Questions

2. (7) (Improper Integrals, Arc Length, Volumes, Surface Area and so on: something from March) (8) Find the area of the surface of solid obtained by revolving the curve $y = \tan x$, $0 \leqslant x \leqslant \pi/4$ about the x-axis.

Solved problems 3_10

length of curve between $x=a$ and $x=b$ is

$$\int_a^b \sqrt{\dot{x}^2+\dot{y}^2}\,dx$$

Surface about x: $2\pi\int_a^b y(t)\sqrt{\dot{x}^2+\dot{y}^2}\,dt$

about y: $2\pi\int_a^b x(t)\sqrt{\dot{x}^2+\dot{y}^2}\,dt$

okay, let's think that I pass it.

1. $y=\tan(x)$ $\quad 0\le x\le \pi/4$ $\quad$ about x, surface $\qquad \dot{y}=\tan(x)'=\sec^2(x)$

$$S = 2\pi\int_0^{\pi/4} y(t)\sqrt{\dot{x}^2+\dot{y}^2} = 2\pi\int_0^{\pi/4}\tan(x)\sqrt{1+1/\cos^4(x)}\,dx = \left\{\begin{matrix} t=\cos(x) & 0\to 1 \\ dt=-\sin(x)dx & \frac{\pi}{4}\to\frac{\sqrt2}{2} \\ dx=-\csc(x)dt & \end{matrix}\right\} =$$

$$= -2\pi\int_1^{\frac{\sqrt2}{2}} \frac{\cancel{\sin(x)}}{\cos(x)}\cdot\sqrt{1+1/\cos^4(x)}\cdot\frac{1}{\cancel{\sin(x)}}\,dt = 2\pi\int_{\frac{\sqrt2}{2}}^1 \frac{1}{t}\sqrt{1+1/t^4}\,dt = 2\pi\int_{\frac{\sqrt2}{2}}^1\frac1t\sqrt{\frac{t^4+1}{t^4}}\,dt = 2\pi\int_{\frac{\sqrt2}{2}}^2\frac{1}{t^3}\sqrt{t^4+1}\,dt =$$

" $\int_a^b f(x)dx = -\int_b^a f(x)dx$ "

$$= \left\{\begin{matrix} u=1/t^4 & \frac{\sqrt2}{2}\to 4 \\ du=-4\frac{1}{t^5}dt & \\ dt=-\frac14 t^5 dt & 1\to1\end{matrix}\right\} = \frac{\pi}{2}\int_1^4\frac{\sqrt{u+1}}{u}\,du = \left\{\begin{matrix} l=\sqrt{u+1} & 1\to\sqrt2 \\ dl=\frac{1}{2\sqrt{u+1}}du & 4\to\sqrt5 \\ du=2\sqrt{u+1}\,dl & \end{matrix}\right\} = \frac{\pi}{2}\int_{\sqrt2}^{\sqrt5}\frac{\sqrt{u+1}}{u}\left(2\sqrt{u+1}\right)dl =$$

midstep

$$= \pi\int_{\sqrt2}^{\sqrt5}\frac{u+1}{u}\,dl = \qquad \pi\int_{\sqrt2}^{\sqrt5}\frac{l^2}{l^2-1}\,dl = \pi\int_{\sqrt2}^{\sqrt5}\left[1+\frac{1}{l^2-1}\right]dl = \pi\underbrace{\int_{\sqrt2}^{\sqrt5}dl}_{(1)} + \pi\underbrace{\int_{\sqrt2}^{\sqrt5}\frac{1}{l^2-1}\,dl}_{(2)} \;=$$

midstep, just for you, it's not accurate to write like this at exam

↑correct step

(1) $\pi\int_{\sqrt2}^{\sqrt5} dl = \pi(l)\Big|_{\sqrt2}^{\sqrt5} = \pi(\sqrt5-\sqrt2)$

