2. (HW) Find the relative extrema and saddle points of the function

$$f(x, y) = x^3 - 3x + y^4 - 2y^2.$$

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$$\begin{matrix} f_x: \\ f_y: \end{matrix} \begin{cases} 3x^2 - 3 = 0 \\ 4y^3 - 4y = 0 \end{cases} \Rightarrow \begin{cases} x = \pm 1 \\ y = \pm 1, 0 \end{cases}$$

$p_1 = (-1, 0)$ $p_2 (1, 0)$ $p_3 (-1\ -1)$

$p_4 (1, -1)$ $p_5 (-1, 1)$ $p_6 (1, 1)$

$f_{xx} = 6x$ $f_{xy} = 0$

$f_{yy} = 12y^2 - 4$

notice, that if $6(x_i) < 0 \Rightarrow$

$\Rightarrow (12y_i - 4)(6x_i)$ depends only on $12y_i - 4$

$$H(p_i) = \begin{pmatrix} 6(x_i) & 0 \\ 0 & 12y_i^2 - 4 \end{pmatrix}$$

, then if $\begin{cases} 6(x_i) < 0 \\ 12y_i^2 - 4 > 0 \end{cases} \rightarrow$ min

$\begin{cases} 6(x_i) > 0 \\ 12y_i^2 - 4 > 0 \end{cases} \rightarrow$ max otherwise indef.

$$H = \begin{pmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{pmatrix}$$

$\begin{cases} 6(x_i) > 0 \\ 12y_i^2 - 4 < 0 \end{cases} \rightarrow$ saddle or $\begin{cases} x_i < 0 \\ 12y_i - 4 < 0 \end{cases}$

if $\delta_1 = 0 \vee \delta_2 = 0 \Rightarrow$ further analysis is req.

$p_1 : \delta_1 < 0$ $\delta_2 < 0$ saddle

$p_2 : \delta_1 > 0$ $\delta_2 < 0$ saddle

$p_3 : \delta_3 < 0$ $\delta_2 > 0$ max

$p_4 : \delta_1 > 0$ $\delta_2 \geq 0$ min

$p_5 : \delta_1 < 0$ $\delta_2 > 0$ max

$p_6 : \delta_1 > 0$ $\delta_2 > 0$ min

4. (HW) Find the critical points of the function $f(x, y) = x^2 - 8\ln x + 3y^2 - 6\ln y$. Test the nature of the critical points.

$$\begin{matrix} f_x : \\ f_y : \end{matrix} \begin{cases} 2x - \dfrac{8}{x} = 0 \\ 6y - \dfrac{6}{y} = 0 \end{cases} \Rightarrow \begin{bmatrix} P_1(-2,-1) \\ P_2(2,-1) \\ P_3(-2,1) \\ P_4(2,1) \end{bmatrix}$$

$$f_{xx} = 2 + \frac{8}{x^2} \qquad f_{xy} = 0$$

$$f_{yy} = 6 + \frac{6}{y^2}$$

$$H(P_1) = H(P_2) = H(P_3) = H(P_4) = \begin{pmatrix} 4 & 0 \\ 0 & 12 \end{pmatrix}$$

$$|4| > 0 \wedge \begin{vmatrix} 4 & 0 \\ 0 & 12 \end{vmatrix} > 0$$

Thus P_1 P_2 P_3 P_4 are points of local maxima.

6. (HW) Find the critical points of the function $f(x,y) = x^4 + y^4 - 2x^2$. Test the nature of the critical points.

$$f_x = 4x^3 - 4x$$

$$f_y = 4y^3$$

$$f_{xx} = 12x^2 - 4$$

$$f_{yy} = 12y^2$$

$$f_{xy} = 0$$

Candidates: $\begin{cases} x^3 - x = 0 \\ y^3 = 0 \end{cases} \Rightarrow \left[\begin{array}{l} P_1\ (0,0) \\ P_2\ (-1,0) \\ P_3\ (1,0) \end{array} \right.$

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 12x^2-4 & 0 \\ 0 & 12y^2 \end{pmatrix}$$

$$\left.\begin{array}{ll} P_1 : \delta_1 < 0 & \delta_2 = 0 \\ P_2 : \delta_1 > 0 & \delta_2 = 0 \\ P_3 : \delta_1 > 0 & \delta_3 = 0 \end{array}\right\} \Rightarrow \text{Further analysis is required.}$$

8. (HW) Find the critical points of the function $f(x,y,z) = 6x^2 + 5y^2 + z^2 - 8xy - 2xz + 4yz + 2x - 2y + 1$. Test the nature of the critical points.

