第三次作业答案

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目录

第四题

解: (1)

$$N=\int_{-\infty}^{\infty}|\psi(x)|^2dx=\int_{-\infty}^{\infty}e^{-2|x|}dx=\int_{-\infty}^{0}e^{2x}dx+\int_{0}^{\infty}e^{-2x}dx=1$$

得到:

$$\psi'(x)=\frac{1}{\sqrt{N}}\psi(x)=\psi(x)=e^{-|x|}$$

$$\phi(p)=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}e^{-|x|}e^{-\frac{i}{\hbar}px}dx=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{0}e^{\left(1-\frac{ip}{\hbar}\right)x}dx+\frac{1}{\sqrt{2\pi\hbar}}\int_{0}^{\infty}e^{\left(-1-\frac{ip}{\hbar}\right)x}dx=\frac{1}{\sqrt{2\pi\hbar}}\frac{2\hbar^2}{\hbar^2+p^2}$$

第五题

解:

$$N=\int_{-\infty}^{\infty}|\psi(x)|^2dx=\int_{-\infty}^{\infty}e^{-2\frac{x^2}{\sigma^2}}dx=\sigma\sqrt{\frac{\pi}{2}}$$

得到归一化后的波函数:

$$\psi(x)=\frac{1}{\sqrt{\sigma}}\left(\frac{2}{\pi}\right)^{\frac{1}{4}}e^{-\frac{x^2}{\sigma^2}}$$

$$\langle x\rangle=\int_{-\infty}^{\infty}\psi^*(x)x\psi(x)dx=\frac{1}{\sigma}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}xe^{-\frac{2x^2}{\sigma^2}}dx=0$$

$$\langle p\rangle=\int_{-\infty}^{\infty}\psi^*(x)(-i\hbar)\frac{d}{dx}\psi(x)dx=-\frac{2\sqrt{2}}{\pi\sigma^3}\int_{-\infty}^{\infty}xe^{-\frac{2x^2}{\sigma^2}}dx=0$$

注：奇函数积分性质。

$$\langle T\rangle=\int_{-\infty}^{\infty}\psi^*(x)\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x)dx=\frac{1}{\sigma}\sqrt{\frac{2}{\pi}}\frac{\hbar^2}{m\sigma^2}\int_{-\infty}^{\infty}\left(1-\frac{2x^2}{\sigma^2}\right)e^{-\frac{2x^2}{\sigma^2}}dx=\frac{\hbar^2}{2m\sigma^2}$$

第七题

解: 由不确定性关系:

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$
$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$$

因而有:

$$\langle T \rangle = \frac{\langle p^2 \rangle}{2m} \geq \frac{\Delta p^2}{2m} \geq \frac{\hbar^2}{8m\Delta x^2}$$

接下来带入计算即可。

第八题

解: 假设本征值为复数, 记作 $E = E_0 + i\Gamma$, 则定态波函数可以写作:

$$\psi(\vec{x}, t) = \psi(\vec{x}, 0) e^{-i\frac{E_0 + i\Gamma}{\hbar}t}$$

由此我们可以验证, 在初始时刻归一化的波函数, 在 t 时刻是否依然满足归一化条件:

$$\int d^n\vec{x} |\psi(\vec{x}, t)|^2 = \int d^n\vec{x} |\psi(\vec{x}, 0)|^2 e^{\frac{2\Gamma\leftarrow}{\hbar}} = e^{\frac{2\Gamma t}{\hbar}} = 1$$

可见, 为使得归一化条件一直成立, 就要求定态波函数的对应的本征值的虚部为 $\Gamma = 0$, 即本征值为一个实数。

第十题

解:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) + V(x)\psi(x) = E\psi(x)$$

给方程两边同时乘以 $\psi^*(x)$, 并对整个方程在全空间积分, 得到:

$$E = \int dx \psi^*(x) \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) \right) \psi(x) =$$
$$-\frac{\hbar^2}{2m}\psi^*(x)\frac{\partial}{\partial x}\psi(x)\bigg|_{\partial x} + \frac{\hbar^2}{2m}\int dx \frac{\partial}{\partial x}\psi^*(x)\frac{\partial}{\partial x}\psi(x) + \int dx |\psi(x)|^2 V(x) =$$
$$\frac{\hbar^2}{2m}\int dx \frac{\partial}{\partial x}\psi^*(x)\frac{\partial}{\partial x}\psi(x) + \int dx |\psi(x)|^2 V(x)$$
$$\geq \int |\psi(x)|^2 V(x) \geq V_{\min}$$

得证。

第十二题

a.

$$N = \int (\phi_1^*(x) - i\phi_2^*(x))(\phi_1(x) + i\phi_2(x))\, dx = \int dx \left|\phi_1(x)\right|^2 + \left|\phi_2(x)\right|^2 = 2$$

得到:

$$\psi(x,0) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x))$$

b.