(2) $\int_{\sqrt2}^{\sqrt5}\frac{1}{l^2-1}\,dl = \frac12\ln\left(\left|\frac{l-1}{l+1}\right|\right)\Big|_{\sqrt2}^{\sqrt5} = \frac12\ln\left(\frac{\sqrt5-1}{\sqrt5+1}\right)-\frac12\ln\left(\left|\frac{\sqrt2-1}{\sqrt2+1}\right|\right) = \ln\left(\frac{\sqrt{10}+\sqrt5-\sqrt2-1}{2}\right)$

since $\int\frac{1}{x^2-a^2}dx = \frac{1}{2a}\ln\left(\left|\frac{x-a}{x+a}\right|\right)$

$= \pi\left(\sqrt5-\sqrt2+\ln\left(\frac{\sqrt{10}+\sqrt5-\sqrt2-1}{2}\right)\right) \approx 3{,}8391\ldots$

✓ 7. $x(t)=t-\sin(t)$ $\quad y=1-\cos(t)$ $\quad$ area, about y-axis

$\dot{x}(t) = 1-\cos(t)$

$\dot{y}(t) = \sin(t)$

$\dot{x}(t)^2+\dot{y}(t)^2 = 4\sin(\frac t2)$

$a=0 \quad b=2\pi$

$$\frac{A}{4\pi} = \int_a^b (t-\sin(t))\sin(\tfrac t2)\,dt = \left\{\begin{matrix} x=\frac t2 & a\to a \\ dt=2dx & b\to\pi\end{matrix}\right\} = 2\int_0^\pi(2x-\sin(2x))\sin(x)\,dx = 4\int_0^\pi x\sin(x)\,dx - 2\int_0^\pi \sin^2(x)\cos(x)\,dx = 4\pi$$

$$\int_0^\pi x\sin(x)\,dx = \left\{\begin{matrix} u=x & dw=\sin(x) \\ du=dx & w=-\cos(x)\end{matrix}\right\} = -x\cos(x)\Big|_0^\pi + \int_0^\pi\cos(x)dx = -x\cos(x)+\sin(x)\Big|_0^\pi = \pi$$

$$\int_0^\pi\sin^2(x)\cos(x)\,dx = \left\{\begin{matrix} w=\sin(x) & 0\to0 \\ dw=\cos(x)dw & \pi\to0\end{matrix}\right\} = 0$$

✓ 2. $y=\ln(\cos(x))$, $0\le x\le\frac{\pi}{3}$

$$\int_0^{\pi/3}\sqrt{(\ln(\cos(x)))'^2+1}\,dx = \int_0^{\pi/3}\sqrt{1+\tan(x)^2}\,dx = \int_0^{\pi/3}\sqrt{\sec^2(x)}\,dx = \int_0^{\pi/3}\sec(x)\,dx = \ln(|\tan(x)+\sec(x)|)\Big|_0^{\pi/3} = \ln(|\sqrt3+2|)+\ln(0+1) = \ln(2+\sqrt3)$$

✓ 3. $y^2=(x-1)^3$ $\quad 1\le x\le5$, $y>0$

⇓

$y=\pm(x-1)^{3/2}$, since $y\ge0$ $y=(x-1)^{3/2} \Rightarrow y' = \frac32(x-1)^{\frac12}$ $\dot{y}^2=\frac94(x-1)$ | $\dot{y}^2+1 = \frac14(9x-5)$

Thus arc len. $= \int_1^5\sqrt{\frac14(9x-5)}\,dx = \left\{\begin{matrix} u=\frac94x-\frac54 & 1\to1 \\ du=\frac94dx & 5\to10\end{matrix}\right\} = \frac49\int_1^{10}\sqrt{u}\,du = \frac49\cdot\frac23 u^{3/2}\Big|_1^{10} \; \frac{8}{27}(10\sqrt{10}-1)$

4. $x(t)=t\sin(t)$ $\quad y(t)=t\cos(t)$ $\quad 0\le t\le1$, arc?

