2. (HW) Evaluate the improper integral with infinite discontinuities in the interval of integration or prove that it is divergent:

(a) $\int_{-1}^{3} \frac{dx}{\sqrt{x^2+4x+3}}$; (b) $\int_{1}^{e^4} \frac{dx}{x\sqrt{\ln x}}$; (c) $\int_{2}^{3} \frac{x\,dx}{\sqrt{x-2}}$; (d) $\int_{0}^{2} \frac{dx}{\sqrt[3]{(x-1)^2}}$.

Novosad Ivan.

I miss these topics due to illness sorry if I miss our variant off notation.

a) V.A. on $x=-1$

$$\lim_{a\to-1}\left(\int_a^3 \frac{dx}{\sqrt{x^2+4x+3}}\right) = \lim_{a\to-1}\left(\int_a^3 \frac{dx}{\sqrt{(x+2)^2-1}}\right) = \left\{\begin{matrix} u = x+2 \\ du = dx \end{matrix}\right\} = \lim_{a\to-1}\left(\int_a^5 \frac{du}{\sqrt{u^2-1}}\right) = \ln\left(\left|u+\sqrt{u^2-1}\right|\right)\Big]_a^5$$

$$\ln\left(\left|5+\sqrt{24}\right|\right) - \lim_{a\to-1}\left(\ln\left(\left|a-\sqrt{a^2-1}\right|\right)\right) = \ln(5+\sqrt{24}) - \ln(1) = \ln(5+\sqrt{24})$$

b) $\int_1^{e^4} \frac{dx}{x\sqrt{\ln(x)}}$; V. A. at $x=1$

$$\lim_{a\to1}\int_a^{e^4} \frac{dx}{x\sqrt{\ln(x)}} = \left\{\begin{matrix} u = \ln(x) \\ du = \frac{1}{x}dx \end{matrix}\right\} = \lim_{a\to0}\int_a^4 \frac{du}{\sqrt{u}} = 2\sqrt{u}\Big]_0^4 = 4$$

c) $\int_2^3 \frac{x}{\sqrt{x-2}}dx$; VA on $x=2$;

$$\lim_{a\to2}\int_a^3 \frac{x}{\sqrt{x-2}}dx = \lim_{a\to2}\int_a^3 \frac{x-2}{\sqrt{x-2}}dx + \lim_{a\to2}\int_a^3 \frac{2}{\sqrt{x-2}}dx = \left\{\begin{matrix} u = x-2 \\ du = dx \end{matrix}\right\} =$$

$$\lim_{a\to0}\int_a^1 \sqrt{u}\,du + \lim_{a\to0}\int_a^1 \frac{2}{\sqrt{u}}du = \lim_{a\to0} \frac{2}{3}u^{3/2} + 4\sqrt{u}\Big]_a^1 = \frac{2}{3}+4 = \frac{14}{3}$$

d) $\int_0^2 \frac{dx}{(x-1)^{2/3}}$ dis. on $x=1$

$$\lim_{a\to1}\int_0^a (x-1)^{-2/3}dx + \lim_{a\to1}\int_a^2 (x-1)^{-2/3}dx = \left\{\begin{matrix} u = x-1 \\ du = dx \end{matrix}\right\}$$

$$\lim_{a\to0}\int_{-1}^a u^{-2/3}du + \lim_{a\to0}\int_a^1 u^{-2/3}du = 3\sqrt[3]{u}\Big]_{-1}^a + 3\sqrt[3]{u}\Big]_a^1 = 6$$

$\lim_{a\to0}$

4. (HW) Evaluate the improper integral with infinite intervals of integration or prove that it is divergent:

(a) $\displaystyle\int_0^{+\infty} \frac{dx}{x^2+2x+7}$; (b) $\displaystyle\int_{-\infty}^{1} x\cos(2x+5)\,dx$; (c) $\displaystyle\int_0^{+\infty} \frac{dx}{x(\ln^2 x+1)}$; (d) $\displaystyle\int_{-\infty}^{+\infty} \frac{2x\,dx}{1+x^2}$.