$$\begin{aligned} f_x &= 12x - 8y - 2z + 2 = 0 \\ f_y &= 10y - 8x + 4z - 2 = 0 \\ f_z &= 2z - 2x + 4y = 0 \end{aligned} \Rightarrow \underbrace{(x,y,z) = (1,3,-5)}_{P}$$

$$\boxed{\begin{aligned} f_{xx} &= 12 & f_{xy} &= -8 \\ f_{yy} &= 10 & f_{xz} &= -2 \\ f_{zz} &= 2 & f_{zy} &= 4 \end{aligned}}$$

doesn't depends on coordinates

$$H = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{xy} & f_{yy} & f_{yz} \\ f_{xz} & f_{yz} & f_{zz} \end{pmatrix} = \begin{pmatrix} 12 & -8 & -2 \\ -8 & 10 & 4 \\ -2 & 4 & 2 \end{pmatrix}$$

$$q_1 = |12| = 12$$

$$q_2 = \begin{vmatrix} 12 & -8 \\ -8 & 10 \end{vmatrix} = 56 \Rightarrow f(x,y,z) \text{ has rel. maxima at } (x,y,z) = (1,3,-5)$$

$$q_3 = |H| = 8$$

11. (HW) Find the critical points of following function. Test the nature of the critical points:

$$f(x,y) = (3x + y^2)e^{x+2y}.$$

$$f_x = 3e^{x+2y} + 3xe^{x+2y} + y^2e^{x+2y}$$
$$f_y = 2ye^{x+2y} + 6xe^{x+2y} + 2y^2e^{x+2y}$$
$$f_{xx} = 6e^{x+2y} + 3xe^{x+2y} + y^2e^{x+2y}$$
$$f_{xy} = 6e^{x+2y} + 6xe^{x+2y} + 2ye^{x+2y} + 2y^2e^{x+2y}$$
$$f_{yy} = 2e^{x+2y} + 8ye^{x+2y} + 12xe^{x+2y} + 4y^2e^{x+2y}$$

point are solutions of $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$

$$\begin{cases} 3e^{x+2y} + 3xe^{x+2y} + y^2e^{x+2y} = 0 \\ 2ye^{x+2y} + 6xe^{x+2y} + 2y^2e^{x+2y} = 0 \end{cases} \Rightarrow (x,y) = \underbrace{(-4,3)}_{P}$$

$$f_{xx}(P) = e^2(6 - 12 + 9) = 3e^2$$
$$f_{yy}(P) = e^2(2 + 24 - 48 + 36) = 14e^2$$
$$f_{xy}(P) = f_{yx}(P) = e^2(6 - 24 + 6 + 18) = 6e^2$$

Thus $H(P) = \begin{pmatrix} 3e^2 & 6e^2 \\ 6e^2 & 14e^2 \end{pmatrix} \Rightarrow \begin{cases} q_1 = 3e > 0 \\ q_2 = 6e^2 > 0 \end{cases}$

Hence $f(x,y)$ has rel. maxima at $(x,y) = (-4,3)$

(b) (HW) $f(x, y) = 4x^2 + y^2$ subject to the constraint $-\frac{x}{2} - \frac{y}{3} = 1$.

$-\frac{x}{2} - \frac{y}{3} = 1 \Rightarrow y = -\frac{3}{2}x - 3 \Rightarrow -\frac{48}{25}$

$f(x) = 4x^2 + \frac{9}{4}x^2 - 9x + 9$

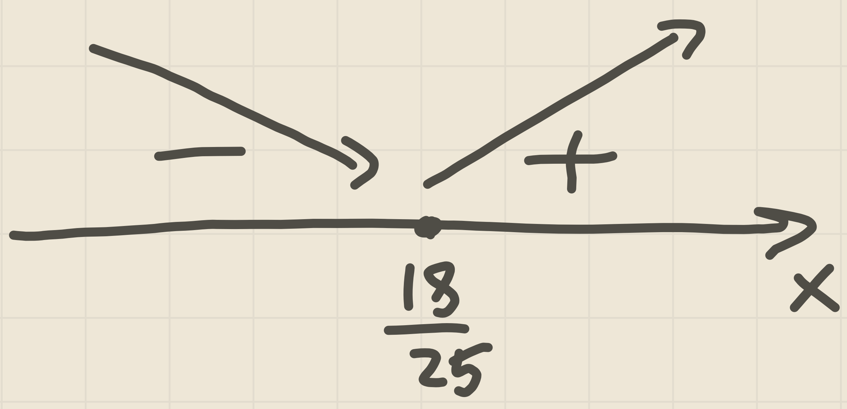
$(x,y) = \left(-\frac{18}{25}, \frac{-48}{25}\right)$ is rel. minima

$f(x) = \frac{25}{4}x^2 - 9x + 9$

$f(x,y) = \frac{144}{25}$ is min value.

