1. Vector Spaces

LAaG List of Examples for ICWT 2

1. **Vector Spaces**

1.1. Find the change of basis matrix for two given ordered bases.

For example:

Let

$$\mathcal{A}=\left(\overset{a_1}{\begin{bmatrix}1&0\\0&1\end{bmatrix}},\overset{a_2}{\begin{bmatrix}0&1\\1&1\end{bmatrix}},\overset{a_3}{\begin{bmatrix}-1&0\\0&0\end{bmatrix}}\right)\quad\text{and}\quad\mathcal{B}=\left(\overset{b_1}{\begin{bmatrix}1&1\\1&1\end{bmatrix}},\overset{b_2}{\begin{bmatrix}1&0\\0&-1\end{bmatrix}},\overset{b_3}{\begin{bmatrix}1&0\\0&0\end{bmatrix}}\right)$$

be two ordered bases for the vector space of all symmetric matrices of size two over the field of reals. Then, find $C(\mathcal{A},\mathcal{B})$ and $C(\mathcal{B},\mathcal{A})$.

1.1)

$$b_1 = a_2 - a_3 \Rightarrow \begin{bmatrix}0\\1\\-1\end{bmatrix}$$

$$b_2 = -a_1 - 2a_3 \Longrightarrow \begin{bmatrix}-1\\0\\-2\end{bmatrix}$$

$$b_3 = -a_3 \Rightarrow \begin{bmatrix}0\\0\\-1\end{bmatrix}$$

$$\Rightarrow C(A,B) = \begin{bmatrix}0&-1&0\\1&0&0\\-1&-2&-1\end{bmatrix}$$

$$C(B,A) = C(A,B)^{-1} \Rightarrow \left[\begin{array}{ccc|ccc}0&-1&0&1&0&0\\1&0&0&0&1&0\\-1&-2&-1&0&0&1\end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|ccc}1&0&0&0&1&0\\0&1&0&-1&0&0\\0&0&1&2&-1&-1\end{array}\right]$$

$$\underbrace{\begin{bmatrix}0&1&0\\-1&0&0\\2&-1&-1\end{bmatrix}}_{=\,C(B,A)}$$

ii. $\mathbb{U}_1 = \{p(x)\in\mathbb{R}[x;4]\,|\,p(-1)=0 \text{ and } p(2)=0\}$ and $\mathbb{U}_2 = \{p(x)\in\mathbb{R}[x;4]\,|\,p(1)=0\}$.

1.2.2) U_1 is a polynomials of degree at most 4 with roots (-1) and (2), U_2 - same but with root (1), so $U_1 \cap U_2$ is a polynomials $\deg \leq 4$ and $(-2)(1)(-1)$ are roots,

so $U_1\cap U_2 = \{p(x)\in\mathbb{R}[x,4]\,|\,p(1)=0 \wedge p(-1)=0 \wedge p(-2)=0\}$

So $\langle (x-1)(x+1)(x+2), (x-1)(x+1)(x+2)x \rangle$ basis

1.3. Find a basis for a direct complement of a given subspace.

For example:

Find a basis for a direct complement to $\mathbb{U}$ if:

i. $\mathbb{U} = \left\langle \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-2\\1\end{bmatrix} \right\rangle \subset \mathbb{R}^3$;

ii. $\mathbb{U} = \{(x_1,x_2,x_3,x_4)\in\mathbb{R}^4\,|\,x_1+x_2-x_3+x_4=0 \text{ and } x_1-x_2-x_3=0\}$.

1.2. Find a basis for the intersection of two given subspaces.

For example:

Find a basis for the intersection $\mathbb{U}_1\cap\mathbb{U}_2$ of the following subspaces:

i. $\mathbb{U}_1 = \left\langle \begin{bmatrix}2\\1\\0\\-1\end{bmatrix}, \begin{bmatrix}5\\5\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\-3\\2\\-2\end{bmatrix} \right\rangle \subset \mathbb{R}^4$ and $\mathbb{U}_2 = \left\langle \begin{bmatrix}-1\\3\\1\\1\end{bmatrix}, \begin{bmatrix}-2\\7\\1\\1\end{bmatrix}, \begin{bmatrix}0\\7\\-1\\2\end{bmatrix} \right\rangle \subset \mathbb{R}^4$;

ii. $\mathbb{U}_1 = \{p(x)\in\mathbb{R}[x;4]\,|\,p(-1)=0 \text{ and } p(2)=0\}$ and $\mathbb{U}_2 = \{p(x)\in\mathbb{R}[x;4]\,|\,p(1)=0\}$.

