Novosad Ivan

2. (HW) Find Taylor expansion for $2x^3 - 3xy^2 + 5x^2 + 4y + 2$ about the point $P(2,1)$.

$f(x,y) = 2x^3 - 3xy^2 + 5x^2 + 4y + 2$

$f_x = 6x^2 - 3y^2 + 10x \qquad f_y = -6xy + 4$

$f_{xx} = 12x + 10 \qquad f_{yy} = -6x$

$f_{xxx} = 12 \qquad f_{yyy} = 0$

$f_{xy} = -6y \qquad f_{yyx} = -6 \qquad f_{xxy} = 0$

$df = (6x^2 - 3y^2 + 10x)dx + (-6xy + 4)dy \Rightarrow df(2,1) = 41dx - 8dy$

$d^2f = (12x+10)dx^2 - 6x\,dy^2 - (12y)dx\,dy \Rightarrow d^2f(2,1) = 34dx^2 - 12dy^2 - 12dx\,dy$

$d^3f = 12dx^3 - 18dx\,dy^2 \Rightarrow d^3f(2,1) = 12dx^3 - 18\,dx\,dy^2$

$f(x,y) = 36 + 41(x-2) - 8(y-1) + 17(x-2)^2 - 6(y-1)^2 - 6(x-2)(y-1) + 2(x-2)^3 - 3(x-2)(y-1)^2$

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5. (HW) Find the first and the second total differentials of the function $z = \sqrt{\operatorname{th} x + y}$ at the point $P(0, 1)$. Find Taylor expansion for this function about the point P with $o(x^2 + (y-1)^2)$.

thx = thanks? $f(x,y_1) = \sqrt{th(x)+y} = z$

$f_x = \left(2\cosh^2(x)\sqrt{\tanh(x)+y}\right)^{-1}$ $f_y = \dfrac{\sqrt{e^{2x}+1}}{2\sqrt{e^{2x}-1+ye^{2x}+y}}$

$$f_{xx} = -\frac{4\sinh^2(x)+4\sinh(x)\cosh(x)y+1}{4\cosh^3(x)\sqrt{\tanh(x)+y}\,(\sinh(x)+\cosh(x)y)}$$

$$f_{xy} = -\frac{e^x}{\left|e^x+\frac{-1+y}{e^x}+ye^x\right|\sqrt{e^{4x}+ye^{4x}+2ye^{2x}+y-1}}$$

$$f_{yy} = \frac{\sqrt{e^{2x}+1}\,(e^{2x}+1)}{\sqrt{e^{2x}-1+ye^{2x}+y}\,(4e^{2x}-4+4ye^{2x}+4y)}$$

$$df = \left(2\cosh^2(x)\sqrt{\tanh(x)+y}\right)^{-1}dx + \left(\frac{\sqrt{e^{2x}+1}}{2\sqrt{e^{2x}-1+ye^{2x}+y}}\right)dy$$

$$df(p) = \frac{1}{2}dx + \frac{1}{2}dy$$

$$d^2f = -\frac{4\sinh^2(x)+4\sinh(x)\cosh(x)y+1}{4\cosh^3(x)\sqrt{\tanh(x)+y}\,(\sinh(x)+\cosh(x)y)}dx^2 + \frac{\sqrt{e^{2x}+1}\,(e^{2x}+1)}{\sqrt{e^{2x}-1+ye^{2x}+y}\,(4e^{2x}-4+4ye^{2x}+4y)}dy^2 - \frac{e^x}{\left|e^x+\frac{-1+y}{e^x}+ye^x\right|\sqrt{e^{4x}+ye^{4x}+2ye^{2x}+y-1}}dxdy$$

$$d^2f(p) = \frac{1}{5}dx^2 + \frac{\sqrt{2}}{2}dy^2 - \frac{2}{5}dxdy$$

$$f(p) = 1$$

$$f(x,y) = 1 + \frac{1}{2}dx + \frac{1}{2}dy + \frac{1}{10}dx^2 + \frac{\sqrt{2}}{4}dy^2 - \frac{1}{5}dxdy + o(\rho^2)$$

$$f(x,y) = 1 + \frac{1}{2}(x) + \frac{1}{2}(y-1) + \frac{1}{10}(x^2) + \frac{\sqrt{2}}{4}(y-1)^2 - \frac{1}{5}(x)(y-1) + o(x^2+(y-1)^2)$$

6. (HW) Find the first and the second total differentials of the function $z = \ln(1 + e^x \ln y)$ at the point $P(1, e)$. Find Taylor expansion for this function about the point P with $o((x-1)^2 + (y-e)^2)$.

let $z = f(x,y)$, so f_x is $\frac{\partial z}{\partial x}$, an so on

$f_x = \frac{\ln(y)e^x}{1+\ln(y)e^x}$ $\qquad f_y = \frac{e^x}{y+ye^x\ln(y)}$

$f_{xx} = \frac{\ln(y)e^x}{(1+\ln(y)e^x)^2}$ $\qquad f_{yy} = -\frac{e^x + e^{2x}\ln(y) + e^{2x}}{(y+ye^x\ln(y))^2}$

$f_{xy} = \frac{e^x}{y(1+e^x\ln(y))^2}$

$f_x(p) = \frac{e}{1+e}dx$ $\qquad f_{xy}(p) = \frac{1}{1+2e+e^2}dx\,dy$ $\qquad f_{yy}(p) = -\frac{1+2e}{1+e}dy^2$

