LAaG Homework #18 Group: 231 (M+P+)

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In this HW, you can transform any matrix into REF/RREF by using a machine.

1. (1 point per item) Determine which of the following functions are linear transformations and which are not.

   Note that if you want to prove that a given function is a linear transformation, use the fact that a function $\varphi\colon \mathbb{V} \to \mathbb{W}$, then where $\mathbb{V}$ and $\mathbb{W}$ are two vector spaces over the *same* field $\mathbb{F}$, is a linear transformation *if and only if* $\varphi(a \cdot \mathbf{x} + b \cdot \mathbf{y}) = a \cdot \varphi(\mathbf{x}) + b \cdot \varphi(\mathbf{y})$, for every $\mathbf{x}, \mathbf{y} \in \mathbb{V}$ and every $a, b \in \mathbb{F}$; if you want to prove that a given function is not a linear transformation, it suffices to show that some of the properties of linear transformations are not satisfied by the given function (also, see Problem 1 from Seminar 18).

   LT (a) $\varphi\colon \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x+y \\ x-y \\ 3x-2y \end{bmatrix}$, for every $[x, y]^{\mathrm{T}} \in \mathbb{R}^2$;

   ¬LT (b) $\varphi\colon \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x \\ x+y+1 \\ x+2y \end{bmatrix}$, for every $[x, y]^{\mathrm{T}} \in \mathbb{R}^2$;

   ¬LT (c) $\varphi\colon A \mapsto \det(A)$, for every square matrix A of size n;
   [**hint:** do not forget about $n = 1$]

   LT (d) $\varphi\colon A \mapsto \operatorname{tr}(A)$, for every square matrix A of size n;

   LT (e) $\varphi\colon p(x) \mapsto p(x+1)$, for every polynomial $p(x) \in \mathbb{R}[x; n]$.

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√ a) $\varphi(ax+by) = \varphi\left(\begin{bmatrix} ax_1 \\ ay_1 \end{bmatrix} + \begin{bmatrix} bx_2 \\ by_2 \end{bmatrix}\right) = \varphi\left(\begin{bmatrix} ax_1+bx_2 \\ ay_1+by_2 \end{bmatrix}\right) = \begin{bmatrix} a(x_1+y_1)+b(x_2+y_2) \\ a(x_1-y_1)+b(x_2-y_2) \\ a(3x_1-2y_1)+b(3x_2-2y_2) \end{bmatrix}$

$a\varphi\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + b\varphi\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} ax_1+ay_1 \\ ax_1-ay_1 \\ 3ax_1-2ay_1 \end{bmatrix} + \begin{bmatrix} bx_2+by_2 \\ bx_2-by_2 \\ 3bx_2-2by_2 \end{bmatrix} = \begin{bmatrix} a(x_1+y_1)+b(x_2+y_2) \\ a(x_1-y_1)+b(x_2-y_2) \\ a(3x_1-2y_1)+b(3x_2-2y_2) \end{bmatrix}$ $\Rightarrow$ hence LT.

√ b) $\varphi(\bar{0}) \neq \bar{0}$; $\varphi\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\Rightarrow$ not LT.

√ c) $\varphi(A+B) = \varphi(A) + \varphi(B)$ √

but $\varphi(A+\lambda B) \neq \varphi(A) + \lambda\varphi(B)$; in fact $\det(\lambda B) \neq \lambda\det(B)$

ie. $\det \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B} = 1$; $2B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $\det(2B) = 4 \Rightarrow \det(2B) \neq 2\det(B)$ $\Rightarrow$ hence not LT.

in fact $2 \neq 4$.

√ d) $\operatorname{tr}(A+B) = \operatorname{tr}(A) + \operatorname{tr}(B)$ by statement 2.3 point 1

$\operatorname{tr}(A+\lambda B) = \operatorname{tr}(A) + \lambda\operatorname{tr}(B)$ by statement 2.3 point 1 ∧ 2 $\Rightarrow$ hence it's LT.

√ e) $\varphi : p(x) \to p(x+1)$ $\quad R[x, n]$ for $\forall\, n$

$\varphi(f(x)) + \varphi(g(x)) = f(x+1) + g(x+1)$

$\varphi(f(x) + g(x)) = \varphi(f(x)) + \varphi(g(x)) = f(x+1) + g(x+1)$ $\Rightarrow$ hence LT.