$\dot{x}(t)=\sin(t)+t\cos(t)$ $\quad \dot{y}(t)=\cos(t)-\sin(t)t$

$\dot{x}(t)^2 = \sin^2(t)+\sin(2t)t+t^2\cos^2(t)$

$\dot{y}(t)^2 = \cos^2(t)-\sin(2t)t+\sin^2(t)t^2$ $\qquad \dot{x}(t)^2+\dot{y}(t)^2 = t^2+1$

$$\text{Thus len} = \int_0^1\sqrt{t^2+1}\,dt = \left\{\begin{matrix} t=\tan(\theta) & 1\to\frac{\pi}{4} \\ t^2+1=\tan^2(\theta)+1=\sec^2(\theta) & \\ dt=\sec^2(\theta) & 0\to0\end{matrix}\right\} = \int_0^{\pi/4}\sec^3(\theta)\,d\theta = \left\{\begin{matrix} u=\sec(\theta) & dw=\sec^2(\theta) \\ du=\sec(\theta)\tan(\theta) & w=\tan(\theta)\end{matrix}\right\} =$$

$$= \sec(\theta)\tan(\theta)\Big|_0^{\pi/4} - \int\sec\theta\tan^2(\theta)\,d\theta = \sec(\theta)\tan(\theta)\Big|_0^{\pi/4} - \int_0^{\pi/4}\sec(\theta)(\sec^2\theta-1)\,d\theta = \sec(\theta)\tan(\theta)\Big|_0^{\pi/4} + \int_0^{\pi/4}\sec\theta\,d\theta - \int_0^{\pi/4}\sec^3(\theta)\,d\theta =$$

$$\Rightarrow \int_0^{\pi/4}\sec^3(\theta)\,d\theta = \frac12\left(\sec(\theta)\tan(\theta)\right)\Big|_0^{\pi/4} + \frac12\ln(|\tan(\theta)+\sec(\theta)|)\Big|_0^{\pi/4} = \frac12(0-\sqrt2)+\frac12(\ln(1)-\ln(1+\sqrt2)) = -\frac{\sqrt2}{2}+\frac14\ln(2)$$

5. $x(t)=\cosh(\ln(t))$ $\quad y(t)=\sinh(\ln(t))$ $\quad z(t)=\ln(t)$ $\quad 1\le t\le2$

$$\dot{x}(t) = \frac{\sinh(\ln(t))}{t} = \frac{e^{\ln(t)}-e^{-\ln(t)}}{2}/t = \frac{t-\frac1t}{2}/t = \frac{\frac{t^2-1}{t}}{\frac{2}{1}}/t = \frac{\frac{t^2-1}{2t}}{t} = \frac{t^2-1}{2t^2}$$

$$\dot{y}(t) = \frac{\cosh(\ln(t))}{t} = \frac{t^2+1}{2t^2}$$

$$\dot{z}(t) = \frac1t$$

$$\dot{x}(t)^2 = \frac{(t^2-1)^2}{4t^4} \quad \dot{y}(t)^2 = \frac{(t^2+1)^2}{4t^4} \quad \dot{z}(t) = \frac{1}{t^2}$$

$$\text{sum: } \frac{(t^2-1)^2+(t^2+1)^2}{4t^4}+\frac{1}{t^2} = \frac{t^4+1}{2t^4}+\frac{1}{t^2} = \frac{t^4+2t^2+1}{2t^4} = \frac{(t^2+1)^2}{2t^4} = \left(\frac{t^2+1}{\sqrt2 t^2}\right)^2 = \left(\frac{1}{\sqrt2}\left(1+\frac{1}{t^2}\right)\right)^2$$

$$\text{Thus len} = \frac{1}{\sqrt2}\int_1^2[1+t^{-2}]\,dt = \frac{1}{\sqrt2}\left(t-\frac1t\right)\Big|_1^2 = \frac{t^2-1}{\sqrt2 t}\Big|_1^2 = -0+\frac{4-1}{\sqrt2\,2} = \frac{3}{2\sqrt2}$$

