2. (HW) Evaluate the indefinite integral using integration by parts:

(a) $\int (x^2+x+4)\cos 3x\,dx$; (b) $\int \frac{x}{\cos^2 x}\,dx$; (c) $\int x\ln^2(x+1)\,dx$;

(d) $\int \frac{\arcsin\sqrt{x}}{\sqrt{1-x}}\,dx$; (e) $\int \sin(\ln x)\,dx$; (f) $\int\sqrt{3-x^2}\,dx$; (g) $\int e^{2x}\cdot\sin(3x)\,dx$.

a) $$\int (x^2+x+4)\cos(3x)\,dx = \left|\begin{matrix} u=x^2+x+4 & dw=\sin(3x)dx \\ du=(2x+1)dx & w=\frac{1}{3}\cos(3x)\end{matrix}\right| = -\frac{1}{3}\cos(3x)(x^2+x+4)+\frac{1}{3}\int(2x+1)\cos(3x)\,dx = \left|\begin{matrix} u=2x+1 & dw=\cos(3x)dx \\ du=2dx & w=\frac{1}{3}\sin(3x)\end{matrix}\right|$$

$$= -\frac{1}{3}\cos(3x)(x^2+x+4)+\frac{1}{9}\sin(3x)(2x+1)-\frac{2}{9}\int\sin(3x)\,dx = \frac{(x^2+x+4)\sin(3x)}{3}+\frac{(2x+1)\cos(3x)}{9}-\frac{2}{27}\sin(3x)+C$$

b) $$\int\frac{x}{\cos^2(x)}\,dx = \left|\begin{matrix} u=x & dw=\frac{dx}{\cos^2(x)} \\ du=dx & w=\tan(x)\end{matrix}\right| = x\tan(x)-\int\tan(x)\,dx = x\tan(x)+\ln(|\cos(x)|)+C$$

$$\int\tan(x)\,dx = \left|\begin{matrix} t=\cos(x) \\ \sin(x)dx=-dt\end{matrix}\right| = -\int\frac{1}{t}\,dt = -\ln(|\cos(x)|)+C$$

c) $$\int x\ln^2(x+1)\,dx = \left|\begin{matrix} t=x+1 \\ dt=dx\end{matrix}\right| = \int(t-1)\ln^2(t)\,dt = \int t\ln^2(t) - \int\ln^2(t)\,dt = \frac{t^2}{2}\left(\ln^2(t)-\ln(t)+\tfrac{1}{2}\right)+t(-\ln^2(t)+2\ln(t)-2)+C =$$

$$= \frac{(x+1)^2}{2}\left(\ln^2(x+1)-\ln(x+1)+\tfrac{1}{2}\right)+(x+1)\left(-\ln^2(x+1)+2\ln(x+1)-2\right)+C$$

$$\int t\ln^2(t)\,dt = \left|\begin{matrix} u=\ln^2(t) & dw=t\,dt \\ du=\frac{2\ln(t)}{t}dt & w=\frac{1}{2}t^2\end{matrix}\right| = \frac{t^2}{2}\ln^2(t)-\int t\ln(t)\,dt = \frac{t^2}{2}\ln^2(t)-\frac{t^2}{2}\ln(t)+\frac{t^2}{4}+C = \frac{t^2}{2}\left(\ln^2(t)-\ln(t)+\tfrac{1}{2}\right)+C$$

$$-\int t\ln(t)\,dt = \left|\begin{matrix} u=\ln(t) & dw=t\,dt \\ du=\frac{dt}{t} & w=\frac{1}{2}t^2\end{matrix}\right| = -\frac{t^2}{2}\ln(t)+\frac{1}{2}\int t\,dt = -\frac{t^2}{2}\ln(t)+\frac{1}{4}t^2+C$$