$\varphi(f(x) + \lambda g(x)) = \varphi(f(x)) + \varphi(\lambda g(x)) = \varphi(f(x)) + \lambda\varphi(g(x))$

2. (0.5 points per item) Let a linear transformation $\varphi\colon \mathbb{R}^5 \to \mathbb{R}^3$ be defined as

$$\varphi\colon \begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} \mapsto \begin{bmatrix}1&-2&2&-1&8\\2&-4&1&1&7\\1&-2&0&1&2\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix},$$

for every $[x_1, x_2, x_3, x_4, x_5]^{\mathrm{T}} \in \mathbb{R}^5$. Then

(a) find a basis for $\mathrm{Ker}(\varphi)$;

(b) find a basis for $\mathrm{Im}(\varphi)$.

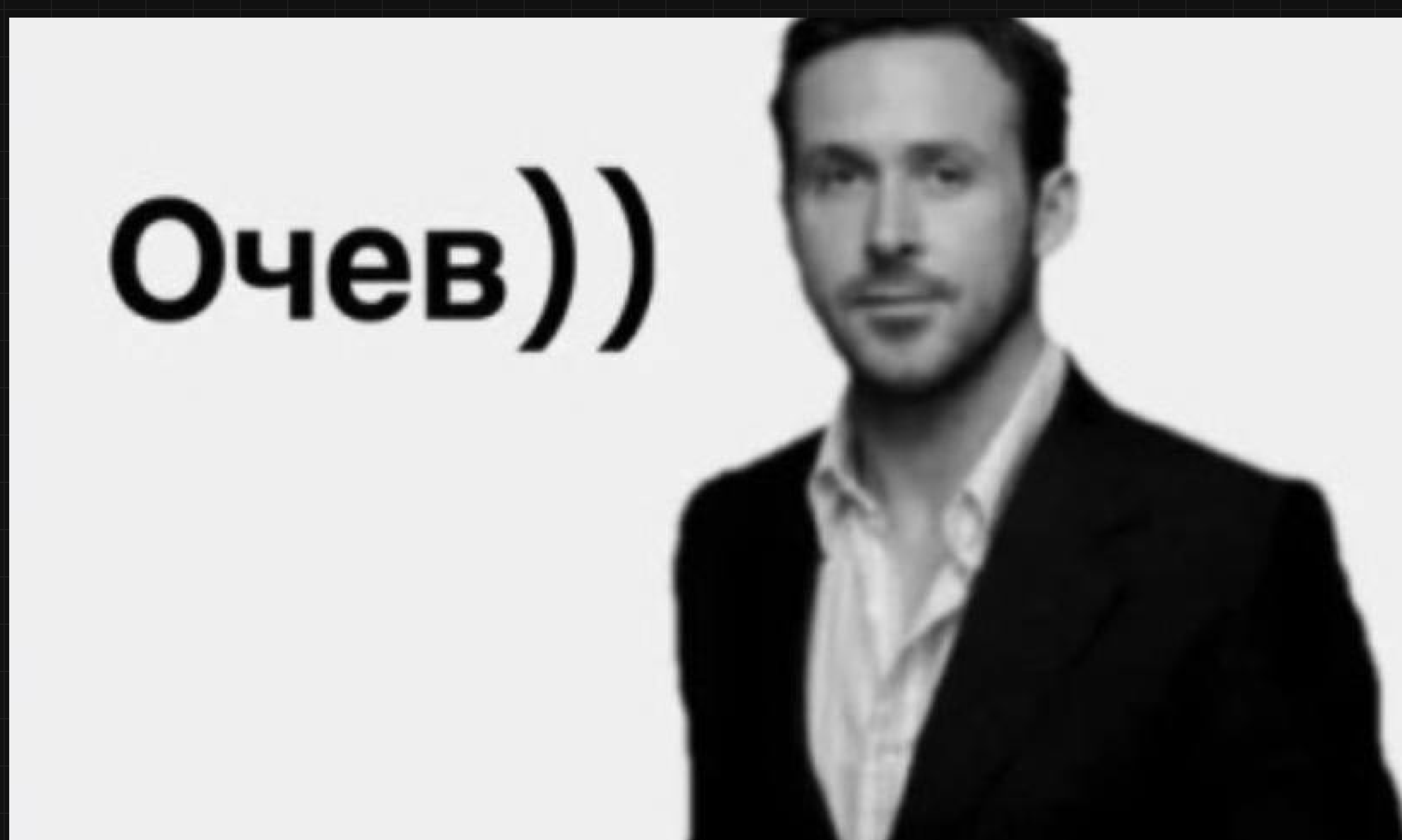
~~[**hint:** see Problem 2 from Seminar 18; note that, due to Theorem 18.1, you should have~~ $\cancel{\dim(\mathrm{Ker}(\varphi)) + \dim(\mathrm{Im}(\varphi)) = \dim(\mathbb{R}^5) = 5}$~~]~~

hints for weak students, isn't it?

