Novosad Ivan

1. We are given a system of equations with real coefficients:

$$(*) \begin{cases} x^2 + 2y^2 = 3 \\ x^2 + xy + y^2 = 3 \end{cases}$$

Solve the system using the following steps:

(a) Find a Gröbner basis $G$ of $\{x^2+2y^2-3, x^2+xy+y^2-3\}$ using Lex$(x,y)$. It turns out that solving the initial system is the same as solving the system $g_1 = 0, \ldots, g_k = 0$ for $g_i \in G$.

(b) In the system $g_1 = 0, \ldots, g_k = 0$ for $g_i \in G$, solve the equations depending of $y$ only.

(c) Solve the system $g_1 = 0, \ldots, g_k = 0$ for $g_i \in G$. This will give you the solution of the initial system.

a) Using Buchberger's algorithm:

1. $f_1 = x^2 + 2y^2 - 3$

   $f_2 = x^2 + xy + y^2 - 3$

2. Compute S-polynomials:

$$S(f,g) = \frac{LCM(LT(f), LT(g))}{LT(f)} f - \frac{LCM(LT(f), LT(g))}{LT(g)} g$$

where $LT(f)$ is the leading monomial (term) of $f$.

Thus for $f_1$ and $f_2$ the leading terms are both $x^2$. Thus, the S-polynomial is:

$$S(f_1, f_2) = \frac{x^2}{x^2} f_1 - \frac{x^2}{x^2} f_2 = (x^2 + 2y^2 - 3) - (x^2 + xy + y^2 - 3) = y^2 - xy$$

3. Reduce the S-Polynomial with respect to $f_1$ and $f_2$

   However, since $y^2 - xy$ does not contain $x^2$, it's already a remainder

4. Add the Reduced S-Polynomial to the Basis

   Now, our basis is $\{f_1, f_2, y^2 - xy\}$

5. Check for New Polynomial:

   We need to check the S-polynomial of new basis elements:

   - $S(f_1, y^2 - xy)$
   - $S(f_2, y^2 - xy)$

   However, since $y^2 - xy$ is already reduced with respect do $f_1$ and $f_2$, no new S-polynomials will be that are not already in the ideal

b) Solve the Equations Depending only on $y$:

From the Grobner basis $G$, we have the polynomial

$$y^2 - xy \Rightarrow y^2 - xy = 0 \Leftrightarrow y(y-x) = 0 \Rightarrow \left[\begin{array}{l} y = 0 \\ y = x \end{array}\right.$$

b) Solve the Equations Depending only on $y$:

From the Grobner basis $G$, we have the polynomial

$$y^2 - xy \Rightarrow y^2 - xy = 0 \Leftrightarrow y(y-x) = 0 \Rightarrow \left[\begin{array}{l} y = 0 \\ y = x \end{array}\right.$$

c) Solve the system $g_1 = 0, \dots g_k = 0$ for $g_i \in G$

Now, we substitute the solutions for $y$ back into the original equations $\circledast$

Case $y = 0$: $x^2 = 3 \Rightarrow x = \pm\sqrt{3}$

So, the solutions for $y=0$ are: $\left[\begin{array}{l} (x,y) = (0, \sqrt{3}) \\ (x,y) = (0, -\sqrt{3}) \end{array}\right.$

Case $y = x$: $x^2 + 2x^2 = 3 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

So, the solutions for $y = x$ are: $\left[\begin{array}{l} (x,y) = (1, 1) \\ (x,y) = (-1,-1) \end{array}\right.$

Thus all (combined) solutions are: $\left[\begin{array}{l} (x,y) = (0, \sqrt{3}) \\ (x,y) = (0, -\sqrt{3}) \\ (x,y) = (1, 1) \\ (x,y) = (-1,-1) \end{array}\right.$

2. We are given $\mathbb{R}[x,y,z]$, $I = (2y^2 + yz, xy + z)$, $g_1 = xz^3 + 4yz^2$, and $g_2 = xz^2 + 4yz^3$. Does $g_1$ or $g_2$ belong to $I$?