6. $x(t)=t-\sin(t)$ $\quad y(t)=1-\cos(t)$ $\quad$ area, about x-axis

$\dot{x}(t) = 1-\cos(t)$

$\dot{y}(t) = \sin(t)$ $\qquad$ Since $\sin^2(\theta)=\frac12(1-\cos(2\theta)) \Rightarrow 1-\cos(\theta) = 2\sin^2(\theta/2)$

$\dot{x}(t)^2+\dot{y}(t)^2 = 1-2\cos(t)+\cos^2(t)+\sin^2(t) = 2(1-\cos(t)) = 4\sin^2(\frac t2)$

$$2\pi\int_0^{2\pi}(1-\cos(t))\sqrt{4\sin^2(t/2)}\,dt = 2\pi\int_0^{2\pi}(1-\cos(t))(2\sin(t/2))\,dt = 2\pi\int_0^{2\pi}2\sin^2(\tfrac t2)2\sin(\tfrac t2)\,dt = 8\pi\int_0^{2\pi}\sin(\tfrac t2)(1-\cos^2(t/2))\,dt = \left\{\begin{matrix} \cos(\frac t2)=x \\ dx=\frac12\sin(\frac t2)dt \\ 0\to1 \quad \frac{\pi}{2}\to-1\end{matrix}\right\} =$$

$$= -16\pi\int_{-1}^1[1-x^2]\,dt = +16\pi\left(x-\frac13x^3\right)\Big|_{-1}^1 = +16\pi\left(\frac43\right) = +\frac{64\pi}{3}$$

solved problems "4–6"

3. (7) (Functions of Several Variables) (a) Find the directional derivative of $f = 2x^3y - 3y^2z$ at $P(1,2,-1)$ in a direction toward $Q(3,-1,5)$. (b) In what direction from P is the directional derivative a maximum? (c) What is the magnitude of the maximum directional derivative?

✓1. $f(x,y,z) = -2x^2 + xy + yz - z^2$, where $x = t$ $y = 2t^2$ $z = t + t^3$

$f_x = -4x\,dx + y\,dx = -2x + y = -4t + 2t^2$

$f_y = x\,dy + z\,dy = 4tx + 4tz = 4t^2 + 4t(1+t^3) = 4t^2 + 4t^4 + 4t$

$f_z = y\,dz - 2z\,dz = 3t^2y - 6t^2z = 6t^4 - 6t^2(1+t^3) = 6t^4 - 6t^5 - 6t^2$

$df = f_x + f_y + f_z = -6t^5 + 10t^4$

✓3. $x = u\cos(\alpha) - v\sin(\alpha)$, $y = u\sin(\alpha) + v\cos(\alpha)$; find $f_u^2 + f_v^2$

$f_u = f_x(x_u) + f_y(y_u) = f_x\cos(\alpha) + f_y\sin(\alpha)$

$f_v = f_x(x_v) + f_y(y_v) = -f_x\sin(\alpha) + f_y\cos(\alpha)$

So $f_x^2 + f_y^2 = f_u^2 + f_v^2$

✓4. Find the directional derivative of the function $f(x,y) = x^2y$ at $P(1,2)$ in the direction determined by unit vector $u = (\sqrt{3}/2, 1/2)$

$f_x = 2xy$

$f_y = x^2$

$\Rightarrow \nabla f = (2xy, x^2) \Rightarrow \nabla f(P) = (4,1)$

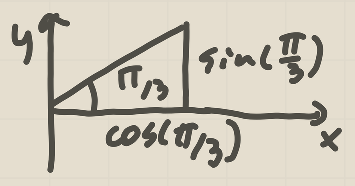
thus $\frac{\partial f}{\partial u}(P) = \frac{\nabla f(P)\cdot u}{\underbrace{\|u\|}_{1\text{ - given}}} = (4,1)\cdot\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = 2\sqrt{3} + \frac{1}{2}$

✓5. $f(x,y) = e^{xy}$ at $P(-2,1)$ in the direction of u that makes angle of $\theta = \pi/3$ with pos. x-axis

