Release: 18.04.2024
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In this HW, you can perform all arithmetic operations on matrices (e.g. multiplication, transforming into RREF, finding the inverse, finding the determinant, etc) by a machine.

1. (1 point) Let $\mathbb{V}$ be a 3-dimensional vector space over the field of reals; let $\mathcal{A}$ be an ordered basis for $\mathbb{V}$. Then, find all $\lambda \in \mathbb{R}$ such that the quadratic form

$$q(\mathbf{x}) = x_1^2 + x_2^2 + 3\lambda x_3^2 + 2\lambda x_1 x_2 - 2x_1 x_3 - 2x_2 x_3,$$

where $[\mathbf{x}]_{\mathcal{A}} = [x_1\ x_2\ x_3]^{\mathrm{T}}$, is positive definite (see Definition 28.2).

[**hint**: use Item 1) of Sylvester's Criterion (Theorem 28.4); also see Problem 3 from Seminar 28; 1 is a root of $\det(H(q, \mathcal{A})) = 0$]

$$H(q,A) = \begin{bmatrix} 1 & \lambda & -1 \\ \lambda & 1 & -1 \\ -1 & -1 & 3\lambda \end{bmatrix} \qquad \begin{aligned} \delta_1 &= 1 \\ \delta_2 &= 1-\lambda^2 \\ \delta_3 &= 5\lambda - 3\lambda^3 - 2 \end{aligned}$$

so it's req. to $\begin{cases} 1-\lambda^2 > 0 \\ 5\lambda - 3\lambda^3 - 2 > 0 \end{cases} \Rightarrow \lambda \in \left(\dfrac{-3+\sqrt{33}}{6}, 1 \right)$

positive def.

2. (1 point) Let $\mathbb{V}$ be a 3-dimensional vector space over the field of reals; let $\mathcal{A}$ be an ordered basis for $\mathbb{V}$. Then, find all $\lambda \in \mathbb{R}$ such that the quadratic form

$$q(\mathbf{x}) = \lambda x_1^2 - x_2^2 - x_3^2 + \lambda x_1 x_2 + 2\lambda x_1 x_3,$$

where $[\mathbf{x}]_{\mathcal{A}} = [x_1\ x_2\ x_3]^{\mathrm{T}}$, is negative definite (see Definition 28.2).

[**hint**: use Item 2) of Sylvester's Criterion (Theorem 28.4)]

for real??, okay...

$$H(q, A) = \begin{bmatrix} \lambda & \lambda/2 & \lambda \\ \lambda/2 & -1 & 0 \\ \lambda & 0 & -1 \end{bmatrix} \Rightarrow \begin{aligned} \delta_1 &= \lambda \\ \delta_2 &= -\lambda - \lambda^2/4 \\ \delta_3 &= \frac{\lambda^2}{4} + \lambda + \lambda^3 \end{aligned}$$

So it's req. $\begin{cases} \lambda < 0 \\ \lambda + \lambda^2/4 < 0 \\ \frac{\lambda^2}{4} + \lambda + \lambda^3 > 0 \end{cases} \Rightarrow \lambda \in \varnothing$

So there no such λ, at least in $\mathbb{R}$ - wolframα

3. (1 point) Let $\mathbb{V}$ be a 4-dimensional vector space over the field of reals; let $\mathcal{A}$ be an ordered basis for $\mathbb{V}$. Then, find a normal basis (see Definition 28.1) [1] of the quadratic form

$$q(\mathbf{x}) = 2x_1^2 - 3x_3^2 + 5x_4^2,$$

where $[\mathbf{x}]_{\mathcal{A}} = [x_1\ x_2\ x_3\ x_4]^{\mathrm{T}}$.

[**hint**: since $\mathcal{A}$ is a canonical basis of q, see Problem 1 from Seminar 28; also see Theorem 28.2]

A'

$$\begin{cases} k_1 = \sqrt{2}\,x_1 \\ k_2 = \sqrt{5}\,x_4 \\ k_3 = \sqrt{3}\,x_3 \\ k_4 = x_2 \end{cases} \Rightarrow \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{5} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/\sqrt{2} & 0 & 0 & 0 \\ 0 & 1/\sqrt{5} & 0 & 0 \\ 0 & 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{thus } q(\bar{x}) = k_1^2 + k_2^2 - k_3^2 - 0\cdot k_4^2 \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{So } \left(\begin{bmatrix} 2^{-1/2} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5^{-1/2} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3^{-1/2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

is A' normal basis.

