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Homework

1. Let $\circ\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be given by $(x,y)\mapsto x+y-\frac{\sqrt{\pi}}{2}$. Check if $(\mathbb{R},\circ)$ is a group or not.

1) Closure:

$\forall x \in G\ \forall y \in G:\ \left(x+y-\frac{\sqrt{\pi}}{2}\right) \in G$, since addition and substitution are binary operations

2) Neutral element:

$$x+\underset{\text{n.e.}}{e}-\frac{\sqrt{\pi}}{2}=x \Rightarrow e=\frac{\sqrt{\pi}}{2}$$

So $x+e-\frac{\sqrt{\pi}}{2}=x=e+x-\frac{\sqrt{\pi}}{2}$ $\quad (x,e)=(e,x)$

3) Inverse element

$$\forall x \in G\ \exists x^{-1}\ (x,x^{-1})=e=\frac{\sqrt{\pi}}{2}=(x^{-1},x)$$

$$x+x^{-1}-\frac{\sqrt{\pi}}{2}=\frac{\sqrt{\pi}}{2} \Leftrightarrow x+x^{-1}=\sqrt{\pi} \Leftrightarrow x^{-1}=\sqrt{\pi}-x$$

$$x^{-1}+x-\frac{\sqrt{\pi}}{2}=\frac{\sqrt{\pi}}{2} \Leftrightarrow x^{-1}+x=\sqrt{\pi} \Leftrightarrow x^{-1}=\sqrt{\pi}-x$$

4) Associativity:

$$\forall x,y,z \in G \quad x\circ(y\circ z)=(x\circ y)\circ z:$$

$$x\circ\left(y+z-\frac{\sqrt{\pi}}{2}\right)=\left(x+y-\frac{\sqrt{\pi}}{2}\right)\circ z$$

$$x+y+z-\frac{\sqrt{\pi}}{2}-\frac{\sqrt{\pi}}{2}=x+y-\frac{\sqrt{\pi}}{2}+z-\frac{\sqrt{\pi}}{2}$$

$$x+y+z-\sqrt{\pi}=x+y+z-\sqrt{\pi}$$

5) Commutativity: $x\circ y=y\circ x$:

$x+y-\frac{\sqrt{\pi}}{2}=y+x-\frac{\sqrt{\pi}}{2}$, since addition is commutative. ∎

2. Find all subgroups in $(\mathbb{Z}_7^*, \cdot)$.

$Z_7^* = \{1, 2, 3, 4, 5, 6\} = 6$

$H_1 = \{1\}$, generated by 1

$H_2 = \{1, 2, 4\}$, generated by 2, 4

$H_3 = \{1, 6\}$, generated by 6

$H_4 = \{1, 2, 3, 4, 5, 6\}$, generated by 3, 5

No other subgroups are present, as there two distinct elements of order one: 1 and 6

three: 2 and 4; six: 3, 5

No other elements are present $\Rightarrow$ no more subgroups.

3. Solve the equation $\tau^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ in S_4.

$\tau^2 = (1\ 3)(2\ 4)$

$\text{init} \xrightarrow{\tau} \text{mid} \xrightarrow{\tau^2} \text{result.}$

$$\left.\begin{array}{l} 1 \to x_1 \to 3 \\ 3 \to x_2 \to 1 \\ 2 \to x_3 \to 4 \\ 4 \to x_4 \to 2 \end{array}\right\} \text{ord} = 4$$

$1 \to 2 \to 3 \to 4$, or $1 \to 4 \to 3 \to 2$

So $\tau = (1\ 2\ 3\ 4)$ or $\tau = (1\ 4\ 3\ 2)$

4. For each element of $(\mathbb{Z}_{13}^*, \cdot)$, find its order and the inverse element.

$\mathbb{Z}_{13}^* : \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

Element:	Inverse:	Order:
1	1	1
2	7	12
3	9	3
4	10	6
5	8	4
6	11	12
7	2	12
8	5	4
9	3	3
10	4	6
11	6	12
12	12	2