or writing of options. And we have only seen the very simplest of contracts, there are many, many more complex products to examine. All of these issues are going to be addressed in later chapters.

FURTHER READING

- McMillan (1996) and Options Institute (1995) describe many option strategies used in practice.
- Most exchanges have websites. The London International Financial Futures Exchange website contains information about the money markets, bonds, equities, indices and commodities. See `www.liffe.com`. For information about options and derivatives generally, see `www.cboe.com`, the Chicago Board Options Exchange website. The American Stock Exchange is on `www.amex.com` and the New York Stock Exchange on `www.nyse.com`.
- Derivatives have often had bad press (and there's probably more to come). See Miller (1997) for a discussion of the pros and cons of derivatives.
- The best books on options are Hull (1999) and Cox & Rubinstein (1985), modesty forbids me mentioning others.

CHAPTER 3

predicting the markets? a small digression

The aim of this Chapter...

... is to explain ways in which people supposedly predict future movements in the financial markets. There is little scientific evidence that these methods work in practice but you, a future bond trader perhaps, must know about such matters and eventually decide for yourself.

In this Chapter...

- some of the commonly used technical methods for predicting market direction
- some modern approaches to modeling markets and their microstructure

3.1 INTRODUCTION

People have been making predictions about the future since the dawn of time. And predicting the future of the financial markets has been especially popular. Despite the claims of many 'legendary' investors it is not clear whether there is any validity in any of the methods they use, or whether the claims are examples of survivor bias. The big losers tend to keep quiet.

In this chapter we look at some of the traditional methods for determining trends, technical analysis, and also some of the more recent methods, often emanating from physics. I won't be describing some of the more dubious ideas, such as astrology, but then we Scorpios tend to be skeptical.

In the book generally, I'm taking the view that the markets are best modeled via probabilities. This chapter is very much a digression from the main thrust of the book.

3.2 TECHNICAL ANALYSIS

Technical analysis is a way of predicting future price movements based only on observing the past history of prices. This price history may also include other quantities such as volume of trade. These methods contrast with **fundamental analysis** in which prediction is made based on an examination of the factors underlying the stock or other instrument. This may include general economic or political analysis, or analysis of factors specific to the stock, such as the effect of global warming on snowfall in the Alps, if one is concerned with a travel company. In practice, most traders will use a combination of both technical and fundamental analysis.

Technical analysis is also called **charting** because the graphical representation of prices etc. plays an important part. Technical analysis is thought to be particularly good for timing market moves; fundamental analysis may get the direction right, but not necessarily when the move will happen.

3.2.1 Plotting

The simplest chart types just join together the prices from one day to the next, with time along the horizontal axis. These are the sort of plots we have seen throughout this book. Sometimes a logarithmic scale is used for the vertical price axis to represent return rather than absolute level. Later on we'll see some more complicated types of plotting. Sometimes you will see trading volume on the same graph, this is also used for prediction but I won't go into any details here, see Figure 3.1.

3.2.2 Support and resistance

Resistance is a price level which an asset seems to have difficulty rising above. This may be a previously realized highest value, or it may be a psychologically important (round) number. **Support** is a level below which an asset price seems to be reluctant to fall. There may be sufficient demand at this low price to stop it falling any further. Examples of support and resistance are shown in Figure 3.2.

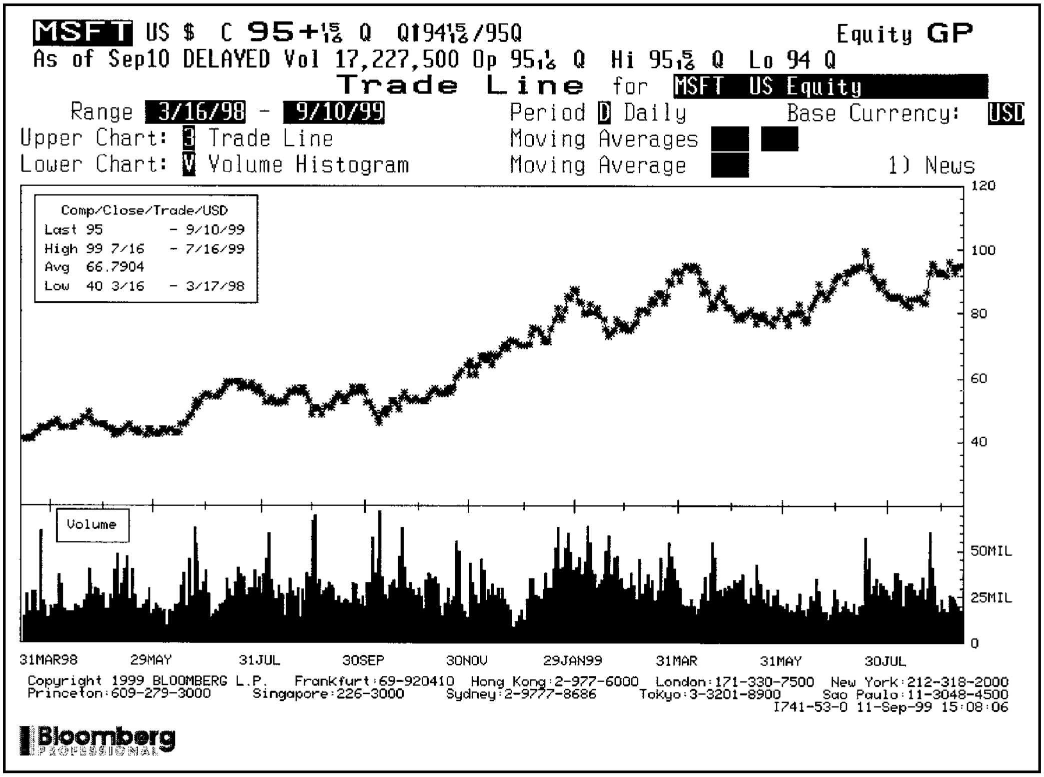


Figure 3.1 Price and volume. Source: Bloomberg L.P.

When a support or resistance level finally breaks it is said to do so quite dramatically.

3.2.3 Trendlines

Similar to support and resistance are **trendlines**. These are formed by joining together successive peaks and/or troughs in the price history to form a rising or falling support or resistance level. An example is shown in Figure 3.3.

3.2.4 Moving averages

Moving averages are calculated in many ways. Different time windows can be used, or even exponentially weighted averages can be calculated. Moving averages are supposed to distill out the basic trend in a price by smoothing the random noise.

Sometimes two moving averages are calculated, say a 10-day and a 250-day average. The crossing of these two would signify a change in the underlying trend and a time to buy or sell.

Although I'm not the greatest fan of technical analysis, there is some evidence that there may be predictive power in moving averages.

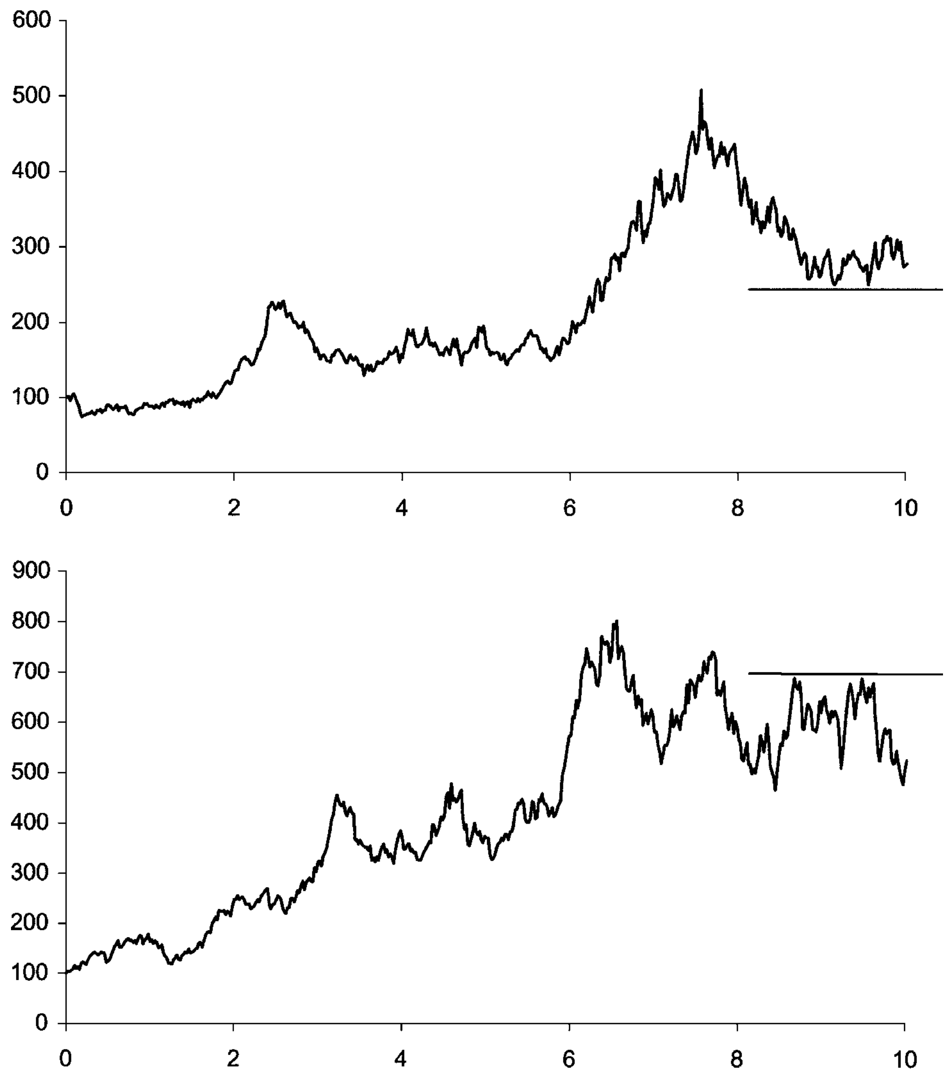


Figure 3.2 Support and resistance.

Figure 3.4 shows a Bloomberg screen with Microsoft share price, 5-day and 15-day moving averages.

3.2.5 Relative strength

The **relative strength index** is the percentage of up moves in the last *N* days. A number higher than 70% is said to be overbought and therefore likely to fall and below 30% is said to be oversold and should rise.

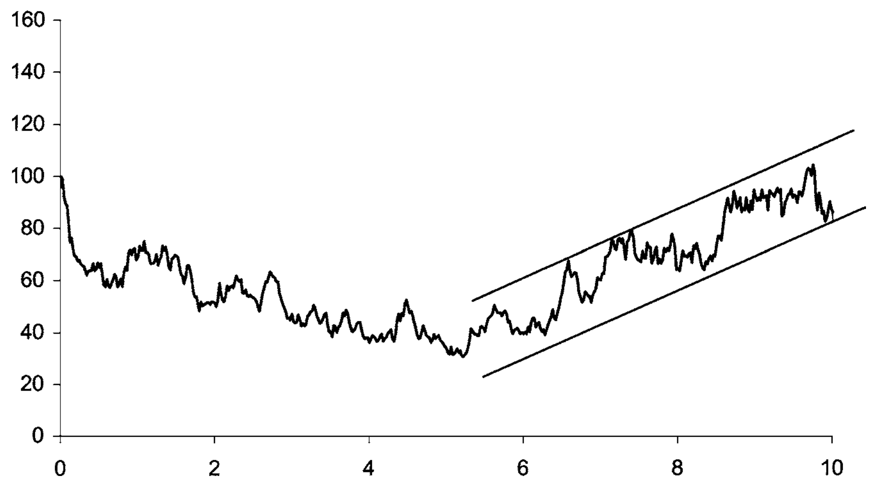


Figure 3.3 A trending stock.

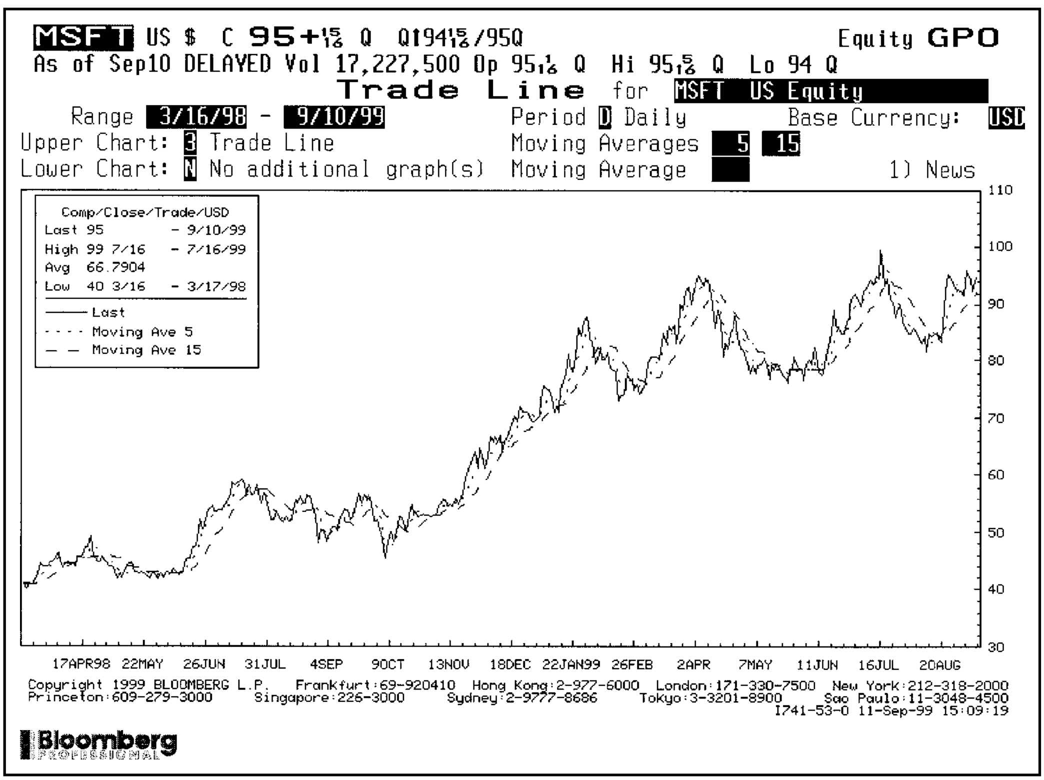


Figure 3.4 Two moving averages. Source: Bloomberg L.P.

3.2.6 Oscillators

An **oscillator** is another indicator of over/underbought conditions. One way of calculating it is as follows.

Define k by

$$100 \times \frac{\text{Current close} - \text{lowest over } n \text{ periods}}{\text{Highest over } n \text{ periods} - \text{lowest over } n \text{ periods}}.$$

Now take a moving average of the last three days, say. This average is plotted against time and any move outside the range 30–70% could be an indication of a move in the asset (Figure 3.5).

3.2.7 Bollinger bands

Bollinger bands are plots of a specified number of standard deviations above and below a specified moving average (Figure 3.6).

3.2.8 Miscellaneous patterns

As well as the ‘quantitative’ side of charting there is also the ‘artistic’ side. Practitioners say that certain patterns anticipate certain future moves. It’s rather like your grandmother reading tea leaves.

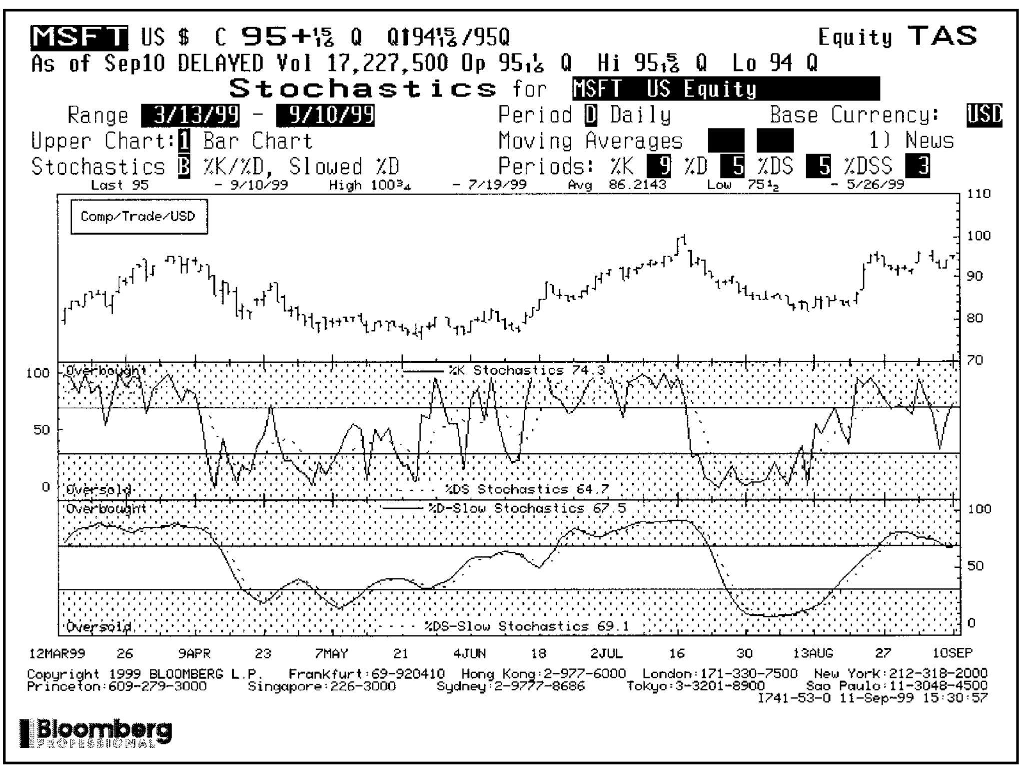


Figure 3.5 Oscillator. Source: Bloomberg L.P.

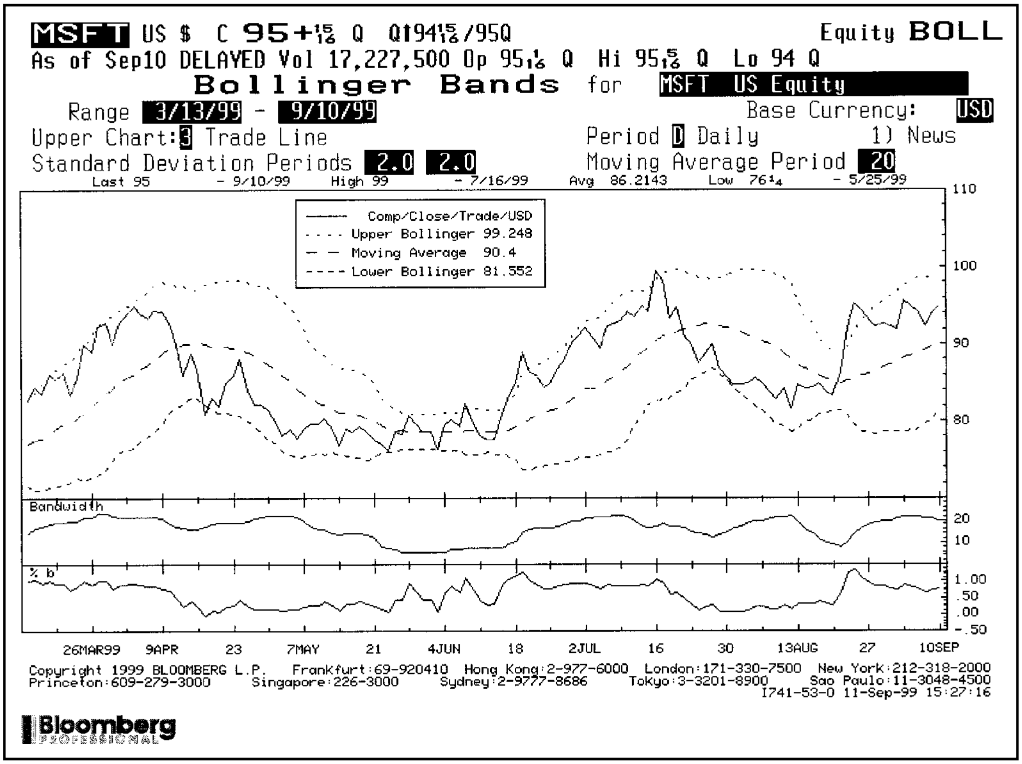


Figure 3.6 Bollinger bands. Source: Bloomberg L.P.

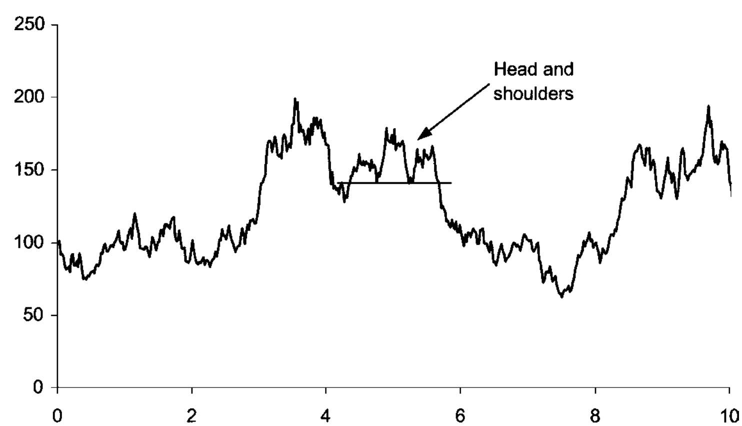


Figure 3.7 Head and shoulders.

Head and shoulders is a common pattern and is best described with reference to Figure 3.7. There are a left and a right shoulder with the head rising above. Following on from the right shoulder should be a dramatic decline in the asset price.

This pattern is supposed to be one of the most reliable predictors. It is also seen in an upside-down formation.

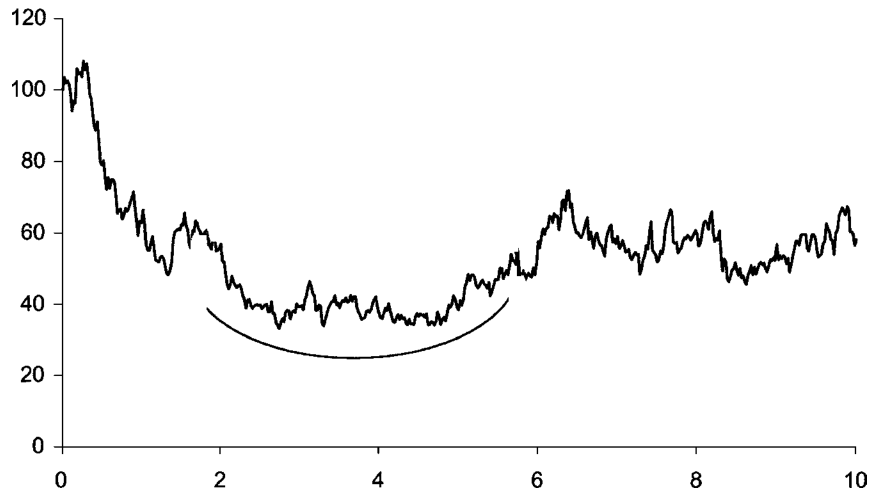


Figure 3.8 Saucer bottom.

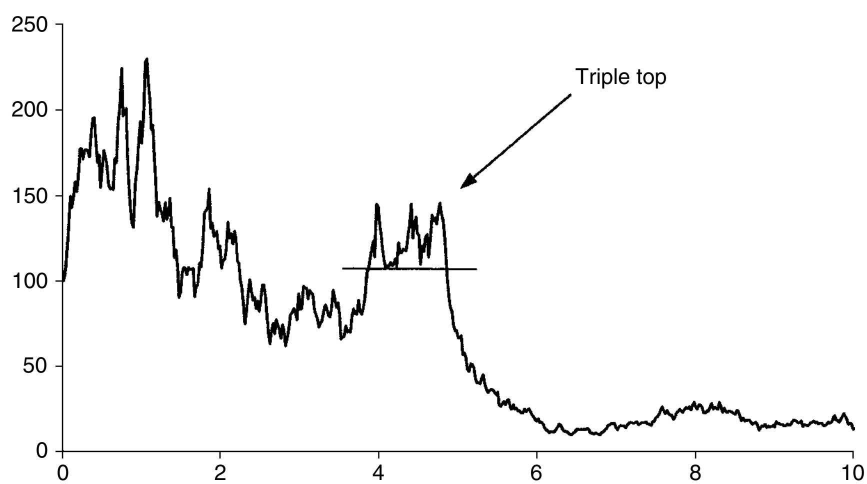


Figure 3.9 A triple top.

Saucer tops and bottoms are also known as **rounding tops** and **bottoms** (Figure 3.8). They are the result of a gradual change in supply and demand. The shape is generally fairly symmetrical as the price rises and falls. These patterns are quite rare. They contain no information about the strength of the new trend.

Double and triple tops and bottoms are quite rare patterns, the triple being even rarer than the double. The double top looks like an 'M' and a double bottom like a 'W.' The triple top is similar but with three peaks, as shown in Figure 3.9. The key point about the peaks and troughs is that they should all be at approximately the same level.

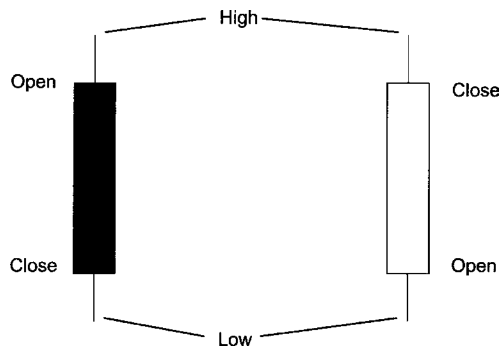


Figure 3.10 Japanese candlesticks.

3.2.9 Japanese candlesticks

Japanese candlesticks contain more information than the simple plots described so far. They record the opening and closing prices as well as the day's high and low. A rectangle is drawn extending from the close to the open, and is colored white if close is above open and black if close is below open. The high-low range is marked by a continuous line.

Certain combinations of candlesticks appearing consecutively have special meanings and names like 'Hanging Man' and 'Upside Gap Two Crows.' See Figure 3.10 for the two types of candlestick and see Figure 3.11 for candlesticks in action. On this chart are shown 'HR' = Bearish Harami, 'D' = Doji (representing indecision), 'BH' = Bullish Harami, 'EL' = Bearish Engulfing Line, and 'H' = Hanging Man (representing reversal after a trend).

Figure 3.12 shows some of the possible candlestick shapes and their interpretation.

3.2.10 Point and figure charts

Point and figure charts are different from the charts described above in that they do not have any explicit timescale on the horizontal axis. Figure 3.13 is an example of a point and figure chart. Each box on the chart represents a prespecified asset price move. The boxes are a way of discretizing asset price moves, instead of discretizing in time. For each consecutive asset price rise of the box size draw an 'X' in the box, in a rising column, one above the other. When this uptrend finishes, and the asset falls, start putting 'O' in a descending column, to the right of the previous rising Xs.

- A long column of Xs denotes demand exceeding supply.
- A long column of Os denotes supply exceeding demand.
- Short up and down columns denote a balance of supply and demand.

3.3 WAVE THEORY

As well as plotting and spotting trends in price movements there have been some theories for price prediction based on market cycles or waves. Below, I briefly mention a couple.

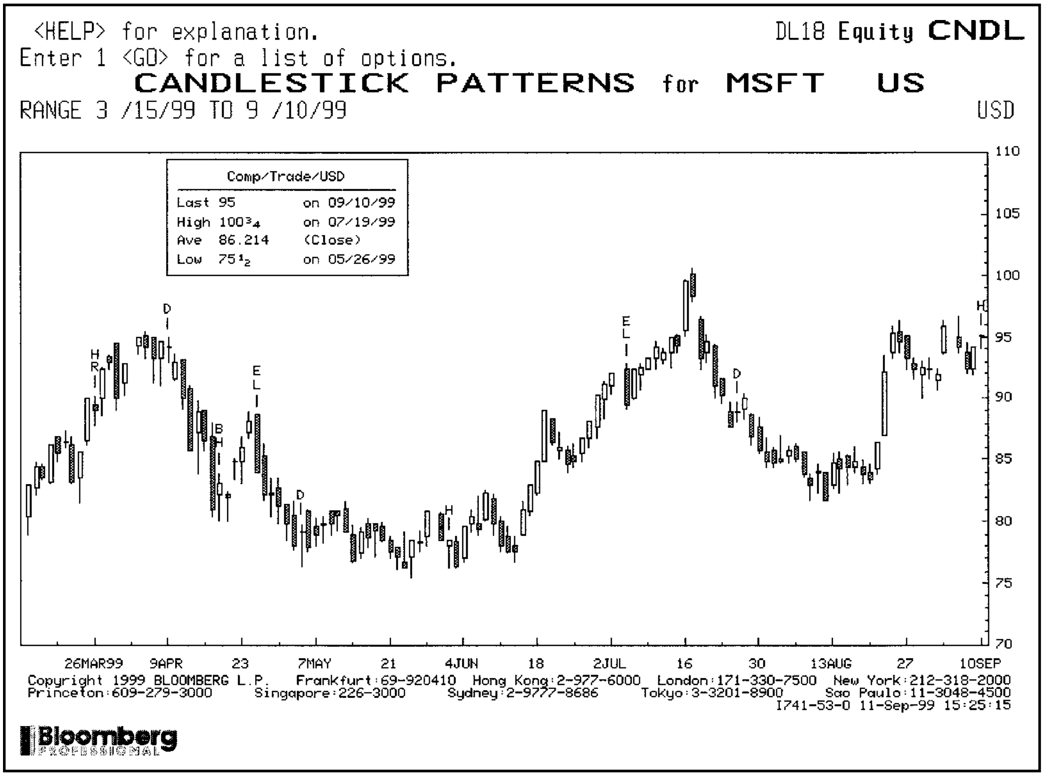
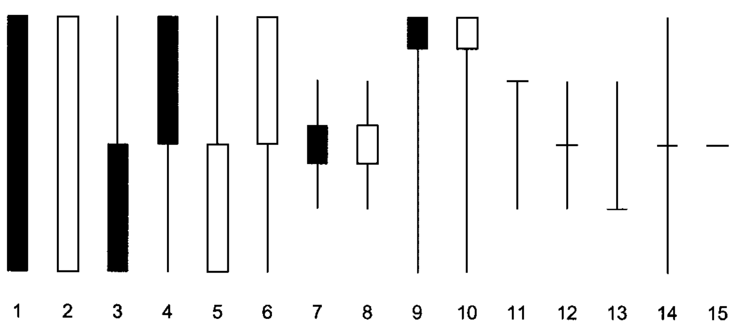


Figure 3.11 A candlestick chart. Source: Bloomberg L.P.



1 Extremely bearish
2 Extremely bullish
3 Bearish
4 Bearish
5 Bullish
6 Bullish
7 Neutral
8 Neutral
9 In downtrend, bullish; in uptrend, bearish
10 In downtrend, bullish; in uptrend, bearish
11 A turning period
12 A turning period
13 End of downtrend
14 A turning period
15 Possible turning period

Figure 3.12 The meanings of the various candlesticks.

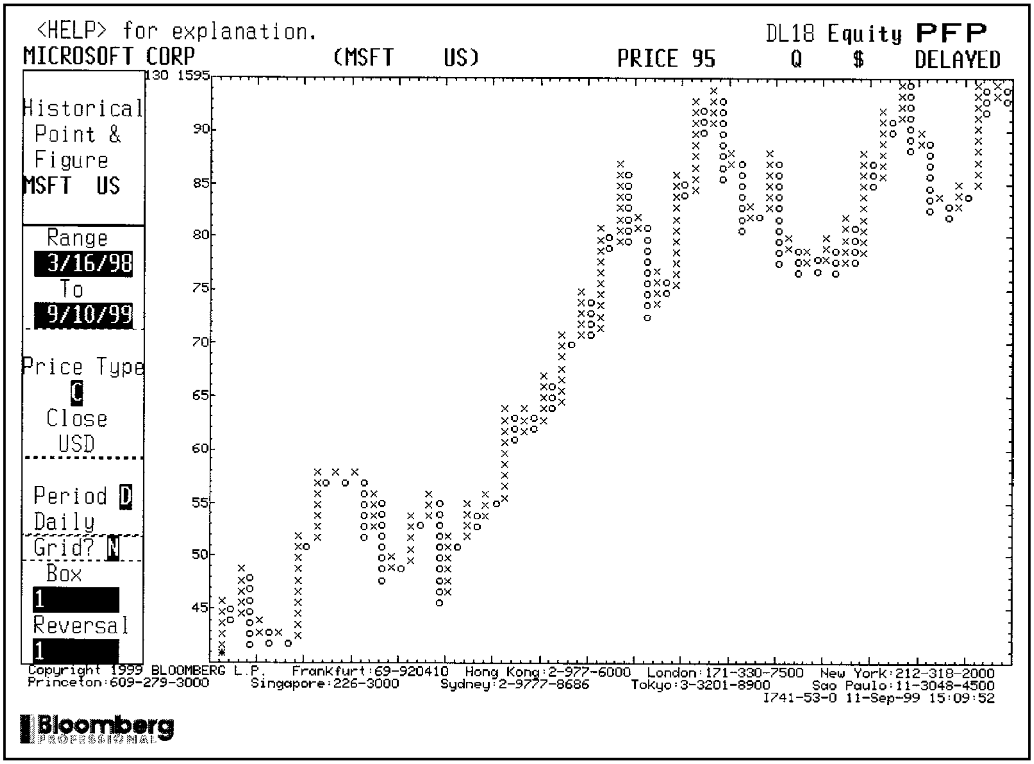


Figure 3.13 A point and figure chart of Microsoft. Source: Bloomberg L.P.

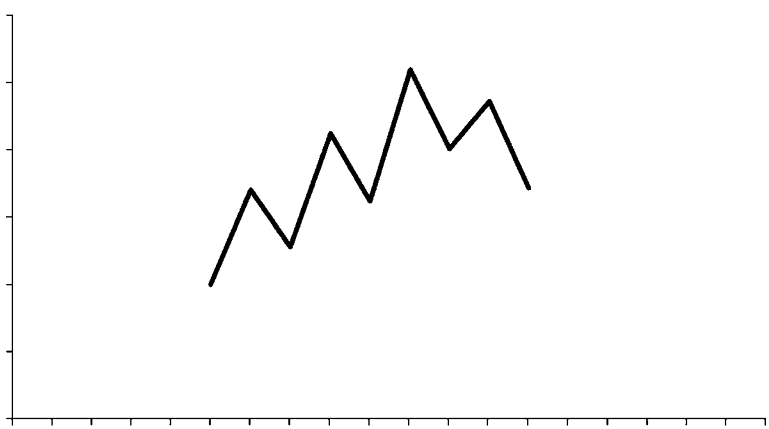


Figure 3.14 Elliot waves.

3.3.1 Elliott waves and Fibonacci numbers

Ralph N. Elliott observed repetitive patterns, waves or cycles in prices. Roughly speaking, there are supposed to be five points in a bullish wave and then three in a bearish one. See Figure 3.14. Within this **Elliott wave theory** there is also supposed to be some predictive

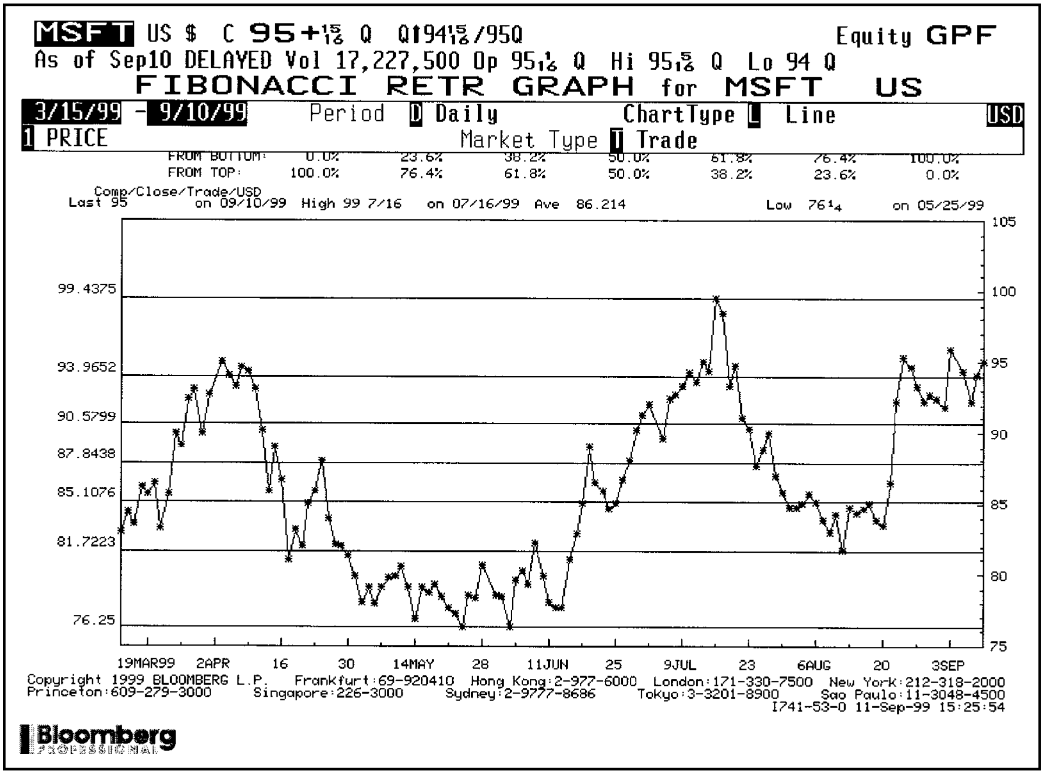


Figure 3.15 Fibonacci lines. Source: Bloomberg L.P.

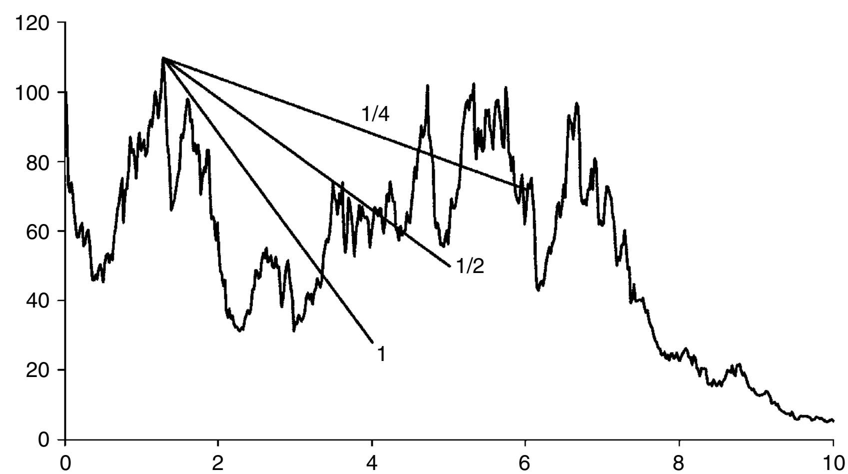


Figure 3.16 Gann charts.

ability in terms of the sizes of the peaks in each wave. For some reason, the ratios of peaks in a trend are supposed to be fairly constant; the ratio of second peak to first should be approximately 1.618 and of the third to the second 2.618. Unfortunately, the number 1.618 is approximately the **Golden ratio** of the ancient Greeks, $\frac{1}{2}\sqrt{5}$. It is also the ratio of successive numbers in the **Fibonacci series** given by $a_n = a_{n-1} + a_{n-2}$ for large n. I say, unfortunately, because people extrapolate wildly from this. And if it's a coincidence then... Figure 3.15 shows the key levels coming from the Fibonacci series.

3.3.2 Gann charts

Figure 3.16 shows a Gann chart. The lines all have slopes which are fractions of the slope of the lowest line. Need I say more?

3.4 OTHER ANALYTICS

There's an almost endless number of ways that chartists analyze data. I'll mention just a couple more before moving on.

Volume is simply the number of contracts traded in a given period. A rising price and high volume means a strong, upwardly trending market. But a rising price with low volume could be a sign that the market is about to turn.

Open interest is the number of still outstanding futures contracts, those which have not been closed out. Because there are equal numbers of buyers and sellers, open interest does not necessarily give any directional info, but an increase in open interest can mean that an existing trend is strong.

3.5 MARKET MICROSTRUCTURE MODELING

The financial markets are made up of many types of players. There are the 'producers' who manufacture or produce or sell various goods and who may be involved in the financial markets for hedging. There are the 'speculators' who try and spot trends in the market, to exploit them and make money. These speculators may be using technical analysis methods, such as those described above, or fundamental analysis, whereby they examine the records and future plans of firms to determine whether stocks are under- or overpriced. Almost all traders use technical analysis at some time. Then there are the market makers who buy and sell financial instruments, holding them for a very short time, often seconds, and profit on bid-offer spreads.

There have been many attempts to model the interaction of these agents, sometimes in a game theoretic way, to try and model the asset price movements that in this book we have taken for granted. For example, can the dynamics induced by the actions of a combination of these three types of agent result in Brownian motion and lognormal random walks?

Below are just a very few examples of work in this area.

3.5.1 Effect of demand on price

Buying and selling assets moves their prices. Market makers respond to demand by increasing price, and reduce prices when the market is selling. If one can model the relationship between demand and price then it should be possible to analyze the effect that various types of technical trading rule have on the evolution of prices. And eventually to model the dynamics of prices.

A common starting point is to assume that there are two types of trader and one market maker. One trader follows a technical trading rule such as watching a moving average and the other is a **noise trader** who randomly buys or sells.

Interesting results follow from such models. For example

- trend followers can induce patterns in asset price time series,
- these artificially induced patterns can only be exploited for gain by someone following a suitably different trend,
- the more people following the same trend as you, the more money you will lose.

There are good reasons for there being genuine trends in the market: There is a slow diffusion of information from the knowledgeable to the less knowledgeable. The piece-by-piece secret acquisition of a company will gradually move a stock price upwards.

On the other hand, if there is no genuine reason for a trend, if it is simply a case of trend followers begetting a trend, then it may be beneficial to be a contrarian.

3.5.2 Combining market microstructure and option theory

Arbitrage does exist, many people make money from its existence. Yet the action of arbitragers will, via a demand/price relationship, remove the arbitrage. But there will be a timescale associated with this removal. What is the optimal way to exploit the arbitrage opportunity while knowing that you actions will to some extent be self-defeating?

3.5.3 Imitation

Another approach to market microstructure modeling is based on the true observation that people copy each other. In these models there are a number of traders who act partly in response to private information about a stock, partly randomly as noise traders, and partly to imitate their nearest neighbors. These models can result in market bubbles or market crashes.

3.6 CRISIS PREDICTION

There has been some work on analyzing data over various timescales to determine the likelihood of a market crash. Some ideas from earthquake modeling have been used to derive a 'Richter'-like measure of market moves. Of course, an effective predictor of market crashes could either

- increase the chance or size of a crash as everyone panics or
- reduce the chance or size of the crash since everyone gets advance warning and can calmy and logically act accordingly.

3.7 SUMMARY

I started out in finance many years ago plotting all of the technical indicators. I was not very successful at it. I could only get directions right for those assets with obvious seasonality effects, such as some commodities.

There is only one technical indicator that I believe in. There is definitely a strong correlation between hemlines and the state of the economy. The shorter the skirts, the better the economy.

FURTHER READING

- The book on technical analysis written by the news agency Reuters (1999) is excellent, as is Meyers (1994).
- Farmer (2000) discusses and models trend following and the creation of trends. He also demonstrates properties of the relationship between demand and price that prevent arbitrage.
- Dewynne & Wilmott (1999) show how to optimally exploit an arbitrage opportunity while moving the market as little as possible.
- Bhamra (2000) has worked on imitation in financial markets.
- Olsen & Associates `www.olsen.ch` are currently working in the area of crisis modeling and prediction.
- Johnson *et al*. (1999) model self-organized segregation of traders, and conclude that cautious traders perform poorly.
- The above is only a brief description of a very few examples from an expanding field. See O'Hara (1995) for a wide-ranging discussion of market microstructure models.
- Bernstein (1998) has a whole chapter on the Golden ratio.
- Elton & Gruber (1995) describe the efficient market hypothesis and criticize technical analysis.
- Prast (2000a,b) discusses 'herding' in the financial markets.

CHAPTER 4

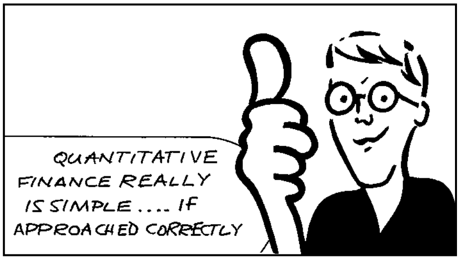
all the math you need. . .and no more (an executive summary)

The aim of this Chapter. . .

. . . is to give you an understanding of almost all the mathematics you need in quantitative finance, and to give you the confidence to read through the more technical parts of this and other books.

In this Chapter. . .

- e
- log
- differentiation and Taylor series
- expectations and variances



4.1 INTRODUCTION

This book is for *everyone* interested in quantitative finance. This subject is increasingly technical. Some people don't have a high level math training but may still be interested in the technical side of things. In this chapter we look at the mathematics that you need to cope with the vast majority of derivatives theory and practice. Although often couched in very high level mathematics almost all finance theory can be interpreted using only the basics that I describe here.

The most useful math is the simplest math. This is particularly true in finance where the beauty of the mathematics can and does lead to people not seeing the wood for the trees. All basic finance theory and a great deal of the advanced research really require only elementary mathematics if approached in the right way. In this chapter I explain this elementary mathematics. I am trying to do for Mathematical Finance what Seuss (1999) did for the English language.

4.2 e

The first bit of math you need to know about, and which you've already seen, is e.

e is

- a number, 2.7183...
- a function when written e^x, this function is a.k.a. $\exp(x)$

The function e^x is just the number 2.7183... raised to the power x; e^2 is just $2.7183\ldots^2 = 7.3891\ldots$, e^1 is 2.7183... and $e^0 = 1$. What about noninteger powers?

The function e^x can be written as an infinite series

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!}.$$

This gets around the noninteger power problem.

A plot of e^x as a function of x is shown in Figure 4.1.

The function e^x has the special property that the slope or gradient of the function is also e^x. Plot this slope as a function of x and for e^x you get the same curve again. It follows that the slope of the slope is also e^x, etc. etc.

4.3 log

Take the plot of e^x, Figure 4.1, and rotate it about a 45° line to get Figure 4.2. This new function is $\ln x$, the Naperian logarithm of x. The relationship between ln and e is

$$e^{\ln x} = x \quad \text{or} \quad \ln(e^x) = x.$$

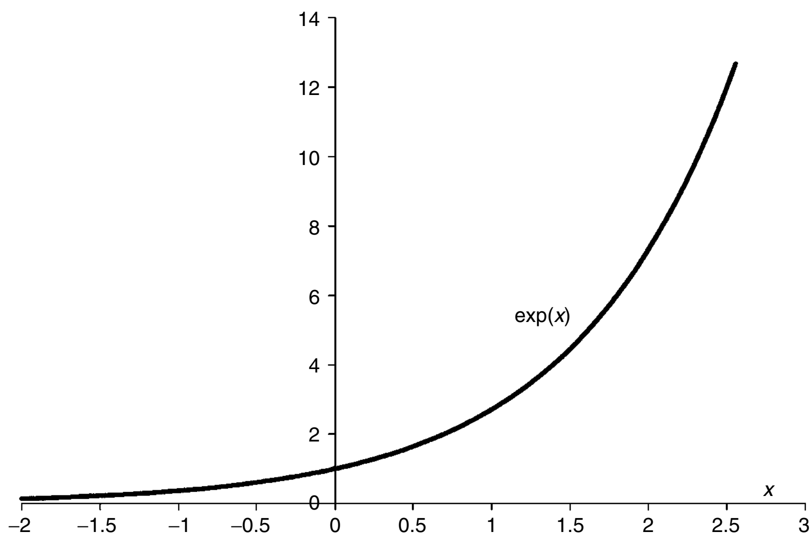


Figure 4.1 The function e^x.

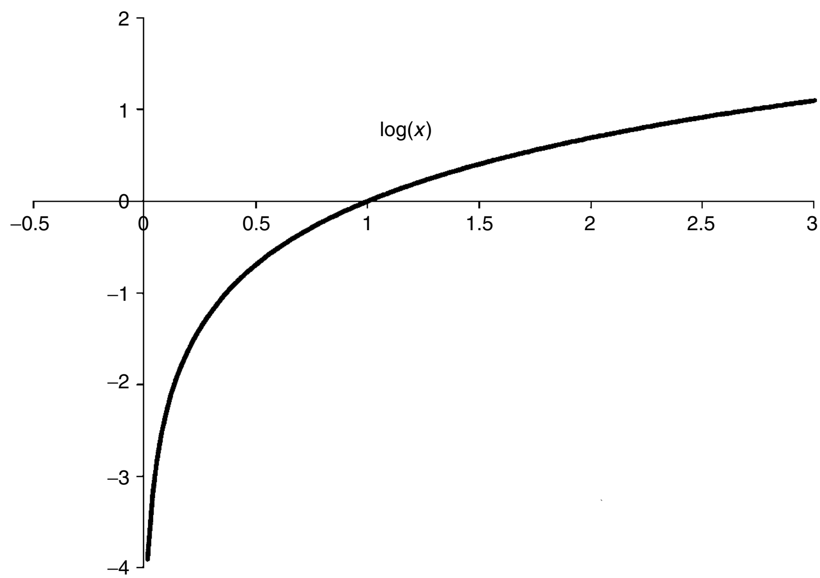


Figure 4.2 The function $\log x$.

So, in a sense, they are inverses of each other.

The function $\ln x$ is also often denoted by $\log x$, as in this book. Sometimes $\log x$ refers to the function with the properties

$$10^{\log x} = x \quad \text{and} \quad \log(10^x) = x.$$

This function would be called 'logarithm base ten.' The most useful logarithm has base $e = 2.7183\ldots$ because of the properties of the gradient of e^x.

The slope of the $\log x$ function is x^{-1}.

From the figure you can see that there don't appear to be any values for $\log x$ for negative x. The function can be defined for these but you'd need to know about complex numbers, something we won't be requiring here.

4.4 DIFFERENTIATION AND TAYLOR SERIES

I've introduced the idea of a gradient or slope in the section above. If we have a function denoted by $f(x)$, then we denote the gradient of this function at the point x by

$$\frac{df}{dx}.$$

Mathematically the slope is defined as

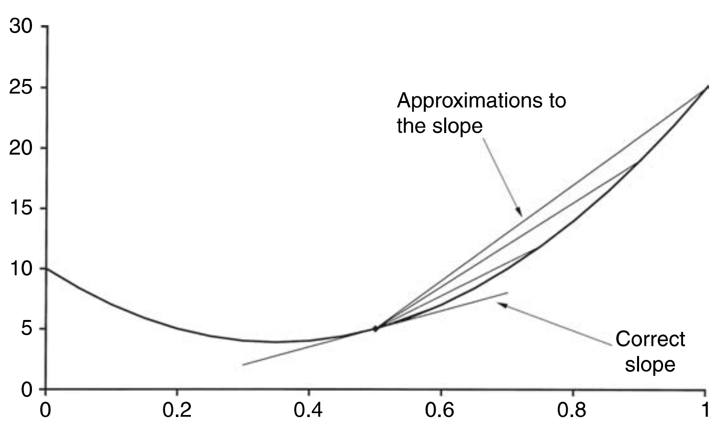
$$\frac{df}{dx} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}.$$

The action of finding the gradient is also called 'differentiating' and the slope can also be called the 'derivative' of the function. This use of 'derivative' shouldn't be confused with the use meaning an option contract.

Time Out...

Differentiation

As I've said before, the slope is exactly the same as the gradient you experience in your car, and represented by signs such as '1-in-10 gradient.' Look at the figure below to see how the limit δx tending to zero might work in practice.



The slope can also be differentiated, resulting in a second derivative of the function $f(x)$. This is denoted by

$$\frac{d^2 f}{dx^2}.$$

We can take this differentiation to higher and higher orders.

Take a look at Figure 4.3. In particular, note the two dots marked on the bold curve. The bold curve is the function $f(x)$. The dot on the left is at the point x on the horizontal axis and the function value is $f(x)$, the distance up the vertical axis. The dot to the right of this is at $x + \delta x$ with function value $f(x + \delta x)$. What can we say about the vertical distance between the two dots in terms of the horizontal distance?

Start with a trivial example. If the distance δx is zero then the vertical distance is also zero. Now consider a very small but nonzero δx.

The straight line tangential to the bold curve $f(x)$ at the point x is shown in the figure. This line has slope df/dx evaluated at x. Notice that the right-hand hollow dot is almost on this bold line. This suggests that a good approximation to the value $f(x + \delta x)$ is

$$f(x + \delta x) \approx f(x) + \delta x \frac{df}{dx}(x).$$

This is a linear relationship between $f(x + \delta x) - f(x)$ and δx. This makes sense since on rearranging we get

$$\frac{df}{dx} \approx \frac{f(x + \delta x) - f(x)}{\delta x}$$

which as δx goes to zero becomes our earlier definition of the gradient.

But the right-hand hollow dot is not *exactly* on the straight line. It is slightly above it. Perhaps a quadratic relationship between $f(x + \delta x) - f(x)$ and δx would be a more accurate approximation. This is indeed true (provided δx is small enough) and we can write

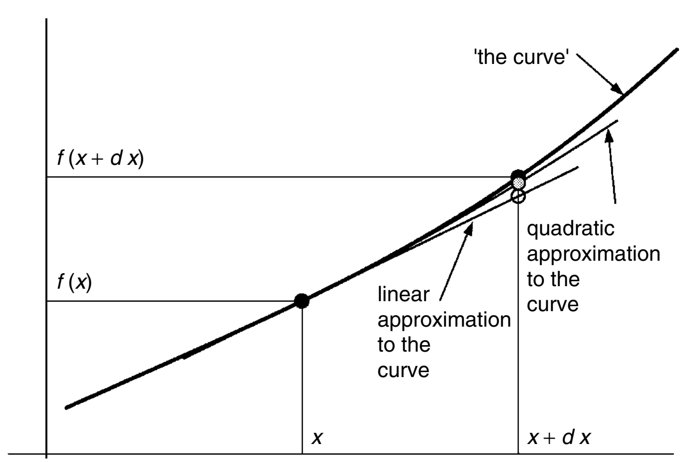


Figure 4.3 A schematic diagram of Taylor series.

$$f(x+\delta x) \approx f(x) + \delta x \frac{df}{dx}(x) + \frac{1}{2}\delta x^2 \frac{d^2 f}{dx^2}(x).$$

This approximation, shown on the figure as the grey dot, is more accurate. One can take this approximation to cubic, quartic, ... The Taylor series representation of $f(x+\delta x)$ is the infinite sum

$$f(x+\delta x) = f(x) + \sum_{i=1}^{\infty} \frac{1}{i!}\delta x^i \frac{d^i f}{dx^i}(x).$$

Taylor series are incredibly useful in derivatives theory. In derivatives theory the function that we are interested in, instead of being f, is V, the value of an option. The independent variable is no longer x but is S, the price of the underlying asset. From day to day the asset price changes by a small, random amount. This asset price change is just δS (instead of δx). The first derivative of the option value with respect to the asset is known as the delta, and the second derivative is the gamma.

The value of an option is not only a function of the asset price S but also the time t: $V(S,t)$. This brings us into the world of partial differentiation.

Think of the function $V(S,t)$ as a surface with coordinates S and t on a horizontal plane. The function V is the height of a hill above sea level with S and t being distances, or map coordinates, in the northerly and westerly directions. The *partial* derivative of $V(S,t)$ with respect to S is written

$$\frac{\partial V}{\partial S}$$

and is defined as

$$\frac{\partial V}{\partial S} = \lim_{\delta S \to 0} \frac{V(S+\delta S, t) - V(S,t)}{\delta S}.$$

Note that in this V is only ever evaluated at time t. This is like measuring the gradient of the function $V(S,t)$ in the S direction along a constant value of t i.e. the slope of our hillside in the northerly direction. Note also that we are now using a special curly ∂ instead of the italic d. The partial derivative of $V(S,t)$ with respect to t is similarly defined as

$$\frac{\partial V}{\partial t} = \lim_{\delta t \to 0} \frac{V(S, t+\delta t) - V(S,t)}{\delta t}$$

and is the slope of the hill in the westerly direction.

Higher-order derivatives are defined in the obvious manner.

The Taylor series expansion of the value of an option is then

$$V(S+\delta S, t+\delta t) \approx V(S,t) + \delta t \frac{\partial V}{\partial t} + \delta S \frac{\partial V}{\partial S} + \frac{1}{2}\delta S^2 \frac{\partial^2 V}{\partial S^2} + \cdots.$$

This series goes on for ever, but I've only written down the largest and most important terms, those which are required for the Black–Scholes analysis.

4.5 DIFFERENTIAL EQUATIONS

Right at the beginning of the book we saw a differential equation. Remember the equation for money in the bank

$$\frac{dM}{dt} = rM?$$

This is an example of an **ordinary partial differential equation**. 'Ordinary' because it involves only derivatives with respect to a single variable; there are no curly ∂s. This equation can be solved to find M as a function of the other variable t.

Throughout this book we'll be seeing **partial differential equations**. These involve the curly ∂s because there are derivatives/slopes/gradients with respect to more than one variable. The Black–Scholes equation is an example. In our mountain example, think of a partial differential equation as relating the height of the mountain to its slope in each direction, and maybe even the slope of the slope. Again, such equations must be solved to find the height of the mountain, the option value. How this is done analytically (if possible) or numerically (always possible) will be discussed later.

4.6 MEAN, STANDARD DEVIATION AND DISTRIBUTIONS

Much of the modeling in finance uses ideas from probability theory. Again you don't need to know that much to understand most of quantitative finance.

The first idea is that of the mean. The mean is the same as the average or the expectation. What is the average shoe size of US males?

If you roll a die there is an equal, $\frac{1}{6}$, probability of each number coming up. What is the expected number or the average number if you roll the die many times? The answer is

$$1 \times \tfrac{1}{6} + 2 \times \tfrac{1}{6} + 3 \times \tfrac{1}{6} + 4 \times \tfrac{1}{6} + 5 \times \tfrac{1}{6} + 6 \times \tfrac{1}{6} = 3\tfrac{1}{2}.$$

Here we just multiply each of the possible numbers that could turn up by the probability of each, and sum. Although $3\frac{1}{2}$ is the expected value, it cannot, of course, be thrown since only integers are possible.

Generally, if we have a random variable X (the number thrown, say) which can take any of the values x_i (1,2,3,4,5,6 in our example) for $i = 1, \ldots, N$ each of which has a probability $P(X = x_i)$ (in the example, $\frac{1}{6}$) then the expected value is

$$E[X] = \sum_{i=1}^{N} x_i P(X = x_i).$$

Expectations have the following properties:

$$E[cX] = cE[X]$$

and

$$E[X + Y] = E[X] + E[Y].$$

If the outcomes of two random events X and Y have no impact on each other they are said to be independent.

If X and Y are independent we have

$$E[XY] = E[X]E[Y].$$

Expectations are important in finance because we often want to know what we can expect to make from an investment on average.

The expectation or mean is also known as the first moment of the distribution of the random variable X. It can be thought of as being a typical value for X. The scatter of values around the mean can be measured by the second moment or the variance:

$$\text{Var}(X) = E\left[(X - E[X])^2\right].$$

Variances have the following property:

$$\text{Var}(cX) = c^2\text{Var}(X).$$

When X and Y are independent

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y).$$

The standard deviation is the square root of the variance and is perhaps more useful as a measure of dispersion since it has the same units as the variable X:

$$\text{Standard deviation}(X) = \sqrt{\text{Var}(X)}.$$

The standard deviation is just a measure of the spread around the mean. The smaller the standard deviation of shoe sizes, the less variation in sizes among the population. If the standard deviation is large, there is a great deal of variation in shoe sizes.

Standard deviations are important in finance because they are often used as a measure of risk in an investment. The higher the standard deviation of investment returns the greater the dispersion of the returns and the greater the risk.

If we were to plot a bar chart (histogram) of number of the population with each shoe size on the vertical axis and shoe size on the horizontal we'd have a frequency distribution of shoe sizes. Now scale the height of each bar with the total number in the population and you've got a probability density function. Why did I ask you to scale the distribution? So that the heights of all the bars add up to one, the probability of having *some* shoe size is one.

That was an example of a discrete distribution. Shoe sizes are a discrete set. Contrast that with foot length. The length of a foot can be any number, not confined to a discrete set. Foot lengths form a continuous distribution. In practice, given a continuous distribution such as foot length you will have to assign each length to a finite-sized 'bucket.' Think of the bucket as being the corresponding shoe size. Figures 4.4 through 4.7 a few examples of probability density functions.

4.7 SUMMARY

The math you need in quantitative finance is surprisingly little. And you've just seen all of it.

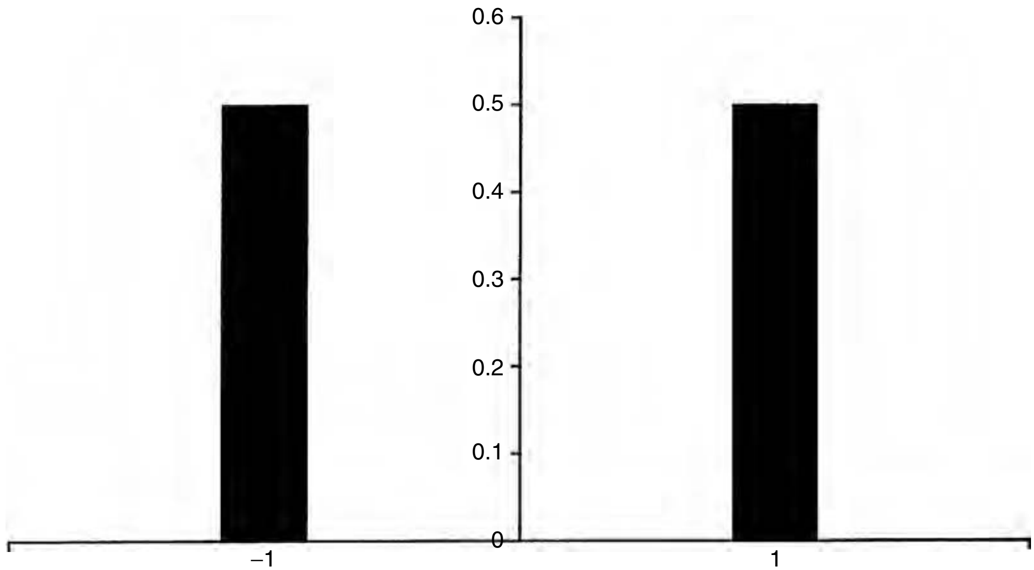


Figure 4.4 Probability density functions for tossing a fair coin; a head wins $1 a tail loses $1. The mean is zero.

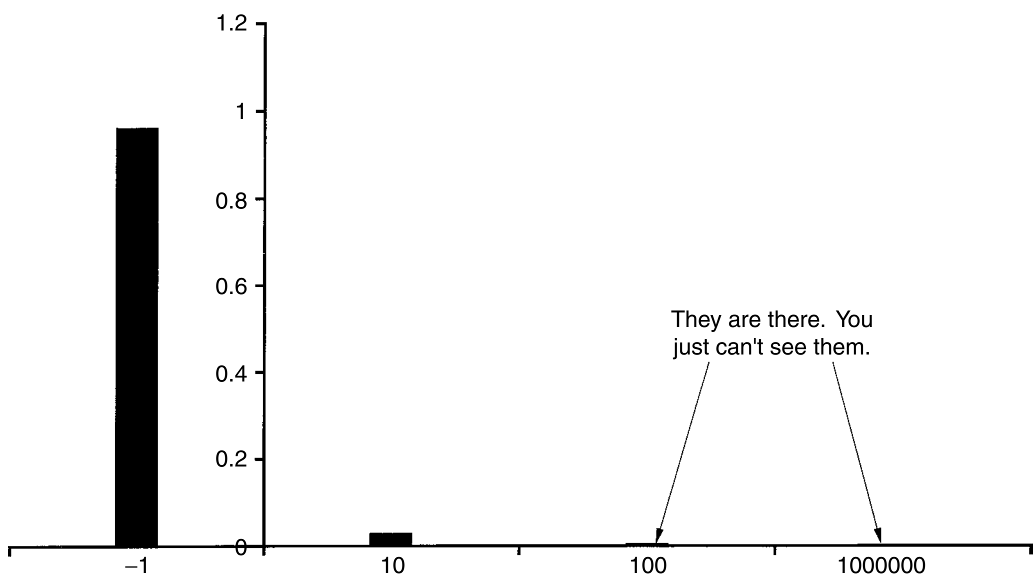


Figure 4.5 Probability density function for a lottery. The mean is negative.

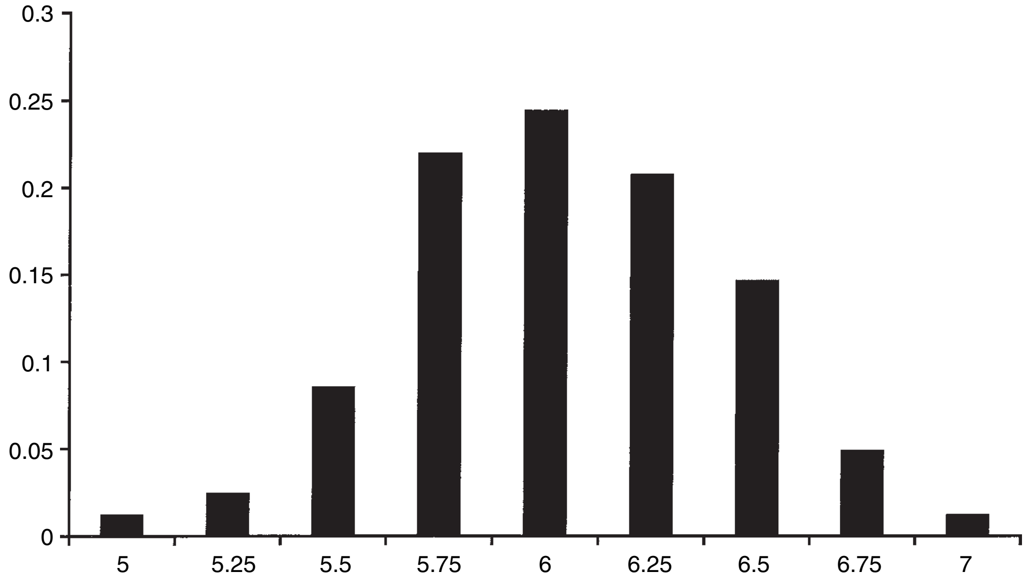


Figure 4.6 Probability density function for right-hand middle finger lengths of female quantitative analysts. Observe that it is a discrete distribution, I didn't have enough data to make it continuous.

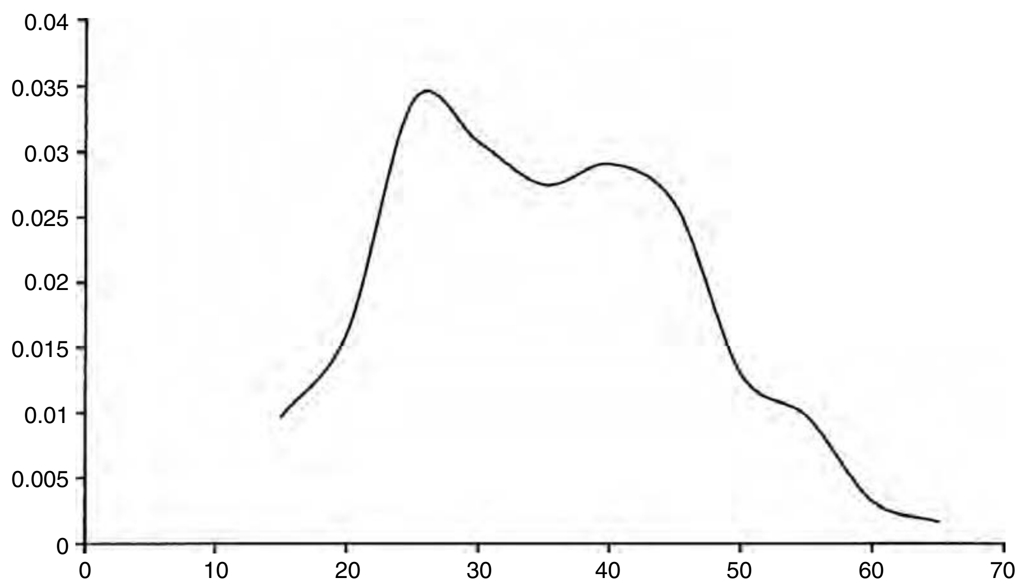


Figure 4.7 Probability density function for the amount I spend on Amazon at 3 am when I can't sleep. A continuous distribution.

CHAPTER 5

the binomial model

The aim of this Chapter...

... is to describe the simplest model for asset prices that can be, and is, used for pricing derivatives, to introduce the two fundamental building blocks of quantitative finance and to explain a very important (but very counterintuitive) financial concept. By the end of this chapter you will be able to write a program for pricing basic derivatives. I hope you won't be too confused by the important but counterintuitive financial concept... even if you are we'll come back to it many more times.

In this chapter...

- a simple model for an asset price random walk
- delta hedging
- no arbitrage
- the basics of the binomial method for valuing options
- risk neutrality

5.1 INTRODUCTION

In this chapter I'm going to present a very simple and popular model for the random behavior of an asset, for the moment think 'equity.' This simple model will allow us to start valuing options. Undoubtedly, one of the reasons for the popularity of this model is that it can be implemented without any higher mathematics (such as differential calculus). This is a positive point; however, the downside is that it is harder to attain greater levels of sophistication or numerical analysis in this setting.

Later we'll be seeing a more sophisticated model, but the ideas we first encounter in this chapter will be seen over and over again. These are the fundamental concepts of hedging and no-arbitrage.

The binomial model is very important because it shows how to get away from a reliance on closed-form solutions. Indeed, it is extremely important to have a way of valuing options that only relies on a simple model and fast, accurate numerical methods. Often in real life a contract may contain features that make analytic solution very hard or impossible. Some of these features may be just a minor modification to some other, easily priced, contract but even minor changes to a contract can have important effects on the value and especially on the method of solution. The classic example is of the American put. Early exercise may seem to be a small change to a contract but the difference between the values of a European and an American put can be large and certainly there is no simple closed-form solution for the American option and its value must be found numerically.

Time Out...

Simplicity itself

The math in this chapter is all very straightforward, addition, subtraction, multiplication and, occasionally, division.

5.2 EQUITIES CAN GO DOWN AS WELL AS UP

In the binomial model we assume that the asset, which initially has the value S, can, during a timestep δt, either rise to a value uS or fall to a value vS, with $0 < v < 1 < u$. Here uS means u multiplied by S, and similarly for vS. The probability of a rise is p and so the probability of a fall is $1 - p$. This behavior is shown in Figure 5.1.

Example: $u = 1.01$, $v = 0.99$, $p = 0.55$ and the current asset price S is 100. So there is a 55% chance that the stock price will next be 101 and a 45% chance it will be 99. What does 'next' mean and how did I choose these numbers?

'Next' means after a small timestep, say one day. So we will be looking at what happens from one day to the next. How I chose u, v and p is something I will come back to shortly.

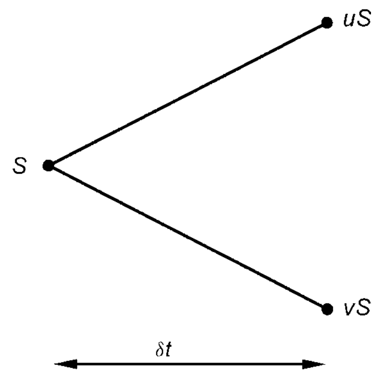


Figure 5.1 A schematic diagram of one timestep in the life of a binomial walk.

Now let's assume that we hold a call option on this asset that is going to expire tomorrow (that's a time δt later). This option has a strike of 100. If the asset rises to 101 we get a payoff of 1, if it falls to 99 we get no payoff, the asset expires out of the money.

Here's a trick, it's called **hedging**. Let's sell short a quantity Δ of the underlying asset so that now we have a portfolio consisting of a long option position and a short stock position. We'll do that calculation again.

If the asset rises to 101 we have a *portfolio* worth

$$\max(101 - 100, 0) - \Delta \times 101 = 1 - 101\Delta.$$

If the asset falls we have

$$\max(99 - 100, 0) - \Delta \times 99 = -99\Delta.$$

This portfolio is risky in the sense that there are two values that it can take, we don't know what the portfolio will be worth tomorrow. Or is it? Suppose we choose Δ such that

$$1 - 101\Delta = -99\Delta,$$

i.e.,

$$\Delta = \frac{1}{2},$$

then whether the asset rises or falls our portfolio has a value of

$$1 - 101\Delta = -99\Delta = -\frac{99}{2}.$$

There is no risk, we are guaranteed this amount of money *irrespective of the behavior of the underlying*. That is hedging. Clever.

We are now halfway to valuing this option today, one day before expiry. The second and final step, is to say that if the portfolio has a guaranteed payoff then the return must be the same as the risk-free rate applied over the one day. If V is the option value today then our portfolio's value is currently

$$V - \Delta \times 100,$$

for some V to be found. The present value of the portfolio value, discounting at an interest rate of r, is

$$-\frac{1}{1+r\,\delta t}\,\frac{99}{2}.$$

This must be the same as the portfolio value today so

$$V - \Delta \times 100 = V - \frac{1}{2}100 = V - 50 = -\frac{1}{1+r/252}\,\frac{99}{2}.$$

Put in the relevant r and calculate V from this.

$$V = 50 - \frac{1}{1+r/252}\,\frac{99}{2}.$$

Simple. Note that I've assumed 252 business days in one year so that $\delta t = \frac{1}{252}$. If $r = 10\%$ then $V = 0.5196$.

The conclusion is that the option value depends on the interest rate, the payoff, the size of the up move, the size of the down move and the timestep. *But it does not depend on the probability of the up move*. p never came into the calculation.

This is completely counterintuitive. Surely the value of an option depends on whether the asset is likely to go up or down. It turns out that this is not the case. I'll expand on this idea further shortly, but first let's do the same calculation in general.

5.3 LET'S GENERALIZE

The three constants u, v and p are chosen to give the binomial walk the same characteristics as the asset we are modeling. I'm going to relate these quantities to the average change in the asset from one day to the next, its drift rate, if you like, and its range of values from one day to the next, its volatility. We will come back to these quantities several times in the next few chapters as we refine our basic asset price model.

First choose a timestep over which the asset move takes place. This should be something short, such as a day. Mathematically we denote this timestep by δt. I'm going to give some expressions now for u, v and p and then explain where they come from:

$$u = 1 + \sigma\sqrt{\delta t},$$
$$v = 1 - \sigma\sqrt{\delta t}$$

and

$$p = \frac{1}{2} + \frac{\mu\sqrt{\delta t}}{2\sigma}. \tag{5.1}$$

I have introduced two new parameters here: μ the drift of the asset and σ the volatility. To see what these mean let's look at the average change in asset price during the timestep and the standard deviation.

5.3.1 Average asset change

The expected asset price after one timestep is

$$puS + (1-p)vS = \left(\frac{1}{2} + \frac{\mu\sqrt{\delta t}}{2\sigma}\right)\left(1+\sigma\sqrt{\delta t}\right)S$$
$$+\left(\frac{1}{2} - \frac{\mu\sqrt{\delta t}}{2\sigma}\right)\left(1-\sigma\sqrt{\delta t}\right)S = (1+\mu\,\delta t)S.$$

So the expected change in the asset is $\mu S\,\delta t$. The average return is thus $\mu\,\delta t$. Something we can easily measure statistically if we have asset data. And we'll be doing just that in the next chapter.

5.3.2 Standard deviation of asset price change

The variance of the change in asset price is

$$S^2\left(p(u-1-\mu\,\delta t)^2 + (1-p)(v-1-\mu\,\delta t)^2\right)$$
$$= S^2\left(\left(\frac{1}{2} + \frac{\mu\sqrt{\delta t}}{2\sigma}\right)\sigma^2\delta t + \left(\frac{1}{2} - \frac{\mu\sqrt{\delta t}}{2\sigma}\right)\sigma^2\delta t\right) = S^2\sigma^2\delta t - S^2\mu^2\delta t^2.$$

So the standard deviation is approximately $S\sigma\sqrt{\delta t}$. The standard deviation of returns is $\sigma\sqrt{\delta t}$.

We will measure μ and σ in the next chapter, and also talk about their significance in option pricing. For the moment pretend that we know these parameters.

5.4 THE BINOMIAL TREE

The binomial model, just introduced, allows the stock to move up or down a prescribed amount over the next timestep. If the stock starts out with value S then it will take either the value uS or vS after the next timestep. We can extend the random walk to the next timestep. After two timesteps the asset will be at either u^2S, if there were two up moves, uvS, if an up was followed by a down or vice versa, or v^2S, if there were two consecutive down moves. After three timesteps the asset can be at u^3S, u^2vS, etc. One can imagine extending this random walk out all the way until expiry. The resulting structure looks like Figure 5.2 where the nodes represent the values taken by the asset. This structure is called the **binomial tree**. Observe how the tree bends due to the geometric nature of the asset growth. Often this tree is drawn as in Figure 5.3 because it is easier to draw, but this doesn't quite capture the correct structure.

The top and bottom branches of the tree at expiry can only be reached by one path each, either all up or all down moves. However, there will be several paths possible for each of the intermediate values at expiry. Therefore the intermediate values are more likely to be reached than the end values if one were doing a simulation. The binomial tree therefore contains within it an approximation to the probability density function for the lognormal random walk.

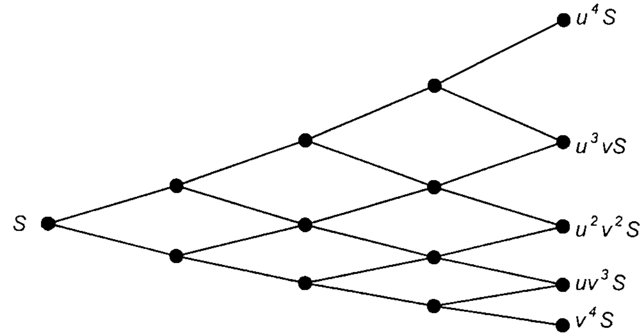


Figure 5.2 The binomial tree.

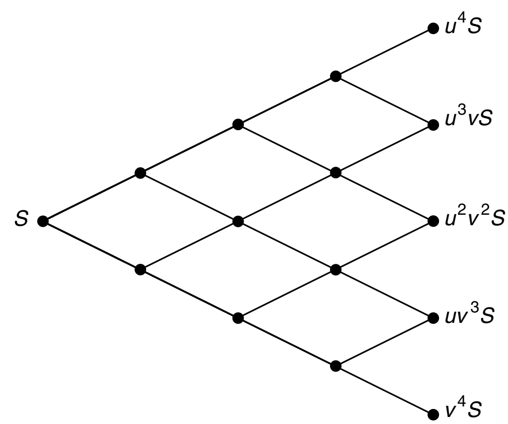


Figure 5.3 The binomial tree: a schematic version.

5.5 THE ASSET PRICE DISTRIBUTION

The probability of reaching a particular node in the binomial tree depends on the number of distinct paths to that node and the probabilities of the up and down moves. Since up and down moves are approximately equally likely and since there are more paths to the interior prices than to the two extremes we will find that the probability distribution of future prices is roughly bell shaped. In Figure 5.4 is shown the number of paths to each node after four timesteps and the probability of getting to each. In Figure 5.5 this is interpreted as probability density functions at a sequence of times.

5.6 AN EQUATION FOR THE VALUE OF AN OPTION

Suppose, for the moment, that we know the value of the option at the time $t + \delta t$. For example, this time may be the expiry of the option, say. Now construct a portfolio at time t consisting of one option and a short position in a quantity Δ of the underlying. At time t this portfolio has value

$$\Pi = V - \Delta S,$$

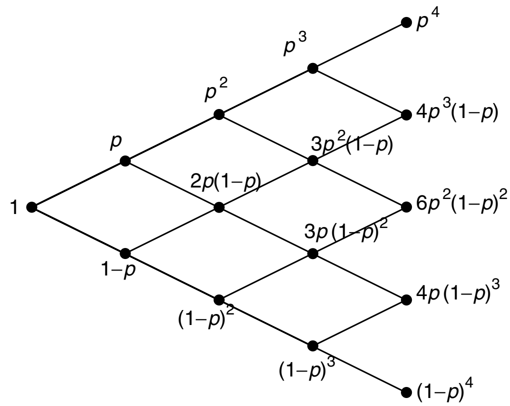


Figure 5.4 Counting paths.

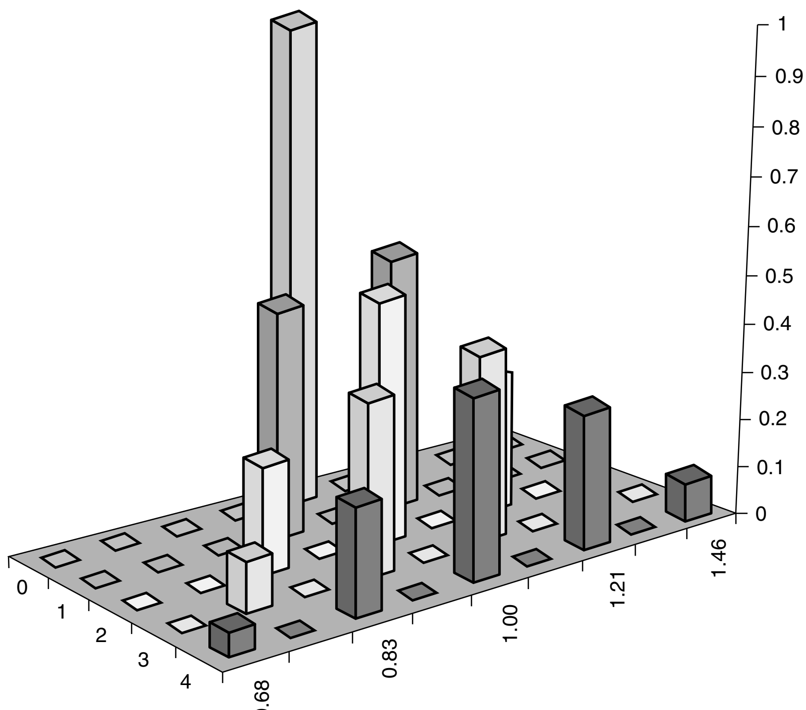


Figure 5.5 The probability distribution of future asset prices.

where the value V is to be determined. You'll recognize this as exactly what we did before, but now we're using symbols instead of numbers.

At time $t + \delta t$ the portfolio takes one of two values, depending on whether the asset rises or falls. These two values are

$$V^+ - \Delta uS \text{ and } V^- - \Delta vS.$$

V^+ is the option value (payoff) if the asset rises and V^- the value if it falls. Since we assume that we know V^+, V^-, u, v, S and Δ, the values of both of these expressions are known, and, in particular, depend on Δ.

5.6.1 Hedging

Having the freedom to choose Δ, we can make the value of this portfolio the same whether the asset rises or falls. This is ensured if we make

$$V^+ - \Delta uS = V^- - \Delta vS.$$

This gives us the choice

$$\Delta = \frac{V^+ - V^-}{(u - v)S}, \tag{5.2}$$

when the new portfolio value is

$$\Pi + \delta\Pi = V^+ - \frac{u(V^+ - V^-)}{(u - v)} = V^- - \frac{v(V^+ - V^-)}{(u - v)}.$$

5.6.2 No arbitrage

Since the value of the portfolio has been guaranteed, we can say that its value must coincide with the value of the original portfolio plus any interest earned at the risk-free rate; this is the no-arbitrage argument. Thus

$$\delta\Pi = r\Pi\,\delta t.$$

Putting everything together we get

$$(1 + r\,\delta t)\left(V - \frac{V^+ - V^-}{(u - v)}\right) = V^- - \frac{v(V^+ - V^-)}{(u - v)}.$$

Rearranging we get

$$(1 + r\,\delta t)V = (1 + r\,\delta t)\frac{V^+ - V^-}{u - v} + \frac{uV^- - vV^+}{(u - v)}. \tag{5.3}$$

This, then, is an equation for V given V^+ and V^-, the option values at the next timestep, and the parameters u and v describing the random walk of the asset.

Equation (5.3) can also be written as

$$(1 + r\,\delta t)V = p'V^+ + (1 - p')V^-, \tag{5.4}$$

where

$$p' = \frac{1}{2} + \frac{r\sqrt{\delta t}}{2\sigma}. \tag{5.5}$$

The right-hand side of Equation (5.4) is just like an expectation; it's the sum of probabilities multiplied by events. If only the expression contained p, the real probability of a stock rise, then this expression would be the expected value at the next timestep.

The left-hand side of Equation (5.4) is the future value of today's option value.

We see that the probability of a rise or fall is irrelevant as far as option pricing is concerned. But what if we interpret p' as a probability? Then we could 'say' that the option price is the present value of an expectation. But not the real expectation.

Let's compare the expression for p' with the expression for the actual probability p:

$$p = \frac{1}{2} + \frac{\mu\sqrt{\delta t}}{2\sigma}.$$

The two expressions differ in that where one has the interest rate r the other has the drift μ, but are otherwise the same. Strange. We call p' the **risk-neutral probability**. It's like the real probability, but the real probability if the drift rate were r instead of μ.

Observe that the risk-free interest plays two roles in option valuation. It's used once for discounting to give present value, and it's used as the drift rate in the asset price risk-neutral random walk.

5.7 WHERE DID THE PROBABILITY *p* GO?

What happened to the probability p and the drift rate μ?

Interpreting p' as a probability, (5.4) is the statement that the option value at any time is the present value of the expected value at any later time. That's because the up move value V^+ is multiplied by a probability and the down move value V^- is multiplied by one minus that probability.

In reading books or research papers on mathematical finance you will often encounter the expression 'risk-neutral' this or that, including the expression risk-neutral probability. You can think of an option value as being the present value of an expectation, only it's not the real expectation. Don't worry we'll come back to this several more times until you get the hang of it.

5.8 OTHER CHOICES FOR *u*, *v* AND *p*

We don't have to choose the above expressions for u, v and p. After all, we only want to ensure that we get the right drift and volatility, so that's three equations for two unknowns essentially. A very common choice for u, v and p is such that $uv = 1$, this ensures that after an up move followed by a down move we are back where we started. In this case we would choose[1]

$$u = \frac{1}{2}\left(e^{-r\,\delta t} + e^{(r+\sigma^2)\delta t}\right) + \frac{1}{2}\sqrt{\left(e^{-r\,\delta t} + e^{(r+\sigma^2)\delta t}\right)^2 - 4}.$$

$$v = \frac{1}{u}$$

and

$$p' = \frac{e^{r\,\delta t} - v}{u - v}.$$

These are the forms used in the following examples, note that μ becomes r when we are valuing options. Remember risk neutrality.

[1] Approximations to these that are good enough for most purposes are $u \approx 1 + \sigma\ \delta t^{1/2} + \frac{1}{2}\sigma^2\delta t$, $\approx 1 - \sigma\ \delta t^{1/2} + \frac{1}{2}\sigma^2\delta t$ and $p' \approx \frac{1}{2} + \frac{\left(r - \frac{1}{2}\sigma^2\right)\delta t^{1/2}}{2\sigma}$.

5.9 VALUING BACK DOWN THE TREE

Supposing that we know V^+ and V^- we can use (5.4) to find V. But do we know V^+ and V^-?

We certainly know V^+ and V^- at expiry, time T, because we know the option value as a function of the asset then, this is the payoff function. If we know the value of the option at expiry we can use Equation (5.4) to find the option value at the time $T - \delta t$ for all values of S on the tree. But knowing these values means that we can find the option values one step further back in time. Thus we work our way back down the tree until we get to the root. This root is the current time and asset value, and thus we find the option value today.

This method is illustrated in Figure 5.6. Here we are valuing a European call option with strike price 100 and maturity in four months' time. Today's asset price is 100, and the volatility is 20%. The risk-free interest rate is 10%.

I use a timestep of one month so that there are four steps until expiry. Using these numbers we have $\delta t = 1/12 = 0.08333$, $u = 1.0604$, $v = 0.9431$ and $p' = 0.5567$. As an example, after one timestep the asset takes either the value $100 \times 1.0604 = 106.04$ or $100 \times 0.9431 = 94.31$. Working back from expiry, the option value at the timestep before expiry when $S = 119.22$ is given by

$$e^{-0.1\times 0.0833}(0.5567 \times 26.42 + (1 - 0.5567) \times 12.44) = 20.05.$$

Working right back down the tree to the present time, the option value when the asset is 100 is 6.13.

In practice, the binomial method is programmed rather than done on a spreadsheet.

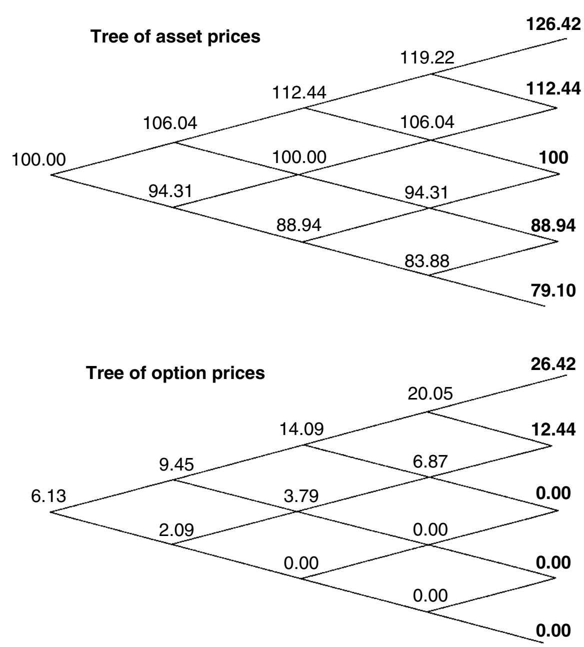
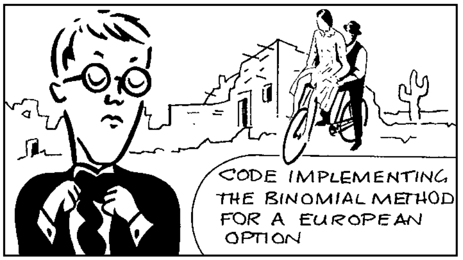


Figure 5.6 The binomial tree and corresponding option prices.

Here is a function that takes inputs for the underlying and the option, using an externally-defined payoff function. Key points to note about this program concern the building up of the arrays for the asset `S()` and the option `V()`. First of all, the asset array is built up only in order to find the final values of the asset at each node at the final timestep, expiry. The asset values on other nodes are never used. Second, the argument `j` refers to how far up the asset is from the lowest node *at that timestep*.




```
Function Price(Asset As Double, Volatility As Double, _
                    IntRate As Double, Strike As _
                    Double, Expiry As Double, _
                    NoSteps As Integer)
ReDim S(0 To NoSteps)
ReDim V(0 To NoSteps)
timestep = Expiry / NoSteps
DiscountFactor = Exp(-IntRate * timestep)
temp1 = Exp((IntRate + Volatility * Volatility) _
                                        * timestep)
temp2 = 0.5 * (DiscountFactor + temp1)
u = temp2 + Sqr(temp2 * temp2 - 1)
d = 1 / u
p = (Exp(IntRate * timestep) - d) / (u - d)

S(0) = Asset
For n = 1 To NoSteps
    For j = n To 1 Step -1
        S(j) = u * S(j - 1)
    Next j
        S(0) = d * S(0)
Next n

For j = 0 To NoSteps
    V(j) = Payoff(S(j), Strike)
Next j

For n = NoSteps To 1 Step -1
    For j = 0 To n - 1
        V(j) = (p * V(j + 1) + (1 - p) * V(j)) _
            * DiscountFactor
    Next j
Next n
Price = V(0)
End Function
```

Here is the externally defined payoff function `Payoff(S, Strike)` for a call.

```
Function Payoff(S, K)
Payoff = 0
If S > K Then Payoff = S - K
End Function
```

Because I never use the asset nodes other than at expiry I could have used only the one array in the above, with the same array being used for both *S* and *V*. I have kept them separate to make the program more transparent. Also, I could have saved the values of *V* at all of the nodes, in the above I have only saved the node at the present time. Saving all the values will be important if you want to see how the option value changes with the asset price and time, if you want to calculate greeks for example.

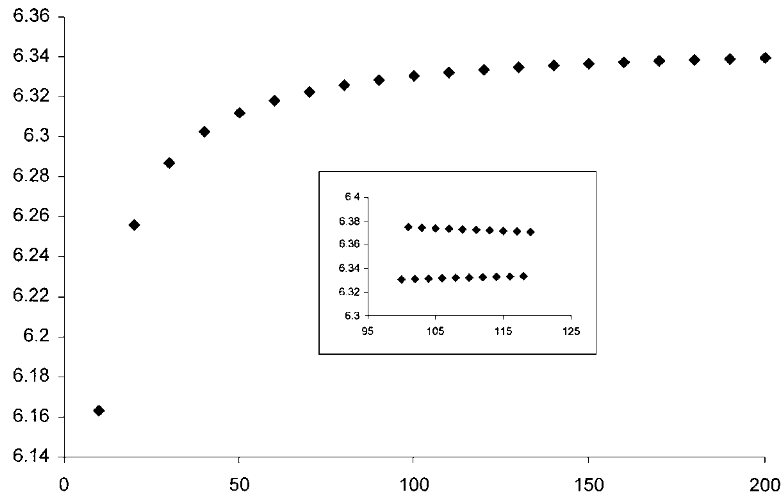


Figure 5.7 Option price as a function of number of timesteps.

In Figure 5.7 I show a plot of the calculated option price against the number of timesteps using this algorithm. The inset figure is a close-up. Observe the oscillation. In this example, an odd number of timesteps gives an answer that is too high and an even an answer that is too low.

5.10 EARLY EXERCISE

American-style exercise is easy to implement in a binomial setting. The algorithm is identical to that for European exercise with one exception. We use the same binomial tree, with the same u, v and p, but there is a slight difference in the formula for V. We must ensure that there are no arbitrage opportunities at any of the nodes.

For reasons which will become apparent, I'm going to change my notation now, making it more complex but more informative. Introduce the notation S_j^n to mean the asset price at the nth timestep, at the node j from the bottom, $0 \leq j \leq n$. This notation is consistent with the code above. In our lognormal world we have

$$S_j^n = Su^j v^{n-j},$$

where S is the current asset price. Also introduce V_j^n as the option value at the same node. Our ultimate goal is to find V_0^0 knowing the payoff, i.e. knowing V_j^M for all $0 \leq j \leq M$ where M is the number of timesteps.

Returning to the American option problem, arbitrage is possible if the option value goes below the payoff at any time. If our theoretical value falls below the payoff then it is time to exercise. If we do then exercise the option its value and the payoff must be the same. If we find that

$$\frac{V_{j+1}^{n+1} - V_j^{n+1}}{u-v} + e^{-r\,\delta t}\frac{uV_j^{n+1} - vV_{j+1}^{n+1}}{u-v} \geq \text{Payoff}(S_j^n)$$

then we use this as our new value. But if

$$\frac{V_{j+1}^{n+1} - V_j^{n+1}}{u-v} + e^{-r\,\delta t}\frac{uV_j^{n+1} - vV_{j+1}^{n+1}}{u-v} < \text{Payoff}(S_j^n)$$

we should exercise, giving us a better value of

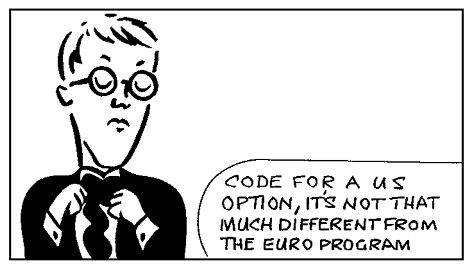
$$V_j^n = \text{Payoff}(S_j^n).$$

We can put these two together to get

$$V_j^n = \max\left(\frac{V_{j+1}^{n+1} - V_j^{n+1}}{u - v} + e^{-r\,\delta t}\frac{uV_j^{n+1} - vV_{j+1}^{n+1}}{u - v},\ \text{Payoff}(S_j^n)\right)$$

instead of (5.3). This ensures that there are no arbitrage opportunities. This modification is easy to code, but note that the payoff is a function of the asset price at the node in question. This is new, not seen in the European problem for which we did not have to keep track of the asset values on each of the nodes.

Below is a function for calculating the value of an American-style option. Note the differences between this program and the one for European-style exercise. The code is the same except that we keep track of more information and the line that updates the option value incorporates the no-arbitrage condition.




```
Function USPrice(Asset As Double, Volatility As _
                      Double, IntRate As Double, _
                      Strike As Double, Expiry As _
                      Double, NoSteps As Integer)
ReDim S(0 To NoSteps, 0 To NoSteps)
ReDim V(0 To NoSteps, 0 To NoSteps)
timestep = Expiry / NoSteps
DiscountFactor = Exp(-IntRate * timestep)
temp1 = Exp((IntRate + Volatility * Volatility) _
    * timestep)
temp2 = 0.5 * (DiscountFactor + temp1)
u = temp2 + Sqr(temp2 * temp2 - 1)
d = 1 / u
p = (Exp(IntRate * timestep) - d) / (u - d)

S(0, 0) = Asset
For n = 1 To NoSteps
    For j = n To 1 Step -1
        S(j, n) = u * S(j - 1, n - 1)
    Next j
        S(0, n) = d * S(0, n - 1)
Next n

For j = 0 To NoSteps
    V(j, NoSteps) = Payoff(S(j, NoSteps), Strike)
Next j

For n = NoSteps To 1 Step -1
    For j = 0 To n - 1
        V(j, n - 1) = max((p * V(j + 1, n) + (1 - p) _
            * V(j, n)) * DiscountFactor, _
            Payoff(S(j, n - 1), Strike))
    Next j
Next n
USPrice = V(0, 0)
End Function
```

5.11 THE CONTINUOUS-TIME LIMIT

Let's examine (5.3) as $\delta t \to 0$, this will lead us to a model that is independent of any timestep.

First of all, we have that

$$u \sim 1 + \sigma\sqrt{\delta t} + \tfrac{1}{2}\sigma^2 \delta t + \cdots$$

and

$$v \sim 1 - \sigma\sqrt{\delta t} + \tfrac{1}{2}\sigma^2 \delta t + \cdots.$$

Now we introduce a function $V(S, t)$ as the option value for any asset price and time. We can write

$$V = V(S, t), \quad V^+ = V(uS, t + \delta t) \quad \text{and} \quad V^- = V(vS, t + \delta t).$$

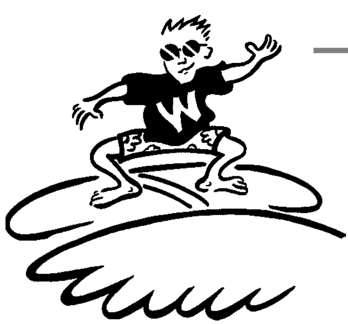
Expanding these expressions in Taylor series for small δt and substituting into (5.2) we find that

$$\Delta \sim \frac{\partial V}{\partial S} \text{ as } \delta t \to 0.$$

Similarly, we can substitute the expressions for V, V^+ and V^- into (5.3) to find

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r\frac{\partial V}{\partial S} - rV = 0.$$

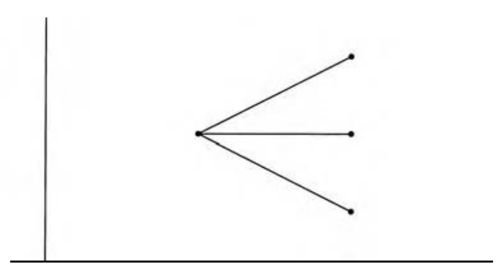
This is the famous Black–Scholes partial differential equation. Don't worry about it for now, we'll be deriving it in a different way in a later chapter and discussing it in detail.



Time Out...

Is it a mathematical model or a numerical method?

We can think of the binomial method as a mathematical model for a stock price, the asset can go to one of two states, up or down. Or we can think of it as a way of solving the continuous-time Black-Scholes partial



differential equation numerically, on a computer. I tend to think of it as the latter. Why? Because the binomial model is clearly not an accurate description of reality, whereas the Black–Scholes equation can be derived, and will be here, from many other assumptions about the behaviour of assets. The Black–Scholes equation is pretty robust in that sense.

If we choose to think of the binomial as a numerical method, then why stop at two branches? Some people use **trinomial trees** such as below for solving financial problems.

5.12 SUMMARY

In this chapter I described the basics of the binomial model, deriving pricing equations and algorithms for both European- and American-style exercise. The method can be extended in many ways, to incorporate dividends, to allow Bermudan exercise, to value path-dependent contracts and to price contracts depending on other stochastic variables such as interest rates. I have not gone into the method in any detail for the simple reason that the binomial method is just a version of an explicit finite-difference scheme. As such it will be discussed in depth later. Finite-difference methods have an obvious advantage over the binomial method, they are far more flexible.

Time Out...

Risk neutrality again

My experience teaching quantitative finance at all levels is that risk neutrality is a very hard concept to grasp. So let's take another look at it.

- Hedging is used to eliminate risk
- In simple models, hedging can be used to eliminate all risk from an option position
- As well as eliminating risk, hedging removes dependence of an option value on the direction of an asset
- If we don't care whether the asset price rises or falls, we shouldn't care about the probability of the rise or fall

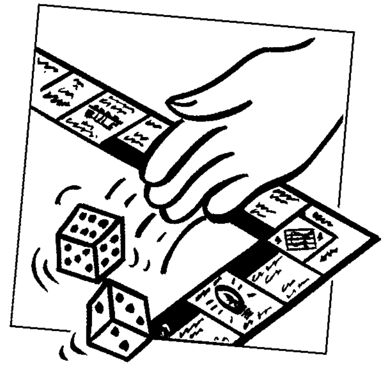
- The risk-neutral random walk is one that has the same volatility as the real asset random walk but a drift rate that is the same as the risk-free interest rate and not the real drift rate
- The punchline is that the option value is the present value of the option payoff under a risk-neutral random walk

FURTHER READING

- The original binomial concept is due to Cox, Ross & Rubinstein (1979).
- Almost every book on options describes the binomial method in more depth than I do. One of the best is Hull (1999).

CHAPTER 6

the random behavior of assets



The aim of this Chapter...

... is to demonstrate mathematical modeling in practice, to take the reader from an analysis of stock price data towards a probabilistic model for the behavior of asset prices. The fundamental model we build up will later be used as the starting point for deriving the famous Black–Scholes model for option prices. By the end of this chapter the reader will feel comfortable performing a simple analysis of any financial data.

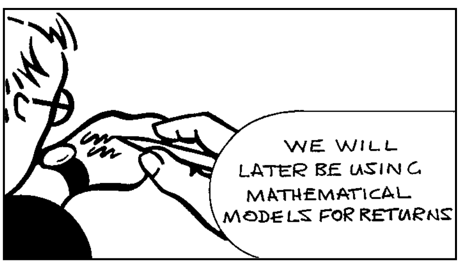
In this Chapter...

- more notation commonly used in mathematical finance
- how to examine time-series data to model returns
- the Wiener process, a mathematical model of randomness
- a simple model for equities, currencies, commodities and indices

6.1 INTRODUCTION

In this chapter I describe a simple continuous-time model for equities and other financial instruments, inspired by our earlier coin-tossing experiment. This takes us into the world of stochastic calculus and Wiener processes. Although there is a great deal of theory behind the ideas I describe, I am going to explain everything in as simple and accessible manner as possible. We will be modeling the behavior of equities, currencies and commodities, but the ideas are applicable to the fixed-income world as well.

The new model we'll be seeing for assets is not unlike the binomial model just described. Watch out for similarities and differences.



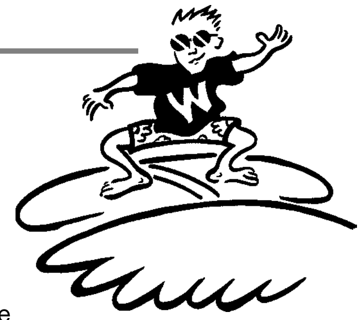
6.2 SIMILARITIES BETWEEN EQUITIES, CURRENCIES, COMMODITIES AND INDICES

When you invest in something, whether it is a stock, commodity, work of art or a racehorse, your main concern is that you will make a comfortable return on your investment. By **return** we tend to mean the percentage growth in the value of an asset, together with accumulated dividends, over some period:

$$\text{Return} = \frac{\text{Change in value of the asset} + \text{accumulated cashflows}}{\text{Original value of the asset}}.$$

I want to distinguish here between the percentage or relative growth and the absolute growth. Suppose we could invest in either of two stocks, both of which grow on average by \$10 *per annum*. Stock A has a value of \$100 and stock B is currently worth \$1000. Clearly the former is a better investment, at the end of the year stock A will probably be worth around \$110 (if the past is anything to go by) and stock B \$1010. Both have gone up by \$10, but A has risen by 10% and B by only 1%. If we have \$1000 to invest we would be better off investing in ten of asset A than one of asset B. This illustrates that when we come to model assets, it is the return that we should concentrate on. In this respect, all of equities, currencies, commodities and stock market indices can be treated similarly. What return do we expect to get from them?

Part of the business of estimating returns for each asset is to estimate how much unpredictability there is in the asset value. In the next section I am going to show that randomness plays a large part in financial markets, and start to build up a model for asset returns incorporating this randomness.



Time Out.

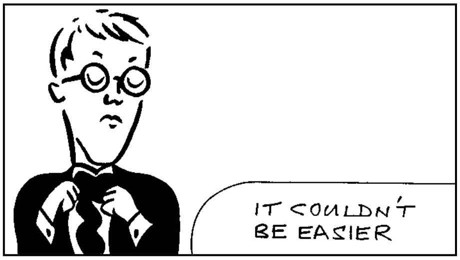
Returns

Here is another way of understanding why returns are more important than actual stock price. Suppose I told you that one stock had a value of 5 and another, 500. You would think nothing of it. Now suppose I told you that one currency had an interest rate of 5% and another had an interest rate of 500%. Whoa. . . you'd be somewhat surprised by the currency with the 500% interest rate, wouldn't you? There's a big clue in such an observation. We don't care about the absolute value of a stock price, only its return. So let's analyze and model returns. When we come to modeling interest rates we won't have such a valuable clue to help us. This makes interest rate modeling harder than equity modeling.

6.3 EXAMINING RETURNS

In Figure 6.1 I show the quoted price of Perez Companc, an Argentinean conglomerate, over the period February 1995 to November 1996. This is a very typical plot of a financial asset. The asset shows a general upward trend over the period but this is far from guaranteed. If you bought and sold at the wrong times you would lose a lot of money. The unpredictability that is seen in this figure is the main feature of financial modeling. Because there is so much randomness, any mathematical model of a financial asset must acknowledge the randomness and have a probabilistic foundation.

Figure 6.1 Perez Companc from February 1995 to November 1996.



Date	Perez	Return				
01-Mar-95	2.11			Average return	0.002916	
02-Mar-95	1.90	-0.1		Standard deviation	0.024521	
03-Mar-95	2.18	0.149906				
06-Mar-95	2.16	-0.010809				
07-Mar-95	1.91	-0.112583	= AVERAGE(C3:C463)			
08-Mar-95	1.86	-0.029851				
09-Mar-95	1.97	0.061538				
10-Mar-95	2.27	0.15		= STDEVP(C3:C463)		
13-Mar-95	2.49	0.099874				
14-Mar-95	2.76	0.108565				
15-Mar-95	2.61	-0.054264				
16-Mar-95	2.67	0.021858				
17-Mar-95	2.64	-0.010695				
20-Mar-95	2.60	-0.016216	=(B13-B12)/B12			
21-Mar-95	2.59	-0.002747				
22-Mar-95	2.59	-0.002755				
23-Mar-95	2.55	-0.012321				
24-Mar-95	2.73	0.069307				
27-Mar-95	2.91	0.064815				
28-Mar-95	2.92	0.002899				
29-Mar-95	2.92	0				
30-Mar-95	3.12	0.069364				
31-Mar-95	3.14	0.005405				
03-Apr-95	3.13	-0.002688				
04-Apr-95	3.24	0.037736				
05-Apr-95	3.25	0.002597				
06-Apr-95	3.28	0.007772				
07-Apr-95	3.21	-0.020566				
10-Apr-95	3.02	-0.060367				
11-Apr-95	3.08	0.019553				
12-Apr-95	3.19	0.035616				
17-Apr-95	3.21	0.007936				
18-Apr-95	3.17	-0.013123				
19-Apr-95	3.24	0.021277				

Figure 6.2 Spreadsheet for calculating asset returns.

Remembering that the returns are more important to us than the absolute level of the asset price, I show in Figure 6.2 how to calculate returns on a spreadsheet. Denoting the asset value on the ith day by S_i, then the return from day i to day $i+1$ is given by

$$\frac{S_{i+1} - S_i}{S_i} = R_i.$$

(I've ignored dividends here, they are easily allowed for, especially since they only get paid two or four times a year typically.) Of course, I didn't need to use data spaced at intervals of a day, I will comment on this later.

In Figure 6.3 I show the daily returns for Perez Companc. This looks very much like 'noise,' and that is exactly how we are going to model it. The mean of the returns distribution is

$$\overline{R} = \frac{1}{M} \sum_{i=1}^{M} R_i \tag{6.1}$$

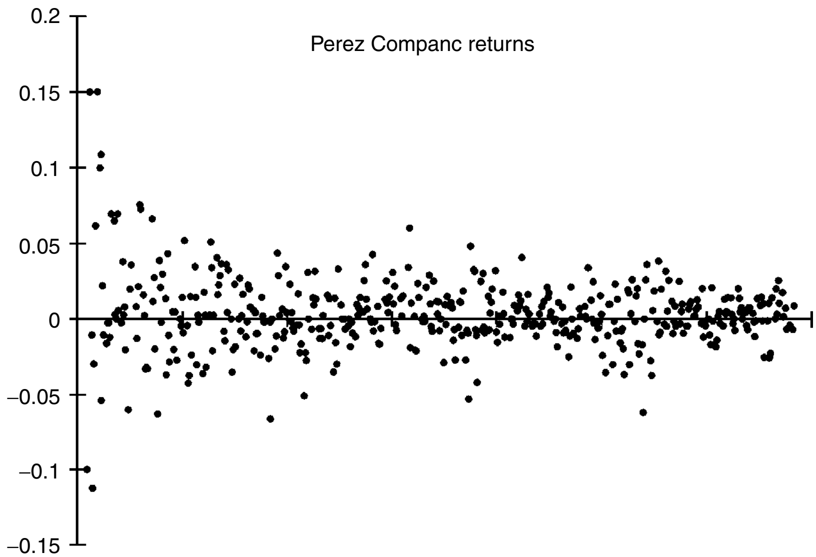


Figure 6.3 Daily returns of Perez Companc.

and the sample standard deviation is

$$\sqrt{\frac{1}{M-1}\sum_{i=1}^{M}(R_i-\overline{R})^2}, \tag{6.2}$$

where M is the number of returns in the sample (one fewer than the number of asset prices). From the data in this example we find that the mean is 0.002916 and the standard deviation is 0.024521.

The frequency distribution of this time series of daily returns is easily calculated, and very instructive to plot. In Excel use Tools | Data Analysis | Histogram. In Figure 6.4 is shown the frequency distribution of daily returns for Perez Companc. This distribution has been scaled and translated to give it a mean of zero, a standard deviation of one and an area under the curve of one. On the same plot is drawn the probability density function for the standardized Normal distribution function

$$\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\phi^2},$$

where ϕ is a standardized Normal variable.[1] The two curves are not identical but are fairly close.

Supposing that we believe that the empirical returns are close enough to Normal for this to be a good approximation, then we have come a long way towards a model. I am going to write the returns as a random variable, drawn from a Normal distribution with a known, constant, nonzero mean and a known, constant, nonzero

[1] Think of a number, any number... funny, that's the number I was thinking of.

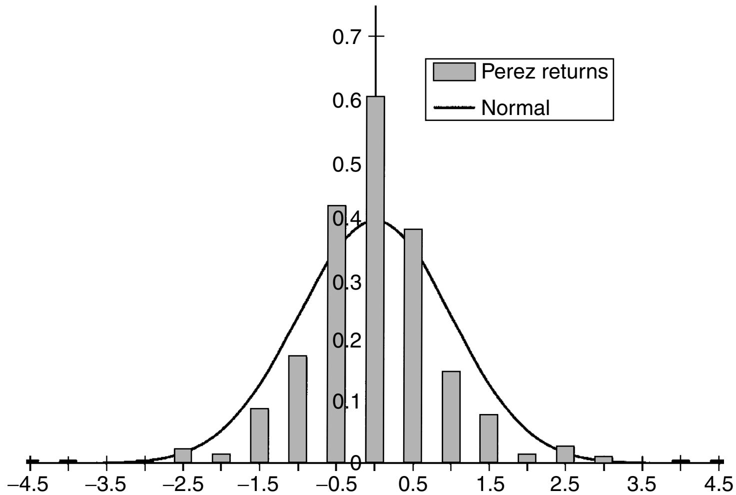


Figure 6.4 Normalized frequency distribution of Perez Companc and the standardized Normal distribution.

standard deviation:

$$R_i = \frac{S_{i+1} - S_i}{S_i} = \text{mean} + \text{standard deviation} \times \phi.$$

Figure 6.5 shows the returns distribution of Glaxo–Wellcome as calculated by Bloomberg. This has not been normalized.

Time Out...

The Normal distribution

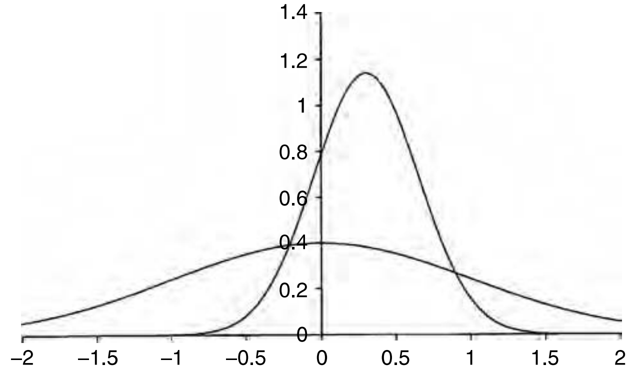
In Excel the bell-shaped standardized Normal probability density function curve is, as a function of X,

1/SQRT(2*PI( ))*EXP(-0.5*X*X).

Mathematically this is denoted by $N(0, 1)$. The Normal distribution having mean m and standard deviation s is denoted by $N(m, s^2)$ and in Excel is

1/SQRT(2*PI( ))/S*EXP(-0.5*(X-M)*(X-M)/S/S).

The figure below shows a couple of Normal distributions, one is the standardized, and the other has a positive mean and quite a small standard deviation.



There are a couple more Time Outs on the Normal distribution in this chapter.

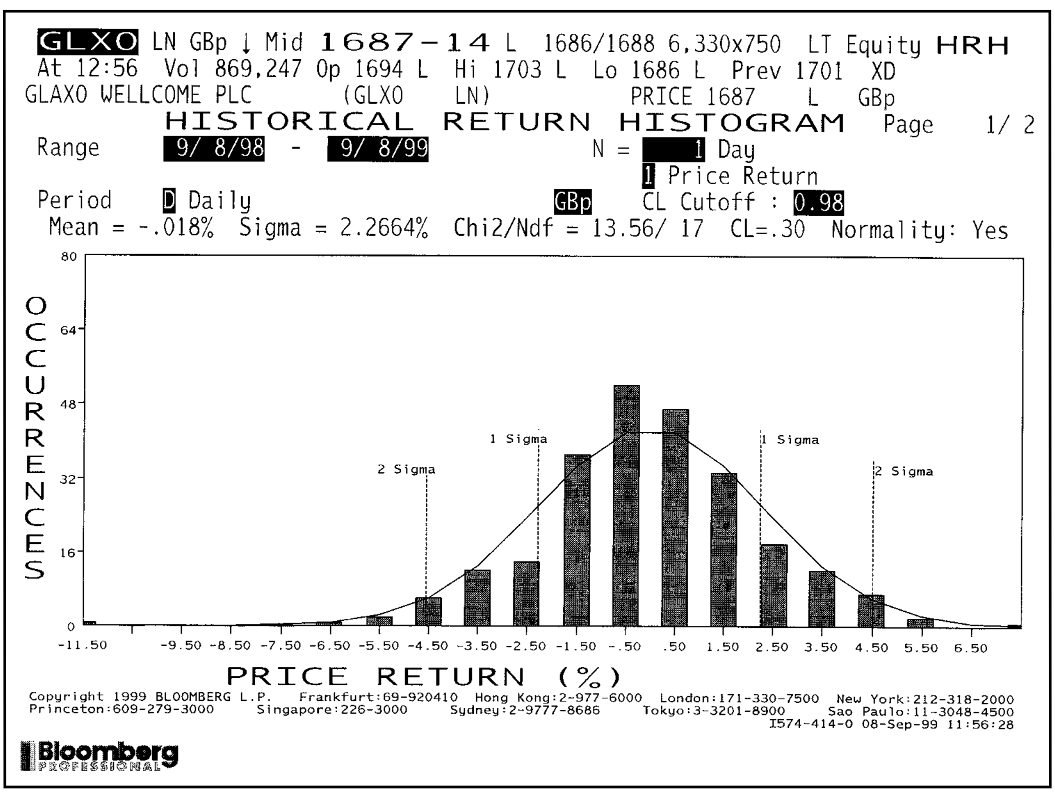


Figure 6.5 Glaxo–Wellcome returns histogram. Source: Bloomberg L.P.

6.4 TIMESCALES

How do the mean and standard deviation of the returns' time series, as estimated by (6.1) and (6.2), scale with the timestep between asset price measurements? In the example the timestep is one day, but suppose I sampled at hourly intervals or weekly, how would this affect the distribution?

Call the timestep δt. The mean of the return scales with the size of the timestep. That is, the larger the time between sampling the more the asset will have moved in the meantime, *on average*. I can write

$$\text{mean } = \mu\ \delta t,$$

for some μ which we will assume to be constant. This is the same μ as in Chapter 5, representing the annualized average return or the drift.

Ignoring randomness for the moment, our model is simply

$$\frac{S_{i+1} - S_i}{S_i} = \mu\ \delta t.$$

Rearranging, we get

$$S_{i+1} = S_i(1 + \mu\ \delta t).$$

If the asset begins at S_0 at time $t = 0$ then after one timestep $t = \delta t$ and

$$S_1 = S_0(1 + \mu\ \delta t).$$

After two timesteps $t = 2\,\delta t$ and

$$S_2 = S_1(1 + \mu\ \delta t) = S_0(1 + \mu\,\delta t)^2,$$

and after M timesteps $t = M\,\delta t = T$ and

$$S_M = S_0(1 + \mu\,\delta t)^M.$$

This is just

$$S_M = S_0\,(1 + \mu\ \delta t)^M = S_0\,e^{M \log(1+\mu\ \delta t)} \approx S_0 e^{\mu M\,\delta t} = S_0 e^{\mu T}.$$

In the limit as the timestep tends to zero with the total time T fixed, this approximation becomes exact. This result is important for two reasons.

First, in the absence of any randomness the asset exhibits exponential growth, just like cash in the bank.

Second, the model is meaningful in the limit as the timestep tends to zero. If I had chosen to scale the mean of the returns distribution with any other power of δt it would have resulted in either a trivial model ($S_T = S_0$) or infinite values for the asset.

The second point can guide us in the choice of scaling for the random component of the return. How does the standard deviation of the return scale with the timestep δt? Again, consider what happens after $T/\delta t$ timesteps each of size δt (i.e. after a total time of T). Inside the square root in expression (6.2) there are a large number of terms, $T/\delta t$ of them. In order for the standard deviation to remain finite as we let δt tend to zero, the individual terms in the expression must each be of $O(\delta t)$. Since each term is a square of a return, the standard deviation of the asset return over a timestep δt must be $O(\delta t^{1/2})$:

$$\text{standard deviation } = \sigma\ \delta t^{1/2},$$

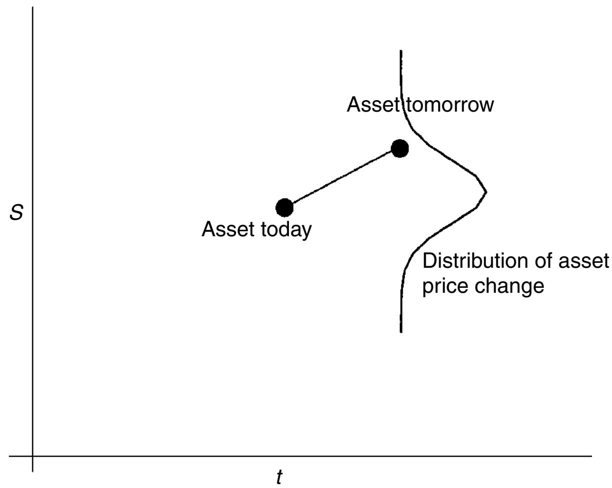


Figure 6.6 A representation of the random walk.

where σ is some parameter measuring the amount of randomness, the larger this parameter the more uncertain is the return. This σ is the same σ we saw in Chapter 5. It is the annualized standard deviation of asset returns.

Putting these scalings explicitly into our asset return model

$$R_i = \frac{S_{i+1} - S_i}{S_i} = \mu\,\delta t + \sigma\phi\,\delta t^{1/2}. \tag{6.3}$$

I can rewrite Equation (6.3) as

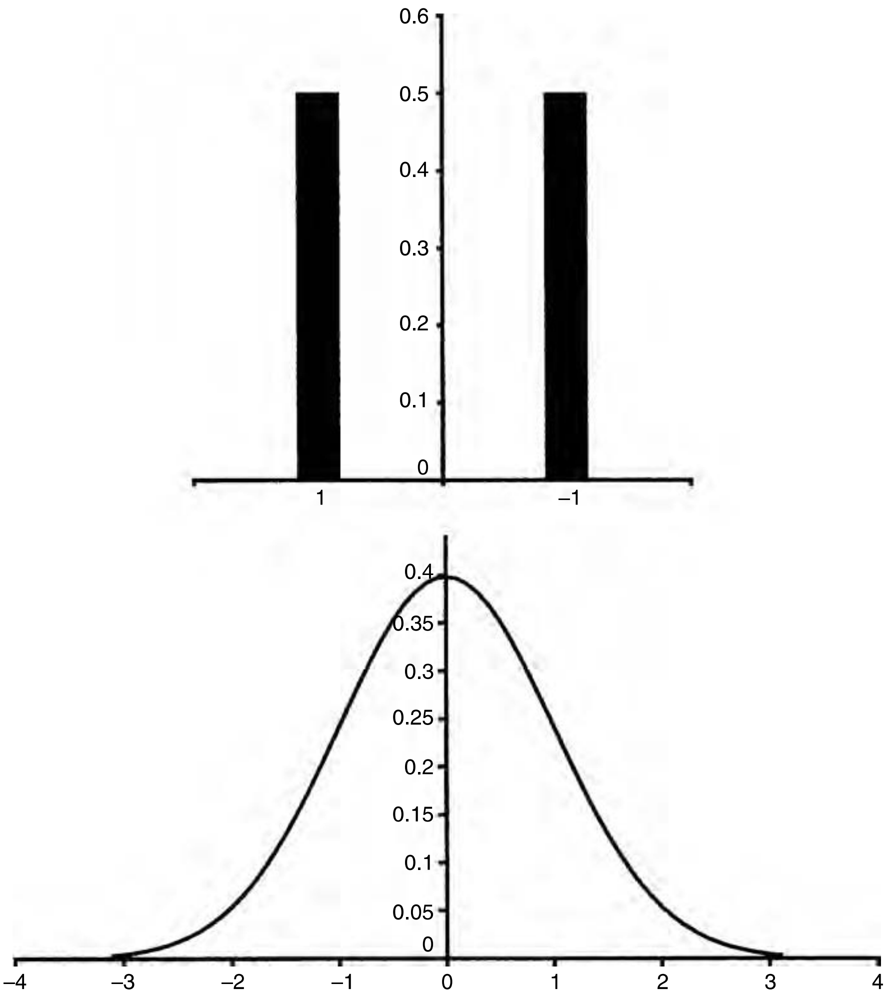
$$S_{i+1} - S_i = \mu S_i\,\delta t + \sigma S_i\phi\,\delta t^{1/2}. \tag{6.4}$$

The left-hand side of this equation is the change in the asset price from timestep i to timestep $i+1$. The right-hand side is the 'model.' We can think of this equation as a model for a **random walk** of the asset price. This is shown schematically in Figure 6.6. We know exactly where the asset price is today but tomorrow's value is unknown. It is distributed about today's value according to (6.4).

Time Out...

Binomial versus Normal

We've been considering two models for the asset return, the binomial and the Normal. The figure below shows what these two look like. Although completely different, they both should have the same mean return and standard deviation.



6.4.1 The drift

The parameter μ is called the **drift rate**, the **expected return** or the **growth rate** of the asset. Statistically it is very hard to measure since the mean scales with the usually small parameter δt. It can be estimated by

$$\mu = \frac{1}{M\,\delta t}\sum_{i=1}^{M} R_i.$$

The unit of time that is usually used is the year, in which case μ is quoted as an *annualized* growth rate.

In the classical option pricing theory the drift plays almost no role. So even though it is hard to measure, this doesn't matter too much.[2]

6.4.2 The volatility

The parameter σ is called the **volatility** of the asset. It can be estimated by

$$\sqrt{\frac{1}{(M-1)\,\delta t}\sum_{i=1}^{M}(R_i-\overline{R})^2}.$$

Again, this is almost always quoted in annualized terms.

The volatility is the most important and elusive quantity in the theory of derivatives. I will come back again and again to its estimation and modeling.

Because of their scaling with time, the drift and volatility have different effects on the asset path. The drift is not apparent over short timescales for which the volatility dominates. Over long timescales, for instance decades, the drift becomes important. Figure 6.7 is a realized path of the logarithm of an asset, together with its expected path and a 'confidence interval.' In this example the confidence interval represents one standard deviation. With the assumption of Normality this means that 68% of the time the asset should be within this range. The mean path is growing linearly in time and the confidence interval grows like the square root of time. Thus over short timescales the volatility dominates.

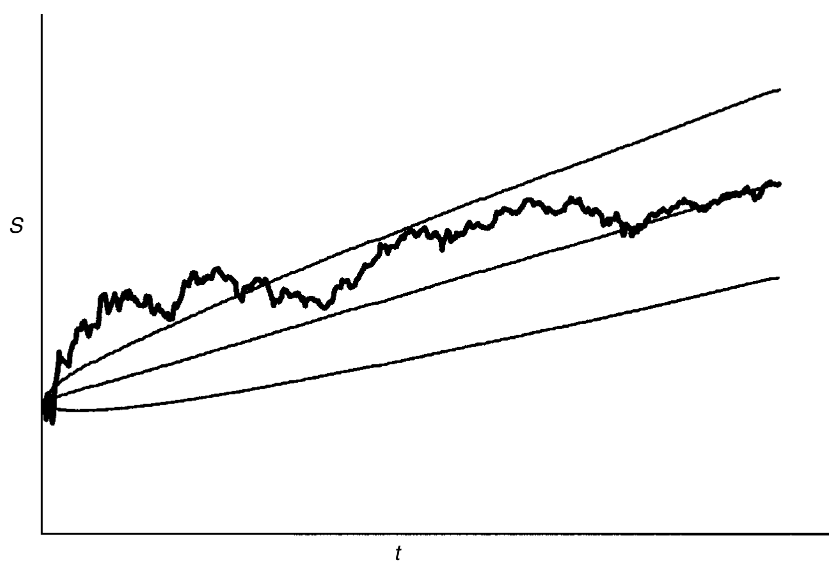


Figure 6.7 Path of the logarithm of an asset, its expected path and one standard deviation above and below.

