additional constraints are imposed on the numerical flux or reconstruction process, often through the use of slope limiters. A limiter function is a technique used to reduce oscillations in the solution, especially near sharp edges or discontinuities, such as shock waves. For this project, we use the minmod limiter, defined as

$$\text{minmod}(a,b) = \begin{cases} \max(0, \min(a,b)) & \text{if } a > 0 \\ \min(0, \max(a,b)) & \text{if } a < 0. \end{cases}$$

Another method we consider is the MUSCL scheme. This method is second-order accurate in both time and space, while the WAF scheme is second-order accurate in space but relies on the time-stepping scheme for temporal accuracy. The MUSCL scheme, combined with slope limiters, ensures the solution satisfies the TVD constraint, effectively preventing oscillations that could occur with high-order spatial discretization near shocks, discontinuities, or sharp changes in the solution. The key idea in MUSCL is that, instead of using cell averages as in the 1D FVM, we reconstruct the solution in each cell using piecewise linear functions. This reconstruction provides a higher-order representation of the solution within each cell. In this project, we use the MUSCL scheme with the TVD criteria to ensure smooth and stable solutions.

3.3 The Finite Volume Method for the spherical 1D Linearized Shallow Water Equations

In this section, we derive the FVM for the 1D LSWE on a sphere, by following the same method as in section 3.1. We begin by stating the integral form of the 1D LSWE in spherical coordinates on a global domain, meaning we now consider a control volume given by

$$V_i^n = [\theta_{i-1/2}, \theta_{i+1/2}] \times [t_n, t_{n+1}],$$

where $\Delta\theta = \theta_{i+1/2} - \theta_{i-1/2}$ is the length of the cell and $\Delta t = t_{n+1} - t_n$ is the time step size. We rewrite the previous derived integral form (2.6.10) to be in global variables:

$$\int_{\theta_{i-1/2}}^{\theta_{i+1/2}} \mathbf{W}(\theta, t_{n+1}) \, \mathrm{d}\theta = \int_{\theta_{i-1/2}}^{\theta_{i+1/2}} \mathbf{W}(\theta, t_n) \, \mathrm{d}\theta - \mathbf{A} \left(\int_{t_n}^{t_{n+1}} \mathbf{W}(\theta_{i+1/2}, t) \, \mathrm{d}t - \int_{t_n}^{t_{n+1}} \mathbf{W}(\theta_{i-1/2}, t) \, \mathrm{d}t \right). \tag{3.3.1}$$

We divide the integral form (3.3.1) with the cell length $\Delta\theta$ to obtain

$$\begin{aligned} \frac{1}{\Delta\theta} \int_{\theta_{i-1/2}}^{\theta_{i+1/2}} \mathbf{W}(\theta, t_{n+1}) \, \mathrm{d}\theta = \frac{1}{\Delta\theta} \int_{\theta_{i-1/2}}^{\theta_{i+1/2}} \mathbf{W}(\theta, t_n) \, \mathrm{d}\theta \\ - \frac{\Delta t}{\Delta\theta} \mathbf{A} \left(\frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \mathbf{W}(\theta_{i-1/2}, t) \, \mathrm{d}t - \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \mathbf{W}(\theta_{i+1/2}, t) \, \mathrm{d}t \right). \end{aligned}$$

Averaging over the terms for a finite volume gives the first-order explicit time-stepping finite volume scheme:

$$\mathbf{W}_i^{n+1} = \mathbf{W}_i^n - \frac{\Delta t}{\Delta\theta} (\mathbf{F}_{i+1/2}^n - \mathbf{F}_{i-1/2}^n). \tag{3.3.2}$$

The scheme uses the cell averages:

$$\mathbf{W}_i^n = \frac{1}{\Delta\theta} \int_{\theta_{i-1/2}}^{\theta_{i+1/2}} \mathbf{W}(\theta, t_n) \, \mathrm{d}\theta$$

