Novosad Ivan 231

1. (2 points) Let

$$A=\begin{bmatrix}1&-2&1&-2&2\\0&2&-2&2&-2\\3&-4&9&-2&5\\-4&6&-8&4&-7\\1&-8&7&-8&-1\end{bmatrix}.$$

Then, find $\det(A)$.
To solve this problem, we strongly recommend you the following approach:

(a) add both row 1 and row 3 to row 4 in the initial determinant, apply 4-row Laplace expansion;
(b) subtract column 2 from column 3 in the resulting determinant from the previous item, apply 3-column Laplace expansion;
(c) add row 2 to row 1 in the resulting determinant from the previous item, apply 1-row Laplace expansion;
(d) now you are on your own!

$$\begin{vmatrix}1&-2&1&-2&2\\0&2&-2&2&-2\\3&-4&9&-2&5\\-4&6&-8&4&-7\\1&-8&7&-8&-1\end{vmatrix} \xrightarrow[l_{4,3,1}]{l_{4,1,1}} \begin{vmatrix}1&-2&1&-2&2\\0&2&-2&2&-2\\3&-4&9&-2&5\\0&0&2&0&0\\1&-8&7&-8&-1\end{vmatrix} = 2(-1)^{4+3}\begin{vmatrix}1&-2&-2&2\\0&2&2&-2\\3&-4&-2&5\\1&-8&-8&-1\end{vmatrix} =$$

$$\overset{l'_{3,2,-1}}{=} -2\begin{vmatrix}1&-2&0&2\\0&2&0&-2\\3&-4&2&5\\1&-8&0&-1\end{vmatrix} = -2\cdot 2\cdot(-1)^{3+3}\begin{vmatrix}1&-2&2\\0&2&-2\\1&-8&-1\end{vmatrix} \xrightarrow{l_{1,2,1}} -4\begin{vmatrix}1&0&0\\0&2&-2\\1&-8&-1\end{vmatrix}$$

$$= -4\cdot 1\cdot(-1)^{1+1}\begin{vmatrix}2&-2\\-8&-1\end{vmatrix} = -4\left(2(-1)-(-2)(-8)\right) = 72$$

2. (1 points) Let

$$A=\begin{bmatrix}1&2&-2\\-2&-2&6\\2&1&-2\end{bmatrix}.$$

Then, find $\operatorname{adj}(A)$.
[**hint:** since A is invertible, one may use the equality $\operatorname{adj}(A)=\det(A)\,A^{-1}$; note that there is no need to calculate $\det(A)$ separately, since it can be done "naturally" in the process of calculating A^{-1} at the point when you transformed A into an upper triangular matrix.]

$$\det(A) = \begin{vmatrix}1&2&-2\\-2&-2&6\\2&1&-2\end{vmatrix} = 1(-2)(-2)+2\cdot 6\cdot 2+(-2)(-2)1 \ominus$$
$$\ominus\left(2(2)(2)+1\cdot 6\cdot 1+(-2)(-2)(2)\right) = 10$$

find A^{-1}:

$$\left[\begin{array}{ccc|ccc}1&2&-2&1&0&0\\-2&-2&6&0&1&0\\2&1&-2&0&0&1\end{array}\right] \xrightarrow{d_{2,1/2}} \left[\begin{array}{ccc|ccc}1&2&-2&1&0&0\\-1&-1&3&0&1/2&0\\2&1&-2&0&0&1\end{array}\right] \xrightarrow{l_{2,1,1}} \left[\begin{array}{ccc|ccc}1&2&-2&1&0&0\\0&1&1&1&1/2&0\\2&1&-2&0&0&1\end{array}\right]$$

$$\Big\downarrow l_{3,2,3},\ l_{1,2,-2}$$

$$\left[\begin{array}{ccc|ccc}1&0&-4&-1&-1&0\\0&1&1&1&1/2&0\\0&0&5&1&3/2&1\end{array}\right] \xrightarrow{d_{3,1/5}} \left[\begin{array}{ccc|ccc}1&0&-4&-1&-1&0\\0&1&1&1&1/2&0\\0&0&1&1/5&3/10&1/5\end{array}\right] \xrightarrow[l_{2,3,-1}]{l_{1,3,4}} \left[\begin{array}{ccc|ccc}1&0&0&-1/5&1/5&4/5\\0&1&0&4/5&1/5&-1/5\\0&0&1&1/5&3/10&1/5\end{array}\right]$$

$$\underbrace{\begin{bmatrix}-1/5&1/5&4/5\\4/5&1/5&-1/5\\1/5&3/10&1/5\end{bmatrix}}_{A^{-1}}$$

then $\operatorname{adj}(A) = 10\begin{bmatrix} -1/5 & 1/5 & 4/5 \\ 4/5 & 1/5 & -1/5 \\ 1/5 & 3/10 & 1/5 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 8 \\ 8 & 2 & -2 \\ 2 & 3 & 2 \end{bmatrix}$