[2] In nonclassical theories and in portfolio management, it *does* often matter, very much.

6.5 ESTIMATING VOLATILITY

The most common estimate of volatility is simply

$$\sqrt{\frac{1}{(M-1)\,\delta t}\sum_{i=1}^{M}(R_i-\overline{R})^2}.$$

If δt is sufficiently small the mean return $\overline{R}$ term can be ignored. For small δt

$$\sqrt{\frac{1}{(M-1)\,\delta t}\sum_{i=1}^{M}\left(\log S(t_i)-\log S(t_{i-1})\right)^2}$$

can also be used, where $S(t_i)$ is the closing price on day t_i.

It is highly unlikely that volatility is constant for any given asset. Changing economic circumstances, seasonality etc. will inevitably result in volatility changing with time. If you want to know the volatility today you must use some past data in the calculation. Unfortunately, this means that there is no guarantee that you are actually calculating *today's* volatility.

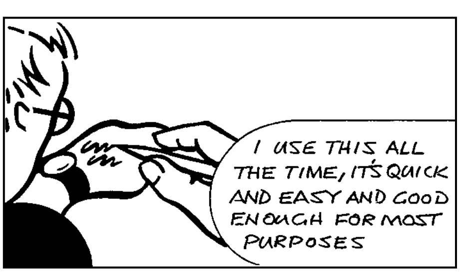
Typically you would use daily closing prices to work out daily returns and then use the past 10, 30, 100, ... daily returns in the formula above. Or you could use returns over longer or shorter periods. Since all returns are equally weighted, while they are in the estimate of volatility, any large return will stay in the estimate of vol until the 10 (or 30 or 100) days have past. This gives rise to a plateauing of volatility, and is totally spurious.

6.6 THE RANDOM WALK ON A SPREADSHEET

The random walk (6.4) can be written as a 'recipe' for generating S_{i+1} from S_i:

$$S_{i+1}=S_i\left(1+\mu\,\delta t+\sigma\phi\,\delta t^{1/2}\right). \tag{6.5}$$

We can easily simulate the model using a spreadsheet. In this simulation we must input several parameters, a starting value for the asset, a timestep δt, the drift rate μ, the volatility σ and the total number of timesteps. Then, at each timestep, we must choose a random number ϕ from a Normal distribution. I will talk about simulations in depth in Chapter 26, for the moment let me just say that an approximation to a Normal variable that is fast in a spreadsheet, and quite accurate, is simply to add up twelve random variables drawn from a uniform distribution over zero to one, and subtract six:



$$\left(\sum_{i=1}^{12} RAND()\right)-6.$$

The Excel spreadsheet function `RAND()` gives a uniformly distributed random variable.

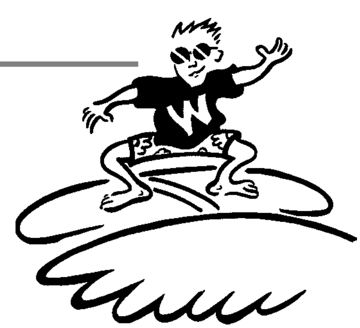
	A	B	C	D	E	F	G	H
1	**Asset**	**100**		**Time**	**Asset**			
2	**Drift**	**0.15**		0	100			
3	**Volatility**	**0.25**		0.01	101.2378			
4	**Timestep**	**0.01**		0.02	103.8329			
5				0.03	106.5909			
6				0.04	110.993			
7				0.05	115.9425			
8				0.06	117.1478			
9				0.07	115.9868			
10				0.08	114.921			
11								
12								
13				0.11	113.3875			
14				0.12	108.5439			
15				0.13	107.4318			
16				0.14	109.092			
17				0.15	110.8794			
18				0.16	113.5328			
19				0.17	116.099			
20				0.18	116.2446			
21				0.19	119.315			
22				0.2	120.0332			
23				0.21	124.337			
24				0.22	128.3446			
25				0.23	125.5112			
26				0.24	128.2683			
27				0.25	124.0548			
28				0.26	125.9068			
29				0.27	122.4632			
30				0.28	122.4472			
31				0.29	121.3325			
32				0.3	124.593			
33				0.31	121.9263			

=D4+$B$4

=E7*(1+$B$2*$B$4+$B$3*SQRT($B$4)*(RAND()+RAND()+RAND()+RAND()
+RAND()+RAND()+RAND()+RAND()+RAND()+RAND()+RAND()+RAND()-6))

Figure 6.8 Simulating the random walk on a spreadsheet.

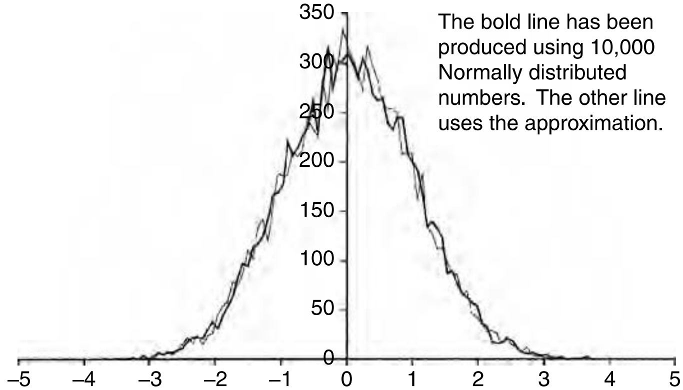
In Figure 6.8 I show the details of a spreadsheet used for simulating the asset price random walk.



Time Out...

Excel and the approximation to the Normal

You can draw Normally distributed random numbers in Excel using NORM-SINV(RAND( )). But this is very slow. Below are the distributions for the real Normal and the approximate Normal using 10,000 random numbers. Not bad?



Why use 12 RANDs? Well, you can use any number, the more you use the closer the approximation will be to Normal, but the longer it will take to compute. The general formula using N uniformly distributed numbers is

$$\sqrt{\frac{12}{N}}\left(\left(\sum_{i=1}^{N} RAND()\right)-\frac{N}{2}\right).$$

This has been scaled to have a mean of zero and unit standard deviation.

6.7 THE WIENER PROCESS

So far we have a model that allows the asset to take any value after a timestep. This is a step forward but we have still not reached our goal of continuous time, we still have a discrete timestep. This section is a brief introduction to the continuous-time limit of equations like (6.3). I will start to introduce ideas from the world of stochastic modeling and Wiener processes, delving more deeply in Chapter 7.

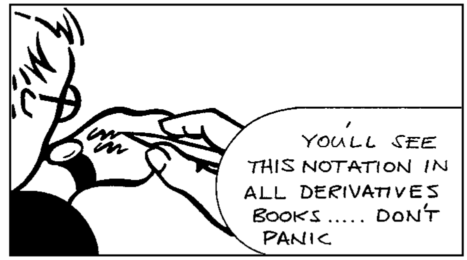
I am now going to use the notation $d\cdot$ to mean 'the change in' some quantity. Thus dS is the 'change in the asset price.' But this change will be in *continuous time*. Thus we will go to the limit $\delta t = 0$. The first δt on the right-hand side of (6.4) becomes dt but the second term is more complicated.

I cannot straightforwardly write $dt^{1/2}$ instead of $\delta t^{1/2}$. If I do go to the zero-timestep limit then any random $dt^{1/2}$ term will dominate any deterministic dt term. Yet in our problem the factor in front of $dt^{1/2}$ has a mean of zero, so maybe it does not outweigh the drift after all. Clearly something subtle is happening in the limit.

It turns out, and we will see this in Chapter 7, that because the *variance* of the random term is $O(\delta t)$ we *can* make a sensible continuous-time limit of our discrete-time model. This brings us into the world of Wiener processes.

I am going to write the term $\phi\ \delta t^{1/2}$ as

$$dX.$$

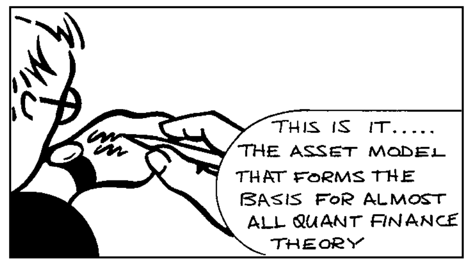


You can think of dX as being a random variable, drawn from a Normal distribution with mean zero and variance dt:

$$E[dX] = 0 \quad \text{and} \quad E[dX^2] = dt.$$

This is not exactly what it is, but it is close enough to give the right idea. This is called a **Wiener process**. The important point is that we can build up a continuous-time theory using Wiener processes instead of Normal distributions and discrete time.

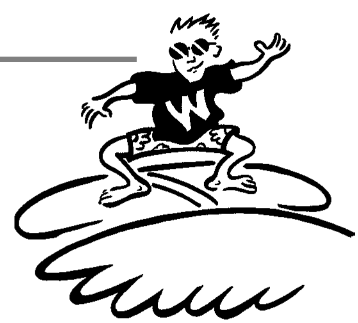
6.8 THE WIDELY ACCEPTED MODEL FOR EQUITIES, CURRENCIES, COMMODITIES AND INDICES



Our asset price model in the continuous-time limit, using the Wiener process notation, can be written as

$$dS = \mu S\ dt + \sigma S\ dX. \tag{6.6}$$

This is our first **stochastic differential equation**. It is a continuous-time model of an asset price. It is the most widely accepted model for equities, currencies, commodities and indices, and the foundation of so much finance theory.



Time Out...

Don't panic!

Stochastic differential equations can be a bit unnerving, especially to any-one used to ordinary or partial differential calculus. But do not worry, keep thinking in terms of simulations and the algorithm we've built up in the spreadsheet.

We've now built up a simple model for equities that we are going to be using quite a lot. You could ask, if the stock market is so random how can fund managers justify their fee? Do they manage to outsmart the market? Are they clairvoyant or aren't the markets random? Well, I won't swear that markets are random but I can say with confidence that fund managers don't outperform the market. In Figure 6.9 is shown the percentage of funds that outperform an index of all UK stocks. Whether we look at a one-, three-, five-

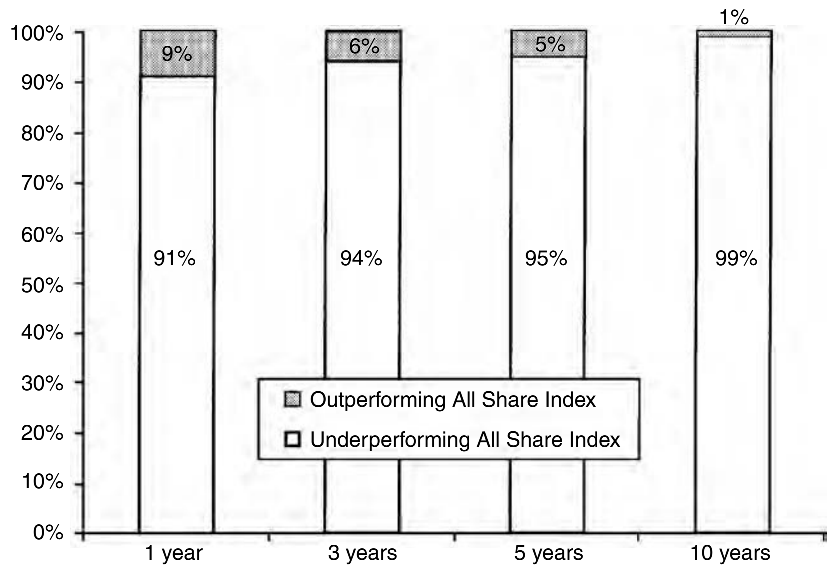


Figure 6.9 Fund performances compared with UK All Share Index. To end December 1998. Data supplied by Virgin Direct.

or ten-year horizon we can see that the vast majority of funds can't even keep up with the market. And statistically speaking, there are bound to be a few that beat the market, but only by chance. Maybe one should invest in a fund that does the opposite of all other funds. Great idea except that the management fee and transaction costs probably mean that that would be a poor investment too. This doesn't prove that markets are random, but it's sufficiently suggestive that most of my personal share exposure is via an index-tracker fund.

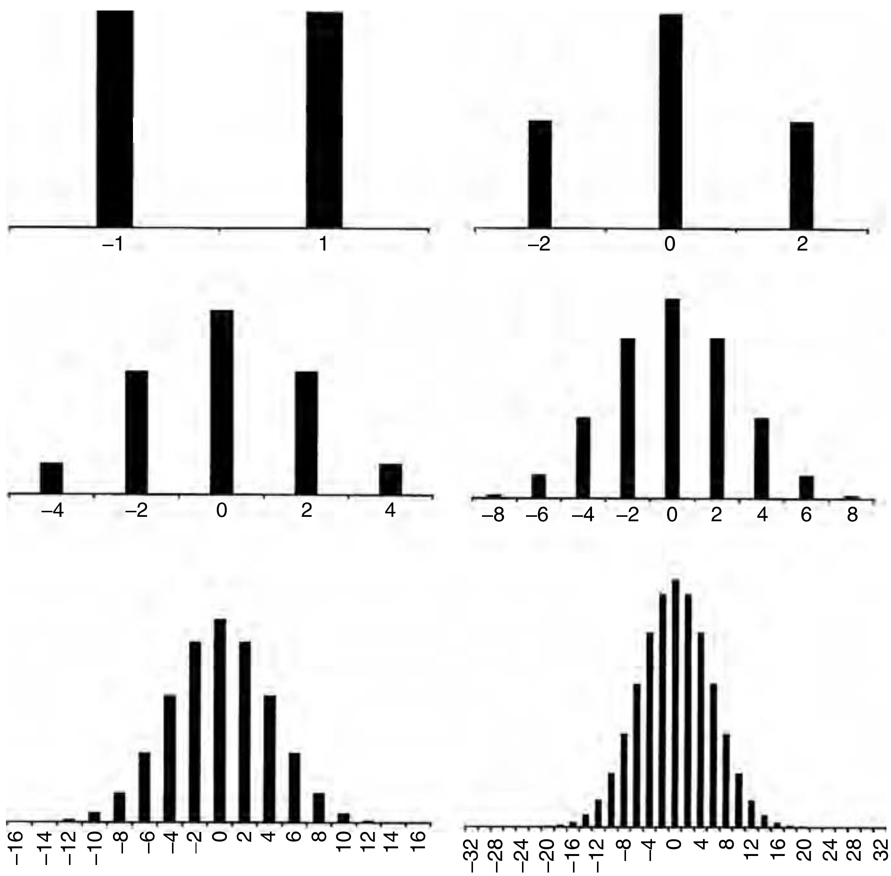
Time Out...

Why do we like the Normal distribution?

The Normal distribution is a special and wonderful distribution. It occurs naturally in many walks of life, and has nice properties. Here's a little experiment that shows why it crops up naturally, and after this I'll give you a theorem.

Back to coin tossing. Toss one coin, heads you win one dollar, tails you lose one dollar. Figure 1 shows the probability distribution.

Now toss two coins, same rules, for each head you get one dollar, but lose one for each tail. The probability density function is shown in Figure 2.



The sequence of figures below shows the probability density function of winnings/losses after an increasing number of tosses. What do you notice? It's starting to look more and more like the bell-shaped Normal distribution.

This is a simple demonstration of the **Central Limit Theorem**: Let X_1, X_2, ... be a sequence of independent identically distributed (i.i.d.) random variables with finite means m and finite nonzero variances s^2 then the sum

$$S_n = X_1 + X_2 + \cdots + X_n$$

in the limit as $n \to \infty$ is distributed Normally with mean nm and variance ns^2. Or if we rescale,

$$S_n' = \frac{X_1 + X_2 + \cdots + X_n - nm}{\sqrt{n}s}$$

tends to the standardized Normal distribution.

The point is that if we add up enough i.i.d. random variables (with finite mean and standard deviation) we end up with something that's Normally distributed. And that's why the Normal distribution occurs all over the place.

6.9 SUMMARY

In this chapter I introduced a simple model for the random walk of an asset. Initially I built the model up in discrete time, showing what the various terms mean, how they scale with the timestep and showing how to implement the model on a spreadsheet.

Most of this book is about continuous-time models for assets. The continuous-time version of the random walk involves concepts such as stochastic calculus and Wiener processes. I introduced these briefly in this chapter and will now go on to explain the underlying theory of stochastic calculus to give the necessary background for the rest of the book.

FURTHER READING

- Mandelbrot (1963) and Fama (1965) did some of the early work on the analysis of financial data.
- Parkinson (1980) derived the high-low estimator and Garman & Klass (1980) derived the high-low-close estimator.
- For an introduction to random walks and Wiener processes see Øksendal (1992) and Schuss (1980).
- Some high frequency data can be ordered through Olsen Associates, `www.olsen.ch`. It's not free, but nor is it expensive.
- The famous book by Malkiel (1990) is well worth reading for its insights into the behavior of the stock market. Read what he has to say about chimpanzees, blindfolds and darts. In fact, if you haven't already Malkiel's book make sure that it is the next book you read after finishing mine.

CHAPTER 7
elementary stochastic calculus

The aim of this Chapter...

... is to develop the theory behind the manipulation of random quantities, in particular stochastic differential equations like the one at the end of the previous chapter. This is very important for a thorough understanding of quantitative finance.

In this Chapter...

- all the stochastic calculus you need to know, and no more
- the meaning of Markov and martingale
- Brownian motion
- stochastic integration
- stochastic differential equations
- Itô's lemma in one and more dimensions

7.1 INTRODUCTION

Stochastic calculus is very important in the mathematical modeling of financial processes. This is because of the underlying random nature of financial markets. Because stochastic calculus is such an important tool I want to ensure that it can be used by everyone. To that end, I am going to try to make this chapter as accessible and intuitive as possible. By the end, I hope that the reader will know what various technical terms mean (and rarely are they very complicated), but, more importantly, will also know how to use the techniques with the minimum of fuss.

Most academic articles in finance have a 'pure' mathematical theme. The mathematical rigor in these works is occasionally justified, but more often than not it only succeeds in obscuring the content. When a subject is young, as is mathematical finance (young*ish*), there is a tendency for technical rigor to feature very prominently in research. This is due to lack of confidence in the methods and results. As the subject ages, researchers will become more cavalier in their attitudes and we will see much more rapid progress.

Time Out...

Aaaaarghhhhhh!

If you don't feel comfortable with messy algebra, just skip to the end of the chapter where I outline the intuition behind stochastic calculus and give you a few rules of thumb to help you use it in practice. Actually, most of this chapter is just groundwork for the important mathematical 'tool' of Itô's lemma.

7.2 A MOTIVATING EXAMPLE

Toss a coin. Every time you throw a head I give you \$1, every time you throw a tail you give me \$1. Figure 7.1 shows how much money you have after six tosses. In this experiment the sequence was THHTHT, and we finished even.

If I use R_i to mean the random amount, either \$1 or $-$\$1, you make on the ith toss then we have

$$E[R_i] = 0, \quad E[R_i^2] = 1 \quad \text{and} \quad E[R_iR_j] = 0.$$

In this example it doesn't matter whether or not these expectations are conditional on the past. In other words, if I threw five heads in a row it does not affect the outcome of the sixth toss. To the gamblers out there, this property is also shared by a fair die, a balanced roulette wheel, but not by the deck of cards in blackjack. In blackjack the same deck is used for game after game, the odds during one game depend on what cards were dealt out from the same deck in previous games. That is why you can in the long run beat the house at blackjack but not roulette.

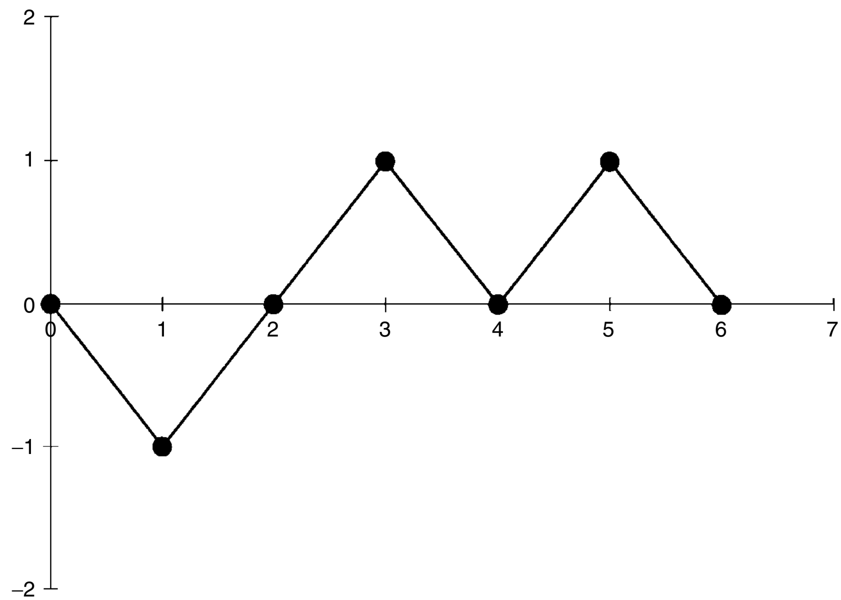


Figure 7.1 The outcome of a coin tossing experiment.

Introduce S_i to mean the total amount of money you have won up to and including the ith toss so that

$$S_i = \sum_{j=1}^{i} R_j.$$

Later on it will be useful if we have $S_0 = 0$, i.e. you start with no money.

Time Out...

Just like coin tossing or the binomial tree

Very similar, but here we have something like an arithmetic random walk rather than geometric... we are adding or subtracting a quantity rather than multiplying.

If we now calculate expectations of S_i it *does* matter what information we have. If we calculate expectations of future events before the experiment has even begun then

$$E[S_i] = 0 \quad \text{and} \quad E[S_i^2] = E[R_1^2 + 2R_1R_2 + \cdots] = i.$$

On the other hand, suppose there have been five tosses already, can I use this information and what can we say about expectations for the sixth toss? This is the **conditional expectation**. The expectation of S_6 conditional upon the previous five tosses gives

$$E[S_6|R_1, \dots, R_5] = S_5.$$

7.3 THE MARKOV PROPERTY

This result is special, the expected value of the random variable S_i conditional upon all of the past events *only depends on the previous value* S_{i-1}. This is the **Markov property**. We say that the random walk has no memory beyond where it is now. Note that it doesn't have to be the case that the expected value of the random variable S_i is the same as the previous value.

This can be generalized to say that given information about S_j for some values of $1 \leq j < i$ then the only information that is of use to us in estimating S_i is the value of S_j for the largest j for which we have information.

Almost all of the financial models that I will show you have the Markov property. This is of fundamental importance in modeling in finance. I will also show you examples where the system has a small amount of memory, meaning that one or two other pieces of information are important. And I will also give a couple of examples where *all* of the random walk path contains relevant information.

7.4 THE MARTINGALE PROPERTY

The toin cossing experiment possesses another property that can be important in finance. You know how much money you have won after the fifth toss. Your expected winnings after the sixth toss, and indeed after any number of tosses if we keep playing, is just the amount you already hold. That is, the conditional expectation of your winnings at any time in the future is just the amount you already hold:

$$E[S_i|S_j, j < i] = S_j.$$

This is called the **martingale property**.

7.5 QUADRATIC VARIATION

I am now going to define the **quadratic variation** of the random walk. This is defined by

$$\sum_{j=1}^{i} \left(S_j - S_{j-1}\right)^2.$$

Because you either win or lose an amount \$1 after each toss, $|S_j - S_{j-1}| = 1$. Thus the quadratic variation is always i:

$$\sum_{j=1}^{i} \left(S_j - S_{j-1}\right)^2 = i.$$

I want to use the coin-tossing experiment for one more demonstration. And that will lead us to a continuous-time random walk.

7.6 BROWNIAN MOTION

I am going to change the rules of my coin-tossing experiment. First of all I am going to restrict the time allowed for the six tosses to a period t, so each toss will take a time $t/6$. Second, the size of the bet will not be \$1 but $\sqrt{t/6}$.

This new experiment clearly still possesses both the Markov and martingale properties, and its quadratic variation measured over the whole experiment is

$$\sum_{j=1}^{6}(S_j - S_{j-1})^2 = 6 \times \left(\sqrt{\frac{t}{6}}\right)^2 = t.$$

I have set up my experiment so that the quadratic variation is just the time taken for the experiment.

I will change the rules again, to speed up the game. We will have n tosses in the allowed time t, with an amount $\sqrt{t/n}$ riding on each throw. Again, the Markov and martingale properties are retained and the quadratic variation is still

$$\sum_{j=1}^{n}(S_j - S_{j-1})^2 = n \times \left(\sqrt{\frac{t}{n}}\right)^2 = t.$$

I am now going to make n larger and larger. All I am doing with my rule changes is to speed up the game, decreasing the time between tosses, with a smaller amount for each bet. But I have chosen my new scalings very carefully, the timestep is decreasing like n^{-1} but the bet size only decreases by $n^{-1/2}$.

In Figure 7.2 I show a series of experiments, each lasting for a time 1, with increasing number of tosses per experiment.

As I go to the limit $n = \infty$, the resulting random walk stays finite. It has an expectation, conditional on a starting value of zero, of

$$E[S(t)] = 0$$

and a variance

$$E[S(t)^2] = t.$$

I use $S(t)$ to denote the amount you have won or the value of the random variable after a time t. The limiting process for this random walk as the timesteps go to zero is called **Brownian motion**, and I will denote it by $X(t)$.

The important properties of Brownian motion are as follows:

- *Finiteness*: Any other scaling of the bet size or 'increments' with timestep would have resulted in either a random walk going to infinity in a finite time, or a limit in which there was no motion at all. It is important that the increment scales with the square root of the timestep.
- *Continuity*: The paths are continuous, there are no discontinuities. Brownian motion is the continuous-time limit of our discrete time random walk.

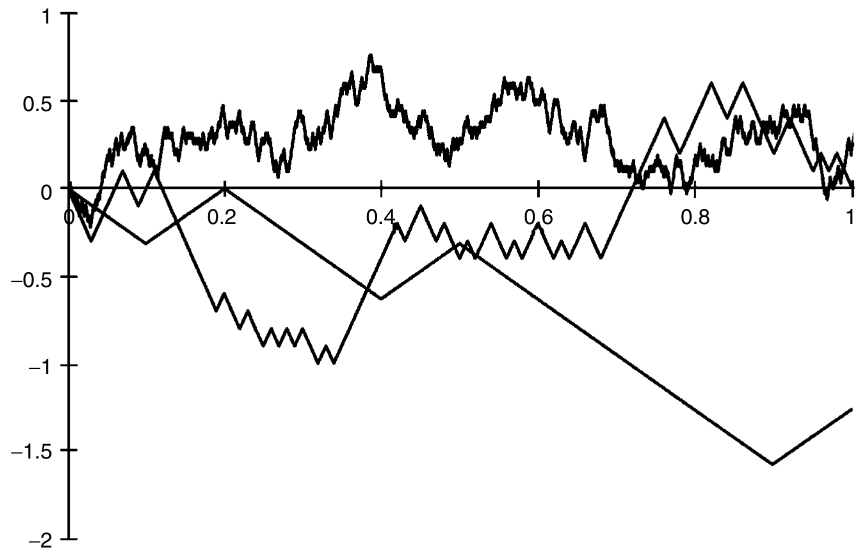


Figure 7.2 A series of coin-tossing experiments, the limit of which is Brownian motion.

- *Markov*: The conditional distribution of $X(t)$ given information up until $\tau < t$ depends only on $X(\tau)$.
- *Martingale*: Given information up until $\tau < t$ the conditional expectation of $X(t)$ is $X(\tau)$.
- *Quadratic variation*: If we divide up the time 0 to t in a partition with $n+1$ partition points $t_i = it/n$ then

$$\sum_{j=1}^{n} \left(X(t_j) - X(t_{j-1})\right)^2 \to t. \quad \text{(technically 'almost surely')}$$

- *Normality*: Over finite time increments t_{i-1} to t_i, $X(t_i) - X(t_{i-1})$ is Normally distributed with mean zero and variance $t_i - t_{i-1}$.

Having built up the idea and properties of Brownian motion from a series of experiments, we can discard the experiments, to leave the Brownian motion that is defined by its properties. These properties will be very important for our financial models.

7.7 STOCHASTIC INTEGRATION

I am going to define a **stochastic integral** by

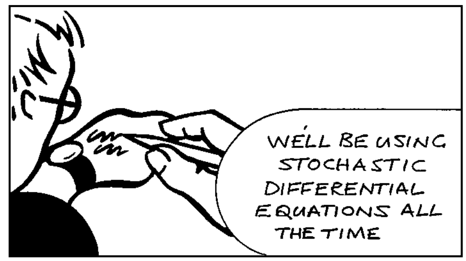
$$W(t) = \int_0^t f(\tau)\, dX(\tau) = \lim_{n\to\infty} \sum_{j=1}^{n} f(t_{j-1}) \left(X(t_j) - X(t_{j-1})\right)$$

with

$$t_j = \frac{jt}{n}.$$

Before I manipulate this is any way or discuss its properties, I want to stress that the function $f(t)$ which I am integrating is evaluated in the summation at the *left-hand point* t_{j-1}. It will be crucially important that each function evaluation does not know about the random increment that multiplies it, i.e. the integration is **nonanticipatory**. In financial terms, we will see that we take some action such as choosing a portfolio and only then does the stock price move. This choice of integration is natural in finance, ensuring that we use no information about the future in our current actions.

7.8 STOCHASTIC DIFFERENTIAL EQUATIONS



Stochastic integrals are important for any theory of stochastic calculus since they can be meaningfully defined. (And in the next section I show how the definition leads to some important properties.) However, it is very common to use a shorthand notation for expressions such as

$$W(t) = \int_0^t f(\tau)\, dX(\tau). \tag{7.1}$$

That shorthand comes from 'differentiating' (7.1) and is

$$dW = f(t)\, dX. \tag{7.2}$$

Think of dX as being an increment in X, i.e. a Normal random variable with mean zero and standard deviation $dt^{1/2}$.

Equations (7.1) and (7.2) are meant to be equivalent. One of the reasons for this shorthand is that the Equation (7.2) looks a lot like an ordinary differential equation. We *do not* go the further step of dividing by dt to make it look exactly like an ordinary differential equation because then we would have the difficult task of defining $\dfrac{dX}{dt}$.

Pursuing this idea further, imagine what might be meant by

$$dW = g(t)\, dt + f(t)\, dX. \tag{7.3}$$

This is simply shorthand for

$$W(t) = \int_0^t g(\tau)\, d\tau + \int_0^t f(\tau)\, dX(\tau).$$

Equations like (7.3) are called **stochastic differential equations**. Their precise meaning comes, however, from the technically more accurate equivalent stochastic integral. In this book I will use the shorthand versions almost everywhere, no confusion should arise.

7.9 THE MEAN SQUARE LIMIT

I am going to describe the technical term **mean square limit**. This is useful in the precise definition of stochastic integration. I will explain the idea by way of the simplest example.

Examine the quantity

$$E\left[\left(\sum_{j=1}^{n}(X(t_j)-X(t_{j-1}))^2-t\right)^2\right] \tag{7.4}$$

where

$$t_j=\frac{jt}{n}.$$

This can be expanded as

$$E\left[\sum_{j=1}^{n}(X(t_j)-X(t_{j-1}))^4+2\sum_{i=1}^{n}\sum_{j<i}(X(t_i)-X(t_{i-1}))^2(X(t_j)-X(t_{j-1}))^2\right.$$
$$\left.-2t\sum_{j=1}^{n}(X(t_j)-X(t_{j-1}))^2+t^2\right].$$

Since $X(t_j)-X(t_{j-1})$ is Normally distributed with mean zero and variance t/n we have

$$E\left[(X(t_j)-X(t_{j-1}))^2\right]=\frac{t}{n}$$

and

$$E\left[(X(t_j)-X(t_{j-1}))^4\right]=\frac{3t^2}{n^2}.$$

Thus (7.4) becomes

$$n\frac{3t^2}{n^2}+n(n-1)\frac{t^2}{n^2}-2tn\frac{t}{n}+t^2=O\left(\frac{1}{n}\right).$$

As $n\to\infty$ this tends to zero. We therefore say that

$$\sum_{j=1}^{n}(X(t_j)-X(t_{j-1}))^2=t$$

in the 'mean square limit.' This is often written, for obvious reasons, as

$$\int_0^t(dX)^2=t.$$

I am not going to use this result, nor will I use the mean square limit technique. However, when I talk about 'equality' in the following 'proof' I mean equality in the mean square sense.

7.10 FUNCTIONS OF STOCHASTIC VARIABLES AND ITÔ'S LEMMA

I am now going to introduce the idea of a function of a stochastic variable. In Figure 7.3 is shown a realization of a Brownian motion $X(t)$ and the function $F(X)=X^2$.

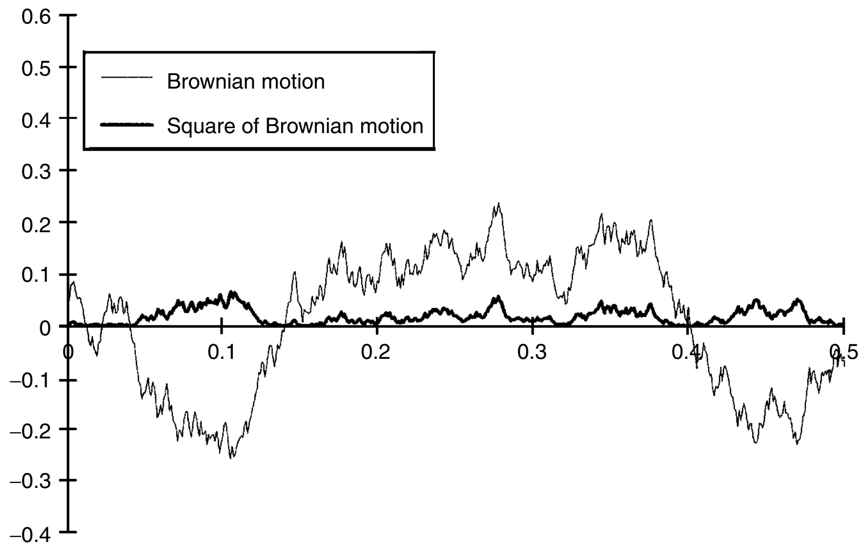


Figure 7.3 A realization of a Brownian motion and its square.

If $F = X^2$ is it true that $dF = 2X\,dX$? No. The ordinary rules of calculus do not generally hold in a stochastic environment. Then what are the rules of calculus?

I am going to 'derive' the most important rule of stochastic calculus, **Itô's lemma**. My derivation is more heuristic than rigorous, but at least it is transparent. I will do this for an arbitrary function $F(X)$.

In this derivation I will need to introduce various timescales. The first timescale is very, very small. I will denote it by

$$\frac{\delta t}{n} = h.$$

This timescale is so small that the function $F(X(t+h))$ can be approximated by a Taylor series:

$$\begin{aligned}&F(X(t+h)) - F(X(t))\\&\quad = (X(t+h) - X(t))\frac{dF}{dX}(X(t)) + \tfrac{1}{2}(X(t+h) - X(t))^2\frac{d^2F}{dX^2}(X(t)) + \cdots.\end{aligned}$$

From this it follows that

$$\begin{aligned}&(F(X(t+h)) - F(X(t))) + (F(X(t+2h)) - F(X(t+h))) + \cdots + (F(X(t+nh))\\&\quad - F(X(t+(n-1)h)))\\&= \sum_{j=1}^{n}(X(t+jh) - X(t+(j-1)h))\frac{dF}{dX}(X(t+(j-1)h))\\&\quad + \frac{1}{2}\frac{d^2F}{dX^2}(X(t))\sum_{j=1}^{n}(X(t+jh) - X(t+(j-1)h))^2 + \cdots.\end{aligned}$$

In this I have used the approximation

$$\frac{d^2F}{dX^2}(X(t+(j-1)h)) = \frac{d^2F}{dX^2}(X(t)).$$

This is consistent with the order of accuracy I require.

The first line in this becomes simply

$$F(X(t+nh)) - F(X(t)) = F(X(t+\delta t)) - F(X(t)).$$

The second is just the definition of

$$\int_t^{t+\delta t} \frac{dF}{dX} dX$$

and the last is

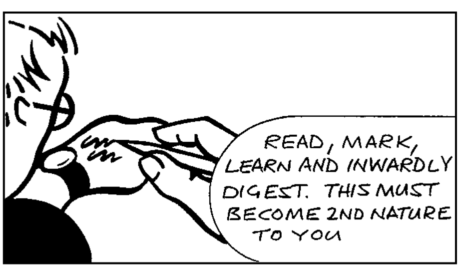
$$\frac{1}{2}\frac{d^2F}{dX^2}(X(t))\delta t,$$

in the *mean square sense*. Thus we have

$$F(X(t+\delta t)) - F(X(t)) = \int_t^{t+\delta t} \frac{dF}{dX}(X(\tau))\,dX(\tau) + \frac{1}{2}\int_t^{t+\delta t} \frac{d^2F}{dX^2}(X(\tau))\,d\tau.$$

I can now extend this result over longer timescales, from zero up to t, over which F *does* vary substantially to get

$$\boxed{F(X(t)) = F(X(0)) + \int_0^t \frac{dF}{dX}(X(\tau))\,dX(\tau) + \frac{1}{2}\int_0^t \frac{d^2F}{dX^2}(X(\tau))\,d\tau.}$$



This is the integral version of **Itô's lemma**, which is usually written as

$$\boxed{dF = \frac{dF}{dX}dX + \frac{1}{2}\frac{d^2F}{dX^2}dt.} \tag{7.5}$$

We can now answer the question, If $F = X^2$ what stochastic differential equation does F satisfy? In this example

$$\frac{dF}{dX} = 2X \quad \text{and} \quad \frac{d^2F}{dX^2} = 2.$$

Therefore Itô's lemma tells us that

$$dF = 2X\,dX + dt.$$

This is *not* what we would get if X were a deterministic variable. In integrated form

$$X^2 = F(X) = F(0) + \int_0^t 2X\,dX + \int_0^t 1\,d\tau = \int_0^t 2X\,dX + t.$$

Therefore

$$\int_0^t X\,dX = \tfrac{1}{2}X^2 - \tfrac{1}{2}t.$$

7.11 ITÔ AND TAYLOR

Having derived Itô's lemma, I am going to give some intuition behind the result and then slightly generalize it.

If we were to do a naive Taylor series expansion of F, completely disregarding the nature of X, and treating dX as a small increment in X, we would get

$$F(X + dX) = F(X) + \frac{dF}{dX}dX + \frac{1}{2}\frac{d^2F}{dX^2}dX^2,$$

ignoring higher-order terms. We could argue that $F(X + dX) - F(X)$ was just the 'change in' F and so

$$dF = \frac{dF}{dX}dX + \frac{1}{2}\frac{d^2F}{dX^2}dX^2.$$

This is very similar to (7.5) (and Taylor series *is* very similar to Itô), with the only difference being that there is a dX^2 instead of a dt. However, since in a sense

$$\int_0^t (dX)^2 = t$$

I could perhaps write

$$dX^2 = dt. \tag{7.6}$$

Although this lacks any rigor (because it's wrong) it does give the correct result. However, on a positive note you can, with little risk of error, use Taylor series with the 'rule of thumb' (7.6) and in practice you will get the right result. Although this is technically incorrect, you almost certainly won't get the wrong result. I will use this rule of thumb almost every time I want to differentiate a function of a random variable.

Time Out...

Intuition behind dX^2 "=" dt

This is subtle. Pay close attention.

We shouldn't really think of dX^2 as being the square of a single Normally distributed random variable, mean zero, variance dt. No, we should think of it as the sum of squares of lots and lots (an infinite number) of independent and identically distributed Normal variables, each one having mean zero and a very, very small (infinitesimal) variance. What happens when you add together lots of i.i.d. variables? In this case we get a quantity with a mean of dt and a variance which goes rapidly to zero as the 'lots' approaches 'infinity.'

To end this section I will generalize slightly. Suppose my stochastic differential equation is

$$dS = a(S)\,dt + b(S)\,dX, \tag{7.7}$$

say, for some functions $a(S)$ and $b(S)$. Here dX is the usual Brownian increment. Now if I have a function of S, $V(S)$, what stochastic differential equation does it satisfy? The answer is

$$dV = \frac{dV}{dS}dS + \tfrac{1}{2}b^2\frac{d^2V}{dS^2}dt.$$

We could derive this properly or just cheat by using Taylor series with $dX^2 = dt$. I could, if I wanted, substitute for dS from (7.7) to get an equation for dV in terms of the pure Brownian motion X:

$$dV = \left(a(S)\frac{dV}{dS} + \tfrac{1}{2}b(S)^2\frac{d^2V}{dS^2}\right) dt + b(S)\frac{dV}{dS}dX.$$

7.12 ITÔ IN HIGHER DIMENSIONS

In financial problems we often have functions of one stochastic variable S and a deterministic variable t, time: $V(S, t)$. If

$$dS = a(S, t)\, dt + b(S, t)\, dX,$$

then the increment dV is given by

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \tfrac{1}{2}b^2\frac{\partial^2 V}{\partial S^2}dt. \tag{7.8}$$

Again, this is shorthand notation for the correct integrated form. This result is obvious, as is the use of partial instead of ordinary derivatives.

Occasionally, we have a function of two, or more, random variables, and time as well: $V(S_1, S_2, t)$. An example would be the value of an option to buy the more valuable out of Nike and Reebok. I will write the behavior of S_1 and S_2 in the general form

$$dS_1 = a_1(S_1, S_2, t)\, dt + b_1(S_1, S_2, t)\, dX_1$$

and

$$dS_2 = a_2(S_1, S_2, t)\, dt + b_2(S_1, S_2, t)\, dX_2.$$

Note that I have *two* Brownian increments dX_1 and dX_2. We can think of these as being Normally distributed with variance dt, but *they are correlated*. The correlation between these two random variables I will call ρ. This can also be a function of S_1, S_2 and t but must satisfy

$$-1 \le \rho \le 1.$$

The 'rules of thumb' can readily be imagined:

$$dX_1^2 = dt, \quad dX_2^2 = dt \quad \text{and} \quad dX_1\, dX_2 = \rho\, dt.$$

Itô's lemma becomes

$$dV = \frac{\partial V}{\partial t}\,dt + \frac{\partial V}{\partial S_1}\,dS_1 + \frac{\partial V}{\partial S_2}\,dS_2 + \tfrac{1}{2}b_1^2\frac{\partial^2 V}{\partial S_1^2}\,dt + \rho b_1 b_2 \frac{\partial^2 V}{\partial S_1\,\partial S_2}\,dt + \tfrac{1}{2}b_2^2\frac{\partial^2 V}{\partial S_2^2}\,dt. \quad (7.9)$$

7.13 SOME PERTINENT EXAMPLES

In this section I am going to introduce a few common random walks and talk about their properties.

7.13.1 Brownian motion with drift

The first example is like the simple Brownian motion but with a drift:

$$dS = \mu\,dt + \sigma\,dX.$$

A realization of this is shown in Figure 7.4. The point to note about this realization is that S has gone negative. This random walk would therefore not be a good model for many financial quantities, such as interest rates or equity prices. This stochastic differential equation can be integrated exactly to get

$$S(t) = S(0) + \mu t + \sigma(X(t) - X(0)).$$

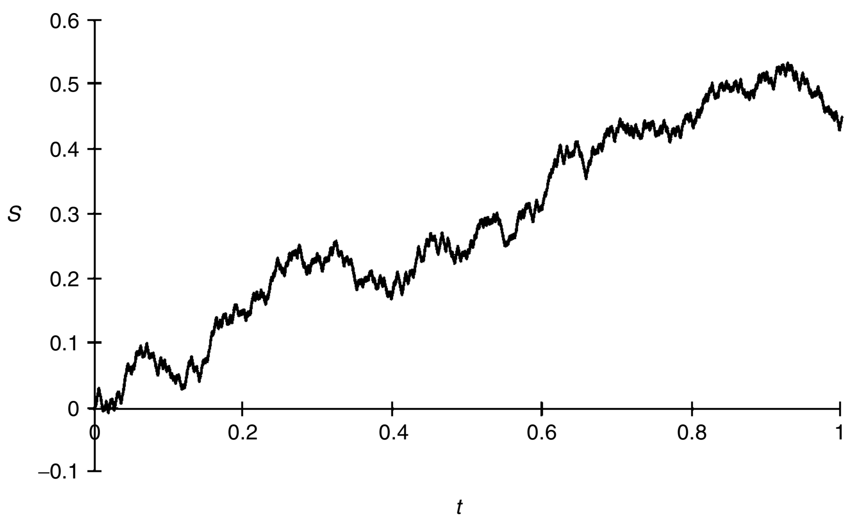


Figure 7.4 A realization of $dS = \mu\,dt + \sigma\,dX$.

Time Out...

Spreadsheet simulation

This random walk is simulated on the following spreadsheet.

	A	B	C	D	E	F
1	**Start**	**100**		**Time**	**S**	
2	μ	**0.1**		0	100	
3	σ	**0.2**		0.01	99.97317	
4				0.02	99.97348	
5	**Timestep**	**0.01**		0.03	99.96162	
6				0.04	99.94905	
7				0.05	99.90528	
8				0.06	99.93077	
9				0.07	99.94513	
10				0.08	99.94401	
11				0.09	99.92764	
12				0.1	99.8889	
13				11	99.84872	
14				12	99.84517	
15				13	99.88342	
16				14	99.90862	
17				0.15	99.92692	
18				0.16	99.9339	
19				0.17	99.91633	
20				0.18	99.92429	
21				0.19	99.90016	
22				0.2	99.88454	

=B1

=D8+$B$5

=E11+$B$2*$B$5+$B$3*SQRT($B$5)*(RAND()+RAND()+RAND()+RAND()+RAND()+RAND()+RAND()+RAND()+RAND()+RAND()+RAND()+RAND()-6)

7.13.2 The lognormal random walk

My second example is similar to the above but the drift and randomness scale with S:

$$dS = \mu S\,dt + \sigma S\,dX. \tag{7.10}$$

A realization of this is shown in Figure 7.5. If S starts out positive it can never go negative; the closer that S gets to zero the smaller the increments dS. For this reason I have had to start the simulation with a nonzero value for S. This property of this random walk is clearly seen if we examine the function $F(S) = \log S$ using Itô's lemma. From Itô we have

$$\begin{aligned} dF &= \frac{dF}{dS}dS + \tfrac{1}{2}\sigma^2 S^2 \frac{d^2F}{dS^2}dt = \frac{1}{S}\left(\mu S\,dt + \sigma S\,dX\right) - \tfrac{1}{2}\sigma^2 dt \\ &= \left(\mu - \tfrac{1}{2}\sigma^2\right) dt + \sigma\,dX. \end{aligned}$$

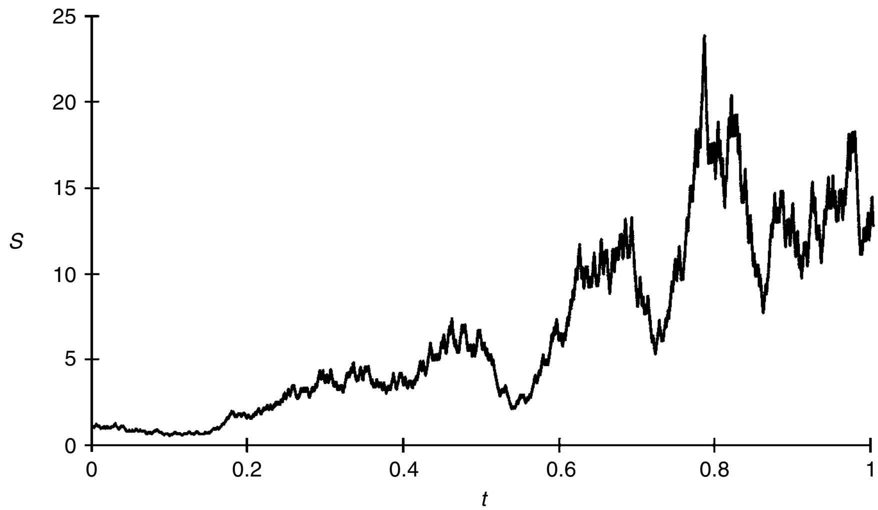


Figure 7.5 A realization of $dS = \mu S\,dt + \sigma S\,dX$.

This shows us that $\log S$ can range between minus and plus infinity but cannot reach these limits in a finite time, therefore S cannot reach zero or infinity in a finite time.

How does the time series in Figure 7.5 which was generated on a spreadsheet using random returns compare qualitatively with the time series in Figure 7.6 which is the real series for Glaxo–Wellcome?

The integral form of this stochastic differential equation follows simply from the stochastic differential equation for $\log S$:

$$S(t) = S(0)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma(X(t)-X(0))}.$$

The stochastic differential equation (7.10) will be particularly important in the modeling of many asset classes. And if we have some function $V(S, t)$ then from Itô it follows that

$$\boxed{dV = \frac{\partial V}{\partial t}\,dt + \frac{\partial V}{\partial S}\,dS + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\,dt.} \tag{7.11}$$

7.13.3 A mean-reverting random walk

The third example is

$$dS = (\nu - \mu S)\,dt + \sigma\,dX.$$

A realization of this is shown in Figure 7.7.

This random walk is an example of a **mean-reverting** random walk. If S is large, the negative coefficient in front of dt means that S will move down on average, if S is small it rises on average. There is still no incentive for S to stay positive in this random walk. With r instead of S this random walk is the Vasicek model for the short-term interest rate.

Figure 7.6 Glaxo–Wellcome share price (volume below). Source: Bloomberg L.P.

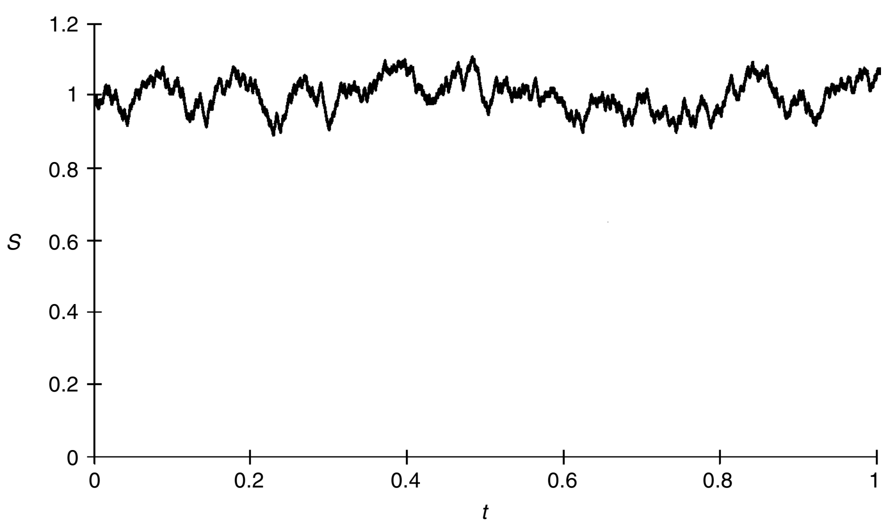


Figure 7.7 A realization of $dS = (\nu - \mu S)\,dt + \sigma\,dX$.

Mean-reverting models are used for modeling a random variable that 'isn't going anywhere.' That's why they are often used for interest rates; Figure 7.8 shows the yield on a Japanese government bond (JGB).

7.13.4 And another mean-reverting random walk

The final example is similar to the third but I am going to adjust the random term slightly:

$$dS = (\nu - \mu S)\, dt + \sigma S^{1/2}\, dX.$$

Now if S ever gets close to zero the randomness decreases, perhaps this will stop S from going negative? Let's play around with this example for a while. And we'll see Itô in practice.

Write $F = S^{1/2}$. What stochastic differential equation does F satisfy? Since

$$\frac{dF}{dS} = \tfrac{1}{2} S^{-1/2} \quad \text{and} \quad \frac{d^2F}{dS^2} = -\tfrac{1}{4} S^{-3/2}$$

we have

$$dF = \left(\frac{4\nu - \sigma^2}{8F} - \tfrac{1}{2}\mu F \right) dt + \tfrac{1}{2}\sigma\, dX.$$

Figure 7.8 Time series of the yield on a JGB. Source: Bloomberg L.P.

I have just turned the original stochastic differential equation with a variable coefficient in front of the random term into a stochastic differential equation with a constant random term. In so doing I have made the drift term nastier. In particular, the drift is now singular at $F = S = 0$. Something special is happening at $S = 0$.

Instead of examining $F(S) = S^{1/2}$, can I find a function $F(S)$ such that its stochastic differential equation has a zero drift term? For this I will need

$$(\nu - \mu S)\frac{dF}{dS} + \tfrac{1}{2}\sigma^2 S\frac{d^2F}{dS^2} = 0.$$

This is easily integrated once to give

$$\frac{dF}{dS} = AS^{-\frac{2\nu}{\sigma^2}}\, e^{\frac{2\mu S}{\sigma^2}} \tag{7.12}$$

for any constant A. I won't take this any further but just make one observation. If

$$\frac{2\nu}{\sigma^2} \geq 1$$

we cannot integrate (7.12) at $S = 0$. This makes the origin **nonattainable**. In other words, if the parameter ν is sufficiently large it forces the random walk to stay away from zero.

This particular stochastic differential equation for S will be important later on, it is the Cox, Ingersoll & Ross model for the short-term interest rate.

These are just four of the many random walks we will be seeing.

7.14 SUMMARY

This chapter introduced the most important tool of the trade, Itô's lemma. Itô's lemma allows us to manipulate functions of a random variable. If we think of S as the value of an asset for which we have a stochastic differential equation, a 'model,' then we can handle functions of the asset, and ultimately value contracts such as options.

If we use Itô as a tool we do not need to know why or how it works, only how to use it. Essentially all we require to successfully use the lemma is a rule of thumb, as explained in the text. Unless we are using Itô in highly unusual situations, then we are unlikely to make any errors.

FURTHER READING

- Neftci (1996) is the only readable book on stochastic calculus for beginners. It does not assume any knowledge about anything. It takes the reader very slowly through the basics as applied to finance.
- Once you have got beyond the basics, move on to Øksendal (1992) and Schuss (1980).

Time Out...

Stochastic calculus for dummies...learning by using

To use stochastic calculus successfully only really requires a little bit of intuition. Then, with use, familiarity breeds, if not contempt, sufficient confidence to make you believe that you understand. (And that's how I've learned the subject.) So, here is that intuition.

- Stochastic differential equations are like recipes for generating random walks, just as we saw in the previous chapter using Excel.
- If you have some quantity, let's call it S, that follows such a random walk, then any function of S is also going to follow a random walk. For example, if S is moving about randomly, then so is S^2.
- The question then becomes, What is the random walk for this function of S? That is, what is its recipe, or what is its stochastic differential equation?
- The answer to that comes from applying something very like Taylor series but with two tricks.
- The first trick is that when you do your Taylor series expansion, only keep terms of size dt or bigger ($dt^{1/2}$).
- The second trick is that every time you see a dX^2 term replace it with dt. Why? Because dX is really made up of lots of Normally distributed random variables, and so dX^2 becomes its expected value dt.

Now, if you are feeling brave, take a peek at this chapter, but just read the words and ignore the math.

CHAPTER 8

the Black–Scholes model

The aim of this Chapter...

... is to explain in as simple and nontechnical a manner as possible the original breakthrough in quantitative finance that led to such a growth in the industry and the development of the subject. By now you will know all the mathematical tools to follow this chapter, and by the end of the chapter will be ready to apply the ideas to new situations.

In this Chapter...

- the foundations of derivatives theory: delta hedging and no arbitrage
- the derivation of the Black–Scholes partial differential equation
- the assumptions that go into the Black–Scholes equation
- how to modify the equation for commodity and currency options

8.1 INTRODUCTION

This is, without doubt, the most important chapter in the book. In it I describe and explain the basic building blocks of derivatives theory. These building blocks are delta hedging and no arbitrage. They form a moderately sturdy foundation to the subject and have performed well since 1973 when the ideas became public.

In this chapter I begin with the stochastic differential equation model for equities and exploit the correlation between this asset and an option on this asset to make a perfectly risk-free portfolio. I then appeal to no arbitrage to equate returns on all risk-free portfolios to the risk-free interest rate, the so-called 'no free lunch' argument.

These ideas are identical to those we saw in Chapter 5, it's just that the math is different.

The arguments are trivially modified to incorporate dividends on the underlying and also to price commodity and currency options and options on futures.

This chapter is quite theoretical, yet all of the ideas contained here are regularly used in practice. Even though all of the assumptions can be shown to be wrong to a greater or lesser extent, the Black–Scholes model is profoundly important both in theory and in practice.

8.2 A VERY SPECIAL PORTFOLIO

In Chapter 2 I described some of the characteristics of options and options markets. I introduced the idea of call and put options, amongst others. The value of a call option is clearly going to be a function of various parameters in the contract, such as the strike price E and the time to expiry $T-t$, T is the date of expiry, and t is the current time. The value will also depend on properties of the asset itself, such as its price, its drift and its volatility, as well as the risk-free rate of interest.[1] We can write the option value as

$$V(S, t; \sigma, \mu; E, T; r).$$

Notice that the semicolons separate different types of variables and parameters:

- S and t are variables;
- σ and μ are parameters associated with the asset price;
- E and T are parameters associated with the details of the particular contract;
- r is a parameter associated with the currency in which the asset is quoted.

I'm not going to carry all the parameters around, except when it is important. For the moment I'll just use $V(S, t)$ to denote the option value.

One simple observation is that a call option will rise in value if the underlying asset rises, and will fall if the asset falls. This is clear since a call has a larger payoff the greater the value of the underlying at expiry. This is an example of **correlation** between two financial instruments, in this case the correlation is positive. A put and the underlying have a negative correlation. We can exploit these correlations to construct a very special portfolio.

[1] Actually, I'm lying. One of these parameters does not affect the option value.

Use Π to denote the value of a portfolio of one long option position and a short position in some quantity Δ, **delta**, of the underlying:

$$\Pi = V(S, t) - \Delta S. \tag{8.1}$$

The first term on the right is the option and the second term is the short asset position. Notice the minus sign in front of the second term. The quantity Δ will for the moment be some constant quantity of our choosing. We will assume that the underlying follows a lognormal random walk

$$dS = \mu S\, dt + \sigma S\, dX.$$

It is natural to ask how the value of the portfolio changes from time t to $t + dt$. The change in the portfolio value is due partly to the change in the option value and partly to the change in the underlying:

$$d\Pi = dV - \Delta\, dS.$$

Notice that Δ has not changed during the timestep; we have not anticipated the change in S. From Itô we have

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt.$$

Thus the portfolio changes by

$$d\Pi = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt - \Delta\, dS. \tag{8.2}$$

Time Out...

Just like the binomial

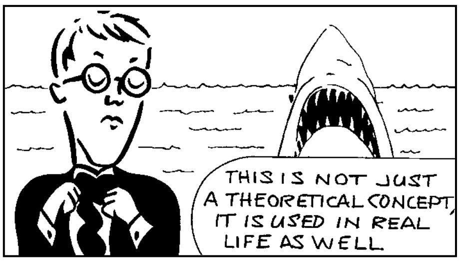
Many people feel more at home with the binomial analysis than with the stochastic analysis of the Black–Scholes model. Well, in principle they are nearly identical, it's just that the math is a little bit more abstract with the Black–Scholes model.

For example, all that Equation (8.2) says is that our special portfolio takes different values depending on what the asset does over the next timestep. In the binomial model there were two different values that the portfolio could take, represented by the up and down movements of the asset. In the Black–Scholes model there's a whole spectrum of possible values represented by the dS terms... so the dS terms represent the risk in the portfolio. And just as in the binomial model we're going to make these terms disappear.

From a technical point of view, in the binomial model we did lots of modeling, hedging etc. first before arriving at the Black–Scholes partial differential equation by performing a Taylor series expansion. In the Black–Scholes analysis the Taylor series expansion, in its stochastic form, comes first and the hedging etc. comes later.

8.3 ELIMINATION OF RISK: DELTA HEDGING

The right-hand side of (8.2) contains two types of terms, the deterministic and the random. The deterministic terms are those with the dt, and the random terms are those with the dS. Pretending for the moment that we know V and its derivatives then we know everything about the right-hand side of (8.2) *except for the value of dS*. And this quantity we can never know in advance.



These random terms are the risk in our portfolio. Is there any way to reduce or even eliminate this risk? This can be done in theory (and *almost* in practice) by carefully choosing Δ. The random terms in (8.2) are

$$\left(\frac{\partial V}{\partial S} - \Delta\right) dS.$$

If we choose

$$\Delta = \frac{\partial V}{\partial S} \tag{8.3}$$

then the randomness is reduced to zero.

Any reduction in randomness is generally termed **hedging**, whether that randomness is due to fluctuations in the stock market or the outcome of a horse race. The perfect elimination of risk, by exploiting correlation between two instruments (in this case an option and its underlying) is generally called **delta hedging**.

Delta hedging is an example of a **dynamic hedging** strategy. From one timestep to the next the quantity $\partial V/\partial S$ changes, since it is, like V, a function of the ever-changing variables S and t. This means that the perfect hedge must be continually rebalanced. In later chapters we will see examples of static hedging, where a hedging position is *not* changed as the variables evolve.

Delta hedging was effectively first described by Thorp & Kassouf (1967) but they missed the crucial (Nobel prize winning) next step. (We will see more of Thorp when we look at casino blackjack as an investment in Chapter 19.)

8.4 NO ARBITRAGE

After choosing the quantity Δ as suggested above, we hold a portfolio whose value changes by the amount

$$d\Pi = \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt. \tag{8.4}$$

This change is completely *riskless*. If we have a completely risk-free change $d\Pi$ in the portfolio value Π then it must be the same as the growth we would get if we put the equivalent amount of cash in a risk-free interest-bearing account:

$$d\Pi = r\Pi\, dt. \tag{8.5}$$

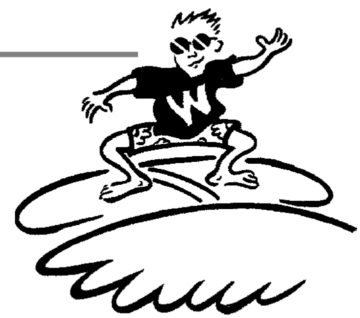
This is an example of the **no arbitrage** principle.

To see why this should be so, consider in turn what might happen if the return on the portfolio were, first, greater and, second, less than the risk-free rate. If we were guaranteed to get a return of greater than r from the delta-hedged portfolio then what we could do is borrow from the bank, paying interest at the rate r, invest in the risk-free option/stock portfolio and make a profit. If, on the other hand the return were less than the risk-free rate we should go short the option, delta hedge it, and invest the cash in the bank. Either way, we make a riskless profit in excess of the risk-free rate of interest. At this point we say that, all things being equal, the action of investors buying and selling to exploit the arbitrage opportunity will cause the market price of the option to move in the direction that eliminates the arbitrage.

Time Out...

The money-in-the-bank equation

Equation (8.5) is the same as 'our first differential equation' for money in the bank. The notation has changed from M to Π.



8.5 THE BLACK–SCHOLES EQUATION

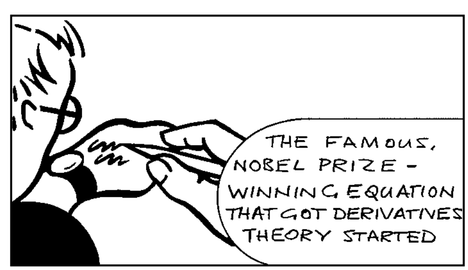
Substituting (8.1), (8.3) and (8.4) into (8.5) we find that

$$\left(\frac{\partial V}{\partial t}+\tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt = r\left(V - S\frac{\partial V}{\partial S}\right)dt.$$

On dividing by dt and rearranging we get

$$\boxed{\frac{\partial V}{\partial t}+\tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0} \qquad (8.6)$$

This is the **Black–Scholes equation**. The equation was first written down in 1969, but a few years passed, with Fischer Black and Myron Scholes justifying the model, before it was published. The derivation of the equation was finally published in 1973, although the call and put formulas had been published a year earlier.[2]



The Black–Scholes equation is a **linear parabolic partial differential equation**. In fact, almost all partial differential equations in finance are of a similar form. They are almost always linear, meaning that if you have two solutions of the equation then the sum of these is itself also a solution. Financial equations are also usually parabolic, meaning that

[2] The pricing formulas were being used even earlier by Ed Thorp, to make money.

they are related to the heat or diffusion equation of mechanics. One of the good things about this is that such equations are relatively easy to solve numerically.

The Black–Scholes equation contains all the obvious variables and parameters such as the underlying, time, and volatility, but there is no mention of the drift rate μ. Why is this? Any dependence on the drift dropped out at the same time as we eliminated the dS component of the portfolio. The economic argument for this is that since we can perfectly hedge the option with the underlying we should not be rewarded for taking unnecessary risk; only the risk-free rate of return is in the equation. This means that if you and I agree on the volatility of an asset we will agree on the value of its derivatives *even if we have differing estimates of the drift*.

Another way of looking at the hedging argument is to ask what happens if we hold a portfolio consisting of just the stock, in a quantity Δ, and cash. If Δ is the partial derivative of some option value then such a portfolio will yield an amount at expiry that is simply that option's payoff. In other words, we can use the same Black–Scholes argument to **replicate** an option just by buying and selling the underlying asset. This leads to the idea of a **complete market**. In a complete market an option can be replicated with the underlying, thus making options redundant. Why buy an option when you can get the same payoff by trading in the asset? Many things conspire to make markets incomplete such as transaction costs.

Time Out...

Slopes, gradients etc.

The Black–Scholes partial differential equation is a relationship between the option value, the gradient in the S and t directions and the gradient of the gradient in the S direction. This sounds complicated. I can understand why. But it is really very simple when you actually come to solve the equation numerically. Here's a foretaste of what's in Chapter 25.

Imagine you're at expiry of a call option. At that time do you know the option value as a function of the underlying asset S? Yes, of course, it's just the payoff function

$$\max(S - E, 0).$$

So you know one term in Equation (8.6), the last one.

Do you know the slope of the option value in the S direction at expiry? You certainly do. It's zero for $S < E$ and one for $S > E$. (Let's not worry about what the value is *at* $S = E$, we'll leave that to others to lose sleep over.) So you know the second-to-last term in the equation. Mathematically, this is represented by the Heaviside function, $\mathcal{H}(\cdot)$, zero when its argument is negative and one when it is positive.

So

$$\frac{\partial V}{\partial S} = \mathcal{H}(S - E).$$

What about the slope of the slope in the S direction? Well, if the slope is zero or one, the slope of the slope is zero. So you know the second term in the equation.

$$\frac{\partial^2 V}{\partial S^2} = 0.$$

To recap., we've got

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \times 0 + rS\mathcal{H}(S-E) - r\max(S-E,0) = 0.$$

This is an equation for $\frac{\partial V}{\partial t}$. For example, if $S < E$ we have

$$\frac{\partial V}{\partial t} = 0.$$

If $S > E$ we have

$$\frac{\partial V}{\partial t} = -rS + rS - RE = -rE.$$

And the significance of this?

If we know $\partial V/\partial t$ then we know the slope of the option value in the t direction. If we know this slope then we can find the option value at the time *just before* expiry. If we are at time $T - \delta t$, where δt is small, then the option value will be approximately

$$V = 0 \quad \text{for} \quad S < E$$

and

$$V = S - E + rE\,\delta t \quad \text{for} \quad S > E.$$

See how we have found the option value one timestep before expiry? We can keep repeating this procedure over and over, working backwards in time until we get to the present. And as the timestep gets smaller, so this approximation to the option value gets more accurate.

One, not so minor point. How does the option value ever become nonzero for $S < E$? I guess we should worry about what happens at $S = E$ after all. This'll sort itself out later on, don't worry. What I've described here is the basis for the important numerical method known as the explicit finite-difference method, which we'll be seeing lots of later on.

8.6 THE BLACK–SCHOLES ASSUMPTIONS

What are the 'assumptions' in the Black–Scholes model? Here is a partial list, together with some discussion.

- *The underlying follows a lognormal random walk:* This is not entirely necessary. To find explicit solutions we will need the random term in the stochastic differential equation for S to be proportional to S. The 'factor' σ does not need to be constant to find solutions, but it must only be time dependent. As far as the validity of the equation is concerned it doesn't matter if the volatility is also asset-price dependent, but then the equation will either have very messy explicit solutions, if it has any at all, or have to be solved numerically. Then there is the question of the drift term μS. Do we need this term to take this form, after all it doesn't even appear in the equation? There is a technicality here that whatever the stochastic differential equation for the asset S, the domain over which the asset can range must be zero to infinity. This is a technicality I am not going into, but it amounts to another elimination of arbitrage. It is possible to choose the drift so that the asset is restricted to lie within a range, such a drift would not be allowed.

- *The risk-free interest rate is a known function of time:* This restriction is just to help us find explicit solutions again. If r were constant this job would be even easier. In practice, the interest rate is often taken to be time dependent but known in advance. Explicit formulas still exist for the prices of simple contracts. In reality the rate r is not known in advance and is itself stochastic, or so it seems from data. I will discuss stochastic interest rates later. We've also assumed that lending and borrowing rates are the same.
- *There are no dividends on the underlying:* I will drop this restriction in a moment.
- *Delta hedging is done continuously:* This is definitely impossible. Hedging must be done in discrete time. Often the time between rehedges will depend on the level of transaction costs in the market for the underlying; the lower the costs, the more frequent the rehedging.
- *There are no transaction costs on the underlying:* The dynamic business of delta hedging is in reality expensive since there is a bid-offer spread on most underlyings. In some markets this matters and in some it doesn't.
- *There are no arbitrage opportunities:* This is a beauty. Of course there are arbitrage opportunities, a lot of people make a lot of money finding them.[3] It is extremely important to stress that we are ruling out model-dependent arbitrage. This is highly dubious since it depends on us having the correct model in the first place, and that is unlikely. I am happier ruling out model-independent arbitrage, i.e. arbitrage arising when two identical cashflows have different values. But even that can be criticized.

There are many more assumptions but the above are the most important.

8.7 FINAL CONDITIONS

The Black–Scholes Equation (8.6) knows nothing about what kind of option we are valuing, whether it is a call or a put, or what is the strike and the expiry. These points are dealt with by the **final condition**. We must specify the option value V as a function of the underlying at the expiry date T. That is, we must prescribe $V(S,T)$, the payoff.

For example, if we have a call option then we know that

$$V(S,T) = \max(S-E,0).$$

For a put we have

$$V(S,T) = \max(E-S,0),$$

[3] Life, and everything in it, is based on arbitrage opportunities and their exploitation.

for a binary call

$$V(S, T) = \mathcal{H}(S - E)$$

and for a binary put

$$V(S, T) = \mathcal{H}(E - S),$$

where $\mathcal{H}(\cdot)$ is the **Heaviside function**, which is zero when its argument is negative and one when it is positive.

The imposition of the final condition will be explained in Chapters 9 and 10, and implemented numerically in later chapters.

As an aside, observe that both the asset, S, and 'money in the bank,' e^{rt}, satisfy the Black–Scholes equation.

8.8 OPTIONS ON DIVIDEND-PAYING EQUITIES

The first generalization we discuss is how to value options on stocks paying dividends. This is just about the simplest generalization of the Black–Scholes model. To keep things simple let's assume that the asset receives a continuous and constant dividend yield, D. Thus in a time dt each asset receives an amount $DS\,dt$. This must be factored into the derivation of the Black–Scholes equation. I take up the Black–Scholes argument at the point where we are looking at the change in the value of the portfolio:

$$d\Pi = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt - \Delta\, dS - D\Delta S\, dt.$$

The last term on the right-hand side is simply the amount of the dividend per asset, $DS\,dt$, multiplied by the number of the asset held, $-\Delta$. The Δ is still given by the rate of change of the option value with respect to the underlying, but after some simple substitutions we now get

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} - rV = 0. \qquad (8.7)$$

8.9 CURRENCY OPTIONS

Options on currencies are handled in exactly the same way. In holding the foreign currency we receive interest at the foreign rate of interest r_f. This is just like receiving a continuous dividend. I will skip the derivation but we readily find that

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - r_f)S\frac{\partial V}{\partial S} - rV = 0. \qquad (8.8)$$

8.10 COMMODITY OPTIONS

The relevant feature of commodities requiring that we adjust the Black–Scholes equation is that they have a **cost of carry**. That is, the storage of commodities is not without cost. Let us introduce q as the fraction of the value of a commodity that goes towards paying the cost of carry. This means that just holding the commodity will result in a gradual loss of wealth even if the commodity price remains fixed. To be precise, for each unit of the commodity held an amount $qS\,dt$ will be required during short time dt to finance the holding. This is just like having a negative dividend and so we get

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r+q)S\frac{\partial V}{\partial S} - rV = 0. \tag{8.9}$$

8.11 EXPECTATIONS AND BLACK–SCHOLES

In the Black–Scholes equation there is no mention of the drift rate of the underlying asset μ. It seems that whether the asset is rising or falling in the long run, it doesn't affect the value of an option. This is highly counterintuitive. But we saw exactly the same thing happening in the binomial model of Chapter 5. *At the same time as hedging away exposure to randomness, we hedge away exposure to direction*.[4]

We also saw in Chapter 5 that an option value can be thought of as being an expectation. But a very special expectation. In words:

> The fair value of an option is the
> present value of the expected payoff at expiry
> under a risk-neutral random walk for the underlying

We can write

$$\text{option value} = e^{-r(T-t)}E[\text{payoff}(S)]$$

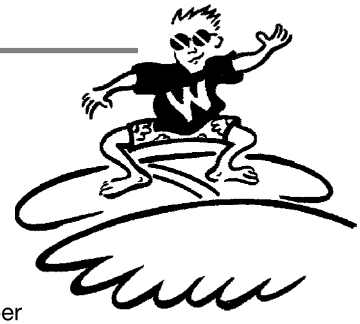
provided that the expectation is with respect to the risk-neutral random walk, not the *real* one.

But what do 'real' and 'risk neutral' mean exactly?

Real refers to the actual random walk as seen, as realized. It has a certain volatility σ and a certain drift rate μ. We can simulate this random walk on a spreadsheet very easily, and calculate expected future option payoffs for example. Typical parameters might be $\sigma = 30\%$ and $\mu = 18\%$.

Risk neutral refers to an artificial random walk that has little to do with the path an asset is actually following. That is not strictly true, both the real and the risk-neutral random asset paths have the same volatility. The difference is in the drift rates. The risk-neutral random walk has a drift that is the same as the risk-free interest rate, r. So simulate risk-free random walks to calculate expectations if you want to work out theoretical option values. In Chapter 26 we will see how this is done in practice.

[4] We are all in debt to the poetical Jeanette Winterson for the rebirth, regrowth and reinstatement of the italic font.



Time Out...

Real and risk neutral

This idea is probably more confusing than anything else in quantitative finance, but is extremely important. I will use the phrase 'risk-neutral (random walk)' several times in this book. Watch out for it, and remember that all it means is that you must pretend that the random walk of the underlying has a drift rate that is the same as the risk-free interest rate.

But remember also that such risk-neutral valuation is only valid *when hedging can be used to eliminate all risk*. If hedging is impossible, risk-neutral valuation is meaningless.

8.12 SOME OTHER WAYS OF DERIVING THE BLACK–SCHOLES EQUATION

The derivation of the Black–Scholes equation above is the classical one, and similar to the original Black & Scholes derivation. There are other ways of getting to the same result. Here are a few, without any of the details. The details, and more examples, are contained in the final reference in the Further reading.

8.12.1 The martingale approach

The value of an option can be shown to be an expectation, not a real expectation but a special, risk-neutral one. This is a useful result, since it forms the basis for pricing by simulation, see Chapter 26. The concepts of hedging and no arbitrage are obviously still used in this derivation. The major drawback with this approach is that it requires a probabilistic description of the financial world. Modern models are moving away from this simplistic idea, and the martingale method may no longer be useful.

8.12.2 The binomial model

The binomial model is a discrete time, discrete asset price model for underlyings and again uses hedging and no arbitrage to derive a pricing algorithm for options. We have seen this in detail in Chapter 5. In taking the limit as the timestep shrinks to zero we get the continuous-time Black–Scholes equation.

8.12.3 CAPM/utility

We'll be seeing the Capital Asset Pricing Model later, for the moment you just need to know that it is a model for the behavior of risky assets and a principle and algorithm for defining and finding optimal ways to allocate wealth among the assets. Portfolios are described in terms of their risk (standard deviation of returns) and reward (expected growth). If you include options in this framework then the possible combinations of risk and reward are not increased. This is because options are, in a sense, just functions of their underlyings. This is market completeness. The risk and reward on an option and on its underlying are related and the Black–Scholes equation follows.

8.13 NO ARBITRAGE IN THE BINOMIAL, BLACK–SCHOLES AND 'OTHER' WORLDS

With the Black–Scholes continuous-time model, as with the binomial discrete-time model, we have been able to eliminate uncertainty in the value of a portfolio by a judicious choice of a hedge. In both cases we find that it does not matter how the underlying asset moves, the resulting value of the portfolio is the same. This is especially clear in the binomial model. This hedging is only possible in these two simple, popular models. For consider a trivial generalization: the trinomial random walk.

In Figure 8.1 we see a representation of a trinomial random walk. After a timestep δt the asset could have risen to uS, fallen to vS or not moved from S.

What happens if we try to hedge an option under this scenario? As before, we can 'hedge' with $-\Delta$ of the underlying but this time we would like to choose Δ so that the value of the portfolio (of one option and $-\Delta$ of the asset) is the same at time $t + \delta t$ no matter to which value the asset moves. In other words, we want the portfolio to have the same value for all *three* possible outcomes. Unfortunately, we cannot choose a value for Δ that ensures this to be the case: this amounts to solving two equations (first portfolio value = second portfolio value = third portfolio value) with just one unknown (the delta). Hedging is not possible in the trinomial world. Indeed, perfect hedging, and thus the application of the 'no-arbitrage principle' is only possible in the two special cases: the Black–Scholes continuous time/continuous asset world, and the binomial world. And in the far more complex 'real' world, delta hedging is *not* possible.[5]

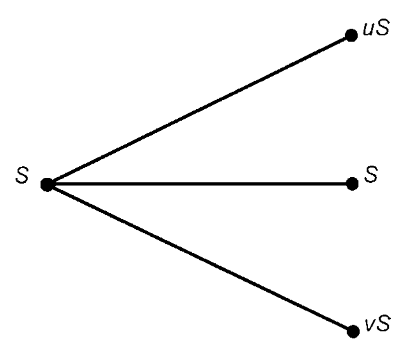


Figure 8.1 The trinomial tree. Perfect risk-free hedging is not possible under this scenario.

[5] Is it good for the popular models to have such an unrealistic property? These models are at least a good *starting* point.

8.14 FORWARDS AND FUTURES

Can we find values for forward and future contracts? How do they fit into the Black–Scholes framework? With ease. Let's look at the simpler forward contract first.

8.14.1 Forward contracts

Notation first. $V(S, t)$ will be the value of the forward contract at any time during its life on the underlying asset S, and maturing at time T. I'll assume that the delivery price is known and then find the forward contract's value. At the end of this section I'll turn this on its head to find the forward price. If you can't remember the differences between all these terms, take a look at Chapter 1 again.

Set up the portfolio of one long forward contract and short Δ of the underlying asset:

$$\Pi = V(S, t) - \Delta S.$$

This changes by an amount

$$d\Pi = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt - \Delta\, dS$$

from t to $t + dt$. Choose

$$\Delta = \frac{\partial V}{\partial S}$$

to eliminate risk. By applying the no-arbitrage argument we end up with exactly the Black–Scholes partial differential equation again.

The final condition for the equation is simply the difference between the asset price S and the fixed delivery price $\overline{S}$, say. So

$$V(S, T) = S - \overline{S}.$$

The solution of the equation with this final condition is

$$V(S, t) = S - \overline{S}e^{-r(T-t)}.$$

This is the forward contract's value during its life.

How does this relate to the setting of the delivery price in the first place, and the newspaper-quoted forward price?

The delivery price is set initially at $t = t_0$ as the price that gives the forward contract zero value. If the underlying asset is S_0 at t_0 then

$$0 = S_0 - \overline{S}e^{-r(T-t_0)}$$

or

$$\overline{S} = S_0 e^{r(T-t_0)}.$$

And the forward price, as quoted? This (see Chapter 1 for a reminder) is the delivery price, as varying from day to day. So the forward price for the contract maturing at T is

$$\text{Forward price} = Se^{r(T-t)}.$$

8.15 FUTURES CONTRACTS

Emboldened by the above, let's try and calculate the futures price. This is more subtle, that's why I calculate it second. Use $F(S, t)$ to denote the futures price.

Remember that the value of the futures contract during its life is always zero because the change in value is settled daily. This cashflow must be taken into account in our analysis.

Set up a portfolio of one long futures contract and short Δ of the underlying:

$$\Pi = -\Delta S.$$

Where is the value of the futures contract? Is this a mistake? No, because it has no value it doesn't appear in the portfolio valuation equation. How does the portfolio change in value?

$$d\Pi = dF - \Delta\, dS.$$

The dF represents the cashflow due to the continual settlement. Applying Itô's lemma,

$$d\Pi = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S}dS + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2}dt - \Delta\, dS.$$

Choose

$$\Delta = \frac{\partial F}{\partial S}$$

to eliminate risk. Set

$$d\Pi = r\Pi\, dt$$

to get

$$\frac{\partial F}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + rS\frac{\partial F}{\partial S} = 0.$$

Observe that there are only three terms in this, it is not the same as the Black–Scholes equation.

The final condition is

$$F(S, T) = S,$$

the futures price and the underlying must have the same value at maturity.

The solution is just

$$F(S, t) = Se^{r(T-t)}.$$

8.15.1 When interest rates are known, forward prices and futures prices are the same

We've just seen that the forward price and the futures price are the same when interest rate are constant. They are still the same when rates are known functions of time. Matters are more subtle when interest rates are stochastic. But we'll have to wait a few chapters to investigate this problem.

8.16 OPTIONS ON FUTURES

The final modification to the Black–Scholes model in this chapter is to value options on futures. Recalling that the future price of a nondividend-paying equity F is related to the spot price by

$$F = e^{r(T_F - t)}S$$

where T_F is the maturity date of the futures contract. We can easily change variables, and look for a solution $V(S,t) = \mathcal{V}(F,t)$. We find that

$$\frac{\partial \mathcal{V}}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 \mathcal{V}}{\partial F^2} - r\mathcal{V} = 0. \tag{8.10}$$

The equation for an option on a future is actually simpler than the Black–Scholes equation.

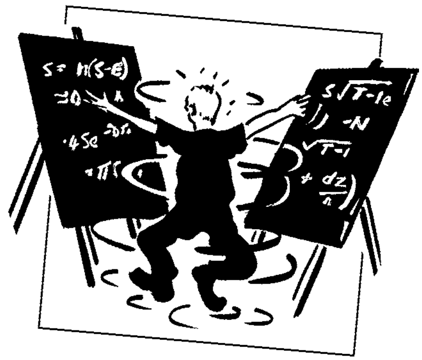
8.17 SUMMARY

This was an important but not too difficult chapter. In it I introduced some very powerful and beautiful concepts such as delta hedging and no arbitrage. These two fundamental principles led to the Black–Scholes option pricing equation. Everything from this point on is based on, or is inspired by, these ideas.

FURTHER READING

- The history of option theory, leading up to Black–Scholes is described in Briys *et al*. (1998).
- The story of the derivation of the Black–Scholes equation, written by Bob Whaley, can be found in the 10th anniversary issue of *Risk* magazine, published in December 1997.
- Of course, you must read the original work, Black & Scholes (1973) and Merton (1973).
- See Black (1976) for the details of the pricing of options on futures, and Garman & Kohlhagen (1983) for the pricing of FX options.
- For details of other ways to derive the Black–Scholes equation see Andreasen, Jensen & Poulson (1998).

CHAPTER 9
partial differential equations



The aim of this Chapter...

... is to compare the Black–Scholes equation with mathematical models in other walks of life and so instill in the reader confidence in the relevance of partial differential equations, and to demonstrate some of the more useful solution methods... although we won't really be needing any of them.

In this Chapter...

- properties of the parabolic partial differential equation
- the meaning of terms in the Black–Scholes equation
- some solution techniques

9.1 INTRODUCTION

The analysis and solution of partial differential equations is a BIG subject. We can only skim the surface in this book. If you don't feel comfortable with the subject, then the list of books at the end should be of help. However, to understand finance, and even to solve partial differential equations numerically, does not require any great depth of understanding. The aim of this chapter is to give just enough background to the subject to permit any reasonably numerate person to follow the rest of the book; I want to keep the entry requirements to the subject as low as possible.

9.2 PUTTING THE BLACK–SCHOLES EQUATION INTO HISTORICAL PERSPECTIVE

The Black–Scholes partial differential equation is in two dimensions, S and t. It is a parabolic equation, meaning that it has a second derivative with respect to one variable, S, and a first derivative with respect to the other, t. Equations of this form are more colloquially known as **heat** or **diffusion equations**.

The equation, in its simplest form, goes back to almost the beginning of the nineteenth century. Diffusion equations have been successfully used to model

- diffusion of one material within another, smoke particles in air
- flow of heat from one part of an object to another
- chemical reactions, such as the Belousov–Zhabotinsky reaction which exhibits fascinating wave structure
- electrical activity in the membranes of living organisms, the Hodgkin–Huxley model
- dispersion of populations, individuals move both randomly and to avoid overcrowding
- pursuit and evasion in predator-prey systems
- pattern formation in animal coats, the formation of zebra stripes
- dispersion of pollutants in a running stream

In most of these cases the resulting equations are more complicated than the Black–Scholes equation.

The simplest heat equation for the temperature in a bar is usually written in the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

where u is the temperature, x is a spatial coordinate and t is time. This equation comes from a heat balance. Consider the flow into and out of a small section of the bar. The flow of heat along the bar is proportional to the spatial gradient of the temperature

$$\frac{\partial u}{\partial x}$$

and thus the derivative of this, the *second* derivative of the temperature, is the heat retained by the small section. This retained heat is seen as a rise in the temperature, represented mathematically by

$$\frac{\partial u}{\partial t}.$$

The balance of the second x-derivative and the first t-derivative results in the heat equation. (There would be a coefficient in the equation, depending on the properties of the bar, but I have set this to one.)

9.3 THE MEANING OF THE TERMS IN THE BLACK–SCHOLES EQUATION

The Black–Scholes equation can be accurately interpreted as a reaction-convection-diffusion equation. The basic diffusion equation is a balance of a first-order t derivative and a second-order S derivative:

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}.$$

If these were the only terms in the Black–Scholes equation it would still exhibit the smoothing-out effect, that any discontinuities in the payoff would be instantly diffused away. The only difference between these terms and the terms as they appear in the basic heat or diffusion equation, is that the diffusion coefficient is a function of one of the variables S. Thus we really have diffusion in a nonhomogeneous medium.

The first-order S-derivative term

$$rS\frac{\partial V}{\partial S}$$

can be thought of as a convection term. If this equation represented some physical system, such as the diffusion of smoke particles in the atmosphere, then the convective term would be due to a breeze, blowing the smoke in a preferred direction.

The final term

$$-rV$$

is a reaction term. Balancing this term and the time derivative would give a model for decay of a radioactive body, with the half-life being related to r. (A better description, for which I am indebted to a delegate on a training course but whose name I've forgotten, is that this term is a 'passive smoking' effect.)

Putting these terms together and we get a reaction-convection-diffusion equation. An almost identical equation would be arrived at for the dispersion of pollutant along a flowing river with absorption by the sand. In this, the dispersion is the diffusion, the flow is the convection, and the absorption is the reaction.

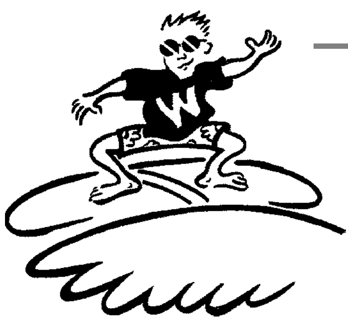
9.4 BOUNDARY AND INITIAL/FINAL CONDITIONS

To uniquely specify a problem we must prescribe **boundary conditions** and an **initial** or **final condition**. Boundary conditions tell us how the solution must behave for all time at certain values of the asset. In financial problems we usually specify the behavior of the solution at $S = 0$ and as $S \to \infty$. We must also tell the problem how the solution begins. The Black–Scholes equation is a backward equation, meaning that the signs of the t derivative and the second S derivative in the equation are the same when written on the same side of the equals sign. We therefore have to impose a final condition. This is usually the payoff function at expiry.

The Black–Scholes equation in its basic form is linear; add together two solutions of the equation and you will get a third. This is not true of nonlinear equations. Linear diffusion equations have some very nice properties. Even if we start out with a discontinuity in the final data, due to a discontinuity in the payoff, this *immediately* gets smoothed out, this is due to the diffusive nature of the equation. Another nice property is the uniqueness of the solution. Provided that the solution is not allowed to grow too fast as S tends to infinity the solution will be unique. This precise definition of 'too fast' need not worry us, we will not have to worry about uniqueness for any problems we encounter.

9.5 SOME SOLUTION METHODS

We are not going to spend much time on the exact solution of the Black–Scholes equation. Such solution is important, but current market practice is such that models have features which preclude the exact solution. The few explicit, closed-form solutions that are used by practitioners will be covered in the next chapter.



Time Out...

Do I need to know this ?

No. You probably won't need to find explicit solutions in practice. Indeed, very rarely can explicit solutions be found to realistic financial problems. That's why I focus more on numerical methods in this book. Unless you are doing a heavily math-orientated course, you can safely skip the rest of this chapter.

9.5.1 Transformation to constant coefficient diffusion equation

It can sometimes be useful to transform the basic Black–Scholes equation into something a little bit simpler by a change of variables. If we write

$$V(S,t) = e^{\alpha x+\beta\tau}U(x,\tau),$$

where

$$\alpha = -\frac{1}{2}\left(\frac{2r}{\sigma^2}-1\right), \quad \beta = -\frac{1}{4}\left(\frac{2r}{\sigma^2}+1\right)^2, \quad S = e^x \quad \text{and} \quad t = T-\frac{2\tau}{\sigma^2},$$

then $U(x,\tau)$ satisfies the basic diffusion equation

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2}. \tag{9.1}$$

This simpler equation is easier to handle than the Black–Scholes equation. Sometimes that can be important, for example when seeking closed-form solutions, or in some simple numerical schemes. We shall not pursue this any further.

9.5.2 Green's functions

One solution of the Black–Scholes equation is

$$V'(S,t) = \frac{e^{-r(T-t)}}{\sigma S'\sqrt{2\pi(T-t)}} \exp\left(-\frac{(\log(S/S') + (r-\frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)}\right) \tag{9.2}$$

for any S'. (You can verify this by substituting back into the equation, but we'll also be seeing it derived in the next chapter.) This solution is special because as $t \to T$ it becomes zero everywhere, except at $S = S'$. In this limit the function becomes what is known as a **Dirac delta function**. Think of this as a function that is zero everywhere except at one point where it is infinite, in such a way that its integral is one. How is this of help to us?

Expression (9.2) is a solution of the Black–Scholes equation for any S'. Because of the linearity of the equation we can multiply (9.2) by any constant, and we get another solution. But then we can also get another solution by adding together expressions of the form (9.2) but with different values for S'. Putting this together, and thinking of an integral as just a way of adding together many solutions, we find that

$$\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_0^\infty \exp\left(-\frac{(\log(S/S') + (r-\frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)}\right) f(S')\frac{dS'}{S'}$$

is also a solution of the Black–Scholes equation for any function $f(S')$. (If you don't believe me, substitute it into the Black–Scholes equation.)

Because of the nature of the integrand as $t \to T$ (i.e. that it is zero everywhere except at S' and has integral one), if we choose the arbitrary function $f(S')$ to be the payoff function then this expression becomes the solution of the problem:

$$V(S,t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_0^\infty \exp\left(-\frac{(\log(S/S') + (r-\frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)}\right) \text{Payoff}(S')\frac{dS'}{S'}.$$

The function $V'(S,t)$ given by (9.2) is called the **Green's function**.

9.5.3 Series solution

Sometimes we have boundary conditions at two finite (and nonzero) values of S, S_u and S_d, say (we see examples in Chapter 13). For this type of problem, we postulate that the required solution of the Black–Scholes equation can be written as an infinite sum of special functions. First of all, transform to the nicer basic diffusion equation in x and τ. Now write the solution as

$$e^{\alpha x+\beta\tau}\sum_{i=0}^{\infty} a_i(\tau)\sin(i\omega x) + b_i(\tau)\cos(i\omega x),$$

for some ω and some functions a and b to be found. The linearity of the equation suggests that a sum of solutions might be appropriate. If this is to satisfy the Black–Scholes equation then we must have

$$\frac{da_i}{d\tau} = -i^2\omega^2 a_i(\tau) \quad \text{and} \quad \frac{db_i}{d\tau} = -i^2\omega^2 b_i(\tau).$$

You can easily show this by substitution. The solutions are thus

$$a_i(\tau) = A_i e^{-i^2\omega^2\tau} \quad \text{and} \quad b_i(\tau) = B_i e^{-i^2\omega^2\tau}.$$

The solution of the Black–Scholes equation is therefore

$$e^{\alpha x+\beta\tau} \sum_{i=0}^{\infty} e^{-i^2\omega^2\tau}(A_i \sin(i\omega x) + B_i \cos(i\omega x)). \tag{9.3}$$

We have solved the equation, all that we need to do now is to satisfy boundary and initial conditions.

Consider the example where the payoff at time $\tau = 0$ is $f(x)$ (although it would be expressed in the original variables, of course) but the contract becomes worthless if ever $x = x_d$ or $x = x_u$.[1]

Rewrite the term in brackets in (9.3) as

$$C_i \sin\left(i\omega' \frac{x - x_d}{x_u - x_d}\right) + D_i \cos\left(i\omega' \frac{x - x_d}{x_u - x_d}\right).$$

To ensure that the option is worthless on these two x values, choose $D_i = 0$ and $\omega' = \pi$. The boundary conditions are thereby satisfied. All that remains is to choose the C_i to satisfy the final condition:

$$e^{\alpha x} \sum_{i=0}^{\infty} C_i \sin\left(i\omega' \frac{x - x_d}{x_u - x_d}\right) = f(x).$$

This also is simple. Multiplying both sides by

$$\sin\left(j\omega' \frac{x - x_d}{x_u - x_d}\right),$$

and integrating between x_d and x_u we find that

$$C_j = \frac{2}{x_u - x_d} \int_{x_d}^{x_u} f(x) e^{-\alpha x} \sin\left(j\omega' \frac{x - x_d}{x_u - x_d}\right) dx.$$

This technique, which can be generalized, is the **Fourier series method**. There are some problems with the method if you are trying to represent a discontinuous function with a sum of trigonometrical functions. The oscillatory nature of an approximate solution with a finite number of terms is known as **Gibbs phenomenon**.

9.6 SIMILARITY REDUCTIONS

Apart from the Green's function, we're not going to use any of the above techniques in this book; rarely will we even find explicit solutions. But one technique that we will find useful is the **similarity reduction**. I will demonstrate the idea using the simple diffusion equation, we will later use it in many other, more complicated problems.

[1] This is an example of a double knock-out option, see Chapter 13.

The basic diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{9.4}$$

is an equation for the function u which depends on the two variables x and t. Sometimes, in very, very special cases we can write the solution as a function of just one variable. Let me give an example. Verify that the function

$$u(x,t) = \int_0^{x/t^{1/2}} e^{-\xi^2/4}\, d\xi$$

satisfies (9.4). But in this function x and t only appear in the combination

$$\frac{x}{t^{1/2}}.$$

Thus, in a sense, u is a function of only one variable.

A slight generalization, but also demonstrating the idea of similarity solutions, is to look for a solution of the form

$$u = t^{-1/2} f(\xi) \tag{9.5}$$

where

$$\xi = \frac{x}{t^{1/2}}.$$

Substitute (9.5) into (9.4) to find that a solution for f is

$$f = e^{-\xi^2/4},$$

so that

$$t^{-1/2} e^{-x^2/4t}$$

is also a special solution of the diffusion equation.

Be warned, though. You can't always find similarity solutions, not only must the equation have a particularly nice structure but also the similarity form must be consistent with any initial condition or boundary conditions.

9.7 OTHER ANALYTICAL TECHNIQUES

The other two main solution techniques for linear partial differential equations are Fourier and Laplace transforms. These are such large and highly technical subjects that I really cannot begin to give an idea of how they work, space is far too short. But be reassured that it is probably not worth your while learning the techniques, in finance they can be used to solve only a very small number of problems. If you want to learn something useful then move on to the next section.

9.8 NUMERICAL SOLUTION

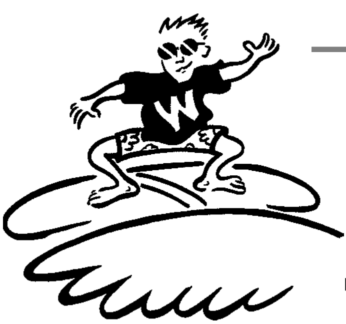
Even though there are several techniques that we can use for finding solutions, in the vast majority of cases we must solve the Black–Scholes equation numerically. But we are lucky. Parabolic differential equations are just about the easiest equations to solve numerically. Obviously, there are any number of really sophisticated techniques, but if you stick with the simplest then you can't go far wrong. I want to stress that I am going to derive many partial differential equations from now on, and I am going to assume you trust me that we will at the end of the book see how to solve them.

9.9 SUMMARY

This short chapter is only intended as a primer on partial differential equations. If you want to study this subject in depth, see the books and articles mentioned below.

FURTHER READING

- Grindrod (1991) is all about reaction-diffusion equations, where they come from and their analysis. The book includes many of the physical models described above.
- Murray (1989) also contains a great deal on reaction-diffusion equations, but concentrating on models of biological systems.
- Wilmott & Wilmott (1990) describe the diffusion of pollutant along a river with convection and absorption by the river bed.
- The classical reference works for diffusion equations are Crank (1989) and Carslaw & Jaeger (1989). But also see the book on partial differential equations by Sneddon (1957) and the book on general applied mathematical methods by Strang (1986).



Time Out...

The main solution methods

We have seen, and will be seeing more of, the three main mathematical approaches to derivative pricing: differential equations; binomial trees; expectations. All of these methods are based on pretty much the same assumptions. All of them will therefore give the same values for a contract, if all parameter values are the same. This is, of course, subject to the accuracy of numerical methods.

Speaking of numerical methods, each of the three approaches has its own associated numerical method. Differential equations, and the Black–Scholes equation, in particular, can be solved by finite-difference methods. The whole of Chapter 25 is devoted to this subject. The binomial tree model is, interestingly, also its own numerical method, and we've seen that in some detail already in Chapter 5. Finally, pricing by calculating risk-neutral expectations is one of the subjects in Chapter 26, on Monte Carlo simulations.

CHAPTER 10

the Black–Scholes formulas and the 'greeks'

The aim of this Chapter...

... is to show how the basic Black–Scholes formulas are derived from the Black–Scholes equation, and to introduce more sophisticated hedging strategies. The chapter contains lots of useful formulas which are also summarized at the end.

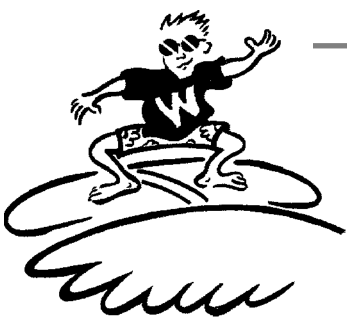
In this Chapter...

- the derivation of the Black–Scholes formulas for calls, puts and simple digitals
- the meaning and importance of the 'greeks,' delta, gamma, theta, vega and rho
- the difference between differentiation with respect to variables and to parameters
- formulas for the greeks for calls, puts and simple digitals

10.1 INTRODUCTION

The Black–Scholes equation has simple solutions for calls, puts and some other contracts. In this chapter I'm going to walk you through the derivation of these formulas step by step. This is one of the few places in the book where I do derive formulas. The reason that I don't often derive formulas is that the majority of contracts do not have explicit solutions for their theoretical value. Instead much of my emphasis will be placed on finding numerical solutions of the Black–Scholes equation.

We've seen how the quantity 'delta,' the first derivative of the option value with respect to the underlying, occurs as an important quantity in the derivation of the Black–Scholes equation. In this chapter I describe the importance of other derivatives of the option price, with respect to the variables (the underlying asset and time) and with respect to some of the parameters. These derivatives are important in the hedging of an option position, playing key roles in risk management. It can be argued that it is more important to get the hedging correct than to be precise in the pricing of a contract. The reason for this is that if you are accurate in your hedging you will have reduced or eliminated future uncertainty. This leaves you with a profit (or loss) that is set the moment that you buy or sell the contract. But if your hedging is inaccurate, then it doesn't matter, within reason, what you sold the contract for initially, future uncertainty could easily dominate any initial profit. Of course, life is not so simple, in reality we are exposed to model error, which can make a mockery of anything we do. However, this illustrates the importance of good hedging, and that's where the 'greeks' come in.



Time Out...

Close your eyes until I tell you to open them

Unless you are doing a highly mathsy course, you won't need to know all the manipulations that follow.

10.2 DERIVATION OF THE FORMULAS FOR CALLS, PUTS AND SIMPLE DIGITALS

The Black–Scholes equation is

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{10.1}$$

This equation must be solved with final condition depending on the payoff: each contract will have a different functional form prescribed at expiry $t = T$, depending on whether it is a call, a put or something more fancy. This is the final condition that must be imposed to make the solution unique. We'll worry about final conditions later, for the moment concentrate on manipulating (10.1) into something we can easily solve.

The first step in the manipulation is to change from present value to future value terms. Recalling that the payoff is received at time T but that we are valuing the option at time t this suggests that we write

$$V(S,t) = e^{-r(T-t)}U(S,t).$$

This takes our differential equation to

$$\frac{\partial U}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S} = 0.$$

The second step is really trivial. Because we are solving a backward equation, discussed in Chapter 9, we'll write

$$\tau = T - t.$$

This now takes our equation to

$$\frac{\partial U}{\partial \tau} = \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S}.$$

When we first started modeling equity prices we used intuition about the asset price *return* to build up the stochastic differential equation model. Let's go back to examine the return and write

$$\xi = \log S.$$

With this as the new variable, we find that

$$\frac{\partial}{\partial S} = e^{-\xi}\frac{\partial}{\partial \xi} \quad \text{and} \quad \frac{\partial^2}{\partial S^2} = e^{-2\xi}\frac{\partial^2}{\partial \xi^2} - e^{-2\xi}\frac{\partial}{\partial \xi}.$$

Now the Black–Scholes equation becomes

$$\frac{\partial U}{\partial \tau} = \tfrac{1}{2}\sigma^2 \frac{\partial^2 U}{\partial \xi^2} + \left(r - \tfrac{1}{2}\sigma^2\right)\frac{\partial U}{\partial \xi}.$$

What has this done for us? It has taken the problem defined for $0 \le S < \infty$ to one defined for $-\infty < \xi < \infty$. But more importantly, the coefficients in the equation are now all constant, independent of the underlying. This is a big step forward, made possible by the lognormality of the underlying asset. We are nearly there.

The last step is simple, but the motivation is not so obvious. Write

$$x = \xi + \left(r - \tfrac{1}{2}\sigma^2\right)\tau,$$

and $U = W(x, \tau)$. This is just a 'translation' of the coordinate system. It's a bit like using the forward price of the asset instead of the spot price as a variable. After this change of variables the Black–Scholes becomes the simpler

$$\frac{\partial W}{\partial \tau} = \tfrac{1}{2}\sigma^2 \frac{\partial^2 W}{\partial x^2}. \tag{10.2}$$

To summarize,

$$\begin{aligned} V(S,t) &= e^{-r(T-t)}U(S,t) = e^{-r\tau}U(S,T-\tau) = e^{-r\tau}U(e^{\xi},T-\tau) \\ &= e^{-r\tau}U\left(e^{x-(r-\frac{1}{2}\sigma^2)\tau}, T-\tau\right) = e^{-r\tau}W(x,\tau). \end{aligned}$$

To those of you who already know the Black–Scholes formulas for calls and puts the variable x will ring a bell:

$$x = \xi + \left(r - \tfrac{1}{2}\sigma^2\right)\tau = \log S + \left(r - \tfrac{1}{2}\sigma^2\right)(T-t).$$

Having turned the original Black–Scholes equation into something much simpler, let's take a break for a moment while I explain where we are headed.

I'm going to derive an expression for the value of any option whose payoff is a known function of the asset price at expiry. This includes calls, puts and digitals. This expression will be in the form of an integral. For special cases, I'll show how to rewrite this integral in terms of the cumulative distribution function for the Normal distribution. This is particularly useful since the function can be found on spreadsheets, calculators and in the backs of books. But there are two steps before I can write down this integral.

The first step is to find a special solution of (10.2), called the fundamental solution. This solution has useful properties. The second step is to use the linearity of the equation and the useful properties of the special solution to find the *general solution* of the equation. Here we go.

I'm going to look for a special solution of (10.2) of the following form:

$$W(x,\tau) = \tau^{\alpha} f\left(\frac{(x-x')}{\tau^{\beta}}\right), \tag{10.4}$$

where x' is an arbitrary constant. And I'll call this special solution $W_f(x,\tau;x')$. Note that the unknown function depends on only *one* variable $(x-x')/\tau^{\beta}$. As well as finding the function f we must find the constant parameters α and β. We can expect that if this approach works, the equation for f will be an ordinary differential equation since the function only has one variable. This reduction of dimension is an example of a similarity reduction, discussed in Chapter 9.

Substituting expression (10.3) into (10.2) we get

$$\tau^{\alpha-1}\left(\alpha f - \beta\eta\frac{df}{d\eta}\right) = \tfrac{1}{2}\sigma^2\tau^{\alpha-2\beta}\frac{d^2f}{d\eta^2}, \tag{10.4}$$

where

$$\eta = \frac{x-x'}{\tau^{\beta}}.$$

Examining the dependence of the two terms in (10.4) on both τ and η we see that we can only have a solution if

$$\alpha - 1 = \alpha - 2\beta \quad \text{i.e.} \quad \beta = \tfrac{1}{2}.$$

I want to ensure that my 'special solution' has the property that its integral over all ξ is independent of τ, for reasons that will become apparent. To ensure this, I require

$$\int_{-\infty}^{\infty} \tau^{\alpha} f((x-x')/\tau^{\beta})\, dx$$

to be constant. I can write this as

$$\int_{-\infty}^{\infty} \tau^{\alpha+\beta} f(\eta)\, d\eta$$

and so I need

$$\alpha = -\beta = -\tfrac{1}{2}.$$

The function f now satisfies

$$-f - \eta \frac{df}{d\eta} = \sigma^2 \frac{d^2 f}{d\eta^2}.$$

This can be written

$$\sigma^2 \frac{d^2 f}{d\eta^2} + \frac{d(\eta f)}{d\eta} = 0,$$

which can be integrated once to give

$$\sigma^2 \frac{df}{d\eta} + \eta f = a,$$

where a is a constant. For my special solution I'm going to choose $a = 0$. This equation can be integrated again to give

$$f(\eta) = b \exp\left(-\frac{\eta^2}{2\sigma^2}\right).$$

I will choose the constant b such that the integral of f from minus infinity to plus infinity is one:

$$f(\eta) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{\eta^2}{2\sigma^2}\right).$$

This is the special solution I have been seeking:[1]

$$W(x, \tau) = \frac{1}{\sqrt{2\pi\tau}\,\sigma} \exp\left(-\frac{(x - x')^2}{2\sigma^2\tau}\right).$$

Now I will explain why it is useful in our quest for the Black–Scholes formulas.

In Figure 10.1 is plotted W as a function of x' for several values of τ. Observe how the function rises in the middle but decays at the sides. As $\tau \to 0$ this becomes more pronounced. The 'middle' is the point $x' = x$. At this point the function grows unboundedly and away from this point the function decays to zero as $\tau \to 0$. Although the function is increasingly confined to a narrower and narrower region its area remains fixed at one. These properties of decay away from one point, unbounded growth at that point and constant area, result in a **Dirac delta function** $\delta(x' - x)$ as $\tau \to 0$. The delta function has one important property, namely

$$\int \delta(x' - x) g(x')\, dx' = g(x)$$

where the integration is from any point below x to any point above x. Thus the delta function 'picks out' the value of g at the point where the delta function is singular i.e. at $x' = x$. In

[1] It is just the probability density function for a Normal random variable with mean zero and standard deviation σ.

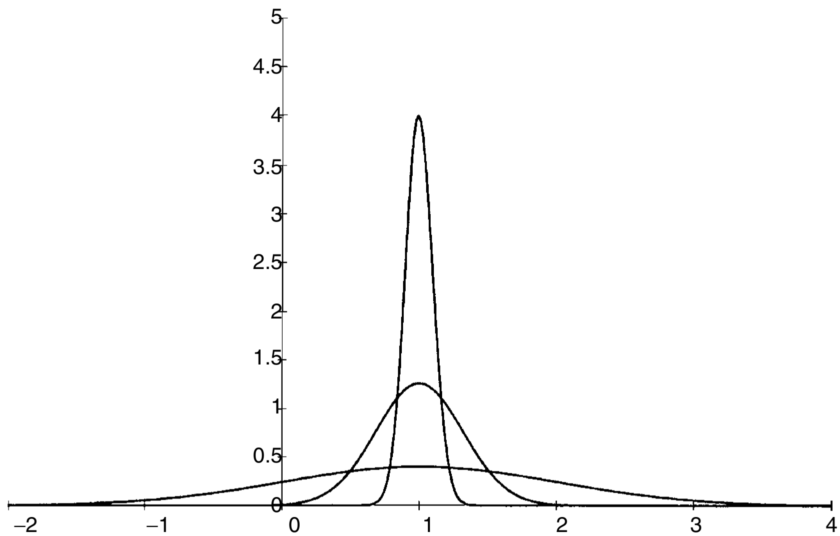


Figure 10.1 The fundamental solution.

the limit as $\tau \to 0$ the function W becomes a delta function at $x = x'$. This means that

$$\lim_{\tau \to 0} \frac{1}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x'-x)^2}{2\sigma^2\tau}\right) g(x')\,dx' = g(x).$$

This property of the special solution, together with the linearity of the Black–Scholes equation are all that are needed to find some explicit solutions.

Now is the time to consider the payoff. Let's call it

$$\text{Payoff}(S).$$

This is the value of the option at time $t = T$. It is the final condition for the function V, satisfying the Black–Scholes equation:

$$V(S, T) = \text{Payoff}(S).$$

In our new variables, this final condition is

$$W(x, 0) = \text{Payoff}(e^x). \tag{10.5}$$

I claim that the solution of this for $\tau > 0$ is

$$W(x, \tau) = \int_{-\infty}^{\infty} W_f(x, \tau; x')\text{Payoff}(e^{x'})\,dx'. \tag{10.6}$$

To show this, I just have to demonstrate that the expression satisfies the Equation (10.2) and the final condition (10.5). Both of these are straightforward. The integration with respect to x' is similar to a summation, and since each individual component satisfies the equation so does the sum/integral. Alternatively, differentiate (10.6) under the integral sign to see that it satisfies the partial differential equation. That it satisfies the condition (10.5) follows from the special properties of the fundamental solution W_f.

Retracing our steps to write our solution in terms of the original variables, we get

$$V(S,t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_0^{\infty} \exp\left(-\frac{\left(\log(S/S') + \left(r - \frac{1}{2}\sigma^2\right)(T-t)\right)^2}{2\sigma^2(T-t)} \right) \text{Payoff}(S') \frac{dS'}{S'}, \tag{10.7}$$

where I have written $x' = \log S'$.

This is the exact solution for the option value in terms of the arbitrary payoff function. In the next sections I will manipulate this expression for special payoff functions.

10.2.1 Formula for a call

The call option has the payoff function

$$\text{Payoff}(S) = \max(S - E, 0).$$

Expression (10.7) can then be written as

$$\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_E^{\infty} \exp\left(-\frac{\left(\log(S/S') + \left(r - \frac{1}{2}\sigma^2\right)(T-t)\right)^2}{2\sigma^2(T-t)} \right) (S' - E) \frac{dS'}{S'}.$$

Return to the variable $x' = \log S'$, to write this as

$$\begin{aligned}
&\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{\log E}^{\infty} \exp\left(-\frac{(-x' + \log S - (r - \frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)} \right) (e^{x'} - E)\, dx' \\
&= \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{\log E}^{\infty} \exp\left(-\frac{(-x' + \log S - (r - \frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)} \right) e^{x'}\, dx' \\
&\quad - E \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{\log E}^{\infty} \exp\left(-\frac{(-x' + \log S - (r - \frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)} \right) dx'.
\end{aligned}$$

Both integrals in this expression can be written in the form

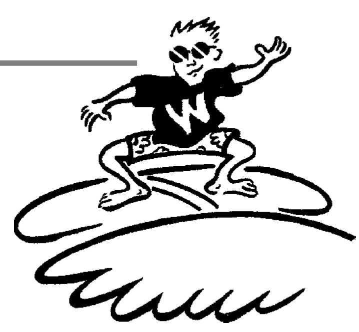
$$\int_d^{\infty} \exp\left(-\tfrac{1}{2}x'^2\right) dx'$$

for some d (the second is just about in this form already, and the first just needs a completion of the square).

Time Out...

You can open your eyes now

From now on I just quote formulas without giving derivations.



Apart from a couple of minor differences, this integral is just like the cumulative distribution function for the standardized Normal distribution[2] defined by

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\tfrac{1}{2}\phi^2\right) \, d\phi.$$

This function, plotted in Figure 10.2, is the probability that a Normally distributed variable is less than x.

Thus the option price can be written as two separate terms involving the cumulative distribution function for a Normal distribution:

$$\text{Call option value} = SN(d_1) - Ee^{-r(T-t)}N(d_2)$$

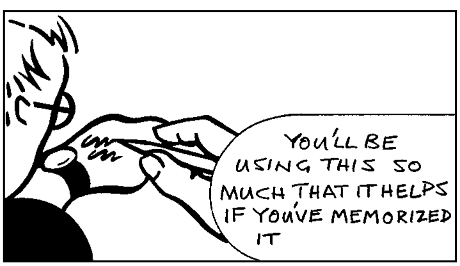
where

$$d_1 = \frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = \frac{\log(S/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

When there is continuous dividend yield on the underlying, or it is a currency, then



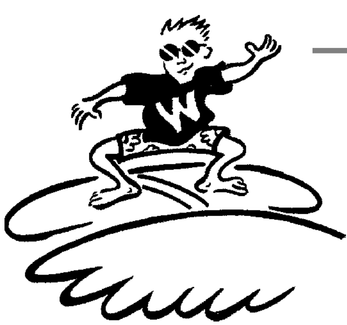
Call option value

$$Se^{-D(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2)$$

$$d_1 = \frac{\log(S/E) + (r - D + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\log(S/E) + (r - D - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$= d_1 - \sigma\sqrt{T-t}$$



Time Out...

In Excel

In Excel the cumulative distribution function for the standardized Normal distribution is NORMSDIST( ). The natural or Naperian logarithm is LN( ).

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[2] I.e. having zero mean and unit standard deviation.

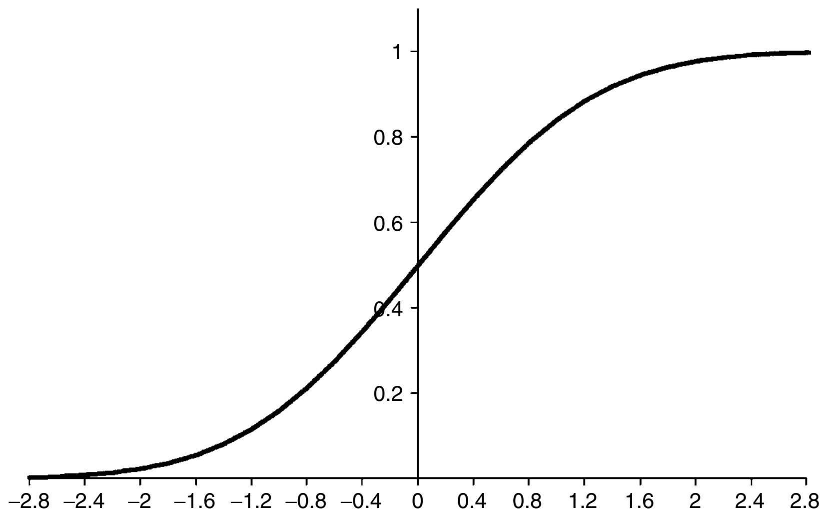


Figure 10.2 The cumulative distribution function for a standardized Normal random variable, $N(x)$.

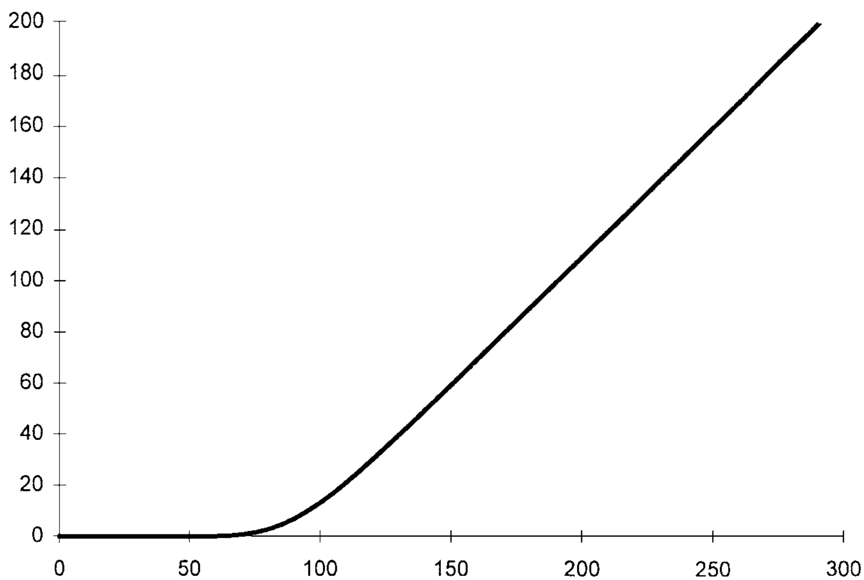


Figure 10.3 The value of a call option as a function of the underlying asset price at a fixed time to expiry.

The option value is shown in Figure 10.3 as a function of the underlying asset at a fixed time to expiry. In Figure 10.4 the value of the at-the-money option is shown as a function of time, expiry is $t = 1$. In Figure 10.5 is the call value as a function of both the underlying and time.

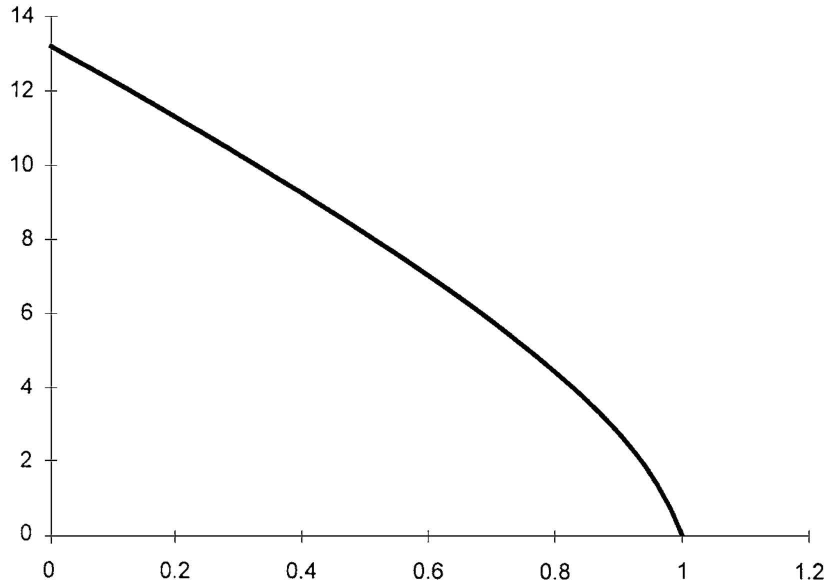


Figure 10.4 The value of an at-the-money call option as a function of time.

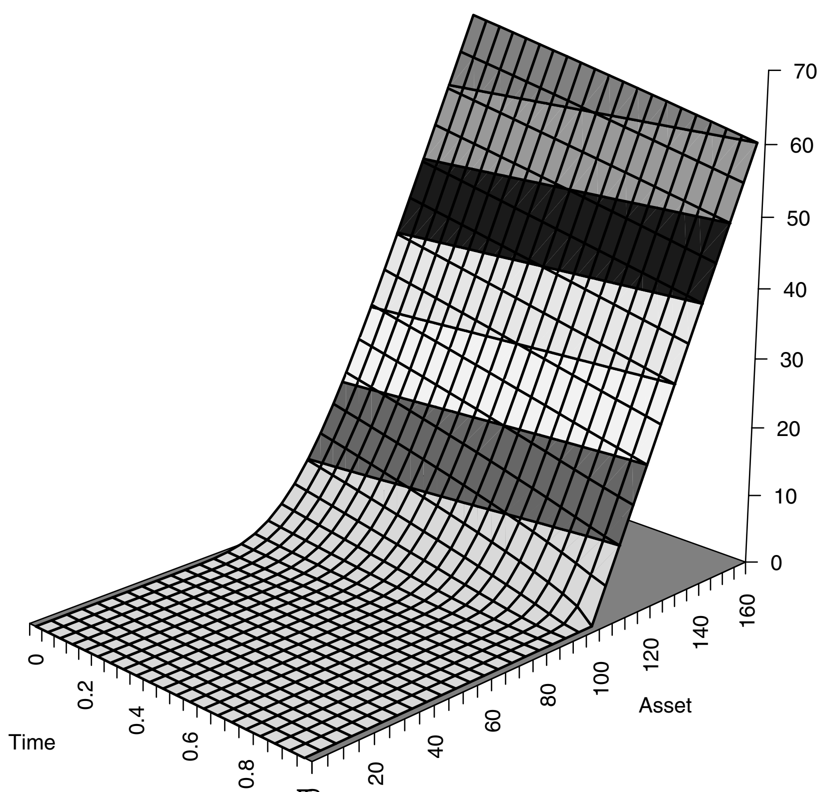


Figure 10.5 The value of a call option as a function of asset and time.

When the asset is 'at-the-money forward,' i.e. $S = Ee^{-(r-D)(T-t)}$, then there is a simple approximation for the call value (Brenner & Subrahmanyam, 1994):

$$\text{Call} \approx 0.4\, Se^{-D(T-t)}\sigma\sqrt{T-t}.$$

10.2.2 Formula for a put

The put option has payoff

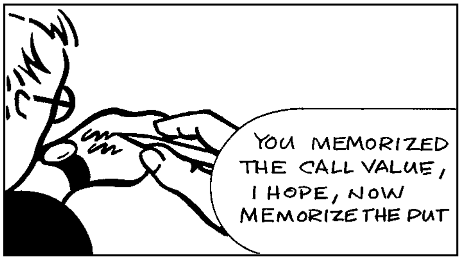
$$\text{Payoff}(S) = \max(E - S, 0).$$

The value of a put option can be found in the same way as above, or using put-call parity

$$\text{Put option value} = -SN(-d_1) + Ee^{-r(T-t)}N(-d_2),$$

with the same d_1 and d_2.

When there is continuous dividend yield on the underlying, or it is a currency, then



Put option value

$$-Se^{-D(T-t)}N(-d_1) + Ee^{-r(T-t)}N(-d_2)$$

The option value is shown in Figure 10.6 against the underlying asset and in Figure 10.7 against time. In Figure 10.8 is the option value as a function of both the underlying asset and time.

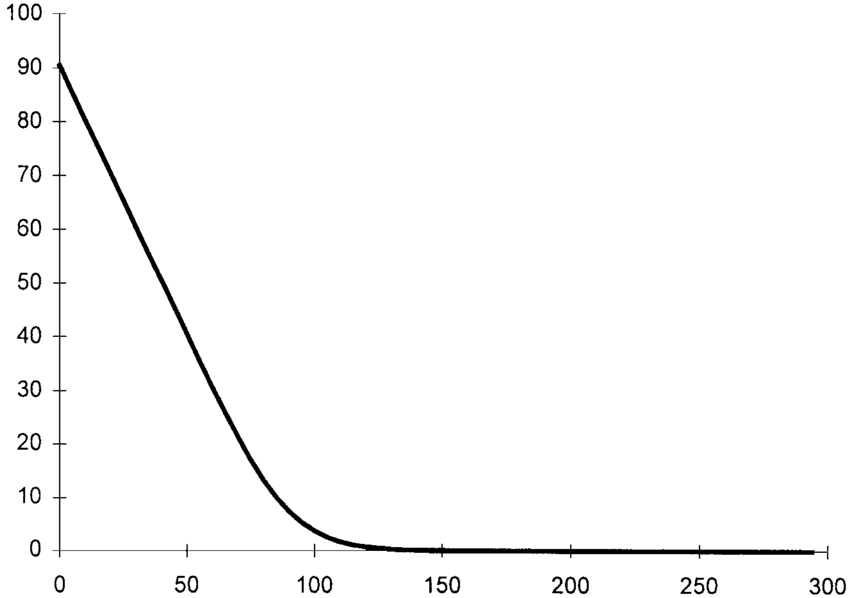


Figure 10.6 The value of a put option as a function of the underlying asset at a fixed time to expiry.

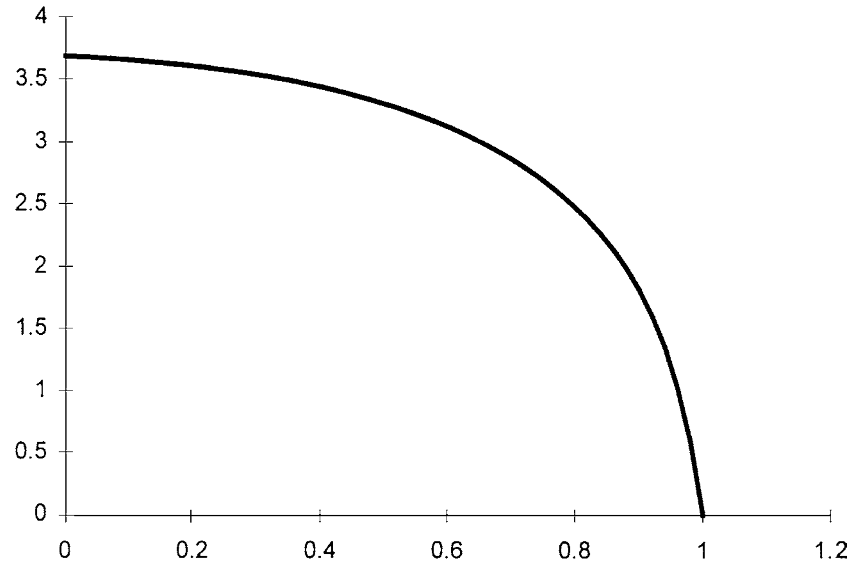


Figure 10.7 The value of an at-the money put option as a function of time.

