2. (HW) Evaluate the indefinite integral involving fractional powers:

✓ (a) $\int \frac{15x+3}{2\sqrt{x+3}}\,dx$; ✓ (b) $\int \frac{\sqrt{x}}{\sqrt{x}+1}\,dx$; ✓ (c) $\int \frac{\sqrt{x}}{x-\sqrt[3]{x^2}}\,dx$.

✓ (d) $\int \frac{\sqrt{x}-1}{\sqrt{x}+1}\,dx$; ✓ (e) $\int \frac{dx}{\sqrt[3]{x}-\sqrt[4]{x}}$; ✓ (f) $\int \frac{2x^2-3x}{\sqrt{x^2-2x+5}}\,dx$.

Novosad Ivan

✓ a) $\int \frac{15x+3}{2\sqrt{x+3}}dx = \left| \begin{matrix} t = x+3 \\ dt = dx \end{matrix} \right| = \int \frac{15t-42}{2\sqrt{t}}dt = \frac{5}{2}\int\sqrt{t}\,dt - 21\int\frac{dt}{\sqrt{t}} = \frac{5}{3}t^{3/2} - 42\sqrt{t} + c =$

$= \frac{5}{3}(x+3)^{3/2} - 42\sqrt{x+3} + c$

✓ b) $\int \frac{\sqrt{x}}{\sqrt{x}+1}dx = \int \frac{\sqrt{x}+1}{\sqrt{x}+1}dx - \int\frac{dx}{\sqrt{x}+1} = \underline{x - 2\sqrt{x} + 2\ln(\sqrt{x}+1) + c}$

$-\int\frac{dx}{\sqrt{x}+1} = \left| \begin{matrix} t = \sqrt{x}+1 \\ dt = \frac{1}{2\sqrt{x}} \end{matrix} \right| = -\int\frac{2t-2}{t}dt = -2\int dt + 2\int\frac{1}{t}dt = -2t + 2\ln(|t|) + c =$

$= -2\sqrt{x} - 2 + 2\ln(\sqrt{x}+1) + c$

using long pol. div.

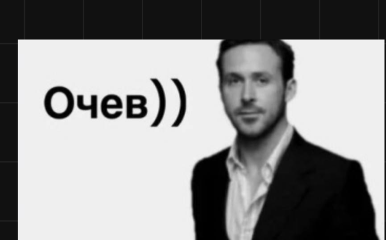
✓ c) $\int \frac{\sqrt{x}}{x - \sqrt[3]{x^2}}dx \left| \begin{matrix} t^6 = x \\ dx = 6t^5dt \end{matrix} \right| = \int \frac{t^3 \cdot 6t^5}{t^6 - t^4}dt = 6\int\frac{t^8}{t^4(t^2-1)}dt = 6\int\frac{t^4}{t^2-1}dt = 6\int\left(t^2 + \frac{1}{2(t-1)} - \frac{1}{2(t+1)} + 1\right)dt = 2t^3 + 6t + 3\ln(|t-1|) - 3\ln(|t+1|) + c =$

$= 2\sqrt{x} + 6\sqrt[6]{x} + 3\ln(|\sqrt[6]{x} - 1|) - 3\ln(\sqrt[6]{x} + 1) + c$

✓ d) $\int\frac{\sqrt{x}-1}{\sqrt{x}+1}dx = \int\frac{\sqrt{x}+1}{\sqrt{x}+1}dx - 2\int\frac{dx}{\sqrt{x}+1} = x - 2\int\frac{dx}{\sqrt{x}+1} = x - 4\sqrt{x} + 4\ln(\sqrt{x}+1) + c$

$-2\int\frac{dx}{\sqrt{x}+1} = \left| \begin{matrix} t = \sqrt{x}+1 \\ dx = 2\sqrt{x}dt \end{matrix} \right| = -4\int\frac{t-1}{t}dt = -4\int dt + 4\int\frac{dt}{t} = -4t + 4\ln(|t|) + c = -4\sqrt{x} - 4 + 4\ln(|\sqrt{x}+1|) = -4\sqrt{x} + 4\ln(\sqrt{x}+1) + c$

