3. (HW) Evaluate the integrals

(a) $\int_{-1}^{1} \frac{d}{dx}\left(\frac{1}{1+2^{1/x}}\right) dx$; (b) $\int_{1/e}^{e} |\ln x|\, dx$.

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a) $\int_{-1}^{1} \frac{d}{dx}\left(\frac{1}{1+2^{1/x}}\right) dx = \int_{-1}^{1} \frac{2^{1/x}\ln(2)}{x^2(1+2^{1/x})^2} dx = \frac{1}{3}+\frac{1}{3} = \frac{2}{3}$

since there is a essential disc. on $x=0$, we will split the integral

$$\int_{-1}^{0} \frac{\sqrt[x]{2}\,\ln(2)}{x^2(1+\sqrt[x]{2})^2} dx = \ln(2)\int_{-1}^{0} \frac{\sqrt[x]{2}\cdot dx}{x^2(1+\sqrt[x]{2})^2} = \left\{\begin{array}{l} u = \sqrt[x]{2}+1 \\ du = -\frac{\sqrt[x]{2}\,\ln(2)}{x^2} dx \end{array}\right\} = -\frac{\ln(2)}{\ln(2)}\int_{3/2}^{1} \frac{1}{u^2} du = \int_{1}^{3/2} u^{-2} du = -\frac{1}{u}\Big|_{1}^{3/2} = -\frac{2}{3}+1 = \frac{1}{3}$$

$$\int_{0}^{1} \frac{\sqrt[x]{2}\,\ln(2)}{x^2(1+\sqrt[x]{2})^2} dx = \ln(2)\int_{0}^{1} \frac{\sqrt[x]{2}\, dx}{x^2(1+\sqrt[x]{2})^2} = \left\{\begin{array}{l} u = \sqrt[x]{2}+1 \\ du = -\frac{\sqrt[x]{2}\ln(2)}{x^2} dx \end{array}\right\} \to -\frac{\ln(2)}{\ln(2)}\int \frac{1}{u^2} du = \frac{1}{u}+C \to \frac{1}{\sqrt[x]{2}+1}\Big|_{0}^{1} = \frac{1}{2+1} + \overset{\nearrow 0}{\frac{1}{2^{\infty}+1}} = \frac{1}{3}$$

Same sense we can obtain if: $f(x) = \frac{d}{dx}\left(\frac{1}{1+\sqrt[x]{2}}\right) \Rightarrow \begin{cases} F_1(x) + \lim_{x\to 0^-}(f(x)) & ; \text{ on } [-1;0] \\ F_2(x) + \lim_{x\to 0^+}(f(x)) & ; \text{ on } [0;1] \end{cases} \Rightarrow \int_{-1}^{1} f(x)dx = F_2(1) - F_1(-1)$

b) $\int_{1/e}^{e} |\ln(x)|\, dx = -\int_{1/e}^{1} \ln(x)dx + \int_{1}^{e} \ln(x)\,dx = 1 - \frac{2}{e} + 1 = 2 - \frac{2}{e}$

$$-\int_{1/e}^{1} \ln(x)\,dx = \left|\begin{array}{ll} u = \ln(x) & dw = dx \\ du = \frac{dx}{x} & w = x \end{array}\right| = -x\ln(x)\Big|_{1/e}^{1} + \int dx = -x\ln(x)+x\Big|_{1/e}^{1} = 1 - \frac{1}{e} - \frac{1}{e} = 1 - \frac{2}{e}$$

$$\int_{1}^{e} \ln(x)dx = \left|\begin{array}{ll} u = \ln(x) & dw = dx \\ du = \frac{dx}{x} & w = x \end{array}\right| = \ln(x)\,x\Big|_{1}^{e} - \int dx = \ln(x)\,x - x\Big|_{1}^{e} = e - e - 0 + 1 = 1$$

5. (HW) Using the integral $\int_0^1 \frac{dx}{1+x^2}$ prove that

$$\lim_{n\to\infty} n\left(\frac{1}{n^2+1^2} + \frac{1}{n^2+2^2} + \cdots + \frac{1}{2n^2}\right) = \frac{\pi}{4}.$$