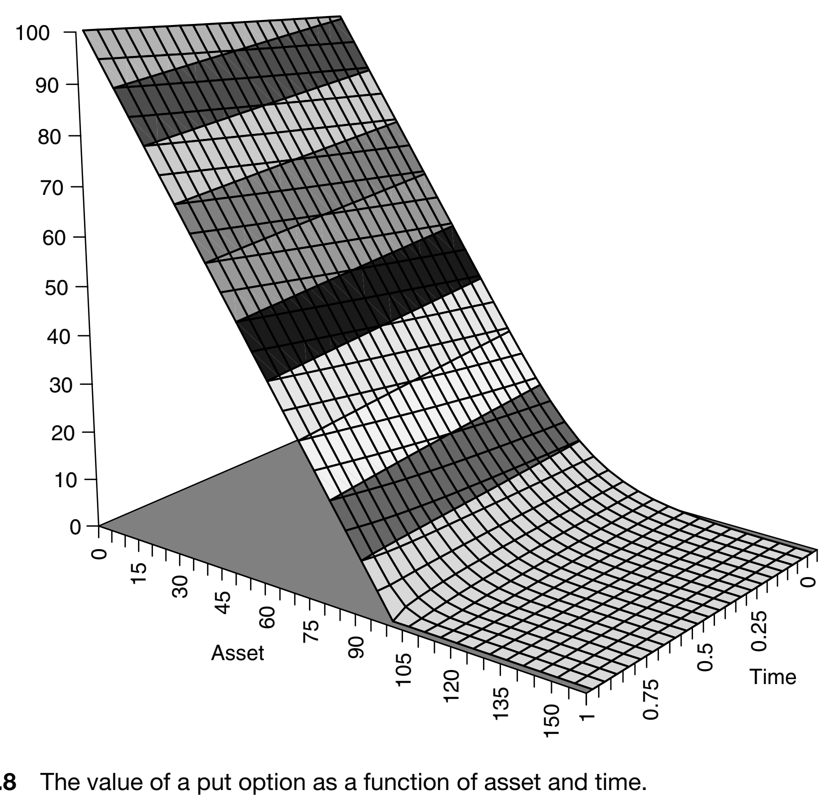


Figure 10.8 The value of a put option as a function of asset and time.

When the asset is at-the-money forward the simple approximation for the put value (Brenner & Subrahmanyam, 1994) is

$$\text{Put} \approx 0.4\, Se^{-D(T-t)}\sigma\sqrt{T-t}.$$

10.2.3 Formula for a binary call

The binary call has payoff

$$\text{Payoff}(S) = \mathcal{H}(S-E),$$

where $\mathcal{H}$ is the Heaviside function taking the value one when its argument is positive and zero otherwise.

Incorporating a dividend yield, we can write the option value as

$$\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_{\log E}^{\infty}\exp\left(-\frac{(-x'-\log S-(r-D-\frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)}\right)dx'.$$

This term is just like the second term in the call option equation and so

Binary call option value

$$e^{-r(T-t)}N(d_2)$$

The option value is shown in Figure 10.9.

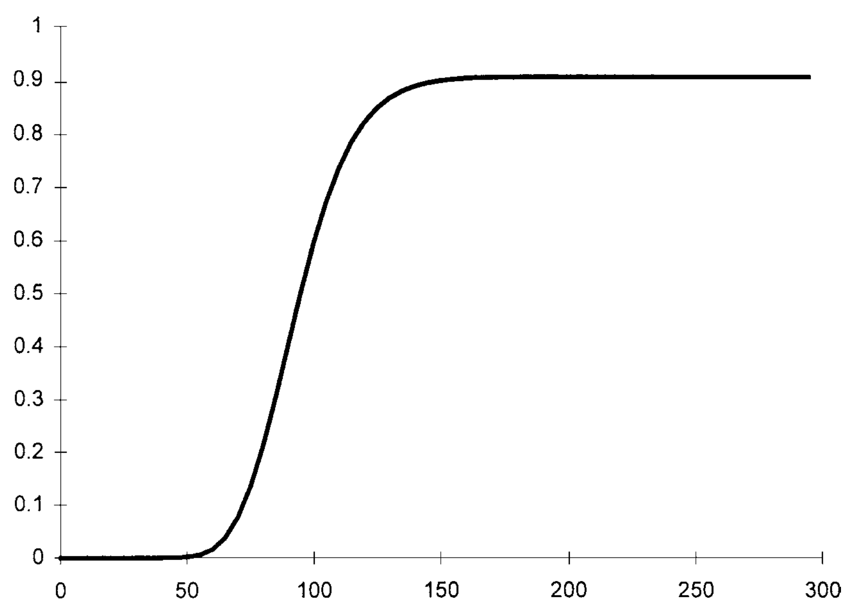


Figure 10.9 The value of a binary call option.

10.2.4 Formula for a binary put

The binary put has a payoff of one if $S < E$ at expiry. It has a value of

Binary put option value

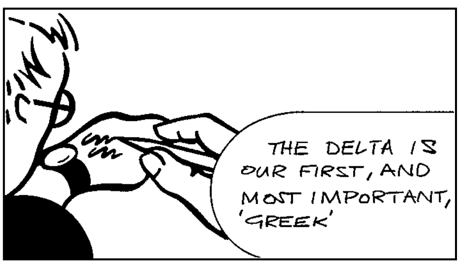
$$e^{-r(T-t)}(1 - N(d_2))$$

since a binary call and a binary put must add up to the present value of \$1 received at time T. The option value is shown in Figure 10.10.

10.3 DELTA

The **delta** of an option or a portfolio of options is the sensitivity of the option or portfolio to the underlying. It is the rate of change of value with respect to the asset:

$$\Delta = \frac{\partial V}{\partial S}$$



Here V can be the value of a single contract or of a whole portfolio of contracts. The delta of a portfolio of options is just the sum of the deltas of all the individual positions.

The theoretical device of delta hedging, introduced in Chapter 8, for eliminating risk is far more than that, it is a very important practical technique.

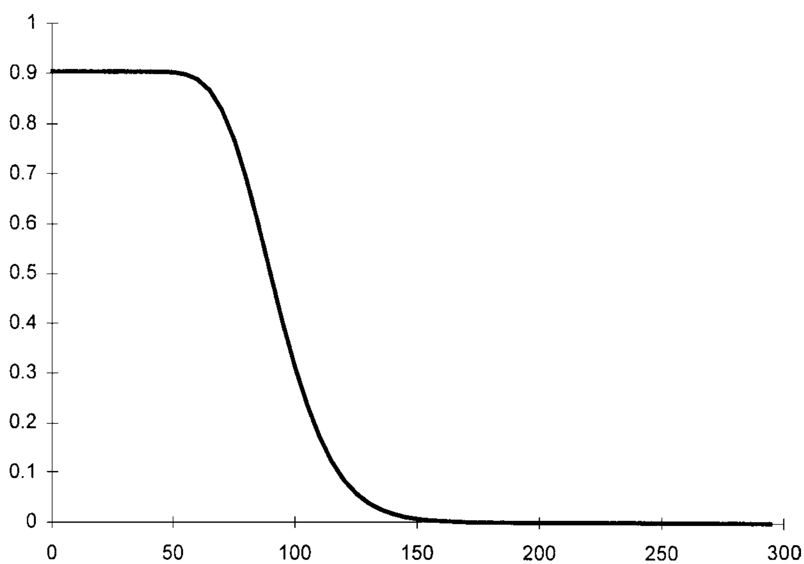


Figure 10.10 The value of a binary put option.

Roughly speaking, the financial world is divided up into speculators and hedgers. The speculators take a view on the direction of some quantity such as the asset price (or more abstract quantities such as volatility) and implement a strategy to take advantage of their view. Such people may not hedge at all.

Then there are the hedgers. There are two kinds of hedger: the ones who hold a position already and want to eliminate some very specific risk (usually using options) and the ones selling (or buying) the options because they believe they have a better price and can make money by hedging away *all* risk. It is the latter type of hedger that is delta hedging. They can only guarantee to make a profit by selling a contract for a high value if they can eliminate all of the risk due to the random fluctuation in the underlying.

Delta hedging means holding one of the option and short a quantity Δ of the underlying. Delta can be expressed as a function of S and t, I'll give some formulas later in this section. This function varies as S and t vary. This means that the number of assets held must be continuously changed to maintain a **delta neutral** position, this procedure is called **dynamic hedging**. Changing the number of assets held requires the continual purchase and/or sale of the stock. This is called **rehedging** or **rebalancing** the portfolio.

This delta hedging may take place very frequently in highly liquid markets where it is relatively costless to buy and sell. Thus the Black–Scholes assumption of continuous hedging may be quite accurate. In less liquid markets, you lose a lot on bid-offer spread and will therefore hedge less frequently. Moreover, you may not even be able to buy or sell in the quantities you want. Even in the absence of costs, you cannot be sure that your model for the underlying is accurate. There will certainly be some risk associated with the model. These issues make delta hedging less than perfect and in practice the risk in the underlying cannot be hedged away perfectly.

Some contracts have a delta that becomes very large at special times or asset values. The size of the delta makes delta hedging impossible; what can you do if you find yourself with a theoretical delta requiring you to buy more stock than exists? In such a situation the basic foundation of the Black–Scholes world has collapsed and you would be right to question the validity of any pricing formula. This happens at expiry close to the strike for binary options. Although I've given a formula for their price above and a formula for their delta below, I'd be careful using them if I were you.

Here are some formulas for the deltas of common contracts (all formulas assume that the underlying pays dividends or is a currency):

Deltas of common contracts

Call	$e^{-D(T-t)}N(d_1)$
Put	$e^{-D(T-t)}(N(d_1)-1)$
Binary call	$\dfrac{e^{-r(T-t)}N'(d_2)}{\sigma S\sqrt{T-t}}$
Binary put	$-\dfrac{e^{-r(T-t)}N'(d_2)}{\sigma S\sqrt{T-t}}$

$$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-(1/2)x^2}$$

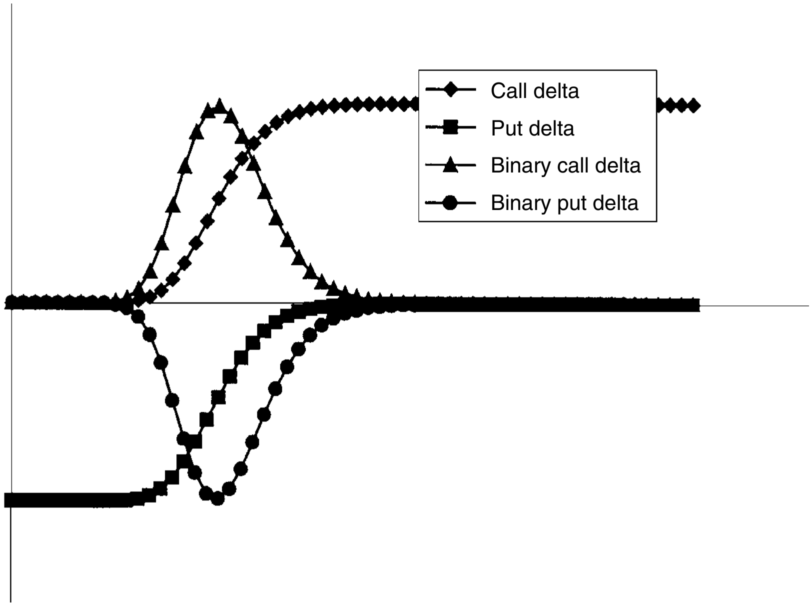
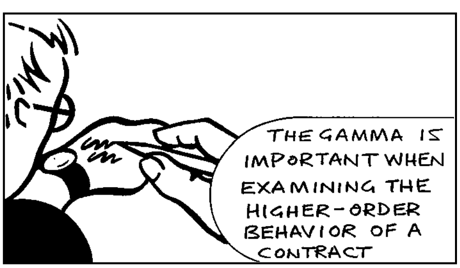


Figure 10.11 The deltas of a call, a put, a binary call and a binary put option. (Binary values scaled to a maximum value of one.)

Examples of these functions are plotted in Figure 10.11, with some scaling of the binaries.

10.4 GAMMA

The **gamma**, Γ, of an option or a portfolio of options is the second derivative of the position with respect to the underlying:



$$\boxed{\Gamma = \frac{\partial^2 V}{\partial S^2}}$$

Since gamma is the sensitivity of the delta to the underlying it is a measure of by how much or how often a position must be rehedged in order to maintain a delta-neutral position. Although the delta also varies with time this effect is dominated by the Brownian nature of the movement in the underlying.

In a delta-neutral position the gamma is partly responsible for making the return on the portfolio equal to the risk-free rate, the no-arbitrage condition of Chapter 8. The rest of this task falls to the time derivative of the option value, discussed below. Actually, the situation is far more complicated than this because of the necessary discreteness in the hedging, there is a finite time between rehedges. In any delta-hedged position you make money on some hedges and lose some on others. In a long gamma position ($\Gamma > 0$) you make money on the large moves in the underlying and lose it on the small moves. The net effect is to get the risk-free rate of return on the portfolio.

Gamma also plays an important role when there is a mismatch between the market's view of volatility and the actual volatility of the underlying.

Because costs can be large and because one wants to reduce exposure to model error it is natural to try to minimize the need to rebalance the portfolio too frequently. Since gamma is a measure of sensitivity of the hedge ratio Δ to the movement in the underlying, the hedging requirement can be decreased by a gamma-neutral strategy. This means buying or selling more *options*, not just the underlying. Because the gamma of the underlying (its second derivative) is zero, we cannot add gamma to our position just with the underlying. We can have as many options in our position as we want, we choose the quantities of each such that both delta and gamma are zero. The minimal requirement is to hold two different types of option and the underlying. In practice, the option position is not readjusted too often because, if the cost of transacting in the underlying is large, then the cost of transacting in its derivatives is even larger.

Here are some formulas for the gammas of common contracts:

Gammas of common contracts

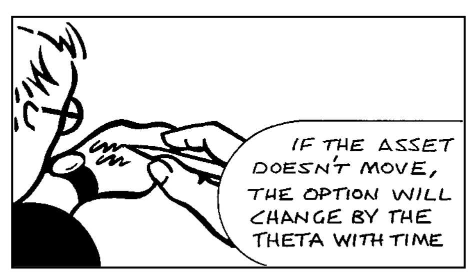
Call	$\dfrac{e^{-D(T-t)}N'(d_1)}{\sigma S\sqrt{T-t}}$
Put	$\dfrac{e^{-D(T-t)}N'(d_1)}{\sigma S\sqrt{T-t}}$
Binary call	$-\dfrac{e^{-r(T-t)}d_1N'(d_2)}{\sigma^2S^2(T-t)}$
Binary put	$\dfrac{e^{-r(T-t)}d_1N'(d_2)}{\sigma^2S^2(T-t)}$

Examples of these functions are plotted in Figure 10.12, with some scaling for the binaries.

10.5 THETA

Theta, Θ, is the rate of change of the option price with time.

$$\Theta = \frac{\partial V}{\partial t}$$



The theta is related to the option value, the delta and the gamma by the Black–Scholes equation. In a delta-hedged portfolio the theta contributes to ensuring that the portfolio earns the risk-free rate. But it contributes in a completely certain way, unlike the gamma which contributes the right amount *on average*.

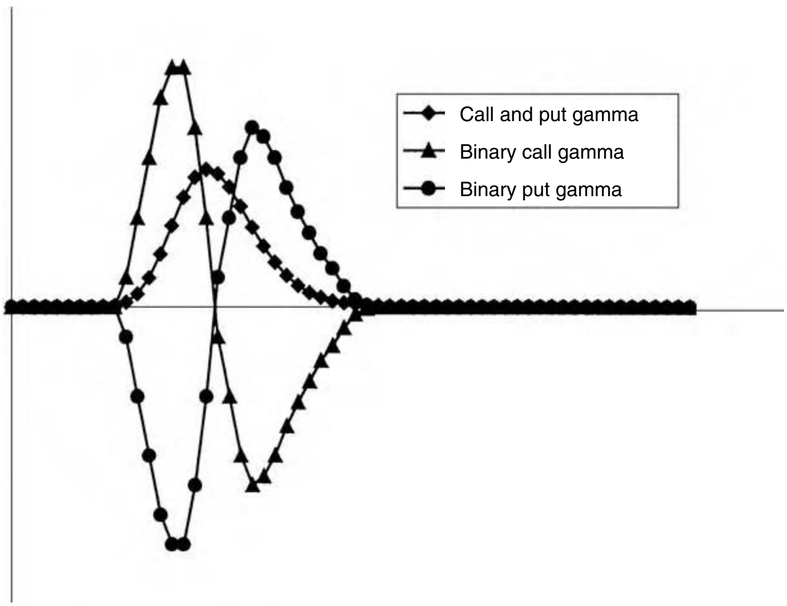


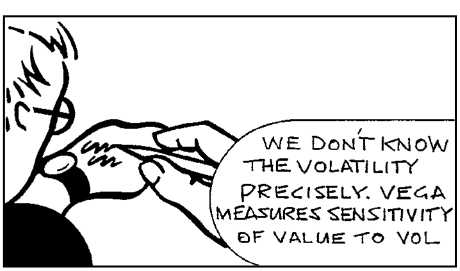
Figure 10.12 The gammas of a call, a put, a binary call and a binary put option.

Here are some formulas for the thetas of common contracts:

Thetas of common contracts

Call	$-\dfrac{\sigma S e^{-D(T-t)} N'(d_1)}{2\sqrt{T-t}} + DSN(d_1)e^{-D(T-t)} - rEe^{-r(T-t)}N(d_2)$
Put	$-\dfrac{\sigma S e^{-D(T-t)} N'(-d_1)}{2\sqrt{T-t}} - DSN(-d_1)e^{-D(T-t)} + rEe^{-r(T-t)}N(-d_2)$
Binary call	$re^{-r(T-t)}N(d_2) + e^{-r(T-t)}N'(d_2)\left(\dfrac{d_1}{2(T-t)} - \dfrac{r-D}{\sigma\sqrt{T-t}}\right)$
Binary put	$re^{-r(T-t)}(1-N(d_2)) - e^{-r(T-t)}N'(d_2)\left(\dfrac{d_1}{2(T-t)} - \dfrac{r-D}{\sigma\sqrt{T-t}}\right)$

These functions are plotted in Figure 10.13.



10.6 VEGA

Vega, a.k.a. zeta and kappa, is a very important but confusing quantity. It is the sensitivity of the option price to volatility.

$$\text{Vega} = \frac{\partial V}{\partial \sigma}$$

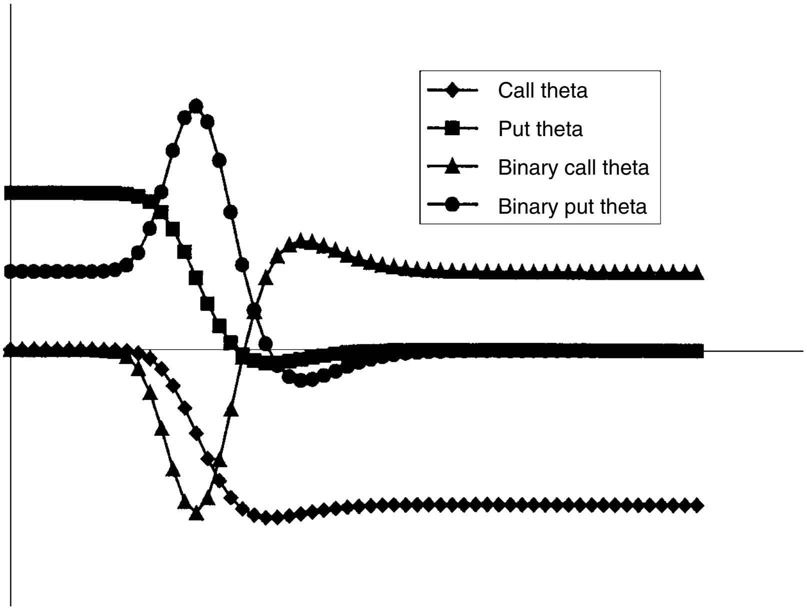
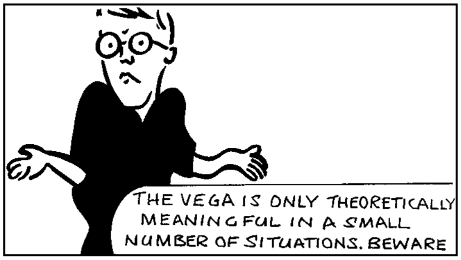


Figure 10.13 The thetas of a call, a put, a binary call and a binary put option.

This is completely different from the other greeks[3] since it is a derivative with respect to a parameter and not a variable. This makes something of a difference when we come to finding numerical solutions for such quantities.

In practice, the volatility of the underlying is not known with certainty. Not only is it very difficult to measure at any time, it is even harder to predict what it will do in the future. Suppose that we put a volatility of 20% into an option pricing formula, how sensitive is the price to that number? That's the vega.

As with gamma hedging, one can vega hedge to reduce sensitivity to the volatility. This is a major step towards eliminating some model risk, since it reduces dependence on a quantity that, to be honest, is not known very accurately.



There is a downside to the measurement of vega. It is only really meaningful for options having single-signed gamma everywhere. For example it makes sense to measure vega for calls and puts but not binary calls and binary puts. I have included the formulas for the vega of such contracts below, but they should be used with care, if at all. The reason for this is that call and put values (and options with single-signed gamma) have values that are monotonic

---

[3] It's not even Greek. Among other things it is an American car, a star (Alpha Lyræ), the real name of Zorro, there are a couple of sixteenth century Spanish authors called Vega, an op art painting by Vasarely and a character in the computer game 'Street Fighter.' And who could forget Vincent, and his brother, and his 'cousin'?

The second derivative with respect to σ has been called 'vomma' and the second-order derivative with respect to the asset and the volatility has been called 'kabanga.' I doubt that they represent what their fans think they represent, and I'm going to make no further mention of them.

in the volatility: increase the volatility in a call and its value increases everywhere. Contracts with a gamma that changes sign may have a vega measured at zero because as we increase the volatility the price may rise somewhere and fall somewhere else. Such a contract is very exposed to volatility risk but that risk is not measured by the vega.

Here are formulas for the vegas of common contracts:

Vegas of common contracts

Call	$S\sqrt{T-t}e^{-D(T-t)}N'(d_1)$
Put	$S\sqrt{T-t}e^{-D(T-t)}N'(d_1)$
Binary call	$-e^{-r(T-t)}N'(d_2)\left(\sqrt{T-t}+\dfrac{d_2}{\sigma}\right)$
Binary put	$e^{-r(T-t)}N'(d_2)\left(\sqrt{T-t}+\dfrac{d_2}{\sigma}\right)$

In Figure 10.14 is shown the value of an at-the-money call option as a function of the volatility. There is one year to expiry, the strike is 100, the interest rate is 10% and there are no dividends. No matter how far in or out of the money this curve is always monotonically increasing for call options and put options, uncertainty adds value to the contract. The slope of this curve is the vega.

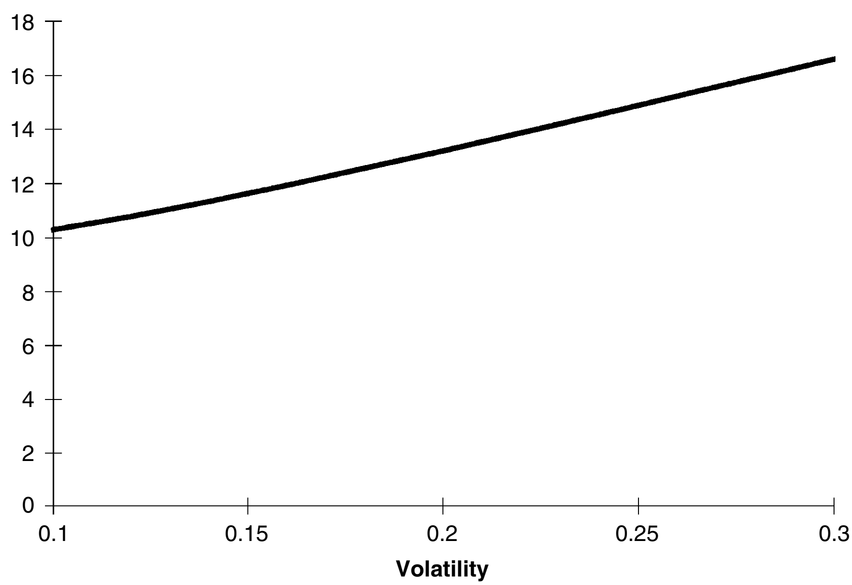


Figure 10.14 The value of an at-the-money call option as a function of volatility.

10.7 RHO

Rho, ρ, is the sensitivity of the option value to the interest rate used in the Black–Scholes formulas:

$$\boxed{\rho = \frac{\partial V}{\partial r}}$$

In practice one often uses a whole term structure of interest rates, meaning a time-dependent rate $r(t)$. Rho would then be the sensitivity to the level of the rates assuming a parallel shift in rates at all times. Again, you must be careful for which contracts you measure rho.

Here are some formulas for the rhos of common contracts:

Rhos of common contracts

Call	$E(T-t)e^{-r(T-t)}N(d_2)$
Put	$-E(T-t)e^{-r(T-t)}N(-d_2)$
Binary call	$-(T-t)e^{-r(T-t)}N(d_2) + \frac{\sqrt{T-t}}{\sigma}e^{-r(T-t)}N'(d_2)$
Binary put	$-(T-t)e^{-r(T-t)}(1-N(d_2)) - \frac{\sqrt{T-t}}{\sigma}e^{-r(T-t)}N'(d_2)$

The sensitivities of common contracts to the dividend yield or foreign interest rate are given by the following formulas:

Sensitivity to dividend for common contracts

Call	$-(T-t)Se^{-D(T-t)}N(d_1)$
Put	$(T-t)Se^{-D(T-t)}N(-d_1)$
Binary call	$-\frac{\sqrt{T-t}}{\sigma}e^{-r(T-t)}N'(d_2)$
Binary put	$\frac{\sqrt{T-t}}{\sigma}e^{-r(T-t)}N'(d_2)$

10.8 IMPLIED VOLATILITY

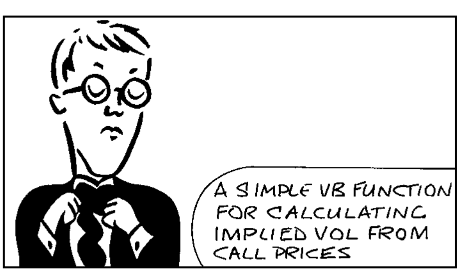
The Black–Scholes formula for a call option takes as input the expiry, the strike, the underlying and the interest rate *together with the volatility* to output the price. All but the volatility are easily measured. How do we know what volatility to put into the formulas? A trader can see on his screen that a certain call option with four months until expiry and a strike of 100 is trading at 6.51 with the underlying at 101.5 and a short-term interest rate of 8%. Can we use this information in some way?

Turn the relationship between volatility and an option price on its head, if we can see the price at which the option is trading, we can ask 'What volatility must I use to get the correct market price?' This is called the **implied volatility**. The implied volatility is the volatility of the underlying which when substituted into the Black–Scholes formula gives a theoretical price equal to the market price. In a sense it is the market's view of volatility over the life of the option. Assuming that we are using call prices to estimate the implied volatility then provided the option price is less than the asset and greater than zero then we can find a unique value for the implied volatility. (If the option price is outside these bounds then there's a very extreme arbitrage opportunity.) Because there is no simple formula for the implied volatility as a function of the option value we must solve the equation

$$V_{BS}(S_0, t_0; \sigma, r; E, T) = \text{known value}$$

for σ, where V_{BS} is the Black–Scholes formula. Today's asset price is S_0, the date is t_0 and everything is known in this equation except for σ. Below is an algorithm for finding the implied volatility from the market price of a call option to any required degree of accuracy. The method used is **Newton–Raphson** which uses the derivative of the option price with respect to the volatility (the vega) in the calculation. This method is particularly good for such a well-behaved function as a call value.

```
Function ImpVolCall(MktPrice As Double, Strike As _
         Double, Expiry As Double, _
         Asset As Double, IntRate As Double, _
         error As Double)
         Volatility = 0.2
dv = error + 1
While Abs(dv) > error
    d1 = Log(Asset / Strike) + (IntRate + 0.5 * _
 Volatility * Volatility) * Expiry
    d1 = d1 / (Volatility * Sqr(Expiry))
    d2 = d1 - Volatility * Sqr(Expiry)
    PriceError = Asset * cdf(d1) - Strike * _
 Exp(-IntRate * Expiry) * cdf(d2) - MktPrice
    Vega = Asset * Sqr(Expiry / 3.1415926 / 2) * _
   Exp(-0.5 * d1 * d1)
    dv = PriceError / Vega
    Volatility = Volatility - dv
Wend
ImpVolCall = Volatility
End Function
```



In this we need the cumulative distribution function for the Normal distribution. The following is a simple algorithm which gives an accurate, and fast, approximation to the cumulative distribution function of the standardized Normal:

$$\text{For } x \geq 0 \quad N(x) \approx 1 - \frac{1}{\sqrt{2\pi}} e^{-(1/2)x^2} (a_1 d + a_2 d^2 + a_3 d^3 + a_4 d^4 + a_5 d^5)$$

where

$$d = \frac{1}{1 + 0.2316419x}$$

and

$a1 = 0.31938153, \quad a2 = -0.356563782, \quad a3 = 1.781477937, \quad a4 = -1.821255978$
$a5 = 1.330274429.$

For $x < 0$ use the fact that $N(x) + N(-x) = 1$.

```
Function cdf(x As Double) As Double
Dim d As Double
Dim temp as Double
Dim a1 As Double
Dim a2 As Double
Dim a3 As Double
Dim a4 As Double
Dim a5 As Double
d = 1 / (1 + 0.2316419 * Abs(x))
a1 = 0.31938153
a2 = -0.356563782
a3 = 1.781477937
a4 = -1.821255978
a5 = 1.330274429
temp = a5
temp = a4 + d * temp
temp = a3 + d * temp
temp = a2 + d * temp
temp = a1 + d * temp
temp = d * temp
cdf = 1 - 1 / Sqr(2 * 3.1415926) * Exp(-0.5 * x * x) * temp
If x < 0 Then cdf = 1 - cdf
End Function
```

In practice if we calculate the implied volatility for many different strikes and expiries on the same underlying then we find that *the volatility is not constant*. A typical result is that of Figure 10.15 which shows the implied volatilities for the S&P500 on 9th September 1999 for options expiring later in the month. The implied volatilities for the calls and puts should be identical, because of put-call parity. The differences seen here could be due to bid-offer spread or calculations performed at slightly different times.

This shape is commonly referred to as the **smile**, but it could also be in the shape of a **frown**. In this example it's a rather lopsided wry grin. Whatever the shape, it tends to persist with time, with certain shapes being characteristic of certain markets.

The dependence of the implied volatility on strike and expiry can be interpreted in many ways. The easiest interpretation is that it represents the market's view of future volatility in some complex way.

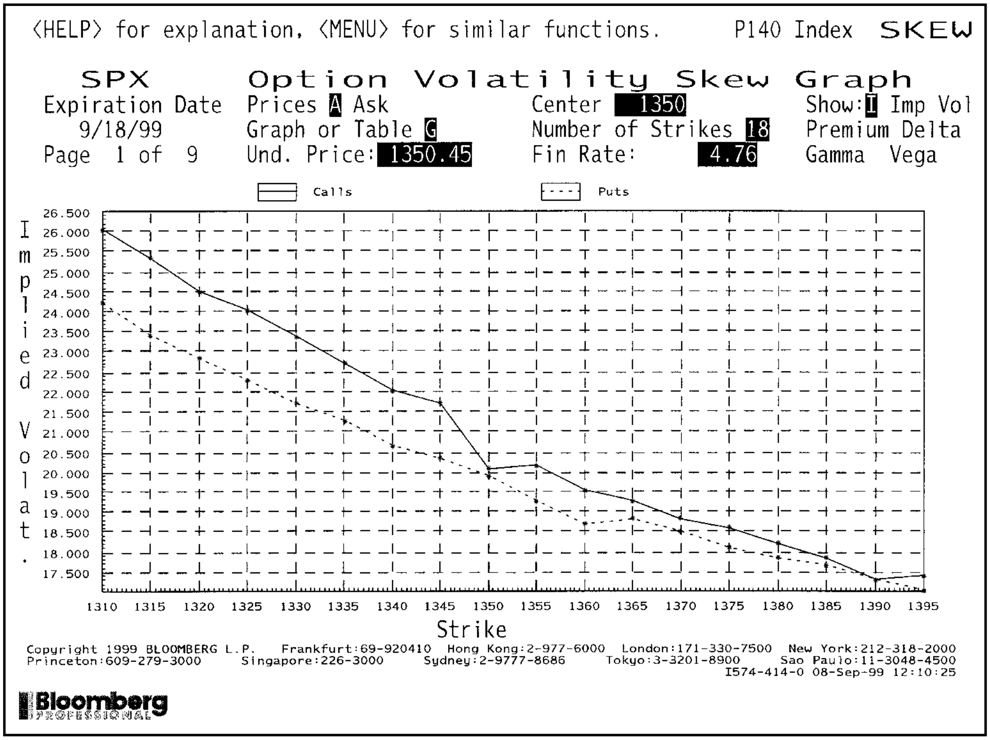
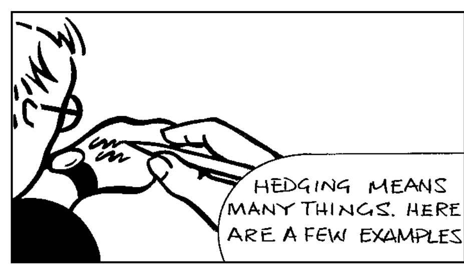


Figure 10.15 Implied volatilities for the S&P500. Source: Bloomberg L.P.

10.9 A CLASSIFICATION OF HEDGING TYPES



10.9.1 Why hedge?

'Hedging' in its broadest sense means the reduction of risk by exploiting relationships or correlation between various risky investments (or bets). The concept is used widely in horse racing, other sports betting and, of course, high finance. The reason for hedging is that it can lead to an improved risk/return. In the classical Modern Portfolio Theory framework (Chapter 19), for example, it is usually possible to construct many portfolios having the same expected return but with different variance of returns ('risk'). Clearly, if you have two portfolios with the same expected return the one with the lower risk is the better investment.

10.9.2 The two main classifications

Probably the most important distinction between types of hedging is between model-independent and model-dependent hedging strategies.

Model-independent hedging: An example of such hedging is put-call parity. There is a simple relationship between calls and puts on an asset (when they are both European and with the same strikes and expiries), the underlying stock and a zero-coupon bond with the same maturity. This relationship is completely independent of how the underlying asset changes in value. Another example is spot-forward parity. In neither case do we have to specify the dynamics of the asset, not even its volatility, to find a possible hedge. Such model-independent hedges are few and far between.

Model-dependent hedging: Most sophisticated finance hedging strategies depend on a model for the underlying asset. The obvious example is the hedging used in the Black–Scholes analysis that leads to a whole theory for the value of derivatives. In pricing derivatives we typically need to at least know the volatility of the underlying asset. If the model is wrong then the option value and any hedging strategy will also be wrong.

10.9.3 Delta hedging

One of the building blocks of derivatives theory is **delta hedging**. This is the theoretically perfect elimination of all risk by using a very clever hedge between the option and its underlying. Delta hedging exploits the perfect correlation between the changes in the option value and the changes in the stock price. This is an example of 'dynamic' hedging; the hedge must be continually monitored and frequently adjusted by the sale or purchase of the underlying asset. Because of the frequent rehedging, any dynamic hedging strategy is going to result in losses due to transaction costs. In some markets this can be very important.

10.9.4 Gamma hedging

To reduce the size of each rehedge and/or to increase the time between rehedges, and thus reduce costs, the technique of **gamma hedging** is often employed. A portfolio that is delta hedged is insensitive to movements in the underlying as long as those movements are quite small. There is a small error in this due to the convexity of the portfolio with respect to the underlying. Gamma hedging is a more accurate form of hedging that theoretically eliminates these second-order effects. Typically, one hedges one, exotic, say, contract with a vanilla contract and the underlying. The quantities of the vanilla and the underlying are chosen so as to make both the portfolio delta and the portfolio gamma instantaneously zero.

10.9.5 Vega hedging

As I said above, the prices and hedging strategies are only as good as the model for the underlying. The key parameter that determines the value of a contract is the volatility of the underlying asset. Unfortunately, this is a very difficult parameter to measure or even estimate. Nor is it usually a constant as assumed in the simple theories. Obviously, the value of a contract depends on this parameter, and so to ensure that our portfolio value is insensitive to this parameter we can **vega hedge**. This means that we hedge one option with both the underlying and another option in such a way that both the delta and the vega, the sensitivity of the portfolio value to volatility, are zero. This is often quite satisfactory in practice but is usually theoretically inconsistent; we should not use a constant volatility (basic Black–Scholes) model to calculate sensitivities to parameters that are assumed not to vary. The distinction between variables (underlying asset price and time) and parameters (volatility, dividend yield, interest rate) is extremely important here. It is justifiable to rely on sensitivities of prices to variables, but usually not sensitivity to parameters. To get around this problem it is possible to independently model volatility etc. as variables themselves. In such a way it is possible to build up a consistent theory.

10.9.6 Static hedging

There are quite a few problems with delta hedging, on both the practical and the theoretical side. In practice, hedging must be done at discrete times and is costly. Sometimes one has to buy or sell a prohibitively large number of the underlying in order to follow the theory. This is a problem with barrier options and options with discontinuous payoff. On the theoretical side, the model for the underlying is not perfect, at the very least we do not know parameter values accurately. Delta hedging alone leaves us very exposed to the model, this is model risk. Many of these problems can be reduced or eliminated if we follow a strategy of **static hedging** as well as delta hedging: buy or sell more liquid traded contracts to reduce the cashflows in the original contract. The static hedge is put into place now, and left until expiry. In the extreme case where an exotic contract has all of its cashflows matched by cashflows from traded options then its value is given by the cost of setting up the static hedge; a model is not needed.

10.9.7 Margin hedging

Often what causes banks, and other institutions, to suffer during volatile markets is not the change in the paper value of their assets but the requirement to suddenly come up with a large amount of cash to cover an unexpected margin call. Recent examples where margin has caused significant damage are Metallgesellschaft and Long-Term Capital Management. Writing options is very risky. The downside of buying an option is just the initial premium, the upside may be unlimited. The upside of writing an option is limited, but the downside could be huge. For this reason, to cover the risk of default in the event of an unfavorable outcome, the clearing houses that register and settle options insist on the deposit of a margin by the writers of options. Margin comes in two forms, the initial margin and the maintenance margin. The initial margin is the amount deposited at the initiation of the contract. The total amount held as margin must stay above a prescribed maintenance margin. If it ever falls below this level then more money (or equivalent in bonds, stocks etc.) must be deposited. The amount of margin that must be deposited depends on the particular contract. A dramatic market move could result in a sudden large margin call that may be difficult to meet. To prevent this situation it is possible to **margin hedge**. That is, set up a portfolio such that a margin calls on one part of the portfolio are balanced by refunds from other parts. Usually over-the-counter contracts have no associated margin requirements and so won't appear in the calculation.

10.9.8 Crash (Platinum) hedging

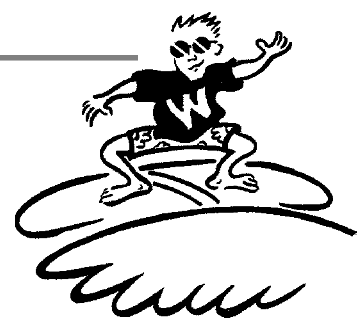
The final variety of hedging that we discuss is specific to extreme markets. Market crashes have at least two obvious effects on our hedging. First of all, the moves are so large and rapid that they cannot be traditionally delta hedged. The convexity effect is not small. Second, normal market correlations become meaningless. Typically all correlations become one (or minus one). **Crash** or **Platinum hedging** exploits the latter effect in such a way as to minimize the worst possible outcome for the portfolio. The method, called CrashMetrics (Chapter 23), does not rely on difficult to measure parameters such as volatilities and so is a very robust hedge. Platinum hedging comes in two types: hedging the paper value of the portfolio and hedging the margin calls.

10.10 SUMMARY

In this chapter we went through the derivation of some of the most important formulas. We also saw the definitions and descriptions of the hedge ratios. Trading in derivatives would be no more than gambling if you took away the ability to hedge. Hedging is all about managing risk and reducing uncertainty.

FURTHER READING

- See Taleb (1997) for a lot of detailed analysis of vega.
- See Press *et al*. (1992) for more routines for finding roots, i.e. for finding implied volatilities.
- There are many 'virtual' option pricers on the internet. See, for example, `www.cboe.com`.
- I'm not going to spend much time on deriving or even presenting formulas. There are 1001 books that contain option formulas, there is even one book with 1001 formulas (Haug, 1997).



Time Out...

Another look at Black–Scholes

Let's take another look at the Black–Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

The option value $V(S,t)$ depends on (or 'is a function of') the asset price S and the time t.

The first derivative of the option value with respect to time is called the theta:

$$\Theta = \frac{\partial V}{\partial t}.$$

Notice that this is a partial derivative and so theta is the gradient of the option value in the direction of changing time, asset price fixed. It measures the rate of change of the option value with time if the asset price doesn't move, hence the other name 'time decay.'

The first derivative of the option value with respect to the asset price is called the delta:

$$\Delta = \frac{\partial V}{\partial S}.$$

This is the slope in the S direction with time fixed. Asset prices change very rapidly and so the dominant change in the option value from moment to moment is the delta multiplied by the change in the asset price. This is just a simple application of Taylor series; the difference between the option price at time t when the asset is at S and a later time $t + \delta t$ when the asset price is $S + \delta S$ is given by

$$V(S + \delta S, t + \delta t) - V(S, t) = \Delta\, \delta S + \cdots.$$

The $\cdots$ are terms which are, generally speaking, smaller than the leading term. They are still important, as we'll see in a moment.

Because the change in option value and the change in asset price are so closely linked we can see that holding a quantity Δ of the underlying asset short we can eliminate, to leading order, fluctuations in our net portfolio value. This is the basis of delta hedging.

The second derivative of the option value with respect to the asset price is called the gamma:

$$\Gamma = \frac{\partial^2 V}{\partial S^2}.$$

This is also just the S derivative of the delta. If the asset changes by an amount δS then the delta changes by an amount $\Gamma \delta S$. Thus the gamma is a measure of how much one might have to rehedge, and gives a measure of the amount of transaction costs from delta hedging.

Now we can interpret all the terms in the Black–Scholes equation, but what does the equation itself mean?

Written in terms of the greeks, the Black–Scholes equation is

$$\Theta + \tfrac{1}{2}\sigma^2 S^2 \Gamma + rS\Delta - rV = 0.$$

Reordering this we have

$$\Theta = rV - rS\Delta - \tfrac{1}{2}\sigma^2 S^2 \Gamma = r(V - S\Delta) - \tfrac{1}{2}\sigma^2 S^2 \Gamma.$$

When we have a delta hedged position we hold the option with value V and are short Δ of the underlying asset. Thus our portfolio value is at any time

$$V - S\Delta.$$

So we can write the Black–Scholes equation in words as

Time decay $=$ interest received on cash equivalent of portfolio value $- \tfrac{1}{2}\sigma^2 S^2$gamma.

The option value grows by an equivalent of interest that would have been received by a riskless pure cash position. But the delta hedged option is not a cash position. That's where the final, gamma, term comes in.

Ignoring the interest on the cash equivalent, the theta and gamma terms add up to zero. Of course, you can't ignore this interest unless the portfolio has zero value or rates are zero.

The delta hedge is only accurate to leading order. If one is hedging with finite time intervals between rehedges then there is inevitably a little bit of randomness that we can't hedge away. We can see this if we go to higher order in the Taylor series expansion of $V(S + \delta S, t + \delta t)$:

$$V(S + \delta S, t + \delta t) - V(S, t) = \Delta\, \delta S + \Theta\, \delta t + \tfrac{1}{2}\Gamma\, \delta S^2 \cdots.$$

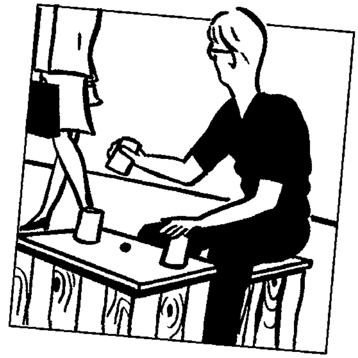
The Θ term is predictable if we know the time δt between hedges (and it has already appeared in the Black–Scholes equation). But the Γ term is multiplied by the random δS^2. We can't hedge this away perfectly. It is, in practice, the source of hedging errors. However, if we rehedge sufficiently frequently (i.e. δt is very small) then the combined effect of the gamma terms is via an *average* of the δS^2. And this average is $\sigma^2 S^2\, \delta t$. Why is it the average that matters? It's like betting on the toss of a biased coin. If you have an advantage then you can exploit it by betting a small amount but very, very often. In the long run you will certainly win. (In terms of standard deviations, as the time between hedges decreases so does the standard deviation of the hedging error accumulated over the life of the option.)

We can now see that the gamma term in the Black–Scholes equation is to balance the higher-order fluctuations in the option value. Naturally, it therefore depends on the magnitude of these fluctuations, the volatility of the underlying asset.

	Call	**Put**	**Binary Call**	**Binary Put**
Value V Black–Scholes value	$Se^{-D(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2)$	$-Se^{-D(T-t)}N(-d_1) + Ee^{-r(T-t)}N(-d_2)$	$e^{-r(T-t)}N(d_2)$	$e^{-r(T-t)}(1-N(d_2))$
Delta $\frac{\partial V}{\partial S}$ Sensitivity to underlying	$e^{-D(T-t)}N(d_1)$	$e^{-D(T-t)}(N(d_1)-1)$	$\frac{e^{-r(T-t)}N'(d_2)}{\sigma S\sqrt{T-t}}$	$-\frac{e^{-r(T-t)}N'(d_2)}{\sigma S\sqrt{T-t}}$
Gamma $\frac{\partial^2 V}{\partial S^2}$ Sensitivity of delta to underlying	$\frac{e^{-D(T-t)}N'(d_1)}{\sigma S\sqrt{T-t}}$	$\frac{e^{-D(T-t)}N'(d_1)}{\sigma S\sqrt{T-t}}$	$-\frac{e^{-r(T-t)}d_1N'(d_2)}{\sigma^2S^2(T-t)}$	$\frac{e^{-r(T-t)}d_1N'(d_2)}{\sigma^2S^2(T-t)}$
Theta $\frac{\partial V}{\partial t}$ Sensitivity to time	$-\frac{\sigma Se^{-D(T-t)}N'(d_1)}{2\sqrt{T-t}} + DSN(d_1)e^{-D(T-t)} - rEe^{-r(T-t)}N(d_2)$	$-\frac{\sigma Se^{-D(T-t)}N'(-d_1)}{2\sqrt{T-t}} - DSN(-d_1)e^{-D(T-t)} + rEe^{-r(T-t)}N(-d_2)$	$re^{-r(T-t)}N(d_2) + e^{-r(T-t)}N'(d_2) \times \left(\frac{d_1}{2(T-t)} - \frac{r-D}{\sigma\sqrt{T-t}}\right)$	$re^{-r(T-t)}(1-N(d_2)) - e^{-r(T-t)}N'(d_2) \times \left(\frac{d_1}{2(T-t)} - \frac{r-D}{\sigma\sqrt{T-t}}\right)$
Vega $\frac{\partial V}{\partial \sigma}$ Sensitivity to volatility	$S\sqrt{T-t}e^{-D(T-t)}N'(d_1)$	$S\sqrt{T-t}e^{-D(T-t)}N'(d_1)$	$-e^{-r(T-t)}N'(d_2) \times \left(\sqrt{T-t} + \frac{d_2}{\sigma}\right)$	$e^{-r(T-t)}N'(d_2) \times \left(\sqrt{T-t} + \frac{d_2}{\sigma}\right)$
Rho (r) $\frac{\partial V}{\partial r}$ Sensitivity to interest rate	$E(T-t)e^{-r(T-t)}N(d_2)$	$-E(T-t)e^{-r(T-t)}N(-d_2)$	$-(T-t)e^{-r(T-t)}N(d_2) + \frac{\sqrt{T-t}}{\sigma}e^{-r(T-t)}N'(d_2)$	$-(T-t)e^{-r(T-t)}(1-N(d_2)) - \frac{\sqrt{T-t}}{\sigma}e^{-r(T-t)}N'(d_2)$
Rho (D) $\frac{\partial V}{\partial D}$ Sensitivity to dividend yield	$-(T-t)Se^{-D(T-t)}N(d_1)$	$(T-t)Se^{-D(T-t)}N(-d_1)$	$-\frac{\sqrt{T-t}}{\sigma}e^{-r(T-t)}N'(d_2)$	$\frac{\sqrt{T-t}}{\sigma}e^{-r(T-t)}N'(d_2)$

$$d_1 = \frac{\log\left(\frac{S}{E}\right) + \left(r - D + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}},\quad d_2 = \frac{\log\left(\frac{S}{E}\right) + \left(r - D - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t},\quad N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}\xi^2}d\xi \quad \text{and} \quad N'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$$

CHAPTER 11
multi-asset options



The aim of this Chapter...

... is to introduce the idea of correlation between many different assets and so develop a theory for derivatives that depend on several different assets simultaneously.

In this Chapter...

- how to model the behavior of many assets simultaneously
- estimating correlation between asset price movements
- how to value and hedge options on many underlying assets in the Black–Scholes framework
- the pricing formula for European non-path-dependent options on dividend-paying assets

11.1 INTRODUCTION

In this chapter I introduce the idea of higher dimensionality by describing the Black–Scholes theory for options on more than one underlying asset. This theory is perfectly straightforward; the only new idea is that of correlated random walks and the corresponding multifactor version of Itô's lemma.

Although the modeling and mathematics is easy, the final step of the pricing and hedging, the 'solution,' can be extremely hard indeed. I explain what makes a problem easy, and what makes it hard, from the numerical analysis point of view.

11.2 MULTIDIMENSIONAL LOGNORMAL RANDOM WALKS

The basic building block for option pricing with one underlying is the lognormal random walk

$$dS = \mu S\,dt + \sigma S\,dX.$$

This is readily extended to a world containing many assets via models for each underlying

$$dS_i = \mu_i S_i\,dt + \sigma_i S_i\,dX.$$

Here S_i is the price of the ith asset, $i = 1, \ldots, d$, and μ_i and σ_i are the drift and volatility of that asset respectively and dX_i is the increment of a Wiener process. We can still continue to think of dX_i as a random number drawn from a Normal distribution with mean zero and standard deviation $dt^{1/2}$ so that

$$E[dX_i] = 0 \quad \text{and} \quad E[dX_i^2] = dt$$

but the random numbers dX_i and dX_j are **correlated**:

$$E[dX_i\,dX_j] = \rho_{ij}dt.$$

Here ρ_{ij} is the correlation coefficient between the ith and jth random walks. The symmetric matrix with ρ_{ij} as the entry in the ith row and jth column is called the **correlation matrix**. For example, if we have seven underlyings $d = 7$ and the correlation matrix will look like this:

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & \rho_{12} & \rho_{13} & \rho_{14} & \rho_{15} & \rho_{16} & \rho_{17} \\ \rho_{21} & 1 & \rho_{23} & \rho_{24} & \rho_{25} & \rho_{26} & \rho_{27} \\ \rho_{31} & \rho_{32} & 1 & \rho_{34} & \rho_{35} & \rho_{36} & \rho_{37} \\ \rho_{41} & \rho_{42} & \rho_{43} & 1 & \rho_{45} & \rho_{46} & \rho_{47} \\ \rho_{51} & \rho_{52} & \rho_{53} & \rho_{54} & 1 & \rho_{56} & \rho_{57} \\ \rho_{61} & \rho_{62} & \rho_{63} & \rho_{64} & \rho_{65} & 1 & \rho_{67} \\ \rho_{71} & \rho_{72} & \rho_{73} & \rho_{74} & \rho_{75} & \rho_{76} & 1 \end{pmatrix}$$

Note that $\rho_{ii} = 1$ and $\rho_{ij} = \rho_{ji}$. The correlation matrix is positive definite, so that $\mathbf{y}^T\mathbf{\Sigma}\mathbf{y} \geq 0$. The **covariance matrix** is simply

$$\mathbf{M\Sigma M},$$

where $\mathbf{M}$ is the matrix with the σ_i along the diagonal and zeros everywhere else.

To be able to manipulate functions of many random variables we need a multidimensional version of Itô's lemma. If we have a function of the variables $S_1, \ldots, S_d$ and t, $V(S_1, \ldots, S_d, t)$, then

$$dV = \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \sigma_i \sigma_j \rho_{ij} S_i S_j \frac{\partial^2 V}{\partial S_i \, \partial S_j} \right) dt + \sum_{i=1}^{d} \frac{\partial V}{\partial S_i} dS_i.$$

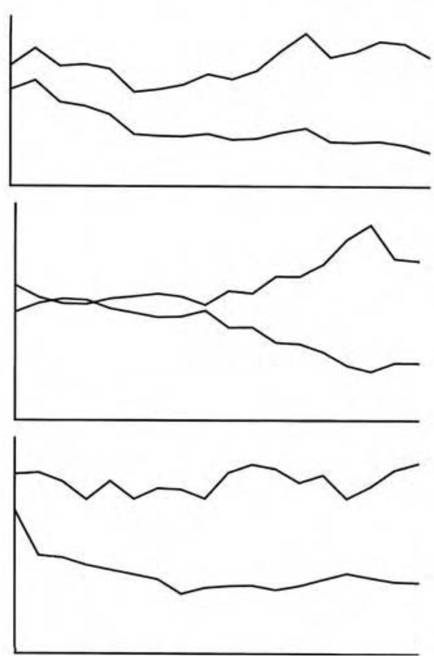
We can get to this same result by using Taylor series and the rules of thumb:

$$dX_i^2 = dt \quad \text{and} \quad dX_i \, dX_j = \rho_{ij} \, dt.$$

Time Out...

Correlation

Correlation is a measure of the relationship or dependence between two or more random quantities. This is most easily explained by reference to the following series of figures. In the first figure we see two random walks that are perfectly correlated, $\rho = 1$. They may be moving



apart overall, but that is a long-term phenomenon. In the short term, and correlation is a characteristic of random walks over small periods of time, you can see that each up move in one random walk is matched by an up move in the other.

In the second figure we see a correlation ρ of -1. Now each up move in one walk is matched by a down in the other. The third figure shows two uncorrelated random walks, there is no relationship between the up and down moves in the two walks.

Generally, the correlation can be anywhere between -1 and $+1$. What would two random walks with a correlation of 0.5 look like?

P.S. I don't believe in correlations among financial assets.

11.3 MEASURING CORRELATIONS

If you have time series data at intervals of δt for all d assets you can calculate the correlation between the returns as follows. First, take the price series for each asset and calculate the return over each period. The return on the ith asset at the kth data point in the time series is simply

$$R_i(t_k) = \frac{S_i(t_k + \delta t) - S_i(t_k)}{S_i(t_k)}.$$

The historical volatility of the ith asset is

$$\sigma_i = \sqrt{\frac{1}{\delta t(M-1)} \sum_{k=1}^{M} (R_i(t_k) - \overline{R}_i)^2}$$

where M is the number of data points in the return series and $\overline{R}_i$ is the mean of all the returns in the series.

The covariance between the returns on assets i and j is given by

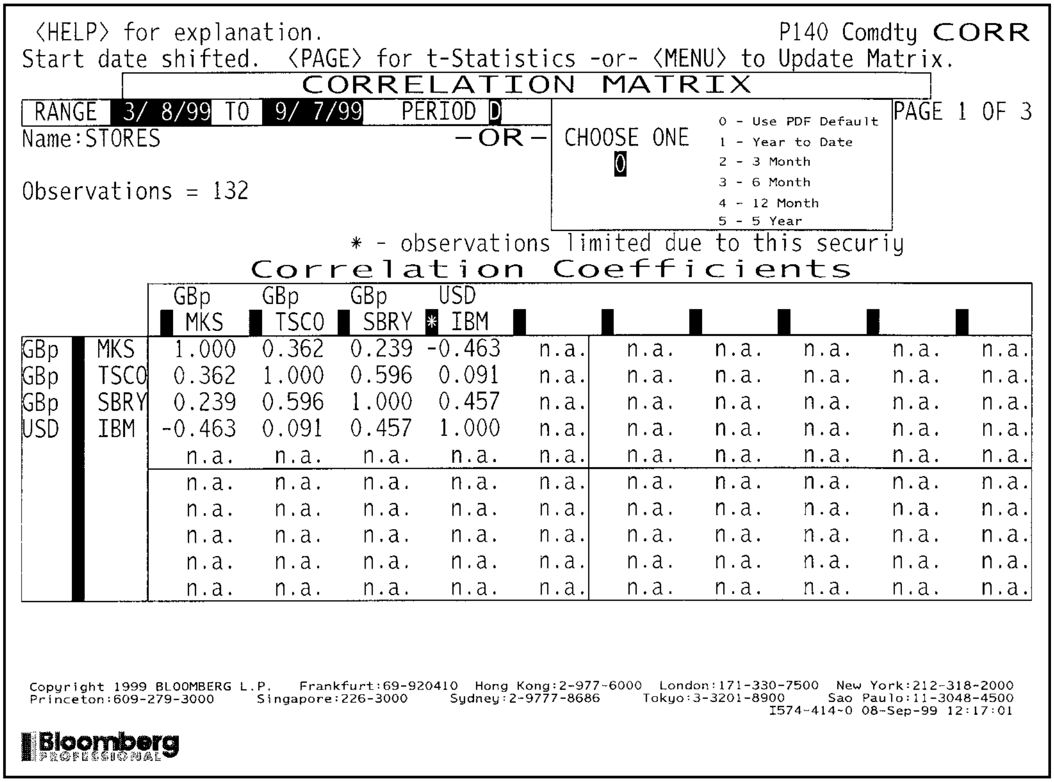
$$\frac{1}{\delta t(M-1)} \sum_{k=1}^{M} (R_i(t_k) - \overline{R}_i)(R_j(t_k) - \overline{R}_j).$$

The correlation is then

$$\frac{1}{\delta t(M-1)\sigma_i\sigma_j} \sum_{k=1}^{M} (R_i(t_k) - \overline{R}_i)(R_j(t_k) - \overline{R}_j).$$

In Excel correlation between two time series can be found using the CORREL worksheet function, or Tools | Data Analysis | Correlation.

Figure 11.1 shows the correlation matrix for Marks & Spencer, Tesco, Sainsbury and IBM.



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CORRELATION MATRIX

RANGE 3/ 8/99 TO 9/ 7/99 PERIOD D PAGE 1 OF 3
Name:STORES -OR- CHOOSE ONE 0
0 - Use PDF Default
1 - Year to Date
2 - 3 Month
3 - 6 Month
4 - 12 Month
5 - 5 Year

Observations = 132

* - observations limited due to this securiy

Correlation Coefficients

		GBp MKS	GBp TSCO	GBp SBRY	USD * IBM						
GBp	MKS	1.000	0.362	0.239	-0.463	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.
GBp	TSCO	0.362	1.000	0.596	0.091	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.
GBp	SBRY	0.239	0.596	1.000	0.457	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.
USD	IBM	-0.463	0.091	0.457	1.000	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.
		n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.
		n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.
		n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.
		n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.
		n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.
		n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.	n.a.

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Bloomberg PROFESSIONAL

Figure 11.1 Some correlations. Source: Bloomberg L.P.

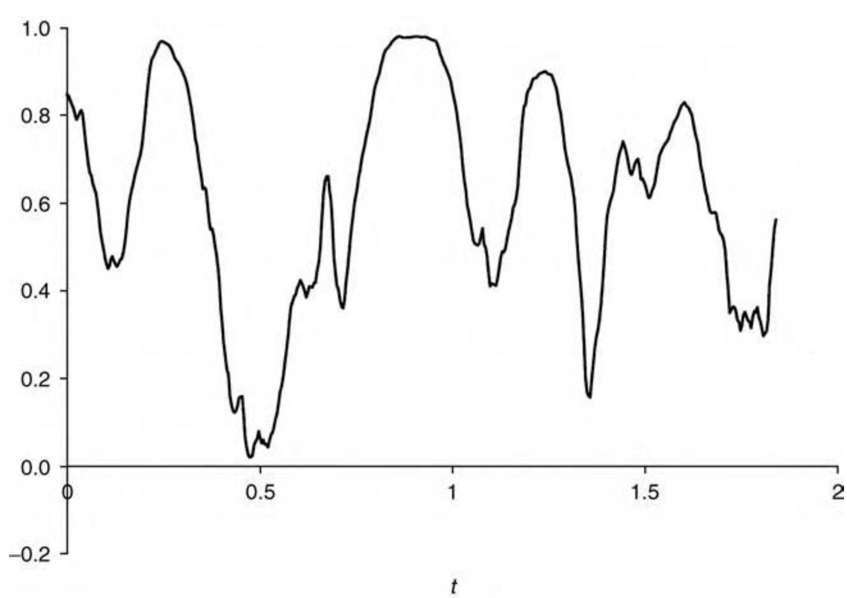


Figure 11.2 A correlation time series.

Correlations measured from financial time series data are notoriously unstable. If you split your data into two equal groups, up to one date and beyond that date, and calculate the correlations for each group you may find that they differ quite markedly. You could calculate a 60-day correlation, say, from several years' data and the result would look something like Figure 11.2. You might want to use a historical 60-day correlation if you have a contract of that maturity. But, as can be seen from the figure, such a historical correlation should be used with care; correlations are even more unstable than volatilities.

The other possibility is to back out an **implied correlation** from the quoted price of an instrument. The idea behind that approach is the same as with implied volatility, it gives an estimate of the market's perception of correlation.

Time Out...

On a spreadsheet

The following spreadsheet shows how to simulate two correlated random walks on a spread-sheet. Both of these random walks are lognormal, but notice the correlation between them.

	A	B	C	D	E	F	G	H	I
1	Asset1	Asset2		Time	Random1	Random2	Asset1	Asset2	
2	100	80		0	0.892325	-1.823526	100	80	
3				0.01	-0.05225	-0.826921	99.9955	78.37858	
4	Drift1	Drift2		0.02	1.450597	-1.056731	102.9966	78.08891	
5	0.1	0.2	=D3+$B$12	.03	-0.644218	-0.17235	101.7725	77.14083	
6				0.04	0.070245	-1.412904	102.0173	74.54468	
7	Vol1	Vol2		0.05	1.710178	-0.08226	105.6086	76.44673	
8	0.2	0.3	=RAND()+RAN	6	-0.355846	-0.994373	104.9626	74.2166	
9			D()+RAND()+R	7	0.220388	-0.455272	105.5302	73.73252	
10	Correl.	0.5	AND()+RAND()	8	-0.765802	-0.902387	104.0406	71.31545	
11			+RAND()+RAN	9	0.313413	-0.001138	104.7968	71.79123	
12	Timestep	0.01	D()+RAND()+R	1	-0.277696	1.202352	104.3195	73.87839	
13			AND()+RAND()	1	0.869986	-0.103287	106.239	74.79199	
14	Sqrt(1-correl^2)		+RAND()+RAN	2	-0.359709	-1.470363	105.5809	71.68089	
15	0.866025		D()-6	3	-0.37802	-0.958738	104.8883	69.63231	
16				4	-1.110287	0.822666	102.664	70.10019	
17				0.15	0.347478	-1.684565	103.4802	67.53774	
18	=SQRT(1-B10*B10)			0.16	1.325781	1.112296	106.3275	70.96764	
19				0.17	-1.18274	-0.616839	103.9187	68.71321	
20				0.18	-0.708953	-0.933997	102.5491	66.45253	
21	=G19*(1+A$5*$B$12+A$8*SQRT($B$12)*E20)				67	-1.564002	101.2759	63.21659	
22				0.2	-0.07179	0.522096	101.2318	64.13245	
23				0.21	0.130124	0.542272	101.5965	65.28943	
24				0.22	-1.581902	2.461052	98.48376	68.04539	
25				0.23	1.005466	0.055309	100.5627	69.30552	
26				0.24	0.635218	1.452914	101.9408	72.72063	
27	=H24*(1+B$5*$B$12+B$8*SQRT($B$12)*($B$10*E25+$A$15*F25))						68971	69.33961	
28				0.26	-1.193529	0.403454	97.40975	68.96373	
29				0.27	1.919892	1.149419	101.2475	73.14715	
30				0.28	0.893661	0.678735	103.1584	75.56386	
31				0.29	-0.81773	-0.312339	101.5744	74.17494	
32				0.3	-0.781891	1.663137	100.0876	76.6584	

11.4 OPTIONS ON MANY UNDERLYINGS

Options with many underlyings are called **basket options**, **options on baskets** or **rainbow options**. The theoretical side of pricing and hedging is straightforward, following the Black–Scholes arguments but now in higher dimensions.

Set up a portfolio consisting of one basket option and short a number Δ_i of each of the assets S_i:

$$\Pi = V(S_1, \ldots, S_d, t) - \sum_{i=1}^{d} \Delta_i S_i.$$

The change in this portfolio is given by

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \sigma_i \sigma_j \rho_{ij} S_i S_j \frac{\partial^2 V}{\partial S_i \, \partial S_j} \right) dt + \sum_{i=1}^{d} \left(\frac{\partial V}{\partial S_i} - \Delta_i \right) dS_i.$$

If we choose

$$\Delta_i = \frac{\partial V}{\partial S_i}$$

for each i, then the portfolio is hedged, is risk-free. Setting the return equal to the risk-free rate we arrive at

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \sigma_i \sigma_j \rho_{ij} S_i S_j \frac{\partial^2 V}{\partial S_i \, \partial S_j} + r \sum_{i=1}^{d} S_i \frac{\partial V}{\partial S_i} - rV = 0. \tag{11.1}$$

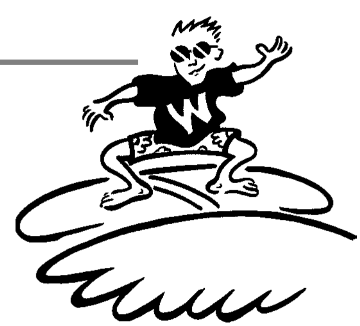
This is the multidimensional version of the Black–Scholes equation. The modifications that need to be made for dividends are obvious. When there is a dividend yield of D_i on the ith asset we have

$$\boxed{\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \sigma_i \sigma_j \rho_{ij} S_i S_j \frac{\partial^2 V}{\partial S_i \, \partial S_j} + \sum_{i=1}^{d} (r - D_i) S_i \frac{\partial V}{\partial S_i} - rV = 0}$$

Time Out...

Here we go again

Risk neutrality means that the drift rates of the assets do not appear in the pricing equation.



11.5 THE PRICING FORMULA FOR EUROPEAN NON-PATH-DEPENDENT OPTIONS ON DIVIDEND-PAYING ASSETS

Because there is a Green's function for this problem (see Chapter 9) we can write down the value of a European non-path-dependent option with payoff of $\text{Payoff}(S_1, \dots, S_d)$ at time T:

$$
\boxed{
\begin{aligned}
V &= e^{-r(T-t)}\,(2\pi(T-t))^{-d/2}\,(\operatorname{Det}\boldsymbol{\Sigma})^{-1/2}(\sigma_1\cdots\sigma_d)^{-1} \\
&\quad \int_0^\infty \cdots \int_0^\infty \frac{\text{Payoff}(S_1'\cdots S_d')}{S_1'\cdots S_d'}\exp\left(-\frac{1}{2}\boldsymbol{\alpha}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}\right)dS_1'\cdots dS_d' \\
\alpha_i &= \frac{1}{\sigma_i(T-t)^{1/2}}\left(\log\left(\frac{S_i}{S_i'}\right)+\left(r-D_i-\frac{\sigma_i^2}{2}\right)(T-t)\right)
\end{aligned}
}
\tag{11.2}
$$

This has included a constant continuous dividend yield of D_i on each asset.

Time Out...

Ouch! But don't worry

This formula looks horrible. But it has a simple interpretation as the present value of an expectation of the payoff. Part of the integrand (the bit inside the integral) is the payoff, and part represents the probability density function. The numerical integration of this expression is actually quite straightforward as we'll see in Chapter 26.

As always, it's the risk-neutral expectation that matters. Do you see any μs anywhere in the formula? No.

11.6 EXCHANGING ONE ASSET FOR ANOTHER: A SIMILARITY SOLUTION

An **exchange option** gives the holder the right to exchange one asset for another, in some ratio. The payoff for this contract at expiry is

$$\max(q_1S_1 - q_2S_2, 0),$$

where q_1 and q_2 are constants.

The partial differential equation satisfied by this option in a Black–Scholes world is

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_i\sigma_j\rho_{ij}S_iS_j\frac{\partial^2 V}{\partial S_i\,\partial S_j}+\sum_{i=1}^{2}(r-D_i)S_i\frac{\partial V}{\partial S_i}-rV=0.$$

A dividend yield has been included for both assets. Since there are only two underlyings the summations in these only go up to two.

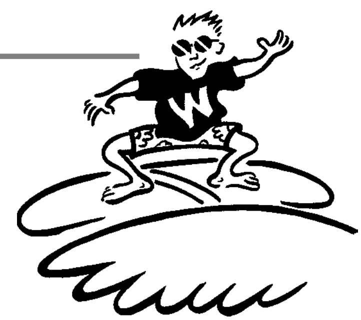
This contract is special in that there is a similarity reduction. Let's postulate that the solution takes the form

$$V(S_1,S_2,t)=q_1S_2H(\xi,t),$$

where the new variable is

$$\xi=\frac{S_1}{S_2}.$$

If this is the case, then instead of finding a function V of three variables, we only need find a function H of two variables, a much easier task.



Time Out...

Similarity reductions

If you skipped much of Chapter 9, as I advised some of you to do, you won't have read about similarity reductions. This is just a useful trick, not one you can often use, but when you can, you should.

Sometimes it is possible to reduce the number of dimensions in a problem by exploiting the 'nice' form of the problem. The example here is typical.

Changing variables from S_1, S_2 to ξ we must use the following for the derivatives:

$$\frac{\partial}{\partial S_1}=\frac{1}{S_2}\frac{\partial}{\partial\xi},\qquad \frac{\partial}{\partial S_2}=-\frac{\xi}{S_2}\frac{\partial}{\partial\xi},$$

$$\frac{\partial^2}{\partial S_1^2}=\frac{1}{S_2^2}\frac{\partial^2}{\partial\xi^2},\qquad \frac{\partial^2}{\partial S_2^2}=\frac{\xi^2}{S_2^2}\frac{\partial^2}{\partial\xi^2}+\frac{2\xi}{S_2^2}\frac{\partial}{\partial\xi},\qquad \frac{\partial^2}{\partial S_1\,\partial S_2}=-\frac{\xi}{S_2^2}\frac{\partial^2}{\partial\xi^2}-\frac{1}{S_2^2}\frac{\partial}{\partial\xi}.$$

The time derivative is unchanged. The partial differential equation now becomes

$$\frac{\partial H}{\partial t}+\frac{1}{2}\sigma'^2\xi^2\frac{\partial^2 H}{\partial\xi^2}+(D_2-D_1)\xi\frac{\partial H}{\partial\xi}-D_2H=0.$$

where

$$\sigma'=\sqrt{\sigma_1^2-2\rho_{12}\sigma_1\sigma_2+\sigma_2^2}.$$

You will recognize this equation as being the Black–Scholes equation for a single stock with D_2 in place of r, D_1 in place of the dividend yield on the single stock and with a volatility of σ'.

From this it follows, retracing our steps and writing the result in the original variables, that

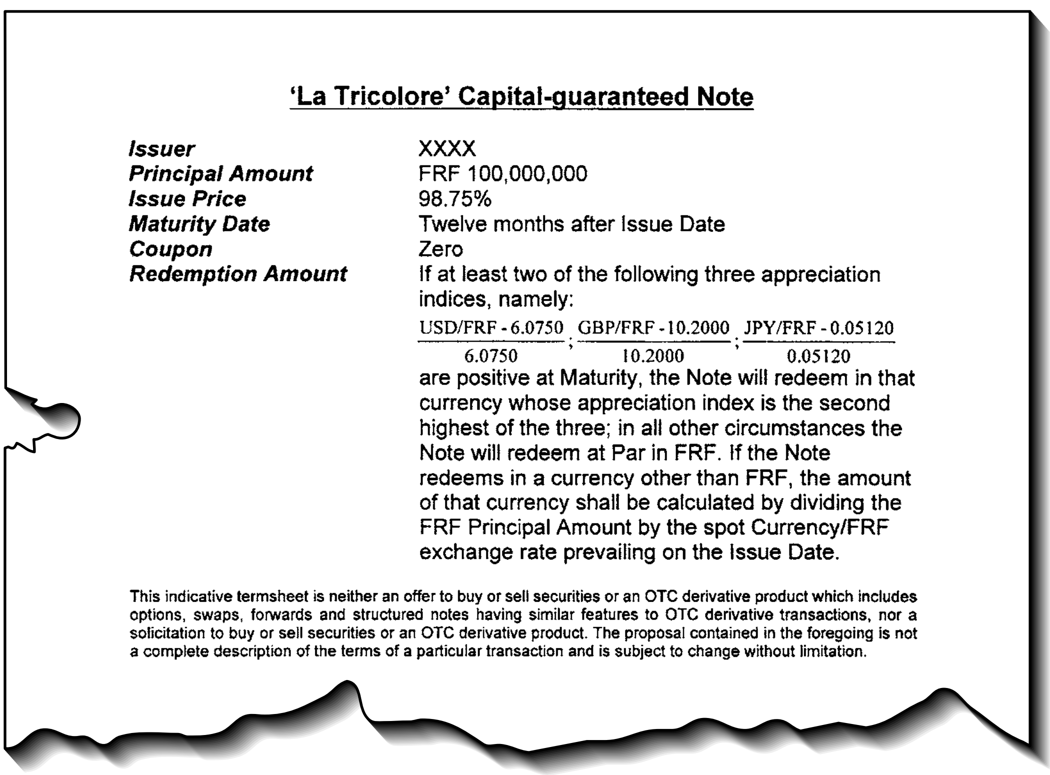
$$V(S_1, S_2, t) = q_1 S_1 e^{-D_1(T-t)} N(d_1') - q_2 S_2 e^{-D_2(T-t)} N(d_2')$$

where

$$d_1' = \frac{\log(q_1 S_1/q_2 S_2) + \left(D_2 - D_1 + \frac{1}{2}\sigma'^2\right)(T-t)}{\sigma'\sqrt{T-t}} \quad \text{and} \quad d_2' = d_1' - \sigma'\sqrt{T-t}.$$

11.7 TWO EXAMPLES

In Figure 11.3 is shown the term sheet for ‘La Tricolore’ capital-guaranteed note. This contract pays off the *second* best performing of three currencies against the French franc, but only if the second-best performing has appreciated against the franc, otherwise it pays off at par. This contract does not have any unusual features, and has a value that can be written as a three-dimensional integral, of the form (11.2). But what would the



‘La Tricolore’ Capital-guaranteed Note

Issuer	XXXX
Principal Amount	FRF 100,000,000
Issue Price	98.75%
Maturity Date	Twelve months after Issue Date
Coupon	Zero
Redemption Amount	If at least two of the following three appreciation indices, namely: $\frac{\text{USD/FRF} - 6.0750}{6.0750}$; $\frac{\text{GBP/FRF} - 10.2000}{10.2000}$; $\frac{\text{JPY/FRF} - 0.05120}{0.05120}$ are positive at Maturity, the Note will redeem in that currency whose appreciation index is the second highest of the three; in all other circumstances the Note will redeem at Par in FRF. If the Note redeems in a currency other than FRF, the amount of that currency shall be calculated by dividing the FRF Principal Amount by the spot Currency/FRF exchange rate prevailing on the Issue Date.

This indicative termsheet is neither an offer to buy or sell securities or an OTC derivative product which includes options, swaps, forwards and structured notes having similar features to OTC derivative transactions, nor a solicitation to buy or sell securities or an OTC derivative product. The proposal contained in the foregoing is not a complete description of the terms of a particular transaction and is subject to change without limitation.

Figure 11.3 Term sheet for ‘La Tricolore’ capital-guaranteed note.

payoff function be? You wouldn't use a partial differential equation to price this contract. Instead you would estimate the multiple integral directly by the methods of Chapter 26.

The next example, whose term sheet is shown in Figure 11.4, is of basket equity swap. This rather complex, high-dimensional contract, is for a swap of interest payment based on three-month LIBOR and the level of an index. The index is made up of the weighted

International Pharmaceutical Basket Equity Swap

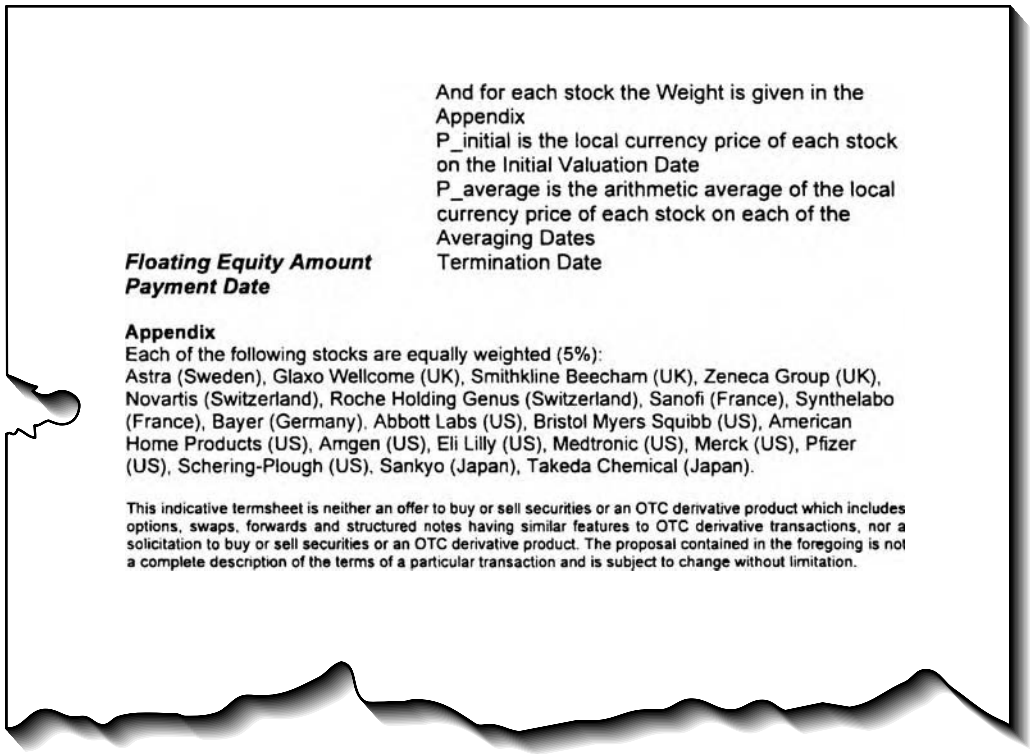
Indicative terms	
Trade Date	[ ]
Initial Valuation Date	[ ]
Effective Date	[ ]
Final Valuation Date	26th September 2002
Averaging Dates	The monthly anniversaries of the Initial Valuation Date commencing 26th March 2002 and up to and including the Expiration Date
Notional Amount	US$25,000,000
Counterparty floating amounts (US$ LIBOR)	
Floating Rate Payer	[ ]
Floating Rate Index	USD-LIBOR
Designated Maturity	Three months
Spread	Minus 0.25%
Day Count Fraction	Actual/360
Floating Rate Payment Dates	Each quarterly anniversary of the Effective Date
Initial Floating Rate Index	[ ]
The Bank Fixed and Floating Amounts (Fee, Equity Option)	
Fixed Amount Payer	XXXX
Fixed Amount	1.30% of Notional Amount
Fixed Amount Payment Date	Effective Date
Basket	A basket comprising 20 stocks and constructed as described in attached Appendix
Initial Basket Level	Will be set at 100 on the Initial Valuation Date
Floating Equity Amount Payer	XXXX
Floating Equity Amount	Will be calculated according to the performance of the basket of stocks in the following way:

$$\text{Notional Amount} * \max\left[0, \left(\frac{\text{BASKET}_{\text{average}} - 100}{100}\right)\right]$$

where

$$\text{BASKET}_{\text{average}} = 100 * \sum_{120\ \text{stocks}} \left(\text{Weight} * \frac{P_{\text{average}}}{P_{\text{initial}}}\right)$$

Figure 11.4 Term sheet for a basket equity swap.



And for each stock the Weight is given in the Appendix
P_initial is the local currency price of each stock on the Initial Valuation Date
P_average is the arithmetic average of the local currency price of each stock on each of the Averaging Dates

Floating Equity Amount Payment Date	Termination Date

Appendix
Each of the following stocks are equally weighted (5%):
Astra (Sweden), Glaxo Wellcome (UK), Smithkline Beecham (UK), Zeneca Group (UK), Novartis (Switzerland), Roche Holding Genus (Switzerland), Sanofi (France), Synthelabo (France), Bayer (Germany), Abbott Labs (US), Bristol Myers Squibb (US), American Home Products (US), Amgen (US), Eli Lilly (US), Medtronic (US), Merck (US), Pfizer (US), Schering-Plough (US), Sankyo (Japan), Takeda Chemical (Japan).

This indicative termsheet is neither an offer to buy or sell securities or an OTC derivative product which includes options, swaps, forwards and structured notes having similar features to OTC derivative transactions, nor a solicitation to buy or sell securities or an OTC derivative product. The proposal contained in the foregoing is not a complete description of the terms of a particular transaction and is subject to change without limitation.

Figure 11.4 (*Continued*).

average of 20 pharmaceutical stocks. To make matters even more complex, the index uses a time averaging of the stock prices.

11.8 REALITIES OF PRICING BASKET OPTIONS

The factors that determine the ease or difficulty of pricing and hedging multi-asset options are

- existence of a closed-form solution
- number of underlying assets, the dimensionality
- path dependency
- early exercise

We have seen all of these except path dependency, we'll see this shortly.

The solution technique that we use will generally be one of

- finite-difference solution of a partial differential equation
- numerical integration
- Monte Carlo simulation

These methods are the subjects of later chapters.

11.8.1 Easy problems

If we have a closed-form solution then our work is done; we can easily find values and hedge ratios. This is provided that the solution is in terms of sufficiently simple functions for which there are spreadsheet functions or other libraries. If the contract is European with no path-dependency then the solution may be of the form (11.2). If this is the case, then we often have to do the integration numerically. This is not difficult. Several methods are described in Chapter 26, including Monte Carlo integration and the use of low-discrepancy sequences.

11.8.2 Medium problems

If we have low dimensionality, less than three or four, say, the finite-difference methods are the obvious choice. They cope well with early exercise and many path-dependent features can be incorporated, though usually at the cost of an extra dimension.

For higher dimensions, Monte Carlo simulations are good. They cope with all path-dependent features. Unfortunately, they are not very efficient for American-style early exercise.

11.8.3 Hard problems

The hardest problems to solve are those with both high dimensionality, for which we would like to use Monte Carlo simulation, and with early exercise, for which we would like to use finite-difference methods. There is currently no numerical method that copes well with such a problem.

11.9 REALITIES OF HEDGING BASKET OPTIONS

Even if we can find option values and the Greeks, they are often very sensitive to the level of the correlation. But as I have said, the correlation is a very difficult quantity to measure. So the hedge ratios are very likely to be inaccurate. If we are delta hedging then we need accurate estimates of the deltas. This makes basket options very difficult to delta hedge successfully.

When we have a contract that is difficult to delta hedge we can try to reduce sensitivity to parameters, and the model, by hedging with other derivatives. This was the basis of vega hedging, mentioned in Chapter 10. We could try to use the same idea to reduce sensitivity to the correlation. Unfortunately, that is also difficult because there just aren't enough contracts traded that depend on the right correlations.

11.10 CORRELATION VERSUS COINTEGRATION

The correlations between financial quantities are notoriously unstable. One could easily argue that a theory should not be built up using parameters that are so unpredictable. I would tend to agree with this point of view. One could propose a stochastic correlation model, but that approach has its own problems.

An alternative statistical measure to correlation is **cointegration**. Very loosely speaking, two time series are cointegrated if a linear combination has constant mean and standard deviation. In other words, the two series never stray too far from one another. This is probably a more robust measure of the linkage between two financial quantities but as yet there is little derivatives theory based on the concept.

11.11 SUMMARY

The new ideas in this chapter were the multifactor, correlated random walks for assets, and Itô's lemma in higher dimensions. These are both simple concepts, and we will use them often, especially in interest-rate-related topics.

FURTHER READING

- See Hamilton (1994) for further details of the measurement of correlation and cointegration.
- The first solution of the exchange option problem was by Margrabe (1978).
- For analytical results, formulas or numerical algorithms for the pricing of some other multifactor options see Stulz (1982), Johnson (1987), Boyle *et al*. (1989), Boyle & Tse (1990), Rubinstein (1991) and Rich & Chance (1993).
- For details of cointegration, what it means and how it works see the papers by Alexander & Johnson (1992, 1994).

CHAPTER 12
an introduction to exotic and path-dependent options

The aim of this Chapter...

... is to give an overview of many of the exciting derivatives above and beyond the basic vanillas. By the end of the chapter you should be able to compare and contrast different sorts of derivative contracts.

In this Chapter...

- how to classify options according to important features
- how to think about derivatives in a way that makes it easy to compare and contrast different contracts
- the names and contract details for many basic types of exotic options

12.1 INTRODUCTION

The contracts we have seen so far are the most basic, and most important, derivative contracts but they only hint at the features that can be found in the more interesting products.

It is an impossible task to classify all options. The best that we can reasonably achieve is a rough characterization of the most popular of the features to be found in derivative products. I list some of these features in this chapter and give several examples. The features that I describe now are discrete cashflows, early exercise, weak path dependence and strong path dependence, time dependence and dimensionality. Finally, I comment on the 'order' of an option.

Exotic options are interesting for several reasons. They are harder to price, sometimes being very model dependent. The risks inherent in the contracts are usually more obscure and can lead to unexpected losses. Careful hedging becomes important, whether delta hedging or some form of static hedging to minimize cashflows. Actually, how to hedge exotics is all that really matters. A trader may have a good idea of a reasonable price for an instrument, either from experience or by looking at the prices of similar instruments. But he may not be so sure about the risks in the contract or how to hedge them away successfully.

12.2 DISCRETE CASHFLOWS

Imagine a contract that pays the holder an amount q at time t_q. The contract could be a bond and the payment a coupon. If we use $V(t)$ to denote the contract value (ignoring any dependence on any underlying asset) and t_q^- and t_q^+ to denote just before and just

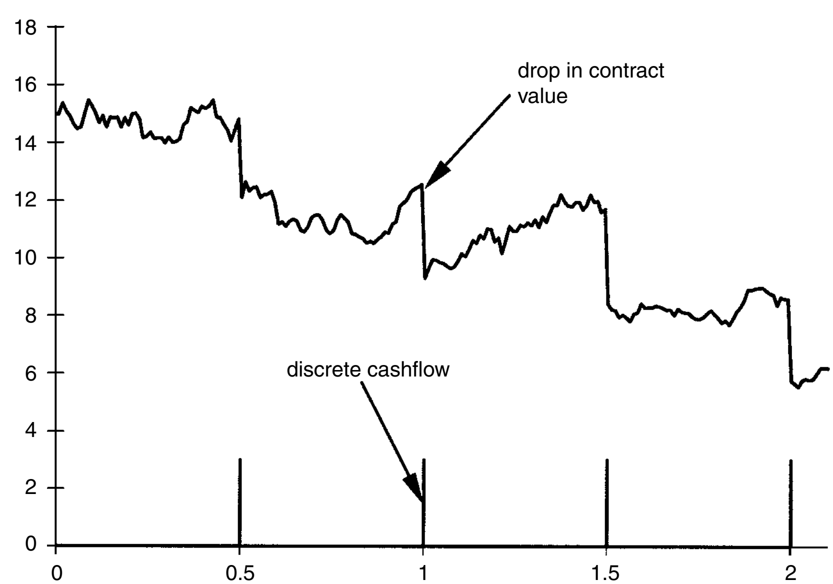


Figure 12.1 A discrete cashflow and its effect on a contract value.

after the cashflow date then simple arbitrage considerations lead to

$$V(t_q^-) = V(t_q^+) + q.$$

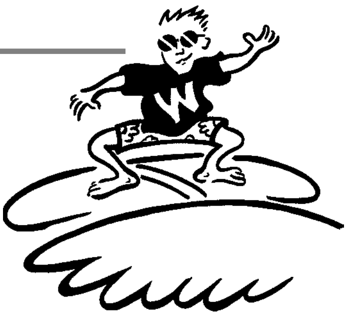
This is a jump condition. The value of the contract jumps by the amount of the cashflow. If this were not the case then there would be an arbitrage opportunity. The behavior of the contract value across the payment date is shown in Figure 12.1.

If the contract is contingent on an underlying variable so that we have $V(S, t)$ then we can accommodate cashflows that depend on the level of the asset S, i.e. we could have $q(S)$. Furthermore, this also allows us to lump all our options on the same underlying into one large portfolio. Then, across the expiry of each option, there will be a jump in the value of our whole portfolio of the amount of the payoff for that option.

Time Out...

In terms of the binomial model?

When you work back down the tree you must add in the relevant cashflow to the option value on those nodes where the payment is received.



12.3 EARLY EXERCISE

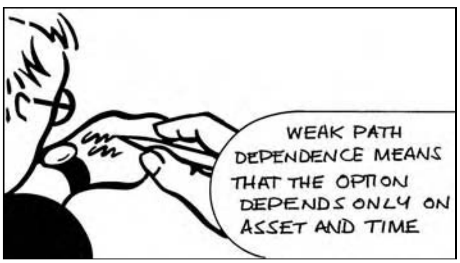
We have seen early exercise in the American option problem. Early exercise is a common feature of other contracts, perhaps going by other names. For example, the conversion of convertible bonds is mathematically identical to the early exercise of an American option. The key point about early exercise is that the holder of this valuable right should ideally act *optimally*, i.e. they must decide *when* to exercise or convert. In the partial differential equation framework that has been set up, this optimality is achieved by solving a free boundary problem, with a constraint on the option value, together with a smoothness condition. It is this smoothness condition, that the derivative of the option value with respect to the underlying is continuous, that ensures optimality, i.e. maximization of the option value with respect to the exercise or conversion strategy. It is perfectly possible for there to be more than one early-exercise region.

One rarely mentioned aspect of American options, and, generally speaking, contracts with early exercise-type characteristics, is that they are path dependent. Whether the owner of the option still holds the option at expiry depends on whether or not he has exercised the option, and thus on the path taken by the underlying. For American-type options this path dependence is weak, in the sense that the partial differential equation to be solved has no more independent variables than a similar, but European, contract.

Time Out...

In terms of the binomial model?

You've seen this already in Chapter 5. Just make sure that the option value stays above the payoff at all times that exercise is allowed.



12.4 WEAK PATH DEPENDENCE

The next most common reason for weak path dependence in a contract is a **barrier**. Barrier (or knock-in, or knock-out) options are triggered by the action of the underlying hitting a prescribed value at some time before expiry. For example, as long as the asset remains below 150, the contract will have a call payoff at expiry. However, should the asset reach this level before expiry then the option becomes worthless; the option has 'knocked out.' This contract is clearly path dependent, for consider the two paths in Figure 12.2; one has a payoff at expiry because the barrier was not triggered, the other is worthless, yet both have the same value of the underlying at expiry.

We shall see in Chapter 13 that such a contract is only weakly path dependent: we still solve a partial differential equation in the two variables, the underlying and time.

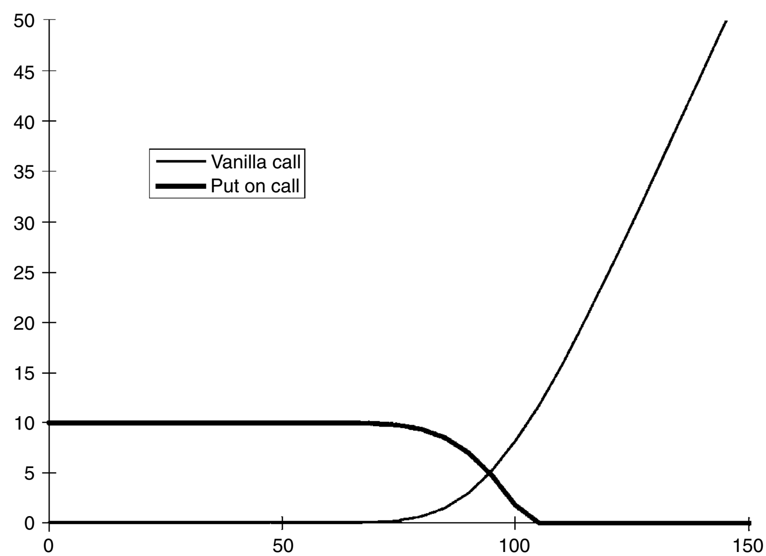


Figure 12.2 Two paths, having the same value at expiry, but with completely different payoffs.

Time Out...

In terms of the binomial model?

This is usually quite straightforward to implement. But not necessarily very accurate though.

12.5 STRONG PATH DEPENDENCE

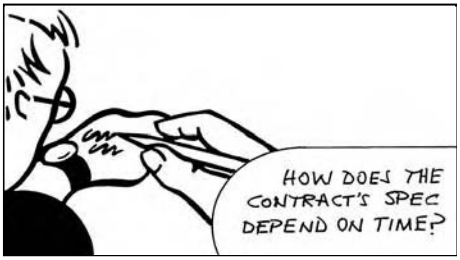
Of particular interest, mathematical and practical, are the strongly path-dependent contracts. These have payoffs that depend on some property of the asset price path in addition to the value of the underlying at the present moment in time; in the equity option language, we cannot write the value as $V(S, t)$. The contract value is a function of at least one more independent variable. This is best illustrated by an example.

The Asian option has a payoff that depends on the average value of the underlying asset from inception to expiry. We must keep track of more information about the asset price path than simply its present position. The extra information that we need is contained in the 'running average.' This is the average of the asset price from inception until the present, when we are valuing the option. No other information is needed. This running average is then used as a new independent variable, the option value is a function of this as well as the usual underlying and time, and a derivative of the option value with respect to the running average appears in the governing equation.

Time Out...

In terms of the binomial model?

This is rather outside the scope of this book. But what you'll have to do is introduce a two-dimensional tree. There will be branches orthogonal to the S branches to keep track of the path-dependent quantity.



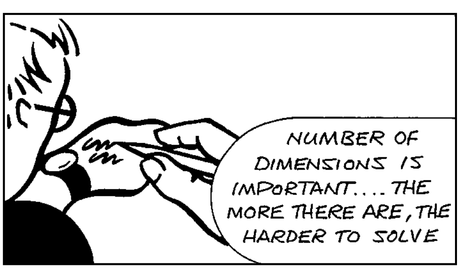
12.6 TIME DEPENDENCE

We have seen time dependence in parameters, and have shown how to apply the Black–Scholes formulas when interest rates, dividends and volatility vary in time (in a known, deterministic, way). Here we are concerned with time dependence in the option contract. We can add such time dependence to any of the features described above. For example, early exercise might only be permitted on certain dates or during certain periods. This intermittent early exercise is a characteristic of **Bermudan options**. Similarly, the position of the barrier in a knock-out option may change with time. Every month it may be reset at a higher level than the month before. Or we can readily imagine a knock-out option in which the barrier is only active (i.e. can be triggered) during the last week of every month. These contracts are referred to as time inhomogeneous.

Time Out...

In terms of the binomial model?

Quantities such as interest rate and volatility will vary with position in the tree.



12.7 DIMENSIONALITY

Dimensionality refers to the number of underlying independent variables. The vanilla option has two independent variables, S and t, and is thus two-dimensional. The weakly path-dependent contracts have the same number of dimensions as their non-path-dependent cousins, i.e. a barrier call option has the same two dimensions as a vanilla call. For these contracts the roles of the asset dimension and the time dimension are quite different from each other, as discussed in Chapter 9 on the diffusion equation. This is because the governing equation, the Black–Scholes equation, contains a second asset-price derivative but only a first time derivative.

We can have two types of three-dimensional problem. The first occurs when we have a second source of randomness, such as a second underlying asset. We might, for example, have an option on the maximum of two equities. Both of these underlyings are stochastic, each with a volatility, and there will be a correlation between them. In the governing equation we will see a second derivative of the option value with respect to each asset. We say that there is diffusion in both S_1 and S_2.

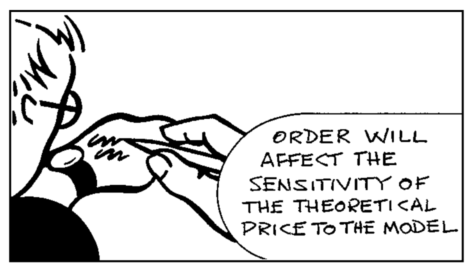
The other type of problem that is also three-dimensional is the strongly path-dependent contract. We will see examples of these in later chapters. Typically, the new independent variable is a measure of the path-dependent quantity on which the option is contingent. The new variable may be the average of the asset price to date, say. In this case, derivatives of the option value with respect to this new variable are only of the first order. Thus the new variable acts more like another time-like variable.

Time Out...

In terms of the binomial model?

As with path dependency you will have to have a higher-dimensional tree structure to model the new dependent variables.

12.8 THE ORDER OF AN OPTION



The next classification that we make is the **order** of an option. Not only is this a classification but the idea also introduces fundamental modeling issues.

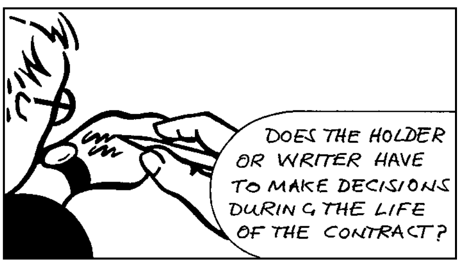
The basic, vanilla options are of first order. Their payoffs depend only on the underlying asset, the quantity that we are *directly* modeling. Other, path-dependent, contracts can still be of first order if the payoff only depends only on properties of the asset price path. ‘Higher order’ refers to options whose payoff, and hence value, is contingent on the value of *another* option. The obvious second-order options are compound options, for example, a call option giving the holder the right to buy a put option. The compound option expires at some date T_1 and the option on which it is contingent, expires at a later time T_2. Technically speaking, such an option is weakly path dependent. The *theoretical* pricing of such a contract is straightforward as we shall see.

From a practical point of view, the compound option raises some important modeling issues: the payoff for the compound option depends on the *market* value of the underlying option, and not on the theoretical price. If you hold a compound option, and want to exercise the first option then you must take possession of the underlying option. If that option is worth less than you think it should (because your model says so) then there is not much you can do about it. High-order option values are very sensitive to the basic pricing model and should be handled with care.

Time Out...

In terms of the binomial model?

You will need one tree structure to model the 'underlying option' and this will feed into the tree for the higher-order option.



12.9 DECISIONS, DECISIONS

Holding an American option you are faced with the decision whether and when to exercise your rights. The American option is the most common contract that contains within it a decision feature. Other contracts require more subtle and interesting decisions to be made. For example, the passport option is an option on a trading account. You buy and sell some asset, if you are in profit on the expiry of the option you keep the money, if you have made a loss it is written off. The decisions to be made here are when to buy, sell or hold, and how much to buy, sell or hold.

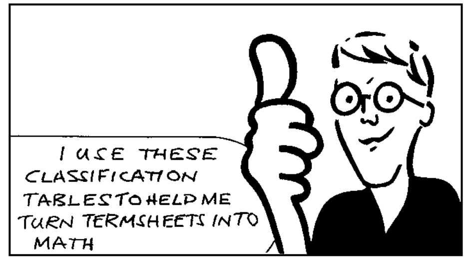
Time Out...

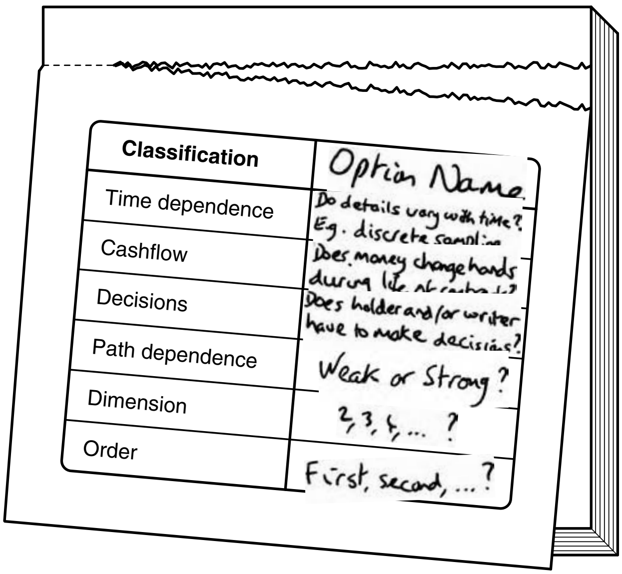
In terms of the binomial model?

Whether this is easy or not depends on the nature of the contract. But as with the American option, decision making requires a process for ensuring optimality. Mathematically this usually boils down to a "$\geq$" sign somewhere.

12.10 CLASSIFICATION TABLES

Watch out for tables like the following for the classification of special contracts.





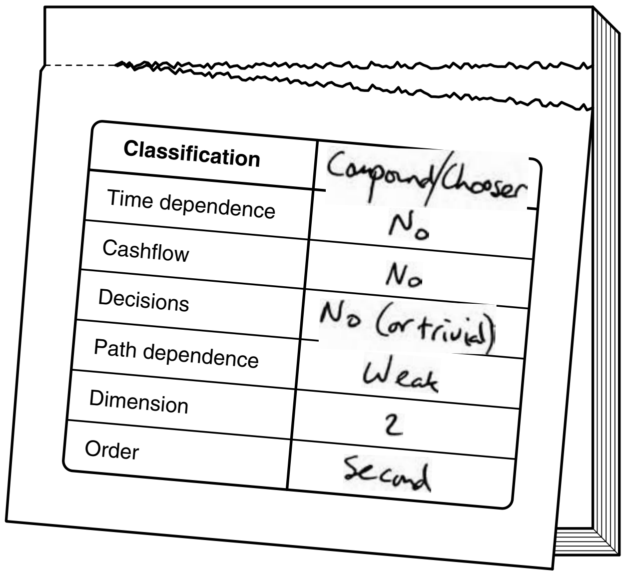
12.11 COMPOUNDS AND CHOOSERS

Compound and **chooser options** are simply options on options. The compound option gives the holder the right to buy (call) or sell (put) another option. Thus we can imagine owning a call on a put, for example. This gives us the right to buy a put option for a specified amount on a specified date. If we exercise the option then we will own a put option which gives us the right to sell the underlying. This compound option is second order because the compound option gives us rights over another derivative. Although the Black–Scholes model can theoretically cope with second-order contracts it is not so clear that the model is completely satisfactory in practice; when we exercise the contract we get an option at the market price, not at the theoretical price.

It is possible to find analytical formulas for the price of basic compound options in the Black–Scholes framework when volatility is constant. These formulas involve the cumulative distribution function for a bivariate Normal variable. However, because of the second-order nature of compound options and thus their sensitivity to the precise nature of the asset price random walk, these formulas are dangerous to use in practice.

Chooser options are similar to compounds in that they give the holder the right to buy a further option. With the chooser option the holder can choose whether to receive a call or a put, for example.

The practical problems with pricing choosers are the same as for compounds.



In Figure 12.3 is shown the values of a vanilla call and a vanilla put some time before expiry. In the same figure is the payoff for a call on the best of these two options (less an exercise price). This is the final condition for the Black–Scholes partial differential equation .

Extendible options are very, very similar to compounds and choosers. At some specified time the holder can choose to accept the payoff for the original option or to extend the option's life and even change the strike. Sometimes it is the writer who has these powers of extension. The reader has sufficient knowledge to be able to model these contracts in the Black–Scholes framework.

Figures 12.4 and 12.5 show the Bloomberg screens for valuing chooser options.

12.12 RANGE NOTES

Range notes are very popular contracts, existing on the 'lognormal' assets such as equities and currencies, and as fixed-income products. In its basic, equity derivative, form the range note pays at a rate of L all the time that the underlying lies within a given range, $S_l \le S \le S_u$. That is, for every dt that the asset is in the range you receive $L\,dt$. Introducing $\mathcal{I}(S)$ as the function taking the value 1 when $S_l \le S \le S_u$ and zero otherwise, the range note satisfies

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV + L\mathcal{I}(S) = 0.$$

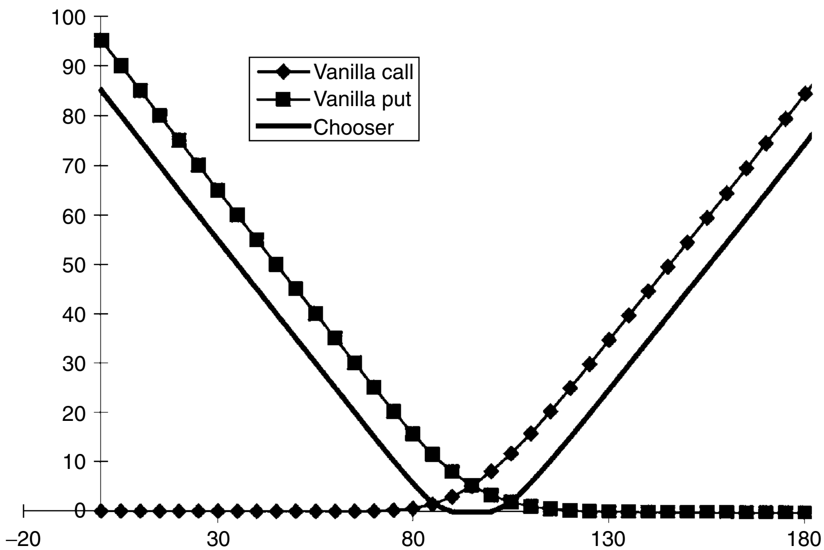


Figure 12.3 The value of a vanilla call option and a vanilla put option some time before expiry and the payoff for the best of these two.

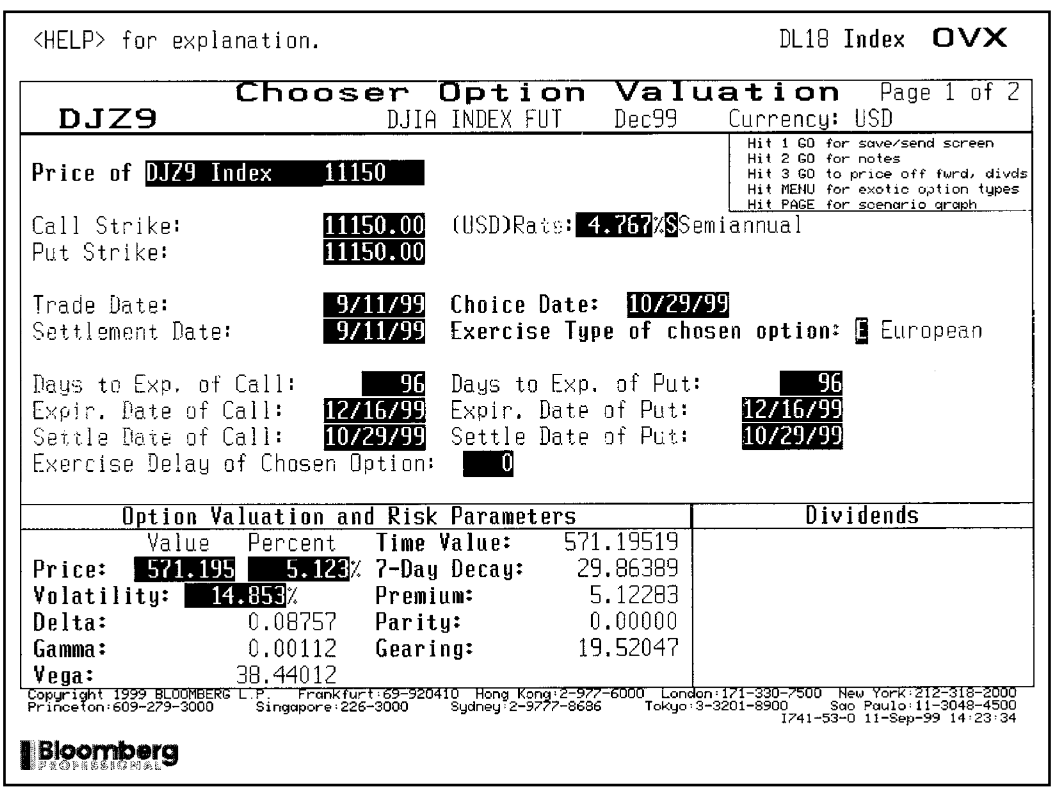
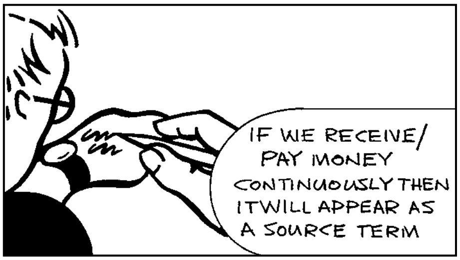
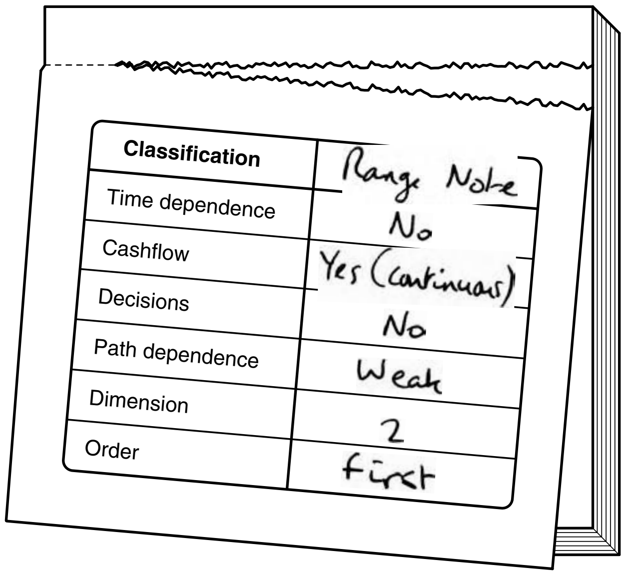


Figure 12.4 Bloomberg chooser option valuation screen. Source: Bloomberg L.P.



In Figure 12.6 is shown the term sheet for a range note on the Mexican peso, US dollar exchange rate. This contract pays out the positive part of the difference between the number of days the exchange rate is inside the range less the number of days outside the range. This payment is received at expiry. Most range notes are weakly path dependent. This one is strongly path dependent. Why?



12.13 BARRIER OPTIONS

Barrier options have a payoff that is contingent on the underlying asset reaching some specified level before expiry. The critical level is called the barrier, there may be more than one. Barrier options are weakly path dependent. Barrier options are discussed in depth in Chapter 13.

Barrier options come in two main varieties, the ‘in’ barrier option (or **knock-in**) and the ‘out’ barrier option (or **knock-out**). The former only have a payoff if the barrier level is reached before expiry and the latter only have a payoff if the barrier is *not* reached before expiry. These contracts are weakly path dependent, meaning that the price depends only on the current level of the asset and the time to expiry. They satisfy the Black–Scholes equation, with special boundary conditions as we shall see.

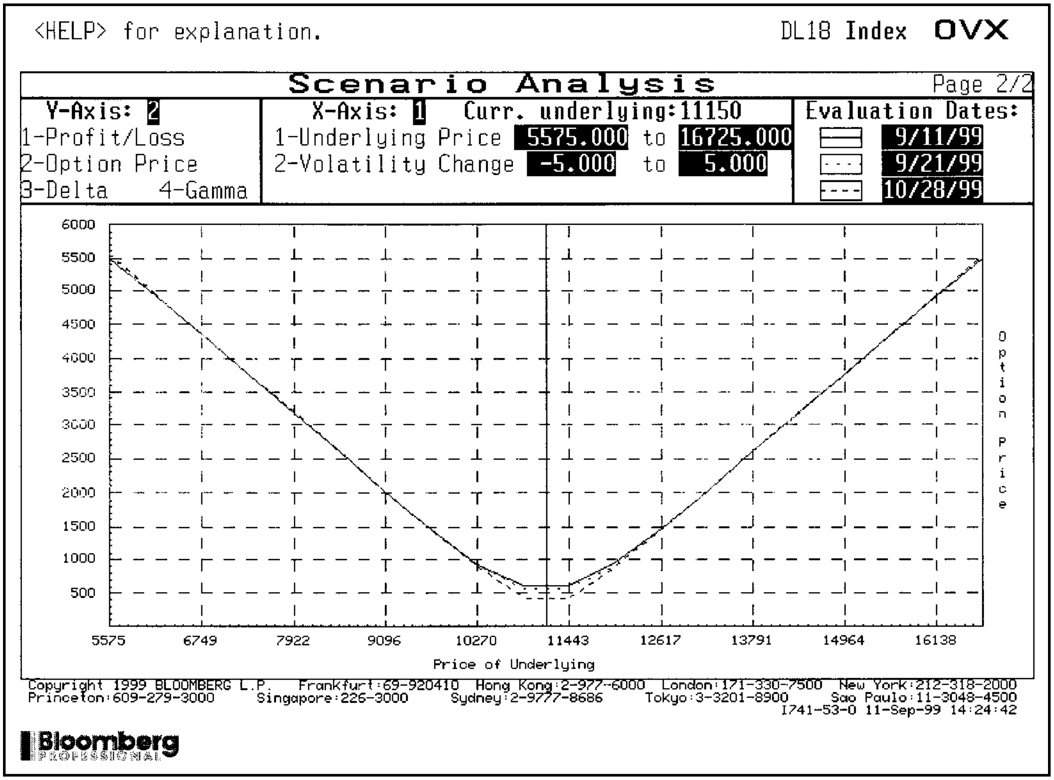
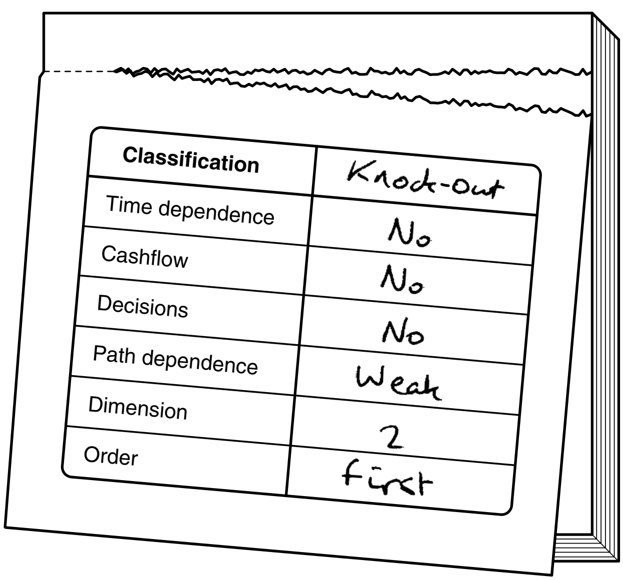
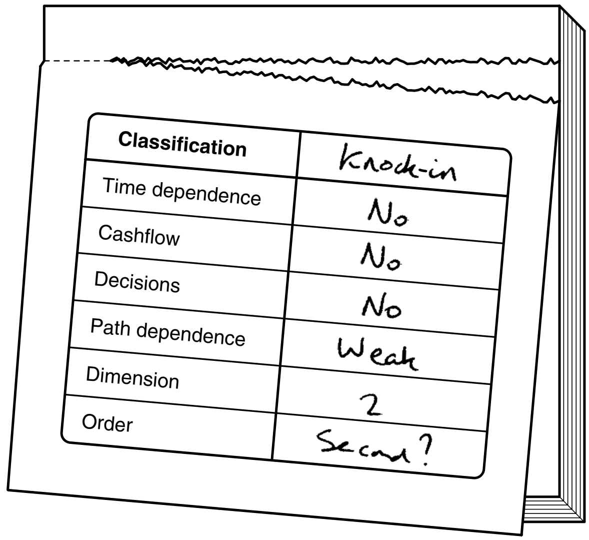


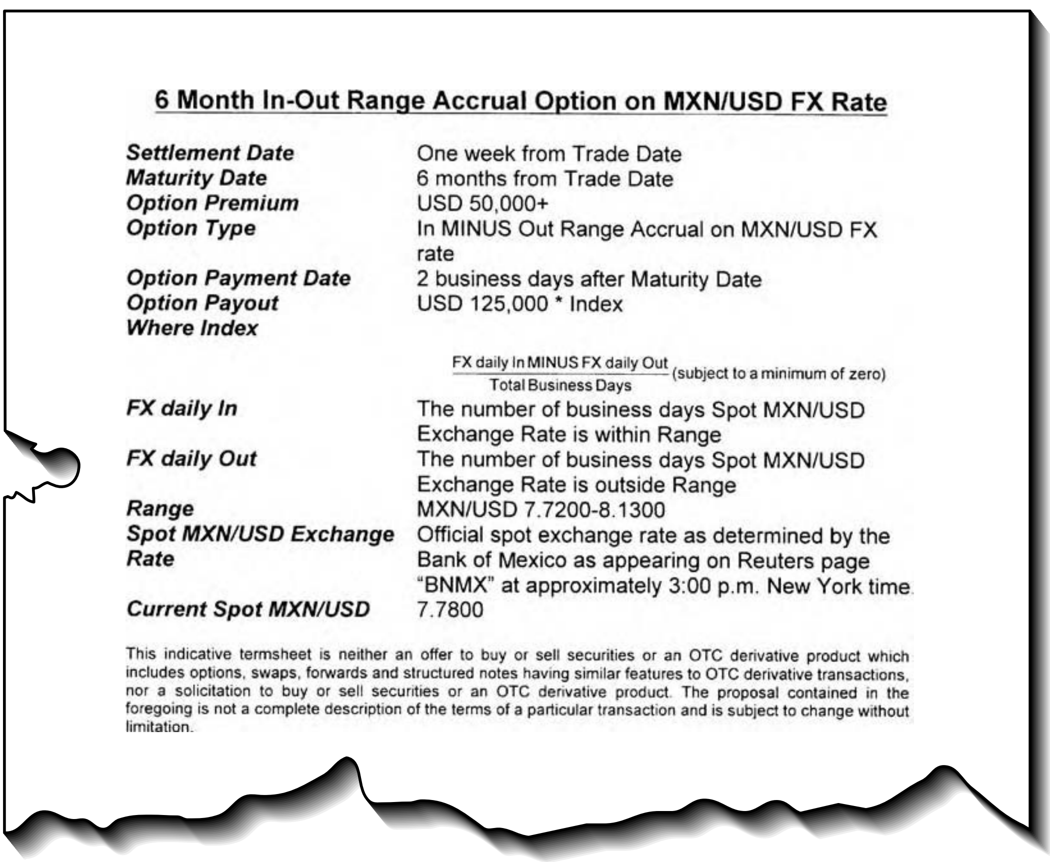
Figure 12.5 Bloomberg scenario analysis for a chooser. Source: Bloomberg L.P.



Classification	Knock-out
Time dependence	No
Cashflow	No
Decisions	No
Path dependence	Weak
Dimension	2
Order	first



Classification	Knock-in
Time dependence	No
Cashflow	No
Decisions	No
Path dependence	Weak
Dimension	2
Order	Second?



6 Month In-Out Range Accrual Option on MXN/USD FX Rate

Settlement Date	One week from Trade Date
Maturity Date	6 months from Trade Date
Option Premium	USD 50,000+
Option Type	In MINUS Out Range Accrual on MXN/USD FX rate
Option Payment Date	2 business days after Maturity Date
Option Payout	USD 125,000 * Index
Where Index	$\frac{\text{FX daily In MINUS FX daily Out}}{\text{Total Business Days}}$ (subject to a minimum of zero)
FX daily In	The number of business days Spot MXN/USD Exchange Rate is within Range
FX daily Out	The number of business days Spot MXN/USD Exchange Rate is outside Range
Range	MXN/USD 7.7200-8.1300
Spot MXN/USD Exchange Rate	Official spot exchange rate as determined by the Bank of Mexico as appearing on Reuters page "BNMX" at approximately 3:00 p.m. New York time.
Current Spot MXN/USD	7.7800

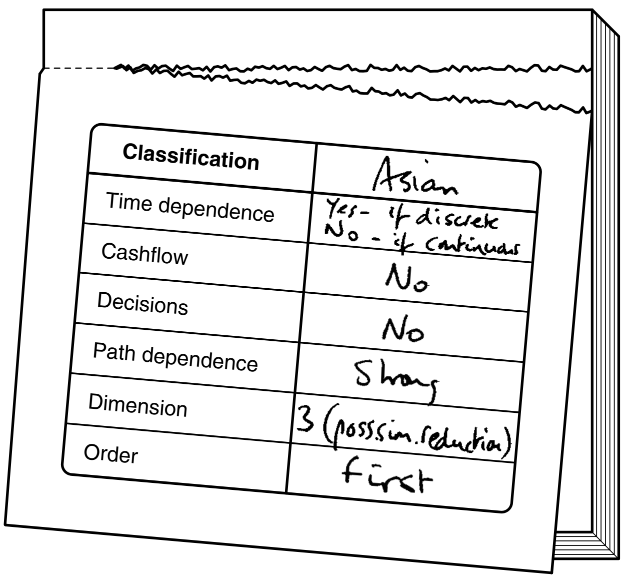
This indicative termsheet is neither an offer to buy or sell securities or an OTC derivative product which includes options, swaps, forwards and structured notes having similar features to OTC derivative transactions, nor a solicitation to buy or sell securities or an OTC derivative product. The proposal contained in the foregoing is not a complete description of the terms of a particular transaction and is subject to change without limitation.

Figure 12.6 Term sheet for an in-out range accrual note on MXN/USD.

12.14 ASIAN OPTIONS

Asian options have a payoff that depends on the average value of the underlying asset over some period before expiry. They are the first strongly path-dependent contract we examine. They are strongly path dependent because their value prior to expiry depends on the path taken and not just on where they have reached. Their value depends on the *average to date* of the asset. This average to date will be very important to us, we introduce something like it as a new state variable. We shall see how to derive a partial differential equation for the value of this Asian contract, but now the differential equation will have *three* independent variables.

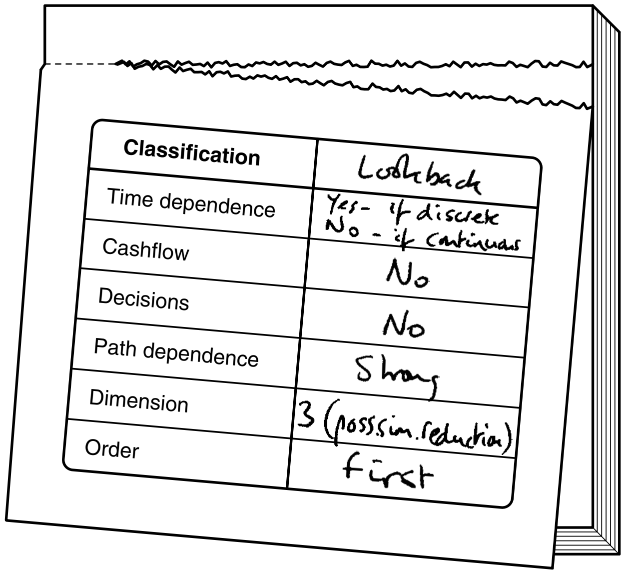
The average used in the calculation of the option's payoff can be defined in many different ways. It can be an arithmetic average or a geometric average, for example. The data could be continuously sampled, so that every realized asset price over the given period is used. More commonly, for practical and legal reasons, the data is usually sampled discretely; the calculated average may only use every Friday's closing price, for example.



12.15 LOOKBACK OPTIONS

Lookback options have a payoff that depends on the realized maximum or minimum of the underlying asset over some period prior to expiry. An extreme example, that captures the flavor of these contracts is the option that pays off the difference between that maximum realized value of the asset and the minimum value over the next year. Thus it enables the holder to buy at the lowest price and sell at the highest, every trader's dream. Of course, this payoff comes at a price. And for such a contract that price would be very high.

Again the maximum or minimum can be calculated continuously or discretely, using every realized asset price or just a subset. In practice the maximum or minimum is measured discretely.



12.16 SUMMARY

This chapter suggests ways to think about derivative contracts that make their analysis simpler. To be able to make comparisons between different contracts is a big step forward in understanding them.

FURTHER READING

- Geske (1979) discusses the valuation of compound options.
- See Taleb (1997) for more details of classifications of the type I have described. This book is an excellent, and entertaining read.
- The book by Zhang (1997) is a discussion of many types of exotic options, with many formulas.

SOME FORMULAS FOR ASIAN OPTIONS

There are very few nice formulas for the values of Asian options. The most well known are for average rate calls and puts when the average is a continuously as sampled, geometrical average.

The geometric average rate call This option has payoff

$$\max(A - E, 0),$$

where A is the continuously sampled geometric average. This option has a Black–Scholes value of

$$e^{-r(T-t)}\left(G\exp\left(\frac{(r-D-\sigma^2/2)(T-t)^2}{2T}+\frac{\sigma^2(T-t)^3}{6T^2}\right)N(d_1)-EN(d_2)\right)$$

where

$$I=\int_0^t \log(S(\tau))\,d\tau,$$

$$G=e^{I/T}S^{(T-t)/T},$$

$$d_1=\frac{T\log(G/E)+(r-D-\sigma^2/2)(T-t)^2/2+\sigma^2(T-t)^3/3T}{\sigma\sqrt{(T-t)^3/3}}$$

and

$$d_2=\frac{T\log(G/E)+(r-D-\sigma^2/2)(T-t)^2/2}{\sigma\sqrt{(T-t)^3/3}}.$$

The geometric average of a lognormal random walk is itself lognormally distributed, but with a reduced volatility.