✓ e) $\int\frac{dx}{\sqrt[3]{x} - \sqrt[4]{x}} = \left| \begin{matrix} t^{12} = x \\ dx = 12t^{11} \end{matrix} \right| = 12\int\frac{t^{11}}{t^4 - t^3}dt = 12\int\frac{t^8}{t-1}dt = 12\int\left[t^7 + t^6 + t^5 + \ldots + t + 1 + \frac{1}{t-1}\right]dt = \frac{12}{8}t^8 + \frac{12}{7}t^7 + \frac{12}{6}t^6 + \frac{12}{5}t^5 + \frac{12}{4}t^4 + \frac{12}{3}t^3 + \frac{12}{2}t^2 + 12 + 12\ln(|t-1|) + c$



all answers are blue

$= \frac{12}{8}x^{8/12} + \frac{12}{7}x^{7/12} + \frac{12}{6}x^{6/12} + \frac{12}{5}x^{5/12} + \frac{12}{4}x^{4/12} + \frac{12}{3}x^{\frac{3}{12}} + \frac{12}{2}x^{2/12} + 12x^{1/12} + 12\ln(|x^{1/12} - 1|) + c$

$\frac{3}{2}x^{2/3} + \frac{12}{7}x^{7/12} + 2x^{1/2} + \frac{12}{5}x^{5/12} + 3x^{1/3} + 4x^{1/4} + 6x^{1/6} + 12x^{1/12} + 12\ln(1 - x^{1/12}) + c$ iff $x \in \mathbb{R}$

✓ f) $\int\frac{2x^2-3x}{\sqrt{x^2-2x+5}}dx = (Ax+B)\sqrt{x^2-2x+5} + \lambda\int\frac{dx}{\sqrt{x^2-2x+5}} \Rightarrow \int\frac{2x^2-3x}{\sqrt{x^2-2x+5}}dx = x\sqrt{x^2-2x+5} - 5\int\frac{dx}{\sqrt{x^2-2x+5}} = x\sqrt{x^2-2x+5} - 5\int\frac{dx}{\sqrt{(x-1)^2+4}} =$

$\frac{d}{dx}\left(\int\frac{2x^2-3x}{\sqrt{x^2-2x+5}}dx = (Ax+B)\sqrt{x^2-2x+5} + \lambda\int\frac{dx}{\sqrt{x^2-2x+5}}\right) =$

$= x\sqrt{x^2-2x+5} - 5\ln(|x - 1 + \sqrt{x^2-2x+5}|) + c$

$= \frac{2x^2-3x}{\sqrt{x^2-2x+5}} = \frac{2Ax^2 - 3Ax + 5A + Bx - B + \lambda}{\sqrt{x^2-2x+5}} = 2x^2 - 3x = x^2(2A) + x(-3A + B) + (5A - B + \lambda)$

$\Updownarrow$

$$\begin{cases} 2A = 2 \\ -3A + B = -3 \\ 5A - B + \lambda = 0 \end{cases} \Rightarrow (A, B, \lambda) = (1, 0, -5)$$

4. (HW) Find the indefinite integral using trigonometric or hyperbolic substitution:

$n^2 \cdot \sqrt{1-n^2}\,dn$

✓ (a) $\int x^2 \cdot \sqrt{1-x^2}\,dx$; ✓ (b) $\int \sqrt{x^2+9}\,dx$.

√ a) $\int n^2 \cdot \sqrt{1-n^2}\,dn = \left| \begin{array}{l} n = \sin(x) \\ dn = \cos(x)d\theta \end{array} \right| = \int \cos^2(x)\sin^2(x)\,dx = \int \sin^2(x)(1-\sin^2(x))\,dx = \int (\sin^2(x) - \sin^4(x))\,dx \ominus$

sorry for n instead of x

√ $\int \sin^2(x)\,dx = \frac{1}{2}\int [1-\cos(2x)]\,dx = \frac{x}{2} - \frac{1}{4}\sin(2x) + c$

