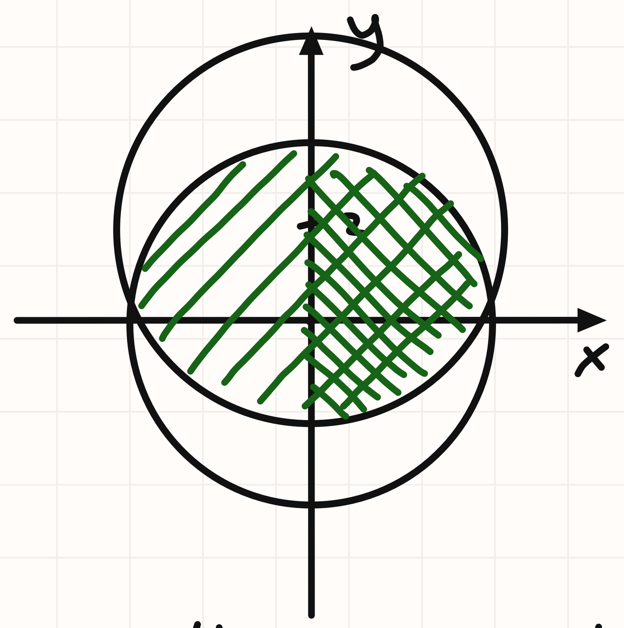
2. (HW) Find the area of the region bounded by the given curves:

(a) $y = 3x^2 - 4x + 8,\ y = 0,\ x = -1,\ x = 2;$ (b) $x^2 + y^2 = 4,\ x^2 + (y-2)^2 = 8.$

a) $y = 3x^2 - 4x + 8 \quad y = 0 \quad x = -1 \quad x = 2$

$$\int_{-1}^{2} (3x^2 - 4x + 8)\,dx = x^3 - 2x^2 + 8x \Big|_{-1}^{2} = 16 + 1 + 2 + 8 = 27$$

b) $x^2 + y^2 = 4$; $x^2 + (y-2)^2 = 8$



$y = \pm\sqrt{4 - x^2} \qquad y = \pm\sqrt{8 - x^2} + 2$

it's obvious that function are symetrical with respect to the y-axis

then we can find only area to the right of y-axis and double it.

let's find their intersection: $\sqrt{4-x^2} = -\sqrt{8-x^2} + 2 \iff \begin{cases} x = 2 \\ x = -2 \end{cases}$

$$\int_0^2 \sqrt{4-x^2}\,dx + \int_0^2 \left[\sqrt{8-x^2} - 2\right]dx = 2(\pi + \pi + 2 - 4) = 4\pi - 4$$ Answer for intersection area.

$$\int_0^2 \sqrt{4-x^2}\,dx \to \begin{Bmatrix} x = 2\sin(\theta) \\ d\theta = 2\cos(\theta) \end{Bmatrix} \to \int 2\cos(\theta)\sqrt{4 - 4\sin^2(\theta)}\,d\theta = 4\int \cos^2(\theta)\,d\theta = 2\int 1 + \cos(2\theta) = 2\theta + \sin(2\theta) + C \to$$

$$= 2\theta + 2\sin(\theta)\cos(\theta) + C \to \int_0^2 \sqrt{4-x^2}\,dx = 2\arcsin\left(\frac{x}{2}\right) + 2x\sqrt{4-x^2}\,\Big|_0^2 = 2\arcsin(1) + 4\sqrt{4-4} = 2\cdot\frac{\pi}{2} = \pi.$$

$$\int_0^2 \sqrt{8-x^2}\,dx \to \begin{Bmatrix} x = \sqrt{8}\sin(\theta) \\ dx = \sqrt{8}\cos(\theta) \end{Bmatrix} \to \int \sqrt{8}\cos(\theta)\sqrt{8 - 8\sin^2(\theta)}\,d\theta = \int \sqrt{8}\cos(\theta)\sqrt{8}\cos(\theta)\,d\theta = 8\int \cos^2(\theta)\,d\theta =$$

$$= 4\theta + 4\sin(\theta)\cos(\theta) + C \to \int_0^2 \sqrt{8-x^2}\,dx = 4\arcsin\left(\frac{x}{\sqrt{8}}\right) + \frac{1}{2}x\sqrt{8-x^2}\,\Big|_0^2 = 4\arcsin\left(\frac{2}{\sqrt{8}}\right) + 2 = 4\arcsin\left(\frac{\sqrt{2}}{2}\right) + 2 = \pi + 2$$

$$\int_0^2 2\,dx = 2x\Big|_0^2 = 4$$

b) But I'm pretty sure now, what we need to find are benice large circle and above small one.