let me introduce new (lazy) notation:

a) consider "a" as $\lim\limits_{a\to+\infty}$

then solve im proper integral using MIT lazy notation:

$$\int_0^a \frac{dx}{x^2+2x+7} = \int_0^a \frac{dx}{(x+1)^2+6} = \left\{\begin{matrix} u=x+1 & 0\to 1 \\ du=dx & \infty\to\infty \end{matrix}\right\} =$$

$$= \int_1^a \frac{dx}{u^2+6} = \frac{1}{\sqrt6}\arctan\left(\frac{u}{\sqrt6}\right)\Bigg]_1^a \ominus$$

$$\ominus \lim_{a\to\infty^+}\left(\frac{1}{\sqrt6}\arctan\left(\frac{a}{\sqrt6}\right)\right) - \frac{1}{\sqrt6}\arctan\left(\frac{1}{\sqrt6}\right) = \frac{1}{\sqrt6}\left(\frac{\pi}{2} - \arctan\left(\frac{\sqrt6}{6}\right)\right)$$

b) $\displaystyle\int_{-\infty}^{1} x\cos(2x+5)\,dx$; consider a as $\lim \to -\infty$:

$$\int_a^1 x\cos(2x+5)\,dx = \left\{\begin{matrix} u=x & dw=\cos(2x+5)dx \\ du=dx & w=\frac12\sin(2x+5) \end{matrix}\right\} =$$

$$= \frac{x}{2}\sin(2x+5)\Big]_a^1 - \frac12\int \sin(2x+5)\,dx = \frac{x}{2}\sin(2x+5) + \frac14\cos(2x+5)\Big]_a^1 =$$

$$= \frac12\sin(7) + \frac14\cos(7) - \lim_{x\to-\infty}\left(\frac{2x\sin(2x+5)+\cos(2x+5)}{4}\right)$$ ← undifined $\sin(-\infty) \wedge \cos(-\infty)$

$\Rightarrow$ integral is diverges

c) $\displaystyle\int_0^{+\infty} \frac{dx}{x(\ln^2(x)+1)}$ consider a as $\lim \to 0$ $\wedge$ b as $\lim \to +\infty$

$$\int_0^a \frac{dx}{x(\ln^2(x)+1)} = \left\{\begin{matrix} u=\ln(x) & b\to b \\ du=\frac{1}{x}dx & a\to-\infty \end{matrix}\right\}$$ then c is $\lim c\to-\infty$:

$$= \int_c^b \frac{du}{u^2+1} = \arctan(u)\Big]_c^b = \lim_{b\to\infty}(\arctan(b)) - \lim_{c\to-\infty}(\arctan(c)) = \pi.$$

d) $\displaystyle\int_{-\infty}^{\infty} \frac{2x\,dx}{1+x^2}$;

to avoid notation like that: $\displaystyle\lim_{a\to-\infty}\left(\lim_{b\to\infty}\left(\int_a^b \frac{2x\,dx}{x^2+1}\right)\right)$, we will split it:

and if we just solve them independently we will obtain DNE

let a be $\lim\to-\infty$ $\wedge$ b $\lim\to\infty$

or just $0-\infty = -\infty$

$$\int_a^0 \frac{2x}{x^2+1}dx = \left\{\begin{matrix} u=x^2+1 \\ du=2x\,dx \\ a\to b;\ 0\to 1 \end{matrix}\right\} = \int_b^1 \frac{du}{u} = \lim_{b\to\infty}\left(\overset{\text{"}0}{\ln(|1|)} - \underset{\text{"}\infty}{\ln(|b|)}\right) = \lim_{b\to\infty}\left(\ln\left(\left|\tfrac{1}{b}\right|\right)\right)\xrightarrow{}0 = -\infty$$ ①

$$\int_0^b \frac{2x}{x^2+1}dx = \left\{\begin{matrix} u=x^2+1 \\ du=2x\,dx \\ 0\to 1\ \ b\to b \end{matrix}\right\} = \int_1^b \frac{du}{u} = \lim_{b\to\infty}\left(\ln(|u|) - \ln(|1|)\right) = \lim_{b\to\infty}\left(\ln(|b|)\right) = \infty$$

$\Rightarrow$ DNE $(\infty - \infty)$

① and at this moment we can notice that (plz prove me why not?)

since $\displaystyle\int_a^b \frac{2x\,dx}{x^2+1} = \int_a^0 \frac{2x\,dx}{x^2+1} + \int_0^b \frac{2x\,dx}{x^2+1} = \int_b^1 \frac{du}{u} + \int_1^b \frac{du}{u} = -\int_1^b \frac{du}{u} + \int_1^b \frac{du}{u} = 0$

⊗ if we consider our integral as:

$\displaystyle\lim_{a\to\infty}\int_{-a}^{a} \frac{2x\,dx}{x^2+1}$, then by cauchy principal value it's equal 0

6. (HW) Check the following improper integrals for convergence:

(a) $\int_1^{+\infty} \frac{\sin(1/x)}{x}\,dx$; (b) $\int_0^{+\infty} \frac{x\,dx}{\sqrt[3]{x^5+2}}$; (c) $\int_0^{+\infty} \frac{\sin^2 x}{x^2}\,dx$.

