Novosad

1. Let $S = \langle x,\ x+1,\ x^3 \rangle$ be a subspace of the Euclidean space $\left(\mathbb{R}[x;3], \langle\ |\ \rangle\right)$ with the scalar product

$$\langle f | g \rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x, \quad \text{for every } f, g \in \mathbb{R}[x;3].$$

Then:

(a) (2 points) using the Gram-Schmidt orthogonalization process, transform the basis $\{x,\ x+1,\ x^3\}$ of S into an *orthogonal* basis of S;

(b) (1 point) find an *orthonormal* basis for S;

[**hint:** see Theorem 30.2]

(c) (2 points) for the vector $x^2+1 \in \mathbb{R}[x;3]$, find its projection on S (that is, $\mathrm{pr}_S(x^2+1)$) and its rejection from S (that is, $\mathrm{rj}_S(x^2+1)$).

[**hint:** see the formulas from Proposition 30.2; note that these formulas hold true only for an *orthogonal* basis]

a) $f_1 = x$

$$f_2 = (x+1) - \frac{\langle x+1 | x \rangle}{\langle x | x \rangle} x = x+1 - \frac{5/6}{1/3} x = -\frac{3}{2}x + 1$$

$$f_3 = x^3 - \frac{\langle x^3 | x \rangle}{\langle x | x \rangle} x - \frac{\langle x^3 | -3/2 x + 1 \rangle}{\langle -3/2 x + 1 | -\frac{3}{2}x + 1 \rangle}\left(-\frac{3}{2}x + 1\right) = x^3 - \frac{1/5}{1/3} x + \frac{1/20}{1/4}\left(-\frac{3}{2}x + 1\right) = x^3 - \frac{9}{10}x + \frac{1}{5}$$

Computations:

$$\langle x | x \rangle = \int_0^1 x^2\,dx = \frac{1}{3}$$

$$\langle x+1 | x \rangle = \int_0^1 [x^2 + x]\,dx = \frac{5}{6}$$

$$\langle x^3 | x \rangle = \int_0^1 x^4\,dx = 1/5$$

$$\langle x^3 | -\tfrac{3}{2}x + 1 \rangle = \int_0^1 \left(-\frac{3}{2}x^4 + x^3\right)dx = -\frac{1}{20}$$

$$\langle -\tfrac{3}{2}x + 1 | -\tfrac{3}{2}x + 1 \rangle = \int_0^1 \left(-\frac{3}{2}x + 1\right)^2 dx = \frac{1}{4}$$

$\Rightarrow$ $\boxed{(f_1\ f_2\ f_3) \text{ is orthogonal basis for } S}$

$$\langle f_1 | f_2 \rangle = \langle f_1 | f_3 \rangle = \langle f_2 | f_3 \rangle = 0$$

$$\int_0^1 \left(x^3 - \frac{9}{10}x + \frac{1}{5}\right)\left(-\frac{3}{2}x + 1\right)dx = 0$$

$$\int_0^1 \left(x^4 - \frac{9}{10}x^2 + \frac{x}{5}\right)dx = 0$$

$$\int_0^1 \left(-\frac{3}{2}x^2 + x\right)dx = 0$$

b) $\left(\underbrace{\frac{f_1}{\|f_1\|}}_{\tilde{f}_1}, \underbrace{\frac{f_2}{\|f_2\|}}_{\tilde{f}_2}, \underbrace{\frac{f_3}{\|f_3\|}}_{\tilde{f}_3} \right)$ is orthonormal basis for S.

$$\frac{f_1}{\|f_1\|} = \frac{f_1}{\sqrt{\langle f_1 | f_1 \rangle}} = \frac{x}{1/\sqrt{3}} = \sqrt{3}\,x$$

$$\frac{f_2}{\|f_2\|} = \frac{f_2}{\sqrt{\langle f_2 | f_2 \rangle}} = \frac{-3/2 x + 1}{1/2} = -3x + 2$$

$$\frac{f_3}{\|f_3\|} = \frac{f_3}{\sqrt{\langle f_3 | f_3 \rangle}} = \frac{x^3 - 9/10 x + 1/5}{\sqrt{9/700}} = \frac{10\sqrt{7}x^3 - 9\sqrt{7}x + 2\sqrt{7}}{3}$$

