Chapter 2

Theory

In this chapter we will derive the shallow water equations (SWE) in conservative form, and present the equations in vector form for 1D and 2D.

2.1 Notation

Before deriving the shallow water equations (SWE), we will introduce the notation that will be used throughout this report. In both the 1D and 2D cases of SWE, we use cartesian coordinates (x, y, z) with time denoted by t. Given that linear algebra is a fundamental tool used in this report, we first establish the relevant notation. Lowercase bold letters represent vectors, while uppercase bold letters represent matrices. For instance, $\mathbf{a}$ is a vector of size $1 \times r, r \in \mathbb{R}$, and $\mathbf{A}$ is a matrix of size $m \times n, m, n \in \mathbb{R}$. The identity matrix, denoted by $\mathbf{I}$, is a square matrix with ones along the diagonal and zeros elsewhere. For example, the 3×3 identity matrix is given by:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We use the following notation for partial derivatives:

$$f_x = \frac{\partial f}{\partial x}, \quad f_y = \frac{\partial f}{\partial y}, \quad f_z = \frac{\partial f}{\partial z}.$$

The gradient operator, denoted by ∇, gives the gradient of a scalar function f as a vector:

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix}.$$

Given two vectors $\mathbf{a} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}^\top$ and $\mathbf{b} = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}^\top$, the dot product of $\mathbf{a}$ and $\mathbf{b}$ is given by:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

The dot product can also be written as a matrix product:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^\top \mathbf{b}.$$

The divergence operator, represented as $\nabla \cdot$, gives the divergence of a vector $\mathbf{a}$ as:

$$\nabla \cdot \mathbf{a} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z} = a_{1x} + a_{2y} + a_{3z}.$$

The tensor product of two vectors $\mathbf{a}$ and $\mathbf{b}$, denoted as $\mathbf{a} \otimes \mathbf{b}$, is a matrix where each element is the product of the elements of $\mathbf{a}$ and $\mathbf{b}$, i.e.,

$$\mathbf{a} \otimes \mathbf{b} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}.$$

2.2 The SWE with conservative variables

In this section we will derive the shallow water equations (SWE) in conservative form. The derivation follows four steps: First we consider the conservation laws for mass and momentum, and then we consider the boundary conditions for a free surface problem. Afterwards we make some neccessary assumptions and finally we use the boundary conditions to integrate the conservation laws over depth. The derivation follows the methods outlined in [1] and [2].

2.2.1 Conservation laws

The conservation laws for mass and momentum can be expressed generally as follows (see eq. (2.1) and (2.2) in [1]):

$$\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{2.2.1}$$

$$(\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v} + p\mathbf{I} - \mathbf{T}) = \rho \mathbf{g}, \tag{2.2.2}$$

where ρ is the fluid density, $\mathbf{v} = \begin{bmatrix} u & v & w \end{bmatrix}^\top$ is the fluid velocity in the x, y and $z-$direction respectively; p is the pressure, $\mathbf{I}$ is the identity matrix, and vector $\mathbf{g} = \begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix}^\top$ represents body forces including gravity. In these equations, the density ρ and the pressure p are dependent of x, y, z and t, but later we will introduce some assumptions that simplify the equations. The matrix $\mathbf{T}$ is the viscous stress tensor, given by

$$\mathbf{T} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}.$$

However, in this project the viscous stress tensor $\mathbf{T}$ is neglected, since we assume the function $\tau(x, y, z)$ is constant. The matrix $\mathbf{v} \otimes \mathbf{v}$ represents the tensor product of the velocity vector $\mathbf{v}$ with itself, i.e.,

$$\mathbf{v} \otimes \mathbf{v} = \begin{bmatrix} u^2 & uv & uw \\ vu & v^2 & vw \\ wu & wv & w^2 \end{bmatrix}.$$

Note that $\mathbf{v} \otimes \mathbf{v} = \mathbf{v}\mathbf{v}^\top$. Putting this together, we can rewrite (2.2.2) as

$$(\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v}\mathbf{v}^\top + p\mathbf{I}) - \rho \mathbf{g} = 0. \tag{2.2.3}$$

In this project we consider incompressible fluids, meaning that the fluid density ρ is independent of the pressure p. We also assume that the fluid density only depends on temperature and salinity, and thus is independent of t, x, y and z. Additionally, we assume $\rho \neq 0$. Using these assumptions and the product rule for differentiation, we can simplify the mass conservation equation (2.2.1) to

$$\rho_t + \rho(u_x + v_y + w_z) + u\rho_x + v\rho_y + w\rho_z = 0,$$

implying that

$$u_x + v_y + w_z = 0. \tag{2.2.4}$$

Applying the divergence operator $\nabla\cdot$, the momentum conservation equation (2.2.3) can be written out as:

$$\rho_t \mathbf{v} + \rho \mathbf{v}_t + \rho \begin{bmatrix} (u^2+p)_x + (uv)_y + (uw)_z \\ (vu)_x + (v^2+p)_y + (vw)_z \\ (wu)_x + (wv)_y + (w^2+p)_z \end{bmatrix} - \rho \mathbf{g} = 0. \tag{2.2.5}$$