3. (0.5 points for (a) and (b), 1 point for (c)) Assuming that $\operatorname{adj}(A) \in \operatorname{Mat}_n(\mathbb{R})$ and $\det(A) \neq 0$ are given, find:

   (a) $\operatorname{adj}(A^{\mathrm{T}})$;

   (b) $\operatorname{adj}(\lambda A)$, for every $\lambda \in \mathbb{R}$;

   (c) $\det(\operatorname{adj}(A))$.

   [**hint:** for (a) and (b) use Definition 8.2; for (c) take a look at Statement 8.1 and Theorem 7.2.]

a) $\operatorname{adj}(A^T) = \operatorname{adj}(A)^T$

Prove: $(A^T)^{-1} = \dfrac{\operatorname{adj}(A^T)}{\det(A^T)}$ (since $A^{-1} = \dfrac{\operatorname{adj}(A)}{\det(A)}$) $\Rightarrow$

$\Rightarrow \operatorname{adj}(A^T) = \det(A^T) \cdot (A^T)^{-1} \Rightarrow$ (since $\det(A^T) = \det(A)$)

$\Rightarrow \operatorname{adj}(A^T) = \det(A) \cdot (A^{-1})^T \Rightarrow \operatorname{adj}(A^T) = \det(A) \cdot \left(\dfrac{\operatorname{adj}(A)^T}{\det(A)}\right) \Rightarrow$

$\Rightarrow \operatorname{adj}(A^T) = \dfrac{\det(A) \cdot \operatorname{adj}(A)^T}{\det(A)} \Leftrightarrow \operatorname{adj}(A^T) = \operatorname{adj}(A)^T$

b) $\operatorname{adj}(\lambda A) = \lambda^{(n-1)} \operatorname{adj}(A) \quad \forall \lambda \in \mathbb{R}$

Prove:

$(\lambda A)^{-1} = \dfrac{\operatorname{adj}(\lambda A)}{\det(\lambda A)}$

$\operatorname{adj}(\lambda A) = \det(\lambda A) \cdot (\lambda A)^{-1} \Leftrightarrow \operatorname{adj}(\lambda A) = \underline{\lambda}^n \det(A)\, \underline{\lambda}^{-1} \cdot A^{-1} \Leftrightarrow$

$\Leftrightarrow \operatorname{adj}(\lambda A) = {}^{n-1}\det(A) \cdot \left(\dfrac{\operatorname{adj}(A)}{\det(A)}\right) \Leftrightarrow \operatorname{adj}(\lambda A) = \lambda^{n-1} \cdot \operatorname{adj}(A)$

c) $\det(\operatorname{adj}(A)) = \det(A)^{n-1}$

$A^{-1} = \dfrac{\operatorname{adj}(A)}{\det(A)} \Rightarrow A^{-1} \cdot \det(A) = \operatorname{adj}(A) \Rightarrow A \cdot A^{-1} \operatorname{adj}(A) = A \cdot \operatorname{adj}(A) \Leftrightarrow$

$\Leftrightarrow I \det(A) = A \cdot \operatorname{adj}(A) \Leftrightarrow \det(I \cdot \det(A)) = \det(A \cdot \operatorname{adj}(A)) \Leftrightarrow$

$\Leftrightarrow (\det(A))^n = \det(A) \cdot \det(\operatorname{adj}(A)) \Leftrightarrow (\det(A))^{n-1} = \det(\operatorname{adj}(A))$

(since $|k \cdot I_{n \times n}| = k^n$)

done

4. Following the instructions, for every $n \in \mathbb{N}$, find the value of the following determinant of size n:

$$\lambda_n = \begin{vmatrix} 3 & 1 & 0 & 0 & \dots & 0 & 0 \\ 2 & 3 & 1 & 0 & \dots & 0 & 0 \\ 0 & 2 & 3 & 1 & \dots & 0 & 0 \\ 0 & 0 & 2 & 3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 3 & 1 \\ 0 & 0 & 0 & 0 & \dots & 2 & 3 \end{vmatrix}.$$