1.3.1) $\begin{bmatrix}1&1&1&0&0\\1&-2&0&1&0\\0&1&0&0&1\end{bmatrix} \xrightarrow{RREF} \begin{bmatrix}1&0&0&1&2\\0&1&0&0&1\\0&0&1&-1&-3\end{bmatrix}$, so $\begin{bmatrix}0\\0\\1\end{bmatrix}$ is a DC.

1.3.2) $\left(\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\1\\2\end{bmatrix}\right)$ is a basis fo U, so $\begin{bmatrix}1&0&1&0&0&0\\0&-1&0&1&0&0\\1&1&0&0&1&0\\0&2&0&0&0&1\end{bmatrix} \xrightarrow{RREF} \begin{bmatrix}I_4 & \boxed{*}\end{bmatrix}$ so $\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right)$ is a DC

1.2.1)

$$\begin{bmatrix}2&1&0&-1&2&1&0&-1\\5&5&-1&0&5&5&-1&0\\1&-3&2&-2&1&-3&2&-2\\-1&3&1&1&0&0&0&0\\-2&7&1&1&0&0&0&0\\0&7&-1&2&0&0&0&0\end{bmatrix} \xrightarrow{l_{2,5,1}} \begin{bmatrix}2&1&0&-1&2&1&0&-1\\5&5&-1&0&5&5&-1&0\\1&-3&2&-2&1&-3&2&-2\\-1&3&1&1&0&0&0&0\\0&8&1&0&2&1&0&-1\\0&7&-1&2&0&0&0&0\end{bmatrix} \xrightarrow{l_{3,4,1}} \begin{bmatrix}2&1&0&-1&2&1&0&-1\\5&5&-1&0&5&5&-1&0\\1&-3&2&-2&1&-3&2&-2\\0&0&3&-1&1&-3&2&-2\\0&8&1&0&2&1&0&-1\\0&7&-1&2&0&0&0&0\end{bmatrix} \xrightarrow{d,5,1/8}$$

$$\xrightarrow{d\,5,1/8} \begin{bmatrix}2&1&0&-1&2&1&0&-1\\5&5&-1&0&5&5&-1&0\\1&-3&2&-2&1&-3&2&-2\\0&0&3&-1&1&-3&2&-2\\0&1&\frac{1}{8}&0&2/8&1/8&0&-1/8\\0&7&-1&2&0&0&0&0\end{bmatrix} \xrightarrow{l_{5,6,-7}} \begin{bmatrix}2&1&0&-1&2&1&0&-1\\5&5&-1&0&5&5&-1&0\\1&-3&2&-2&1&-3&2&-2\\0&0&3&-1&1&-3&2&-2\\0&1&\frac{1}{8}&0&2/8&1/8&0&-1/8\\0&0&-\frac{15}{8}&2&-14/8&-7/8&0&7/8\end{bmatrix} \xrightarrow{l_{3,1,-2}} \begin{bmatrix}0&7&-4&3&0&7&-4&3\\5&5&-1&0&5&5&-1&0\\1&-3&2&-2&1&-3&2&-2\\0&0&3&-1&1&-3&2&-2\\0&1&\frac{1}{8}&0&2/8&1/8&0&-1/8\\0&0&-\frac{15}{8}&2&-14/8&-7/8&0&7/8\end{bmatrix} \xrightarrow{l_{3,2,-5}}$$

