GROUP 231. SEMINAR 10. COMPUTATION OF LIMITS

Novosad Ivan
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Two functions $f(x)$ and $g(x)$ are **equivalent** ($f \sim g$) as $x \to c$ if $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = 1$.

Table of Equivalences as $x \to 0$				
$\sin x \sim x$	$\tan x \sim x$	$1 - \cos x \sim x^2/2$	$\arcsin x \sim x$	$\arctan x \sim x$
$\ln(1+x) \sim x$	$\log_a(1+x) \sim x/\ln a$	$e^x - 1 \sim x$	$a^x - 1 \sim x \ln a$	$(1+x)^a - 1 \sim ax$

Theorem. If $a(x) \sim a_1(x)$ and $b(x) \sim b_1(x)$ as $x \to c$, then $\lim\limits_{x\to c} \dfrac{a(x)}{b(x)} = \lim\limits_{x\to c} \dfrac{a_1(x)}{b_1(x)}$ if these limits exist.

2. (HW) Find the following limits:

(a) $\lim\limits_{x\to 0} \dfrac{1-\cos 3x}{x \cdot \arctan 8x}$; (b) $\lim\limits_{x\to 0} \dfrac{\ln(1-5x)}{\sqrt[3]{7x+8}-2}$; (c) $\lim\limits_{x\to 2} (2-x)\log_{-1+x} 6$;

(d) $\lim\limits_{x\to 0} \dfrac{\ln^3(1+2x)}{x \cdot (e^{x^2}-1)}$; (e) $\lim\limits_{x\to 0} \dfrac{\arcsin(\ln(1+x))}{2x+x^2}$; (f) $\lim\limits_{x\to 0} \dfrac{8^x - 6^x}{x}$.

a) $$\lim_{x\to 0}\left(\frac{1-\cos(3x)}{x\cdot \arctan(8x)}\right) = \left(\frac{0}{0}\right) = \lim_{x\to 0}\left(\frac{3\sin(3x)}{\arctan(8x) + \frac{8}{1+64x^2}}\right) =$$

$$\lim_{x\to 0}\left(\frac{3\sin(3x)}{\frac{(1+64x^2)\arctan(8x)+8x}{1+64x^2}}\right) = \lim_{x\to 0}\left(\frac{(1+64x^2)(3\sin(3x))}{(1+64x^2)\arctan(8x)+8x}\right) =$$

$$= \lim_{x\to 0}\left(\frac{(1+64x^2)(3\sin(3x))}{\arctan(8x)+64x^2\arctan(8x)+8x}\right) = \lim_{x\to 0}\left(\frac{9\cos(3x)(1+64x^2)+384x\sin(3x)}{16+128x\arctan(8x)}\right)$$

$$= \frac{9\cdot\cos(0)(1+64\cdot 0)+384\cdot 0\cdot\sin(0)}{16+0(128\arctan(0))} = \frac{9}{16}$$

⊛Note:

$$((1+64x^2)(3\sin(3x)))' = (1+64x^2)'(3\sin(3x)) + (1+64x^2)(3\sin(3x))' =$$

$$= 384x\sin(3x) + (1+64x^2)(9\cos(3x))$$

$$(\arctan(8x) + 64x^2\arctan(8x) + 8x)' =$$

$$= \arctan(8x)' + (64x^2)'\arctan(8x) + (64x^2)(\arctan(8x))' + 8x' =$$

$$= \frac{1}{1+(8x)^2}8 + 128x\arctan(8x) + \frac{64x^2 \cdot 8}{1+(8x)^2} =$$

(b) $\lim_{x\to 0} \frac{\ln(1-5x)}{\sqrt[3]{7x+8}-2}$;

$$= 16 + 128x \cdot \arctan(8x)$$

$$b)\ \lim_{x\to 0}\left(\frac{\ln(1-5x)}{\sqrt[3]{7x+8}-2}\right) = \frac{0}{0} \overset{(1)}{=} \lim_{x\to 0}\left(\frac{\frac{-5}{1-5x}}{\frac{7}{3\sqrt[3]{7x+8}^2}}\right) =$$

$$= \lim_{x\to 0}\left(\frac{-15\sqrt[3]{7x+8}^2}{7-35x}\right) = \frac{-15\sqrt[3]{0+8}^2}{7-35\cdot 0} = \frac{-15\cdot 4}{7} = -\frac{60}{7}$$

Note:

(1) $\ln(1-5x)' = \frac{1}{1-5x}(-5) = -\frac{5}{1-5x}$

$$f(x) = \left(\sqrt[3]{7x+8} - 2\right)' = \left((7x+8)^{\frac{1}{3}}\right)' = \frac{1}{3}(7x+8)^{-\frac{2}{3}} \cdot 7 = \frac{7}{3}\left(\left(\sqrt[3]{7x+8}\right)^2\right)^{-1} =$$

$$\frac{7}{3\sqrt[3]{7x+8}^{\,2}}$$

(c) $\lim\limits_{x\to 2}(2-x)\log_{-1+x} 6;$

c) $$\lim_{x\to 2}\left((2-x)\log_{x-1}(6)\right) = \lim_{x\to 2}\left(\frac{\ln(6^{x-1})}{\ln(x-1)}\right) \text{ does not exist}$$

(d) $\lim\limits_{x\to 0}\frac{\ln^3(1+2x)}{x\cdot(e^{x^2}-1)}$

$$= \frac{0}{0} \overset{(1)}{=} \lim_{x\to 0}\left(\frac{\frac{8x^3}{8x^3}\ln^3(1+2x)}{x(e^{x^2}-1)}\right) =$$

$$= \lim_{x\to 0}\left(\frac{8x^{\cancel{3}\,2}\ln^3\left((1+2x)^{\frac{1}{2x}}\right)}{\cancel{x}(e^{x^2}-1)}\right) = \lim_{x\to 0}\left(\frac{8x^2(\ln^3(e))}{e^{x^2}-1}\right) = \lim_{x\to 0}\left(\frac{8x^2}{e^{x^2}-1}\right)$$

$$= 8 \lim_{x \to 0} \left(\frac{1}{\frac{e^{x^2}-1}{x^2}} \right) = 8 \left(\lim_{x \to 0} (1) : \lim_{x \to 0} \left(\frac{e^{x^2}-1}{x^2} \right) \right) = 8 \cdot (1 : e^{0^2}) = 8 \cdot 1 = 8$$

Note:

$$\lim_{x \to 0} \left(\frac{e^{x^2}-1}{x^2} \right) = \left(\frac{0}{0} \right) = \lim_{x \to 0} \left(\frac{e^{x^2} \cancel{2x}}{\cancel{2x}} \right) = \lim_{x \to 0} (e^{x^2}) = e^{0^2} = 1$$

(e) $\lim\limits_{x \to 0} \dfrac{\arcsin(\ln(1+x))}{2x+x^2}$;

$$= \left(\frac{0}{0} \right) = \lim_{x \to 0} \left(\frac{\arcsin(\overbrace{\ln(1+x)}^{\sim x})}{2x+x^2} \right) =$$

$$\lim_{x \to 0} \left(\frac{\arcsin(x)}{2x+x^2} \right) = \lim_{x \to 0} \left(\frac{\arcsin(x)}{x(2+x)} \right) = \lim_{x \to 0} \left(\frac{1}{2+x} \right) \left(\text{since } \arcsin(x) \underset{x \to 0}{\sim} x \right)$$

$$= \frac{1}{2}$$

(f) $\lim\limits_{x \to 0} \dfrac{8^x - 6^x}{x}$.

$$= \left(\frac{0}{0} \right) = \lim_{x \to 0} \left(\frac{\ln(8) 8^x - \ln(6) 6^x}{1} \right) =$$

$$= \ln(8) - \ln(6) = \ln\left(\frac{8}{6}\right) = \ln\left(\frac{4}{3}\right)$$

5. (HW) Use L'Hospital's rule, if applicable, to evaluate the following limits:

(a) $\lim_{x\to 3} \frac{x^3 - 7x - 6}{\ln(x^2 - 8)}$; (b) $\lim_{x\to 1} \frac{\sqrt[6]{x} - 5/6 - x/6}{\sqrt[8]{x} - 7/8 - x/8}$; (c) $\lim_{x\to 0} \frac{\sin x - x}{3x^3}$;

(d) $\lim_{x\to 0} \frac{x^2 \sin\frac{1}{x}}{\sin x}$; (e) $\lim_{x\to 0+} x \ln x$; (f) $\lim_{x\to 0} \frac{x(e^x + 1) - 2(e^x - 1)}{x^3}$.