$f_x = ye^{xy}$

$f_y = xe^{xy}$

$\Rightarrow \nabla f(P) = \left(\frac{1}{e^2}; -\frac{2}{e^2}\right) \Rightarrow$



$\Rightarrow u = (\cos(\pi/3), \sin(\pi/3)) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

Thus $\frac{\partial f}{\partial u}(P) = \nabla f(P)\cdot u = (e^{-2}; -2e^{-2})\left(\frac{1}{2}; \frac{\sqrt{3}}{2}\right) = \frac{1}{2e^2} - \frac{\sqrt{3}}{e^2}$

✓6. $f = \log(x^2+y^2)$, $v = (5,12)$ $P = (1,2)$, comput in the direction of v

$f_x = \frac{2x}{x^2+y^2}$

$f_y = \frac{2y}{x^2+y^2}$

$\Rightarrow \nabla f(P) = \left(\frac{2}{5}; \frac{4}{5}\right)$ $u = \frac{v}{\|v\|} = \frac{(5,12)}{13} = \left(\frac{5}{13}, \frac{12}{13}\right)$

$\frac{\partial f}{\partial u}(P) = \left(\frac{2}{5}, \frac{4}{5}\right)\left(\frac{5}{13}, \frac{12}{13}\right) = \frac{2}{13} + \frac{48}{65} = \frac{58}{65}$

$f = 2x^3y - 3y^2z$ $P(1,2,-1)$ $Q(3,-1,5)$

$\nabla f = (f_x, f_y, f_z) = (6x^2y, 2x^3 - 6yz, -3y^2)$

$\nabla f(P) = (f_x(P), f_y(P), f_z(P)) = (12, 14, -12)$

$\frac{\nabla f(P)}{\|\nabla f(P)\|} = \frac{(12,14,-12)}{22} = \left(\frac{6}{11}\ \frac{7}{11}\ -\frac{6}{11}\right)$

directional of fastest growth

$\|\nabla f(P)\| = \sqrt{12^2 + 14^2 + (-12)^2} = \sqrt{484} = 22$

magnitude of fastest growth

$\frac{\nabla f(P)\cdot Q}{\|Q\|}$ is the directional derivative towards Q

$\frac{\nabla f(P)\cdot Q}{\|Q\|} = \frac{(12,14,-12)\cdot(3,-1,5)}{\sqrt{35}} = \frac{1}{\sqrt{35}}(36 - 14 - 60) = -\frac{38}{\sqrt{35}}$

✓7. $f = xy\sqrt{1+z^2}$, $v = (1,-2,2)$ and $P = (1,1,2)$

$f_x = y\sqrt{1+z^2}$

$f_y = x\sqrt{1+z^2}$

$f_z = \frac{xyz}{\sqrt{1+z^2}}$

$\nabla f(P) = \left(\sqrt{5}; \sqrt{5}, -\frac{2}{\sqrt{5}}\right)$

$\|v\| = \sqrt{1+4+4} = 3 \Rightarrow u = \left(\frac{1}{3}\ -\frac{2}{3}\ \frac{2}{3}\right)$

$\frac{\partial f}{\partial u}(P) = \frac{\sqrt{5}}{3} - \frac{2\sqrt{5}}{3} - \frac{4}{3\sqrt{5}} = -\frac{1}{3\sqrt{5}}$

8. $f = 2x^2 + 5y^2$ at $P(3,-2)$ in the direction from P to $Q(6,-1)$

$f_x = 4x$

$f_y = 10y$

$\Rightarrow \nabla f(P) = (12, -20)$

from P to Q : $\vec{PQ} = Q - P = (3,1)$ $\|Q - P\| = \sqrt{10}$

thus $\frac{\partial f}{\partial u}(P) = \frac{(12,-20)(3,1)}{\sqrt{10}} = \frac{36 - 20}{\sqrt{10}} = \frac{16}{\sqrt{10}}$

9. The temp of metal plate is $T(x,y) = 20 - 4x^2 - y^2$, where $x\,y$ are measured in centimeters

I what direction from $P(2,-3)$ does the t. increase most rapidly? What's the rate?