$$-\int\ln^2(t)\,dt = \left|\begin{matrix} u=\ln^2(t) & dw=dt \\ du=\frac{2\ln(t)}{t}dt & w=t\end{matrix}\right| = -t\ln^2(t)+2\int\ln(t)\,dt = t(-\ln^2(t)+2\ln(t)-2)+C$$

$$2\int\ln(t)\,dt = \left|\begin{matrix} u=\ln(t) & dw=dt \\ du=\frac{dt}{t} & w=t\end{matrix}\right| = 2t\ln(t)-\int dt = 2t\ln(t)-2t+C$$

to check it, I can give you an advise:

diff it: $f(t)=\frac{t^2}{2}\left(\ln^2(t)-\ln(t)+\tfrac{1}{2}\right)+t(-\ln^2(t)+2\ln(t)-2)+C$

it's much easier than check the initial one

d) $$\int\frac{\arcsin(\sqrt{x})}{\sqrt{1-x}}\,dx = \left|\begin{matrix} u=\arcsin(\sqrt{x}) & dw=\frac{dx}{\sqrt{1-x}} \\ du=\frac{dx}{2\sqrt{x-x^2}} & w=-2\sqrt{1-x}\end{matrix}\right| = -2\sqrt{1-x}\arcsin(\sqrt{x})+\int\frac{\sqrt{1-x}}{\sqrt{x-x^2}}\,dx = -2\sqrt{1-x}\arcsin(\sqrt{x})+2\sqrt{x}+C$$

$$\int\frac{\sqrt{1-x}}{\sqrt{x-x^2}}\,dx = \int\sqrt{\frac{1-x}{x(1-x)}}\,dx = \int\frac{dx}{\sqrt{x}} = 2\sqrt{x}+C$$

e) $$\sin(\ln(x))\,dx = \left|\begin{matrix} u=\sin(\ln(x)) & dw=dx \\ du=\frac{\cos(\ln(x))}{x}dx & w=x\end{matrix}\right| = x\sin(\ln(x))-\int\cos(\ln(x))\,dx = \left|\begin{matrix} u=\cos(\ln(x)) & dw=dx \\ du=\frac{-\sin(\ln(x))}{x}dx & w=x\end{matrix}\right| =$$

$$= x\sin(\ln(x))-x\cos(\ln(x))-\int\sin(\ln(x))\,dx \Rightarrow \int\sin(\ln(x))\,dx = \frac{x}{2}\left(\sin(\ln(x))-\cos(\ln(x))\right)+C$$

f) $$\int\sqrt{3-x^2}\,dx = \left|\begin{matrix} u=\sqrt{3-x^2} & dw=dx \\ du=\frac{-x}{\sqrt{3-x^2}}dx & w=x\end{matrix}\right| = x\sqrt{3-x^2}-\int\frac{-x^2}{\sqrt{3-x^2}}\,dx = x\sqrt{3-x^2}+\frac{1}{2}\left(3\arcsin\left(\frac{x}{\sqrt{3}}\right)-x\sqrt{3-x^2}\right)+C$$

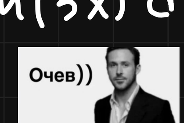
$$-\int\frac{-x^2}{\sqrt{3-x^2}}\,dx = -\int\frac{-x^2-3+3}{\sqrt{3-x^2}}\,dx = -\int\frac{3-x^2}{\sqrt{3-x^2}}\,dx-\int\frac{-3}{\sqrt{3-x^2}}\,dx = -x\sqrt{3-x^2}-\int\frac{x^2}{\sqrt{3-x^2}}\,dx+3\arcsin\left(\frac{x}{\sqrt{3}}\right)+C$$

$$-\int\frac{3-x^2}{\sqrt{3-x^2}}\,dx = \int-\sqrt{3-x^2}\,dx = \left|\begin{matrix} u=-\sqrt{3-x^2} & dw=dx \\ du=\frac{x}{\sqrt{3-x^2}}dx & w=x\end{matrix}\right| = -x\sqrt{3-x^2}-\int\frac{x^2}{\sqrt{3-x^2}}\,dx$$

$$-\int\frac{-3}{\sqrt{3-x^2}}\,dx = 3\int\frac{dx}{\sqrt{3-x^2}} = 3\arcsin\left(\frac{x}{\sqrt{3}}\right)+C$$

