2. (HW) Use the definition to find the derivatives of the following functions:

(a) $f(x) = x^3$; (b) $f(x) = 2\sqrt{x+4}$ at $x_0 = 5$; (c) $\cos x$.

Novosad Ivan

231

a) $f(x) = x^3$ $\quad f'(x_0) = \lim\limits_{\Delta x \to 0} \dfrac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$

a) $f(x_0) = \lim\limits_{\Delta x \to 0} \left(\dfrac{(x_0+\Delta x)^3 - x_0^3}{\Delta x} \right) = \lim\limits_{\Delta x \to 0} \left(\dfrac{x_0^3 + \Delta x^3 + 3x_0^2 \Delta x + 3\Delta x x_0^2 - x_0^3}{\Delta x} \right) =$

$= \lim\limits_{\Delta x \to 0} \left(\dfrac{\Delta x^3 + 3x_0^2 \Delta x + 3\Delta x^2 x_0}{\Delta x} \right) = \lim\limits_{\Delta x \to 0} \left(\Delta x^2 + 3x_0^2 + 3x_0 \Delta x \right) =$

$= 3x^2$

b) $f(x) = 2\sqrt{x+4}$

$f'(x_0) = \lim\limits_{\Delta x \to 0} \left(\dfrac{2\sqrt{x_0+\Delta x+4} - 2\sqrt{x_0+4}}{\Delta x} \right) =$

$= \lim\limits_{\Delta x \to 0} \left(\dfrac{4(x_0+\Delta x+4) - 4(x_0+4)}{\Delta x(2\sqrt{x_0+\Delta x+4} - 2\sqrt{x_0+\Delta x}} \right) =$

$= \lim\limits_{\Delta x \to 0} \left(\dfrac{4x_0 + 4\Delta x + 16 - 4x_0 - 16}{\Delta x(2\sqrt{x_0+\Delta x+4} - 2\sqrt{x_0+\Delta x})} \right) =$

$\lim\limits_{\Delta x \to 0} \left(\dfrac{4\Delta \cancel{x}}{\cancel{\Delta x}(2\sqrt{x_0+\Delta x+4} - 2\sqrt{x_0+\Delta x})} \right) = \dfrac{4}{4\sqrt{x_0+4}} =$

$= \dfrac{1}{\sqrt{x_0+4}}$

c) $f(x) = \cos(x)$

$$f'(x) = \lim_{\Delta x \to 0} \left(\frac{\cos(\Delta x + x_0) - \cos(x_0)}{\Delta x} \right) =$$

$$\lim_{\Delta x \to 0} \left(\frac{\cos(x_0)\cos(\Delta x) - \sin(x_0)\sin(\Delta x) - \cos(x_0)}{\Delta x} \right)$$

$$= \lim_{\Delta x \to 0} \left(\frac{\cos(x_0)\cos(\Delta x) - \cos(x_0)}{\Delta x} - \frac{\sin(x_0)\sin(\Delta x)}{\Delta x} \right) =$$

$$= \lim_{\Delta x \to 0} \left(\cos(x_0) \frac{\cos(\Delta x) - 1}{\Delta x} - \sin(x_0) \frac{\sin(\Delta x)}{\Delta x} \right) =$$

$$= \cos(x_0) \left(\lim_{\Delta x \to 0} \left(\frac{\cos(\Delta x) - 1}{\Delta x} \right) \right) - \sin(x_0) \left(\lim_{\Delta x \to 0} \left(\frac{\sin(\Delta x)}{\Delta x} \right) \right)$$

$$= \cos(x_0)\, 0 - \sin(x_0)\, 1 = -\sin(x_0)$$

Quick note:

$$\lim_{x \to 0} \left(\frac{\cos(x) - 1}{x} \right) = \lim_{x \to 0} \left(\frac{\cos^2(x) - 1}{x(\cos(x) + 1)} \right) = \lim_{x \to 0} \left(\frac{-\sin^2(x)}{x(\cos(x) + 1)} \right) =$$

$$= -\lim_{x \to 0} \left(\frac{\sin(x)}{x} \right) \lim_{x \to 0} \left(\frac{\sin(x)}{\cos(x) + 1} \right) = -1 \cdot \frac{0}{2} = 0$$

