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1. Use implicit differentiation to find y':

(a) $4xy^5 - 3x + 5y^2 + 8 = 0$; (b) $\ln(x^2 - y) - 3x + 2y^3 = 0$;
(c) (HW) $\cos(x + 2y) + \cos y + x^2 y = 2$.

c) $F(x,y) = \cos(x+2y) + \cos(y) + x^2 y - 2$

$F_x = -\sin(x+2y) + 2xy$

$F_y = -2\sin(x+2y) - \sin(y) + x^2$

$$y' = -\frac{F_x}{F_y} = \frac{2xy - \sin(x+2y)}{2\sin(x+2y) + \sin(y) - x^2}$$

3. (HW) (a) Can the equation $\sqrt{x^2 + y^2 + z^2} - \sqrt{2} \cdot \cos z = 0$ be solved uniquely for y in terms of x, z in a neighborhood of $(1, 1, 0)$? (b) Can it be solved uniquely for z in terms of x, y in such a neighborhood?

In each case, if yes find partial derivatives of the implicit function at the given point.

$F(x,y,z) = \sqrt{x^2+y^2+z^2} - \sqrt{2}\cos(z)$

$F_x = \dfrac{x}{\sqrt{x^2+y^2+z^2}}$ $F_y = \dfrac{y}{\sqrt{x^2+y^2+z^2}}$ $F_z = \dfrac{z}{\sqrt{x^2+y^2+z^2}} + \sqrt{2}\sin(z)$

$F_x(p) = \dfrac{\sqrt{2}}{2}$ $F_y(p) = \dfrac{\sqrt{2}}{2}$ $F_z(p) = 0$

So for y - can, for z - can't, since $F_z'(p) = 0$

Thus: $y_x(p) = -\dfrac{F_x(p)}{F_y(p)} = -1$

$y_z(p) = 0$

5. (HW) (a) Can the system of equations

$$xy^2 + xzu + yv^2 = 3, \qquad u^3yz + 2xv + u^2v^2 = 4$$

be uniquely solved for u, v in terms of x, y, z in a neighborhood of $(x_0, y_0, z_0, u_0, v_0) = (1, 1, 1, 1, 1)$?

(b) Can it be solved uniquely for x, y in terms of u, v, z in such a neighborhood?

(c) Can it be solved uniquely for x, z in terms of u, v, y in such a neighborhood?

Denote: $F_1 = xy^2 + xzu + yv^2 - 3$

$F_2 = u^3yz + 2xv + u^2v^2 - 4$

a) Then Jacobi matrix is: $\begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix}$

$\frac{\partial F_1}{\partial x} = F_{1x} = y^2 + zu$ $\qquad \frac{\partial F_2}{\partial x} = 2v$

$\frac{\partial F_1}{\partial y} = F_{1y} = 2xy + v^2$ $\qquad \frac{\partial F_2}{\partial y} = u^3 z$

Thus at the point p: $\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \Rightarrow \det = -4 \neq 0 \Rightarrow$

$\Rightarrow$ system can be solved.

b) $\frac{\partial F_1}{\partial x} = y^2 + zu$ $\qquad \frac{\partial F_2}{\partial x} = 2v$

$\frac{\partial F_1}{\partial z} = xu$ $\qquad \frac{\partial F_2}{\partial z} = u^3 y$

So Jaccobian matrix is $\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \Rightarrow \det\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} = 0 \Rightarrow$

$\Rightarrow$ System can not be uniquely solved for x, z in terms of u, v, y in neighborhood of p.

7. (HW) Find y' and y'' for functions defined by the following equations:

(a) $xy + \ln x + \ln y = 0$; (b) $e^{x-y} = x + y$.

