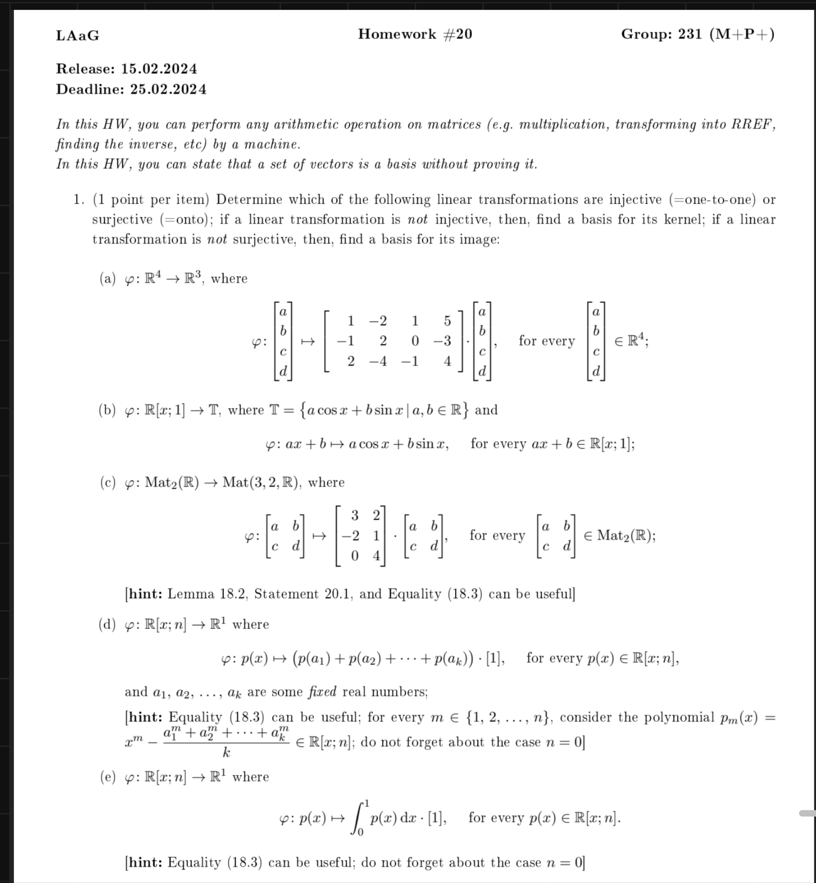


LAaG | Homework #20 | Group: 231 (M+P+)

Release: 15.02.2024
Deadline: 25.02.2024

In this HW, you can perform any arithmetic operation on matrices (e.g. multiplication, transforming into RREF, finding the inverse, etc) by a machine.
In this HW, you can state that a set of vectors is a basis without proving it.

1. (1 point per item) Determine which of the following linear transformations are injective (=one-to-one) or surjective (=onto); if a linear transformation is *not* injective, then, find a basis for its kernel; if a linear transformation is *not* surjective, then, find a basis for its image:

(a) $\varphi\colon \mathbb{R}^4 \to \mathbb{R}^3$, where

$$\varphi\colon \begin{bmatrix} a\\ b\\ c\\ d \end{bmatrix} \mapsto \begin{bmatrix} 1 & -2 & 1 & 5\\ -1 & 2 & 0 & -3\\ 2 & -4 & -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} a\\ b\\ c\\ d \end{bmatrix}, \quad \text{for every } \begin{bmatrix} a\\ b\\ c\\ d \end{bmatrix} \in \mathbb{R}^4;$$

(b) $\varphi\colon \mathbb{R}[x;1] \to \mathbb{T}$, where $\mathbb{T} = \{a\cos x + b\sin x \mid a, b \in \mathbb{R}\}$ and

$$\varphi\colon ax+b \mapsto a\cos x + b\sin x, \quad \text{for every } ax+b \in \mathbb{R}[x;1];$$

(c) $\varphi\colon \mathrm{Mat}_2(\mathbb{R}) \to \mathrm{Mat}(3,2,\mathbb{R})$, where

$$\varphi\colon \begin{bmatrix} a & b\\ c & d \end{bmatrix} \mapsto \begin{bmatrix} 3 & 2\\ -2 & 1\\ 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} a & b\\ c & d \end{bmatrix}, \quad \text{for every } \begin{bmatrix} a & b\\ c & d \end{bmatrix} \in \mathrm{Mat}_2(\mathbb{R});$$

