1. (1). $f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right\}$

$\begin{cases} Z = \frac{Y-\rho X}{\sqrt{1-\rho^2}} \\ W = X \end{cases}$ 即 $\begin{cases} X = W \\ Y = \sqrt{1-\rho^2}Z + \rho W \end{cases}$

$$|J| = \left| \frac{\partial(X,Y)}{\partial(Z,W)} \right|$$

$$= \left\| \begin{matrix} 0 & 1 \\ \sqrt{1-\rho^2} & \rho \end{matrix} \right\| = \sqrt{1-\rho^2}$$

$$f(z,w) = f(w, \sqrt{1-\rho^2}z+\rho w)\cdot\sqrt{1-\rho^2}$$

$$= \frac{1}{2\pi}\exp\left\{-\frac{z^2+w^2}{2}\right\}$$

$$f_1(z) = \int_{-\infty}^{+\infty} f(z,w)\,dw = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)$$

$$f_2(w) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{w^2}{2}\right)$$

$\Rightarrow f(z,w) = f_1(z)\cdot f_2(w) \Rightarrow Z, W$ 相互独立

(2). $P(XY<0) = 1 - P(XY \geq 0)$

$$= 1 - 2P(X\geq 0, Y\geq 0)$$

$$P(X\geq 0, Y\geq 0) = P\left(W\geq 0, Z\geq -\frac{\rho}{\sqrt{1-\rho^2}}W\right)$$

$$= \frac{1}{2\pi}\iint_D \exp\left\{-\frac{z^2+w^2}{2}\right\}dw\,dz$$

$D := \{(z,w) \mid \sqrt{1-\rho^2}z + \rho w \geq 0,\ w\geq 0\}$

设 $r = \sqrt{w^2+z^2}$ $\quad D_{(r,\theta)} = \{(r,\theta) \mid -\theta_0 < \theta < \frac{\pi}{2}\}$ $\quad \tan\theta_0 = \frac{\rho}{\sqrt{1-\rho^2}}$

$$原式 = \frac{1}{2\pi}\int_{-\theta_0}^{\pi/2}\int_0^{+\infty} e^{-\frac{1}{2}r^2}\cdot r\,d\theta\,dr$$

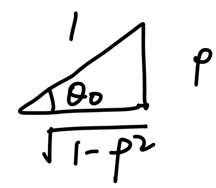
$$= \frac{1}{2\pi}\left(\frac{\pi}{2}+\theta_0\right)\left(-e^{-\frac{1}{2}r^2}\right)\Big|_0^{+\infty} = \frac{1}{4} + \frac{\theta_0}{2\pi}$$

$$P(XY<0) = 1- \quad 2\cdot\left(\frac{1}{4}+\frac{\theta_0}{2\pi}\right)$$

$$= 1-\frac{1}{2}-\frac{\theta_0}{\pi}.$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2}-\arcsin p\right)$$

$$= \frac{1}{\pi}\arccos p$$



2. $P(X=4) = p^4(1-p)^0 + (1-p)^4\cdot p^0 = p^4+(1-p)^4$

$P(X=5) = C_4^3 p^3 (1-p)^1\cdot p + C_4^3(1-p)^3\cdot p\cdot(1-p) = 4p(1-p)[p^3+(1-p)^3]$

$P(X=6) = C_5^3 p^3 (1-p)^2\cdot p + C_5^3 (1-p)^3\cdot p^2 (1-p)$

$= 10p^2(1-p)^2(p^2+(1-p)^2)$

$P(X=7) = C_6^3 p^3 (1-p)^3\cdot p + C_6^3\cdot(1-p)^3\cdot p^3(1-p)$

$= 20p^3(1-p)^3.$

其余 均有 $P(X=k)=0$

$p=0.5$

4	5	6	7
0.125	0.25	0.3125	0.3125

$EX = 4\times0.125 + 5\times0.25 + 6\times0.3125 + 7\times0.3125$

$= 5.8125$

$p=0.6$

4	5	6	7
0.1552	0.2688	0.29952	0.27648

$EX = 4\times0.1552 + 5\times0.2688 + 6\times0.29952 + 7\times0.27648$

$= 5.69728$

3. (1). $EX = \sum_{n=1}^{\infty} nP(X=n)$

$= P(X=1) + 2P(X=2) + \cdots$

$= (P(X=1) + P(X=2) + \cdots) +$

$(P(X=2) + P(X=3) + \cdots) +$

$\cdots\cdots$

$= \sum_{n=1}^{\infty} P(X \geq n)$

$= \sum_{n=0}^{\infty} P(X > n).$

(2). $EX = \int_0^{+\infty} x f(x)\, dx.$

右边 $= \int_0^{+\infty} (1 - F(x))\, dx = \int_0^{+\infty} \left(1 - \int_0^x f(t)\, dt\right) dx.$

$= \int_0^{+\infty} \int_x^{+\infty} f(t)\, dt\, dx.$ 交换积分次序

$= \int_0^{+\infty} \left(\int_0^t f(t)\, dx\right) dt.$

$= \int_0^{+\infty} t f(t)\, dt. = EX$

(3). 若 X 是非负连续型随机变量

由 (2) 可得 (3).