then the solution for that:

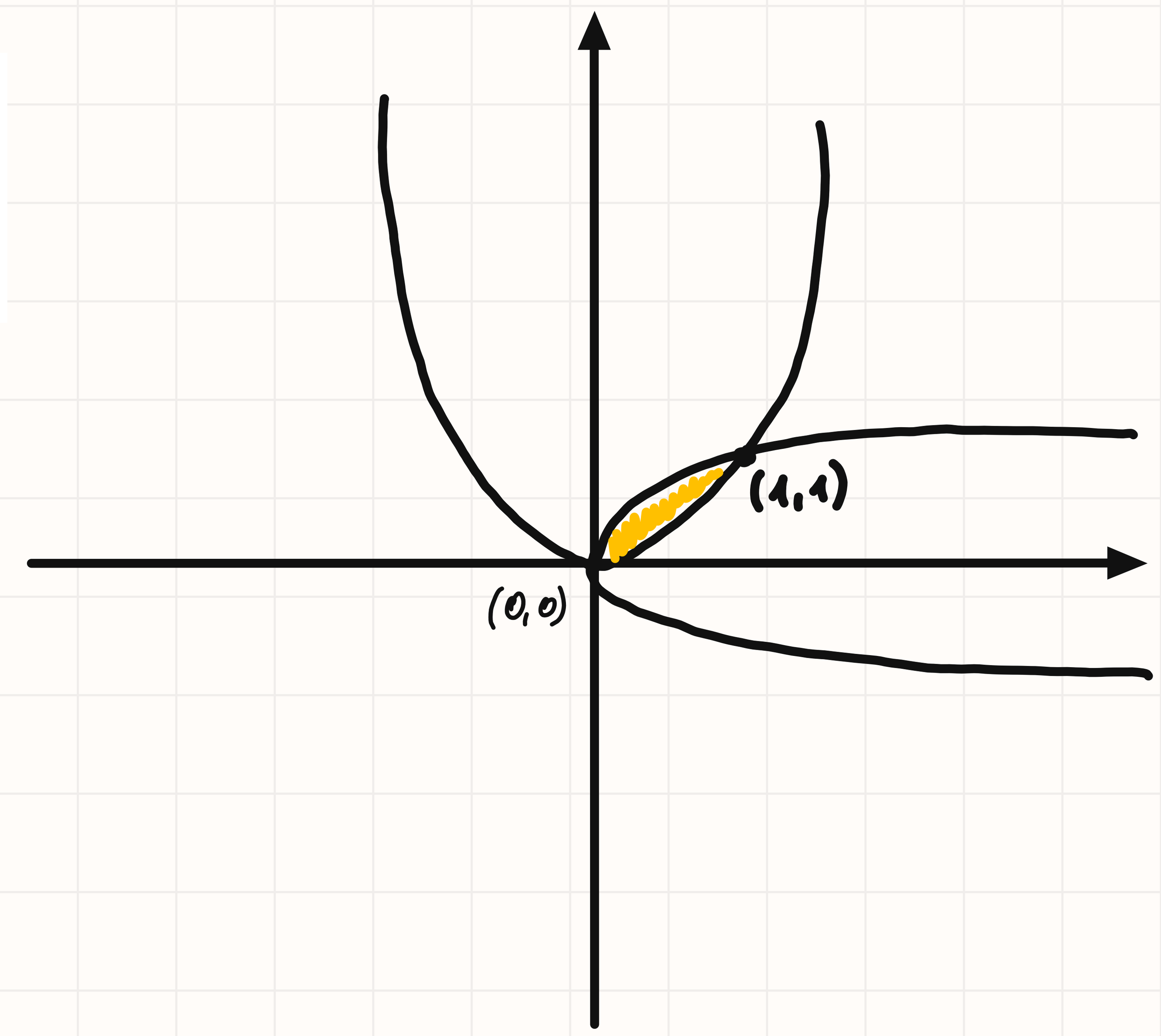
$$\int_{-2}^{2} \left(\sqrt{4-x^2} - \sqrt{8-x^2} + 2\right)dx = 2\int_0^2 \sqrt{4-x^2} - 2\int_0^2 \sqrt{8-x^2}\,dx + 4\int dx = 2\pi - 2\pi - 4 + 8 = 4$$ — I think that's right answer!