4. (HW) Investigate the following functions for differentiability at each point of the real line:

(a) $f(x) = |\sin x|$; (b) $f(x) = x|x|$.

$$f'_+(\pi k) = \lim_{\Delta x \to 0^+} \left(\frac{|\sin(\pi k + \Delta x)| - |\sin(\pi k)|}{\Delta x} \right) = \lim_{\Delta x \to 0^+} \left(\frac{|\sin(\Delta x)|}{\Delta x} \right) =$$

$$= \lim_{\Delta x \to 0^+} \left(\frac{\sin(\Delta x)}{\Delta x} \right) = 1$$

$$f'_-(\pi k) = \lim_{\Delta x \to 0^-}\left(\frac{|\sin(\pi k + \Delta x)| - |\sin(\pi k)|}{\Delta x}\right) = \lim_{\Delta x \to 0^-}\left(\frac{|\sin(\Delta x)|}{\Delta x}\right) =$$

$$= -\lim_{\Delta x \to 0^-}\left(\frac{\sin(\Delta x)}{\Delta x}\right) = -1$$

Since $f'_+(\pi k) \neq f'_-(\pi k)$ the function $\sin|x|$ is not differentiable at points $x_0 = \pi k,\ k \in \mathbb{Z}$

b) If $x > 0$, then the function can be represented as $f(x) = x^2$, and it differentiable. If $x < 0$, then the function can be represented as $f(x) = -x^2$, and also differentiable.

then find derivative at the point $x = 0$, we will use the definition:

$$f'_+(0) = \lim_{\Delta x \to +0}\left(\frac{\Delta x^2}{\Delta x}\right) = 0 \quad \wedge \quad f'_-(0) = \lim_{\Delta x \to 0^-}\left(\frac{-\Delta x^2}{\Delta x}\right) = 0$$

As $f'_+(0) = f'_-(0) = 0$, it follows that the function is differentiable at $x = 0$ and $f'(x) = 0$.

6. (HW) For which values of a and b is the function $f(x) = \begin{cases} ax + b & \text{if } x > 1 \\ bx^2 + 2 & \text{if } x \leqslant 1 \end{cases}$ differentiable at $x = 1$

First of all, the function $f(x)$ must be continuous at $x = 1$, so $\lim_{x \to 1^-}(bx^2 + 2) = \lim_{x \to 1^-}(ax + b) \Rightarrow b + 2 = a + b \Rightarrow a = 2$

Next find the left and right derivatives of the function at $x = 1$

$$f'_-(1) = \lim_{\Delta x \to 0^-}\left(\frac{b(\Delta x + x_0)^2 + 2 - (b(x_0)^2 + 2)}{\Delta x}\right) =$$

$$= \lim_{\Delta x \to 0^-} \left(\frac{b\Delta x^2 + 2b\Delta x x_0 + b x_0^2 + 2 - b x_0^2 - 2}{\Delta x} \right) = \lim_{\Delta x \to 0^-} \left(\frac{b \Delta x^2 + 2b\Delta x x_0}{\Delta x} \right) =$$

$$= \lim_{\Delta x \to 0} \left(\frac{b\Delta x + 2b x_0}{1} \right) = 2b x_0 = 2b \text{ (for } x_0 = 1)$$

$$f'_+(1) = \lim_{\Delta x \to 0^+} \left(\frac{2(x_0 + \Delta x) + b - 2x_0 - b}{\Delta x} \right) =$$

$$\lim_{\Delta x \to 0^+} \left(\frac{2x_0 - 2x_0 + \Delta x + b - b}{\Delta x} \right) = \lim_{\Delta x \to 0^+} \left(\frac{\Delta x}{\Delta x} \right) = 1$$

For differentiability, these limits must be equal, so we get $1 = 2b \Rightarrow b = \frac{1}{2}$

8. (HW) Compute the derivatives of the following functions using the table and basic differentiation rules:

(a) $f(x) = 5^{\cos x} \cdot \ln x + \dfrac{x^2 + \sin x}{\sqrt{5x^2 + 3x - 7}}$; (b) $f(x) = (x^2 + 3) \cdot \tan \sqrt{x} + \dfrac{5^x}{7x - \ln x}$.

a) $$f(x)' = \left(5^{\cos(x)} \cdot \ln(x)\right)' + \left(\frac{x^2 + \sin(x)}{\sqrt{5x^2 + 3x - 7}} \right)' =$$

$$= \left(5^{\cos(x)}\right)' \ln(x) + 5^{\cos(x)} (\ln(x))' + \frac{(x^2 + \sin(x))' \sqrt{5x^2 + 3x - 7} - (x^2 + \sin(x))\left(\sqrt{5x^2 + 3x - 7}\right)'}{\sqrt{5x^2 + 3x - 7}^{\,2}}$$

Note: $\left(5^{\cos(x)}\right)' = 5^g \cdot (\cos(x))' = \ln(5)\, 5^g \cdot (-\sin(x)) = -\ln(5)\, 5^{\cos(x)} \cdot \sin(x)$

$\ln(x)' = \frac{1}{x}$; $x^2 + \sin(x)' = 2x + \cos(x)$, $\sqrt{5x^2 + 3x - 7}' = \frac{1}{2\sqrt{5x^2 + 3x - 7}} (10x + 3) = \frac{10x + 3}{2\sqrt{5x^2 + 3x - 7}}$

$$= -\ln(5)\, 5^{\cos(x)} \cdot \sin(x)\, \ln(x) + \frac{5^{\cos(x)}}{x} + \frac{(2x + \cos(x))\sqrt{5x^2 + 3x - 7} - (x^2 + \sin(x))(10x + 3)}{\left(2\sqrt{5x^2 + 3x - 7}\right)(5x^2 + 3x - 7)}$$

b) $$f(x)' = (x^2 + 3)' \cdot \tan(\sqrt{x}) + (x^2 + 3)\left(\tan(\sqrt{x})\right)' + \frac{(5^x)'(7x - \ln(x)) - 5^x(7x - \ln(x))'}{(7x - \ln(x))^2}$$

Note:

$(x^2+3)' = 2x$

$$\tan(\sqrt{x})' = \frac{1}{\cos^2(\sqrt{x})} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}\cos^2(\sqrt{x})}$$

$5^x = \ln(5)\, 5^x$

$$(7x - \ln(x))' = 7 - \frac{1}{x}$$

$$2x\tan(\sqrt{x}) + \frac{x^2+3}{2\sqrt{x}\cos^2(\sqrt{x})} + \frac{(\ln(5)5^x)(7x-\ln(x)) - 5^x(7-\frac{1}{x})}{(7x-\ln(x))^2}$$

10. (HW) Compute the derivatives of the following functions:

(a) $f(x) = (\arctan x)^{\cos^2 x}$; (b) $f(x) = x^{x^x}$; (c) $f(x) = \dfrac{e^{\arccos x}(x+7)^9}{(1+x^2)^4}$.

a) $$f(x)' = \left((\arctan(x))^{\cos^2(x)}\right)' = \left(e^{\ln(\arctan(x))^{\cos^2(x)}}\right)'$$

$$= e^{\cos^2(x)\cdot\ln(\arctan(x))} =$$

$$= e^{\cos^2(x)\cdot\ln(\arctan(x))} \cdot (\cos^2(x)\cdot\ln(\arctan(x)))' =$$

$$= e^{\cos^2(x)\cdot\ln(\arctan(x))} \cdot \left(\cos^2(x)'\ln(\arctan(x)) + \cos^2(x)\ln(\arctan(x))'\right)$$