To determine whether $g_1$ or $g_2$ belongs to ideal $I$, generated by $2y^2+yz$ and $xy+z$ in the polynomial ring we can use Gröbner basis reduction approach:

1. Define the generators of the ideal $I$:

$$f_1 = 2y^2+yz \qquad f_2 = xy+z$$

2. Compute S polynomials:

$$S_{f_1f_2} = x(2y^2+yz) - 2y(xy+z) = 2xy^2 + xyz - 2xy^2 - 2yz = xyz - 2yz$$

3. Reduce it with respect to $G$:

change nothing in ideal

$$xyz - 2yz - z(xy+z) = xyz - 2yz - xyz - z^2 = -2yz - z^2 \approx 2yz + z^2$$

Now it's irreducible with respect to $f_1$ and $f_2$, so add it to basis

Repeat again

1. $S_{f_1f_3} = z(2y^2+yz) - y(2yz+z^2) = yz^2 - yz^2 = 0$

$S_{f_2f_3} = 2z(xy+z) - x(2yz+z^2) = 2z^2 - xz^2 \approx xz^2 - 2z^2$

2. Reduce $S_{f_2f_3}$ with respect to $G$, but it's already reduced, so add it to $G$

Repeat:

1. $f_1$ and $f_4$ are co-prime

$S_{f_2f_4} = z^2(xy+z) - y(xz^2 - 2z^2) = z^3 + 2yz^2$

$S_{f_3f_4} = xz(2yz+z^2) - 2y(xz^2 - 2z^2) = xz^3 + 4yz^2$

2. Reduce $S_{f_2f_4}$ and $S_{f_3f_4}$ with respect to $\{2y^2+yz,\ xy+z,\ 2yz+z^2,\ xz^2-2z^2\}$

$$2yz^2 + z^3 - z(2yz+z^2) = 0$$

$$xz^3 + 4yz^2 - z(xz^2 - 2z^2) = 4yz^2 + 2z^3 - 2z(2yz+z^2) = 0$$

Thus $\{2y^2+yz,\ xy+z,\ 2yz+z^2,\ xz^2-2z^2\}$ is a Gröbner Basis.

Now we need to reduce $g_1$ and $g_2$ with respect to $G$, if it's remainder is 0, then it belongs to $I$;

1. Reduce $g_1$: $xz^3 + 4yz^2 - z(xz^2 - 2z^2) \to 4yz^2 + 2z^3 - 2z(2yz+z^2) = 0$

Thus $g_1$ belongs to $I$

2. Reduce $g_2$: $xz^2 + 4yz^3 - (xz^2 - 2z^2) \to 4yz^3 + 2z^2 - 2z^2(2yz+z^2) \longrightarrow$

$\longrightarrow -2z^4 + 2z^2$, and it's no longer reducible, thus $g_2$ doesn't belong to $I$

3. We are given $\mathbb{R}[x,y,z]$ and an ideal $I = (xz^3+1, yz - z^2)$. Find generators of the ideal $I \cap \mathbb{R}[x,y]$.

Calculations

Let use Lex(z, x, y), and constain a Gröbner basis.

First of all, let's reduce $f_1$ by $f_2$ (that wont change the ideal)

$$xz^3+1 - xz(z^2-yz) \Rightarrow xyz^2+1 - xy(z^2-yz) = xy^2z+1$$

For now $G_1 = \{f_1, f_2\} = \{xy^2z+1, z^2-yz\}$

$$S_{12} = z(xy^2z+1) - xy^2(z^2-yz) = z + xy^3z$$

Reduction:

$$xy^3z + z - y(xy^2z+1) = z - y$$

So $G_2 = \{xy^2z+1; z^2-yz; z-y\}$

$$S_{13} = (xy^2z+1) - xy^2(z-y) = xy^3+1$$

$$S_{23} = (z^2-yz) - z(z-y) = 0$$

Thus $G_3 = \{xy^2z+1; z^2-yz; z-y; xy^3+1\}$ is Gröbner basis

since $f_1$ $f_2$ $f_3$ are co-prime with $f_4 = xy^3+1$

Answer

Thus the only generator of $I \cap \mathbb{R}[x,y]$ is $xy^3+1$.

Also, it's not obligatory, but we may notice, that:

Extra Part

$$I = (xy^2z+1; z^2-yz; z-y; xy^3+1) = (z-y; xy^3+1)$$

Because we can reduce $xy^2z+1$ to zero using $f_3$ $f_4$

$$(xy^2z+1) - xy^2(z-y) = 1 + xy^3 - xy^3+1 \Rightarrow 0$$

as well as $z^2-yz$:

$$z^2-yz - z(z-y) = 0$$

but $z-y$ ($f_3$) and $xy^3+1$ ($f_4$) are co-prime, thus we cannot reduce them.