a) $\int_1^{+\infty} \frac{\sin(1/x)}{x}dx$;

First approach:

$$\lim_{a\to\infty} \int_1^a \frac{\sin(1/x)}{x}dx = \left\{\begin{matrix} t = \frac{1}{x} & 1\to 1 \\ dt = -\frac{dx}{x^2} & a\to 0 \end{matrix}\right\}$$

$$= \lim_{a\to\infty} \int_1^{\frac{1}{a}} \frac{-\sin(t)\,t}{t^2}dt = \lim_{a\to 0}\int_a^1 \frac{\sin(t)}{t}dt =$$

since $0 \le \frac{\sin(t)}{t} \le \sec^2(t)$ on $t\in(0,1]$

and $\int_0^1 \sec^2(t)\,dt = \tan(1)$ (check 6c)

$\int_0^1 \frac{\sin(t)}{t}dt$ is convergent, then

$\int_1^\infty \frac{\sin(\frac{1}{x})}{x}dx$ is also convergent.

Second approach:

$\sin(\frac{1}{x}) \le \frac{1}{x} \quad \forall x\in[1,\infty)$

Proof: $\sin(\frac{1}{x}) = \int_0^{1/x} \cos(t)dt$ also

$\int_0^{1/x} \cos(t)dt \le \int_0^{1/x} 1\,dt = \frac{1}{x}$

(since $\cos(t)\le 1$) $\Rightarrow \sin(\frac{1}{x}) \le \frac{1}{x}, x\ge 1$

So, since $0\le \sin(\frac{1}{x}) \le \frac{1}{x}$ on $x\in[1,\infty)$

$0 \le \frac{\sin(\frac{1}{x})}{x} \le \frac{1}{x^2}$.

So, since $\int_1^\infty \frac{1}{x^2}dx$ is a p-integral, it's convergent, hence $\int_1^\infty \frac{\sin(1/x)}{x}dx$ is also convergent.

b) $\int_0^{+\infty} \frac{x}{\sqrt[3]{x^5+2}}dx = \int_0^1 \frac{x}{\sqrt[3]{x^5+2}}dx + \int_1^{+\infty} \frac{x}{\sqrt[3]{x^5+2}}dx$

since $\int_0^1 \frac{x}{\sqrt[3]{x^5+2}}dx$ is proper it's convergent.

for $\int_1^{+\infty} \frac{x}{\sqrt[3]{x^5+2}}dx$ we apply limit test:

since $\frac{x}{\sqrt[3]{x^5+2}} \sim \frac{x}{x^{5/3}} = x^{-2/3}$ as $x\to\infty$

and p-integral of the first kind ($\int_a^\infty \frac{1}{x^p}dx \ \forall a>0$)

diverges for $p = 2/3$, then $\int_0^{+\infty} \frac{x}{\sqrt[3]{x^5+2}}dx$ is diverges

(also we can split it to 0-2 2-∞, first is proper, second is greater then $\frac{1}{x} \Rightarrow$ diverges)

c) $\int_0^\infty \frac{\sin^2(x)}{x^2}dx \int_0^1 \frac{\sin^2(x)}{x^2}dx + \int_1^\infty \frac{\sin^2(x)}{x^2}dx$

$\int_0^1 \frac{\sin^2(x)}{x^2}dx$ is proper integral; proper integral is convergent

Since $\lim_{x\to 0}\left(\frac{\sin^2(x)}{x^2}\right) = 1 \Rightarrow 0$ isn't disc. point $\Rightarrow \int_0^1 \frac{\sin^2(x)}{x^2}dx$ convergent.

(anoter appreach is to say, that $0 \le \frac{\sin^2(x)}{x^2} \le \sec^2(x)$ on $[0,1]$; since $\int_0^1 \sec^2(x)dx = \tan(x)\Big]_0^1 = \tan(1) < \inf \Rightarrow \int_0^1 \frac{\sin^2(x)}{x^2}$ is convergent)

$\int_1^\infty \frac{\sin^2(x)}{x^2}$ is convergent, since $0 \le \frac{\sin^2(x)}{x^2} \le \frac{1}{x^2}$ on $x\in[1,\infty)$

and $\int_1^\infty \frac{1}{x^2}dx$ is p-integral and it's convergent $\Rightarrow \int_1^\infty \frac{\sin^2(x)}{x^2}dx$ is also convergent.

then $\int_0^\infty \frac{\sin^2(x)}{x^2}dx$ is convergent. (to $\frac{\pi}{2}$ ..., but tsss...)

