1. For the following linear operators, find $\operatorname{Spec}(\varphi)$ (see Definition 21.9) and, for every $\lambda \in \operatorname{Spec}(\varphi)$, find a basis for $\mathrm{E}_\varphi(\lambda)$ (see Definition 21.10):

(a) (1 point) $\varphi\colon \mathbb{R}[x;n] \to \mathbb{R}[x;n]$ where

$$\varphi\colon p(x) \mapsto xp(x)', \qquad \text{for every } p(x) \in \mathbb{R}[x;n];$$

(for example, if $p(x) = 2x^3 - 7x + 9$, then, $\varphi(p(x)) = x(2x^3 - 7x + 9)' = x(6x^2 - 7) = 6x^3 - 7x$)

[hint: use the fact that if $\mathbb{V}$ is finite dimensional and $\varphi\colon \mathbb{V} \to \mathbb{V}$ is a linear operator, then, $|\operatorname{Spec}(\varphi)| \leqslant \dim(\mathbb{V})$ (we will prove this fact later)]

(b) (1 point) $\varphi\colon \mathrm{Mat}_3(\mathbb{R}) \to \mathrm{Mat}_3(\mathbb{R})$ where

$$\varphi\colon A \mapsto A^{\mathrm{T}}, \qquad \text{for every } A \in \mathrm{Mat}_3(\mathbb{R});$$

[hint: if φ is a linear operator and $\varphi(\mathbf{x}) = \lambda\mathbf{x}$ then $\varphi \circ \varphi(\mathbf{x}) = ?$; in our case, $\varphi \circ \varphi(A) = ?$]

(c) (1 point) $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ where

$$\varphi\colon \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 4 & -5 & 2 \\ 5 & -7 & 3 \\ 6 & -9 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad \text{for every } [x,\, y,\, z]^{\mathrm{T}} \in \mathbb{R}^3;$$

[hint: see Problems 2 and 3 from Seminar 21; also you may take a look at the remark on the next page]

(d) (2 points) $\varphi\colon \mathbb{R}[x;n] \to \mathbb{R}[x;n]$ where

$$\varphi\colon p(x) \mapsto p(ax+b), \qquad \text{for every } p(x) \in \mathbb{R}[x;n],$$

and $a \neq \pm 1$, b are some fixed real numbers.

(for example, if $p(x) = 3x^2 - 2x + 9$, then, $\varphi(p(x)) = 3(ax+b)^2 - 2(ax+b) + 9 = 3a^2x^2 + (6ab - 2a)x + 3b^2 - 2b + 9$)

[hint: suppose that $\lambda \in \operatorname{Spec}(\varphi)$ and $x + B \in \mathrm{E}_\varphi(\lambda)$, then, $\varphi(x+B) = \lambda(x+B) = (ax+b) + B$, therefore, $\lambda = ?$ and $B = ?$; then what can you say about $\varphi((x+B)^k)$?; do not forget that $|\operatorname{Spec}(\varphi)| \leqslant \dim(\mathbb{V})$]

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a) coordinate matrix of φ is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 4 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & 0 & \cdots & n+1 \end{bmatrix} \Rightarrow \operatorname{Spec}(\varphi) = \{1, 2, 3, \dots, n, n+1\}$$

basis for λ for $\operatorname{Spec}(\varphi) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \lambda \\ \vdots \\ 0 \end{bmatrix} \leftarrow \lambda$'s position

i.e. $E_3 = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ (n+1 rows)

b) we have only two types of such matrices:

- symmetric (eigenval: 1)
- skew-symmetric (eigenval: -1)

$\Rightarrow \operatorname{Spec}(\varphi) = \{-1, 1\}$

$$E_{-1} = \left\langle \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\rangle$$

$$E_1 = \left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\rangle$$

c)
$$\begin{vmatrix} 4-\lambda & -5 & 2 \\ 5 & -7-\lambda & 3 \\ 6 & -9 & 4-\lambda \end{vmatrix} = \lambda^2 - \lambda^3 = 0 \Rightarrow \left[\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= 1 \end{aligned}\right.$$

$\Rightarrow \operatorname{Spec}(\varphi) = \{0, 1\}$

$$\Rightarrow \begin{bmatrix} 4 & -5 & 2 \\ 5 & -7 & 3 \\ 6 & -9 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 4 & -5 & 2 & 0 \\ 5 & -7 & 3 & 0 \\ 6 & -9 & 4 & 0 \end{array}\right] \end{array}$$

$$\Downarrow$$

$$E_0 = \left\langle \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\rangle \Leftarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c/3 \\ 2c/3 \\ c \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & -5 & 2 & 0 \\ 5 & -8 & 3 & 0 \\ 6 & -9 & 3 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c \\ c \\ c \end{bmatrix} \Rightarrow E_1 = \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle$$

d) we can find coordinate matrix of φ:

$$A = \begin{bmatrix} 1 & b & b^2 & b^3 & \cdots & b^n \\ 0 & a & ab & 3ab^2 & & nab^{n-1} \\ 0 & 0 & a^2 & 3a^2b & & na^2b^{n-2} \\ 0 & 0 & 0 & a^3 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & na^{n-1}b \\ 0 & 0 & 0 & 0 & & a^n \end{bmatrix}$$

