一. **(1)** $f(t,x)=0$时, 用达朗贝尔公式

$$u=\frac{1}{2}\left[(x+3t)^3+(x-3t)^3\right]+\frac{1}{2\times 3}\int_{x-3t}^{x+3t}\sin\xi d\xi$$

$$=x^3+27t^2x+\frac{1}{3}\sin x\sin 3t \qquad (6分)$$

(2)$f(t,x)=x+xt$时，利用叠加原理设$u=u_1+u_2$, 其中

$$\begin{cases} u_{1tt}=9u_{1xx}, & (t>0,\ -\infty<x<+\infty) \\ u_1(0,x)=x^3,\ \ u_{1t}(0,x)=\sin x\,. \end{cases}$$

$$\begin{cases} u_{2tt}=9u_{2xx}+x+xt, & (t>0,\ -\infty<x<+\infty) \\ u_2(0,x)=0,\ \ u_{2t}(0,x)=0\,. \end{cases}$$

利用冲量原理：

$$u_2=\frac{1}{2\times 3}\int_0^t\int_{x-3(t-\tau)}^{x+3(t-\tau)}(\xi+\xi\tau)d\xi d\tau=x\int_0^t\left[(t-\tau)(1+\tau)\right]d\tau=\frac{1}{2}xt^2+\frac{1}{6}xt^3$$

也可以用特解法求出u_2

最后

$$u=u_1+u_2=x^3+27t^2x+\frac{1}{3}\sin x\sin 3t+\frac{1}{2}xt^2+\frac{1}{6}xt^3 \qquad (12分)$$

二.设 $u=u_1+u_2$, 先解u_1(齐次问题)

固有值和固有函数为:

$$\lambda_{\mathbf{n}}=\left[\frac{(\mathbf{2n+1})\pi}{\mathbf{2l}}\right]^{\mathbf{2}},\quad \mathbf{X_n(x)}=\sin\frac{(\mathbf{2n+1})\pi\mathbf{x}}{\mathbf{2l}} \qquad n=0,\ 1,\ \cdots(7分)$$

设级数解为

$$u_1(t,x)=\sum_{n=0}^{+\infty}[C_n\cos\frac{(2n+1)\pi at}{2l}+D_n\sin\frac{(2n+1)\pi at}{2l}]\sin\frac{(2n+1)\pi x}{2l}$$

由初始条件得

$$C_n=\frac{2}{l}\int_0^l\varphi(\xi)\sin\frac{(2n+1)\pi\xi}{2l}d\xi$$

$$D_n=\frac{4}{(2l+1)\pi a}\int_0^l\psi(\xi)\sin\frac{(2n+1)\pi\xi}{2l}d\xi \qquad (10分)$$

对于u_2(非齐次问题)

$$1=\sum_{n=0}^{+\infty}f_n\sin\frac{(2n+1)\pi x}{2l}, \qquad f_n=\frac{4}{(2n+1)\pi}$$

$$u_2=\sum_{n=0}^{+\infty}T_n(t)\sin\frac{(2n+1)\pi x}{2l}$$

$T_n(t)$ 满足 $T''+\lambda_n a^2T=\dfrac{4}{(2n+1)\pi},\quad T(0)=T'(0)=0$

$$T_n(t)=\frac{16l^2}{(2n+1)^3a^2\pi^3}-\frac{16l^2}{(2n+1)^3a^2\pi^3}\cos\frac{(2n+1)\pi at}{2l}\quad (14分)$$

最后

$$u=u_1+u_2$$

本题也可以用特解法和齐次化原理法求解

三. 极坐标系下,

$$r^2\frac{\partial^2u}{\partial^2r}+r\frac{\partial u}{\partial r}+\frac{\partial^2u}{\partial^2\theta}=0$$

$$u=R(r)\Theta(\theta)$$

$$\begin{cases}r^2R''+rR'+\lambda R=0, & (1<x<e)\\ R(1)=R(e)=0,\end{cases}$$

$$\lambda_n=(n\pi)^2,\quad R_n(r)=\sin(n\pi lnr)\qquad (8分)$$

$$\Theta_n(\theta)=C_n\cosh n\pi\theta+D_n\sinh n\pi\theta$$

$$u(r,\theta)=\sum_{n=1}^{+\infty}(C_n\sinh n\pi\theta+D_n\cosh n\pi\theta)\sin(n\pi lnr)$$

$$C_n=0,\quad D_n=\frac{2}{\sinh\frac{n\pi^2}{3}}\int_1^e\varphi(r)\sin(n\pi lnr)\frac{1}{r}dr\qquad (8分)$$

四 使用柱坐标，并注意到泛定方程和定解条件不显含θ, 可设$u=u(r,z)$, 对应柱标方程为

$$\frac{\partial^2u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2u}{\partial z^2}=0$$