The flux $\mathbf{F}_{i-1/2}^n$ is the average flux across the line $\theta = \theta_{i-1/2}$ from time t_n to t_{n+1}:

$$\mathbf{F}_{i-1/2}^n = \frac{1}{\Delta t} \mathbf{A} \int_{t_n}^{t_{n+1}} (\mathbf{W}(\theta_{i-1/2}, t)) \, \mathrm{d}t,$$

and correspondingly the flux $\mathbf{F}_{i+1/2}^n$ is the average flux across the line $\theta = \theta_{i+1/2}$ from time t_n to t_{n+1}:

$$\mathbf{F}_{i+1/2}^n = \frac{1}{\Delta t} \mathbf{A} \int_{t_n}^{t_{n+1}} (\mathbf{W}(\theta_{i+1/2}, t)) \, \mathrm{d}t.$$

The Explicit Runge-Kutta 4th Order Method

We will now introduce how we can use the explicit Runge-Kutta 4th order (ERK4) method to solve the 1D spherical LSWE. The ERK4 method solves the general initial value problem (IVP) for ordinary differential equations (ODEs):

$$\frac{dy(t)}{dt} = f(y(t), t), \quad y(t_0) = y_0. \tag{3.3.3}$$

To solve the IVP (3.3.3) using ERK4 we choose a step size $\Delta t > 0$ and define the update formula:

$$\begin{aligned} y_{n+1} &= y_n + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4), \\ t_{n+1} &= t_n + \Delta t, \quad n = 0, 1, \ldots, N-1, \end{aligned}$$

where the stages k_1, k_2, k_3 and k_4 are given by [31]:

$$\begin{aligned} k_1 &= f(t_n, y_n), \\ k_2 &= f(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta t}{2}k_1), \\ k_3 &= f(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta t}{2}k_2), \\ k_4 &= f(t_n + \Delta t, y_n + \Delta t k_3). \end{aligned}$$

This means that to use ERK4 to solve the 1D spherical LSWE, we must bring the equations in the same form as the IVP (3.3.3). Recall, that the 1D LSWE in spherical coordinates are given by

$$\left.\begin{aligned} \frac{\partial h'}{\partial t} &= -\frac{h_0}{r\cos\phi}\frac{\partial u}{\partial \theta}, \\ \frac{\partial u}{\partial t} &= -\frac{g}{r\cos\phi}\frac{\partial h'}{\partial \theta}, \end{aligned}\right\} \tag{3.3.6}$$

where h' is the perturbation in water height, u is the velocity in the $\theta-$direction, h_0 is the mean water depth, g is the gravitational acceleration, r is the Earth's radius, and ϕ is the fixed latitude. Since the ERK4 method solves ODEs, and the LSWE are PDEs, we must use the method of lines (MOL) to bring the LSWE in the form of the IVP (3.3.3). The MOL is a technique for solving PDEs by discretizing all variables but one (usually time), transforming the PDE into a system of ODEs. The system of ODEs can then be solved using standard ODE solvers, such as the ERK4 method. We discretize the spatial dimension θ into N cells, and approximate the spatial derivatives using central differences:

$$\left.\begin{aligned} \frac{\partial u}{\partial \theta} &\approx \frac{u_{i+1} - u_{i-1}}{2\Delta\theta}, \\ \frac{\partial h'}{\partial \theta} &\approx \frac{h'_{i+1} - h'_{i-1}}{2\Delta\theta}, \end{aligned}\right\} \tag{3.3.7}$$

for $i = 0, 1, \ldots, N-1$. To perform the time integration, we use the ERK4 method. This method is particularly effective for time-stepping and is an alternative approach to traditional time integration schemes. The ERK4 method provides higher accuracy in time while relying on the same spatial discretization, such as the FVM. Substituing the spatial derivatives (3.3.7) into the LSWE (3.3.6) gives the system of ODEs:

$$\begin{aligned} \frac{dh'}{dt} &= -\frac{h_0}{r\cos\phi}\frac{u_{i+1} - u_{i-1}}{2\Delta\theta}, \\ \frac{du}{dt} &= -\frac{g}{r\cos\phi}\frac{h'_{i+1} - h'_{i-1}}{2\Delta\theta}, \end{aligned} \tag{3.3.8}$$

for $i = 0, 1, \ldots, N-1$. Hence, we can write the 1D spherical LSWE in the form of the IVP (3.3.3) by defining

$$y(t) = \begin{bmatrix} h' \\ u \end{bmatrix}, \quad f(y(t), t) = \begin{bmatrix} -\frac{h_0}{r\cos\phi}\frac{u_{i+1} - u_{i-1}}{2\Delta\theta} \\ -\frac{g}{r\cos\phi}\frac{h'_{i+1} - h'_{i-1}}{2\Delta\theta} \end{bmatrix} \quad \text{for } i = 0, 1, \ldots, N-1. \tag{3.3.9}$$