$T_x = -8x$

$T_y = -2y$

$\Rightarrow \nabla T(P) = (-16, 6)$ ↙ fastest direction, but not unit vector

So $\|\nabla T(P)\| = \sqrt{16^2 + 36} = \sqrt{292} \Rightarrow \left(-\frac{16}{\sqrt{292}}; \frac{6}{\sqrt{292}}\right)$ is the fastest direction.

$\|\nabla T(P)\|$ = rate of growth $\Rightarrow (292)^{1/2}$ is the req. rate.

Improper Integrals

1. $\int_0^{+\infty}\sin(x)\,dx = \lim_{a\to\infty}\left(\int_0^a \sin(x)dx\right) = \lim_{a\to\infty}\left(-\cos(x)\Big|_0^a\right) = -\cos(0) + \lim_{a\to\infty}(\cos(x)) = -1 + DNE \Rightarrow$ divergent

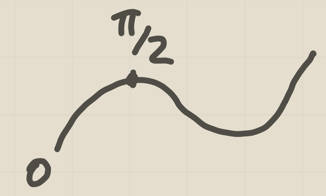
2. $\int_{-\infty}^0 xe^{2x}\,dx$ (let a be lim of lower bound and b be lim of upper bound) $= \int_0^b x e^{2x}\,dx = \left\{\begin{matrix} u=x & dv=e^{2x} \\ du=dx & v=\frac12 e^{2x}\end{matrix}\right\} = \frac12 e^{2x}x\Big|_a^0 - \frac12\int_a^0 e^{2x}dx = e^{2x}\left(\frac x2-\frac14\right)\Big|_a^0 = e^0\left(-\frac14\right) - e^{2a}\left(\frac a2 - \frac14\right) = -\frac14$

$\lim_{a\to-\infty}\left(e^{2a}\left(\frac a2-\frac14\right)\right) = 0$

enought?

3. $\int_0^{+\infty}\frac{dx}{4+x^2} = \int_0^b \frac{dx}{4+x^2} = \frac12\arctan\left(\frac x2\right)\Big|_0^b = \frac12 \underbrace{\arctan\left(\frac b2\right)}_{\to\frac\pi2} - \frac12\overbrace{\arctan(0)}^{=0} = \frac\pi4$

4. $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \int_0^b \frac{dx}{\sqrt{1-x^2}} = \arcsin(x)\Big|_0^b = \arcsin(1) - \arcsin(0) = \arcsin(1) = \frac\pi2$



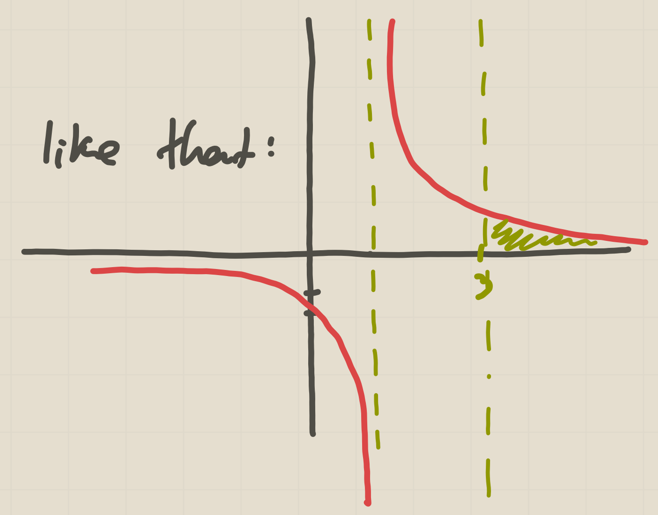
5. $\int_0^1 \ln(x)\,dx = \left\{\begin{matrix}u=\ln(x) & dv=dx\\ du=\frac1x & v=x\end{matrix}\right\} = \ln(x)x - \int_a^b \frac xx\,dx = \ln(x)x - x\Big|_a^b = \ln(1)(1)-(1)-\ln(0)(0)-(0) = -1$

