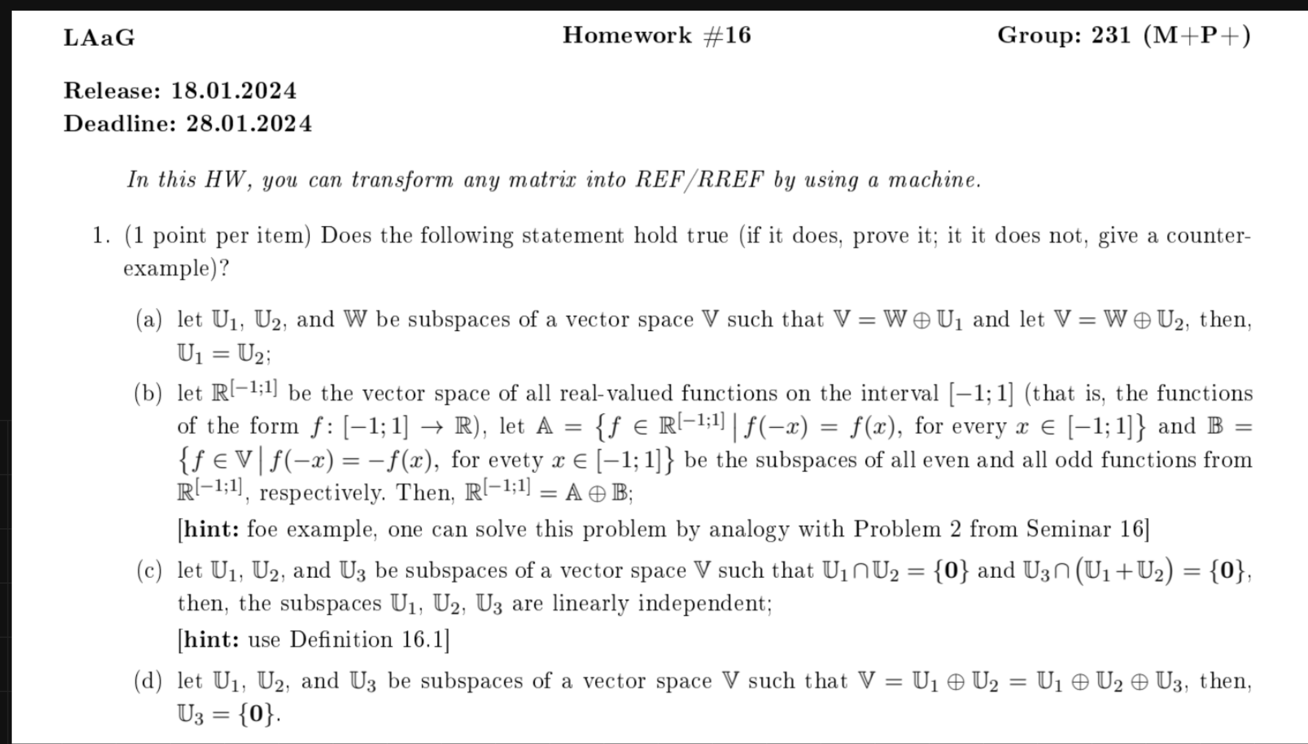

LAaG — Homework #16 — Group: 231 (M+P+)

Release: 18.01.2024
Deadline: 28.01.2024

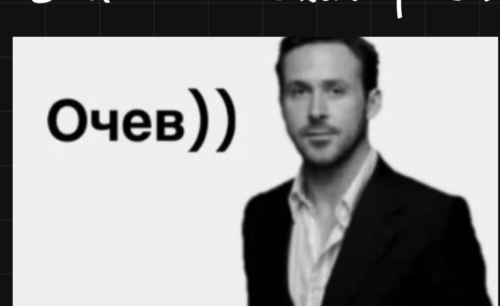
In this HW, you can transform any matrix into REF/RREF by using a machine.

