In this HW, you can perform all arithmetic operations on matrices (e.g. multiplication, transforming into RREF, finding the inverse, etc) by a machine.

1. (1 point per item) Let $\varphi\colon [x,\, y,\, z]^{\mathrm{T}} \mapsto A \cdot [x,\, y,\, z]^{\mathrm{T}}$ be a linear operator on $\mathbb{R}^3$, then, for every $\lambda \in \operatorname{Spec}(\varphi)$, find the algebraic and geometric multiplicity of λ if

   (a) $A = \begin{bmatrix} 7 & 8 & -8 \\ -8 & -13 & 16 \\ -4 & -8 & 11 \end{bmatrix}$;

   [**hint:** see Problem 1 from Seminar 22]

   (b) $A = \begin{bmatrix} 9 & 8 & -6 \\ -11 & -13 & 13 \\ -6 & -8 & 9 \end{bmatrix}$.

a) $\begin{vmatrix} 7-x & 8 & -8 \\ -8 & -13-x & 16 \\ -4 & -8 & 11-x \end{vmatrix} = 0 \iff -9-3x+5x^2-x^3=0 \iff -(x+1)(x-3)^2=0$

then $\operatorname{Spec}_\varphi = \{-1, 3\}$; a.m $(3) = 2$; a.m$(-1) = 1$;

$\begin{bmatrix} 7+1 & 8 & -8 \\ -8 & -13+1 & 16 \\ -4 & -8 & 11+1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow E_\varphi(-1) = \left\langle \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\rangle \Rightarrow$ g.m.$(-1) = 1$

similary for (2): $E_\varphi(2) = \left\langle \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\rangle \Rightarrow$ g.m. $(2) = 2$;

$\Rightarrow$ diagonazia ble

ofc, from span it's not follow, but it's also a basises for the lambdas.

b) $\begin{vmatrix} 9-x & 8 & -6 \\ -11 & -13-x & 13 \\ -6 & -8 & 9-x \end{vmatrix} = 0 \iff -9-3x+5x^2-x^3=0 \iff (x+1)(x-3)^2 = 0$

hence a.m$(-1) = 1$; a.m.$(3) = 2$

but, if we substitute $x \in \operatorname{Spec}_\varphi \iff x \in \{-1, 3\}$ and find basis for kernels, we obtain:

$E_\varphi(-1) = \left\langle \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\rangle$ basis for $E_\varphi(-1) \Rightarrow$ g.m.$(-1) = 1$

$\Rightarrow$ not diagonazible.

$E_\varphi(3) = \left\langle \begin{bmatrix} -1 \\ 3/2 \\ 1 \end{bmatrix} \right\rangle$ basis for $E_\varphi(3) \Rightarrow$ g.m.$(3) = 1$

2. (1 point) Describe all $a, b \in \mathbb{R}$ such that the matrix $A = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \in \mathrm{Mat}_2(\mathbb{R})$ is *diagonalizable* (that is, there exists a square matrix $C \in \mathrm{Mat}_2(\mathbb{R})$ such that $A = C^{-1}DC$, where D is a diagonal matrix).

[**hint:** use a criteria for diagonalizability (see notes of Seminar 22)]

$$\begin{vmatrix} -x & a \\ b & -x \end{vmatrix} = 0 \iff x^2 - ba = 0 \underset{\text{if } ba \geq 0}{\iff} (x - \sqrt{ba})(x + \sqrt{ba}) = 0$$

$\Downarrow$ then $\mathrm{Spec}\,\varphi = \{-\sqrt{ba}; \sqrt{ba}\}$

$(\varphi : \bar{x} \to A\bar{x})$

$\dim\left(\left\langle \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\rangle\right) = 2 \Rightarrow$ if $ab > 0 \Rightarrow$ we have two eigenvalues $\Rightarrow$

