1. 解：(1) $E\hat{\theta}_1 = E(\bar{X}+a_n) = E(\frac{1}{n}\sum_{i=1}^{n}X_i + a_n) = EX_1 + a_n$

又 $EX = \int_{-\infty}^{+\infty} x f(x;\theta)dx = \int_{\theta}^{+\infty} x\cdot e^{-(x-\theta)}dx = -\int_{\theta}^{+\infty} x\, de^{-(x-\theta)}$

$= [-xe^{-(x-\theta)}]\Big|_{\theta}^{+\infty} + \int_{\theta}^{+\infty} e^{-(x-\theta)}dx$

$= \theta - e^{-(x-\theta)}\Big|_{\theta}^{+\infty} = \theta+1$

故 $E\hat{\theta}_1 = \theta+1+a_n = \theta \Rightarrow a_n = -1$

令 $X_{(1)} = \min\{X_1, X_2, \cdots, X_n\}$，则

$E\hat{\theta}_2 = E(X_{(1)}+b_n) = EX_{(1)} + b_n$

又 $F(x) = P(X\le x) = \int_{\theta}^{x} e^{-(u-\theta)}du = -e^{-(u-\theta)}\Big|_{\theta}^{x} = -e^{-(x-\theta)}+1$，$x\ge\theta$.

故 $F_{X(1)}(x) = 1-[1-F(x)]^n = 1-e^{-n(x-\theta)}$，$x\ge\theta$

则 $f_{X(1)}(x) = ne^{-n(x-\theta)} I_{[\theta,+\infty)}(x)$

故 $EX_{(1)} = \int_{\theta}^{+\infty} x n e^{-n(x-\theta)}dx = [-xe^{-n(x-\theta)}]\Big|_{\theta}^{+\infty} + \int_{\theta}^{+\infty} e^{-n(x-\theta)}dx$

$= \theta + (-\frac{1}{n}e^{-n(x-\theta)})\Big|_{\theta}^{+\infty} = \theta+\frac{1}{n}$

则 $E\hat{\theta}_2 = \theta+\frac{1}{n}+b_n = \theta \Rightarrow b_n = -\frac{1}{n}$

(2) $\hat{\theta}_1 = \bar{X}-1$，$\hat{\theta}_2 = X_{(1)}-\frac{1}{n}$，则

$Var(\hat{\theta}_1) = Var(\bar{X}) = Var(\frac{1}{n}\sum_{i=1}^{n}X_i) = \frac{1}{n^2}\sum_{i=1}^{n}Var(X_i) = \frac{1}{n}Var(X_1)$

又 $EX^2 = \int_{\theta}^{+\infty} x^2 e^{-(x-\theta)}dx = [-x^2e^{-(x-\theta)}]\Big|_{\theta}^{+\infty} + \int_{\theta}^{+\infty} 2xe^{-(x-\theta)}dx$

$= \theta^2 + 2EX = \theta^2+2\theta+2 = (\theta+1)^2+1$

故 $Var(X) = EX^2-(EX)^2 = 1$

则 $Var(\hat{\theta}_1)=\frac{1}{n}$

又 $EX_{(1)}^2=\int_\theta^{+\infty}x^2\cdot ne^{-n(x-\theta)}dx=[-x^2e^{-n(x-\theta)}]\Big|_\theta^{+\infty}+\int_\theta^{+\infty}2xe^{-n(x-\theta)}dx$

$=\theta^2+\frac{2}{n}EX_{(1)}=\theta^2+\frac{2\theta}{n}+\frac{2}{n^2}=(\theta+\frac{1}{n})^2+\frac{1}{n^2}$

故 $Var(X_{(1)})=EX_{(1)}^2-(EX_{(1)})^2=\frac{1}{n^2}$

则 $Var(\hat{\theta}_2)=Var(X_{(1)}-\frac{1}{n})=Var(X_{(1)})=\frac{1}{n^2}\leq Var(\hat{\theta}_1)$

故 θ_2 比 θ_1 更有效.

2. 解: (1) $\lim\limits_{x\to0^+}F(x;\theta)=\lim\limits_{x\to0^+}1-e^{-x^2/\theta}=0=\lim\limits_{x\to0^-}F(x;\theta)$

故 $F(x;\theta)$ 连续. 则

$$f(x;\theta)=\begin{cases}\frac{2x}{\theta}e^{-x^2/\theta}, & x\geq0\\ 0, & \text{others}\end{cases}$$

故 $EX=\int_0^{+\infty}xf(x;\theta)dx=\int_0^{+\infty}\frac{2}{\theta}x^2e^{-x^2/\theta}dx$

$=\sqrt{\theta}\int_0^{+\infty}(\frac{x^2}{\theta})^{\frac{1}{2}}e^{-x^2/\theta}d(x^2/\theta)$

$=\sqrt{\theta}\,\Gamma(\frac{1}{2}+1)=\sqrt{\theta}\cdot\frac{1}{2}\Gamma(\frac{1}{2})=\frac{1}{2}\sqrt{\pi\theta}$

