1. (0.5 points per item) Find the algebraic form of the following complex numbers:

(a) $(2+i)(3+7i)-(1+2i)(5+3i)$;

(b) $\dfrac{(3-i)(1-4i)}{1-2i}$;

(c) $\dfrac{(1-i)^9}{(1+i)^6}$;

(d) $1+i+i^2+i^3+\cdots+i^{2077}$.

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a) $(2+i)(3+7i)-(1+2i)(5+3i)=$
$= 6+17i-7-(5+13i-6) = -1+17i-13i+1 = 4i$

b) $\dfrac{(3-i)(1-4i)}{1-2i} = \dfrac{3-13i-4}{1-2i} = \dfrac{-1-13i}{1-2i} =$

$= \dfrac{(-1-13i)(1+2i)}{(1-2i)(1+2i)} = \dfrac{-1-2i-13i+26}{1-4i^2} = \dfrac{25-15i}{1+4} =$

$\dfrac{25-15i}{5} = 5-3i$

c) $\dfrac{(1-i)^9}{(1+i)^6} = \left(\begin{array}{l} r=\sqrt{1^2+(-1)^2}=\sqrt{2} \\ \theta=\frac{-1}{1} = -\frac{\pi}{4} \end{array} \right) =$

$= \dfrac{\left(\sqrt{2}\left(\cos\left(\frac{7\pi}{4}\right)+i\sin\left(\frac{7\pi}{4}\right)\right)\right)^9}{\left(\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right)\right)^6} =$

$= \dfrac{2^{\frac{9}{2}}\left(\cos\left(\frac{7\pi}{4}+8\cdot\frac{7\pi}{4}\right)+i\sin\left(\frac{7\pi}{4}+8\frac{7\pi}{4}\right)\right)}{2^{\frac{6}{2}}\left(\cos\left(\frac{6\pi}{4}\right)+i\sin\left(\frac{6\pi}{4}\right)\right.} =$

$$\frac{16\sqrt{2}\left(\cos\left(\frac{7\pi}{4}\right)+i\left(\sin\left(\frac{7\pi}{4}\right)\right)\right)}{8\left(\cos\left(\frac{3\pi}{2}\right)+i\sin\left(\frac{3\pi}{2}\right)\right)} =$$

$$= \frac{2\sqrt{2}\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)}{0+(-1)i} = \frac{2-2i}{i} = \frac{2i-2i^2}{i^2}$$

$$= -2-2i$$

d) $\underline{i} + i^2 + i^3 + \dots + i^{2077}$

$$i^{4k+1} + i^{4k+3} = 0 \quad (\forall k \in \mathbb{N})$$

$$i^{4k+2} + i^{4k} = 0 \quad (\forall k \in \mathbb{N}) \left(\begin{array}{l} i^{4k+2} = -1 \\ i^{4k} = 1 \end{array} \right)$$

then $i^{4k} + i^{4k+1} + i^{4k+2} + i^{4k+3} = 0 \Rightarrow$

$$\Rightarrow i + i^2 + i^3 \dots i^{2076} = 0 \Rightarrow$$

$$\Rightarrow i^{2077} = i^{4 \cdot 519 + 1} = i^1 = i$$

a) $1-\sqrt{3}i \Rightarrow \begin{cases} R=\sqrt{1^2+(\sqrt{3})^2}=\sqrt{4}=2 \\ \tan(\theta)-\frac{\sqrt{3}}{1} \Leftrightarrow \theta=-\frac{\pi}{3} \end{cases} \Rightarrow 2\left(\cos\left(\frac{5\pi}{3}\right)+i\sin\left(\frac{5\pi}{3}\right)\right)$

b) $-2i \Rightarrow \begin{cases} R=\sqrt{0+2^2}=2 \\ \theta=\frac{3\pi}{2} \end{cases} \Rightarrow 2\left(\cos\left(\frac{3\pi}{2}\right)+i\sin\left(\frac{3\pi}{2}\right)\right)$

c) $\sqrt{3}+i \Rightarrow \begin{cases} R=\sqrt{3+1}=2 \\ \tan(\theta)=\frac{1}{\sqrt{3}} \Rightarrow \theta=\frac{\pi}{6} \end{cases} \Rightarrow 2\left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right)$

