1. (0.5 point per item) Which of the following functions are bilinear forms and which are not (you need to justify your answer):

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+ (a) $f_1(z_1, z_2) = \mathrm{Im}(z_1 \cdot \bar{z}_2)$[1], where $z_1, z_2 \in \mathbb{V}$, and $\mathbb{V}$ is the 2-dimensional vector space of complex numbers over the field of reals;

− (b) $f_2(A, B) = \mathrm{tr}(A + B)$, where $A, B \in \mathrm{Mat}_n(\mathbb{R})$, and $\mathbb{V} = \mathrm{Mat}_n(\mathbb{R})$ is the vector space of all square matrices of size n over the field of reals;

+ (c) $f_3(A, B) = [AB](i, j)$[2], where $A, B \in \mathrm{Mat}_n(\mathbb{R})$, and $\mathbb{V} = \mathrm{Mat}_n(\mathbb{R})$ is the vector space of all square matrices of size n over the field of reals;

+ (d) $f_4(f, g) = \int\limits_a^b f(x)g(x)e^{x^2}\,\mathrm{dx}$, where $f, g \in C([a; b])$, and $\mathbb{V} = C([a; b])$ is the vector space of all continuous functions on the interval $[a; b]$ over the field of reals.

a) Suppose $z_1\ z_2\ z_3 \in \mathbb{V} \wedge a \in \mathbb{R}$

1) $f_1(z_1 + z_2, z_3) = \mathrm{Im}((z_1+z_2)\cdot\bar{z}_3) = \mathrm{Im}(z_1\cdot\bar{z}_3 + z_2\cdot\bar{z}_3) =$
$= \mathrm{Im}(z_1\cdot\bar{z}_3) + \mathrm{Im}(z_1\cdot\bar{z}_3) = f_1(z_1, z_3) + f(z_2, z_3)$

2) $f_1(a z_1; z_2) = \mathrm{Im}(a z_1\cdot\bar{z}_2) = a\,\mathrm{Im}(z_1\cdot\bar{z}_2) = a f(z_1 z_2)$

3) $f_1(z_1, z_2+z_3) = \mathrm{Im}(z_1\cdot(\overline{z_2+z_3})) = \mathrm{Im}(z_1\cdot(\bar{z}_2+\bar{z}_3)) =$
$= \mathrm{Im}(z_1\cdot\bar{z}_2 + z_1\cdot\bar{z}_3) = \mathrm{Im}(z_1\cdot\bar{z}_2) + \mathrm{Im}(z_1\cdot\bar{z}_3) = f_1(z_1 z_2) + f(z, z_3)$

4) $f_1(z_1 a z_2) = \mathrm{Im}(z_1 a\cdot\bar{z}_2) = a\,\mathrm{Im}(z_1\cdot\bar{z}_2) = a f_1(z_1 z_2)$

Hence f_1 is a bilinear map.

b) Suppose $A, B, C \in \mathrm{Mat}_n(\mathbb{R})$ and $a \in \mathbb{R}$

$f_2(A+B, C) = tr(A+B+C) = tr(A+B) + tr(C) \neq tr(A+C) + tr(B+C)$

Hence f_2 isn't a bilinear map.

c) Suppose $A, B, C \in \mathrm{Mat}_n(\mathbb{R})$ and $a \in \mathbb{R}$

1) $f_3(A+B, C) = [(A+B)C](i,j) = [AC + BC](i,j) =$
$= [AC](i,j) + [BC](i,j) = f_3(A, C) + f_3(B, C)$

2) $f_3(aA, B) = [aAB](i j) = a[AB](i j) = a f_3(A,B)$

3) $f_3(A, B+C) = [A(B+C)](i j) = [AB + AC](i j) =$
$= [AB](i j) + [AC](i j) = f_3(A,B) + f_3(A,C)$

4) $f_3(A, aB) = [A, aB](i j) = a[A, B](i j) = a f_3(A,B)$

Thus f_3 is a bilinear map.

d) let $f(x), g(x), k(x) \in C([a,b])$, $a \in \mathbb{R}$ and $l(x) = e^{x^2}$

1) $f_4(f+k, g) = \int_a^b (f(x) + k(x))g(x)\,l(x)\,dx =$
$\int_a^b [f(x)g(x)\,l(x) + k(x)\,g(x)\,l(x)]\,dx =$
$= \int_a^b f(x)g(x)\,l(x)\,dx + \int_a^b k(x)g(x)\,l(x)\,dx = f_4(f,g) + f_4(k,g)$

