1. Operations with matrices. There are some matrices and you ought to perform some operations with them. Possible operations: matrix addition, matrix scalar multiplication, matrix multiplication, matrix transposition, the trace, the determinant.
For example:
Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 17 & 29 & 54 \\ 39 & 13 & 19 \\ 78 & -24 & -23 \end{bmatrix}.$$

Then, find $\det(A^{\mathrm{T}} \cdot B) + \operatorname{tr}(C)$.

$$A^{T} \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{vmatrix} 5 & 5 \\ -3 & -8 \end{vmatrix} = 5\cdot(-8) + 5(-3) = -55$$

$$+ tr(C) = -55 + 7 = 48$$

2. Find a "high" power of a given matrix.
For example:
Find

$$\begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}^{2023}.$$

$k=1$ $\begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}$ $k:3$ $\begin{bmatrix} -1 & 6 \\ 0 & -1 \end{bmatrix}$

$k=2$ $\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$

Our IH: $\forall k \in \mathbb{N} \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}^{k} = \begin{bmatrix} (-1)^k & 2(-1)^{k+1} \cdot k \\ 0 & (-1)^k \end{bmatrix}$

Base ($\varphi(1)$)

Ass: $\varphi(k) = \begin{bmatrix} (-1)^k & 2(-1)^{k+1} \cdot k \\ 0 & (-1)^k \end{bmatrix}$

$$\varphi(k+1) = \begin{bmatrix} (-1)^k & 2k\cdot(-1)^{k+1} \\ 0 & (-1)^k \end{bmatrix} \cdot \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} (-1)^{k+1} & 2(-1)^k(1+k) \\ 0 & (-1)^{k+1} \end{bmatrix} \Rightarrow \varphi(k)$$

$$\Rightarrow k = 2023 \Rightarrow \begin{bmatrix} (-1)^{2023} & 2(-1)^{2023+1} \cdot 2023 \\ 0 & (-1)^{2023} \end{bmatrix} = \begin{bmatrix} -1 & 4046 \\ 0 & -1 \end{bmatrix}$$

3. Find the inverse of a matrix.
For example:
Find

$$\begin{bmatrix} 1 & 1 & 2 \\ -2 & 0 & -4 \\ 1 & -1 & 1 \end{bmatrix}^{-1}.$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ -2 & 0 & -4 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[l_{3,1,-1}]{l_{2,1,2}} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 & 1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[d_{2,\frac{1}{2}},\ d_{3,-1}]{l_{3,2,1}} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{array}\right]$$

$$\downarrow l_{1,3,-2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & \frac{3}{2} & 2 \\ 0 & 1 & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{array}\right] \xleftarrow{l_{1,2,-1}} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & 2 & 2 \\ 0 & 1 & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{array}\right]$$

6. Find a general solution to a parametric system of linear equations.
For example:
For any $\lambda \in \mathbb{R}$, find a general solution to the following system of linear equations

$$\left[\begin{array}{ccccc|c} 2 & -2 & 2 & -1 & 1 & 3 \\ \lambda & -7 & 4 & -4 & 3 & 13 \\ -3 & 5 & -2 & 3 & -2 & -8 \end{array}\right].$$

$$\xrightarrow[t_{2,3}]{t'_{1,5}}$$

$$\begin{array}{ccccc} x_5 & x_2 & x_3 & x_4 & x_1 \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 2 & -1 & 2 & 3 \\ -2 & 5 & -2 & 3 & -3 & 13 \\ 3 & -7 & 4 & -4 & \lambda & -8 \end{array}\right] \xrightarrow[l_{3,1,-3}]{l_{2,1,2}} \left[\begin{array}{ccccc|c} 1 & -2 & 2 & -1 & 2 & 3 \\ 0 & 1 & 2 & 1 & 1 & 7 \\ 0 & -1 & -2 & -1 & \lambda-6 & -17 \end{array}\right] \xrightarrow[l_{3,2,1}]{l_{1,2,2}} \left[\begin{array}{ccccc|c} 1 & 0 & 6 & 1 & 4 & 17 \\ 0 & 1 & 2 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 & \lambda-5 & -10 \end{array}\right]$$

now we have 2 cases: 1) $\lambda = 5$: no solutions

2) $\lambda \neq 5$:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 6 & 1 & 4 & 17 \\ 0 & 1 & 2 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 & \lambda-5 & -10 \end{array}\right] \xrightarrow{d_{3,\lambda-5}} \left[\begin{array}{ccccc|c} 1 & 0 & 6 & 1 & 4 & 17 \\ 0 & 1 & 2 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 & 1 & \frac{-10}{\lambda-5} \end{array}\right] \xrightarrow[l_{2,3,-1}]{l_{1,3,-4}}$$

$$\begin{array}{ccccc} x_5 & x_2 & x_3 & x_4 & x_1 \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 6 & 1 & 0 & 17+\frac{40}{\lambda-5} \\ 0 & 1 & 2 & 1 & 0 & 7+\frac{10}{\lambda-5} \\ 0 & 0 & 0 & 0 & 1 & \frac{-10}{\lambda-5} \end{array}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \frac{-10}{\lambda-5} \\ \left(7+\frac{10}{\lambda-5} - 2\alpha\right) \\ \alpha \\ \beta \\ 17+\frac{40}{\lambda-5} - 6\alpha - \beta \end{bmatrix}, \text{ where } \alpha, \beta \in \mathbb{R}$$