We neglect all body forces in $\mathbf{g}$, expect the gravitational force in the $z-$direction, i.e., $\mathbf{g} = \begin{bmatrix} 0 & 0 & -g \end{bmatrix}$, where g is the gravity acceleration, and use the product rule in (2.2.5). Hence we obtain

$$\rho \begin{bmatrix} u_t \\ v_t \\ w_t \end{bmatrix} + \rho \begin{bmatrix} p_x + uu_x + vu_y + wu_z + u(u_x + v_y + w_z) \\ p_y + uv_x + vv_y + wv_z + v(u_x + v_y + w_z) \\ p_z + uw_x + vw_y + ww_z + w(u_x + v_y + w_z) \end{bmatrix} - \rho \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} = 0. \tag{2.2.6}$$

We apply (2.2.4) to (2.2.6) to remove some terms, we move the pressure terms to the right hand side, and we divide by ρ. Putting it all together, the mass equation (2.2.1) and the momentum equation (2.2.3), split in x, y and $z-$directions, simplify to

$$\left.\begin{aligned} u_x + v_y + w_z &= 0, \\ u_t + uu_x + vu_y + wu_z &= -\frac{1}{\rho} p_x, \\ v_t + uv_x + vv_y + wv_z &= -\frac{1}{\rho} p_y, \\ w_t + uw_x + vw_y + ww_z &= -\frac{1}{\rho} p_z - g. \end{aligned}\right\} \tag{2.2.7}$$

2.2.2 Boundary conditions

In this project, we consider the flow of water with a free surface. To solve the SWE, it is essential to impose boundary conditions at both the bottom of the water column and the free surface. We assumme the bottom boundary is defined by a function

$$z = b(x, y),$$

meaning that the bottom is dependent on x and y, but not on time. The free surface is defined by

$$z = s(x, y, t) \equiv b(x, y) + h(x, y, t),$$

where $h(x, y, t)$ is the water depth at time t. The following illustration helps to visualize the setup:

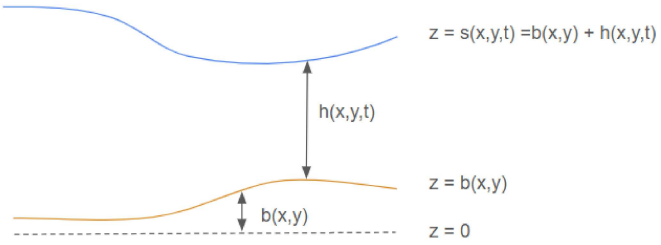

Figure 2.1: Illustration of a water column with a free surface.

We impose boundary conditions at the bottom and at the free surface, adressing both kinematic and dynamical conditions. To describe the boundaries mathematically, we introduce a boundary function $\psi(x, y, z, t)$ that is zero on a boundary:

$$\psi(x, y, z, t) = 0.$$

For the free surface, this boundary is given by

$$\psi(x, y, z, t) \equiv z - s(x, y, t) = 0, \tag{2.2.8}$$

and for the bottom, it is described by

$$\psi(x, y, z, t) \equiv z - b(x, y) = 0. \tag{2.2.9}$$

In the kinematic condition, we assume that fluid particles on the boundary remain on the boundary over time. Mathematically this is expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(x, y, z, t) = 0.$$

Recall, that $\frac{\partial\psi}{\partial t}$ is the partial derivative of ψ with respect to t, while the total derivate $\frac{\mathrm{d}\psi}{\mathrm{d}t}$ accounts for both the direct change of ψ with respect to t and the changes due to the movement of the fluid in the x, y and z directions. Hence, the total derivative of ψ wrt. t is given by

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{\partial\psi}{\partial t} + \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial\psi}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\partial\psi}{\partial y} + \frac{\mathrm{d}z}{\mathrm{d}t}\frac{\partial\psi}{\partial z}.$$

We see that $\frac{\mathrm{d}x}{\mathrm{d}t}$ denotes the velocity in the $x-$direction, i.e., u, and correspondingly for y and z. Thus, the kinematic condition is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(x, y, z, t) = \psi_t + u\psi_x + v\psi_y + w\psi_z = 0. \tag{2.2.10}$$

Applying this to the free surface by substituting (2.2.8) into the kinematic condition (2.2.10) yields

$$(s_t + us_x + vs_y - w)|_{z=s} = 0. \tag{2.2.11}$$

Similarly, for the bottom, substitung (2.2.9) into the kinematic condition (2.2.10) gives

$$(ub_x + vb_y - w)|_{z=b} = 0. \tag{2.2.12}$$

The dynamical condition is related to the pressure distribution at the free surface. We assume that the pressure at the free surface is equal to the pressure in the air above the surface, that is, the atmospheric pressure. Since absolute pressure levels are irrelevant (we are primarily concerned with pressure differences), we set the pressure at the free surface to zero. This leads to the following expression for the pressure at the free surface:

$$p(x, y, z, t)|_{z=s(x,y,t)} = 0. \tag{2.2.13}$$

This condition, known as the dynamical condition, relates to the forces acting on the boundaries of the fluid.