(solution is above)

5. (HW) Find the volume of the solid obtained by revolving the region bounded by

$$\text{(a) } y = x^2,\ y^2 = x, \qquad \text{(b) } x^2 + \frac{y^2}{9} = 1$$

about the x-axis.



a) $y = x^2$; $y^2 = x$

$\Rightarrow y = x^2 \wedge y = \sqrt{x} \wedge y = -\sqrt{x}$

intersections on $x=0 \wedge x=1$

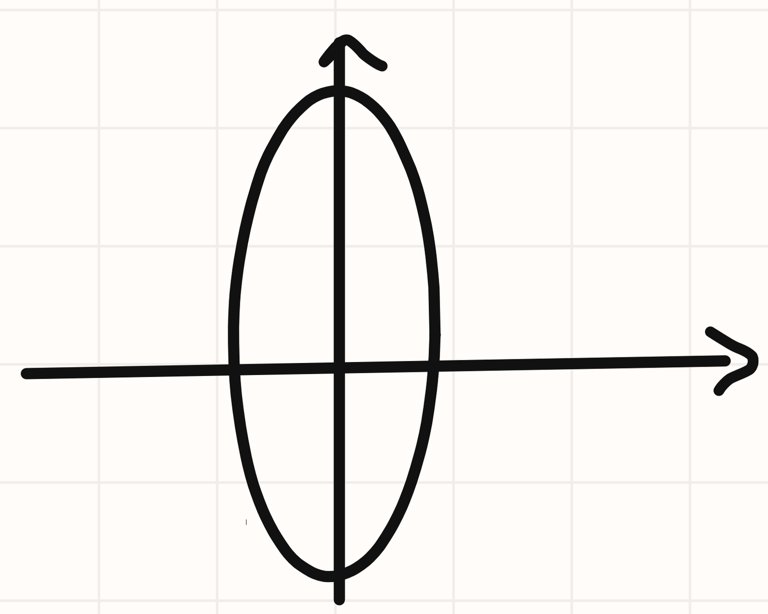
then using disk method (Washer):

$$V = \pi \int_0^1 [x - x^4]\, dx = \pi \left(\frac{x^2}{2} - \frac{x^5}{5} \right) \Big|_0^1 = \pi \left(\frac{1}{2} - \frac{1}{5} \right) = \frac{3\pi}{10}$$

Using shell method

$$V = 2\pi \int_0^1 y(\sqrt{y} - y^2)\, dy = 2\pi \left(\frac{2}{5} y^{5/2} - \frac{1}{4} y^4 \right) \Big|_0^1 = 2\pi \left(\frac{2}{5} - \frac{1}{4} \right) = \frac{3\pi}{10}$$

