1. Let $\mathbb{V}$ be a 3-dimensional vector space over the field of reals; let $\mathcal{A}$ be an ordered basis for $\mathbb{V}$; let

$$q(\mathbf{x}) = -3x_1x_2 + 2x_1x_3 - 7x_2x_3, \tag{1}$$

where $[\mathbf{x}]_{\mathcal{A}} = [x_1\ x_2\ x_3]^{\mathrm{T}}$, be a quadratic form on $\mathbb{V}$. Then:

(a) (3 points) using Lagrange's method for quadratic forms (see Theorem 27.1 and Problem 1 from Seminar 27), find a canonical basis of q (that is, find the change of basis matrix from $\mathcal{A}$ to your canonical basis);

a) $\begin{cases} x_1 = k_1 \\ x_2 = k_2 + k_3 \\ x_3 = k_2 - k_3 \end{cases}$ Rewrite $q(k)$ with respect to these coourdinate.

$$q(\bar{k}) = -3k_1(k_2+k_3) + 2k_1(k_2-k_3) - 7(k_2+k_3)(k_2-k_3) =$$

$$= -3k_1k_2 - 3k_1k_3 + 2k_1k_2 - 2k_1k_3 - 7k_2^2 + 7k_3^2 =$$

$$= -k_1k_2 - 5k_1k_3 - 7k_2^2 + 7k_3^2 = -\left(7k_2^2 + k_1k_2 + \frac{k_1^2}{28}\right) + \frac{k_1^2}{28} - 5k_1k_3 + 7k_3^2$$

$$= -\left(\sqrt{7}k_2 + \frac{1}{2\sqrt{7}}k_1\right)^2 + \frac{k_1^2}{28} - 5k_1k_3 + 7k_3^2 = -\left(\sqrt{7}k_2 + \frac{1}{2\sqrt{7}}k_1\right)^2 + \left(\frac{k_1}{2\sqrt{7}} - 5\sqrt{7}k_3\right)^2 - 168k_3^2$$

So $\begin{cases} y_1 = \sqrt{7}k_2 + \frac{1}{2\sqrt{7}}k_1 \\ y_2 = \frac{k_1}{2\sqrt{7}} - 5\sqrt{7}k_3 \\ y_3 = k_3 \end{cases}$ then $q(\bar{y}) = -y_1^2 + y_2^2 - 168y_3^2$

So canonical quadratic is $-y_1^2 + y_2^2 - 168y_3^2 \iff \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -168 \end{bmatrix}$

Transit matrix from $k \to y$: $\begin{bmatrix} 1/2\sqrt{7} & \sqrt{7} & 0 \\ 1/2\sqrt{7} & 0 & -5\sqrt{7} \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1/2\sqrt{7} & \sqrt{7} & 0 \\ 1/2\sqrt{7} & 0 & -5\sqrt{7} \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 2\sqrt{7} & 70 \\ \frac{\sqrt{7}}{7} & -\frac{\sqrt{7}}{7} & -5 \\ 0 & 0 & 1 \end{bmatrix}$ is for $y \to k$

Transit matrix from $x \to k$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$ is from $k \to x$

Hence

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}}_{[x\to k]^{T}} \underbrace{\begin{bmatrix} \frac{\sqrt{7}}{14} & \frac{\sqrt{7}}{14} & 0 \\ \sqrt{7} & 0 & 0 \\ 0 & -5\sqrt{7} & 1 \end{bmatrix}}_{[k\to y]^{T}} \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -168 \end{bmatrix}}_{H(q,A')} \underbrace{\begin{bmatrix} \frac{\sqrt{7}}{14} & \sqrt{7} & 0 \\ \frac{\sqrt{7}}{14} & 0 & -5\sqrt{7} \\ 0 & 0 & 1 \end{bmatrix}}_{y \leftarrow k} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & -1/2 \end{bmatrix}}_{k \leftarrow x} = H(q, A)$$

So $\begin{bmatrix} \frac{\sqrt{7}}{14} & \sqrt{7} & 0 \\ \frac{\sqrt{7}}{14} & 0 & -5\sqrt{7} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & -1/2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{7}}{14} & \frac{\sqrt{7}}{2} & \frac{\sqrt{7}}{2} \\ \frac{\sqrt{7}}{14} & \frac{-5\sqrt{7}}{2} & \frac{5\sqrt{7}}{2} \\ 0 & 1/2 & -1/2 \end{bmatrix}$ is a transit matrix C