Note:

$$\cos^2(x)' = \cos(x)'\cos(x) + \cos(x)'\cos(x) =$$

$$= -\sin(x)\cos(x) - \sin(x)\cos(x) = -2\sin(x)\cos(x) = -\sin(2x)$$

$$\ln(\arctan(x))' = \frac{1}{\arctan(x)} \cdot \arctan(x)' = \frac{1}{(1+x^2)(\arctan(x))}$$

then:

$$f(x)' = e^{\cos^2(x)\ln(\arctan(x))} \cdot \left(-\sin(2x)\ln(\arctan(x)) + \frac{\cos^2(x)}{(1+x^2)(\arctan(x))}\right)$$

b) $f(x) = x^{x^x} \Leftrightarrow y = x^{x^x} \Rightarrow \ln(y) = \ln(x^{x^x}) \Leftrightarrow \ln(y) = x^x \cdot \ln(x) \Leftrightarrow$

$\Leftrightarrow \ln(\ln(y)) = \ln(x^x \cdot \ln(x)) \Leftrightarrow \ln(\ln(y)) = \ln(x^x) + \ln(\ln(x)) \Leftrightarrow$

$\Leftrightarrow \ln(\ln(y)) = x\ln(x) + \ln(\ln(x)) \Leftrightarrow (\ln(\ln(x)))' = (x\ln(x) + \ln(\ln(x)))' \Leftrightarrow$

$\Leftrightarrow \frac{1/y}{\ln(y)} \cdot \frac{dy}{dx} = 1(\ln(x)) + x\frac{1}{x} + \frac{1/x}{\ln(x)}$ (since $\frac{1/y}{\ln(y)} = \frac{1}{y} \cdot \frac{1}{\ln y} = \frac{1}{y\ln(y)}$) $\Leftrightarrow$

$\Leftrightarrow \frac{1}{\cancel{y\ln(y)}} \cdot \frac{dy}{dy} = \ln(x) + 1 + \frac{1}{x\ln(x)} \quad | \, y\ln(y) \Leftrightarrow$

$\Leftrightarrow \frac{dy}{dx} = y\ln(y)\left(\ln(x) + 1 + \frac{1}{x\ln(x)}\right)$ (note that $y = x^{x^x} \wedge \ln(y) = \ln(x^{x^x}) = x^x(\ln(x))$)

$\Leftrightarrow \frac{dy}{dx} = x^{x^x} \cdot x^x\ln(x)\left(\ln(x) + 1 + \frac{1}{x\ln(x)}\right) \Leftrightarrow$

$\frac{dy}{dx} = x^{x^x + x}\left(\ln^2(x) + \ln(x) + \frac{1}{x}\right)$ **Answer**

c) $f(x) = \frac{e^{\arccos(x)}(x+7)^9}{(1+x^2)^4}$

$$f(x) = \frac{(e^{\arccos(x)}(x+7)^9)'((1+x^2)^4) - (e^{\arccos(x)}(x+7)^9)((1+x^2)^4)'}{(1+x^2)^8} \ominus$$

Note:

$(e^{\arccos(x)}(x+7)^9)' = e^{\arccos(x)'}(x+7)^9 + e^{\arccos(x)}(x+7)^{9'} \ominus$

$e^{\arccos(x)'} = e^{\arccos(x)} \cdot \arccos(x)' = e^{\arccos(x)} \cdot \left(-\frac{1}{\sqrt{1-x^2}}\right) =$

$$= -\frac{e^{\arccos(x)}}{\sqrt{1-x^2}}$$

$$((x+7)^9)' = 9(x+7)^8 \cdot (x+7)' = 9(x+7)^8$$

$$= -\frac{(x+7)^9 e^{\arccos(x)}}{\sqrt{1-x^2}} + e^{\arcos(x)} \cdot 9(x+7)^8$$

$$((1+x^2)^4)' = 4(1+x^2)^3 \cdot (1+x^2)' = 8x(1+x^2)^3$$

$$= \frac{\left(-\frac{(x+7)^9 e^{\arccos(x)}}{\sqrt{1-x^2}} + e^{\arccos(x)} \cdot 9(x+7)^8\right)\left((1+x^2)^4\right) - \left(e^{\arccos(x)}(x+7)^9\right)\left(8x(1+x^2)^3\right)}{(1+x^2)^8}$$

I hope you will have anice day!