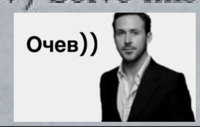
$$-\int\frac{-x^2}{\sqrt{3-x^2}}\,dx+\int\frac{x^2}{\sqrt{3-x^2}}\,dx = -x\sqrt{3-x^2}+3\arcsin\left(\frac{x}{\sqrt{3}}\right)+C \Leftrightarrow \int\frac{x^2}{\sqrt{3-x^2}}\,dx = \frac{1}{2}\left(3\arcsin\left(\frac{x}{\sqrt{3}}\right)-x\sqrt{3-x^2}\right)+C$$

g) $$\int e^{2x}\sin(3x)\,dx = \left|\begin{matrix} u=\sin(3x) & dw=e^{2x}dx \\ du=3\cos(3x)dx & w=\frac{1}{2}e^{2x}\end{matrix}\right| = \frac{1}{2}\sin(3x)e^{2x}-\frac{3}{2}\int e^{2x}\cos(3x)\,dx = \left|\begin{matrix} u=\cos(3x) & dw=e^{2x}dx \\ du=-3\sin(3x)dx & w=\frac{1}{2}e^{2x}\end{matrix}\right| = \frac{1}{2}\sin(3x)e^{2x}-\frac{3}{4}e^{2x}\cos(3x)-\frac{9}{4}\int\sin(3x)e^{2x}\,dx$$

ну это 

$$\Rightarrow \int e^{2x}\sin(3x)\,dx = \frac{4}{13}\left(\frac{1}{2}\sin(3x)e^{2x}-\frac{3}{4}e^{2x}\cos(3x)\right)+C$$

4. (HW) Solve miscellaneous problems:



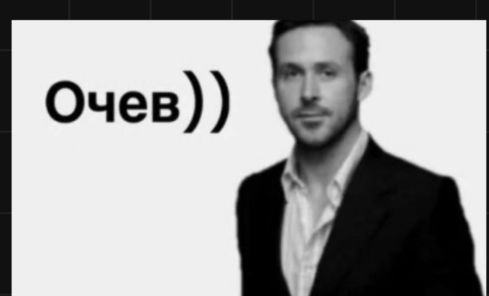
(a) $\int \frac{\ln(\ln x)}{x}\,dx$; (b) $\int \frac{x^2e^x}{(x+2)^2}\,dx$; (c) $\int \sqrt{x}\sin\sqrt{x}\,dx$.

√a) $$\int \frac{\ln(\ln(x))}{x} = \left| \begin{matrix} u = \ln(\ln(x)) & dw = \frac{dx}{x} \\ du = \frac{dx}{\ln(x)\,x} & w = \ln(x) \end{matrix} \right| = \ln(\ln(x))\ln(x) - \int \frac{dx}{x} = \ln(\ln(x))\ln(x) - \ln(x) + C$$

√b) $$\int \frac{x^2e^x}{(x+2)^2}dx = \left| \begin{matrix} u = x^2e^x & dw = \frac{dx}{(x+2)^2} \\ du = e^x(x)(x+2)dx & w = \frac{-1}{x+2} \end{matrix} \right| = -\frac{x^2e^x}{x+2} + \int \frac{e^x(x)(x+2)}{x+2}dx = \frac{-x^2e^x}{x+2} + e^x(x-1) + C$$

$$f(x) = \frac{-x^2e^x}{x+2} + e^x(x-1) + C$$

$$\int \frac{e^x(x)(x+2)}{x+2}dx = \int xe^x dx = \left| \begin{matrix} u = x & dw = e^x dx \\ du = dx & w = e^x \end{matrix} \right| = xe^x - \int e^x dx = xe^x - e^x + C = e^x(x-1) + C$$