用分离变量$u=R(r)Z(z)$, 代入方程和边界条件，得Bessel方程固有值问题

$$\begin{cases}r^2R''+rR'+\lambda r^2R=0\\ R(0)\text{ 有界},\ R(1)=0\end{cases}$$

和方程

$$Z''-\lambda Z=0\,.$$

解固有值问题得到：固有值：$\lambda_n = \omega_n^2$，固有函数$J_0(\omega_n r)$,而ω_n是$J_0(x) = 0$的第n个正根.相应地:$Z_n(z) = C_n ch\omega_n z + D_n sh\omega_n z$.设

$$u(r,z) = \sum_{n=1}^{+\infty} \left(C_n ch\omega_n z + D_n sh\omega_n z\right) J_0(\omega_n r) \qquad (9分)$$

$$u(r,0) = \sum_{n=1}^{+\infty} C_n J_0(\omega_n x) = r - r^2, \;\; u(r,2) = \sum_{n=1}^{+\infty} \left(C_n ch2\omega_n + D_n sh2\omega_n\right) J_0(\omega_n r) = 0$$

这样得到$D_n = -\dfrac{ch2\omega_n}{sh2\omega_n} C_n$, 而

$$C_n = \frac{\int_0^1 r(r-r^2)J_0(\omega_n r)dr}{N_{01}^2} = \frac{1}{N_{01}^2}\frac{1}{\omega_n^2}\int_0^{\omega_n} t\left(\frac{t}{\omega_n} - \frac{t^2}{\omega_n^2}\right)J_0(t)dt$$

$$= \frac{1}{N_{01}^2\omega_n^2}\left[\left(\frac{t}{\omega_n} - \frac{t^2}{\omega_n^2}\right)tJ_1(t)\mid_0^{\omega_n} - \int_0^{\omega_n}\left(\frac{1}{\omega_n} - \frac{2t}{\omega_n^2}\right)tJ_1(t)dt\right]$$

$$= \frac{8}{\omega_n^3 J_1(\omega_n)} - \frac{2}{\omega_n^3 J_1^2(\omega_n)}\int_0^{\omega_n} J_0(t)dt \qquad (14分)$$

五

(1) 设

$$1 + x + x^2 = C_0P_0(x) + C_1P_1(x) + C_2P_2(x)$$

$$P_2 = \frac{2}{3}, \quad P_1 = 1, \quad P_0 = \frac{4}{3} \qquad (7分)$$

(2) $\displaystyle\int_{-1}^{1} P'_{2019}(x)P'_{2021}(x)dx = \int_{-1}^{1} P'_{2019}(x)\; dP_{2021}(x)$

$$= P'_{2019}(x)\;P_{2021}(x)\Big|_{-1}^{1} - \int_{-1}^{1} P'_{2019}(x)P_{2021}(x)dx$$

$$= P'_{2019}(1) + P'_{2019}(-1) - 0 = 2019 \times 2020 = 4078380$$

$$P'_{2019}(1) - P'_{2017}(1) = (2 \times 2019 + 1)P_{2018}(1)$$

六解：(1)

$$\begin{cases} u_t = u_{xx} + 5u_x, \; (t > 0, \, -\infty < x < +\infty) \\ u\mid_{t=0} = \delta(x). \end{cases}$$

作Fourier 变换：

$$\begin{cases}\overline{u}_t = -\lambda^2\overline{u} + 5i\lambda\overline{u}, \quad t > 0,\\ \overline{u}(0,\lambda) = 1,\end{cases}$$

解得

$$\overline{u} = e^{(-\lambda^2+5i\lambda)t} \qquad (6分)$$

由于$F^{-1}[e^{-\lambda^2 t}] = \frac{1}{2\sqrt{\pi t}}\exp\{-\frac{x^2}{4t}\}$, 所以

$$F^{-1}[e^{-\lambda^2 t+5i\lambda t}] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\lambda^2 t+5i\lambda t}e^{i\lambda x}d\lambda = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\lambda^2 t}e^{i\lambda(x+5t)}d\lambda = \frac{1}{2\sqrt{\pi t}}\exp\{-\frac{(x+5t)^2}{4t}\},$$

因此

$$u(t,x) = \varphi(x) * \frac{1}{2\sqrt{\pi t}}e^{-\frac{(x+5t)^2}{4t}} + \int_0^t \frac{1}{2\sqrt{\pi(t-\tau)}}e^{-\frac{(x+5(t-\tau))^2}{4(t-\tau)}} * f(\tau,x)d\tau \qquad (16\ 分)$$

七 (1)利用镜像法，$M_0 = (\xi,\eta,\zeta)$, $M_1 = (2-\xi,\eta,\zeta)$, 而$M = (x,y,z)$.

这样格林函数

$$G = \frac{1}{4\pi r(M,M_0)} - \frac{1}{4\pi r(M,M_1)}$$

$$= \frac{1}{4\pi}\Big[\frac{1}{[(x-\xi)^2+(\eta-y)^2+(\zeta-z)^2]^{\frac{3}{2}}} - \frac{1}{[(x+\xi-2)^2+(\eta-y)^2+(\zeta-z)^2]^{\frac{3}{2}}}\Big]$$

(2)设$z' = \dfrac{z}{3}$

$$\begin{cases} u_{xx}+u_{yy}+u_{z'z'} = 0 \qquad (x>0)\\ u|_{x=0} = \varphi(y,3z'),\end{cases}$$

$$u(\xi,\eta,\zeta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\varphi(y,3z')\xi}{[\xi^2+(\eta-y)^2+(\zeta-z')^2]^{\frac{3}{2}}}dydz'$$