$\Rightarrow$ it's diagonalizable (since $a.m(\sqrt{ab}) = 1 \wedge g.m(\sqrt{ab}) = 1$
$a.m.(-\sqrt{ab}) = 1 \wedge g.m(-\sqrt{ab}) = 1$)

since $-\sqrt{ab} \neq \sqrt{ab}$ if $ab > 0$;

if $ab < 0 \Rightarrow x^2 - ba = 0$ have no roots in $\mathbb{R} \Rightarrow \mathrm{Spec}\,\varphi = \emptyset$ in $\mathbb{R} \Rightarrow A$ isn't diagonalizable.

if $ab = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0 \Rightarrow \mathrm{Spec}\,\varphi = \{0\}$; $\Rightarrow A$ is not diagonalizable

$\Updownarrow$

$(a = 0) \vee (b = 0) \vee (a = 0 \wedge b = 0)$

if $a = 0 \wedge b = 0 \Rightarrow E_\varphi(0) = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle \Rightarrow$ matrix is diagonalizable; i.e. $I \cdot O_2^n \cdot I = A^n$

if $a = 0 \wedge b \neq 0 \Rightarrow \begin{vmatrix} -x & 0 \\ a & -x \end{vmatrix} = 0 \iff x^2 - a.0 = 0 \Rightarrow E_\varphi(0) = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle \Rightarrow \underset{=2}{a.m.}(0) \neq \underset{=1}{g.m.}(0)$

$\Downarrow$

A isn't diagonalizable

if $a \neq 0 \wedge b = 0 \Rightarrow E_\varphi(0) = \left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle \Rightarrow a.m.(0) \neq g.m.(0) \Rightarrow A$ isn't diagonalizable.

3. Let $A = [a_1, a_2, \ldots, a_n]$ be a non-zero 1-by-n matrix with real coefficients; let a linear operator $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ be defined as

$$\varphi\colon \mathbf{x} \mapsto A^{\mathrm{T}} A \mathbf{x}, \quad \text{for every } \mathbf{x} \in \mathbb{R}^n.$$

Then, following the instructions, for every $\lambda \in \mathrm{Spec}(\varphi)$, find the algebraic and geometric multiplicity of λ.

Instructions:

(a) (1 point) write down the following: Formula (21.1), the formula from Definition 21.1, Formula (18.3), Formula (21.12) and the remark which is after it, Definition 22.4; use these facts to obtain a formula for the geometric multiplicity of one of the eigenvalues of φ (the formula should include $\mathrm{rk}(A^{\mathrm{T}}A)$);

(b) (0.5 points) using Theorem 17.2 and the fact that (due to the statement) A is a non-zero matrix, find $\mathrm{rk}(A^{\mathrm{T}}A)$; now find the geometric multiplicity from Item (a);

(c) (0.5 points) note that $A^{\mathrm{T}}AA^{\mathrm{T}} = A^{\mathrm{T}} \cdot (AA^{\mathrm{T}}) = A^{\mathrm{T}} \cdot \lambda = \lambda \cdot A^{\mathrm{T}}$, where $\lambda = AA^{\mathrm{T}}$[1]; find a non-zero $\mathbf{x} \in \mathbb{R}^n$ such that $\varphi(\mathbf{x}) = \lambda \mathbf{x}$; due to Definition 22.4, this implies that $\mathrm{g.m.}(\lambda) \geqslant ?$;

(d) (1 point) note that, due the remark from Lecture 22, Definitions 22.3 and 22.4, and Statement 22.3, we have $\mathrm{g.m.}(\lambda_1) + \mathrm{g.m.}(\lambda_2) \leqslant \dim(\mathbb{V}) = n$, for every two eigenvalues λ_1 and λ_2; use this fact and the results from Items (b) and (c) to find the spectrum and all geometric multiplicities; use Statement 22.3 to find all algebraic multiplicities.