[**hint:** Lemma 18.2, Statement 20.1, and Equality (18.3) can be useful]

(d) $\varphi\colon \mathbb{R}[x;n] \to \mathbb{R}^1$ where

$$\varphi\colon p(x) \mapsto \big(p(a_1) + p(a_2) + \cdots + p(a_k)\big) \cdot [1], \quad \text{for every } p(x) \in \mathbb{R}[x;n],$$

and $a_1, a_2, \ldots, a_k$ are some *fixed* real numbers;
[**hint:** Equality (18.3) can be useful; for every $m \in \{1, 2, \ldots, n\}$, consider the polynomial $p_m(x) = x^m - \frac{a_1^m + a_2^m + \cdots + a_k^m}{k} \in \mathbb{R}[x;n]$; do not forget about the case $n = 0$]

(e) $\varphi\colon \mathbb{R}[x;n] \to \mathbb{R}^1$ where

$$\varphi\colon p(x) \mapsto \int_0^1 p(x)\,\mathrm{d}x \cdot [1], \quad \text{for every } p(x) \in \mathbb{R}[x;n].$$

[**hint:** Equality (18.3) can be useful; do not forget about the case $n = 0$]

Novosad Ivan

a) $\begin{bmatrix} 1 & -2 & 1 & 5\\ -1 & 2 & 0 & -3\\ 2 & -4 & -1 & 4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \begin{bmatrix} 1 & -2 & 0 & 3\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$

$\Rightarrow \varphi$ is not injective; $\Rightarrow \begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4 \end{bmatrix} = \begin{bmatrix} -3a+2t\\ t\\ -2a\\ a \end{bmatrix} \Rightarrow$

$\Rightarrow \left(\begin{bmatrix} 2\\ 1\\ 0\\ 0 \end{bmatrix}, \begin{bmatrix} -3\\ 0\\ -2\\ 1 \end{bmatrix}\right)$ is a basis for $Ker(\varphi)$

$\left(\begin{bmatrix} 1\\ -1\\ 2 \end{bmatrix}, \begin{bmatrix} 1\\ 0\\ -1 \end{bmatrix}\right)$ - basis for $Im(\varphi)$

b) $\left(\begin{bmatrix} 1\\ 0 \end{bmatrix}, \begin{bmatrix} 0\\ 1 \end{bmatrix}\right)$ - basis for $R[x,1]$

$(ax+b) \xrightarrow{\varphi} a\cos(x) + b\sin(x)$

$B = \langle \cos(x), \sin(x) \rangle$

$\begin{bmatrix} 1\\ 0 \end{bmatrix}_A \xrightarrow{\varphi} \begin{bmatrix} 1\\ 0 \end{bmatrix}_B \wedge \begin{bmatrix} 0\\ 1 \end{bmatrix} \xrightarrow{\varphi} \begin{bmatrix} 0\\ 1 \end{bmatrix}_B \Rightarrow \varphi$ is injective since $\dim(Ker(\varphi)) = 0$

$\Rightarrow \dim(Im(\varphi)) = \dim(R[x,1]) = \dim(T) \Rightarrow \varphi$ is bijective.

c) $\begin{bmatrix} 3 & 2\\ -2 & 1\\ 0 & 4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0\\ 0 & 1\\ 0 & 0 \end{bmatrix} \Rightarrow \left(\begin{bmatrix} 3\\ -2\\ 1 \end{bmatrix}, \begin{bmatrix} 2\\ 1\\ 4 \end{bmatrix}\right)$ is a basis for $Im(\varphi) \wedge \varphi$ is not surjective

d) $x^3 - \frac{1^3 + 2^3 + 3^3}{3} = x^3 - 12 \Rightarrow 0(\,p(1) + p(2) + p(3)) \cdot [1] = [0]$ but that just random check. it's not a prove btw

but I provide an exampe, that if $n \geq 0$ then $\dim(Ker(\varphi)) \neq 0 \Rightarrow$

$Ker(\varphi) = \left\{ p(x) \in R[x,n] \mid n \geq 0 \wedge p(x) = x^m - \frac{a_1^m + a_2^m + \ldots a_k^m}{k} \right\}$

φ is not injective

if $n = 0 \Rightarrow \left.\begin{matrix} Ker(\varphi) = \{0\} \\ \dim(Im(\varphi)) = \dim(R^1) = 1 \end{matrix}\right\} \Rightarrow \varphi$ is bijective, since ochev.