4. Find the signature (see Definition 28.1) of the following quadratic forms:

(a) (1 point) $q(A) = \operatorname{tr}(A^2)$, for every $A \in \mathbb{V}$, where $\mathbb{V}$ is the vector space of all symmetric matrices of size 2 over the field of reals;

[**hint**: find the coordinate matrix of q with respect to some ordered basis of $\mathbb{V}$ (for example, see Problem 2 from HW 26); use Equality (28.2) and Theorem 28.2 (in this case, Jacobi's theorem works for almost any basis of $\mathbb{V}$; if it fails for your basis, just try another one)]

a) $\left(\underset{e_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{e_2}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}, \underset{e_3}{\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}} \right) = A'$, basis for V

so $(H(q, A')) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$, where $[i\,j]$ el. is $B(e_i\, e_j)$, $B(A,B) = tr(A,B)$

Hence $\delta_1 = 1$ $\delta_2 = 1$ $\delta_3 = 3$; Hence signature is $(3,0,0)$

(b) (2 points) $q(P) = \frac{\mathrm{d}}{\mathrm{d}x}(P^2)(-1)$, for every $P \in \mathbb{R}[x;2]$, where $\mathbb{R}[x;2] = \{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\}$ is the vector space of all polynomials with real coefficients of degree at most two.

For example, if $P(x) = 2x^2 - 3x + 1$, then, $q(P) = \left((2x^2-3x+1)^2\right)'\big|_{x=-1} = (2x^2-3x+1)(4x-3)\big|_{x=-1} = \left(2(-1)^2 - 3(-1) + 1\right)\left(4(-1) - 3\right) = (2+3+1)(-4-3) = 6 \cdot (-7) = -42$.

[**hint**: since $\det(H(q,\mathcal{A})) = 0$, for every ordered basis $\mathcal{A}$ of $\mathbb{R}[x;2]$ (do you understand why?), Jacobi's theorem (Theorem 28.1) cannot be apply to solve this problem; using Lagrange's Method (Theorem 27.1) or Symmetric Gaussian Elimination, we can always find a normal form of q which, due to Sylvester's Law of Inertia (Theorem 28.3), basically gives us the signature, but, this approach

[1] That is, find the change of basis matrix from $\mathcal{A}$ to a normal basis.

Now, since $q(x) = ((ax^2+bx+c)^2)' \Rightarrow 4a^2x^3 + 6abx^2 + 2b^2x + 4acx + 2bc \Rightarrow$

$= -4a^2 + 6ab - 2b^2 - 4ac + 2bc$

Hence $\begin{bmatrix} -4 & 3 & -2 \\ 3 & -2 & 1 \\ -2 & 1 & 0 \end{bmatrix}$ is matrix rep. of q.

$$\begin{bmatrix} 4 & 3 & -2 \\ 3 & -2 & 1 \\ -2 & 1 & 0 \end{bmatrix} \underset{\hat{l}_{1,3,-1/2}}{\xrightarrow{l_{1,3,-1/2}}} \begin{bmatrix} -4 & 3 & 0 \\ 3 & -2 & -1/2 \\ 0 & -1/2 & 1 \end{bmatrix} \underset{\hat{d}_{1,1/2}}{\xrightarrow{d_{1,1/2}}} \begin{bmatrix} -1 & 3/2 & 0 \\ 3/2 & -2 & -1/2 \\ 0 & -1/2 & 1 \end{bmatrix} \underset{\hat{l}_{1,2,3/2}}{\xrightarrow{l_{1,2,3/2}}} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1/4 & 1/2 \\ 0 & 1/2 & 1 \end{bmatrix}$$

$$\hat{l}_{2,3,-2} \downarrow l_{2,3,-2}$$

$$\text{Signature } \Leftarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underset{\hat{l}_{2,3,2}}{\xleftarrow{l_{2,3,2}}} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{bmatrix} \underset{\hat{d}_{3,2}}{\xleftarrow{d_{3,2}}} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Signature
$\shortparallel$
$(2, 0, 1)$

(c) (2 points) $q(P) = P(1) \cdot P(2)$, for every $P \in \mathbb{R}[x;n]$, where $\mathbb{R}[x;n]$ is the vector space of all polynomials with real coefficients of degree at most n.

For example, if $P(x) = 3x^2 + x - 7$, then, $q(P) = P(1) \cdot P(2) = (3 \cdot 1^2 + 1 - 7) \cdot (3 \cdot 2^2 + 2 - 7) = -21$.

[**hint**: if you are struggling with this problem, try to solve it first for, say, $n = 3$, then, the general case should be clear; find the associated bilinear form of q (see Statement 26.1); using the associated bilinear form, find the coordinate matrix of q with respect to the ordered basis $\mathcal{A} = \left(1, (x-1), (x-1)^2, \ldots, (x-1)^n\right)$ (see Definitions 26.2 and 25.2); using Symmetric Gaussian Elimination, find a canonical form of q]

$$\beta_q(P_1, P_2) = \frac{1}{2}\left(q(P_1 + P_2) - q(P_1) - q(P_2)\right)$$

$$\mathcal{A} = \left(1, (x-1), (x-1)^2, \ldots, (x-1)^n\right)$$

$$H(\beta, \mathcal{A}) = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1n} \\ h_{21} & h_{22} & \cdots & h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n+1\,1} & h_{n+1\,2} & \cdots & h_{n+1\,n+1} \end{bmatrix}.$$