The geometric average rate put This option has payoff

$$\max(E - A, 0),$$

where A is the continuously sampled geometric average. This option has a Black–Scholes value of

$$e^{-r(T-t)}\left(EN(-d_2)-G\exp\left(\frac{(r-D-\sigma^2/2)(T-t)^2}{2T}+\frac{\sigma^2(T-t)^3}{6T^2}\right)N(d_1)\right).$$

SOME FORMULAS FOR LOOKBACK OPTIONS

Floating strike lookback call The continuously sampled version of this option has a payoff

$$\max(S - M, 0) = S - M,$$

where M is the realized minimum of the asset price. In the Black–Scholes world the value is

$$Se^{-D(T-t)}N(d_1)-Me^{-r(T-t)}N(d_2)$$
$$+Se^{-r(T-t)}\frac{\sigma^2}{2(r-D)}\left(\left(\frac{S}{M}\right)^{-\frac{2(r-D)}{\sigma^2}}N\left(-d_1+\frac{2(r-D)\sqrt{T-t}}{\sigma}\right)-e^{(r-D)(T-t)}N(-d_1)\right),$$

where

$$d_1 = \frac{\log(S/M) + \left(r - D + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

Floating strike lookback put The continuously sampled version of this option has a payoff

$$\max(M - S, 0) = M - S,$$

where M is the realized maximum of the asset price. The value is

$$Me^{-r(T-t)}N(-d_2) - Se^{-D(T-t)}N(-d_1)$$
$$+ Se^{-r(T-t)}\frac{\sigma^2}{2(r-D)}\left(-\left(\frac{S}{M}\right)^{-\frac{2(r-D)}{\sigma^2}} N\left(d_1 - \frac{2(r-D)\sqrt{T-t}}{\sigma}\right) + e^{(r-D)(T-t)}N(d_1)\right),$$

where

$$d_1 = \frac{\log(S/M) + \left(r - D + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

Fixed strike lookback call This option has a payoff given by

$$\max(M - E, 0)$$

where M is the realized maximum. For $E > M$ the fair value is

$$Se^{-D(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2)$$
$$+ Se^{-r(T-t)}\frac{\sigma^2}{2(r-D)}\left(-\left(\frac{S}{E}\right)^{-\frac{2(r-D)}{\sigma^2}} N\left(d_1 - \frac{2(r-D)\sqrt{T-t}}{\sigma}\right) + e^{(r-D)(T-t)}N(d_1)\right),$$

where

$$d_1 = \frac{\log(S/E) + \left(r - D + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

When $E < M$ the value is

$$(M - E)e^{-r(T-t)} + Se^{-D(T-t)}N(d_1) - Me^{-r(T-t)}N(d_2)$$
$$+ Se^{-r(T-t)}\frac{\sigma^2}{2(r-D)}\left(-\left(\frac{S}{M}\right)^{-\frac{2(r-D)}{\sigma^2}} N\left(d_1 - \frac{2(r-D)\sqrt{T-t}}{\sigma}\right) + e^{(r-D)(T-t)}N(d_1)\right),$$

where

$$d_1 = \frac{\log(S/M) + \left(r - D + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

Fixed strike lookback put This option has a payoff given by

$$\max(E - M, 0)$$

where M is the realized minimum. For $E < M$ the fair value is

$$Ee^{-r(T-t)}N(-d_2) - Se^{-D(T-t)}N(-d_1) + Se^{-r(T-t)}\frac{\sigma^2}{2(r-D)}\left(\left(\frac{S}{E}\right)^{-\frac{2(r-D)}{\sigma^2}} N\left(-d_1 + \frac{2(r-D)\sqrt{T-t}}{\sigma}\right) - e^{(r-D)(T-t)}N(-d_1)\right),$$

where

$$d_1 = \frac{\log(S/E) + \left(r - D + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

When $E > M$ the value is

$$(E-M)e^{-r(T-t)} - Se^{-D(T-t)}N(-d_1) + Me^{-r(T-t)}N(-d_2) + Se^{-r(T-t)}\frac{\sigma^2}{2(r-D)}\left(\left(\frac{S}{M}\right)^{-\frac{2(r-D)}{\sigma^2}} N\left(-d_1 + \frac{2(r-D)\sqrt{T-t}}{\sigma}\right) - e^{(r-D)(T-t)}N(-d_1)\right),$$

where

$$d_1 = \frac{\log(S/M) + \left(r - D + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

CHAPTER 13

barrier options

The aim of this Chapter...

... is to describe and classify barrier options, to show how they can easily be put into a partial differential equation framework for later solution by numerical methods. Such a framework is ideal for pricing barrier options with complex modern volatility models.

In this Chapter...

- the different types of barrier options
- how to price many of them in the partial differential equation framework
- some of the practical problems with the pricing and hedging of barriers

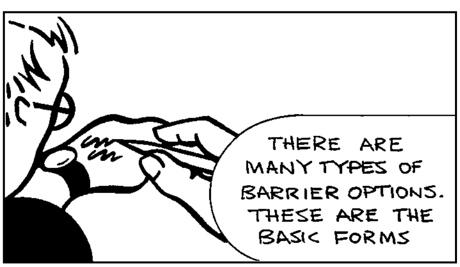
13.1 INTRODUCTION

I mentioned barrier options briefly in the previous chapter. In this chapter we study them in detail, from both a theoretical and a practical perspective. **Barrier options** are path-dependent options. They have a payoff that is dependent on the realized asset path via its level; certain aspects of the contract are triggered if the asset price becomes too high or too low. For example, an up-and-out call option pays off the usual $\max(S - E, 0)$ at expiry unless at any time previously the underlying asset has traded at a value S_u or higher. In this example, if the asset reaches this level (from below, obviously) then it is said to 'knock out,' becoming worthless. Apart from 'out' options like this, there are also 'in' options which only receive a payoff if a level is reached, otherwise they expire worthless.

Barrier options are popular for a number of reasons. Perhaps the purchaser uses them to hedge very specific cashflows with similar properties. Usually, the purchaser has very precise views about the direction of the market. If he wants the payoff from a call option but does not want to pay for all the upside potential, believing that the upward movement of the underlying will be limited prior to expiry, then he may choose to buy an up-and-out call. It will be cheaper than a similar vanilla call, since the upside is severely limited. If he is right and the barrier is not triggered he gets the payoff he wanted. The closer that the barrier is to the current asset price then the greater the likelihood of the option being knocked out, and thus the cheaper the contract.

Conversely, an 'in' option will be bought by someone who believes that the barrier level will be realized. Again, the option is cheaper than the equivalent vanilla option.

13.2 DIFFERENT TYPES OF BARRIER OPTIONS



There are two main types of barrier option:

- The **out option**, that only pays off if a level is *not* reached. If the barrier is reached then the option is said to have **knocked out**.
- The **in option**, that pays off as long as a level is reached before expiry. If the barrier is reached then the option is said to have **knocked in**.

Then we further characterize the barrier option by the position of the barrier relative to the initial value of the underlying:

- If the barrier is above the initial asset value, we have an **up** option.
- If the barrier is below the initial asset value, we have a **down** option.

Finally, we describe the payoff received at expiry:

- The payoffs are all the usual suspects, call, put, binary, etc.

The above classifies the commonest barrier options. In all of these contracts the position of the barrier could be time dependent. The level may begin at one level and then rise, say. Usually the level is a piecewise-constant function of time.

USD/MXN Double Knock-Out Note

Principal Amount	USD 10,000,000
Issuer	XXXX
Maturity	6 months from Trade Date
Issue Price	100%
Coupon	If the USD/MXN spot exchange rate trades above the Upper Barrier or below the Lower Barrier at any time during the term of the Note: Zero Otherwise: $400\% \times \max\left(0, \frac{8.2500 - FX}{FX}\right)$ where FX is the USD/MXN spot exchange rate at Maturity
Redemption Amount	100%
Upper Barrier Level	8.2500
Lower Barrier Level	7.4500

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Figure 13.1 Term sheet for a USD/MXN double knock-out note.

Another style of barrier option is the **double barrier**. Here there is both an upper and a lower barrier, the first above and the second below the current asset price. In a double 'out' option the contract becomes worthless if *either* of the barriers is reached. In a double 'in' option one of the barriers must be reached before expiry, otherwise the option expires worthless. Other possibilities can be imagined, one barrier is an 'in' and the other an 'out,' at expiry the contract could have either an 'in' or an 'out' payoff.

Sometimes a **rebate** is paid if the barrier level is reached. This is often the case for 'out' barriers in which case the rebate can be thought of as cushioning the blow of losing the rest of the payoff. The rebate may be paid as soon as the barrier is triggered or not until expiry.

In Figure 13.1 is shown the term sheet for a double knock-out option on the Mexican peso, US dollar exchange rate. The upper barrier is set at 8.25 and the lower barrier at 7.45. If the exchange rate trades inside this range until expiry then there is a payment. This is a very vanilla example of a barrier contract.

13.3 PRICING BARRIERS IN THE PARTIAL DIFFERENTIAL EQUATION FRAMEWORK

Barrier options are path dependent. Their payoff, and therefore value, depends on the path taken by the asset up to expiry. Yet that dependence is weak. We only have to know whether or not the barrier has been triggered, we do not need any other information about the path. This is in contrast to some of the contracts we will be seeing shortly, such as the Asian option, that are strongly path dependent. I use $V(S,t)$ to denote the value of the barrier contract *before the barrier has been triggered*. This value still satisfies the Black–Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

The details of the barrier feature come in through the specification of the boundary conditions.

Time Out...

The framework

Partial differential equations are the natural framework for pricing barrier options. When combined with the kind of numerical method escribed in Chapter 25 the solution is relatively straightforward. Indeed, in many cases it is easier to solve a barrier option numerically than a vanilla option.

Once you are working in this framework it is very easy to incorporate modern volatility models. As hinted at later in this chapter, barrier options can be very sensitive to the assumptions made about volatility and very rarely are such options priced using the assumption of constant vol. In such a situation it's nice to have a framework that can be easily adapted to sophisticated volatility models. This is that framework.

However, in the following I will also give a few pointers to valuing via trees.

13.3.1 'Out' barriers

If the underlying asset reaches the barrier in an 'out' barrier option then the contract becomes worthless. This leads to the boundary condition

$$V(S_u, t) = 0 \quad \text{for } t < T,$$

for an up-barrier option with the barrier level at $S = S_u$. We must solve the Black–Scholes equation for $0 \le S \le S_u$ with this condition on $S = S_u$ and a final condition corresponding to the payoff received if the barrier is not triggered. For a call option we would have

$$V(S, T) = \max(S - E, 0).$$

If we have a down-and-out option with a barrier at S_d then we solve for $S_d \le S < \infty$ with

$$V(S_d, t) = 0,$$

and the relevant final condition at expiry.

The boundary conditions are easily changed to accommodate rebates. If a rebate of R is paid when the barrier is hit then

$$V(S_d, t) = R.$$

Time Out...

In terms of the binomial model?

To price out barrier options in the binomial framework you have to specify on which nodes the option becomes valueless. The option value is then set to zero on these nodes, and the usual algorithm is used for all other nodes. (Of course, there is no need to find option values beyond the barrier, so the tree is usually smaller than for a vanilla option.)

13.3.2 'In' barriers

An 'in' option only has a payoff if the barrier is triggered. If the barrier is not triggered then the option expires worthless

$$V(S, T) = 0.$$

The value in the option is in the potential to hit the barrier. If the option is an up-and-in contract then on the upper barrier the contract must have the same value as a vanilla contract:

$$V(S_u, t) = \text{value of vanilla contract, a function of } t.$$

Using the notation $V_v(S, t)$ for value of the equivalent vanilla contract (a vanilla call, if we have an up-and-in call option) then we must have

$$V(S_u, t) = V_v(S_u, t) \quad \text{for } t < T.$$

A similar boundary condition holds for a down-and-in option.

The contract we receive when the barrier is triggered is a derivative itself, and therefore the 'in' option is a second-order contract.

In solving for the value of an 'in' option completely numerically we must solve for the value of the vanilla option first, before solving for the value of the barrier option. The solution therefore takes roughly twice as long as the solution of the 'out' option.[1]

When volatility is constant we can solve for the theoretical price of many types of barrier contract. Some examples are given at the end of the chapter.

[1] And, of course, the vanilla option must be solved for $0 \leq S < \infty$.

Time Out...

In terms of the binomial model?

The in barrier option can be thought of as a second-order contract. So you would need one tree for pricing the underlying option and the results of this will be passed to the in barrier tree. To make things simple, make sure that the two trees have the same structure.

13.4 EXAMPLES

Down-and-out call option As the first example, consider the down-and-out call option with barrier level S_d below the strike price E. The value of this option is shown as a function of S in Figure 13.2.

Down-and-in call option In the absence of any rebates the relationship between an 'in' barrier option and an 'out' barrier option (with same payoff and same barrier level) is very simple:

$$\text{in} + \text{out} = \text{vanilla}.$$

If the 'in' barrier is triggered then so is the 'out' barrier, so whether or not the barrier is triggered we still get the vanilla payoff at expiry.

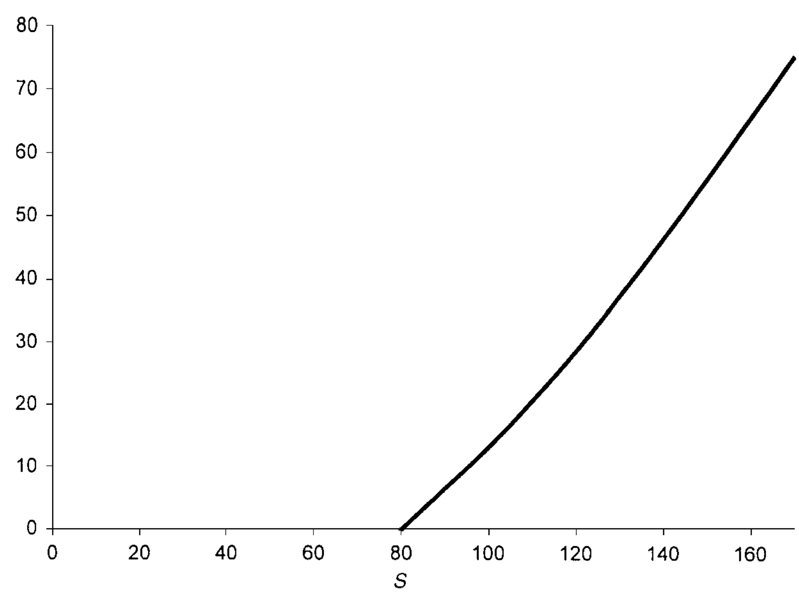


Figure 13.2 Value of a down-and-out call option.

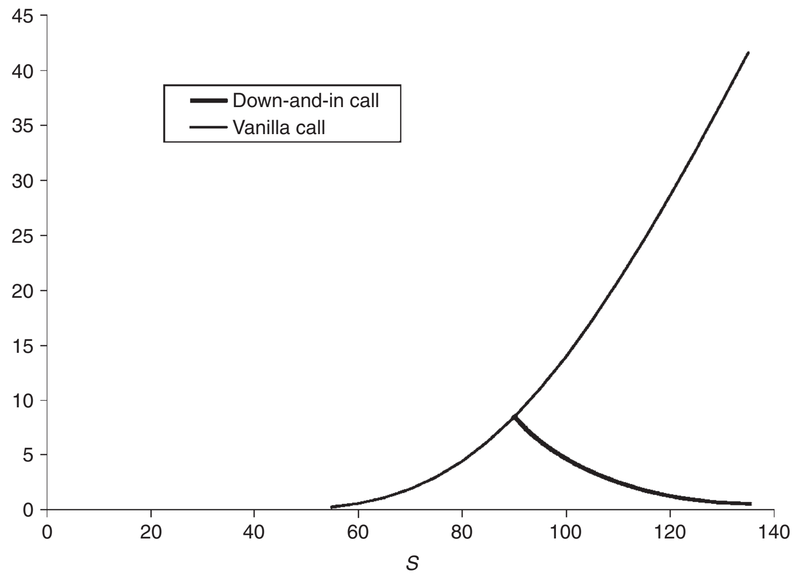


Figure 13.3 Value of a down-and-in call option.

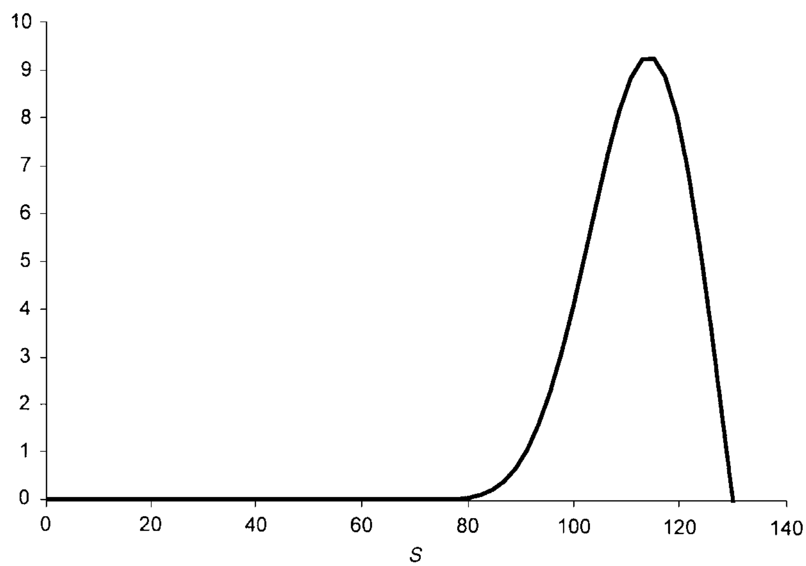


Figure 13.4 Value of an up-and-out call option.

The value of this option is shown as a function of S in Figure 13.3. Also shown is the value of the vanilla call. Note that the two values coincide at the barrier.

Up-and-out call option The barrier S_u for an up-and-out call option must be above the strike price E (otherwise the option would be valueless).

The value of this option is shown as a function of S in Figure 13.4. In Figure 13.5 is shown the delta.

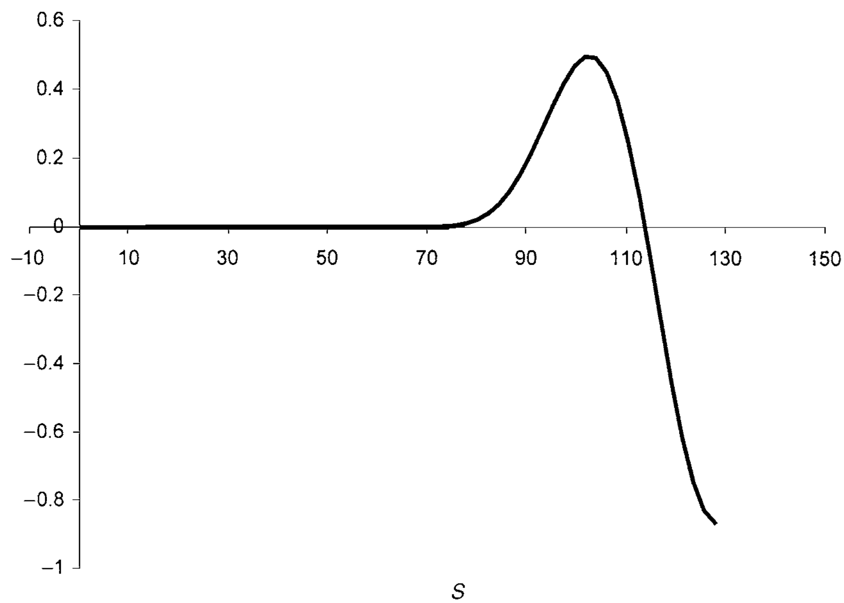
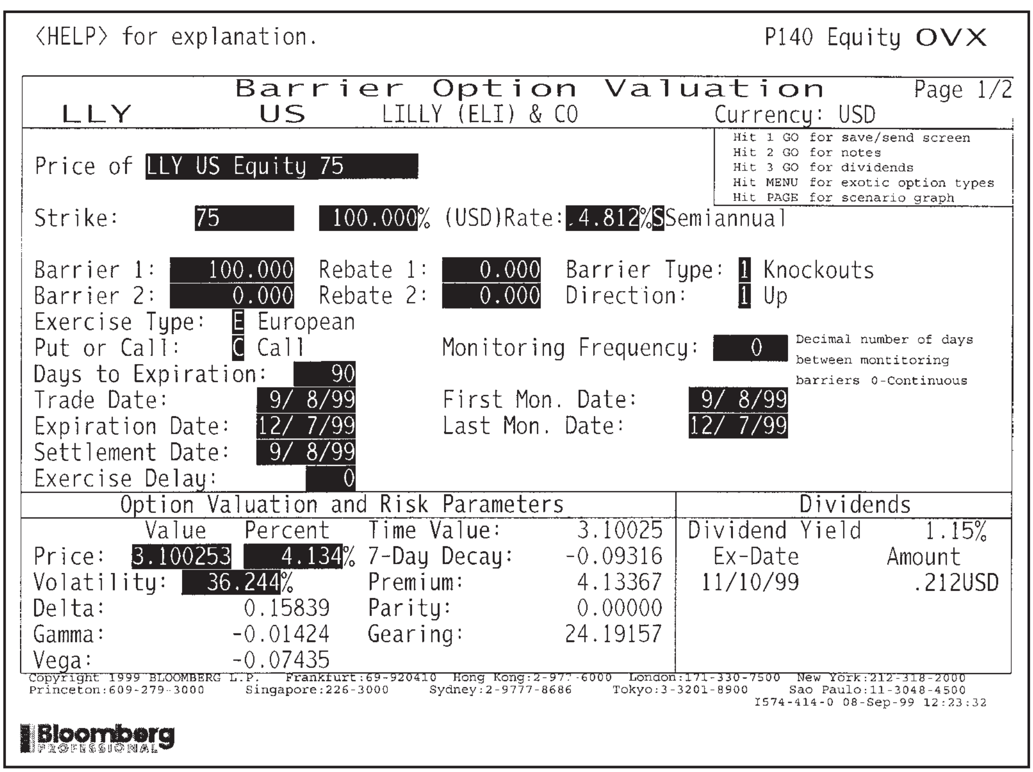


Figure 13.5 Delta of an up-and-out call option.



<HELP> for explanation. P140 Equity OVX

Barrier Option Valuation Page 1/2

LLY US LILLY (ELI) & CO Currency: USD

Hit 1 GO for save/send screen
Hit 2 GO for notes
Hit 3 GO for dividends
Hit MENU for exotic option types
Hit PAGE for scenario graph

Price of LLY US Equity 75

Strike: 75 100.000% (USD)Rate: .4.812% S Semiannual

Barrier 1: 100.000 Rebate 1: 0.000 Barrier Type: 1 Knockouts
Barrier 2: 0.000 Rebate 2: 0.000 Direction: 1 Up
Exercise Type: E European
Put or Call: C Call Monitoring Frequency: 0 Decimal number of days between montitoring barriers 0-Continuous
Days to Expiration: 90
Trade Date: 9/ 8/99 First Mon. Date: 9/ 8/99
Expiration Date: 12/ 7/99 Last Mon. Date: 12/ 7/99
Settlement Date: 9/ 8/99
Exercise Delay: 0

Option Valuation and Risk Parameters

	Value	Percent		
Price:	3.100253	4.134%	Time Value:	3.10025
Volatility:	36.244%		7-Day Decay:	-0.09316
Delta:	0.15839		Premium:	4.13367
Gamma:	-0.01424		Parity:	0.00000
Vega:	-0.07435		Gearing:	24.19157

Dividends

Dividend Yield 1.15%

Ex-Date	Amount
11/10/99	.212USD

Copyright 1999 BLOOMBERG L.P. Frankfurt:69-920410 Hong Kong:2-977-6000 London:171-330-7500 New York:212-318-2000
Princeton:609-279-3000 Singapore:226-3000 Sydney:2-9777-8686 Tokyo:3-3201-8900 Sao Paulo:11-3048-4500
I574-414-0 08-Sep-99 12:23:32

Bloomberg PROFESSIONAL

Figure 13.6 An up-and-out call again. Source: Bloomberg L.P.

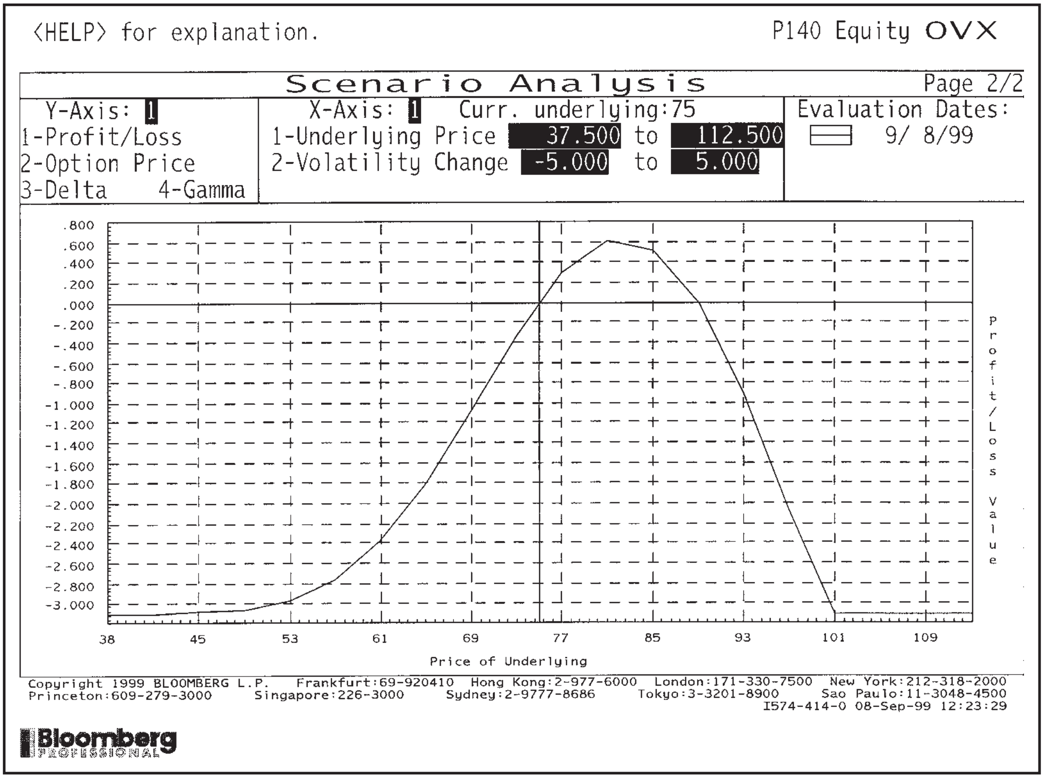


Figure 13.7 Profit/loss for an up-and-out call. Source: Bloomberg L.P.

Figure 13.6 shows the Bloomberg barrier option calculator and Figure 13.7 shows the option *profit/loss* against asset price.

13.4.1 Some more examples

Figures 13.8 through 13.13 are all taken from Bloomberg, who use the formulas below, for the pricing.

13.5 OTHER FEATURES IN BARRIER-STYLE OPTIONS

Not so long ago barrier options were exotic, the market for them was small and few people were comfortable pricing them. Nowadays they are heavily traded and it is only the contracts with more unusual features that can rightly be called exotic. Some of these features are described below.

13.5.1 Early exercise

It is possible to have American-style early exercise. The contract must specify what the payoff is if the contract is exercised before expiry. As always, early exercise is a simple constraint on the value of the option.

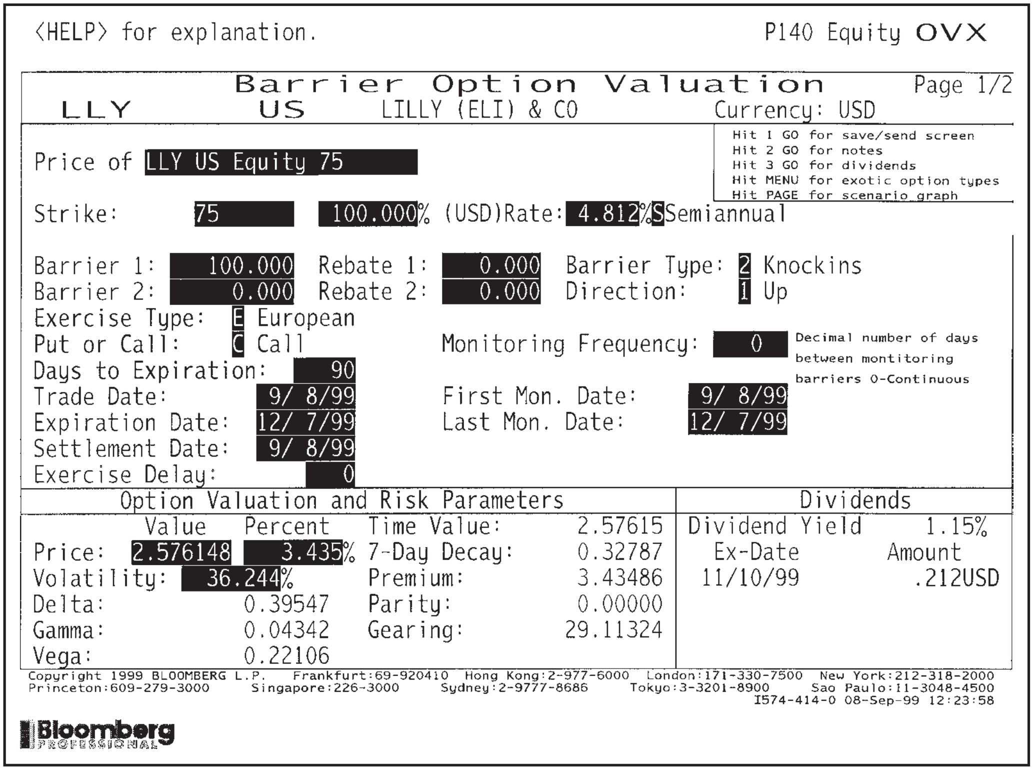


Figure 13.8 Calculator for an up-and-in call. Source: Bloomberg L.P.

In Figure 13.14 is the term sheet for a knock-out installment premium option on the US dollar, Japanese yen exchange rate. This knocks out if the exchange rate ever goes above 140. If the option expires without ever hitting this level there is a vanilla call payoff. I mention this contract in the section on early exercise because it has a similar feature. To keep the contract alive the holder must pay in installments, every month another payment is due. The question is when to stop paying the installments? This can be done optimally.

13.5.2 Repeated hitting of the barrier

The double barrier that we have seen above can be made more complicated. Instead of only requiring one hit of either barrier we could insist that *both* barriers are hit before the barrier is triggered.

This contract is easy to value. Observe that the first time that one of the barriers is hit the contract becomes a vanilla barrier option. Thus on the two barriers we solve the Black–Scholes equation with boundary conditions that our double-barrier value is equal to an up-barrier option on the lower barrier and a down-barrier option on the upper barrier.

13.5.3 Resetting of barrier

Another type of barrier contract that can be priced by the same two-step (or more-step) procedure as ‘in’ barriers is the reset barrier. When the barrier is hit the contract turns into

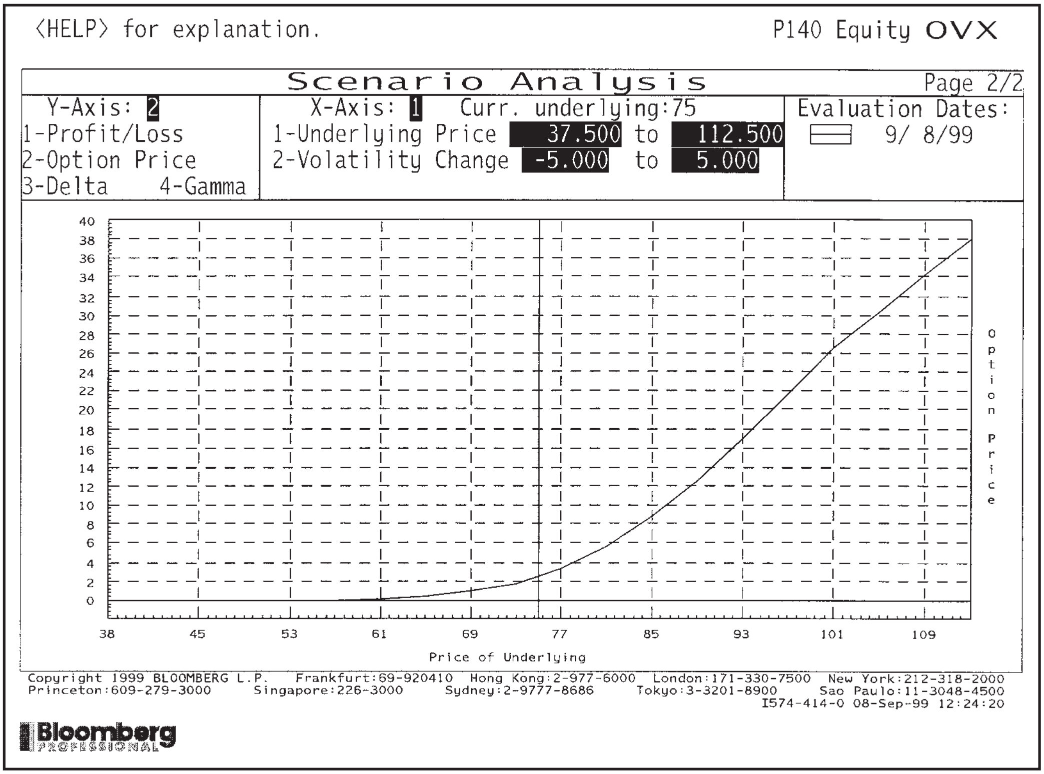


Figure 13.9 Value of an up-and-in call. Source: Bloomberg L.P.

another barrier option with a different barrier level. The contract may be time dependent in the sense that if the barrier is hit before a certain time we get a new barrier option, if it is hit after a certain time we get the vanilla payoff.

Related to these contracts are the **roll-up** and **roll-down options**. These begin life as vanilla options, but if the asset reaches some predefined level they become a barrier option. For example, with a roll-up put if the roll-up strike level is reached the contract becomes an up-and-out put with the roll-up strike being the strike of the barrier put. The barrier level will then be at a prespecified level.

13.5.4 Outside barrier options

Outside or **rainbow barrier options** have payoffs or a trigger feature that depends on a second underlying. Thus the barrier might be triggered by one asset, with the payoff depending on the other. These products are clearly multifactor contracts.

13.5.5 Soft barriers

The **soft barrier option** allows the contract to be gradually knocked in or out. The contract specifies two levels, an upper and a lower. In the knock-out option a proportion of the contract is knocked out depending on the distance that the asset has reached between

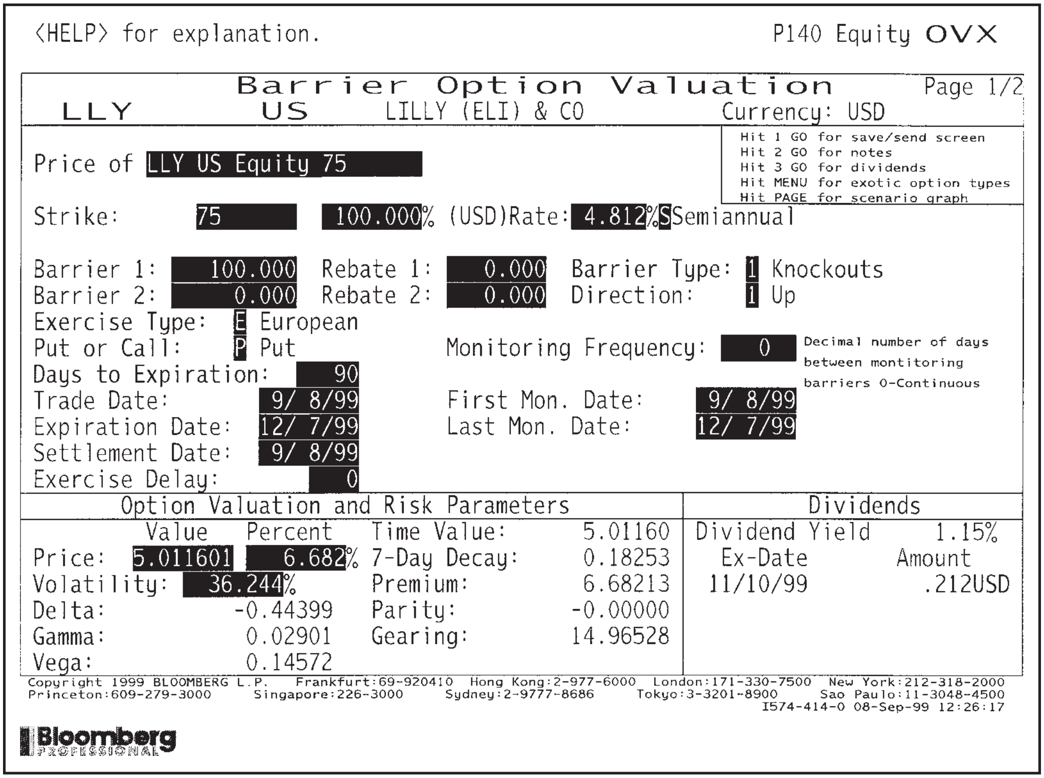
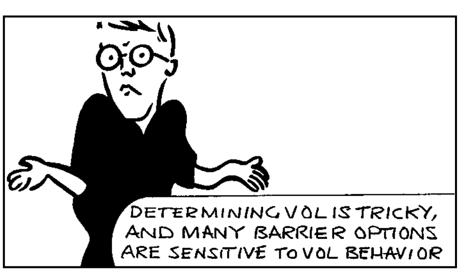


Figure 13.10 Calculator for an up-and-out put. Source: Bloomberg L.P.

the two barriers. For example, suppose that the option is an up and out with a soft barrier range of 100 to 120. If the maximum asset value reached before expiry is 105 then 5/20 or 25% of the payoff is lost.

13.5.6 Parisian options

Parisian options have barriers that are triggered only if the underlying has been beyond the barrier level for more than a specified time. This additional feature reduces the possibility of manipulation of the trigger event and makes the dynamic hedging easier. However, this new feature also increases the dimensionality of the problem.



13.6 MARKET PRACTICE: WHAT VOLATILITY SHOULD I USE?

Practitioners do not price contracts using a single, constant volatility. Let us see some of the pitfalls with this, and then see what practitioners do.

In Figure 13.15 we see a plot of the value of an up-and-out call option using three different volatilities, 15%,

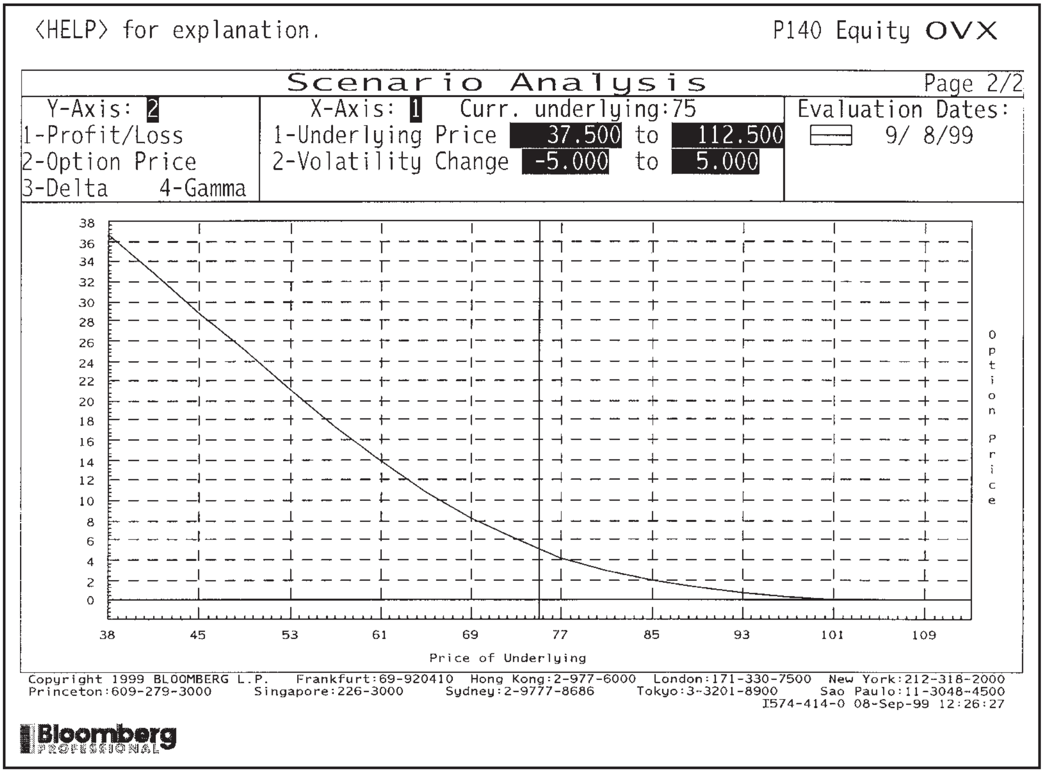
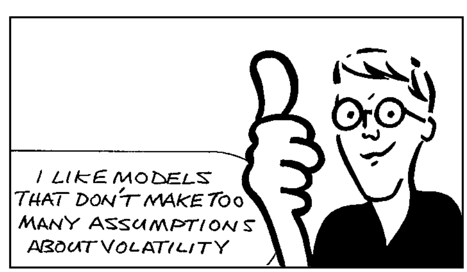


Figure 13.11 Value of an up-and-out put. Source: Bloomberg L.P.

20% and 25%. I have chosen three very different values to make a point. If we are unsure about the value of the volatility (as we surely are) then which value do we use to price the contract? Observe that at approximately $S = 100$ the option value seems to be insensitive to the volatility, the vega is zero. If S is greater than this value perhaps we should only sell the contract for a volatility of 15% to be on the safe side. If S is less than this, perhaps we should sell the contract for 25%, again to play it safe. Now ask the question, Do I believe that volatility will be one of 15%, 20% or 25%, and will it be fixed at that level? Or do I believe that volatility could move around between 15% and 25%? Clearly the latter is closer to the truth. But the measurement of vega, and the plots in Figure 13.15 assume that volatility is fixed until expiry. If we are concerned with playing it safe we should assume that the behavior of volatility will be that which gives us the lowest value if we are buying the contract. The worst outcome for volatility is for it to be low around the strike price, and high around the barrier. Financially, this means that if we are near the strike we get a small payoff, but if we are near the barrier we are likely to hit it. Mathematically, the 'worst' choice of volatility path depends on the sign of the gamma at each point. If gamma is positive then low volatility is bad, if gamma is negative then high volatility is bad. When the gamma is not single-signed,



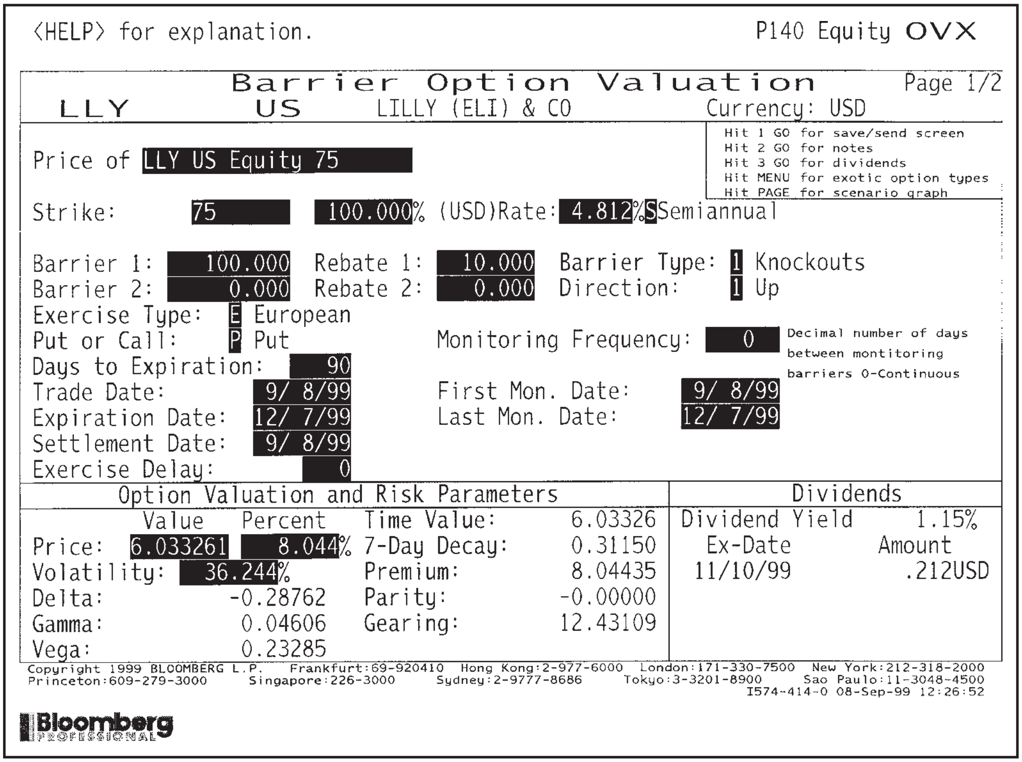


Figure 13.12 Calculator for an up-and-out put with a rebate on the upper barrier. Source: Bloomberg L.P.

the measurement of vega can be meaningless. Barrier options with non-single-signed gamma include the up-and-out call, down-and-out put and many double-barrier options.

Figures 13.16 through 13.19 show the details of a double knockout put contract, its price versus the underlying, its gamma versus the underlying and its price versus volatility. This is a contract with a gamma that changes sign as can be seen from Figure 13.18. You must be very careful when pricing such a contract as to what volatility to use. Suppose you wanted to know the implied volatility for this contract when the price was 3.2, what value would you get? Refer to Figure 13.19.

To accommodate problems like this, practitioners have invented a number of ‘patches.’ One is to use two different volatilities in the option price. For example, one can calculate implied volatilities from vanilla options with the same strike, expiry and payoff as the barrier option and also from American-style one-touch options with the strike at the barrier level. The implied volatility from the vanilla option contains the market’s estimate of the value of the payoff, but including all the upside potential that the call has but which is irrelevant for the up-and-out option. The one-touch volatility, however, contains the market’s view of the likelihood of the barrier level being reached. These two volatilities can be used to price an up-and-out call by observing that an ‘out’ option is the same as a vanilla minus an ‘in’ option. Use the vanilla volatility to price the vanilla call and the one-touch volatility to price the ‘in’ call.

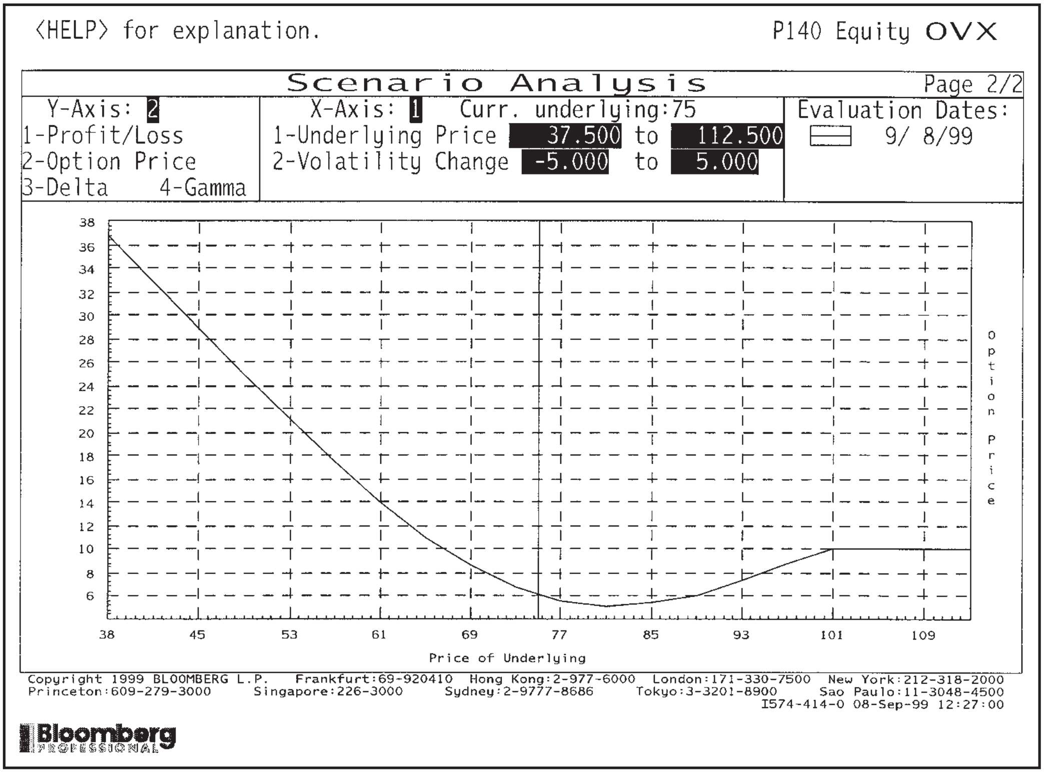


Figure 13.13 Value of an up-and-out put with a rebate on the upper barrier. Source: Bloomberg L.P.

13.7 HEDGING BARRIER OPTIONS

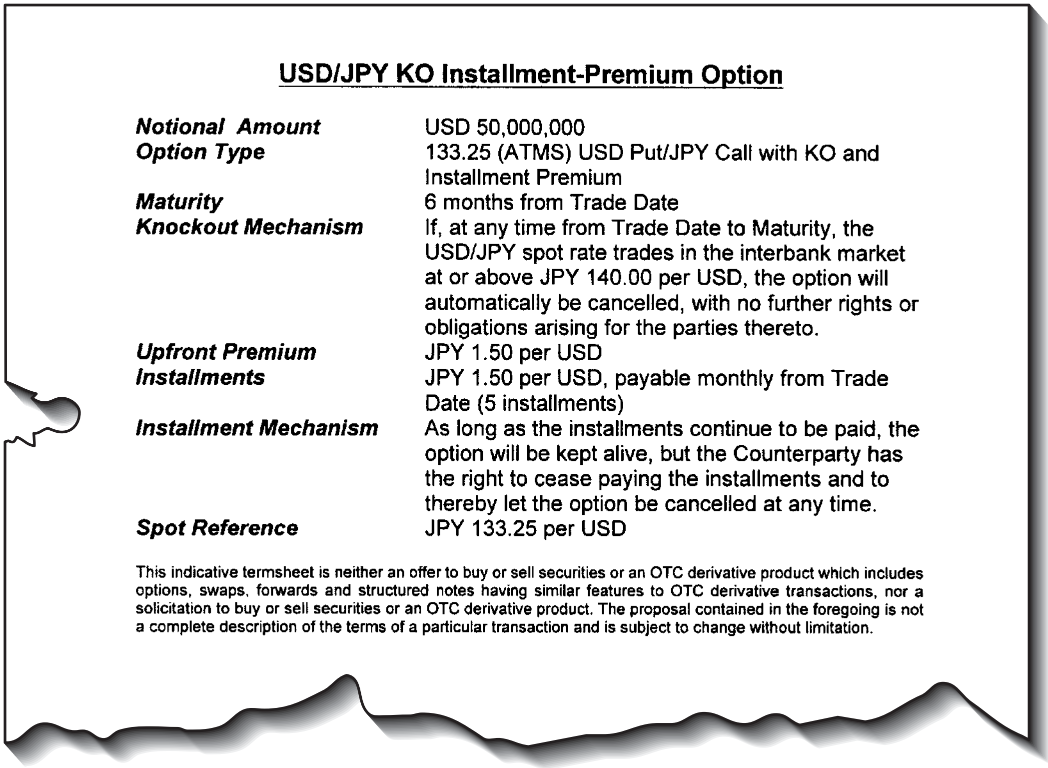
Barrier options have discontinuous delta at the barrier. For a knock-out, the option value is continuous, decreasing approximately linearly towards the barrier then being zero beyond the barrier. This discontinuity in the delta means that the gamma is instantaneously infinite at the barrier. Delta hedging through the barrier is virtually impossible, and certainly very costly. This raises the issue of whether there are improvements on delta hedging for barrier options.

There have been a number of suggestions made for ways to *statically* hedge barrier options. These methods try to mimic as closely as possible the value of a barrier option with vanilla calls and puts, or with binary options. A very common practice for hedging a short up-and-out call is to buy a long call with the same strike and expiry. If the option does knock out then you are fortunate in being left with a long call position.

I now describe another simple but useful technique, based on the **reflection principle** and **put-call symmetry**. This technique only really works if the barrier and strike lie in the correct order, as we shall see. The method gives an approximate hedge only.

The simplest example of put-call symmetry is actually put-call parity. At all asset levels we have

$$V_C - V_P = S - Ee^{-r(T-t)},$$



USD/JPY KO Installment-Premium Option

Notional Amount	USD 50,000,000
Option Type	133.25 (ATMS) USD Put/JPY Call with KO and Installment Premium
Maturity	6 months from Trade Date
Knockout Mechanism	If, at any time from Trade Date to Maturity, the USD/JPY spot rate trades in the interbank market at or above JPY 140.00 per USD, the option will automatically be cancelled, with no further rights or obligations arising for the parties thereto.
Upfront Premium	JPY 1.50 per USD
Installments	JPY 1.50 per USD, payable monthly from Trade Date (5 installments)
Installment Mechanism	As long as the installments continue to be paid, the option will be kept alive, but the Counterparty has the right to cease paying the installments and to thereby let the option be cancelled at any time.
Spot Reference	JPY 133.25 per USD

This indicative termsheet is neither an offer to buy or sell securities or an OTC derivative product which includes options, swaps, forwards and structured notes having similar features to OTC derivative transactions, nor a solicitation to buy or sell securities or an OTC derivative product. The proposal contained in the foregoing is not a complete description of the terms of a particular transaction and is subject to change without limitation.

Figure 13.14 Term sheet for a USD/JPY knock-out installment premium option.

where E is the strike of the two options, and C and P refer to call and put. Suppose we have a down-and-in call, how can we use this result? To make things simple for the moment, let's have the barrier and the strike at the same level. Now hedge our down-and-in call with a short position in a vanilla put with the same strike. If the barrier is reached we have a position worth

$$V_C - V_P.$$

The first term is from the down-and-in call and the second from the vanilla put. This is exactly the same as

$$S - Ee^{-r(T-t)} = E(1 - e^{-r(T-t)}),$$

because of put-call parity and since the barrier and the strike are the same. If the barrier is not touched then both options expire worthless. If the interest rate were zero then we would have a perfect hedge. If rates are nonzero what we are left with is a one-touch option with small and time-dependent value on the barrier. Although this leftover cashflow is nonzero, it is small, bounded and more manageable than the original cashflows.

Now suppose that the strike and the barrier are distinct. Let us continue with the down-and-in call, now with barrier below the strike. The static hedge is not much more complicated than the previous example. All we need to know is the relationship between the value of a call option with strike E when $S = S_d$ and a put option with strike S_d^2/E.

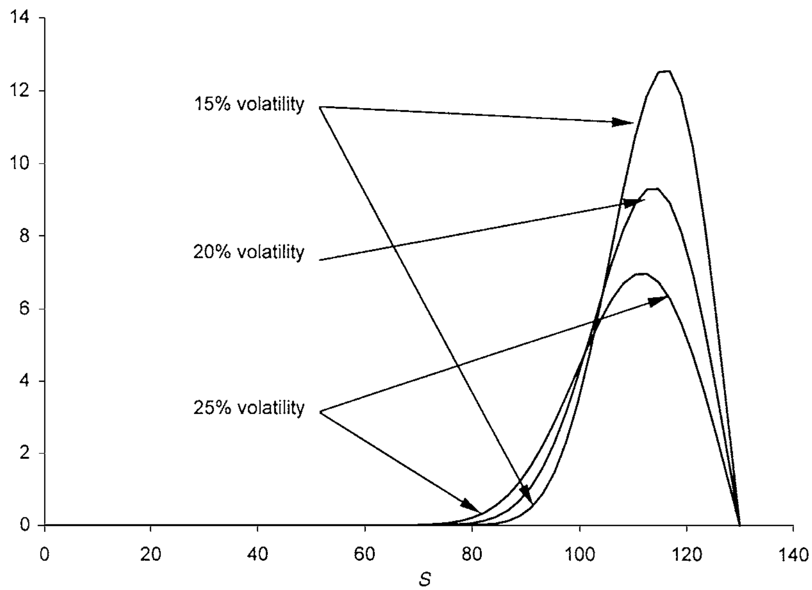


Figure 13.15 Theoretical up-and-out call price with three different volatilities.

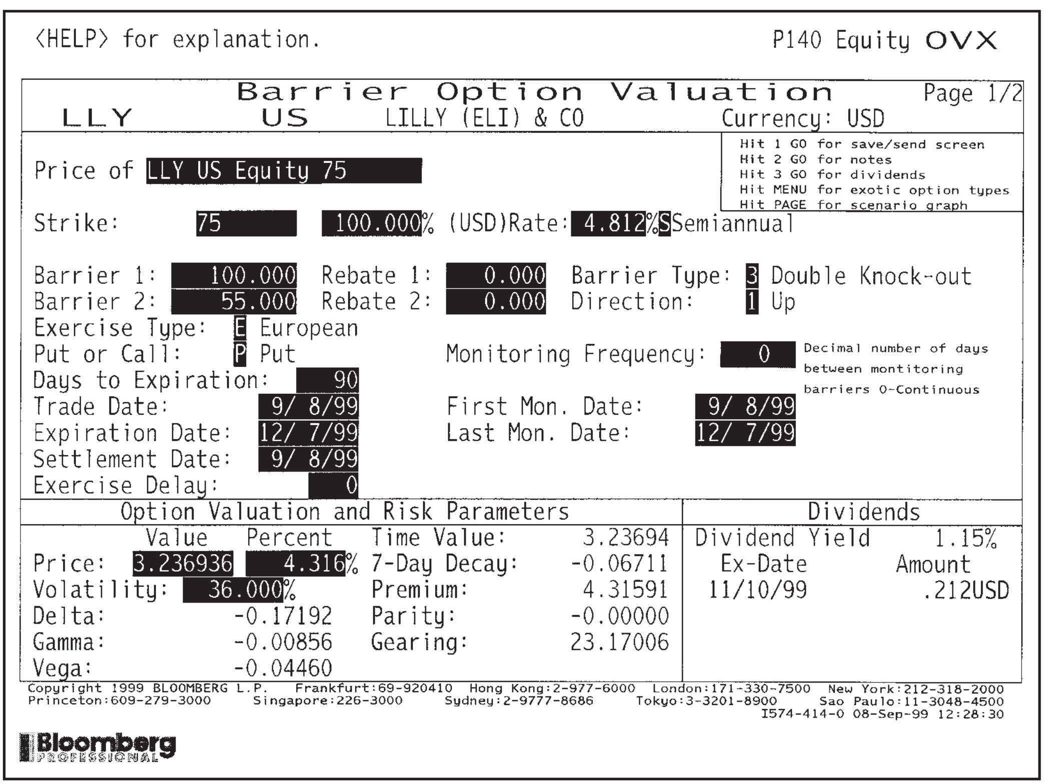


Figure 13.16 Details of a double knock-out put. Source: Bloomberg L.P.

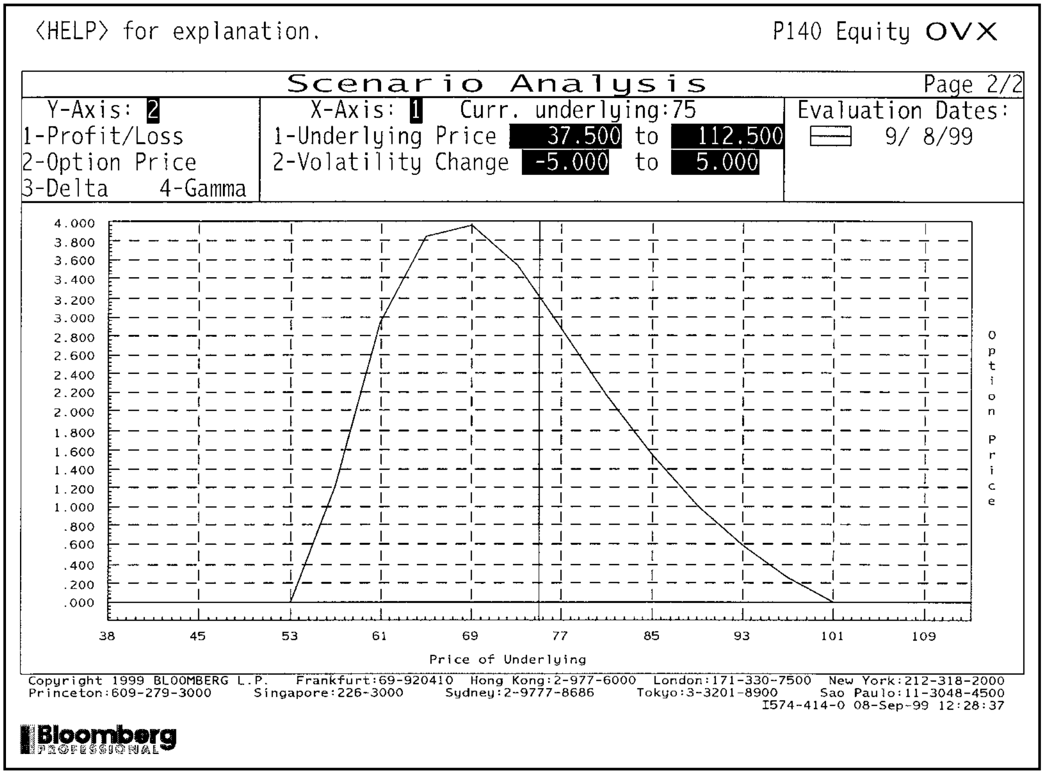


Figure 13.17 Price of the double knock-out put. Source: Bloomberg L.P.

It is easy to show from the formulas for calls and puts that if interest rates are zero, the value of this call at $S = S_d$ is equal to a number E/S_d of the puts, valued at S_d. We would therefore hedge our down-and-in call with E/S_d puts struck at S_d^2/E. Note that the geometric average of the strike of the call and the strike of the put is the same as the barrier level, this is where the idea of 'reflection' comes in. The strike of the hedging put is at the reflection in the barrier of the call's strike. When rates are nonzero there is some error in this hedge, but again it is small and manageable, decreasing as we get closer to expiry. If the barrier is not touched then both options expire worthless (the strike of the put is below the barrier remember).

If the barrier level is above the strike, matters are more complicated since if the barrier is touched we get an in-the-money call. The reflection principle does not work because the put would also be in the money at expiry if the barrier is not touched.

13.7.1 Slippage costs

The delta of a barrier option is discontinuous at the barrier, whether it is an in or an outoption. This presents a particular problem to do with **slippage** or **gapping**. Should the underlying move significantly as the barrier is triggered it is likely that it will not be possible to continuously hedge through the barrier. For example, if the contract is knocked out then

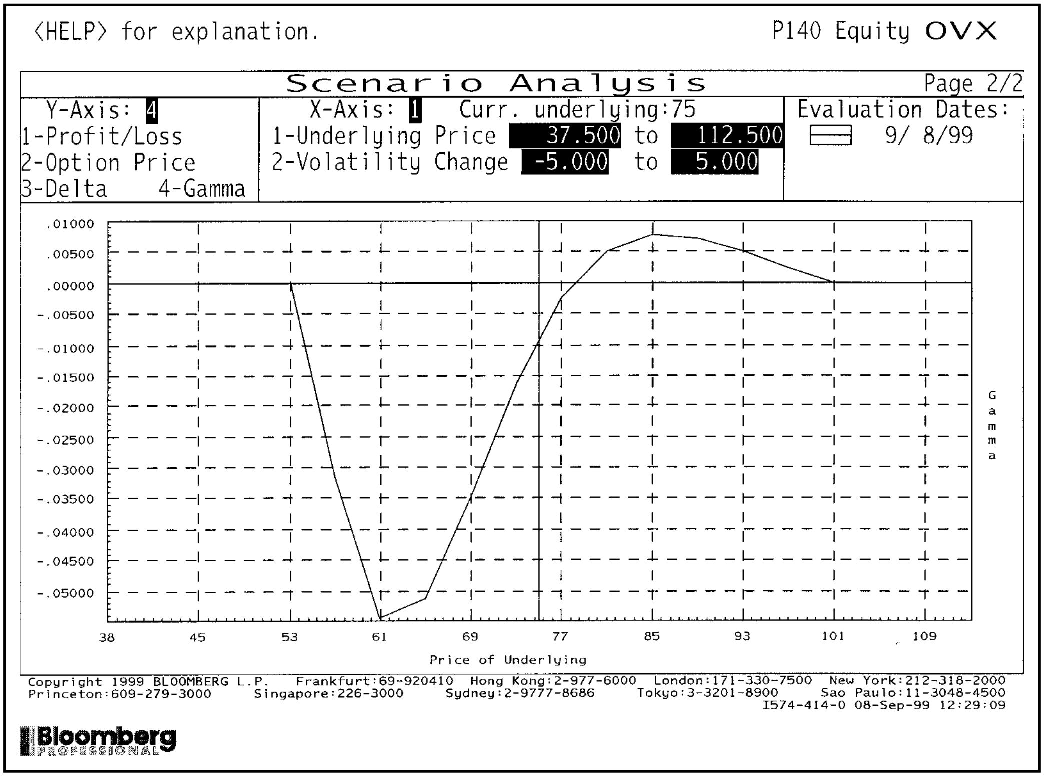


Figure 13.18 Gamma of the double knock-out put. Source: Bloomberg L.P.

one finds oneself with a $-\Delta$ holding of the underlying that should have been offloaded sooner. This can have a significant effect on the hedging costs.

It is not too difficult to allow for the *expected* slippage costs, and all that is required is a slight modification to the apparent barrier level.

At the barrier we hold $-\Delta$ of the underlying. The value of this position is $-\Delta X$, since $S = X$ is the barrier level. Suppose that the asset moves by a small fraction k before we can close out our asset position, or equivalently, that there is a transaction charge involved in closing. We thus lose

$$-k\Delta X$$

on the trigger event.

Now refer to Figure 13.20 where we'll look at the specific example of a down-and-out option. Because we lose $-k\Delta X$ we should use the boundary condition

$$V(X, t) = -k\Delta X.$$

After a little bit of Taylor series we find that this is approximately the same as

$$V((1 + k)X, t) = 0.$$

In other words, we should apply the boundary condition at a slightly higher value of S and so slightly reduce the option's value.

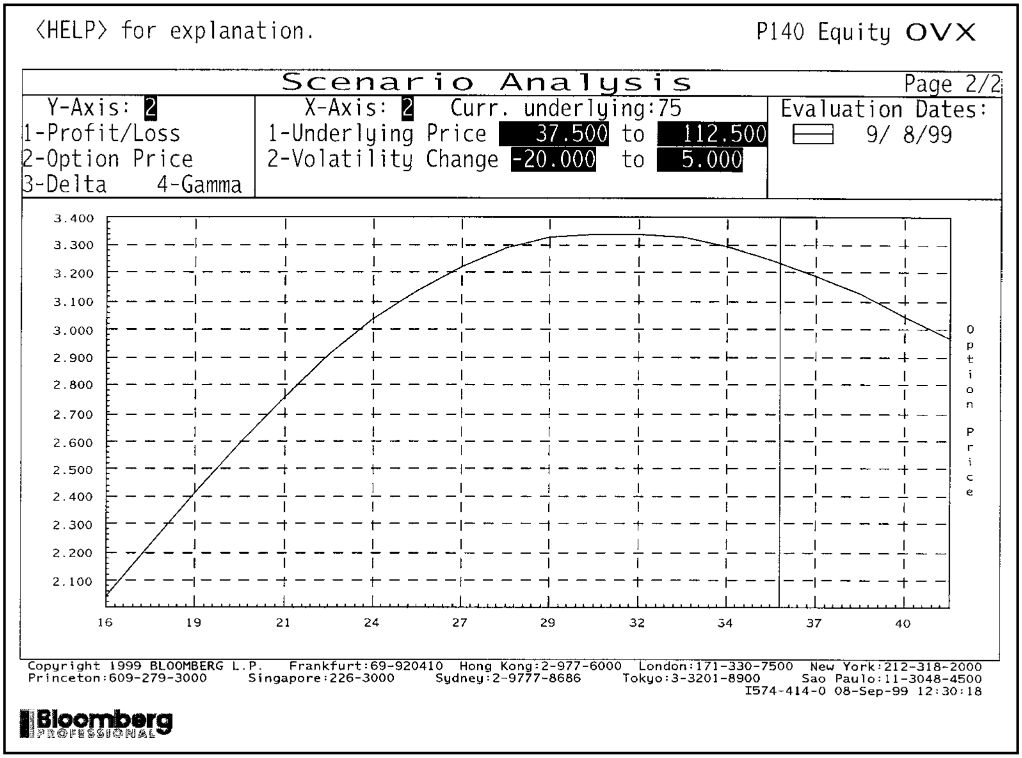


Figure 13.19 Option price versus volatility for the double knock-out put. Source: Bloomberg L.P.

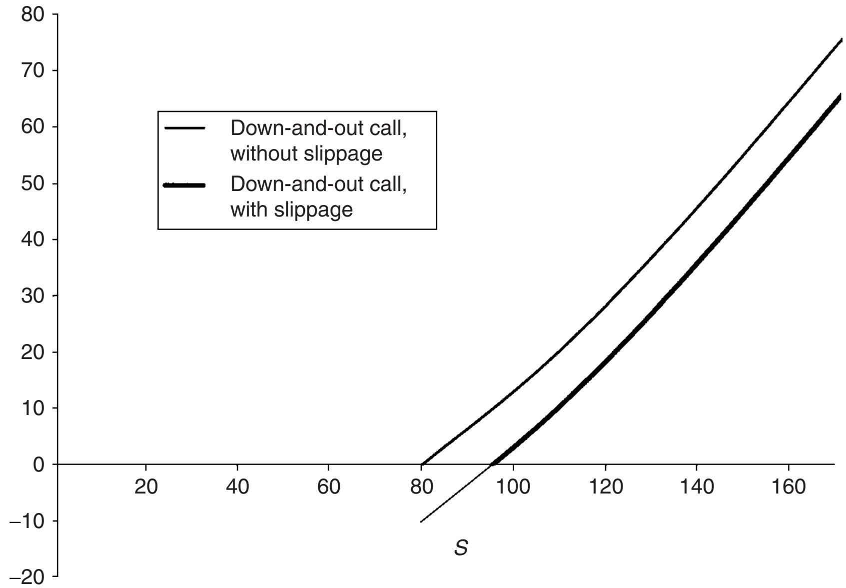


Figure 13.20 Incorporating slippage.

13.8 SUMMARY

In this chapter we have seen a description of many types of barrier option. We have seen how to put these contracts into the partial differential equation framework. Many of these contracts have simple pricing formulas. Unfortunately, the extreme nature of these contracts make them very difficult to hedge in practice and in particular, they can be very sensitive to the volatility of the underlying. Worse still, if the gamma of the contract changes sign we cannot play safe by adding a spread to the volatility. Practitioners seem to be most comfortable statically hedging as much of the barrier contract as possible using traded vanillas.

Some formulas

In the following I use $N(\cdot)$ to denote the cumulative distribution function for a standardized Normal variable. The dividend yield on stocks or the foreign interest rate for FX are denoted by q. Also

$$a = \left(\frac{S_b}{S}\right)^{-1+\frac{2(r-q)}{\sigma^2}},$$

$$b = \left(\frac{S_b}{S}\right)^{1+\frac{2(r-q)}{\sigma^2}},$$

where S_b is the barrier position (whether S_u or S_d should be obvious from the example).

$$d_1 = \frac{\log(S/E) + (r - q + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\log(S/E) + (r - q - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_3 = \frac{\log(S/S_b) + (r - q + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_4 = \frac{\log(S/S_b) + (r - q - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_5 = \frac{\log(S/S_b) - (r - q - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_6 = \frac{\log(S/S_b) - (r - q + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_7 = \frac{\log(SE/S_b^2) - (r - q - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_8 = \frac{\log(SE/S_b^2) - (r - q + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

Up-and-out call

$$Se^{-q(T-t)}\left(N(d_1)-N(d_3)-b(N(d_6)-N(d_8))\right)$$
$$-Ee^{-r(T-t)}\left(N(d_2)-N(d_4)-a(N(d_5)-N(d_7))\right).$$

Up-and-in call

$$Se^{-q(T-t)}\left(N(d_3)+b(N(d_6)-N(d_8))\right)-Ee^{-r(T-t)}\left(N(d_4)+a(N(d_5)-N(d_7))\right).$$

Down-and-out call

1. $E > S_b$:

$$Se^{-q(T-t)}\left(N(d_1)-b(1-N(d_8))\right)-Ee^{-r(T-t)}\left(N(d_2)-a(1-N(d_7))\right).$$

2. $E < S_b$:

$$Se^{-q(T-t)}\left(N(d_3)-b(1-N(d_6))\right)-Ee^{-r(T-t)}\left(N(d_4)-a(1-N(d_5))\right).$$

Down-and-in call

1. $E > S_b$:

$$Se^{-q(T-t)}b(1-N(d_8))-Ee^{-r(T-t)}a(1-N(d_7)).$$

2. $E < S_b$:

$$Se^{-q(T-t)}\left(N(d_1)-N(d_3)+b(1-N(d_6))\right)-Ee^{-r(T-t)}\left(N(d_2)-N(d_4)+a(1-N(d_5))\right).$$

Down-and-out put

$$-Se^{-q(T-t)}\left(N(d_3)-N(d_1)-b(N(d_8)-N(d_6))\right)$$
$$+Ee^{-r(T-t)}\left(N(d_4)-N(d_2)-a(N(d_7)-N(d_5))\right).$$

Down-and-in put

$$-Se^{-q(T-t)}\left(1-N(d_3)+b(N(d_8)-N(d_6))\right)+Ee^{-r(T-t)}\left(1-N(d_4)+a(N(d_7)-N(d_5))\right).$$

Up-and-out put

1. $E > S_b$:

$$-Se^{-q(T-t)}\left(1-N(d_3)-bN(d_6)\right))+Ee^{-r(T-t)}\left(1-N(d_4)-aN(d_5)\right).$$

2. $E < S_b$:

$$-Se^{-q(T-t)}\left(1-N(d_1)-bN(d_8)\right)+Ee^{-r(T-t)}\left(1-N(d_2)-aN(d_7))\right).$$

Up-and-in put

1. $E > S_b$:

$$-Se^{-q(T-t)}\left(N(d_3) - N(d_1) + bN(d_6)\right) + Ee^{-r(T-t)}\left(N(d_4) - N(d_2) + aN(d_5)\right).$$

2. $E < S_b$:

$$-Se^{-q(T-t)}bN(d_8) + Ee^{-r(T-t)}aN(d_7).$$

FURTHER READING

- Many of the original barrier formulas are due to Reiner & Rubinstein (1991).
- The formulas above are explained in Taleb (1997) and Haug (1997). Taleb discusses barrier options in great detail, including the reality of hedging that I have only touched upon.
- The article by Carr (1995) contains an extensive literature review as well as a detailed discussion of protected barrier options and rainbow barrier options.
- See Derman *et al*. (1997) for a full description of the static replication of barrier options with vanilla options.
- See Carr (1994) for more details of put-call symmetry.

Time Out...

Binomial model revisited

Using trees to price barrier options can be a bit of a chore. This is because it is tedious to line up nodes with the barrier level. You can be faced with the question of on which node to set the option value to zero. Interpolation methods make the most sense. Those fond of trees will go to extraordinary lengths to justify various *ad hoc* tree modifications. In the finite-difference method the barrier is trivial — the mathematician's favorite word — to incorporate.

CHAPTER 14
fixed-income products and analysis: yield, duration and convexity

The aim of this Chapter...

... is to introduce the most common contracts of the fixed-income world and to show simple ways for their analysis. The big assumption of this chapter is that interest rates are deterministic. Greater levels of sophistication are needed for pricing more complex fixed-income contracts such as derivatives, but these will be reached in later chapters.

In this Chapter...

- the names and properties of the basic and most important fixed-income products
- the definitions of features commonly found in fixed-income products
- simple ways to analyze the market value of the instruments: yield, duration and convexity
- how to construct yield curves and forward rates

14.1 INTRODUCTION

This chapter is an introduction to some basic instruments and concepts in the world of fixed income, that is, the world of cashflows that are in the simplest cases independent of any stocks, commodities etc. I will describe the most elementary of fixed-income instruments, the coupon-bearing bond, and show how to determine various properties of such bonds to help in their analysis.

This chapter is self-contained, and does not require any knowledge from earlier chapters. A lot of it is also not really necessary reading for anything that follows. The reason for this is that, although the concepts and techniques I describe here are used in practice and are *useful* in practice, it is difficult to present a completely coherent theory for more sophisticated products in this framework.

14.2 SIMPLE FIXED-INCOME CONTRACTS AND FEATURES

14.2.1 The zero-coupon bond

The **zero-coupon bond** is a contract paying a known fixed amount, the **principal**, at some given date in the future, the **maturity** date T. For example, the bond pays $100 in 10 years' time. We're going to scale this payoff, so that in future all principals will be $1.

This promise of future wealth is worth something now: it cannot have zero or negative value. Furthermore, except in extreme circumstances, the amount you pay initially will be smaller than the amount you receive at maturity.

We discussed the idea of time value of money in Chapter 1. This is clearly relevant here and we will return to this in a moment.

14.2.2 The coupon-bearing bond

A **coupon-bearing bond** is similar to the above except that as well as paying the principal at maturity, it pays smaller quantities, the coupons, at intervals up to and including the maturity date. These coupons are usually prespecified fractions of the principal. For example, the bond pays $1 in 10 years and 2%, i.e. 2 cents, every six months. This bond is clearly more valuable than the bond in the previous example because of the coupon payments. We can think of the coupon-bearing bond as a portfolio of zero-coupon bearing bonds: one zero-coupon bearing bond for each coupon date with a principal being the same as the original bond's coupon, and then a final zero-coupon bond with the same maturity as the original.

Figure 14.1 is an excerpt from *The Wall Street Journal Europe* of 5th January 2000 showing US Treasury bonds, notes and bills. Observe that there are many different 'rates' or coupons, and different maturities. The values of the different bonds will depend on the size of the coupon, the maturity and the market's view of the future behavior of interest rates.

14.2.3 The money market account

Everyone who has a bank account has a **money market account**. This is an account that accumulates interest compounded at a rate that varies from time to time. The rate at which interest accumulates is usually a short-term and unpredictable rate. In the sense that money held in a money market account will grow at an unpredictable rate, such an

U.S. TREASURY ISSUES

Tuesday, January 4, 2000

Representative and Indicative Over-the-Counter quotations based on transactions of $1 million or more.

Treasury bond, note and bill quotes are as of mid-afternoon. Colons in bond and note bid-and-asked quotes represent 32nds; 101:01 means 101 1/32. Net changes in 32nds. Treasury bill quotes in hundredths, quoted on terms of a rate of discount. Days to maturity calculated from settlement date. All yields are based on a one-day settlement and calculated on the offer quote. Currrent 13-week and 26-week bills are boldfaced.

For bonds callable prior to maturity, yields are computed to the earliest call date for issues quoted above par and to the maturity date for issues quoted below par. n-Treasury note.wi-When issued; daily change is expressed in basis points.

Source: Dow Jones Telerate/Cantor Fitzgerald

U.S. Treasury strips as of 3 p.m. Eastern time, also based on transactions of $1 million or more. Colons in bid-and-asked quotes represent 32nds; 101:01 means 101 1/32. Net changes in 32nds. Yields calculated on the asked quotation. ci-stripped coupon interest. bp-Treasury bond, stripped principal. np-Treasury note, stripped principal. For bonds callable prior to maturity, yields are computed to the earliest call date for issues quoted above par and to the maturity date for issues below par.

Source: Bear, Stearns & Co. via Street Software Technology Inc.

GOVT. BONDS & NOTES

Maturity Rate	Mo/Yr	Bid	Asked	Chg.	Ask Yld.
6 3/8	Jan 00n	99:30	100:00	.....	6.19
5 3/8	Jan 00n	99:31	100:01	+1	4.82
7 3/4	Jan 00n	100:04	100:06	.....	4.92
5 7/8	Feb 00n	100:00	100:02	.....	5.19
8 1/2	Feb 00n	100:09	100:11	.....	5.22
5 1/2	Feb 00n	99:31	100:01	.....	5.19
7 1/8	Feb 00n	100:07	100:09	.....	5.12
5 1/2	Mar 00n	99:31	100:01	+1	5.29
6 7/8	Mar 00n	100:09	100:11	+1	5.30
5 1/2	Apr 00n	99:30	100:00	.....	5.43
5 5/8	Apr 00n	99:30	100:00	+1	5.57
6 3/4	Apr 00n	100:09	100:11	+1	5.58
6 3/8	May 00n	100:06	100:08	+1	5.62
8 7/8	May 00n	101:06	101:08	.....	5.27
5 1/2	May 00n	99:27	99:29	.....	5.71
6 1/4	May 00n	100:04	100:06	.....	5.74
5 3/8	Jun 00n	99:25	99:27	.....	5.70
5 7/8	Jun 00n	100:01	100:03	.....	5.67
5 3/8	Jul 00n	99:22	99:24	+1	5.82
6 1/8	Jul 00n	100:03	100:05	.....	5.83
6	Aug 00n	99:31	100:01	.....	5.93
8 3/4	Aug 00n	101:20	101:22	.....	5.89
5 1/8	Aug 00n	99:13	99:15	+1	5.96
6 1/4	Aug 00n	100:04	100:06	+1	5.94
4 1/2	Sep 00n	98:27	98:29	+1	6.03
6 1/8	Sep 00n	100:00	100:02	+1	6.02
4	Oct 00n	98:10	98:12	+1	6.06
5 3/4	Oct 00n	99:23	99:25	+2	6.02
5 3/4	Nov 00n	99:21	99:23	+1	6.08
8 1/2	Nov 00n	102:00	102:02	+2	5.99
4 5/8	Nov 00n	98:22	98:24	+2	6.06
5 5/8	Nov 00n	99:17	99:19	+1	6.09
4 5/8	Dec 00n	98:16	98:18	+2	6.15
5 1/2	Dec 00n	99:11	99:13	+1	6.13
4 1/2	Jan 01n	98:08	98:10	+1	6.15
5 1/4	Jan 01n	99:02	99:04	+2	6.10
5 3/8	Feb 01n	99:03	99:05	+2	6.16
7 3/4	Feb 01n	101:19	101:21	+1	6.18
11 3/4	Feb 01	105:27	105:29	.....	6.17
5	Feb 01n	98:21	98:23	+2	6.16
5 5/8	Feb 01n	99:11	99:13	+2	6.16
4 7/8	Mar 01n	98:11	98:13	+1	6.23
6 3/8	Mar 01n	100:03	100:05	+1	6.23
5	Apr 01n	98:14	98:16	+2	6.20
6 1/4	Apr 01n	100:00	100:02	+2	6.19
5 5/8	May 01n	99:05	99:07	+2	6.23
8	May 01n	102:06	102:08	+1	6.24

Maturity Rate	Mo/Yr	Bid	Asked	Chg.	Ask Yld.
3 5/8	Jul 02i	98:31	99:00	+1	4.04
6	Jul 02n	99:09	99:11	+6	6.28
6 3/8	Aug 02n	100:01	100:03	+5	6.33
6 1/4	Aug 02n	99:24	99:26	+7	6.32
5 7/8	Sep 02n	98:27	98:29	+8	6.31
5 3/4	Oct 02n	98:16	98:18	+7	6.31
11 5/8	Nov 02	113:10	113:14	+7	6.40
5 3/4	Nov 02n	98:13	98:15	+8	6.33
5 5/8	Dec 02n	98:00	98:02	+7	6.35
5 1/2	Jan 03n	97:18	97:20	+8	6.36
6 1/4	Feb 03n	99:17	99:19	+7	6.39
10 3/4	Feb 03	111:24	111:28	+7	6.47
5 1/2	Feb 03n	97:14	97:16	+7	6.39
5 1/2	Mar 03n	97:11	97:13	+7	6.40
5 3/4	Apr 03n	98:00	98:02	+7	6.40
10 3/4	May 03	112:18	112:22	+8	6.48
5 1/2	May 03n	97:06	97:08	+8	6.41
5 3/8	Jun 03n	96:27	96:29	+9	6.38
5 1/4	Aug 03n	96:08	96:10	+9	6.41
5 3/4	Aug 03n	97:25	97:27	+8	6.43
11 1/8	Aug 03	114:14	114:18	+9	6.53
4 1/4	Nov 03n	92:22	92:24	+7	6.40
11 7/8	Nov 03	117:25	117:29	+10	6.54
4 3/4	Feb 04n	94:04	94:06	+9	6.38
5 7/8	Feb 04n	98:03	98:05	+9	6.39
5 1/4	May 04n	95:18	95:20	+9	6.41
7 1/4	May 04n	102:27	102:29	+11	6.47
12 3/8	May 04	121:18	121:24	+11	6.55
6	Aug 04n	98:07	98:08	+9	6.44
7 1/4	Aug 04n	102:30	103:00	+11	6.49
13 3/4	Aug 04	127:30	128:04	+11	6.58
5 7/8	Nov 04n	97:26	97:27	+10	6.40
7 7/8	Nov 04n	105:18	105:20	+13	6.50
11 5/8	Nov 04	120:18	120:24	+13	6.56
7 1/2	Feb 05n	104:04	104:06	+13	6.52
6 1/2	May 05n	99:27	99:29	+14	6.52
8 1/4	May 00-05	100:24	100:26	+1	5.88
12	May 05	123:30	124:04	+13	6.58
6 1/2	Aug 05n	99:26	99:28	+14	6.52
10 3/4	Aug 05	119:02	119:06	+15	6.60
5 7/8	Nov 05n	96:30	97:00	+15	6.50
5 5/8	Feb 06n	95:16	95:18	+15	6.51
9 3/8	Feb 06	113:28	114:00	+17	6.56
6 7/8	May 06n	101:18	101:20	+16	6.56
7	Jul 06n	102:06	102:08	+16	6.57
6 1/2	Oct 06n	99:16	99:18	+16	6.58
3 3/8	Jan 07i	94:05	94:06	+3	4.34
6 1/4	Feb 07n	98:05	98:07	+16	6.57

Mat.		Type	Bid	Asked	Chg.	Ask Yld.
9 7/8	Nov	15	128:26	129:00	+36	6.85
9 1/4	Feb	16	123:01	123:07	+34	6.85
7 1/4	May	16	104:05	104:07	+30	6.82
7 1/2	Nov	16	106:19	106:21	+32	6.83
8 3/4	May	17	118:31	119:03	+35	6.85
8 7/8	Aug	17	120:11	120:17	+36	6.85
9 1/8	May	18	123:12	123:18	+36	6.85
9	Nov	18	122:11	122:17	+36	6.85
8 7/8	Feb	19	121:05	121:11	+36	6.85
8 1/8	Aug	19	113:15	113:19	+35	6.85
8 1/2	Feb	20	117:22	117:26	+37	6.85
8 3/4	May	20	120:15	120:21	+38	6.85
8 3/4	Aug	20	120:18	120:24	+38	6.85
7 7/8	Feb	21	111:11	111:15	+36	6.84
8 1/8	May	21	114:04	114:08	+36	6.84
8 1/8	Aug	21	114:06	114:10	+36	6.85
8	Nov	21	112:31	113:03	+36	6.84
7 1/4	Aug	22	104:31	105:01	+35	6.81
7 5/8	Nov	22	109:07	109:09	+35	6.82
7 1/8	Feb	23	103:22	103:24	+34	6.80
6 1/4	Aug	23	93:28	93:30	+32	6.77
7 1/2	Nov	24	108:21	108:23	+37	6.77
7 5/8	Feb	25	110:08	110:12	+37	6.76
6 7/8	Aug	25	101:11	101:13	+35	6.76
6	Feb	26	91:00	91:02	+31	6.73
6 3/4	Aug	26	99:31	100:01	+34	6.75
6 1/2	Nov	26	97:00	97:02	+34	6.74
6 5/8	Feb	27	98:19	98:21	+34	6.73
6 3/8	Aug	27	95:18	95:20	+34	6.72
6 1/8	Nov	27	92:17	92:19	+32	6.72
3 5/8	Apr	28i	89:04	89:05	+5	4.29
5 1/2	Aug	28	84:28	84:30	+30	6.69
5 1/4	Nov	28	81:30	82:00	+29	6.66
5 1/4	Feb	29	82:06	82:07	+24	6.64
3 7/8	Apr	29i	93:06	93:07	+4	4.28
6 1/8	Aug	29	94:19	94:20	+26	6.54

TREASURY BILLS

Maturity	Days to Mat.	Bid	Asked	Chg	Ask Yld.
Jan 06 '00	1	4.21	4.13	+0.19	4.19
Jan 13 '00	8	5.31	5.23	+0.05	5.31
Jan 20 '00	15	5.22	5.14	+0.02	5.22
Jan 27 '00	22	4.82	4.74	−0.13	4.82
Feb 03 '00	29	4.88	4.80	−0.24	4.89
Feb 10 '00	36	5.00	4.96	−0.12	5.05
Feb 17 '00	43	5.00	4.96	−0.12	5.06

Figure 14.1 *The Wall Street Journal Europe* of 5th January 2000, Treasury Bonds, Notes and Bills. Reproduced by permission of Dow Jones & Company, Inc.

account is risky when compared with a one-year zero-coupon bond. On the other hand, the money market account can be closed at any time but if the bond is sold before maturity there is no guarantee how much it will be worth at the time of the sale.

14.2.4 Floating rate bonds

In its simplest form a **floating interest rate** is the amount that you get on your bank account. This amount varies from time to time, reflecting the state of the economy and in response to pressure from other banks for your business. This uncertainty about the interest rate you receive is compensated by the flexibility of your deposit, it can be withdrawn at any time.

The most common measure of interest is **London Interbank Offer Rate** or **LIBOR**. LIBOR comes in various maturities, one month, three month, six month etc., and is the rate of interest offered between Eurocurrency banks for fixed-term deposits.

Sometimes the coupon payment on a bond is not a prescribed dollar amount but depends on the level of some 'index,' measured at the time of the payment or before. Typically, we cannot know at the start of the contract what level this index will be at when the payment is made. We will see examples of such contracts in later chapters.

14.2.5 Forward rate agreements

A **forward rate agreement** (**FRA**) is an agreement between two parties that a prescribed interest rate will apply to a prescribed principal over some specified period in the future. The cashflows in this agreement are as follows: party A pays party B the principal at time T_1 and B pays A the principal plus agreed interest at time $T_2 > T_1$. The value of this exchange at the time the contract is entered into is generally not zero and so there will be a transfer of cash from one party to the other at the start date.

14.2.6 Repos

A **repo** is a repurchase agreement. It is an agreement to sell some security to another party and buy it back at a fixed date and for a fixed amount. The price at which the security is bought back is greater than the selling price and the difference implies an interest rate called the **repo rate**. The commonest repo is the overnight repo in which the agreement is renegotiated daily. If the repo agreement extends for 30 days it is called a **term repo**.

A **reverse repo** is the borrowing of a security for a short period at an agreed interest rate.

Repos can be used to lock in future interest rates. For example, buy a six-month Treasury bill today and repo it out for three months. There is no cash flow today since the bond has been paid for (money out) and then repoed (same amount in). In three months time you will have to repurchase the bill at the agreed price, this is an outflow of cash. In six months you receive the principal. Money out in three months, money in six months, for there to be no arbitrage the equivalent interest rate should be that currently prevailing between three and six months time.

14.2.7 STRIPS

STRIPS stands for Separate Trading of Registered Interest and Principal of Securities. The coupons and principal of normal bonds are split up, creating artificial zero-coupon bonds of longer maturity than would otherwise be available.

14.2.8 Amortization

In all of the above products I have assumed that the principal remains fixed at its initial level. Sometimes this is not the case, the principal can **amortize** or decrease during the life of the contract. The principal is thus paid back gradually and interest is paid on the amount of the principal outstanding. Such amortization is arranged at the initiation of the contract and may be fixed, so that the rate of decrease of the principal is known beforehand, or can depend on the level of some index, if the index is high the principal amortizes faster for example. We see an example of a complex amortizing structure in Chapter 17.

14.2.9 Call provision

Some bonds have a **call provision**. The issuer can call back the bond on certain dates or at certain periods for a prescribed, possibly time-dependent, amount. This lowers the value of the bond. The mathematical consequences of this are discussed in Chapter 16.

14.3 INTERNATIONAL BOND MARKETS

14.3.1 United States of America

In the US, bonds of maturity less than one year are called **bills** and are usually zero coupon. Bonds with maturity 2–10 years are called **notes**. They are coupon bearing with coupons every six months. Bonds with maturity greater than 10 years are called **bonds**. Again they are coupon bearing. In this book I tend to call all of these 'bonds,' merely specifying whether or not they have coupons.

Bonds traded in the United States foreign bond market but which are issued by non-US institutions are called **Yankee** bonds.

Since the beginning of 1997 the US government has also issued bonds linked to the rate of inflation.

14.3.2 United Kingdom

Bonds issued by the UK government are called **gilts**. Some of these bonds are callable, some are irredeemable, meaning that they are perpetual bonds having a coupon but no repayment of principal. The government also issues convertible bonds which may be converted into another bond issue, typically of longer maturity. Finally, there are index-linked bonds having the amount of the coupon and principal payments linked to a measure of inflation, the Retail Price Index (RPI).

14.3.3 Japan

Japanese goverment bonds (**JGB**s) come as short-term treasury bills, medium-term, long-term (10-year maturity) and super long-term (20-year maturity). The long-term and super long-term bonds have coupons every six months. The short-term bonds have no coupons and the medium-term bonds can be either coupon-bearing or zero-coupon bonds.

Yen-denominated bonds issued by non-Japanese institutions are called **Samurai** bonds.

14.4 ACCRUED INTEREST

The market price of bonds quoted in the newspapers are **clean prices**. That is, they are quoted without any **accrued interest**. The accrued interest is the amount of interest that has built up since the last coupon payment:

$$\text{accrued interest} = \text{interest due in full period} \times \frac{\text{number of days since last coupon date}}{\text{number of days in period between coupon payments}}.$$

The actual payment is called the **dirty price** and is the sum of the quoted clean price and the accrued interest.

14.5 DAY-COUNT CONVENTIONS

Because of such matters as the accrual of interest between coupon dates there naturally arises the question of how to accrue interest over shorter periods. Interest is accrued between two dates according to the formula

$$\frac{\text{number of days between the two dates}}{\text{number of days in period}} \times \text{ interest earned in reference period.}$$

There are three main ways of calculating the 'number of days' in the above expression:

- **Actual/Actual** Simply count the number of calendar days
- **30/360** Assume there are 30 days in a month and 360 days in a year
- **Actual/360** Each month has the right number of days but there are only 360 days in a year

14.6 CONTINUOUSLY AND DISCRETELY COMPOUNDED INTEREST

To be able to compare fixed-income products we must decide on a convention for the measurement of interest rates. So far, we have used a continuously compounded rate, meaning that the present value of \$1 paid a time T in the future is

$$e^{-rT} \times \$1$$

for some r. We have seen how this follows from the cash-in-the-bank or money market account equation

$$dM = rM\,dt.$$

This is the convention used in the options world.

Another common convention is to use the formula

$$\frac{1}{(1+r')^T} \times \$1,$$

for present value, where r' is some interest rate. This represents discretely compounded interest and assumes that interest is accumulated *annually* for T years. The formula is derived from calculating the present value from a single-period payment, and then compounding this for each year. This formula is commonly used for the simpler type of instruments such as coupon-bearing bonds. The two formulas are identical, of course, when

$$r = \log(1 + r').$$

This gives the relationship between the continuously compounded interest rate r and the discrete version r'. What would the formula be if interest was discretely compounded twice per year?

Throughout the book we use the continuous definition of interest rates.

Time Out...

Which is better?

Would you rather get 10% once a year or 5% twice a year? With the former \$1 would be worth \$1.10 at the end of the year, whereas the two instalments of 5% would give you

$$(1 + 0.05)^2 = 1.1025.$$

It is very important to know what kind of interest payment you are getting.

What interest rate paid continuously is equivalent to a one-off 10%? The answer is the r which satisfies

$$e^r = 1.1$$

i.e.

$$r = \log(1.1) = 0.09531018\ldots\,.$$

So, about 9.53% on an annualized basis. Remember this logarithm is the natural or Naperian logarithm, also denoted (as in Excel) by $\ln(\cdot)$.

14.7 MEASURES OF YIELD

There are such a variety of fixed-income products, with different coupon structures, amortization, fixed and/or floating rates, that it is necessary to be able to consistently compare different products. One way to do this is through measures of how much each contract earns, there are several measures of this all coming under the name **yield**.

14.7.1 Current yield

The simplest measurement of how much a contract earns is the **current yield**. This measure is defined by

$$\text{current yield} = \frac{\text{annual \$ coupon income}}{\text{bond price}}.$$

For example, consider the 10-year bond that pays 2 cents every six months and \$1 at maturity. This bond has a total income per annum of 4 cents. Suppose that the quoted market price of this bond is 88 cents. The current yield is simply

$$\frac{0.04}{0.88} = 4.5\%.$$

This measurement of the yield of the bond makes no allowance for the payment of the principal at maturity, nor for the time value of money if the coupon payment is reinvested, nor for any capital gain or loss that may be made if the bond is sold before maturity. It is a relatively unsophisticated measure, concentrating very much on short-term properties of the bond.

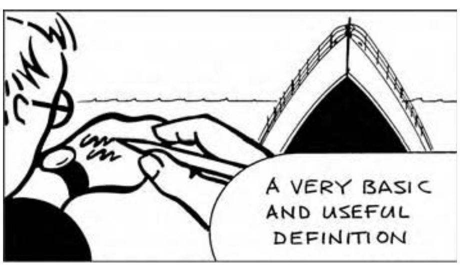
14.7.2 The yield to maturity (YTM) or internal rate of return (IRR)

Suppose that we have a zero-coupon bond maturing at time T when it pays one dollar. At time t it has a value $Z(t;T)$. Applying a constant rate of return of y between t and T, then one dollar received at time T has a present value of $Z(t;T)$ at time t, where

$$Z(t;T) = e^{-y(T-t)}.$$

It follows that

$$y = -\frac{\log Z}{T-t}.$$



Let us generalize this. Suppose that we have a coupon-bearing bond. Discount all coupons and the principal to the present by using some interest rate y. The present value of the bond, at time t, is then

$$V = Pe^{-y(T-t)} + \sum_{i=1}^{N} C_i e^{-y(t_i-t)}, \tag{14.1}$$

where P is the principal, N the number of coupons, C_i the coupon paid on date t_i.

If the bond is a traded security then we know the price at which the bond can be bought. If this is the case then we can calculate the **yield to maturity** or **internal rate of return** as the value y that we must put into Equation (14.1) to make V equal to the traded price of the bond. This calculation must be performed by some trial and error/iterative procedure. For example, in the bond in Table 14.1 we have a principal of \$1 paid in five years and coupons of 3 cents (3%) paid every six months.

Suppose that the market value of this bond is 96 cents. We ask 'What is the internal rate of return we must use to give these cash flows a total present value of 96 cents?' This value is the yield to maturity. In the fourth column in this table is the present value (PV) of each of the cashflows using a rate of 6.8406%: since the sum of these present values is 96 cents the YTM or IRR is 6.8406%.

This yield to maturity is a valid measure of the return on a bond if we intend to hold it to maturity.

Table 14.1 An example of a coupon-bearing bond.

Time	Coupon	Principal repayment	PV (discounting at 6.8406%)
0			0
0.5	.03		0.0290
1.0	.03		0.0280
1.5	.03		0.0270
2.0	.03		0.0262
2.5	.03		0.0253
3.0	.03		0.0244
3.5	.03		0.0236
4.0	.03		0.0228
4.5	.03		0.0220
5.0	.03	1.00	0.7316
		Total	0.9600

To calculate the yield to maturity of a portfolio of bonds simply treat all the cashflows as if they were from the one bond and calculate the value of the whole portfolio by adding up the market values of the component bonds.

14.8 THE YIELD CURVE

The plot of yield to maturity against time to maturity is called the **yield curve**. For the moment assume that this has been calculated from zero-coupon bonds and that these bonds have been issued by a perfectly creditworthy source.

If the bonds have coupons then the calculation of the yield curve is more complicated and the 'forward curve,' described below, is a better measure of the interest rate pertaining at some time in the future. Figure 14.2 shows the yield curve for US Treasuries as it was on 9th September 1999.

Time Out...

Discount factors

Once you've calculated the yield curve you can use the results to work out the present value of any fixed-rate cashflows. All you have to do is work out the discount factor for each cashflow and multiply the cashflow by that amount. Typically, you'll find yourself in the situation of having a cashflow on a certain date but no yield associated with that maturity. Then you'll have to interpolate between the two yields either side to get an estimate for the required maturity.

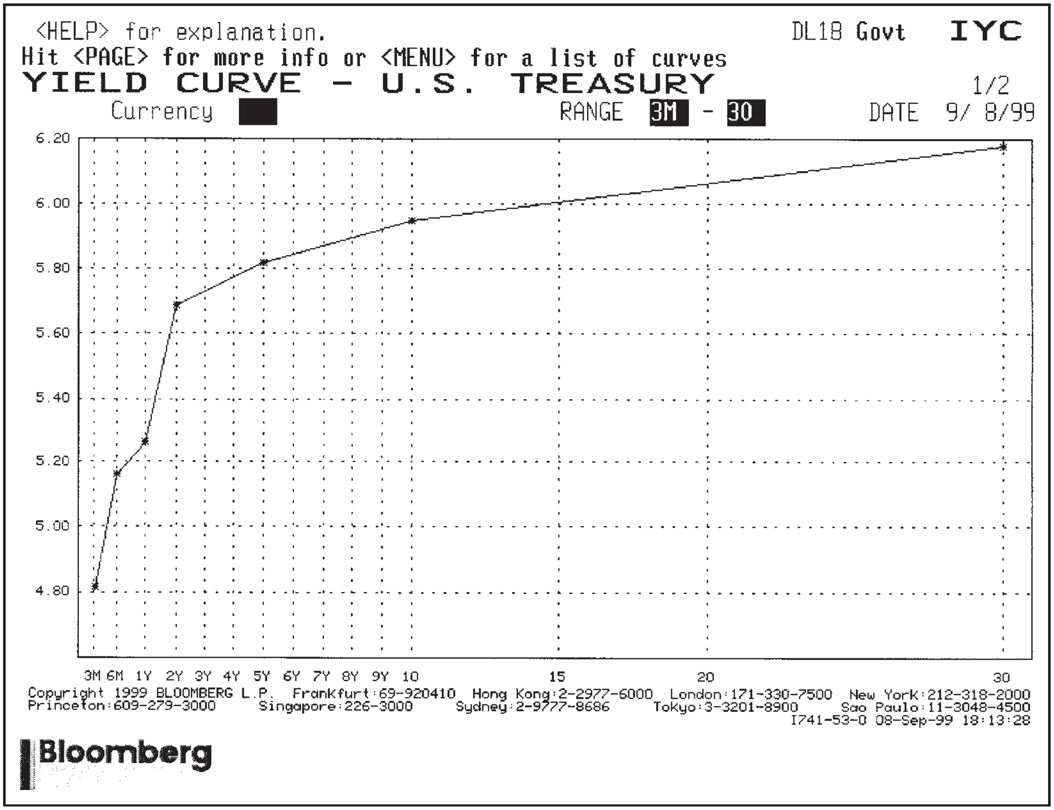


Figure 14.2 Yield curve for US Treasuries. Source: Bloomberg L.P.

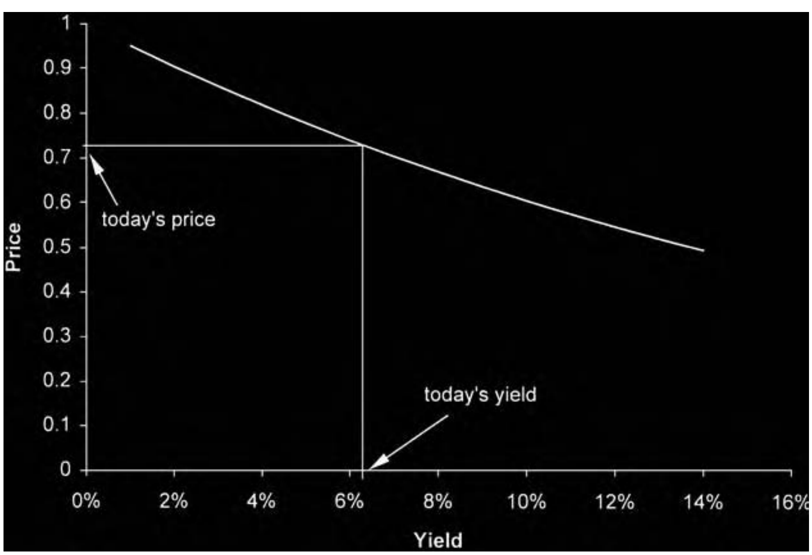


Figure 14.3 The price/yield relationship.

14.9 PRICE/YIELD RELATIONSHIP

From Equation (14.1) we can easily see that the relationship between the price of a bond and its yield is of the form shown in Figure 14.3 (assuming that all cash flows are positive). On this figure is marked the current market price and the current yield to maturity.

Since we are often interested in the sensitivity of instruments to the movement of certain underlying factors it is natural to ask how the price of a bond varies with the yield, or vice versa. To a first approximation this variation can be quantified by a measure called the duration.

Figure 14.4 shows the price/yield relationship for a specific five-year US Treasury.

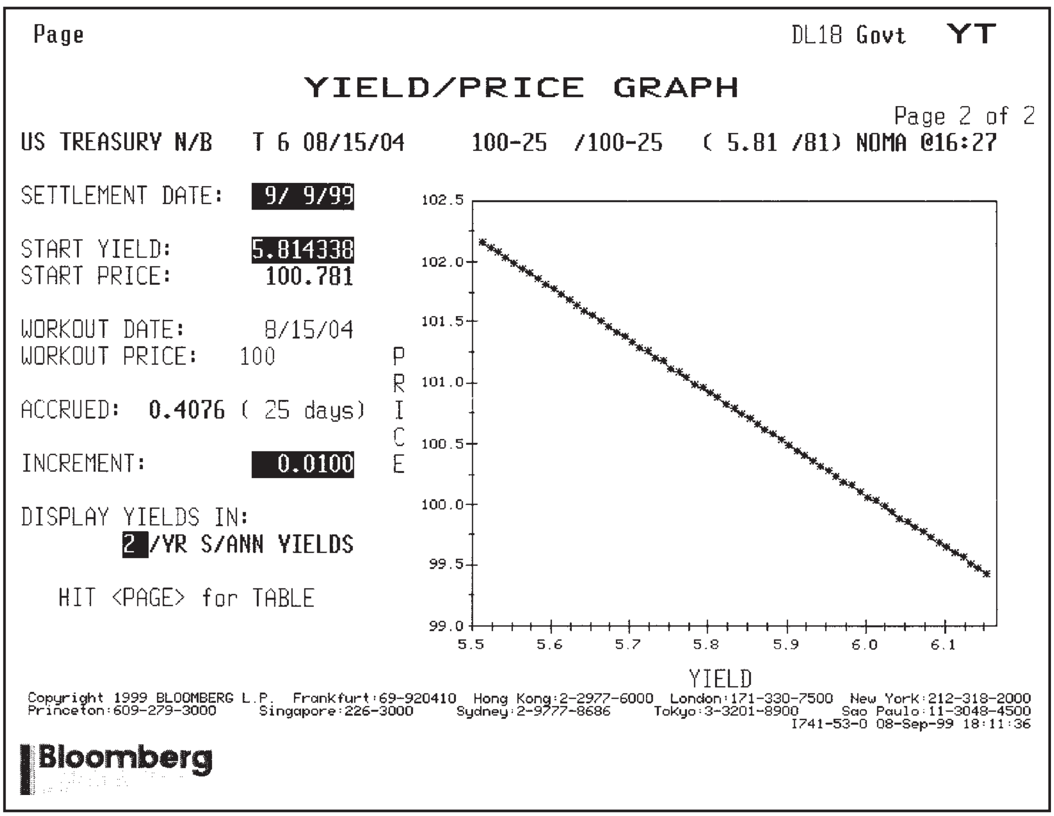
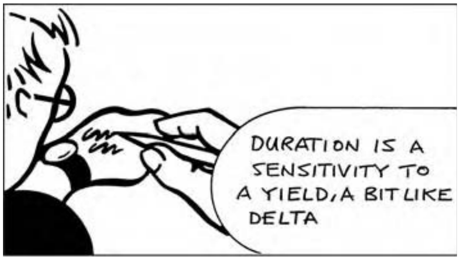


Figure 14.4 Bloomberg's price/yield graph. Source: Bloomberg L.P.

14.10 DURATION



From Equation (14.1) we find that

$$\frac{dV}{dy} = -(T-t)Pe^{-y(T-t)} - \sum_{i=1}^{N} C_i(t_i - t)e^{-y(t_i-t)}.$$

This is the slope of the price/yield curve. The quantity

$$-\frac{1}{V}\frac{dV}{dy}$$

is called the **Macaulay duration**. (The **modified duration** is similar but uses the discretely compounded rate.) In the expression for the duration the time of each coupon payment is weighted by its present value. The higher the value of the present value of the coupon the more it contributes to the duration. Also, since y is measured in units of inverse time, the units of the duration are time. The duration is a measure of the average life of the bond. It is easily shown that the Macaulay duration for a zero-coupon bond is the same as its maturity.

For small movements in the yield, the duration gives a good measure of the change in value with a change in the yield. For larger movements we need to look at higher-order terms in the Taylor series expansion of $V(y)$.

One of the most common uses of the duration is in plots of yield *versus* duration for a variety of instruments. An example is shown in Figure 14.5. Look at the bond marked 'CPU.' This bond has a coupon of 4.75% paid twice per year, callable from June 1998 and maturing in June 2000. We can use this plot to group together instruments with the same or similar durations and make comparisons between their yields. Two bonds having the same duration but with one bond having a higher yield might be suggestive of value for money in the higher-yielding bond, or of credit risk issues. However, such indicators of relative value must be used with care. It is possible for two bonds to have

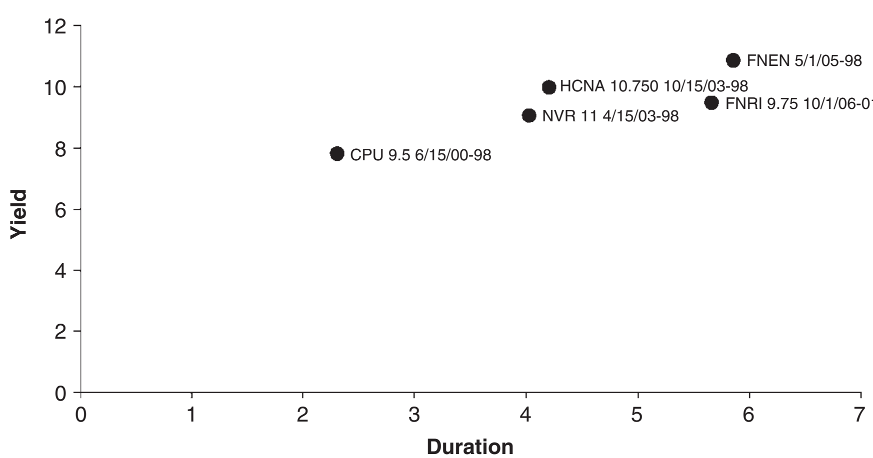


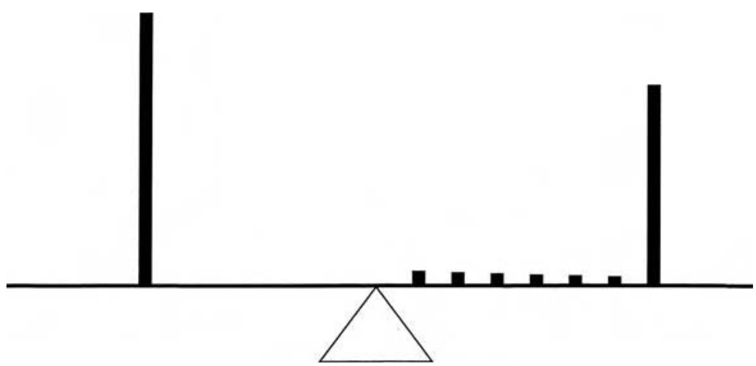
Figure 14.5 Yield versus duration: measuring the relative value of bonds.

vastly different cashflow profiles yet have the same duration; one may have a maturity of 30 years but an average life and hence a duration of seven years, whereas another may be a 7-year zero-coupon bond. Clearly, the former has 23 years more risk than the latter.

Time Out...

Duration = Average life. Why?

The figure below shows a lot of arrows balancing over a fulcrum. On the right-hand side the arrows are the cashflows associated with the bond we are analyzing. The height of each arrow represents the present value of the cashflow, its distance along the balance from the fulcrum represents the time to the cashflow. The sum of all the heights will therefore represent the known value of the bond. Question, where should an arrow be placed on the left-hand side, having the same height as the sum of all the right-hand arrows, to make a perfect balance? Answer, the distance along from the fulcrum representing the duration of the bond. So, it's rather like an average distance of all the bond cashflows, or the average life.



14.11 CONVEXITY

The Taylor series expansion of V gives

$$\frac{dV}{V} = \frac{1}{V}\frac{dV}{dy}\delta y + \frac{1}{2V}\frac{d^2V}{dy^2}(\delta y)^2 + \cdots,$$

where δy is a change in yield. For very small movements in the yield, the change in the price of a bond can be measured by the duration. For larger movements we must take account of the curvature in the price/yield relationship.

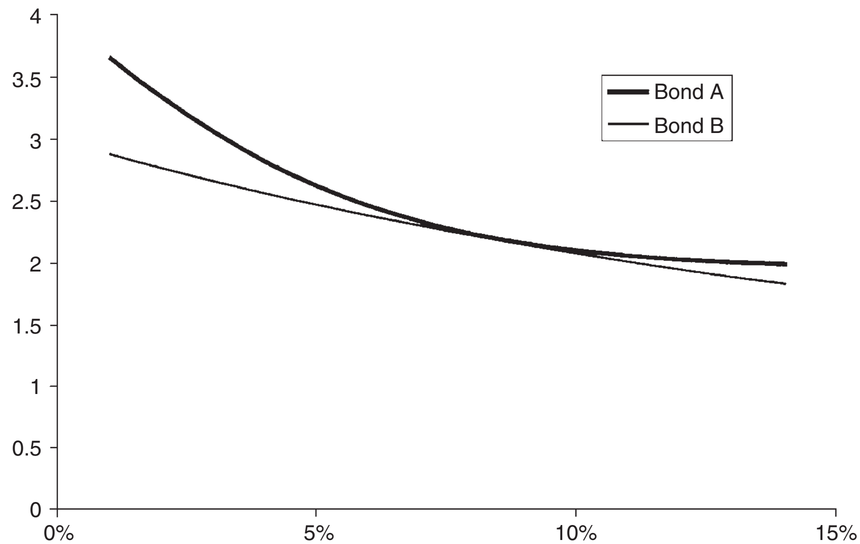


Figure 14.6 Two bonds with the same price and duration but different convexities.

The **dollar convexity** is defined as

$$\frac{d^2V}{dy^2} = (T-t)^2 P e^{-y(T-t)} + \sum_{i=1}^{N} C_i (t_i - t)^2 e^{-y(t_i-t)}.$$

and the **convexity** is

$$\frac{1}{V}\frac{d^2V}{dy^2}.$$

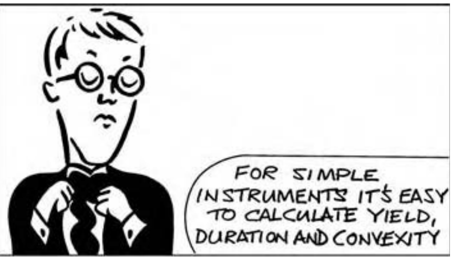
To see how these can be used, examine Figure 14.6.

In this figure we see the price/yield relationship for two bonds having the same value and duration when the yield is around 8%, but then they have different convexities. Bond A has a greater convexity than bond B. This figure suggests that bond A is better value than B because a small change in the yields results in a higher value for A. Convexity becomes particularly important in a hedged portfolio of bonds.

The calculation of yield to maturity, duration and convexity are shown in the following simple spreadsheet (Figure 14.7). Inputs are in the grey boxes.

14.12 AN EXAMPLE

Figure 14.8 shows the yield analysis screen from Bloomberg. The yield, duration and convexity have been calculated for a specific US Treasury. Figures 14.9 and 14.10 show time series of the price and yield respectively.



	A	B	C	D	E	F	G	H	I	J	K
1					Date	Coupon	Principal	PVs	Time	Time^2	
2									wtd	wtd	
3		YTM	4.95%					0.0195	0.0098	0.0049	
4		Mkt price	0.921					0.0190	0.0190	0.0190	
5		Th. Price	0.921					0.0186	0.0279	0.0418	
6		Error	1.4E-08					0.0181	0.0362	0.0725	
7		Duration	8.2544					0.0177	0.0442	0.1104	
8		Convexity	76.8728					0.0172	0.0517	0.1551	
9					3.5	2%		0.0168	0.0589	0.2060	
10					4	2%		0.0164	0.0656	0.2625	
11					4.5	2%		0.0160	0.0720	0.3241	
12					5	2%		0.0156	0.0781	0.3903	
13					5.5	2%		0.0152	0.0838	0.4607	
14					6	2%		0.0149	0.0892	0.5349	
15					6.5	2%		0.0145	0.0942	0.6124	
16					7	2%		0.0141	0.0990	0.6929	
17					7.5	2%		0.0138	0.1035	0.7760	
18					8	2%		0.0135	0.1077	0.8613	
19					8.5	2%		0.0131	0.1116	0.9485	
20					9	2%		0.0128	0.1153	1.0374	
21					9.5	2%		0.0125	0.1187	1.1276	
22					10	2%		0.6216	6.2161	62.1614	
23											
24											
25											
26											
27											
28											
29											
30											
31											
32											
33											
34											
35											
36											
37											

= SUM(H3:H22)

= C4-C5

= SUM(I3:I22)/C5

= SUM(J3:J22)/C5

= F20*EXP(-E20*$C$3)

= (G22+F22)*EXP(-E22*$C$3)

= E20*H20

= E20*I20

Figure 14.7 A spreadsheet showing the calculation of yield, duration and convexity.

14.13 HEDGING

In measuring and using yields to maturity, it must be remembered that the yield is the rate of discounting that makes the present value of a bond the same as its market value. A yield is thus identified with each individual instrument. It is perfectly possible for the yield

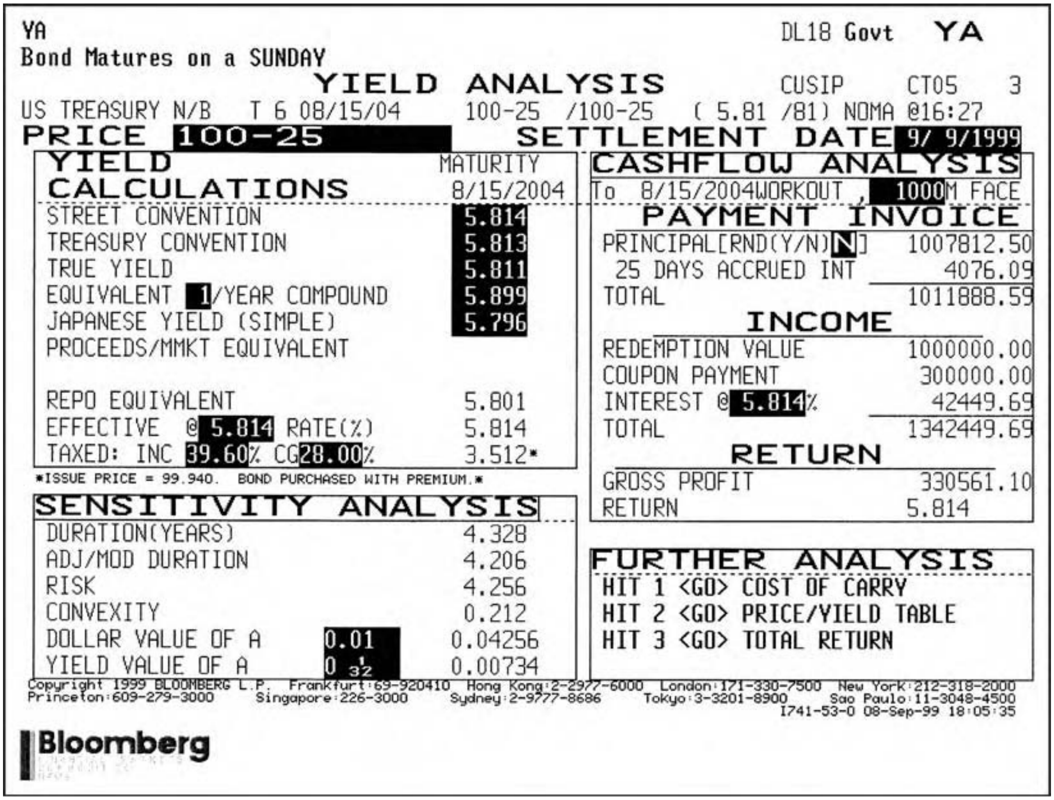


Figure 14.8 Yield analysis. Source: Bloomberg L.P.

on one instrument to rise while another falls, especially if they have significantly different maturities or durations. Nevertheless, one often wants to hedge movements in one bond with movements in another. This is commonly achieved by making one big assumption about the relative movements of yields on the two bonds. Bond A has a yield of 6.12%, bond B has a yield of 6.5%, they have different maturities and durations but we will assume that a move of x% in A's yield is accompanied by a move of x% in B's yield. This is the assumption of **parallel shifts** in the yield curve. If this is the case, then if we hold A bonds and B bonds in the inverse ratio of their durations (with one long position and one short) we will be leading-order hedged:

$$\Pi = V_A(y_A) - \Delta V_B(y_B),$$

with the obvious notation for the value and yield of the two bonds. The change in the value of this portfolio is

$$\delta\Pi = \frac{\partial V_A}{\partial y_A}x - \Delta\frac{\partial V_B}{\partial y_B}x + \text{ higher-order terms.}$$

Choose

$$\Delta = \frac{\partial V_A}{\partial y_A}\bigg/\frac{\partial V_B}{\partial y_B}$$

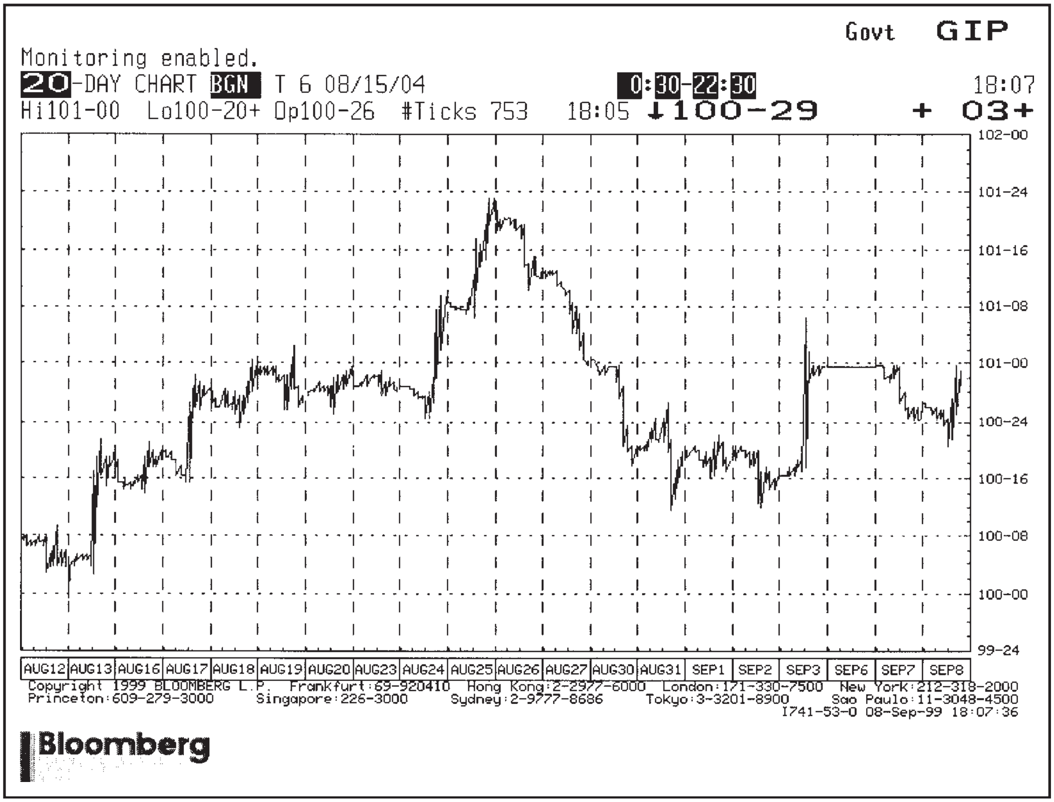


Figure 14.9 Price time series. Source: Bloomberg L.P.

to eliminate the leading-order risk. The higher-order terms depend on the convexity of the two instruments.

Of course, this is a simplification of the real situation; there may be little relationship between the yields on the two instruments, especially if the cash flows are significantly different. In this case there may be twisting or arching of the yield curve.

14.14 TIME-DEPENDENT INTEREST RATE

In this section we examine bond pricing when we have an interest rate that is a known function of time. The interest rate we consider will be what is known as a **short-term interest rate** or **spot interest rate** $r(t)$. This means that the rate $r(t)$ is to apply at time t: interest is compounded at this rate at each moment in time but *this rate may change*, generally we assume it to be time dependent.

If the spot interest rate $r(t)$ is a known function of time, then the bond price is also a function of time only: $V = V(t)$. (The bond price is, of course, also a function of maturity date T, but I suppress that dependence except when it is important.) We begin with a zero-coupon bond example. Because we receive 1 at time $t = T$ we know that $V(T) = 1$. I now derive an equation for the value of the bond at a time before maturity, $t < T$.

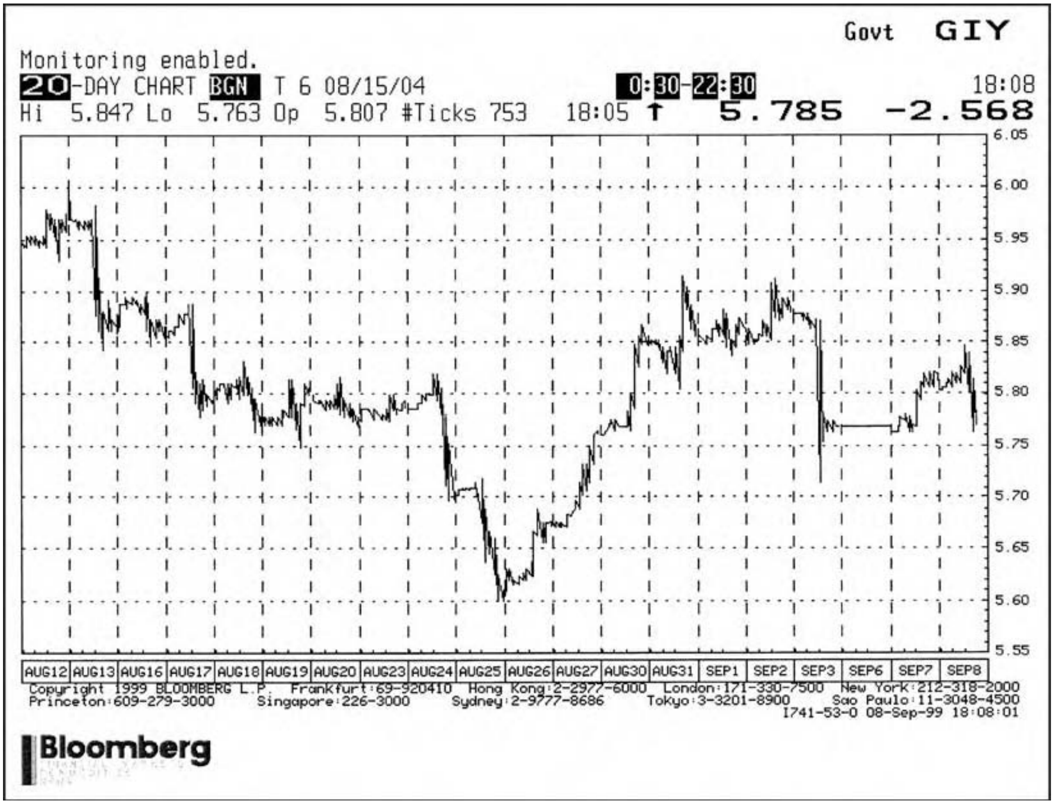


Figure 14.10 Yield time series. Source: Bloomberg L.P.

Suppose we hold one bond. The change in the value of that bond in a timestep dt (from t to $t + dt$) is

$$\frac{dV}{dt}\,dt.$$

Arbitrage considerations again lead us to equate this with the return from a bank deposit receiving interest at a rate $r(t)$:

$$\frac{dV}{dt} = r(t)V.$$

The solution of this equation is

$$V(t;T) = e^{-\int_t^T r(\tau)d\tau}. \tag{14.2}$$

Now let's introduce coupon payments. If during the period t to $t + dt$ we have received a coupon payment of $K(t)\,dt$, which may be either in the form of continuous or discrete payments or a combination, our holdings including cash change by an amount

$$\left(\frac{dV}{dt} + K(t)\right)dt.$$

Again setting this equal to the risk-free rate $r(T)$ we conclude that

$$\frac{dV}{dt} + K(t) = r(t)V. \tag{14.3}$$

The solution of this ordinary differential equation is easily found to be

$$V(t) = e^{-\int_t^T r(\tau)d\tau}\left(1 + \int_t^T K(t')\, e^{\int_{t'}^T r(\tau)d\tau} dt'\right); \tag{14.4}$$

the arbitrary constant of integration has been chosen to ensure that $V(T) = 1$.

