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Homework

1. Find all homomorphisms from $\mathbb{Z}_{16}$ to $\mathbb{Z}_{20}$. For each homomorphism find its kernel and image.

Since $\mathbb{Z}_n$ is cyclic there exist a generator

Namely, 1

Hence, any homorphism must satisfy:

$f: 1 \in \mathbb{Z}_{16} \to m \in \mathbb{Z}_{20} : 16 \cdot m \equiv 0\ (20) \Leftrightarrow 4 \cdot m \equiv 0\ (5)$

since f preserves the neutral element.

$\Rightarrow$ only 5, 10, 15 are suitable for this map.

So there are 3 distinct non-trivial homorphisms:

4 including trivial one.

$f_1 : \mathbb{Z}_{16} \to 0$ (trivial one) : $\ker(f_1) = \mathbb{Z}_{16}$, $\mathrm{Im}(f_1) = \{0\}$

$f_2 : 1 \to 5$ $(5 \cdot k \bmod 20)$: $\ker(f_2) = \{0, 4, 8, 12\}$, $\mathrm{Im}(f_2) = \{0, 5, 10, 15\}$

$f_3 : 1 \to 10$ $(10 \cdot k \bmod 20)$: $\ker(f_3) = \{0, 2, 4, 6, 8, 10, 12, 14\}$; $\mathrm{Im}(f_3) = \{0, 10\}$

$f_4 : 1 \to 15$ $(15 \cdot k \bmod 20)$: $\ker(f_4) = \{0, 4, 8, 12\}$ $\mathrm{Im}(f_4) = \{0, 5, 10, 15\}$

2. Find all elements of order 4 in the group $\mathbb{Z}_6 \times \mathbb{Z}_{20} \times \mathbb{Z}_{15}$.

The order of an element (a,b,c) in the d. product $\mathbb{Z}_6 \times \mathbb{Z}_{20} \times \mathbb{Z}_{15}$ is at least lcm of the orders in $\mathbb{Z}_6$, $\mathbb{Z}_{20}$ and $\mathbb{Z}_{15}$. An element has the order 4 iff it generates a cyclic subgroup of size 4 within the group

- Element of order 4 in each group:

  $\mathbb{Z}_6$ has no such elements

  $\mathbb{Z}_{20}$: $\{5, 15\}$

  $\mathbb{Z}_{15}$: has no such elements

The elements of order 4 in $\mathbb{Z}_6 \times \mathbb{Z}_{20} \times \mathbb{Z}_{15}$ are combined where the $\mathbb{Z}_{20}$ component is either 5 or 15, and the other components are such that they do not affect the overall order to be more than 4. Since $\mathbb{Z}_6$ and $\mathbb{Z}_{15}$ have no element of order 4, we focus on $\mathbb{Z}_{20}$ for el. ord. 4. :

we need element from $\mathbb{Z}_6$ and $\mathbb{Z}_{15}$ which order $\mid 4$, So :

neutral element and 3 for $\mathbb{Z}_6$, (since $\operatorname{ord}(3) = 2 \mid 4$)

So answer is: $(0,5,0)\,(0,15,0)\,(3,5,0)\,(3,15,0)$

3. Find all finite abelian groups of order 360.

Since $360 = 2^3 \cdot 3^2 \cdot 5$

$\mathbb{Z}_{360} \cong \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

$\mathbb{Z}_{360} \cong \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5$

$\mathbb{Z}_{360} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5$

$\mathbb{Z}_{360} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

$\mathbb{Z}_{360} \cong \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

$\mathbb{Z}_{360} \cong \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

4. Suppose $G = (\mathbb{C}^*, \cdot)$ and $H = (\mathbb{R}^*, \cdot)$. Check if the map $\varphi\colon G \to H$ is a homomorphism of groups:

(a) $\varphi(z) = |z|$.

(b) $\varphi(z) = 2|z|$.

(c) $\varphi(z) = \frac{1}{|z|}$.

$\varphi(g_1 \cdot g_2) = \varphi(g_1) \cdot \varphi(g_2)$, $z_1 = a + b \cdot i \quad z_2 = c + d \cdot i$

a) $\varphi(z_1 \cdot z_2) = |z_1 \cdot z_2| = \sqrt{(ac+bd)^2 + (ad+cb)^2} = \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2}$

$\parallel$

$\varphi(z_1) \cdot \varphi(z_2) = |z_1| \cdot |z_2| = \sqrt{a^2+b^2} \cdot \sqrt{c^2+d^2} = \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2}$

homomonorphism.

b) $\varphi(z_1 \cdot z_2) = 2|z_1 \cdot z_2| = 2|z_1| \cdot |z_2|$

$\varphi(z_1) \cdot \varphi(z_2) = 2|z_1| \cdot 2|z_2| = 4|z_1||z_2|$

not a homomorphism

c) $\varphi(z_1 \cdot z_2) = \dfrac{1}{|z_1 \cdot z_2|} = \dfrac{1}{|z_1||z_2|}$

$\varphi(z_1) \cdot \varphi(z_2) = \dfrac{1}{|z_1|} \cdot \dfrac{1}{|z_2|} = \dfrac{1}{|z_1||z_2|}$

Homomorphism.