Computations:

$$\|f_1\|^2 = \int_0^1 x^2\,dx = \frac{1}{3}$$

$$\|f_2\|^2 = \int_0^1 \left(-\frac{3}{2}x + 1\right)^2 dx = \frac{1}{4}$$

$$\|f_3\|^2 = \int_0^1 \left(x^3 - \frac{9}{10}x + \frac{1}{5}\right)^2 dx = \frac{9}{700}$$

$$\langle \tilde{f}_1\, \tilde{f}_1 \rangle = 1$$

$$\langle \tilde{f}_2 | \tilde{f}_2 \rangle = 1$$

$$\langle \tilde{f}_3 | \tilde{f}_3 \rangle = 1$$

3) $u = x^2 + 1$

$$\mathrm{pr}_S(u) = \sum_{i=1}^{3} \frac{\langle u | f_i \rangle}{\langle f_i | f_i \rangle} f_i = \sum_{i=1}^{3} \langle u | \tilde{f}_i \rangle \tilde{f}_i \quad (\text{since } \langle \tilde{f}_i\ \tilde{f}_i \rangle = 1)$$

$$\mathrm{pr}_S(u) = \sqrt{3}\int_0^1 (x^2+1)(x)\,dx\,(\sqrt{3}x) + \int_0^1 (-3x+2)(x^2+1)\,dx\,(3x+2) + \int_0^1 \frac{10\sqrt{7}x^3 - 9\sqrt{7}x + 2\sqrt{7}}{3}(x^2+1)\,dx\left(\frac{10\sqrt{7}x^3 - 9\sqrt{7}x + 2\sqrt{7}}{3}\right) =$$

$$= \frac{3\sqrt{3}}{4}(\sqrt{3}x) + \frac{5}{12}(3x+2) + \frac{\sqrt{7}}{36}\left(\frac{10\sqrt{7}x^3 - 9\sqrt{7}x + 2\sqrt{7}}{3}\right) = \frac{70x^3 + 315x + 104}{108}$$

since $\mathrm{pr}_S(u) + \mathrm{rj}_S(u) = u \Rightarrow \mathrm{rj}_S(u) = u - \mathrm{pr}_S(u) = x^2 + 1 - \dfrac{70x^3 + 315x + 104}{108} = -\dfrac{35x^3}{54} + x^2 - \dfrac{35}{12}x + \dfrac{1}{27}$

2. (2 points) Let $(\mathbb{R}^4, \langle\,|\,\rangle)$ be a Euclidean space with the scalar product

$$\langle \mathbf{x}|\mathbf{y}\rangle = x_1y_1 + x_1y_2 + x_2y_1 + 3x_2y_2 + 2x_3y_3 + x_4y_4,$$

for every $\mathbf{x} = [x_1\ x_2\ x_3\ x_4]^{\mathrm{T}}$, $\mathbf{y} = [y_1\ y_2\ y_3\ y_4]^{\mathrm{T}} \in \mathbb{R}^4$ (since one of the coefficients is 3, this scalar product is *not* the standard one). Then, find a basis of the orthogonal complement of the solution set $S \subseteq \mathbb{R}^4$ of the following system of linear equations

$$\left[\begin{array}{rrrr|r} -3 & 3 & 4 & -18 & 0 \\ 2 & -2 & -2 & 10 & 0 \end{array}\right].$$

[**hint:** find a basis of S; see Problem 2 from Seminar 30]

Definition I Let $(\mathbb{R}^n, \langle\,|\,\rangle)$ be a Euclidean space with the *standard* scalar product[1]. Then, a matrix $Q \in \mathrm{Mat}_n(\mathbb{R})$ is called orthogonal (or orthonormal) if the set of its columns is *orthonormal* (with respect to $\langle\,|\,\rangle$).

For example, the matrix

$$Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

is orthogonal (since the set of its columns is orthonormal in $(\mathbb{R}^4, \langle\,|\,\rangle)$).

$$\left[\begin{array}{rrrr|r} -3 & 3 & 4 & -18 & 0 \\ 2 & -2 & -2 & 10 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} -1 & -1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \Bigg(\underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{e_1}, \underbrace{\begin{bmatrix} -2 \\ 0 \\ 3 \\ 1 \end{bmatrix}}_{e_2} \Bigg)$$ is basis for ker, let $\langle e_1, e_2 \rangle = S$

check: $\langle e_1 | e_2 \rangle = 0 \Rightarrow$ orthogonal

then $\langle f_1, f_2 \rangle$ will be $S^\perp$, for that we need to solve:

$$\begin{cases} \langle \tilde{x} | e_1 \rangle = 0 \\ \langle \tilde{x} | e_2 \rangle = 0 \end{cases}$$

$$\langle \tilde{x} | e_1 \rangle = 2\tilde{x}_1 + \tilde{x}_2 + 3\tilde{x}_2 = 0$$

$$\langle \tilde{x} | e_2 \rangle = -2\tilde{x}_1 - 2\tilde{x}_2 + 6\tilde{x}_3 + \tilde{x}_4 = 0$$

$$\Updownarrow$$

$$\underbrace{\left[\begin{array}{rrrr|r} 2 & 4 & 0 & 0 & 0 \\ -2 & -2 & 6 & 1 & 0 \end{array}\right]}_{M} \Rightarrow \tilde{x} : \begin{bmatrix} \alpha + 6\beta \\ -\frac{\alpha}{2} - 3\beta \\ \beta \\ \alpha \end{bmatrix} \Rightarrow \left(\begin{bmatrix} 6 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \end{bmatrix} \right)$$

basis for ker (M)

basis for $S^\perp$

3. Let $(\mathbb{R}^n, \langle \mid \rangle)$ be a Euclidean space with the standard scalar product and let $Q \in \mathrm{Mat}_n(\mathbb{R})$ be an orthogonal matrix. Then

(a) (0.5 points) find $Q^{\mathrm{T}}Q$;

[**hint:** use Definition I]

[1]That is $\langle \mathbf{x}|\mathbf{y}\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n$ for every $\mathbf{x} = [x_1\ x_2\ \ldots\ x_n]^{\mathrm{T}}$, $\mathbf{y} = [y_1\ y_2\ \ldots\ y_n]^{\mathrm{T}} \in \mathbb{R}^n$.