14.15 DISCRETELY PAID COUPONS

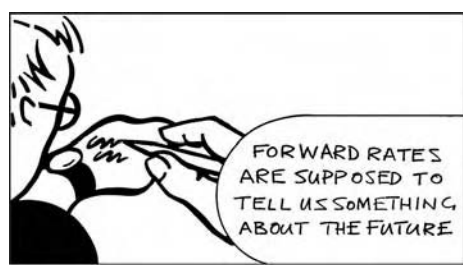
Equation (16.1) allows for the payment of a coupon. But what if the coupon is paid discretely, as it is in practice, for example, every six months? We can arrive at this result by a financial argument that will be useful later. Since the holder of the bond receives a coupon, call it K_c, at time t_c there must be a jump in the value of the bond across the coupon date. That is, the values before and after this date differ by K_c:

$$V(t_c^-) = V(t_c^+) + K_c.$$

This will be recognized as a jump condition. This time the realized bond price is *not* continuous. After all, there is a discrete payment at the coupon date. This jump condition will still apply when we come to consider stochastic interest rates.

Having built up a simple framework in which interest rates are time dependent I now show how to derive information about these rates from the market prices of bonds.

14.16 FORWARD RATES AND BOOTSTRAPPING



The main problem with the use of yield to maturity as a measure of interest rates is that it is not consistent across instruments. One five-year bond may have a different yield from another five-year bond if they have different coupon structures. It is therefore difficult to say that there is a single interest rate associated with a maturity.

One way of overcoming this problem is to use **forward rates**.

Forward rates are interest rates that are assumed to apply over given periods *in the future* for *all* instruments. This contrasts with yields which are assumed to apply up to maturity, with a different yield for each bond.

Let us suppose that we are in a perfect world in which we have a continuous distribution of zero-coupon bonds with all maturities T. Call the prices of these at time t $Z(t;T)$.

The **implied forward rate** is the curve of a time-dependent spot interest rate that is consistent with the market price of instruments. If this rate is $r(\tau)$ at time τ then it satisfies

$$Z(t;T) = e^{-\int_t^T r(\tau)d\tau}.$$

On rearranging and differentiating this gives

$$r(T) = -\frac{\partial}{\partial T}(\log Z(t;T)).$$

This is the forward rate for time T as it stands today, time t. Tomorrow the whole curve (the dependence of r on the future) may change. For that reason we usually denote the forward rate at time t applying at time T in the future as $F(t;T)$ where

$$F(t;T) = -\frac{\partial}{\partial T}(\log Z(t;T)).$$

In the less-than-perfect real world we must do with only a discrete set of data points. We continue to assume that we have zero-coupon bonds but now we will only have a discrete set of them. We can still find an implied forward rate curve as follows.

Rank the bonds according to maturity, with the shortest maturity first. The market prices of the bonds will be denoted by Z_i^M where i is the position of the bond in the ranking.

Using only the first bond, ask the question 'What interest rate is implied by the market price of the bond?' The answer is given by f_1, the solution of

$$Z_1^M = e^{-f_1(T_1-t)},$$

i.e.

$$f_1 = -\frac{\log(Z_1^M)}{T_1 - t}.$$

This rate will be the rate that we use for discounting between the present and the maturity date T_1 of the first bond. And it will be applied to *all* instruments whenever we want to discount over this period.

Now move on to the second bond having maturity date T_2. We know the rate to apply between now and time T_1, but at what interest rate must we discount between dates T_1 and T_2 to match the theoretical and market prices of the second bond? The answer is f_2 which solves the equation

$$Z_2^M = e^{-f_1(T_1-t)}e^{-f_2(T_2-T_1)},$$

i.e.

$$f_2 = -\frac{\log\left(Z_2^M/Z_1^M\right)}{T_2 - T_1}.$$

Generally,

$$f_i = -\frac{\log\left(Z_i^M/Z_{i-1}^M\right)}{T_i - T_{i-1}}.$$

By this method of **bootstrapping** we can build up the forward rate curve. Note how the forward rates are applied between two dates, for which period we have assumed they are constant: the forward rate between 5 and 7 years is 9.85%, say. Figure 14.11, gives an example.

This method can easily be extended to accommodate coupon-bearing bonds. Again rank the bonds by their maturities, but now we have the added complexity that we may only have one market value to represent the sum of several cashflows. Thus one often has to make some assumptions to get the right number of equations for the number of unknowns.

Given the market price of zero-coupon bonds it is very easy to calculate yields and forward rates, as shown in the spreadsheet. Inputs are in the grey boxes.

The yields and forward rates for this data are shown in Figures 14.12 and 14.13. Note that in each case the yield begins at zero maturity and extends up to the maturity of each bond. The forward rates pick up where the last forward rates left off.

	A	B	C	D	E	F
1	**Time to**	**Market**	**Yield to**	**Forward**		
2	**Maturity**	**price z-c b**	**maturity**	**rate**		
3	0.25	0.9809	7.71%	7.71%		
4	0.5	0.9612	7.91%	8.12%		
5	1	0.9194	8.40%	8.89%		
6	2	0.8436	8.50%	8.60%		
7	3	0.7772	8.40%	8.20%		
8	5	0.644	8.80%	9.40%		
9	7	0.5288	9.10%	9.85%		
10	10	0.3985	9.20%	9.43%		
11						
12	= -LN(B10)/A10					
13						
14						
15		= (C10*A10-C9*A9)/(A10-A9)				
16						
17						
18						
19						
20						
21						
22						
23						

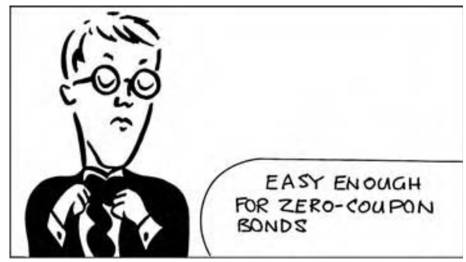


Figure 14.11 A spreadsheet showing the calculation of yields and forward rates from zero-coupon bonds.

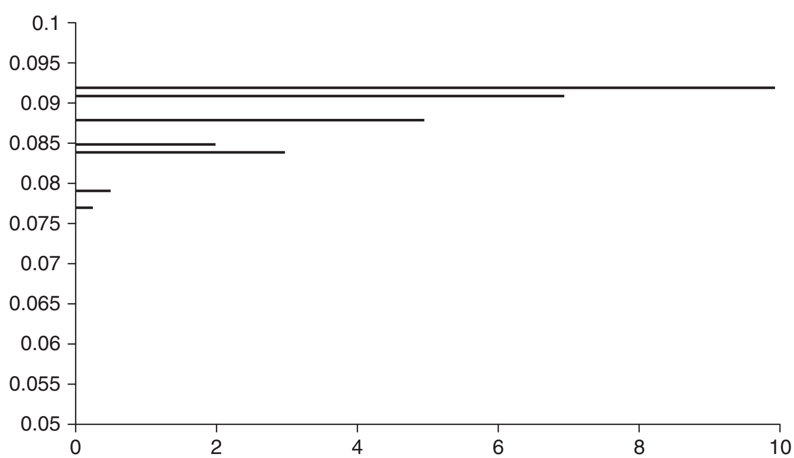


Figure 14.12 Yield to maturities.

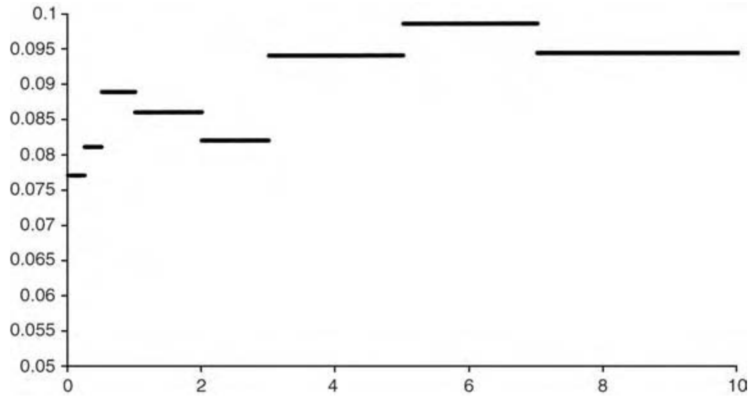


Figure 14.13 Forward rates.

There are far more swaps of different maturities than there are bonds, so that in practice swaps are used to build up the forward rates by bootstrapping. Fortunately, there is a simple decomposition of swaps prices into the prices of zero-coupon bonds so that bootstrapping is still relatively straightforward. Swaps are discussed in more detail in Chapter 15.

14.17 INTERPOLATION

We have explicitly assumed in the previous section that the forward rates are piecewise constant, jumping from one value to the next across the maturity of each bond. Other methods of 'interpolation' are also possible. For example, the forward rate curve could be made continuous, with piecewise constant gradient. Some people like to use cubic splines. The correct way of 'joining the dots' (for there are only a finite number of market prices) has been the subject of much debate. If you want to know what rate to apply to a $2\frac{1}{2}$-year cashflow and the nearest bonds are at two and three years then you will have to make some assumptions; there is no 'correct' value. Perhaps the best that can be done is to bound the rate.

14.18 SUMMARY

There are good and bad points about the interest rate model of this chapter. First, I mention the advantages.

Compare the simplicity of the mathematics in this chapter with that in previous chapters on option pricing. Clearly there is benefit in having models for which the analysis is so simple. Computation of many values and hedging can be performed virtually instantaneously on even slow computers. Moreover, it may be completely unnecessary to have a more complex model. For example, if we want to compare simple cashflows it may be possible to directly value one bond by summing other bonds, if their cashflows can be made to match. Such a situation, although uncommon, is market-independent modeling. Even if exact cashflow matches are not possible, there may be sufficiently close agreement for the differences to be estimated or at least bounded; large errors are easily avoided.

On the other hand, it is common experience that interest rates are unpredictable, random, and for complex products the movement of rates is the most important factor in their pricing. To assume that interest rates follow forward rates would be financial suicide in such cases. There is therefore a need for models more closely related to the stochastic models we have seen in earlier chapters.

In this chapter we saw simple yet powerful ways to analyze simple fixed-income contracts. These methods are used very frequently in practice, far more frequently than the complex methods we later discuss for the pricing of interest rate derivatives. The assumptions underlying the techniques, such as deterministic forward rates, are only relevant to simple contracts. As we have seen in the options world, more complex products with nonlinear payoffs, require a model that incorporates the stochastic nature of variables. Stochastic interest rates will be the subject of later chapters.

FURTHER READING

- The work of Macaulay (1938) on duration wasn't used much prior to the 1960s, but now it is considered fundamental to fixed-income analysis.
- See Fabozzi (1996) for a discussion of yield, duration and convexity in greater detail. He explains how the ideas are extended to more complicated instruments.
- The argument about how to join the yield curve dots is as meaningless as the argument between the Little-Endians and Big-Endians of Swift (1726).

CHAPTER 15
swaps

The aim of this Chapter...

... is to introduce the reader to the important world of swaps, one of the most important and fundamental financial contracts. We will also be seeing the simple relationship between swaps and bonds.

In this Chapter...

- the specifications of basic interest rate swap contracts
- the relationship between swaps and zero-coupon bonds
- exotic swaps

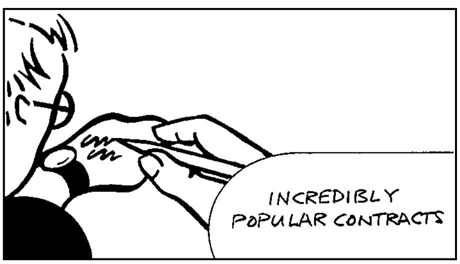
15.1 INTRODUCTION

A **swap** is an agreement between two parties to exchange, or swap, future cashflows. The size of these cashflows is determined by some formulas, decided upon at the initiation of the contract. The swaps may be in a single currency or involve the exchange of cashflows in different currencies.

The swaps market is big. The total notional principal amount is, in US dollars, currently comfortably in *14* figures. This market really began in 1981 although there were a small number of swap-like structures arranged in the 1970s. Initially the most popular contracts were currency swaps, discussed below, but very quickly they were overtaken by the interest rate swap.

15.2 THE VANILLA INTEREST RATE SWAP

In the **interest rate swap** the two parties exchange cashflows that are represented by the interest on a notional principal. Typically, one side agrees to pay the other a fixed interest rate and the cashflow in the opposite direction is a **floating rate**. The parties to a swap are shown schematically in Figure 15.1. One of the commonest floating rates used in a swap agreement is LIBOR, London Interbank Offer Rate.



Commonly in a swap, the exchange of the fixed and floating interest payments occur every six months. In this case the relevant LIBOR rate would be the six-month rate. At the maturity of the contract the principal is *not* exchanged.

Let me give an example of how such a contract works.

Example Suppose that we enter into a five-year swap on 8th July 2000, with semi-annual interest payments. We will pay to the other party a rate of interest fixed at 6% on a notional principal of \$100 million, the counterparty will pay us six-month LIBOR. The cashflows in this contract are shown in Figure 15.2. The straight lines denote a fixed rate of interest and thus a known amount, the curly lines are floating rate payments.

The first exchange of payments is made on 8th January 2001, six months after the deal is signed. How much money changes hands on that first date? We must pay $0.03 \times \$100,000,000 = \$3,000,000$. The cashflow in the opposite direction will be at six-month LIBOR, *as quoted six months previously*, i.e. at the initiation of the contract. This is a very important point. The LIBOR rate is set six months before it is paid, so that in the first exchange of payments the floating side is known. This makes the first exchange special.

The second exchange takes place on 8th July 2001. Again we must pay \$3,000,000, but now we receive LIBOR, as quoted on 8th January 2001. Every six months there is an

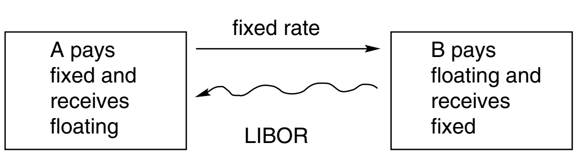


Figure 15.1 The parties to an interest rate swap.

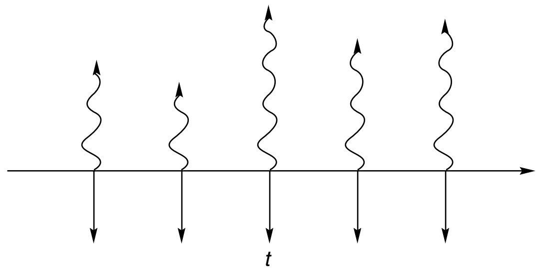


Figure 15.2 A schematic diagram of the cashflows in an interest rate swap.

exchange of such payments, with the fixed leg always being known and the floating leg being known six months before it is paid. This continues until the last date, 8th July 2005.

Why is the floating leg set six months before it is paid? This ‘minor’ detail makes a large difference to the pricing of swaps, believe it or not. It is no coincidence that the time between payments is the same as the maturity of LIBOR that is used, six months in this example. This convention has grown up because of the meaning of LIBOR, it is the rate of interest on a fixed-term maturity, set now and paid at the end of the term. Each floating leg of the swap is like a single investment of the notional principal six months prior to the payment of the interest. Hold that thought, we return to this point in a couple of sections, to show the simple relationship between a swap and bonds.

There is also the **LIBOR in arrears swap** in which the LIBOR rate paid on the swap date is the six-month rate set that day, not the rate set six months before.

15.3 COMPARATIVE ADVANTAGE

Swaps were first created to exploit **comparative advantage**. This is when two companies who want to borrow money are quoted fixed and floating rates such that by exchanging payments between themselves they benefit, at the same time benefitting the intermediary who puts the deal together. Here’s an example.

Two companies A and B want to borrow \$50MM, to be paid back in two years. They are quoted the interest rates for borrowing at fixed and floating rates shown in Table 15.1.

Note that both must pay a premium over LIBOR to cover risk of default, which is perceived to be greater for company B.

Ideally, company A wants to borrow at floating and B at fixed. If they each borrow directly then they pay the following:

The total interest they are paying is

$$\text{six-month LIBOR} + 30\text{ bps} + 8.2\% = \text{six-month LIBOR} + 8.5\%.$$

Table 15.1 Borrowing rates for companies A and B.

	Fixed	Floating
A	7%	six-month LIBOR + 30 bps
B	8.2%	six-month LIBOR + 100 bps

Table 15.2 Borrowing rates with no swap involved.

A	six-month LIBOR + 30 bps (floating)
B	8.2% (fixed)

If only they could get together they'd only be paying

$$\text{six-month LIBOR} + 100 \text{ bps} + 7\% = \text{six-month LIBOR} + 8\%.$$

That's a saving of 0.5%.

Let's suppose that A borrows fixed and B floating, even though that's not what they want. Their total interest payments are six-month LIBOR plus 8%. Now let's see what happens if we throw a swap into the pot.

*A is currently paying 7% and B six-month LIBOR plus 1%. They enter into a swap in which A pays LIBOR to B and B pays 6.95% to A. They have swapped interest payments.

Looked at from A's perspective they are paying 7% and LIBOR while receiving 6.95%, a net floating payment of LIBOR plus 5 bps. Not only is this floating, as A originally wanted, but it is 25 bps better than if they had borrowed directly at the floating rate. There's still another 25 bps missing, and, of course, B gets this. B pays LIBOR plus 100 bps and also 6.95% to A while receiving LIBOR from A. This nets out at 7.95%, which is fixed, as required, and 25 bps less than the original deal.

Where did I get the 6.95% from? Let's do the same calculation with 'x' instead of 6.95.

Go back to $*$. A is currently paying 7% and B six-month LIBOR plus 1%. They enter into a swap in which A pays LIBOR to B and B pays x% to A. They have swapped interest payments.

Looked at from A's perspective they are paying 7% and LIBOR while receiving x%, a net floating payment of LIBOR plus $7 - x$%. Now we want A to benefit by 25 bps over the original deal, this is half the 50 bps advantage. (I've just unilaterally decided to divide the advantage equally, 25 bps each.) So...

$$\text{LIBOR} + 7 - x + 0.25 = \text{LIBOR} + 0.3,$$

i.e.

$$x = 6.95\%.$$

Not only does A now get floating, as originally wanted, but it is 25 bps better than if they had borrowed directly at the floating rate. There's still another 25 bps missing, and, of course, B gets this. B pays LIBOR plus 100 bps and also 6.95% to A while receiving LIBOR from A. This nets out at 7.95%, which is fixed, as required, and 25 bps less than the original deal.

In practice the two counterparties would deal through an intermediary who would take a piece of the action.

Although comparative advantage was the original reason for the growth of the swaps market, it is no longer the reason for the popularity of swaps. Swaps are now very vanilla products existing in many maturities and more liquid than simple bonds.

Given the ubiquity of swaps you would expect the comparative advantage argument to have been arbed away. This is true. However, the arbitrage still exists in special circumstances. For example, floating loans usually come with provision for reviewing the spread over LIBOR every few months. If the company has become less creditworthy between reviews the spread will be increased. This is difficult to model or anticipate and so is outside the no-arbitrage concept.

15.4 THE SWAP CURVE

When the swap is first entered into it is usual for the deal to have no value to either party. This is done by a careful choice of the fixed rate of interest. In other words, the 'present value,' let us say, of the fixed side and the floating side both have the same value, netting out to zero. Consider the two extreme scenarios, very high fixed leg and very low fixed leg. If the fixed leg is very high the receiver of fixed has a contract with a high value. If the fixed leg is low the receiver has a contract that is worth a negative amount. Somewhere in between is a value that makes the deal valueless. The fixed leg of the swap is chosen for this to be the case.

Such a statement throws up many questions: How is the fixed leg decided upon? Why should both parties agree that the deal is valueless?

There are two ways to look at this. One way is to observe that a swap can be decomposed into a portfolio of bonds (as we see shortly) and so its value is not open

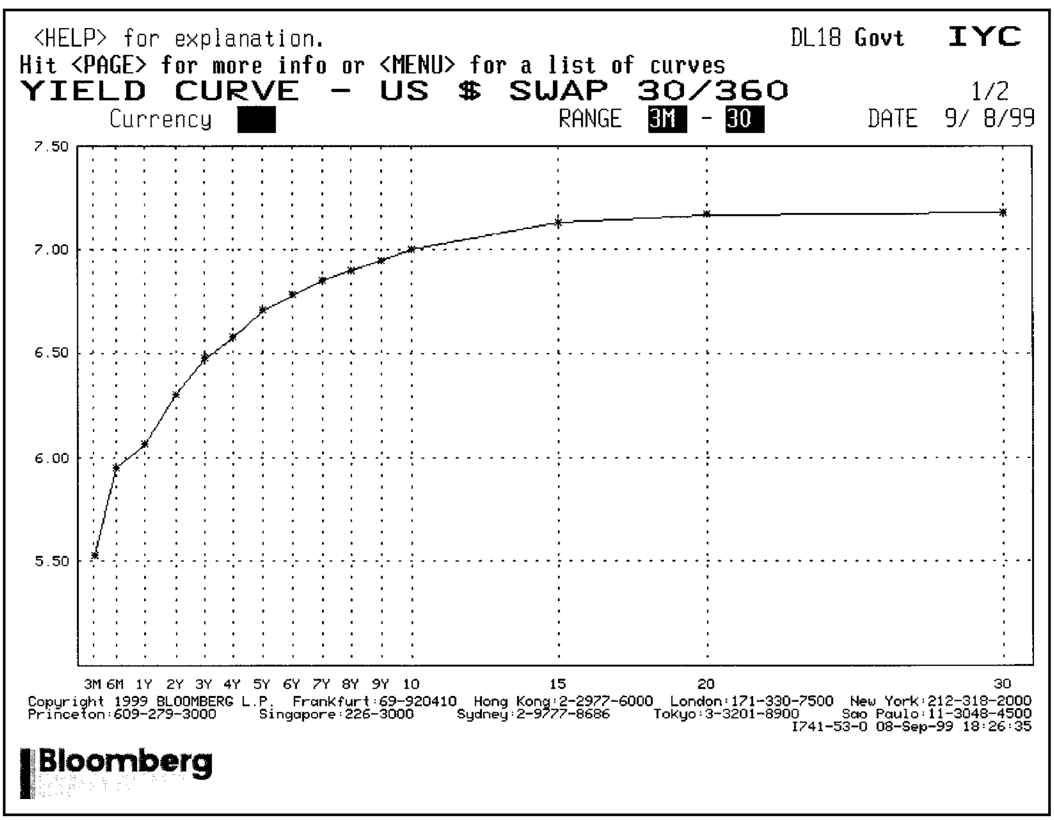


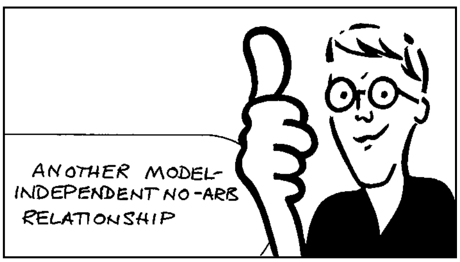
Figure 15.3 The swap curve. Source: Bloomberg L.P.

to question if we are given the yield curve. However, in practice the calculation goes the other way. The swaps market is so liquid, at so many maturities, that it is the prices of swaps that drive the prices of bonds. The fixed leg of a **par swap** (having no value) is determined by the market.

The rates of interest in the fixed leg of a swap are quoted at various maturities. These rates make up the **swap curve**, see Figure 15.3.

15.5 RELATIONSHIP BETWEEN SWAPS AND BONDS

There are two sides to a swap, the fixed-rate side and the floating-rate side. The fixed interest payments, since they are all known in terms of actual dollar amount, can be seen as the sum of zero-coupon bonds. If the fixed rate of interest is r_s then the fixed payments add up to



$$r_s \sum_{i=1}^{N} Z(t;T_i).$$

This is the value today, time t, of all the fixed-rate payments. Here there are N payments, one at each T_i. Of course, this is multiplied by the notional principal, but assume that we have scaled this to one.

To see the simple relationship between the floating leg and zero-coupon bonds I draw some schematic diagrams and compare the cashflows. A single floating leg payment is shown in Figure 15.4. At time T_i there is payment of r_τ of the notional principal, where r_τ is the period τ rate of LIBOR, set at time $T_i - \tau$. I add and subtract \$1 at time T_i to get the second diagram. The first and the second diagrams obviously have the same present value. Now recall the precise definition of LIBOR. It is the interest rate paid on a fixed-term deposit. Thus the $\$1 + r_\tau$ at time T_i is the same as \$1 at time $T_i - \tau$. This gives the third diagram. It follows that the single floating rate payment is equivalent to two zero-coupon bonds. A single floating leg of a swap at time T_i is *exactly* equal to a deposit of \$1 at time $T_i - \tau$ and a withdrawal of \$1 at time τ.

Now add up all the floating legs as shown in Figure 15.5, note the cancellation of all \$1 (dashed) cashflows except for the first and last. This shows that the floating side of the swap has value

$$1 - Z(t;T_N).$$

Bring the fixed and floating sides together to find that the value of the swap, to the receiver of the fixed side, is

$$r_s \sum_{i=1}^{N} Z(t;T_i) - 1 + Z(t;T_N).$$

This result is *model independent*. This relationship is independent of any mathematical model for bonds or swaps.

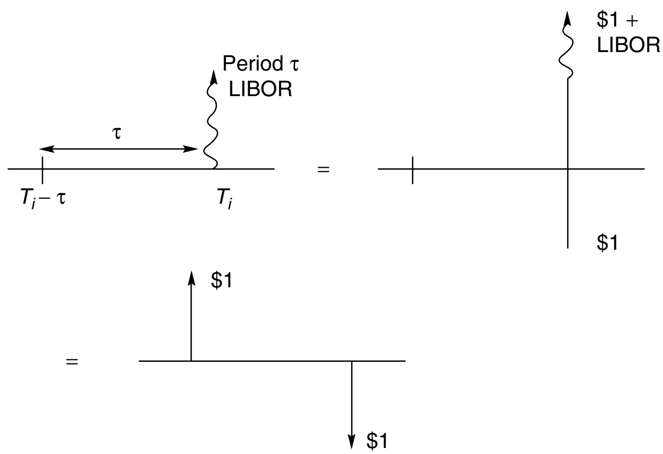


Figure 15.4 A schematic diagram of a single floating leg in an interest rate swap and equivalent portfolios.

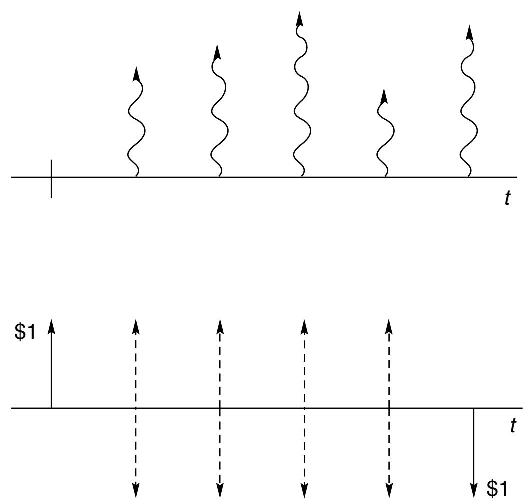


Figure 15.5 A schematic diagram of all the floating legs in a swap.

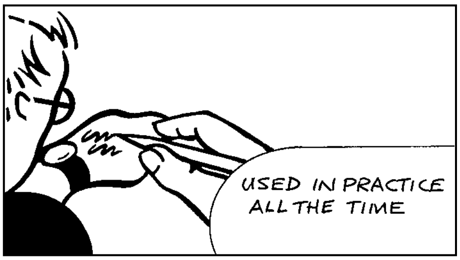
At the start of the swap contract the rate r_s is usually chosen to give the contract par value, i.e. zero value initially. Thus

$$\boxed{r_s = \frac{1 - Z(t;T_N)}{\sum_{i=1}^{N} Z(t;T_i)}.} \tag{15.1}$$

This is the quoted swap rate.

15.6 BOOTSTRAPPING

Swaps are now so liquid and exist for an enormous range of maturities that their prices determine the yield curve and not vice versa. In practice one is given $r_s(T_i)$ for many maturities T_i and one uses (15.1) to calculate the prices of zero-coupon bonds and thus the yield curve. For the first point on the discount-factor curve we must solve



$$r_s(T_1) = \frac{1 - Z(t; T_1)}{Z(t; T_1)},$$

i.e.

$$Z(t; T_1) = \frac{1}{1 + r_s(T_1)}.$$

After finding the first j discount factors the $(j + 1)$ th is found from

$$\boxed{Z(t : T_{j+1}) = \frac{1 - r_s(T_{j+1}) \sum_{i=1}^{j} Z(t; T_i)}{1 + r_s(T_{j+1})}}$$

Figure 15.6 shows the forward curve derived from the data in Figure 15.3 by bootstrapping.

15.7 OTHER FEATURES OF SWAPS CONTRACTS

The above is a description of the vanilla interest rate swap. There any many features that can be added to the contract that make it more complicated, and most importantly, model dependent. A few of these features are mentioned here.

Callable and puttable swaps A **callable** or **puttable swap** allows one side or the other to close out the swap at some time before its natural maturity. If you are receiving fixed and the floating rate rises more than you had expected you would want to close the position. Mathematically we are in the early exercise world of American-style options. The problem is model dependent and is discussed in Chapter 17.

Extendible swaps The holder of an **extendible swap** can extend the maturity of a vanilla swap at the original swap rate.

Index amortizing rate swaps The principal in the vanilla swap is constant. In some swaps the principal declines with time according to a prescribed schedule. The index amortizing rate swap is more complicated still with the amortization depending on the level of some index, say LIBOR, at the time of the exchange of payments. We will see this contract in some detail in Chapter 17.

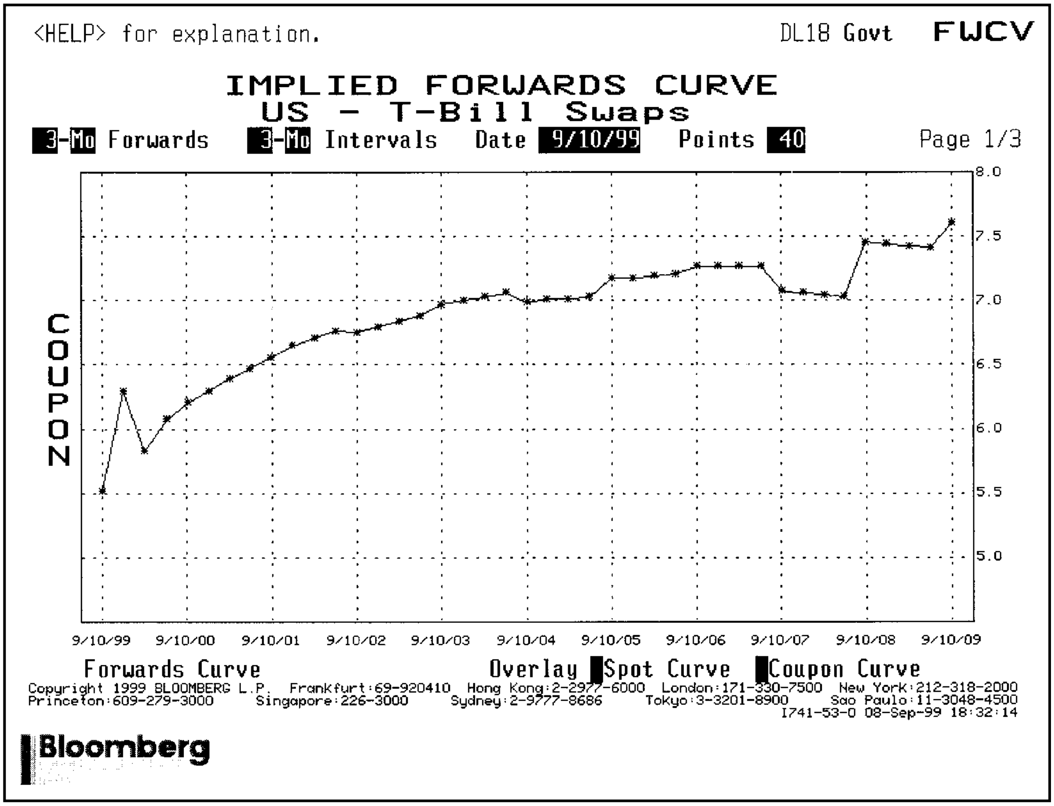


Figure 15.6 Forward rates derived from the swap curve by bootstrapping. Source: Bloomberg L.P.

15.8 OTHER TYPES OF SWAP

15.8.1 Basis rate swap

In the **basis rate swap** the floating legs of the swap are defined in terms of two distinct interest rates. For example, the prime rate versus LIBOR. A bank may have outstanding loans based on this prime rate but itself may have to borrow at LIBOR. It is thus exposed to **basis risk** and can be reduced with a suitable basis rate swap.

15.8.2 Equity swaps

The basic **equity swap** is an agreement to exchange two payments, one being an agreed interest rate (either fixed or floating) and the other depending on an equity index. This equity component is usually measured by the total return on an index, both capital gains and dividend are included. The principal is not exchanged.

The **equity basis swap** is an exchange of payments based on *two* different indices.

15.8.3 Currency swaps

A **currency swap** is an exchange of interest payments in one currency for payments in another currency. The interest rates can both be fixed, both floating or one of each. As well as the exchange of interest payments there is also an exchange of the principals (in two different currencies) at the beginning of the contract and at the end.

To value the fixed-to-fixed currency swap we need to calculate the present values of the cashflows in each currency. This is easily done, requiring the discount factors for the two currencies. Once this is done we can convert one present value to the other currency using the current *spot* exchange rate. If floating interest payments are involved we first decompose them into a portfolio of bonds (if possible) and value similarly.

15.9 SUMMARY

The need and ability to be able to exchange one type of interest payment for another is fundamental to the running of many businesses. This has put swaps among the most liquid of financial contracts. This enormous liquidity makes swaps such an important product that one has to be very careful in their pricing. In fact, swaps are so liquid that you do not price them in any theoretical way, to do so would be highly dangerous. Instead they are almost treated like an 'underlying' asset. From the market's view of the value we can back out, for example, the yield curve. We are helped in this by the fine detail of the swaps structure, the cashflows are precisely defined in a way that makes them exactly decomposable into zero-coupon bonds. And this can be done in a completely model-independent way. To finish this chapter I want to stress the importance of not using a model when a set of cashflows can be perfectly, statically and model-independently, hedged by other cashflows. Any mispricing, via a model, no matter how small could expose you to large and risk-free losses.

FURTHER READING

- Two good technical books on swaps are by Das (1994) and Miron & Swannell (1991).
- The pocketbook by Ungar (1996) describes the purpose of the swaps market, how it works and the different types of swaps, with no mathematics.

CHAPTER 16

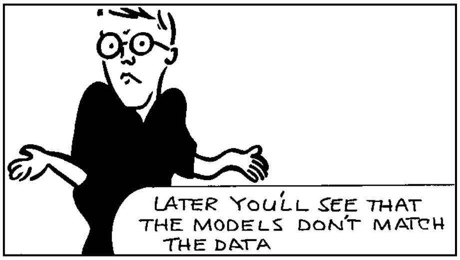
one-factor interest rate modeling

The aim of this Chapter...

... is to model interest rates as random walks and bring together the instruments of the fixed-income world and the modeling ideas of Black and Scholes. You will see many familiar ideas and a few new ones that are not seen in the context of equity derivatives.

In this Chapter...

- stochastic models for interest rates
- how to derive the bond pricing equation for many fixed-income products
- the structure of many popular interest rate models



16.1 INTRODUCTION

Until now I have assumed that interest rates are either constant or a known function of time. This may be a reasonable assumption for short-dated equity contracts. But for longer-dated contracts the interest rate must be more accurately modeled. This is not an easy task. In this chapter I introduce the ideas behind modeling interest rates using a single source of randomness. This is **one-factor interest rate modeling**. The model will allow the short-term interest rate, the spot rate, to follow a random walk. This model leads to a parabolic partial differential equation for the prices of bonds and other interest rate derivative products.

The 'spot rate' that we will be modeling is a very loosely defined quantity, meant to represent the yield on a bond of infinitesimal maturity. In practice one should take this rate to be the yield on a liquid finite-maturity bond, say one of one month. Bonds with one *day* to expiry do exist but their price is not necessarily a guide to other short-term rates. I will continue to be vague about the precise definition of the spot interest rate. We could argue that if we are pricing a complex product that is highly model dependent then the *exact* definition of the independent variable will be relatively unimportant compared with the choice of model.

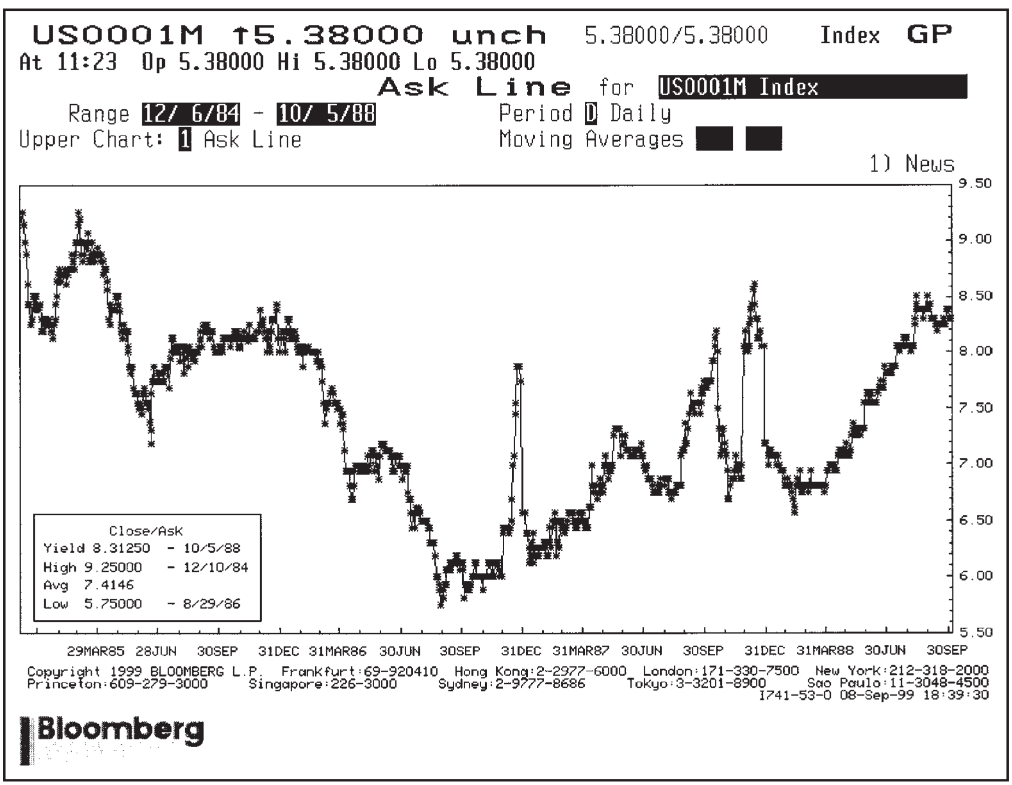


Figure 16.1 One-month interest rate time series. Source: Bloomberg L.P.

16.2 STOCHASTIC INTEREST RATES

Since we cannot realistically forecast the future course of an interest rate, it is natural to model it as a random variable. We are going to model the behavior of r, the interest rate received by the shortest possible deposit. From this we will see the development of a model for all other rates. The interest rate for the shortest possible deposit is commonly called the **spot interest rate**.

Figure 16.1 shows the time series of a one-month US interest rate. We will often use the one-month rate as a proxy for the spot rate.

Earlier I proposed a model for the asset price as a stochastic differential equation, the lognormal random walk. Now let us suppose that the interest rate r is governed by another stochastic differential equation of the form

$$dr = u(r, t)\, dt + w(r, t)\, dX. \tag{16.1}$$

The functional forms of $u(r, t)$ and $w(r, t)$ determine the behavior of the spot rate r. For the present I will not specify any particular choices for these functions. We use this random walk to derive a partial differential equation for the price of a bond using similar arguments to those in the derivation of the Black–Scholes equation. Later I describe functional forms for u and w that have become popular with practitioners.

Time Out...

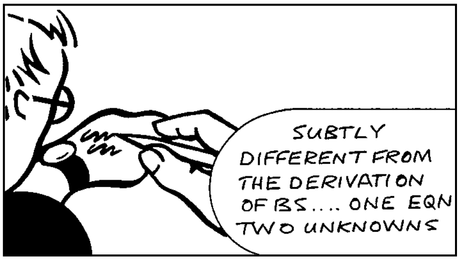
Intuition behind stochastic interest rates

Equation (16.1) is just another recipe for generating random numbers. Until now we've concentrated on the lognormal random walk as the model for asset prices. But there's no reason why interest rates should behave like stock prices, there's no reason why we should use the same model for interest rates as for equities. In fact, such a model would be a very poor one; interest rates certainly do not exhibit the long-term exponential growth seen in the equity markets.

So, we need another model. But we're going to use the same mathematical, stochastic framework, with subtly and suitably different forms. Modeling interest rates in this framework amounts to choosing functional forms for the dt and dX coefficients in our random walk recipe.

From a model for the short-term interest rate r will follow a model for bonds of all maturities and hence interest rates for all maturities. In other words, the spot interest rate model leads to a model for the whole forward curve.

I'll be taking the stochastic calculus and differential equation approach to the pricing of interest rate products. But it can all be done in a binomial or trinomial setting. Actually, trinomial is the more popular for interest rate products. The principle is the same as in the equity tree model. I'll give some details shortly.



16.3 THE BOND PRICING EQUATION FOR THE GENERAL MODEL

When interest rates are stochastic a bond has a price of the form $V(r, t; T)$. The reader should think for the moment in terms of simple bonds, but the governing equation will be far more general and may be used to price many other contracts. That's why I'm using the notation V rather than our earlier Z, for zero-coupon bonds.

Pricing a bond presents new technical problems, and is in a sense harder than pricing an option since *there is no underlying asset with which to hedge*. We are therefore not modeling a *traded* asset; the traded asset (the bond, say) is a derivative of our independent variable r. The only way to construct a hedged portfolio is by hedging one bond with a bond of a different maturity. We set up a portfolio containing two bonds with different maturities T_1 and T_2. The bond with maturity T_1 has price $V_1(r, t; T_1)$ and the bond with maturity T_2 has price $V_2(r, t; T_2)$. We hold one of the former and a number $-\Delta$ of the latter. We have

$$\Pi = V_1 - \Delta V_2. \tag{16.2}$$

The change in this portfolio in a time dt is given by

$$d\Pi = \frac{\partial V_1}{\partial t}dt + \frac{\partial V_1}{\partial r}dr + \tfrac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2}dt - \Delta\left(\frac{\partial V_2}{\partial t}dt + \frac{\partial V_2}{\partial r}dr + \tfrac{1}{2}w^2\frac{\partial^2 V_2}{\partial r^2}dt\right), \tag{16.3}$$

where we have applied Itô's lemma to functions of r and t. Which of these terms are random? Once you've identified them you'll see that the choice

$$\Delta = \frac{\partial V_1}{\partial r}\bigg/\frac{\partial V_2}{\partial r}$$

eliminates all randomness in $d\Pi$. This is because it makes the coefficient of dr zero. We then have

$$\begin{aligned} d\Pi &= \left(\frac{\partial V_1}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2} - \left(\frac{\partial V_1}{\partial r}\bigg/\frac{\partial V_2}{\partial r}\right)\left(\frac{\partial V_2}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V_2}{\partial r^2}\right)\right)dt \\ &= r\Pi\,dt = r\left(V_1 - \left(\frac{\partial V_1}{\partial r}\bigg/\frac{\partial V_2}{\partial r}\right)V_2\right)dt, \end{aligned}$$

where we have used arbitrage arguments to set the return on the portfolio equal to the risk-free rate. This risk-free rate is just the spot rate.

Collecting all V_1 terms on the left-hand side and all V_2 terms on the right-hand side we find that

$$\left(\frac{\partial V_1}{\partial r}\right)^{-1}\left(\frac{\partial V_1}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2} - rV_1\right) = \left(\frac{\partial V_2}{\partial r}\right)^{-1}\left(\frac{\partial V_2}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V_2}{\partial r^2} - rV_2\right)$$

At this point the distinction between the equity and interest-rate worlds starts to become apparent. This is *one* equation in *two* unknowns. Fortunately, the left-hand side is a function of T_1 but not T_2 and the right-hand side is a function of T_2 but not T_1. The only way for this to be possible is for both sides to be independent of the maturity date. Dropping the subscript from V, we have

$$\left(\frac{\partial V}{\partial r}\right)^{-1}\left(\frac{\partial V}{\partial t}+\tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}-rV\right)=a(r,t)$$

for some function $a(r,t)$. I shall find it convenient to write

$$a(r,t)=w(r,t)\lambda(r,t)-u(r,t);$$

for a given $u(r,t)$ and nonzero $w(r,t)$ this is always possible. The function $\lambda(r,t)$ is as yet unspecified.

The bond pricing equation is therefore

$$\boxed{\frac{\partial V}{\partial t}+\tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}+(u-\lambda w)\frac{\partial V}{\partial r}-rV=0} \qquad (16.4)$$

Time Out...

Is this like Black-Scholes?

Pretty much, yes. Mathematically, it's of the same form as the Black–Scholes equation, but with different coefficients in front of two of the partial derivative terms. That's why I like to teach people about BS before interest rates... the math is almost identical but there are no problems with one equation for two unknowns.

The downside of this kind of modeling for interest rates is rather severe. Finding the best (correct?) form for w and $u-\lambda w$ is not easy. And it's not even possible to determine $u-\lambda w$ from observing time series for r, since that time series depends on u and w not on λ.

To find a unique solution of (16.4) we must impose one final and two boundary conditions. The final condition corresponds to the payoff on maturity and so for a zero-coupon bond

$$V(r,T;T)=1.$$

Boundary conditions depend on the form of $u(r,t)$ and $w(r,t)$ and are discussed later for a special model.

It is easy to incorporate coupon payments into the model. If an amount $K(r, t)\,dt$ is received in a period dt then

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV + K(r, t) = 0.$$

When this coupon is paid discretely, arbitrage considerations lead to jump condition

$$V(r, t_c^-; T) = V(r, t_c^+; T) + K(r, t_c),$$

where a coupon of $K(r, t_c)$ is received at time t_c.

Time Out...

Pricing by binomial and trinomial trees

Remember how we built up the binomial tree in Chapter 5 for equities? The process is the same for interest rate products, after all, the pricing differential equation is mathematically very similar to the Black–Scholes equation. Here's how the binomial method works. There are several stages.

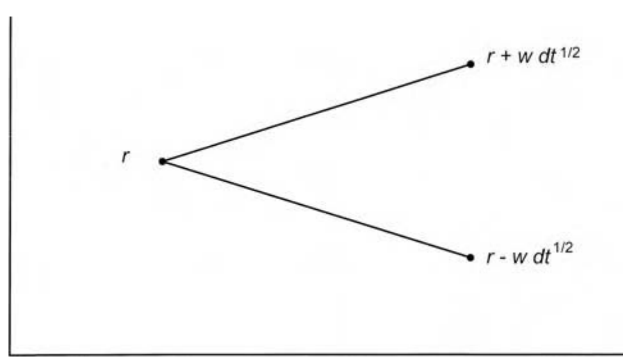
Stage 1: Build your tree There are several possibilities for this, just as there were when building up the equity tree. The simplest is to put all the diffusion into the up and down moves. For example, the interest rate r goes to

$$r + w\,\delta t^{1/2}$$

on an up move, or

$$r - w\,\delta t^{1/2}$$

on a down move. See the figure below.



Stage 2: Define the risk-neutral probabilities Simple. The probability of an up move is

$$\frac{1}{2} + \frac{u\,\delta t^{1/2}}{2w}.$$

But the *risk-neutral* probability is

$$\frac{1}{2} + \frac{(u - \lambda w)\,\delta t^{1/2}}{2w}.$$

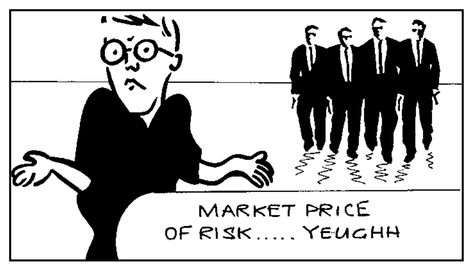
It's the risk-neutral probability you will use when working out expected values.

Stage 3: Discounting Discount at the rate r at the base of the two branches.

Now you just follow the same procedure as in Chapter 5 to work out contract values. You could even modify the VB code in that chapter for interest rate products.

Often trinomial models are used because of the extra degree of freedom they allow in choosing parameters — you are still only going to fit the volatility and risk-neutral drift.

16.4 WHAT IS THE MARKET PRICE OF RISK?



I now give an interpretation of the function $\lambda(r, t)$. Imagine that you hold an unhedged position in one bond with maturity date T. In a time-step dt this bond changes in value by

$$dV = w\frac{\partial V}{\partial r}dX + \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + u\frac{\partial V}{\partial r}\right)dt.$$

From (16.4) this may be written as

$$dV = w\frac{\partial V}{\partial r}dX + \left(w\lambda\frac{\partial V}{\partial r} + rV\right)dt,$$

or

$$dV - rV\,dt = w\frac{\partial V}{\partial r}(dX + \lambda\,dt). \tag{16.5}$$

The right-hand side of this expression contains two terms: a deterministic term in dt and a random term in dX. The presence of dX in (16.5) shows that this is not a riskless portfolio. The deterministic term may be interpreted as the excess return above the risk-free rate for accepting a certain level of risk. In return for taking the extra risk the portfolio profits by an extra $\lambda\,dt$ per unit of extra risk, dX. The function λ is therefore called the **market price of risk**.

16.5 INTERPRETING THE MARKET PRICE OF RISK, AND RISK NEUTRALITY

The bond pricing Equation (16.4) contains references to the functions $u - \lambda w$ and w. The former is the coefficient of the first-order derivative with respect to the spot rate, and the latter appears in the coefficient of the diffusive, second-order derivative. The four terms in the equation represent, in order as written, time decay, diffusion, drift and discounting. We can interpret the solution of the bond pricing equation as the expected present value of all cashflows. As with equity options, this expectation is not with respect to the *real* random variable, but instead with respect to the *risk-neutral* variable. There is this difference because the drift term in the equation is not the drift of the real spot rate u, but the drift of another rate, called the **risk-neutral spot rate**. This rate has a drift of $u - \lambda w$. When pricing interest rate derivatives (including bonds of finite maturity) it is important to model, and price, using the risk-neutral rate. This rate satisfies

$$dr = (u - \lambda w)\,dt + w\,dX.$$

We need the new market-price-of-risk term because our modeled variable, r, is not traded.

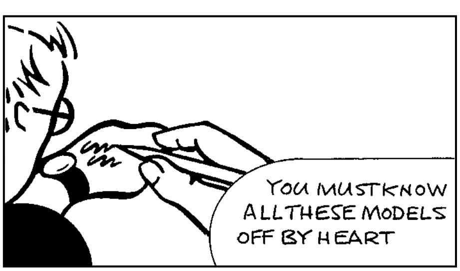
Because we can't observe the function λ, except possibly via the whole yield curve, I tend to think of it as a great big carpet under which we can brush all kinds of nasty, inconvenient things.

Time Out...

Contract values as expectations

Everything I said about the relationship between equity option values and expectations is the same in the interest rate world, with one minor extra subtlety. There is a simple relationship between expected payoffs and values. Here it is. *The value of an instrument is the risk-neutral expectation of the present value of the payoff.* Did you notice the subtle difference between the equity and fixed-income cases? In the world of random interest rates you have to take the present value before the expectation, because the discount factor will depend on the path of rates, not just the payoff. In the equity world it doesn't matter whether you PV first or last.

We exploit this relationship in Chapter 26, and see exactly how to price via simulations.



16.6 NAMED MODELS

There are many interest rate models, associated with the names of their inventors. The stochastic differential Equation (16.1) for the risk-neutral interest rate process incorporates the models of Vasicek, Cox, Ingersoll & Ross, Ho & Lee, and Hull & White.

16.6.1 Vasicek

The Vasicek model takes the form

$$dr = (\eta - \gamma r)\,dt + \beta^{1/2} dX.$$

This model is so 'tractable' that there are explicit formulas for many interest rate derivatives. The value of a zero-coupon bond is given by

$$e^{A(t;T)-rB(t;T)}$$

where

$$B = \frac{1}{\gamma}(1 - e^{-\gamma(T-t)})$$

and

$$A = \frac{1}{\gamma^2}(B(t;T) - T + t)\left(\eta\gamma - \frac{1}{2}\beta\right) - \frac{\beta B(t;T)^2}{4\gamma}.$$

The model is mean reverting to a constant level, which is a good property, but interest rates can easily go negative, which is a very bad property.

In Figure 16.2 are shown three types of yield curves predicted by the Vasicek model, each uses different parameters. (It is quite difficult to get the humped yield curve with reasonable numbers.)

16.6.2 Cox, Ingersoll & Ross

The CIR model takes the form

$$dr = (\eta - \gamma r)\,dt + \sqrt{\alpha r}\,dX.$$

The spot rate is mean reverting and if $\eta > \alpha/2$ the spot rate stays positive. There are some explicit solutions for interest rate derivatives, although typically involving integrals of the noncentral chi-squared distribution. The value of a zero-coupon bond is

$$e^{A(t;T)-rB(t;T)}$$

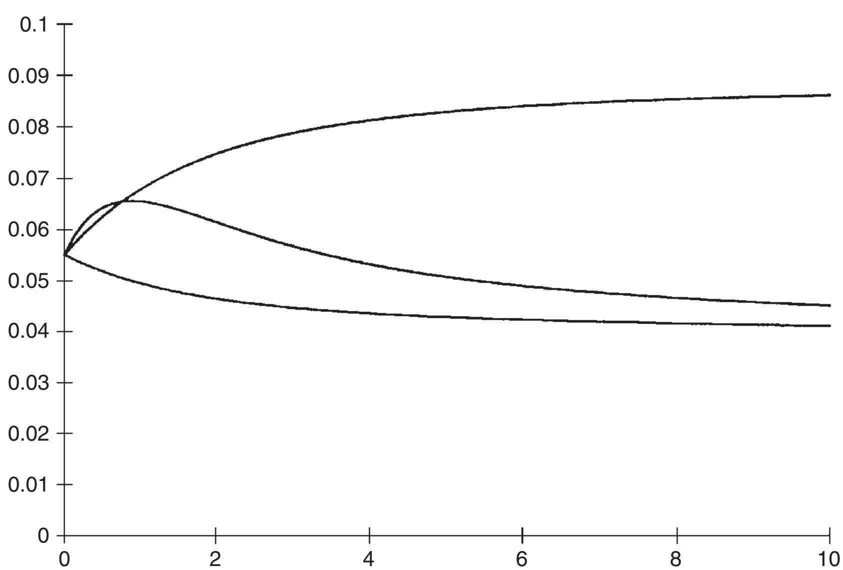


Figure 16.2 Three types of yield curve given by the Vasicek model.

where A and B are given by

$$\frac{\alpha}{2}A = a\psi_2 \log(a - B) + \psi_2 b \log((B + b)/b) - a\psi_2 \log a,$$

and

$$B(t;T) = \frac{2(e^{\psi_1(T-t)} - 1)}{(\gamma + \psi_1)(e^{\psi_1(T-t)} - 1) + 2\psi_1}$$

where

$$\psi_1 = \sqrt{\gamma^2 + 2\alpha} \quad \text{and} \quad \psi_2 = \frac{\eta}{a + b}$$

and

$$b, a = \frac{\pm\gamma + \sqrt{\gamma^2 + 2\alpha}}{\alpha}.$$

In Figure 16.3 are simulations of the Vasicek and CIR models using the same random numbers. The parameters have been chosen to give similar mean and standard deviations for the two processes.

16.6.3 Ho & Lee

Ho & Lee takes the form

$$dr = \eta(t)\,dt + \beta^{1/2}dX.$$

Note the function of time in this. The value of zero-coupon bonds is given by

$$e^{A(t;T) - rB(t;T)}$$

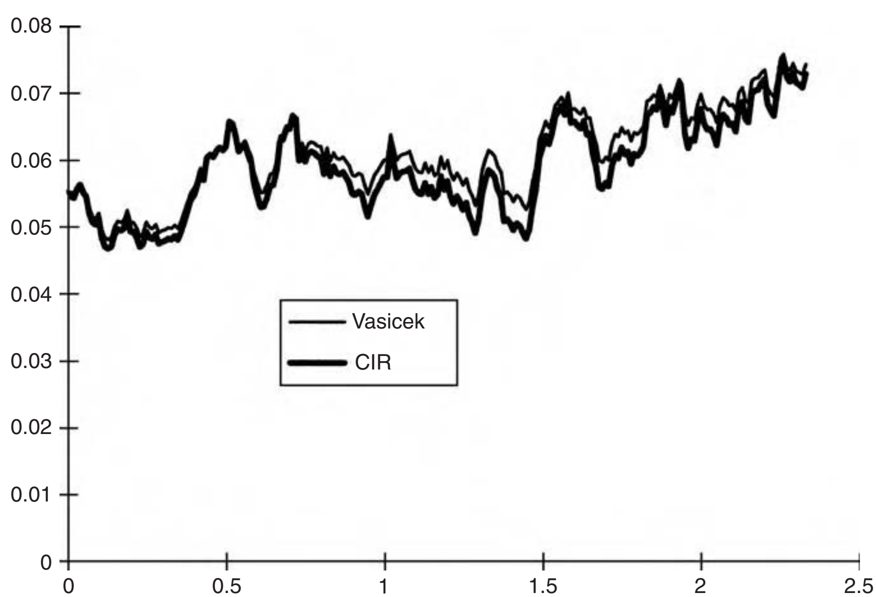


Figure 16.3 A simulation of the Vasicek and CIR models using the same random numbers.

where

$$B = T - t$$

and

$$A = -\int_t^T \eta(s)(T-s)\,ds + \tfrac{1}{6}\beta(T-t)^3.$$

This model was the first 'no-arbitrage model' of the term structure of interest rates. By this is meant that the careful choice of the function $\eta(t)$ will result in theoretical zero-coupon bond prices, output by the model, which are the same as market prices. This technique is also called **yield curve fitting**. This careful choice is

$$\eta(t) = -\frac{\partial^2}{\partial t^2}\log Z_M(t^*;t) + \beta(t-t^*)$$

where today is time $t = t^*$. In this $Z_M(t^*;T)$ is the market price today of zero-coupon bonds with maturity T. Clearly this assumes that there are bonds of all maturities and that the prices are twice differentiable with respect to the maturity.

16.6.4 Hull & White

Hull & White have extended both the Vasicek and the CIR models to incorporate time-dependent parameters. This time dependence again allows the yield curve (and even a volatility structure) to be fitted.

16.7 EQUITY AND FX FORWARDS AND FUTURES WHEN RATES ARE STOCHASTIC

Recall from Chapter 8 that forward prices and futures prices are the same if rates are constant? How does this change, if at all, when rates are stochastic? We must repeat the analysis of that chapter but now with

$$dS = \mu S\,dt + \sigma S\,dX_1$$

and

$$dr = u(r,t)\,dt + w(r,t)\,dX_2.$$

We are in the world of correlated random walks, as described in Chapter 11. The correlation coefficient is ρ.

16.7.1 Forward contracts

$V(S,r,t)$ will be the value of the forward contract at any time during its life on the underlying asset S, and maturing at time T. As in Chapter 8, I'll assume that the delivery price is known and then find the forward contract's value.

Set up the portfolio of one long forward contract and short Δ of the underlying asset, and Δ_1 of a risk-free bond:

$$\Pi = V(S,t) - \Delta S - \Delta_1 Z.$$

I won't go through all the details, because the conclusion is the obvious one:

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma S w \frac{\partial^2 V}{\partial S\,\partial r} + \tfrac{1}{2}w^2 \frac{\partial^2 V}{\partial r^2} + rS\frac{\partial V}{\partial S} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0.$$

The final condition for the equation is simply the difference between the asset price S and the fixed delivery price $\overline{S}$. So

$$V(S, r, T) = S - \overline{S}.$$

The solution of the equation with this final condition is

$$V(S, r, t) = S - \overline{S}Z.$$

At this point Z is not just any old risk-free bond, it is a zero-coupon bond having the same maturity as the forward contract. This is the forward contract's value during its life.

Remember that the delivery price is set initially at $t = t_0$ as the price that gives the forward contract zero value. If the underlying asset is S_0 at t_0 then

$$0 = S_0 - \overline{S}Z$$

or

$$\overline{S} = \frac{S_0}{Z}.$$

The quoted forward price is therefore

$$\text{Forward price } = \frac{S}{Z}.$$

Remember that Z satisfies

$$\frac{\partial Z}{\partial t} + \tfrac{1}{2}w^2 \frac{\partial^2 Z}{\partial r^2} + (u - \lambda w)\frac{\partial Z}{\partial r} - rZ = 0$$

with

$$Z(r, T) = 1.$$

16.8 FUTURES CONTRACTS

Use $F(S, r, t)$ to denote the futures price.

Set up a portfolio of one long futures contract and short Δ of the underlying, and Δ_1 of a risk-free bond:

$$\Pi = -\Delta S - \Delta_1 Z.$$

(Remember that the futures contract has no value.)

$$d\Pi = dF - \Delta\, dS - \Delta_1\, dZ.$$

Following the usual routine we get

$$\frac{\partial F}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + \rho\sigma S w \frac{\partial^2 F}{\partial S\,\partial r} + \tfrac{1}{2}w^2 \frac{\partial^2 F}{\partial r^2} + rS\frac{\partial F}{\partial S} + (u - \lambda w)\frac{\partial F}{\partial r} = 0.$$

The final condition is

$$F(S, r, T) = S.$$

Let's write the solution of this as

$$F(S, r, t) = \frac{S}{p(r, t)}.$$

Why? Two reasons. First a similarity solution is to be expected, the price should be proportional to the asset price. Second, I want to make a comparison between the futures price and the forward price. The latter is

$$\frac{S}{Z}.$$

So it's natural to ask, how similar are Z and p?

It turns out that p satisfies

$$\frac{\partial p}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 p}{\partial r^2} + (u - \lambda w)\frac{\partial p}{\partial r} - rp\underline{-w^2\frac{\left(\frac{\partial p}{\partial r}\right)^2}{q} + \rho\sigma\beta\frac{\partial p}{\partial r}} = 0. \qquad (16.6)$$

(Just plug the similarity form into the equation to see this.)

The final condition is

$$p(r, T) = 1.$$

The differences between the p and Z equations are in the underlined terms in Equation (16.6).

16.8.1 The convexity adjustment

There is clearly a difference between the prices of forwards and futures when interest rates are stochastic. From Equation (16.6) you can see that the difference depends on the volatility of the spot interest rate, the volatility of the underlying and the correlation between them. Provided that $\rho \geq 0$ the futures price is always greater than the equivalent forward price. Should the correlation be zero then the volatility of the stock is irrelevant. If the interest rate volatility is zero then rates are deterministic and forward and futures prices are the same.

Since the difference in price between forwards and futures depends on the spot rate volatility, market practitioners tend to think in terms of **convexity adjustments** to get from one to the other. Clearly, the convexity adjustment will depend on the precise nature of the model. For the popular models described above, the p Equation (16.6) still has simple solutions.

16.9 SUMMARY

In this chapter I introduced the idea of a random interest rate. The interest rate that we modeled was the 'spot rate,' a short-term interest rate. Several popular spot rate models were described. These models were chosen because simple forms of the coefficients make the solution of the basic bond pricing equation straightforward analytically.

FURTHER READING

- See the original interest rate models by Vasicek (1977), Dothan (1978), Cox *et al*. (1985), Ho & Lee (1986) and Black *et al*. (1990).
- For details of the general affine model see the papers by Pearson & Sun (1989), Duffie (1992), Klugman (1992) and Klugman & Wilmott (1994).
- The comprehensive book by Rebonato (1996) describes all of the popular interest rate models in detail.

CHAPTER 17

interest rate derivatives

The aim of this Chapter...

... is to examine some of the more important fixed-income products and to explain various ways in which to approach their pricing. You will see how it is common market practice to use simple Black–Scholes pricing formulas in novel ways.

In this Chapter...

- common fixed-income contracts such as bond options, caps and floors
- how to price interest rate products in the consistent partial differential equation framework
- how to price contracts the market way
- path dependency in interest rate products, such as the index amortizing rate swap

17.1 INTRODUCTION

In the first part of this book I derived a theory for pricing and hedging many different types of options on equities, currencies and commodities. In Chapter 16 I presented the theory for zero-coupon bonds, boldly saying that the model may be applied to other contracts.

In the equity options world we have seen different degrees of complexity. The simple contracts have no path dependency. These include the vanilla calls and puts and contracts having different final conditions such as binaries or straddles. At the next stage of complexity we find the path-dependent contracts such as American options or barriers for which, technically speaking, the path taken by the underlying is important. Many of these ideas are mirrored in the theory of interest rate derivatives.

In this chapter we delve deeper into the subject of fixed-income contracts by considering interest rate derivatives such as bond options, caps and floors, swaptions, captions and floortions, and more complicated and path-dependent contracts such as the index amortizing rate swap.

Time Out...

Pricing methodologies

I'm going to give you some insight into the two main methods for pricing interest rate derivatives. One way is consistent across all instruments but not necessarily accurate, the other is the exact opposite.

The former method is to price contracts using the same stochastic differential equation model for the spot rate and the resulting partial differential equation. Different contracts have different final and boundary conditions. The problem with this approach is that the basic model may not be that accurate. It can therefore be highly dangerous to use this method for pricing popular, highly liquid contracts.

The other approach is to bend and squeeze the instrument so as to make the 'underlying' (suitably defined) look like a lognormal asset. From then on, just apply the basic Black–Scholes formulas. The main positive point about this rather artificial method is that it is common practice in the market.

17.2 CALLABLE BONDS

As a gentle introduction to more complex fixed-income products, consider the **callable bond**. This is a simple coupon-bearing bond, but one that the issuer may call back on specified dates for a specified amount. The amount for which it may be called back may be time dependent. This feature reduces the value of the bond; if rates are low, so that the bond value is high, the issuer will call the bond back.

The callable bond satisfies

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0,$$

with

$$V(r, T) = 1,$$

and

$$V(r, t_c^-) = V(r, t_c^+) + K_c,$$

across coupon dates. If the bond can be called back for an amount $C(t)$ then we have the constraint on the bond's value

$$V(r, t) \leq C(t),$$

together with continuity of $\partial V/\partial r$.

17.3 BOND OPTIONS

The stochastic model for the spot rate presented in Chapter 16 allows us to value contingent claims such as bond options. A **bond option** is identical to an equity option except that the underlying asset is a bond. Both European and American versions exist.

As a simple example, we derive the differential equation satisfied by a call option, with exercise price E and expiry date T, on a zero-coupon bond with maturity date $T_B \geq T$. Before finding the value of the option to buy a bond we must find the value of the bond itself.

Let us write $Z(r, t; T_B)$ for the value of the bond. Thus, Z satisfies

$$\frac{\partial Z}{\partial t} + \tfrac{1}{2}w^2 \frac{\partial^2 Z}{\partial r^2} + (u - \lambda w)\frac{\partial Z}{\partial r} - rZ = 0 \tag{17.1}$$

with

$$Z(r, T_B; T_B) = 1$$

and suitable boundary conditions. Now write $V(r, t)$ for the value of the call option on this bond. Since V also depends on the random variable r, it too must satisfy Equation (17.1). The only difference is that the final value for the option is

$$V(r, T) = \max(Z(r, t; T_B) - E, 0).$$

This payoff is shown in Figure 17.1.

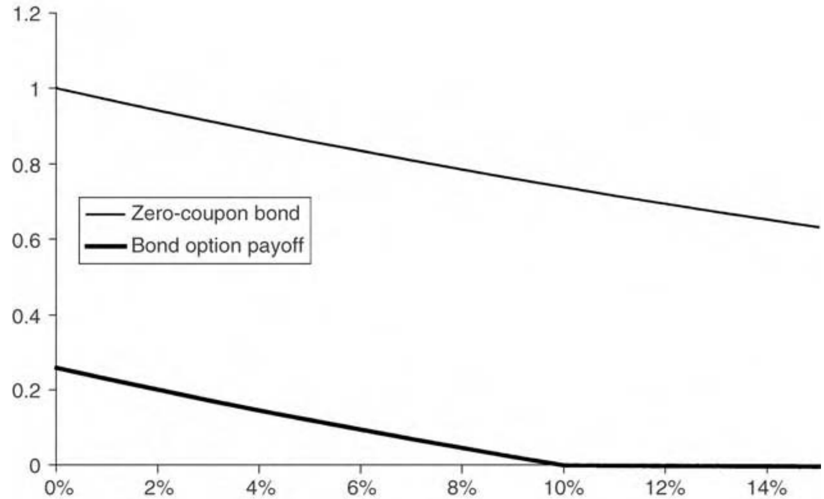


Figure 17.1 Zero-coupon bond price as a function of spot, and the payoff for a call option on the bond.

Figure 17.2 shows the Bloomberg option calculator for bond options. In this case the model used is Black, Derman & Toy.

Time Out...

Must we really solve the partial differential equation?

No. There are lots of other things you can do. Here's a brief summary of your choices.

- If there's no 'optionality' in the contract, and all cashflows are fixed or floating as in swaps, you should price by discounting using the yield curve. But first convert floating cashflows to fixed as described in Chapter 15.
- Use the above PDE approach if you feel comfortable with such concepts, and are happy to solve numerically by finite-difference methods. I hope you will feel happy with this by the end of the book.
- You can use trees, as hinted at in the previous chapter. Easy to understand, but not very sophisticated from a numerical point of view.
- Risk-neutral expectations are always there for you to fall back on. All the details are covered in Chapter 26.
- Finally, perhaps most popular and robust, fudge. This means pretend that you've got an equity derivative and not a fixed-income derivative and use a Black–Scholes-type formula. Some examples are given below.

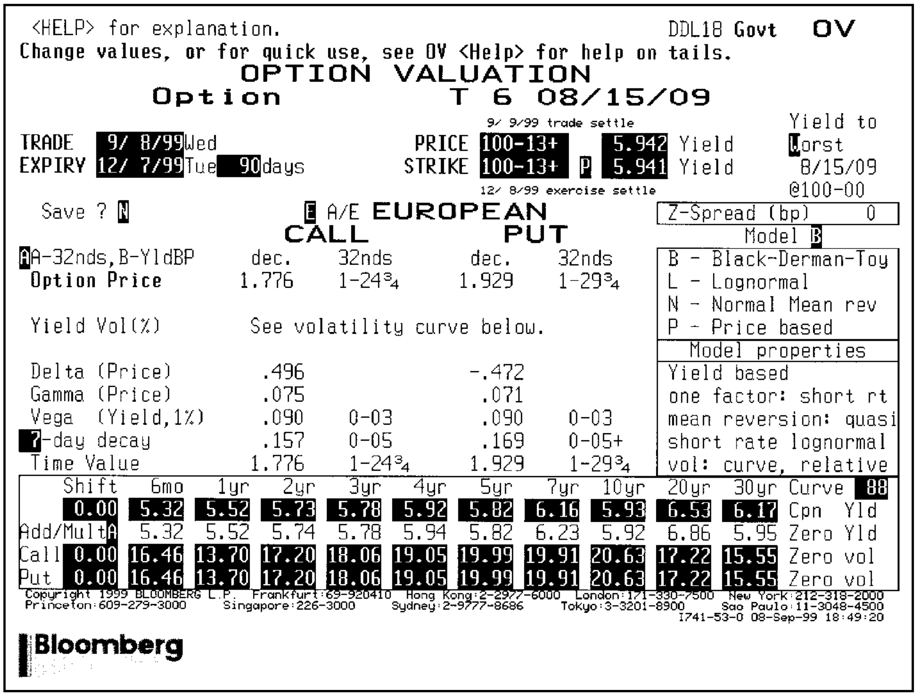


Figure 17.2 Bond option valuation. Source: Bloomberg L.P.

17.3.1 Market practice

The above is all well and good, but suffers from the problem that any inaccuracy in the model is magnified by the process of solving once for the bond and then again for the bond option. This makes the contract second order, see Chapter 12. When the time comes to exercise the option the amount you receive will, for a call, be the difference between the *actual* bond price and the exercise price, not the difference between the *theoretical* bond price and the exercise price. So the model had better be correct for the bond price. Of course, this model can never be correct, and so we must treat the pricing of bond options with care. Practitioners tend to use an approach that is internally inconsistent but which is less likely to be very wrong. They use the Black–Scholes equity option pricing equation and formulas assuming that the underlying is the bond. That is, they assume that the bond behaves like a lognormal asset. This requires them to estimate a volatility for the bond, either measured statistically or implied from another contract, and an interest rate for the lifetime of the bond option. This will be a good model provided that the expiry of the bond option is much shorter than the maturity of the underlying bond. Over short time periods, well away from maturity, the bond does behave stochastically, with a measurable volatility.

The price of a European bond call option in this model is

$$e^{-r(T-t)}(FN(d_1') - EN(d_2')),$$

and the put has value

$$e^{-r(T-t)}(EN(-d_2') - FN(-d_1')),$$

where F is the forward price of the bond at expiry of the option and

$$d_1' = \frac{\log(F/X) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2' = d_1' - \sigma\sqrt{T-t}.$$

This model should not be used when the life of the option is comparable to the maturity of the bond, because then there is an appreciable **pull to par**, that is, the value of the bond at maturity is the principal plus last coupon; the bond cannot behave lognormally close to maturity because we know where it must end up, this contrasts greatly with the behavior of an equity for which there is no date in the future on which we know its value for certain. This pull to par is shown in Figure 17.3.

Another approach used in practice is to model the yield to maturity of the underlying bond. The usual assumption is that this yield follows a lognormal random walk. By modeling the yield and then calculating the bond price based on this yield, we get a bond that behaves well close to its maturity; the pull to par is incorporated.

There is one technical point about the definition of the bond option concerning the meaning of 'price.' One must be careful to use whichever of the clean or dirty price is correct for the option in question. This amounts to knowing whether or not accrued interest should be included in the payoff, see Chapter 14.

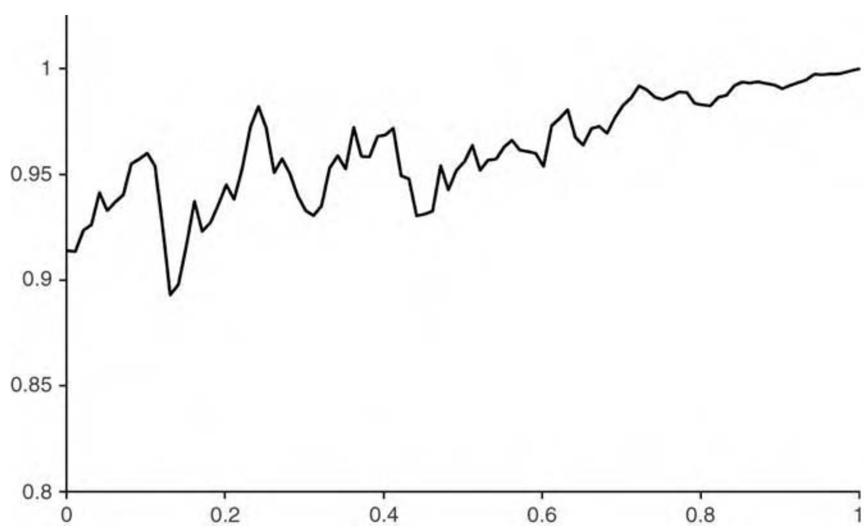


Figure 17.3 The pull to par for a zero-coupon bond.

17.4 CAPS AND FLOORS

A **cap** is a contract that guarantees to its holder that otherwise floating rates will not exceed a specified amount; the variable rate is thus capped.

A typical cap contract involves the payment at times t_i, each quarter, say, of a variable interest on a principal with the cashflow taking the form

$$\max(r_L - r_c, 0),$$

multiplied by the principal. Here r_L is the basic floating rate, for example three-month LIBOR if the payments are made quarterly, and r_c is the fixed cap rate. These payments continue for the lifetime of the cap. The rate r_L to be paid at time t_i is set at time t_{i-1}. Each of the individual cashflows is called a **caplet**, a cap is thus the sum of many caplets.

The cashflow of a caplet is shown in Figure 17.4.

If we assume that the actual floating rate is the spot rate, i.e. $r_L \approx r$ (and this approximation may not be important), then a single caplet may be priced by solving

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0, \tag{17.2}$$

with

$$V(r, T) = \max(r - r_c, 0).$$

Mathematically, this is similar to a call option on the floating rate r.

Figure 17.5 shows the Bloomberg calculator for caps.

A **floor** is similar to a cap except that the floor ensures that the interest rate is bounded below, by r_f. A floor is made up of a sum of floorlets, each of which has a cashflow of

$$\max(r_f - r_L, 0).$$

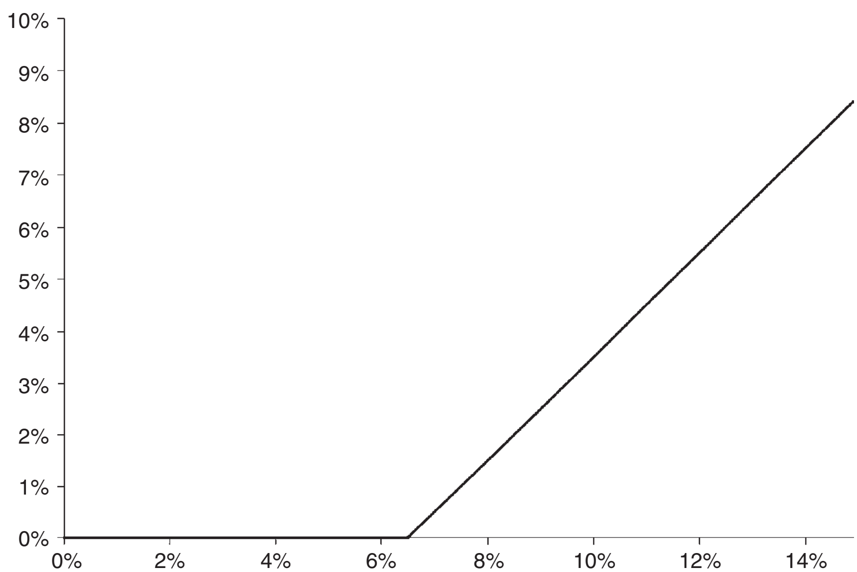


Figure 17.4 The dependence of the payment of a caplet on LIBOR.

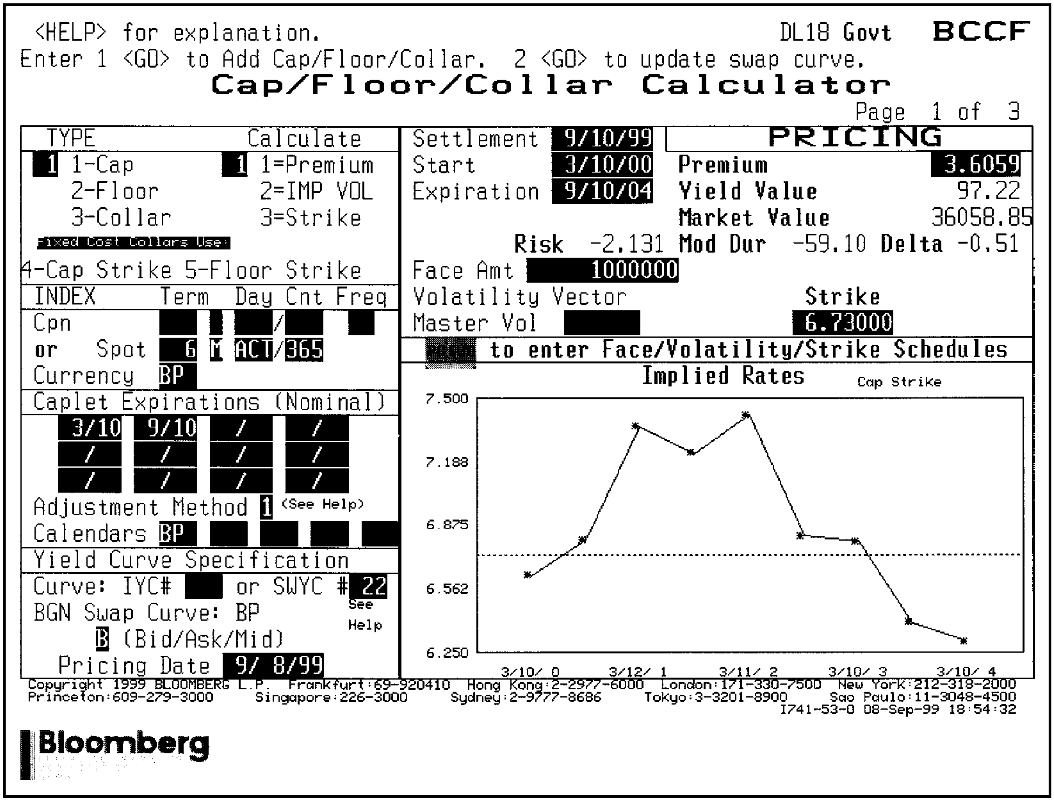


Figure 17.5 Cap/floor/collar calculator. Source: Bloomberg L.P.

We can approximate r_L by r again, in which case the floorlet satisfies the bond pricing equation but with

$$V(r, T) = \max(r_f - r, 0).$$

A floorlet is thus a put on the spot rate.

17.4.1 Cap/floor parity

A portfolio of a long caplet and a short floorlet (with $r_c = r_f$) has the cashflow

$$\max(r_L - r_c, 0) - \max(r_c - r_L, 0) = r_L - r_c.$$

This is the same cashflow as one payment of a swap. Thus there is the model-independent no-arbitrage relationship

$$\text{cap} = \text{floor} + \text{swap}.$$

17.4.2 The relationship between a caplet and a bond option

A caplet has the following cashflow:

$$\max(r_L - r_c, 0).$$

This is received at time t_i but the rate r_L is set at t_{i-1}. This cashflow is *exactly* the same as the cashflow

$$\frac{1}{1+r_L}\max(r_L - r_c, 0)$$

received at time t_{i-1}, after all, that is the definition of r_L. We can rewrite this cashflow as

$$\max\left(1 - \frac{1+r_c}{1+r_L}, 0\right).$$

But

$$\frac{1+r_c}{1+r_L}$$

is the price at time t_{i-1} of a bond paying $1 + r_c$ at time t_i. We can conclude that a caplet is equivalent to a put option expiring at time t_{i-1} on a bond maturing at time t_i.

17.4.3 Market practice

Again, because the Black–Scholes formulas are so simple to use, it is common to use them to price caps and floors. This is done as follows. Each individual caplet (or floorlet) is priced as a call (or put) on a lognormally distributed interest rate. The inputs for this model are the volatility of the interest rate, the strike price r_c (or r_f), the time to the cashflow $t_i - t$, and *two* interest rates. One interest rate takes the place of the stock price and will be the current forward rate applying between times t_{i-1} and t_i. The other interest rate, used for discounting to the present is the yield on a bond maturing at time t_i. For a caplet the relevant formula is

$$e^{-r^*(t_i-t)}\left(F(t, t_{i-1}, t_i)N(d_1') - r_c N(d_2')\right).$$

Here $F(t, t_{i-1}, t_i)$ is the forward rate today between t_{i-1} and t_i, r^* is the yield to maturity for a maturity of $t_i - t$,

$$d_1' = \frac{\log(F/r_c) + \frac{1}{2}\sigma^2 t_{i-1}}{\sigma\sqrt{t_{i-1}}} \quad \text{and} \quad d_2' = d_1' - \sigma\sqrt{t_{i-1}}.$$

σ is the volatility of the $(t_i - t_{i-1})$ interest rate. The floorlet value is

$$e^{-r^*(t_i-t)}\left(-F(t, t_{i-1}, t_i)N(-d_1') + r_c N(-d_2')\right).$$

17.4.4 Collars

A **collar** places both an upper and a lower bound on the interest payments. It can be valued as a long cap and a short floor.

17.4.5 Step-up swaps, caps and floors

Step-up swaps etc. have swap (cap etc.) rates that vary with time in a prescribed manner.

17.5 RANGE NOTES

The **range note** pays interest on a notional principal for every day that an interest rate lies between prescribed lower and upper bounds. Let us assume that the relevant interest rate is our spot rate r. In this case we must solve

$$\frac{\partial V}{\partial t}+\tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}+(u-\lambda w)\frac{\partial V}{\partial r}-rV+\mathcal{I}(r)=0,$$

with

$$V(r,T)=0,$$

where

$$\mathcal{I}(r)=r \quad \text{if} \quad r_l<r<r_u \quad \text{and is zero otherwise.}$$

This is only an approximation to the correct value since in practice the relevant interest rate will have a finite (not infinitesimal) maturity.

17.6 SWAPTIONS, CAPTIONS AND FLOORTIONS

A swaption has a strike rate, r_E, that is the fixed rate that will be swapped against floating if the option is exercised. In a call swaption or **payer swaption** the buyer has the right to become the fixed rate payer, in a put swaption or receiver swaption the buyer has the right to become the payer of the floating leg.

Captions and **floortions** are options on caps and floors respectively. These contracts can be put into the partial differential equation framework with little difficulty. However, these contracts are second order, meaning that their value depends on another, more basic, contract, see Chapter 12. Although the partial differential equation approach is possible, and certainly consistent across instruments, it is likely to be time consuming computationally and prone to serious mispricings because of the high order of the contracts.

17.6.1 Market practice

With some squeezing the Black–Scholes formulas can be used to value European swaptions. Perhaps this is not entirely consistent, but it is certainly easier than solving a partial differential equation.

The underlying is assumed to be the fixed leg of a par swap with maturity T_S, call this r_f. It is assumed to follow a lognormal random walk with a measurable volatility. If at time T the par swap rate is greater than the strike rate r_E then the option is in the money. At this time the value of the swaption is

$$\max(r_f-r_E,0)\times \text{present value of all future cashflows.}$$

It is important that we are 'modeling' the par rate because the par rate measures the rate at which the present value of the floating legs is equal to the present value of the fixed legs. Thus in this expression we only need worry about the excess of the par rate over the strike rate. This expression looks like a typical call option payoff, all we need to price the swaption in the Black–Scholes framework are the volatility of the par rate, the times

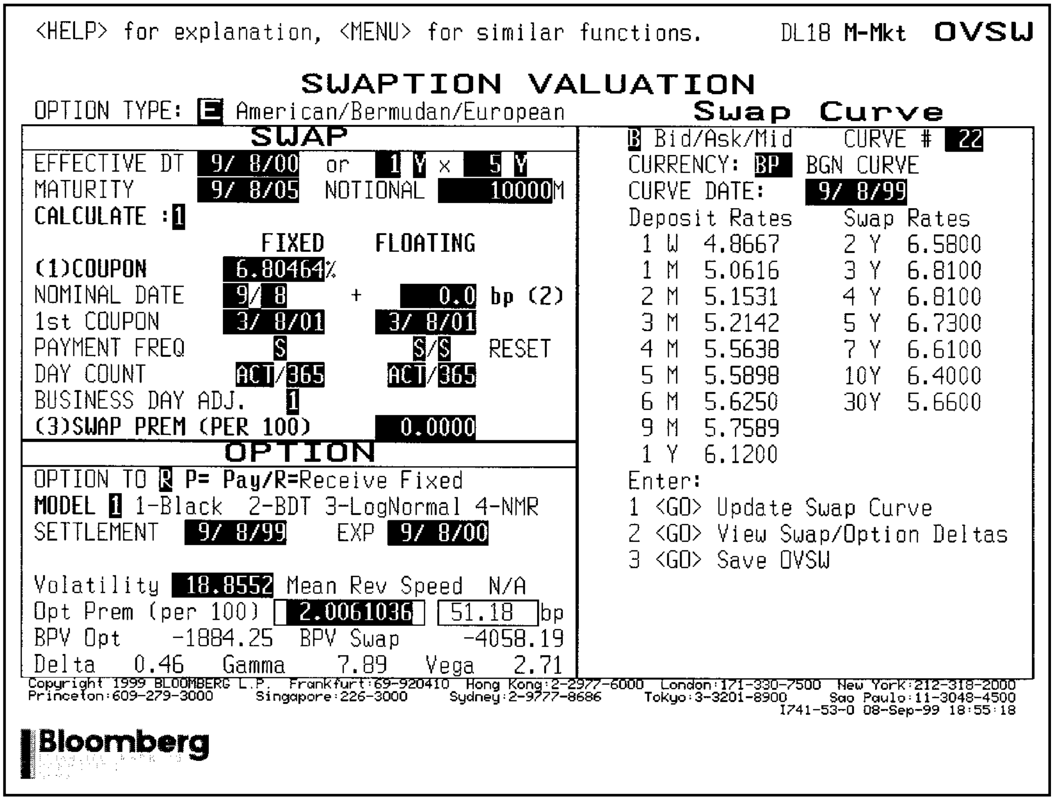


Figure 17.6 Swaption valuation. Source: Bloomberg L.P.

to exercise and maturity and the correct discount factors. The payer swaption formula in this framework is

$$\frac{1}{F}e^{-r(T-t)}\left(1-\left(1+\tfrac{1}{2}F\right)^{-2(T_S-T)}\right)\left(FN(d_1')-EN(d_2')\right)$$

and the receiver swaption formula is

$$\frac{1}{F}e^{-r(T-t)}\left(1-\left(1+\tfrac{1}{2}F\right)^{-2(T_S-T)}\right)\left(EN(-d_2')-FN(-d_1')\right)$$

where F is the forward rate of the swap, T_S is the maturity of the swap and

$$d_1' = \frac{\log(F/X)+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2' = d_1' - \sigma\sqrt{T-t}.$$

These formulas assume that interest payments in the swap are exchanged every six months.

Figure 17.6 shows the Bloomberg swaption valuation page. They use the Black model for pricing.

17.7 SPREAD OPTIONS

Spread options have a payoff that depends on the difference between two interest rates. In the simplest case the two rates both come from the same yield curve, more generally the spread could be between rates on different but related curves (yield curve *versus* swap curve, LIBOR *versus* Treasury bills), risky and riskless curves or rates in different currencies.

Can we price this contract in the framework we have built up? No. The contract crucially depends on the tilting of the yield curve. In our one-factor world all rates are correlated and there is little room for random behavior in the spread. One way to price such a contract is to use a two-factor interest rate model that captures both the overall rising and falling of the yield curve and also any titling.

Another method for pricing this contract is to squeeze it into the Black–Scholes-type framework. This amounts to modeling the spread directly as a lognormal (or Normal) variable and choosing suitable rates for discounting. This latter method is the market practice and although intellectually less satisfying it is also less prone to major errors.

17.8 INDEX AMORTIZING RATE SWAPS

A swap is an agreement between two parties to exchange interest payments on some principal, usually one payment is at a fixed rate and the other at a floating rate. In an index amortizing rate (IAR) swap the amount of this principal decreases, or **amortizes**, according to the behavior of an 'index' over the life of the swap; typically, that index is a short-term interest rate. The easiest way to understand such a swap is by example, which I keep simple.

Example Suppose that the principal begins at $10,000,000 with interest payments being at 5% from one party to the other, and r%, the spot interest rate, in the other direction. These payments are to be made quarterly.[1] At each quarter, there is an exchange of $(r-5)$% of the principal. However, at each quarter the principal may also amortize according to the level of the spot rate at that time. In Table 17.1 we see a typical amortizing schedule.

Table 17.1 Typical amortizing schedule.

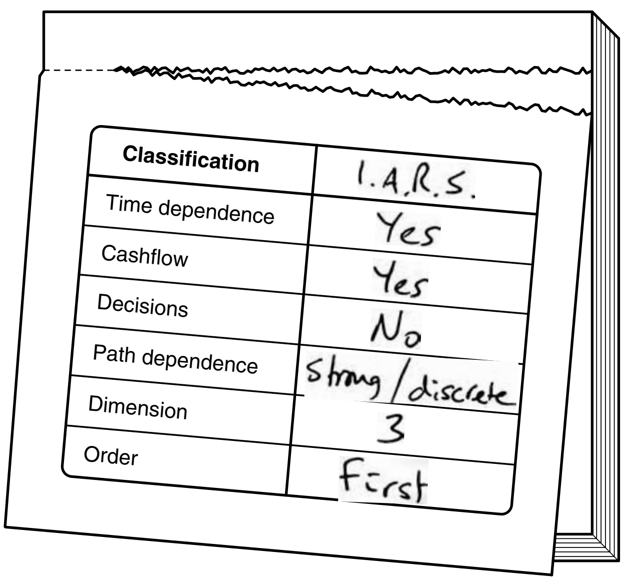
Spot rate	**Principal reduction**
less than 3%	100%
4%	60%
5%	40%
6%	20%
8%	10%
over 12%	0%

[1] In which case, r would, in practice, be a three-month rate and not the spot rate. The IAR swap is so path dependent that this difference will not be of major importance.

We read this table as follows. First, on a reset date, each quarter, there is an exchange of interest payments on the principal as it then stands. What happens next depends on the level of the spot rate. If the spot interest rate (or whatever index the amortization schedule depends on) is below 3% on the date of the exchange of payments then the principal on which future interest is paid is then amortized 100%; in other words, this new level of the principal is zero and thus no further payments are made. If the spot rate is 4% then the amortization is 60%, i.e. the principal falls to just 40% of its level before this reset date. If the spot rate is 8% then the principal amortizes by just 10%. If the rate is over 12% there is no amortization at all and the principal remains at the same level. For levels of the rate between the values in the first column of the table the amount of amortization is a linear interpolation. This interpolation is shown in Figure 17.7 and the function of r that it represents I call $g(r)$.

So, although the principal begins at $10,000,000, it can change after each quarter. This feature makes the index amortizing rate swap path-dependent.

The party receiving the fixed rate payments will suffer if rates rise because he will pay out a rising floating rate while the principal does not decrease. If rates fall the principal amortizes and so his lower floating rate payments are unfortunately on a lower principal. Again, he suffers. Thus the receiver of the fixed rate wants rates to remain steady and is said to be selling volatility.



Classification option table for IARs.

In Figure 17.8 is shown the term sheet for a USD IAR swap. In this contract there is an exchange every six months of a fixed rate and six-month LIBOR. This is a vanilla IAR swap with no extra features and can be priced in the way described above. Terms in square brackets would be set at the time that the deal was made.

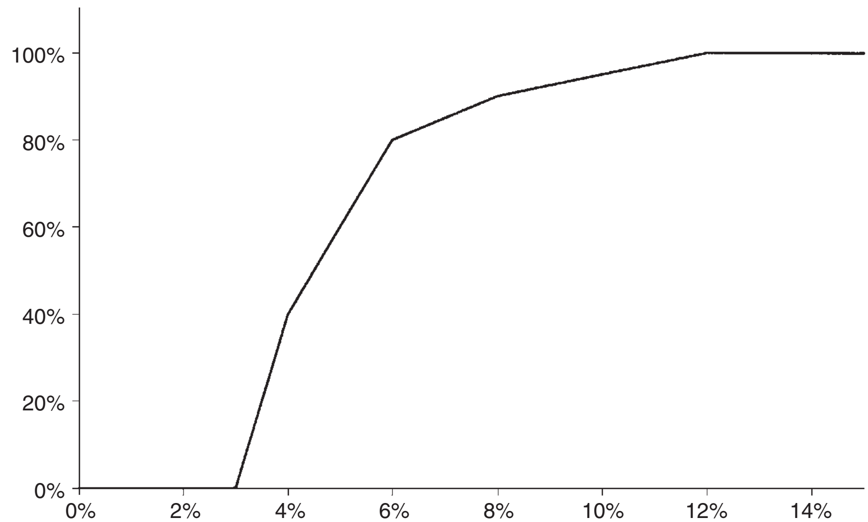


Figure 17.7 A typical amortizing schedule.

Time Out...

Is there a simple math model?

Not exactly simple, but then not too difficult. This is beyond the scope of this book since it involves path dependency. Let me just say that there is a differential equation formulation, and a tree formulation, and it can also be valued by simulation.

17.8.1 Other features in the index amortizing rate swap

Lockout period There is often a 'lockout' period, usually at the start of the contract's life, during which there are no reductions in the principal. During this period the interest payments are like those of a vanilla swap. Mathematically, we can model this feature by allowing the amortizing schedule, previously $g(r)$, to be time-dependent: $g(r, i)$. In this case the amount of amortizing depends on the reset date, t_i, as well as the spot interest rate. Such a model can be used for far more sophisticated structures than the simple lockout period.

Cleanup Some contracts have that if the principal ever falls to a specified percentage of the original principal then it is reduced all the way to zero.

USD Index Amortizing Swap

Counterparties	XXXX The Customer
Notional Amount	USD 50 millions, subject Amortization Schedule
Settlement Date	Two days after Trade Date
Maximum Maturity Date	Five years after Trade Date
Early Maturity Date	On any Fixing Date leading to a Notional Amount equal to 0
Payments made by Customer	**USD 6m LIBOR** paid semiannually, A/360
Payments made by XXXX	**In USD X% p.a.** paid semiannually, 30/360
Index Rate	USD 6m LIBOR
Base Rate	[ ]%
Amortisation Schedule (after 1st coupon period)	see table below

USD 6m LIBOR – Base Rate	Amortization
−3%	−[ ]%
−2%	−[ ]%
−1%	−[ ]%
0	−[ ]%
1%	0%
2%	0%

NB If the observed difference falls between two entries of this schedule, the amortization amount is interpolated

Fixing Dates	2 business days before each coupon period
USD 6m LIBOR	The USD 6m LIBOR rate as seen on Telerate page 3750 at noon, London time, on each Fixing Date
Documentation	ISDA
Governing Law	English

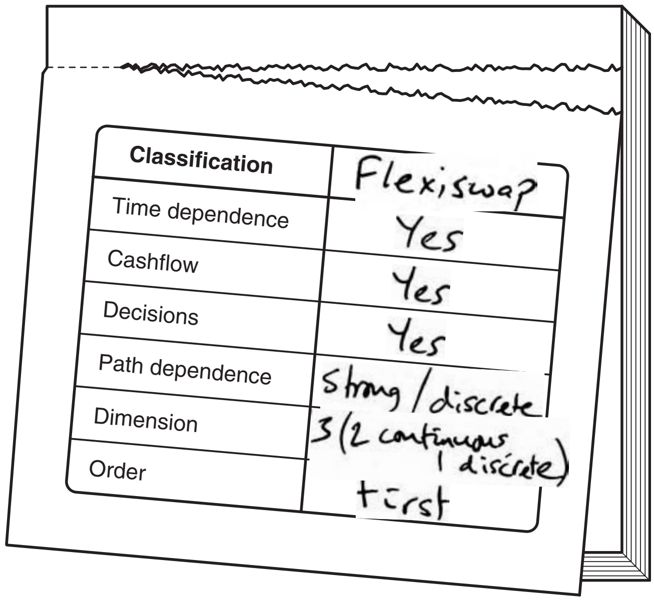
This indicative termsheet is neither an offer to buy or sell securities or an OTC derivative product which includes options, swaps, forwards and structured notes having similar features to OTC derivative transactions, nor a solicitation to buy or sell securities or an OTC derivative product. The proposal contained in the foregoing is not a complete description of the terms of a particular transaction and is subject to change without limitation.

Figure 17.8 Term sheet for a USD index amortizing swap.

17.9 CONTRACTS WITH EMBEDDED DECISIONS

The following contract is interesting because it requires the holder to make a complex series of decisions on whether to accept or reject cashflows. The contract is path dependent.

This contract, called a **flexiswap**, is a swap with M cashflows of floating minus fixed during its life. The catch is that the holder must choose to accept exactly $m \leq M$ of these cashflows. At each cashflow date the holder must say whether they want the cashflow or not, they cannot later change their mind. When they have taken *m* of them they can take no more.



17.10 SOME MORE EXOTICS

All of the following require a stochastic interest rate model for their pricing, since they are model dependent.

Ratchets and one-way floaters are floating rate notes where the amount of the periodic payments is reset, usually in a monotonically increasing (or decreasing) manner. The amount of the reset will depend on a specified floating interest rate.

Triggers are just like barrier options in that payments are received until (or after) a specified financial asset trades above or below a specified level. For example, the trigger swap is like a plain vanilla swap of fixed and floating until the reference LIBOR rate fixes above/below a specified rate. You can imagine that they comein in and out, up and down varieties.

Bermudan swaptions are like vanilla swaptions in that they give the holder the right to pay (payer swaption) or receive (receiver swaption) the fixed leg of a swap. The Bermudan characteristic allows the holder to exercise into this at specified dates.

LIBOR-in-arrears swap is an interest rate swap in which the floating payment is made at the same time that it is set. In the plain vanilla swap the rate is fixed prior to the payment, so that a swap with six-monthly payments of six-month LIBOR has the floating rate set six months before it is paid. This subtle difference means that the LIBOR-in-arrears swap cannot be decomposed into bonds and the pricing is not model independent. Because the difference depends on the slope of the forward curve, the LIBOR-in-arrears swap is often thought of as a play on the steepening or flattening on the yield curve.

17.11 SOME EXAMPLES

The term sheet in Figure 17.9 shows details of a Sterling/Deutschmark deconvergence swap. This contract allows the counterparty to express the view that rates between Germany and the UK will widen. Pricing this contract requires models for both UK and German interest rates and the correlation between them.

Figure 17.10 shows a one-year USD fixed rate note with redemption linked to World Bank bonds. The interesting point about this contract is that the issuer of the bond gets to choose whether to redeem at par or to redeem using a choice of three World Bank bonds. Obviously the issuer chooses whichever will be cheapest to deliver at the time of redemption. Hence this is an example of a **cheapest-to-deliver** bond.

The term sheet in Figure 17.11 is of a GBP two-year chooser range accrual note linked to GBP LIBOR. The contract pays a daily coupon equivalent to an annual six-month PLIBOR + 100 bps. But this is only paid while LIBOR is within an 80 bps range. The novel feature about this range note is that the holder chooses the 80 bps range at the start of each coupon period.

Sterling/Deutschmark Deconvergence Swap

Start Date	10th March 1999
Maturity Date	10th March 2003
Counterparty 1 payments	
Floating rate	3m DEM LIBOR plus SPREAD
SPREAD	2.35%
Counterparty 2 payments	
Floating rate	3m GBP LIBOR

This indicative termsheet is neither an offer to buy or sell securities or an OTC derivative product which includes options, swaps, forwards and structured notes having similar features to OTC derivative transactions, nor a solicitation to buy or sell securities or an OTC derivative product. The proposal contained in the foregoing is not a complete description of the terms of a particular transaction and is subject to change without limitation.

Figure 17.9 Term sheet for a Sterling/Deutschmark deconvergence swap.

USD 1YR Fixed Rate Note with Redemption linked to World Bank Zero-Coupon Bonds

The note either redeems at par or the investor is delivered World Bank zero-coupon bonds maturing on either 1st March 2009, 15th April 2009 or 15th July 2009. The choice of redemption is at the Issuer's option. If World Bank bonds are delivered, this will be at effective prices of 57.24% for the March bond, 56.85% for the April bond and 56.05% for the July bond.

Issue Date	15 April 2002
Maturity Date	15 April 2003
Issue price	100%
Coupon	7.00%

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Figure 17.10 Term sheet for a USD fixed rate note.

GBP 2YR Chooser Range Accrual Note Linked to 6 month GBP LIBOR

The note pays a coupon based on the number of days that 6-month LIBOR sets within an 80bps range. The range is chosen by the buyer at the beginning of each coupon period.

Issue Date	24th March 2000
Maturity Date	24th March 2002
Issue Price	100%
Coupon	[6 month LIBOR + 1.00%] x N/D
N	Number of business days that 6 month LIBOR is within the RANGE
D	Number of business days in the OBSERVATION PERIOD
RANGE	Determined by the buyer two days prior to the beginning of each OBSERVATION PERIOD
OBSERVATION PERIOD	Period 1: 24th March 2000—24th September 2000 Period 2: 24th September 2000—24th March 2001 Period 3: 24th March 2001—24th September 2001 Period 4: 24th September 2001—24th March 2002

This indicative termsheet is neither an offer to buy or sell securities or an OTC derivative product which includes options, swaps, forwards and structured notes having similar features to OTC derivative transactions, nor a solicitation to buy or sell securities or an OTC derivative product. The proposal contained in the foregoing is not a complete description of the terms of a particular transaction and is subject to change without limitation.

Figure 17.11 Term sheet for a chooser range accrual note.

17.12 SUMMARY

There are a vast number of contracts in the fixed-income world. It is an impossible task to describe and model any but a small quantity of these. In this chapter I have tried to show two of the possible approaches to the modeling in a few special cases. These two approaches to the modeling are the consistent way via a partial differential equation or the practitioner way via the Black–Scholes equity model and formulas. The former is nice because it can be made consistent across all instruments, but is dangerous to use for liquid, and high-order contracts. Save this technique for the more complex, illiquid and path-dependent contracts. The alternative approach is, as everyone admits, a fudge, requiring a contract to be squeezed and bent until it looks like a call or a put on something vaguely lognormal. Although completely inconsistent across instruments it is far less likely to lead to serious mispricings.

The reader is encouraged to find out more about the pricing of products in these two distinct ways. Better still, the reader should model new contracts for himself as well.

FURTHER READING

- Black (1976) models the value of bond options assuming the bond is a lognormal asset.
- See Hull & White (1996) for more examples of pricing index amortizing rate swaps.
- Everything by Jamshidian on the pricing of interest rate derivatives is popular with practitioners. See the bibliography for some references.
- The best technical book on interest rate derivatives, their pricing and hedging, is by Rebonato (1998).

CHAPTER 18

Heath, Jarrow and Morton

The aim of this Chapter...

... is to explain in as simple a way as possible the breakthrough in interest rate modeling known as the Heath, Jarrow & Morton model. Unfortunately, even the simplest explanation is tough going and I would understand if you skipped all the math in this chapter.

In this Chapter...

- the Heath, Jarrow & Morton forward rate model
- the relationship between HJM and spot rate models
- the advantages and disadvantages of the HJM approach
- how to decompose the random movements of the forward rate curve into its principal components
- the Brace, Gatarek & Musiela model

18.1 INTRODUCTION

The **Heath, Jarrow & Morton** approach to the modeling of the whole forward rate curve was a major breakthrough in the pricing of fixed-income products. They built up a framework that encompassed all of the models we have seen so far (and many that we haven't). Instead of modeling a short-term interest rate and deriving the forward rates (or, equivalently, the yield curve) from that model, they boldly start with a model for the whole forward rate curve. Since the forward rates are known today, the matter of yield-curve fitting is contained naturally within their model, it does not appear as an afterthought. Moreover, it is possible to take *real data* for the random movement of the forward rates and incorporate them into the derivative-pricing methodology.

Two things spoil this otherwise wonderful model. First, in its basic form, there is no guarantee that rates will stay positive or finite. Second, it can be slow to price derivatives.

18.2 THE FORWARD RATE EQUATION

The key concept in the HJM model is that we model the evolution of the whole forward rate curve, not just the short end. Write $F(t;T)$ for the forward rate curve at time t. Thus the price of a zero-coupon bond at time t and maturing at time T, when it pays \$1, is

$$Z(t;T) = e^{-\int_t^T F(t;s)\,ds}. \tag{18.1}$$

Let us assume that all zero-coupon bonds evolve according to

$$dZ(t;T) = \mu(t,T)Z(t;T)\,dt + \sigma(t,T)Z(t;T)\,dX. \tag{18.2}$$

This is not much of an assumption, other than to say that it is a one-factor model, and I will generalize that later. In this $d\cdot$ means that time t evolves but the maturity date T is fixed. Note that since $Z(t;t) = 1$ we must have $\sigma(t,t) = 0$. From (18.1) we have

$$F(t;T) = -\frac{\partial}{\partial T}\log Z(t;T).$$

Differentiating this with respect to t and substituting from (18.2) results in an equation for the evolution of the forward curve:

$$dF(t;T) = \frac{\partial}{\partial T}\left(\tfrac{1}{2}\sigma^2(t,T) - \mu(t,T)\right)dt - \frac{\partial}{\partial T}\sigma(t,T)\,dX. \tag{18.3}$$

In Figure 18.1 is shown the forward rate curve today, time t^*, and a few days later. The whole curve has moved according to (18.3).

Where has this got us? We have an expression for the drift of the forward rates in terms of the volatility of the forward rates. There is also a μ term, the drift of the bond. We have seen many times before how such drift terms disappear when we come to pricing derivatives, to be replaced by the risk-free interest rate r. Exactly the same will happen here.

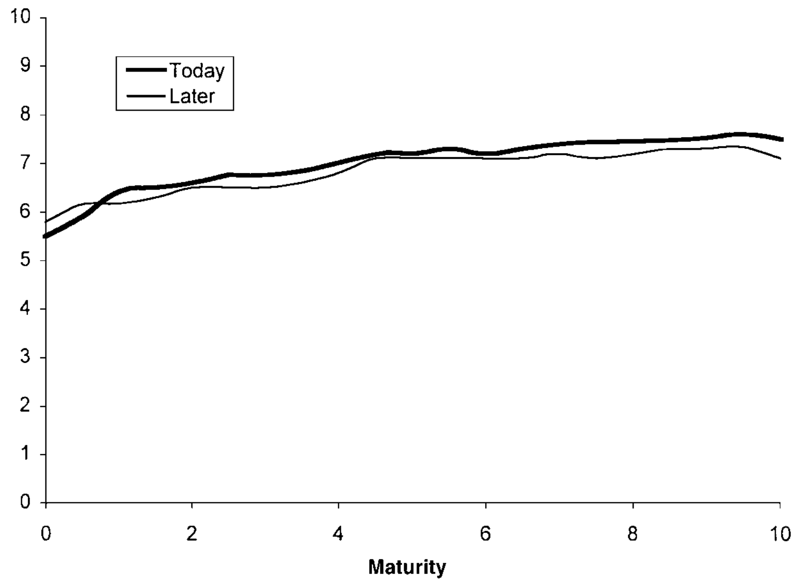


Figure 18.1 The forward rate curve today and a few days later.

18.3 THE SPOT RATE PROCESS

The spot interest rate is simply given by the forward rate for a maturity equal to the current date, i.e.

$$r(t) = F(t;t).$$

In this section I am going to manipulate this expression to derive the stochastic differential equation for the spot rate. In so doing we will begin to see why the HJM approach can be slow to price derivatives.

Suppose today is t^* and that we know the whole forward rate curve today, $F(t^*;T)$. We can write the spot rate for *any* time t in the future as

$$r(t) = F(t;t) = F(t^*;t) + \int_{t^*}^{t} dF(s;t).$$

From our earlier expression (18.3) for the forward rate process for F we have

$$r(t) = F(t^*;t) + \int_{t^*}^{t} \left(\sigma(s,t)\frac{\partial \sigma(s,t)}{\partial t} - \frac{\partial \mu(s,t)}{\partial t} \right) ds - \int_{t^*}^{t} \frac{\partial \sigma(s,t)}{\partial t} dX(s).$$

Differentiating this with respect to time t we arrive at the stochastic differential equation for r:

$$dr = \left(\frac{\partial F(t^*;t)}{\partial t} - \left.\frac{\partial \mu(t,s)}{\partial s}\right|_{s=t} + \int_{t^*}^{t} \left(\sigma(s,t)\frac{\partial^2 \sigma(s,t)}{\partial t^2} + \left(\frac{\partial \sigma(s,t)}{\partial t}\right)^2 - \frac{\partial^2 \mu(s,t)}{\partial t^2} \right) ds \right.$$
$$\left. \underline{- \int_{t^*}^{t} \frac{\partial^2 \sigma(s,t)}{\partial t^2} dX(s)} \right) dt - \left.\frac{\partial \sigma(t,s)}{\partial s}\right|_{s=t} dX.$$

Time Out...

Eeeek!

This is not a pretty sight. The idea is simple but the math is not. Hold the thought that in the HJM model we move/model *the whole of the forward rate curve*, and not just one end of it. Since we start with today's forward rate curve we don't have to worry about getting discount factors correct initially, they are automatically correct.

If you find the math daunting, just read the words for the rest of this chapter.

18.3.1 The non-Markov nature of HJM

The details of this expression are not important. I just want you to observe one point. Compare this stochastic differential equation for the spot rate with any of the models in Chapter 16. Clearly, it is more complicated, there are many more terms. All but the last one are deterministic, the last is random. The important point concerns the nature of these terms. In particular, the term underlined depends on the history of σ from the date t^* to the future date t, and *it depends on the history of the stochastic increments* dX. This term is thus highly path dependent. Moreover, for a general HJM model it makes the motion of the spot rate **non-Markov**. In a **Markov process** it is only the present state of a variable that determines the possible future (albeit random) state. Having a non-Markov model may not matter to us if we can find a small number of extra state variables that contain all the information that we need for predicting the future. Unfortunately, the general HJM model requires an infinite number of such variables to define the present state; if we were to write the HJM model as a partial differential equation we would need an infinite number of independent variables.

At the moment we are in the real world. To price derivatives we need to move over to the risk-neutral world. The first step in this direction is to see what happens when we hold a hedged portfolio.

18.4 **THE MARKET PRICE OF RISK**

In the one-factor HJM model all stochastic movements of the forward rate curve are perfectly correlated. We can therefore hedge one bond with another of a different maturity. Such a hedged portfolio is

$$\Pi = Z(t;T_1) - \Delta Z(t;T_2).$$

The change in this portfolio is given by

$$d\Pi = dZ(t;T_1) - \Delta\, dZ(t;T_2) = Z(t;T_1)\left(\mu(t,T_1)\,dt + \sigma(t,T_1)\,dX\right)$$
$$- \Delta Z(t;T_2)\left(\mu(t,T_2)\,dt + \sigma(t,T_2)\,dX\right).$$

If we choose

$$\Delta = \frac{\sigma(t, T_1)Z(t; T_1)}{\sigma(t, T_2)Z(t; T_2)}$$

then our portfolio is hedged, is risk free. Setting its return equal to the risk-free rate $r(t)$ and rearranging we find that

$$\frac{\mu(t, T_1) - r(t)}{\sigma(t, T_1)} = \frac{\mu(t, T_2) - r(t)}{\sigma(t, T_2)}.$$

The left-hand side is a function of T_1 and the right-hand side is a function of T_2. This is only possible if both sides are independent of the maturity date T:

$$\mu(t, T) = r(t) + \lambda(t)\sigma(t, T).$$

As before, $\lambda(t)$ is the market price of risk (associated with the one factor).

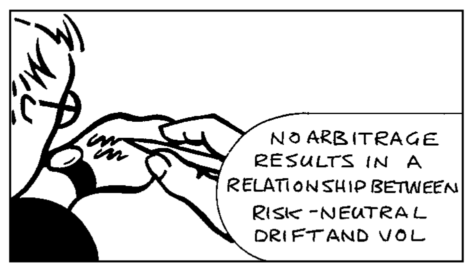
18.5 REAL AND RISK NEUTRAL

We are almost ready to price derivatives using the HJM model. But first we must discuss the real and risk-neutral worlds, relating them to the ideas in previous chapters.

All of the variables I have introduced above have been *real* variables. But when we come to pricing derivatives we must do so in the risk-neutral world. In the present HJM context, risk-neutral 'means' $\mu(t, T) = r(t)$. This means that in the risk-neutral world the return on any traded investment is simply $r(t)$. We can see this in (18.2). The risk-neutral zero-coupon bond price satisfies

$$dZ(t; T) = r(t)Z(t; T)\,dt + \sigma(t, T)Z(t; T)\,dX.$$

The deterministic part of this equation represents exponential growth of the bond at the risk-free rate. The form of the equation is very similar to that for a risk-neutral equity, except that here the volatility will be much more complicated.



18.5.1 The relationship between the risk-neutral forward rate drift and volatility

Let me write the stochastic differential equation for the *risk-neutral* forward rate curve as

$$dF(t; T) = m(t, T)\,dt + \nu(t, T)\,dX.$$

From (18.3)

$$\nu(t, T) = -\frac{\partial}{\partial T}\sigma(t, T)$$

is the forward rate volatility and, from (18.3), the drift of the forward rate is given by

$$\frac{\partial}{\partial T}\left(\tfrac{1}{2}\sigma^2(t, T) - \mu(t, T)\right) = \nu(t, T)\int_t^T \nu(t, s)\,ds - \frac{\partial}{\partial T}\mu(t, T),$$

where we have used $\sigma(t, t) = 0$. In the risk-neutral world we have $\mu(t, T) = r(t)$, and so the drift of the risk-neutral forward rate curve is related to its volatility by

$$m(t, T) = \nu(t, T) \int_t^T \nu(t, s)\, ds. \tag{18.4}$$

18.6 PRICING DERIVATIVES

Pricing derivatives is all about finding the expected present value of all cashflows in a risk-neutral framework. If we are lucky then this calculation can be done via a low-dimensional partial differential equation. The HJM model, however, is a very general interest rate model and in its full generality one cannot write down a finite-dimensional partial differential equation for the price of a derivative.

Because of the non-Markov nature of HJM in general a partial differential equation approach is unfeasible. This leaves us with two alternatives. One is to estimate directly the necessary expectations by simulating the random evolution of, in this case, the risk-neutral forward rates. The other is to build up a tree structure.

18.7 SIMULATIONS

If we want to use a Monte Carlo method, then we must simulate the evolution of the whole forward rate curve, calculate the value of all cashflows under each evolution and then calculate the present value of these cashflows by *discounting at the realized spot rate $r(t)$*.

To price a derivative using a Monte Carlo simulation perform the following steps. I will assume that we have chosen a model for the forward rate volatility, $\nu(t, T)$. Today is t^* when we know the forward rate curve $F(t^*; T)$.

1. Simulate a realized evolution of the whole risk-neutral forward rate curve for the necessary length of time, until T^*, say. This requires a simulation of

   $$dF(t; T) = m(t, T)\, dt + \nu(t, T)\, dX,$$

   where

   $$m(t, T) = \nu(t, T) \int_t^T \nu(t, s)\, ds.$$

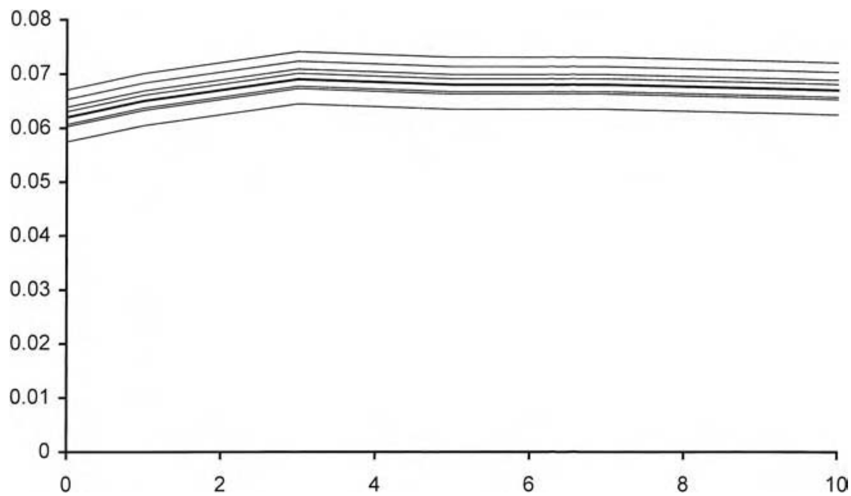
   After this simulation we will have a realization of $F(t; T)$ for $t^* \le t \le T^*$ and $T \ge t$. We will have a realization of the whole forward rate path.
2. At the end of the simulation we will have the realized prices of all maturity zero-coupon bonds at every time up to T^*.
3. Using this forward rate path calculate the value of all the cashflows that would have occurred.
4. Using the realized path for the spot interest rate $r(t)$ calculate the present value of these cashflows. Note that we discount at the continuously compounded risk-free rate, not at any other rate. In the risk-neutral world all assets have an expected return of $r(t)$.

5. Return to Step 1 to perform another realization, and continue until you have a sufficiently large number of realizations to calculate the expected present value as accurately as required.

Time Out...

Evolution of the forward curve

In this figure we see how the forward curve may have evolved during one set of simulations.



The disadvantage of the HJM model is that a Monte Carlo simulation such as this can be very slow. On the plus side, because the whole forward rate curve is calculated the bond prices at all maturities are trivial to find during this simulation.

18.8 **TREES**

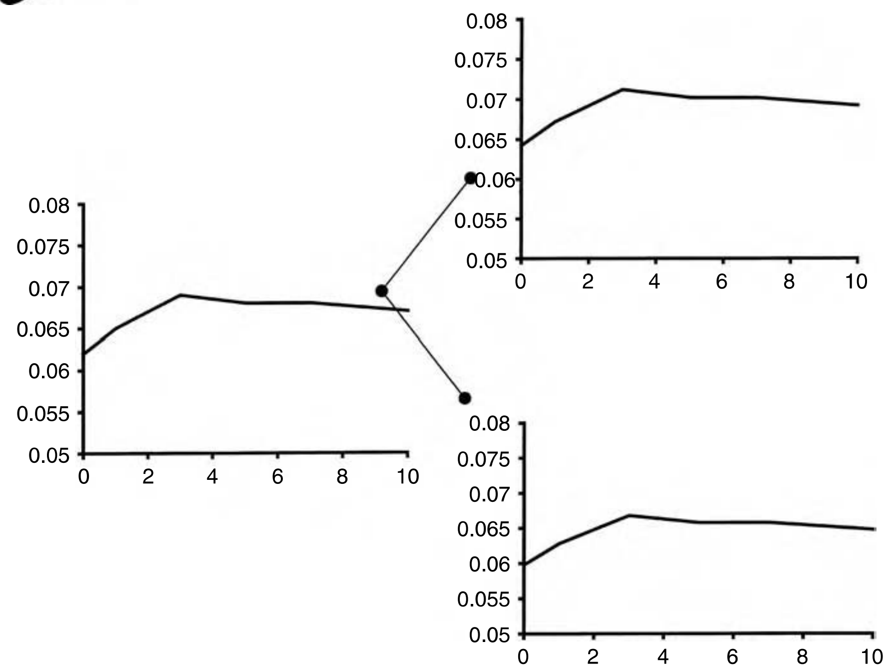
If we are to build up a tree for a non-Markov model then we find ourselves with the unfortunate result that the forward curve after an up move followed by a down is *not* the same as the curve after a down followed by an up. The equivalence of these two paths in the Markov world is what makes the binomial method so powerful and efficient. In the non-Markov world our tree structure becomes 'bushy,' and grows *exponentially* in size with the addition of new timesteps.

If the contract we are valuing is European with no early exercise then we don't need to use a tree, a Monte Carlo simulation can be immediately implemented. However, if the contract has some American feature then to correctly price in the early exercise we don't have much choice but to use a tree structure. The exponentially large tree structure will make the pricing problem very slow.

Time Out...

Trees for the whole forward rate curve

The figures below show how the forward rate curve might evolve in a treelike version of the HJM model.



18.9 THE MUSIELA PARAMETRIZATION

Often in practice the model for the volatility structure of the forward rate curve will be of the form

$$\nu(t, T) = \overline{\nu}(t, T - t),$$

meaning that we will model the volatility of the forward rate at each maturity, one, two, three years, and not at each maturity date, 2001, 2002, 2003. If we write τ for the maturity

period $T-t$ then it is a simple matter to find that $\overline{F}(t;\tau)=F(t,t+\tau)$ satisfies

$$d\overline{F}(t;\tau)=\overline{m}(t,\tau)\,dt+\overline{\nu}(t,\tau)\,dX,$$

where

$$\overline{m}(t,\tau)=\overline{\nu}(t,\tau)\int_0^\tau \overline{\nu}(t,s)\,ds+\frac{\partial}{\partial\tau}\overline{F}(t,\tau).$$

It is much easier in practice to use this representation for the evolution of the risk-neutral forward rate curve.

18.10 MULTIFACTOR HJM

Often a single-factor model does not capture the subtleties of the yield curve that are important for particular contracts. The obvious example is the spread option, that pays off the difference between rates at two different maturities. We then require a multifactor model. The multifactor theory is identical to the one-factor case, so we can simply write down the extension to many factors.

If the risk-neutral forward rate curve satisfies the N-dimensional stochastic differential equation

$$dF(t,T)=m(t,T)\,dt+\sum_{i=1}^{N}\nu_i(t,T)\,dX_i,$$

where the dX_i are uncorrelated, then

$$m(t,T)=\sum_{i=1}^{N}\nu_i(t,T)\int_t^T \nu_i(t,s)\,ds.$$

18.11 A SIMPLE ONE-FACTOR EXAMPLE: HO & LEE

In this section we make a comparison between the spot rate modeling of Chapter 16 and HJM. One of the key points about the HJM approach is that the yield curve is fitted, by default. The simplest yield-curve fitting spot rate model is Ho & Lee, so we draw a comparison between this and HJM.

In Ho & Lee the risk-neutral spot rate satisfies

$$dr=\eta(t)\,dt+c\,dX,$$

for a constant c. The prices of zero-coupon bonds, $Z(r,t;T)$, in this model satisfy

$$\frac{\partial Z}{\partial t}+\tfrac{1}{2}c^2\frac{\partial^2 Z}{\partial r^2}+\eta(t)\frac{\partial Z}{\partial r}-rZ=0$$

with

$$Z(r,T;T)=1.$$

The solution is easily found to be

$$Z(r,t;T)=\exp\left(\tfrac{1}{6}c^2(T-t)^3-\int_t^T \eta(s)(T-s)\,ds-(T-t)r\right).$$

In the Ho & Lee model $\eta(t)$ is chosen to fit the yield curve at time t^*. In forward rate terms this means that

$$F(t^*;T) = r(t^*) - \tfrac{1}{2}c^2(T-t^*)^2 + \int_{t^*}^{T} \eta(s)\,ds,$$

and so

$$\eta(t) = \frac{\partial F(t^*;t)}{\partial t} + c^2(t-t^*).$$

At any time later than t^*

$$F(t;T) = r(t) - \tfrac{1}{2}c^2(T-t)^2 + \int_{t}^{T} \eta(s)\,ds.$$

From this we find that

$$dF(t;T) = c^2(T-t)\,dt + c\,dX.$$

In our earlier notation, $\sigma(t,T) = -c(T-t)$ and $\nu(t,T) = c$. This is the evolution equation for the risk-neutral forward rates. It is easily confirmed for this model that Equation (18.4) holds. This is the HJM representation of the Ho & Lee model. Most of the popular models have HJM representations.

18.12 PRINCIPAL COMPONENT ANALYSIS

There are two main ways to use HJM. One is to choose the volatility structure $\nu_i(t,T)$ to be sufficiently 'nice' to make a tractable model, one that is Markov. This usually leads us back to the 'classical' popular spot-rate models. The other way is to choose the volatility structure to match data. This is where principal component analysis (PCA) comes in.

In analyzing the volatility of the forward rate curve one usually assumes that the volatility structure depends only on the time to maturity, i.e.

$$\nu = \overline{\nu}(T-t).$$

I will assume this but examine a more general multifactor model:

$$dF(t;T) = m(t,T)\,dt + \sum_{i=1}^{N} \overline{\nu}_i(T-t)\,dX_i.$$

From time series data we can determine the functions $\overline{\nu}_i$ empirically, this is **principal component analysis**. I will give a loose description of how this is done, with more details in the spreadsheets.