a) $$\left[\begin{array}{ccccc|c}1&-2&2&-1&8&0\\2&-4&1&1&7&0\\1&-2&0&1&2&0\end{array}\right] \xrightarrow{\text{RREF}} \begin{array}{c}\begin{array}{cccccc}v_1&v_2&v_3&v_4&v_5&\end{array}\\ \left[\begin{array}{ccccc|c}1&-2&0&1&2&0\\0&0&1&-1&3&0\\0&0&0&0&0&0\end{array}\right]\end{array} \Rightarrow \begin{bmatrix}v_1\\v_2\\v_3\\v_4\\v_5\end{bmatrix} = \begin{bmatrix}2\alpha-\beta-2\varphi\\ \alpha\\ \beta-3\varphi\\ \beta\\ \varphi\end{bmatrix}$$

$$\ker(\varphi) < \begin{bmatrix}2&-1&-2\\1&0&0\\0&1&-3\\0&1&0\\0&0&1\end{bmatrix} > \Rightarrow \left(\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\-3\\0\\1\end{bmatrix}\right)$$ - basis for $\ker(\varphi)$

b) $$\begin{bmatrix}1&-2&2&-1&8\\2&-4&1&1&7\\1&-2&0&1&2\end{bmatrix} \xrightarrow{\text{RREF}} \begin{array}{c}\begin{array}{cccccc}v_1&v_2&v_3&v_4&v_5&\end{array}\\ \left[\begin{array}{ccccc|c}1&-2&0&1&2&0\\0&0&1&-1&3&0\\0&0&0&0&0&0\end{array}\right]\end{array} \Rightarrow \left(\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}\right)$$ - basis for $\mathrm{Im}(\varphi)$



3. (1 point per item) Does there exist a *linear* transformation $\varphi\colon \mathbb{V} \to \mathbb{W}$ such that it satisfies the following conditions? If it does, then find a matrix A such that $\varphi(\mathbf{x}) = A\mathbf{x}$, for every $\mathbf{x} \in \mathbb{V}$, if it does not, prove it.

(a) $\mathbb{V} = \mathbb{W} = \mathbb{R}^2$ and $\operatorname{Ker}(\varphi) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 = 1 \right\}$;

~~[hint: take a look at Proposition 18.1]~~

(b) $\mathbb{V} = \mathbb{W} = \mathbb{R}^3$ and

$$\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \overset{\varphi}{\mapsto} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} \overset{\varphi}{\mapsto} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 6 \\ -1 \\ -1 \end{bmatrix} \overset{\varphi}{\mapsto} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix};$$

~~[hint: see Problem 3 from Seminar 18]~~

(c) $\mathbb{V} = \mathbb{W} = \mathbb{R}^2$ and $\operatorname{Ker}(\varphi) = \left\langle \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\rangle$;

~~[hint: take a look at Item (b) of this problem]~~

$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{matrix} x = 0 \\ y = 0 \end{matrix}$ $\quad 0^2 + 0^2 \neq 1$;

$\Downarrow \in \ker(\varphi)$ $\quad$ 1 $\quad$ $\Downarrow \notin \ker(\varphi)$ $\quad \Rightarrow$ not exist.

another proof:

$V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $V_1 V_2 \in \ker(\varphi)$;

$\underset{\ker(\varphi)}{\underset{\cap}{V_1}} + \underset{\ker \varphi}{\underset{\cap}{V_2}} = \underline{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \Rightarrow$ 1

$\Downarrow \notin \ker(\varphi)$

b) $A \begin{bmatrix} -1 & 3 & 6 \\ 3 & -2 & -1 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ $\qquad \begin{bmatrix} -1 & 3 & 6 \\ 3 & -2 & -1 \\ 1 & -1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 3 & -9 \\ -2 & 5 & -17 \\ 1 & -2 & 7 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 6 \\ 3 & -2 & -1 \\ 1 & -1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -9 \\ -2 & 5 & -17 \\ 1 & -2 & 7 \end{bmatrix} = \underline{\begin{bmatrix} 0 & 1 & -2 \\ -1 & 3 & -10 \\ 2 & -4 & 15 \end{bmatrix}}$

c) let $u = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$; then $\left\langle \begin{bmatrix} -1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle = \mathbb{R}^2$

