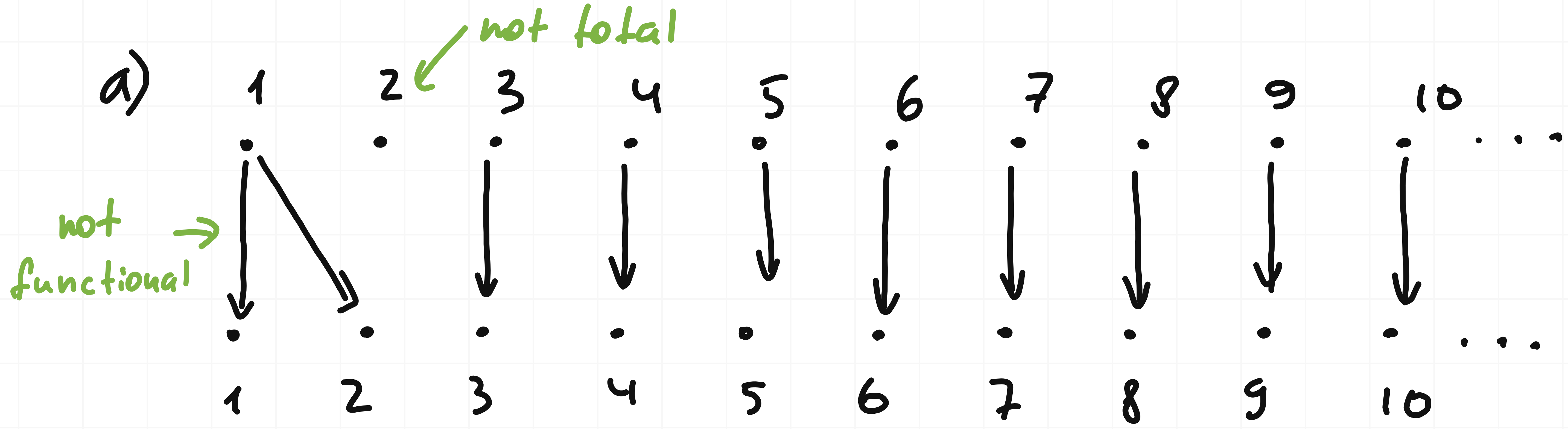
Novosad Ivan

1. Give an example of a binary relation $P \subseteq \mathbb{R} \times \mathbb{R}$ such that:

a) P is not functional, injective, not total, and surjective;

b) P is functional, not injective, total, and not surjective.

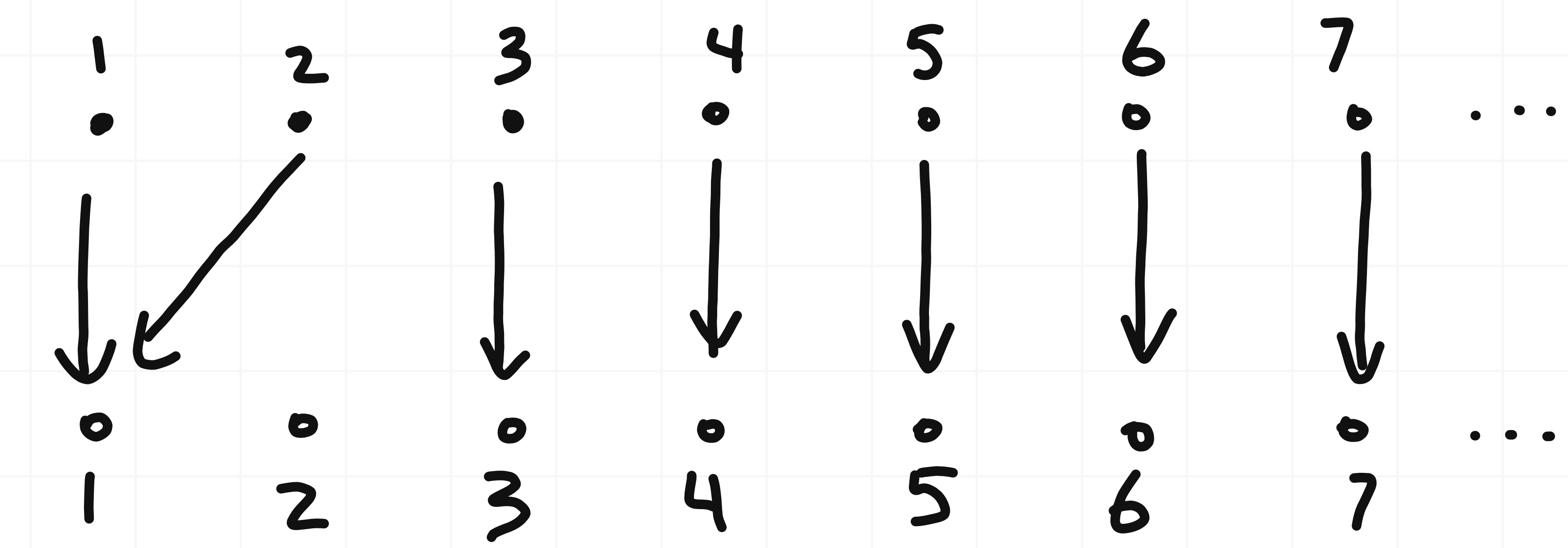
a)