If we have forward rate time series data going back a few years we can calculate the covariances between the *changes* in the rates of different maturities. We may have, for example, rates for 1, 3 and 6 months; 2, 3, 5, 7, 10 and 30 years. The covariance matrix would then be a 10×10 symmetric matrix with the variances of the rates along the diagonal and the covariances between rates off the diagonal.

	A	B	C	D	E	F	G	H	I	J	K	L
1	**Forward rates:**				**Changes in rates:**							
2		**1 month**	**3 month**	**6 month**	**1 month**	**3 month**	**6 month**					
3	22-Sep-88	8.25000	8.31250	8.56250								
4	23-Sep-88	8.25000	8.31250	8.56250	0.00000	0.00000	0.00000					
5	26-Sep-88	8.31250	8.37500	8.62500	0.06250	0.06250	0.06250	= COVAR(E4:E1721,F4:F1721)				
6	27-Sep-88	8.31250	8.43750	8.6	= B4-B3 0000	0.06250	0.06250		**1 month**	**3 month**	**6 month**	
7	28-Sep-88	8.42188	8.50000	8.81250	0.10938	0.06250	0.12500	**1 month**	0.007501			
8	29-Sep-88	8.37500	8.68750	8.81250	−0.04688	0.18750	0.00000	**3 month**	0.003831	0.004225		
9	30-Sep-88	8.31250	8.62500	8.75000	−0.06250	−0.06250	−0.06250	**6 month**	0.003628	0.004020	0.004997	
10	3-Oct-88	8.31250	8.62500	8.68750	0.00000	0.00000	−0.06250					
11	4-Oct-88	8.31250	8.56250	8.68750	0.00000	−0.06250	0.00000	**Scaled covariance matrix:**				
12	5-Oct-88	8.31250	8.56250	8.68750	0.00000	0.00000	0.00000		**1 month**	**3 month**	**6 month**	
13	6-Oct-88	8.31250	8.56250	8.68750	0.00000	0.00000	0.00000	**1 month**	0.000189			
14	7-Oct-88	8.31250	8.62500	8.75000	0.00000	0.06250	0.06250	**3 month**	0.000097	0.000106		
15	10-Oct-88	8.25000	8.56250	8.56250	−0.06250	−0.06250	−0.18750	**6 month**	0.000091	0.000101	0.000126	
16	11-Oct-88	8.25000	8.56250	8.62500	0.00000	0.00000	0.06250					
17	12-Oct-88	8.31250	8.62500	8.68750	0.06250	0.06250	0.06250					
18	13-Oct-88	8.31250	8.64063	8.68750	0.00000	0.01563	0.00000		= I8*252/10000			
19	14-Oct-88	8.31250	8.62500	8.62500	0.00000	−0.01563	−0.06250					
20	17-Oct-88	8.31250	8.62500	8.62500	0.00000	0.00000	0.00000					
21	18-Oct-88	8.31250	8.62500	8.62500	0.00000	0.00000	0.00000					
22	19-Oct-88	8.31250	8.62500	8.62500	0.00000	0.00000	0.00000					
23	20-Oct-88	8.37500	8.68750	8.68750	0.06250	0.06250	0.06250					
24	21-Oct-88	8.37500	8.68750	8.68750	0.00000	0.00000	0.00000					
25	24-Oct-88	8.37500	8.68750	8.75000	0.00000	0.00000	0.06250					
26	25-Oct-88	8.37500	8.68750	8.75000	0.00000	0.00000	0.00000					
27	26-Oct-88	8.37500	8.68750	8.75000	0.00000	0.00000	0.00000					
28	27-Oct-88	8.37500	8.68750	8.68750	0.00000	0.00000	−0.06250					

Figure 18.2 One-, three- and sixth-month rates and the changes.

In Figure 18.2 is shown a spreadsheet of daily one-, three- and sixth-month rates, and the day-to-day changes. The covariance matrix for these changes is also shown.

PCA is a technique for finding common movements in the rates, for essentially finding eigenvalues and eigenvectors of the matrix. We expect to find, for example, that a large part of the movement of the forward rate curve is common between rates, that a parallel shift in the rates is the largest component of the movement of the curve in general. The next most important movement would be a twisting of the curve, followed by a bending.

Suppose that we have found the covariance matrix, $\mathbf{M}$, for the changes in the rates mentioned above. This 10×10 matrix will have ten eigenvalues, λ_i, and eigenvectors, $\mathbf{v}_i$ satisfying

$$\mathbf{M}\mathbf{v}_i = \lambda_i \mathbf{v}_i;$$

$\mathbf{v}_i$ is a column vector.

The eigenvector associated with the largest eigenvalue is the first principal component. It gives the dominant part in the movement of the forward rate curve. Its first entry represents the movement of the one-month rate, the second entry is the three-month rate etc. Its eigenvalue is the variance of these movements. In Figure 18.3 we see the entries in this first principal component plotted against the relevant maturity. This curve is relatively flat, when compared with the other components. This indicates that, indeed, a parallel shift of the yield curve is the dominant movement. Note that the eigenvectors are orthogonal, there is no correlation between the principal components.

In this figure are also plotted the next two principal components. Observe that one gives a twisting of the curve and the other a bending.

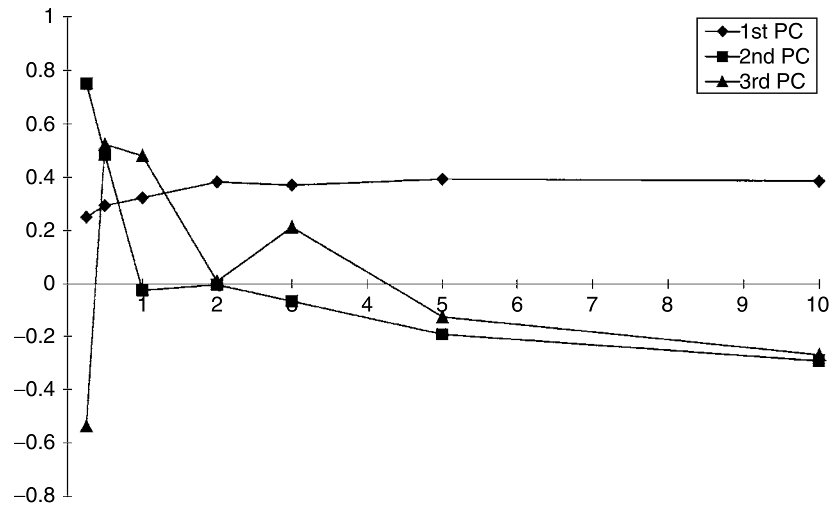


Figure 18.3 The first three principal components for the US forward rate curve. The data runs from 1988 until 1996.

The result of this analysis is that the volatility factors are given by

$$\overline{\nu}_i(\tau_j) = \sqrt{\lambda_i}(\mathbf{v}_i)_j.$$

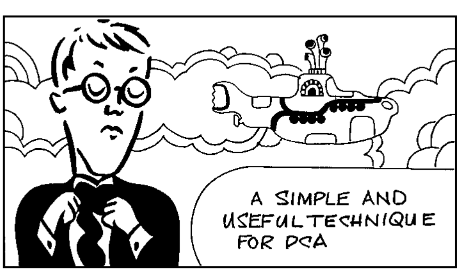
Here τ_j is the maturity, i.e. 1/12, 1/4 etc., and $(\mathbf{v}_i)_j$ is the jth entry in the vector $\mathbf{v}_i$. To get the volatility of other maturities will require some interpolation.

The calculation of the covariance matrix is simple, discussed in Chapter 11. The calculation of the eigenvalues and vectors is also simple if you use the following algorithm.

18.12.1 The power method

I will assume that all the eigenvalues are distinct, a reasonable assumption given the empirical nature of the matrix. Since the matrix is symmetric positive definite (it is a covariance matrix) we have all the nice properties we need. The eigenvector associated with the *largest* eigenvalue is easily found by the following iterative procedure. First make an initial guess for the eigenvector, call it $\mathbf{x}^0$. Now iterate using

$$\mathbf{y}^{k+1} = \mathbf{M}\mathbf{x}^k,$$



for $k = 0, \ldots,$ and

$$\beta^{k+1} = \text{element of } \mathbf{y}^{k+1} \text{ having largest modulus}$$

followed by

$$\mathbf{x}^{i+1} = \frac{1}{\beta^{k+1}}\mathbf{y}^{k+1}.$$

As $k \to \infty$, $\mathbf{x}^k$ tends to the eigenvector and β^k to the eigenvalue λ. In practice you would stop iterating once you had reached some set tolerance. Thus we have found the first principal component. It is standard to normalize the vector, and this is our $\mathbf{v}_1$.

To find the next principal component we must define a new matrix by

$$\mathbf{N} = \mathbf{M} - \lambda_1 \mathbf{v}_1 \mathbf{v}_1^T.$$

Now use the power method on this new matrix $\mathbf{N}$ to find the second principal component. This process can be repeated until all (ten) components have been found.

18.13 OPTIONS ON EQUITIES ETC.

The pricing of options contingent on equities, currencies, commodities, indices etc. is straightforward in the HJM framework. All that we need to know are the volatility of the asset and its correlations with the forward rate factors. The Monte Carlo simulation then uses the risk-neutral random walk for both the forward rates and the asset, i.e. zero-coupon bonds and the asset have a drift of $r(t)$. Of course, there are the usual adjustment to be made for dividends, foreign interest rate or cost of carry, amounting to a change to the drift rate for the asset.

The only fly in the ointment is that American-style exercise is difficult to accommodate. If we have early exercise then we may have to build up a bushy tree.

18.14 NONINFINITESIMAL SHORT RATE

One of the problems with HJM is that there is no guarantee that interest rates will stay positive, nor that the money market account will stay finite. These problems are associated with the use of a continuously compounded interest rate; all rates can be deduced from the evolution of this rate, but the rate itself is not actually observable. Modeling rates with a finite accruals period, such as three-month LIBOR, for example, has two advantages: the rate is directly observable, and positivity and finiteness can be guaranteed. Let's see how this works. I use the Musiela parametrization of the forward rates.

I have said that $\overline{F}(t, \tau)$ satisfies

$$d\overline{F}(t, \tau) = \cdots + \overline{\nu}(t, \tau)\, dX.$$

A reasonable choice for the volatility structure might be

$$\overline{\nu}(t, \tau) = \gamma(t, \tau)\overline{F}(t, \tau)$$

for finite, nonzero $\gamma(t, \tau)$. At first sight, this is a good choice for the volatility, after all, lognormality is a popular choice for random walks in finance. Unfortunately, this model leads to exploding interest rates. Yet we would like to retain some form of lognormality of rates, recall that market practice is to assume lognormality of just about everything.

We can get around the explosive rates by defining an interest rate $j(t, \tau)$ that is accrued m times *per annum*. The relationship between the new $j(t, \tau)$ and the old $\overline{F}(t, \tau)$ is then

$$\left(1 + \frac{j(t, \tau)}{m}\right)^m = e^{\overline{F}(t, \tau)}.$$

Now what happens if we choose a lognormal model for the rates? If we choose

$$dj(t,\tau) = \cdots + \gamma(t,\tau) j(t,\tau)\, dX,$$

it can be shown that this leads to a stochastic differential equation for $\overline{F}(t,\tau)$ of the form

$$d\overline{F}(t,\tau) = \cdots - m\gamma(t,\tau)\left(1 - e^{\overline{F}(t,\tau)/m}\right) dX.$$

The volatility structure in this expression is such that all rates stay positive and no explosion occurs.

If we specify the quantity m, we can of course still do PCA to find out the best form for the function $\gamma(t,\tau) = \gamma(\tau)$.

18.15 THE BRACE, GATAREK & MUSIELA MODEL

Continuing with the theme of observable variables, we now take a quick look at the Brace, Gatarek & Musiela model. It's difficult to trace the originator of the idea behind this model.

We will work in terms of observable, discrete forward rates, i.e. rates that really are quoted in the market, rather than the unrealistic continuous forward curve. With $Z(t;T)$ being the price of a zero-coupon bond at time t that matures at time T we have the forward rates at time t, separated by the timestep τ, given by

$$1 + \tau F(t; T_i, T_{i+1}) = 1 + \tau F(t; T_i, T_i + \tau) = 1 + \tau F_i(t) = \frac{Z(t; T_i)}{Z(t; T_{i+1})}.$$

Note that here we have used the discrete compounding definition of an interest rate, consistent with that introduced in Chapter 14, rather than our usual continuous definition.

Let's suppose that we can write the dynamics of each forward rate as

$$dF_i = \mu_i F_i\, dt + \sigma_i F_i\, dX_i.$$

This looks like a lognormal model, but, of course, the μs and σs could be hiding more Fs. Similarly suppose that the zero-coupon bond dynamics are given by

$$dZ_i = rZ_i\, dt + Z_i \sum_{j=1}^{i-1} a_{ij}\, dX_j$$

where

$$Z_i(t) = Z(t; T_i).$$

There are several points to note about this expression. First of all, we are clearly in a risk-neutral world with the drift of the *traded* asset Z being the risk-free rate. Actually, we'll see that we don't need a model for r, it will drop out of the analysis shortly. Also, it looks like a lognormal model but again the as could be hiding more Zs. Finally, the zero-coupon bond volatility is only given in terms of the volatilities of forward rates of shorter maturities.

We can write

$$Z_i = (1 + \tau F_i) Z_{i+1}.$$

Applying Itô's lemma to this we get

$$dZ_i = (1 + \tau F_i)\, dZ_{i+1} + \tau Z_{i+1}\, dF_i + \tau\sigma_i F_i Z_{i+1} \sum_{j=1}^{i} a_{i+1,j}\, \rho_{ij}\, dt,$$

where ρ_{ij} is the correlation between dX_i and dX_j. Equating coefficients of dX_i in this we get

$$0 = (1 + \tau F_i) a_{i+1,i} Z_{i+1} + \tau Z_{i+1} \sigma_i F_i,$$

($a_{ii} = 0$ remember) that is

$$a_{i+1,i} = -\frac{\sigma_i F_i \tau}{1 + \tau F_i}.$$

Equating the other random terms gives

$$a_{ij} Z_i = (1 + \tau F_i) Z_{i+1} a_{i+1,j},$$

that is

$$a_{i+1,j} = a_{ij} \quad \text{for } j < i.$$

It follows that

$$a_{i+1,j} = -\frac{\sigma_j F_j \tau}{1 + \tau F_j} \quad \text{for } j < i.$$

Equating the dt terms we get

$$rZ_i = (1 + \tau F_i) r Z_{i+1} + \tau Z_{i+1} \mu_i F_i + \tau \sigma_i F_i Z_{i+1} \sum_{j=1}^{i} a_{i+1,j}\, \rho_{ij}.$$

From the definition of F_i the terms including r cancel, leaving

$$\mu_i = -\sigma_i \sum_{j=1}^{i} a_{i+1,j} \rho_{ij} = \sigma_i \sum_{j=1}^{i} \frac{\sigma_j F_j \tau \rho_{ij}}{1 + \tau F_j}.$$

And we are done. Assuming that we can measure the volatilities of the forward rates σ_i and the correlations between them ρ_{ij} then we have found the correct risk-neutral drift. We are on familiar territory, Monte Carlo simulations using these volatilities, correlations and drifts can be used to price interest rate derivatives.

18.16 SUMMARY

The HJM approach to modeling the whole forward rate curve in one go is very powerful. For certain types of contract it is easy to program the Monte Carlo simulations. For example, bond options can be priced in a straightforward manner. On the other hand, the market has its own way of pricing most basic contracts, such as the bond option, as we discussed in Chapter 17. It is the more complex derivatives for which a model is needed. Some of these are suitable for HJM. But the, in a sense trivial, introduction of early exercise to a contract makes it difficult to price in the HJM setting, since

Monte Carlo simulations cannot easily handle American-style exercise. If there is a partial differential equation formulation of such a contract then that is the natural way forward. Interest rate modeling and the pricing and hedging of interest rate products is still a relatively new subject. Perhaps no longer in its infancy, the subject is certainly in its troublesome teens.

FURTHER READING

- See the original paper by Heath *et al*. (1992) for all the technical details for making their model rigorous.
- For details of the finite maturity interest rate process model see Sandmann & Sondermann (1994) and Brace *et al*. (1997).

CHAPTER 19
portfolio management

The aim of this Chapter...

... is to move beyond the world of risk-free hedging and enter the exciting and dangerous world of gambling, also known as investing, in risky assets. You will see some simple ideas for deciding how to allocate your money between all the possible investments on offer.

In this Chapter...

- the Kelly criterion
- Modern Portfolio Theory and the Capital Asset Pricing Model
- optimizing your portfolio
- alternative methodologies such as cointegration
- how to analyze portfolio performance

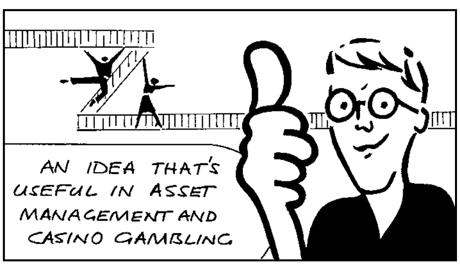
19.1 INTRODUCTION

The theory of derivative pricing is a theory of deterministic returns: we hedge our derivative with the underlying to eliminate risk, and our resulting risk-free portfolio then earns the risk-free rate of interest. Banks make money from this hedging process; they sell something for a bit more than it's worth and hedge away the risk to make a guaranteed profit.

But not everyone is hedging. Fund managers buy and sell assets (including derivatives) with the aim of beating the bank's rate of return. In so doing they take risk. In this chapter I explain some of the theories behind the risk and reward of investment. Along the way I show the benefits of diversification, how the return and risk on a portfolio of assets is related to the return and risk on the individual assets, and how to optimize a portfolio to get the best value for money.

For the most part, the assumptions are as follows:

- We hold a portfolio for 'a single period,' examining the behavior after this time
- During this period returns on assets are Normally distributed
- The return on assets can be measured by an expected return (the drift) for each asset, a standard deviation of return (the volatility) for each asset and correlations between the asset returns



19.2 THE KELLY CRITERION

To get us into the spirit of asset choice, consider the following real-life example. You have $1000 to invest and the only investments available to you are in a casino playing blackjack or roulette. We will concentrate on blackjack. Optimal strategies at blackjack were first described by Thorp (1962).

If you play blackjack with no strategy you will lose your money quickly. The odds, as ever, are in favor of the house. If your strategy is to copy the dealer's rules then there is a house edge of between five and six percent. This is because when you bust you lose, even if the dealer busts later. There is, however, an optimal strategy. The best strategy involves knowing when to hit or stand, when to split, double down, take insurance (pretty much never) etc. This decision will be based on the two cards you hold and the dealer's face-up card. If you play the best strategy you can cut the odds down to about evens, the exact figure depending on the rules of the particular casino. To consistently win at blackjack takes two things: patience and the ability to count cards. The latter only means keeping track of, for example, the number of aces and ten-count cards left in the deck. Aces and tens left in the deck improve your odds of winning. If you follow the optimal strategy and simultaneously bet high when there are a lot of aces and tens left, and low otherwise, then you will in the long run do well. If there are any casino managers reading this, I'd like to reassure them that I have never mastered the technique of card counting, so it's not worth them banning me. On the other hand, I always seem to win, but that may just be selective memory.

What does this have to do with investing?

Let me introduce the following notation: ϕ is the random variable denoting the outcome of a bet, μ is its mean and σ its standard deviation. In blackjack and roulette ϕ will take discrete values. Suppose I bet a fraction f of my \$1000, how much will I have after the hand? The amount will be

$$1000(1+f\phi_1),$$

where the subscript 1 denotes the first hand. I will consistently bet a constant fraction f of my holdings each hand, so that after two hands I have an amount

$$1000(1+f\phi_1)(1+f\phi_2).$$

This is not quite what one does when counting cards, since one will change the amount f. After M hands I have

$$1000\prod_{i=1}^{M}(1+f\phi_i).$$

How should I choose the amount f? I am going to choose it to maximize my expected long-term growth rate. This growth rate is

$$\frac{1}{M}\log\left(1000\prod_{i=1}^{M}(1+f\phi_i)\right)=\frac{1}{M}\sum_{i=1}^{M}\log((1+f\phi_i))+\frac{1}{M}\log(1000).$$

Assuming that the outcome of each hand is independent, an assumption not true for blackjack of course, then the expected value of this is

$$E[\log(1+f\phi_i)],$$

ignoring the scaling factor $\log(1000)$. Expanding the argument of the logarithm in Taylor series, assuming that the mean is small but that the standard deviation is not, we get approximately

$$f\mu-\tfrac{1}{2}f^2\sigma^2.$$

This is maximized by the choice

$$f^*=\frac{\mu}{\sigma^2},$$

giving an expected growth rate of

$$\frac{\mu^2}{2\sigma^2}$$

per hand. If $\mu>0$ then $f>0$ and we stand to make a profit, in the long term. If $\mu<0$, as it is for roulette or if you follow a naive blackjack strategy, then you should invest a negative amount, i.e. own the casino. If you must play roulette, put all your money you would gamble in your lifetime on a color, and play once. Not only do you stand an almost 50% chance of doubling your money, you will gain an invaluable reputation as a serious player. The long-run growth rate maximization and the optimal amount to invest is called the **Kelly criterion**.

If you can play M times in an evening you would expect to make

$$\frac{\mu^2 M}{2\sigma^2}. \tag{19.1}$$

This illustrates one possible way of choosing a portfolio, which asset to invest in (blackjack) and how much to invest (f^*). Faced with other possible investments, then you could argue in favor of choosing the one with the highest (19.1), depending on the mean of the return, its standard deviation and how often the investment opportunity comes your way. These ideas are particularly important to the technical analyst or chartist who trades on the basis of signals such as golden crosses, saucer bottoms, and head and shoulder patterns. Not only do the risk and return of these signals matter, but so does their frequency of occurrence.

Time Out...

An example

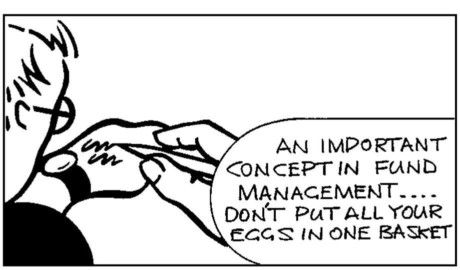
Let's suppose that you are a card counter playing blackjack and at some stage you reckon you have an edge of one percent. How much you bet?

The standard deviation of returns at blackjack is approximately one, since most of the time you either win or lose. Standoffs and three-to-two payments for a natural don't affect this too much. So you should bet one percent of your cash. If you have a thousand dollars you should bet \$10. As your holdings, you hope, increase, you will bet more and more but always sticking to the same percentage. If the cards left in the deck are against you then you should cut down bets to the table minimum. The trick is to vary your bet size without being spotted as a card counter.

19.3 DIVERSIFICATION

In this section I introduce some more notation, and show the effects of diversification on the return of the portfolio.

We hold a portfolio of N assets. The value today of the ith asset is S_i and its random return is R_i over our time horizon T. The Rs are Normally distributed with mean $\mu_i T$ and standard deviation $\sigma_i\sqrt{T}$. The correlation between the returns on the ith and jth assets is ρ_{ij} (with $\rho_{ii} = 1$). The parameters μ, σ and ρ correspond to the drift, volatility and correlation that we are used to. Note the scaling with the time horizon.



If we hold w_i of the ith asset, then our portfolio has value

$$\Pi = \sum_{i=1}^{N} w_i S_i.$$

At the end of our time horizon the value is

$$\Pi + \delta\Pi = \sum_{i=1}^{N} w_i S_i (1 + R_i).$$

We can write the relative change in portfolio value as

$$\frac{\delta \Pi}{\Pi} = \sum_{i=1}^{N} W_i R_i, \tag{19.2}$$

where

$$W_i = w_i S_i \bigg/ \sum_{i=1}^{N} w_i S_i.$$

The weights W_i sum to one.

From (19.2) it is simple to calculate the expected return on the portfolio

$$\mu_\Pi = \frac{1}{T} E\left[\frac{\delta \Pi}{\Pi}\right] = \sum_{i=1}^{N} W_i \mu_i \tag{19.3}$$

and the standard deviation of the return

$$\sigma_\Pi = \frac{1}{\sqrt{T}} \sqrt{\text{var}\left[\frac{\delta \Pi}{\Pi}\right]} = \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{N} W_i W_j \rho_{ij} \sigma_i \sigma_j}. \tag{19.4}$$

In these, we have related the parameters for the individual assets to the expected return and the standard deviation of the entire portfolio.

19.3.1 Uncorrelated assets

Suppose that we have assets in our portfolio that are uncorrelated, $\rho_{ij} = 0$, $i \neq j$. To make things simple assume that they are equally weighted so that $W_i = 1/N$. The expected return on the portfolio is represented by

$$\mu_\Pi = \frac{1}{N} \sum_{i=1}^{N} \mu_i,$$

the average of the expected returns on all the assets, and the volatility becomes

$$\sigma_\Pi = \sqrt{\frac{1}{N^2} \sum_{i=1}^{N} \sigma_i^2}.$$

This volatility is $O(N^{-1/2})$ since there are N terms in the sum. As we increase the number of assets in the portfolio, the standard deviation of the returns tends to zero. It is rather extreme to assume that all assets are uncorrelated but we will see something similar when I describe the Capital Asset Pricing Model below, diversification reduces volatility without hurting expected return.

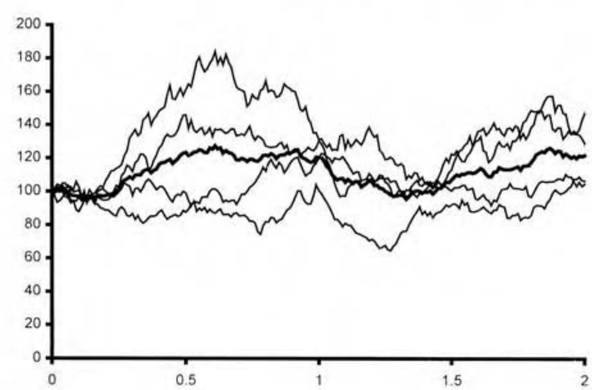
I am now going to refer to volatility or standard deviation as **risk**, a bad thing to be avoided (within reason), and the expected return as **reward**, a good thing that we want as much of as possible.

Time Out...

Spreadsheet test

In the following spreadsheet you can see the effect of investing all your cash in one risky asset, or of spreading it across four equally risky but uncorrelated assets. A convincing case for diversification... but watch out for those market crashes when all correlations become one.

	A	B 0.1	C	D	E	F	G	H	I
1	Drift			Time	Asset 1	Asset 2	Asset 3	Asset 4	Basket
2	Volatility	0.25		0	100	100	100	100	100
3	Timestep	0.01		0.01	97.33751	100.213	98.64893	101.8083	99.50194
4			=D2+$B$3	0.02	93.43911	102.775	100.8176	104.2861	100.3294
5				0.03	94.49886	99.72662	102.5955	103.5345	100.0889
6				0.04	96.46828	103.2703	104.6897	99.27946	100.927
7	=E6*(1+$B$1*$B$3+$B$2*SQRT($B$3)*(RAND()+RAND()+RAND()+RAND()+RAND()+RAND()+RAND()+RAND()+RAND()+RAND()+RAND()+RAND()-6))			.05	99.08164	101.4862	104.7989	96.73786	100.5261
8				.06	97.35307	103.5856	102.4137	95.97761	99.8325
9				.07	94.84761	100.3929	102.7862	92.99146	97.75454
10				.08	=AVERAGE(E8:H8)		99.60708	94.34865	97.37433
11				.09	94.17034	105.8131	100.0633	88.8536	97.22508
12							7	87.92572	96.94641
13							2	90.92774	95.06596
14							9	94.15164	97.39325
15							1	93.97522	94.57292
16							1	97.09774	97.24903
17							1	95.05194	97.13403
18							5	92.15589	95.06844
19							9	92.36177	97.58334
20							5	90.13494	97.51209
21							8	89.61827	97.1136
22							9	88.69484	97.47218
23							6	88.81852	97.87265
24							7	87.16549	98.36276
25							6	85.84099	98.82896
26							4	85.21653	100.173
27							8	89.34063	101.6623
28							3	90.61221	104.4723
29							5	89.08832	106.7648
30				0.28	115.6777	124.3511	103.8864	85.515	107.3576
31				0.29	114.677	132.0356	103.1644	85.08131	108.7396
32				0.3	111.8899	134.3804	97.47574	86.72787	107.6185
33				0.31	117.2041	137.0603	101.6789	83.23628	109.7949



19.4 MODERN PORTFOLIO THEORY

We can use the above framework to discuss the 'best' portfolio. The definition of 'best' was addressed very successfully by Nobel Laureate Harry Markowitz. His model provides a way of defining portfolios that are **efficient**. An efficient portfolio is one that has the highest reward for a given level of risk, or the lowest risk for a given reward. To see how this works imagine that there are four assets in the world, A, B, C , D and E with reward and risk as shown in Figure 19.1. If you could buy any one of these (but as yet you are not allowed more than one), which would you buy? Would you choose D? No, because it has the same risk as B but less reward. It has the same reward as C but for a higher risk. We can rule out D. What about B or C? They are both appealing when set against D, but against each other it is not so clear. B has a higher risk, but gets a higher reward. However, comparing them both with A we see that there is no contest. A is the preferred choice. B, C and D cannot be efficient portfolios. If we introduce asset E with the same risk as B and a higher reward than A, then we cannot objectively say which out of A and E is the better, this is a subjective choice and depends on an investor's **risk preferences**.

Now suppose that I have the two assets A and E of Figure 19.1, and I am now allowed to combine them in my portfolio, what effect does this have on my risk/reward?

From (19.3) and (19.4) we have

$$\mu_\Pi = W\mu_A + (1 - W)\mu_E$$

and

$$\sigma_\Pi^2 = W^2\sigma_A^2 + 2W(1 - W)\rho\sigma_A\sigma_E + (1 - W)^2\sigma_E^2.$$

Here W is the weight of asset A, and remembering that the weights must add up to one, the weight of asset E is $1 - W$.

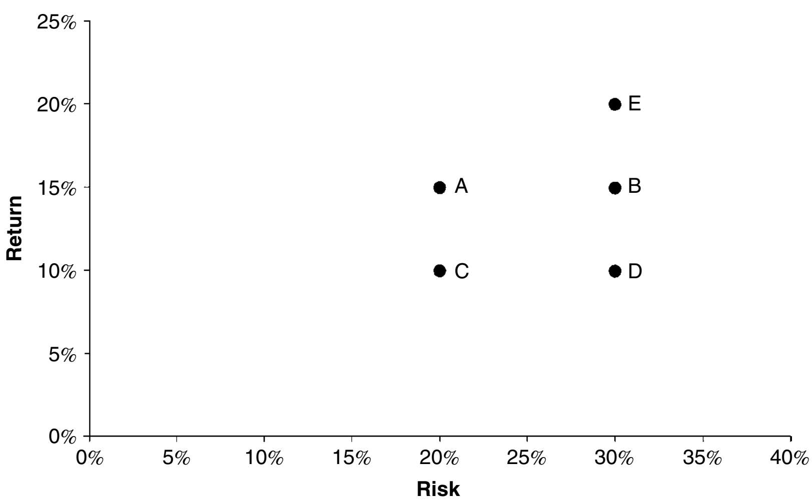


Figure 19.1 Risk and reward for five assets.

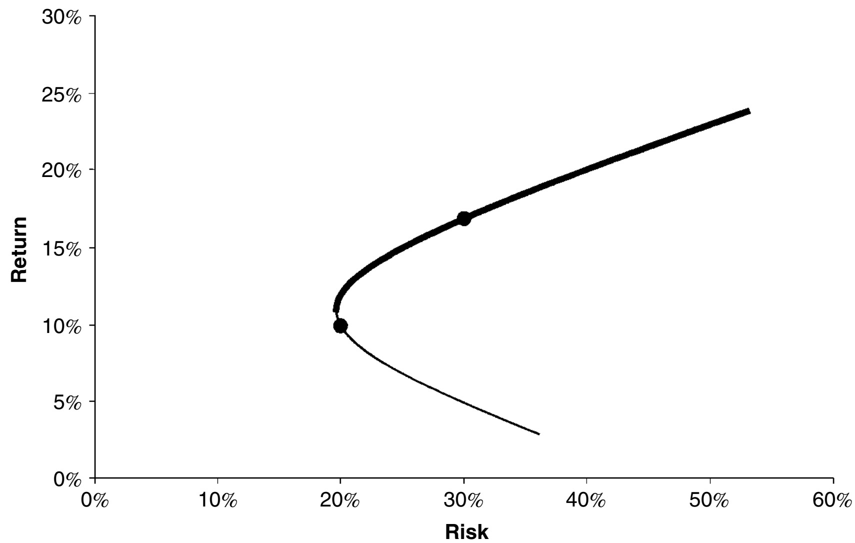


Figure 19.2 Two assets and any combination.

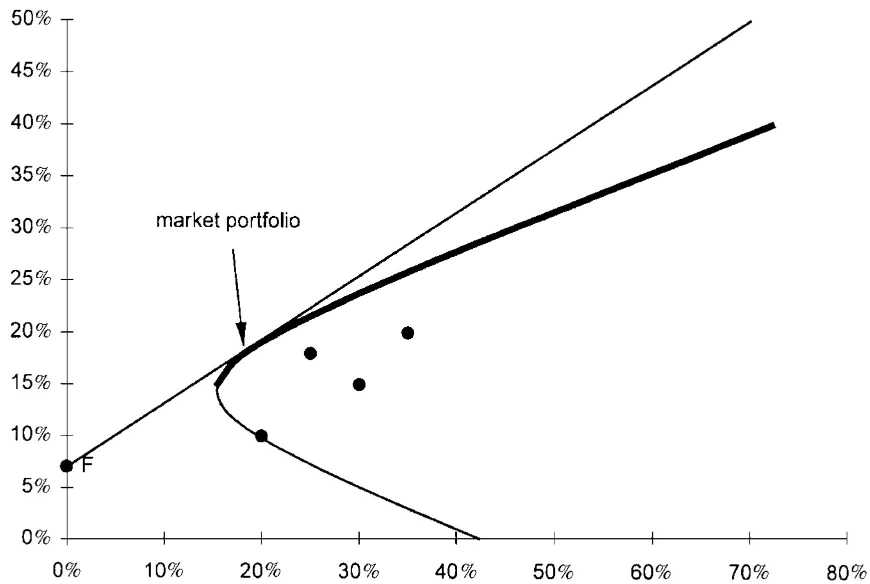


Figure 19.3 Portfolio possibilities and the efficient frontier.

As we vary W, so the risk and the reward change. The line in risk/reward space that is parametrized by W is a hyperbola, as shown in Figure 19.2. The part of this curve in bold is efficient, and is preferable to the rest of the curve. Again, an individual's risk preferences will say where he wants to be on the bold curve. When one of the volatilities is zero the line becomes straight. Anywhere on the curve between the two dots requires a long position in each asset. Outside this region, one of the assets is sold short to finance

the purchase of the other. Everything that follows assumes that we can sell short as much of an asset as we want. The results change slightly when there are restrictions.

If we have many assets in our portfolio we no longer have a simple hyperbola for our possible risk/reward profiles, instead we get something like that shown in Figure 19.3. In this figure we can see the **efficient frontier** marked in bold. Given any choice of portfolio we would choose to hold one that lies on this efficient frontier.

The calculation of the risk for a given return is demonstrated in the spreadsheet in Figure 19.4. This spreadsheet can be used to find the efficient frontier if it is used many times for different target returns.

19.4.1 Including a risk-free investment

A risk-free investment earning a guaranteed rate of return r would be the point F in Figure 19.3. If we are allowed to hold this asset in our portfolio, then since the volatility of this asset is zero, we get the new efficient frontier which is the straight line in the figure. The portfolio for which the straight line touches the original efficient frontier is called the **market portfolio**. The straight line itself is called the **capital market line**.

19.5 WHERE DO I WANT TO BE ON THE EFFICIENT FRONTIER?

Having found the efficient frontier we want to know whereabouts on it we should be. This is a personal choice, the efficient frontier is objective, given the data, but the 'best' position on it is subjective.

The following is a way of interpreting the risk/reward diagram that may be useful in choosing the best portfolio.

The return on portfolio Π is Normally distributed because it is comprised of assets which are themselves Normally distributed. It has mean μ_Π and standard deviation σ_Π (I have ignored the dependence on the horizon T).

The slope of the line joining the portfolio Π to the risk-free asset is

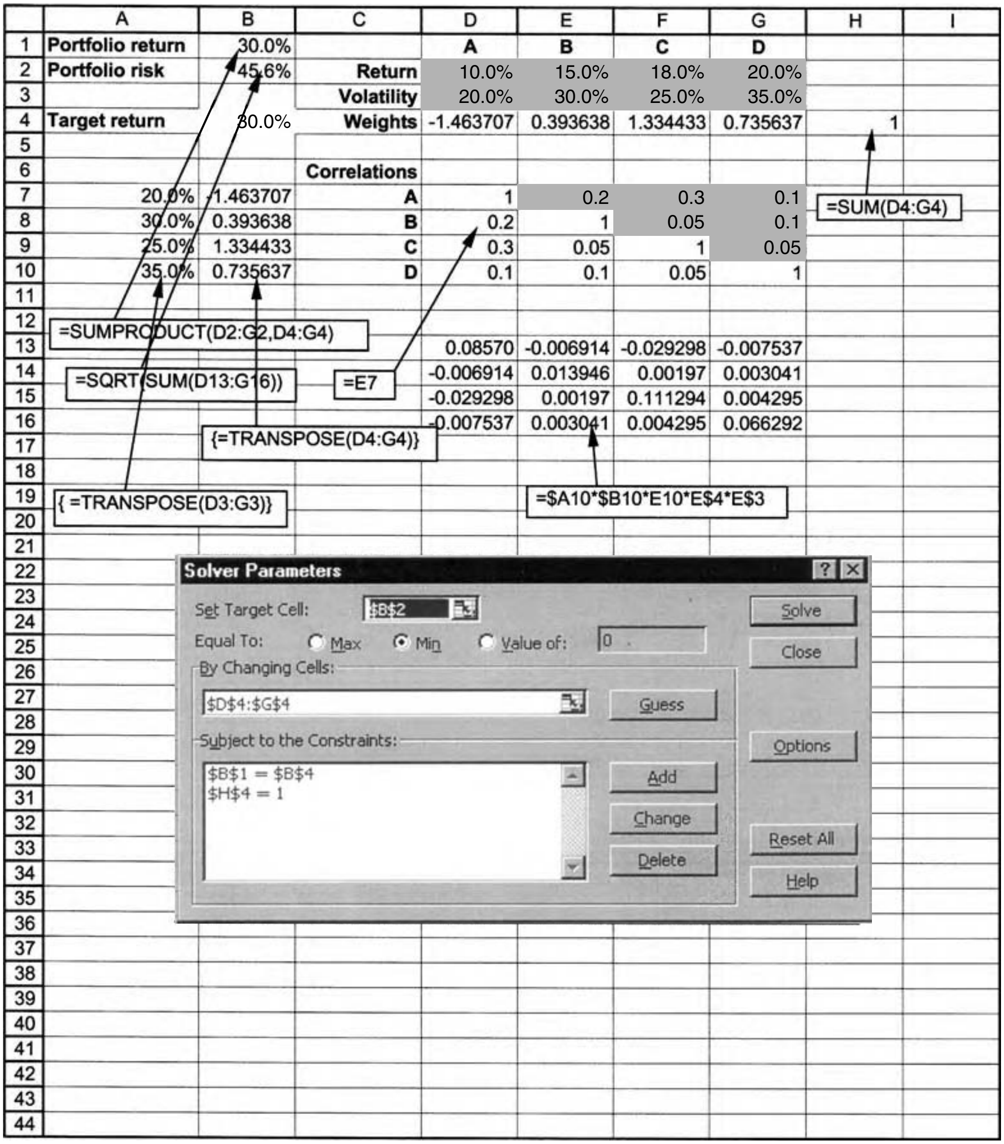
$$s = \frac{\mu_\Pi - r}{\sigma_\Pi}.$$

This is an important quantity, it is a measure of the likelihood of Π having a return that exceeds r. If $C(\cdot)$ is the cumulative distribution function for the standardized Normal distribution then $C(s)$ is the probability that the return on Π is at least r. More generally

$$C\left(\frac{\mu_\Pi - r^*}{\sigma_\Pi}\right)$$

is the probability that the return exceeds r^*. This suggests that if we want to minimize the chance of a return of less than r^* we should choose the portfolio from the efficient frontier set Π_{eff} with the largest value of the slope

$$\frac{\mu_{\Pi_{\mathrm{eff}}} - r^*}{\sigma_{\Pi_{\mathrm{eff}}}}.$$



	A	B	C	D	E	F	G	H	I
1	**Portfolio return**	30.0%		**A**	**B**	**C**	**D**		
2	**Portfolio risk**	45.6%	**Return**	10.0%	15.0%	18.0%	20.0%		
3			**Volatility**	20.0%	30.0%	25.0%	35.0%		
4	**Target return**	30.0%	**Weights**	-1.463707	0.393638	1.334433	0.735637	1	
5									
6			**Correlations**						
7	20.0%	-1.463707	**A**	1	0.2	0.3	0.1		
8	30.0%	0.393638	**B**	0.2	1	0.05	0.1		
9	25.0%	1.334433	**C**	0.3	0.05	1	0.05		
10	35.0%	0.735637	**D**	0.1	0.1	0.05	1		
11									
12									
13				0.08570	-0.006914	-0.029298	-0.007537		
14				-0.006914	0.013946	0.00197	0.003041		
15				-0.029298	0.00197	0.111294	0.004295		
16				-0.007537	0.003041	0.004295	0.066292		
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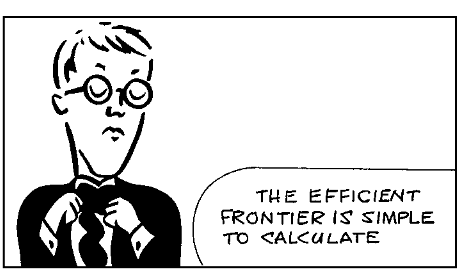
Figure 19.4 Spreadsheet for calculating one point on the efficient frontier.



Conversely, if we keep the slope of this line fixed at s then we can say that with a confidence of $C(s)$ we will lose no more than

$$\mu_{\Pi_{\text{eff}}} - s\sigma_{\Pi_{\text{eff}}}.$$

Our portfolio choice could be determined by maximizing this quantity. These two strategies are shown schematically in Figure 19.5.

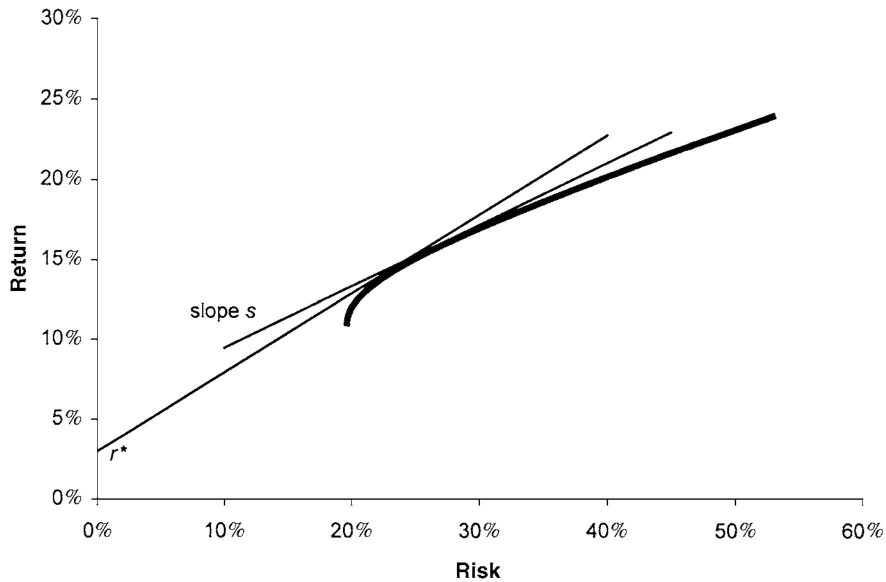


Figure 19.5 Two simple ways for choosing the best efficient portfolio.

Neither of these methods gives satisfactory results when there is a risk-free investment among the assets and there are unrestricted short sales, since they result in infinite borrowing.

Another way of choosing the optimal portfolio is with the aid of a **utility function**. This approach is popular with economists. In Figure 19.6 I show **indifference curves** and the

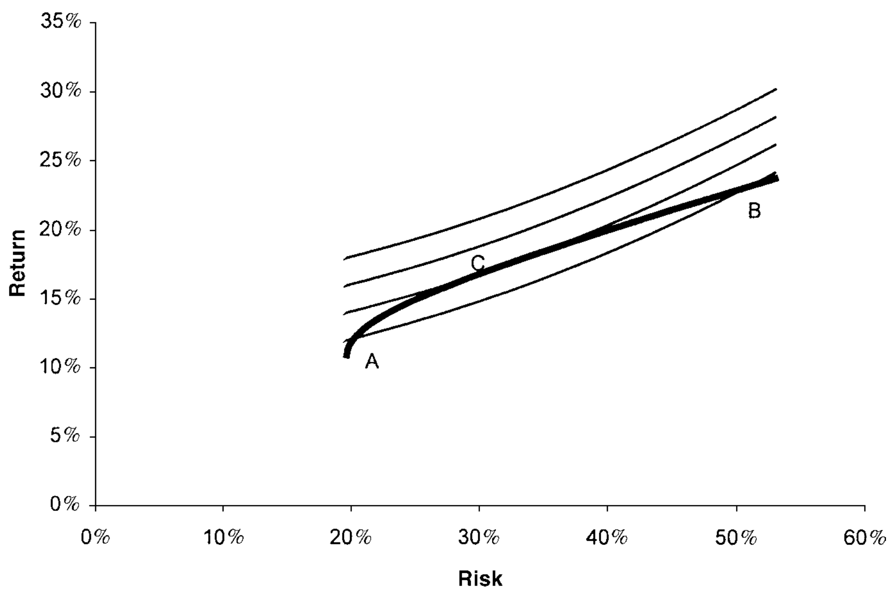
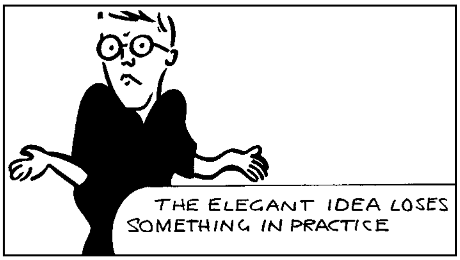


Figure 19.6 The efficient frontier and indifference curves.

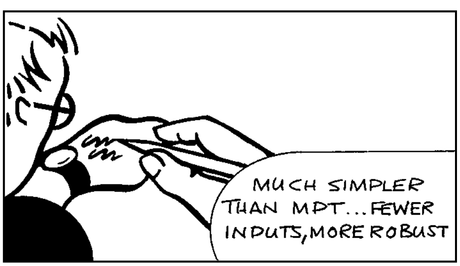
efficient frontier. The curves are called by this name because they are meant to represent lines along which the investor is indifferent to the risk/reward trade-off. An investor wants high return, and low risk. Faced with portfolios A and B in the figure, he sees A with low return and low risk, but B has a better reward at the cost of greater risk. The investor is indifferent between these two. However, C is better than both of them, being on a preferred curve.



19.6 MARKOWITZ IN PRACTICE

The inputs to the Markowitz model are expected returns, volatilities and correlations. With N assets this means $N + N + N(N-1)/2$ parameters. Most of these cannot be known accurately (do they even exist?), only the volatilities are at all reliable. Having input these parameters, we must optimize over all weights of assets in the portfolio: choose a portfolio risk and find the weights that make the return on the portfolio a maximum subject to this volatility. This is a very time-consuming process computationally unless one only has a small number of assets.

The problem with the practical implementation of this model was one of the reasons for development of the simpler model of the next section.



19.7 CAPITAL ASSET PRICING MODEL

Before discussing the **Capital Asset Pricing Model** or **CAPM** we must introduce the idea of a security's beta. The beta, β_i, of an asset relative to a portfolio M is the ratio of the covariance between the return on the security and the return on the portfolio to the variance of the portfolio. Thus

$$\beta_i = \frac{\mathrm{Cov}[R_i R_M]}{\mathrm{Var}[R_M]}.$$

19.7.1 The single-index model

I will now build up a **single-index model** and describe extensions later. I will relate the return on all assets to the return on a representative index, M. This index is usually taken to be a wide-ranging stock market index in the single-index model. We write the return on the ith asset as

$$R_i = \alpha_i + \beta_i R_M + \varepsilon_i.$$

Using this representation we can see that the return on an asset can be decomposed into three parts: a constant drift, a random part common with the index M and a random part uncorrelated with the index, ε_i. The random part ε_i is unique to the ith asset, and has mean zero. Notice how all the assets are related to the index M but

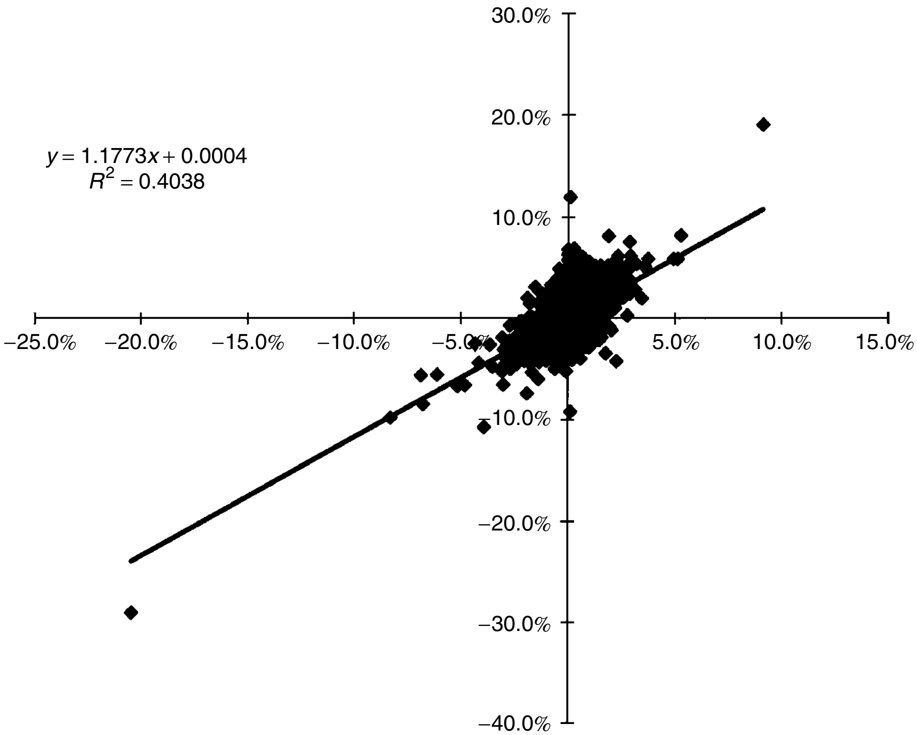


Figure 19.7 Returns on Walt Disney stock against returns on the S&P500.

are otherwise completely uncorrelated. In Figure 19.7 is shown a plot of returns on Walt Disney stock against returns on the S&P500; α and β can be determined from a linear regression analysis. The data used in this plot ran from January 1985 until almost the end of 1997.

The expected return on the index will be denoted by μ_M and its standard deviation by σ_M. The expected return on the ith asset is then

$$\mu_i = \alpha_i + \beta_i \mu_M$$

and the standard deviation

$$\sigma_i = \sqrt{\beta_i^2 \sigma_M^2 + e_i^2}$$

where e_i is the standard deviation of ε_i.

If we have a portfolio of such assets then the return is given by

$$\frac{\delta \Pi}{\Pi} = \sum_{i=1}^{N} W_i R_i = \left(\sum_{i=1}^{N} W_i \alpha_i \right) + R_M \left(\sum_{i=1}^{N} W_i \beta_i \right) + \sum_{i=1}^{N} W_i \varepsilon_i.$$

From this it follows that

$$\mu_\Pi = \left(\sum_{i=1}^{N} W_i\alpha_i\right) + E[R_M]\left(\sum_{i=1}^{N} W_i\beta_i\right).$$

Let us write

$$\alpha_\Pi = \sum_{i=1}^{N} W_i\alpha_i \quad \text{and} \quad \beta_\Pi = \sum_{i=1}^{N} W_i\beta_i,$$

so that

$$\mu_\Pi = \alpha_\Pi + \beta_\Pi E[R_M] = \alpha_\Pi + \beta_\Pi \mu_M.$$

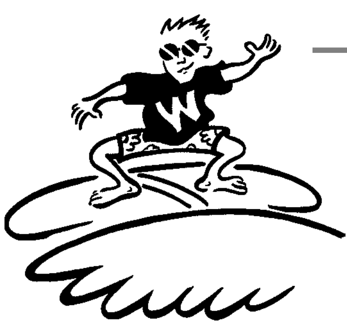
Similarly the risk in Π is measured by

$$\sigma_\Pi = \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} W_i W_j \beta_i \beta_j \sigma_M^2 + \sum_{i=1}^{N} W_i^2 e_i^2}.$$

If the weights are all about the same, N^{-1}, then the final terms inside the square root are also $O(N^{-1})$. Thus this expression is, to leading order as $N \to \infty$,

$$\sigma_\Pi = \left|\sum_{i=1}^{N} W_i\beta_i\right| \sigma_M = |\beta_\Pi|\sigma_M.$$

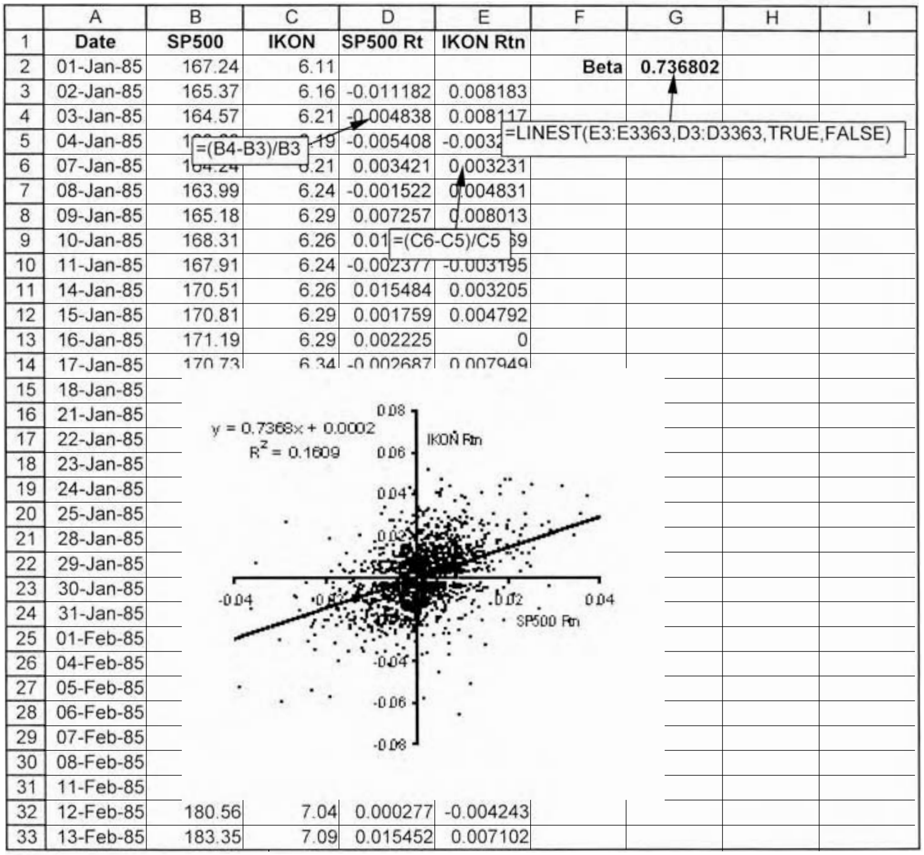
Observe that the contribution from the uncorrelated εs to the portfolio vanishes as we increase the number of assets in the portfolio: the risk associated with the εs is called **diversifiable risk**. The remaining risk, which is correlated with the index, is called **systematic risk**.



Time Out...

Finding beta

It's really easy to find beta from asset returns and index returns data using Excel. See the spreadsheet below. I've used the Add Trend-line option when drawing the plot. This finds the best-fit straight line through the data, the slope of which is the beta.



	A	B	C	D	E	F	G	H	I
1	Date	SP500	IKON	SP500 Rt	IKON Rtn				
2	01-Jan-85	167.24	6.11			Beta	0.736802		
3	02-Jan-85	165.37	6.16	-0.011182	0.008183				
4	03-Jan-85	164.57	6.21	-0.004838	0.008117				
5	04-Jan-85	[illegible]	[illegible]	-0.005408	-0.0032[illegible]				
6	07-Jan-85	[illegible]	[illegible]	0.003421	0.003231				
7	08-Jan-85	163.99	6.24	-0.001522	0.004831				
8	09-Jan-85	165.18	6.29	0.007257	0.008013				
9	10-Jan-85	168.31	6.26	0.01[illegible]	[illegible]				
10	11-Jan-85	167.91	6.24	-0.002377	-0.003195				
11	14-Jan-85	170.51	6.26	0.015484	0.003205				
12	15-Jan-85	170.81	6.29	0.001759	0.004792				
13	16-Jan-85	171.19	6.29	0.002225	0				
14	17-Jan-85	170.73	6.34	-0.002687	0.007949				
15	18-Jan-85								
16	21-Jan-85								
17	22-Jan-85								
18	23-Jan-85								
19	24-Jan-85								
20	25-Jan-85								
21	28-Jan-85								
22	29-Jan-85								
23	30-Jan-85								
24	31-Jan-85								
25	01-Feb-85								
26	04-Feb-85								
27	05-Feb-85								
28	06-Feb-85								
29	07-Feb-85								
30	08-Feb-85								
31	11-Feb-85								
32	12-Feb-85	180.56	7.04	0.000277	-0.004243				
33	13-Feb-85	183.35	7.09	0.015452	0.007102				

19.7.2 Choosing the optimal portfolio

The principal is the same as the Markowitz model for optimal portfolio choice. The only difference is that there are a lot fewer parameters to be input, and the computation is a lot faster.

The procedure is as follows. Choose a value for the portfolio return μ_Π. Subject to this constraint, minimize σ_Π. Repeat this minimization for different portfolio returns to obtain efficient frontier. The position on this curve is then a subjective choice.

19.8 **THE MULTI-INDEX MODEL**

The model presented above is a single-index model. The idea can be extended to include further representative indices. For example, as well as an index representing the stock market one might include an index representing bond markets, an index representing currency markets or even an economic index if it is believed to be relevant in linking assets. In the multi-index model we write each asset's return as

$$R_i = \alpha_i + \sum_{j=1}^{n} \beta_{ji} R_j + \varepsilon_i,$$

where there are n indices with return R_j. The indices can be correlated to each other. Similar results to the single-index model follow.

It is usually not worth having more than three or four indices. The fewer the parameters, the more robust will be the model. At the other extreme is the Markowitz model with one index per asset.

19.9 COINTEGRATION

Whether you use MPT or CAPM you will always worry about the accuracy of your parameters. Both of these methods are only as accurate as the input data, CAPM being more reliable than MPT generally speaking, because it has fewer parameters. There is another method which is gaining popularity, and which I will describe here briefly. It is unfortunately a complex technique requiring sophisticated statistical analysis (to do it properly) but which at its core makes a lot of sense. Instead of asking whether two series are correlated we ask whether they are **cointegrated**.

Two stocks may be perfectly correlated over short timescales yet diverge in the long run, with one growing and the other decaying. Conversely, two stocks may follow each other, never being more than a certain distance apart, but with any correlation, positive, negative or varying. If we are delta hedging then maybe the short timescale correlation matters, but not if we are holding stocks for a long time in an unhedged portfolio. To see whether two stocks stay close together we need a definition of **stationarity**. A time series is stationary if it has finite and constant mean, standard deviation and autocorrelation function. Stocks, which tend to grow, are not stationary. In a sense, stationary series do not wander too far from their mean.

We can see the difference between stationary and nonstationary with our first coin tossing experiment. The time series given by 1 every time we throw a head and -1 every time we throw a tail is stationary. It has a mean of zero, a standard deviation of 1 and an autocorrelation function that is zero for any nonzero lag. But what if we add up the results, as we might do if we are betting on each toss? This time series is nonstationary. This is because the standard deviation of the sum grows like the square root of the number of throws. The mean may be zero but the sum is wandering further and further away from that mean.

Testing for the stationarity of a time series X_t involves a linear regression to find the coefficients a, b and c in

$$X_t = aX_{t-1} + b + ct.$$

If it is found that $|a| > 1$ then the series is unstable. If $-1 \le a < 1$ then the series is stationary. If $a = 1$ then the series is nonstationary. As with all things statistical, we can only say that our value for a is accurate with a certain degree of confidence. To decide whether we have got a stationary or nonstationary series requires us to look at the Dickey–Fuller statistic to estimate the degree of confidence in our result. From this point on the subject of cointegration gets complicated.

How is this useful in finance? Even though individual stock prices might be non-stationary it is possible for a linear combination (i.e. a portfolio) to be stationary. Can we find λ_i, with $\sum_{i=1}^{N} \lambda_i = 1$, such that

$$\sum_{i=1}^{N} \lambda_i S_i$$

is stationary? If we can, then we say that the stocks are cointegrated.

For example, suppose we find that the S&P500 is cointegrated with a portfolio of 15 stocks. We can then use these 15 stocks to **track the index**. The error in this tracking portfolio will have constant mean and standard deviation, so should not wander too far from its average. This is clearly easier than using all 500 stocks for the tracking (when, of course, the tracking error would be zero).

We don't have to track the index, we could track anything we want, such as $e^{0.2t}$ to choose a portfolio that gets a 20% return. Clearly there are similarities with MPT and CAPM in concepts such as means and standard deviations. The important difference is that cointegration assumes far fewer properties for the individual time series. Most importantly, volatility and correlation do not appear explicitly.

19.10 PERFORMANCE MEASUREMENT

If one has followed one of the asset allocation strategies outlined above, or just traded on gut instinct, can one tell how well one has done? Were the outstanding results because of an uncanny natural instinct, or were the awful results simply bad luck?

The ideal performance would be one for which returns outperformed the risk-free rate, but *in a consistent* fashion. Not only is it important to get a high return from portfolio management, but one must achieve this with as little randomness as possible.

The two commonest measures of 'return per unit risk' are the **Sharpe ratio** of 'reward to variability' and the **Treynor ratio** of 'reward to volatility.' These are defined as follows:

$$\text{Sharpe ratio} = \frac{\mu_\Pi - r}{\sigma_\Pi}$$

and

$$\text{Treynor ratio} = \frac{\mu_\Pi - r}{\beta_\Pi}.$$

In these μ_Π and σ_Π are the *realized* return and standard deviation for the portfolio over the period. The β_Π is a measure of the portfolio's volatility. The Sharpe ratio is usually used when the portfolio is the whole of one's investment and the Treynor ratio when one is examining the performance of one component of the whole firm's portfolio, say. When the portfolio under examination is highly diversified the two measures are the same (up to a factor of the market standard deviation).

In Figure 19.8 we see the portfolio value against time for a good manager and a bad manager.

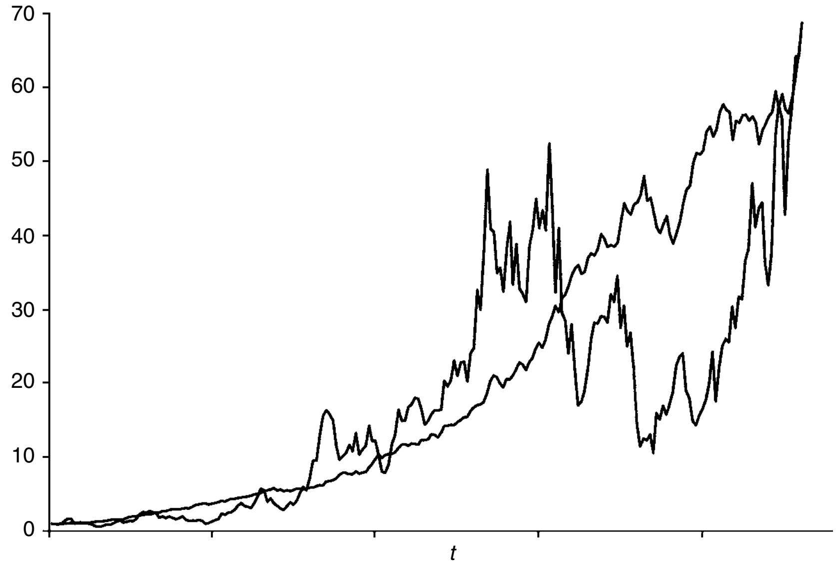


Figure 19.8 A good manager and a bad manager: same returns, different variability.

19.11 SUMMARY

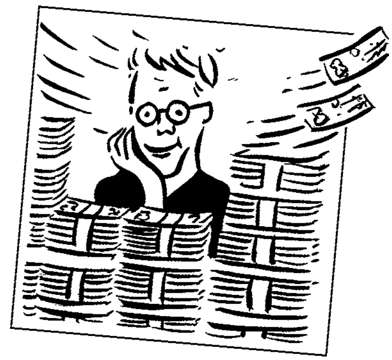
Portfolio management and asset allocation are about taking risks in return for a reward. The questions are how to decide how much risk to take, and how to get the best return. But derivatives theory is based on not taking any risk at all, and so I have spent little time on portfolio management in the book. On the other hand, as I have stressed, there is so much uncertainty in the subject of finance that elimination of risk is well-nigh impossible and the ideas behind portfolio management should be appreciated by anyone involved in derivatives theory or practice. I have tried to give the flavor of the subject with only the easiest-to-explain mathematics, the following sources will prove useful to anyone wanting to pursue the subject further.

FURTHER READING

- The classic reference texts on blackjack are by Thorp (1962) and Wong (1981).
- See Markowitz's original book (1959) for all the details of MPT.
- One of the best texts on investments, including chapters on portfolio management, is Sharpe (1985).
- For a description of cointegration and other techniques in econometrics see Hamilton (1994) and Hendry (1995).
- See Farrell (1997) for further discussion of portfolio performance.
- I have not discussed the subject of continuous-time asset allocation here, but the elegant subject is explained nicely in the collection of Robert Merton's papers (1992).

- Transaction costs can have a big effect on portfolios that are supposed to be continuously rebalanced. See Morton & Pliska (1995) for a model with costs, and Atkinson & Wilmott (1995), Atkinson *et al*. (1997) and Atkinson & Al-Ali (1997) for asymptotic results.
- For a description of chaos-based methods in finance, and how they won the First International Nonlinear Financial Forecasting Competition, see Alexander & Giblin (1997).
- For a review of current thinking in risk management see Alexander (1998).

CHAPTER 20
Value at Risk



The aim of this Chapter...

... is to explain one common way of analyzing risk, to describe the basic methodology and also to hint at some of the difficulties associated with the methodology in practice. Many of the problems arise from the assumptions of stable parameters such as volatilities and correlations, and the absence of large jumps in the asset prices.

In this Chapter...

- the meaning of VaR
- how VaR is calculated in practice
- some of the difficulties associated with VaR for portfolios containing derivatives

20.1 INTRODUCTION

It is the mark of a prudent investor, be they a major bank with billions of dollars' worth of assets or a pensioner with just a few hundred, that they have some idea of the possible losses that may result from the typical movements of the financial markets. Having said that, there have been well-publicized examples where the institution had no idea what might result from some of their more exotic transactions, often involving derivatives.

As part of the search for more transparency in investments, there has grown up the concept of Value at Risk as a measure of the possible downside from an investment or portfolio.

20.2 DEFINITION OF VALUE AT RISK

One of the definitions of **Value at Risk** (VaR), and the definition now commonly intended, is the following.

> Value at Risk is an estimate,
> with a given degree of confidence,
> of how much one can lose from one's
> portfolio over a given time horizon.

The portfolio can be that of a single trader, with VaR measuring the risk that he is taking with the firm's money, or it can be the portfolio of the entire firm. The former measure will be of interest in calculating the trader's efficiency and the latter will be of interest to the owners of the firm who will want to know the likely effect of stock market movements on the bottom line.

The degree of confidence is typically set at 95%, 97.5%, 99% etc. The time horizon of interest may be one day, say, for trading activities or months for portfolio management. It is supposed to be the timescale associated with the orderly liquidation of the portfolio, meaning the sale of assets at a sufficiently low rate for the sale to have little effect on the market. Thus the VaR is an estimate of a loss that can be realized, not just a 'paper' loss.

As an example of VaR, we may calculate (by the methods to be described here) that over the next week there is a 95% probability that we will lose no more than \$10 m. We can write this as

$$\text{Prob}\{\delta V \le -\$10\text{ m}\} = 0.05,$$

where δV is the change in the portfolio's value. (I use $\delta\cdot$ for 'the change in' to emphasize that we are considering changes over a finite time.) In symbols,

$$\text{Prob}\{\delta V \le -\text{VaR}\} = 1 - c,$$

where the degree of confidence is c, 95% in the above example.

VaR is calculated assuming normal market circumstances, meaning that extreme market conditions such as crashes are not considered, or are examined separately. Thus, effectively, VaR measures what can be expected to happen during the day-to-day operation of an institution.

The calculation of VaR requires at least having the following data: the current prices of all assets in the portfolio and their volatilities and the correlations between them. If the assets are traded we can take the prices from the market (**marking to market**). For OTC contracts we must use some 'approved' model for the prices, such as a Black–Scholes-type model, this is then **marking to model**. Usually, one assumes than the movement of the components of the portfolio are random and drawn from Normal distributions. I make that assumption here, but make a few general comments later on.

For more information about VaR and data sets for volatilities and correlations see `www.jpmorgan.com` and links therein.

20.3 VaR FOR A SINGLE ASSET

Let us begin by estimating the VaR for a portfolio consisting of a single asset.

We hold a quantity Δ of a stock with price S and volatility σ. We want to know with 99% certainty what is the maximum we can lose over the next week. I am deliberately using notation similar to that from the derivatives world, but there Δ would be the number held *short* in a hedged portfolio. Here Δ is the number held long.

In Figure 20.1 is the distribution of possible returns over the time horizon of one week. How do we calculate the VaR? First of all we are assuming that the distribution is Normal. Since the time horizon is so small, we can reasonably assume that the mean is zero. The standard deviation of the stock price over this time horizon is

$$\sigma S \left(\frac{1}{52}\right)^{1/2},$$

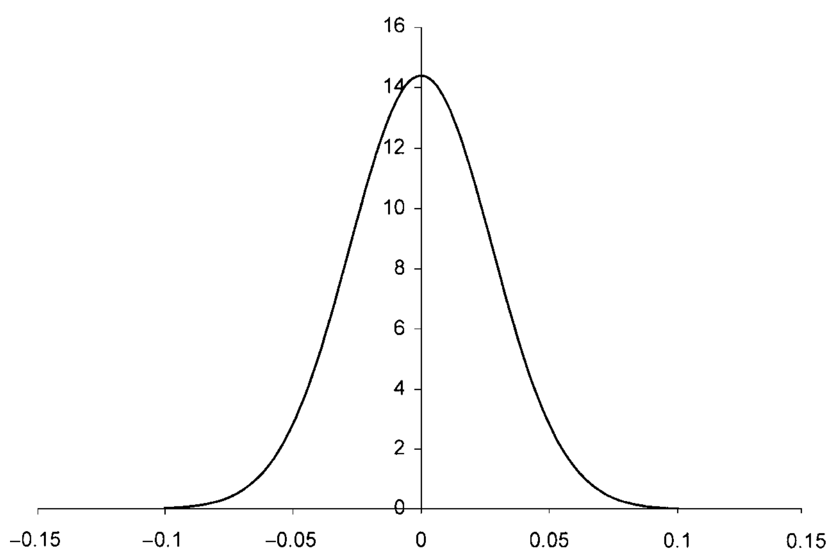


Figure 20.1 The distribution of future stock returns.

Table 20.1 Degree of confidence and the relationship with deviation from the mean.

Degree of confidence	Number of standard deviations from the mean
99%	2.326342
98%	2.053748
97%	1.88079
96%	1.750686
95%	1.644853
90%	1.281551

since the timestep is 1/52 of a year. Finally, we must calculate the position of the extreme left-hand tail of this distribution corresponding to $1\% = (100 - 99)\%$ of the events. We only need do this for the standardized Normal distribution because we can get to any other Normal distribution by scaling. Referring to Table 20.1, we see that the 99% confidence interval corresponds to 2.33 standard deviations from the mean. Since we hold a number Δ of the stock, the VaR is given by $2.33\sigma\,\Delta S(1/52)^{1/2}$.

Time Out...

Degree of confidence in Excel

How to get the number of standard deviations from the mean for a specified degree of confidence on a spreadsheet is shown below.

	A	B	C	D	E	F
1	**Mean**	**0.4**				
2	**Std dev**	**1.7**				
3						
4	**Conf.**	**No. SDs**	**Left-hand tail**			
5	99%	2.326342	-3.554781			
6	98%	2.053748	-3.091372			
7	97%	1.88079	-2.797342			
8	96%	1.750686	-2.576167			
9	95%	1.644853	-2.39625			
10	90%	1.281551	-1.778636			
11						
12						
13						
14						
15						

=$B$1-B8*$B$2

=NORMSINV(A10)

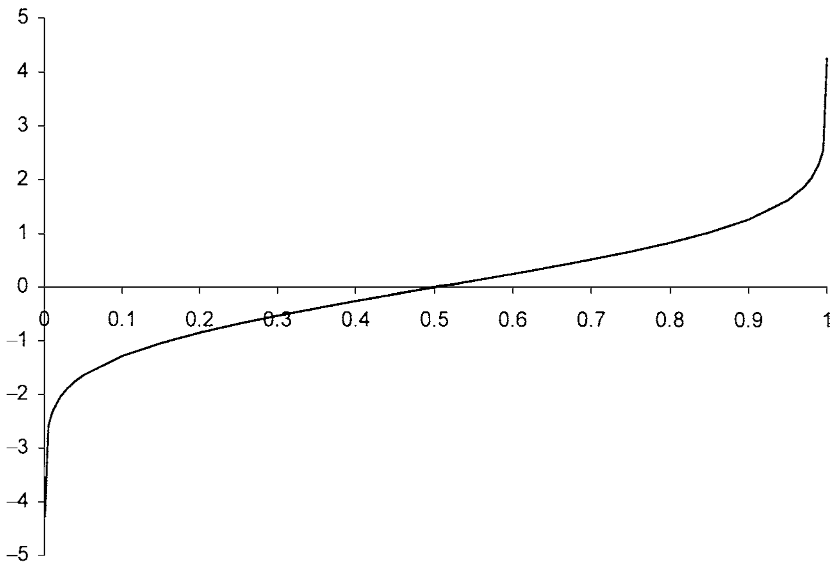


Figure 20.2 The inverse cumulative distribution function for the standardized Normal distribution.

More generally, if the time horizon is δt and the required degree of confidence is c, we have

$$\mathrm{VaR} = -\sigma\,\Delta S(\delta t)^{1/2}\alpha(1-c), \tag{20.1}$$

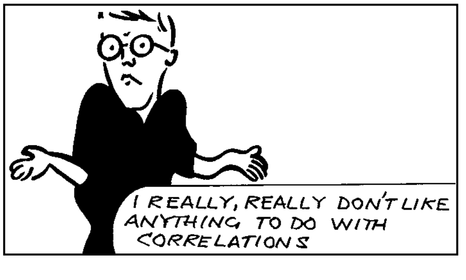
where is $\alpha(\cdot)$ is the inverse cumulative distribution function for the standardized Normal distribution, shown in Figure 20.2.

In (20.1) we have assumed that the return on the asset is Normally distributed *with a mean of zero*. The assumption of zero mean is valid for short time horizons: the standard deviation of the return scales with the square root of time but the mean scales with time itself. For longer time horizons, the return is shifted to the right (one hopes) by an amount proportional to the time horizon. Thus for longer timescales, expression (20.1) should be modified to account for the drift of the asset value. If the rate of this drift is μ then (20.1) becomes

$$\mathrm{VaR} = \Delta S(\mu\,\delta t - \sigma\,\delta t^{1/2}\alpha(1-c)).$$

Note that I use the *real* drift rate and not the *risk-neutral*. I shall not worry about this adjustment for the rest of this chapter.

20.4 VaR FOR A PORTFOLIO



If we know the volatilities of all the assets in our portfolio and the correlations between them then we can calculate the VaR for the whole portfolio.

If the volatility of the of the ith asset is σ_i and the correlation between the ith and jth assets is ρ_{ij} (with $\rho_{ii} = 1$), then the VaR for the portfolio consisting of M assets with a holding

of Δ_i of the ith asset is

$$-\alpha(1-c)\delta t^{1/2}\sqrt{\sum_{j=1}^{M}\sum_{i=1}^{M}\Delta_i\Delta_j\sigma_i\sigma_j\rho_{ij}S_iS_j}.$$

Several obvious criticisms can be made of this definition of VaR: returns are not Normal, volatilities and correlations are notoriously difficult to measure, and it does not allow for derivatives in the portfolio. We discuss the first criticism later; I now describe in some detail ways of incorporating derivatives into the calculation.

20.5 VaR FOR DERIVATIVES

The key point about estimating VaR for a portfolio containing derivatives is that, even if the change in the underlying *is* Normal, the essential nonlinearity in derivatives means that the change in the derivative can be far from Normal. Nevertheless, if we are concerned with very small movements in the underlying, for example over a very short time horizon, we may be able to approximate for the sensitivity of the portfolio to changes in the underlying by the option's delta. For larger movements we may need to take a higher-order approximation. We see these approaches and pitfalls next.

20.5.1 The delta approximation

Consider for a moment a portfolio of derivatives with a single underlying, S. The sensitivity of an option, or a portfolio of options, to the underlying is the delta, Δ. If the standard deviation of the distribution of the underlying is $\sigma S\,\delta t^{1/2}$ then the standard deviation of the distribution of the option position is

$$\sigma S\,\delta t^{1/2}\Delta.$$

Δ must here be the delta of the whole position, the sensitivity of all of the relevant options to the particular underlying, i.e. the sum of the deltas of all the option positions on the same underlying.

It is but a small, and obvious, step to the following estimate for the VaR of a portfolio containing options:

$$-\alpha(1-c)\,\delta t^{1/2}\sqrt{\sum_{j=1}^{M}\sum_{i=1}^{M}\Delta_i\Delta_j\sigma_i\sigma_j\rho_{ij}S_iS_j}.$$

Here Δ_i is the rate of change of the *portfolio* with respect to the ith asset.

20.5.2 The delta/gamma approximation

The delta approximation is satisfactory for small movements in the underlying. A better approximation may be achieved by going to higher order and incorporating the gamma or convexity effect.

I demonstrate this by example. Suppose that our portfolio consists of an option on a stock. The relationship between the change in the underlying, δS, and the change in the value of the option, δV, is

$$\delta V = \frac{\partial V}{\partial S}\delta S + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(\delta S)^2 + \frac{\partial V}{\partial t}\delta t + \cdots.$$

Since we are assuming that

$$\delta S = \mu S\,\delta t + \sigma S\,\delta t^{1/2}\phi,$$

where ϕ is drawn from a standardized Normal distribution, we can write

$$\delta V = \frac{\partial V}{\partial S}\sigma S\,\delta t^{1/2}\phi + \delta t\left(\frac{\partial V}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\phi^2 + \frac{\partial V}{\partial t}\right) + \cdots.$$

This can be rewritten as

$$\delta V = \Delta\sigma S\,\delta t^{1/2}\phi + \delta t\,(\Delta\mu S + \tfrac{1}{2}\Gamma\sigma^2 S^2\phi^2 + \Theta) + \cdots. \qquad (20.2)$$

To leading order, the randomness in the option value is simply proportional to that in the underlying. To the next order there is a deterministic shift in δV due to the deterministic drift of S and the theta of the option. More importantly, however, the effect of the gamma is to introduce a term that is nonlinear in the random component of δS.

In Figure 20.3 are shown three pictures. First, there is the assumed distribution for the change in the underlying. This is a Normal distribution with standard deviation $\sigma S\,\delta t^{1/2}$, drawn in bold in the figure. Second is shown the distribution for the change in the

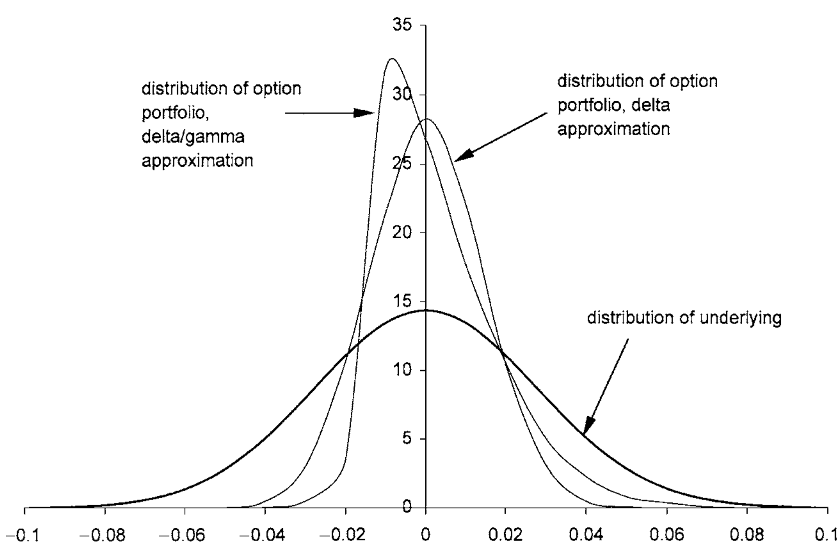


Figure 20.3 A Normal distribution for the change in the underlying (bold), the distribution for the change in the option assuming the delta approximation (another Normal distribution) and the distribution for the change in the option assuming the delta/gamma approximation (definitely not a Normal distribution).

option assuming the delta approximation only. This is a Normal distribution with standard deviation $\Delta\sigma S\,\delta t^{1/2}$. Finally, there is the distribution for the change in the underlying assuming the delta/gamma approximation.

From this figure we can see that the distribution for the delta/gamma approximation is far from Normal. In fact, because expression (20.2) is quadratic in ϕ, δV must satisfy the following constraint.

$$\delta V \geq -\frac{\Delta^2}{2\Gamma} \quad \text{if} \quad \Gamma > 0$$

or

$$\delta V \leq -\frac{\Delta^2}{2\Gamma} \quad \text{if} \quad \Gamma < 0.$$

The extreme value is attained when

$$\phi = -\frac{\Delta}{\sigma S \Gamma\,\delta t^{1/2}}.$$

The question to ask is then 'Is this critical value for ϕ in the part of the tail in which we are interested?' If it is not then the delta approximation may be satisfactory, otherwise it will not be. If we cannot use an approximation we may have to run simulations using valuation formulas.

One obvious conclusion to be drawn is that positive gamma is good for a portfolio and negative gamma is bad. With a positive gamma the downside is limited, but with a negative gamma it is the upside that is limited.

20.5.3 Use of valuation models

The obvious way around the problems associated with nonlinear instruments is to use a simulation for the random behavior of the underlyings and then use valuation formulas or algorithms to deduce the distribution of the changes in the whole portfolio. This is the ultimate solution to the problem but has the disadvantage that it can be very slow. After all, we may want to run tens of thousands of simulations but if we must solve a multifactor partial differential equation each time then we find that it will take far too long to calculate the VaR.

20.5.4 Fixed-income portfolios

When the asset or portfolio has interest rate dependence then it is usual to treat the yield to maturity on each instrument as the Normally distributed variable. Yields on different instruments are then suitably correlated. The relationship of price to change in yield is via duration (and convexity at higher order). So our fixed-income asset can be thought of as a derivative of the yield. The VaR is then estimated using duration in place of delta (and convexity in place of gamma) in the obvious way.

20.6 SIMULATIONS

The two simulation methods described in this book are **Monte Carlo**, based on the generation of Normally distributed random numbers, and **bootstrapping** using actual asset price movements taken from historical data.

Within these two simulation methods, there are two ways to generate future scenarios, depending on the timescale of interest and the timescale for one's model or data. If one is interested in a horizon of one year and one has a model or data for returns with this same horizon, then this is easily used to generate a distribution of future scenarios. On the other hand, if the model or data is for a shorter timescale, a stochastic differential equation or daily data, say, and the horizon is one year, then the model must be used to build up a one-year distribution by generating whole year-long paths of the asset. This is more time consuming but is important for path-dependent contracts when the whole path taken must obviously be modeled.

Remember, the simulation must use *real* returns and not *risk-neutral*.

20.6.1 Monte Carlo

Monte Carlo simulation is the generation of a distribution of returns and/or asset price paths by the use of random numbers. This subject is discussed in great depth in Chapter 26. The technique can be applied to VaR: using numbers, ϕ, drawn from a Normal distribution, to build up a distribution of future scenarios. For each of these scenarios use some pricing methodology to calculate the value of a portfolio (of the underlying asset and its options) and thus directly estimate its VaR.

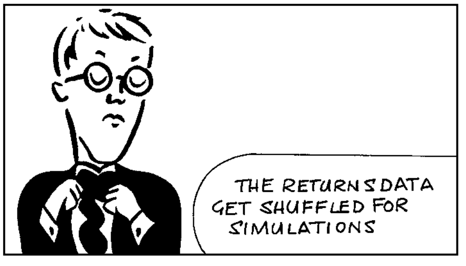
20.6.2 Bootstrapping

Another method for generating a series of random movements in assets is to use historical data. Again, there are two possible ways of generating future scenarios: a one-step procedure if you have a model for the distribution of returns over the required time horizon, or a multistep procedure if you only have data/model for short periods and want to model a longer time horizon.

The data that we use will consist of daily returns, say, for all underlying assets going back several years. The data for each day is recorded as a vector, with one entry per asset.

Suppose we have real time-series data for N assets and that our data is daily data stretching back four years, resulting in approximately 1000 daily *returns* for each asset. We are going to use these returns for simulation purposes. This is done as follows.

Assign an 'index' to each daily change. That is, we assign 1000 numbers, one for each vector of returns. To visualize this, imagine writing the returns for all of the N assets on the blank pages of a notebook. On page 1 we write the changes in asset values that occurred from 8th July 1998 to 9th July 1998. On page 2 we do the same, but for the changes from 9th July to 10th July 1998. On page 3... from 10th to 11th July etc. We will fill 1000 pages if we have 1000 data sets. Now, draw a number from 1 to 1000, uniformly distributed; it is 534. Go to page 534 in the notebook. Change all assets from today's value by the vector of returns given on the page. Now draw another number between 1 and 1000 at random and repeat the process. Increment this new value again using one of the vectors. Continue this process until the required time horizon has been reached. This is one realization of the path of the assets. Repeat this simulation to generate many, many possible realizations to get an accurate distribution of all future prices.



	A	B	C	D	E	F	G	H	I	J	K	L	M
1	**Index**	**Prob.**	**TELEBRAS**	**ELETROBRAS**	**PETROBRAS**	**CVDR**	**USIMINAS**	**YPF**	**TAR**	**TEO**	**TGS**	**PEREZ**	**TELMEX**
2	1	0.001	−0.0152210	0.0180185	0.0118345	−0.0240975	−0.0111733	0.0355909	0.0185764	0.0071943	0.0121214	−0.0182190	−0.0235305
3	2	0.001	0.0091604	0.0072214	−0.0046130	0.0001039	−0.0226244	0.0207620	0.0121953	0.0106953	0.0000000	−0.0026393	0.0327898
4	3	0.001	−0.0498546	−0.0397883	−0.0324353	−0.0246926	−0.0115608	0.0068260	−0.0121953	−0.0106953	0.0119762	−0.0067506	−0.0139213
5	4	0.001	−0.0357762	−0.0494712	−0.0358569	−0.1001323	−0.0602100	−0.0068260	−0.0123458	−0.0254097	0.0000000	−0.0144520	−0.0284379
6	5	0.001	0.0033058	−0.0204013	0.0112413	0.0264184	0.0114388	−0.0068729	0.0061920	0.0036697	−0.0240976	0.0096238	0.0000000
7	6	0.001	0.0424306	0.0260111	−0.0255043	−0.0022909	0.0004155	0.0067340	0.0452054	0.0438519	0.0115608	0.0140863	0.0421608
8	7	0.001	0.0279317	0.0006216	0.0006216	0.0006216	−0.0234759	0.0040080	−0.0208341	0.0000000	−0.0114287	0.0000000	0.0123205
9	8	0.001	−0.0139801	−0.0383626	−0.0346393	0.0161040	0.0122250	0.0000000	0.0259755	0.0176996	0.0114287	−0.0020299	0.0439192
10	9	0.001	−0.0052219	−0.0036563	0.0403377	0.0142520	0.0229896	0.0063898	0.0101524	0.0085349	0.0114287	−0.0019638	0.0000000
11	10	0.001	−0.0130693	0.0000000	−0.0116961	−0.0047282	0.0112996	0.0000000	0.0100503	0.0112677	0.0112996	0.0081304	−0.0267702
12	11	0.001	−0.0106656	−0.0148639	0.0069306	−0.0048541	−0.0114030	0.0000000	−0.0100503	−0.0169496	−0.0112996	−0.0166294	−0.0116960
13	12	0.001	0.0079035	0.0184167	−0.0023392	−0.0240975	0.0112996	0.0000000	−0.0050633	−0.0085837	0.0335227	−0.0170073	0.0039139
14	13	0.001	0.0000000	−0.0072225	−0.0284021	0.0001035	−0.0226248	0.0000000	−0.0257084	−0.0380719	−0.0110498	−0.0063762	0.0000000
15	14	0.001	0.0000000	−0.0042230	0.0242792	−0.0004134	−0.0126835	−0.0061920	0.0100001	0.0057471	0.0109291	0.0144183	0.0411581
16	15	0.001	−0.0475377	−0.0138620	−0.0251059	−0.0444106	−0.0254146	−0.0062305	−0.0150379	−0.0057471	0.0000000	0.0300138	−0.0080972
17	16	0.001	0.0297428	0.0102652	0.0114445	0.0714950	−0.0008256	0.0415490	0.0277796	0.0309303	0.0408220	−0.0035537	0.0419109
18	17	0.001	−0.0037045	−0.0192060	−0.0248187	0.0078541	−0.0129914	−0.0116280	−0.0045977	0.0052771	0.0000000	0.0007187	−0.0075758
19	18	0.001	−0.0208341	−0.0205823	−0.0264421	−0.0021947	−0.0267507	0.0231224	0.0091744	−0.0052771	−0.0304592	0.0152095	−0.0269766
20	19	0.001	0.0568874	0.0828276	0.1052577	0.0477150	0.0612661	0.0271019	0.0753494	0.0746435	0.0388398	0.0599762	0.0498324
21	20	0.001	0.0655911	0.0209154	0.0209922	0.0420456	0.0224756	−0.0173917	0.0165293	0.0223058	0.0000000	0.0232360	0.0036036
22	21	0.001	−0.0091109	0.0309987	0.0615672	−0.0428678	0.0221646	0.0058309	0.0122201	0.0145988	0.0392207	0.0139353	−0.0592766
23	22	0.001	0.0231470	−0.0036810	−0.0003084	0.0158210	0.0099482	0.0342891	0.0160646	0.0264121	0.0204089	0.0208262	0.0679507
24	23	0.001	0.0157998	−0.0014713	−0.0122804	−0.0094697	−0.0014375	0.0277026	0.0504962	0.0440396	0.0492710	0.0285456	−0.0073260
25	24	0.001	−0.0045147	−0.0054003	−0.0008614	−0.0020500	0.0379553	−0.0054795	−0.0191210	−0.0136988	0.0000000	0.0101176	−0.0073801
26	25	0.001	0.0270287	0.0275012	0.0583711	0.0001022	0.0361421	0.0280130	−0.0038536	0.0069849	0.0000000	0.0034104	−0.0109690
27	26	0.001	−0.0089286	0.0068362	0.0038686	0.0248968	−0.0087868	0.0000000	−0.0116506	−0.0093241	0.0000000	0.0110554	−0.0036832
28	27	0.001	0.0133632	0.0232183	0.0512933	0.0202027	0.0436750	−0.0167135	−0.0117880	−0.0285055	−0.0099503	−0.0080555	0.0000000
29	28	0.001	−0.0110341	0.0417395	0.0176995	0.0506932	0.0085108	0.0000000	−0.0240012	−0.0268636	0.0000000	0.0005324	0.0000000
30	29	0.001	−0.0044544	−0.0289144	0.0069770	−0.0002423	−0.0002043	−0.0056023	0.0000000	0.0024907	0.0000000	−0.0037901	0.0000000
31	30	0.001	0.0044544	0.0001022	−0.0344781	−0.0075315	0.0086859	0.0056023	0.0202027	0.0049628	0.0204089	0.0147792	0.0038241
32	31	0.001	−0.0044544	−0.0065956	−0.0225751	0.0151071	−0.0173439	0.0111112	−0.0040080	0.0024722	0.0396091	0.0090663	−0.0115164
33	32	0.001	0.0088685	−0.0067409	0.0284793	0.0204063	0.0261791	0.0096270	0.0043197	−0.0052632	0.0000000	0.0184492	−0.0076923
34	33	0.001	−0.0178163	−0.0084174	0.0036982	0.0183391	−0.0186661	−0.0242436	−0.0350913	−0.0240332	−0.0202027	−0.0280518	−0.0234386
35	34	0.001	−0.0205219	−0.0174839	−0.0199091	−0.0849882	−0.0120831	−0.0061539	−0.0134834	−0.0108697	0.0000000	−0.0075047	0.0039448
36	35	0.001	−0.0092167	0.0172806	0.0323986	0.0769500	−0.0145534	−0.0313505	−0.0368705	−0.0221003	0.0202027	−0.0096608	−0.0320027
37	36	0.001	−0.0256977	−0.0470936	−0.0464976	−0.0325785	−0.0460766	−0.0128207	−0.0189579	−0.0340942	−0.0100503	−0.0547012	−0.0081633

Figure 20.4 Spreadsheet showing bootstrap data.

By this method we generate a distribution of possible future scenarios based on historical data.

Note how we keep together all asset changes that happen on a certain date. By doing this we ensure that we capture any correlation that there may be between assets.

This method of bootstrapping is very simple to implement. The advantages of this method are that it naturally incorporates any correlation between assets, and any non-Normality in asset price changes. It does not capture any autocorrelation in the data, but then neither does a Monte Carlo simulation in its basic form. The main disadvantage is that it requires a lot of historical data that may correspond to completely different economic circumstances than those that currently apply.

In Figure 20.4 is shown the daily historical returns for several stocks and the 'index' used in the random choice.

20.7 USE OF VaR AS A PERFORMANCE MEASURE

One of the uses of VaR is in the measurement of performance of banks, desks or individual traders. In the past, 'trading talent' has been measured purely in terms of profit; a trader's bonus is related to that profit. This encourages traders to take risks, think of tossing a coin with you receiving a percentage of the profit but without the downside (which is taken by the bank), how much would you bet? A better measure of trading talent might take into account the risk in such a bet, and reward a good return-to-risk ratio. The ratio

$$\frac{\text{Return in excess of risk-free}}{\text{volatility}} = \frac{\mu - r}{\sigma},$$

the **Sharpe ratio**, is such a measure. Alternatively, use VaR as the measure of risk and profit/loss as the measure of return:

$$\frac{\text{daily P\&L}}{\text{daily VaR}}.$$

20.8 SUMMARY

The estimation of downside potential in any portfolio is clearly very important. Not having an idea of this could lead to the disappearance of a bank, and has. In practice, it is more important to the managers in banks, and not the traders. What do they care if their bank collapses as long as they can walk into a new job?

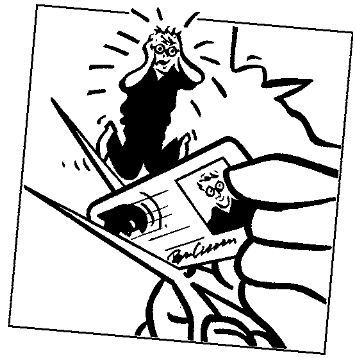
I have shown the simplest ways of estimating Value at Risk, but the subject gets much more complicated. 'Complicated' is not the right word, 'messy' and 'time-consuming' are better. And currently there are many software vendors, banks and academics touting their own versions in the hope of becoming the market standard. In Chapter 22 we'll see a few of these in more detail.

FURTHER READING

- See Lawrence (1996) for the application of the Value at Risk methodology to derivatives.
- See Chew (1996) for a wide-ranging discussion of risk management issues and details of important real-life VaR 'crises.'

- See Jorion (1997) for further information about the mathematics of Value at Risk.
- The allocation of bank capital is addressed in Matten (1996).
- Alexander & Leigh (1997) and Alexander (1997a) discuss the estimation of covariance matrices for VaR.
- Artzner *et al*. (1997) discuss the properties that a sensible VaR model must possess.

CHAPTER 21
credit risk



The aim of this Chapter...

... is to introduce the concept of credit worthiness and default into our finance modeling. We will also be seeing a new random process, the Poisson process, which is useful for modeling sudden, unexpected changes of state.

In this Chapter...

- models for instantaneous and exogenous risk of default
- stochastic risk of default and implied risk of default
- credit ratings
- how to model change of rating

21.1 INTRODUCTION

In this chapter I describe some ways of looking at default. A recent approach to this subject is to model default as a completely exogenous event, i.e. a bit like the tossing of a coin or the appearance of zero on a roulette wheel, and having nothing to do with how well the company or country is doing. Typically, one then infers from risky bond prices the probability of default as perceived by the market. I'm not wild about this idea but it is very popular.[1]

Later in this chapter I describe the rating service provided by Standard & Poor's and Moody's, for example. These ratings provide a published estimate of the relative creditworthiness of firms.

21.2 RISKY BONDS

If you are a company wanting to expand, but without the necessary capital, you could borrow the capital, intending to pay it back with interest at some time in the future. There is a chance, however, that before you pay off the debt the company may have got into financial difficulties or even gone bankrupt. If this happens, the debt may never be paid off. Because of this risk of default, the lender will require a significantly higher interest rate than if he were lending to a very reliable borrower such as the US government.

The real situation is, of course, more complicated than this. In practice it is not just a case of all or nothing. This brings us to the ideas of the seniority of debt and the partial payment of debt.

Firms typically issue bonds with a legal ranking, determining which of the bonds take priority in the event of bankruptcy or inability to repay. The higher the priority, the more likely the debt is to be repaid, the higher the bond value and the lower its yield. In the event of bankruptcy there is typically a long, drawn out battle over which creditors get paid. It is usual, even after many years, for creditors to get *some* of their money back. Then the question is how much and how soon? It is also possible for the repayment to be another risky bond, this would correspond to a refinancing of the debt. For example, the debt could not be paid off at the planned time so instead a new bond gets issued entitling the holder to an amount further in the future.

In Figure 21.1 is shown the yield versus duration, calculated by the methods of Chapter 14, for some risky bonds. In this figure the bonds have been ranked according to their estimated riskiness. We will discuss this later, for the moment you just need to know that Ba3 is considered to be less risky than Caa1 and this is reflected in its smaller yield spread over the risk-free curve.

The problem that we examine in this chapter is the modeling of the risk of default and thus the fair value of risky bonds. Conversely, if we know the value of a bond, does this tell us anything about the likelihood of default?

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[1] Bankers are paid such obscene amounts of money, you'd hope they'd feel obliged to think originally once in a while, wouldn't you? Apparently, this is not the case.

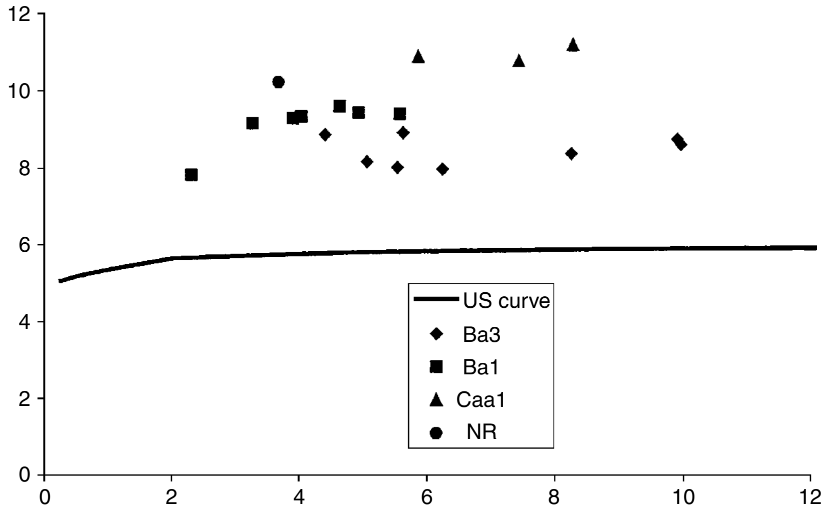


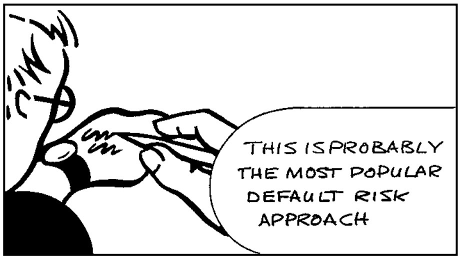
Figure 21.1 Yield versus duration for some risky bonds.

21.3 MODELING THE RISK OF DEFAULT

The models that I describe assume that the likelihood of default is exogenous. These instantaneous risk of default models are simple to use and are therefore the most popular type of credit risk models. In its simplest form the time at which default occurs is completely exogenous. For example, we could roll a die once a month, and when a 1 is thrown the company defaults. This illustrates the exogeneity of the default and also its randomness, a Poisson process is a typical choice for the time of default. We will see that when the intensity of the Poisson process is constant (as in the die example), the pricing of risky bonds amounts to the addition of a time-independent spread to the bond yield. We will also see models for which the intensity is itself a random variable.

A refinement of the modeling that we also consider is the regrading of bonds. There are agents, such as Standard & Poor's and Moody's, who classify bonds according to their estimate of their risk of default. A bond may start life with a high rating, for example, but may find itself regraded due to the performance of the issuing firm. Such a regrading will have a major effect on the perceived risk of default and on the price of the bond. I will describe a simple model for the rerating of risky bonds.

21.4 THE POISSON PROCESS AND THE INSTANTANEOUS RISK OF DEFAULT



One approach to the modeling of credit risk is via the **instantaneous risk of default**, p. If at time t the company has not defaulted and the instantaneous risk of default is p then the probability of default between times t and $t + dt$ is

$p\,dt$. This is an example of a Poisson process; nothing happens for a while, then there is a *sudden* change of state. This is a continuous-time version of our earlier model of throwing a die.

Time Out...

Poisson processes

The basic building block for the random walks we have considered so far is continuous Brownian motion based on the Normally distributed increment. Another stochastic process that is useful in finance is the Poisson process.

A Poisson process dq is defined by

$$dq = \begin{cases} 0 & \text{with probability } 1 - \lambda\,dt \\ 1 & \text{with probability } \lambda\,dt. \end{cases}$$

There is therefore a probability $\lambda\,dt$ of a jump in q in the timestep dt. The parameter λ is called the **intensity** of the Poisson process. The scaling of the probability of a jump with the size of the timestep is important in making the resulting process 'sensible', i.e. there being a finite chance of a jump occurring in a finite time, with q not becoming infinite.

The simplest example to start with is to take p constant. In this case we can easily determine the risk of default before time T. We do this as follows.

Let $P(t;T)$ be the probability that the company does not default before time T given that it has not defaulted at time t. The probability of default between later times t' and $t'+dt'$ is the product of $p\,dt$ and the probability that the company has not defaulted up until time t'. Thus,

$$P(t'+dt', T) - P(t', T) = p\,dt\,P(t', T).$$

Expanding this for a small timestep results in the ordinary differential equation representing the rate of change of the required probability:

$$\frac{\partial P}{\partial t'} = pP(t';T).$$

If the company starts out not in default then $P(T;T) = 1$. The solution of this problem is

$$e^{-p(T-t)}.$$

The value of a zero-coupon bond paying \$1 at time T could therefore be modeled by taking the present value of the *expected* cashflow. This results in a value of

$$e^{-p(T-t)}Z, \tag{21.1}$$

where Z is the value of a riskless zero-coupon bond of the same maturity as the risky bond. Note that this does not put any value on the risk taken. The yield to maturity on this bond is now given by

$$-\frac{\log(e^{-p(T-t)}Z)}{T-t} = -\frac{\log Z}{T-t} + p.$$

Thus the effect of the risk of default on the yield is to add a spread of p. In this simple model, the spread will be constant across all maturities.

Now we apply this to derivatives, including risky bonds. We will assume that the spot interest rate is stochastic. For simplicity we will assume that there is no correlation between the diffusive change in the spot interest rate and the Poisson process.

Construct a 'hedged' portfolio:

$$\Pi = V(r, p, t) - \Delta Z(r, t).$$

Consider how this changes in a timestep. See Figure 21.2 for a diagram illustrating the analysis below.

There is a probability of $(1 - p\,dt)$ that the bond does not default. In this case the change in the value of the portfolio during a timestep is

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}\right)dt + \frac{\partial V}{\partial r}dr - \Delta\left(\left(\frac{\partial Z}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2}\right)dt + \frac{\partial Z}{\partial r}dr\right). \qquad (21.2)$$

Choose Δ to eliminate the risky dr term.

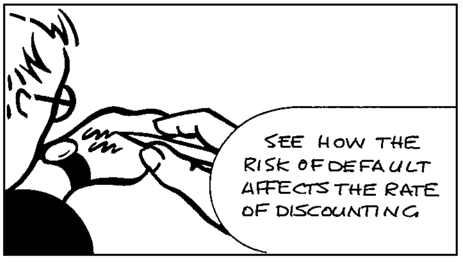
On the other hand, if the bond defaults, with a probability of $p\,dt$, then the change in the value of the portfolio is

$$d\Pi = -V + O(dt^{1/2}). \qquad (21.3)$$

This is due to the sudden loss of the risky bond, the other terms are small in comparison.

Taking expectations and using the bond-pricing equation for the riskless bond, we find that the value of the risky bond satisfies

$$\boxed{\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - (r + p)V = 0. \qquad (21.4)}$$



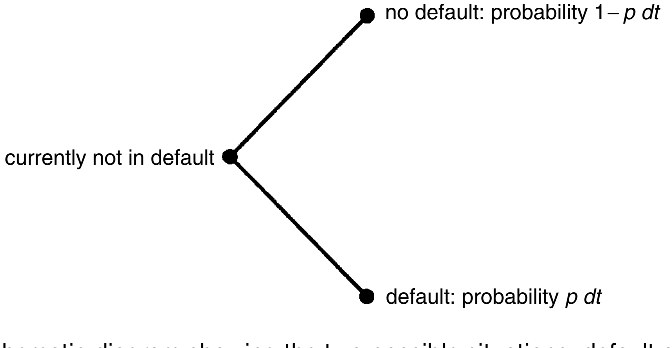


Figure 21.2 A schematic diagram showing the two possible situations: default and no default.

Observe that the spread has been added to the discounting term.

Time Out...

$$r \to r + p$$

The risk of default has been added to the discounting term. Thus all cashflows are going to be discounted at a rate of $r + p$ instead of r. We can interpret p as a 'spread.' With p being positive this means that all positive cashflows have less present value than if there were no credit risk. The simplicity of this model is the reason for its popularity. (It is not popular because it is a good model.)

Example A four-year US government risk-free zero-coupon bond has a principal of \$100 and a current value of \$81.11. A similar bond issued by a less reliable source has a value of \$77.72 . What do these numbers tell you? The yield to maturity of the US government bond is 5.2%. The yield to maturity of the risky bond is 6.3%. The spread of 1.1% between these two means that the market expects (ish) that there is a $1 - e^{-0.011} \approx 1.11\%$ chance per year of the company defaulting.

How would you change this model so that negative cashflows are not affected? After all, why would a company refuse a payment in its favor?

The portfolio is only hedged against moves in the interest rate (assuming that the market price of risk is known, big assumption) and not the event of default. I'll come back to this point below.

This model is the most basic for the instantaneous risk of default. It gives a very simple relationship between a risk-free and a risky bond. There is only one new parameter to estimate, p.

To see whether this is a realistic model for the expectations of the market we take a quick look at the valuation of Brady bonds. In particular we examine the market price of Latin American par bonds, described in full later in this chapter. For the moment, we just need to know that these bonds have interest payments and final return of principal denominated in US dollars. If the above is a good model of market expectations with constant p then we would find a very simple relationship between interest rates in the US and the value of Brady bonds. To find the Brady bond value perform the following:

1. Find the risk-free yield for the maturity of each cashflow in the risky bond.
2. Add a constant spread, p, to each of these yields.
3. Use this new yield to calculate the present value of each cashflow.
4. Sum all the present values.

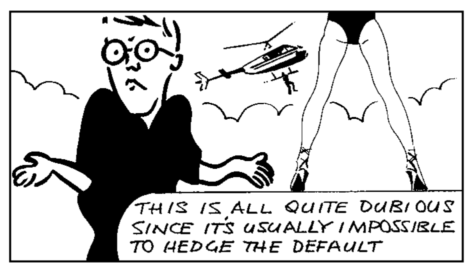
Conversely, the same procedure can be used to determine the value of p from the market price of the Brady bond: this would be the **implied risk of default**. In Figure 21.3 are shown the implied risks of default for the par bonds of Argentina, Brazil, Mexico and Venezuela using the above procedure and assuming a constant p.

In this simple model we have assumed that the instantaneous risk of default is constant (different for each country) through time. However, from Figure 21.3 we can see that, if we believe the market prices of the Brady bonds, this assumption is incorrect: the market prices are inconsistent with a constant p. This will be our motivation for the stochastic risk of default model which we will see in a moment. Nevertheless, supposing that the figure represents, in some sense, the views of the market (and this constant p model *is* used in practice) we draw a few conclusions from this figure before moving on.

The first point to notice in the graph is the perceived risk of Venezuela, which is consistently greater than the three other countries. Venezuela's risk peaked in July 1994, nine months before the rest of South America, but this had absolutely no effect on the other countries.

The next, and most important, thing to notice is the 'tequila effect' in all the Latin markets. The tequila crisis began with a 50% devaluation of the Mexican peso in December 1994. Markets followed suit by plunging. Before December 1994 we can see a constant spread between Mexico and Argentina and a contracting spread between Brazil and Argentina. The consequences of tequila were felt through all the first quarter of 1995 and had a knock-on effect throughout South America. In April 1995 the default risks peaked in all the countries apart from Venezuela, but by late 1996 the default risk had almost returned to pre-tequila levels in all four countries. By this time, Venezuela's risk had fallen to the same order as the other countries.

21.4.1 A note on hedging



In the above we have not hedged the event of default. This can sometimes be done (kinda), provided we can hedge with other bonds that will default at exactly the same time, and later we'll see how this introduces a market price of default risk term, as might be expected. Usually, though, there are so few risky bonds that hedging default is not possible. For the moment I'm going to work in terms of real expectations on the understanding that in the world of default risk, perfect hedging is rarely possible.

21.5 TIME-DEPENDENT INTENSITY AND THE TERM STRUCTURE OF DEFAULT

Suppose that a company issues risky bonds of different maturities. We can deduce from the market's prices of these bonds how the risk of default is perceived to depend on time. To make things as simple as possible let's assume that the company issues zero-coupon bonds and that in the event of default in one bond, all other outstanding bonds also default with no recovery rate.

If the risk of default is time dependent, $p(t)$, and uncorrelated with the spot interest rate, then the real expected value of a risky bond paying \$1 at time T is just

$$Ze^{-\int_t^T p(\tau)d\tau}.$$

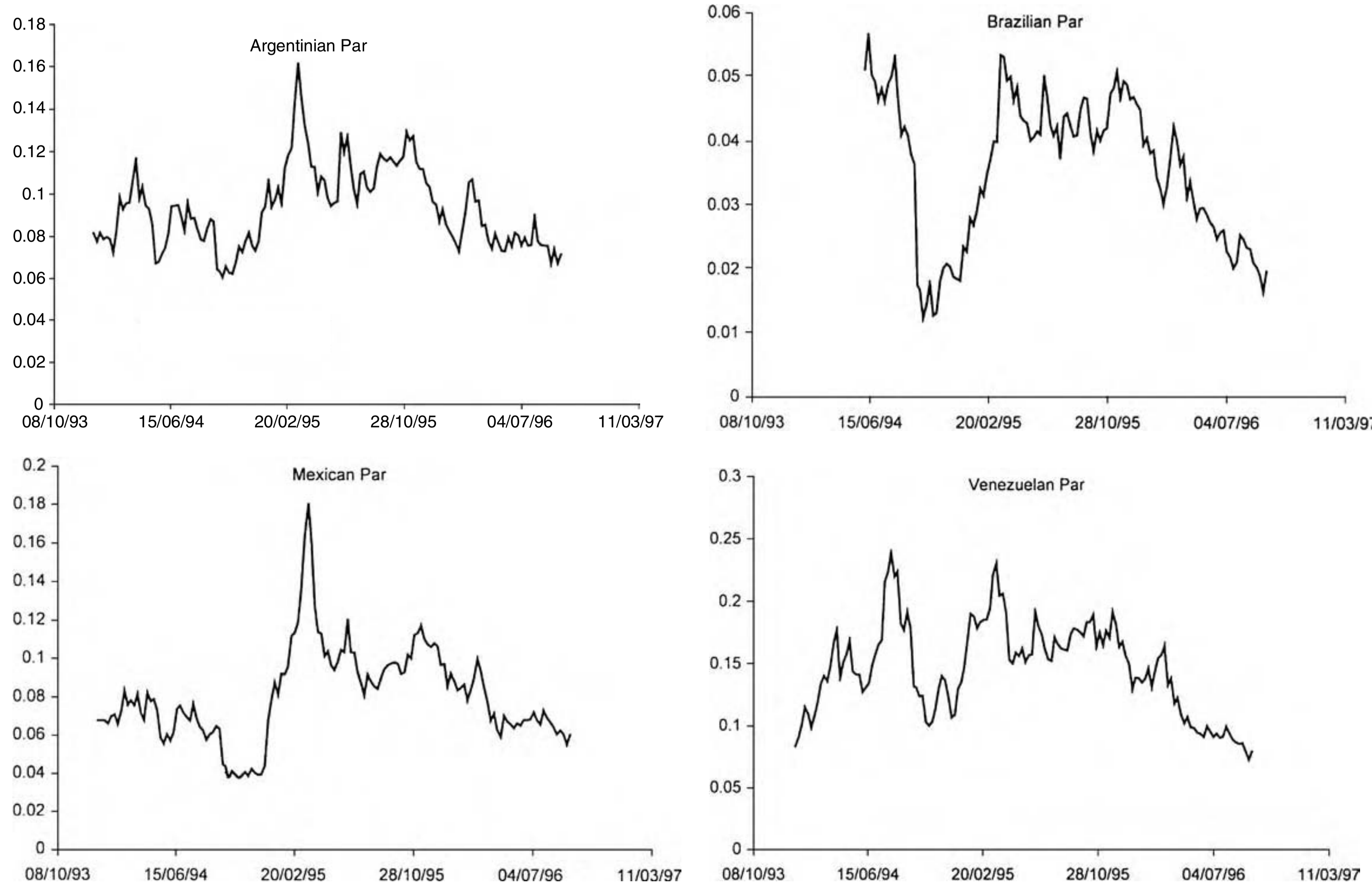


Figure 21.3 The implied risk of default for the par bonds of Argentina, Brazil, Mexico and Venezuela assuming constant *p*.

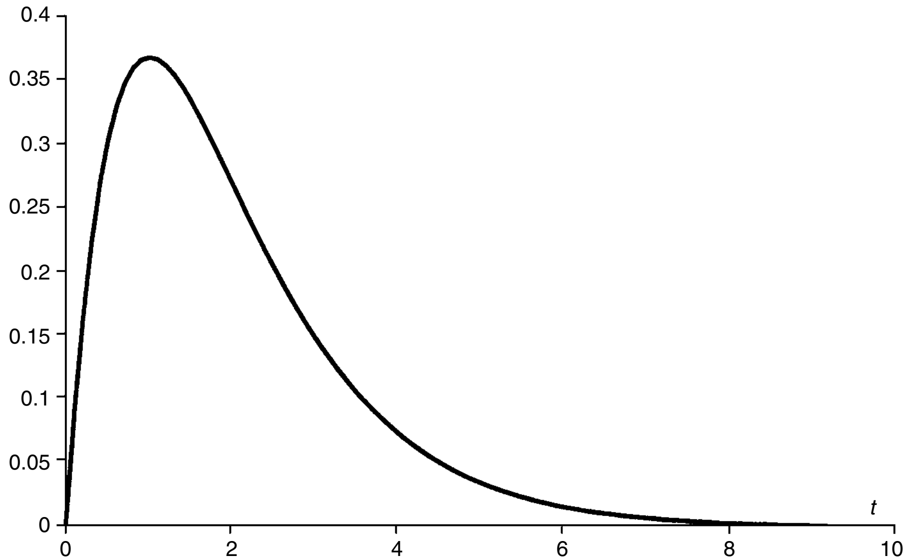


Figure 21.4 A plausible structure for a time-dependent hazard rate.

If the market value of the risky bond is Z^* then we can write

$$\int_t^T p(\tau)\, d\tau = \log\left(\frac{Z}{Z^*}\right).$$

Differentiating this with respect to T gives the market's view at the current time t of the **hazard rate** or risk of default at time T. A plausible structure for such a hazard rate is given in Figure 21.4. This figure shows a very small chance of default initially, rising to a maximum before falling off. The company is clearly expected to be around for at least a little while longer, and in the long term it will either have already expired or become very successful. If the area under the curve is finite then there is a finite probability of the company never going bankrupt.

21.6 STOCHASTIC RISK OF DEFAULT

To 'improve' the model, and make it consistent with market prices, we now consider a model in which the instantaneous probability of default is itself random. We assume that it follows a random walk given by

$$dp = \gamma(r, p, t)\, dt + \delta(r, p, t)\, dX_1,$$

with interest rates still given by

$$dr = u(r, t)\, dt + w(r, t)\, dX_2.$$

It is reasonable to expect some interest rate dependence in the risk of default, but not the other way around.

To value our risky zero-coupon bond we construct a portfolio with one of the risky bond, with value $V(r, p, t)$ (to be determined), and short Δ of a riskless bond, with value $Z(r, t)$ (satisfying our earlier bond pricing equation):

$$\Pi = V(r, p, t) - \Delta Z(r, t).$$

In the next timestep either the bond is defaulted or it is not. There is a probability of default of $p\,dt$. We must consider the two cases: default and no default in the next timestep. As in the two models above, we take expectations to arrive at an equation for the value of the risky bond.

First, suppose that the bond does not default; this has a probability of $(1 - p\,dt)$. In this case the change in the value of the portfolio during a timestep is

$$d\Pi = \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + \rho w\delta\frac{\partial^2 V}{\partial r\,\partial p} + \tfrac{1}{2}\delta^2\frac{\partial^2 V}{\partial p^2}\right)dt + \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial p}dp$$
$$- \Delta\left(\left(\frac{\partial Z}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2}\right)dt + \frac{\partial Z}{\partial r}dr\right),$$

where ρ is the correlation between dX_1 and dX_2. Choose Δ to eliminate the risky dr term.

On the other hand, if the bond defaults, with a probability of $p\,dt$, then the change in the value of the portfolio is

$$d\Pi = -V + O(dt^{1/2}).$$

This is due to the sudden loss of the risky bond, the other terms are small in comparison. It is at this point that we could put in a recovery rate, discussed in the next section, as it stands here default means no return whatsoever.

Taking expectations and using the bond-pricing equation for the riskless bond, we find that the value of the risky bond satisfies

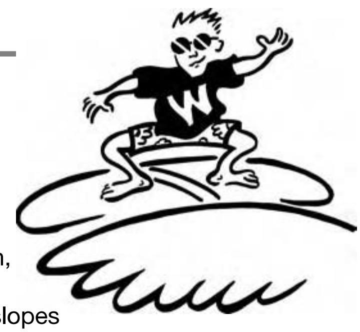
$$\boxed{\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + \rho w\delta\frac{\partial^2 V}{\partial r\,\partial p} + \tfrac{1}{2}\delta^2\frac{\partial^2 V}{\partial p^2} + (u - \lambda w)\frac{\partial V}{\partial r} + \gamma\frac{\partial V}{\partial p} - (r + p)V = 0.} \qquad (21.5)$$

This equation has final condition

$$V(r, p, T) = 1,$$

if the bond is zero coupon with a principal repayment of \$1.

Equation (21.5) again shows the similarity between the spot interest rate, r, and the hazard rate, p. The equation is remarkably symmetrical in these two variables, the only difference is in the choice of the model for each. In particular, the final term includes a discounting at rate r and also at rate p. These two variables play similar roles in credit risk equations.



Time Out...

Two-factor models

There are three independent variables in this partial differential equation, t, r and p. And one dependent variable V. If we wanted to think of the meaning of such an equation in terms of our usual mountainside, with slopes in the northerly and westerly directions, then we'd have our job cut out. But the principle is the same. Recall our discussion of the explicit finite-difference method on page 144. The idea works even in this higher-dimensional problem.

As a check on this result, return to the simple case of constant p. In the new framework this case is equivalent to $\gamma = \delta = 0$. The solution of (21.5) is easily seen to be

$$e^{-p(T-t)}Z(r,t),$$

as derived earlier.

If γ and δ are independent of r and the correlation coefficient ρ is zero then we can write

$$V(r,p,t) = Z(r,t)H(p,t),$$

where H satisfies

$$\frac{\partial H}{\partial t} + \frac{1}{2}\delta^2\frac{\partial^2 H}{\partial p^2} + \gamma\frac{\partial H}{\partial p} - pH = 0,$$

with

$$H(p,T) = 1.$$

In this special, but important case, the default risk decouples from the bond pricing.

21.7 POSITIVE RECOVERY

In default there is usually *some* payment, not all of the money is lost. In Table 21.1, produced by Moody's from historical data, is shown the mean and standard deviations for recovery according to the seniority of the debt. These numbers emphasize the fact that the rate of recovery is itself very uncertain. How can we model a positive recovery?

Suppose that on default we know that we will get an amount Q. This will change the partial differential equation. To see this we return to the derivation of Equation (21.4). If there is no default we still have Equation (21.2). However, on default Equation (21.3) becomes instead

$$d\Pi = -V + Q + O(dt^{1/2});$$

Table 21.1 Rate of recovery. Source: Moody's.

Class	Mean (%)	Std Dev. (%)
Senior secured	53.80	26.86
Senior unsecured	51.13	25.45
Senior subordinated	38.52	23.81
Subordinated	32.74	20.18
Junior subordinated	17.09	10.90

we lose the bond but get Q. Taking expectations results in

$$\frac{\partial V}{\partial t}+\tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}+\rho w\delta\frac{\partial^2 V}{\partial r\,\partial p}+\tfrac{1}{2}\delta^2\frac{\partial^2 V}{\partial p^2}+(u-\lambda w)\frac{\partial V}{\partial r}+\gamma\frac{\partial V}{\partial p}-(r+p)V+pQ=0.$$

Now you are faced with the difficult task of estimating Q, or modeling it as another random variable. It could even be a fraction of V. The last is probably the most sensible approach.

21.8 HEDGING THE DEFAULT

In the above we used riskless bonds to hedge the random movements in the spot interest rate. Can we introduce another risky bond or bonds into the portfolio to help with hedging the default risk? To do this we must assume that default in one bond automatically means default in the other.

Assuming that the risk of default p is constant for simplicity, consider the portfolio

$$\Pi=V-\Delta Z-\Delta_1 V_1,$$

where both V and V_1 are risky.

The choices

$$\Delta_1=\frac{V}{V_1}\quad\text{and}\quad\Delta=\frac{V_1\dfrac{\partial V}{\partial r}-V\dfrac{\partial V_1}{\partial r}}{V_1\dfrac{\partial Z}{\partial r}}$$

eliminate both default risk and spot rate risk. The analysis results in

$$\frac{\partial V}{\partial t}+\tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}+(u-\lambda w)\frac{\partial V}{\partial r}-(r+\lambda_1(r,t)p)V=0.$$

Observe that the 'market price of default risk' λ_1 is now where the probability of default appeared before, thus we have a risk-neutral probability of default. This equation is the risk-neutral version of Equation (21.4).

Can you imagine what happens if the risk of default is stochastic? There are actually three sources of randomness:

- the spot rate (the random movement in r)
- the probability of default (the random movement in p)
- the event of default (the Poisson process kicking in)

This means that we must hedge with *three* bonds, two other risky bonds and a risk-free bond, say. Where will you find market prices of risk? Can you derive a risk-neutral version of Equation (21.5)?

21.9 CREDIT RATING

There are many **credit rating agencies** who compile data on individual companies or countries and estimate the likelihood of default. The most famous of these are **Standard & Poor's** and **Moody's**. These agencies assign a **credit rating** or **grade** to firms as an estimate of their creditworthiness. Standard & Poor's rate businesses as one of AAA, AA, A, BBB, BB, B, CCC or Default. Moody's use Aaa, Aa, A, Baa, Ba, B, Caa, Ca, C. Both of these companies also have finer grades within each of these main categories. The Moody grades are described in Table 21.2.

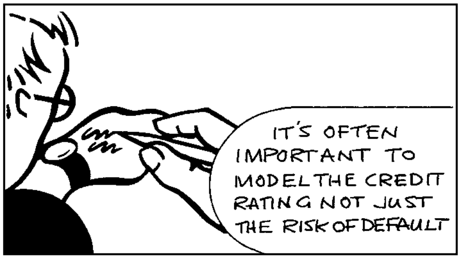
In Figure 21.5 is shown the percentage of defaults over the past eighty years, sorted according to their Moody's credit rating.

The credit rating agencies continually gather data on individual firms and will, depending on the information, grade/regrade a company according to well-specified criteria. A change of rating is called a **migration** and has an important effect on the price of bonds issued by the company. Migration to a higher rating will increase the value of a bond and decrease its yield, since it is seen as being less likely to default.

Clearly there are two stages to modeling risky bonds under the credit-rating scenario. First we must model the migration of the company from one grade to another and second we must price bonds taking this migration into account.

Figure 21.6 shows the credit rating for Eastern European countries by rating agency.

21.10 A MODEL FOR CHANGE OF CREDIT RATING



Company XYZ is currently rated A by Standard & Poor's. What is the probability that in one year's time it will still be rated A? Suppose that it is 91.305%. Now what is the probability that it will be rated AA or even AAA, or in default?

Table 21.2 The meaning of Moody's ratings.

Aaa	Bonds of best quality. Smallest degree of risk. Interest payments protected by a large or stable margin.
Aa	High quality. Margin of protection lower than Aaa.
A	Many favorable investment attributes. Adequate security of principal and interest. May be susceptible to impairment in the future.
Baa	Neither highly protected nor poorly secured. Adequate security for the present. Lacking outstanding investment characteristics. Speculative features.
Ba	Speculative elements. Future not well assured.
B	Lack characteristics of a desirable investment.
Caa	Poor standing. May be in default or danger with respect to principal or interest.
Ca	High degree of speculation. Often in default.
C	Lowest-rated class. Extremely poor chance of ever attaining any real investment standing.