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Consider $F(x) = \frac{1}{1+x^2}$ on $[0,1]$ and $\|P\| = \frac{1}{n}$

s.t. $0 = \frac{0}{n} < \frac{1}{n} < \ldots < \frac{n}{n} = 1 \Rightarrow \varepsilon_i = \frac{i}{n}$

$$F(\varepsilon_i) = \frac{1}{1+\left(\frac{i}{n}\right)^2} = \frac{n^2}{n^2+i^2} \Rightarrow \sum_{i=1}^{n} F(\varepsilon_i)\frac{1}{n} = \sum_{i=1}^{n} \frac{n}{n^2+i^2} \Rightarrow \int_0^1 \frac{dx}{1+x^2} = \lim_{n\to\infty}\left(\sum_{i=1}^{n} n\left(\frac{1}{n^2+i^2}\right)\right)$$

$$\int_0^1 \frac{1}{x^2+1} dx = \arctan(x)\Big|_0^1 = \frac{\pi}{4} \blacksquare$$

6. (HW) Using definite integral, find

$$\lim_{n\to\infty} \frac{1}{n}\left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \cdots + \sin\frac{(n-1)\pi}{n}\right).$$

$$\lim_{n\to\infty}\left(\frac{1}{n}\left(\sin\left(\frac{\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right) + \ldots + \sin\left(\frac{(n-1)\pi}{n}\right)\right)\right) =$$

$$= \lim_{n\to\infty}\left(\sum_{i=1}^{n-1} \underbrace{\sin\left(\frac{i\pi}{n}\right)}_{f(c_i)} \underbrace{\frac{1}{n}}_{\Delta x}\right) \Rightarrow \int_0^1 \sin(x\pi)\,dx = -\frac{1}{\pi}\cos(\pi x)\Big|_0^1 = \frac{2}{\pi}$$

8. (HW) Prove the following inequalities:

(a) $\frac{2}{3} < \int_0^1 \sqrt{x}\, e^x \, dx < e - 1$; (b) $\ln 2 < \int_0^{3/4} \frac{2^x}{\sqrt{1+x^2}} \, dx < \frac{1}{\ln 2}$.

a) $0 \le \sqrt{x} \le 1$; $1 \le e^x \le e \quad \forall x \in [0,1]$

$\Rightarrow \sqrt{x} \le \sqrt{x}\, e^x \le e^x \quad \forall x \in [0,1]$

$\Rightarrow \int_0^1 \sqrt{x}\, dx < \int_0^1 \sqrt{x}\, e^x dx < \int_0^1 e^x \Rightarrow \frac{2}{3} < \int_0^1 \sqrt{x}\, e^x dx < e-1$

$\int_0^1 \sqrt{x}\, dx = \frac{2}{3} x^{3/2} \Big|_0^1 = \frac{2}{3}$; $\int_0^1 e^x = e^x \Big|_0^1 = e-1$

b) $1 \le 2^x \le \sqrt[4]{8}$; $1 \le \sqrt{1+x^2} \le \frac{5}{4} \quad \forall x \in [0, 3/4]$

$\frac{1}{\sqrt{1+x^2}} \le \frac{2^x}{\sqrt{1+x^2}} \le 2^x \quad \forall x \in [0,1] \Rightarrow \int_0^1 \frac{1}{\sqrt{1+x^2}} dx < \int_0^1 \frac{2^x}{\sqrt{1+x^2}} dx < \int_0^1 2^x dx$

$\int_0^1 \frac{1}{\sqrt{1+x^2}} dx = \ln(|x+\sqrt{x^2+1}|) \Big|_0^1 = \ln(1+\sqrt{2})$; $\int_0^1 2^x dx = \frac{2^x}{\ln(2)} \Big|_0^1 = \frac{1}{\ln(2)}$