2.2.3 Assumptions

To derive the SWE it is neccessary to make some assumptions. The shallow water equations are an approximation to the full free-surface problem and result from the assumption that the vertical compononent of the acceleration is negligible. We begin by assuming that the vertical acceleration, represented by the total derivate of the vertical velocity component w with respect to time, is negligible. This assumption leads to the condition

$$\frac{\mathrm{d}w}{\mathrm{d}t} = w_t + uw_x + vw_y + ww_z = 0.$$

Applying $\frac{\mathrm{d}w}{\mathrm{d}t} = 0$ in the $z-$momentum conservation equation (2.2.7) simplifies it to

$$p_z = -\rho g.$$

By using properties of an integral, together with (2.2.13) we get

$$\int_{b(x,y)}^{z} -\rho g \ \mathrm{d}t + \int_{z}^{s(x,y,t)} -\rho g \ \mathrm{d}t = \int_{b(x,y)}^{s(x,y,t)} -\rho g \ \mathrm{d}t = p|_{z=s(x,y,t)} = 0.$$

This implies the pressure distribution follows

$$p = \int_{b(x,y)}^{z} -\rho g \ \mathrm{d} = \int_{z}^{s(x,y,t)} \rho g \ \mathrm{d}z = \rho g(s-z), \tag{2.2.14}$$

where s is the surface height. Differentiating (2.2.14) with respect to x and y yields

$$p_x = \rho g s_x, \quad p_y = \rho g s_y.$$

Substituting these expressions into the $x-$ and $y-$momentum conservation equations (2.2.7) leads to

$$u_t + uu_x + vu_y + wu_z = -gs_x, \tag{2.2.15}$$

and

$$v_t + uv_x + vv_y + wv_z = -gs_y. \tag{2.2.16}$$

These are the simplified momentum equations for the shallow water equations. We realize that p_x and p_y are both independent of z, implying that $\frac{\mathrm{d}\,u}{\mathrm{d}\,t}$ and $\frac{\mathrm{d}\,v}{\mathrm{d}\,t}$are also independent of z. Hence $u_z = v_z = 0$, implying that (2.2.15) and (2.2.16) can be simplified to

$$u_t + uu_x + vu_y = -gs_x, \tag{2.2.17}$$

and

$$v_t + uv_x + vv_y = -gs_y. \tag{2.2.18}$$

2.2.4 Integration over depth

The next step in deriving the SWE is to integrate the equations over the vertical direction. We integrate the mass conservation equation (2.2.4) and the x and $y-$momentum conservation equations in (2.2.7), from the bottom, $z = b(x,y)$ to the free surface, $z = s(x,y,t)$. Starting with the mass conservation equation (2.2.4), we have

$$\int_{b}^{s} u_x + v_y + w_z \ \mathrm{d}z = 0,$$

implying that, using linearity of the integral:

$$\int_{b}^{s} u_x \ \mathrm{d}z + \int_{b}^{s} v_y \ \mathrm{d}z + w|_{z=s} - w|_{z=b} = 0. \tag{2.2.19}$$

We will use Leibniz's integral rule [3], which is stated as follows:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{a(x)}^{b(x)} f(x,t) \ \mathrm{d}t = \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t) \ \mathrm{d}t + f(x,b(x))\frac{\mathrm{d}}{\mathrm{d}x}b(x) - f(x,a(x))\frac{\mathrm{d}}{\mathrm{d}x}a(x), \tag{2.2.20}$$

to integrate the first two terms in (2.2.19), which yields

$$\left.\begin{aligned} \int_{b}^{s} u_x \,\mathrm{d}z &= \frac{\mathrm{d}}{\mathrm{d}x}\int_{b}^{s} u \,\mathrm{d}z - u|_{z=s}\frac{\mathrm{d}s}{\mathrm{d}x} + u|_{z=b}\frac{\mathrm{d}b}{\mathrm{d}x}, \\ \int_{b}^{s} v_y \,\mathrm{d}z &= \frac{\mathrm{d}}{\mathrm{d}y}\int_{b}^{s} v \,\mathrm{d}z - v|_{z=s}\frac{\mathrm{d}s}{\mathrm{d}y} + v|_{z=b}\frac{\mathrm{d}b}{\mathrm{d}y}. \end{aligned}\right\} \tag{2.2.21}$$

Note that since a change in x does not affect the y-component of the bottom or surface, we have that $\frac{\mathrm{d}s}{\mathrm{d}x} = s_x$ and $\frac{\mathrm{d}b}{\mathrm{d}x} = b_x$, and correspondingly for s_y and b_y. Likewise we can substitute $\frac{\mathrm{d}}{\mathrm{dx}}$ with $\frac{\partial}{\partial x}$ in the integrals, since the integrals are with respect to z, and u and v are independent of z. Inserting these results in (2.2.21) gives

$$\left.\begin{aligned} \int_b^s u_x \,\mathrm{d}z &= \frac{\partial}{\partial x}\int_b^s u \,\mathrm{d}z - u|_{z=s}s_x + u|_{z=b}b_x, \\ \int_b^s v_y \,\mathrm{d}z &= \frac{\partial}{\partial y}\int_b^s v \,\mathrm{d}z - v|_{z=s}s_y + v|_{z=b}b_y. \end{aligned}\right\} \tag{2.2.22}$$

We can now insert the integrals (2.2.22) into the integrated mass conservation equation (2.2.19) to get

$$\frac{\partial}{\partial x}\int_b^s u \,\mathrm{d}z - u|_{z=s}s_x + u|_{z=b}b_x + \frac{\partial}{\partial y}\int_b^s v \,\mathrm{d}z - v|_{z=s}s_y + v|_{z=b}b_y + w|_{z=s} - w|_{z=b} = 0. \tag{2.2.23}$$