d) $\left(\frac{-1+\sqrt{3}i}{1-i}\right)^{50}$ note: $\left[\begin{array}{l} R_1=\sqrt{1+3}=2 \\ \tan(\theta_1)=-\sqrt{3} \Leftrightarrow \theta_1=-\frac{\pi}{3} \Leftrightarrow \frac{2\pi}{3} \\ R_2=\sqrt{1+1}=\sqrt{2} \\ \tan(\theta_2)=-1 \Leftrightarrow \theta_2=-\frac{\pi}{4} \Leftrightarrow \frac{7\pi}{4} \end{array}\right] =$

$$= \frac{2\left(\cos\left(\frac{2\pi}{3}50\right)+i\sin\left(\frac{2\pi\cdot 50}{3}\right)\right)}{\sqrt{2}\left(\cos\left(\frac{7\cdot 50\pi}{4}\right)+i\sin\left(\frac{7\cdot 50\pi}{4}\right)\right)} = \frac{2\left(\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right)}{\sqrt{2}\left(\cos(\pi/2)+i\sin(\pi/2)\right)}$$

$$= \frac{2\left(-1/2+\frac{\sqrt{3}}{2}i\right)}{\sqrt{2}(0+i)} = \frac{-1+\sqrt{3}i}{\sqrt{2}i} = \frac{-i-\sqrt{3}}{-\sqrt{2}} =$$

$$\frac{\sqrt{3}+i}{\sqrt{2}} = \frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}i$$

3. (1 point) Find *all* complex numbers z such that $z^3=1$ and draw them on the complex plane.

[**hint:** use the polar form $z=|z|(\cos\varphi+i\sin\varphi)$; note that there should be more than one such number]

$$|z|^3(\cos(\varphi)+i\sin(\varphi))^3=1$$

$$|z|^3(\cos(\varphi)+i\sin(\varphi))^3=1\left(\cos\left(0+\frac{2\pi k}{3}\right)+i\sin\left(0+\frac{2\pi k}{3}\right)\right)\ (\forall k\in\mathbb{N})$$

$$|z|(\cos(3\varphi)+i\sin(3\varphi))=1\left(\cos\left(0+\frac{2\pi k}{3}\right)+i\sin\left(0+\frac{2\pi k}{3}\right)\right)\ (\forall k\in\mathbb{N})$$

$$\Downarrow$$

$$z = \cos\left(0 + \frac{2\pi k}{3}\right) + i\sin\left(0 + \frac{2\pi k}{3}\right) \quad \forall k \in \mathbb{N}$$

5. (1 point) Find all $n \in \mathbb{N}$ such that $(1+i)^n = (1-i)^n$.

[**hint:** you can use the polar form *or* you can multiply both sides by $(1+i)^n$ and then notice that $(1+i)^2 = ?$]

$$(1+i)^n = (1-i)^n$$

$$\sqrt{2}^n\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)^n = \sqrt{2}^n\left(\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right)^2 \Longleftrightarrow$$

since $\left(2^{\frac{n}{2}} > 0\right)$

$$\Longleftrightarrow \cos\left(\frac{n\pi}{4}\right) + i\sin\left(\frac{n\pi}{4}\right) = \cos\left(\frac{7n\pi}{4}\right) + i\sin\left(\frac{7n\pi}{4}\right)$$

$$\begin{cases} \cos\left(\frac{n\pi}{4}\right) = \cos\left(\frac{7n\pi}{4}\right) \\ \sin\left(\frac{n\pi}{4}\right) = \sin\left(\frac{7\pi n}{4}\right) \end{cases}$$

$n = k \mid n = \frac{4k}{3}$

$\Longleftrightarrow$

$n = \frac{1}{2} + k \quad \frac{4k}{3}$

$$\Longleftrightarrow \begin{cases} 2\sin(\pi n)\sin\left(\frac{3\pi n}{4}\right) = 0 \\ 2\cos(\pi n)\sin\left(\frac{3\pi n}{4}\right) = 0 \end{cases} \Longleftrightarrow$$

$$\begin{cases} \left[\begin{array}{l} \sin(\pi n) = 0 \\ \sin\left(\frac{3\pi n}{4}\right) = 0 \end{array}\right. \\ \left[\begin{array}{l} \cos(\pi n) = 0 \\ \sin\left(\frac{3\pi n}{4}\right) = 0 \end{array}\right. \end{cases} \Longleftrightarrow \begin{cases} \left[\begin{array}{l} n = k \\ n = \frac{4k}{3} \end{array}\right. \\ \left[\begin{array}{l} n = \frac{1}{2} + k \\ n = \frac{4k}{3} \end{array}\right. \end{cases} \quad k \in \mathbb{Z}$$