(using Pascal's triangle i.e.) (for $R[x,n]$)

hence, since it's upper triangular matrix, $\det(A) = (1 \cdot a \cdot a^2 \cdot \ldots \cdot a^n)$ then, to find all eigent values:

$$\det(A - \lambda I_{n+1}) = (1-\lambda)(a-\lambda)(a^2-\lambda)\cdot(a^3-\lambda)\ldots(a^n-\lambda) = 0 :$$

for $\lambda \in \operatorname{Spec}(\varphi)$

then $\operatorname{Spec}(\varphi) = \{1, a, a^2, a^3, a^4, \dots a^n\}$ for $R[x,n]$

for each eigent value: $E_\varphi(\lambda) = \left\langle \begin{bmatrix} 1-\lambda \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} b \\ a-\lambda \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} b^2 \\ 2ab \\ a^2-\lambda \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} b^n \\ nab^{n-1} \\ na^2b^{n-2} \\ \vdots \\ na^{n-1}b \\ a^n-\lambda \end{bmatrix} \right\rangle$

Is hint ignoring okay?

2. (1 point) Find *all* φ-invariant subspaces (see Def. 21.6) of the linear operator

$$\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad \varphi\colon \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}.$$

[**hint:** let $\mathbb{U}$ be a φ-invariant subspaces of $\mathbb{R}^2$; since $\mathbb{R}^2$ is a two dimensional vector space, the dimension of $\mathbb{U}$ is either 0, 1, or 2; if the dimension of $\mathbb{U}$ is 0 or 2 then it is clear that ?; if the dimension of $\mathbb{U}$ is 1, then, $\mathbb{U} = \langle \mathbf{x} \rangle$, for some $\mathbf{x} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, and ?]

Ivatiant subspaces of $\mathbb{V}$ under φ is:

1) $\{\bar{0}\}$ 3) $\ker(\varphi)$

2) $\mathbb{V}$ 4) $\mathrm{Im}(\varphi)$

In case if φ is a linear operator 1) = 3) 2) = 4);

$$\ker(\varphi): \left[\begin{array}{cc|c} 5 & -6 & 0 \\ 1 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \Rightarrow \ker(\varphi) = \{\bar{0}\}$$

$\Rightarrow$ only φ invariant subspace of $\mathbb{V}$ is $\mathbb{V}$ is self.

3. (2 points) Let

$$A = \frac{1}{6} \cdot \begin{bmatrix} -57 & -15 & 222 \\ -72 & -18 & 276 \\ -21 & -5 & 80 \end{bmatrix} \in \mathrm{Mat}_3(\mathbb{R}).$$

Then, find $\lim\limits_{n \to +\infty} A^n$.

[**hint:** see Problems 4 and 3 from Seminar 21; in this problem, one can use a machine to find $\det(A - \lambda I_3)$]

first of all: $\det(A - xI_3) = 0 \iff \begin{vmatrix} -\frac{19}{2} - x & -5/2 & 37 \\ -12 & -3-x & 46 \\ -\frac{7}{2} & -5/6 & \frac{40}{3} - x \end{vmatrix} = 0 \Rightarrow$

$\Rightarrow 750 + \frac{1}{3}x + \frac{5}{6}x^2 - x^3 - \frac{4501}{6} = 0 \Rightarrow \left[\begin{array}{l} x = -1/2 \\ x = 1/3 \\ x = 1 \end{array}\right. \Rightarrow \mathrm{spec}(\varphi) = \{1, 1/3, -1/2\}$

let's think about A as a matrix representation of some linear transformation (φ) $(\varphi: \bar{x} \to A\bar{x})$

then $\mathcal{E}_\varphi(1) = \left\langle \begin{bmatrix} 11/4 \\ 13/4 \\ 1 \end{bmatrix} \right\rangle$; $\mathcal{E}_\varphi(-1/2) = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$; $\mathcal{E}_\varphi(1/3) = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$

basis I mean

(It's ochev! I mean finding the eigenvectors: just sub λ and find ker)

then $B = \begin{bmatrix} 11/4 & 3 & 3 \\ 13/4 & 4 & 3 \\ 1 & 1 & 1 \end{bmatrix}$ (called eigen-basis for $\mathbb{R}^3$ under φ)

$\Rightarrow \underset{\overset{\parallel}{D}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}}$ hence $\overset{A}{\overset{\parallel}{\begin{bmatrix} -19/2 & -5/2 & 37 \\ -12 & -3 & 46 \\ -7/2 & -5/6 & 40/3 \end{bmatrix}}}^n = \overset{B}{\overset{\parallel}{\begin{bmatrix} 11/4 & 3 & 3 \\ 13/4 & 4 & 3 \\ 1 & 1 & 1 \end{bmatrix}}} \overset{D}{\overset{\parallel}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}}}^n \overset{B^{-1}}{\overset{\parallel}{\begin{bmatrix} 11/4 & 3 & 3 \\ 13/4 & 4 & 3 \\ 1 & 1 & 1 \end{bmatrix}}}^{-1}$