a) $$\lim_{x\to 3}\left(\frac{x^3 - 7x - 6}{\ln(x^2-8)}\right) = \left(\frac{27 - 21 - 6}{\ln(9-8)}\right) = \left(\frac{0}{0}\right) = \lim_{x\to 3}\left(\frac{3x^2 - 7}{\frac{2x}{x^2-8}}\right) =$$

$$= \lim_{x\to 3}\left(\frac{(3x^2-7)(x^2-8)}{2x}\right) = \frac{(27-7)(9-8)}{6} = \frac{20}{6} = \frac{10}{3} = 3\frac{1}{3}$$

b) $$\lim_{x\to 1}\left(\frac{\sqrt[6]{x} - 5/6 - x/6}{\sqrt[8]{x} - 7/8 - x/8}\right) = \left(\frac{0}{0}\right) = \lim_{x\to 1}\left(\frac{\frac{1}{6\sqrt[6]{x^5}} - \frac{1}{6}}{\frac{1}{8\sqrt[8]{x^7}} - \frac{1}{8}}\right) \Leftrightarrow$$

Note:

$$\left(\frac{1}{6\sqrt[6]{x^5}}\right)' = \left(\frac{1}{6}\cdot x^{-\frac{5}{6}}\right)' = \frac{1}{6}\cdot\left(-\frac{5}{6}\right)\cdot x^{-\frac{5}{6}-1} = \frac{-5}{36\sqrt[6]{x^{11}}}$$

$$\left(\frac{1}{8\sqrt[8]{x^7}}\right)' = \left(\frac{1}{8}\cdot x^{-\frac{7}{8}}\right)' = \frac{1}{8}\cdot\left(-\frac{7}{8}\right)\cdot x^{-\frac{7}{8}-1} = \frac{1}{8}\cdot\left(-\frac{7}{8}\right)x^{-\frac{15}{8}} =$$

$$= \frac{-7}{64\sqrt[8]{x^{15}}}$$

$$= \lim_{x\to 1}\left(\frac{\dfrac{-5}{36\sqrt[6]{x^{11}}}}{\dfrac{-7}{64\sqrt[8]{x^{15}}}}\right) = \frac{\dfrac{-5}{36}}{\dfrac{-7}{64}} = \frac{5\cdot 64}{36\cdot 7} = \frac{80}{63}$$

(c) $\lim\limits_{x\to 0} \dfrac{\sin x - x}{3x^3}$;

$$= \lim_{x\to 0}\left(\frac{\sin(x)-x}{3x^3}\right) = \left(\frac{0}{0}\right) = \lim_{x\to 0}\left(\frac{\cos(x)-1}{9x^2}\right)$$

$$= \left(\frac{0}{0}\right) = \lim\left(\frac{-\sin(x)}{\quad}\right) = \left(\frac{0}{0}\right) = \lim\left(\frac{-\cos(x)}{18}\right) = -\frac{1}{18}$$

$18x$

(d) $\lim_{x\to 0} \frac{x^2 \sin \frac{1}{x}}{\sin x}$; $= \left(\frac{0}{0}\right)$ (since $x^2 \underset{x\to 0}{=} 0 \wedge \sin\left(\frac{1}{x}\right)$ - bounded) $=$

$= \lim_{x\to 0}\left(\frac{\sin\left(\frac{1}{x}\right) x}{\frac{\sin(x)}{x}}\right) = \lim_{x\to 0}\left(\sin\left(\frac{1}{x}\right) x\right) \cdot \underbrace{\lim_{x\to 0}\left(\frac{\sin(x)}{x}\right)}_{=1} =$

$= \lim_{x\to 0}\left(\sin\left(\frac{1}{x}\right)\cdot x\right) = \lim_{x\to 0}\left(\sin\left(\frac{1}{x}\right)\right) \lim_{x\to 0}(x) = 0 \cdot \lim_{x\to 0}\left(\underbrace{\sin\left(\frac{1}{x}\right)}_{\text{bounded}}\right) = 0$

(since infinitesimal $\cdot$ bounded = infinitesimal)

(e) $\lim_{x\to 0+} x \ln x$; $= \lim_{x\to 0^+}\left(\frac{\ln(x)}{\frac{1}{x}}\right) = \left(\frac{-\infty}{0}\right)$ (since $\lim_{x\to 0^+}(\ln(x)) = -\infty$)

$= \lim_{x\to 0^+}\left(\frac{\frac{1}{x}}{\frac{1}{\dots}}\right) = \lim_{x\to 0^+}\left(-\frac{x^2}{x}\right) = \lim_{x\to 0^+}\left(-\frac{x}{1}\right) = -0 = 0$

$$x^2$$

(f) $\lim_{x\to 0} \frac{x(e^x+1)-2(e^x-1)}{x^3}$.