$EX^2=\int_0^{+\infty}\frac{2x^3}{\theta}e^{-x^2/\theta}dx=\theta\int_0^{+\infty}(\frac{x^2}{\theta})^{2-1}e^{-x^2/\theta}d(x^2/\theta)$

$=\theta\,\Gamma(2)=\theta$

(2) $L(\theta)=\prod_{i=1}^{n}\frac{2x_i}{\theta}e^{-x_i^2/\theta}\Rightarrow l(\theta)=\sum_{i=1}^{n}\ln 2x_i-\sum_{i=1}^{n}\ln\theta-\sum_{i=1}^{n}\frac{x_i^2}{\theta}$

故 $\frac{\partial l(\theta)}{\partial\theta}=\sum_{i=1}^{n}\frac{-1}{\theta}+\frac{1}{\theta^2}\sum_{i=1}^{n}x_i^2=\frac{-n}{\theta}+\frac{1}{\theta^2}\sum_{i=1}^{n}x_i^2$

令 $\frac{\partial L(\theta)}{\partial\theta}=0 \Rightarrow \frac{1}{\theta^2}\sum_{i=1}^{n}x_i^2=\frac{n}{\theta} \Rightarrow \hat{\theta}=\frac{1}{n}\sum_{i=1}^{n}x_i^2$

(3) 由大数定律，当 $n\to\infty$ 时，

$$\hat{\theta}=\frac{1}{n}\sum_{i=1}^{n}x_i^2 \xrightarrow{P} EX^2=\theta$$

故 $\alpha=\theta$.

3. 解：证明：令 $X_{(n)}=\max\{X_1,X_2,\cdots,X_n\}$，又 $X\sim U(0,\theta)$，则

$$f_{X_{(n)}}(x)=\frac{nx^{n-1}}{\theta^n}I_{(0,\theta)}(x).$$

故 $P(|\hat{\theta}-\theta|>\varepsilon)=\int_0^{\theta-\varepsilon}\frac{nx^{n-1}}{\theta^n}dx=\frac{(\theta-\varepsilon)^n}{\theta^n}=(1-\frac{\varepsilon}{\theta})^n$

则 $\lim\limits_{n\to\infty}P(|\hat{\theta}-\theta|>\varepsilon)=\lim\limits_{n\to\infty}(1-\frac{\varepsilon}{\theta})^n=0$

即 $\hat{\theta}$ 为 θ 的相合估计.

又 $E\hat{\theta}=EX_{(n)}=\int_0^{\theta}x\frac{nx^{n-1}}{\theta^n}dx=\frac{n}{n+1}\left.\frac{x^{n+1}}{\theta^n}\right|_0^{\theta}=\frac{n\theta}{n+1}\neq\theta$

故 $\hat{\theta}$ 不是 θ 的无偏估计.

4. 解：(1) 总体均值已知的情况下，枢轴变量为

$$\frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\sigma^2}\sim\chi_n^2$$

故 $\sigma^2\in[\sum_{i=1}^{10}(X_i-\mu)^2/\chi_{10}^2(\alpha/2),\ \sum_{i=1}^{10}(X_i-\mu)^2/\chi_{10}^2(1-\alpha/2)]$

又 $\sum_{i=1}^{10}(X_i-\mu)^2=2.9$，$\alpha=1-0.95=0.05$

查表知，$\chi_{10}^2(0.975)=3.247$，$\chi_{10}^2(0.025)=20.483$

则 $\sigma^2 \in [2.9/20.483, 2.9/3.247] = [0.142, 0.893]$

(2) μ未知，则枢轴变量为 $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

又 $\bar{X} = \frac{1}{10}\sum_{i=1}^{10} X_i = 50.08$

故 $S^2 = \frac{1}{9}\sum_{i=1}^{10}(X_i - \bar{X})^2 = \frac{1}{9} \times 2.836 = 0.315$

又查表知，$\chi^2_9(0.975) = 2.700$，$\chi^2_9(0.025) = 19.023$

故 $\sigma^2 \in [(n-1)S^2/\chi^2_9(0.025), (n-1)S^2/\chi^2_9(0.975)]$

$= [2.836/19.023, 2.836/2.700]$

$= [0.149, 1.050]$

5. 解：令 $X_{(n)} = \max\{X_1, X_2, \cdots, X_n\}$，则

$f_{X_{(n)}}(x) = \frac{nx^{n-1}}{\theta^n} I_{(0,\theta)}(x)$，

由题，$P(X_{(n)} \leq \theta \leq c_n X_{(n)}) = 1 - \alpha$

又 $P(X_{(n)} \leq \theta \leq c_n X_{(n)}) = P(\frac{\theta}{c_n} \leq X_{(n)} \leq \theta)$

$= \int_{\theta/c_n}^{\theta} \frac{nx^{n-1}}{\theta^n} dx = \left(\frac{x}{\theta}\right)^n \Big|_{\theta/c_n}^{\theta} = 1 - c_n^{-n}$

故 $1 - \alpha = 1 - c_n^{-n} \Rightarrow c_n = \alpha^{-1/n}$