To simplify this equation further, we consider the boundary conditions. From (2.2.12) we have

$$w|_{z=b} = (ub_x + vb_y)|_{z=b}, \tag{2.2.24}$$

and from (2.2.11) we have

$$w|_{z=s} = (s_t + us_x + vs_y)|_{z=s}. \tag{2.2.25}$$

Realizing that $s = b + h$ and hence $s_t = h_t$, as the bottom is fixed. Recall that u and v are independent of z, and the water depth is $h = s - b$, meaning we have

$$\int_b^s u \,\mathrm{d}z = u(s-b) = hu, \quad \int_b^s v \,\mathrm{d}z = v(s-b) = hv.$$

Putting it all together (2.2.23) simplifies to

$$h_t + (hu)_x + (hv)_y = 0, \tag{2.2.26}$$

which is also the first equation in the SWE in conservative form. When integrating the momentum equations (2.2.17) and (2.2.18) over the vertical direction, we see that since the equations are independent of z, the resulting equations are simply

$$h(u_t + uu_x + vu_y + gs_x) = 0, \tag{2.2.27}$$

$$h(v_t + uv_x + vv_y + gs_y) = 0. \tag{2.2.28}$$

We multiply (2.2.26) with u and v respectively, and add the two equations to (2.2.27) and (2.2.28) correspondingly. Recall that $s = h + b$. By using the product rule for differentiation and collecting terms, we obtain the momentum equations in conservative form:

$$(hu)_t + (hu^2 + \frac{1}{2}gh^2)_x + (huv)_y = -ghb_x, \tag{2.2.29}$$

and

$$(hv)_t + (huv)_x + (hv^2 + \frac{1}{2}gh^2)_y = -ghb_y. \tag{2.2.30}$$

The three partial differential equations in (2.2.26), (2.2.29) and (2.2.30) are the shallow water equations in conservative form.

2.2.5 The SWE in vector form for 1D and 2D

The SWE can also be written in differential conservation law form as a vector equation

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x + \mathbf{G}(\mathbf{U})_y = \mathbf{S}(\mathbf{U}), \tag{2.2.31}$$

where

$$\mathbf{U} = \begin{bmatrix} h \\ hu \\ hv \end{bmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \end{bmatrix}, \quad \mathbf{G}(\mathbf{U}) = \begin{bmatrix} hv \\ huv \\ hv^2 + \frac{1}{2}gh^2 \end{bmatrix} \quad \text{and} \quad \mathbf{S}(\mathbf{U}) = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -ghb_x \\ -ghb_y \end{bmatrix}.$$

We call $\mathbf{U}$ the vector of conserved variables, $\mathbf{F}(\mathbf{U})$ and $\mathbf{G}(\mathbf{U})$ the flux vectors in the x and y direction, and $\mathbf{S}(\mathbf{U})$ the source term vector.

Homogeneous 1D case:
In this project, we will begin by considering the homogenneous one-dimensional case of the shallow water equations. In the homogeneous case we assume that $\mathbf{S}(\mathbf{U}) = 0$. The vector form of the SWE in 1D is then given by

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0,$$

where

$$\mathbf{U} = \begin{bmatrix} h \\ hu \end{bmatrix}, \quad \text{and} \quad \mathbf{F}(\mathbf{U}) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix}.$$

In the 1D case, we only consider flow in the x-direction.

Inhomogeneous 1D case:
The inhomoegeneous one-dimensional case of the shallow water equations in vector form is given by

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{S}(\mathbf{U}), \tag{2.2.32}$$

where

$$\mathbf{U} = \begin{bmatrix} h \\ hu \end{bmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix} \quad \text{and} \quad \mathbf{S}(\mathbf{U}) = \begin{bmatrix} 0 \\ -ghb_x \end{bmatrix}. \tag{2.2.33}$$

If $\mathbf{S}(\mathbf{U}) = 0$, the equation (2.2.32) is called a conservation law, and otherwise it is called a balance law.

2.2.6 The SWE in integral form

Often it is more convenient to work with the integral form of the SWE, since the integral form of equations of the form (2.2.32) and (2.2.33) allows discountinuous solutions. An integral form of (2.2.32) is stated in [4]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_L}^{x_R} \mathbf{U}(x,t)\ \mathrm{d}x = \mathbf{F}(\mathbf{U}(x_L,t)) - \mathbf{F}(\mathbf{U}(x_R,t)) + \int_{x_L}^{x^R} \mathbf{S}(\mathbf{U})(x,t)\ \mathrm{d}x, \tag{2.2.34}$$

when integrated over the control volume V in the $x-t$ plane

$$V = [x_L, x_R] \times [t_1, t_2].$$

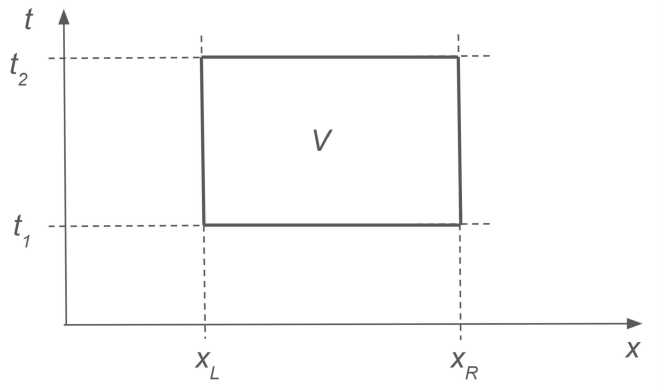

Figure 2.2: Illustration of a control volume V in the $x-t$ plane. Illustration modified from [4].