$$= \lim_{x\to 0}\left(\frac{x(e^x+1)-2(e^x-1)}{x^3}\right) = \left(\frac{0}{0}\right) =$$

$$\lim_{x\to 0}\left(\frac{e^x(x+1)-2e^x}{3x^2}\right) = \left|\frac{0}{0}\right| \circledcirc \quad (e^x(x+1))' = e^x(x+1)+e^x = e^x(x+2)$$

$$\circledcirc \lim_{x\to 0}\left(\frac{e^x(x+2)-2e^x}{6x}\right) = \left(\frac{0}{0}\right) = \lim_{x\to 0}\left(\frac{e^x(x+3)-2e^x}{6}\right) =$$

$$= \frac{1(3)-2\cdot 1}{6} = \frac{3-2}{6} = \frac{1}{6}$$

7. (HW) Use L'Hospital's rule, if applicable, to evaluate the following limits:

(a) $\lim_{x\to 1} x^{\frac{x+1}{x-1}}$; (b) $\lim_{x\to \pi/2} (\tan x)^{\cos x}$; (c) $\lim_{x\to 0} \left(2\sqrt[4]{x}+x\right)^{1/\ln x}$.

$$a) \lim_{x\to 1}\left(x^{\frac{x+1}{x-1}}\right) = \lim_{x\to 1}\left(\left(1+x-1\right)^{\frac{1}{x-1}(x+1)}\right) = \lim_{x\to 1}\left(e^{x+1}\right) = e^2$$

b) $\lim_{x \to \pi/2} \left(\tan(x)\right)^{\cos(x)} = \left(\text{let } t = x - \frac{\pi}{2} \wedge x = t + \frac{\pi}{2} \quad x \to \frac{\pi}{2} \Rightarrow t \to 0 \right) = \lim_{t \to 0} \left(\frac{1}{(-\cot(t))^{\sin(t)}} \right) =$

$= \lim_{t \to 0} \left(\frac{1}{-\left(\frac{\cos(t)}{\sin(t)}\right)^{\sin(t)}} \right) = \lim_{t \to 0} \left(\frac{1}{\left(-\frac{\cos(t)}{\frac{\sin(t)}{t}\, t}\right)^{t \frac{\sin(t)}{t}}} \right) \overset{\cdot}{=} \left(\text{since } \lim_{t \to 0} \left(\frac{\sin(t)}{t} \right) = 1 \right)$

$\overset{\cdot}{=} \lim_{t \to 0} \left(\frac{1}{\left(\frac{-\cos(t)}{t}\right)^{t}} \right) =$ ~~~ I try..

c) $\lim_{x \to 0} \left(2\sqrt[4]{x} + x \right)^{1/\ln(x)} = \lim_{x \to 0} \left(e^{\frac{\ln(x + 2\sqrt[4]{x})}{\ln(x)}} \right)$

$\left(\text{since } \alpha^{\beta} = e^{\beta \ln(\alpha)} \right)$

$= e^{\lim_{x \to 0} \left(\frac{\ln(x + 2\sqrt[4]{x})}{\ln(x)} \right)} =$ evaluating only limit for now.

$$\to \lim_{x\to 0}\left(\frac{\ln(x+2\sqrt[4]{x})}{\ln(x)}\right) \xrightarrow[\text{joke}]{\text{L'hospital}} \lim_{x\to 0}\left(\frac{\left(\frac{1}{2\sqrt[4]{x^3}}+1\right)x}{x+2\sqrt[4]{x}}\right) = 1$$

since $\ln(x+2\sqrt[4]{x})' = \frac{1}{x+2\sqrt[4]{x}}\cdot\left(1+\frac{2}{4\sqrt[4]{x^3}}\right) = \frac{1+\frac{2}{4\sqrt[4]{x^3}}}{x+2\sqrt[4]{x}}$ $\wedge$

$\wedge$ $\ln(x)' = \frac{1}{x}$

$$= \left(\begin{matrix}\text{Let } x^4 = t & \wedge\ t = \sqrt[4]{x} \\ x\to 0 & \Rightarrow t\to 0\end{matrix}\right) \Rightarrow \lim_{t\to 0}\left(\frac{\frac{1}{2t^3}+1}{t^4+2t}\,t^4\right) = \lim_{t\to 0}\left(\frac{\frac{(2t^3+1)\cancel{t}}{2}}{(t^3+2)\cancel{t}}\right) =$$

factorization

$$= \lim_{t\to 0}\left(\frac{2t^3+1}{2(t^3+2)}\right) = \frac{2\cdot 0^3+1}{2(0^3+2)} = \frac{1}{4} \Rightarrow = e^{\frac{1}{4}}$$

thx for checking