$f_{xx}(p) = \frac{e}{(1+e)^2}dx^2$ $\qquad f_y(p) = \frac{1}{1+e}dy$

Hence

$df(p) = \frac{e}{1+e}dx + \frac{1}{1+e}dy$; $f(p) = \ln(1+e)$

$d^2f(p) = \frac{e}{(1+e)^2}dx^2 - \frac{1+2e}{1+e}dy^2 + \frac{2}{1+2e+e^2}dx\,dy$

$f(x,y) = z = \ln(e+1) + \frac{e}{1+e}(x-1) + \frac{1}{1+e}(y-e) + \frac{e}{2(1+e^2)}(x-1)^2 - \frac{1+2e}{2+2e}(y-e)^2 + \frac{1}{1+2e+e^2}(x-1)(y-e) + \bar{o}$

8. (HW) Let $u = f(xy)$. Show that $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0$.

$$x\frac{du}{dx} - y\frac{du}{dy} = f'_p \cdot \underline{y \cdot x} - f'_p \, \underline{x \cdot y} = 0$$

10. (HW) Let $h(x, y) = f(e^x \sin y, e^x \cos y)$ where $f(u, v)$ is a function having second order continuous derivatives. Compute $h_{xx} + h_{yy}$ in terms of the derivatives of f.

$$h_x = f_u \cdot e^x \sin(y) + f_v e^x \cos(y)$$

$$h_y = f_u e^x \cos(y) - f_v e^x \sin(y)$$

$$h_{xx} = \frac{d^2u}{dx} e^x \sin(y) + f_u \cdot e^x \sin(y) + \frac{dv}{dx} e^x \cos(y) + f_v e^x \cos(y) =$$

$$= e^x(f_u \sin(y) + f_v \cos(y) + e^x(f_{uu} \sin^2(y) + 2 f_{uv} \sin(y)\cos(y) + f_{vv} \cos^2(y)$$

$$h_{yy} = e^x(-f_u \sin(y) + f_v \cos(y) + e^x(f_{uu} \cos^2(y) - f_{uu} \sin(y)\cos(y) + f_{vv} \sin^2(y)))$$

Thus $h_{xx} + h_{yy} = e^{2x}(f_{uu} + f_{vv})$

12. (HW) Prove that if the function $f(u)$ is differentiable, then $\varphi(x,y) = xy + xf\left(\frac{y}{x}\right)$ satisfies the following equation:

$$x\frac{\partial\varphi}{\partial x} + y\frac{\partial\varphi}{\partial y} = xy + \varphi.$$

$$\frac{d\varphi}{dx} = y + f\left(\frac{y}{x}\right) - \frac{y}{x} f'\left(\frac{y}{x}\right)$$

$$\frac{d\varphi}{dy} = x + f'\left(\frac{y}{x}\right)$$

$$x \cdot \frac{dy}{dx} + y\frac{d\varphi}{dy} = xy + xf\left(\frac{y}{x}\right) - yf'\left(\frac{y}{x}\right) + xy + yf'\left(\frac{y}{x}\right) =$$

$$= xy + \varphi(x,y)$$

14. (HW) Prove that if the function $f(u,v)$ is differentiable, then $\varphi(x,y,z) = f\left(\dfrac{x-y}{xy},\ (x-y)e^{-z^2/2}\right)$ satisfies the following equation:

$$x^2 z\frac{\partial \varphi}{\partial x} + y^2 z\frac{\partial \varphi}{\partial y} + (x+y)\frac{\partial \varphi}{\partial z} = 0.$$

$$\frac{d\varphi}{dx} = f_u\left(\frac{du}{dx}\right) + f_v\left(\frac{dv}{dx}\right) = \frac{1}{x^2} f_u + e^{-z^2/2} f_v$$

$$\frac{d\varphi}{dy} = f_u\left(\frac{du}{dy}\right) + f_v\left(\frac{dv}{dy}\right) = -\frac{1}{y^2} f_u - e^{-z^2/2} f_v$$

$$\frac{d\varphi}{dz} = f_u\left(\frac{du}{dy}\right) + f_v\left(\frac{dv}{dz}\right) = 0 - z(x-y)e^{-z^2/2} f_v$$

$$x^2 z\frac{d\varphi}{dx} + y^2 z\frac{d\varphi}{dy} + (x+y)\frac{d\varphi}{dz} =$$

$$= z f_u + x^2 z e^{-z^2/2} f_v - z f_u - y^2 z e^{-z^2/2} f_v - x^2 z e^{-z^2/2} f_v + y^2 z e^{-z^2/2} f_v = 0$$

15*. (HW) Find Taylor expansion about the origin for $f(x,y) = \frac{1}{\sqrt{1-x^2-y^2}}$ with $o(\rho^{2m})$, where $\rho = \sqrt{x^2+y^2}$, $m \in \mathbb{N}$.

$$f(x,y) = \frac{1}{\sqrt{1-x^2-y^2}} = \frac{1}{\sqrt{1-(x^2+y^2)}} = \frac{1}{\sqrt{1-\rho^2}} = (1-\rho^2)^{-1/2}$$

|| Macloren expansion

$$1+\sum_{k=1}^{m} \frac{(-1)^k (2k-1)!!}{2^k k!} (-\rho^2)^k + o(\rho^{2m}) =$$

$$= 1+\sum_{k=1}^{m} \frac{(2k-1)!!}{2^k k!} \rho^{2k} + o(\rho^{2m}) =$$

$$= 1+\sum_{k=1}^{m} \frac{(2k-1)!!}{2^k k!} (x^2+y^2)^k + o(\rho^{2m})$$

|| Binominal formula

$$1+\sum_{k=1}^{m} \left(\frac{(2k-1)!!}{2^k k!} \sum_{e=0}^{k} C\binom{e}{k} x^{2k-2e} y^{2e} \right) + o(\rho^{2m})$$

Tnx for checking