1. (1 point per item) Does the following statement hold true (if it does, prove it; it it does not, give a counter-example)?
   (a) let $\mathbb{U}_1$, $\mathbb{U}_2$, and $\mathbb{W}$ be subspaces of a vector space $\mathbb{V}$ such that $\mathbb{V} = \mathbb{W} \oplus \mathbb{U}_1$ and let $\mathbb{V} = \mathbb{W} \oplus \mathbb{U}_2$, then, $\mathbb{U}_1 = \mathbb{U}_2$;
   (b) let $\mathbb{R}^{[-1;1]}$ be the vector space of all real-valued functions on the interval $[-1;1]$ (that is, the functions of the form $f\colon [-1;1] \to \mathbb{R}$), let $\mathbb{A} = \{f \in \mathbb{R}^{[-1;1]} \mid f(-x) = f(x),\ \text{for every } x \in [-1;1]\}$ and $\mathbb{B} = \{f \in \mathbb{V} \mid f(-x) = -f(x),\ \text{for evety } x \in [-1;1]\}$ be the subspaces of all even and all odd functions from $\mathbb{R}^{[-1;1]}$, respectively. Then, $\mathbb{R}^{[-1;1]} = \mathbb{A} \oplus \mathbb{B}$;
   [**hint:** foe example, one can solve this problem by analogy with Problem 2 from Seminar 16]
   (c) let $\mathbb{U}_1$, $\mathbb{U}_2$, and $\mathbb{U}_3$ be subspaces of a vector space $\mathbb{V}$ such that $\mathbb{U}_1 \cap \mathbb{U}_2 = \{\mathbf{0}\}$ and $\mathbb{U}_3 \cap (\mathbb{U}_1 + \mathbb{U}_2) = \{\mathbf{0}\}$, then, the subspaces $\mathbb{U}_1$, $\mathbb{U}_2$, $\mathbb{U}_3$ are linearly independent;
   [**hint:** use Definition 16.1]
   (d) let $\mathbb{U}_1$, $\mathbb{U}_2$, and $\mathbb{U}_3$ be subspaces of a vector space $\mathbb{V}$ such that $\mathbb{V} = \mathbb{U}_1 \oplus \mathbb{U}_2 = \mathbb{U}_1 \oplus \mathbb{U}_2 \oplus \mathbb{U}_3$, then, $\mathbb{U}_3 = \{\mathbf{0}\}$.

a) Let V be a vector space of n dimention, then
Let W be a vector space of $n-k$ dimention, hence
U_1 and U_2 are vector spaces of k dimention, hence
since they are direct complements, their linear spans
are equal. $\Rightarrow U_1 = U_2$

b) since odd function it's a function s.t. $-f(x) = -f(-x)$ $\wedge$ even $f(x) = f(-x)$
and $f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}$ $\to$ simple algebraic trick; it follows
that any function can be expressed using odd and even functions.
A is a subspace of even functions $\wedge$ B is a subspace of odd functions
$\Rightarrow A \oplus B = \mathbb{R}^{[-1;1]}$

c) counter-example:

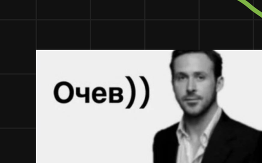
d) 

; $V = U_1 \oplus U_2 \ \wedge\ V = U_1 \oplus U_2 \oplus U_3$
$V = U_1 \oplus U_2 \oplus U_3 \Rightarrow V = V \oplus U_3 \Rightarrow$
$\Rightarrow \dim(V) = \dim(V) + \dim(U_3) \Rightarrow \dim(U_3) = 0 \Leftrightarrow U_3 = \{0\}$

② a) $U - \begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{bmatrix}$; $S - \begin{bmatrix} 0 & k & l & m \\ -k & 0 & n & o \\ -l & n & 0 & p \\ -m & o & p & 0 \end{bmatrix}$

then basis for $U = \langle e_{11}; e_{12}; e_{13}; e_{14}, e_{22}; e_{23}; e_{24}, e_{33}; e_{34}, e_{44} \rangle$
hence $\dim(U) = 10$, where e_{ij} = is a Mat_4 with all zeros except ij element
and also $S = \langle l_{12}; l_{13}, l_{14}; l_{23}, l_{24}; l_{34} \rangle$
, where l_{ij} is a Mat_4 with all zeros except ij element and ji element $\underline{-a_{ji} = a_{ji}}$

hence $\dim(S) = 6 \Rightarrow \dim(\mathbb{V}) = \dim(U) + \dim(S) \Leftrightarrow \mathbb{V} = U \oplus S$ $\wedge$

$\wedge$ since $\dim(\mathbb{V}) = \dim(U) + \dim(S) \Rightarrow \mathbb{V} = U \oplus S$ 

b) since "lower part" of Mat A we can obtain only from U, we already know, that:

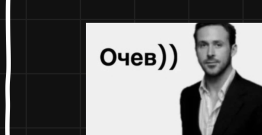
$C = \begin{bmatrix} 0 & 2 & -3 & -2 \\ -2 & 0 & 3 & 1 \\ 3 & -3 & 0 & 1 \\ 2 & -1 & -1 & 0 \end{bmatrix}$ and now it ought to be clear that Mat $B = \begin{bmatrix} 1 & -2 & 5 & 3 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$, that's it

$$\begin{bmatrix} 0 & 2 & -3 & -2 \\ -2 & 0 & 3 & 1 \\ 3 & -3 & 0 & 1 \\ 2 & -1 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 5 & 3 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 1 \\ -2 & 2 & 3 & -1 \\ 3 & -3 & 1 & 1 \\ 2 & -1 & -1 & 5 \end{bmatrix}$$