$$\xrightarrow{l_{3,2,-5}} \begin{bmatrix}0&7&-4&3&0&7&-4&3\\0&20&-11&10&0&20&-11&10\\1&-3&2&-2&1&-3&2&-2\\0&0&3&-1&1&-3&2&-2\\0&1&\frac{1}{8}&0&2/8&1/8&0&-1/8\\0&0&-\frac{15}{8}&2&-14/8&-7/8&0&7/8\end{bmatrix} \xrightarrow{l_{5,3,3}} \begin{bmatrix}0&7&-4&3&0&7&-4&3\\0&20&-11&10&0&20&-11&10\\1&0&19/8&-2&14/8&-15/8&2&-19/8\\0&0&3&-1&1&-3&2&-2\\0&1&\frac{1}{8}&0&2/8&1/8&0&-1/8\\0&0&-\frac{15}{8}&2&-14/8&-7/8&0&7/8\end{bmatrix} \xrightarrow{l_{5,1,-7}} \begin{bmatrix}0&0&\frac{-39}{8}&3&-14/8&\frac{49}{8}&-4&31/8\\0&20&-11&10&0&20&-11&10\\1&0&19/8&-2&14/8&-15/8&2&-19/8\\0&0&3&-1&1&-3&2&-2\\0&1&\frac{1}{8}&0&2/8&1/8&0&-1/8\\0&0&-\frac{15}{8}&2&-14/8&-7/8&0&7/8\end{bmatrix} \xrightarrow{l_{5,2,-20}}$$

$$\xrightarrow{l_{5,2,-20}} \begin{bmatrix}0&0&\frac{-39}{8}&3&-14/8&\frac{49}{8}&-4&31/8\\0&0&-\frac{108}{8}&10&\frac{-20}{8}&\frac{140}{8}&-11&\frac{60}{8}\\1&0&19/8&-2&14/8&-15/8&2&-19/8\\0&0&3&-1&1&-3&2&-2\\0&1&\frac{1}{8}&0&2/8&1/8&0&-1/8\\0&0&-\frac{15}{8}&2&-14/8&-7/8&0&7/8\end{bmatrix} \xrightarrow[\substack{t_{2,5}\\ d_{4,1/8}\\ d_{5,\frac{1}{4}}}]{t_{1,3}} \frac{1}{8}\begin{bmatrix}8&0&19&-16&14&-15&16&-19\\0&8&1&0&2&1&0&-1\\0&0&-39&24&-14&45&-32&31\\0&0&3&-1&1&-3&2&-2\\0&0&-54&20&-5&35&-22&16\\0&0&-15&16&-14&-7&0&7\end{bmatrix} \xrightarrow{l_{4,6,-5}} \frac{1}{8}\begin{bmatrix}8&0&19&-16&14&-15&16&-19\\0&8&1&0&2&1&0&-1\\0&0&-39&24&-14&45&-32&31\\0&0&3&-1&1&-3&2&-2\\0&0&-54&20&-5&35&-22&16\\0&0&0&21&-19&8&-10&17\end{bmatrix} \xrightarrow{l_{4,5,19}}$$

$$\xrightarrow{l_{4,6,18}} \frac{1}{8}\begin{bmatrix}8&0&19&-16&14&-15&16&-19\\0&8&1&0&2&1&0&-1\\0&0&-39&24&-14&45&-32&31\\0&0&3&-1&1&-3&2&-2\\0&0&0&2&-5&-19&14&-21\\0&0&0&21&-19&8&-10&17\end{bmatrix} \xrightarrow{REF} \left[\begin{array}{cccc|cccc}1&0&0&0&&&&\\0&1&0&0&&&*&\\0&0&1&0&&&&\\0&0&0&1&&&&\\0&0&0&0&0&-1&1&1\\0&0&0&0&-1&-4&2&-1\end{array}\right]$$

$\left(\begin{bmatrix}0\\-1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\-4\\2\\-1\end{bmatrix}\right)$ – is a basis for $U_1\cap U_2$

$\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right)$ – basis for $U_1\cup U_2$

1.3.1) We just transfom matrix into vector, using one-to-one formula, for example, just $\begin{bmatrix}a&b\\c&d\end{bmatrix} = \begin{bmatrix}a\\b\\c\\d\end{bmatrix}$