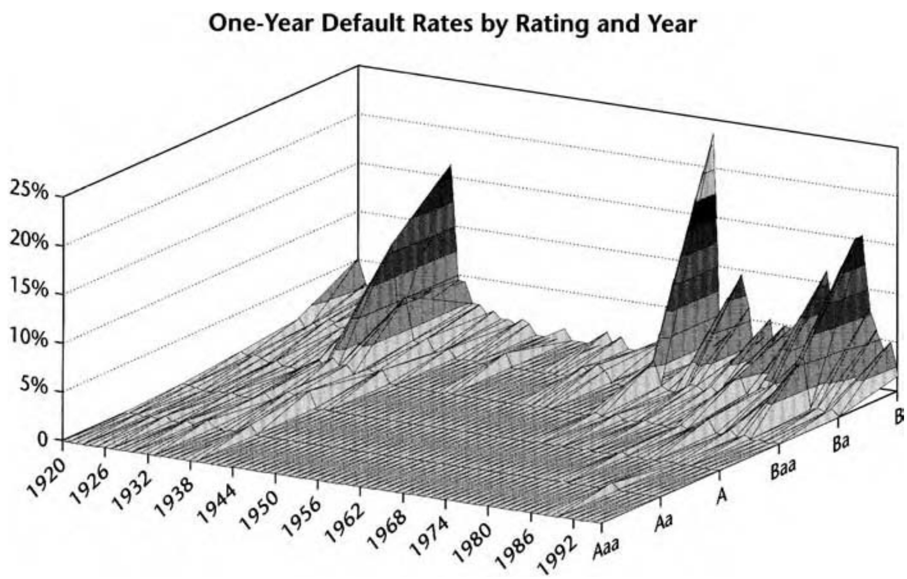


Figure 21.5 Percentage of defaults according to rating. Reproduced by permission of Moody's investors services.

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Foreign Currency LT Debt
Region – Eastern Europe

	MOODY'S	S&P	DCR	FI
Bulgaria	1) B2	15) B	29) NR	43) B+
Croatia	2) Baa3	16) BBB-	30) NR	44) BB+
Cyprus	3) A2	17) A+	31) NR	45) NR
Czech Republic	4) Baa1	18) A-	32) A-	46) BBB+
Estonia	5) Baa1	19) BBB+	33) NR	47) BBB
Hungary	6) Baa1	20) BBB	34) BBB	48) BBB
Latvia	7) Baa2	21) BBB	35) NR	49) BBB
Lithuania	8) Ba1	22) BBB-	36) NR	50) BB+
Moldova	9) B2 *-	23) NR	37) NR	51) B
Poland	10) Baa1	24) BBB	38) BBB-	52) BBB+
Romania	11) B3	25) B-	39) NR	53) B-
Russia	12) B3	26) SD	40) B- *-	54) CCC
Slovakia	13) Ba1	27) BB+	41) NR	55) BB+
Slovenia	14) A3	28) A	42) NR	56) A-

COLOR DENOTES A RATING CHANGE WITHIN THE LAST 30 DAYS (Pos/Neg/Neutral)

Bloomberg

Figure 21.6 Ratings for Eastern European countries by rating agency. Source: Bloomberg L.P.

Table 21.3 An example of a transition matrix.

	AAA	AA	A	BBB	BB	B	CCC	Default
AAA	0.90829	0.08272	0.00736	0.00065	0.00066	0.00014	0.00006	0.00012
AA	0.00665	0.90890	0.07692	0.00583	0.00064	0.00066	0.00029	0.00011
A	0.00092	0.02420	0.91305	0.05228	0.00678	0.00227	0.00009	0.00041
BBB	0.00042	0.00320	0.05878	0.87459	0.04964	0.01078	0.00110	0.00149
BB	0.00039	0.00126	0.00644	0.07710	0.81159	0.08397	0.00970	0.00955
B	0.00044	0.00211	0.00361	0.00718	0.07961	0.80767	0.04992	0.04946
CCC	0.00127	0.00122	0.00423	0.01195	0.02690	0.11711	0.64479	0.19253
Default	0	0	0	0	0	0	0	1

We can represent these probabilities over the one-year time horizon by a **transition matrix**. An example is shown in Table 21.3.

This table is read as follows. Today the company is rated A. The probability that in one year's time it will be at another rating can be seen by reading across the A row in the table. Thus the probability of being rated AAA is 0.092%, AA 2.42%, A 91.305% etc. The highest probability is of no migration. By reading down the rows, this table can be interpreted as either a representation of the probabilities of migration of *all* companies from one grade to another, or of company XYZ had it started out at other than A. Whatever the grade today, the company must have some rating at the end of the year even if that rating is default. Therefore the probabilities reading across each row must sum to one. And once a company is in default, it cannot leave that state, therefore the bottom row must be all zeros except for the last number which represents the probability of going from default to default, i.e. 1.

This table or matrix represents probabilities over a finite horizon. But during that time a bond may have gone from A to BBB to BB, how can we model this sequence of migrations? This is done by introducing a transition matrix over an infinitesimal time period. We can model continuous-time transitions between states via **Markov chains**. This is quite an advanced topic, and I won't be going into this subject any more here.

21.11 SUMMARY

As can be seen from this chapter, credit risk modeling is a very big subject. I have shown some of the popular approaches, but they are by no means the only possibilities. To aid with the assessment of credit risk, the bank JP Morgan has created CreditMetrics, a methodology for assessing the impact of default events on a portfolio. This is described in Chapter 22.

As a final thought for this chapter, suppose that a company issues just the one risky bond so that there is no way of hedging the default. If you believe that the market is underpricing the bond because it overestimates the risk of default then you might decide to buy it. If you intend holding it until expiry, then the market price in the future is only relevant in so far as you may change your mind. But you really do care about the likelihood of default and will pay very close attention to news about the company. On the other hand, if you buy the bond with the intention of only holding it for a short time your main concern should be for how the market is behaving and the real risk of default is irrelevant.

You may still watch out for news about the company, but now your concern will be for how the market reacts to the news, not the news itself.

FURTHER READING

- See Black & Scholes (1973), Merton (1974), Black & Cox (1976), Geske (1977) and Chance (1990) for a treatment of the debt of a firm as an option on the assets of that firm. See Longstaff & Schwartz (1994) for more recent work in this area.
- The articles by Cooper & Martin (1996) and Schönbucher (1998) are general reviews of the state of credit risk modeling.
- See Apabhai *et al*. (1998) for more details of the company and debt valuation model, especially for final and boundary conditions for various business strategies. Epstein *et al*. (1997a, b) describe the firm valuation models in detail, including the effects of advertising and market research.
- The classic reference, and a very good read, for firm-valuation modeling is Dixit & Pindyck (1994). For a nontechnical POV see Copeland *et al*. (1990).
- See Kim (1995) for the application of the company valuation model to the question of company mergers and some suggestions for how it can be applied to problems in company relocation and tax status.
- Important work on the instantaneous risk of default model is by Jarrow & Turnbull (1990, 1995), Litterman & Iben (1991), Madan & Unal (1994), Lando (1994a), Duffie & Singleton (1994a, b) and Schönbucher (1996).
- See Blauer & Wilmott (1998) for the instantaneous risk of default model and an application to Latin American Brady bonds.
- See Duffee (1995) for other work on the estimation of the instantaneous risk of default in practice.
- The original work on change of credit rating was due to Lando (1994b), Jarrow *et al*. (1997) and Das & Tufano (1994). Cox & Miller (1965) describe Markov chains in a very accessible manner.
- See Ahn, Khadem & Wilmott (1998) for the rather sensible use of utility theory in credit risk modeling.
- Current market conditions and prices for Brady bonds can be found at `www.bradynet.com`.
- See `www.emgmkts.com` for financial news from emerging markets.

CHAPTER 22

RiskMetrics and CreditMetrics

The aim of this Chapter...

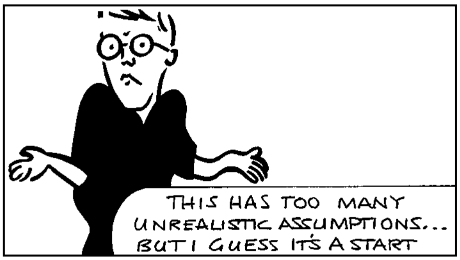
... is to describe JP Morgan's two popular methodologies for estimating risk associated with general movements of the financial markets, and with credit issues. We will be seeing how parameters are estimated and how the methodologies are implemented.

In this chapter...

- the methodology of RiskMetrics for measuring value at risk
- the methodology of CreditMetrics for measuring a portfolio's exposure to default events

22.1 INTRODUCTION

In Chapter 20 I described the concept of the Value at Risk (VaR) of a portfolio. I repeat the definition of VaR here: VaR is 'an estimate, with a given degree of confidence, of how much one can lose from one's portfolio over a given time horizon.' In that chapter I showed ways of calculating VaR (and some of the pitfalls of such calculations). Typically the data required for the calculations are parameters for the 'underlyings' and measures of a portfolio's current exposure to these underlyings. The parameters include volatilities and correlations of the assets, and, for longer time horizons, drift rates. The exposure of the portfolio is measured by the deltas, and, if necessary, the gammas (including cross derivatives) and the theta of the portfolio. The sensitivities of the portfolio are obviously best calculated by the owner of the portfolio, the bank. However, the asset parameters can be estimated by anyone with the right data. In October 1994 the American bank JP Morgan introduced the system **RiskMetrics** as a service for the estimation of VaR parameters. Some of this service is free, the data sets are available at `www.riskmetrics.com`, but the accompanying risk management software is not.



JP Morgan has also proposed a similar approach, together with a data service, for the estimation of risks associated with risk of default: **CreditMetrics**. CreditMetrics has several aims, two of these are the creation of a benchmark for measuring credit risk and the increase in market liquidity. If the former aim is successful then it will become possible to measure risks systematically across instruments and, at the very least, to make relative value judgments. From this follows the second aim. Once instruments, and in particular the risks associated with them, are better understood they will become less frightening to investors, promoting liquidity.

22.2 THE RISKMETRICS DATASETS

The RiskMetrics datasets are extremely broad and comprehensive. They are distributed over the internet. They consist of three types of data: one used for estimating risk over the time horizon of one day, the second having a one-month time horizon and the third has been designed to satisfy the requirements in the latest proposals from the Bank for International Settlements on the use of internal models to estimate market risk. The datasets contain estimates of volatilities and correlations for almost 400 instruments, covering foreign exchange, bonds, swaps, commodities and equity indices. Term-structure information is available for many currencies.

22.3 CALCULATING PARAMETERS THE RISKMETRICS WAY

A detailed technical description of the method for estimating financial parameters can be found at the JP Morgan web site. Here I only give a brief outline of major points.

22.3.1 Estimating volatility

The volatility of an asset is measured as the annualized standard deviation of returns. There are many ways of taking this measurement. The simplest is to take data going back a set period, three months, say, calculate the return for each day (or over the typical timescale at which you will be rehedging) and calculate the sample standard deviation of this data. This will result in a time series of three-month volatility.[1] This approach gives equal weight to all of the observations over the previous three months. This estimate of volatility on day i is calculated as

$$\sigma_i = \sqrt{\frac{1}{\delta t\,(M-1)} \sum_{j=i-M+1}^{i} (R_j - \overline{R})^2},$$

where δt is the timestep (typically one day), M is the number of days in the estimate (approximately 63 in three months), R_j is the return on day j and $\overline{R}$ is the average return over the previous M days. If δt is small then we can in practice neglect $\overline{R}$.

This measurement of volatility has two major drawbacks. First, it is not clear how many days of data we should use; what happened three months ago may not be relevant today. But the more data we have the smaller will be the sampling error if the volatility really has not changed in that period. Second, a large positive or negative return on one day will be felt in this historical volatility for the next three months. At the end of this period the volatility will apparently drop suddenly, yet there will have been no underlying change in market conditions; the drop will be completely spurious. Thus, the volatility measured in this way will show 'plateauing.' A typical thirty-day volatility plot, with plateauing, is shown in Figure 22.1.

In RiskMetrics the volatility is measured as the square root of a variance that is an exponential moving average of the square of price returns. This ensures that any individual return has a gradually decreasing effect on the estimated volatility, and plateauing does

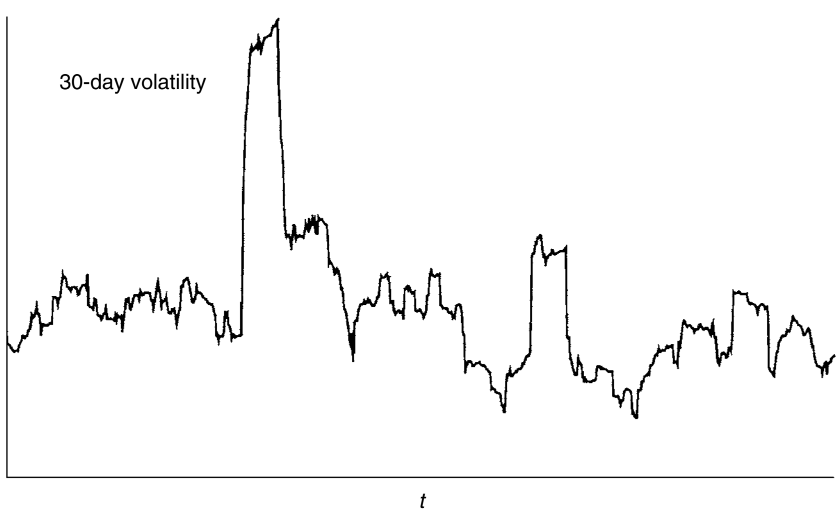


Figure 22.1 Thirty-day volatility.

[1] Which may or may not bear any resemblance to three-month implied volatility.

not occur. This volatility is estimated according to

$$\sigma_i = \sqrt{\frac{1-\lambda}{\delta t}\sum_{j=-\infty}^{i}\lambda^{i-j}R_j^2},$$

where λ represents the weighting attached to the past volatility versus the present return (and we have neglected the mean of R, assuming that the time horizon is sufficiently small). This difference in weighting is more easily seen if we write the above as

$$\sigma_i^2 = \lambda\sigma_{i-1}^2 + (1-\lambda)\frac{R_i^2}{\delta t}.$$

The parameter λ has been chosen by JP Morgan as either 0.94 for a horizon of one day and 0.97 for a horizon of one month. Another possibility is to choose λ to minimize the difference between the squares of the historical volatility and an implied volatility.

The spreadsheet in Figure 22.2 shows how to calculate such an exponentially weighted volatility. As can be seen from the plot, it is much 'better behaved' than the uniformly weighted version.

22.3.2 Correlation

The estimation of correlation is similar to that of volatility. To calculate the covariance σ_{12} between assets 1 and 2 we can take an equal weighting of returns from the two assets over the last M days:

$$\sigma_{12_i} = \sqrt{\frac{1}{\delta t\,(M-1)}\sum_{j=i-M+1}^{i}(R_{1_j}-\overline{R}_1)(R_{2_j}-\overline{R}_2)}.$$

Again, this measure shows spurious sudden rises and falls because of the equal weighting of all the returns.

Alternatively, we can use an exponentially weighted estimate

$$\sigma_{12_i}^2 = \lambda\sigma_{12_{i-1}}^2 + (1-\lambda)\frac{R_{1_i}R_{2_i}}{\delta t}.$$

There are problems with the estimation of covariance due to the synchronicity of asset movements and measurement. Two assets may be perfectly correlated but because of their measurement at different times they may appear to be completely uncorrelated. This is a problem when using data from markets in different time zones. Moreover, there is no guarantee that the exponentially weighted covariances give a positive-definite matrix.

22.4 THE CREDITMETRICS DATASET

The CreditMetrics dataset is available free of charge from `www.jpmorgan.com`. The CreditMetrics methodology is also described in great detail at that site. The dataset consists of four data types: yield curves, spreads, transition matrices and correlations. Before reading the following sections the reader should be comfortable with the concept of credit rating, see Chapter 21.

	A	B	C	D	E	F	G	H
1	**Start volatility**	**0.3**		**Date**	**Stock**	**Returns**	**Vol^2**	**Volatility**
2	**Lambda**	**0.97**		1-Jan-85	218.32		0.09	0.3
3				2-Jan-85	217.16	-0.005313	0.087513	0.295827
4		=(E3-E2)/E2		3-Jan-85	215.24	-0.008841	0.085479	0.292368
5				4-Jan-85	215.24	0.000000	0.082915	0.287949
6		=B1*B1		7-Jan-85	217.16	0.008920	0.081029	0.284655
7				8-Jan-85	220.25	0.014229	0.080129	0.28307
8	=$B$2*G7+(1-$B$2)*F8*F8*252			9-Jan-85	224.87	0.020976	0.081051	0.284695
9				10-Jan-85	224.87	0.000000	0.07862	0.280392
10				11-Jan-85	224.1	-0.003424	0.07635	0.276314
11		=SQRT(G11)		14-Jan-85	219.09	-0.022356	0.077838	0.278994
12				15-Jan-86	219.09	0.000000	0.075502	0.274777
13				16-Jan-85	222.17	0.014058	0.074731	0.273371
14				17-Jan-85	226.02	0.017329	0.07476	0.273422
15				18-Jan-85	226.02	0.000000	0.072517	0.26929
16				21-Jan-85	226.79	0.003407	0.070429	0.265385
17				22-Jan-85	234.49	0.033952	0.077031	0.277545
18					57	-0.008188	0.075227	0.274275
19					55	0.013243	0.074296	0.272573
20					27	0.019605	0.074973	0.273812
21					.5	-0.003205	0.072802	0.269818
22					72	-0.024134	0.075021	0.273899
23					35	0.041203	0.085605	0.292583
24					12	0.003164	0.083112	0.288292
25					04	0.007865	0.081086	0.284757
26					04	0.000000	0.078654	0.280453
27					39	0.015648	0.078145	0.279545
28					97	-0.007683	0.076247	0.276129
29				7-Feb-85	247.97	0.000000	0.07396	0.271956
30				8-Feb-85	249.12	0.004638	0.071904	0.268149
31				11-Feb-85	243.35	-0.023162	0.073802	0.271666
32				12-Feb-85	238.34	-0.020588	0.074792	0.273482
33				13-Feb-85	239.5	0.004867	0.072728	0.269681
34				14-Feb-85	238.34	-0.004843	0.070723	0.265938
35				15-Feb-85	235.65	-0.011286	0.069565	0.263751
36				18-Feb-85	232.57	-0.013070	0.068769	0.262239
37				19-Feb-85	234.49	0.008256	0.067221	0.259271
38				20-Feb-85	235.65	0.004947	0.06539	0.255714
39				21-Feb-85	236.42	0.003268	0.063509	0.252009
40				22-Feb-85	235.65	-0.003257	0.061684	0.248362
41				25-Feb-85	234.49	-0.004923	0.060016	0.244982
42				26-Feb-85	235.65	0.004947	0.058401	0.241663
43				27/02/85	232.57	-0.013070	0.05794	0.240708

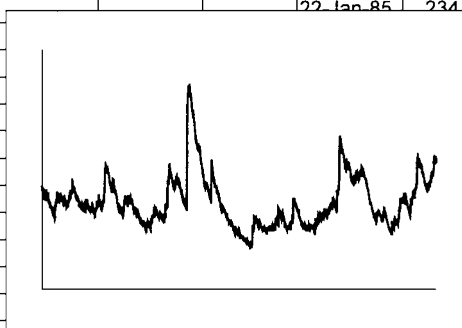
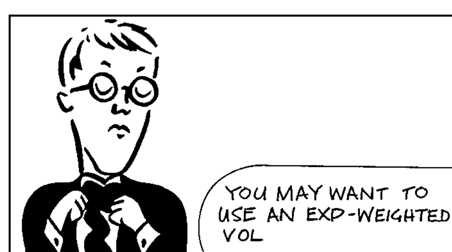


Figure 22.2 Spreadsheet to calculate an exponentially weighted volatility.



22.4.1 Yield curves

The CreditMetrics yield curve dataset consists of the *risk-free* yield to maturity for major currencies. In Figure 22.3 is shown an example of these risk-free yields. The dataset contains yields for maturities of 1, 2, 3, 5, 7, 10 and 30 years. For example, from the yield curve dataset we have information such as the yield to maturity for a three-year US dollar bond is 6.12%.

22.4.2 Spreads

For each credit rating, the dataset gives the spread above the riskless yield for each maturity. In Figure 22.3 is shown a typical riskless US yield curve, and the yield on AA and BBB bonds. For example, we may be given that the spread for an AA bond is 0.54% above the riskless yield for a three-year bond. Observe that the riskier the bond the higher the yield; the yield on the BBB bond is everywhere higher than that on the AA bond which is in turn higher than the risk-free yield. This higher yield for risky bonds is compensation for the possibility of not receiving future coupons or the principal.

22.4.3 Transition matrices

The concept of the transition matrix has been discussed in Chapter 21. In the CreditMetrics framework, the transition matrix has as its entries the probability of a change of credit rating at the end of a given time horizon, for example, the probability of an upgrade from AA to AAA might be 5.5%. The time horizon for the CreditMetrics dataset is one year. Unless the time horizon is very long, the largest probability is typically for the bond to remain at its initial rating, let's say that the probability of staying at AA is 87% in this example. We discussed transition matrices in depth in Chapter 21.

22.4.4 Correlations

In the risk-free yield, the spreads and the transition matrix, there is sufficient information for the CreditMetrics method to derive distributions for the possible future values of a

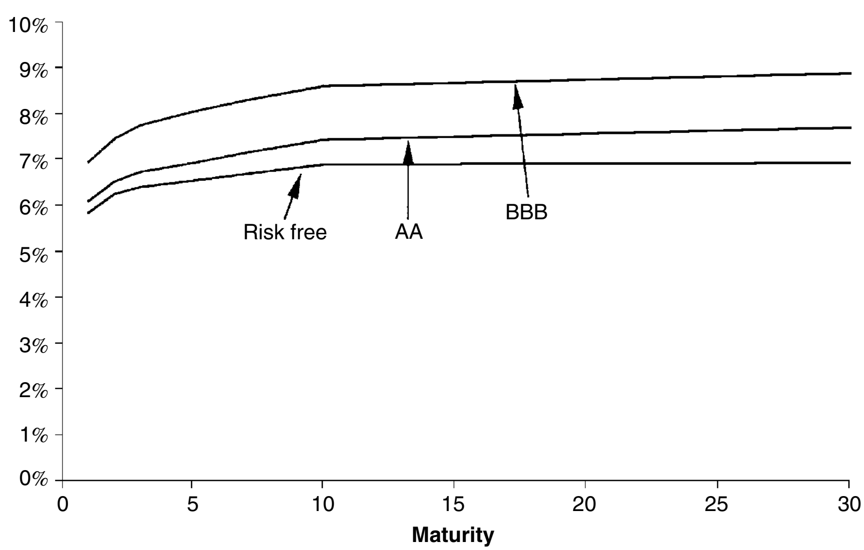


Figure 22.3 Risk-free and two risky yield curves.

single bond. I show how this is done in the next section. However, when we come to examine the behavior of a portfolio of risky bonds, we must consider whether there is any relationship between the rerating or default of one bond and the rerating or default of another. In other words, are bonds issued by different companies or governments in some sense correlated? This is where the CreditMetrics correlation dataset comes in. This dataset gives the correlations between major indices in many countries.

Each company issuing bonds has the return on its stock decomposed into parts correlated with these indices and a part which is specific to the company. By relating all bond issuers to these indices we can determine correlations between the companies in our portfolio. We will see how this is used in practice later in this chapter.

22.5 THE CREDITMETRICS METHODOLOGY

The CreditMetrics methodology is about calculating the possible values of a risky portfolio at some time in the future (the time horizon) and estimating the probability of such values occurring. Let us consider just a single risky bond currently rated AA. Suppose that the bond is zero-coupon, with a maturity of three years and we want to know the possible state of this investment in one year's time. The yield to maturity on this instrument might be 6.12%, for a three-year riskless bond, plus 0.54% for the spread for a three-year AA-rated bond. The total yield is therefore 6.66%, giving a price of 0.819.

The value of the bond will fluctuate between now and one year's time for each of three reasons: the passage of time, the evolution of interest rates and the possible regrading of the bond. Let us take these three points in turn.

First, because of the passage of time our three-year bond will be a two-year bond in one year. But what will be the yield on a two-year bond in one year's time? This is the second point. The assumption that is made in CreditMetrics is that the forward rates do not change between today and the time horizon ('rolling down the curve'). From the yields that we have today we can calculate the forward rates that apply between now and one year, between one and two years, between two and three years etc. This calculation is described in Chapter 14. We can calculate the value of the bond after one year, suppose it is 0.882. But why should the bond still be rated AA at that time? This is the third point. From our transition matrix we see that the probability of the bond's rating staying AA is 87%. So, there is an 87% chance that the bond's value will be 0.882. We can similarly work out the value of the bond in one year if it is rated AAA, A, BBB etc. using the relevant forward rates and spreads that we assume will apply in one year's time. And each of these has a probability of occurring that is given in the transition matrix. A probability distribution of the possible bond values is shown in Figure 22.4.

This, highly skewed, distribution tells us all we need to know to determine the risk in this particular bond. We can, for example, calculate the expected value of the bond.

22.6 A PORTFOLIO OF RISKY BONDS

We have seen how to apply the CreditMetrics methodology to a single risky bond, to apply the ideas to a portfolio of risky bonds is significantly harder since it requires the knowledge of any relationship between the different bonds. This is most easily measured by some sort of correlation.

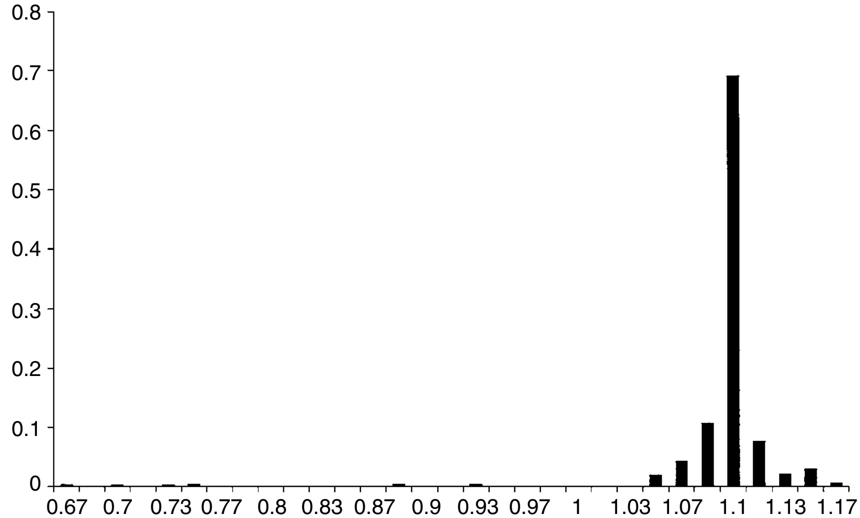


Figure 22.4 The probability distribution for the bond's value after one year.

Suppose that we have a portfolio of two bonds. One, issued by ABC, is currently rated AA and the other, issued by XYZ, is BBB. We can calculate, using the method above, the value of each of these bonds at our time horizon for each of the possible states of the two bonds. If we assume that each bond can be in one of eight states (AAA, AA, ..., CCC, Default) there are $8^2 = 64$ possible joint states at the time horizon. To calculate the expected value of our portfolio and standard deviation we need to know the probability of each of these joint states occurring. This is where the correlation comes in.

There are two stages to determining the probability of any particular future joint state:

1. Calculate the correlations between bonds.
2. Calculate the probability of any joint state.

Stage 1 is accomplished by decomposing the return on the stock of each issuing company into parts correlated with the major indices.

22.7 CREDITMETRICS MODEL OUTPUTS

CreditMetrics is, above all, a way of measuring risk associated with default issues. From the CreditMetrics methodology one can calculate the risk, measured by standard deviation, of the risky portfolio over the required time horizon. Because of the risk of default the distribution of returns from a portfolio exposed to credit risk is highly skewed, as in Figure 22.4. The distribution is far from being Normal. Thus ideas from simple portfolio theory must be used with care. Although, it may not be a good absolute measure of risk in the classical sense, the standard deviation is a good indicator of relative risk between instruments or portfolios.

22.8 SUMMARY

This chapter has outlined some of the methodologies for competing and complementary Value at Risk measures. With something as important as Value at Risk there is an obvious case to be made for exploring all of the possible VaR measures to build up as accurate a profile as possible of the dangers lurking in your portfolio. In the next chapter we look at measuring and reducing the risk in stock market crashes.

FURTHER READING

- Download the datasets and very detailed descriptions of the RiskMetrics and CreditMetrics methodologies from `www.riskmetrics.com`.
- Alexander (1996a) is a critique of RiskMetrics as a risk measurement tool.
- Shore (1997) describes and implements the CreditMetrics methodology.

CHAPTER 23
CrashMetrics

The aim of this Chapter...

. . . is to explain the analysis of risk associated with market crashes. The methodology is orthogonal to that explained in the previous chapter and is used in conjunction with these other VaR methods and not instead of.

In this Chapter...

- the methodology of CrashMetrics for measuring a portfolio's exposure to sudden, unhedgeable market movements
- Platinum hedging
- crash coefficients
- margin hedging
- counterparty risk
- the CrashMetrics Index for measuring the magnitude of crashes

23.1 INTRODUCTION

The final piece of the jigsaw for estimating risk in a portfolio is **CrashMetrics**. If Value at Risk is about normal market conditions then CrashMetrics is the opposite side of the coin, it is about 'fire sale' conditions and the far-from-orderly liquidation of assets in far-from-normal conditions. CrashMetrics is a dataset and methodology for estimating the exposure of a portfolio to extreme market movements or crashes. It assumes that the crash is unhedgeable and then finds the worst outcome for the value of the portfolio. The method then shows how to mitigate the effects of the crash by the purchase or sales of derivatives in an optimal fashion, so-called Platinum hedging. Derivatives have sometimes been thought of as being a dangerous component in a portfolio, in the CrashMetrics methodology they are put to a benign use.

23.2 WHY DO BANKS GO BROKE?

There are two main reasons why banks get into serious trouble. The first reason is the lack of suitable or sufficient control over the traders. Through misfortune, negligence or dishonesty, large and unmanageable positions can be entered into. The consequences are either that the trader concerned becomes a hero and the bank makes a fortune, or the bank loses a fortune, the trader makes a run for it and the bank goes under. The odds are fifty-fifty. The second causes of disaster are the extreme, unexpected and unhedgeable moves in the stock market, the crashes.

23.3 MARKET CRASHES

In typical market conditions one's portfolio will fluctuate rapidly, but not dramatically. That is, it will rise and fall, minute by minute, day by day, but will not collapse. There are times, say once a year on average, when that fluctuation is dramatic... and usually in the downward direction. These are extreme market movements or market crashes. VaR can tell us nothing about these and they must be analyzed separately.

What's special about a crash? Two things spring to mind. Obviously a crash is a sudden fall in market prices, too rapid for the liquidation of a portfolio. But a crash isn't just a rise in volatility. It is also characterized by a special relationship between individual assets. During a crash, all assets fall together. There is no such thing as a crash where half the stocks fall and the rest stay put. Technically this means that all assets become perfectly correlated. In normal market conditions there may be some relationship between stocks, especially those in the same sector, but this connection may not be that strong. Indeed, it is the weakness of these relationships that allows diversification. A small insurance company will happily insure your car, because they can diversify across individuals. Insuring against an earthquake is a different matter. A high degree of correlation makes diversification impossible. This is where traditional VaR falls down, at exactly the time when it is needed most. Figure 23.1 shows the behavior of the correlation of several constituents of the S&P500 around the time of the 1987 stock market crash.

All is not lost. VaR is a very recent concept, created during the 1990s and fast becoming a market standard, with known drawbacks. Many researchers in universities and in banks

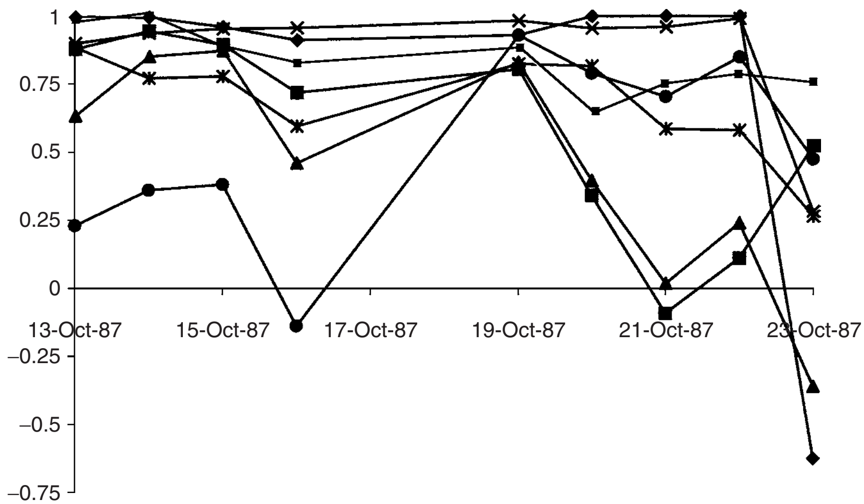


Figure 23.1 The correlation between several assets for a few days before and after the 1987 crash. When the correlation is close to one all assets move in the same direction.

are turning their thoughts to analyzing and protecting against crashes. Some of these researchers are physicists who concern themselves with examining the tails of returns distributions; are crashes more likely than traditional theory predicts? The answer is a definite yes.

My personal preference though is for models that don't make any assumptions about the likelihood of a crash. One line of work is that of 'worst-case scenarios.' Given that a crash could wipe out your portfolio, wouldn't you like to know what is the worst that could realistically happen, or would you be happy knowing what you would lose on average? CrashMetrics is used to analyze worst cases, and provide advice about how to hedge or insure against a crash.

23.4 CRASHMETRICS

CrashMetrics is a methodology for evaluating portfolio performance in the event of extreme movements in financial markets. It is not part of the JP Morgan family of performance measures. We will see how in CrashMetrics the portfolio of financial instruments is valued under a worst-case scenario with few assumptions about the size of the market move or its timing. The only assumptions made are that the market move, the 'crash' is limited in size and that the number of such crashes is limited in some way. There are no assumptions about the probability distribution of the size of the crash or its timing.

This method, used for day-to-day portfolio protection, is concerned with the extreme market movements that may occur when we are not watching, or that cannot be hedged away. These are the fire sale conditions. This is the method I will explain for the rest of this chapter. There are many nice things about the method such as its simplicity and ease of generalization, and no explicit dependence on the annoying parameters volatility and correlation.

23.5 CRASHMETRICS FOR ONE STOCK

To introduce the ideas, let's consider a portfolio of options on a single underlying asset. For the moment think in terms of a stock, although we could equally well talk about currencies, commodities or interest rate products.

If the stock changes dramatically by an amount δS how much does the portfolio of options on that stock behave? There will be a relationship between the change in the portfolio value $\delta\Pi$ and δS:

$$\delta\Pi = F(\delta S).$$

The function $F(\cdot)$ will simply be the sum of all the formulas, expressions, numerical solutions... for each of the individual contracts in the portfolio. Think of it as the sum of lots of Black–Scholes formulas with lots of different strikes, expiries, payoffs. If there is no change in the asset price there will be no change in the portfolio, so $F(0) = 0$. (There will be a small time decay, which we'll come back to later.) Figure 23.2 shows a possible portfolio change against underlying change.

If we are lucky, and we are not too near to the expiries and strikes of the options then we could approximate the portfolio by the Taylor series in the change in the underlying asset:

$$\delta\Pi = \Delta\ \delta S + \tfrac{1}{2}\Gamma\ \delta S^2. \tag{23.1}$$

In practice this won't be a good enough approximation. Imagine having some knock-out options in the portfolio, we really will have to use the relevant formula or numerical method to capture the sudden drop in value of this contract at the barrier. A simple delta/gamma approximation is not going to work.

However, as far as the math is concerned I'm going to show you both the general CrashMetrics methodology and the simple Taylor series version.

Now let's ask what is the worst that could happen to the portfolio overnight say? We want to find the minimum of $F(\delta S)$.

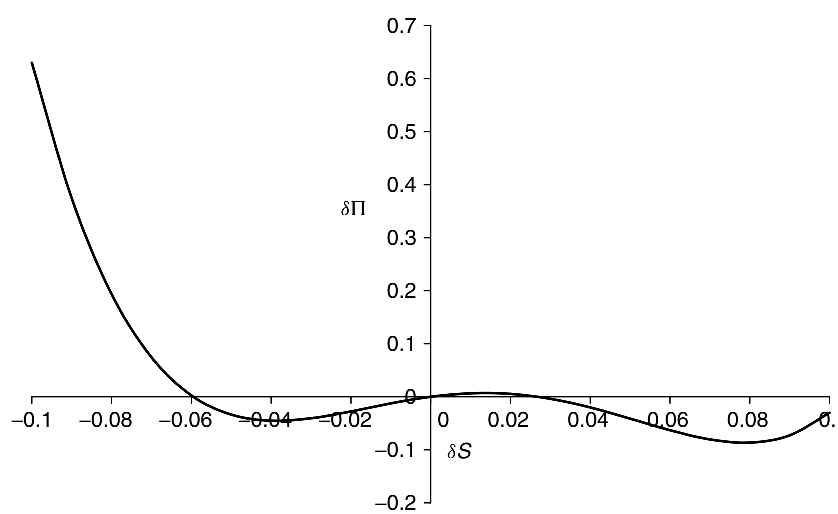


Figure 23.2 Size of portfolio change against change in the underlying.

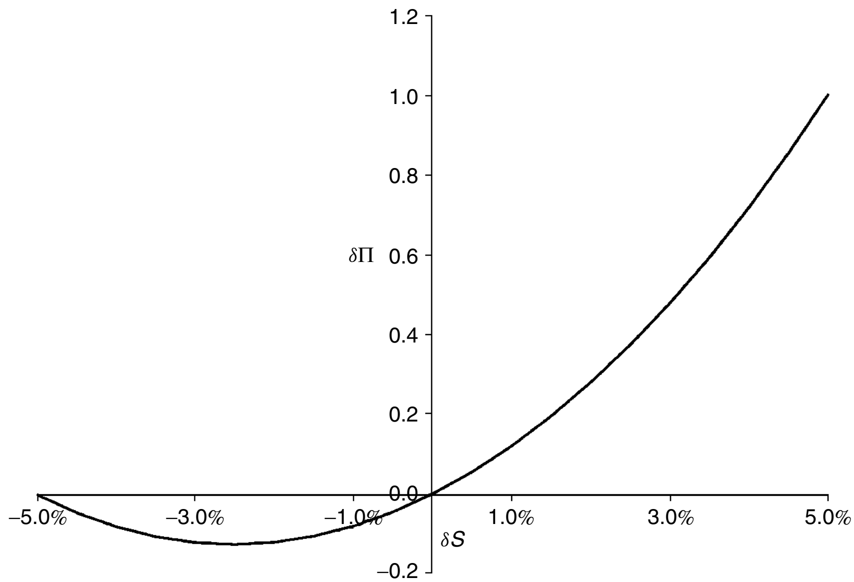


Figure 23.3 Size of portfolio change against change in the underlying, Taylor approximation.

In Figure 23.3 we see a plot of the change in the portfolio against δS assuming for the moment that a Taylor approximation is valid. Note that it is zero at $\delta S = 0$. If the gamma is positive the portfolio change (23.1) has a minimum at

$$\delta S = -\frac{\Delta}{\Gamma}.$$

The portfolio change in this worst-case scenario is

$$\delta\Pi_{\text{worst}} = -\frac{\Delta^2}{2\Gamma}.$$

This is the worst case given an arbitrary move in the underlying. If the gamma is small or negative the worst case will be a fall to zero or a rise to infinity, both far too unrealistic. For this reason we may want to constrain the move in the underlying by

$$-\delta S^- < \delta S < \delta S^+.$$

Now the portfolio fall is restricted.

If we can't use the greek approximation (Taylor series) then we're looking for

$$\min_{-\delta S^- < \delta S < \delta S^+} F(\delta S).$$

Figure 23.2 shows an example where there is one local minimum as well as a global one; it's the global one we want.

23.5.1 Portfolio optimization and the Platinum hedge

Having found a technique for finding out what could happen in the worst case, it is natural to ask how to make that worst case not so bad. This can be done by optimal static hedging.

To start with, I'll assume the Taylor expansion and then generalize.

Suppose that there is a contract available with which to hedge our portfolio. This contract has a bid-offer spread, a delta and a gamma. I will call the delta of the hedging contract Δ^*, meaning the sensitivity of the hedging contract to the underlying asset. The gamma is similarly Γ^*. Denote the bid-offer spread by $C > 0$, meaning that if we buy (sell) the contract and immediately sell (buy) it back we lose this amount.

Imagine that we add a number λ of the hedging contract to our original position. Our portfolio now has a first-order exposure to the crash of

$$\delta S\left(\Delta + \lambda\Delta^*\right)$$

and a second-order exposure of

$$\tfrac{1}{2}\,\delta S^2\left(\Gamma + \lambda\Gamma^*\right).$$

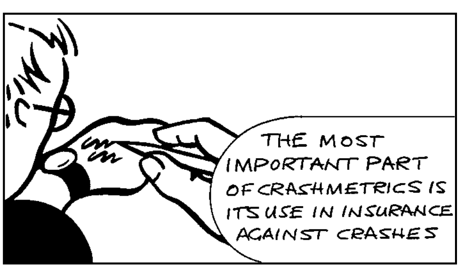
Not only does the portfolio change by these amounts for a crash of size δS but also it loses a *guaranteed* amount

$$|\lambda|\,C$$

just because we cannot close our new position without losing out on the bid-offer spread.

The total change in the portfolio with the static hedge in place is now

$$\delta\Pi = \delta S\left(\Delta + \lambda\Delta^*\right) + \tfrac{1}{2}\,\delta S^2\,\left(\Gamma + \lambda\Gamma^*\right) - |\lambda|\,C.$$



In general, the optimal choice of λ is such that the worst value of this expression for $-\delta S^- \leq \delta S \leq \delta S^+$ is as high as possible. Thus we are exchanging a guaranteed loss (due to bid-offer spread) for a reduced worst-case loss. This is simply insurance and the optimal choice gives the **Platinum hedge**, named for the plastic card that comes after green and gold.[1] For the optimal choice of the λ Figure 23.4 shows the change in the portfolio value as a function of δS. Note that it no longer goes through $(0, 0)$.

In Figure 23.5 is shown a simple spreadsheet for finding the worst-case scenario and the Platinum hedge when there is a single asset.

If we can't use the Taylor approximation, as will generally be the case, we must look for the worst case of

$$F(\delta S) + \lambda F^*(\delta S) - |\lambda| C.$$

Here $F^*(\cdot)$ is the 'formula' for the change in value of the hedging contract.

Having found the worst case, we just make this as painless as possible by optimizing over the hedge ratio λ.

[1] Thank you AmEx for taking the hint last time. . . any chance of an upgrade to Black?

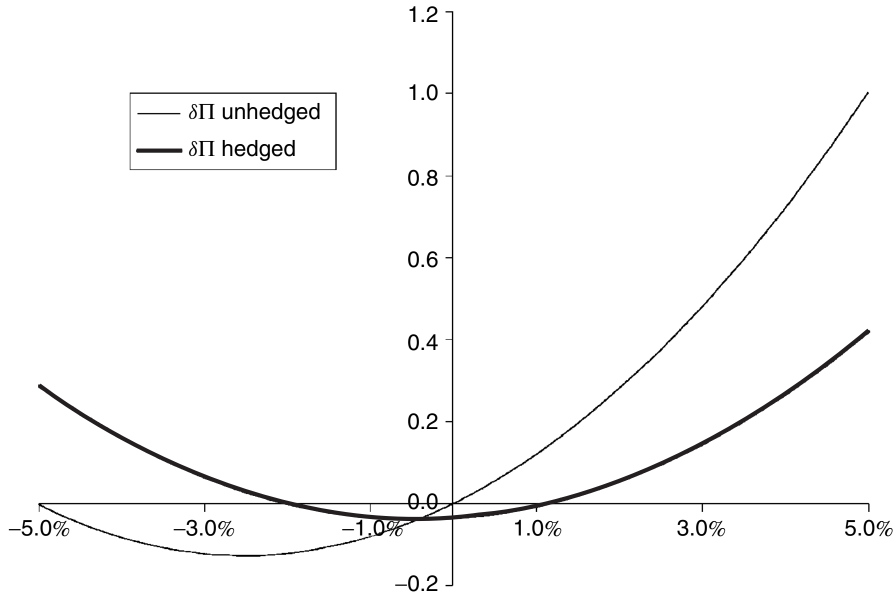


Figure 23.4 Size of portfolio change against δS after optimal hedging, Taylor approximation.

Of course, there won't just be the one option with which to statically hedge, there will be many. How does this change the optimization? We'll find out soon.

23.6 THE MULTI-ASSET/SINGLE-INDEX MODEL

A bank's portfolio has many underlyings, not just the one. How does CrashMetrics handle them? This is done via an index or benchmark.

We can measure the performance of a portfolio of assets and options on these assets by relating the magnitude of extreme movements in any one asset to one or more **benchmarks** such as the S&P500. The relative magnitude of these movements is measured by the **crash coefficient** for each asset relative to the benchmark. If the benchmark moves by $x\%$ then the ith asset moves by $\kappa_i x\%$. Estimates of the κ_i for the constituents of the S&P 500, with that index as the benchmark, may be downloaded free of charge from `www.crashmetrics.com`. Note that the benchmark need not be an index containing the assets, but can be any representative quantity. Unlike the RiskMetrics and CreditMetrics datasets, the CrashMetrics dataset does not have to be updated frequently because of the rarity of extreme market movements.

Tables 23.1 through 23.5 give the crash coefficients for a few constituents of major indices in several countries. The crash coefficients have been estimated using the tails of the daily return distributions from the beginning of 1985 until the end of 1997, and so include the Black Monday crash of October 1987 and the rice/dragon/sake/Asian 'flu' effect starting in October 1997. For example, in Table 23.1 we see the 10 largest positive and negative daily returns in the S&P500 during that period. In this table are also shown the returns on the same days for several constituents of the index.

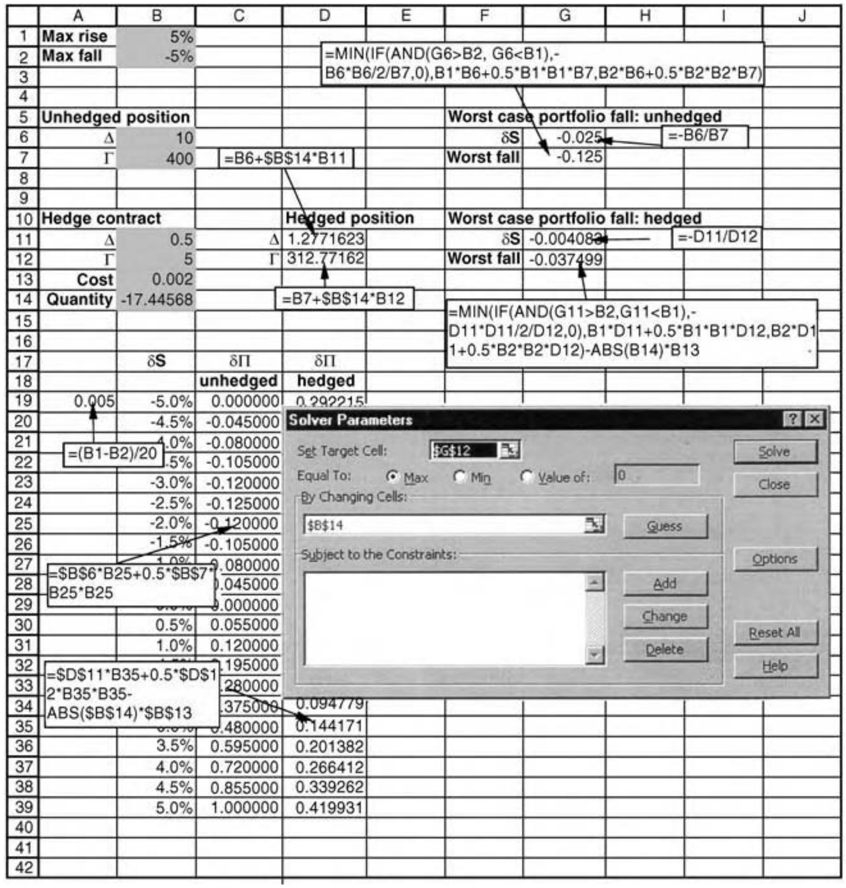
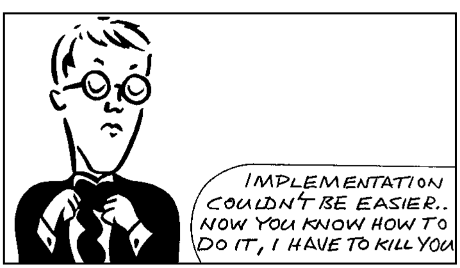


Figure 23.5 Spreadsheet for implementing basic CrashMetrics in one asset.



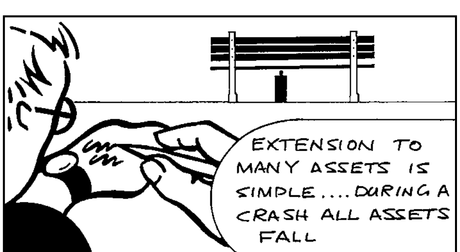


Figure 23.6 uses the same data as used in Chapter 19 for the calculation of the beta for Disney. The fine line in this figure has slope beta. On the figure is shown the line with zero intercept that fits the largest 20 rises and falls in the S&P500, this is the bold line. For small returns, the data points are scattered widely, for large returns there is a greater degree of correlation between the index and the asset returns. I call this the **rings-of-Saturn effect**.

In Figure 23.7 are the returns on the Hong Kong and Shanghai Hotel group versus returns returns on the Hang Seng and in Figure 23.8 are the 40 extreme moves in Daimler–Benz versus returns returns on the Dax. It is important to note at this stage that the crash coefficient is not the same as the asset's beta with respect to the index. Not only is the number different, but preliminary results suggest that the crash coefficient is more stable than the beta. Moreover, for large moves in the index the stock and the

Table 23.1 The 10 largest positive and negative moves in several constituents of the S&P500 against the moves in the S&P500 on the same days.

Date	S&P-500	% change	ABBOTT LABS.	ADOBE SYS.	ADVD.MICR. DEVC.	AEROQUIP-VICKERS	AETNA	AHMANSON (H.F.)
19-Oct-87	225	−20.4	−10.5	−22.2	−36.1	−36.6	−15.3	−20.8
26-Oct-87	228	−8.3	−7.3	−20.0	−14.3	−15.2	−4.5	−4.3
27-Oct-97	877	−6.9	−5.3	−6.1	−19.8	−6.7	−8.5	−6.5
08-Jan-88	243	−6.8	−3.8	−14.3	−6.8	−13.5	−7.1	−6.2
13-Oct-89	334	−6.1	−8.2	−12.5	−5.8	−9.3	−5.5	−3.7
16-Oct-87	283	−5.2	−4.6	0.0	−5.3	−6.4	−1.5	−1.3
11-Sep-86	235	−4.8	−5.2	−50.0	−4.7	−5.3	−3.3	−2.9
14-Apr-88	260	−4.4	−4.0	−12.5	−5.6	−2.6	−4.4	−4.8
30-Nov-87	230	−4.2	−6.7	−16.7	−1.4	−9.8	−2.7	−6.1
22-Oct-87	248	−3.9	−4.6	0.0	−5.9	−6.5	−1.9	−1.4
21-Oct-87	258	9.1	4.3	0.0	4.1	11.5	8.0	9.5
20-Oct-87	236	5.3	−0.6	−14.3	6.5	14.3	−3.7	3.3
28-Oct-97	921	5.1	5.2	4.3	19.0	−0.6	4.3	7.4
29-Oct-87	244	4.9	2.4	33.3	10.3	15.5	−1.9	3.2
17-Jan-91	327	3.7	4.3	0.0	4.6	8.0	2.8	3.0
04-Jan-88	255	3.6	0.8	0.0	7.6	−0.4	1.9	−0.7
31-May-88	262	3.4	2.7	0.0	5.3	0.9	3.9	2.5
27-Aug-90	321	3.2	5.4	8.3	4.5	2.2	1.4	3.1
02-Sep-97	927	3.1	3.8	2.6	3.3	0.1	2.5	2.3
21-Aug-91	391	2.9	3.1	4.2	3.5	0.0	−3.3	2.6

Table 23.2 The 10 largest positive and negative moves in several constituents of the FTSE100 against the moves in the FTSE100 on the same days.

Date	FTSE100	% change	ALLIED DOMECQ	ASDA FOODS	ASSD. BRIT.	BAA	BANK OF SCOTLAND	BARCLAYS
20-Oct-87	1802	−12.2	−7.1	−9.0	−10.1	−7.1	−12.7	−13.2
19-Oct-87	2052	−10.8	−11.5	−9.6	−3.1	−3.7	−3.8	−11.4
26-Oct-87	1684	−6.2	−7.6	−2.4	−3.0	−3.1	−9.0	−7.5
22-Oct-87	1833	−5.7	−4.0	−7.7	−4.7	−2.6	−1.8	−1.0
30-Nov-87	1580	−4.3	−3.0	−4.3	−2.3	−3.8	−2.1	−6.3
05-Oct-92	2446	−4.1	−2.0	−2.9	2.3	−3.4	−6.8	−4.6
03-Nov-87	1653	−4.0	−3.0	−1.7	−2.0	−0.8	−2.0	−7.8
09-Nov-87	1565	−3.4	−3.9	−5.5	−1.0	−4.5	−1.6	−2.2
29-Dec-87	1730	−3.4	−2.3	−1.8	−2.3	−2.1	−0.6	−2.6
16-Oct-89	2163	−3.2	−3.2	−5.0	−3.4	−3.3	0.0	−2.2
21-Oct-87	1943	7.9	7.7	4.9	2.6	2.7	1.5	8.7
10-Apr-92	2572	5.6	8.2	3.3	3.5	7.1	13.1	7.6
17-Sep-92	2483	4.4	3.5	3.6	2.4	2.5	9.6	15.5
11-Nov-87	1639	4.2	2.3	1.8	4.4	2.1	1.0	3.4
30-Oct-87	1749	4.0	1.5	5.0	6.9	3.2	4.6	2.1
12-Nov-87	1702	3.9	1.8	−0.6	−1.0	4.1	0.8	2.2
05-Oct-90	2143	3.6	5.9	0.0	−1.0	3.2	9.3	9.8
18-Sep-92	2567	3.3	6.7	3.4	4.9	3.0	4.2	3.0
26-Sep-97	5226	3.2	1.8	−0.3	1.9	3.2	9.5	8.9
31-Dec-91	2493	3.0	4.0	7.8	1.7	1.3	4.5	2.7

Table 23.3 The 10 largest positive and negative moves in several constituents of the Hang Seng against the moves in the Hang Seng on the same days.

Date	Hang Seng	% change	AMOY PROPS	BANK OF E. ASIA	CHEUNG KONG.	CHINA LT.& POW.	FIRST PACIFIC	GREAT EAGLE
26-Oct-87	2241	−33.3	−37.4	−37.0	−29.2	−32.2	−28.3	−57.3
05-Jun-89	2093	−21.7	−36.4	−22.9	−27.0	−16.6	−28.6	−41.3
28-Oct-97	9059	−13.7	−4.8	−12.7	−9.8	−10.8	−14.7	−19.3
19-Oct-87	3362	−11.1	−18.9	0.0	−9.6	−10.5	−10.7	−20.2
22-May-89	2806	−10.8	−18.6	−8.2	−10.7	−6.2	−14.9	−15.4
23-Oct-97	10426	−10.4	−14.0	−8.9	−13.4	−7.4	−26.9	−6.3
25-May-89	2752	−8.5	−16.6	−7.6	−10.8	−5.0	−9.5	−14.9
19-Aug-91	3722	−8.4	−11.6	−4.7	−6.8	−6.8	−10.2	−9.4
03-Dec-92	4978	−8.0	−2.4	−13.4	−4.8	−9.9	−7.5	−11.3
06-Aug-90	3107	−7.4	−10.3	−6.1	−6.1	−8.2	−6.9	−6.0
29-Oct-97	10765	18.8	9.2	4.9	17.8	20.2	17.3	18.0
23-May-89	3067	9.3	11.4	7.5	9.9	5.2	11.4	14.3
06-Nov-87	2113	7.8	8.5	1.2	12.2	5.4	2.2	9.7
12-Jun-89	2440	7.6	11.7	8.7	10.5	7.0	7.0	16.9
03-Sep-97	14713	7.1	6.0	6.2	5.3	11.9	11.2	5.0
24-Oct-97	11144	6.9	4.1	2.0	9.5	5.9	12.6	7.1
27-Oct-87	2395	6.9	20.1	−2.0	3.8	9.0	−11.4	1.8
03-Nov-97	11255	5.9	5.3	7.8	7.4	−2.9	13.7	6.8
14-Jan-94	10774	5.9	6.1	3.7	6.6	3.6	3.9	4.9
19-Jun-85	1510	5.8	0.0	6.1	6.1	6.2	0.0	13.2

Table 23.4 The 10 largest positive and negative moves in several constituents of the Nikkei against the moves in the Nikkei on the same days.

Date	Nikkei	% change	AJINO-MOTO	ALL NIPPON AIRWAYS	AOKI	ASAHI BREW.	ASAHI CHEM.	ASAHI DENKA KOGYO
20-Oct-87	21910	−14.9	−14.4	−17.9	−18.9	−18.1	−15.7	−10.8
02-Apr-90	28002	−6.6	−3.2	−6.3	−13.7	−5.0	−1.2	−5.8
19-Aug-91	21456	−6.0	−12.1	−7.9	−8.4	−1.6	−5.5	−4.7
23-Aug-90	23737	−5.8	−7.6	−12.0	−9.6	−5.4	−8.4	−7.3
23-Jan-95	17785	−5.6	−8.0	−7.2	−4.9	−1.9	−5.3	−5.2
19-Nov-97	15842	−5.3	−9.0	−3.5	−16.3	−3.4	−3.2	−5.6
24-Jan-94	18353	−4.9	−4.3	−6.0	−6.8	−1.6	−4.1	−7.6
23-Oct-87	23201	−4.9	−4.0	−6.8	−0.9	−4.1	−6.8	−3.4
26-Sep-90	22250	−4.7	−5.2	−2.4	−5.4	−1.6	−2.7	−0.6
03-Apr-95	15381	−4.7	−2.2	−4.1	−2.3	−2.0	−8.4	−6.6
02-Oct-90	22898	13.2	16.0	14.7	9.8	11.4	10.0	15.8
21-Oct-87	23947	9.3	13.5	16.3	11.6	14.7	9.3	5.2
17-Nov-97	16283	8.0	7.3	6.1	0.0	7.4	11.8	9.9
31-Jan-94	20229	7.8	8.5	3.6	13.6	4.2	4.6	9.6
10-Apr-92	17850	7.5	10.5	3.9	13.3	0.0	6.1	3.5
07-Jul-95	16213	6.3	9.8	6.3	9.2	3.0	5.0	1.2
21-Aug-92	16216	6.2	7.9	3.3	21.2	5.6	2.9	2.5
27-Aug-92	17555	6.1	5.6	−1.0	9.4	10.1	5.8	1.8
06-Jan-88	22790	5.6	5.0	6.3	6.4	2.7	6.6	2.4
15-Aug-90	28112	5.4	2.2	6.9	5.4	3.2	3.8	3.8

Table 23.5 The 10 largest positive and negative moves in several constituents of the Dax against the moves in the Dax on the same days.

Date	Dax	% change	ALLIANZ HLDG.	BASF	BAYER	BAYER HYPBK.	BAYERISCHE VBK.	BMW
16-Oct-89	1385	−12.8	−11.3	−10.0	−7.0	−16.1	−13.8	−13.1
19-Aug-91	1497	−9.4	−9.9	−4.8	−5.8	−11.2	−11.2	−10.0
19-Oct-87	1321	−9.4	−10.4	−9.4	−8.5	−7.0	−3.6	−8.2
28-Oct-97	3567	−8.0	−4.7	−8.3	−9.6	−8.6	−7.8	−14.8
26-Oct-87	1193	−7.7	−10.2	−4.0	−5.3	−9.0	−5.5	−6.4
28-Oct-87	1142	−6.8	−7.1	−2.6	−3.2	−8.9	−5.7	−7.5
22-Oct-87	1287	−6.7	−4.2	−4.6	−6.9	−4.6	−5.8	−7.5
10-Nov-87	945	−6.5	−8.9	−4.8	−4.1	−7.4	−6.6	−7.8
04-Jan-88	943	−5.6	−9.9	−6.9	−5.7	−2.1	−5.6	−3.3
06-Aug-90	1740	−5.4	−3.4	−4.3	−6.1	−5.5	−5.2	−6.4
17-Jan-91	1422	7.6	7.8	5.3	6.9	5.8	9.6	9.3
12-Nov-87	1061	7.4	16.6	4.7	7.6	7.6	10.8	6.4
30-Oct-87	1177	6.6	10.2	3.6	8.1	7.8	2.7	7.4
17-Oct-89	1475	6.5	5.8	3.7	2.5	5.9	9.2	5.9
01-Oct-90	1420	6.4	7.8	7.4	9.0	6.0	9.5	7.0
05-Jan-88	1004	6.4	8.6	5.1	4.7	1.7	4.6	4.4
29-Oct-97	3791	6.3	3.5	8.4	8.2	5.9	6.8	12.0
27-Aug-90	1654	6.1	3.9	8.7	5.7	5.0	2.5	6.1
21-Oct-87	1379	5.9	10.9	1.5	8.8	8.7	4.1	4.2
08-Oct-90	1465	5.3	9.1	5.0	5.2	5.1	1.8	3.7

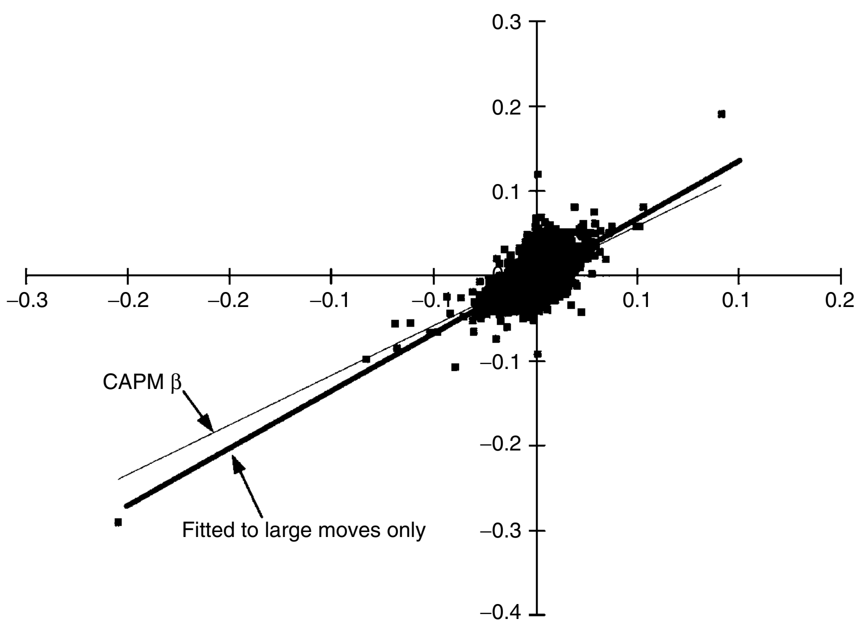


Figure 23.6 The returns on Disney versus returns on the S&P500. Also shown are the line with slope β, fitted to all points, and the line with slope κ fitted to the 40 extreme moves and having zero intercept.

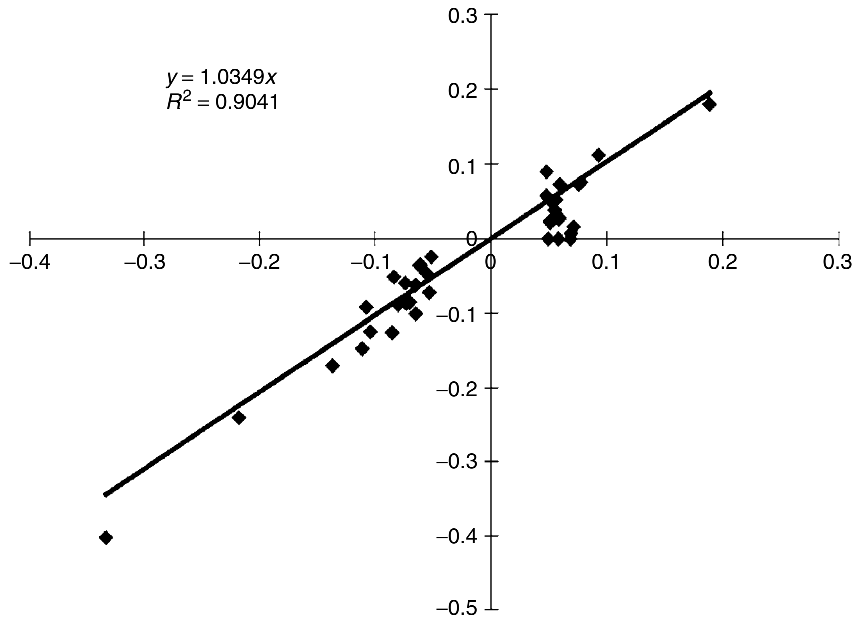


Figure 23.7 The returns on the Hong Kong and Shanghai Hotel group versus returns returns on the Hang Seng. Also shown is the line with slope κ fitted to the 40 extreme moves and having zero intercept.

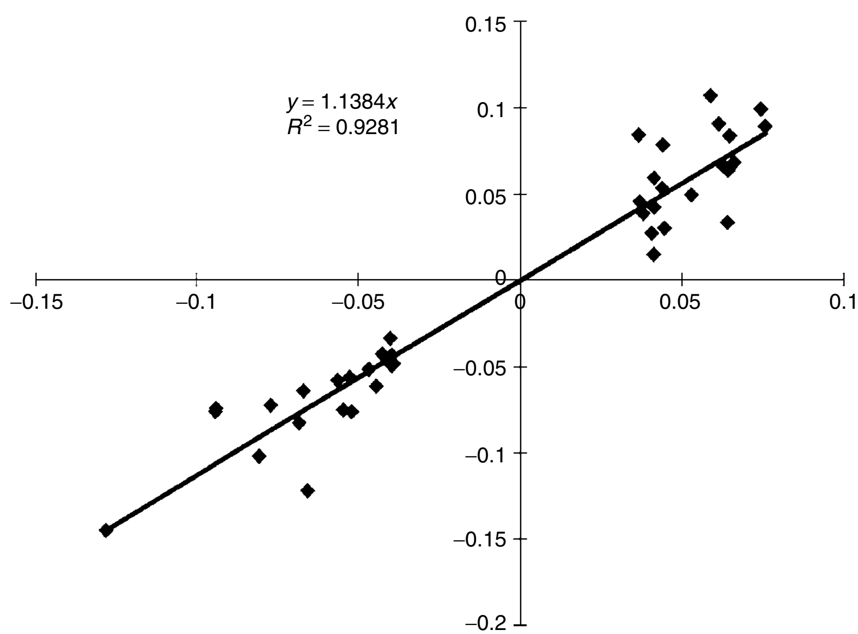


Figure 23.8 The returns on Daimler–Benz versus returns on the Dax. Also shown is the line with slope κ fitted to the 40 extreme moves and having zero intercept.

index are far more closely correlated than under normal market conditions. In other words, when there is a crash all stocks move together.

Let's use these ideas, first assuming a Taylor expansion for the portfolio change.

In the single-index, multi-asset model we can write the change in the value of the portfolio as

$$\delta\Pi = \sum_{i=1}^{N} \Delta_i \, \delta S_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \Gamma_{ij} \, \delta S_i \, \delta S_j \tag{23.2}$$

with the obvious notation. (In particular, observe the cross gammas.) We assume that the percentage change in each asset can be related to the percentage change in the benchmark, x, when there is an extreme move:

$$\delta S_i = \kappa_i x S_i.$$

This simplifies (23.2) to

$$\begin{aligned}\delta\Pi &= x \sum_{i=1}^{N} \Delta_i \kappa_i S_i + \tfrac{1}{2} x^2 \sum_{i=1}^{N} \sum_{j=1}^{N} \Gamma_{ij} \kappa_i S_i \kappa_j S_j \\ &= xD + \tfrac{1}{2} x^2 G.\end{aligned}$$

Observe how this contains a first- and a second-order exposure to the crash. The first-order coefficient D is the **crash delta** and the second-order coefficient G is the **crash gamma**.

Now we constrain the change in the benchmark by

$$-x^- \le x \le x^+.$$

The worst-case portfolio change occurs at one of the endpoints of this range or at the internal point

$$x = -\frac{D}{G}.$$

In this last case the extreme portfolio change is

$$\delta\Pi_{\text{worst}} = -\frac{D^2}{2G}.$$

We can also calculate the crash delta and gamma at this worst point.

All of the ideas contained in the single-asset model described above carry over to the multi-asset model, we just use x instead of δS to determine the worst that can happen to our portfolio.

If we can't use the delta/gamma Taylor series expansion then we must look for the worst case of an expression such as

$$\delta\Pi = F(\delta S_1, \ldots, \delta S_N) = F(\kappa_1 x S_1, \ldots, \kappa_N x S_N).$$

This is not hard, or even time consuming as long as we have formulas for the options in our portfolio.

23.6.1 Portfolio optimization and the Platinum hedge in the multi-asset model

Suppose that there are M contracts available with which to hedge our portfolio. Let us call the deltas of the kth hedging contract Δ_i^k, meaning the sensitivity of the contract to the ith asset, $k = 1, \ldots, M$. The gammas are similarly Γ_{ij}^k. Denote the bid-offer spread by $C_k > 0$, meaning that if we buy (sell) the contract and immediately sell (buy) it back we lose this amount.

Imagine that we add a number λ_k of each of the available hedging contracts to our original position. Our portfolio now has a first-order exposure to the crash of

$$x\left(D + \sum_{k=1}^{M} \lambda_k \sum_{i=1}^{N} \Delta_i^k \kappa_i S_i\right)$$

and a second-order exposure of

$$\tfrac{1}{2}x^2\left(G + \sum_{k=1}^{M} \lambda_k \sum_{i=1}^{N}\sum_{j=1}^{N} \Gamma_{ij}^k \kappa_i S_i \kappa_j S_j\right).$$

Not only does the portfolio change by these amounts for a crash of size x but also it loses a guaranteed amount

$$\sum_{k=1}^{M} |\lambda_k|\, C_k$$

just because we cannot close our new positions without losing out on the bid-offer spread.

The total change in the portfolio with the static hedge in place is now

$$\delta\Pi = x\left(D + \sum_{k=1}^{M} \lambda_k \sum_{i=1}^{N} \Delta_i^k \kappa_i S_i\right) + \tfrac{1}{2}x^2\left(G + \sum_{k=1}^{M} \lambda_k \sum_{i=1}^{N}\sum_{j=1}^{N} \Gamma_{ij}^k \kappa_i S_i \kappa_j S_j\right) - \sum_{k=1}^{M} |\lambda_k|\, C_k.$$

And if we can't use the Taylor series expansion? We must examine

$$\delta\Pi = F(\kappa_1 x S_1, \ldots, \kappa_N x S_N) + \sum_{k=1}^{M} \lambda_k F_k(\kappa_1 x S_1, \ldots, \kappa_N x S_N) - \sum_{k=1}^{M} |\lambda_k|\, C_k.$$

Here F is the original portfolio and the F_ks are the available hedging contracts.

From now on I'll stick to the delta/gamma approximation and leave it to you to do the more robust and realistic whole-formulas approach.

23.6.2 The marginal effect of an asset

We can separate the contribution to the portfolio movement in the worst case into components due to each of the underlyings:

$$\delta\Pi_i = x^* \Delta_i\, \delta S_i + \tfrac{1}{2}x^{*2} \sum_{j=1}^{N} \Gamma_{ij}\, \delta S_i\, \delta S_j,$$

where x^* is the value of x in the worst case. This has, rather arbitrarily, divided up the parts with exposure to two assets (when the cross gamma is nonzero) equally between those assets. The ratio

$$\frac{\delta \Pi_i}{\delta \Pi_{\text{worst}}}$$

measures the contribution to the crash from the ith asset.

23.7 THE MULTI-INDEX MODEL

In the same way that the CAPM model can accommodate multiple indices, so we can have a multiple-index CrashMetrics model. I will skip most of the details, the implementation is simple.

We fit the extreme returns in each asset to the extreme returns in the indices according to

$$\delta S_i = \sum_{j=1}^{n} \kappa_i^j x_j,$$

where the n indices are denoted by the j sub/superscript.

The change in value of our portfolio of stocks and options is now quadratic in all of the x_js. At this point we must decide over what range of index returns do we look for the worst case. Consider just the two-index case, because it is easy to draw the pictures. One possibility is to allow x_1 and x_2 to be independent, to take any values in a given range. This would correspond to looking for the minimum of the quadratic function over the rectangle in Figure 23.9. Note that there is no correlation in this between the two indices, fortunately this difficult-to-measure parameter is irrelevant. Alternatively if you believe that there is some relationship between the size of the crash in one index and the size of the crash in the other you may want to narrow down the area that you explore for the worst case. An example is given in the figure.

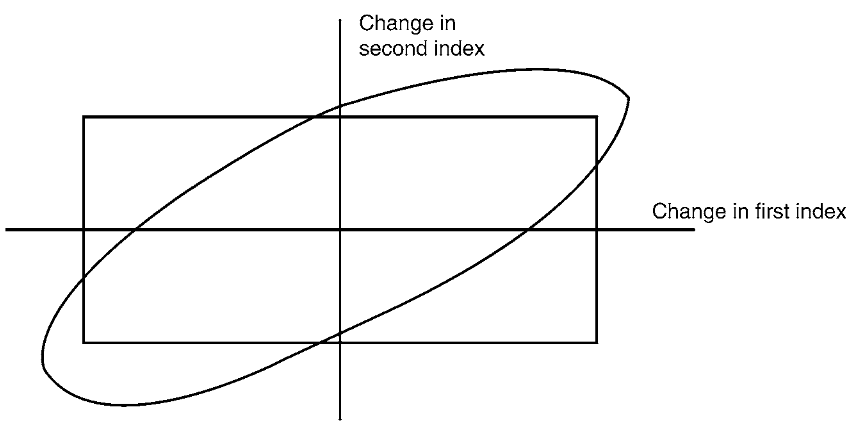


Figure 23.9 Regions of interest in the two-index model.

23.8 INCORPORATING TIME VALUE

Generally we are interested in the behavior over a longer period than overnight. Can we examine the worst case over a finite time horizon? We can expand the portfolio change in a Taylor series in both δS and δt, the time variable, to get

$$\delta\Pi - r\Pi\,\delta t = (\Theta - r\Pi)\,\delta t + \Delta\,\delta S + \tfrac{1}{2}\Gamma\,\delta S^2. \tag{23.3}$$

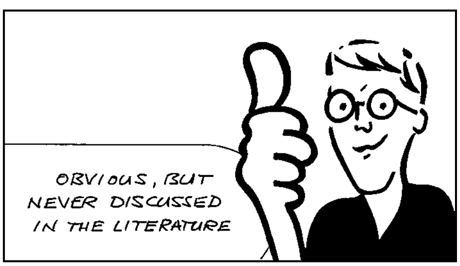
Observe that we examine the portfolio change in excess of the return at the risk-free rate. We must now determine the lowest value taken by this for

$$0 < \delta t < \tau \quad \text{and} \quad -\delta S^- < \delta S < \delta S^+,$$

where τ is the horizon of interest. Since the time and asset changes decouple, the problem for the worst-case asset move is exactly the same as the above, overnight, problem. The worst-case time decay up to the horizon will be

$$\min\left((\Theta - r\Pi)\,\tau, 0\right).$$

The idea of Platinum hedging carries over after a simple modification. The modification we need is to the theta. We must incorporate the Θ^* for each of the hedging contracts, suitably multiplied by the number of contracts.



23.9 MARGIN CALLS AND MARGIN HEDGING

Stock market crashes are more common than one imagines, if one defines a crash as any unhedgeable move in prices. Although we have focused on the change in value of our portfolio during a crash this is not what usually causes trouble. One of the reasons for this is that in the long run stock markets rise significantly faster than the rate of interest, and banks are usually net long the market. What causes banks, and other institutions, to suffer is not the paper value of their assets but the requirement to suddenly come up with a large amount of cash to cover an unexpected margin call. Banks can weather extreme markets provided they do not have to come up with large amounts of cash for margin calls. For this reason it makes sense to be 'margin hedged.' Margin hedging is the reduction of future margin calls by buying/selling contracts so that the net margin requirement is insensitive to movements in underlyings. In the worst-case crash scenario discussed here, this means optimally choosing hedging contracts so that the worst-case margin requirement is optimized. Typically, over-the-counter (OTC) options will not play a role in the optimal margin hedge since they do not usually have margin call requirements.

Recent examples where margin has caused significant damage are Metallgesellschaft and Long-Term Capital Management.[2]

[2] The latter suffered after a 'once in a millennium... 10 sigma event.' Unfortunately it happened in only their fourth year of trading.

I now show how the basic CrashMetrics methodology can be easily modified to estimate and ameliorate worst-case margin calls.

23.9.1 What is margin?

Writing options is very risky. The downside of buying an option is just the initial premium, the upside may be unlimited. The upside of writing an option is limited, but the downside could be huge. For this reason, to cover the risk of default in the event of an unfavorable outcome, the clearing houses that register and settle options insist on the deposit of a margin by the writers of options. Clearing houses act as counterparty to each transaction.

Margin comes in two forms, the initial margin and the maintenance margin. The initial margin is the amount deposited at the initiation of the contract. The total amount held as margin must stay above a prescribed maintenance margin. If it ever falls below this level then more money (or equivalent in bonds, stocks etc.) must be deposited. The levels of these margins vary from market to market.

23.9.2 Modeling margin

The amount of margin that must be deposited depends on the particular contract. Obviously, we are not too concerned with the initial margin since this is known in advance of the purchase/sale of the contract. It is the variation margin that will concern us since a dramatic market move could result in a sudden large margin call that may be difficult to meet.

We will model the margin call as a percentage of the change in value of the contract. We denote that percentage by the Greek letter χ. Note that for an over-the-counter (OTC) contract there is usually no margin requirement so $\chi = 0$.

With the same notation as above, the change in the value of a single contract with a single underlying is

$$\delta\Pi = \Delta\ \delta S + \tfrac{1}{2}\Gamma\ \delta S^2.$$

Therefore the margin call would be

$$\delta M = \chi\left(\Delta\ \delta S + \tfrac{1}{2}\Gamma\ \delta S^2\right).$$

The reader can imagine the details of extending this formula to many underlyings and many contracts, and to include time decay.

The final result is simply

$$\delta M - rM\,\delta t = \left(\sum_{i=1}^{N}\overline{\Theta}_i - rM\right)\delta t + \sum_{i=1}^{N}\overline{\Delta}_i\,\delta S_i + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\overline{\Gamma}_{ij}\,\delta S_i\,\delta S_j.$$

This assumes that interest is received by the margin. Here

$$\overline{\Theta}_i = \textbf{margin theta} \text{ of all options with } S_i \text{ as the underlying}$$
$$= \sum \chi\Theta, \text{ where the sum is taken over all options}$$

$$\begin{aligned}
\overline{\Delta}_i &= \textbf{margin delta} \text{ of all options with } S_i \text{ as the underlying}\\
&= \sum \chi\,\Delta, \text{ where the sum is taken over all options}\\
\overline{\Gamma}_{ij} &= \textbf{margin gamma} \text{ of all options with } S_i \text{ and } S_j \text{ as the underlyings}\\
&= \sum \chi\,\Gamma, \text{ where the sum is taken over all options}\\
r &= \text{risk-free interest rate}\\
\delta t &= \text{time horizon}\\
\delta S_i &= \text{change in value of } i\text{th asset}
\end{aligned}$$

The notation is self-explanatory.

The conclusion is that the CrashMetrics methodology will carry over directly to the analysis of margin provided that the Greeks are suitably redefined. We have therefore introduced the new greeks, $\overline{\Theta}$, $\overline{\Delta}$ and $\overline{\Gamma}$, margin theta, margin delta and margin gamma, respectively.

The reader who is aware of the Metallgesellschaft fiasco will recall that they were delta hedged but not margin hedged.

I've assumed a Taylor series cum delta/gamma approximation that almost certainly won't be realistic during a crash. We are lucky when modeling margin that virtually every contract on which there is margin has a nice formula for its price. The complex products which require numerical solution are typically OTC contracts with no margin requirements at all. I leave it to the reader to go through the details when using formulas rather than greek approximations.

23.9.3 The single-index model

As in the original CrashMetrics methodology we relate the change in asset value to the change in a representative index via

$$\delta S_i = \kappa_i x S_i.$$

Thus we have

$$\begin{aligned}
\delta M - rM\,\delta t &= \left(\sum_{i=1}^{N} \overline{\Theta}_i - rM\right)\delta t + x\sum_{i=1}^{N} \overline{\Delta}_i \kappa_i S_i + \tfrac{1}{2}x^2 \sum_{i=1}^{N}\sum_{j=1}^{N} \overline{\Gamma}_{ij}\kappa_i S_i \kappa_j S_j\\
&= \overline{Th}\,\delta t + x\overline{D} + \tfrac{1}{2}x^2\overline{G}.
\end{aligned}$$

Here $\overline{Th}$ is the portfolio margin theta *in excess of the risk-free growth*. Observe how this expression contains a first- and a second-order exposure to the crash. The first-order coefficient $\overline{D}$ is the crash margin delta and the second-order coefficient $\overline{G}$ is the crash margin gamma.

We've seen worst-case scenarios, Platinum hedging and the multi-index model applied to portfolio analysis and hedging. All of these carry over unchanged to the case of margin analysis, provided that the greeks are suitably redefined. In particular Platinum margin hedging is used to optimally reduce the size of margin calls in the event of a market crash.

23.10 COUNTERPARTY RISK

If OTC contracts do not have associated margin calls, they do have another serious kind of risk: counterparty risk. During extreme markets counterparties may go broke, having a knock-on effect on other banks. For this reason, one should divide up one's portfolio by counterparty initially and examine the worst-case scenario counterparty by counterparty. Everything that we have said above about worst-case scenarios and Platinum hedging carries over to the smaller portfolio associated with each counterparty.

23.11 SIMPLE EXTENSIONS TO CRASHMETRICS

In this section I want to briefly outline ways in which CrashMetrics has been extended to other situations and to capture other market effects. Because of the simplicity of the basic form of CrashMetrics, many additional features can be incorporated quite straightforwardly.

First of all, I haven't described how the CrashMetrics methodology can be applied to interest rate products. This is not difficult, simply use a yield (or several) as the benchmark and relate changes in the values of products to changes in the yield via durations and convexities. The reader can imagine the rest.

A particularly interesting topic is what happens to parameter values after a crash. After a crash there is usually a rise in volatility and an increase in bid-offer spread. The rise in volatility can be incorporated into the methodology by including vega terms, dependent also on the size of the crash. This is conceptually straightforward, but requires analysis of option price data around the times of crashes. If you are long vanilla options during a crash, you will benefit from this rise is volatility. Similarly, crash-dependent bid-offer spread can be incorporated but again requires historical data analysis to model the relationship between the size of a crash and the increase in the spread.

Finally, it is common experience that shortly after a crash stocks bounce back, so that the real fall in price is not as bad as it seems. Typically 20% of the sudden loss is recovered shortly afterwards, but this is by no means a hard and fast rule. You can see this in the earlier data tables, a date on which there is a very large fall is followed by a date on which there is a large rise. To incorporate such a dynamic effect into the relatively static CrashMetrics is an interesting task.

23.12 THE CRASHMETRICS INDEX (CMI)

The results and principles of CrashMetrics have been applied to a **CrashMetrics Index** (**CMI**). This is an index that measures the magnitude of market moves and whether or not we are in a crash scenario. It's like a Richter scale for the financial world. Unlike most measures of market movements this one is *not* a volatility index in disguise, it is far more subtle than that. However, being proprietary, I can't tell you how it's defined. Sorry. I can give you some clues. It's based on a logarithmic scale; it has only one timescale (unlike volatility which needs a long timescale such as thirty days, and a short one, a day, say); it exploits the effect shown in Figure 23.1. Figure 23.10 shows a time series of the CMI applied to the S&P500.

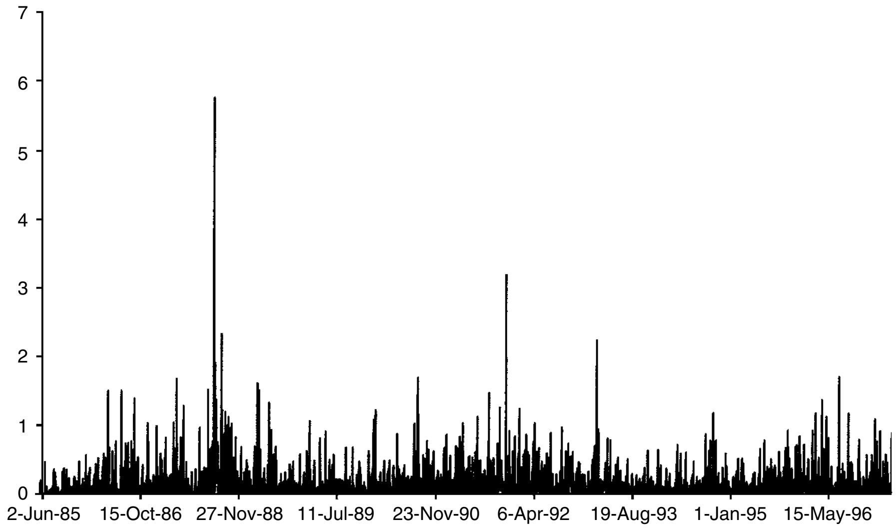


Figure 23.10 The S&P500 CMI.

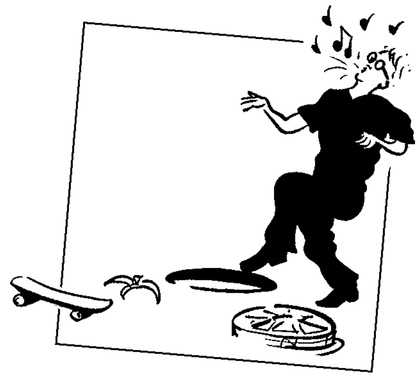
23.13 SUMMARY

This chapter described a VaR methodology that is specifically designed for the analysis of and protection against market crashes. Such analysis is fundamental to the well-being of financial institutions and for that reason I have taken a nonprobabilistic approach to the modeling.

FURTHER READING

- Download the CrashMetrics technical documents, data sets and demonstration software for CrashMetrics from `www.crashmetrics.com`.
- CrashMetrics is currently being turned into commercial software by Xenomorph, see `www.xenomorph.com`.

CHAPTER 24
derivatives **** ups

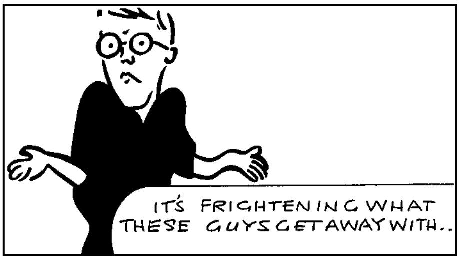


The aim of this Chapter. . .

. . . is to warn the reader of the dangers associated with the naive use of derivatives, and to encourage the application of common sense at all times. The chapter is also a nice break from the intense mathematics of preceding chapters.

In this Chapter. . .

- Orange County
- Procter and Gamble
- Metallgesellschaft
- Gibson Greetings
- Barings
- LTCM



24.1 INTRODUCTION

Derivatives, in the wrong hands, can be dangerous weapons. They can destroy careers and institutions. In this chapter we take a look of some of the more well-publicized cases of derivatives f**k ups. In many cases the details are not in the public domain but where possible I include what *is* known, and offer some analysis.

24.2 ORANGE COUNTY

Orange County is in California. For the first half of the 1990s the County Treasurer was the aptly named Robert Citron. He was in charge of the County investment fund, a pool of money into which went various taxes of the townsfolk. From 1991 until the beginning of 1994 Citron steadily made money, totaling some $750 million, by exploiting low interest rates. This was a very good return on investment, representing approximately 400 basis points above US government rates. How was this possible? He had invested the good people's money in leveraged inverse floating rate notes.

A floating rate note is a bond that pays coupons linked to a floating interest rate such as three- or six-month LIBOR. Normally, the coupon rises when rates rise, not so with inverse floaters. Typically inverse floaters have a coupon of the form

$$\max(\alpha r_f - r_L, 0)$$

where r_f is a fixed rate, r_L some LIBOR rate and $\alpha > 1$ a multiplicative factor. As rates rise, the coupons fall but with a floor at zero; the bond holder never has to return money. Citron bought leveraged inverse floaters[1] having coupons of the form

$$\max(\alpha r_f - \beta r_L, 0)$$

with $\beta > 1$. (If $\beta < 1$ it is a deleveraged note.) While rates are low, coupons are high. This was the situation in the early 1990s, and Citron and Orange County benefitted. But what if rates rise? These notes have a high degree of gearing.

Citron gambled that rates would stay low.

In mid 1994 US rates rose dramatically, by a total of 3%. The leveraging in the notes kicked in big time and Orange County lost some $1.6 billion. Figure 24.1 shows US interest rates during the early 1990s. For about 18 months short-term yields had been below 4%, the horizontal line in the figure. But in early 1994 they started a rise that was dramatic in comparison with the previous period of 'stability.' Citron (and others, see below) were 'caught out' by this rise (to the right of the vertical line in the figure).

Orange County didn't have to declare bankruptcy, they were still in the black and money was still pouring into the fund. But as part of the tactics in their damage suit against the brokers of the deal, they declared bankruptcy on 6th December 1994.

It seems that no one had much of a clue how the losses were piling up. There was no frequent marking to market necessary. Had there been, presumably the losses would have been anticipated and someone would have taken action. At the time of the bankruptcy

[1] Actually, he bought a lot of other things as well. He was not a lucky boy, our Bob.

Figure 24.1 US rates in the early 1990s.

filing S&P's rated Orange County as AA and Moody's as Aa1, very high ratings and thought very unlikely to default.

'Leveraged inverse floating rate note' is a very long name for a very simple instrument. The risks should be obvious. After all, how difficult is it to understand $\max(\alpha r_f - \beta r_L, 0)$?

If Orange County lost who gained? The counterparties selling the floaters were various US government housing agencies. Amusingly, the money flowed from the Orange County taxpayers to the US taxpayers everywhere. Hee, hee.

Citron was found guilty of violating state investment laws and was sentenced to one year of community service. During the sentencing phase psychologists found that he had the math skills of a seventh grader and that he was in the lowest 5% of the population in terms of ability to think and reason.

24.3 PROCTER AND GAMBLE

Procter and Gamble (P&G) is a major multinational company who manufactures beauty and health care products, food and beverages, and laundry and cleaning products. They have a large exposure to interest rates and to exchange rates. To reduce this exposure they use interest rate and currency swaps.

In late 1993 P&G wanted to enter into a swap from fixed to floating, having the view that rates then low would remain low. A vanilla swap would be fine as long as rates didn't rise, but what if they did? Bankers Trust (BT), the counterparty to the deal, suggested some modifications to the swap that satisfied P&G's concerns.

The deal, struck on 2nd November 1993 was a five-year swap on a notional $200 million. It contained something a little out of the ordinary, but not outrageously so. P&G had sold BT something like a put on long-term bond prices.

The deal went like this. BT pays P&G a fixed rate of interest on the $200 million for five years. In return P&G pays BT a fixed rate for the first six months, thereafter a rate defined by

$$r_C - 0.0075 + 0.01 \times \max\left(\frac{98.5}{5.78} Y_5 - P_{30}, 0\right) \tag{24.1}$$

where r_C was the rate on P&G's own corporate bonds, Y_5 the five-year Treasury yield and P_{30} the price of the 30-year Treasury bond. The Treasury yield and price would be known at the time of the first payment, 2nd May 1994, *at which time it would be fixed in the formula*. In other words, the yield and price pertaining on that date would be locked in for the remaining four and a half years.

The best that P&G could achieve would be for rates to stay near the level of November 1993 for just a few more months in which case they would benefit by

$$0.0075 \times \$200 \text{ million} \times 5 = \$7.5 \text{ million}.$$

Not a vast amount in the scheme of things. Five- and 30-year rates had been falling fairly steadily for the whole of the 1990s so far, see Figure 24.1, perhaps they would continue to do so, matching the stability of the short-term yields.

However, if rates were to rise between November and May...

Expression (24.1) increases as the five-year yield increases and decreases if the 30-year bond rises in value. But, of course, if the 30-year yield rises the bond price falls and (24.1) increases. Although there is some small exposure to the slope of the yield curve, the dominant effect is due to the level of the yield curve.

In November 1993 the 6.25% coupon bond maturing in August 2023 had a price of about 103.02, corresponding to a yield of approximately 5.97%. The five-year rate was around 4.95%. With those values expression (24.1) was safely the required $r_C - 0.0075$. However, rates rose at the beginning of 1994 and the potential $7.5 million was not realized, instead P&G lost close to $200 million.

Subsequently, P&G sued BT on the grounds that they failed to disclose pertinent information. The case was settled out of court.

The following was taken from P&G Corporate News Releases

P&G Settles Derivatives Lawsuit With Bankers Trust

May 9, 1996

> CINCINNATI, May 9, 1996 — The Procter & Gamble Company today reached an agreement to settle its lawsuit against Bankers Trust. The suit involves two derivative contracts on which Bankers Trust claimed P&G owed approximately $200 million. Under the terms of the agreement, P&G will absorb $35 million of the amount in dispute, and Bankers Trust will absorb the rest, or about 83% of the total.
>
> 'We are pleased with the settlement and are glad to have this issue resolved,' said John E. Pepper, P&G chairman and chief executive.

It's not difficult to work out the potential losses *a priori* from a shift in the yield curve, and I've done just that in Table 24.1 assuming a parallel shift. P&G start to lose out after about a 70 bps rise in the yield curve. Thereafter they lose about $2.3 million per basis point.

Table 24.1 Effect of parallel shift in yield curve on P&G's losses.

Parallel shift (bps)	0	50	100	150	200
Five-year yield (%)	4.95	5.45	5.95	6.45	6.95
Price of 30-year bond	103.02	97.77	93.04	88.74	84.82
Thirty-year yield (%)	5.97	6.47	6.97	7.47	7.97
Total loss over 4.5 years ($m)	0	0	75	190	302

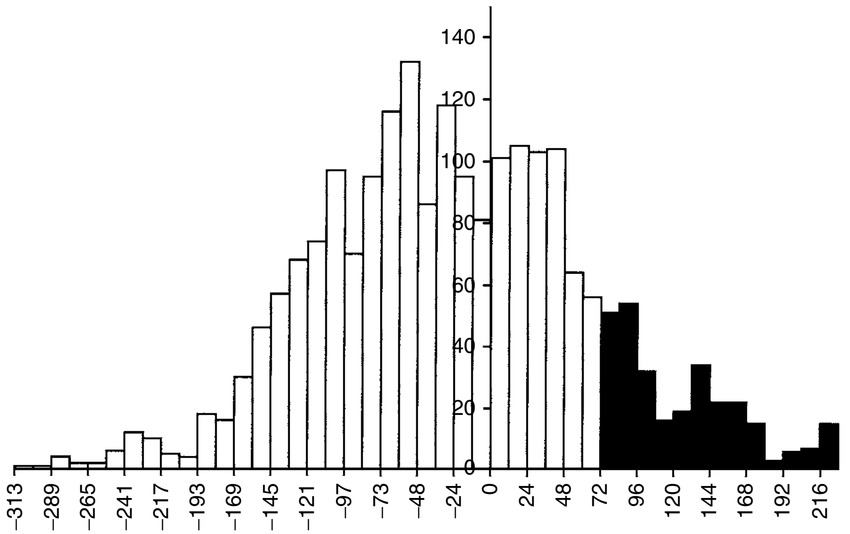


Figure 24.2 Distribution of changes in US five-year rates over a six-month period covering the 10 years prior to November 1993.

On 2nd May 1994 the 5-year and 30-year rates were 6.687% and 7.328% respectively, an average rate rise of over 150 bps.

In Figure 24.2 is shown the distribution of changes in US five-year rates over a six-month period during the ten years prior to November 1993, data readily available at the time that the contract was signed.[2] This historical data suggests that there is a 14% chance of rates rising more than the 70 bps at which P&G start to lose out (the black bars in the figure). There is a 3% chance of a 150 bps or worse rise. Using this data to calculate the expected profit over the five-year period one finds that it is −$8.7 million, rather than the hoped for +$7.5 million.

When you get to the end of this chapter I want you to do a little exercise. Perform the above parallel shift calculation on a spreadsheet. Time yourself to find out how long it would have taken you to save $200 million. (If you use Excel's built-in spreadsheet functions to calculate yields and prices then it should take less than 10 minutes of typing, and that includes switching on your PC and a comfort break.)

The following, taken from the P&G website (`www.pg.com`), seems a decent enough principle (I don't know when it was written):

[2] I've cheated a bit in using overlapping data. Using non-overlapping data gives the same, or slightly worse for P&G, results.

> Integrity: We always try to do the right thing. We are honest and straightforward with each other. We operate within the letter and spirit of the law. We uphold the values and principles of P&G in every action and decision. We are data-based and intellectually honest in advocating proposals, including *recognizing risks*. (The italics are mine.)

24.4 METALLGESELLSCHAFT

Metallgesellschaft (pron. Met Al gazelle shaft, emphasis on Al) is a large German conglomerate with a US subsidiary called MG Refining and Marketing (MGRM). In 1992 MGRM issued forward contracts to its clients, locking in the price of heating oil and gasoline for 10 years. The forward price was fixed at about $3 above the then spot prices. Each month the client received a delivery of oil, paying fixed price. The contract also allowed for both parties to close the position. For example, the client could cancel the contract at any time that the shortest-dated oil futures price exceeded the fixed price. On exercise they would receive 50% of the difference in price between the short future and the fixed price, multiplied by the total volume of oil remaining on the contract. MGRM could also close some of the contracts if the short futures price exceeded some prescribed exit price.

These contracts proved popular with clients because they were the only long-dated contracts available with which to lock in a fixed price. No such contracts existed on an exchange. The total volume of oil in the contracts amounted to some 180 million barrels, the equivalent of 85 days of the output of Kuwait.

Since these contracts were OTC forwards no money changed hands until each delivery. At which time the net cash flow to MGRM was fixed minus spot. The lower the spot price the more that the contracts were of value to MGRM.

MGRM naturally wanted to hedge the oil price risk. But the only exchange traded contracts available were one- to three-month futures. MGRM implemented a strategy of hedging the long-term short OTC forward position with long positions in short-term futures traded on the New York Mercantile Exchange (NYMEX). Because the short-term contracts expired every few months the position had to be rolled over; as one position expired so another was entered into for the next shortest maturity. Theoretically this strategy would have been successful provided that MGRM had a good model for interest rates and the cost of carry. We will return to this point in a moment. Even if MGRM did have a decent model they hit problems because of the important distinction between futures and OTC forwards.

Oil prices fell during the latter half of 1993. Because they held long positions in the futures, a fall in price had to be met on a daily basis by an increase in margin. Futures are marked to market. The extra margin requirements amounted to $900 million during 1993, a large sum of money. MGRM turned to its parent company to help with the funding. Metallgesellschaft responded by taking control of MGRM, installing a new management. In December 1993 the new management closed out half of the short-term contracts. Oil prices then started to rise in early 1994 so that MGRM started to lose out on the long-term contracts. Their response was to cancel the OTC forward contracts and close all positions. By this time losses had amounted to $1.3 billion.

But were they really losses? MGRM were losing out on the short-term futures positions as oil prices fell but remember that these contracts were for hedging purposes. As they lost money on the hedging position they also made money on the OTC forward contracts.

The problem was that because these were forward contracts the profit on them was not realized until the positions expired. Marking to model should have resulted in a net flat position regardless of what the oil price did. Think back to margin hedging, discussed in Chapter 23.

Some say that the MGRM management panicked, they say otherwise.

24.4.1 Basis risk

There was a slight complication in this story due to the behavior of futures prices. The relationship between forward prices and spot prices is not as simple in the commodity markets as it is in the FX markets, for example. Arbitrage considerations lead to the theoretical result

$$F = Se^{(r+q)(T-t)}$$

where T is the maturity, r is the relevant risk-free yield and q is the cost of carry. This relationship leads to forward prices that are higher than spot prices, the graph of forward prices as a function of maturity would be upward sloping. In this case the market is said to be in **contango**.

In practice the strategy required to take advantage of the arbitrage opportunity is so impractical, involving the buying, transporting, storing, transporting, ... of the commodity that the theoretical arbitrage is irrelevant. It is therefore possible, and even common, for the forward curve to be downward sloping. The market is said to be in **backwardation**.

In the Metallgesellschaft story the oil markets were in a state of backwardation initially. By rolling over the futures contract it was possible to make a profit, benefitting from the slope of the forward curve at the short end. During 1993 the spot oil price fell sharply and the market moved into contango. Now the rolling over led to losses, amounting to \$20 million per month. The question remains, did MGRM price into the OTC forward contracts this possible behavior of effectively the cost of carry? If they did (which seems unlikely, or even impossible *a priori*) then their hedging strategy would have been successful had it not been cut off in its prime.

It is difficult to build an accurate interest rate and/or cost of carry model. MGRM was therefore exposed to the risk of hedging one instrument with an imperfectly correlated one. Such a risk is generally termed **basis risk**.

24.5 GIBSON GREETINGS

> I think we should use this as an opportunity. We should just call [the Gibson contract], and maybe chip away at the differential a little more. I mean we told him \$8.1 million when the real number was 14. So now if the real number is 16, we'll tell him that it is 11. You know, just slowly chip away at the differential between what it really is and what we're telling him.
>
> ... when there's a big move, you know, if the market backs up like this, and he is down another 1.3, we can tell him he is down another 2 ... If the market really rallies like crazy, and he's made back a couple of million dollars, you can say you have only made back a half a million.
>
> (February 23, 1994, BT Securities tape of a BT Securities manager discussing the BT internal valuation of the Gibson positions and the valuation given to Gibson.)

Gibson Greetings is a US manufacturer of greetings cards. In May 1991 they issued \$50 million worth of bonds with a coupon of 9.33% and with maturities of between four and ten years. In the early 1990s interest rates fell and Gibson were left paying out a now relatively high rate of interest. To reduce the cost of their debt they entered into vanilla interest rate swaps with Bankers Trust in November 1991.

Swap 1: Notional \$30 million, two-year maturity, BT pay Gibson six-month LIBOR and Gibson pay BT 5.91%.

Swap 2: Notional \$30 million, five-year maturity, BT pay 7.12% and Gibson pay six-month LIBOR.

For the first two years the LIBOR cashflows cancel and Gibson receive 7.12−5.91 = 1.21%.

In July 1992 both swaps were cancelled. Shortly afterwards Gibson entered into a more leveraged swap contract, and so began a sequence of buying and canceling increasingly complex products. Some of these products are described below.

Ratio swap: For five years Gibson were to pay

$$\frac{50}{3} r_L^2$$

to BT every six months and receive 5.5% on a notional of \$30 million. Here r_L is six-month LIBOR. Since the first fixing of LIBOR was at approximately 3.08% the first payment by Gibson was just 1.581%. The second payment was 1.893% corresponding to LIBOR of 3.4%.

Interest rates were starting to rise. And judging by the implied forward curve at the time the market was 'expecting' rates to go higher. Should six-month LIBOR reach 5.7% ($= \sqrt{0.06 \times 0.055}$) the net cashflow would be towards BT from Gibson. Fearing that rates would go beyond this level, the swap contract was amended three times and finally cancelled in April 1993, six months after its initiation.

LIBOR did not rise as rapidly as the forward curve had suggested (it usually doesn't). With hindsight it probably would have been in Gibson's best interests to have retained the swap in its original form.

Periodic floor: For five years BT were to pay 28 bps above six-month LIBOR on a notional of \$30 million. Gibson were to pay

$$r_L$$

where r_L' is six-month LIBOR measured at the previous swap date as long as $r_L > r_L' - 0.15\%$. This is a path-dependent contract. At the end of 1992 BT informed Gibson that the value of the periodic floor was negative and it was cancelled nine months after its initiation. The loss was incorporated into a later contract.

Treasury-linked swap: This swap was entered into in exchange for BT decreasing the maturity of the above ratio swap. Gibson were to pay BT LIBOR and BT were to pay Gibson LIBOR plus 2%. The catch was in the amount of the principal repayment at maturity. Gibson were to pay \$30 million while BT were to pay

$$\min\left(\$30.6 \text{ million}, \$30 \text{ million} \times \left(1 + \frac{103 \times Y_2}{4.88} - \frac{P_{30}}{100}\right)\right),$$

Table 24.2 Ranges for LIBOR in the corridor swap.

6 Aug 1993 — 6 Feb 1994	3.1875–4.3125%
6 Feb 1994 — 6 Aug 1994	3.2500–4.5000%
6 Aug 1994 — 6 Feb 1995	3.3750–5.1250%
6 Feb 1995 — 6 Aug 1995	3.5000–5.2500%

where Y_2 is the two-year Treasury yield and P_{30} is the 30-year Treasury price. Does this formula look familiar?

Knock-out call option: To reduce its exposure under the terms of the Treasury-linked swap due to an anticipated small principal repayment by BT, Gibson entered into a knock-out call option in June 1993. In return for an up-front premium Gibson were to receive at expiry

$$12.5 \times \$25 \text{ million} \times \max(0, 6.876\% - Y_{30})$$

where Y_{30} is the yield on 30-year Treasuries. The downside for Gibson was that the contract expired if the 30-year Treasury yield fell below 6.48%. Long-term rates would have had to stay remarkably stable since the payoff would only be received if $6.48\% < Y_{30} < 6.876\%$ at expiry with Y_{30} *never* having fallen below 6.48%.

The time swap/corridor swap: In August 1993 Gibson entered into a corridor swap with BT. The maturity of the contract was three years and the notional $30 million. BT were to pay Gibson six-month LIBOR plus 1%. Gibson's payments were as follows:

$$r_L + 0.05 \times N\%$$

where r_L is six-month LIBOR and N is the number of days during each six-month calculation period that r_L was outside the range specified in Table 24.2.

After much modification of the multiplier (0.05 initially), the ranges and the termination date, the contract was canceled in January 1994.

In all there were approximately 29 contracts. Gibson's losses amounted to over $20 million, equivalent to almost a year's profits. They sued BT and in an out-of-court settlement agreed in November 1994 they were released from paying BT $14 million outstanding from some swap contracts, only paying BT $6.2 million. Gibson's argument was that they had been misled by BT as to the true value of their contracts. As the tape recording transcribed at the start of this section shows, BT had their own internal models for the value of the contracts but consistently understated the losses to Gibson.

24.6 BARINGS

The Barings story is actually very simple, and the role of derivatives is relatively small. The reasons for all the fuss are the magnitude of the losses ($1.3 billion), that it involved a very staid, 200-year old bank, and that its main protagonist, 28-year-old Nick Leeson, did a runner.

Nick Leeson was a trader for Barings Futures, working in Singapore. In 1994 he was reported to have made $20 million for Barings and was expecting to be rewarded with

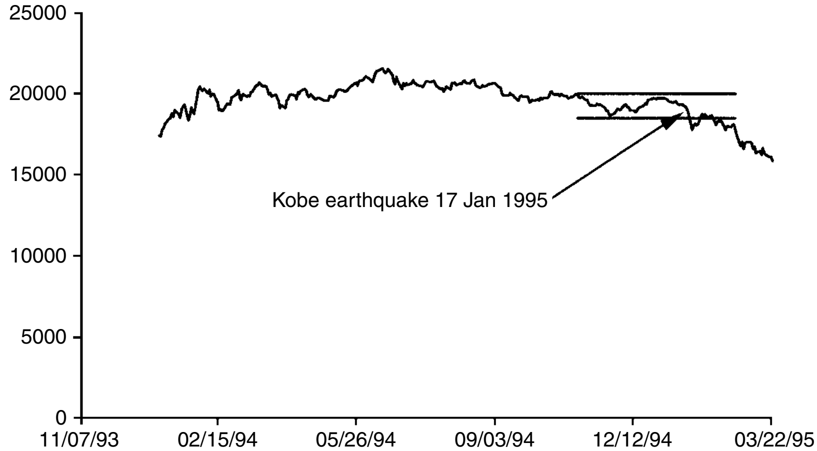


Figure 24.3 The Nikkei 225 index from the beginning of 1994 until March 1995.

a bonus of $1 million on top of his $150,000 basic. It is likely that he had in fact lost a considerable amount already, with the real position only appearing in the infamous 'Error Account 88888.' Leeson had been given too much freedom. At the time, late 1994 and early 1995, Leeson had control over both a trading desk and back office operations; in effect he was allowed to police his own activities.

The trades that caused the downfall of Barings involved the Japanese stock market, specifically futures and options on the Nikkei 225 index. In November and December 1994 Leeson had been selling straddles on the Nikkei with strikes in the region of 18500–20000. The Nikkei index was then trading in a similar range. As long as the index was stable and volatility remained low he would profit from this almost delta-neutral strategy. On 17th January 1995 an earthquake hit Kobe and the Nikkei started to fall, see Figure 24.3. The figure also shows Leeson's trading range. Over the next few days Neeson began buying index futures with March expiry. One of the reasons behind this strategy was the belief that the magnitude of his trades would act to reverse the declining market. If he could bring the Nikkei up to the pre-earthquake levels his option positions would be safe.

Although his trades had a significant impact on the index they could not hold back the fall. Over the next month the index was to fall to 17400. As the index continued its fall, Leeson increased his trades to shore up the index. Margin was required for the futures mark-to-market and vast sums of money were transferred from Barings in London. Finally, the margin calls became too much to cover. On 23rd February 1995 Nick Leeson went on the lam, fleeing Singapore. Finally, he took a flight to Frankfurt, where he was arrested. On December 1st 1995 he pleaded guilty to two offences of deceiving Barings' auditors in a way likely to cause harm to their reputation as well as cheating SIMEX. The next day he was sentenced to six and a half years in prison.

Leeson was released from the Tanah Merah wing of Changi jail on 3rd July 1999. While in prison he developed cancer of the colon, his wife divorced him and remarried, and they made a movie of the story starring Ewan McGregor and Anna Friel.

Meanwhile Barings went broke. The Sultan of Brunei was approached to bail them out but declined. The Dutch bank ING 'was finally persuaded to take on the corpse of Barings,' as Richard Thomson put it, for the grand sum of £1.

24.7 LONG-TERM CAPITAL MANAGEMENT

Long-Term Capital Management (LTCM) is a hedge fund. Hedge funds are supposed to hedge, you'd think. Yet there are few regulations governing their activities. The term 'hedge fund' came about because these funds take short as well as long positions, but this is *not* the same as hedging. LTCM was founded by John Meriwether, ex-Salomon bond arbitrage team, with Nobel laureates Myron Scholes and Robert Merton as partners in the firm. (The excellent book by Lewis is an account of dodgy dealings at Salomon's during Meriwether's time there.)

Edward Thorp (of blackjack fame, see Chapter 19) started one of the first hedge funds in 1969. It has been a very successful hedge fund and Thorp was invited to invest in LTCM in 1994 when it was founded. Thorp declined, 'I didn't want to have anything to do with it because I knew these guys were just dice rollers. It was just a mutual admiration society at Long-Term and nobody was focussing clearly enough on the model.'

If Thorp didn't want to invest, he was in the minority. It was a glamorous line-up and many who should have known better got sucked in. For the first couple of years they made good returns, of the order of 40%. But look at Figure 24.4 *everyone* was making 40%. And in 1997 they made 27%.

The hype surrounding LTCM, its legendary team and the friends they had in high places meant that they had three benefits that other firms would kill to have:

- They were able to leverage $4.8 billion into $100 billion. Its notional position in swaps was at one time $1.25 trillion, 5% of the entire market.
- They were excused collateral on many deals.
- When they finally went under the Federal Reserve organized a bailout.

During 1998 LTCM took huge leveraged bets on the relative value of certain instruments. They were expecting a period of financial calm and convergence of first and emerging

Figure 24.4 The S&P 500.

world interest rates and credit risk. Here are some of the strategies LTCM had in place, what they gambled would happen and what actually did happen. Observe how many of these trades come in pairs; LTCM were making relative value trades.

European government bonds LTCM sold German Bunds and bought other European bonds in the expectation of interest rate convergence in the run-up to EMU. They did ultimately converge but not before first diverging further.

Emerging market bonds and US Treasuries LTCM had long positions in Brazilian and Argentinian bonds and short positions in US Treasuries. They expected credit spreads to decrease, instead they widened to as much as 2000 bps.

Russian GKOs and Japanese bonds They expected Russian yields to fall and Japanese to rise, and so bought Russian GKOs and sold Japanese bonds. They were wrong about the direction of Japanese yields, and Russia defaulted.

Long- and short-term German Bunds LTCM bought 30-year German bonds, sold 10-year bonds, expecting the German yield curve to flatten. Instead, demand for the short-term bonds caused the yield curve to steepen.

Long- and short-dated swaption straddles They bought long-dated swaption straddles and sold short-dated swaption straddles. This is an almost delta-neutral strategy and therefore a volatility trade. Short-term volatility rose, resulting in losses.

The main point to note about these trades is that all seem to have the same view of the world market. They were expecting a period of relative stability, with emerging markets in particular benefitting. In fact, the default by Russia on 17th August 1998 sent markets into a panic, investors 'fled to quality' and all the above trades went wrong. LTCM had most definitely made a big bet on one aspect of the market, and were far from being diversified. Along the way the partners lost 90% of their investment and a couple of Nobel prizewinners had very red faces. They say on Wall Street that if you lose \$5 million you'll never work on the Street again, lose \$50 million and you'll walk into a new job the next day. But lose billions? It's good to have friends.

Figure 24.5 shows the price of the $9\frac{1}{4}$% 2001 USD Argentine bond. Prior to the August Russian default the price was quite stable. But then the price plummeted in the overall panic. As things calmed down the price returned.[3] LTCM may have been correct in their views *long term*, but if you can't weather the storm in the *short term* your correct market view is irrelevant. As hedge fund is a misnomer, so, perhaps, is LTCM... a more accurate name might be STCMM.

On 21st August 1998 they managed to lose \$550 million, and there was more to come.

It was deemed too dangerous for LTCM to be allowed to fail completely, the impact on the US economy could have been disastrous. So the New York Federal Reserve organized, in September 1998, a bailout in which 14 banks invested a total of \$3.6 billion in return for a 90% stake in LTCM.

Much of the LTCM story is still unknown, they were so glamorous that many of their investors didn't even know what they were up to. However, it seems that LTCM were using only rudimentary VaR estimates, with little or no emphasis on stress testing. They

[3] My Argentinian wife, now ex, and I made a tidy profit on that upswing.

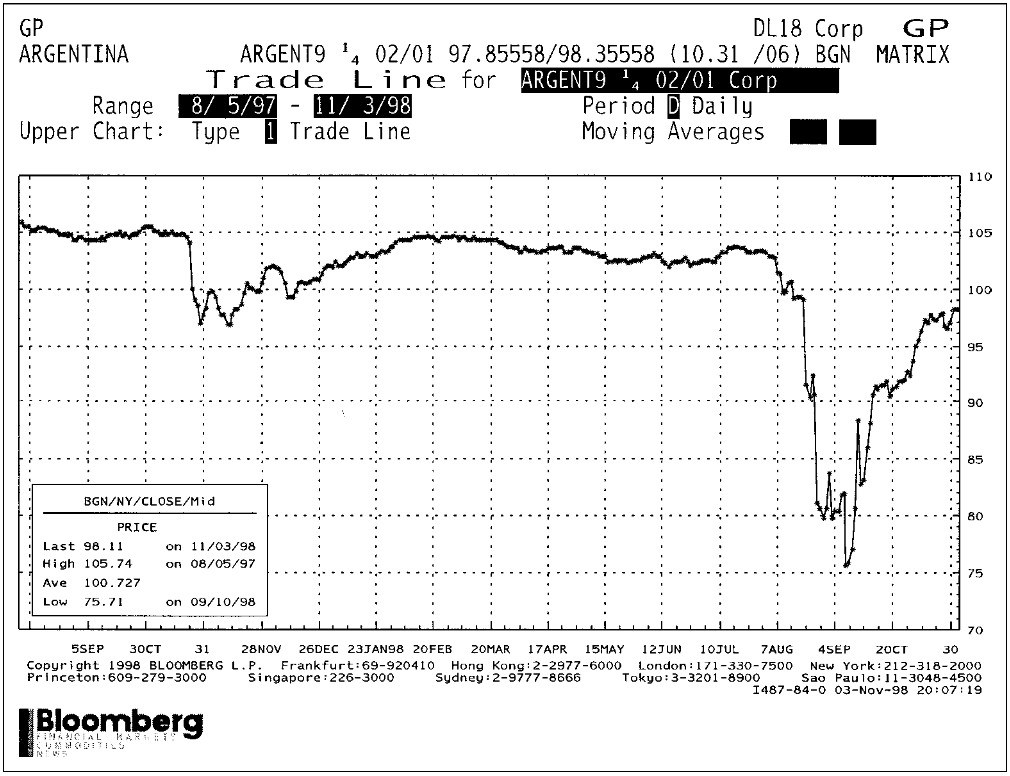


Figure 24.5 Time series of the $9\frac{1}{4}\%$ 2001 USD Argentine bond. Source: Bloomberg L.P.

estimated that daily swings in the portfolio should be of the order of $45 million. But this didn't allow for extreme market moves and problems with liquidity. In trying to reduce risks LTCM even sold off liquid assets, leaving the (theoretically) more profitable illiquid trades on their books. Not a sensible move at times of crisis. Simple VaR as described in Chapter 20 is fine as far as it goes, but cannot deal with global financial meltdowns.

'When we examine banks, we expect them to have systems in place that take account of outsized market moves,' said Alan Greenspan, chairman of the Federal Reserve.

There's another aspect to the LTCM tale that should make anyone using simple quantitative analysis a little bit wary. And that is the matter of liquidity. In August 1998 there was a worldwide drying up of liquidity. Without liquidity it is impossible to offload your positions and the idea of 'value' for any product becomes meaningless. You just have to wait until the market decides to loosen up. Part of the problem was that many of the banks they dealt with knew of LTCM's trades. Many of these were simply copying LTCM's strategies. This wouldn't have mattered if LTCM and all these banks were dealing in small quantities. However, they weren't and in some cases they completely cornered the market. It's one thing to corner the market in one of the necessities of life, such as beer, but to corner the market in something obscure makes you a sitting duck when you want to sell.

24.8 SUMMARY

Most of these stories have similar themes; overconfidence, lack of understanding of the risks, pure speculation at inappropriate times or for inappropriate reasons, overgearing.

We're only human but what is the point of the math modeling when any profits get thrown away by a few individuals who don't know what they are doing? If they *do* know what they are doing, I would assume they are crooks.

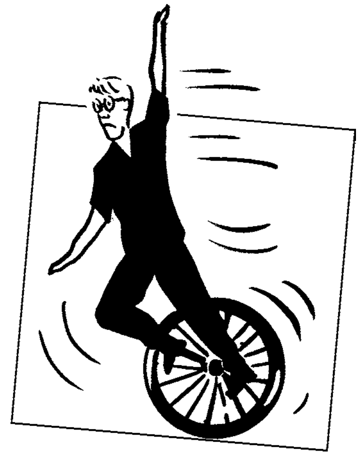
I'll end with a note of caution sounded by Robert Merton in 1993. 'Any virtue can become a vice if taken to extreme, and just so with the application of mathematical models in finance practice. . . At times the mathematics of the models become too interesting and we lose sight of the models' ultimate purpose. The mathematics of the models are precise, but the models are not, being only approximations to the complex, real world. . . The practitioner should therefore apply the models only tentatively, assessing their limitations carefully in each application.' Doh!

FURTHER READING

- Leeson (1997) explains the Barings disaster from his own perspective.
- Chew (1996) is an excellent book covering the risks of derivative transactions. She discusses the technical side of contracts, the legal side and the morality.
- Partnoy (1998) is a cracking good tale of goings-on at Morgan Stanley (coincidentally one of my least favorite banks).
- For background on Meriwether, the LTCM partner, see Lewis (1989).
- Dunbar (1998), Kolman (1999) and Jorion (1999) are nice accounts of LTCM.
- Miller (1997) discusses Metallgesellschaft in detail.
- Thomson (1998) has plenty of inside gen on many of the derivatives stories.
- Read Merton (1995) for his thoughts on the influence of mathematical models on the finance world.

CHAPTER 25

finite-difference methods for one-factor models



The aim of this Chapter...

... is to put in place one of the final pieces of the quantitative finance jigsaw. By the end of this chapter you will know one of the most important methods for solving the equations that we have derived in previous chapters.

In this Chapter...

- finite-difference grids
- how to approximate derivatives of a function
- how to go from the Black–Scholes partial differential equation to a *difference* equation
- the explicit finite-difference method, a generalization of the binomial method

25.1 INTRODUCTION

Rarely can we find closed-form solutions for the values of options. Unless the problem is very simple indeed we are going to have to solve the problem numerically. In an earlier chapter I described the binomial method for pricing options. This used the idea of a finite tree structure branching out from the current asset price and the current time right up to the expiry date. One way of thinking of the binomial method is as a method for solving a partial differential equation. Finite-difference methods are no more than a generalization of this concept, although we tend to talk about **grids** and **meshes** rather than 'trees.' Once we have found an equation to solve numerically then it is much easier to use a finite-difference grid than a binomial tree, simply because the transformation from a differential equation (Black–Scholes) to a difference equation is easier when the grid/mesh/tree is nice and regular. Moreover, there are many, many ways the finite-difference method can be improved upon, making it faster and more accurate. The binomial method is not so flexible. And finally, there is a great deal in the mathematical/numerical analysis literature on these and other methods, it would be such a shame to ignore this. The main difference between the binomial method and the finite-difference methods is that the former contains the diffusion, the volatility, in the tree structure. In the finite-difference methods the 'tree' is fixed but parameters change to reflect a changing diffusion.

To those of you who are new to numerical methods let me start by saying that you will find the parabolic partial differential equation very easy to solve numerically. If you are not new to these ideas, but have been brought up on binomial methods, now is the time to wean yourself off them. On a personal note, for solving problems in practice I would say that I use finite-difference methods about 75% of the time, Monte Carlo simulations 20%, and the rest would be explicit formulas. Those explicit formulas are almost always just the regular Black–Scholes formulas for calls and puts, never for barriers for which it is highly dangerous to use constant volatility. Only once have I ever seriously used a binomial method, and that was more to help with modeling than with the numerical analysis.

In this chapter I'm going to show how to approximate derivatives using a grid and then how to write the Black–Scholes equation as a difference equation using this grid. I'll show how this can be done in many ways, discussing the relative merits of each. In the next chapter I'll also show how to extend the ideas to price contracts with early exercise and to price exotic options.

When I describe the numerical methods I often use the Black–Scholes equation as the example. *But the methods are all applicable to other problems, such as stochastic interest rates*. I am therefore assuming a certain level of intelligence from my reader, that once you have learned the methods as applied to the equity, currency, commodity worlds you can use them in the fixed-income world. I am sure you won't let me down.

25.2 GRIDS

Figure 25.1 is the binomial tree figure from Chapter 5. This is the structure commonly used for pricing simple non-path-dependent contracts. The idea was explained in Chapter 5. In the world of finite differences we use the grid or mesh shown in Figure 25.2. In the former figure the **nodes** are spaced at equal time intervals and at equal intervals in $\log S$. The finite-difference grid usually has equal **timesteps**, the time between nodes, and either

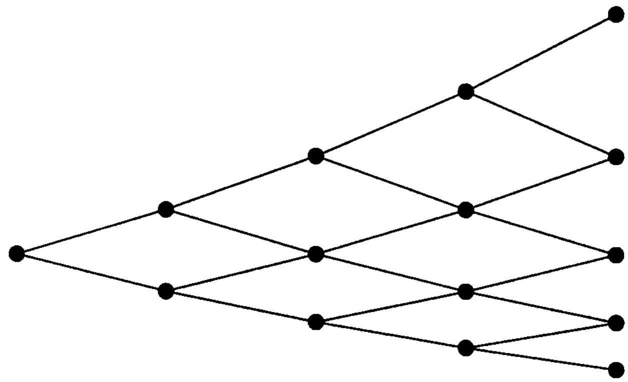


Figure 25.1 The binomial tree.

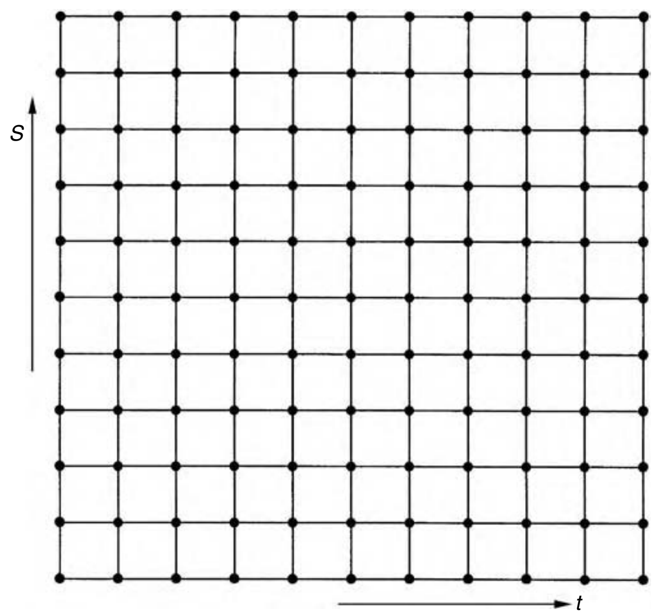


Figure 25.2 The finite-difference grid.

equal S steps or equal $\log S$ steps. If we wanted, we could make the grid any shape we wanted.

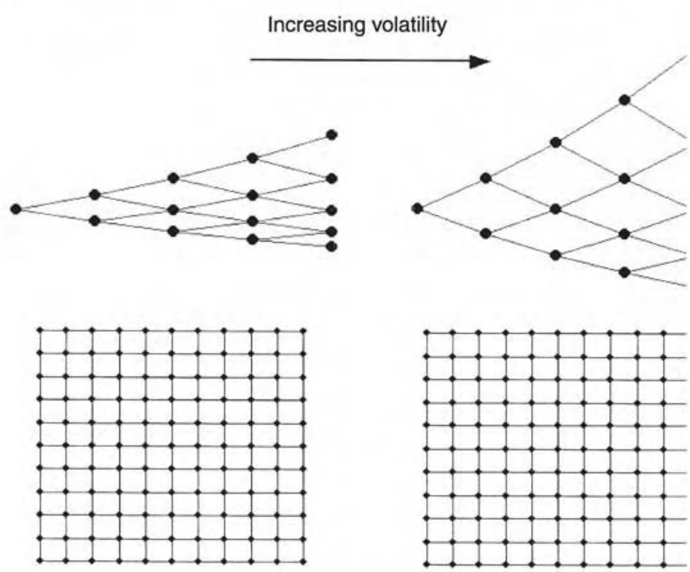
I am only going to describe finite-difference methods with constant time and asset step. There are advantages and disadvantages in this. If we are solving the Black–Scholes equation there is something appealing in having a grid with constant $\log S$ steps, after all, the underlying is following a lognormal random walk. But if you want to use constant $\log S$ steps then it is conceptually simpler to change variables, to write the Black–Scholes equation in terms of the new variable $x = \log S$. Once you have done this then constant $\log S$ step size is equivalent to constant x step size. You could even go so far as to transform to the much neater heat equation as in Chapter 9. One downside to such a transformation is that equal spacing in $\log S$ means that a lot of grid points are spread around small values of S where there is usually not very much happening. The main reason that I rarely do any transforming of the equation when I am solving it numerically is that I like to solve in terms of the real financial variables since terms of the contract are specified using these real variables: transforming to the heat equation could cause problems for contracts such as barrier options. For other problems such a transformation to something nicer is not even possible. Examples would be an underlying with asset- and time-dependent volatility, or an interest rate product.