The integral form of the SWE in 1D is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2}\mathbf{U}(x,t)\,\mathrm{d}x = \mathbf{F}(\mathbf{U}(x_1,t)) - \mathbf{F}(\mathbf{U}(x_2,t)), \tag{2.2.35}$$

meaning that the rate of change of the integral over a domain is equal to the flux through the boundaries of the domain.

The integral form is obtained by integrating the vector form (2.2.32) over a control volume V in the $x-t$ plane. First we integrate over x from x_L to x_R to obtain

$$\int_{x_1}^{x_2}\mathbf{U}_t\,\mathrm{d}x + \int_{x_1}^{x_2}\mathbf{F}(\mathbf{U})_x\,\mathrm{d}x = \int_{x_1}^{x_2}\mathbf{S}(\mathbf{U})\,\mathrm{d}x. \tag{2.2.36}$$

Using the fundamental theorem of calculus, we get that

$$\int_{x_L}^{x_R}\mathbf{F}(\mathbf{U})\,\mathrm{d}x = \mathbf{F}(\mathbf{U}(x_R,t)) - \mathbf{F}(\mathbf{U}(x_L,t)),$$

which we insert in (2.2.36):

$$\int_{x_1}^{x_2}\mathbf{U}_t\,\mathrm{d}x = \mathbf{F}(\mathbf{U}(x_L,t)) - \mathbf{F}(\mathbf{U}(x_R,t)) + \int_{x_1}^{x_2}\mathbf{S}(\mathbf{U})\,\mathrm{d}x. \tag{2.2.37}$$

Then we integrate over time from t_1 to t_2 to get

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2}\mathbf{U}_t\,\mathrm{d}x\mathrm{d}t = \int_{t_1}^{t_2}\mathbf{F}(\mathbf{U}(x_L,t))\,\mathrm{d}t - \int_{t_1}^{t_2}\mathbf{F}(\mathbf{U}(x_R,t))\,\mathrm{d}t + \int_{t_1}^{t_2}\int_{x_1}^{x_2}\mathbf{S}(\mathbf{U})\,\mathrm{d}x\mathrm{d}t.$$

Rewriting the left hand side using the fundamental theorem of calculus, we get

$$\int_{x_L}^{x_R}\mathbf{U}(x,t_2)\,\mathrm{d}x = \int_{x_L}^{x_R}\mathbf{U}(x,t_1)\,\mathrm{d}x + \int_{t_1}^{t_2}\mathbf{F}(\mathbf{U}(x_L,t))\,\mathrm{d}t - \int_{t_1}^{t_2}\mathbf{F}(\mathbf{U}(x_R,t))\,\mathrm{d}t + \int_{t_1}^{t_2}\int_{x_1}^{x_2}\mathbf{S}(\mathbf{U})\,\mathrm{d}x\mathrm{d}t. \tag{2.2.38}$$

2.2.7 SWE in Spherical Coordinates

TBD.

2.3 Finite Volume Method for the Shallow Water Equations

In this section we will describe the Finite Volume Method (FVM) for solving the shallow water equations (SWE). In the FVM, we discretize the domain into finite control volumes

$$V_i = [x_{i-1/2}, x_{i+1/2}] \times [t_n, t_{n+1}],$$

where $\Delta x = x_{i+1/2} - x_{i-1/2}$ is the length of the cell and $\Delta t = t_{n+1} - t_n$ is the time step.

For a finite volume V_i^n, we average the integral form (2.2.38) over the volume to get the explicit conservative formula:

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{i+1/2}^n - \mathbf{F}_{i-1/2}^n\right) + \Delta t \mathbf{S}_i, \tag{2.3.1}$$

where $\mathbf{U}_i^n$ approximates the cell average over the i-th cell at time t_n:

$$\mathbf{U}_i^n \approx \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{U}(x, t_n)\, \mathrm{d}x, \tag{2.3.2}$$

where $\Delta x = x_{i+1/2} - x_{i-1/2}$ is the length of the cell. The value $\mathbf{F}_{i-1/2}^n$ approximates the average flux along the line $x = x_{i-1/2}$ at time t_n:

$$\mathbf{F}_{i-1/2}^n \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \mathbf{F}(\mathbf{U}(x_{i-1/2}, t))\, \mathrm{d}t,$$

and the source term $\mathbf{S}_i$ approximates the average source term over the i-th cell at time t_n:

$$\mathbf{S}_i = \frac{1}{\Delta t \Delta x}\int_{t_n}^{t_{n+1}}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{S}(x, t)\, \mathrm{d}x\mathrm{d}t.$$

In the FVM, we discretize the domain into cells or control volumes. Then we solve the local Riemann problem at the cell interface to obtain the fluxes. Using the computed fluxes, we update the solution in each cell. This way, the FVM allows for discountinuous solutions, as we solve the Riemann problem at the cell interfaces. Therefore it is well suited for hyperbolic conservation laws, such as the shallow water equations.