So $\begin{bmatrix}1&-2\\-2&3\end{bmatrix} \to \begin{bmatrix}1\\-2\\-2\\3\end{bmatrix}$; $\begin{bmatrix}0&-1\\-1&2\end{bmatrix} \to \begin{bmatrix}0\\-1\\-1\\2\end{bmatrix}$

and just add standart basis for V in case of sym. matrixes of size 2, it's $\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}0&0\\0&1\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}$

So combine them into one matrix, like: $\begin{bmatrix}1&0&1&0&0\\-2&-1&0&0&1\\-2&-1&0&0&1\\3&2&0&1&0\end{bmatrix}$

$$\begin{bmatrix}1&0&1&0&0\\-2&-1&0&0&1\\-2&-1&0&0&1\\3&2&0&1&0\end{bmatrix} \xrightarrow{l_{2,3,-1}} \begin{bmatrix}1&0&1&0&0\\-2&-1&0&0&1\\0&0&0&0&0\\3&2&0&1&0\end{bmatrix} \xrightarrow[l_{1,4,-3}]{l_{1,2,2}} \begin{bmatrix}1&0&1&0&0\\0&-1&2&0&1\\0&0&0&0&0\\0&2&-3&1&0\end{bmatrix} \xrightarrow[d_{2,-1}]{l_{2,4,2}} \begin{bmatrix}\boxed{1}&0&1&0&0\\0&\boxed{1}&-2&0&-1\\0&0&0&0&0\\0&0&\boxed{1}&1&2\end{bmatrix}$$

leading el.

so the corresponding vectors are LI, so $\begin{bmatrix}1&0\\0&0\end{bmatrix}$ is LI to our matrixes,

so $\begin{bmatrix}1&0\\0&0\end{bmatrix}$ is a direct compliment.

1.4. Let $\mathbb{U}_1 = \langle \overset{\bar{w}_1}{[0\ 0\ 0\ 1\ 1]^{\mathrm{T}}} \rangle$, $\mathbb{U}_2 = \{[x_1\ x_2\ x_3\ x_4\ x_5]^{\mathrm{T}}\in\mathbb{R}^5\,|\,x_1+x_3+x_5=0 \text{ and } x_2+x_4=0\}$, and $\mathbb{U}_3 = \langle \underset{\bar{a}_1}{[1\ 2\ 3\ 4\ 5]^{\mathrm{T}}} \rangle$ be subspaces of $\mathbb{R}^5$. Then:

i. prove that $\mathbb{R}^5 = \mathbb{U}_1\oplus\mathbb{U}_2\oplus\mathbb{U}_3$;

ii. for $\mathbf{x} = [1\ -5\ 2\ 3\ -2]^{\mathrm{T}}\in\mathbb{R}^5$, find the vectors $\mathbf{x}_1\in\mathbb{U}_1$, $\mathbf{x}_2\in\mathbb{U}_2$, and $\mathbf{x}_3\in\mathbb{U}_3$ such that $\mathbf{x} = \mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3$.

basis for U_2: $\left(\overset{\bar{u}_1}{\begin{bmatrix}0\\1\\0\\-1\\0\end{bmatrix}}, \overset{\bar{u}_2}{\begin{bmatrix}1\\0\\0\\0\\-1\end{bmatrix}}, \overset{\bar{u}_3}{\begin{bmatrix}0\\0\\1\\0\\-1\end{bmatrix}}\right)$

so **1.4.1)** $\begin{bmatrix}0&1&0&0&1\\1&0&0&0&2\\0&0&1&0&3\\-1&0&0&1&4\\0&-1&-1&1&5\end{bmatrix} \xrightarrow{RREF} I_5$, hence they are LI $\Rightarrow$ $\Rightarrow$ It's direct Compl.