$f'(x) = \frac{25}{4}x - 9$

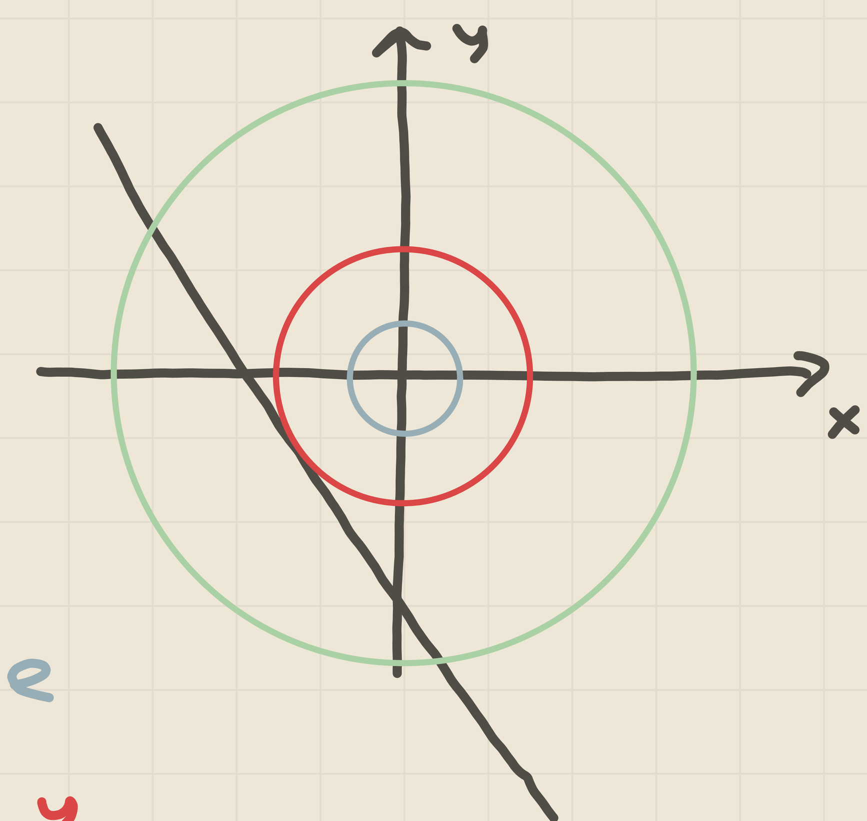
$f'(x) = 0 \iff x = \frac{18}{25} \Rightarrow x = \frac{18}{25}$ is crit. point.



$\Rightarrow x = \frac{18}{25}$ is minimum value point.

Geometrically : $f(x,y)$ is a ellipse with center at $(0,0)$ and $R = c$, Thus we can obtain any R

$-\frac{x}{2} - \frac{y}{3} = 1$ is a line, thus :



we have 3 cases:

then $f(x,y)$ & $-\frac{x}{2} - \frac{y}{3} = 1 \Rightarrow$ nothing to analise

then $f(x,y)$ has exactly one point with $-\frac{x}{2} - \frac{y}{3} = 1$

obv. then $c(R)$ is minimal poss. value, $\Rightarrow f(x,y)$ has min value there.

Then $f(x,y)$ has 2 shared points with $-\frac{x}{2} - \frac{y}{3} = 1$, then $f(x,y) = c$ obv greater then in case 2.

Thus minima value is c from 2^{nd} case, and $(x_0\ y_0)$ s.t. $f(x_0\ y_0) = c$ are point of minima.

to find such c, we can solve the system:

$$\begin{cases} 4x^2 + y^2 = c \\ -\frac{x}{2} - \frac{y}{3} = 1 \end{cases}$$

with condition o c: system ought to have one sol $\Rightarrow$

$\iff D = 0$

$\Downarrow$

$$4x^2 + \left(-\frac{3}{2}x - 3\right)^2 = c \Rightarrow \left[\begin{array}{l} c > \frac{144}{25}, \text{ two real solutions} \\ c = \frac{144}{25}, \text{ one real solution} \\ c < \frac{144}{25}, \text{ No real solutions} \end{array}\right.$$

$\Rightarrow f(x,y)$ has min value $= \frac{144}{25}$ (if we restrict dom. with $-\frac{x}{2} - \frac{y}{3} = 1$)

$$\begin{cases} 4x^2 - y^2 = \frac{144}{25} \\ -\frac{x}{2} - \frac{y}{3} = 1 \end{cases} \Rightarrow (x,y) = \left(-\frac{18}{25}, \frac{-48}{25}\right)$$

13*. (HW) Prove that the function

$$f(x,y) = (y^2 - x)(y^2 - 2x)$$

attains a minimum value at $(0,0)$ along any straight line through the origin, but has no local minimum at $(0,0)$.