$$\varphi(x) = \begin{cases} 1,2 & \text{if } x = 1 \\ x & \text{if } x \neq 1 \land x \neq 2 \end{cases}$$

Surjective since $\forall y \, \exists x \; \varphi(x) = y$; injective since $\forall x \forall z \; (\varphi(x) = y \land \varphi(z) = y) \to x = z$

b)



$$\varphi(x) = \begin{cases} 1 & \text{if } x = 2 \lor x = 1 \\ x & \text{if } x \neq 2 \land x \neq 1 \end{cases}$$

functional since $\forall x \, !\exists y \;\; \varphi(x) = y \;\; (xRy, \; R = \varphi)$

not injective, since $\varphi(1) = 1 \land \varphi(2) = 1$; total since $\forall x \, \exists y \;\; xRy \; (\varphi(x) = y)$

4. Suppose that $f\colon A \to B$ and $g\colon B \to C$. Prove that if $g \circ f$ is an injection, then f is an injection as well.

$g \circ f$ is injective iff $\forall a \in A\ \forall b \in A\ \ g(f(a)) = g(f(b)) \Rightarrow a = b$

Suppose that f isn't injective, then $\exists a \in A\ \exists b \in A\ \ f(a) = f(b) \wedge a \neq b$

$\Rightarrow g(f(a)) = g(f(b)) = g \circ f(a) = g \circ f(b)$ (and $a \neq b$) $\Rightarrow g \circ f$ is not injective as well, then $\perp$.

hence if $g \circ f$ is injective, then so is f.

6. Suppose that $f\colon A \to B$ and $f^{-1}\colon B \to A$. Then f is a bijection from A to B

Given: $f: A \to B \wedge f^{-1}: B \to A$

Prove: $\forall y \in B\ !\exists x \in A\ f(x) = y \wedge \forall x \in A\ !\exists y \in B\ f(x) = y$

suppose f is not bijective, then $\exists y \in B$, so:

1) then $\neg \exists x \in A\ \ f(x) = y\ (\forall x \in A\ f(x) \neq y)$, but $f^{-1}: B \to A$ is total $\Rightarrow$

$\Rightarrow \forall y \in B\ \exists x \in A\ \big((y,x) \in f^{-1} \Leftrightarrow (x,y) \in f\big) \Rightarrow \perp \Rightarrow f$ is surjective

not really need

2) $\forall a \in A\ \forall b \in B\ \ a \neq b\ \ f(a) = f(b) = y$ (means not injective) $\Leftrightarrow$

$\Leftrightarrow \exists y \in B\ \exists a \in A\ \exists b \in A\ \ f^{-1}(y) = a = b \wedge a \neq b$, but f^{-1} is functional

$\Rightarrow \perp \Rightarrow f$ is injective

Since f is injective and surjective $\Leftrightarrow f$ is bijective ∎

3. Suppose that $f\colon A \to B$ and $g\colon A \to B$. Prove that $f \cup g\colon A \to B$ iff $f = g$

1) $\Leftarrow$ if $f = g$, then $f \cup g = f = g : A \to B$ •

2) $\Rightarrow$ Since $f \cup g$ is total $\wedge$ functional $\Rightarrow \forall x \in A\ !\exists y \in B\ (x,y) \in f \cup g$

$\Rightarrow \forall x \in A\ \ \exists b \in B\ \exists c \in B\ \ (x,b) \in f \wedge (x,c) \in g \Rightarrow (x,b) \in f \cup g,\ (x,c) \in f \cup g$

but since $f \cup g$ is functional $b = c$, so $A = A \wedge \forall x \in A\ f(x) = g(x) = b = c \Rightarrow f = g$ •

7. Give an example element from the following sets:

a) $\mathbb{Q}^3$;

b) $\mathbb{R}^{\mathbb{Q}}$;

c) $\mathbb{R}^{\mathbb{R} \times \mathbb{Z}}$.

a) $\{(0, 1/3), (1, 2/3), (2, 6)\}$

b) $f(x) = x$

c) $f((x,y)) = x \cdot y$