1.4.2) $\left[\begin{array}{ccccc|c}0&1&0&0&1&1\\1&0&0&0&2&-5\\0&0&1&0&3&2\\-1&0&0&1&4&3\\0&-1&-1&1&5&-2\end{array}\right] \xrightarrow{RREF} \left[\begin{array}{c|c}I_5&\begin{matrix}-7\\0\\-1\\-8\\1\end{matrix}\end{array}\right]$

Hence $\begin{bmatrix}1\\-5\\2\\3\\-2\end{bmatrix} = \underbrace{-7\bar{u}_1 + 0\bar{u}_2 - \bar{u}_3}_{\bar{x}_2}\underbrace{-8\bar{w}_1}_{\bar{x}_1}\underbrace{+\bar{a}_1}_{\bar{x}_3}$

So $\begin{bmatrix}0\\-7\\-1\\7\\1\end{bmatrix} = \bar{x}_2$ $\begin{bmatrix}0\\0\\0\\-8\\-8\end{bmatrix} = \bar{x}_1$ $\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix} = \bar{x}_3$; that's it.

The Rank of Matrix

2. The Rank of a Matrix

2.1. For every $\lambda \in \mathbb{R}$, find the rank of the matrix:

$$A(\lambda) = \begin{bmatrix} 1 & 1 & \lambda \\ -2 & 1 & -5 \\ -3 & 1 & -6 \end{bmatrix} \in \mathrm{Mat}_3(\mathbb{R}).$$

2.2. Let $A = [a_1, a_2, \ldots, a_n]$ be a 1-by-n matrix with real coefficients, then, find the rank of an n-by-n matrix $A^T A$.

Novosad Ivan 15:57

2.2) Consider a linear maps φ, ψ, γ s.t. for some basis B, K, N $T(\varphi, B) = A^T A$; $T(\psi, K) = A^T$ $T(\gamma, N) = A$, then $\varphi = \psi \circ \gamma$ So, since $rk(\gamma) = 1$, then $rk(\psi \circ \gamma) = 1 \Rightarrow rk(\varphi) = 1 \Rightarrow rk(A^T A) = 1$

Владислав, скажите достаточно аккуратный пруф, или за такое по башке дадут и нужно более формально описывать? 15:58

2.2. Let $A = [a_1, a_2, \ldots, a_n]$ be a 1-by-n matrix with real coefficients, then, find the rank of an n-by-n matrix $A^T A$. 15:58

Vladislav 15:59

ну строго говоря

2.2) Consider a linear maps φ, ψ, γ s.t. for some basis B, K, N $T(\varphi, B) = A^T A$; $T(\psi, K) = A^T$ $T(\gamma, N) = A$, then $\varphi = \psi \circ \gamma$ So, since $rk(\gamma) = 1$, then $rk(\psi \circ \gamma) \leq 1 \Rightarrow rk(\varphi) = 1 \Rightarrow rk(A^T A) = 1$ 15:59

Novosad Ivan 15:59

согласен

что-то ещё? 15:59

Vladislav 15:59

но что он не 0 очевидно

не, все ок 15:59

2.1) $\begin{bmatrix} 1 & 1 & \lambda \\ -2 & 1 & -5 \\ -3 & 1 & -6 \end{bmatrix} \xrightarrow[\hat{t}_{1,2}]{t_{1,3}} \begin{bmatrix} 1 & -3 & -6 \\ 1 & -2 & -5 \\ 1 & 1 & \lambda \end{bmatrix} \xrightarrow[l_{1,3,-1}]{l_{1,2,-1}} \begin{bmatrix} 1 & -3 & -6 \\ 0 & 1 & 1 \\ 0 & 4 & \lambda+6 \end{bmatrix}$

$\xrightarrow[l_{2,3,-4}]{l_{2,1,3}} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & \lambda+2 \end{bmatrix}$, hence if $\lambda = -2 \Rightarrow \mathrm{Rank}(A) = 2$

if $\lambda \neq -2$, $\Rightarrow \mathrm{Rank}(A) = 3$.