b) $x^2 + \frac{y^2}{9} = 1$

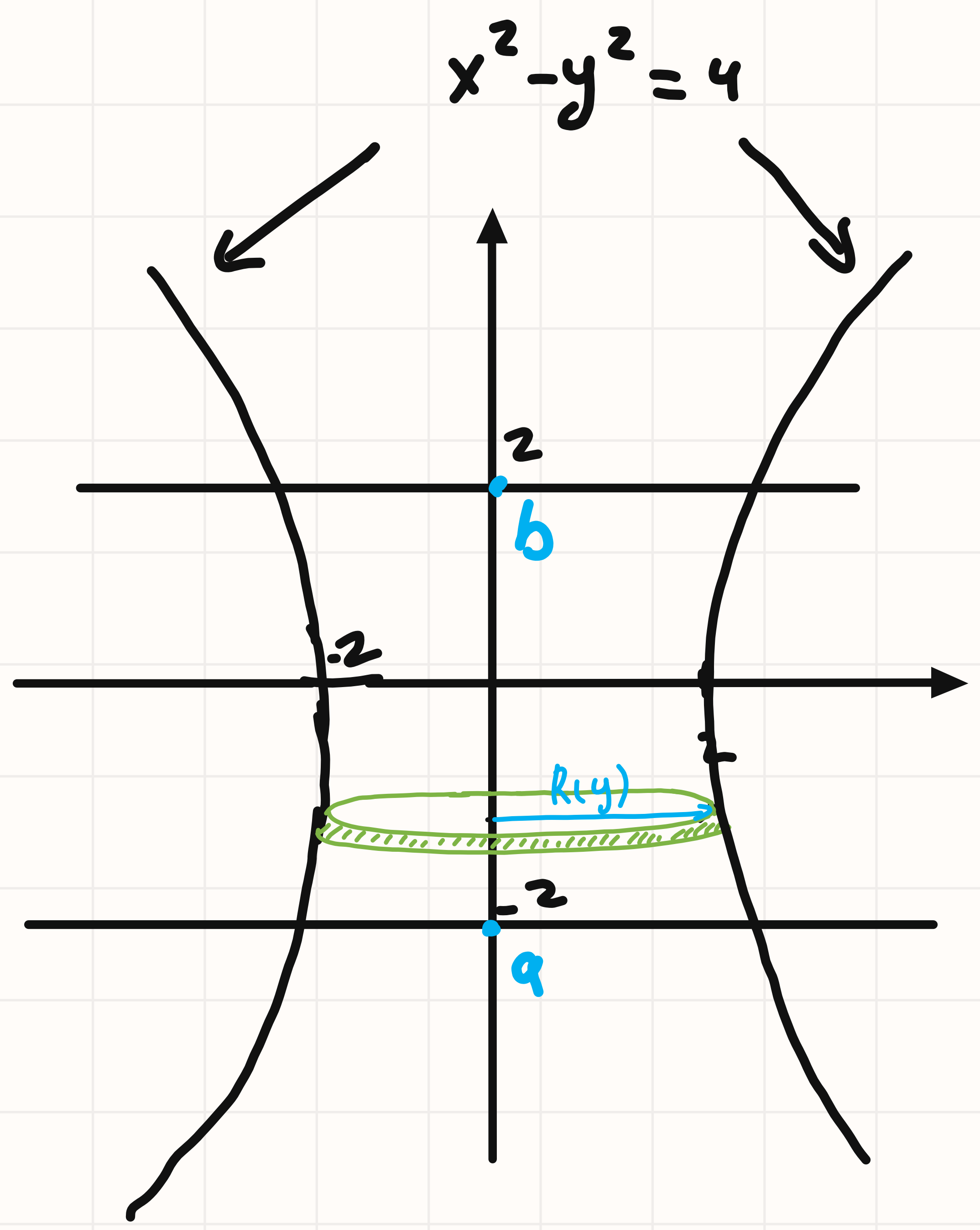


since it's absolutly simetry with respect to the origin, we will consider right part.

$$V = 2\pi \int_0^1 9(1 - x^2)\, dx = 18\pi \left(x - \frac{x^3}{3} \right) \Big|_0^1 = 18\pi \left(1 - \frac{1}{3} \right) = \frac{18 \cdot 2 \cdot \pi}{3} = 12\pi$$

6. (HW) Find the volume of the solid obtained by revolving the region bounded by $x^2 - y^2 = 4$, $y = 2$, $y = -2$, about the y-axis.

$x^2 - y^2 = 4$ from $y = -2$ to $y = 2$



$R(y) = \pm\sqrt{4+y^2}$

$$V = \pi \int_{-2}^{2} R^2(y)\,dy = \pi \int_{-2}^{2} |4+y^2|\,dy = 2\pi \int_0^2 (4+y^2)\,dy = 2\pi \left(4y + \frac{y^3}{3}\right)\Bigg|_0^2 = 2\pi\left(8 + \frac{8}{3}\right) = \frac{64\pi}{3}$$

7. (HW) Find the volume of the solid obtained by revolving the region within the parabola $x = 9 - y^2$ and between $y = x - 7$ and the y-axis, about the y-axis.

intersection points (a and b):