√ $\int \sin^4(x)\,dx = \frac{1}{4}\int (1-\cos(2x))^2\,dx = \frac{1}{4}\int dx - \frac{1}{2}\int \cos(2x)\,dx + \frac{1}{4}\int \cos^2(2x)\,dx = \frac{x}{4} - \frac{1}{4}\sin(2x) + \frac{x}{8} + \frac{1}{32}\sin(4x) + c =$

$= \frac{3}{8}x - \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32} + c$

√ $\frac{1}{4}\int \sin^2(2x)\,dx = \frac{1}{8}\int (1-\cos(4x))\,dx = \frac{x}{8} - \frac{1}{32}\sin(4x) + c$

✓ $\ominus \frac{x}{2} - \frac{1}{4}\sin(2x) - \frac{3}{8}x + \frac{\sin(2x)}{4} - \frac{\sin(4x)}{32} + c = \frac{x}{8} - \frac{\sin(4x)}{32} + c = \frac{x}{8} - \frac{\sin(2x)\cos(2x)}{16} + c =$

$= \frac{x}{8} - \frac{\sin(x)\cos(x)(1-2\sin^2(x))}{8} = \frac{x}{8} - \frac{\sin(x)\cos(x)}{8} + \frac{\sin^3(x)\cos(x)}{4} + c \ominus$

$\ominus \frac{\arcsin(n)}{8} - \frac{n\sqrt{1-n^2}}{8} + \frac{n^3\sqrt{1-n^2}}{4} + c = \frac{\arcsin(n) - (n\sqrt{1-n^2})(1-2n^2)}{8} + c$

√ b) $\int \sqrt{x^2+9}\,dx = \left| \begin{array}{l} x = 3\sinh(\theta) \\ dx = 3\cosh(\theta) \end{array} \right| = 9\int \cosh^2(\theta)\,d\theta = \frac{9}{2}\int (1+\cosh(2\theta))\,d\theta = \frac{9\theta}{2} + \frac{9}{4}\sinh(2\theta) + c =$

$= \frac{9}{2}\theta + \frac{9}{2}\sinh(\theta)\cosh(\theta) + c$

$= \frac{9}{2}\operatorname{arcsinh}\left(\frac{x}{3}\right) + \frac{3}{2}x\sqrt{\frac{x^2}{9}+1} + c$

5. (HW) Solve miscellaneous problems:

$\int \frac{\sin x}{1+\sin x}\,dx$; $\int \frac{(x+3)(x+4)}{(x-2)(x-6)^2}\,dx$; $\int \cos x \cos 2x \cos 3x\,dx$;

$\int \frac{\ln x \cos \ln x}{x}\,dx$; $\int \frac{3x^2-1}{x\sqrt{x}} \arctan x\,dx$.

✓a) $\int \frac{\sin(x)}{1+\sin(x)}dx = \int dx - \int \frac{dx}{1+\sin(x)} = x + \frac{2}{\tan(\frac{x}{2})+1} + c$

$$\int \frac{dx}{1+\sin(x)} = \left| \begin{array}{l} x = 2\arctan(t) \\ dx = \frac{2\,dt}{1+t^2} \\ \sin(x) = \frac{2t}{1+t^2} \end{array} \right| = \int \frac{2\,dt}{(1+t^2)(1+\frac{2t}{1+t^2})} = \int \frac{2\,dt}{\left(\frac{1+t^2+2t}{1+t^2}\right)(1+t^2)} = \int \frac{2\,dt}{t^2+2t+1} =$$

$$= 2\int \frac{dt}{(t+1)^2} = -2\frac{1}{t+1} + c = \frac{2}{\tan(\frac{x}{2})+1} + c$$

✓ $\int \frac{(x+3)(x+4)}{(x-2)(x-6)^2}dx = \int \left[\frac{15}{8(x-2)} - \frac{7}{8(x-6)} + \frac{45}{2(x-6)^2}\right]dx = \frac{15}{8}\ln(|x-2|) - \frac{7}{8}\ln(|x-6|) - \frac{45}{2(x-6)} + c$