(b) (0.5 points) is it correct that Q^{T} is an orthogonal matrix (you need to justify your answer)?

[**hint:** for example, use the result from Item (a) and Theorem 7.4]

Definition II Let $(\mathbb{R}^n, \langle \mid \rangle)$ be a Euclidean space with the standard scalar product. Then, a QR-decomposition of a matrix $A \in \mathrm{Mat}_n(\mathbb{R})$ is an equality of the form $A = QR$, where Q is an orthonormal matrix (see Definition I) and R is an upper triangular matrix.

For example, if

$$A = \begin{bmatrix} -1 & 3 \\ 1 & 5 \end{bmatrix}, \quad Q = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \text{and} \quad R = \sqrt{2}\begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix},$$

then, $A = QR$ is a QR-decomposition of A.

a) $Q^TQ = I_n$

b) $QQ^T = I = Q^TQ \Rightarrow Q^TQ^{TT} = Q^TQ = I = QQ^T = Q^{TT}Q$.

since $Q^T = Q^{-1}$
if Q is orthogonal

so Q^T is orthogonal

another proof:

Q is orthogonal if $(Ax|Ax) = (x,x)\ \forall x \in \mathbb{R}^n$

Note that $\ker(A) = \{0\}$, hence any x has y with $Ax = y$

Then $(A^{-1}x, A^{-1}x) = (A^{-1}Ay, A^{-1}Ay) = (y,y) = (Ax, Ax) = (x,x)$

4. Let $(\mathbb{R}^3, \langle\,|\,\rangle)$ be a Euclidean space with the standard scalar product $\langle \mathbf{x}|\mathbf{y}\rangle = x_1y_1 + x_2y_2 + x_3y_3$, for every $\mathbf{x} = [x_1\ x_2\ x_3]^T$, $\mathbf{y} = [y_1\ y_2\ y_3]^T \in \mathbb{R}^3$. Then, following the following steps, find a QR-decomposition of the matrix

$$A = \begin{bmatrix} 2 & -2 & 1 \\ 1 & 0 & 3 \\ 2 & -1 & 2 \end{bmatrix}$$

Step 1: (1 point) let $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ be the first, the second, and the third *column* of matrix A, correspondingly, then, using the Gram-Schmidt orthogonalization process, transform the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ into an orthogonal one, say $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$;

Step 2: (0.5 points) transform the set $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ from Step 1 into an *orthonormal* one, say, $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$; construct matrix Q using vectors $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$ as its first, second, and third column, correspondingly;

[**hint:** it is not a part or the problem, but it is advisable to verify that Q is indeed an orthogonal matrix]

Step 3: (0.5 points) using the result of Item (a) of Problem 3 and the equality $A = QR$, find matrix R.

[**hint:** it is not a part or the problem, but it is advisable to verify that R is indeed an upper triangular matrix]

$\langle a_2\, b_1\rangle = -6$

$\langle b_1, b_1\rangle = 9$

$\langle a_3\, b_2\rangle = 2$

$\langle b_2\, b_2\rangle = 1$

$\langle a_3\, b_1\rangle = 9$

1. $b_1 = a_1$

$$b_2 = \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix} - \frac{\langle a_2\, b_1\rangle}{\langle b_1, b_1\rangle} b_1 = \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix} + 6/9 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$b_3 = a_3 - \frac{\langle a_3\, b_2\rangle}{\langle b_2\, b_2\rangle} b_2 - \frac{\langle a_3\, b_1\rangle}{\langle b_1\, b_1\rangle} b_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} - 2\begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

$\langle b_1\, b_2\rangle = \langle b_1\, b_3\rangle = \langle b_2\, b_3\rangle = 0$

2. $\left.\begin{aligned} \langle b_1, b_1\rangle &= 9 \\ \langle b_2\, b_2\rangle &= 1 \\ \langle b_3\, b_3\rangle &= 1 \end{aligned}\right| \Rightarrow$ $q_1 = b_1/\|b_1\| = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$ $q_2 = \frac{b_2}{\|b_2\|} = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$ $q_3 = b_3/\|b_3\| = \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix}}_{Q^{-1}} = I$$

3) $A = QR \Leftrightarrow Q^{-1}A = Q^{-1}QR : Q^{-1}A = R \Rightarrow R = Q^T A$

$$\begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 \\ 1 & 0 & 3 \\ 2 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Thx for checking ♡