7. Solve a matrix equation of the form $AX = B$ or $XA = B$, where A is invertible.
For example:
Solve the following matrix equation:

$$\begin{bmatrix} -1 & 3 & -1 \\ -2 & 7 & 0 \\ -2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_4 & x_7 \\ x_2 & x_5 & x_8 \\ x_3 & x_6 & x_9 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

$AX = B$

$A^{-1} \cdot A \cdot X = B \cdot A^{-1}$

$I\ X = B \cdot A^{-1} \Leftrightarrow X = B \cdot A^{-1}$

$$A^{-1} = \begin{bmatrix} -1 & 3 & -1 \\ -2 & 7 & 0 \\ -2 & 7 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -7 & 10 & -7 \\ -2 & 3 & -2 \\ 0 & -1 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 0 & -2 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -7 & 10 & -7 \\ -2 & 3 & -2 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & x_4 & x_7 \\ x_2 & x_5 & x_8 \\ x_3 & x_6 & x_9 \end{bmatrix} = \begin{bmatrix} 4 & -7 & 5 \\ -3 & 3 & -2 \\ -2 & 2 & -1 \end{bmatrix}$$

9. Find numbers $a, b, c, \in \mathbb{R}$ such that the following equality holds true

$$\frac{x^2 + x - 4}{(x+1)(x^2+3)} = \frac{a}{x+1} + \frac{bx + c}{x^2+3}.$$

$$\frac{x^2 + x - 4}{(x+1)(x^2+3)} = \frac{a}{x+1} + \frac{bx+c}{x^2+3}$$

$$\frac{x^2 + x - 4}{(x+1)(x^2+3)} = \frac{ax^2 - bx^2 + bx + cx + c + 3a}{(x+1)(x^2+3)}$$

$$1x^2 + 1x - 4 = x^2(a-b) + x(b-c) + (c+3a)$$

$$\begin{cases} 1 = a - b \\ 1 = b - c \\ -4 = c + 3a \end{cases} \Leftrightarrow \begin{array}{ccc} c & a & b \end{array} \left[\begin{array}{ccc|c} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & 3 & 0 & -4 \end{array}\right] \Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1/2 \\ -3/2 \\ -5/2 \end{bmatrix}$$

8. Find a polynomial by some given values.
For example:
Find a quadratic polynomial (that is, a polynomial of the form $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$) such that

$$f(-2) = 19, \quad f(1) = 4, \quad f(2) = 7.$$

$$\begin{array}{l} f(-2) = \\ f(1) = \\ f(2) = \end{array} \begin{cases} 4a - 2b + c = 19 \\ a + b + c = 4 \\ 4a + 2b + c = 7 \end{cases} \Leftrightarrow \begin{array}{ccc} c & a & b \end{array} \left[\begin{array}{ccc|c} 1 & 4 & -2 & 19 \\ 1 & 1 & 1 & 4 \\ 1 & 4 & 2 & 7 \end{array}\right] \rightarrow (a,b,c) = (2,-3,5)$$

10. Find the sign of a permutation.
For example:
Find the sign of $f \in S_6$ if

$$f = \begin{pmatrix} 2 & 1 & 6 & 4 & 5 & 3 \\ 1 & 5 & 6 & 3 & 4 & 2 \end{pmatrix}.$$

$n = 6$

$(21543)(6)$

$n' = 2$

$(-1)^{6+2} = 1 \Rightarrow sgn(f) = +$

11. Find a composition of two permutations and/or the inverse of a permutation.
For example
Let

$$f = \begin{pmatrix} 2 & 1 & 6 & 4 & 5 & 3 \\ 1 & 5 & 6 & 3 & 4 & 2 \end{pmatrix} \text{ and } g = \begin{pmatrix} 3 & 5 & 1 & 6 & 4 & 2 \\ 6 & 1 & 3 & 4 & 2 & 5 \end{pmatrix}.$$

Then, find $f \circ g^{-1}$.

$$f \circ g^{-1} \begin{pmatrix} 6 & 1 & 3 & 4 & 2 & 5 \\ 2 & 4 & 5 & 6 & 3 & 1 \end{pmatrix}$$