若 X 是非负离散型随机变量.

X 取 $x_1 < x_2 < \cdots < x_n < \cdots$

对应概率 $p_1 \quad p_2 \quad \cdots\cdots \quad p_n \quad \cdots$

$$F(x) = \begin{cases} 0 & 0 < x < x_1 \\ p_1 & x_1 < x < x_2 \\ \sum_{j=1}^{n-1} p_j & x_{n-1} < x < x_n \end{cases}$$

$左 = x_1 + \sum (1-\sum_{j=1}^{i-1} p_j)(x_i - x_{i-1})$

$= x_1 + \sum_{i=2}^{\infty} x_i (1-\sum_{j=1}^{i-1} p_j) - \sum_{i=1}^{\infty} x_i (1-\sum_{j=1}^{i} p_j)$

$= x_1 - x_1(1-p_1) + \sum_{i=2}^{\infty} x_i (1-\sum_{j=1}^{i-1} p_j) - \sum_{i=2}^{\infty} x_i (1-\sum_{j=1}^{i-1} p_j) + \sum_{i=2}^{\infty} x_i p_i$

$= \sum_{i=1}^{\infty} x_i p_i$

$右 = EX = \sum_{i=1}^{\infty} x_i p_i$

故等式左右相等

4. 设集齐 $m-1$ 张后，集齐第 m 张需要买的袋数为 P_m

$1 \le m < +\infty$

$EP_m = 1 \cdot \frac{n-(m-1)}{n} + 2 \cdot (\frac{m-1}{n}) \cdot \frac{n-(m-1)}{n} + \cdots + j \cdot (\frac{m-1}{n})^{j-1} \cdot \frac{n-(m-1)}{n} + \cdots$

$= \sum_{j=1}^{\infty} j \cdot q^{j-1} (1-q)$

$= (1-q) \sum_{j=1}^{\infty} j q^{j-1} \quad 0 < q = \frac{m-1}{n} < 1$

$S = \sum_{j=1}^{\infty} j q^{j-1}$

$qS = \sum_{j=1}^{\infty} j q^{j}$

$\Rightarrow (1-q) S = \lim_{k \to \infty} (\frac{1-q^k}{1-q} - k q^k)$

$= \frac{1}{1-q} = \frac{n}{n-m+1}$

而 $X = P_1 + P_2 + \cdots + P_n \quad P_i \; P_j$ 相互独立.

$\therefore EX = \sum_{i=1}^{n} E(P_i)$

$= n \sum_{m=1}^{n} \frac{1}{n-m+1} = n \sum_{k=1}^{n} \frac{1}{k}$

$= 12 \sum_{k=1}^{12} \frac{1}{k} \approx 37.236 \approx 37$

(2). $\lim_{n \to \infty} E(\frac{X_n}{n \ln n})$

$= \lim_{n \to \infty} \frac{E(X_n)}{n \ln n} = \lim_{n \to \infty} \frac{\sum_{k=1}^{n} \frac{1}{k}}{\ln n}$

又 $\ln n \le \sum_{k=1}^{n} \frac{1}{k} \le 1 + \ln n$

由夹逼定理: $\lim_{n\to\infty} E\left(\frac{X_n}{n\ln n}\right) = 1$

5. 不妨设 $x_{j_1} < x_{j_2} < \cdots \ x_{j_k}$ 对应概率为:

$$p_{j_1} \quad p_{j_2} \quad \cdots \quad p_{j_k}$$

其中 $\{j_1, \cdots \ j_k\} = \{1, 2, \cdots k\}$

$$\lim_{n\to\infty} \frac{E(X^{n+1})}{E(X^n)} = \lim_{n\to\infty} \frac{\sum_{m=1}^{k} p_{j_m} x_{j_m}^{n+1}}{\sum_{m=1}^{k} p_{j_m} x_{j_m}^{n}}$$

$$= \lim_{n\to\infty} \frac{\sum_{m=1}^{k-1} p_{j_m} \left(\frac{x_{j_m}}{x_{j_k}}\right)^n \cdot x_{j_m} + p_{j_k} x_{j_k}}{\sum_{m=1}^{k-1} p_{j_m} \left(\frac{x_{j_m}}{x_{j_k}}\right)^n + p_{j_k}}$$

$$= \lim_{n\to\infty} \frac{p_{j_k} x_{j_k}}{p_{j_k}} = x_{j_k} \quad \left(\because \lim_{n\to\infty} \left(\frac{x_{j_m}}{x_{j_k}}\right)^n = 0\right)$$

$$= \max_{1\le i\le k} x_i$$