$\Rightarrow \ln(1+\sqrt{2}) < \int_0^{3/4} \frac{2^x}{\sqrt{1+x^2}} dx < \frac{1}{\ln(2)} \Rightarrow \ln(2) < \int_0^{3/4} \frac{2^x}{\sqrt{1+x^2}} dx < \frac{1}{\ln(2)}$

since $\ln(2) < \ln(1+\sqrt{2}) < \int_0^{3/4} f(x)\, dx \Rightarrow \ln(2) < \int_0^{3/4} f(x)\, dx$. $\blacksquare$

prove that $\int_0^{\pi/2} \left(\frac{\sin nx}{\sin x}\right)^2 dx = \frac{\pi n}{2}, \qquad n \in \mathbb{N}.$

First approach :

let $I_n = \int_0^{\pi/2} \frac{\sin^2(nx)}{\sin^2(x)} dx \quad I_{n-1} = \int_0^{\pi/2} \frac{\sin^2((n-1)x)}{\sin^2(x)} dx \quad I_{n+1} = \int_0^{\pi/2} \frac{\sin^2((n+1)x)}{\sin^2(x)} dx$

then $I_{n+1} + I_{n-1} - 2I_n = \int_0^{\pi/2} \frac{\sin^2((n-1)x)}{\sin^2(x)} dx + \int_0^{\pi/2} \frac{\sin^2((n-1)x)}{\sin^2(x)} dx - 2\int_0^{\pi/2} \frac{\sin^2(nx)}{\sin^2(x)} dx =$

$$= \int_0^{\pi/2} \frac{\sin^2((n+1)x) - \sin^2(nx) + \sin^2((n-1)x) - \sin^2(nx)}{\sin^2(x)} dx =$$

since $\sin^2(xn+x) - \sin^2(nx) = \sin\big((2n+1)x\big)\sin(x) \ \wedge \ \sin^2(xn-x) - \sin^2(nx) = \sin\big((2n-1)x\big)\sin(x)$

$$= \int_0^{\pi/2} \frac{\sin\big((2n+1)x\big)\sin(x) - \sin\big((2n-1)x\big)\sin(x)}{\sin^2(x)} dx = \int_0^{\pi/2} \frac{\sin(2xn+x) - \sin(2xn-x)}{\sin(x)} dx =$$

$$= \int_0^{\pi/2} \frac{2\sin(x)\cos(2nx)}{\sin(x)} dx = 2\int_0^{\pi/2} \cos(2xn)\, dx = \frac{2}{2n}\sin(2xn)\Big|_0^{\pi/2} = \frac{\sin(\pi n)}{n} - \frac{\sin(0)}{n} = \frac{\sin(\pi n)}{n}$$

$$\Rightarrow I_{n+1} + I_{n-1} - 2I_n = \frac{\overset{=0 \ \forall n \in \mathbb{Z}}{\sin(\pi n)}}{n} = 0 \Rightarrow I_{n+1} + I_{n-1} = 2I_n \Rightarrow I_1, I_2, I_3, \ldots \text{ are in Arithmetic progression;}$$

$$\Rightarrow \left\{ \begin{array}{l} \text{since for } n=1: \int_0^{\pi/2} \frac{\sin^2(x)}{\sin^2(x)} dx = \frac{\pi}{2} \Rightarrow I_1 = \pi/2 \\ \text{since for } n=2: \int_0^{\pi/2} \frac{\sin^2(2x)}{\sin^2(x)} dx = \pi \Rightarrow I_2 = \pi; \end{array} \right\} \Rightarrow I_{n+1} = 2I_n - I_{n-1} \ (\text{i.e. } I_3 = 2\pi - \tfrac{\pi}{2} = 3\pi/2;\ I_4 = 2\pi \text{ and so on})$$

$$\Rightarrow I_n = \frac{\pi n}{2} \quad \forall n \in \mathbb{N} \ \blacksquare$$

draft:

$$\int_0^{\pi/2} \frac{\sin^2(2x)}{\sin(x)} dx = \int_0^{\pi/2} \frac{4\sin^2(x)\cos^2(x)}{\sin^2(x)} dx = 4\int_0^{\pi/2} \cos^2(x) dx = 2\int_0^{\pi/2} [1 + \cos(2x)] dx = 2x + \frac{\sin(2x)}{4}\Big|_0^{\pi/2} = \pi + 0 - 0 - 0 = \pi$$