1. (1 point) Let a bilinear form $\beta(\mathbf{x},\mathbf{y})$ be a scalar product on the arithmetic vector space $\mathbb{R}^n$, then, *describe* all n-by-n matrices A such that $\beta(A\mathbf{x}, A\mathbf{y})$ is also a scalar product on $\mathbb{R}^n$.

[**hint:** use Definition 29.1]

Novosad Ivan

1) $\beta(Ax, Ay) = \sum_{i=1}^{n} [Ax]_i \cdot [Ay]_i = \sum_{i=1}^{n} [Ay]_i [Ax]_i = \beta(Ay, Ax)$

that condition holds for any A.

2) $\beta(Ax, Ax) = \sum_{i=1}^{n} [Ax]_i [Ax]_i = \sum_{i=1}^{n} [Ax]_{(i)}^2 \geq 0$, since $[Ax]_i^2 \geq 0 \ \forall A, \forall i$

that condition also holds for any A.

3) $\beta(Ax, Ax) = 0$, iff $x = \bar{0}$

So $\sum_{i=1}^{n} [Ax]_i^2 = 0 \iff [Ax]_i^2 = 0 \ \forall i \in [n] \iff [Ax]_i = 0 \ \forall i \in [n]$

$[Ax]_i = \sum_{j=1}^{n} [A]_{(i)}^{(j)} [x]_{(i)} = 0$ iff $[x]_{(i)} = 0 \iff$

$\iff \forall i \forall j \in [n]^2 \ [A]_{(i)}^{(j)} \neq 0 \iff$ $\boxed{\text{all element of } A \text{ ought to be non zero.}}$

2. (1 point) Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be some vectors in a Euclidean space $(\mathbb{V}, \langle\cdot\,|\,\cdot\rangle)$. Suppose that $\mathbf{d} = \langle\mathbf{b}|\mathbf{c}\rangle\cdot\mathbf{a} - \langle\mathbf{a}|\mathbf{c}\rangle\cdot\mathbf{b}$, then, find $\langle\mathbf{c}|\mathbf{d}\rangle$.

$$\langle c|d\rangle = \langle c|\langle b|c\rangle a - \langle a|c\rangle b\rangle =$$

$$= \langle c|\langle b|c\rangle a\rangle - \langle c|\langle a|c\rangle\cdot b\rangle =$$

since scalar products $\in \mathbb{F}$

$$= \langle b|c\rangle\langle c|a\rangle - \langle a|c\rangle\cdot\langle c|b\rangle = \langle b|c\rangle\langle c|a\rangle - \langle c|a\rangle\langle b|c\rangle = 0$$

3. (1 point) Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors in a Euclidean space $(\mathbb{V}, \langle\cdot\,|\,\cdot\rangle)$ such that $\|\mathbf{a}\| = 4$, $\|\mathbf{b}\| = 3$, and $\langle\mathbf{a}+2\mathbf{b}|5\mathbf{a}-4\mathbf{b}\rangle = 0$. Then, find the angle between $\mathbf{a}$ and $\mathbf{b}$.

[**hint:** use Definition 29.4]

$$\sqrt{\langle a|a\rangle} = \sqrt{\|a\|^2} = 4 \iff \langle a|a\rangle = 16$$

$$\sqrt{\langle b|b\rangle} = \sqrt{\|b\|^2} = 3 \implies \langle b|b\rangle = 9$$

$$\langle a+2b|5a-4b\rangle = \langle a+2b|5a\rangle - \langle a+2b|4b\rangle =$$

$$= \langle a|5a\rangle + \langle 2b|5a\rangle - \langle a|4b\rangle - \langle 2b|4b\rangle =$$

$$= 5\langle a|a\rangle + 10\langle b|a\rangle - 4\langle a|b\rangle - 8\langle b|b\rangle = 6\langle a|b\rangle + 8 = 0 \implies \langle a|b\rangle = -4/3$$

$$\cos(\theta) = \frac{\langle a|b\rangle}{\|a\|\,\|b\|} = \frac{-4/3}{4\cdot 3} = -\frac{1}{9}$$

ultra-mega teddies task?

4. (2 points) Let $\mathbb{V}$ be the vector space of all square matrices of size two over the field of reals (that is, $\mathbb{V} = \mathrm{Mat}_2(\mathbb{R})$). Suppose that the scalar product on $\mathbb{V}$ is specified by the following formula

$$\langle A|B\rangle = \begin{bmatrix} 1 & -2 \end{bmatrix} \cdot (A + A^{\mathrm{T}}) \cdot (B + B^{\mathrm{T}}) \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \mathrm{tr}(A^{\mathrm{T}}B), \quad \text{for every } A,\, B \in \mathbb{V}.$$

Then, find the angle between matrices $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$.

[**hint:** use Definition 29.4 and direct calculation]

4) $$<A|B> = [1\ -2] \cdot \begin{bmatrix} 1+1 & -1 \\ -1 & 1+1 \end{bmatrix} \begin{bmatrix} 1+1 & 1 \\ 1 & -1-1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + tr\left(\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\right) =$$

$$= [4\ -5] \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + tr\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = [3\ 14] \begin{bmatrix} 1 \\ -2 \end{bmatrix} = -25$$

$$\|A\|^2 = [1\ -2] \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}^2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + tr\left(\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\right) = 41 + tr\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = 44$$

$$\|B\|^2 = [1\ -2] \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}^2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + tr\left(\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\right) = 25 + tr\left(\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\right) = 28$$

Thus: $\theta = \arccos\left(\dfrac{<A|B>}{\|A\|\,\|B\|}\right) = \dfrac{-25}{4\sqrt{77}}$, $\theta \in [0, \pi]$

5. (1 point per item) Let $\mathbb{V}$ be the vector space of all continuous functions on the interval $[0;\pi]$ (that is, $\mathbb{V} = C[0;\pi]$). Suppose that the scalar product on $\mathbb{V}$ is specified by the following formula

$$\langle f|g\rangle = \int_0^\pi f(x)g(x)\,\mathrm{d}x, \quad \text{for every } f,\, g \in \mathbb{V}.$$

Then:

(a) find the Gram matrix $G(x, \sin x, \cos x)$;

[**hint:** use Definition 29.5 and direct calculation]

(b) is it correct that the vectors x, $\sin x$, and $\cos x$ are linearly independent (you need to justify your answer)?