2.3.1 Finite Volume Methods for the 1D SWE

We begin by considering finite volume methods for the SWE in one space dimension. Recall that the homogeneous 1D SWE are given by

$$\begin{aligned} h_t + (hu)_x &= 0, \\ (hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x &= 0. \end{aligned}$$

A finite volume method for the 1D SWW is based on dividing the spatial domain into a set of intervals, also called grid cells. The grid is illustrated in Figure 2.3.

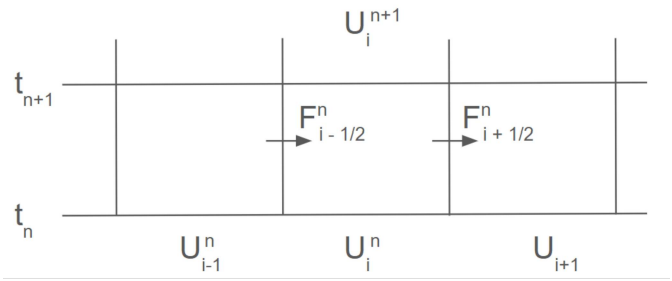

Figure 2.3: Illustration of the grid for the 1D SWE.

This approach (2.3.1) is called a finite volume scheme, since it is based on the integral conservation over finite volumes. Note that the main difference between the FVM and the FDM is that the FVM is based on the integral conservation over finite volumes, whereas the FDM is based on the differential conservation over finite differences. The main idea in the FVM is to define the numerical flux $\mathbf{F}^n_{i+1/2}$ at the cell inerface as a function of the cell averages $\mathbf{U}^n_i$ and $\mathbf{U}^n_{i+1}$. Since only the cell-averages solution is known. This also means that the FVM does not provide pointwise values of the solution, i.e., $\mathbf{U}(x,t)$, but only the cell-averages $\mathbf{U}^n_i$ over the control volume.

Throughout this report, we will assume a uniform grid for simplicity. A key problem is to determine good numerical flux functions, that based on the cell averages, which is our only data, approximates the fluxes reasonable well.

2.3.2 Finite Volume Method for the 1D SWE with source term

Consider the inhomoegeneous non-linear system

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{S}(\mathbf{U}),$$

where $\mathbf{S}(\mathbf{U})$ is a vector of sources.

2.3.3 Finite Volume Method for the 2D SWE

Follows the methods outlined in [5]. Consider a time-dependent two dimensional system of conservation laws

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x + \mathbf{G}(\mathbf{U})_y = 0. \tag{2.3.3}$$

A numerical explicit finite volume scheme to solve (2.3.3) is given by

$$\mathbf{U}^{n+1}_{i,j} = \mathbf{U}^n_{i,j} + \frac{\Delta t}{\Delta x}(\mathbf{F}_{i-1/2,j} - \mathbf{F}_{i+1/2,j}) + \frac{\Delta t}{\Delta y}(\mathbf{G}_{i,j-1/2} - \mathbf{G}_{i,j+1/2}). \tag{2.3.4}$$

This is the unsplit finite volume method, meaning that, in a single step, the cell average $\mathbf{U}^n_{i,j}+$ is updated using the fluxes from all intercell boundaries.

Chapter 3

Methodology

In this chapter we will describe the Finite Volume Method (FVM) for solving the shallow water equations (SWE) and the numerical implementation of the method in Python. That is, we consider the FVM for nonlinear hyperbolic conservation laws, such as the shallow water equations. The nonlinear case is obviously more challenging than the linear case, as stability and convergence theory are more difficult. We are interested in discountinuous solutions, which can capture shock waves and other discontinuities in the solution. Finite volume methods are closely related to finite difference methods, but they differs as they are based on the integral form of the conservation laws. Where finite difference methods tend to break down near discontinuities in the solution, finite volume methods are more suited, since they are based on the integral form of the conservation laws.

In finite volume methods, we discretize the domain into cells or control volumes. Then we solve the local Riemann problem at the cell interface to obtain the fluxes. Using the computed fluxes, we update the solution in each cell. This way, the FVM allows for discountinuous solutions, as we solve the Riemann problem at the cell interfaces. Therefore it is well suited for hyperbolic conservation laws, such as the shallow water equations. We will first define the Riemann problem, since it plays a crucial role in the method. The methods described in this chapter are based on the book by LeVeque [6].

3.1 Riemann problem

In the Riemann problem we distinguish between what we call a wet bed and a dry bed. A wet bed is the case where the water depth is positive everywhere, whereas a dry bed is the case where the water depth is zero in some cells. The special Riemann problem where parts of the bed are dry is dealing with the so-called dry fronts or wet/dry fronts, which are challenging to handle numerically. We will leave these cases for now, and only consider the wet bed problems.

The Riemann problem is defined as the initial-value problem (IVP)

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0, \tag{3.1.1}$$

$$\mathbf{U}(x,0) = \begin{cases} \mathbf{U_L}, & x < 0, \\ \mathbf{U_R}, & x > 0. \end{cases} \tag{3.1.2}$$

We consider the x-split, two-dimensional shallow water equations with the initial states:

$$\mathbf{U_L} = \begin{bmatrix} h_L \\ h_L u_L \\ h_L v_L \end{bmatrix}, \quad \mathbf{U_R} = \begin{bmatrix} h_R \\ h_R u_R \\ h_R v_R \end{bmatrix},$$

which represents the conditions at time $t = 0$ in the left and right states of $x = 0$, respectively. The function $\mathbf{U}$ is piecewise constant, with a discontinuity at $x = 0$. The Riemann problem is a generalisation of the so-called dam-break problem. The difference is that in the Riemann problem the particle velocity components, u_L, u_R, v_L and v_R, are allowed to be distinct from zero, whereas in the dam-break problem, they must be zero.