$$\psi(x,t) = \frac{1}{\sqrt{2}}\left(\phi_1(x)e^{-i\frac{E_1}{\hbar}t} + i\phi_2(x)e^{-i\frac{E_2}{\hbar}t}\right)$$

$$E_n = \frac{(n\pi\hbar)^2}{2ma^2}$$

$$|\psi(x,t)|^2 = \frac{1}{2}\left(\left|\phi_1(x)\right|^2 + \left|\phi_2(x)\right|^2 + 2\phi_1(x)\phi_2(x)\sin(\omega t)\right)$$

$$\omega = \frac{E_2 - E_1}{\hbar} = \frac{3\pi^2\hbar}{2ma^2}$$

c.

$$\langle x\rangle = \frac{1}{2}\int_0^a \left(\left|\phi_1(x)\right|^2 + \left|\phi_2(x)\right|^2 + 2\phi_1(x)\phi_2(x)\sin(\omega t)\right) x dx =$$

$$\frac{\langle x\rangle_1 + \langle x\rangle_2}{2} + \int_0^a \phi_1(x)\phi_2(x)\sin(\omega t) x dx = \frac{a}{2} + \frac{2}{a}\sin(\omega t)\int_0^a \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right) dx =$$

$$\frac{a}{2} - \frac{16a}{9\pi^2}\sin(\omega t)$$

$$\left\langle x^2\right\rangle = \frac{1}{2}\int_0^a \left(\left|\phi_1(x)\right|^2 + \left|\phi_2(x)\right|^2 + 2\phi_1(x)\phi_2(x)\sin(\omega t)\right) x^2 dx =$$

$$\frac{\left\langle x^2\right\rangle_1 + \left\langle x^2\right\rangle_2}{2} + \int_0^a \phi_1(x)\phi_2(x)\sin(\omega t)\Big) x^2 dx =$$

$$\frac{a^2}{3} - \frac{5a^2}{16\pi^2} + \int_0^a \phi_1(x)\phi_2(x)\sin(\omega t)\Big) x^2 dx =$$

$$\frac{a^2}{3}-\frac{5a^2}{16\pi^2}-\frac{16D^2\sin(\omega t)}{9\pi^2}$$

$$\langle p\rangle=\frac{1}{2}\int_0^a\left(\phi_1(x)e^{i\frac{E_1}{\hbar}t}-i\phi_2(x)e^{i\frac{E_2}{\hbar}t}\right)p\left(\phi_1(x)e^{-i\frac{E_1}{\hbar}t}+i\phi_2(x)e^{-i\frac{E_2}{\hbar}t}\right)dx=$$

$$\frac{\langle p\rangle_1+\langle p\rangle_2}{2}+\frac{\hbar}{2}\phi_1\phi_2e^{-i\omega t}\Big|_0^a-\hbar\int_0^a\phi_2\frac{\partial}{\partial x}\phi_1\cos(\omega t)dx=$$

$$-\frac{8\hbar\cos(\omega t)}{3D}$$

$$\left\langle p^2\right\rangle=\frac{1}{2}\int_0^a\left(\phi_1(x)e^{i\frac{E_1}{\hbar}t}-i\phi_2(x)e^{i\frac{E_2}{\hbar}t}\right)p^2\left(\phi_1(x)e^{-i\frac{E_1}{\hbar}t}+i\phi_2(x)e^{-i\frac{E_2}{\hbar}t}\right)dx=$$

$$\frac{\left\langle p^2\right\rangle_1+\left\langle p^2\right\rangle_2}{2}+i\frac{\hbar^2}{2}\left(\int_0^a\phi_2\frac{\partial^2}{\partial x^2}\phi_1e^{i\omega t}-\int_0^a\phi_1\frac{\partial^2}{\partial x^2}\phi_2e^{-i\omega t}\right)=$$

$$\frac{\left\langle p^2\right\rangle_1+\left\langle p^2\right\rangle_2}{2}=\frac{5\hbar^2\pi^2}{2D^2}$$

d.

$$\langle E\rangle=\frac{1}{2}\int_0^a\left(\phi_1(x)e^{i\frac{E_1}{\hbar}t}-i\phi_2(x)e^{i\frac{E_2}{\hbar}t}\right)H\left(\phi_1(x)e^{-i\frac{E_1}{\hbar}t}+i\phi_2(x)e^{-i\frac{E_2}{\hbar}t}\right)dx=$$

$$\frac{1}{2}\int_0^a\left(E_1\left|\phi_1(x)\right|^2+E_2\left|\phi_2(x)\right|^2+iE_1\phi_1\phi_2e^{i\omega t}-iE_2\phi_1\phi_2e^{-i\omega t}\right)dx=$$

$$\frac{E_1+E_2}{2}=\frac{5\pi^2\hbar^2}{4mD^2}$$

$$p\left(E_1\right)=\left|\int_0^a\phi_1(x)\psi(x,t)dx\right|^2=\frac{1}{2}$$

$$p\left(E_2\right)=\left|\int_0^a\phi_2(x)\psi(x,t)dx\right|^2=\frac{1}{2}$$

测得能量为 $E_1=\frac{(\pi h)^2}{2ma^2}$ 及 $E_2=\frac{(2\pi h)^2}{2ma^2}$ 的概率各为 $\frac{1}{2}$