Staight lines through origin are in the form $y = \overset{\in \mathbb{R}}{m}x$ or $x = 0$

This means two cases:

1. $y = mx \Rightarrow (x,y) = (x, mx) \Rightarrow f(x, mx) = g(x) = (m^2x^2 - x)(m^2x^2 - 2x) = m^4x^4 - 3m^2x^3 + 2x^2$ constant

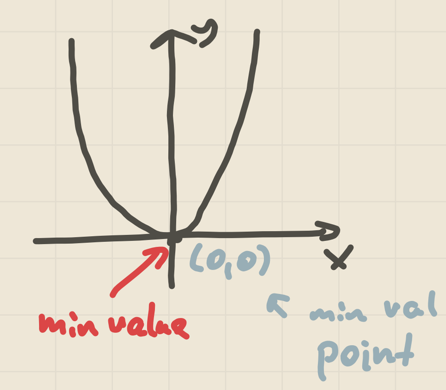
   Thus $g'(x) = 4m^4x^3 - 9m^2x^2 + 4x$

   $g''(x) = 12m^4x^2 - 18m^2x + 4$

   $g'(0) = 0$, thus we have a stationary point, and $g''(0) = 4 > 0$, hence we have a local minimum.

2. $x = 0 \Rightarrow (x,y) = (0,y) \Rightarrow f(0,y) = h(y) = (y^2 - 0)(y^2 - 2\cdot 0) = y^4$ is a parabola of the form

   Hence in this case it's true also.



Hence, (since in both cases it's true), $f(x,y)$ has rel. minima along any straight line which pases origin

okay, we prove first statement

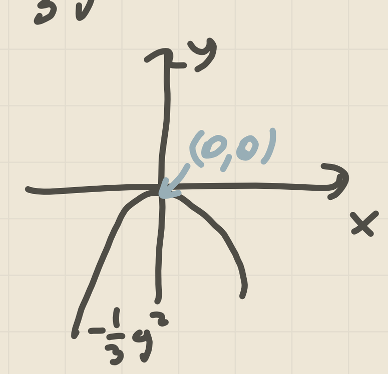
Now we need to check that $f(x,y)$ has no minima at $(0,0)$

Notice that if $f(x,y)$ has rel minima at $(0,0)$ then it's min value along any curve.

Consider a curve $x = \frac{2}{3}y^2$ it's a parabola which clearly passes $(0,0)$

Same trick: let $(x,y) = (\frac{2y^2}{3}, y) \Rightarrow l(y) = (y^2 - \frac{2}{3}y^2)(y^2 - \frac{4}{3}y^2) = -\frac{1}{3}y^2 < 0$

but it's clearly will be rel maxima instead of minima



algebraicly:

$l'(y) = -\frac{2}{3}y \qquad l'(0) = 0 \Rightarrow$ has crit point

$l''(y) = -\frac{2}{3} \qquad l''(0) < 0 \Rightarrow$ rel. maxima

Thus $(0,0)$ is rel. maxima and rel. minima simultaneosly

But the only way that a point can be both a local minimum and local maximum is if it's locally constant

That is, the function takes the value $f(0,0)$ at $(0,0)$ and every point within a small distance of $(0,0)$

but if we consider $\underset{l(y)}{x = \frac{2}{3}y^2}$ and $\underset{k(y)}{x = \frac{3}{2}y^2}$ we will obtain smth like that:

$l(y) = -\frac{1}{3}y^2 \quad k(y) = y^4 \Rightarrow \begin{cases} k(y) \geq 0 & \forall y \in \mathbb{R} \\ l(y) \leq 0 & \forall y \in \mathbb{R} \end{cases}$ and $\begin{cases} k(y) = 0 \Leftrightarrow y = 0 \\ l(y) = 0 \Leftrightarrow y = 0 \end{cases}$

implies that, no matter how close we we come to $(0,0)$ there are at least two point along $x = \frac{2}{3}y^2$ and $x = \frac{3}{2}y^2$ for wich $f(x_0, y_0)$ will be diffrent $\Rightarrow$ it's not a rel. constant.

Hence $(0,0)$ can not be minima and maxima simultaneosly $\Rightarrow \perp \Rightarrow (0,0)$ is not a minimal point of $f(x,y)$ ■

Tnx for checking, have a nice day