$\left(\begin{bmatrix} \frac{\sqrt{7}}{14} \\ \frac{\sqrt{7}}{14} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{7}}{2} \\ \frac{-5\sqrt{7}}{2} \\ 1/2 \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{7}}{2} \\ \frac{5\sqrt{7}}{2} \\ -1/2 \end{bmatrix} \right)$ is a can. basis A', s.t. $H(q, A') = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 168 \end{bmatrix}$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

1. Let $\mathbb{V}$ be a 3-dimensional vector space over the field of reals; let $\mathcal{A}$ be an ordered basis for $\mathbb{V}$; let

$$q(\mathbf{x}) = -3x_1x_2 + 2x_1x_3 - 7x_2x_3,$$

(b) (1 point) find two non-zero vectors $\mathbf{x}, \ \mathbf{y} \in \mathbb{V}$ such that $q(\mathbf{x}) = q(\mathbf{y}) = 0$ and $q(\mathbf{x}+\mathbf{y}) \neq 0$;

[**hint**: it is possible to just guess them, but it is advisable to use a canonical form of q (see Item (a)) to solve the problem]

Just $\bar{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\bar{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, so $q(\bar{x}) = 0 \wedge q(\bar{y}) = 0$, obv.

but $q(\bar{x} + \bar{y}) = 2!$

1. Let $\mathbb{V}$ be a 3-dimensional vector space over the field of reals; let $\mathcal{A}$ be an ordered basis for $\mathbb{V}$; let

$$q(\mathbf{x}) = -3x_1x_2 + 2x_1x_3 - 7x_2x_3, \tag{1}$$

(c) (3 points) find an ordered basis $\mathcal{A}'$ of $\mathbb{V}$ (that is, find $C(\mathcal{A}, \mathcal{A}')$) such that

$$q(\mathbf{x}) = -y_1^2 - 7y_2^2 + 3y_3^2 + 6y_1y_2 + 2y_1y_3, \tag{2}$$

where $[\mathbf{x}]_{\mathcal{A}'} = [y_1\ y_2\ y_3]^{\mathrm{T}}$;[1]

[**hint**: applying Lagrange's method to Expression (2), find another canonical basis of q, say $\mathcal{D}'$; find $C(\mathcal{D}, \mathcal{D}')$, where $\mathcal{D}$ is a canonical basis from Item (a); now it should be clear how to find $C(\mathcal{A}, \mathcal{A}')$; it is not a part of the problem, but it is highly advisable to verify that your matrix $C(\mathcal{A}, \mathcal{A}')$ indeed satisfies the equality $H(q, \mathcal{A}') = C(\mathcal{A}, \mathcal{A}')^{\mathrm{T}} \cdot H(q, \mathcal{A}) \cdot C(\mathcal{A}, \mathcal{A}')$]

$$q(x) = -y_1^2 - 7y_2^2 + 3y_3^2 + 6y_1y_2 + 2y_1y_3 =$$

$$= -y_1^2 - 7y_2^2 + 3y_3^2 + 6y_1y_2 + 2y_1y_3 + \frac{9x^2}{7} - \frac{9x^2}{7} =$$

$$= \frac{2y_1^2}{7} + 2y_1y_3 + 3y_3^2 - 7\left(\frac{9y_1^2}{49} - \frac{6y_1y_2}{7} + y_2^2\right) =$$

$$= \frac{2y_1^2}{7} + 2y_1y_3 + 3y_3^2 - 7\left(-\frac{3}{7}y_1 + y_2\right)^2 + \frac{y_1^2}{3} - \frac{y_1^2}{3} =$$

$$= -\frac{y_1^2}{21} - 7\left(-\frac{3}{7}y_1 + y_2\right)^2 + \left(\frac{y_1^2}{3} + 2y_1y_3 + 3y_3^2\right) =$$

$$= -\frac{1}{21}y_1^2 - 7\left(-\frac{3}{7}y_1 + y_2\right)^2 + 3\left(\frac{y_1^2}{9} + \frac{2y_1y_3}{3} + y_3^2\right) =$$

$$= -\frac{1}{21}y_1^2 - 7\left(y_2 - \frac{3}{7}y_1\right)^2 + 3\left(\frac{1}{3}y_1 + y_3\right)^2$$