(7*) let's split the integral:

$$\int_1^\infty \frac{dx}{x^\alpha \ln^\beta(x)} = \int_1^2 \frac{dx}{x^\alpha \ln^\beta(x)} + \int_2^\infty \frac{dx}{x^\alpha \ln^\beta(x)}$$

(1) consider $\int_1^2 \frac{dx}{x^\alpha \ln^\beta(x)}$ first:

since $\frac{1}{x^\alpha \ln^\beta(x)} \sim \frac{1}{x \ln^\beta(x)}$, consider $\alpha = 1$, $\forall \alpha \in \mathbb{R}$

$$\int_1^2 \frac{dx}{x \ln^\beta(x)} = \left\{ \begin{matrix} t = \ln(x) \\ dt = \frac{dx}{x} \end{matrix} \right\} = \int_0^{\ln(2)} \frac{dt}{t^\beta}$$ it's converges if $\beta < 1$

$\Rightarrow$ diverges for $\beta \geq 1$

Hence, from (1) we can conclude: $\forall \alpha \in \mathbb{R} \wedge \beta < 1$ $\int_1^2 \frac{dx}{x^\alpha \ln^\beta(x)}$ converges

(2) consider $\int_2^\infty \frac{dx}{x^\alpha \ln^\beta(x)}$:

a. $\alpha = 1$: (since it's p-integral)

$$\int_2^\infty \frac{dx}{x \ln^\beta(x)} = \left\{ \begin{matrix} t = \ln(x) \\ dt = \frac{dx}{x} \end{matrix} \right\} = \int_{\ln(2)}^\infty \frac{dt}{t^\beta}$$ it's converge for $\beta > 1$

$\Rightarrow$ diverges for $\beta \leq 1$

b. $\alpha > 1$: (grasp into it, and you'll get it :)

let $\alpha = 1 + 2\delta$, $\delta > 0$

$$\frac{1}{x^\alpha \ln^\beta(x)} = \frac{1}{x^{1+\delta}} \cdot \frac{1}{x^\delta \ln^\beta(x)}$$

(note that $\lim_{x \to \infty} \left(\frac{1}{x^\delta \ln^\beta(x)} \right) = 0$ even for $\beta < 0$, if $\delta > 0$)

more formally:

$$\forall \varepsilon > 0 \ \exists x_0 : \forall x > x_0 \ \frac{1}{x^\delta \ln^\beta(x)} < \varepsilon$$

So $\frac{1}{x^\delta \ln^\beta(x)} < \frac{1}{x^{1+\delta}}$ $\Rightarrow$ converges on $x > 1$

c. $\alpha < 1$

let $\alpha = 1 - 2\delta$, $\delta > 0$, then

$$\frac{1}{x^{\alpha}\ln^{\beta}(x)} = \frac{1}{x^{1-\delta}} \cdot \frac{x^{\delta}}{\ln^{\beta}(x)} \quad \left(\text{note that } \lim_{x\to\infty}\left(\frac{x^{\delta}}{\ln^{\beta}(x)}\right) = \infty, \forall\beta, \forall\delta>0\right)$$

$$\exists x_0 : \forall x > x_0 \quad \frac{x^{\delta}}{\ln^{\beta}(x)} > 1 \quad (\forall\delta>0 \ \forall\beta\in\mathbb{R})$$

moreover $\forall\varepsilon>0 \ \exists x_0 : \forall x > x_0 \ \frac{x^{\delta}}{\ln^{\beta}(x)} > \varepsilon$

$$\frac{1}{x^{\alpha}\ln^{\beta}(x)} > \frac{1}{x^{1-\delta}}, \text{ so } \int_{x_0}^{\infty}\frac{dx}{x^{\alpha}\ln^{\beta}(x)} > \int_{x_0}^{\infty}\frac{dx}{x^{1-\delta}} \ (\text{which is diverges})$$

Hence $\int_{x_0}^{\infty}\frac{dx}{x^{\alpha}\ln^{\beta}(x)}$ diverges, so $\int_{2}^{\infty}\frac{dx}{x^{\alpha}\ln^{\beta}(x)}$ diverges also.

So if $\alpha = 1 \wedge \beta > 1$ integral is converges

also for $\alpha > 1 \ \forall\beta\in\mathbb{R}$ it's converges.

Considering (1) and (2) we get that:

$$\int_{1}^{\infty}\frac{dx}{x^{\alpha}\ln^{\beta}(x)} \text{ converges for } \alpha>1 \wedge \beta<1$$

diverges otherwise it's diverges!

Tnx for checking, have a nice day

Solved by Novosad Ivan