Mathematics and how the world does not work.

Igor Melnikov

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Through centuries conflict between rationalism and idealism has been developed and due to technical and scientific breakthroughs of the last century, common beliefs and attitudes to science have changed significantly. Still, even basic concepts of science may be misinterpreted or paid not enough attention. One of the most common mistakes, to my opinion, is an intuitive attitude to mathematics. As we have been studied mathematics from the very beginning of our lives, a wide variety of complicated notions and theories (even basic ones) seems to be intuitively obvious to us. However, this feeling is ambiguous and may be controversial. In the essay, we will look at some classical paradoxes, that are based on such a phenomenon. Finally, we will take a look at mathematics logic, and the basics of how mathematics can be formalised, to avoid such a contradiction.

To begin with, let's start with the notion of infinity. In philosophy, the concept of infinity is given great ideological significance. Still, the common notion of infinity in mathematics as about ∞ - smth, which is larger than any number is nothing more than a formal symbol for shorter definitions. When you see $\lim\limits_{n\to\infty} x_n = \infty$, it's just a shorter version of $\forall C \exists N \in \mathbf{N} \forall n > N \; |x_n| > C$. Still, the notion of infinity is used in set theory, when we talk about set power. In terms of sets powers, there are finite sets, whose power is just a natural number, and non-finite (infinite sets). Still, infinite sets may differ in their power. For instance, sets, whose elements may enumerate with natural numbers may be called countable sets, which is the smallest infinite set. The other famous set is a continuum ($\mathbf{R}$). It's easy to be proven, that for any set A, 2^A (set of all subsets of A) is larger in terms of power than A(i.e. the is an injection from A to 2^A, but no injection from 2^A to A). As $2^{\mathbf{N}} \sim [0,1]$ (that could be proven by looking at binary notation of a number), $[0,1] \succ \mathbf{N}$. Surprisingly, $[0,1] \sim \mathbf{R}$ (that could be seen by applying inversion).

Now we will look at famous Russell's paradox, to understand, why naive set theory is controversial. Let's assume A is a set, of all sets, which does not contain itself. In this case, if A contains itself, it does not satisfy the condition of not containing itself, so it should not contain itself. Otherwise, if A does not contain itself, it does satisfy the condition of not containing itself, so it should contain itself. To deal with this contradiction, set theory was significantly modernised and formalised by Axioms of Zermelo-Fraenkel. A new notion of classes was introduced. Generally speaking, the main difference, between sets and classes is that for any set A we could make another set by applying any condition C, as $B = \{x$, for x in A, such that A satisfies C$\}$, but we can't do that for classes. In more detail set theory and this paradox are described in Thomas Jech book[2]. For a deeper understanding of the unnaturalness of even basic set theory let's look at the next children's task: Santa Claus is giving candies to children, all candies are enumerated. He starts an hour before Christmas and every half of the remaining time he makes a present. He acts the following way, he asks children to return the candy with the smallest number which they have, and give them 100 new candies in return. The question is, how many candies will the children have after the New Year? It may seem that the number of candies they have is increasing, so they must have an infinite number of candies. However, if you will think, which particular candy will children have after New Year, you will realise that greedy Santa will ask for any candy back. So in the end children won't get any candies.

Now, let us move to the other famous mathematical theory that is widely used in any technical field, but may be misunderstood by a wide variety of people. That is probability theory. Although it is taught in many universities to first-year students, the proper understanding of it requires serious mathematical background. We will start with the famous envelope problem. Assume a stranger come to you and give you an envelope. He said that he put a random amount of money in it, and in one envelope there is twice as much money, as in another one. He asked you if you want to change the envelope. You think, if there are x bitcoins in my envelope, than in other one there are 0.5x or 2x with equal probability, so is you change the envelop you gain $\frac{(2x-x)+(0.5x-x)}{2} = 0.25x > 0$. So you have to change the envelope. Now you could repeat this reasoning, so you have to change envelop again. So you will constantly change the envelopes and your profit will constantly grow. The key problem here is randomly putting money into the envelope. There is no way to choose a random number with equal probability, which is an implication of the fact, that there is no uniform measure on R. A few more words about the measure. Despite the fact, that measure seems to be a very natural notion, there are lots of contradicting facts about it. There is no measure on all subsets of $\mathbf{R}$ or even [0, 1], so in probability theory, we can't always talk about events probabilities. In other words, some sets of possible events could not be counted as an event, and do not have properly defined probability.

Another significant aspect of mathematics application in the real life is our language. If we treat language as something formal we face many difficulties as it allows us to make lots of inappropriate expressions. A good example of such a phenomenon would be the following reasoning. Let's call a sentence good if it describes exactly one integer number. For example sentence "Hypotenuse of a right triangle with cathartics 3 and 4" describes number 5. Let A be a set of all numbers, described by good short sentences. By short I mean sentences with length less than 10^{10}. This set is finite, as the number of good sentences is finite and each of them describes exactly one number. Now, let's take a look at the following phrase (let's denote it by w): "The smallest integer number, that does not belong to the set, $<$description of set from our problem$>$". This phrase is short, and it describe exactly one integer number - let it be number x. According to our statement, it is the smallest number out of A. However it is described by a short phrase, so it belongs to A. But what is the problem with such reasoning? Consider the mapping f from good phrases to numbers - each good phrase is matched by the number it describes. What is the problem with the phrase w? This phrase is referenced to an object, referenced by the phrase. This construction is quite similar to the famous Barber paradox. The barber is the "one who shaves all those, and those only, who do not shave". The question is, does the barber shave himself? Bertrand Russell in his work [1] wrote: «That contradiction [Russell's paradox] is extremely interesting. You can modify its form; some forms of modification are valid and some are not. I once had a form suggested to me which was not valid, namely the question whether the barber shaves himself or not. You can define the barber as "one who shaves all those, and those only, who do not shave themselves". The question is, does the barber shave himself? In this form the contradiction is not very difficult to solve. But in our previous form I think it is clear that you can only get around it by observing that the whole question whether a class is or is not a member of itself is nonsense, i.e. that no class either is or is not a member of itself, and that it is not even true to say that, because the whole form of words is just noise without meaning». Our language allows such incorrect references, but this problem was resolved in mathematics by formalising set theory and denoting first-order logic(which is explained in detail in the book "LANGUAGES AND CALCULATIONS[3]").

To conclude, many mathematics theories are not intuitive and natural, so we should clearly understand, that their application in real life is modelling, which may accurately predict events, but will never describe how the world really behave. For instance, despite the common belief, that physics claims that the world consists of neutrons, protons and electrons, it is just a common misunderstanding of the scientific principles. Such theories are suggesting a hypothesis that is not proven but supported by an experiment. In other words, experiments show, that the theories gave high accuracy predictions, still, it does not prove that the model absolutely accurately describes the world.

References

[1] Bertrand Russell - "The Philosophy of Logical Atomism"

[2] Thomas Jech - "Set Theory"

[3] H. K. Vereshchagin, A. Shen. - "LANGUAGES AND CALCULATIONS"