2.2) Consider a linear maps φ, ψ, γ s.t. for some basis B, K, N

$T(\varphi, B) = A^T A$; $T(\psi, K) = A^T$ $T(\gamma, N) = A$, then $\varphi = \psi \circ \gamma$

So, since $rk(\gamma) = 1$, then $rk(\psi \circ \gamma) \leq 1 \Rightarrow rk(\varphi) = 1 \Rightarrow rk(A^T A) = 1$

LT and LO and Eigen-stuff

3. **Linear Transformations and Linear Operators**

3.1. Find a basis for the kernel and the image of a given linear transformation.

For example:

Find a basis for $\mathrm{Ker}(\varphi)$ and $\mathrm{Im}(\varphi)$ if:

i. a linear transformation $\varphi\colon \mathbb{R}^5 \to \mathbb{R}^2$ is defined as

$$\varphi\colon \begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -2 & 2 & -1 & 1\\ 2 & -4 & 1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5 \end{bmatrix}, \quad \text{for every } [x_1, x_2, x_3, x_4, x_5]^{\mathrm{T}} \in \mathbb{R}^5;$$

3.1.1) $\begin{bmatrix} 1 & -2 & 2 & -1 & 1\\ 2 & -4 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{\ell_{1,2,-2}} \begin{bmatrix} 1 & -2 & 2 & -1 & 1\\ 0 & 0 & -3 & 3 & 0 \end{bmatrix}$

$$\xrightarrow[\ell_{2,1,-2}]{d_{2,-1/3}} \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \\ \begin{bmatrix} \boxed{1} & -2 & 0 & 1 & 1\\ 0 & 0 & \boxed{1} & -1 & 0 \end{bmatrix} \end{array} \Rightarrow \begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5 \end{bmatrix} = \begin{bmatrix} 2\alpha - \beta - \gamma\\ \alpha\\ \beta\\ \beta\\ \gamma \end{bmatrix}, \ \forall\, \beta, \alpha, \gamma \in \mathbb{R}$$

So basis for Image: $\left(\begin{bmatrix} 1\\ 2 \end{bmatrix}, \begin{bmatrix} 2\\ 1 \end{bmatrix}\right)$, for ker $= \left(\begin{bmatrix} 2\\ 1\\ 0\\ 0\\ 0 \end{bmatrix}, \begin{bmatrix} -1\\ 0\\ 1\\ 1\\ 0 \end{bmatrix}, \begin{bmatrix} -1\\ 0\\ 0\\ 0\\ 1 \end{bmatrix}\right)$

ii. a linear transformation $\varphi\colon \mathbb{R}[x;n] \to \mathbb{R}[x;n]$ is defined as

$$\varphi(p(x)) = p(x+1) - p(x), \quad \text{for every } p(x) \in \mathbb{R}[x;n];$$

3.1.2) with respect to standard basis for $\mathbb{R}[x, n]$

$\langle 1, x, x^2, x^3, \dots x^n \rangle$

$1 \to 0$	since $\deg(p(x+1)) = \deg(p(x))$
$x \to 1$	so $\deg(p(x+1) - p(x)) = \deg(p(x)) - 1$
$x^2 \to 2x+1$	Hence $\ker(\varphi) = \langle 1 \rangle = \mathbb{R}$
$x^3 \to 3x + \dots$	$\mathrm{Im}(\varphi) = \{p(x) \in \mathbb{R}[x, n]\}$ i.e. $\langle 1, x, x^2, \dots x^{n-1} \rangle$

3.1.3 iii. A is an n-by-n non-zero matrix such that all its rows are the same, and a linear transformation $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$\varphi\colon \mathbf{x} \mapsto A\mathbf{x}, \quad \text{for every } \mathbf{x} \in \mathbb{R}^n.$$

Since all rows are the same $rk(A) = 1 \Rightarrow \mathrm{Im}(\varphi) = \mathbb{R}$

Basis for $\mathrm{Im}(\varphi) = \left([A]^{(1)}\right)$ - first column of A. , or $\mathrm{Im}(\varphi) = \left(\begin{bmatrix} 1\\ 1\\ \vdots\\ 1 \end{bmatrix}\right)$

$$A = \begin{bmatrix} a_1 & a_2 & \dots & a_n\\ a_1 & a_2 & \dots & a_n\\ \vdots & \vdots & \ddots & \vdots\\ a_1 & a_2 & \dots & a_n \end{bmatrix} \Rightarrow \left[\begin{array}{cccccc|c} a_1 & a_2 & a_3 & \dots & a_n & & 0\\ 0 & 0 & 0 & \dots & 0 & & 0\\ \vdots & \vdots & \vdots & \ddots & & & \vdots\\ 0 & 0 & 0 & \dots & 0 & & 0 \end{array}\right] \Rightarrow \begin{bmatrix} x_1\\ x_2\\ x_3\\ \vdots\\ x_n \end{bmatrix} = \begin{bmatrix} \frac{-a_2 - a_3 \dots - a_n}{a_1}\\ \lambda_1\\ \lambda_2\\ \vdots\\ \lambda_n \end{bmatrix}$$