$\lim\limits_{n \to +\infty} (D^n) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow \lim\limits_{n \to +\infty} (A^n) = \begin{bmatrix} -11 & 0 & 33 \\ -13 & 0 & 39 \\ -4 & 0 & 12 \end{bmatrix}$

4. (2 points) The n-th *Fibonacci number*, denoted by F_n, is defined as following

$$F_n = \begin{cases} 0 & \text{if } n = 0; \\ 1 & \text{if } n = 1; \\ F_{n-1} + F_{n-2} & \text{if } n \geqslant 2. \end{cases}$$

Thus, the beginning of the Fibonacci sequence is:

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89.\ 144,\ \ldots$$

Following the instructions, find a closed form expression for F_n (it is called the Binet formula).

Instructions:

(a) using the definition of F_n, find a matrix A such that

$$\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = A \cdot \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix}, \quad \text{for every } n \geqslant 1;$$

(b) find A^n (see Problems 4 and 3 from Seminar 21; if you have some $\sqrt{5}$ in your formula, it is a good sign);

(c) note that, due to Item (a), we have:

$$\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = A^{n-1} \cdot \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}, \quad \text{for every } n \geqslant 1;$$

a) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix}$

$$\Updownarrow$$

$$\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} F_{n-1} + F_{n-2} \\ F_{n-1} \end{bmatrix}$$

b) $= \begin{vmatrix} 1-x & 1 \\ 1 & -x \end{vmatrix} = 0 \iff x^2 - x - 1 = 0 \iff \left[\begin{array}{l} x = \frac{1-\sqrt{5}}{2} \\ x = \frac{1+\sqrt{5}}{2} \end{array} \right.$

$\Rightarrow E_{\frac{1-\sqrt{5}}{2}} = \left\langle \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} \right\rangle ; E_{\frac{1+\sqrt{5}}{2}} = \left\langle \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} \right\rangle$

$C = \begin{bmatrix} \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \Rightarrow A^n = \begin{bmatrix} \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}^n \begin{bmatrix} \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}^{-1}$

c) I don't really understand that is req. but anyway, maybe you want:

IF $\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = A \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix} \rightarrow \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \iff \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

also we may notice, that $F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}$; $\varphi = \frac{1+\sqrt{5}}{2}$; $\psi = \frac{1-\sqrt{5}}{2}$

5*) $A = \begin{bmatrix} a_1^2 & a_1a_2 & \cdots & a_1a_n \\ a_2a_1 & a_2^2 & \cdots & a_2a_n \\ a_3a_1 & a_3a_2 & \cdots & a_3a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_na_1 & a_na_2 & \cdots & a_n^2 \end{bmatrix}$ if $a_1 = x \wedge a_2 = x^2 \wedge a_3 = x^3 \wedge a_4 = x^4 \dots a_n = x^n$.

for $n = 1$: $\lambda = a_1$

for $n = 2$: $(a_1^2 - \lambda)(a_2^2 - \lambda) - (a_1a_2 - \lambda)^2 = -a_1^2\lambda - a_2^2\lambda + 2a_1a_2\lambda = 0 \Rightarrow \lambda = 0$ if $a_1 \neq a_2$

$\lambda = \mathbb{R}$ if $a_1 = a_2$

for $n = 3$: $\begin{vmatrix} a_1^2 - \lambda & a_1a_2 & a_1a_3 \\ a_2a_1 & a_2^2 - \lambda & a_2a_3 \\ a_3a_1 & a_3a_2 & a_3^2 - \lambda \end{vmatrix} = 0 \iff a_1^2\lambda^2 + a_2^2\lambda^2 + a_3^2\lambda^2 - \lambda^3 = 0 \Rightarrow \left[\begin{array}{l} \lambda = 0 \\ \lambda = a_1^2 + a_2^2 + a_3^2 \end{array}\right.$

---

okay $\dim(\ker(\varphi)) \neq 0$, since $a_1 = x$; $a_2 = x$; $a_3 = x^2$; $a_4 = x^3$; ... $a_n = x^{n-1}$

hence $A = \begin{bmatrix} x^2 & x^2 & x^3 & \cdots & x^n \\ x^2 & x^2 & x^4 & \cdots & x^{n+1} \\ x^3 & x^3 & x^5 & \cdots & x^{n+2} \\ x^4 & x^4 & x^6 & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x^n & x^{n+1} & x^{n+2} & \cdots & x^{2n-2} \end{bmatrix}$, Obviously $\det(A) = 0$, since 1 and 2nd column are equal;

then $\ker(\varphi) \neq \{0\}$; at least if $a_2 = a_1^1$; $a_3 = a_1^2$; $a_4 = a_1^3 \dots a_n = a_1^{n-1}$;