17. If $a^{10}+b^{10}+c^{10}+d^{10}+e^{10}+f^{10}$ is divisible by 11, then $abcdef$ is divisible by 11^6

for any x (for example our a, b, c, d, e, f) we have 2 cases:

1) if $\gcd(x, 11) = 1 \Rightarrow x^{10} \equiv 1 (11)$ by FLT

2) if $\gcd(x, 11) = 11 \Rightarrow x^{10} \equiv 0 (11)$

then each term of given sum is either 0 or 1, then, since given sum is divisible by 11, $a \equiv b \equiv c \equiv d \equiv e \equiv f \equiv 0 (11)$

then $abcdef \equiv 0 (11^6)$ (since $abcdef \equiv 0 (11)$)

18. Solve the equation $19x + 22y = -21$ in integer numbers.

$\gcd(19,22) = 1 \Rightarrow 1 \mid -21 \Rightarrow$ green has infinity many solutions.

Let's find one of them:

$19 = 22 \cdot 0 + 19$

$22 = 19 \cdot 1 + 3$

$19 = 3 \cdot 6 + 1$

$1 = 19 - 3 \cdot 6$

$1 = 19 - (22 - 19)6$

$1 = 19 - 22 \cdot 6 + 19 \cdot 6$

$1 = 7 \cdot 19 - 22 \cdot 6$, hence $\begin{cases} x' = 7 \\ y' = -6 \end{cases}$ is a solution for: $19 \cdot x + 22y = 1$

$19 \cdot 7 - 22 \cdot 6 = 1 \quad | \cdot (-21)$

$19 \cdot \underbrace{7(-21)}_{x} + 22\underbrace{(-6)(-21)}_{y} = -21$

hence $\begin{cases} x = -147 \\ y = 126 \end{cases}$ is one of solution for green

Hence all integer solutions:

$\begin{cases} x = -147 - 22r \\ y = 126 - 19r \end{cases} \quad r \in \mathbb{Z}$ $\leftarrow$ Answer

19. How many distinct solutions does the congruence $39x \equiv 104 \pmod{221}$ have? (*There are just finitely many solutions. Please pay attention to the definition of a solution to a congruence.*)

$39x \equiv 104 \pmod{221}$

has $\underset{\overset{\shortparallel}{13}}{\gcd(39, 221)}$ many solutions

20. Solve the simultaneous congruences:

$$\begin{cases} x \equiv -14 \pmod{12} \\ x \equiv 6 \pmod{11} \\ x \equiv 19 \pmod{5}. \end{cases}$$

$$\begin{cases} x \equiv -14\ (12) \\ x \equiv 6\ (11) \\ x \equiv 19\ (5) \end{cases} \Leftrightarrow \begin{cases} x \equiv 10\ (12) \\ x \equiv 6\ (11) \\ x \equiv 4\ (5) \end{cases} \quad \text{by CRT:}$$

$$x = (10 M_1 M_1^{-1} + 6 M_2 M_2^{-1} + 4 M_3 M_3^{-1}) \bmod M$$

$$M = 12 \cdot 11 \cdot 5 = 60 \cdot 11 = 660 \qquad M_i = \frac{M}{m_i}$$

$$M_1 = \frac{660}{12} = 55 \qquad M_2 = \frac{660}{11} = 60 \qquad M_3 = \frac{660}{5} = 132$$

$$M_i \cdot M_i^{-1} = 1 \bmod m_i$$

$$55 \cdot M_1^{-1} = 1 \bmod 12 \qquad 60 M_2^{-1} = 1 \bmod 11 \qquad 132 M_3^{-1} = 1 \bmod 5$$

$$7 M_1^{-1} = 1 \bmod 12 \qquad 5 M_2^{-1} = 1 \bmod 11 \qquad 2 M_3^{-1} = 1 \bmod 5$$

$$M_1^{-1} = 7 \qquad M_2^{-1} = 9 \qquad M_3^{-1} = 3$$

Hence $x = (10 \cdot 7 \cdot 55 + 6 \cdot 60 \cdot 9 + 19 \cdot 132 \cdot 3) \bmod 660$

$$x = (3850 + 3240 + 1584) \bmod 660$$

$$x = (14614) \bmod 660$$

$$x = 94$$

21. Find a way to compute the number $\gcd(3^{168}-1, 3^{140}-1)$ without using a calculator and compute this number actually.

$\gcd(3^{168}-1, 3^{140}-1) = \gcd(3^{168}-3^{140}, 3^{140}-1) =$ (since $\gcd(A,B) = \gcd(A, B-AK)$ $\forall K \in \mathbb{Z}$)

$= \gcd(3^{28}-1, 3^{140}-1)$ (since $\gcd(168,140) = 28$) $= 3^{28}-1$

22*. Find the remainder after dividing the number $\underbrace{3^{3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}}}_{2020 \text{ occurences of } 3}$ by 46. (*Recall that the notation* a^{b^c} *stands for* $a^{(b^c)}$ *rather than* $(a^b)^c$.)

let's find a cycle:

(0) $3 \equiv 3 \ (46)$

(1) $3^3 \equiv 27 \ (46)$

(2) $27^3 = 19683 \equiv 41 \ (46)$

(3) $41^3 = 68921 \equiv 13 \ (46)$

(4) $13^3 = 2197 \equiv 35 \ (46)$

(5) $35^3 = 42875 \equiv 3 \ (46)$

hence $3^{3^{3^{3^{3}}}}$ 2020 mod 46 = (2020 mod 5 = (0)) = 3