So, $\ker(\varphi) = (\bar{w}_1; \bar{w}_2; \bar{w}_3: \dots \bar{w}_n)$, where $\bar{w}_i = \frac{-a_i}{a_1}\bar{e}_1 + e_i$.

$\bar{e}_i = \begin{bmatrix} 0\\ \vdots\\ 1\\ \vdots\\ 0 \end{bmatrix} \leftarrow i^{th}$ position.

3.2. Find the coordinate matrix of a given linear operator with respect to a given ordered basis.

For example:

Let $\mathbb{V}$ and $\mathbb{W}$ be two vector spaces, let $\mathcal{A}$ and $\mathcal{B}$ be an ordered basis for $\mathbb{V}$ and $\mathbb{W}$, respectively, and let $\varphi\colon \mathbb{V} \to \mathbb{W}$ be a linear transformation. Then, find the coordinate matrix of φ with respect to $\mathcal{A}$ and $\mathcal{B}$ (that is, find $T(\varphi, \mathcal{A}, \mathcal{B})$) if:

i. $\mathbb{V} = \mathbb{W} = \mathbb{R}[x; 2]$, $\mathcal{A} = (1 + x^2, 1 - x, 1 - x + x^2)$, $\mathcal{B} = (2 + x, x^2, 1 + x + x^2)$, and

$$\varphi(ax^2 + bx + c) = bx^2 + cx, \quad \text{for every } ax^2 + bx + c \in \mathbb{R}[x, 2];$$

ii. $\mathbb{V} = \mathbb{W} = \mathrm{Mat}_2(\mathbb{R})$, $\mathcal{A} = \mathcal{B} = \left(\begin{bmatrix} 1 & 1\\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1\\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0\\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0\\ 1 & 1 \end{bmatrix}\right)$, and $\varphi\colon A \mapsto A^{\mathrm{T}}$, for every $A \in \mathrm{Mat}_2(\mathbb{R})$.

$A: \begin{bmatrix} 1 & 1 & 1\\ 0 & -1 & -1\\ 1 & 0 & 1 \end{bmatrix}$

$B: \begin{bmatrix} 2 & 0 & 1\\ 1 & 0 & 1\\ 0 & 1 & 1 \end{bmatrix}$

$$\left.\begin{aligned} \varphi(\bar{a}_1) &= \varphi(1 + x^2) = x,\\ \varphi(\bar{a}_2) &= \varphi(1 - x) = -x^2 + x\\ \varphi(\bar{a}_3) &= \varphi(1 - x + x^2) = -x^2 + x \end{aligned}\right\} \Rightarrow \left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 0 & 0 & 0\\ 1 & 0 & 1 & 1 & 1 & 1\\ 0 & 1 & 1 & 0 & -1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 0 & 0 & 0\\ 1 & 0 & 1 & 1 & 1 & 1\\ 0 & 1 & 1 & 0 & -1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 0 & 0 & 0\\ 0 & 0 & 1 & 2 & 2 & 2\\ 0 & 1 & 1 & 0 & -1 & -1 \end{array}\right]$$

$$\downarrow$$

$$\underbrace{\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & -1\\ 0 & 1 & 0 & -2 & -3 & -3\\ 0 & 0 & 1 & 2 & 2 & 2 \end{array}\right]}_{T(\varphi, A, B)} \leftarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & -1\\ 0 & 0 & 1 & 2 & 2 & 2\\ 0 & 1 & 0 & -2 & -3 & -3 \end{array}\right]$$