Novosad I.

1. (1 point) Let $(\mathbb{R}[x;4], \langle\,|\,\rangle)$ be a Euclidean space with the scalar product

$$\langle f|g\rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x, \quad \text{for every } f,\, g \in \mathbb{R}[x;4].$$

Then, find the distance between the vector $q(x) = x+1 \in \mathbb{R}[x;4]$ and the subspace $\mathbb{W} = \{f \in \mathbb{R}[x;4] \mid f(0) = 0 \text{ and } f(1) = 0\}$.

[hint: use Theorem 31.2]

let $(\underbrace{(x-1)x}_{e_1}, \underbrace{(x-1)x^2}_{e_2}, \underbrace{(x-1)x^3}_{e_3})$ be an ordered basis for W

$(x+1) = w$

$$\langle (x-1)x \mid (x-1)x\rangle = \int_0^1 ((x-1)x)^2 dx = \frac{1}{30}$$

$$\langle (x-1)x \mid (x-1)x^2\rangle = \int_0^1 ((x-1)^2x^3)dx = \frac{1}{60}$$

$$\langle (x-1)x^2 \mid \tfrac{1}{2}x(x-1)(2x-1)\rangle = \int_0^1 \tfrac{1}{2}x^3(x-1)^2(2x-1)dx = \frac{1}{840}$$

$$\langle \tfrac{1}{2}x(x-1)(2x-1) \mid \tfrac{1}{2}x(x-1)(2x-1)\rangle = \int_0^1 (\tfrac{1}{2}x(x-1)(2x-1))^2 dx = \frac{1}{840}$$

Let's orthogonolize it:

$$\tilde{e}_1 = e_1$$

$$\tilde{e}_2 = e_2 - \frac{\langle e_2\, e_1\rangle}{\langle e_1\, e_1\rangle} e_1 = (x-1)x^2 - \frac{1/60}{1/30}(x-1)x = \frac{1}{2}x(x-1)(2x-1)$$

$$\tilde{e}_3 = e_3 - \frac{\langle e_3\, \tilde{e}_2\rangle}{\langle \tilde{e}_2\, \tilde{e}_2\rangle}\tilde{e}_2 - \frac{\langle e_3\, \tilde{e}_1\rangle}{\langle \tilde{e}_1\, \tilde{e}_1\rangle}\tilde{e}_1 = x^3(x-1) - \frac{1/840}{1/840}\frac{1}{2}x(x-1)(2x-1) - \frac{1/105}{1/30}x(x-1) = x(x-1)\left(x^2 - \frac{2x-1}{2} - \frac{2}{7}\right)$$

Since $\langle \tilde{e}_1\, \tilde{e}_2\rangle = \langle \tilde{e}_1\, \tilde{e}_3\rangle = \langle \tilde{e}_2\, \tilde{e}_3\rangle = 0$ it's indeed orthogonal.

$$\langle (x-1)x \mid (x-1)x^3\rangle = \int_0^1 ((x-1)^2x^4)dx = \frac{1}{105}$$

$$\left\langle x(x-1)\left(x^2 - \tfrac{2x-1}{2} - \tfrac{2}{7}\right) \,\middle|\, x(x-1)\left(x^2 - \tfrac{2x-1}{2} - \tfrac{2}{7}\right)\right\rangle = \int_0^1 \left(x(x-1)\left(x^2 - \frac{2x-1}{2} - \frac{2}{7}\right)\right)^2 dx = \frac{1}{17640}$$

$$\langle x+1 \mid \sqrt{30}\,x(x-1)\rangle = \int_0^1 (x-1)(x+1)\sqrt{30}\,x\,dx = -\frac{\sqrt{30}}{4}$$

$$\langle x+1 \mid \sqrt{210}\,x(x-1)(2x-1)\rangle = \int_0^1 \sqrt{210}(x-1)(x)(2x-1)(x+1)dx = -\frac{\sqrt{210}}{60}$$

$$\langle x+1 \mid o_3\rangle = \int_0^1 3\sqrt{10}\,x(x-1)(14x^2-14x+3)(x+1)\,dx = -\frac{3\sqrt{10}}{20}$$