a) $F(x,y) = xy + \ln(x) + \ln(y)$

$$F_x = y + \frac{1}{x} = \frac{yx+1}{x}$$

$$\Rightarrow y' = -\frac{\frac{yx+1}{x}}{\frac{yx+1}{y}} = -\frac{(yx+1)y}{(yx+1)x} = -\frac{y}{x}$$

$$F_y = x + \frac{1}{y} = \frac{xy+1}{y}$$

$$y'' = (y')' = \frac{(-y)'(x) - (-y)(x)'}{x^2} = \frac{\left(\frac{y}{x}\right)x - (-y)(1)}{x^2} = \frac{2y}{x^2}$$

b) $F(x,y) = e^{x-y} - x - y$

$$\left.\begin{aligned} F_x &= e^{x-y} - 1 \\ F_y &= -e^{x-y} - 1 \end{aligned}\right\} \Rightarrow y' = \frac{e^{x-y}-1}{e^{x-y}+1}$$

$$y'' = \frac{(e^{x-y}-1)'(e^{x-y}+1) - (e^{x-y}-1)(e^{x-y}+1)'}{(e^{x-y}+1)^2} =$$

$$= \frac{(e^{x-y})'(e^{x-y}+1 - e^{x-y}+1)}{(e^{x-y}+1)^2} = 2\frac{(e^{x-y})'}{(e^{x-y}+1)^2} = 4\frac{e^{x-y}}{(e^{x-y}+1)^3}$$

let $f(x,y) = e^{x-y}$, so $f_x = e^{x-y}dx$ $f_y = -e^{x-y}dy$

so $f' = (e^{x-y})\underset{\overset{\shortparallel}{1}}{dx} - e^{x-y}\overset{\text{"}y'\text{"}}{dy} =$

$$= (e^{x-y})\left(1 - \frac{e^{x-y}-1}{e^{x-y}+1}\right) = \boxed{\frac{2e^{x-y}}{e^{x-y}+1}}$$

9. (HW) If $xu^2 + v = y^3$, $2yu - xv^3 = 4x$, find u_x, u_y, v_x, v_y.

Differentiate the given equations with respect to x, considerind u and v as functions of x and y. Then

$$u^2 + 2xuu_x + v_x = 0 \qquad 2yu_x - v^3 - 3xv^2\underline{v_x} = 4$$

$$\begin{cases} u_x = -\dfrac{u^2 + v_x}{2xu} \\ u_x = \dfrac{3v_x v^2 x + v^3 + 4}{2y} \end{cases} \qquad \begin{cases} v_x = -u^2 - 2xuu_x \\ v_x = -\dfrac{v^3 - 2yu_x + 4}{3xv^2} \end{cases}$$

Differentiate equations with respect to y, we have:

$$2xuu_y + v_y = 3y^2 \qquad 2u + 2yu_y - 3xv^2\underline{v_y} = 0$$

$$\begin{cases} u_y = \dfrac{3y^2 - v_y}{2xu} \\ u_y = \dfrac{3xv^2 v_y - 2u}{2y} \end{cases} \qquad \begin{cases} v_y = 3y^2 - 2xuu_y \\ v_y \quad \dfrac{2u + 2yu_y}{3xv^2} \end{cases}$$

Input

$$\left\{z = \frac{-h-u^2}{2ux},\ z = \frac{4 + v^3 + 3hv^2x}{2y},\ h = -u^2 - 2uxz,\ h = \frac{-4 - v^3 + 2yz}{3v^2x}\right\}$$

Expanded form

Step-by-step solution

$$\left\{z = -\frac{h}{2ux} - \frac{u}{2x},\ z = \frac{v^3}{2y} + \frac{3hxv^2}{2y} + \frac{2}{y},\ h = -u^2 - 2xzu,\ h = -\frac{v}{3x} + \frac{2yz}{3xv^2} - \frac{4}{3xv^2}\right\}$$

Alternate form

$$\left\{z = -\frac{u^2+h}{2ux},\ z = \frac{v^3 + 3hxv^2 + 4}{2y},\ h = -u(u + 2xz),\ h = -\frac{v^3 - 2yz + 4}{3v^2x}\right\}$$

Solutions

$$u = -i\sqrt{h},\quad hv \neq 0,\quad x = \frac{-v^3-4}{3hv^2},\quad z = 0,\quad v^3y + 4y \neq 0$$

$$u = i\sqrt{h},\quad hv \neq 0,\quad x = \frac{-v^3-4}{3hv^2},\quad z = 0,\quad v^3y + 4y \neq 0$$

$$h + u^2 \neq 0,\quad y = -\frac{ux(3hv^2x + v^3 + 4)}{h + u^2},\quad ux \neq 0,\quad z = \frac{-h-u^2}{2ux},\quad 3hv^3x + v^4 + 4v \neq 0$$