Note: (draft)

$\frac{A}{x-2} + \frac{B}{x-6} + \frac{C}{(x-6)^2} = \frac{(x+3)(x+4)}{(x-2)(x-6)^2}$

$\Updownarrow$

$(x+3)(x+4) = (x-6)^2 A + (x-2)(x-6)B + (x-2)C$

$\Updownarrow$

$x^2+7x+12 = Ax^2 + Bx^2 - 12Ax - 8Bx + Cx + 36A + 12B - 2C$

$$\Rightarrow \begin{cases} 12 = 36A + 12B - 2C \\ 7 = -12A - 8B + C \\ 1 = A + B \end{cases} \Rightarrow (A,B,C) = (15/8; -7/8; 45/2)$$

✓c) $\int \cos(x)\cos(2x)\cos(3x)\,dx = \frac{1}{2}\int(\cos(x)+\cos(3x))\cos(3x)\,dx = \frac{1}{2}\int \cos(x)\cos(3x)dx + \frac{1}{2}\int\cos^2(3x)\,dx = \frac{x}{4} + \frac{\sin(6x)}{24} + \frac{\sin(4x)}{16} + \frac{\sin(2x)}{8} + c$

✓ $\frac{1}{2}\int \cos^2(3x)dx = \{t = 3x\} = \frac{1}{6}\int\cos^2(t)dx = \frac{1}{12}\int(1+\cos(2t))dt = \frac{t}{12} + \frac{\sin(2t)}{24} + c = \frac{x}{4} + \frac{\sin(6x)}{24} + c$

✓ $\frac{1}{2}\int\cos(x)\cos(3x)\,dx = \frac{1}{4}\int[\cos(2x)+\cos(4x)]\,dx = \frac{1}{8}\sin(2x) + \frac{1}{16}\sin(4x) + c$

✓d) $\int \frac{\ln(x)\cos(\ln(x))}{x}dx = \left\{\begin{array}{l} t = \ln(t) \\ dt = \frac{1}{x}\end{array}\right\} = \int t\cos(t)\,dt = \left\{\begin{array}{ll} m = t & dw = \cos(t)dt \\ dm = dt & w = \sin(t)\end{array}\right\} = t\sin(t) - \int\sin(t)dt = t\sin(t) + \cos(t) + c = \ln(x)\sin(\ln(x)) + \cos(\ln(x)) + c$

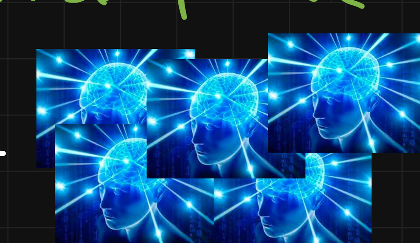
✗ e) $\int \frac{3x^2-1}{x\sqrt{x}}\arctan(x)dx = \{t = \sqrt{x}\} = 2\int \frac{3t^4-1}{t^2}\arctan(t^2)dt = 6\int t^2\arctan(t^2)\,dt - 2\int\frac{\arctan(t^2)}{t^2}dt$

✓ $6\int t^2\arctan(t^2)\,dt = \text{sub. back} = 6\int x\arctan(x)dx = \left\{\begin{array}{ll} \varphi = \arctan(x) & d\psi = x\,dx \\ d\varphi = \frac{dx}{x^2+1} & \psi = \frac{x^2}{2}+\frac{1}{2}\end{array}\right\} = 3(x^2+1)\arctan(x) - 3\int\frac{x^2+1}{x^2+1}dx = 3(x^2+1)\arctan(x) - 3x + c$

? cause why not?