Also let's normolaze them

$$o_1 = \frac{\tilde{e}_1}{\|\tilde{e}_1\|} = \frac{x(x-1)}{1/\sqrt{30}} = \sqrt{30}\,x(x-1)$$

$$o_2 = \frac{\frac{1}{2}x(x-1)(2x-1)}{1/\sqrt{840}} = \sqrt{210}\,x(x-1)(2x-1)$$

$$o_3 = \frac{x(x-1)\left(x^2 - \frac{2x-1}{2} - \frac{2}{7}\right)}{1/\sqrt{17640}} = 3\sqrt{10}\,x(x-1)(14x^2-14x+3)$$

let $\bar{v} = x - 1$

$$\bar{d} = \bar{v} - Pr_o(\bar{v}) = \bar{v} - \langle \bar{v}|o_1\rangle o_1 - \langle \bar{v}|o_2\rangle o_2 - \langle \bar{v}|o_3\rangle o_3 =$$

$$= (x+1) + \frac{\sqrt{30}}{4}\sqrt{30}\,x(x-1) + \frac{\sqrt{210}}{60}\sqrt{210}\,x(x-1)(2x-1) + \frac{3\sqrt{10}}{20}3\sqrt{10}\,x(x-1)(14x^2-14x+3) =$$

$$= \frac{17}{2}x - \frac{13}{2} + 7x^2 - \frac{21}{2}x + \frac{7}{2} + \frac{126x^4 - 252x^3 + 153x^2 - 27x}{2} = \frac{126x^4 - 252x^3 + 167x^2 - 31x}{2} - 3 =$$

$$= 63x^4 - 126x^3 + \frac{167}{2}x^2 - \frac{31}{2}x - 3$$

Thus $\|d\| = \sqrt{\int_0^1 \left(63x^4 - 126x^3 + \frac{167}{2}x^2 - \frac{31}{2}x - 3\right)^2 dx} = \sqrt{\frac{661}{120}}$ is the distance.

2. (1 point) Let $(\mathrm{Mat}_2(\mathbb{R}), \langle\,|\,\rangle)$ be a Euclidean space with the scalar product

$$\langle A|B\rangle = \mathrm{tr}\big(A^{\mathrm{T}}B\big)\,, \quad \text{for every } A,\, B \in \mathrm{Mat}_2(\mathbb{R}).$$

Then, find the 2-volume of the 2-parallelotope $P(A_1, A_2)$, where $A_1 = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix}$ and $A_2 = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$.

[**hint:** use Theorem 31.3]

$$V(P(A_1 A_2)) = \sqrt{\det(G(A_1 A_2)}$$

$$< A_1\ A_1 > = tr\left(\begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix}\right) = 6$$

$$< A_1\ A_2 > = tr\left(\begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}\right) = -5 = < A_2\ A_1 >$$

$$< A_2\ A_2 > = tr\left(\begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}\right) = 7$$

thus $G(A_1 A_2) = \begin{bmatrix} 6 & -5 \\ -5 & 7 \end{bmatrix} \Rightarrow \det(G) = 17 \Rightarrow V = \sqrt{17}$

3. (1 point per item) Let $(\mathbb{R}^3, \langle\ |\ \rangle)$ be a Euclidean space with the standard scalar product

$$\langle \mathbf{x}|\mathbf{y}\rangle = x_1y_1 + x_2y_2 + x_3y_3, \quad \text{for every } \mathbf{x} = [x_1\ x_2\ x_3]^{\mathrm{T}},\ \mathbf{y} = [y_1\ y_2\ y_3]^{\mathrm{T}} \in \mathbb{R}^3.$$

Let

$$\mathbf{a} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -3 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

be three vectors from $\mathbb{R}^3$. Then:

(a) using Item 1 of Theorem 32.1, find $\mathbf{a} \times \mathbf{b}$;