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$h = v_x \qquad z = u_x$

Thus, that's it, 'cause I've no idea that to do next.

if $F_x \neq 0 \Rightarrow y'$ and y'' exist

i.e. $y = x^2 + 2xy^2 + y^4$

then $F(x,y) = x^2 + 3 + x^2 + 2xy^2 + y^4$

$F_x = 4x + 2y^2$ $\quad F_y = +4xy + 4y^3$

$F_{xx} = 4$ $\quad F_{yy} = 4x + 12y^2$

$F_{xy} = 4y$

$\tilde{F}(x,y) = 4xy + 4y^3$

$\tilde{F}'(x,y) = \tilde{F}(x,y)_x + \tilde{F}(x,y)_y$

$\tilde{F}'(x,y) = F_{xx} + F_{xy}$

10*. (HW) Let $y = y(x)$ be a twice continuously differentiable function satisfying $F(x,y) = 0$, where $F(x,y)$ is a function having first two continuous derivatives. Prove that if $F_y \neq 0$, then

$$F_y^3 \cdot y'' = \begin{vmatrix} F_{xx} & F_{xy} & F_x \\ F_{xy} & F_{yy} & F_y \\ F_x & F_y & 0 \end{vmatrix} \begin{matrix} F_{xx} & F_{xy} \\ F_{xy} & F_{yy} \\ F_x & F_y \end{matrix}$$

$$\begin{vmatrix} F_{xx} & F_{xy} & F_x \\ F_{xy} & F_{yy} & F_y \\ F_x & F_y & 0 \end{vmatrix} = 2F_{xy}F_xF_y - F_x^2 F_{yy} - F_y^2 F_{xx}$$

$$y' = -\frac{F_x}{F_y} \quad \text{then} \quad y'' = \left(-\frac{F_x}{F_y}\right)' = -\frac{(F_x)'(F_y) + F_x(F_y)'}{F_y^2}$$

$$(F_x)' = \left(\frac{\partial F}{\partial x}\right)' = \left(\frac{\partial F}{\partial x}\right)\frac{\partial}{\partial x} + \left(\frac{\partial F}{\partial x}\right)\frac{\partial}{\partial y} = \frac{\partial^2 F}{\partial x^2}dx + \frac{\partial^2 F}{\partial x \partial y}dy = \frac{\partial^2 F}{\partial x^2}dx + \frac{\partial^2 F}{\partial x \partial x}\left(-\frac{F_x}{F_y}\right) = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial x \partial y}\left(-\frac{\partial F/\partial x}{\partial F/\partial y}\right) = F_{xx} - F_{xy}\left(\frac{F_x}{F_y}\right)$$

$$(F_y)' = \left(\frac{\partial F}{\partial y}\right)' = \left(\frac{\partial F}{\partial y}\right)\frac{\partial}{\partial x} + \left(\frac{\partial F}{\partial y}\right)\frac{\partial}{\partial y} = \left(\frac{\partial^2 F}{\partial x \partial y}\right)\overset{x'=1}{dx} + \left(\frac{\partial^2 F}{\partial y^2}\right)\overset{\text{"}y'\text{"}}{dy} = \frac{\partial^2 F}{\partial x \partial y} + \frac{\partial^2 F}{\partial y^2}\left(-\frac{F_x}{F_y}\right) = \frac{\partial^2 F}{\partial x \partial y} - \frac{\partial^2 F}{\partial y^2}\left(\frac{\partial F/\partial x}{\partial F/\partial x}\right) = F_{xy} - F_{yy}\left(\frac{F_x}{F_y}\right)$$

$$\text{Thus} \quad y'' = -\frac{\left(F_{xx} - F_{xy}\left(\frac{F_x}{F_y}\right)\right)F_y - \left(F_{xy} - F_{yy}\left(\frac{F_x}{F_y}\right)\right)F_x}{F_y^2} = -\frac{F_yF_{xx} - F_{xy}F_x - F_xF_{xy} + F_{yy}F_x^2/F_y}{F_y^2}$$

$$F_y^3 y'' = -F_y^2F_{xx} + F_{xy}F_xF_y + F_{xy}F_xF_y - F_x^2F_{yy} = 2F_{xy}F_xF_y - F_x^2F_{yy} - F_y^2F_{xx} = \det|\ |$$