[**hint:** use Theorem 29.2]

$$5)\ a)\ G(x, \sin(x), \cos(x)) = \begin{bmatrix} \langle x|x\rangle & \langle x|\sin(x)\rangle & \langle x|\cos(x)\rangle \\ \langle \sin(x)|x\rangle & \langle \sin(x)|\sin(x)\rangle & \langle \sin(x)|\cos(x)\rangle \\ \langle \cos(x)|x\rangle & \langle \cos(x)|\sin(x)\rangle & \langle \cos(x)|\cos(x)\rangle \end{bmatrix} = \begin{bmatrix} \pi^3/3 & \pi & -2 \\ \pi & \pi/2 & 0 \\ -2 & 0 & \pi/2 \end{bmatrix}$$

© all calculation performed by wolfram, since we are not at Calculus.

$$b)\ \det(G(x, \sin(x), \cos(x))) = \frac{\pi^5 - 6\pi^3 - 24\pi}{12} \neq 0 \Rightarrow \text{they are LI}$$

6. (1 point) Using Cauchy-Schwartz inequality, prove that for any interval $[a;b] \subset \mathbb{R}$ and any continuous on $[a;b]$ function f the following inequality holds true:

$$\frac{1}{b-a}\left(\int_a^b f(x)\,\mathrm{d}x\right)^2 \leqslant \int_a^b (f(x))^2\,\mathrm{d}x.$$

[**hint:** $g \equiv 1$ is a continuous on $[a;b]$ function]

$$6)\ \left(\int_a^b f(x)g(x)\,dx\right)^2 \le \int_a^b f^2(x)\,dx \int_a^b g^2(x)\,dx \quad \forall g(x)\ \forall f(x) \in C[a,b]$$

since $g(x) = 1$ is continues on $[a,b]$:

$$\left(\int_a^b f(x)\,dx\right)^2 \le \int_a^b f^2(x)\,dx \int_a^b dx \quad \left(\int_a^b dx = b-a\right)$$

$$\left(\int_a^b f(x)\,dx\right)^2 \le \int_a^b f^2(x)\,dx\,(b-a) \quad (\text{since } a \neq b, \text{ if is it's trivial both equal } 0.)$$

$$\frac{1}{a-b}\left(\int_a^b f(x)\,dx\right)^2 \le \int_a^b f^2(x)\,dx \ \blacksquare$$

7. (2 points) Suppose that a, b and c are real numbers such that $a+b+c=1$. Then, using Cauchy-Schwartz inequality, find the minimum of $a^2+3b^2+c^2$ and all concrete values of a, b, c for which this minimum is attained.

[**hint:** is it correct that the bilinear form $\beta(\mathbf{x},\mathbf{y}) = x_1y_1 + 3x_2y_2 + x_3y_3$ on $\mathbb{R}^3$ satisfies all conditions from Definition 29.1?; consider vectors $\mathbf{x} = [a\ \ b\ \ c]^{\mathrm{T}}$ and $\mathbf{y} = [1\ \ 1/3\ \ 1]^{\mathrm{T}}$]

Consider bilinear form $\beta(x,y) = x_1y_1 + 3x_2y_2 + x_3y_3$. Let's see whether it qualifies to be a scalar product on $\mathbb{R}^3$ or not.

- $\beta(x,y) = x_1y_1 + 3x_2y_2 + x_3y_3 = y_1x_1 + 3y_2x_2 + y_3x_3 = \beta(y,x)$
- $\beta(x,x) = x_1^2 + 3x_2^2 + x_3^2 \geqslant 0$
- $\beta(x,x) = 0 \iff x_1 = 0 \wedge x_2 = 0 \wedge x_3 = 0 \iff x = 0$

So since $\beta(x,y)$ is indeed a scalar product on $\mathbb{R}^3$

let $x = [a,b,c]^T$, $y = [1\ \ 1/3\ \ 1]^T$

Then $\langle x|y\rangle = a+b+c = 1$; $\langle x|x\rangle = a^2+3b^2+c^2$; $\langle y|y\rangle = 1 + 1/3 + 1 = 7/3$

By Cauchy-Schwartz:

$|\langle x|y\rangle| \leq \|x\|\|y\|$

$$1 = a+b+c \leq \sqrt{\langle x|x\rangle}\sqrt{\langle y|y\rangle} = \sqrt{a^2+3b^2+c^2}\sqrt{7/3}$$

so $\sqrt{3/7} \leq \sqrt{a^2+3b^2+c^2} \iff 3/7 \leq a^2+3b^2+c^2$, so $\min(a^2+3b^2+c^2) = 3/7$

Since equality holds iff vectors ar LD:

x and y ought to be LD

$\Downarrow$

$a = k$, $b = k/3$, $c = k$ $\forall k \in \mathbb{R}$, as $a+b+c=1$, $k + \frac{k}{3} + k = 1 \Rightarrow k = 3/7$

Thus there is only one set of values for which this equality holds: $\begin{cases} a = 3/7 \\ b = 1/7 \\ c = 3/7 \end{cases}$

- Novosad Ivan