(b) using Item 5 of Theorem 32.1, find $(\mathbf{a} \times \mathbf{c}) \times \mathbf{b}$.
[**hint:** since the cross product is *not* associative, that is $(\mathbf{a} \times \mathbf{c}) \times \mathbf{b} \neq \mathbf{a} \times (\mathbf{c} \times \mathbf{b})$, first you need to use Item 2 of Theorem 32.1: $(\mathbf{a} \times \mathbf{c}) \times \mathbf{b} = -\mathbf{b} \times (\mathbf{a} \times \mathbf{c})$; pay attention to the order of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$]

positively oriented

$$(e_1\ e_2\ e_3) = \left(\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

orthonormal basis for $\mathbb{R}^3$

a) $$a \times b = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} e_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} e_2 - \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} e_3 =$$

$$= \begin{vmatrix} -2 & 0 \\ 0 & -1 \end{vmatrix} \begin{bmatrix}1\\0\\0\end{bmatrix} - \begin{vmatrix} 3 & 0 \\ -3 & -1 \end{vmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} - \begin{vmatrix} 3 & -2 \\ -3 & 0 \end{vmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}2\\3\\6\end{bmatrix}$$

b) $$a \times c = \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} e_1 - \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} e_1 - \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} e_3 =$$

$$= \begin{vmatrix} -2 & 0 \\ 1 & 1 \end{vmatrix} e_1 - \begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix} e_2 - \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} e_3 = \begin{bmatrix}-2\\-3\\-7\end{bmatrix} = \bar{v}$$

$$(a \times c) \times b = \bar{v} \times b = \begin{vmatrix} -3 & -7 \\ 0 & -1 \end{vmatrix} e_1 - \begin{vmatrix} -2 & -7 \\ -3 & -1 \end{vmatrix} e_2 - \begin{vmatrix} -2 & -3 \\ -3 & 0 \end{vmatrix} e_3 = \begin{bmatrix}3\\19\\9\end{bmatrix}$$

4. (1 point) Let $(\mathbb{V}, \langle\ |\ \rangle)$ be a three dimensional Euclidean space and let $\mathbf{a}, \mathbf{b} \in \mathbb{V}$ be such that $\|\mathbf{a}\| = 5$ and $\|\mathbf{b}\| = 3$. Then, find the value of the expression $|\mathbf{a} \times \mathbf{b}|^2 + \langle \mathbf{a}|\mathbf{b}\rangle^2$.

[**hint:** recall that, for every $\mathbf{x}, \mathbf{y} \in \mathbb{V}$, we have $\langle \mathbf{x}|\mathbf{y}\rangle = \|\mathbf{x}\| \cdot \|\mathbf{y}\| \cos(\measuredangle \mathbf{xy})$ (see Definition 29.4) and $\|\mathbf{x} \times \mathbf{y}\| = \|\mathbf{x}\| \cdot \|\mathbf{y}\| \sin(\measuredangle \mathbf{xy})$ (see seminar)]

$$|a \times b|^2 + \langle a|b\rangle^2 = \|a\|^2 \|b\|^2 = 25 \cdot 9 = 225$$

by Pythagorean identity for 3dim vectors in ES.

If you never ever run into it, here is the proof:

(it's trivial)

(1) $\|a\| = \sqrt{\langle a|a\rangle}$

(2) $\|a \times b\| = \|a\| \|b\| \sin(\angle ab)$ ← well known def.

(3) $\langle a|b\rangle = \|a\| \|b\| \cos(\angle ab)$

(4) Square of magnitudes (norms) of the cross product we get

$$|a \times b|^2 = (\|a\| \|b\| \sin(\overset{\angle ab}{\theta}))^2 = \|a\|^2 \|b\|^2 \sin^2(\theta)$$

$$\langle a|b\rangle^2 = (\|a\| \|b\| \cos(\theta))^2 = \|a\|^2 \|b\|^2 \cos^2(\theta)$$

(5) sum them!