So $\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ -3/7\, y_1 + y_2 \\ 1/3\, y_1 + y_3 \end{bmatrix}$

Hence $\begin{bmatrix} 1 & 0 & 0 \\ -3/7 & 1 & 0 \\ 1/3 & 0 & 1 \end{bmatrix}$ is change of coordinates

$$\begin{bmatrix} 1 & -3/7 & 1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1/21 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3/7 & 1 & 0 \\ 1/3 & 0 & 1 \end{bmatrix}$$

So $\left(\begin{bmatrix} 1 \\ -3/7 \\ 1/3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = D'$, so $C(D, D') = \begin{bmatrix} \frac{\sqrt{7}}{14} & \frac{\sqrt{7}}{2} & \frac{\sqrt{7}}{2} \\ \frac{5\sqrt{7}}{49} & \frac{19\sqrt{7}}{7} & \frac{19\sqrt{7}}{2} \\ -\frac{\sqrt{7}}{42} & \frac{3-\sqrt{7}}{6} & \frac{-\sqrt{7}-3}{6} \end{bmatrix}$

Thus $C(A, A') = C(A, D)\,C(D, D') = \begin{bmatrix} 1 & 0 & 0 \\ -3/7 & 1 & 0 \\ 1/3 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{7}}{14} & \frac{\sqrt{7}}{2} & \frac{\sqrt{7}}{2} \\ \frac{5\sqrt{7}}{49} & \frac{19\sqrt{7}}{7} & \frac{19\sqrt{7}}{2} \\ -\frac{\sqrt{7}}{42} & \frac{3-\sqrt{7}}{6} & \frac{-\sqrt{7}-3}{6} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{7}}{14} & \frac{\sqrt{7}}{2} & \frac{\sqrt{7}}{2} \\ \frac{\sqrt{7}}{14} & \frac{5\sqrt{7}}{2} & \frac{65\sqrt{7}}{7} \\ 0 & 1/2 & -1/2 \end{bmatrix}$

$$\begin{array}{ccc} \overline{\bar{a}_1'} & \overline{\bar{a}_2'} & \overline{\bar{a}_3} \end{array}$$

So $\langle(\bar{a}_1'\ \bar{a}_2'\ \bar{a}_3')\rangle$ is A'

(d) (1 point) using Expression (2) and Jacobi's theorem (see Problem 2 from Seminar 27), find a canonical form of q;[2]

(e) (2 points) is there an ordered basis $\mathcal{A}''$ of $\mathbb{V}$ such that

$$q(\mathbf{x}) = 3z_1^2 + 9z_2^2 + 5z_3^2 - 10z_1z_2 + 2z_1z_3 \tag{3}$$

where $[\mathbf{x}]_{\mathcal{A}''} = [z_1\ z_2\ z_3]^{\mathrm{T}}$?

[**hint**: using Expression (3) and Jacobi's theorem, find a canonical form of q; take a look at Item (b)]

d) $q(x) = -y_1^2 - 7y_2^2 + 3y_3^2 + 6y_1y_2 + 2y_1y_3$

$$H(q,B) = \begin{bmatrix} -1 & 3 & 1 \\ 3 & -7 & 0 \\ 1 & 0 & 3 \end{bmatrix}, \quad \delta_1 = -1,\ \delta_2 = -2,\ \delta_3 = 1$$

so $H(q,B') = \begin{bmatrix} \frac{\delta_1}{1} & 0 & 0 \\ 0 & \frac{\delta_2}{\delta_1} & 0 \\ 0 & 0 & \frac{\delta_3}{\delta_2} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ - is a can. of $q(x)$

can. form of $q(x) = -x_1^2 + 2x_2^2 - 2x_3^2$

e) $\begin{bmatrix} 3 & -5 & 1 \\ -5 & 9 & 0 \\ 1 & 0 & 5 \end{bmatrix}$, so $\delta_1 = 3,\ \delta_2 = 2,\ \delta_3 = 1$, so $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$ is a can. form of q