③ a) $\begin{cases} x_1 + x_3 + x_5 = 0 \\ x_2 + x_4 = 0 \end{cases} \Leftrightarrow \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right] \end{array} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -\alpha - \varphi \\ -\beta \\ \alpha \\ \beta \\ \varphi \end{bmatrix} \Rightarrow \mathbb{U}_2 = \left\langle \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle$

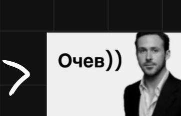
hence if all five vectors from basis are LI, then
by it's automaticly true. $\begin{bmatrix} -1 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 1 & 5 \end{bmatrix} \xrightarrow{RREF} I_5$, hence they are LI. ∎

b) $\left[\begin{array}{ccccc|c} -1 & 0 & -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 2 & -5 \\ 1 & 0 & 0 & 0 & 3 & 2 \\ 0 & 1 & 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & 1 & 5 & -2 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -8 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow X = -\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 7\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - 8\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \Rightarrow$

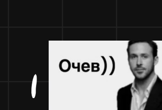
$\Rightarrow X = \underbrace{\begin{bmatrix} 0 \\ -7 \\ -1 \\ 7 \\ 1 \end{bmatrix}}_{x_1} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ -8 \\ -8 \end{bmatrix}}_{x_2} + \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}}_{x_3}$ 

, that $x_1 \in \mathbb{U}_1 \wedge x_2 \in \mathbb{U}_2 \wedge x_3 \in \mathbb{U}_3$

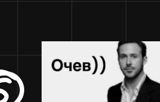
④ find a basis for $\mathbb{U}$: $\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 - x_4 = 0 \end{cases} \Leftrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right] \Leftrightarrow$

$\Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\alpha + \beta \\ \alpha \\ -2\beta \\ \beta \end{bmatrix} \Rightarrow \mathbb{U} = \left\langle \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{u_2} \right\rangle$ 

then $\mathbb{R}^4 = \langle u_1; u_2; e_1; e_2; e_3; e_4 \rangle$, thus $\begin{bmatrix} 1 & 1 & 1 & & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & \\ 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \end{array}$

hence direct complement to $\mathbb{U} = \underbrace{\left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\rangle}_{Dr}$, 

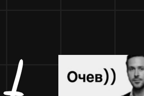
that $\mathbb{R}^4 = \mathbb{U} \oplus Dr$

⑤ 

Let $\mathbb{U}_1 + \dots \mathbb{U}_k$ all be subspaces of a vector space V over a field $\mathbb{F}$. def LI.
Then, the subspaces $\mathbb{U}_1 + \dots + \mathbb{U}_k$ are called LI if any $u_i \in \mathbb{U}_i$; $i \in [k]$, the equality $u_1 + \dots u_k = \bar{0}$

Proof that $\forall i \in [k]$, $\mathbb{U}_i \cap (\mathbb{U}_1 + \dots \mathbb{U}_{i-1} + \mathbb{U}_{i+1} + \dots \mathbb{U}_k) = \{0\} \Leftrightarrow \mathbb{U}_1, \mathbb{U}_2 \dots \mathbb{U}_k$ are LI
Due to Definition, we need to prove that for any $u_i \in \mathbb{U}_i$; $i \in [k]$ the equality
$u_1 + \dots + u_k = 0$ implies $u_i = \bar{0}$ for all $i \in [k]$. if $u_1 + \dots + u_k = \bar{0}$, then for any $i \in [k]$
we can write $\underset{\in \mathbb{U}_i}{u_i}$ as $\underset{\in \mathbb{U}_1 + \dots \mathbb{U}_{i-1} + \mathbb{U}_{i+1} \dots \mathbb{U}_k}{\underline{-(u_1 + \dots u_{i-1} + u_{i+1} + \dots u_k)}}$

Thus, $u_i \in \mathbb{U}_i \cap (\mathbb{U}_1 + \dots + \mathbb{U}_{i-1} + \mathbb{U}_{i+1} + \dots + \mathbb{U}_k) = \{0\}$, that is, $u_i = \bar{0}$ for all $i \in [k]$

Hence $\mathbb{U}_i \cap \left(\sum_{j=1}^{i-1} \mathbb{U}_j\right) = \{0\}$, $i \in \{2, \dots k\}$ it's just 
part of nesessary condition.

it just sum up to $(i-1)$, like this part

Thus, $u_i \in \mathbb{U}_i \cap (\underbrace{\mathbb{U}_1 + \dots + \mathbb{U}_{i-1}}_{\text{up to there!}} + \mathbb{U}_{i+1} + \dots + \mathbb{U}_k) = \{0\}$, that is, $u_i = \bar{0}$ for all $i \in [k]$