2) $f_4(af, g) = \int_a^b a f(x)\,g(x)\,l(x)\,dx = a\int_a^b f(x)g(x)\,l(x)\,dx = a f_4(f,g)$

3) $f_4(f, g+k) = \int_a^b f(x)(g(x)+k(x))\,l(x)\,dx = \int_a^b [f(x)g(x)\,l(x) + f(x)\,k(x)\,l(x)]\,dx =$
$= \int_a^b f(x)\,g(x)\,l(x)\,dx + \int_a^b f(x)\,k(x)\,l(x)\,dx = f_4(f,g) + f_4(f,k)$

4) $f_4(f, ag) = \int_a^b f(x)\,a\cdot g(x)\,l(x)\,dx = a\int_a^b f(x)\,g(x)\,l(x)\,dx = a f_4(f,g)$

So f_4 is bilinear map.

2. For a given bilinear form β on a vector space $\mathbb{V}$:

1) (1 point per item) find the coordinate matrix of a bilinear form β with respect to a given ordered basis $\mathcal{A}$ (see Definition 25.2);

2) (1 point per item) using Formula 25.5, for another given ordered basis $\mathcal{A}'$ of $\mathbb{V}$, find the coordinate matrix of β with respect to $\mathcal{A}'$.

(a) $\mathbb{V} = \mathbb{R}^3$; $\beta(\mathbf{x},\mathbf{y}) = 2 \cdot x_1y_1 - x_2y_1 + 3 \cdot x_2y_3 + 7 \cdot x_3y_1$, where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, $[\mathbf{x}]_{\mathcal{A}} = [x_1, x_2, x_3]^{\mathrm{T}}$, $[\mathbf{y}]_{\mathcal{A}} = [y_1, y_2, y_3]^{\mathrm{T}}$; $\mathcal{A} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$; $\mathcal{A}' = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ and $\mathbf{f}_1 = 2\mathbf{e}_1 + \mathbf{e}_3$, $\mathbf{f}_2 = -\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$, $\mathbf{f}_3 = 3\mathbf{e}_2 + \mathbf{e}_3$;

(b) $\mathbb{V} = \mathbb{R}[x;2] = \{ax^2 + bx + c \,|\, a, b, c \in \mathbb{R}\}$; $\beta(p,q) = \dfrac{\mathrm{d}}{\mathrm{d}x}(p \cdot q)(-1)$, where $p, q \in \mathbb{R}[x;2]$; $\mathcal{A} = (x+1, x, x^2+x-2)$; $\mathcal{A}' = (-1, x+1, x^2-2)$;

(c) $\mathbb{V}$ is the vector space of all symmetric matrices of size 2 over the field of reals; $\beta(A,B) = \mathrm{tr}(A^{\mathrm{T}}MB)$, where $A, B \in \mathbb{V}$, and $M = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \in \mathrm{Mat}_2(\mathbb{R})$ is a fixed matrix;

$$\mathcal{A} = \left(\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right);\quad \mathcal{A}' = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right).$$

1) $[H(\beta,A)](i;j) \overset{def}{=} \beta(\bar{e}_i;\bar{e}_j)$, where $\bar{e}_k$ is k-th vector in A $\forall i\ \forall j \in [n]$ (n is $\dim(A)$)

2) $H(\beta,A') = C(A,A')^T C(\beta,A)\, C(A,A')$, where $C(A,A')$ is the change of basis from A to A'.

a) $$[x_1\ x_2\ x_3]\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 2x_1y_1 - x_2y_1 + 3x_2y_3 + 7x_3y_1$$

$$\Rightarrow \begin{cases} a_1 = 2 \\ a_4 = -1 \\ a_6 = 3 \\ a_7 = 7 \end{cases} \Rightarrow H(\beta,A) = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 0 & 3 \\ 7 & 0 & 0 \end{bmatrix}$$

Since $f_1\ f_2\ f_3$ are already represented as L.C. of vectors from A.