$-2\int\frac{\arctan(t^2)}{t^2}dt = -2\int\frac{\arctan(x)}{x}dx = \left\{\begin{array}{ll} \varphi = \arctan(x) & d\psi = \frac{dx}{x} \\ d\varphi = \frac{dx}{x^2+1} & \psi = \ln(x)\end{array}\right\} = -2\ln(x)\arctan(x) + 2\int\frac{\ln(x)}{x^2+1}dx =$

un computable



✓e) $\int\frac{3x^2-1}{x\sqrt{x}}\arctan(x)dx = \left\{\begin{array}{ll} \varphi = \arctan(x) & d\psi = \frac{3x^2-1}{x\sqrt{x}}dx \\ d\varphi = \frac{dx}{x^2+1} & \psi = 2x^{3/2} + 2x^{-1/2}\end{array}\right| = 2(x\sqrt{x} + \frac{1}{\sqrt{x}})\arctan(x) - 2\int\frac{x\sqrt{x} + 1/\sqrt{x}}{x^2+1}dx =$

$$= 2\left(\frac{x^2+1}{\sqrt{x}}\right)\arctan(x) - 2\int\frac{\frac{x^2+1}{\sqrt{x}}}{x^2+1}dx = 2\left(\frac{x^2+1}{\sqrt{x}}\right)\arctan(x) - 2\int\frac{(x^2+1)}{(x^2+1)\sqrt{x}}dx = 2\left(\frac{x^2+1}{\sqrt{x}}\right)\arctan(x) - 4\sqrt{x} + c$$

draft:

$\int\frac{3x^2-1}{x\sqrt{x}}dx = \int 3x^{1/2}dx - \int x^{-3/2}dx = 2x^{3/2} + 2x^{-1/2} + c$

6*. (HW) For the integrals $J_n = \int \frac{x^n\,dx}{\sqrt{ax^2+bx+c}}$, $n \in \mathbb{N}$, $n > 1$, prove the following recursive formula:

$$J_n = \frac{1}{na}\left(x^{n-1}\sqrt{ax^2+bx+c} - \frac{b}{2}(2n-1)J_{n-1} - c(n-1)J_{n-2}\right).$$

Use this formula to find the integral

$$\int \frac{x^3}{\sqrt{1+2x-x^2}}\,dx.$$

using MIP

base $(n=2)$:

$$J_2 = \frac{1}{2a}\left(x\sqrt{ax^2+bx+c} - \frac{3b}{2}\left(\int\frac{x\,dx}{\sqrt{ax^2+bx+c}}\right) - c\int\frac{dx}{\sqrt{ax^2+bx+c}}\right)$$

$$\frac{d}{dx}\left(\frac{1}{2a}\left(x\sqrt{ax^2+bx+c} - \frac{3b}{2}\left(\int\frac{x\,dx}{\sqrt{ax^2+bx+c}}\right) - c\int\frac{dx}{\sqrt{ax^2+bx+c}}\right)\right)$$

$$\parallel$$

$$\frac{1}{2a}\left(\frac{2(ax^2+bx+c) + x(2ax+b) - 3bx - 2c}{2\sqrt{ax^2+bx+c}}\right)$$

$$\parallel$$

$$\frac{2ax^2+2bx+2c+2ax^2+bx-3bx-2c}{4a\sqrt{ax^2+bx+c}} = \frac{4ax^2}{4a\sqrt{ax^2+bx+c}} = \frac{x^2}{\sqrt{ax^2+bx+c}} = J_2$$

IH: J_{n-1}

IS: $$J_n = \frac{1}{na}\left(x^{n-1}\sqrt{ax^2+bx+c} - \frac{b}{2}(2n-1)\int\frac{x^{n-1}\,dx}{\sqrt{ax^2+bx+c}} - c(n-1)\int\frac{x^{n-2}}{\sqrt{ax^2+bx+c}}\,dx\right)$$

$$\frac{d}{dx}\left(\frac{1}{na}\left(x^{n-1}\sqrt{ax^2+bx+c} - \frac{b}{2}(2n-1)\int\frac{x^{n-1}\,dx}{\sqrt{ax^2+bx+c}} - c(n-1)\int\frac{x^{n-2}}{\sqrt{ax^2+bx+c}}\,dx\right)\right)$$

$$\frac{1}{na}\left(\frac{2anx^n + 2bnx^{n-1} - bx^{n-1} + 2cnx^{n-2} - 2cx^{n-2} - 2bnx^{n-1} + bx^{n-1} - 2cnx^{n-2} + 2cx^{n-2}}{2\sqrt{ax^2+bx+c}}\right)$$

$$\parallel$$

$$\frac{2anx^n}{2an\sqrt{ax^2+bx+c}} = \frac{x^n}{\sqrt{ax^2+bx+c}}$$

In fact I think it's not necessary to use MIP there!

