a) prove that $\frac{2}{\sqrt{1+\pi}} < \int_0^{\pi} \frac{\sin(x)}{\sqrt{1+x}}dx < 2$

since min $(\sqrt{1+x})$ on $[0,\pi]$ is 1:

$$\int_0^{\pi} \frac{\sin(x)}{\sqrt{1-x}}dx < \int_0^{\pi} \frac{\sin(x)}{1}dx = 2$$

respectivly max $(\sqrt{1+x})$ is $\sqrt{1+\pi}$;

$$\underbrace{\int_0^{\pi} \frac{\sin(x)}{\sqrt{1+\pi}}dx}_{= \frac{2}{\sqrt{1+\pi}}} < \int_0^{\pi} \frac{\sin(x)}{\sqrt{1+x}}dx$$

6) $xy=5;\ x+y=6$

$\Updownarrow$

$\overset{\text{low}}{y=\frac{5}{x}};\ \overset{\text{up}}{y=6-x}$

$\frac{5}{x} = 6-x \iff 5-6x+x^2=0$

$(x-1)(x-5)=0$

$\Downarrow$

$$S = \int_1^5 \left((6-x)-\frac{5}{x}\right)dx = \int_1^5 (x^2-6x+5)dx =$$

$$= \frac{1}{3}x^3-3x^2+5x\Big|_1^5 = \frac{125}{3}-75+25-\frac{1}{3}-3+5 = \frac{124}{3}+55 = \frac{289}{3}$$

3) a) $\int (x^2+3x-4)e^x dx = \left\{ \begin{matrix} u = x^2+3x-4 & dw = e^x dx \\ du = (2x+3)dx & w = e^x \end{matrix} \right\} = (x^2+3x-4)e^x - \int (2x+3)e^x dx = \left\{ \begin{matrix} u = 2x+3 & dw = e^x \\ du = 2dx & w = e^x dx \end{matrix} \right\}$

$(x^2+3x-4)e^x - (2x+3)e^x + 2\int e^x dx = e^x(x^2+x-5)+c$

b) $\int \frac{x^2-x+1}{(x+1)(x^2+1)}dx = \frac{3}{2}\int \frac{dx}{x+1} + -\frac{1}{2}\int \frac{x+1}{x^2+1}dx = \frac{3}{2}\ln(|x+1|) - \frac{1}{4}\ln(x^2+1) - \frac{1}{2}\arctan(x)+c$

draft: $\frac{A}{x+1} + \frac{Bx+C}{x^2+1} = \frac{x^2-x+1}{(x+1)(x^2+1)} \iff ax^2+a+bx^2+bx+cx+c$

$$\Rightarrow \begin{cases} a+b=1 \\ b+c=-1 \\ a+c=1 \end{cases} \Rightarrow \begin{cases} b-c=0 \\ b+c=-1 \\ a=1-c \end{cases} \Rightarrow (a,b,c) = (3/2;\ -1/2;\ -1/2)$$

$$-\frac{1}{2}\int \frac{x+1}{x^2+1}dx = -\frac{1}{4}\int \frac{2x}{x^2+1}dx \overset{\{u=x^2+1\}}{=} -\frac{1}{2}\int \frac{dx}{x^2+1} = -\frac{1}{4}\ln(x^2+1) - \frac{1}{2}\arctan(x)+c$$

4) a) $\int \sqrt{25-x^2}\,dx \left| \begin{matrix} x = 5\sin(\theta) \\ dx = 5\cos(\theta) \end{matrix} \right| = \int 25\cos^2(\theta)\,d\theta = \frac{25}{2}\int (1+\cos(2\theta))\,d\theta = \frac{25\theta}{2} + \frac{25\sin(2\theta)}{4}+c =$

$= \frac{25\theta}{2} + \frac{25\sin(\theta)\cos(\theta)}{2}+c = \frac{25\arcsin\left(\frac{x}{5}\right)}{2} + \frac{25x\sqrt{25-x^2}}{2}+c$

b) $\int \tan^3(x)\,dx = \int (\sec^2(x)-1)\tan(x)\,dx = \int \sec^2(x)\tan(x)\,dx - \int \frac{\sin(x)}{\cos(x)}dx = \frac{1}{2}\tan(x)^2 + \ln(|\cos(x)|)+c$

5) a) $\int_1^6 \frac{x+3}{\sqrt{x+1}}dx = \{u=x+1\} = \int_2^7 \frac{u+2}{\sqrt{u}}du = \frac{2}{3}u^{3/2}+4\sqrt{u}\Big|_2^7 = \frac{14}{3}\sqrt{7}+4\sqrt{7}-\frac{4}{3}\sqrt{2}-4\sqrt{2}$

b) $\int_0^1 \arctan(x)\,dx = \left\{ \begin{matrix} u=\arctan(x) & dw=dx \\ du=\frac{dx}{1+x^2} & w=x \end{matrix} \right\} = x\arctan(x)\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}dx = x\arctan(x) - \frac{1}{2}\ln(1+x^2)\Big|_0^1 = \arctan(1) - \frac{\ln(2)}{2} = \frac{\pi-\ln(4)}{4}$

1) def. of integral sum: ← integral sum

$$\int_a^b f(x)\,dx = \cancel{\lim_{n\to\infty}}\left(\sum_{i=1}^n f(c_i)\Delta x\right)$$

$c_i = a + i\Delta x$; $\Delta x = \frac{b-a}{n}$;

2) too hard (we will bet for luck)

3) Let $f(x)$ be bounded on $[a,b]$

then let $\{[x_1 x_2][x_2 x_3]\dots[x_{i-1} x_i]\}$

be sub intervals of $[a,b]$ s.t.