As we consider a circle as the domain, we must impose periodic boundary conditions. We do this by setting $h'_{-1} = h'_{N-1}$, $h'_N = h'_0$, $u_{-1} = u_{N-1}$, and $u_N = u_0$. The inital conditions for y are a Gaussian function in water height and zero velocity:

$$y_0 = \begin{bmatrix} \exp(-\frac{(\theta_i - \theta_0)^2}{\sigma}) \\ 0 \end{bmatrix}, \quad i = 0, 1, \ldots, N-1. \tag{3.3.10}$$

where θ_0 is the center of the Gaussian, and σ is the width of the Gaussian. This means, that we can employ the four-stage time-stepping scheme in the ERK4 method to update the solution for h' and u. The final values of h' and u are obtained by combining the contributions from all four stages, by using the general update formula:

$$\mathbf{W}_i^{n+1} = \mathbf{W}_i^n + \frac{\Delta t}{\Delta \theta} \frac{1}{6} \left(k_1 + 2k_2 + 2k_3 + k_4\right),$$

where the state variable $\mathbf{W}_i = \begin{bmatrix} h'_i \\ u_i \end{bmatrix}$ represents the pertubation height and velocity at the i'th cell. The intermediate stages k_1, k_2, k_3 and k_4 are computed at each time step. The scheme integrates the equations in time while relying on flux differences to update the solution. This approach allows the method to achieve high temporal accuracy while maintaining the spatial resolution provided by the finite volume discretization.

3.4 The Riemann problem

We will now define the Riemann problem, since it plays a crucial role in the finite volume method. The Riemann problem for the SWE with a zero source term, i.e. $\mathbf{S}(\mathbf{U}) = 0$, is defined as the initial-value problem (IVP) [7]:

$$\begin{aligned} \text{PDEs: } & \mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0, \\ \text{ICs: } & \mathbf{U}(x,0) = \begin{cases} \mathbf{U_L}, & \text{if } x < x_0, \\ \mathbf{U_R}, & \text{if } x > x_0. \end{cases} \end{aligned} \tag{3.4.1}$$

The vectors $\mathbf{U}$ and $\mathbf{F}(\mathbf{U})$ in (3.4.1) are given by

$$\mathbf{U} = \begin{bmatrix} h \\ hu \\ hv \end{bmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ hvu \end{bmatrix}, \tag{3.4.2}$$

and the initial conditions $\mathbf{U_L}$ and $\mathbf{U_R}$ are

$$\mathbf{U_L} = \begin{bmatrix} h_L \\ h_L u_L \\ h_L v_L \end{bmatrix}, \quad \mathbf{U_R} = \begin{bmatrix} h_R \\ h_R u_R \\ h_R v_R \end{bmatrix},$$

which represents the conditions at time $t = 0$ s in the left and right states of $x = x_0$. The function $\mathbf{U}$ is piecewise constant, with a discontinuity at $x = x_0$. The Riemann problem is solved at the interface between two cells, where the left and right states are known, and the solution is used to update the solution at each timestep in the FVM. The Riemann problem can be solved either exactly or approximately. When solving the Riemann problem exactly we distinguish between what we call a wet bed and a dry bed. A wet bed is the case where the water depth is positive everywhere, whereas a dry bed is the case where the water depth is zero in some cells. In this project, we focus on approximate Riemann solvers, which are able to solve the Riemann problem with high accuracy and efficiency. Various approximate Riemann solvers exist, based on finding an approximate solution to the Riemann problem, and some of these solvers will be considered in the next section. An interesting example of a Riemann problem is the so-called dam break problem, which is presented next.