6. $\int_0^1 \frac{\arccos(x)}{\sqrt{1-x^2}}\,dx = \left\{\begin{matrix} t=\arccos(x)\\ dt=-\frac{1}{\sqrt{1-x^2}}\,dx\end{matrix}\right\} = -\int_{a'}^{b'} t\,dt = -\frac12 t^2\Big|_{a'}^{b'} = -\frac12\arccos^2(x)\Big|_{-1}^0 = -\frac12\frac{\pi^2}{4}+\frac12\pi^2 = -\frac{\pi^2}{8}+\frac{\pi^2}{2} = \frac{3\pi^2}{8}$

7. Find the area of region lying to the right of $x=3$ and between the curve $y=\frac{1}{x^2-1}$ and x-axis

$\int_3^{+\infty}\frac{1}{x^2-1}\,dx = \frac12\lim_{b\to\infty}\left(\ln\left(\left|\frac{x-1}{x+1}\right|\right)\Big|_3^b\right) = \frac12\ln\left(\frac12\right) - \frac12\lim_{b\to\infty}\left(\ln\left|\underbrace{\frac{b-1}{b+1}}_{\to1}\right|\right) = \frac12\ln\left(\frac12\right)=$

s mth like that:



8. Prove the inequality:

$$0 \underset{(1)}{<} \int_2^{+\infty}\frac{\sqrt{x^3-x^2+3}}{x^5+x^2+1}\,dx \underset{(2)}{<} \frac{1}{10\sqrt2}$$

(1) since $\sqrt{x^3-x^2+3}\ge0\ \forall x>2$ and $x^5+x^2+1\ge0\ \forall x>2$ $\Big\} \Rightarrow f(x)>0\ \forall x\in[2,\infty) \Rightarrow \int_2^\infty f(x)\,d\,dx>0$

(2) since $\frac{\sqrt{x^3-x^2+3}}{x^5+x^2+1} < \frac{\sqrt{x^3}}{x^5}\quad \forall x\ge2$, do I need to prove it?

$\Rightarrow \int_2^{+\infty}\frac{\sqrt{x^3-x^2+3}}{x^5+x^2+1} < \int_2^{+\infty}\frac{\sqrt{x^3}}{x^5}\,dx = \int_2^{+\infty}x^{-7/2}\,dx =$

$\int_2^{+\infty}x^{-7/2}\,dx = -\frac25 x^{-5/2}\Big|_2^b = -\frac25\cdot\lim_{b\to\infty}\left(\frac{1}{\sqrt[5]{b^2}}\right)^{\to0} + \frac25\cdot\frac{1}{\sqrt[2]{2^5}} = \frac{2}{5\cdot4\sqrt2} = \frac{1}{10\sqrt2}$ ■?

$b\to\infty$

9. $\int_0^1\frac{\ln(x)}{\sqrt x}\,dx = \left\{\begin{matrix}\ln(x)=u & dv=\frac1{\sqrt x}\\ du=\frac1x du & v=2\sqrt x\,dv\end{matrix}\right\} = 2\ln(x)\sqrt x\Big|_a^1 - 2\int\frac{\sqrt x}{x}\,dx = 2\ln(x)\sqrt x - 4\sqrt x\Big|_a^1 = 2\sqrt x\left(\ln(x)-2\right)\Big|_a^1 \ominus$

$a\to0$

$\ominus\ 2(\ln(1)-2) - \lim_{x\to0}\left(\overbrace{2x^{-1/2}\ln(x)}^{=0}-\overbrace{2}^{=-2}\right) = -6$

how to prove?

4. (7) (Functions of Several Variables) Find the first and the second total differentials of the function $z = \ln(1 + e^x \ln y)$ at the point $P(1, e)$. Find Taylor expansion for this function about the point P with $o((x-1)^2 + (y-e)^2)$.