$a = x_1 < x_2 < x_3 < \dots < x_{i-1} < x_i = b$

then $m_i = \inf\{f(\varepsilon) \mid \varepsilon \in [x_{i-1} x_i]\}$

and $M_i = \sup\{f(\varepsilon) \mid \varepsilon \in [x_{i-1} x_i]\}$

then $\sum_{i=1}^n m_i(x_i - x_{i-1})$ - be lower Darboux sum;

and $\sum_{i=1}^n M_i(x_i - x_{i-1})$ - be upper Darboux sum;

then $L \le \int_a^b f(x) \le M_i$ also as $\|P\| \to 0$
$M - m \to 0$

5) Mean value Th. for integrals:

suppose $f(x)$ is integrable on $[a,b]$, then

$\exists c \in [a,b]$ $f(c)(b-a) = \int_a^b f(x)dx$.

let

1) Suppose $f(x)$ is integrable on $[a,b]$

then let $a = x_0 < x_1 < x_2 < x_3 < x_4 < \dots < x_n = b$

then let ε_i be some point in $[x_{i-1}\, x_i]$

then $\sum_{i=1}^n f(\varepsilon_i)\underline{\Delta x_i}$ is a integral sum.

$\Delta x_i = x_i - x_{i-1}$

5) The integral with variable upper limit and it's properties:

$$F(x) = \int_a^x f(t)dt \iff F(x)' = f(x)$$

7) FTC: $F(x) = \int_a^x f(t) \iff F(x)' = f(x)$

$\int_a^b f(x)dx = F_1(b) - F_2(a)$; $F_1' = f(x) \wedge F_2' = f(x)$

let $f(x)$ be integrable on $[a,b]$

then let consider n subintervals of $[a,b]$

s.t. $a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$

then $M_i = \sup\{f(c) \mid c \in [x_{i-1}, x_i]\}$

$m_i = \inf\{f(c) \mid c \in [x_{i-1}, x_i]\}$

$\sum_{i=1}^n m_i \Delta x_i$ is a lower Darboux sum.

$\sum_{i=1}^n M_i \Delta x_i$ is an uppre Darboux sum.

1) $\sum_{i=1}^n m_i\Delta x_i \le \overset{\delta}{\overset{\shortparallel}{\sum_{i=1}^n f(\varepsilon_i)\Delta x_i}} \le \overset{M}{\overset{\shortparallel}{\sum_{i=1}^n M_i\Delta x_i}}$

2) $\forall \varepsilon > 0\ \exists n \in \mathbb{N}$ s.t. $M - \delta \le \varepsilon \wedge \delta - m \le \varepsilon$

3) suppose we have n sub intervals s.t. $a = x_0 < x_1 < \dots < x_n = b$

then denote M_n as upper Darboux sum in that case (res. m_n)

then let us split an arbitrary subinterval into two intervals (that's it adding new partition)

then $M_n \geqslant M_{n+1} \wedge m_n \le m_{n+1}$

4) for any new partition $m \le M$

Darboux sum criterian:

Suppose $f(x)$ in integrable, then

let's consider n partitions s.t.

$a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$

then $[x_{i-1}, x_i]$ is a subintervals of $[a,b]$

$M_i = \sup\{f(c) \mid c \in$

$$2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}$$

$$\int_a^b \sqrt{1 + (f'(x))^2}\,dx$$

$\sin\left(\frac{1}{x}\right)$

$\int_{-1}^{\frac{1}{10}} \sin\left(\frac{1}{x}\right)$

Improper integrals:

Suppose $f(x)$ and $g(x)$ are continious and $0 \le g(x) \le f(x)$ on $[a,\infty)$

then:

a) if $\int_a^{+\infty} f(x)dx$ is convergent then $\int_a^{+\infty} g(x)dx$ is also convergent

b) if $\int_a^{+\infty} g(x)dx$ is divergent then $\int_a^{+\infty} f(x)dx$ is also divergent

2) Comparison test of converseuse of improper integrals

Suppose that $f(x)$ and $g(x)$ are continuous functions on $[a,\infty)$

s.t. $g(x) \sim f(x)$ that is $\lim_{x\to\infty^+} \frac{f(x)}{g(x)} = 1$; then

$$\int_a^{+\infty} f(x)dx \text{ and } \int_a^{+\infty} g(x)dx$$

are both convergent or divergent.