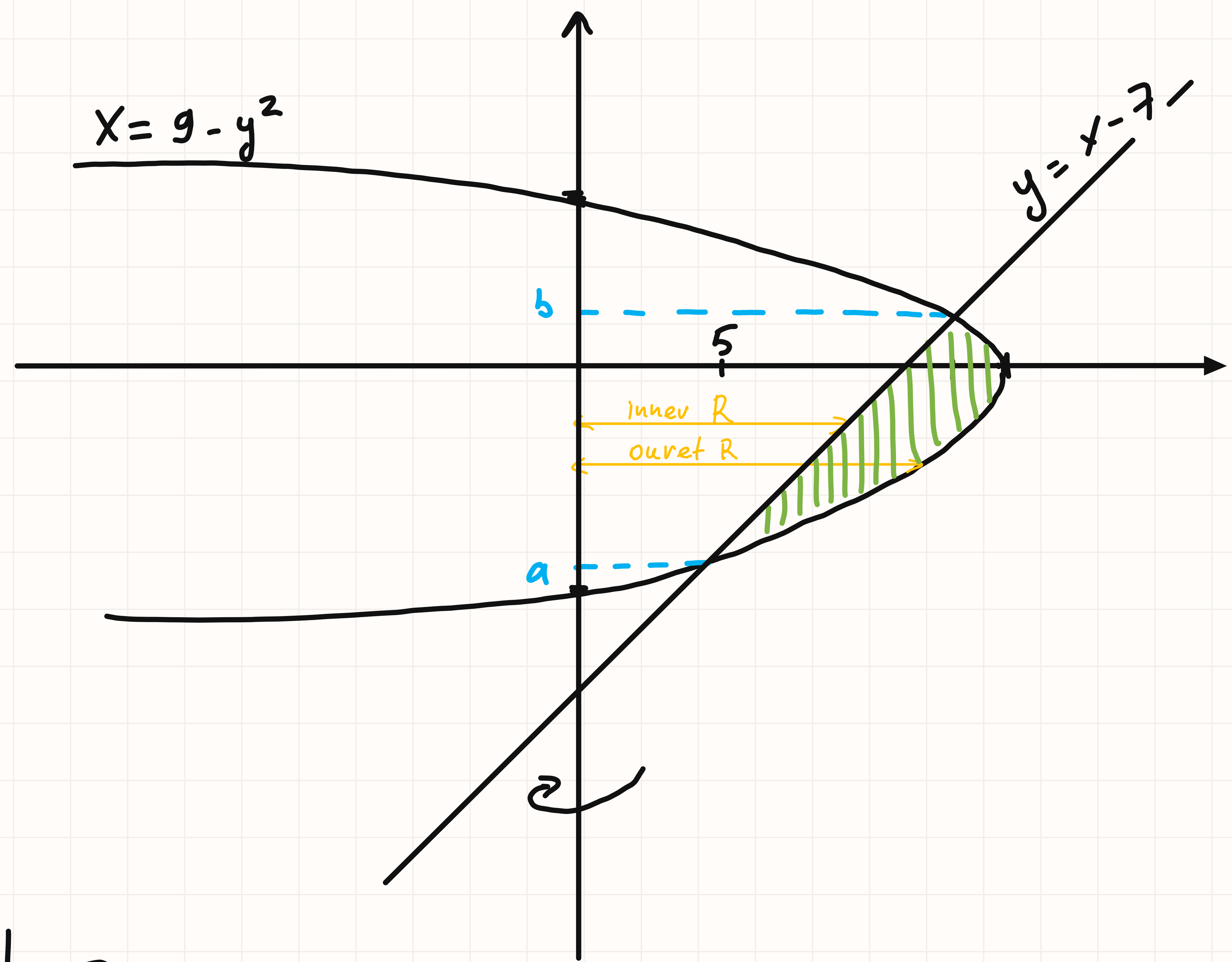
$$-y^2 = x - 9 \qquad R_{out}(y) = 9 - y^2$$