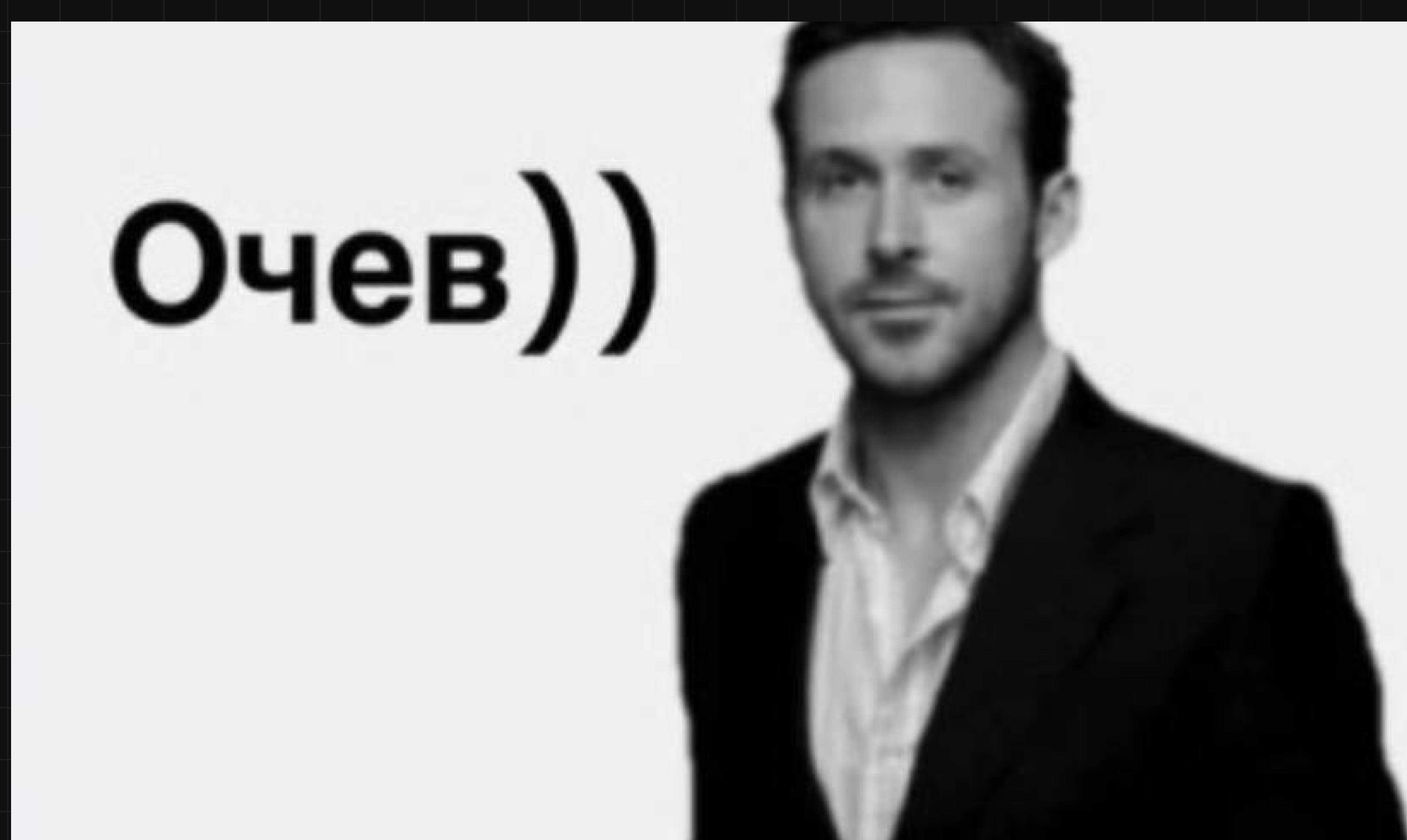


I used the mind techique of differentiating, yes. and trivial sub $u = x-1$ in $\int (x+2)^{-2}dx$