$$\varphi: \mathbb{R}^n \to \mathbb{R} \to \mathbb{R}^n \Rightarrow rk(\varphi) = 1; \Rightarrow 0 \in Spec(\varphi) \ (\text{since } \exists \bar{v} \in \mathbb{V}, \ \varphi(\bar{v}) = \bar{0}\})$$

$$\Rightarrow Im(\varphi) = \left\langle \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \right\rangle$$

$$\varphi(x) = A^T A \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A^T [a_1 x_1 + a_2 x_2 + \ldots + a_n x_n] = \begin{bmatrix} a_1 (\sum_{k=1}^n a_k x_k) \\ a_2 (\sum_{k=1}^n a_k x_k) \\ \vdots \\ a_n (\sum_{k=1}^n a_k x_k) \end{bmatrix} = \sum_{k=1}^n a_k x_k \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}$$

$$\text{if } \bar{x}_0 = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad \varphi(\bar{x}_0) = \sum_{k=1}^n a_k^2 \bar{x}_0$$

$$\text{then} \quad Spec(\varphi) = \left\{ \sum_{k=1}^n a_k^2, 0 \right\}$$

$$\text{Since } \dim(Im(\varphi)) = 1 \wedge \dim(\mathbb{V}) = n \Rightarrow \dim(ker(\varphi)) = n-1 \Rightarrow$$

$$\Rightarrow a.m.(0) = n-1 \Rightarrow a.m.\left(\sum_{k=1}^n a_k^2\right) = 1 \ \left(\text{since } \sum a.m(\lambda_i) = n \ \forall \lambda_i \in spec(\varphi)\right)$$

$$\Rightarrow \text{since } 1 \leq g.m.(\lambda_i) \leq a.m.(\lambda_i) \quad g.m.\left(\sum_{k=1}^n a_k^2\right) = 1$$

$$\Rightarrow g.m(0) = n - rk(\varphi) = 1 \ .$$

4 A linear operator $\varphi\colon \mathbb{V} \to \mathbb{V}$ is called a nilpotent linear operator of index k if $\varphi^k = \mathcal{O}$[2] and $\varphi^{k-1} \neq \mathcal{O}$.[3]

Let $\mathbb{V}$ be an n-dimensional vector space over a field $\mathbb{F}$; let φ be a nilpotent linear operator of index k. Then:

(a) (1 point) prove that $k \leqslant n$;

[**hint:** if $k = 1$ then $\varphi = \mathcal{O}$ and the statement is obvious; thus we can assume that $k \geqslant 2$; note that, since $\varphi^{k-1} \neq \mathcal{O}$, there exists $\mathbf{x} \in \mathbb{V}$ such that $\varphi^{k-1}(\mathbf{x}) \neq \mathbf{0}$; you want to prove that the k-element set $\{\mathbf{x}, \varphi(\mathbf{x}), \ldots, \varphi^{k-1}(\mathbf{x})\}$ is linearly independent (if you do not understand why you want to prove this, take a look at Theorem 14.2); one ought to use the mathematical induction to do it; to prove the base case, apply φ^{k-1} to a linear combination $\alpha_0\mathbf{x} + \alpha_1\varphi(\mathbf{x}) + \cdots + \alpha_{k-1}\varphi^{k-1}(\mathbf{x}) = \mathbf{0}$; since $\varphi^{k-1+j}(\mathbf{x}) = ?$, for every $j \in \mathbb{N}$, we have $\alpha_0 = ?$; use inductive hypothesis to prove the step]

(b) (1.5 points) find the characteristic polynomial of φ, $\chi_\varphi(x)$;

[**hint:** let $A = T(\varphi, \mathcal{A})$, where $\mathcal{A}$ is an ordered basis for $\mathbb{V}$, then $A^k = ?$; consider the equality

$$-(xI_n)^k = (A - xI_n)\big(A^{k-1} + A^{k-2}xI_n + A^{k-3}(xI_n)^2 + \cdots + A(xI_n)^{k-2} + (xI_n)^{k-1}\big)$$

where $x \in \mathbb{F}$ and I_n is the identity matrix of size n (to prove the equality, just expand the brackets; if you do not want to do this, just say that it is obvious); take the determinants of both sides of the equality; use Theorems 7.2, 22.5 and the well-known fact that $\deg(\chi_\varphi(x)) = n$ (see the remark from Lecture 22)]

(c) (1.5 points) if $k = n$, for every $\lambda \in \mathrm{Spec}(\varphi)$, find the algebraic and geometric multiplicity of λ.