$\begin{bmatrix} x & y \\ a & b \end{bmatrix} \underline{\begin{bmatrix} -1 \\ 3 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} -x + 3y = 0 \\ -a + 3b = 0 \end{cases}$ $\quad \begin{cases} -1 + 3y = 0 \\ 3b = 0 \end{cases}$ $\quad A = \begin{bmatrix} 1 & 1/3 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} x & y \\ a & b \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} x = 1 \\ a = 0 \end{cases}$ $\quad b = 0 \Rightarrow y = 1/3$

d)

(d) $\mathbb{V} = \mathbb{R}^3$, $\mathbb{W} = \mathbb{R}^2$, and

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \overset{\varphi}{\mapsto} \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \overset{\varphi}{\mapsto} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} \overset{\varphi}{\mapsto} \begin{bmatrix} -4 \\ 5 \end{bmatrix}.$$

[hint: Git Gud] ← Thx

$A \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 5 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -4 \\ -1 & 1 & 5 \end{bmatrix}$

$\begin{bmatrix} x & y & z \\ a & b & c \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \Rightarrow \begin{cases} x - y + z = 2 \\ a - b + c = -1 \end{cases}$

$\begin{bmatrix} x & y & z \\ a & b & c \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{cases} x + y + z = 0 \\ a + b + c = 1 \end{cases}$

$\begin{bmatrix} x & y & z \\ a & b & c \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 5 \end{bmatrix} \Rightarrow \begin{cases} x + 5y + z = -4 \\ a + 5b + c = 5 \end{cases}$

$\Rightarrow \begin{cases} \begin{cases} x - y + z = 2 \\ x + y + z = 0 \\ x + 5y + z = -4 \end{cases} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 - \lambda \\ -1 \\ \lambda \end{bmatrix} \\ \begin{cases} a - b + c = -1 \\ a + b + c = 1 \\ a + 5b + c = 5 \end{cases} \Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -m \\ 1 \\ m \end{bmatrix} \end{cases}$

$\Rightarrow \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} = A$;

$\underbrace{\begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 5 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -4 \\ -1 & 1 & 5 \end{bmatrix}$

4. ($*$; 2 points) Let a linear transformation $\varphi\colon \mathbb{R}[x;n] \to \mathbb{R}[x;n]$ be defined as $\varphi\colon p(x) \mapsto p(x) - p'(x)$, for every polynomial $p(x) \in \mathbb{R}[x;n]$ (for example, $\varphi(x^3 - 5x^2 + 2x - 9) = x^3 - 5x^2 + 2x - 9 - (x^3 - 5x^2 + 2x - 9)' = x^3 - 8x^2 + 12x - 11$). Then, find a linear transformation $\psi\colon \mathbb{R}[x;n] \to \mathbb{R}[x;n]$ such that $\psi \circ \varphi\colon p(x) \mapsto p(x)$, for every polynomial $p(x) \in \mathbb{R}[x;n]$ (that is, find a linear transformation ψ such that the composition $\psi \circ \varphi$ is the identity linear transformation).

$$R[x,n] = \langle I_{n+1} \rangle$$

$$R[x,n]' = \left\langle \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\rangle$$

$$\varphi(p(x)): \begin{array}{c} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 0 & 0 & \cdots & \vdots \\ 0 & 0 & 1 & -3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \ddots & -n \\ 0 & 0 & 0 & 0 & \cdots & & 1 \end{bmatrix} n+1 = A \\ n+1 \end{array}$$

check: $$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -9 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -9 & -2 & 0 & 0 \\ 0 & 2 & 10 & 0 \\ 0 & 0 & -5 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ true;

then $$\psi: \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 0 & 0 & \cdots & \vdots \\ 0 & 0 & 1 & -3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \ddots & -n \\ 0 & 0 & 0 & 0 & \cdots & & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & 2 & 6 & 24 & 120 & \cdots & n! \backslash 0! \\ 0 & 1 & 2 & 6 & 24 & 120 & \cdots & n! \backslash 1! \\ 0 & 0 & 1 & 3 & 12 & 60 & \cdots & n! \backslash 2! \\ 0 & 0 & 0 & 1 & 4 & 20 & \cdots & n! \backslash (3)! \\ 0 & 0 & 0 & 0 & 1 & 5 & \cdots & n! \backslash (n-3)! \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & n! \backslash (n-2)! \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & n! \backslash (n-1)! \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = B$$

it ought to be clear, that $A \cdot B = I_{n+1}$ = basis for $R[x,n]$

$$\Updownarrow$$

$$\psi \circ \varphi = id$$