I'm also going to concentrate on backward parabolic equations. Every partial differential equation or numerical analysis book explains methods with reference to the forward equation. But of course, the difference between forward and backward is no more than a change of the sign of the time (but make sure you apply initial conditions to forward equations and final conditions to backward).

Time Out...

Varying volatillity

Here's a simple reason why finite differences beat binomial trees. Look at the figures below. They show the effect of increasing vol in a tree and in a grid. As vol increases the tree spreads out more. The finite-difference mesh stays fixed. This is particularly important when you want to price contracts with important features (such as barriers) at specified asset levels.



25.3 DIFFERENTIATION USING THE GRID

Let's introduce some notation. The timestep will be δt and the asset step δS, both of which are constant. Thus the grid is made up of the points at asset values

$$S = i\,\delta S$$

and times

$$t = T - k\,\delta t$$

where $0 \le i \le I$ and $0 \le k \le K$. This means that we will be solving for the asset value going from zero up to the asset value $I\,\delta S$. Remembering that the Black–Scholes equation is to be solved for $0 \le S < \infty$ then $I\,\delta S$ is our approximation to infinity. In practice, this upper bound does not have to be too large. Typically it should be three or four times the value of the exercise price, or more generally, three or four times the value of the asset at which there is some important behavior. In a sense barrier options are easier to solve numerically because you don't need to solve over all values of S; for an up-and-out option there is no need to make the grid extend beyond the barrier.

I will write the option value at each of these grid points as

$$V_i^k = V(i\,\delta S, T - k\,\delta t),$$

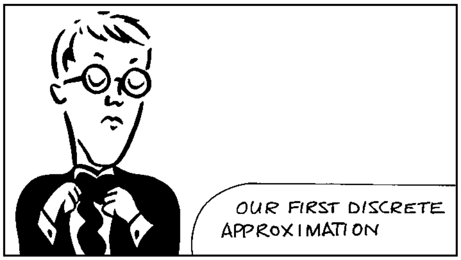
so that the superscript is the time variable and the subscript the asset variable. Notice how I've changed the direction of time, as k increases so real time decreases.

Suppose that we know the option value at each of the grid points, can we use this information to find the derivatives of the option value with respect to S and t? That is, can we find the terms that go into the Black–Scholes equation?

25.4 APPROXIMATING θ

The definition of the first time derivative of V is simply

$$\frac{\partial V}{\partial t} = \lim_{h\to 0} \frac{V(S, t+h) - V(S, t)}{h}.$$



It follows naturally that we can approximate the time derivative from our grid of values using

$$\frac{\partial V}{\partial t}(S, t) \approx \frac{V_i^k - V_i^{k+1}}{\delta t}. \tag{25.1}$$

This is our approximation to the option's theta. It uses the option value at the two points marked in Figure 25.3.

How accurate is this approximation? We can expand the option value at asset value S and time $t - \delta t$ in a Taylor series about the point S, t as

$$V(S, t - \delta t) = V(S, t) - \delta t \frac{\partial V}{\partial t}(S, t) + O(\delta t^2).$$

In terms of values at grid points this is just

$$V_i^k = V_i^{k+1} + \delta t \frac{\partial V}{\partial t}(S, t) + O(\delta t^2).$$

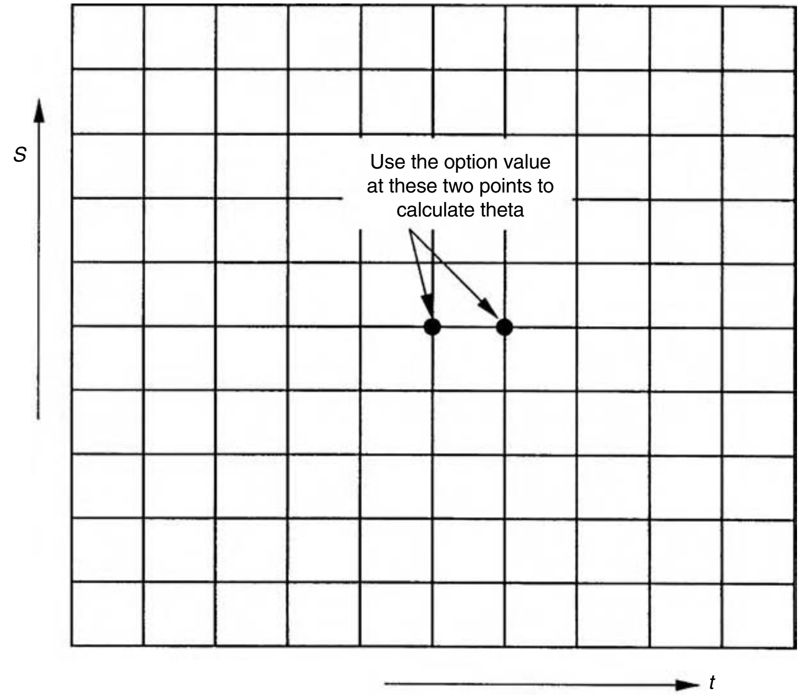


Figure 25.3 An approximation to the theta.

Which, upon rearranging, is

$$\frac{\partial V}{\partial t}(S, t) = \frac{V_i^k - V_i^{k+1}}{\delta t} + O(\delta t).$$

Our question is answered, the error is $O(\delta t)$. It is possible to be more precise than this, the error depends on the magnitude of the second t derivative. But I won't pursue the details here.

There are other ways of approximating the time derivative of the option value, but this one will do for now.

25.5 APPROXIMATING Δ

The same idea can be used for approximating the first S derivative, the delta. But now I *am* going to present some choices.

Let's examine a cross section of our grid at one of the timesteps. In Figure 25.4 is shown this cross section. The figure shows three things: the function we are approximating (the curve), the values of the function at the grid points (the dots) and three possible approximations to the first derivative (the three straight lines). These three approximations are

$$\frac{V_{i+1}^k - V_i^k}{\delta S}, \quad \frac{V_i^k - V_{i-1}^k}{\delta S} \quad \text{and} \quad \frac{V_{i+1}^k - V_{i-1}^k}{2\,\delta S}.$$

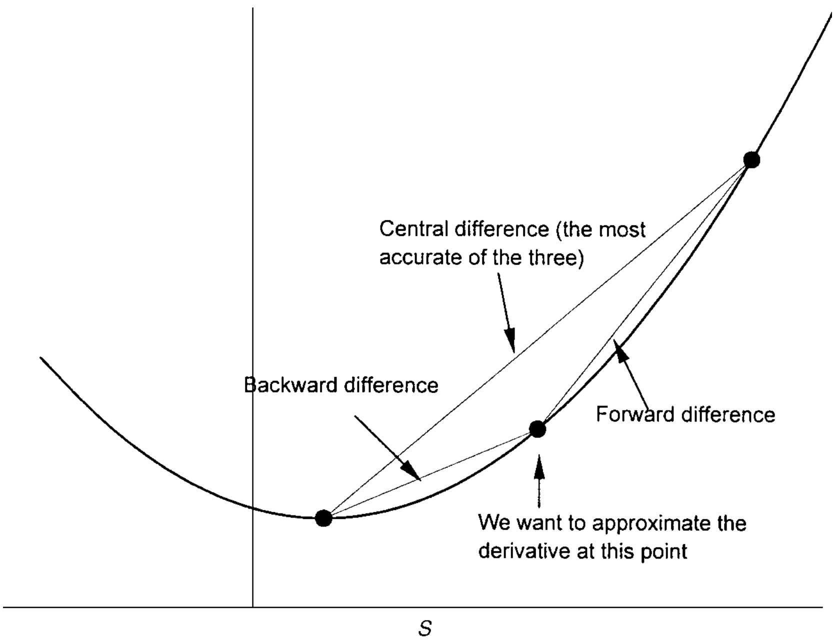


Figure 25.4 Approximations to the delta.

These are called a **forward difference**, a **backward difference** and a **central difference** respectively.

One of these approximations is better than the others, and it is obvious from the diagram which it is. From a Taylor series expansion of the option value about the point $S + \delta S, t$ we have

$$V(S+\delta S,t) = V(S,t) + \delta S \frac{\partial V}{\partial S}(S,t) + \tfrac{1}{2}\delta S^2 \frac{\partial^2 V}{\partial S^2}(S,t) + O(\delta S^3).$$

Similarly,

$$V(S-\delta S,t) = V(S,t) - \delta S \frac{\partial V}{\partial S}(S,t) + \tfrac{1}{2}\delta S^2 \frac{\partial^2 V}{\partial S^2}(S,t) + O(\delta S^3).$$

Subtracting one from the other, dividing by $2\,\delta S$ and rearranging gives

$$\frac{\partial V}{\partial S}(S,t) = \frac{V_{i+1}^k - V_{i-1}^k}{2\,\delta S} + O(\delta S^2).$$

The central difference has an error of $O(\delta S^2)$ whereas the error in the forward and backward differences are both much larger, $O(\delta S)$. The central difference is that much more accurate because of the fortunate cancelation of terms, due to the symmetry about S in the definition of the difference. The central difference calculated at S requires knowledge of the option value at $S + \delta S$ and $S - \delta S$. However, there will be occasions when we do not know one of these values, for example if we are at the extremes of our region, i.e. at $i = 0$ or $i = I$. Then there are times when it may be beneficial to use a one-sided derivative for reasons of stability, an important point which I will come back to.

If we do need to use a one-sided derivative, must we use the simple forward or backward difference or is there something better?

The simple forward and backward differences use only two points to calculate the derivative, if we use three points we can get a better order of accuracy. To find the best approximations using three points we need to use Taylor series again.

Suppose I want to use the points S, $S+\delta S$ and $S+2\,\delta S$ to calculate the option's delta, how can I do this as accurately as possible? First, expand the option value at the points $S+\delta S$ and $S+2\,\delta S$ in a Taylor series:

$$V(S+\delta S,t)=V(S,t)+\delta S\frac{\partial V}{\partial S}(S,t)+\tfrac{1}{2}\delta S^2\frac{\partial^2 V}{\partial S^2}(S,t)+O(\delta S^3)$$

and

$$V(S+2\,\delta S,t)=V(S,t)+2\,\delta S\frac{\partial V}{\partial S}(S,t)+2\,\delta S^2\frac{\partial^2 V}{\partial S^2}(S,t)+O(\delta S^3).$$

If I take the combination

$$-4V(S+\delta S,t)+V(S+2\,\delta S,t)$$

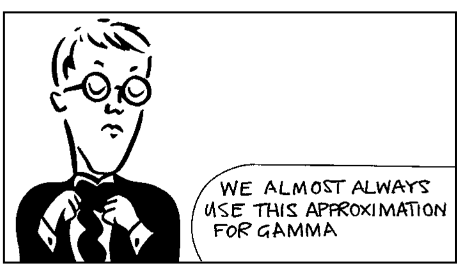
I get

$$-3V(S,t)-2\,\delta S\frac{\partial V}{\partial S}(S,t)+O(\delta S^3),$$

since the second derivative, $O(\delta S^2)$, terms both cancel. Thus

$$\frac{\partial V}{\partial S}(S,t)=\frac{-3V(S,t)+4V(S+\delta S,t)-V(S+2\,\delta S,t)}{2\,\delta S}+O(\delta S^2).$$

This approximation is of the same order of accuracy as the central difference, but of better accuracy than the simple forward difference. It uses no information about V for values below S.



If we want to calculate the delta using a better *backward* difference then we would choose

$$\frac{\partial V}{\partial S}(S,t)=\frac{3V(S,t)-4V(S-\delta S,t)+V(S-2\,\delta S,t)}{2\,\delta S}+O(\delta S^2).$$

25.6 APPROXIMATING Γ

The gamma of an option is the second derivative of the option with respect to the underlying. The natural approximation for this is

$$\frac{\partial^2 V}{\partial S^2}(S,t)\approx\frac{V_{i+1}^k-2V_i^k+V_{i-1}^k}{\delta S^2}.$$

Again, this comes from a Taylor series expansion. The error in this approximation is also $O(\delta S^2)$. I'll leave the demonstration of this as an exercise for the reader.

Time Out...

Some intuition before the math

In terms of the greeks, the Black–Scholes equation can be written

$$\Theta = -\frac{1}{2}\sigma^2 S^2 \Gamma - rS\Delta + rV.$$

We are going to use the equation in this form, with the theta on the left-hand side and the other terms on the right, in the numerical solution. I'll go through all the details shortly, but here's a foretaste.

Imagine we already know the option value as a function of S, at expiry say. In terms of Visual Basic code, call this `VOld(i)`. The distance between asset mesh points is `deltaS`. We can calculate the greeks as follows.

```
Delta = (VOld(i + 1) - VOld(i - 1)) / 2 / deltaS,
```

and

```
Gamma = (VOld(i + 1) - 2 * VOld(i) + VOld(i - 1)) / deltaS / deltaS.
```

So, the Black–Scholes equation gives us

```
Theta = - 0.5 * vol * vol * (i * delta S) * (i * deltaS) * Gamma _
        - intrate * i * deltaS * Delta + intrate * VOld(i)
```

We can now step back, to find the next value for V at the next timestep, call this `VNew(i)`, using

```
VNew(i) = VOld(i) - Theta * deltat.
```

That's all there is to the explicit finite-difference method. Let's go back to the main thrust of the analysis, but rest assured that the ultimate method is very, very easy.

25.7 BILINEAR INTERPOLATION

Suppose that we have an estimate for the option value, or its derivatives, on the mesh points, how can we estimate the value at points in between? The simplest way to do this is to use a two-dimensional interpolation method called **bilinear interpolation**. This method is most easily explained via the following schematic diagram, Figure 25.5.

We want to estimate the value of the option, say, at the interior point in the figure. The values of the option at the four nearest neighbors are called V_1, V_2, V_3 and V_4, simplifying earlier notation just for this brief section. The areas of the rectangles made by the four corners and the interior point, are labeled A_1, A_2, A_3 and A_4. But note that the subscripts for the areas correspond to the subscripts of the option values at the corners *opposite*.

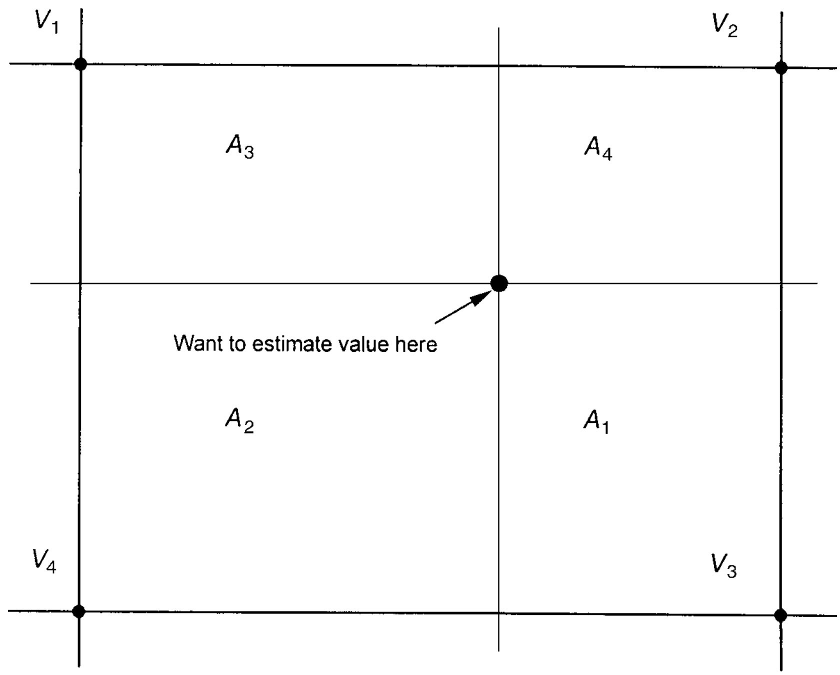


Figure 25.5 Bilinear interpolation.

The approximation for the option value at the interior point is then

$$\sum_{i=1}^{4} A_i V_i \Bigg/ \sum_{i=1}^{4} A_i.$$

25.8 FINAL CONDITIONS AND PAYOFFS

We know that at expiry the option value is just the payoff function. This means that we don't have to solve anything for time T. At expiry we have

$$V(S,T) = \text{Payoff}(S)$$

or, in our finite-difference notation,

$$V_i^0 = \text{Payoff}(i\,\delta S).$$

The right-hand side is a known function. For example, if we are pricing a call option we put

$$V_i^0 = \max(i\,\delta S - E, 0).$$

This final condition will get our finite-difference scheme started. It will be just like working down the tree in the binomial method.

25.9 BOUNDARY CONDITIONS

When we come to solving the Black–Scholes equation numerically in the next section, we will see that we must specify the option value at the extremes of the region. That is, we must prescribe the option value at $S = 0$ and at $S = I\,\delta S$. What we specify will depend on the type of option we are solving. I will give some examples.

Example 1 Suppose we want to price a call option. At $S = 0$ we know that the value is always zero, therefore we have

$$V_0^k = 0.$$

Example 2 For large S the call value asymptotes to $S - Ee^{-r(T-t)}$ (plus exponentially small terms). Thus our upper boundary condition could be

$$V_I^k = I\,\delta S - Ee^{-rk\,\delta t}.$$

This would be slightly different if we had a dividend.

Example 3 For a put option we have the condition at $S = 0$ that $V = Ee^{-r(T-t)}$. This becomes

$$V_0^k = Ee^{-rk\,\delta t}.$$

Example 4 The put option becomes worthless for large S and so

$$V_I^k = 0.$$

Example 5 A useful boundary condition to apply at $S = 0$ for most contracts (including calls and puts) is that the diffusion and drift terms 'switch off.' This means that on $S = 0$ the payoff is guaranteed, resulting in the condition

$$\frac{\partial V}{\partial t}(0, t) - rV(0, t) = 0.$$

Numerically, this becomes

$$V_0^k = (1 - r\,\delta t)V_0^{k-1}.$$

Example 6 When the option has a payoff that is at most linear in the underlying for large values of S then you can use the upper boundary condition

$$\frac{\partial^2 V}{\partial S^2}(S, t) \to 0 \quad \text{as } S \to \infty.$$

Almost all common contracts have this property. The finite-difference representation is

$$V_I^k = 2V_{I-1}^k - V_{I-2}^k.$$

This is particularly useful because it is independent of the contract being valued meaning that your finite-difference program does not have to be too intelligent.[1]

[1] Sometimes I even use this condition for small values of S, not taking the grid down to $S = 0$.

Often there are natural boundaries at finite, nonzero values of the underlying, which means that the domain in which we are solving either does not extend down to zero or up to infinity. Barrier options are the most common form of such contracts.

By way of example, suppose that we want to price an up-and-out call option. This option will be worthless if the underlying ever reaches the value S_u. Clearly,

$$V(S_u, t) = 0.$$

If we are solving this problem numerically how do we incorporate this boundary condition?

The ideal thing to do first of all is to choose an asset step size such that the barrier $S = S_u$ is a grid point, i.e $S_u/\delta S$ should be an integer. This is to ensure that the boundary condition

$$V_I^k = 0$$

is an accurate representation of the correct boundary condition. Note that we are no longer solving over an asset price range that extends to large S. The upper boundary at $S = S_u$ may be close to the current asset level. In a sense this makes barrier problems easier to solve, the solution region is always smaller than the region over which you would solve a nonbarrier problem.

Sometimes it is not possible to make your grid match up with the barrier. This would be the case if the barrier were moving, for example. If this is the case then you are going to have to find an approximation to the boundary condition. There is something that you must not do, and that is to set V equal to zero at the nearest grid point to the barrier. Such an approximation is very inaccurate, of $O(\delta S)$, and will ruin your numerical solution. The trick that we can use to overcome such problems is to introduce a **fictitious point**. This is illustrated in Figure 25.6.

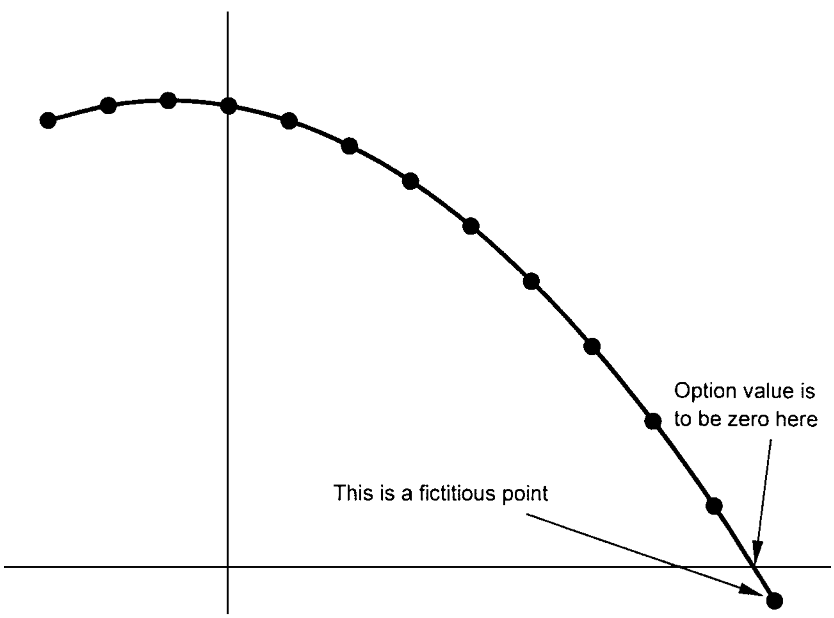


Figure 25.6 A fictitious point, introduced to ensure accuracy in a barrier option boundary condition.

The point $i = I - 1$ is a real point, within the solution region. The point $i = I$ is just beyond the barrier.

Example 7 Suppose that we have the condition that

$$V(S_u, t) = f(t).$$

If we have an 'out' option then f would be either zero or the value of the rebate. If we have an 'in' option then f is the value of the option into which the barrier option converts.

This condition can be approximated by ensuring that the straight line connecting the option values at the two grid points straddling the barrier has the value f at the barrier. Then a good discrete version of this boundary condition is

$$V_I^k = \frac{1}{\alpha}\left(f - (1-\alpha)V_{I-1}^k\right)$$

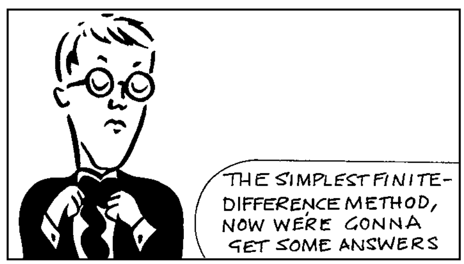
where

$$\alpha = \frac{S_u - (I-1)\,\delta S}{\delta S}.$$

This is accurate to $O(\delta S^2)$, the same order of accuracy as in the approximation of the S derivatives.

I have set up all the foundations for us to begin solving some equations. Remember, there has been nothing difficult in what we have done so far, everything is a simple application of Taylor series.

25.10 THE EXPLICIT FINITE-DIFFERENCE METHOD



The Black–Scholes equation is

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

I'm going to write this as

$$\frac{\partial V}{\partial t} + a(S,t)\frac{\partial^2 V}{\partial S^2} + b(S,t)\frac{\partial V}{\partial S} + c(S,t)V = 0$$

to emphasize the wide applicability of the finite-difference methods. The only constraint we must pose on the coefficients is that if we are solving a backward equation, i.e. imposing final conditions, we must have $a > 0$.

I'm going to take the approximations to the derivatives, explained above, and put them into this equation:

$$\frac{V_i^k - V_i^{k+1}}{\delta t} + a_i^k\left(\frac{V_{i+1}^k - 2V_i^k + V_{i-1}^k}{\delta S^2}\right) + b_i^k\left(\frac{V_{i+1}^k - V_{i-1}^k}{2\,\delta S}\right) + c_i^k V_i^k = O(\delta t, \delta S^2).$$

Points to note:

- The time derivative uses the option values at 'times' k and $k+1$, whereas the other terms all use values at k.

- The gamma term is a central difference, in practice one never uses anything else.
- The delta term uses a central difference. There are often times when a one-sided derivative is better. We'll see examples later.
- The asset- and time-dependent functions a, b and c have been valued at $S_i = i\,\delta S$ and $t = T - k\,\delta t$ with the obvious notation.
- The error in the equation is $O(\delta t, \delta S^2)$.

I am going to rearrange this **difference equation** to put all of the $k+1$ terms on the left-hand side:

$$V_i^{k+1} = A_i^k V_{i-1}^k + (1 + B_i^k) V_i^k + C_i^k V_{i+1}^k \tag{25.2}$$

where

$$A_i^k = \nu_1 a_i^k - \tfrac{1}{2}\nu_2 b_i^k,$$
$$B_i^k = -2\nu_1 a_i^k + \delta t\, c_i^k$$

and

$$C_i^k = \nu_1 a_i^k + \tfrac{1}{2}\nu_2 b_i^k$$

where

$$\nu_1 = \frac{\delta t}{\delta S^2} \quad \text{and} \quad \nu_2 = \frac{\delta t}{\delta S}.$$

The error in this is $O(\delta t^2, \delta t\,\delta S^2)$, I will come back to this in a moment. The error in the approximation of the differential equation is called the **local truncation error**.

Equation (25.2) only holds for $i = 1, \ldots, I-1$, i.e. for interior points, since V_{-1}^k and V_{I+1}^k are not defined. Thus there are $I-1$ equations for the $I+1$ unknowns, the V_i^k. The remaining two equations come from the two boundary conditions on $i = 0$ and $i = I$. The two endpoints are treated separately.

If we know V_i^k for all i then Equation (25.2) tells us V_i^{k+1}. Since we know V_i^0, the payoff function, we can easily calculate V_i^1, which is the option value one timestep before expiry. Using these values we can work step by step back down the grid as far as we want. Because the relationship between the option values at timestep $k+1$ is a simple function of the option values at timestep k this method is called the **explicit finite-difference method**. The relationship between the option values in Equation (25.2) is shown in Figure 25.7.

Equation (25.2) is only used to calculate the option value for $1 \le i \le I-1$ since the equation requires knowledge of the option values at $i-1$ and $i+1$. This is where the boundary conditions come in. Typically we either have a V_i^k being prescribed at $i = 0$ and $i = I$ or, as suggested above, we might prescribe a relationship between the option value at an endpoint and interior values. This idea is illustrated in the following Visual Basic code fragment. This code fragment does not have all the declarations etc. at the top, nor the return of any answers. I will give a full function shortly, for the moment I want you to concentrate on setting up the final condition and the finite-difference time loop.

The array `V(i, k)` holds the option values. Unless we wanted to store all option values for all timesteps this would not be the most efficient way of writing the program, I will describe a better way in a moment.

First set up the final condition, the payoff.

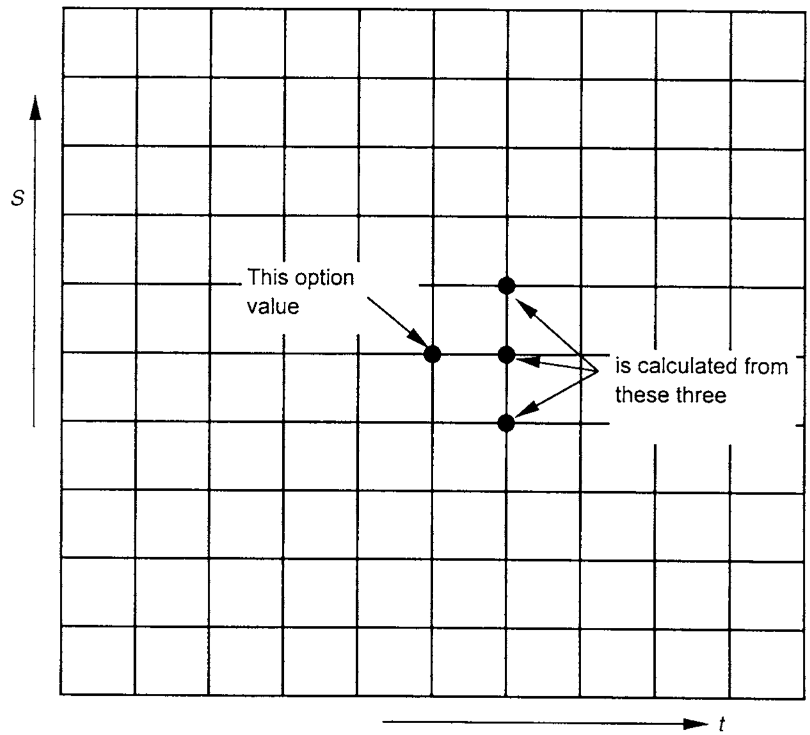
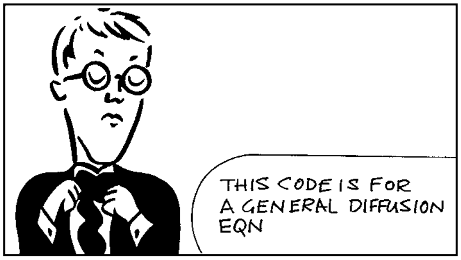


Figure 25.7 The relationship between option values in the explicit method.

```
For i = 0 To NoAssetSteps
    S(i) = i * AssetStep
    V(i, 0) = CallPayoff(S(i)) ' Set up final condition
Next i
```

Now we can work backwards in time using the following time loop.




```
' Time loop
For k = 1 To NoTimesteps
    RealTime = Expiry - k * Timestep
    For i = 1 To NoAssetSteps - 1
    V(i, k + 1) = A(S(i), RealTime) * V(i - 1, k) _
                 + B(S(i), RealTime) * V(i, k) _
                 + C(S(i), RealTime) * V(i + 1, k)
    Next i
' BC at S = 0
    V(0, k + 1) = 0
' BC at "infinity"
    V(NoAssetSteps, k + 1) = 2 * V(NoAssetSteps - 1, k + 1) _
                     - V(NoAssetSteps - 2, k + 1)
Next k
```

The explicit finite-difference algorithm is simple, the functions `A(S(i), RealTime)`, `B(S(i), RealTime)` and `C(S(i), RealTime)` are defined elsewhere and are in terms of the asset price `S(i)` and the time `RealTime`. Since I am valuing a call option here the boundary condition at $S = 0$ is simply `V(0, k + 1) = 0` but the boundary condition I have used at the upper boundary `i = NoAssetSteps` is that the gamma is zero.

25.10.1 The Black–Scholes equation

For the Black–Scholes equation with dividends the coefficients above simplify to

$$A_i^k = \tfrac{1}{2}\left(\sigma^2 i^2 - (r - D)i\right)\delta t,$$
$$B_i^k = -\left(\sigma^2 i^2 + r\right)\delta t$$

and

$$C_i^k = \tfrac{1}{2}\left(\sigma^2 i^2 + (r - D)i\right)\delta t.$$

This uses $S = i\,\delta S$. If the volatility, interest rate and dividend are constant then there is no time or k dependence in these coefficients.

25.10.2 Convergence of the explicit method

I can write the value of the option at any i point at the final timestep K as

$$V_i^K = V_i^0 + \sum_{k=0}^{K-1}(V_i^{k+1} - V_i^k).$$

Each of the terms in this summation is in error by $O(\delta t^2, \delta t\,\delta S^2)$. This means that the error in the final option value is

$$O(K\,\delta t^2, K\,\delta t\,\delta S^2)$$

since there are K terms in the summation. If we value the option at a finite value of T then $K = O(\delta t^{-1})$ so that the error in the final option value is $O(\delta t, \delta S^2)$.

Although the explicit method is simple to implement it does not always converge. Convergence of the method depends on the size on the timestep, the size of the asset step and the size of the coefficients a, b and c.

The technique often used to demonstrate convergence is quite fun, so I'll show you how it is done. The method, as I describe it, is not rigorous but it can be made so.

Ask the question, 'If a small error is introduced into the solution, is it magnified by the numerical method or does it decay away?' If a small error, introduced by rounding errors for example, becomes a large error then the method is useless. The usual way to analyze such stability is to look for a solution of the equation, Equation (25.2), of the form[2]

$$V_i^k = \alpha^k e^{2\pi i\sqrt{-1}/\lambda}. \tag{25.3}$$

In other words, I'm going to look for an oscillatory solution with a wavelength of λ. If I find that $|\alpha| > 1$ then there is instability. Note that I am not worried about how the oscillation gets started, I could interpret this special solution as part of a Fourier series analysis.

[2] This is the only place in the book that I use $\sqrt{-1}$. Normally I'd write this as i but I need i for other quantities.

Substituting (25.3) into (25.2) I get

$$\alpha = \left(1 + c_i^k\,\delta t + 2a_i^k \nu_1 (\cos(2\pi/\lambda) - 1)\right) + \sqrt{-1}\, b_i^k \nu_2 \sin(2\pi/\lambda).$$

It turns out that to have $|\alpha| < 1$, for stability, we require

$$c_i^k \leqslant 0,$$

$$2\nu_1 a_i^k - \delta t\, c_i^k \leqslant 1$$

and

$$\tfrac{1}{2}\nu_2 |b_i^k| \leq \nu_1 a_i^k.$$

To get this result, I have assumed that all the coefficients are slowly varying over the δS length scales.

In financial problems we almost always have a negative c, often it is simply $-r$ where r is the risk-free interest rate. The other two constraints are what really limit the applicability of the explicit method.

Typically we choose ν_1 to be $O(1)$ in which case the second constraint is approximately

$$\nu_1 \leq \frac{1}{2a}$$

(ignoring subscripts and superscripts on a). This is a serious limitation on the size of the timestep,

$$\delta t \leq \frac{\delta S^2}{2a}.$$

If we want to improve accuracy by halving the asset step, say, we must reduce the timestep by a factor of four. The computation time then goes up by a factor of *eight*. The improvement in accuracy we would get from such a reduction in step sizes is a factor of four since the explicit finite-difference method is accurate to $O(\delta t, \delta S^2)$.

In the Black–Scholes equation this timestep constraint becomes

$$\delta t \leq \frac{\delta S^2}{2a} = \frac{\delta S^2}{\sigma^2 S^2} = \frac{1}{\sigma^2}\left(\frac{\delta S}{S}\right)^2.$$

This constraint depends on the asset price. Since δt will be independent of S the constraint is most restrictive for the largest S in the grid. If there are I equally spaced asset grid points then the constraint is simply

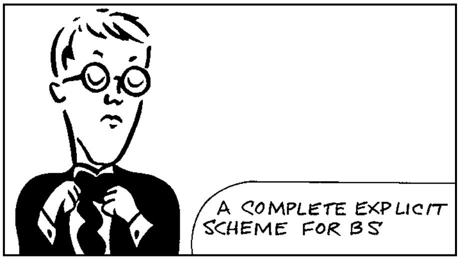
$$\delta t \leq \frac{1}{\sigma^2 I^2}.$$

If the timestep constraint is not satisfied, if it is too large, then the instability is obvious from the results. The instability is so serious, and so oscillatory, that it is easily noticed. It is unlikely that you will get a false but believable result if you use the explicit method.

The final constraint can also be a serious restriction. It can be written as

$$\delta S \leq \frac{2a}{|b|}. \tag{25.4}$$

If we are solving the Black–Scholes equation this restriction does not make much difference in practice unless the volatility is very small. It can be important in other problems though and I will show how to get past this restriction after the next Visual Basic program.




```
Function OptionValue(Asset As Double, Strike As _
                        Double, Expiry As Double, _
                        Volatility As Double, IntRate _
                        As Double, param As Integer, _
                        NoAssetSteps)
Dim VOld(0 To 100) As Double
Dim VNew(0 To 100) As Double
Dim Delta(0 To 100) As Double
Dim Gamma(0 To 100) As Double
Dim Theta(0 To 100) As Double
Dim S(0 To 100) As Double
Dim Ssqd(0 To 100) As Double
halfvolsqd = 0.5 * Volatility * Volatility
AssetStep = 2 * Strike / NoAssetSteps
NearestGridPt = Int(Asset / AssetStep)
dummy = (Asset - NearestGridPt * AssetStep) / AssetStep
Timestep = AssetStep * AssetStep / Volatility / _
         Volatility / (4 * Strike * Strike)
NoTimesteps = Int(Expiry / Timestep) + 1
Timestep = Expiry / NoTimesteps
For i = 0 To NoAssetSteps
    S(i) = i * AssetStep
    Ssqd(i) = S(i) * S(i)
    VOld(i) = Application.Max(S(i) - Strike, 0)
Next i
For j = 1 To NoTimesteps
    For i = 1 To NoAssetSteps - 1
        Delta(i) = (VOld(i + 1) - VOld(i - 1)) / _
           (2 * AssetStep)
        Gamma(i) = (VOld(i + 1) - 2 * VOld(i) _
           + VOld(i - 1)) / (AssetStep * AssetStep)
        VNew(i) = VOld(i) + Timestep * (halfvolsqd * _
            Ssqd(i) * Gamma(i) + IntRate * S(i) * _
            Delta(i) - IntRate * VOld(i))
    Next i
        VNew(0) = 0
        VNew(NoAssetSteps) = 2 * _
           VNew(NoAssetSteps - 1) _
           - VNew(NoAssetSteps - 2)
For i = 0 To NoAssetSteps
    Theta(i) = (VOld(i) - VNew(i)) / Timestep
    VOld(i) = VNew(i)
Next i
Next j
For i = 1 To NoAssetSteps - 1
    Delta(i) = (VOld(i + 1) - VOld(i - 1)) / _
        (2 * AssetStep)
    Gamma(i) = (VOld(i + 1) - 2 * VOld(i) + _
        VOld(i - 1)) / (AssetStep * AssetStep)
Next i
If param = 0 Then OptionValue = (1 - dummy) *  _
           VOld(NearestGridPt) + dummy * _
           VOld(NearestGridPt + 1)
If param = 1 Then OptionValue = (1 - dummy) * _
           Delta(NearestGridPt) + dummy * _
           Delta(NearestGridPt + 1)
If param = 2 Then OptionValue = (1 - dummy) * _
```

```
                Gamma(NearestGridPt) + dummy * _
                Gamma(NearestGridPt + 1)
If param = 3 Then OptionValue = (1 - dummy) * _
                Theta(NearestGridPt) + dummy * _
                Theta(NearestGridPt + 1)
End Function
```

This program will output the value of a call option, its delta, gamma or theta depending on whether the parameter `param` is 0, 1, 2 or 3. The program is fairly self-explanatory, the only points I want to comment on are the use of `VOld(i)` and `VNew(i)`, the size of the timestep and the interpolation to give the final answer.

If we wanted to keep values of the option at all values of the asset for all times up to expiry we would need a two-dimensional array. If we just want the option value today we only need two one-dimensional arrays. One of these is to store the values used on the right-hand side of (25.2) (`VOld(i)`) and one for the left-hand side (`VNew(i)`). After finding the new values for the option value at the new timestep we update the old values, setting them equal to the new. This is the line `VOld(i) = VNew(i)`. If we choose to do this then we save a great deal of storage space.

The timestep has been chosen so that it just about satisfies the constraint on ν_1.

Finally, the asset grid has been set up so that the strike price is a grid point. This means in general the current level of the asset `Asset` will not lie on a grid point. To find the option value at the asset value I have used a simple linear interpolation between the two option values either side of the current asset level. The nearest grid point on the left-hand side is `NearestGridPt` and the ratio of the distance of the current asset value from that point to the size of the asset step is `dummy`.

The error between the results of this explicit finite-difference program and the exact Black–Scholes formula as a function of the underlying is shown in Figure 25.8 for 50 asset

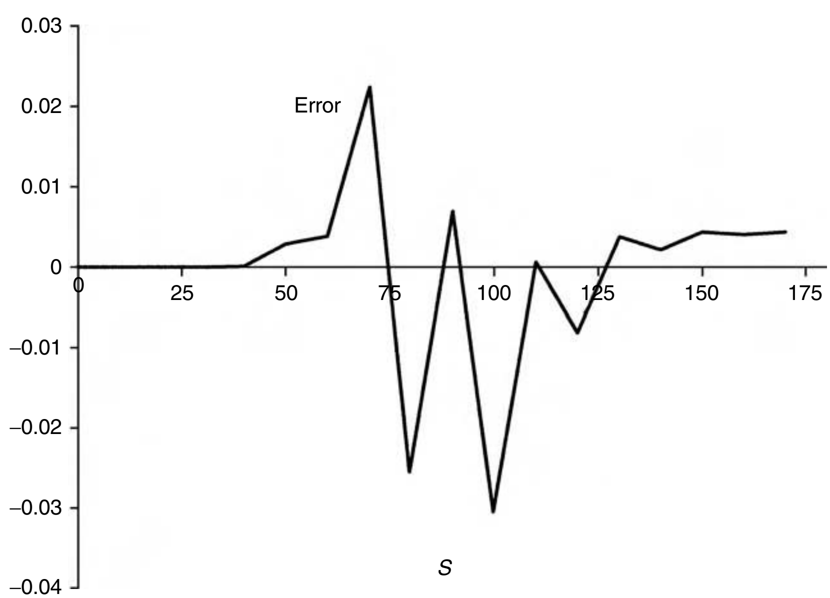


Figure 25.8 Error as a function of the underlying using the finite-difference scheme.

points, a volatility of 20%, and interest rate of 10%, an expiry of one year and a strike of 100.

The logarithm of the absolute error as a function of the logarithm of the square of the asset step size is shown in Figure 25.9. The $O(\delta S^2)$ behavior is very obvious. In these calculations I have kept ν_1 constant.

In Figure 25.10 is shown the error as a function of calculation time, time units will depend on your machine.

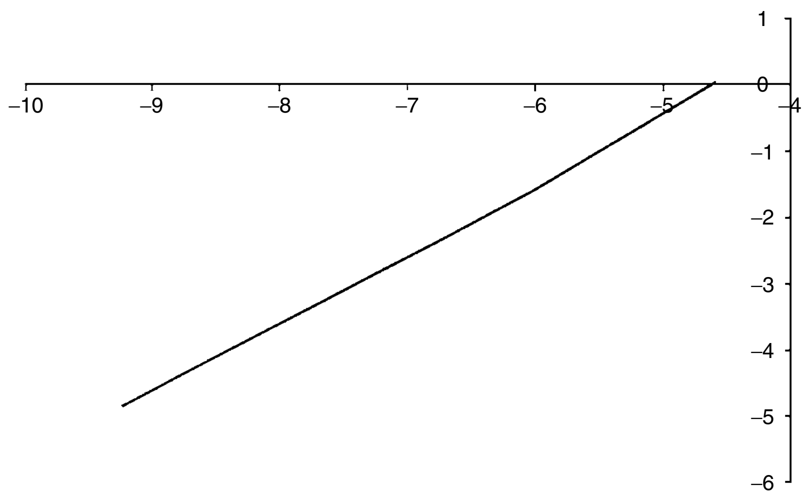


Figure 25.9 Log(Error) as a function of log(δS^2) using the finite-difference scheme.

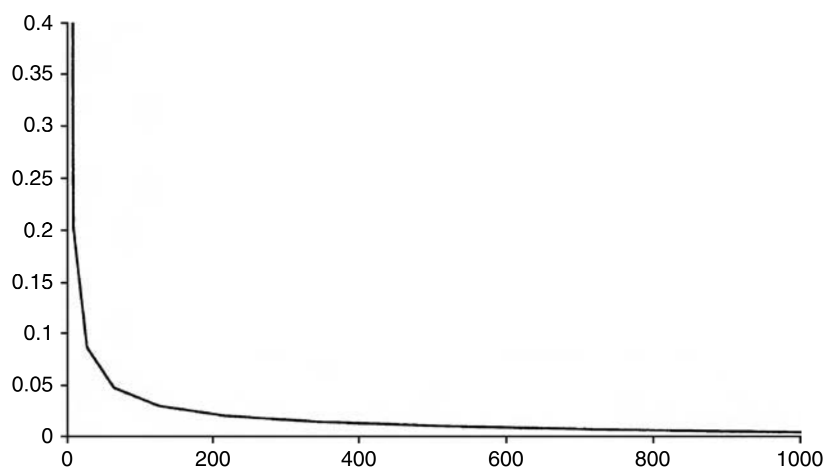


Figure 25.10 Absolute error as a function of the calculation time using the finite-difference scheme.

Time Out...

Breakdown of code

Perhaps I was rushing by just quoting some explicit finite-difference VB code. Here is that code again, but with comments scattered throughout.

Declaring and defining stuff...

```
Function OptionValue(Asset As Double, Strike As Double, Expiry As _
                       Double, Volatility As Double, IntRate As _
                       Double, param As Integer, NoAssetSteps)
Dim VOld(0 To 100) As Double
Dim VNew(0 To 100) As Double
Dim Delta(0 To 100) As Double
Dim Gamma(0 To 100) As Double
Dim Theta(0 To 100) As Double
Dim S(0 To 100) As Double
Dim Ssqd(0 To 100) As Double
halfvolsqd = 0.5 * Volatility * Volatility 'This will reduce _
             computing time a little bit
AssetStep = 2 * Strike / NoAssetSteps
NearestGridPt = Int(Asset / AssetStep) 'Used in the interpolation
dummy = (Asset - NearestGridPt * AssetStep) / AssetStep
Timestep = AssetStep * AssetStep / Volatility / Volatility / _
           (4 * Strike * Strike) 'We want a stable scheme
NoTimesteps = Int(Expiry / Timestep) + 1
Timestep = Expiry / NoTimesteps 'Expiry should be an integer number _
           of timesteps from now
```

Setting up the asset array and payoff...

```
For i = 0 To NoAssetSteps
    S(i) = i * AssetStep
    Ssqd(i) = S(i) * S(i)
    VOld(i) = Application.Max(S(i) - Strike, 0) 'Payoff for a call
Next i
```

The time loop, the finite-difference 'engine,' including boundary conditions and updating...

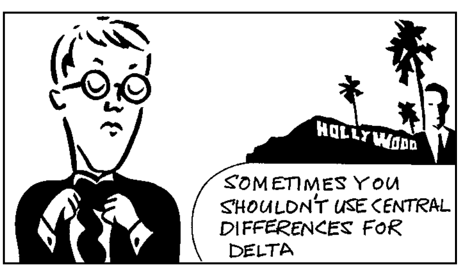
```
For j = 1 To NoTimesteps
    For i = 1 To NoAssetSteps - 1
        Delta(i) = (VOld(i + 1) - VOld(i - 1)) / (2 * AssetStep)
        Gamma(i) = (VOld(i + 1) - 2 * VOld(i) + VOld(i - 1)) / _
                   (AssetStep * AssetStep)
        VNew(i) = VOld(i) + Timestep * (halfvolsqd * Ssqd(i) _
                  * Gamma(i) + IntRate * S(i) * Delta(i) - IntRate _
                  * VOld(i)) 'The Black-Scholes equation
```

```
    Next i
        VNew(0) = 0 'This boundary condition would be different for _
                  a put option
        VNew(NoAssetSteps) = 2 * VNew(NoAssetSteps - 1) _
                  - VNew(NoAssetSteps - 2) 'This works for most payoffs
For i = 0 To NoAssetSteps
    Theta(i) = (VOld(i) - VNew(i)) / Timestep
    VOld(i) = VNew(i) 'Updating
Next i
Next j
```

Output. . .

```
For i = 1 To NoAssetSteps - 1
    Delta(i) = (VOld(i + 1) - VOld(i - 1)) / (2 * AssetStep)
    Gamma(i) = (VOld(i + 1) - 2 * VOld(i) + VOld(i - 1)) / _
               (AssetStep * AssetStep)
Next i
If param = 0 Then OptionValue = (1 - dummy) * VOld(NearestGridPt) _
                                + dummy * VOld(NearestGridPt + 1)
If param = 1 Then OptionValue = (1 - dummy) * Delta(NearestGridPt) _
                                + dummy * Delta(NearestGridPt + 1)
If param = 2 Then OptionValue = (1 - dummy) * Gamma(NearestGridPt) _
                                + dummy * Gamma(NearestGridPt + 1)
If param = 3 Then OptionValue = (1 - dummy) * Theta(NearestGridPt) _
                                + dummy * Theta(NearestGridPt + 1)
End Function
```

25.11 UPWIND DIFFERENCING



The constraint (25.4) can be avoided if we use a one-sided difference instead of a central difference for the first derivative of the option value with respect to the asset. As I said in Chapter 9 the first S derivative represents a drift term. This drift has a direction associated with it, as t decreases, moving away from expiry, so the drift is towards smaller S. In a sense, this makes the forward price of the asset a better variable to use. Anyway, the numerical scheme can make use of this by using a one-sided difference. That is the situation for the Black–Scholes equation . More generally, the approximation that we use for delta in the equation could depend on the sign of b. For example, use the following:

$$\text{if } b(S,t) \geq 0 \quad \text{then} \quad \frac{\partial V}{\partial S}(S,t) = \frac{V_{i+1}^k - V_i^k}{\delta S}$$

but if

$$b(S,t) < 0 \quad \text{then} \quad \frac{\partial V}{\partial S}(S,t) = \frac{V_i^k - V_{i-1}^k}{\delta S}.$$

This removes the limitation (25.4) on the asset step size, improving stability. However, since these one-sided differences are only accurate to $O(\delta S)$ the numerical method is less accurate.

The use of one-sided differences that depend on the sign of the coefficient b is called **upwind differencing**.[3] There is a small refinement of the technique in the choice of the value chosen for the function b:

$$\text{if } b(S,t) \geq 0 \quad \text{then} \quad b(S,t)\frac{\partial V}{\partial S}(S,t) = b_{i+(1/2)}^k \frac{V_{i+1}^k - V_i^k}{\delta S}$$

but if

$$b(S,t) < 0 \quad \text{then} \quad b(S,t)\frac{\partial V}{\partial S}(S,t) = b_{i-(1/2)}^k \frac{V_i^k - V_{i-1}^k}{\delta S}.$$

Notice how I have used the midpoint value for b.[4]

Below is a Visual Basic code fragment that uses a one-sided difference, which one depending on the sign of the drift term. This code fragment can be used for interest rate products, since it is solving

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma(r)^2\frac{\partial^2 V}{\partial r^2} + (\mu(r) - \lambda(r)\sigma(r))\frac{\partial V}{\partial r} - rV = 0.$$

Note the arbitrary $\sigma(r) =$ `Volatility(IntRate(i))` and $\mu(r) - \lambda(r)\sigma(r) =$ `RiskNeutralDrift(IntRate(i))`.

This fragment of code is just the timestepping, above it would go declarations and setting up the payoff. Below it would go the outputting. It does not implement any boundary conditions at `i = 1` or at `i = NoIntRateSteps`, these would depend on the contract being valued.

```
For i = 1 To NoIntRateSteps - 1
    If RiskNeutralDrift(IntRate(i)) > 0 Then
         Delta(i) = (VOld(i + 1) - VOld(i)) / dr
         RNDrift = RiskNeutralDrift(IntRate(i) + 0.5 * dr)
    Else
         Delta(i) = (VOld(i) - VOld(i - 1)) / dr
         RNDrift = RiskNeutralDrift(IntRate(i) - 0.5 * dr)
    End If
    Gamma(i) = (VOld(i + 1) - 2 * VOld(i) + VOld(i - 1)) / (dr * dr)
    VNew(i) = VOld(i) + Timestep * (0.5 * Volatility(IntRate(i)) _
              * Volatility(IntRate(i)) * Gamma(i) + RNDrift _
              * Delta(i) - IntRate(i) * VOld(i))
Next i
```

[3] That's 'wind' as in breeze, not as in to wrap or coil.

[4] This choice won't make much difference to the result but it does help to make the numerical method 'conservative,' meaning that certain properties of the partial differential equation are retained by the solution of the difference equation. Having a conservative scheme is important in computational fluid dynamics applications, otherwise the scheme will exhibit mass 'leakage.'

To get back the $O(\delta S^2)$ accuracy of the central difference with a one-sided difference you can use the approximations described in Section 25.5.

We have seen that the explicit finite-difference method suffers from restrictions in the size of the grid steps. The explicit method is similar in principle to the binomial tree method, and the restrictions can be interpreted in terms of risk-neutral probabilities. The terms A, B and C are related to the risk-neutral probabilities of reaching the points $i-1$, i and $i+1$. Instability is equivalent to one of these being a negative quantity, and we can't allow negative probabilities. More sophisticated numerical methods exist that do not suffer from such restrictions.

The advantages of the explicit method

- It is very easy to program and hard to make mistakes
- When it does go unstable it is usually obvious
- It copes well with coefficients that are asset and/or time dependent
- It is easy to incorporate accurate one-sided differences

The disadvantage of the explicit method

- There are restrictions on the timestep so the method can be slower than other schemes

25.12 SUMMARY

The diffusion equation has been around for a *long* time. Numerical schemes for the solution of the diffusion equation have been around quite a while too, not as long as the equation itself but certainly a lot longer than the Black–Scholes equation and the binomial method. This means that there is a great deal of academic literature on the efficient solution of parabolic equations in general.

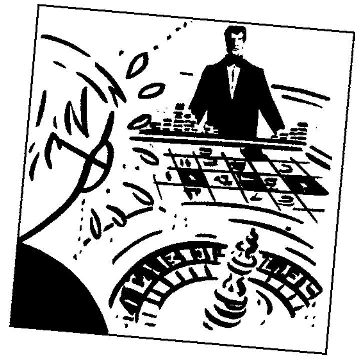
FURTHER READING

- For general numerical methods see Johnson & Riess (1982) and Gerald & Wheatley (1992).
- The first use of finite-difference methods in finance was due to Brennan & Schwartz (1978). For its application in interest rate modeling see Hull & White (1990b).

Over and over again

Once you're happy with writing explicit finite-difference code you'll find yourself using it for many pricing problems. In fact, you may even use the same piece of core code for many types of instruments, it's that robust. But even writing the code from scratch will become second nature. Eventually, expect to take no more than 10 minutes to write basic pricing code *with no mistakes*.

CHAPTER 26
Monte Carlo simulation and related methods



The aim of this Chapter...

... is to explain what is possibly the most useful technique for pricing derivatives and also for examining future scenarios in a probabilistic manner, the Monte Carlo simulation. You will see how the idea is applied to pricing hedged contracts and also to estimating the risks in unhedged positions.

In this Chapter...

- the relationship between option values and expectations for equities, currencies, commodities and indices
- the relationship between derivative products and expectations when interest rates are stochastic
- how to do Monte Carlo simulations to calculate derivative prices and to see the results of speculating with derivatives
- how to do simulations in many dimensions using Cholesky factorization
- how to do numerical integration in high dimensions to calculate the price of options on baskets

26.1 INTRODUCTION

The foundation of the theory of derivative pricing is the random walk of asset prices, interest rates etc. We have seen this foundation in Chapter 6 for equities, currencies and commodities, and the resulting option pricing theory from Chapter 8 onwards. This is the Black–Scholes theory leading to the Black–Scholes parabolic partial differential equation. In Chapter 8 I mentioned the relationship between option prices and expectations. In this chapter we exploit this relationship, and see how derivative prices can be found from special simulations of the asset price, or interest rate, random walks. Briefly, the value of an option is the expected present value of the payoff. The catch in this is the precise definition of 'expected.'

I am going to distinguish between the valuation of options having equities, indices, currencies, or commodities as their underlying with interest rates assumed to be deterministic and those products for which it is assumed that interest rates are stochastic. First, I show the relationship between derivative values and expectations with deterministic interest rates.

26.2 RELATIONSHIP BETWEEN DERIVATIVE VALUES AND SIMULATIONS: EQUITIES, INDICES, CURRENCIES, COMMODITIES

Recall from Chapter 8 that the fair value of an option in the Black–Scholes world is the present value of the expected payoff at expiry under a *risk-neutral* random walk for the underlying.

The risk-neutral random walk for S is

$$dS = rS\,dt + \sigma S\,dX.$$

We can therefore write

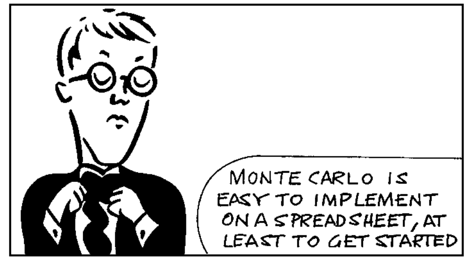
$$\text{option value} = e^{-r(T-t)}E\left[\text{payoff}(S)\right]$$

provided that the expectation is with respect to the risk-neutral random walk, not the *real* one.

This result leads to an estimate for the value of an option by following these simple steps:

1. Simulate the risk-neutral random walk as discussed below, starting at today's value of the asset S_0, over the required time horizon. This time period starts today and continues until the expiry of the option. This gives one realization of the underlying price path.
2. For this realization calculate the option payoff.
3. Perform many more such realizations over the time horizon.
4. Calculate the average payoff over all realizations.
5. Take the present value of this average, this is the option value.

The initial part of this algorithm requires first of all the generation of random numbers from a standardized Normal distribution (or some suitable approximation). We discuss this



	A	B	C	D	E	F	G	H	I
1	**Asset**	100		**Time**	**Sim 1**	**Sim 2**	**Sim 3**	**Sim 4**	**Sim 5**
2	**Drift**	5%		0	100.00	100.00	100.00	100.00	100.00
3	**Volatility**	20%		0.01	98.20	103.18	101.93	97.94	101.55
4	**Timestep**	0.01		0.02	97.68	102.92	102.16	101.54	102.47
5	**Int.rate**	5%		0.03	95.34	103.35	107.30	102.88	104.87
6				0.04	96.27	103.96	104.42	101.73	103.69
7				0.05	97.80	104.52	106.21	101.43	108.72
8				0.06	98.48	104.08	103.54	95.11	111.33
9				0.07	97.96	108.45	105.37	93.58	111.57
10				[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
11				0.09	105.50	106.21	105.04	95.46	111.38
12				0.1	108.75	107.27	105.21	97.19	109.58
94				0.92	104.29	95.02	97.41	80.70	106.94
95				0.93	105.32	93.68	100.96	80.02	106.62
96				0.94	105.70	89.05	100.41	79.22	107.92
97				0.95	102.43	87.65	97.81	81.36	110.04
98				0.96	103.09	86.48	96.50	82.56	110.65
99				0.97	101.29	86.84	96.40	83.87	112.59
100				[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
101				0.99	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
102				1	96.12	87.05	96.36	79.67	120.56
103									
104	**Strike**	105	**CALL**	**Payoff**	0.00	0.00	0.00	0.00	15.56
105			**Mean**	8.43					
106			**PV**	8.02					
107			**PUT**	**Payoff**	8.88	17.95	8.64	25.33	0.00
108			**Mean**	8.31					
109			**PV**	7.9					
110									
111									

= D3+$B$4

=E3*EXP(($B$5-0.5*$B$3*$B$3)*$B$4+$B$3*SQRT($B$4)*NORMSINV(RAND()))

= MAX($B$104-F102,0)

= MAX(G102-$B$104,0)

= AVERAGE(E104:IV104)

=D105*EXP(-$B$5*$D$102)

Figure 26.1 Spreadsheet showing a Monte Carlo simulation to value a call and a put option.

issue below, but for the moment assume that we can generate such a series in sufficient quantities. Then one has to update the asset price at each timestep using these random increments. Here we have a choice how to update S.

An obvious choice is to use

$$\delta S = rS\,\delta t + \sigma S\sqrt{\delta t}\,\phi,$$

where ϕ is drawn from a standardized Normal distribution. This discrete way of simulating the time series for S is called the **Euler method**. Simply put the latest value for S into the right-hand side to calculate δS and hence the next value for S. This method is easy to apply to any stochastic differential equation. This discretization method has an error of $O(\delta t)$.[1]

For the lognormal random walk, however, we are lucky that we can find a simple, and exact, timestepping algorithm. We can write the risk-neutral stochastic differential equation for S in the form

$$d(\log S) = \left(r - \tfrac{1}{2}\sigma^2\right) dt + \sigma\, dX.$$

This can be integrated exactly to give

$$S(t) = S(0) \exp\left(\left(r - \tfrac{1}{2}\sigma^2\right) t + \sigma \int_0^t dX\right).$$

Or, over a timestep δt,

$$S(t + \delta t) = S(t) + \delta S = S(t) \exp((r - \tfrac{1}{2}\sigma^2)\delta t + \sigma\sqrt{\delta t}\,\phi). \tag{26.1}$$

Note that δt need not be small, since the expression is exact. Because this expression is exact and simple it is the best timestepping algorithm to use. Because it is exact, if

	A	B	C	D	E	F	G	H	I
1	**Asset**	100		**Time**	**Sim 1**	**Sim 2**	**Sim 3**	**Sim 4**	**Sim 5**
2	**Drift**	5%		0	100.00	100.00	100.00	100.00	100.00
3	**Volatility**	20%		0.01	100.88	100.74	99.07	100.73	100.03
4	**Timestep**	0.01		0.02	103.01	98.50	100.73	101.71	98.72
5	**Int. rate**	5%		0.03	103.47	97.56	98.73	103.43	99.96
6				0.04	103.92	98.75	97.64	99.90	102.33
7				0.05	106.31	97.94	96.76	100.82	101.43
8				0.06	105.57	99.37	98.38	100.14	97.93
9				[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
10				[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
11				0.09	104.59	97.27	98.31	101.32	94.26
12				0.1	103.80	95.63	100.89	103.75	92.99
13				0.11	101.61	97.06	99.58	107.24	94.21
93				0.91	82.50	95.63	105.93	119.52	97.35
94				0.92	79.69	95.58	105.05	122.86	97.07
95				0.93	78.91	93.11	105.41	119.11	98.27
96				0.94	79.10	92.92	106.84	121.56	100.98
97				0.95	75.42	92.06	107.37	123.28	102.66
98				[illegible]	[illegible]	92.58	107.40	121.94	102.34
99				0.97	75.47	91.01	106.90	122.09	101.44
100				0.98	76.47	89.98	[illegible]	[illegible]	[illegible]
101				[illegible]	77.47	87.15	[illegible]	[illegible]	[illegible]
102				1	76.30	86.68	104.48	122.35	100.30
103									
104				**Average**	91.27	86.49	95.40	111.87	98.31
105									
106	**Strike**	105	**ASIAN**	**Payoff**	0.00	0.00	0.00	6.87	0.00
107				**Mean**	3.18				
108				**PV**	3.02				
109									
110									

= D3+$B$4

= E3*EXP(($B$5-0.5*$B$3*$B$3)*$B$4+$B$3*SQRT($B$4)*NORMSINV(RAND()))

=AVERAGE(E2:E102)

=AVERAGE(E106:IV106)

= MAX(G104-$B$106,0)

=D107*EXP(-$B$5*$D$102)

Figure 26.2 Spreadsheet showing a Monte Carlo simulation to value an Asian option.

[1] There are better approximations, for example the Milstein method which has an error of $O(\delta t^2)$.

we have a payoff that only depends on the final asset value, i.e. is European and path independent, then we can simulate the final asset price in one giant leap, using a timestep of T.

Time Out...

Recap...risk neutrality

Remember, when using simulations for pricing *you must simulate the risk-neutral random walk*. The real random walk is theoretically irrelevant as far as option pricing is concerned. But once you've set up the code or spreadsheet for pricing one contract it can often be modified quite easily to price other, even path-dependent, contracts.

The above algorithm is illustrated in the spreadsheet of Figure 26.1. The stock begins at time $t = 0$ with a value of 100, it has a volatility of 20%. The spreadsheet simultaneously calculates the values of a call and a put option. They both have an expiry of one year and a strike of 105. The interest rate is 5%. In this spreadsheet we see a small selection of a large number of Monte Carlo simulations of the random walk for S, *using a drift rate of 5%*. The timestep was chosen to be 0.01. For each realization the final stock price is shown in row 102 (rows 13–93 have been hidden). The option payoffs are shown in rows 104 and 107. The mean of all these payoffs, over all the simulations, is shown in row 105 and 108. In rows 106 and 109 we see the present values of the means, these are the option values. For serious option valuation you would not do such calculations on a spreadsheet. For the present example I took a relatively small number of sample paths.

The method is particularly suitable for path-dependent options. In the spreadsheet in Figure 26.2 I show how to value an Asian option. This contract pays of an amount $\max(A - 105, 0)$ where A is the average of the asset price over the one-year life of the contract. The remaining details of the underlying are as in the previous example. How would the spreadsheet be modified if the average were only taken of the last six months of the contract's life?

26.3 ADVANTAGES OF MONTE CARLO SIMULATION

Now that we have some idea of how Monte Carlo simulations are related to the pricing of options, I'll give you some of the benefits of using such simulations:

- The mathematics that you need to perform a Monte Carlo simulation can be very basic.

- Correlations can be easily modeled.
- There is plenty of software available, at the least there are spreadsheet functions that will suffice for most of the time.
- To get a better accuracy, just run more simulations.
- The effort in getting *some* answer is very low.
- The models can often be changed without much work.
- Complex path dependency can often be easily incorporated.
- People accept the technique, and will believe your answers.

26.4 **USING RANDOM NUMBERS**

The Black–Scholes theory as we have seen it, has been built on the assumption of either a simple up-or-down move in the asset price, the binomial model, or a Normally distributed return. When it comes to simulating a random walk for the asset price it doesn't matter very much what distribution we use for the random increments as long as the timestep is small and thus that we have a large number of steps from the start to the finish of the asset price path. All we need are that the variance of the distribution must be finite and constant. (The constant must be such that the *annualized* volatility, i.e. the annualized standard deviation of returns, is the correct value. In particular, this means that it must scale with $\delta t^{1/2}$.) In the limit as the size of the timestep goes to zero the simulations have the same probabilistic properties over a finite timescale regardless of the nature of the distribution over the infinitesimal timescale. This is a result of the central limit theorem.

Nevertheless, the most accurate model is the lognormal model with Normal returns. Since one has to worry about simulating sufficient paths to get an accurate option price one would ideally like not to have to worry about the size of the timestep too much. As I said above, it is best to use the exact expression (26.1) and then the choice of timestep does not affect the accuracy of the random walk. In some cases we can take a single timestep since the timestepping algorithm is exact. If we do use such a large timestep then we must generate Normally distributed random variables. I will discuss this below, where I describe the Box–Muller method.

If the size of the timestep is δt then, for more complicated products, such as path-dependent ones, we may still introduce errors of $O(\delta t)$ by virtue of the discrete approximation to continuous events. An example would be of a continuous barrier. If we have a finite timestep we miss the possibility of the barrier being triggered between steps. Generally speaking, the error due to the finiteness of the timestep is $O(\delta t)$.

Because we are only simulating a finite number of an infinite number of possible paths, the error due to using N, say, realizations of the asset price paths is $O(N^{-1/2})$.

The total number of calculations required in the estimation of a derivative price is then $O(N/\delta t)$. This is then also a measure of the time taken in the calculation of the price. The error in the price is

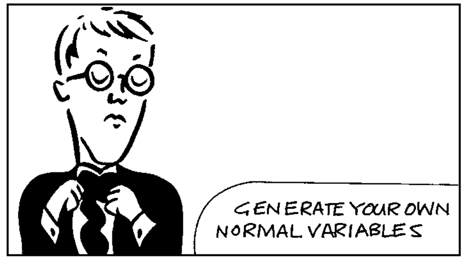
$$O\left(\max\left(\delta t, \frac{1}{\sqrt{N}}\right)\right),$$

i.e. the worst out of either the error due to the discreteness of the timestep or the error in having only a finite number of realizations. To minimize this quantity, while keeping the total computing time fixed such that $O(N/\delta t) = K$, we must choose

$$N = O(K^{2/3}) \quad \text{and} \quad \delta t = O(K^{-1/3}).$$

26.5 GENERATING NORMAL VARIABLES

Some random number generators are good, others are bad, repeating themselves after a finite number of samples, or showing serial autocorrelation. Then they can be fast or slow. A particularly useful distribution that is easy to implement on a spreadsheet, and is fast, is the following approximation to the Normal distribution:



$$\left(\sum_{i=1}^{12}\psi_i\right)-6,$$

where the ψ_i are independent random variables, drawn from a uniform distribution over zero to one. This distribution is close to Normal, having a mean of zero, a standard deviation of one, and a third moment of zero. It is in the fourth and higher moments that the distribution differs from Normal. I would use this in a spreadsheet when generating asset price paths with smallish timesteps.

If you need to generate decent Normally distributed random numbers then the simplest technique is the **Box–Muller method**. This method takes uniformly distributed variables and turns them into Normal. The basic uniform numbers can be generated by any number of methods, see Press *et al*. (1995) for some algorithms. The Box–Muller method takes two uniform random numbers x_1 and x_2 between zero and one and combines them to give two numbers y_1 and y_2 that are both Normally distributed:

$$y_1=\sqrt{-2\ln x_1}\cos(2\pi x_2)\quad\text{and}\quad y_2=\sqrt{-2\ln x_1}\sin(2\pi x_2).$$

Here is a Visual Basic function that outputs a Normally distributed variable.

```
Function BoxMuller()
Randomize
Do
        x = 2 * Rnd() - 1
        y = 2 * Rnd() - 1
        dist = x * x + y * y
Loop Until dist < 1
BoxMuller = x * Sqr(-2 * Log(dist) / dist)
End Function
```

In Figure 26.3 is the approximation to the Normal distribution using 500 points from the uniform distribution and the Box–Muller method.

26.6 REAL VERSUS RISK NEUTRAL, SPECULATION VERSUS HEDGING

In Figure 26.4 are shown several realizations of a risk-neutral asset price random walk with 5% interest rate and 20% volatility. These are the thin lines. The bold lines in this figure are the corresponding *real* random walks using the same random numbers but here with a drift of 20% instead of the 5% interest rate drift. Although I am here emphasizing the use of Monte Carlo simulations in valuing options we can of course use them to estimate

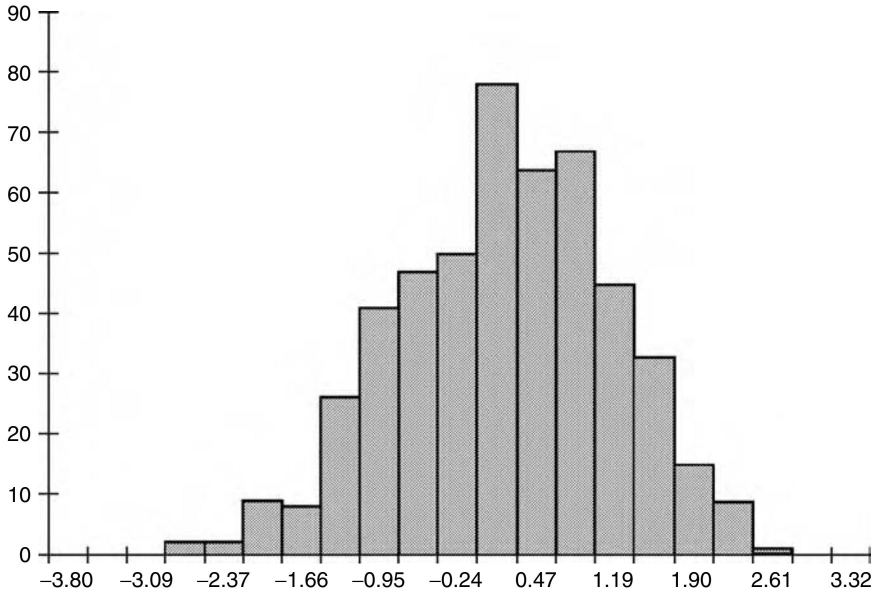


Figure 26.3 The approximation to the Normal distribution using 500 uniformly distributed points and the Box–Muller method.

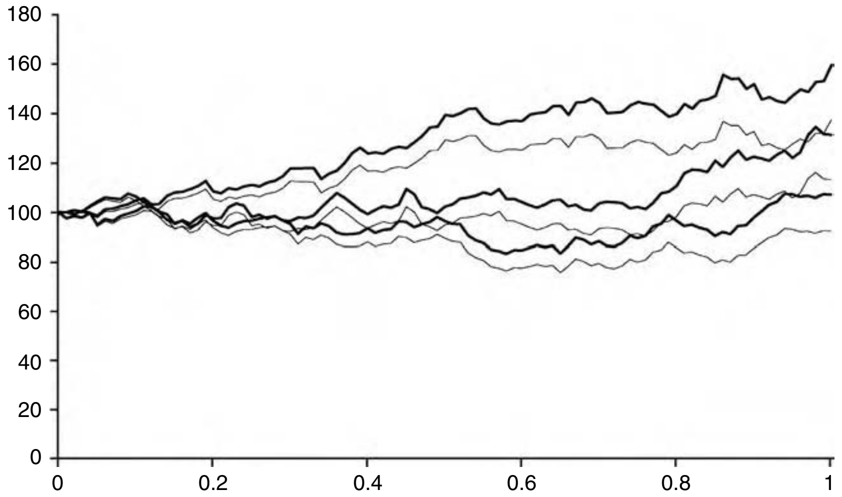


Figure 26.4 Several realizations of an asset price random walk.

the payoff distribution from holding an *unhedged* option position. In this situation we are interested in the whole distribution of payoffs (and their present values) and not just the average or expected value. This is because in holding an unhedged position we cannot guarantee the return that we (theoretically) get from a hedged position. It is therefore valid and important to have the real drift as one of the parameters; it would be incorrect to estimate the probability density function for the return from an unhedged position using the risk-neutral drift.

In Figure 26.5 I show the estimated probability density function and cumulative distribution function for a call with expiry one year and strike 105 using Monte Carlo simulations with $\mu = 20\%$ and $\sigma = 20\%$. The probability density function curve does not show the

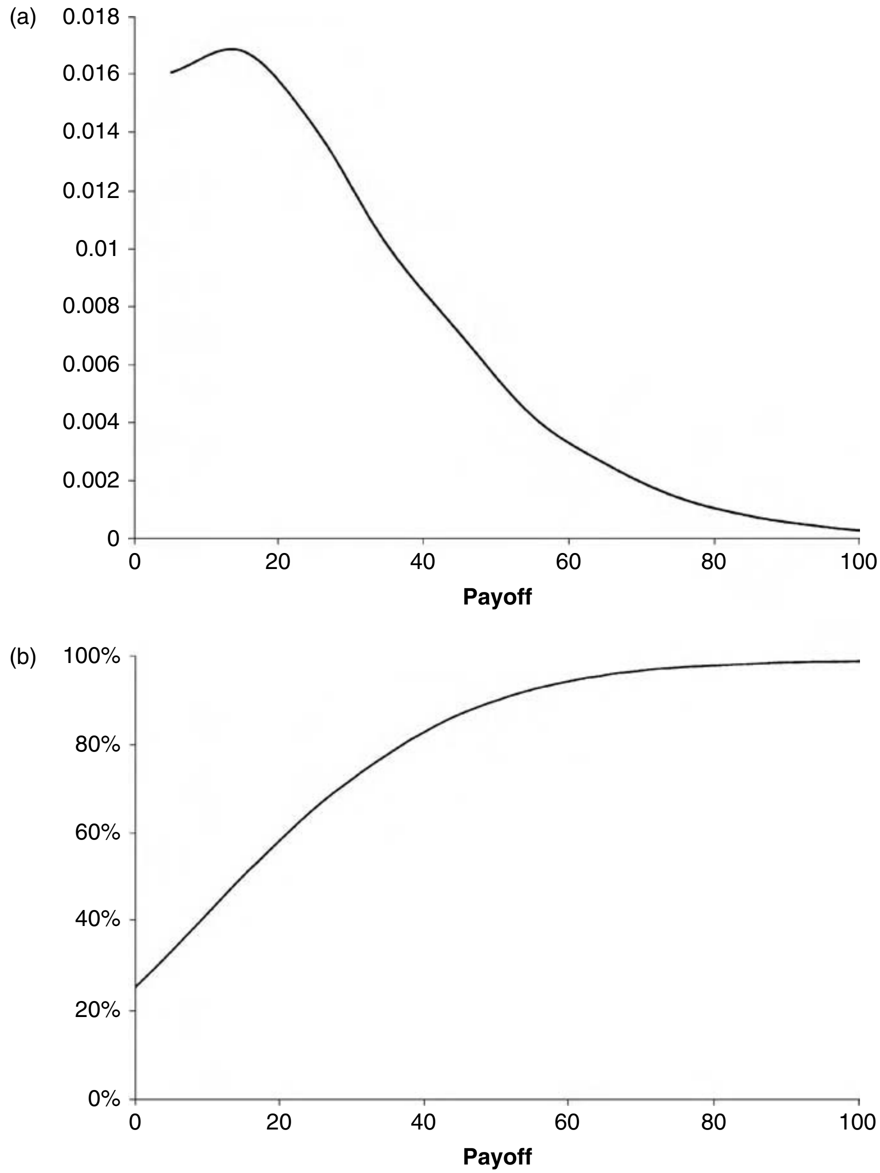


Figure 26.5 Real probability density function (top) and cumulative distribution function (bottom) for the payoff for a call.

zero payoffs. The probability of expiring out of the money and receiving no payoff is approximately 25%.

In Figure 26.6 I show the estimated probability density function and cumulative distribution function for a put with the same expiry and strike, again using Monte Carlo simulations with $\mu = 20\%$ and $\sigma = 20\%$. The probability density function curve does not show the zero payoffs. The probability of expiring out of the money and receiving no payoff is approximately 75%.

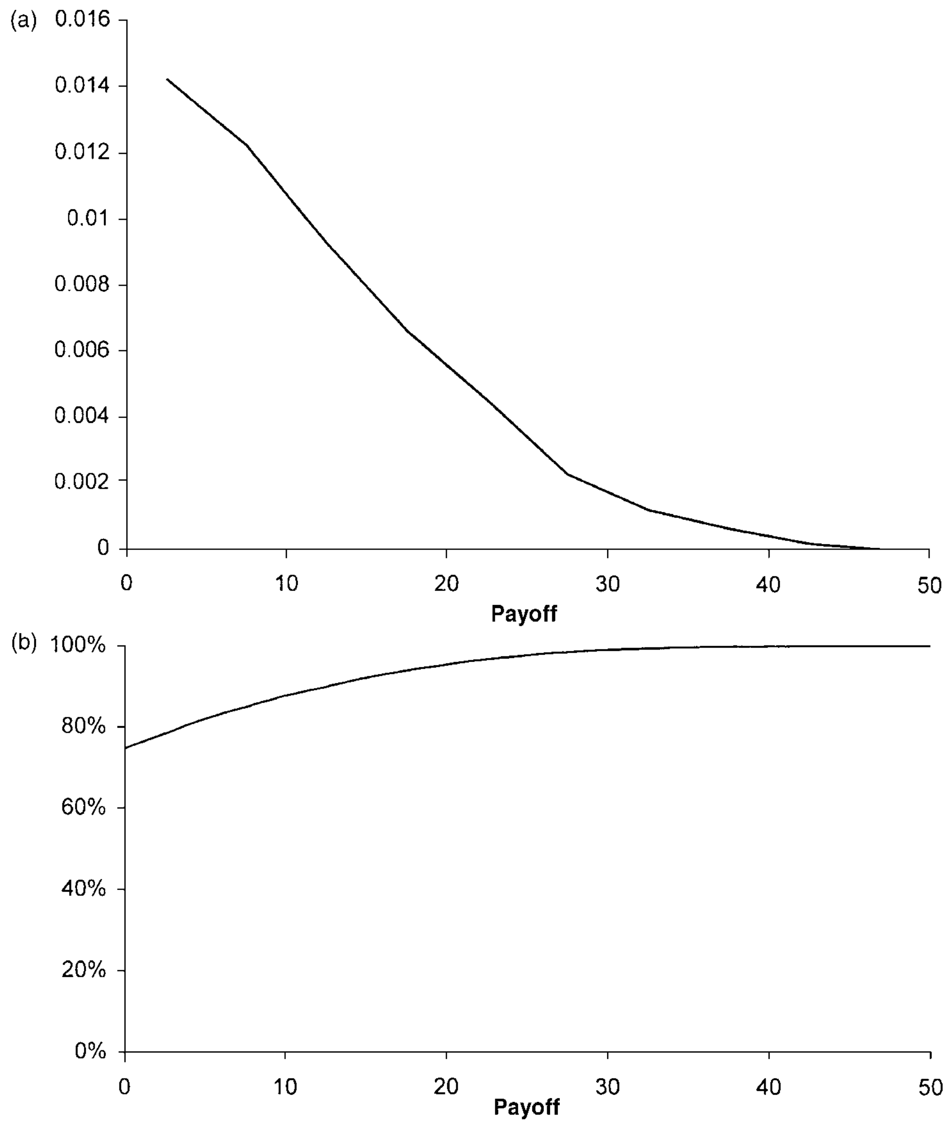


Figure 26.6 Real probability density function (top) and cumulative distribution function (bottom) for the payoff for a put.

Time Out...

Recap...reality

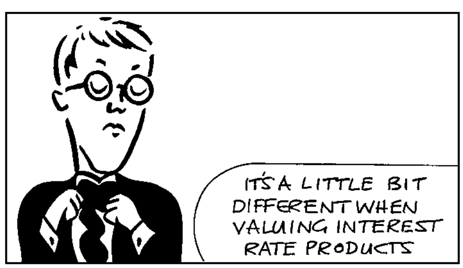
If you use simulations to get an idea of what may happen in the future to unhedged positions *use the real random walk for the underlying*. You can estimate probability density functions, real expectations and standard deviations. These are all useful in risk management.

26.7 INTEREST RATE PRODUCTS

The relationship between expected payoffs and option values when the short-term interest rate is stochastic is slightly more complicated because there is the question of what rate to use for discounting.

The correct way to estimate option value with stochastic interest rates is as follows:

1. Simulate the random walk for the risk-adjusted spot interest rate r, as discussed below, starting at today's value of the spot rate, over the required time horizon. This time period starts today and continues until the expiry of the option. This gives one realization of the spot rate path.
2. For this realization calculate two quantities, the payoff and the *average* interest rate realized up until the payoff is received.
3. Perform many more such realizations.
4. For each realization of the r random walk calculate the present value of the payoff for this realization discounting at the average rate for this realization.
5. Calculate the average present value of the payoffs over all realizations, this is the option value.



In other words,

$$\text{option value } = E\left[e^{-\int_t^T r(\tau)\,d\tau}\ \text{payoff}(r)\right].$$

Why is this different from the deterministic interest rate case? Why discount at the average interest rate? We discount all cashflows at the average rate because this is the interest rate received by a money market account, and in the risk-neutral world all assets have the same risk-free growth rate. Recall that cash in the bank grows according to

$$\frac{dM}{dt} = r(t)M.$$

The solution of which is

$$M(t) = M(T)e^{-\int_t^T r(\tau)d\tau}.$$

This contains the same discount factor as in the option value.

The choice of discretization of spot rate models is usually limited to the Euler method

$$\delta r = (u(r,t) - \lambda(r,t)w(r,t))\,dt + w(r,t)\sqrt{\delta t}\,\phi.$$

Rarely can the spot rate equations be exactly integrated.

In the next spreadsheet Figure 26.7 I demonstrate the Monte Carlo method for a contract with payoff $\max(r - 10\%, 0)$. Maturity is in one year. The model used to perform the simulations is Vasicek with constant parameters. The spot interest rate begins at 6%. The option value is the average present value in the last row.

	A	B	C	D	E	F	G	H
1	**Spot rate**	10%		**Time**	**Sim 1**	**Sim 2**	**Sim 3**	**Sim 4**
2	**Mean rate**	8%		0	10.00%	10.00%	10.00%	10.00%
3	**Reversion rate**	0.2		0.01	9.96%	9.97%	10.01%	10.06%
4	**Volatility**	0.007		0.02	10.06%	10.01%	10.01%	10.05%
5	**Timestep**	0.01		0.03	10.07%	10.04%	9.99%	10.14%
6				0.04	10.10%	10.12%	10.08%	10.13%
7				0.05	10.18%	10.09%	10.06%	10.21%
8				0.06	10.10%	10.18%	9.98%	10.27%
9				0.07	10.24%	10.23%	9.92%	10.28%
10				0.08	10.26%	10.24%	9.90%	10.33%
11				0.09	10.29%	10.29%	9.82%	10.36%
12							3%	10.37%
13							0%	10.39%
14							8%	10.33%
15				0.13	10.26%	10.13%	9.57%	10.23%
95				0.93	9.75%	9.70%	9.38%	8.88%
96				0.94	9.92%	9.74%	9.39%	8.94%
97				0.95	9.92%	9.75%	9.37%	9.04%
98				0.96	9.87%	9.70%	9.39%	9.06%
99				0.97	9.95%	9.69%	9.37%	9.06%
100				0.98	9.86%	9.63%	9.39%	9.14%
101				0.99	9.92%	9.68%	9.52%	9.15%
102				1	9.94%	9.68%	9.54%	9.21%
103								
104				**Mean rate**	10.25%	9.94%	9.62%	9.32%
105								
106	**Strike**	10%		**Payoff**	0.0000	0.0000	0.0000	0.0000
107			**PV'd**		0.0000	0.0000	0.0000	0.0000
108			**Mean**	0.001014				
109								
110								
111								
112								
113								
114								

= D3+$B$5

=F5+$B$3*($B$2–F5)*$B$5+$B$4*SQRT($B$5)*(RAND()+RAND()+RAND()+RAND()+RAND()+(RAND()+RAND()+RAND()+RAND()+RAND()+RAND()+RAND()–6)

= AVERAGE(E2:E102)

= MAX(E102-$B$106,0)

=E106*EXP(-$D$102*E104)

=AVERAGE(E107:IV107)

Figure 26.7 Spreadsheet showing a Monte Carlo simulation to value a contract with a payoff $\max(r - 10\%, 0)$.

Time Out...

Recap...

For pricing use the risk-neutral random walk, for analyzing an unhedged future, use the real.

26.8 CALCULATING THE GREEKS

The simplest way to calculate the delta of an option using Monte Carlo simulation is to estimate the option's value twice. The delta of the option is the derivative of the option with respect to the underlying

$$\Delta = \lim_{h\to 0} \frac{V(S+h,t) - V(S-h,t)}{2h}.$$

This is a central difference, discussed in Chapter 25. This is an accurate estimate of the first derivative, with an error of $O(h^2)$. However, the error in the measurement of the two option values at $S+h$ and $S-h$ can be much larger than this for the Monte Carlo simulation. These Monte Carlo errors are then magnified when divided by h, resulting in an error of $O(1/hN^{1/2})$. To overcome this problem, estimate the value of the option at $S+h$ and $S-h$ *using the same values for the random numbers*. In this way the errors in the Monte Carlo simulation will cancel each other out. The same principal is used to calculate the gamma and the theta of the option.

Another way to calculate the delta is to exploit the differential equation satisfied by the delta. Differentiate the Black–Scholes equation with respect to S. This gives

$$\frac{\partial \Delta}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 \Delta}{\partial S^2} + (r+\sigma^2)S\frac{\partial \Delta}{\partial S} = 0.$$

For a vanilla call option the delta at expiry is

$$\Delta(S,T) = \mathcal{H}(S-E),$$

the Heaviside function. We can estimate the value of the delta today by a Monte Carlo simulation in which we calculate the expected value of the final delta using the following random walk for S:

$$dS = (r+\sigma^2)S\,dt + \sigma S\,dX.$$

Since there is no discounting term in the partial differential equation there is no need to take the present value.

26.9 HIGHER DIMENSIONS: CHOLESKY FACTORIZATION

Monte Carlo simulation is a natural method for the pricing of European-style contracts that depend on many underlying assets. Supposing that we have an option paying off some function of S_1, S_2,. . .,S_d then we could, in theory, write down a partial differential equation in $d+1$ variables. Such a problem would be horrendously time consuming to compute. The simulation methods discussed above can easily be extended to cover such a problem. All we need to do is to simulate

$$S_i(t+\delta t) = S_i(t) \exp\left(\left(r - \tfrac{1}{2}\sigma_i^2\right)\delta t + \sigma_i\sqrt{\delta t}\,\phi_i\right).$$

The catch is that the ϕ_i are correlated,

$$E[\phi_i\phi_j] = \rho_{ij}.$$

How can we generate *correlated* random variables? This is where **Cholesky factorization** comes in.

Let us suppose that we can generate *d uncorrelated* Normally distributed variables ε_1, ε_2,. . ., ε_d. We can use these variables to get correlated variables with the transformation

$$\boldsymbol{\phi} = \mathbf{M}\boldsymbol{\varepsilon} \tag{26.2}$$

where $\boldsymbol{\phi}$ and $\boldsymbol{\varepsilon}$ are the column vectors with ϕ_i and ε_i in the ith rows. The matrix $\mathbf{M}$ is special and must satisfy

$$\mathbf{M}\mathbf{M}^T = \boldsymbol{\Sigma}$$

with $\boldsymbol{\Sigma}$ being the correlation matrix.

It is easy to show that this transformation will work. From (26.2) we have

$$\boldsymbol{\phi}\boldsymbol{\phi}^T = \mathbf{M}\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^T\mathbf{M}^T. \tag{26.3}$$

Taking expectations of each entry in this matrix equation gives

$$E\left[\boldsymbol{\phi}\boldsymbol{\phi}^T\right] = \mathbf{M}E\left[\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^T\right]\mathbf{M}^T = \mathbf{M}\mathbf{M}^T = \boldsymbol{\Sigma}.$$

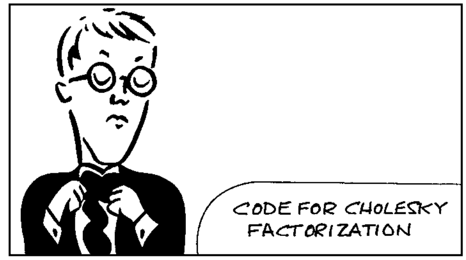
We can take expectations through the matrix multiplication in this because the right-hand side of (26.3) is linear in the terms $\varepsilon_i\varepsilon_j$.

This decomposition of the correlation matrix into the product of two matrices is not unique. The Cholesky factorization gives one way of choosing this decomposition. It results in a matrix $\mathbf{M}$ that is lower triangular. Here is an algorithm for the factorization.

The matrix `Sigma` contains the correlation matrix with dimension `n`. The output matrix is contained in `M`.

```
Function cholesky(Sigma As Object)
Dim n As Integer
Dim k As Integer
Dim i As Integer
Dim j As Integer
Dim x As Double
Dim a() As Double
```

```
Dim M() As Double
n = Sigma.Columns.Count
ReDim a(1 To n, 1 To n)
ReDim M(1 To n, 1 To n)
For i = 1 To n
    For j = 1 To n
        a(i, j) = Sigma.Cells(i, j).Value
        M(i, j) = 0
     Next j
Next i
For i = 1 To n
    For j = i To n
    x = a(i, j)
    For k = 1 To (i - 1)
        x = x - M(i, k) * M(j, k)
    Next k
    If j = i Then
        M(i, i) = Sqr(x)
    Else
        M(j, i) = x / M(i, i)
    End If
    Next j
Next i
cholesky = M
End Function
```



26.10 SPEEDING UP CONVERGENCE

Monte Carlo simulation is inefficient, compared with finite-difference methods, in dimensions less than about three. It is natural, therefore, to ask how can one speed up the convergence. There are several methods in common use, two of which I now describe.

26.10.1 Antithetic variables

In this technique one calculates two estimates for an option value using the one set of random numbers. We do this by using our Normal random numbers to generate one realization of the asset price path, an option payoff and its present value. Now take the same set of random numbers but change their signs, thus replace ϕ with $-\phi$. Again simulate a realization, and calculate the option payoff and its present value. Our estimate for the option value is the average of these two values. Perform this operation many times to get an accurate estimate for the option value.

This technique works because of the symmetry in the Normal distribution. This symmetry is guaranteed by the use of the antithetic variable.

26.10.2 Control variate technique

Suppose we have two similar derivatives, the former is the one we want to value by simulations and the second has a similar (but 'nicer') structure such that we have an explicit formula for its value. Use the *one* set of realizations to value *both* options. Call the values estimated by the Monte Carlo simulation V_1' and V_2'. If the accurate value of the second option is V_2 then a better estimate than V_1' for the value of the first option is

$$V_1' - V_2' + V_2.$$

The argument behind this method is that the error in V_1' will be the same as the error in V_2', and the latter is known.

A refinement of this technique is **martingale variance reduction**. In this method, one simulates one or more new dependent variables at the same time as the path of the underlying. This new stochastic variable is chosen so as to have an *expected value of zero* after each time step. This new variable, the 'variate,' is then added on to the value of the option. Since it has an expected value of zero it cannot make the estimate any worse, but if the variate is chosen carefully it can reduce the variance of the error significantly.

Let's see how this is done in practice using a single variate. Suppose we simulate

$$\delta S = \mu S\,\delta t + \sigma S\sqrt{\delta t}\,\phi$$

to price our contract. Now introduce the variate y, satisfying

$$\delta y = f(S,t)\,(\delta S - E[\delta S])\,,$$

with zero initial value. Note that this has zero expectation. The choice of $f(S,t)$ will be discussed in a moment. The new estimate for the option value is simply

$$\overline{V} - \alpha e^{-r(T-t)}\overline{y},$$

where $\overline{V}$ is our usual Monte Carlo estimate and $\overline{y}$ is the average over all the realizations of the new variate at expiry. The choice of α is simple, choose it to minimize the variance of the error, i.e. to minimize

$$E\left[(V - \alpha e^{-r(T-t)}y)^2\right].$$

I leave the details to the reader.

And the function $f(S,t)$? The natural choice is the delta of an option that is closely related to the option in question, one for which there is a closed-form solution. Such a choice corresponds to an approximate form of delta hedging, and thus reduces the fluctuation in the contract value along each path.

26.11 PROS AND CONS OF MONTE CARLO SIMULATIONS

The Monte Carlo technique is clearly very powerful and general. The concept readily carries over to exotic and path-dependent contracts, just simulate the random walk and the corresponding cash flows, estimate the average payoff and take its present value.

The main disadvantages are twofold. First, the method is slow when compared with the finite-difference solution of a partial differential equation. Generally speaking this is true for problems of up to three or four dimensions. When there are four or more stochastic or path-dependent variables the Monte Carlo method becomes relatively more efficient. Second, the application to American options is far from straightforward. The reason for the problem with American options is to do with the optimality of early exercise. To know when it is optimal to exercise the option one must calculate the option price *for all values of S and t up to expiry* in order to check that at no time is there any arbitrage opportunity.

However, the Monte Carlo method in its basic form is only used to estimate the option price at one point in (S, t) space, now and at today's value.

Because Monte Carlo simulation is based on the generation of a finite number of realizations using series of random numbers, the value of an option derived in this way will vary each time the simulations are run. Roughly speaking, the error between the Monte Carlo estimate and the correct option price is of the order of the inverse square root of the number of simulations. More precisely, if the standard deviation in the option value using a single simulation is ε then the standard deviation of the error after N simulations is $\varepsilon/\sqrt{N}$. To improve our accuracy by a factor of 10 we must perform 100 times as many simulations.

26.12 AMERICAN OPTIONS

Applying Monte Carlo methods to the valuation of European contracts is simple, but applying them to American options is very, very hard. The problem is to do with the time direction in which we are solving. We have seen how it is natural in the partial differential equation framework to work backwards from expiry to the present. If we do this numerically then we find the value of a contract at every mesh point between now and expiry. This means that along the way we can ensure that there is no arbitrage, and in particular ensure that the early-exercise constraint is satisfied.

When we use the Monte Carlo method in it basic form for valuing a European option we only ever find the option's value at the one point, the current asset level and the current time. We have no information about the option value at any other asset level or time. So if our contract is American we have no way of knowing whether or not we violated the early-exercise constraint somewhere in the future.

In principle, we could find the option value at each point in asset-time space using Monte Carlo. For every asset value and time that we require knowledge of the option value we start a new simulation. But when we have early exercise we have to do this at a large number of points in asset-time space, keeping track of whether the constraint is violated. If we find a value for the option that is below the payoff then we mark this point in asset-time space as one where we must exercise the option. And then for every other path that goes through this point we must exercise at this point, if not before. Such a procedure is possible, but the time taken grows exponentially with the number of points at which we value the option. At the time of writing there were few totally convincing ways around this problem.

26.13 NUMERICAL INTEGRATION

Often the fair value of an option can be written down analytically as an integral. This is certainly the case for non-path-dependent European options contingent upon d lognormal underlyings, for which we have

$$V = e^{-r(T-t)}\,(2\pi(T-t))^{-d/2}\,(\mathrm{Det}\boldsymbol{\Sigma})^{-1/2}(\sigma_1\cdots\sigma_d)^{-1}$$
$$\int_0^\infty\cdots\int_0^\infty \frac{\mathrm{Payoff}(S_1'\cdots S_d')}{S_1'\cdots S_d'}\exp\left(-\tfrac{1}{2}\boldsymbol{\alpha}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}\right)dS_1'\cdots dS_d'$$

where

$$\alpha_i = \frac{1}{\sigma_i (T-t)^{1/2}} \left(\log\left(\frac{S_i}{S_i'}\right) + \left(r - D_i - \tfrac{1}{2}\sigma_i^2\right)(T-t) \right)$$

and $\boldsymbol{\Sigma}$ is the correlation matrix for the d assets and Payoff$(\cdots)$ is the payoff function. Sometimes the value of path-dependent contracts can also be written as a multiple integral. American options, however, can rarely be expressed so simply.

If we do have such a representation of an option's value then all we need do to value it is to estimate the value of the multiple integral. Let us see how this can be done.

Time Out...

What is this formula?

It is nothing more complicated than the present value of something. The something is just an expected value, but, of course, the risk-neutral expected value. Interestingly, it's easy to calculate the value of this quantity than to understand all the terms in it... as you'll see below.

26.14 REGULAR GRID

We can do the multiple integration by evaluating the function on a uniform grid in the d-dimensional space of assets. There would thus be $N^{1/d}$ grid points in each direction where N is the total number of points used. Supposing we use the trapezium or midpoint rule, the error in the estimation of the integral will be $O(N^{-2/d})$ and the time taken approximately $O(N)$ since there are N function evaluations. As the dimension d increases this method becomes prohibitively slow. Note that because the integrand is generally not smooth there is little point in using a higher-order method than a midpoint rule unless one goes to the trouble of finding out the whereabouts of the discontinuities in the derivatives. To overcome this 'curse of dimensionality' we can use Monte Carlo integration or low-discrepancy sequences.

26.15 BASIC MONTE CARLO INTEGRATION

Suppose that we want to evaluate the integral

$$\int \cdots \int f(x_1, \ldots, x_d)\, dx_1 \ldots dx_d,$$

over some volume. We can very easily estimate the value of this by Monte Carlo simulation. The idea behind this is that the integral can be rewritten as

$$\int \cdots \int f(x_1, \dots, x_d)\, dx_1 \dots dx_d = \text{ volume of region of integration } \times \text{ average } f,$$

where the average of f is taken over the whole of the region of integration. To make life simple we can rescale the region of integration to make it the unit hypercube. Assuming that we have done this

$$\int_0^1 \cdots \int_0^1 f(x_1, \dots, x_d)\, dx_1 \dots dx_d = \text{ average } f$$

because the volume is one. Such a scaling will obviously be necessary in our financial problems because the range of integration is typically from zero to infinity. I will return to this point later.

We can sample the average of f by Monte Carlo sampling using uniformly distributed random numbers in the d-dimensional space. After N samples we have

$$\text{average } f \approx \frac{1}{N} \sum_{i=1}^{N} f(x_i) \tag{26.4}$$

where x_i is the vector of values of $x_1, \dots, x_d$ at the ith sampling. As N increases, so the approximation improves. Expression (26.4) is only an approximation. The size of the error can be measured by the standard deviation of the correct average about the sampled average, this is

$$\sqrt{\frac{1}{N}\left(\overline{f^2} - \bar{f}^2\right)}$$

(which must be multiplied by the volume of the region), where

$$\bar{f} = \frac{1}{N} \sum_{i=1}^{N} f(x_i)$$

and

$$\overline{f^2} = \frac{1}{N} \sum_{i=1}^{N} f^2(x_i).$$

Thus the error in the estimation of the value of an integral using a Monte Carlo simulation is $O(N^{-1/2})$ where N is the number of points used, and is independent of the number of dimensions. Again there are N function evaluations and so the computational time is $O(N)$. The accuracy is much higher than that for a uniform grid if we have five or more dimensions.

I have explained Monte Carlo integration in terms of integrating over a d-dimensional unit hypercube. In financial problems we often have integrals over the range zero to infinity. The choice of transformation from zero-one to zero-infinity should be suggested by the problem under consideration. Let us suppose that we have d assets following correlated random walks. The risk-neutral value of these assets at a time t can be written as

$$S_i(T) = S_i(t) \exp\left(\left(r - D_i - \tfrac{1}{2}\sigma_i^2\right)(T - t) + \sigma_i \phi_i \sqrt{T - t}\right),$$

in terms of their initial values at time t. The random variables ϕ_i are Normally distributed and correlated. We can now write the value of our European option as

$$e^{-r(T-t)} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \text{Payoff}(S_1(T), \ldots, S_d(T)) p(\phi_1, \ldots, \phi_d)\, d\phi_1 \ldots d\phi_d,$$

where $p(\phi_1, \ldots, \phi_d)$ is the probability density function for d correlated Normal variables with zero mean and unit standard deviation. I'm not going to write down p explicitly since we won't need to know its functional form *as long as we generate numbers from this distribution*. In effect, all that I have done here is to transform from lognormally distributed values of the assets to Normally distributed returns on the assets.

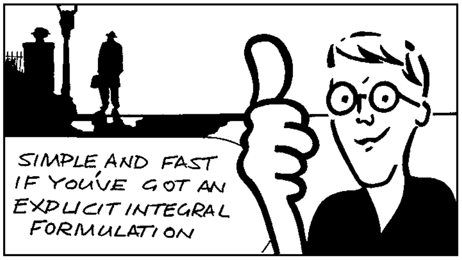
Now to value the option we must generate suitable Normal variables. The first step is to generate uncorrelated variables and then transform them into correlated variables. Both of these steps have been explained above; use Box–Muller and then Cholesky. The option value is then estimated by the average of the payoff over all the randomly generated numbers.

Here is a very simple code fragment for calculating the value of a European option in `NDim` assets using `NoPts` points. The interest rate is `IntRate`, the dividend yields are `Div(i)`, the volatilities are `Vol(i)`, time to expiry `Expiry`. The initial values of the assets are `Asset(i)`. The Normally distributed variables are the `x(i)`, and the `S(i)` are the lognormally distributed future asset values.

```
a = Exp(-IntRate * Expiry) / NoPts
suma = 0
For k = 1 To NoPts
For i = 1 To NDim
If test = 0 Then
    Do
        y = 2 * Rnd() - 1
        z = 2 * Rnd() - 1
        dist = y * y + z * z
    Loop Until dist < 1
x(i) = y * Sqr(-2 * Log(dist) / dist)
test = 1
Else
x(i) = z * Sqr(-2 * Log(dist) / dist)
test = 0
End If
Next i
For i = 1 To NDim
    S(i) = Asset(i) * Exp((IntRate - Div(i) - _
                 0.5 * Vol(i) * Vol(i)) * Expiry + _
                     Vol(i) * x(i) * Sqr(Expiry))
Next i
term = Payoff(S(1), S(2), S(3), S(4), S(5))
suma = suma + term
Next k
Value = suma * a
```

This code fragment is Monte Carlo in its most elementary form, and does not use any of the tricks described below. Some of these tricks are trivial to implement, especially those that are independent of the particular option being valued. And in this example the assets are all uncorrelated.

26.16 LOW-DISCREPANCY SEQUENCES



An obvious disadvantage of the basic Monte Carlo method for estimating integrals is that we cannot be certain that the generated points in the *d*-dimensional space are 'nicely' distributed. Indeed, there is inevitably a good deal of clumping. One way around this is to use a nonrandom series of points with better distributional properties.

Let us motivate the low-discrepancy sequence method by a Monte Carlo example. Suppose that we want to calculate the value of an integral in two dimensions and we use a Monte Carlo simulation to generate a large number of points in two dimensions at which to sample the integrand. The choice of points may look something like Figure 26.8. Notice how the points are not spread out evenly.

Now suppose we want to add a few hundred more points to improve the accuracy. Where should we put the new points? If we put the new points in the gaps between others then we increase the accuracy of the integral. If we put the points close to where there are already many points then we could make matters worse.

The above shows that we want a way of choosing points such that they are not too bunched, but nicely spread out. At the same time we want to be able to add more

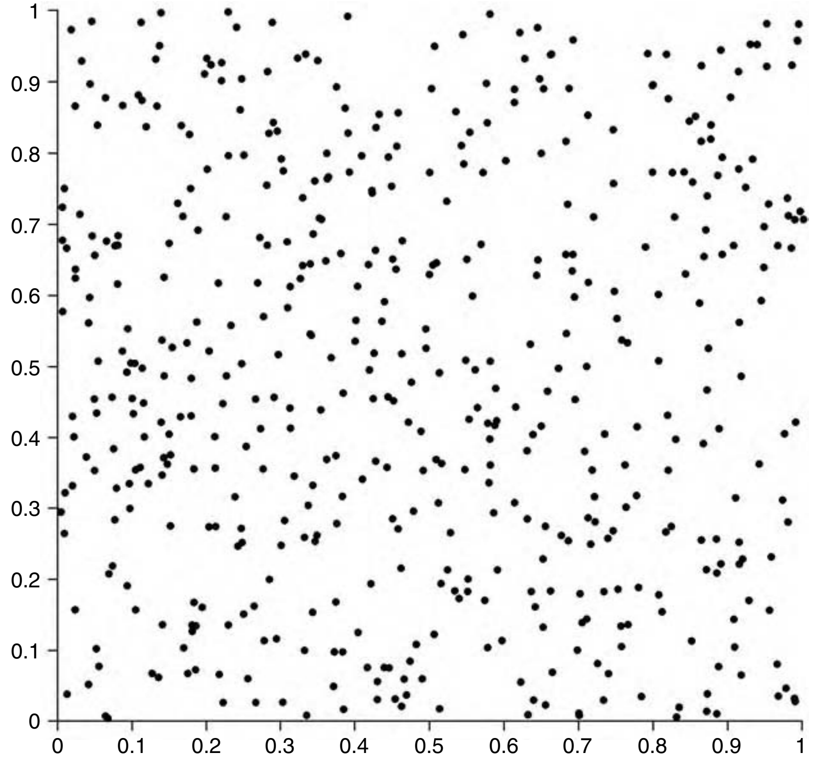


Figure 26.8 A Monte Carlo sample in two dimensions.

points later without spoiling our distribution. Clearly Monte Carlo is bad for evenness of distribution, but a uniform grid does not stay uniform if we add an arbitrary number of extra points. **Low-discrepancy sequences** or **quasi-random sequences** have the properties we require.[2]

There are two types of low-discrepancy sequences, open and closed. The open sequences are constructed on the assumption that we may want to add more points later. The closed sequences are optimized for a given size of sample, to give the best estimate of the integral for the number of points. The regular grid is an example of a closed low-discrepancy sequence. I will describe the open sequences here.

The first application of these techniques in finance was by Barrett, Moore & Wilmott (1992).[3] There are many such sequences with names such as **Sobol'**, **Faure**, **Haselgrove** and **Halton**. I shall describe the Halton sequence here, it is by far the easiest to describe.

The Halton sequence is a sequence of numbers $h(i; b)$ for $i = 1, 2, \ldots$. The integer b is the base. The numbers all lie between zero and one.[4] The numbers are constructed as follows. First choose your base. Let us choose 2. Now write the positive integers in ascending order in base 2, i.e. 1, 10, 11, 100, 101, 110, 111 etc. The Halton sequence base 2 is the reflection of the positive integers in the decimal point, i.e.

Integers base 10	**Integers base 2**	**Halton sequence base 2**	**Halton number base 10**
1	1	$1 \times \frac{1}{2}$	0.5
2	10	$0 \times \frac{1}{2} + 1 \times \frac{1}{4}$	0.25
3	11	$1 \times \frac{1}{2} + 1 \times \frac{1}{4}$	0.75
4	100	$0 \times \frac{1}{2} + 0 \times \frac{1}{4} + 1 \times \frac{1}{8}$	0.125
...	...	...	...

This has been called reflecting the numbers about the decimal point. If you plot the Halton points successively you will see that the next number in the sequence is always as far as possible from the previous point. Generally, the integer n can be written as

$$i = \sum_{j=1}^{m} a_j b^j$$

in base b, where $0 \le a_j < b$. The Halton numbers are then given by

$$h(i; b) = \sum_{j=1}^{m} a_j b^{-j-1}.$$

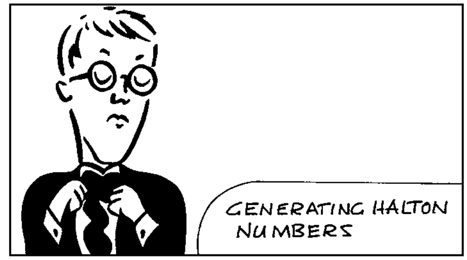
Here is an algorithm for calculating Halton numbers of arbitrary base; the `n`th term in a Halton sequence of base `b` is given by `Halton(n, b)`.

[2] There is actually nothing random about quasi-random sequences.

[3] Andy Morton says that this has been my best piece of work, knowing full well that the numerical analysts John Barrett and Gerald Moore should have all the credit.

[4] So we must map our integrand onto the unit hypercube.

```
Function Halton(n, b)
Dim n0, n1, r As Integer
Dim h As Double
Dim f As Double
    n0 = n
    h = 0
    f = 1 / b
While (n0 > 0)
    n1 = Int(n0 / b)
    r = n0 - n1 * b
    h = h + f * r
    f = f / b
    n0 = n1
Wend
Halton = h
End Function
```



The resulting sequence is nice because as we add more and more numbers, more and more 'dots,' we fill in the range zero to one at finer and finer levels.

In Figure 26.9 is the approximation to the Normal distribution using 500 points from a Halton sequence and the Box–Muller method. Compare this distribution with that in Figure 26.3.

When distributing numbers in two dimensions choose, for example, Halton sequences of bases 2 and 3 so that the integrand is calculated at the points $(h(i, 2), h(i, 3))$ for $i = 1, \ldots, N$. The bases in the two sequences should be prime numbers. The distribution of these points is shown in Figure 26.10, compare the distribution with that in the previous figure.

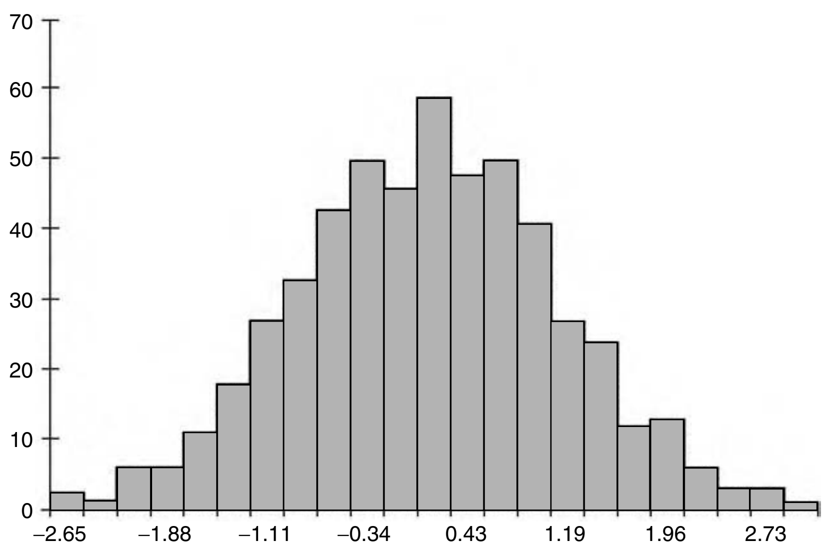


Figure 26.9 The approximation to the Normal distribution using 500 points from a Halton sequence and the Box–Muller method.

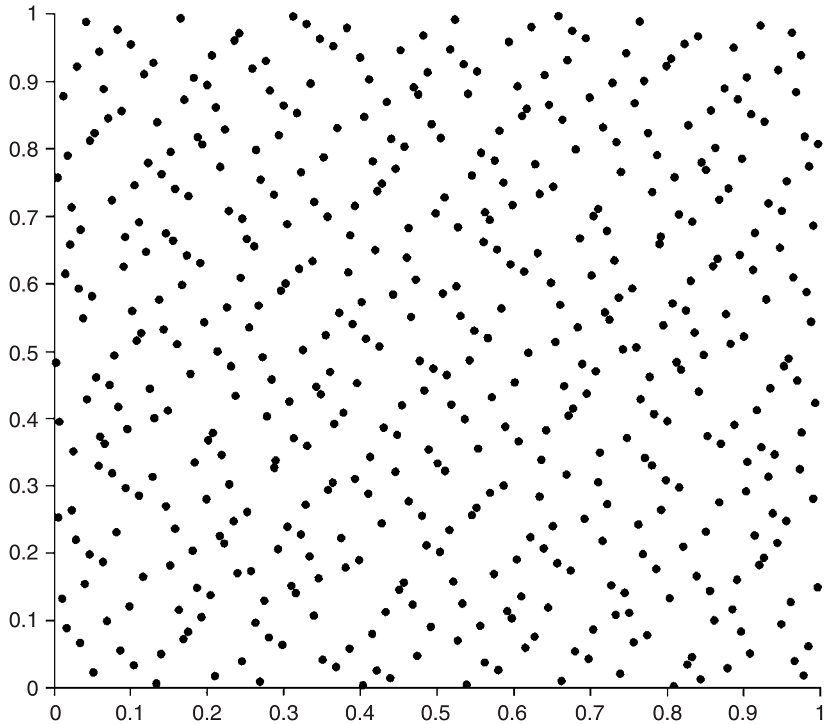


Figure 26.10 Halton points in two dimensions.

The estimate of the d-dimensional integral

$$\int_0^1 \cdots \int_0^1 f(x_1, \ldots, x_d)\, dx_1, \ldots, dx_d$$

is then

$$\frac{1}{N} \sum_{i=1}^{N} f(h(i, b_1), \ldots, h(i, b_n)),$$

where b_j are distinct prime numbers.

The error in these quasi-random methods is

$$O((\log N)^d N^{-1})$$

and is even better than Monte Carlo at all dimensions. The coefficient in the error depends on the particular low-discrepancy series being used. Sobol' is generally considered to be about the best sequence to use... but it's much harder to explain. See Press *et al*. (1995) for code to generate Sobol' points or download code from `www.netlib.org/toms`. In three or more dimensions the method beats the uniform grid. The time taken is $O(N)$.

The error is clearly sensitive to the number of dimensions d. To fully appreciate the inverse relationship to N can require an awful lot of points. However, even with fewer points, in practice the method at its worst has the same error as Monte Carlo.

In Figure 26.11 is shown the relative error in the estimate of value of a five-dimensional contract as a function of the number of points used. The inverse relationship with the Halton sequence is obvious.

26.17 ADVANCED TECHNIQUES

There are several sophisticated techniques that can be used to improve convergence of Monte Carlo and related numerical integration methods. They can be generally classified as techniques for the **reduction of variance**, and hence for the increase in accuracy. None of these methods improve the speed of convergence with respect to N (for example, the error remains $O(N^{-1/2})$ for Monte Carlo) but they can significantly reduce the coefficient in the error term.

The method of **antithetic variables** described above is *very* easily applied to numerical integration. It should always be used since it can do no harm and is completely independent of the product being valued.

Control variates can also be used in exactly the same way as described above. As with the pathwise simulation for pricing, the method depends on there being a good approximation to the product having an analytic formula.

The idea behind **importance sampling** is to change variables so that the resulting integrand is as close as possible to being constant. In the extreme case, when the integrand becomes exactly constant, the 'answer' is simply the volume of the region in the new variables. Usually, it is not possible to do so well. But the closer one gets to having a constant integrand, then the better the accuracy of the result. The method is rarely used in finance.

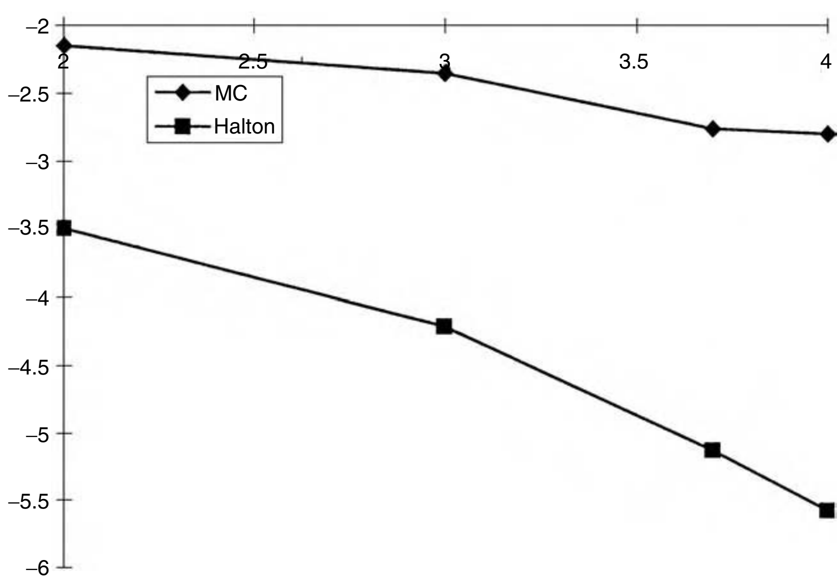


Figure 26.11 Estimate of the error in the value of a five-dimensional contract using basic Monte Carlo and a low-discrepancy sequence.

Stratified sampling involves dividing the region of integration into smaller subregions. The number of sampling points can then be allocated to the subregions in an optimal fashion, depending on the variance of the integral in each subregion. In more than one dimension it is not always obvious how to bisect the region, and can amount to laying down a grid, so defeating the purpose of Monte Carlo methods. The method can be improved upon by **recursive stratified sampling** in which a decision is made whether to bisect a region, based on the variance in the regions. Stratified sampling is rarely used in finance.

26.18 SUMMARY

Simulations are at the very heart of finance. With simulations you can explore the unknown future, and act accordingly. Simulations can also be used to price options, although the future is uncertain, the result of hedging an option is theoretically guaranteed.

In this chapter I have tried to give a flavor of the potential of Monte Carlo and related methods. The reader should now be in a position to begin to use these methods in practice. The subject is a large, and growing one, and the reader is referred to the section below for more information.

FURTHER READING

- See Boyle (1977) for the original application of Monte Carlo simulations to pricing derivatives.
- Duffie (1992) describes the important theory behind the validity of Monte Carlo simulations and also gives some clues about how to make the method efficient.
- The subject of Monte Carlo simulations is described straightforwardly and in detail by Vose (1997).
- For a review of Monte Carlo methods applied to American options see Boyle *et al*. (1995).
- See Sloan & Walsh (1990) and Stetson *et al*. (1995) for details of how to optimize a grid.
- See Barrett *et al*. (1992) for details of the Haselgrove method applied to options. And see Haselgrove (1961) for more details of the method in abstract.
- For a practical example of pricing mortgage-backed securities see Ninomiya & Tezuka (1996)
- For more financial examples see Paskov & Traub (1995), Paskov (1996) and Traub & Wozniakowski (1994).
- See Niederreiter (1992) for an in-depth discussion of low-discrepancy sequences.
- See the amazing Press *et al*. (1992) for samples of code for random number generation and numerical integration. Make sure you use the latest edition, the random number generators in the first edition are not so good. They also describe more advanced integration techniques.

APPENDIX A
a trading game

A.1 INTRODUCTION

A lot of people reading this book will never have traded options or even stocks. In this chapter I describe a very simple trading game so that a group of people can try out their skill without losing their shirts. The game is based on that by one of my ex-students David Epstein.

A.2 AIMS

The aims of this game are to familiarize students with the basic market-traded derivative contracts and to promote an understanding of the concepts involved in trading, such as bid, offer, arbitrage and liquidity.

A.3 OBJECT OF THE GAME

To make more money than your opponents. After the final round of trading, each player sums up their profits and losses. The player who has made the most profit is the winner.

A.4 RULES OF THE GAME

1. One person (possibly a lecturer) is the game organizer and in charge of choosing the types of contracts available for trading, the number and length of the trading rounds, judging any disputes and jollying the game along during slack periods.
2. The trading game takes place over a number of rounds. At the end of each round, a six-sided die is thrown. After the last round, the ‘share price’ is deemed to be the sum of all the die rolls.

3. Traded contracts may include some or all of forwards, calls and puts at the discretion of the organizer (Figure A.1). The organizer must also decide what exercise prices are available for call or put options.
4. All contracts expire at the end of the final round. The settlement value of each traded contract can then be determined by substituting the share price into the appropriate formula. A player's profits and losses on each trade can then be calculated and the resultant profit/loss is their final score.
5. During a round, players can offer to buy or sell any of the traded contracts. If another player chooses to take them up on their offer, then the deal is agreed and both parties must record the transaction on their trading sheet.
6. A deal on a contract must include the following information:
   - Forward: forward price and quantity
   - Call or put: type of option (call or put), exercise price, cost and quantity

   The organizer chooses the types of contracts available and the strike prices.
   The forward price or option cost and the quantity in a deal are chosen by the players.

For beginners, play three games in succession, with the following structures:

1. Play with just the forward contract.
2. Play with the forward contract and the call option with exercise price 15.
3. Play with the forward and the call and put options with exercise price 15.

All three games take place over five rounds, each five minutes in length.

A.5 NOTES

1. Depending on the level of prior knowledge of the players, the organizer may need to explain the characteristics of the various traded contracts. It will be instructive to emphasize that the forward contract has no cost initially.
2. There will probably be times when the organizer has to act as a 'market-maker' and promote trading, for instance, asking the group if anyone wants to buy shares or at what price someone is willing to do so.
3. For more advanced students, consider introducing some of the following ideas:
   - Increase the number of rounds
   - Decrease the length of each round
   - Include extra calls and puts with different exercise prices or which either come into existence or expire at different times. You must fix the details of these extra contracts in advance of the game.
   - Include other contracts, e.g. Asian options or barriers.
   - Include a second die for a second underlying share price.
4. Including futures with 'daily' marking to market can be tried, but slows down the game. Nevertheless, it does illustrate the importance of margin, especially if the students have a limit on how much 'in debt' they are allowed to become.

A.6 HOW TO FILL IN YOUR TRADING SHEET

A.6.1 During a trading round

In the 'Contract' column, fill in the specifications of the instrument that you have bought/sold. Specify the forward price or exercise price if applicable (e.g. if there is more than one contract of this type in the game).

In the 'Buy/sell' column, fill in whether you have bought or sold the contract and the quantity.

In the 'Cost per contract' column, fill in the cost of a single contract.

A.6.2 At the end of the game

In the 'Settlement value' column, fill in the value of a single contract with the final share price.

In the 'Profit/loss per contract' column, fill in the profit/loss for a single contract.

In the 'Total profit/loss' fill in the total profit/loss for the trade ($=$ profit/loss $\times$ quantity).

Example During a round, your transactions are:

Buy 10 call options, with exercise price 20, at a cost of 2 each. Sell 1 put option, with exercise price 15, at a cost of $1. Buy 5 forwards, with forward price 19.

Your trading sheet should be filled in as below.

Contract	Buy/sell	Cost per contract	Total cost	Settlement value	Profit/loss
Call 20	Buy 10	2			
Put 15	Sell 1	1			
Forward 19	Buy 5	—			

At the end of the game, the final share price is 21. Consequently, the trading sheet is completed as follows.

Contract	Buy/sell	Cost per contract	Total cost	Settlement value	Profit/loss
Call 20	Buy 10	2	$21 - 20 = 1$	$1 - 2 = -1$	$-1 \times 10 = -10$
Put 15	Sell 1	1	0	$1 - 0 = +1$	$+1 \times 1 = +1$
Forward 19	Buy 5	—	$21 - 19 = 2$	$+2$	$+2 \times 5 = +10$

The total profit and loss for the trader is therefore $-10 + 1 + 10 = +1$.

Remember that:

If you buy a contract, your profit/loss = settlement value – cost per contract

If you sell a contract, your profit/loss = cost per contract – settlement value

Trading sheet

Contract	Buy/sell	Cost per contract	Total cost	Settlement value	Profit/loss

The Trading Game — designed by David Epstein, 1999.

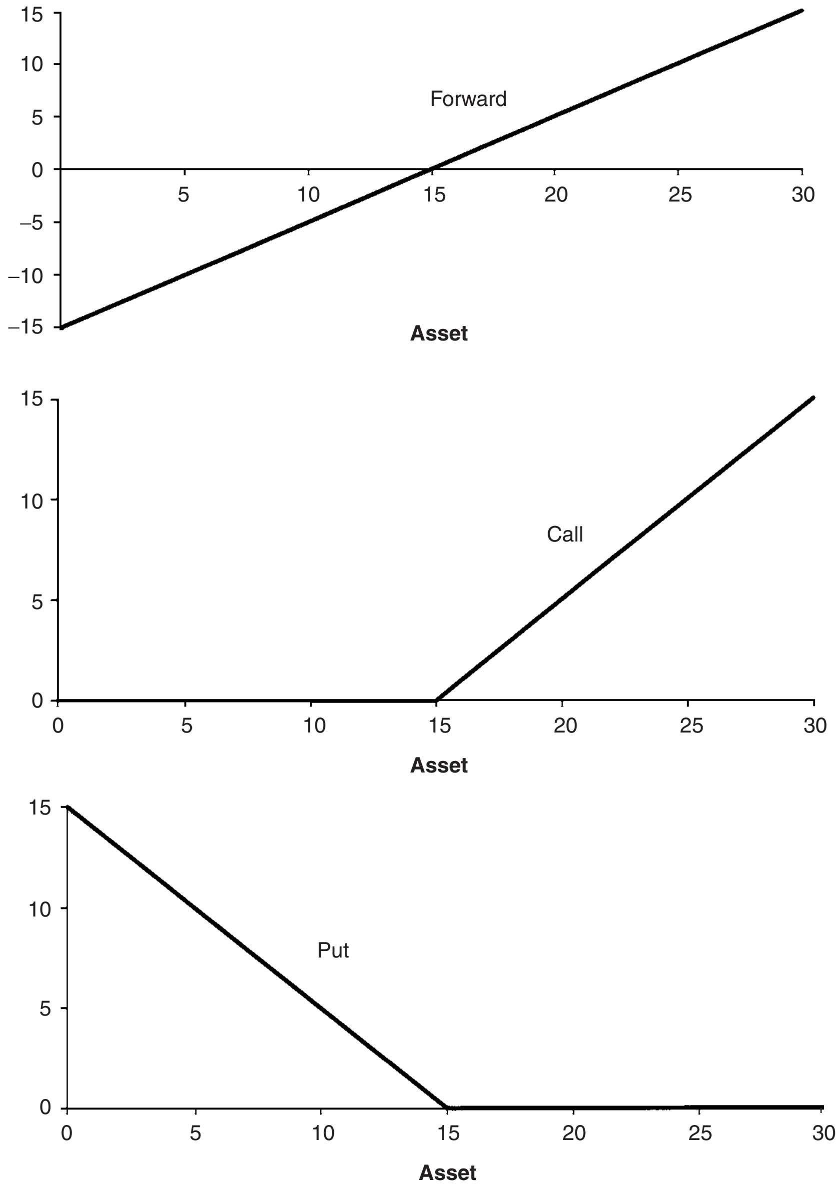


Figure A.1 Available contracts.

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