Thus reduced gröbner basis for $I$ is $\{z-y; xy^3+1\}$

I'm not sure about this exact term.

4. Suppose we are given a system of equations

$$\begin{cases} x+y+z=3 \\ x^2+y^2+z^2=3 \\ x^3+y^3+z^3=1 \end{cases}$$

Compute the value of $x^4+y^4+z^4$ using the following method:

(a) Compute the Gröbner basis $G$ of the set $\{x+y+z-3, x^2+y^2+z^2-3, x^3+y^3+z^3-1\}$ using Lex$(x,y,z)$.

(b) Compute the remainder of $x^4+y^4+z^4$ with respect to $G$. It turns out that the remainder is the value of $x^4+y^4+z^4$.

a) $G_1 = \{x+y+z-3;\ x^2+y^2+z^2-3;\ x^3+y^3+z^3-1\}$

$S_{12} = x(x+y+z-3) - (x^2+y^2+z^2-3) = -xy-xz+3x+y^2+z^2-3$

$-xy-xz+3x+y^2+z^2-3+y(x+y+z-3) =$

$= -xz+3x+2y^2+z^2-3+yz-3y \quad -xz+3x+2y^2+z^2-3+yz-3y+z(x+y+z-3)$

$= 3x+2y^2+2z^2-3+2yz-3y-3z = 3x+2y^2+2z^2-3+2yz-3y-3z-3(x+y+z-3) =$

$= 2y^2+2z^2+6+2yz-6y-6z \to y^2+z^2+3+yz-3y-3z = f_3$

$S_{23} = x^3+y^2+z^3-1 - x(x^2+y^2+z^2-3) = y^3+z^3-1-xy^2-xz^2+3x$

$y^3+z^3-1-xy^2-xz^2+3x+y^2(x+y+z-3) = 2y^3+z^3-1-xz^2+3x+y^2z-3y^2$

$2y^3+z^3-1-xz^2+3x+y^2z-3y^2+z^2(x+y+z-3) = 2y^3+2z^3-1+3x+y^2z-3y^2+yz^2-3z^2$

$2y^3+2z^3-1+3x+y^2z-3y^2+yz^2-3z^2-3(x+y+z-3) = 2y^3+2z^3+8+y^2z-3y^2+yz^2-3z^2-3y-3z$

$2y^3+2z^3+8+y^2z-3y^2+yz^2-3z^2-3y-3z-2y(y^2+z^2+3+yz-3y-3z) =$

$= -y^2z+3y^2-yz^2+6yz-9y+2z^3-3z^2-3z+8$

$-y^2z+3y^2-yz^2+6yz-9y+2z^3-3z^2-3z+8+zf_4 = 3y^2+3yz-9y+3z^3-6z^2+8$

$3y^2+3yz-9y+3z^3-6z^2+8-3f_4 = 3z^3-9z^2+9z-1 = f_5$

Thus $G = \{x+y+z-3;\ x^2+y^2+z^2-3;\ x^3+y^3+z^3-1;\ y^2+z^2+3+yz-3y-3z;\ 3z^3-9z^2+9z-1\}$

is a Gröbnev basis (but we can reduce it, as shown in problem 3, to $\{f_1, f_4, f_5\}$) ← doesn't metter

b) $x^2+y^2+z = (x+y+z)^2-2xy-2yz-2xz = 3$

$3^2-2xy-2yz-2xz = 3$

$9-2(xy+yz+xz) = 3$

$xy+yz+xz = 3$

$x^3+y^3+z^3-3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-xz)$

$1-3xyz = 3(3-3)$

$-3xyz = -1$

$xyz = \frac{1}{3}$

$(xy+yz+xz)^2 = x^2y^2+y^2z^2+x^2z^2+2xyz(x+y+z)$

$9 = x^2y^2+y^2z^2+x^2z^2+2$

$x^2y^2+y^2z^2+x^2z^2 = 7$

$x^4+y^4+z^4 = (x^2+y^2+z^2)^2-2(x^2y^2+y^2z^2+x^2z^2)$

$x^4+y^4+z^4 = 9-14 = -5$

I choose this metod instead of reduction, 'cause it's too teddias...

- Tnx for all ♡