1. let $f(x,y)$ be Z, then

$$f_x = \frac{\ln(y)\, e^x}{1+e^x \ln(y)}$$

$$f_y = \frac{e^x}{y(1+e^x \ln(y))}$$

$$f_{xy} = \frac{(\ln(y)\, e^x)'(1+e^x\ln(y)) - (\ln(y)\, e^x)(1+e^x \ln(y))'}{(1+e^x \ln(y))^2} =$$

$$\left.\begin{aligned}(\ln(y) e^x)' &= \frac{e^x}{y}\\ (1+e^x \ln(y))' &= \frac{e^x}{y}\end{aligned}\right\} \Rightarrow f_{xy} = \frac{e^x}{y(1+e^x \ln(y))^2}$$

Thus $df = \dfrac{2e^x}{y(1+e^x \ln(y))^2} + \dfrac{e^x(y\ln(y)+1)}{y(1+e^x\ln(y))} =$

$=$

5. (7) (Functions of Several Variables) Find y' and y'' for functions defined by the equation $\ln\sqrt{x^2+y^2} = \operatorname{arctg}\left(\frac{y}{x}\right)$.

$$\ln(\sqrt{x^2+y^2}) = \arctan\left(\frac{y}{x}\right) \Rightarrow f(x,y) = \ln\left(\sqrt{x^2+y^2}\right) - \arctan\left(\frac{x}{y}\right)$$

$$f_x = \frac{x-y}{x^2+y^2}$$

$$f_y = \frac{x+y}{x^2+y^2}$$

$$\Rightarrow y' = -\frac{\frac{x-y}{x^2+y^2}}{\frac{x+y}{x^2+y^2}} = -\frac{x-y}{x+y}$$

$$y'' = (y')' = -\frac{(x-y)'(x+y)-(x-y)(x+y)'}{(x+y)^2} = \frac{2(x-y)}{(x+y)^2}$$

6. (7) (Functions of Several Variables) Find the critical points of the functions $f = e^{2x+3y}\left(8x^2 - 6xy + 3y^2\right)$. Test the nature of the critical points.

it's too teddias to calcutate.

1. Lecture 25 (09.04).
Theorems 1, 2 (Bolzano-Weierstrass theorems)

2. Lecture 26 (16.04).
Theorems 1, 2, 3.

3. Lecture 27 (23.04).
Theorems 1, 2.

4. Lecture 28 (27.04).
Theorem 1.

5. Lecture 29 (30.04).
Theorem 1.

6. Lecture 31 (25.05).
Theorem 1, 3.

7. Lecture 32 (01.06).
Theorem 1.

1. (Bolzano-Weirstrass theorem for sequences of points in $\mathbb{R}^2$)

st. Every bonded sequence of points in $\mathbb{R}^2$ has at least one limit point.

proof: let $\{P_n = (x_n\ y_n)\}$ be a bounded sequence of points. Then exist M s.t. $\forall n$ $\sqrt{x_n^2+y_n^2} < M \Rightarrow |x_n| < M \wedge |y_n| < M$

then by Bolzano-W. th. for seq. of real numbers, there exist a subsequence $\{x_{n_k}\}$ that converge to number a $\{x_{n_k}\}$ is also bounded by M.

So applying B.W th. for seq. of reals we can conclude that there is a subsequence $\{y_{n_k}\}$ that is convergent to some number b

So sequence of $\{(x_{n_k}, y_{n_k})\}$ converges to $(a,b) \Rightarrow (a,b)$ is a limit point.

s.t. (B-W th for sets in $\mathbb{R}^2$).

Every bonded infinite set of points in $\mathbb{R}^2$ has at least one limit point

proof.

let G be bounded infinite set of points in $\mathbb{R}^2$. Then there exist a sequence $P_n \in G$ s.t. $P_n \neq P_m$ for $m \neq n$. Accordint to th. 1, $\{P_n\}$ has a limit point Q that is a limit point of G

3.2. The gradient vector ∇F at $P \subset S$ is perpendicular to the tangent vector to any curve γ on S that passes through P. In other words the gradient vector ∇F at $P \subset S$ is orthogonal to the surface $F(x,y,z)=0$ at point P.