(a) (1 point) using the Laplace expansion, find a recursive equation of the form $\lambda_n = a\lambda_{n-1} + b\lambda_{n-2}$, for some $a, b \in \mathbb{R}$ and all $n \geqslant 3$ (see a similar problem from the seminar);

a)

$$\underset{n}{\begin{vmatrix} 3&1&0&0&\cdots&0&0\\ 2&3&1&0&\cdots&0&0\\ 0&2&3&1&\cdots&0&0\\ 0&0&2&3&\cdots&0&0\\ \vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\ 0&0&0&0&\cdots&3&1\\ 0&0&0&0&\cdots&2&3\end{vmatrix}_n} = 3\cdot(-1)^{1+1}\underset{n-1}{\begin{vmatrix}3&1&0&\cdots&0&0\\2&3&1&\cdots&0&0\\0&2&3&\cdots&0&0\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\cdots&3&1\\0&0&0&\cdots&2&3\end{vmatrix}_{n-1}} + 2\cdot(1)^{n+1}\begin{vmatrix}1&0&\cdots&0&0\\2&3&\cdots&0&0\\ \vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\cdots&3&1\\0&0&\cdots&2&3\end{vmatrix}$$

$$\Rightarrow \lambda_n = 3\lambda_{n-1} - 2\lambda_{n-2}$$

(b) (0,5 points) find a 2-by-2 matrix A such that

$$A \cdot \begin{bmatrix} \lambda_{n-1} \\ \lambda_{n-2} \end{bmatrix} = \begin{bmatrix} \lambda_n \\ \lambda_{n-1} \end{bmatrix}, \quad \text{for every} \quad n \geqslant 3;$$

(just guess A, it should be easy)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \lambda_{n-1} \\ \lambda_{n-1} \end{bmatrix} = \begin{bmatrix} a\lambda_{n-1} + b\lambda_{n-2} \\ c\lambda_{n-1} + d\lambda_{n-2} \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} a\lambda_{n-1} + b\lambda_{n-2} = 3\lambda_{n-1} - 2\lambda_{n-2} \Rightarrow a = 3;\ b = -2 \\ (2)\ c\lambda_{n-1} + d\lambda_{n-2} = 3\lambda_{n-2} - 2\lambda_{n-3} \end{cases}$$

$$x = \lambda_{n-1} \quad y = \lambda_{n-2} \quad z = \lambda_{n-3}$$

$$(2)\ cx + dy = 3y - 2z$$

then $A = \begin{bmatrix} 3 & -2 \\ \dfrac{3\lambda_{n-2}}{\lambda_{n-1}} & \dfrac{-2\lambda_{n-3}}{\lambda_{n-2}} \end{bmatrix}$

(c) (0,5 points) find λ_1 and λ_2; since, due to the previous item, we have

$$A^{n-2} \cdot \begin{bmatrix} \lambda_2 \\ \lambda_1 \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}, \quad \text{for every} \quad n \geqslant 3,$$

it (essentially) remains to find A^n, for every $n \in \mathbb{N}$;

$$A_{n-2} = \begin{bmatrix} 3 & -2 \\ \dfrac{3\lambda_1}{\lambda_2} & \dfrac{-2\lambda_0}{\lambda_1} \end{bmatrix}\begin{bmatrix} \lambda_2 \\ \lambda_1 \end{bmatrix} = \begin{bmatrix} 3\lambda_2 - 2\lambda_1 \\ 3\lambda_1 - 2\lambda_0 \end{bmatrix} = \begin{bmatrix} \lambda_3 \\ \lambda_2 \end{bmatrix}$$

(d) (1 point for α and β; 1 point for C) to find A^n, it (essentially) suffices to find matrices C and J such that $A = C^{-1}JC$ and $J = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$ (indeed, if $A = C^{-1}JC$, then, $A^2 = (C^{-1}JC) \cdot (C^{-1}JC) = C^{-1}J^2C$, and so on, $A^n = A^{n-1}A = C^{-1}J^?C$).

To find α and β, use equalities $\det(A) = \det(C^{-1}JC)$ and $\mathrm{tr}(A) = \mathrm{tr}(C^{-1}JC)$ (do not forget that $\det(AB) = \det(A)\det(B)$, for every square matrices A and B of the same size, and $\mathrm{tr}(ABC) = \mathrm{tr}(BCA)$, for every square matrices A, B, C of the same size).

To find matrix C, let $C = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$, rewrite $A = C^{-1}JC$ as $CA = JC$ and solve 4-by-4 system of linear equations with variables x, y, z, t (it should not be that bad; note that the system has *infinitely* many solutions and you can use any solutions such that the matrix C is invertible).

it's not worth it.

I wanna walk out and touch some grass

TNX for checking and...