e) if $n = 0 \Rightarrow \left.\begin{matrix} Ker(\varphi) = \{0\} \\ \dim(Im(\varphi)) = \dim(R^1) = 1 \end{matrix}\right\} \Rightarrow \varphi$ is bijective, since ochev.

if $n > 0$: $\Rightarrow Ker(\varphi) = \left\{ p \in R[x,n] \mid \int_0^1 p(x)\,dx = 0 \right\} \wedge$ it's obviously not empty since: $\int_0^1 [-x^2 + x]\,dx = 0$

moreover basis for $Ker(\varphi) = Mat_n(R)$ s.t. $tr(Mat_n(R)) = 0 \Rightarrow$ i.e. $\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1 \end{bmatrix}$ for $n = 4$

only for pol.

$\wedge\ Im(\varphi) = R$; φ is not injective; φ is surjective;

2. (1 point per item) Let a linear transformation $\varphi\colon \mathbb{R}[x;2] \to \mathbb{R}[x;2]$ be defined as

$$\varphi\colon p(x) \mapsto (\lambda x - 1)p(x)' - 2p(x), \qquad \text{for every } p(x) \in \mathbb{R}[x;2].$$

Then:

(a) find all $\lambda \in \mathbb{R}$ such that φ is *not* an isomorphism;

(b) for every λ from Item (a), find a basis for $\operatorname{Ker}(\varphi)$ and $\operatorname{Im}(\varphi)$.

[**hint:** see Problem 2 from Seminar 20]

$p(x) = 1: \quad 1 \xrightarrow{\varphi} (\lambda x - 1)0 - 2 = -2$

$p(x) = x: \quad x \xrightarrow{\varphi} (\lambda x - 1) - 2x = x(\lambda - 2) - 1$

$p(x) = x^2: \quad x^2 \xrightarrow{\varphi} 2(\lambda x - 1)x - 2x^2 = x^2(2\lambda - 2) - 2x$

$$\varphi : \begin{bmatrix} -2 & -1 & 0 \\ 0 & \lambda - 2 & -2 \\ 0 & 0 & 2\lambda - 2 \end{bmatrix}$$

a) hence φ is not isomorphic iff: $\lambda = 1 \vee \lambda = 2$

b) case $\lambda = 1$:

$$\left[\begin{array}{ccc|c} -2 & -1 & 0 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1c \\ -2c \\ c \end{bmatrix} \Rightarrow \left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right) \quad \operatorname{Ker}(\varphi) \text{ if } \lambda = 1;$$

case $\lambda = 2$:

$$\begin{bmatrix} -2 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -c \\ 2c \\ 0 \end{bmatrix} \Rightarrow \left(\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}\right) \quad \operatorname{ker}(\varphi) \text{ if } \lambda = 1;$$

3. (1 point per item) Find an isomorphism of vector spaces $\mathbb{V}$ and $\mathbb{W}$, say $\varphi\colon \mathbb{V} \to \mathbb{W}$, if

(a) $\mathbb{V} = \mathbb{R}[x;2]$ and $\mathbb{W}$ is the vector space of all skew-symmetric matrices of size 3;

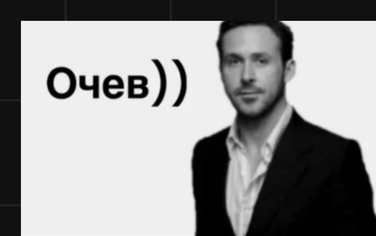
(b) $\mathbb{V} = \{p(x) \in \mathbb{R}[x;3] \mid p(3) = 0\}$ and $\mathbb{W}$ is the solution set of the following homogeneous system of linear equations

$$\left[\begin{array}{cccccc|c} 1 & -2 & 2 & 1 & 3 & 4 & 0 \\ -1 & 2 & -3 & -2 & -1 & -1 & 0 \\ 2 & -4 & 3 & 1 & 9 & 12 & 0 \\ -2 & 4 & -7 & -5 & 1 & 2 & 0 \end{array}\right].$$

[**hint:** use Theorem 20.3]

a) $V = \langle \underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{a}, \underbrace{\begin{bmatrix}0\\1\\0\end{bmatrix}}_{b}, \underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{c} \rangle$ $W = \langle \underbrace{\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix}}_{w_1}, \underbrace{\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix}}_{w_2}, \underbrace{\begin{bmatrix}0&0&0\\0&0&1\\0&-1&0\end{bmatrix}}_{w_3} \rangle$