$$\Rightarrow C(A,A') = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}, \text{ so } \underbrace{\begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}}_{C(A,A')^T}\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ -1 & 0 & 3 \\ 7 & 0 & 0 \end{bmatrix}}_{H(\beta,A)}\underbrace{\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}}_{C(A,A')} = H(\beta,A')$$

b) $[H(\beta,A)](1,1) = \frac{d}{dx}\left((x+1)^2\right)$ for $x=-1$ $= 2(x)+2$ for $x=-1 \Rightarrow 0$

$[H(\beta,A)](1,2) = \frac{d}{dx}\left((x+1)x\right)$ for $x=-1$ $= 2x+1$ for $x=-1 \Rightarrow -1$

similiarly:

$[H(\beta,A)](1,3) = -2$ $\qquad [H(\beta,A)](2,3) = -1$

$[H(\beta,A)](3,1) = [H(\beta,A)](1,2) = -1$ $\qquad [H(\beta,A)](3,1) = [H(\beta,A)](1,3) = -2$

$[H(\beta,A)](2,2) = -2$ $\qquad [H(\beta,A)](3,2) = [H(\beta,A)](2,3) = -1$

$[H(\beta,A)](3,3) = 4$

Hence $H(\beta,A) = \begin{bmatrix} 0 & -1 & -2 \\ -1 & -2 & -1 \\ -2 & -1 & 4 \end{bmatrix}$ $\uparrow$ since function mult. is commutative.

Now, let's associate A with $\begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and A' with $\begin{bmatrix} -1 & 1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$C(A,A')^{(1)}: \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$C(A,A')^{(2)}: \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C(A,A')^{(3)}: \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$C(A,A') = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence $$H(\beta,A') = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & -2 \\ -1 & -2 & -1 \\ -2 & -1 & 4 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

c) $$[H(\beta,A)](1,1) = \mathrm{tr}\left(\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right) = 4$$

$[H(\beta,A)](1,2) = 2$ $\qquad [H(\beta,A)](2,3) = 3$

$[H(\beta,A)](1,3) = 3$ $\qquad [H(\beta,A)](3,3) = 6$

$[H(\beta,A)](2,2) = 2$

Since Mat. are symmetric and $\mathrm{tr}(AB) = \mathrm{tr}(BA)$

Hence $H(\beta,A) = \begin{bmatrix} 4 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 6 \end{bmatrix}$

$$\alpha\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + \beta\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \gamma\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\alpha\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + \beta\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \gamma\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\alpha\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + \beta\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \gamma\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Thus $C(A,A') = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} = C(A,A')^T$

So $$H(\beta,A') = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 4 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 6 \end{bmatrix}\begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & -1 \\ 2 & 4 & -3 \\ -1 & -3 & 4 \end{bmatrix}$$

3. (1 point) Let β be a bilinear form on *odd-dimensional* vector space $\mathbb{V}$ over the field of reals. Suppose that there is an ordered basis $\mathcal{A}$ of $\mathbb{V}$ such that the coordinate matrix $H(\beta, \mathcal{A})$ is invertible, then, is it possible that we have $H(\beta, \mathcal{A}') = -H(\beta, \mathcal{A})$ for some other ordered basis $\mathcal{A}'$ of $\mathbb{V}$ (you need to justify your answer)?