12. Using Gaussian Elimination, evaluate a determinant of order 4.
For example:
Using Gaussian Elimination, evaluate the following determinant:

$$\begin{vmatrix} 2 & -1 & 3 & 4 \\ -2 & 0 & -5 & -1 \\ 4 & -1 & 13 & 6 \\ 6 & -3 & 14 & 12 \end{vmatrix}.$$

$$\begin{vmatrix} 2 & -1 & 3 & 4 \\ -2 & 0 & -5 & -1 \\ 4 & -1 & 13 & 6 \\ 6 & -3 & 14 & 12 \end{vmatrix} \xrightarrow[\substack{l_{3,1,-2} \\ l_{4,1,-3}}]{l_{2,1,1}} \begin{vmatrix} 2 & -1 & 3 & 4 \\ 0 & -1 & -2 & 3 \\ 0 & 1 & 7 & -2 \\ 0 & 0 & 5 & 0 \end{vmatrix} \xrightarrow{l_{2,3,2}} \begin{vmatrix} 2 & -1 & 3 & 4 \\ 0 & 1 & 12 & -1 \\ 0 & 1 & 7 & -2 \\ 0 & 0 & 5 & 0 \end{vmatrix}$$

$$\downarrow l_{3,2,-1}$$

$$10 = (2 \cdot 1 \cdot (-5)(-1)) = \begin{vmatrix} 2 & -1 & 3 & 4 \\ 0 & 1 & 12 & -1 \\ 0 & 0 & -5 & -1 \\ 0 & 0 & 0 & -1 \end{vmatrix} \xleftarrow{l_{4,3,1}} \begin{vmatrix} 2 & -1 & 3 & 4 \\ 0 & 1 & 12 & -1 \\ 0 & 0 & -5 & -1 \\ 0 & 0 & 5 & 0 \end{vmatrix}$$

13. Find all $\lambda \in \mathbb{R}$ such that a given determinant of order 3 is equal to zero.
For example:
Find all $\lambda \in \mathbb{R}$ such that

$$\begin{vmatrix} 2 & -1 & 2 \\ -2 & 2\lambda+7 & 0 \\ -4 & \lambda+5 & \lambda-7 \end{vmatrix} = 0.$$

$$\begin{array}{ccccc} \text{I} & \text{II} & \text{III} & \text{I} & \text{II} \end{array}$$

$$\begin{vmatrix} 2 & -1 & 2 & 2 & -1 \\ 2 & 2\lambda+7 & 0 & 2 & 2\lambda+7 \\ -4 & \lambda+5 & \lambda-7 & -4 & \lambda+5 \end{vmatrix}$$

$$= 4\lambda^2 - 4\lambda - 48 \Rightarrow \lambda_{12} = 3,4$$

solved in our minds "Using Sarrus rule and Photomas app"

14. Find the value of a determinant of size n whose entries are the elements of a given linear recurrence relation.
For example:
Let $b_1 = 1$, $b_2 = 3$, and $b_n = 2b_{n-1} + b_{n-2}$, for any $n \geqslant 3$; let

$$A = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_n \\ b_{n+1} & b_{n+2} & b_{n+3} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n^2-n+1} & b_{n^2-n+2} & b_{n^2-n+3} & \cdots & b_{n2} \end{bmatrix}.$$

Then, for any n, find $\det(A)$.

$$\det(A) = \begin{vmatrix} b_1 & b_2 & b_3 & \cdots & b_n \\ b_{n+1} & b_{n+2} & b_{n+3} & \cdots & b_{2n} \\ b_{2n+1} & b_{2n+2} & b_{2n+3} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n^2-n+1} & b_{n^2-n+2} & b_{n^2-n+3} & \cdots & b_{n^2} \end{vmatrix} \xrightarrow[l'_{3,1,-1}]{l'_{3,2,-2}} \begin{vmatrix} x & x & b_3 - (2b_2 + b_1) & -x \\ & & \| & \\ & & 0 & \end{vmatrix}$$

15. Find the determinant of a square Vandermonde matrix.
For example:
Evaluate the following determinant:

$$\begin{vmatrix} 1 & 2 & 2^2 & 2^3 & 2^4 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 3 & 3^2 & 3^3 & 3^4 \\ 1 & 5 & 5^2 & 5^3 & 5^4 \\ 1 & 2^2 & 2^4 & 2^6 & 2^8 \end{vmatrix}.$$

$$= \begin{vmatrix} 1 & 2 & 2^2 & 2^3 & 2^4 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 3 & 3^2 & 3^3 & 3^4 \\ 1 & 5 & 5^2 & 5^3 & 5^4 \\ 1 & 2^2 & 2^4 & 2^6 & 2^8 \end{vmatrix} = (4-3)(4+1)(4-2)(4-5) \cdot (5-3)(5+1)(5-2) \cdot (3+1)(3-2) \cdot (-1-2)$$

16. Using the Laplace Expansion, evaluate a given determinant.
For example:
Using the Laplace Expansion, find the determinant (we suggest you to start with the second row)

$$\begin{vmatrix} 1 & 2 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{vmatrix}.$$

$$\begin{vmatrix} 1 & 2 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{vmatrix} = 1(-1)^{2+2} \begin{vmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 1 \cdot (-1)^{3+3} \begin{vmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = (1)(-1)^{2+2} \begin{vmatrix} 1 & \alpha \\ 1 & 1 \end{vmatrix} = -(\alpha - 1)$$