$$y^2 = 9 - x \qquad R_{in}(y) = x = y + 7$$

$$y = \pm\sqrt{9-x} \Rightarrow a: -\sqrt{9-x} = x - 7$$

$$\Downarrow$$

$$x = 5 \Rightarrow a = -2$$



$$b: \sqrt{9-x} = x - 7 \Rightarrow x = 8 \Rightarrow b = 1$$

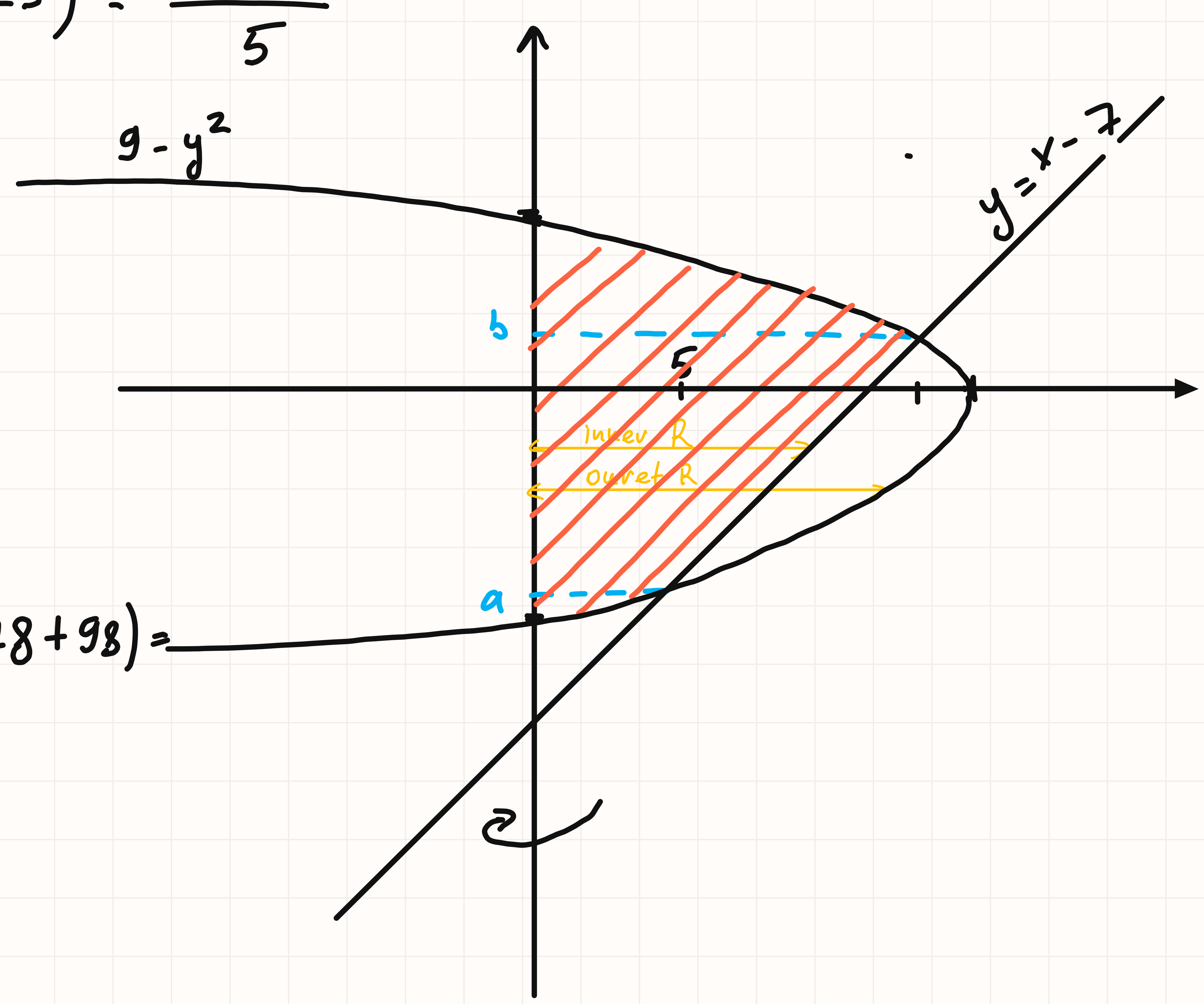
$$V = \pi \int_{-2}^{1} \left(\underbrace{(9-y^2)^2}_{\text{out}} - \underbrace{(y+7)^2}_{\text{in}} \right) dy \Rightarrow$$

$$V = \pi \int_{-2}^{1} [81 - 18y^2 + y^4 - y^2 - 14y - 49]\, dy =$$

$$= \pi \left(81y - 6y^3 + \frac{1}{5}y^5 - \frac{y^3}{3} - 7y^2 - 49y \right) \Big|_{-2}^{1} = \pi \left(32y - \frac{19}{3}y^3 + \frac{1}{5}y^5 - 7y^2 \right) \Big|_{-2}^{1} =$$

$$= \pi \left(32 - \frac{19}{3} + \frac{1}{5} - 7 + 64 + \frac{19 \cdot 8}{3} + \frac{32}{5} + 28 \right) = \frac{333\pi}{5}$$