$$H(\beta, A') = C(A, A')^T H(\beta, A)\, C(A, A') = -H(\beta, A)$$

Since $H(B, A') = -H(B, A)$: $\det(H(B, A')) = \det(-H(B, A)) \Leftrightarrow$

$$\Leftrightarrow (-1)^n \det(H(B, A))$$

$$\det(H(B, A')) = \det(C(A, A')^T H(B, A)\, C(A, A'))$$

$$\Updownarrow$$

$$(-1)^n \det(H(B, A)) = \det(C(A, A')^T) \det(H(B, A)) \det(C(A, A'))$$

$$\Updownarrow$$

$$(-1)^n \det(H(B, A)) = \det(C(A, A')) \det(H(B, A)) \det(C(A, A'))$$

now let's divide both sides on $\det(H(B, A))$, since it's inv. $\det \neq 0$

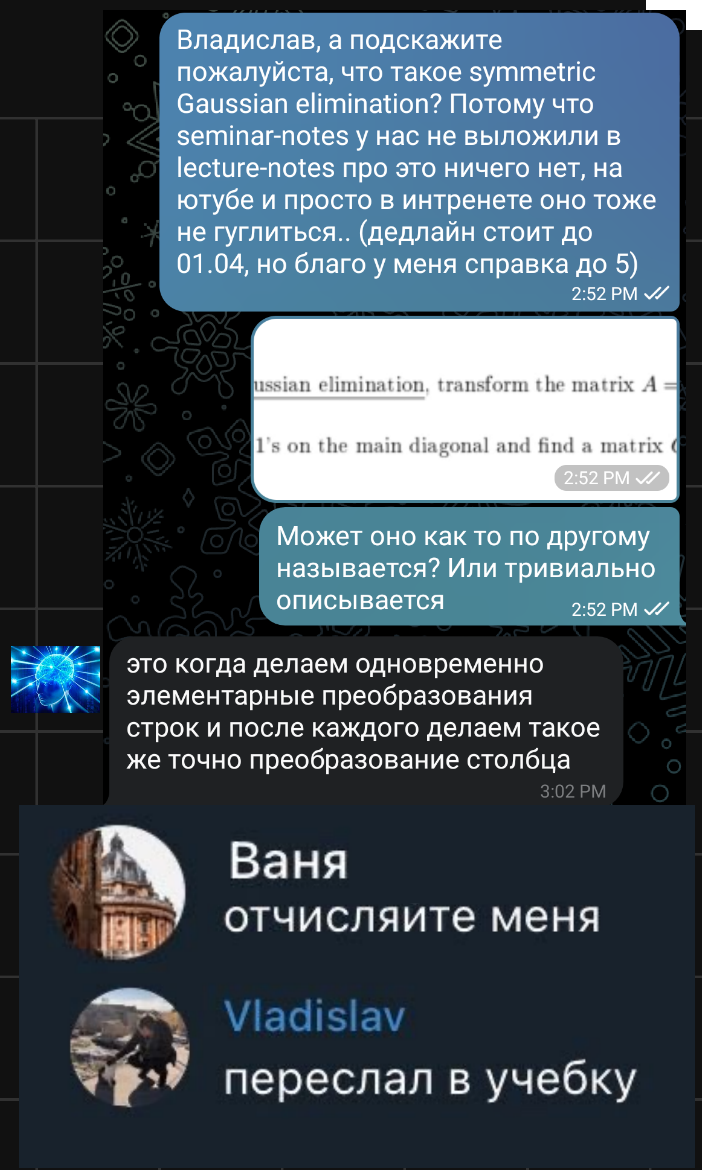
$$(-1)^n = \det(C(A, A')) \det(C(A, A'))$$

$$\Updownarrow$$

$-1 = \det(C(A, A'))^2 \Rightarrow \perp$, Hence such basis cannot exist over $\mathbb{R}$.

since n is odd

4. (1 point) Using symmetric Gaussian elimination, transform the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ into a diagonal matrix D with only 1's and -1's on the main diagonal and find a matrix C such that $D = C^{\mathrm{T}} A C$.



$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow[\hat{l}_{2,3,1}\ \hat{e}_1]{e_1\ l_{2,3,1}} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix} \xrightarrow[\hat{l}_{1,3,-1}\ \hat{e}_2]{e_2\ l_{1,3,-1}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \xrightarrow[\hat{l}_{1,2,-1}\ \hat{e}_3]{e_3\ l_{1,2,-1}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -2 \\ 1 & -2 & -3 \end{bmatrix}$$

$$\xrightarrow[\hat{l}_{1,3,-1}\ \hat{e}_4]{e_4\ l_{1,3,-1}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & -4 \end{bmatrix} \xrightarrow[\hat{d}_{3,-1/2}\ \hat{e}_5]{e_5\ d_{3,-1/2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow[\hat{t}_{2,3}\ \hat{e}_6]{e_6\ t_{2,3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow[\hat{l}_{2,3,1}\ \hat{e}_7]{e_7\ l_{2,3,1}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So $C = \hat{e}_6 \circ \hat{e}_5 \circ \ldots \hat{e}_1 \circ I_3$ and $C^T = e_6 \circ e_5 \circ e_4 \circ \ldots e_1 \circ I_3$

So $C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1/2 & 1/2 \\ 0 & -1/2 & -1/2 \end{bmatrix}$ and $C^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1/2 & -1/2 \\ 0 & 1/2 & -1/2 \end{bmatrix}$

Hence $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1/2 & -1/2 \\ 0 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1/2 & 1/2 \\ 0 & -1/2 & -1/2 \end{bmatrix}$

If you have some questions, ask