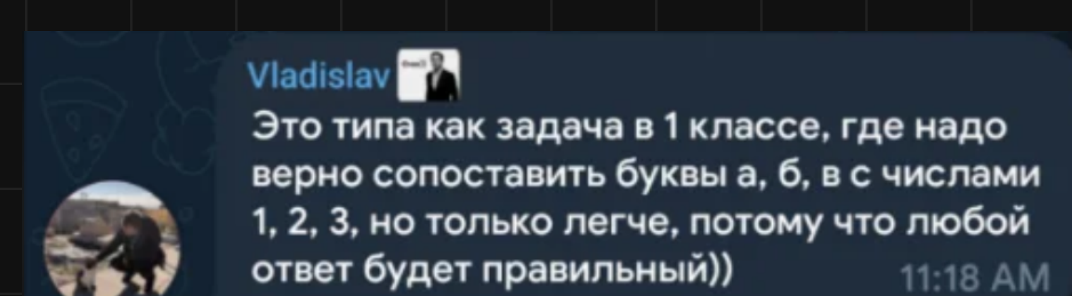
$\varphi: \begin{bmatrix}x\\y\\z\end{bmatrix} \longrightarrow \begin{bmatrix}0&x&y\\-x&0&z\\-y&-z&0\end{bmatrix}$; ochev that $a \xrightarrow{\varphi} w_1$, $b \xrightarrow{\varphi} w_2$, $c \xrightarrow{\varphi} w_3$



b) $V = \langle (x-3), (x-3)x, (x-3)x^2 \rangle$:

$$\left[\begin{array}{cccccc|c} 1 & -2 & 2 & 1 & 3 & 4 & 0 \\ -1 & 2 & -3 & -2 & -1 & -1 & 0 \\ 2 & -4 & 3 & 1 & 9 & 12 & 0 \\ -2 & 4 & -7 & -5 & 1 & 2 & 0 \end{array}\right] \xrightarrow{RREF} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \left[\begin{array}{cccccc|c} 1 & -2 & 0 & -1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow A \begin{bmatrix}-3\\1\\0\\0\end{bmatrix} = \begin{bmatrix}2\\1\\0\\0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{bmatrix} = \begin{bmatrix}2a+b-3c\\a\\-b+c\\b\\-c\\c\end{bmatrix} \Rightarrow \left(\begin{bmatrix}2\\1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\0\\-1\\1\end{bmatrix}\right) \text{ - basis for } W$$



$$\begin{bmatrix}-3\\1\\0\\0\end{bmatrix} \xrightarrow{\varphi} \begin{bmatrix}2\\1\\0\\0\\0\\0\end{bmatrix}; \quad \begin{bmatrix}0\\-3\\1\\0\end{bmatrix} \xrightarrow{\varphi} \begin{bmatrix}1\\0\\-1\\1\\0\\0\end{bmatrix}; \quad \begin{bmatrix}0\\0\\-3\\1\end{bmatrix} \xrightarrow{\varphi} \begin{bmatrix}-3\\0\\1\\0\\-1\\1\end{bmatrix}$$

4. (1 point) Let

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

be two matrices from the set $\mathrm{Mat}_3(\mathbb{R})$. Is there exist a matrix $A \in \mathrm{Mat}_3(\mathbb{R})$ such that $AB = C$ (you need to justify your answer)?

[**hint:** do not use the direct approach (with 9-by-9 system of linear equations); let $\mathcal{A}$ be any ordered basis for $\mathbb{R}^3$ (say the standard one), let $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ and $\psi\colon \mathbb{R}^3 \to \mathbb{R}^3$ be two linear transformations with the coordinate matrices $T(\varphi, \mathcal{A}, \mathcal{A}) = C$ and $T(\psi, \mathcal{A}, \mathcal{A}) = B$ (do you understand why such linear transformations exist?); then use Problem 5 (for $\mathbb{V} = \mathbb{W} = \mathbb{Z} = \mathbb{R}^3$) and Theorem 19.2]

$A = T(\psi \circ \varphi, \mathcal{A}, \mathcal{A})^{-1}$; $AB = C \Rightarrow A = CB^{-1}$; $\wedge$ since B is not invertable $(\det(B) = 0)$ .. A is not exist.