[**hint:** algebraic multiplicity follows directly from Item (b); to find the geometric multiplicity, use the fact that there is $\mathbf{x} \in \mathbb{V}$ such that $\mathcal{B} = \big(\mathbf{x}, \varphi(\mathbf{x}), \ldots, \varphi^{n-1}(\mathbf{x})\big)$ is an ordered basis for $\mathbb{V}$ (since $k = n$, this fact follows directly from your proof of Item (a)); it ought to be (almost) obvious (see Definitions 16.5 and 17.1) that $\mathrm{rk}(T(\varphi, \mathcal{B})) = ?$; use Statement 22.2]

1) if $k=1 \Rightarrow \varphi = \Theta$;

if $k \geq 1$:

since $\varphi^{k-1} \neq 0$ by assumption,

$\exists \bar{v} \in \mathbb{V}$ s.t. $\varphi^{k-1}\bar{v} \neq 0$, Now we set

$\bar{e}_1 = \varphi^{k-1}\bar{v}$, $\bar{e}_2 = \varphi^{k-2}\bar{v}$, ... $e_k = \bar{v}$

Clearly $\varphi\bar{e}_1 = \varphi \circ \varphi^{-1}\bar{v} = \varphi\ \bar{v} = \bar{0}$ $(\varphi(\bar{e}_k) = \bar{e}_{k-1})$

if $c_1\bar{e}_1 + c_2\bar{e} + \ldots c_k\bar{e}_k = 0$ | apply φ^{k-1} $\Rightarrow c_k\bar{e}_k = 0 \Rightarrow c_k = 0$ (since $\bar{e}_k \neq 0$)

Similary we show that:

$c_{k-1} = c_{k-2} = \ldots c_1 = 0 \Rightarrow \{\bar{e}_1\ \bar{e}_2, \ldots \bar{e}_k\}$ are LI.

(apply φ^{n-2} on $c_1\bar{e}_1 + c_2\bar{e}_2 + \ldots c_{k-1}\bar{e}_{k-1} + \underset{\overset{\|}{0}}{c_k}\bar{e}_k = 0$)

then by Theorem 14.2 $k \leq n$. ■

2) since only 0 is in spec(φ): $\chi_\varphi(x) = x^n$, since $\deg(\chi_\varphi) = \dim(T(\varphi, A)) = n$

$\wedge$ 0 is only root of $\chi_\varphi(x)$.

$\mathrm{Spec}(\varphi) = \{0\}$. Proof: suppose $\lambda \neq 0 \wedge \lambda \in \mathrm{Spec}(\varphi)$,

then $\varphi(\bar{v}) = \lambda\bar{v}$ and $\varphi(\lambda\bar{v}) = \varphi \circ \varphi(\bar{v}) = \varphi^2(\bar{v}) = \lambda^2\bar{v}$;

then $\varphi^k(\bar{v}) = \lambda^k\bar{v}$; but $\varphi^k(\bar{v}) = \bar{0} \Rightarrow \lambda^k \underset{\overset{\neq}{\bar{0}}}{\bar{v}} = \bar{0} \Rightarrow \lambda^k = 0 \Rightarrow \perp$

3) a.m.(0) $= n$ it directly follows from $\chi_\varphi(x)$

g.m(0) $= n$ iff φ is $\mathcal{O}$, cause if g.m.(0) = a.m.(0) $= n \Rightarrow T(\varphi, A)$ is diagonazible $\Rightarrow T(\varphi, A) = C\,D^n\,C^{-1}$

where $D = \begin{bmatrix} \lambda & & & \\ & \lambda & & \\ & & \lambda & \\ & & & \ddots\ \lambda \end{bmatrix} = \mathcal{O}_n \Rightarrow \varphi = \Theta$

g.m.(0) $= n - \mathrm{rk}(\varphi - 0 \cdot I) = n - \mathrm{rk}(\varphi) = n - k$

If you have some questions, plz ask me for defense.

- Novosad ♥