6*. (HW) For the integrals $J_n = \int \frac{x^n\,dx}{\sqrt{ax^2+bx+c}}$, $n \in \mathbb{N}$, $n > 1$, prove the following recursive formula:

$$J_n = \frac{1}{na}\left(x^{n-1}\sqrt{ax^2+bx+c} - \frac{b}{2}(2n-1)J_{n-1} - c(n-1)J_{n-2}\right).$$

Use this formula to find the integral

$$\int \frac{x^3}{\sqrt{1+2x-x^2}}\,dx.$$

$$\int \frac{x^3}{\sqrt{1+2x-x^2}}dx = \int \frac{x^3}{\sqrt{-1x^2+2x+1}}dx = -\frac{1}{3}\left(x^2\sqrt{-1x^2+2x+1} - 5\int\frac{x^2}{\sqrt{-1x^2+2x+1}}dx - 2\int\frac{x}{\sqrt{-1x^2+2x+1}}dx\right)$$

$$-\frac{1}{3}\left(x^2\sqrt{-1x^2+2x+1} + \frac{5}{2}\left(x\sqrt{-1x^2+2x+1} - 3\int\frac{x}{\sqrt{-1x^2+2x+1}}dx - \int\frac{dx}{\sqrt{-1x^2+2x+1}}\right)\right) - 2\int\frac{x}{\sqrt{-1x^2+2x+1}}dx =$$

✓ $$\int\frac{x}{\sqrt{-1x^2+2x+1}}dx = \int\frac{x}{\sqrt{2-(x-1)^2}}dx = \{m = x-1\} = \int\frac{m+1}{\sqrt{2-m^2}}dm = \int\frac{m}{\sqrt{2-m^2}}dm + \int\frac{1}{\sqrt{2-m^2}}dm =$$

$$= -\sqrt{2-m^2} + \arcsin\left(\frac{m}{\sqrt{2}}\right) + c = -\sqrt{-x^2+2x+1} + \arcsin\left(\frac{x-1}{\sqrt{2}}\right) + c$$

✓ $$\int\frac{m}{\sqrt{2-m^2}}dm = \left\{\begin{aligned} k &= \sqrt{2-m^2} \\ dk &= -\frac{m}{\sqrt{2-m^2}} \end{aligned}\right\} = -\int dk = -k + c = -\sqrt{2-m^2} + c$$

✓ $$\int\frac{1}{\sqrt{2-m^2}}dm = \arcsin\left(\frac{m}{\sqrt{2}}\right) + c$$

✓ $$\int\frac{dx}{\sqrt{-1x^2+2x+1}} = \int\frac{dx}{\sqrt{2-(x-1)^2}} = \arcsin\left(\frac{x-1}{\sqrt{2}}\right) + c$$

$$= -\frac{1}{3}\left(x^2\sqrt{-1x^2+2x+1} + \frac{5}{2}\left(x\sqrt{-1x^2+2x+1} - 3\int\frac{x}{\sqrt{-1x^2+2x+1}}dx - \int\frac{dx}{\sqrt{-1x^2+2x+1}}\right)\right) - 2\int\frac{x}{\sqrt{-1x^2+2x+1}}dx$$

$$-\frac{1}{3}\left(x^2\sqrt{-x^2+2x+1} + \frac{5}{2}\left(x\sqrt{-x^2+2x+1} + 3\sqrt{-x^2+2x+1} - 4\arcsin\left(\frac{x-1}{\sqrt{2}}\right)\right) + 2\sqrt{-x^2+2x+1} - 2\arcsin\left(\frac{x-1}{\sqrt{2}}\right)\right) + c$$

ans!

Tnx for checking!

solved by Novosad Ivan