$$h_{km} = \left(\left((x-1)^{k-1} + (x-1)^{m-1}\right)\Big|_{x=1}\left((x-1)^{k-1} + (x-1)^{m-1}\right)\Big|_{x=2} - (x-1)^{k-1}\Big|_{x=1} \cdot (x-1)^{k-1}\Big|_{x=2} - (x-1)^{m-1}\Big|_{x=1} \cdot (x-1)^{m-1}\Big|_{x=2} =$$

$$= \frac{1}{2}\left((0^{k-1} + 0^{m-1})(1^{k-1} + 1^{m+1}) - 0^{k-1} \cdot 1^{k-1} - 0^{m-1} \cdot 1^{m-1}\right) = \frac{1}{2}\left(0^{k-1} + 0^{m-1}\right.$$

Thus:
$$\begin{cases} h = 0 & \text{if } k \neq 1 \wedge m \neq 1 \\ h = 1/2 & \text{if } k = 1 \wedge m \neq 1 \vee k \neq 1 \wedge m = 1 \\ h = 1 & \text{if } k = 1 \wedge m = 1 \end{cases} \Rightarrow H(\beta_q, \mathcal{A}) = \begin{bmatrix} 1 & 1/2 & \cdots & 1/2 \\ 1/2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1/2 & 0 & \cdots & 0 \end{bmatrix} = H(q, \mathcal{A}) \text{ by def.}$$

Using sym. gaus. alg.:

$$\begin{bmatrix} 1 & 1/2 & \cdots & 1/2 \\ 1/2 & & & \\ \vdots & & \boxed{0} & \\ 1/2 & & & \end{bmatrix} \xrightarrow[\ell_{n+1,n,-1}]{\ell_{n+1,n,-1}} \begin{bmatrix} 1 & 1/2 & \cdots & 1/2 & 0 \\ 1/2 & & & & \\ \vdots & & \boxed{0} & & \\ 1/2 & & & & \\ 0 & & & & \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 1/2 & 0 & \cdots & 0 \\ 1/2 & & & & \\ 0 & & \boxed{0} & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix} \rightarrow$$

$$\xrightarrow[\ell_{2,1,-1/2}]{\ell_{2,1,-1/2}} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$ is a canonical Form of q, so signature is $(1, 0, n)$

5. (2 points) Let $\mathbb{V}$ be a 3-dimensional vector space over the field of reals; let $\mathcal{A} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be an ordered basis for $\mathbb{V}$; let

$$q(\mathbf{x}) = 2x_1^2 + x_3^2 + 2x_1x_2 - x_1x_3, \tag{1}$$

where $[\mathbf{x}]_{\mathcal{A}} = [x_1\ x_2\ x_3]^{\mathrm{T}}$, be a quadratic form on $\mathbb{V}$. Then, find a basis $\mathcal{A}' = (\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3')$ which is mentioned in the statement of Theorem 28.1.

[**hint**: due to Item 1) of Theorem 28.1, we have $\mathbf{e}_1' = \mathbf{e}_1$, $\mathbf{e}_2' = \mathbf{e}_2 + a \cdot \mathbf{e}_1$, and $\mathbf{e}_3' = \mathbf{e}_3 + b \cdot \mathbf{e}_1 + c \cdot \mathbf{e}_2$, for some $a,\ b,\ c \in \mathbb{R}$; to find these a, b, and c, perform all steps from the proof of Theorem 28.1 for Form (1); it is not a part of this problem but, after you found $\mathcal{A}'$, it is advisable to verify that $H(q, \mathcal{A}') = C(\mathcal{A}, \mathcal{A}')^{\mathrm{T}} \cdot H(q, \mathcal{A}) \cdot C(\mathcal{A}, \mathcal{A}')$]

$$H(q,A) = \begin{bmatrix} 2 & 1 & -1/2 \\ 1 & 1 & 0 \\ -1/2 & 0 & 0 \end{bmatrix}, \quad \begin{cases} e_1' = e_1 \\ e_2' = e_2 + ae_1 \\ e_3' = e_3 + be_1 + ce_2 \end{cases} \quad \begin{cases} \delta_1 = 2 \\ \delta_2 = 1 \\ \delta_3 = -1/4 \end{cases}$$

$$H(q,A')[1,1] = \beta(e_1', e_1') = \frac{\delta_1}{1} = 2 = \frac{1}{2}\left(q(e_1) \overset{?}{=} q(e_2) - q(e_1) - q(e_2)\right)$$

$$H(q,A')[2,2] = \beta(e_2', e_2') = \frac{\delta_2}{\delta_1} = 1/2$$

$$H(q,A')[3,3] = \beta(e_3', e_3') = \frac{\delta_3}{\delta_2} = -1/4$$

$$H(q,A')[1,2] = H(q,A')[1,3] = H(q,A')[2,3] = 0$$

Hence $H(q,A') = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -1/4 \end{bmatrix}$