The Riemann problem can be solved both exactly and approximately. There are several approximate Riemann solvers, such as the HLL and Roe solvers, which are based on the approximate solution of the Riemann problem. We will consider some of these solvers later in the thesis.

3.1.1 Waves in the Riemann problem

To get a better understanding of the flow in shallow water, we provide some very short background information about waves. In particular the wave structure in the solution of the Riemann problem (3.1.1).

In the solution of the Riemann problem (3.1.1) there are four possible wave patterns outcomes, which are combinations of shock waves and rarefaction waves. In each case there are three waves, the left and right waves correspond to the one-dimensional SWE, and the middle wave aries from the $y-$momentum equation in (3.1.1) and is always a shear wave. The left and right waves are either shock waves or rarefaction waves. The four possible wave patterns are illustrated in Figure XX and are as follows:

1. Left rarefaction, right shock
2. Left shock, right rarefaction
3. Both left and right rarefaction
4. Both left and right shock

Hence the structure of the solution in general is shown in the figure below. From the figure we see that the solution consists of three waves, a left wave, a middle wave and a right wave, which together seperate four regions, described by the vector

$$\mathbf{W} = \begin{bmatrix} h \\ hu \\ hv \end{bmatrix}.$$

The four regions are describes by $\mathbf{W}_L$ (left data), $\mathbf{W}_R$ (right data), $\mathbf{W}_{*L}$ and $\mathbf{W}_{*R}$, which both denote star region data. We are interested in the star region data, since these are the unknowns. We know that the left and right waves are always either a shock wave or a rarefaction wave, and the middle wave is always a shear wave. Based on the given initial conditions, we must determine the types of waves. Second, it is known that across the left and right waves, both h and u change but v remains constant. Whereas across the middle wave, v changes but h and u remain constant. Thus the water depth and particle velocity are constant in the star region and are denoted by h_* and u_*, respectively.

3.1.2 Exact Riemann solver

The exact Riemann solver presented in this section is very efficient and leads to Gudonov methods, that are only slightly more expensive than those based on approximate Riemann solvers [1]. Important to find the exact solution to the Riemann problem in the early stage of development, before moving on to more complex applications.

In the solution of the Riemann problem (3.1.1), there are four possible wave patterns outcomes, which are combinations of shock waves and rarefaction waves. In each case there are three waves, the left and right waves correspond to the one-dimensional SWE, and the middle wave aries from the $y-$momentum equation in (3.1.1). The left and right waves are either shock waves or rarefaction waves, and the middle wave is always a shear wave.

But for now, we will consider the case where the solution consists of a single non-trivial wave and all other waves are assumed to have zero strenght. This is enough to solve the Riemann problem as it is always possible to solve the Riemann problem by considering one wave at a time.

We denote the constant values of the water depth and particle velocity in the star region by h_* and u_*, respectively.

3.1.3 Approximate Riemann solvers

We will study two approximate Riemann solvers, that is, the HLL and Roe.

HLL solver

The HLL (Harten, Lax and van Leer) approach assumes a two-wave structure of the Riemann problem, see Figure. The solver is based on the data $\mathbf{U}_L \equiv \mathbf{U}_i^n, \mathbf{U}_R \equiv \mathbf{U}_{i+1}^n$ and fluxes $\mathbf{F}_L \equiv \mathbf{F}(\mathbf{U}_L), \mathbf{F}_R \equiv \mathbf{F}(\mathbf{U}_R)$.

The HLL flux is given by

$$\mathbf{F}_{1+\frac{1}{2}} = \begin{cases} \mathbf{F}_L & \text{if } S_L \geq 0, \\ \mathbf{F}^{hll} \equiv \frac{S_R\mathbf{F}_L - S_L\mathbf{F}_R + S_L S_R(\mathbf{U}_R - \mathbf{U}_L)}{S_R - S_L} & \text{if } S_L \leq 0 \leq S_R, \\ \mathbf{F}_R & \text{if } S_R \leq 0. \end{cases}$$

The wave speeds S_L and S_R must be estimated in some way, and one possibility is to use

$$S_L = u_L - a_L q_L, \quad S_R = u_R + a_R q_R.$$

Roe solver

Consider the non-linear Riemann problem in (3.1.1):

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x \equiv \mathbf{U}_t + \mathbf{A}\mathbf{U}_x = 0,$$

where $\mathbf{A}$ is the Jacobian matrix of $\mathbf{F}$. The Roe solver is based on an approximation of the Jacobian matrix $\mathbf{A}$ by a Roe matrix $\tilde{\mathbf{A}}$, which is a constant coefficient matrix, to obtain the linear system

$$\mathbf{U}_t + \tilde{\mathbf{A}}\mathbf{U}_x = 0.$$

This means that the Roe solver solves the approximated Riemann problem

$$\mathbf{U}_t + \tilde{\mathbf{A}}\mathbf{U}_x = 0.$$
$$\mathbb{U}(x,0) = \begin{cases} \mathbf{U}_L, & x < 0, \\ \mathbf{U}_R, & x > 0. \end{cases}$$

exact. That is, the original non-linear conservation laws are replaced by a linearised system with constant coefficients.