√c) $$\int \sqrt{x}\sin(\sqrt{x})dx = \left| \begin{matrix} t = \sqrt{x} \\ dt = \frac{1}{2\sqrt{x}}dx \end{matrix} \right| = 2\int t^2\sin(t)\,dt = -2t^2\cos(t) + 4t\sin(t) + 4\cos(t) \overset{+C}{=} -2x\cos(\sqrt{x}) + 4\sqrt{x}\sin(\sqrt{x}) + 4\cos(\sqrt{x}) + C$$

$$2\int t^2\sin(t)\,dt = \left| \begin{matrix} u = t^2 & dw = \sin(t)\,dt \\ du = 2t\,dt & w = -\cos(t) \end{matrix} \right| = -2t^2\cos(t) + 4\int t\cos(t)\,dt = -2t^2\cos(t) + 4t\sin(t) + 4\cos(t)$$

$$4\int t\cos(t)\,dt = \left| \begin{matrix} u = t & dw = \cos(t)\,dt \\ du = dt & w = \sin(t) \end{matrix} \right| = 4t\sin(t) - 4\int \sin(t)\,dt = 4t\sin(t) + 4\cos(t) + C$$



5*. (HW) Using integration by parts, for the integral

$$J_n = \int \frac{dx}{(x^2+a^2)^n}, \qquad n \in \mathbb{N}, \quad a \neq 0,$$

prove the following recursive formula:

$$J_{n+1} = \frac{1}{2na^2}\left(\frac{x}{(x^2+a^2)^n} + (2n-1)J_n\right).$$

$$J_{n+1} = \int \frac{dx}{(x^2+a^2)^{n+1}} = \frac{1}{a^2}\int \frac{(x^2+a^2)-x^2}{(x^2+a^2)^{n+1}}\,dx = \frac{1}{a^2}\int \frac{(x^2+a^2)}{(x^2+a^2)^{n+1}}\,dx - \frac{1}{a^2}\int \frac{x^2}{(x^2+a^2)^{n+1}}\,dx = \frac{1}{a^2}J_n - \frac{1}{2a^2}\int \frac{2x^2}{(x^2+a^2)^n}\,dx$$

since $\int \frac{2x^2}{(x^2+a^2)^n}\,dx = \left\{ \begin{matrix} u = x & dw = \frac{2x}{(x^2+a^2)^n}\,dx \\ du = dx & w = -\frac{1}{(n-1)(x^2+a^2)^{n-1}} \end{matrix} \right\}$ $\qquad \int \frac{2x}{(x^2+a^2)^n}\,dx = \{ m = x^2+a^2 \;;\; dm = 2x\,dx \} = \int \frac{1}{m^n}\,dm = -\frac{1}{(n-1)m^{n-1}} = -\frac{1}{(n-1)(x^2+a^2)^{n-1}}$

$$J_{n+1} = \frac{1}{a^2}J_n - \frac{1}{2a^2}\left(\frac{-x}{n(x^2+a^2)^n} + \frac{1}{n}\int \frac{dx}{(x^2+a^2)^n}\right) = \frac{1}{a^2}J_n - \frac{1}{2a^2}\left(\frac{-x}{n(x^2+a^2)^n} + \frac{1}{n}J_n\right) =$$

$$= \frac{x}{2a^2(n)(x^2+a^2)^n} + \frac{1}{a^2}\left(1 - \frac{1}{2n}\right)J_n = \frac{x}{2a^2 n(x^2+a^2)^n} + \frac{1}{a^2}J_n - \frac{1}{2na^2}J_n =$$

$$= \frac{1}{2a^2 n}\left(\frac{x}{(x^2+a^2)^n} + (2n-1)J_n\right) \blacksquare$$