$$\Longrightarrow n = \frac{4k}{3} \quad (\forall k \in \mathbb{Z})$$

$$z = a + bi$$

$$\begin{cases} \sqrt{a^2+b^2} = 1 \\ \sqrt{(a+\sqrt{2})^2+b^2} = 1 \\ \sqrt{(a+\sqrt{2})^2+b^2} = \sqrt{a^2+b^2} \end{cases} \iff \begin{cases} a^2+b^2 = 1 \\ (a+\sqrt{2})^2+b^2 = 1 \\ (a+\sqrt{2})^2+b^2 = a^2+b^2 \end{cases} \iff$$

$$\iff \begin{cases} a^2+b^2 = 1 \\ a^2+2\sqrt{2}a+2+b^2 = 1 \\ a^2+2\sqrt{2}a+2 = a^2 \end{cases} \iff \begin{cases} a^2 = 1-b^2 \\ a^2+2\sqrt{2}a = -b^2-1 \\ 2\sqrt{2}a = -2 \end{cases} \iff$$

$$\iff \begin{cases} a^2 = 1-b^2 \\ a^2+2\sqrt{2}a = -1-b^2 \\ a = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \end{cases} \iff \begin{cases} \left(\frac{-\sqrt{2}}{2}\right)^2 = 1-b^2 \\ \left(\frac{-\sqrt{2}}{2}\right)^2 - \frac{2\sqrt{2}^2}{2} = -1-b^2 \end{cases}$$

$$\iff \begin{cases} \frac{2}{4} = 1-b^2 \\ \frac{2}{4} - \frac{4}{2} = -1-b^2 \end{cases} \iff \begin{cases} b^2 = \frac{1}{2} \\ b^2 = -1+\frac{3}{2} \end{cases} \iff$$

$$\iff b^2 = \frac{1}{2} \iff \left[\, b = \frac{\sqrt{2}}{2} \right.$$

$$b = -\frac{\sqrt{2}}{2}$$

$$\Rightarrow \left[\begin{array}{l} z = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \\ z = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \end{array}\right. \Leftrightarrow z = -\frac{\sqrt{2}}{2}(1 \pm i)$$

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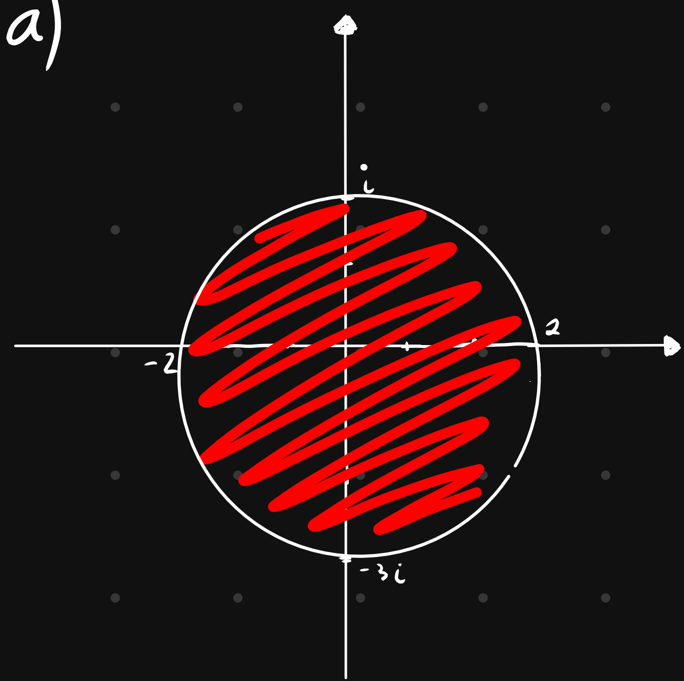
4. (1 point per item) Sketch the following sets of complex numbers in the complex plane

   (a) $|z+i| \leqslant 2$;

   (b) $1 \leqslant \mathrm{Re}(z) \leqslant \mathrm{Im}(z) \leqslant 2$;
   [**hint:** if a complex number z is given by its *algebraic form* $z = a + bi$, where $a, b \in \mathbb{R}$, then $\mathrm{Re}(z) \overset{\text{def}}{=} a$ is called the real part of z and $\mathrm{Im}(z) \overset{\text{def}}{=} b$ is called the imaginary part of z (note that both $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are real numbers); sketch the sets $1 \leqslant \mathrm{Re}(z)$, $\mathrm{Re}(z) \leqslant \mathrm{Im}(z)$, $\mathrm{Im}(z) \leqslant 2$ and then consider their

a)



b)