The main idea in the Roe solver is to find average values $\tilde{h}, \tilde{a}, \tilde{u}$ and $\tilde{\psi}$ for the depth h, the celerity a (??), the velocity component u and the scalar ψ. The method thus use the following Roe averages:

$$\begin{cases} \tilde{h} = \sqrt{h_L h_R}, \\ \tilde{u} = \dfrac{u_L\sqrt{h_L} + u_R\sqrt{h_R}}{\sqrt{h_L} + \sqrt{h_R}}, \\ \tilde{a} = \sqrt{\dfrac{1}{2}(a_L^2 + a_R^2)}, \\ \tilde{\psi} = \dfrac{\psi_L\sqrt{h_L} + \psi_R\sqrt{h_R}}{\sqrt{h_L} + \sqrt{h_R}}. \end{cases} \tag{3.1.3}$$

The average eigenvalues (of what?) are

$$\tilde{\lambda}_1 = \tilde{u} - \tilde{a}, \quad \tilde{\lambda}_2 = \tilde{u}, \quad \tilde{\lambda}_3 = \tilde{u} + \tilde{a},$$

with the correspinding right eigenvectors

$$\tilde{\mathbf{R}}^{(1)} = \begin{bmatrix} 1 \\ \tilde{u} - \tilde{a} \\ \tilde{\psi} \end{bmatrix}, \quad \tilde{\mathbf{R}}^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \tilde{\mathbf{R}}^{(3)} = \begin{bmatrix} 1 \\ \tilde{u} + \tilde{a} \\ \tilde{\psi} \end{bmatrix}.$$

The wave strenghts $\tilde{\alpha}_j$ described by Roe averages are given by

$$\begin{cases} \tilde{\alpha}_1 = \dfrac{1}{2}\left[\Delta h - \dfrac{\tilde{h}}{\tilde{a}}\Delta u\right], \\ \tilde{\alpha}_2 = \dfrac{1}{2}\left[\tilde{h}\Delta\psi\right], \\ \tilde{\alpha}_3 = \dfrac{1}{2}\left[\Delta h + \dfrac{\tilde{h}}{\tilde{a}}\Delta u\right]. \end{cases} \tag{3.1.4}$$

Applying theory of linear systems with constant coefficients. The numerical flux is

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{2}\left(\mathbf{F}_L + \mathbf{F}_R\right) - \frac{1}{2}\sum_{j=1}^{3}\tilde{\alpha}_j\left|\tilde{\lambda}_j\right|\tilde{\mathbf{R}}^{(j)}. \tag{3.1.5}$$

The Roe flux (3.1.5) is used in the explicit conservative scheme to solve the SWE in 1D.

Entropy fix?

3.1.4 Implementation of the FVM in 1D

The code to solve the SWE in 1D using FVM is based on the Godunov scheme with the exact Riemann solver. The exact solution of the Riemann problem is found by using the Riemann invariants and the Rankine-Hugoniot conditions [7].

The true solution is found by solving the Riemann problem exact, with 5000 cells, and distinguishing between the wetbed or drybed case, and also identifying the shock and rarefaction waves.

3.1.5 Godunov's Method

We consider the Godunov Upwind method, which is a first-order accurate metod to solve non-linear systems of hyperbolic conservation laws [4]. Godunov's method is a fundamental starting point. In the method we solve the non-linear Riemann problem at each cell interface.

Consider the initial-boundary value problem (IBVP) for a system of N nonlinear hyperbolic conservation (balance?) laws

$$\begin{cases} \text{PDEs:} & \mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{S}(\mathbf{U}), \quad x \in [a,b], \quad t > 0, \\ \text{ICs:} & \mathbf{U}(x,0) = \mathbf{U}^{(0)}(x), \quad x \in [a,b], \\ \text{BCs:} & \mathbf{U}(a,t) = \mathbf{B}_L(t), \quad \mathbf{U}(b,t) = \mathbf{B}_R(t), \quad t \geq 0. \end{cases} \tag{3.1.6}$$

The vectors $\mathbf{B}_L(t)$ and $\mathbf{B}_R(t)$ denote the boundary conditions at the left and right boundaries, respectively. The Godunov Upwind method in conservative form (2.3.1) solves the IBVP (3.1.6).

3.1.6 Implementation of the FVM in 2D

3.1.7 Data generation

The data generation is done by solving the SWE in 1D using the FVM with the Godunov scheme and the exact Riemann solver. We use Gaussian functions with parametric extension [8] to generate the initial conditions, that is, functions on the form

$$h(x,0) = a \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right),$$

where a is the amplitude of the Gaussian, μ is the mean value, and σ is the standard deviation. We solve the 1D SWE with the following parameters:

- $N = 200$ cells,
- From $t = 0.0$ to $t_{\text{end}} = 1.0$,
- $x \in [0, 1]$,
- $u(x, 0) = 0$,
- $b(x) = 0.0$,
- $g = 9.81$ and
- $\sigma = 0.1$.

The value of μ is varied to generate different initial conditions, as seen in Figure 3.1.