Just in case:



$$V = \pi \int_{-2}^{1} (y+7)^2\, dy = \pi \int_{-2}^{1} [y^2 + 14y + 49]\, dy =$$

$$= \pi \left(\frac{1}{3}y^3 + 7y^2 + 49y \right) \Big|_{-2}^{1} = \pi \left(1/3 + 7 + 49 + \frac{8}{3} - 28 + 98 \right) =$$

$$= \underline{\underline{125\pi}}$$

In fact I think that's an answer for the task!

8*. Find the integral

$$J_{\alpha,n} = \int_0^1 x^\alpha \ln^n x \, dx, \qquad \alpha > 0, \quad n \in \mathbb{N}.$$

$$\int_0^1 x^\alpha \ln^n(x) dx = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n}\right) = \lim_{n\to\infty}\left(\sum_{i=1}^{n}\left(\frac{i}{n}\right)^\alpha \ln^n\left(\frac{i}{n}\right) dx\right) =$$

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$$= \lim_{n\to\infty}\left(\sum_{i=1}^{n}\left(\frac{i}{n}\right)^\alpha \ln^n\left(\frac{i}{n}\right) dx\right) =$$

since $\ln(x) < 0 \quad \forall x \in (0,1]$

$$J_{\alpha,n} = \int_0^1 x^\alpha (-\ln(x))^n dx = \left\{\begin{matrix} u = (-\ln(x))^n & dw = x^\alpha dx \\ du = \frac{-n(\ln(x))^{n-1}}{x} dx & w = \frac{x^{\alpha+1}}{\alpha+1}\end{matrix}\right\} =$$

$$= \left.\frac{(x^{\alpha+1})(\ln(x))^n(-1)^n}{\alpha+1}\right|_0^1 - \frac{n}{\alpha+1}\int_0^1 \frac{x^{\alpha+1}\cdot\ln(x)^{n-1}}{x} dx =$$

$$= (-1)^n\frac{(x^{\alpha+1})\ln^n(x)}{\alpha+1} - \frac{n}{\alpha+1}\int_0^1 x^\alpha \ln(x)^{n-1} dx$$

(still minus, 'cause $-n$ vanish - from formula for IBP, but $(\ln(x))^{n-1}$ goes with minus)

$$\Rightarrow J_{\alpha,n} = (-1)^n\frac{(x^{\alpha+1})\ln^n(x)}{\alpha+1} + \frac{n}{\alpha+1}J_{\alpha\, n-1} \Rightarrow$$

$$\Rightarrow J_{\alpha,n} = \frac{-n}{\alpha+1}J_{\alpha\, n-1} \quad \Big| \text{since} \quad (-1)^n\left.\frac{x^{\alpha+1}\ln^n(x)}{\alpha+1}\right|_0^1 = (-1)^n\frac{(1)^{\alpha+1}\overbrace{\ln(1)^n}^{=0}}{\alpha+1} - (-1)^n\frac{\overbrace{(0)^{\alpha+1}\ln(0)^n}^{=0}}{\alpha+1} = 0$$

$$\Rightarrow J_{\alpha,n} = (-1)^n\frac{n!}{(\alpha+1)^n}J_{\alpha,0} \Rightarrow J_{\alpha\, n} = (-1)^n\frac{n!}{(\alpha+1)^{n+1}} \quad //$$

$$\text{since} \quad J_{\alpha,0} = \int_0^1 x^\alpha dx = \left.\frac{x^{\alpha+1}}{\alpha+1}\right|_0^1 = \frac{1}{\alpha+1} - \frac{0}{\alpha+1} = \frac{1}{\alpha+1}$$

thx for checking, Nastia ♥ - Novosad