$$|a \times b|^2 + \langle a|b\rangle^2 = \|a\|^2 \|b\|^2 \sin^2(\theta) + \|a\|^2 \|b\|^2 \cos^2(\theta) =$$

$$= \|a\|^2 \|b\|^2 \underbrace{(\sin^2(\theta) + \cos^2(\theta))}_{= 1 \text{, by main trig identity}} = \|a\|^2 \|b\|^2 \quad \blacksquare$$

5. (1 point per item) Let $(\mathbb{V}, \langle \, | \, \rangle)$ be a Euclidean space with ordered bases $\mathcal{A} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ and $\mathcal{B} = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$, where $\mathbf{b}_1 = -\mathbf{a}_1 + 3\mathbf{a}_2 + \mathbf{a}_3$, $\mathbf{b}_2 = -2\mathbf{a}_1 + 5\mathbf{a}_2 + \mathbf{a}_3$, $\mathbf{b}_3 = \mathbf{a}_2 - \mathbf{a}_3$.

Then:

(a) if the ordered basis $\mathcal{A}$ is positively oriented, determine whether $\mathcal{B}$ is positively oriented or negatively oriented (with respect to $\mathcal{A}$);

[**hint:** see Definitions 32.1 and 32.2]

(b) find $\|\mathbf{b}_1 \times \mathbf{b}_2\|$ if $\|\mathbf{a}_1\| = 2$, $\|\mathbf{a}_2\| = 3$, $\|\mathbf{a}_3\| = 5$, $\measuredangle \mathbf{a}_1\mathbf{a}_2 = \pi/4$, $\measuredangle \mathbf{a}_1\mathbf{a}_3 = \pi/3$, and $\measuredangle \mathbf{a}_2\mathbf{a}_3 = \pi/6$;

[**hint:** use Items 2), 3), and 4) of Theorem 32.1 and equality $\|\mathbf{x} \times \mathbf{y}\| = \|\mathbf{x}\| \cdot \|\mathbf{y}\| \sin(\measuredangle \mathbf{xy})$, for every $\mathbf{x}, \mathbf{y} \in \mathbb{V}$]

a) $C(A,B) = \begin{bmatrix} -1 & -2 & 0 \\ 3 & 5 & 1 \\ 1 & 1 & -1 \end{bmatrix} = -2 \Rightarrow$ B have opposite orientations with A,

Thus B is negative oriented

b) $b_1 = -a_1 + 3a_2 - a_3$

$b_2 = -2a_1 + 5a_2 + a_3$

$b_1 \times b_2 = (-a_1 + 3a_2 - a_3) \times (-2a_1 + 5a_2 + a_3) =$

since $(a \times a) = 0 \quad \forall a \in V$
'cause $\sin(\measuredangle aa) = 0$

$= 2\underbrace{(a_1 \times a_1)}_{=0} - 5(a_1 \times a_2) - (a_1 \times a_3) - 6(a_2 \times a_1) +$

$+ 15\underbrace{(a_2 \times a_2)}_{=0} + 3(a_2 \times a_3) + 2(a_3 \times a_1) - 5(a_3 \times a_2) - \underbrace{(a_3 \times a_3)}_{=0} =$

$= -5(a_1 \times a_2) + 6(a_1 \times a_2) - (a_1 \times a_3) - 2(a_1 \times a_3) + 3(a_2 \times a_3) + 5(a_2 \times a_3) =$

$= (a_1 \times a_2) - 3(a_1 \times a_3) + 8(a_1 \times a_3) = \frac{3\pi}{2} - 10\pi + 20\pi = \frac{23\pi}{2}$

$a_1 \times a_2 = 2 \cdot 3 \cdot \frac{\pi}{4} = \frac{3\pi}{2}$

$a_1 \times a_3 = 2 \cdot 5 \cdot \frac{\pi}{3} = \frac{10\pi}{3}$

$a_2 \times a_3 = 3 \cdot 5 \cdot \frac{\pi}{6} = \frac{5\pi}{2}$

6. Let $(\mathbb{V}, \langle\ |\ \rangle)$ be a three dimensional Euclidean space. Then, following instructions, for every $\mathbf{a}, \mathbf{b} \in \mathbb{V}$, describe all $\mathbf{x} \in \mathbb{V}$ such that

$$\mathbf{a} \times \mathbf{x} = \mathbf{b}. \tag{1}$$

That is, describe the solution set of Equation (1).

Instructions:

(a) (0.5 points) describe the solution set of Equation (1) if $\mathbf{a} = \mathbf{0}$;

[**hint:** for example, use Item 3 of Theorem 32.1]

(b) (0.5 points) describe the solution set of Equation (1) if the vectors $\mathbf{a}$ and $\mathbf{b}$ are *not* orthogonal (that is, $\langle \mathbf{a}|\mathbf{b}\rangle \neq 0$);

[**hint:** use Definition 32.3]

(c) (1 point) assuming that $\mathbf{a} \neq \mathbf{0}$ and the vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (that is, $\langle \mathbf{a}|\mathbf{b}\rangle = 0$), verify that every vector of the form

$$\mathbf{x} = \frac{\mathbf{b} \times \mathbf{a}}{\|\mathbf{a}\|^2} + t\mathbf{a}, \quad \text{where } t \in \mathbb{R}, \tag{2}$$

is a solution to Equation (1);

[**hint:** substitute (2) into (1), use Items 3 and 5 of Theorem 32.1 and the assumption $\langle \mathbf{a}|\mathbf{b}\rangle = 0$]

(d) (1 point) assuming that $\mathbf{a} \neq \mathbf{0}$, show that every solution $\mathbf{x}$ of Equation (1) is of the Form (2).

[**hint:** if $\mathbf{x}$ is a solution to Equation (1), then, equality $\mathbf{a} \times \mathbf{x} = \mathbf{b}$ implies that $(\mathbf{a} \times \mathbf{x}) \times \mathbf{a} = \mathbf{b} \times \mathbf{a}$ (do you understand why?); using the last equality and Items 5 of Theorem 32.1, show that $\mathbf{x}$ ought to be of the form (2)]

a) $\bar{o} \times X = b \iff \|\bar{o}\| \|x\| \sin(\angle \bar{o} x) = b \Rightarrow \begin{cases} b = 0 \\ \forall x \in \mathbb{V} \end{cases}$

b) $\begin{cases} \langle a \mid a \times X \rangle = 0 \\ \langle x \mid a \times X \rangle = 0 \end{cases}$ that's cross-product of $\bar{a} \times \bar{b}$ is orthogonato to $\bar{a}$ and $\bar{b}$.

$\Downarrow$

Therefore b must be orthogonal to a, if not there no solutions for X.

c) $a \times X = a \times \left(\frac{b \times a}{\langle a | a \rangle} + ta \right) \overset{(1)}{=} a \times \left(\frac{b \times a}{\langle a | a \rangle} \right) + t\underbrace{(a \times a)}_{=0} = a \times \left(\frac{b \times a}{\langle a | a \rangle} \right) \ominus$

using the vector triple product identity $u \times (v \times w) = (u \cdot w) v - (u \cdot v) w$

$\ominus \quad \frac{\langle a | a \rangle b - \overbrace{\langle a | b \rangle}^{=0} a}{\langle a | a \rangle} = \frac{\langle a | a \rangle b}{\langle a | a \rangle} = b \Rightarrow$ it's indeed a root

d) tricky way:

Since, if $\langle a | b \rangle = 0$ and $a \neq 0$, then, from (1) it follows that $x = \frac{b \times a}{\|a\|^2} = \frac{b \times a}{\langle a | a \rangle}$ is a particular solution, let's denote it x_p.

Thus, to show that (2) is the general solution, consider any vector X that satisfies $a \times X = b$.

Any such X can be written as the sum of a particular solution X_p and vector parallel to a:

$$X = X_p + a$$

But we've already shown that $X_p = \frac{b \times a}{\langle a | a \rangle}$ is a particular solution. Therefore, the general solution is

$X = \frac{b \times a}{\langle a | a \rangle} + ta$, where $t \in \mathbb{R}$ ■