3. (HW) Compute the first- and the second-order total differentials of the following functions:

(a) $f(x,y,z)=\sqrt{x^2+y^2+z^2}$; (b) $f(x,y)=e^{y\sqrt{1+x^2}}$.

a) $f_x = \dfrac{x\,dx}{\sqrt{x^2+y^2+z^2}}$ $f_y = \dfrac{y\,dy}{\sqrt{x^2+y^2+z^2}}$ $f_z = \dfrac{z\,dz}{\sqrt{x^2+y^2+z^2}}$

$f_{xx} = \dfrac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}}dx^2$

$f_{yy} = \dfrac{x^2+z^2}{(x^2+y^2+z^2)^{3/2}}dy^2$

$f_{zz} = \dfrac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}}dz^2$

$\left.\begin{aligned} f_{xy}&=f_{yx}\\ f_{xz}&=f_{zx}\\ f_{yz}&=f_{zy}\end{aligned}\right\}$ since $f(x,y,z)$ satisfy Schwarz's th. or Young's theorem

$f_{xy}=f_{yx} = -\dfrac{xy}{(x^2+y^2+z^2)^{3/2}}dx\,dy$ $f_{xz}=f_{zx} = -\dfrac{xz}{(x^2+y^2+z^2)^{3/2}}dx\,dz$

$f_{yz}=f_{zy} = -\dfrac{yz}{(x^2+y^2+z^2)^{3/2}}dy\,dz$

So $df = (f_x+f_y+f_z)f = \dfrac{x\,dx+y\,dy+z\,dz}{\sqrt{x^2+y^2+z^2}}$

$d^2f = (f_{xx}+f_{yy}+f_{zz}+2f_{xy}+2f_{xz}+2f_{yz})f =$

$= \left((y^2+z^2)dx^2+(x^2+z^2)dy^2+(y^2+x^2)dz^2-2xy\,dx\,dy-2xz\,dx\,dz-2yz\,dy\,dz\right)(x^2+y^2+z^2)^{-3/2}$

b) $f(x,y) = e^{y\sqrt{1+x^2}}$; $f_x = \dfrac{xye^{y\sqrt{1+x^2}}}{\sqrt{1+x^2}}dx$; $f_y = \sqrt{1+x^2}\,e^{y\sqrt{1+x^2}}dy$

$f_{xx} = \dfrac{ye^{y\sqrt{1+x^2}}+x^2y^2\sqrt{1+x^2}\,e^{y\sqrt{1+x^2}}}{(1+x^2)^{3/2}}dx^2$

$f_{yy} = (e^{y\sqrt{1+x^2}}+x^2e^{y\sqrt{1+x^2}})dy^2$

$f_{yx} = xe^{y\sqrt{1+x^2}}\left(\dfrac{1}{\sqrt{1+x^2}}+1\right)dy\,dx = f_{xy} = xe^{y\sqrt{1+x^2}}\left(\dfrac{1}{\sqrt{1+x^2}}+1\right)dx\,dy$

hence $df = \dfrac{xye^{y\sqrt{1+x^2}}}{\sqrt{1+x^2}}dx + \sqrt{1+x^2}\,e^{y\sqrt{1+x^2}}dy$

$d^2f = \dfrac{ye^{y\sqrt{1+x^2}}+x^2y^2\sqrt{1+x^2}\,e^{y\sqrt{1+x^2}}}{(1+x^2)^{3/2}}dx^2 + (e^{y\sqrt{1+x^2}}+x^2e^{y\sqrt{1+x^2}})dy^2 + 2xe^{y\sqrt{1+x^2}}\left(\dfrac{1}{\sqrt{1+x^2}}+1\right)dx\,dy$

4. (HW) Find the first- and the second-order total differentials:

(a) of the function $f(x,y) = x\cos(xy)$ at the point $(\pi/2, -1)$; **(p)**

(b) of the function $f(x,y) = (\sin x)^{\cos y}$ at the point $(\pi/6, \pi/2)$. **(c)**

a) $f_x = (\cos(xy) - xy \cdot \sin(xy))\,dx$

$f_y = -x^2 \sin(xy)\,dy$

$f_{xx} = (-2y\sin(xy) - xy^2\cos(xy))\,dx^2$

$f_{yy} = -x^3\cos(xy)\,dy^2$

$f_{yx} = (-2x\sin(xy) - x^2 y\cos(xy))\,dy\,dx$

$f_x(p) = -\frac{\pi}{2}dx \qquad f_y(p) = -\frac{\pi^2}{4}dy$

$f_{xx}(p) = -2dx^2 \qquad f_y(p) = 0$

$f_{xy} = \pi\,dx\,dy = \pi\,dy\,dx = f_{yx}$

So $df(p) = -\frac{\pi}{2}dx - \frac{\pi^2}{4}dy$ and $d^2f(p) = -2dx^2 + 2\pi\,dx\,dy$

b) $f(x,y) = (\sin(x))^{\cos(y)}$

$f_x = \cos(y)\sin(x)^{\cos(y)-1}\cos(x)\,dx$

$f_{xx} = (\sin(x)^{\cos(y)-2}\cos(y)^2\cos(x)^2 - \sin(x)^{\cos(y)-2}\cos(y))\,dx^2$

$f_y = -\ln(\sin(x))\sin(x)^{\cos(y)}\sin(y)\,dy$

$f_{yy} = (\ln(\sin(x))^2\sin(x)^{\cos(y)}\sin^2(y) - \ln(\sin(x))\sin(x)^{\cos(y)}\cos(y))\,dy^2$

$f_{xy} = (-\cos(x)\sin(y)\sin(x)^{\cos(y)-1} - \cos(x)\cos(y)\ln(\sin(x))\sin(x)^{\cos(y)-1}\sin(y))\,dxy$

$f_x(c) = 0 \qquad f_y(c) = \ln(2)dy \qquad f_{xy} = -\sqrt{3}\,dx\,dy$

$f_{xx}(c) = 0 \qquad f_{yy}(c) = \ln(2)^2 dy^2$

Thus $df(c) = \ln(2)dy \quad d^2f(c) = \ln(2)^2dy^2 - 2\sqrt{3}\,dx\,dy$

7. (HW) Find $d\varphi$ if

$$\text{(a) } \varphi = f(u,v), \quad u = y^2, \quad v = \arctan\left(\frac{y}{x}\right);$$
$$\text{(b) } \varphi = f(u,v,w), \quad u = x^2+y^2+z^2, \quad v = x+y+z, \quad w = xyz.$$

a) $d\varphi = \left(f_u \cdot 0 + f_v \frac{1}{1+y^2/x^2}\left(-y/x^2\right)\right)dx + \left(f_u\, 2y + f_v \frac{1}{1+y^2/x^2} \cdot \frac{1}{x}\right)dy =$

$= -\frac{f_v\, dx}{x^2/y + y} + 2 f_u y + f_v \frac{1}{x^2 + y^2/x}\Big) dy$

b) $\varphi = f(u,v,w)$, $u = x^2+y^2+z^2$, $v = x+y+z$, $w = xyz$

$d\varphi = \left(f_u\, 2x + f_v + f_w\, yz\right)dx + \left(f_u\, 2y + f_v + f_w\, xy\right)dy + \left(f_u\, 2z + f_v + f_w xy\right)dz$

9. (HW) Let f be a twice-differentiable function. Find the second-order total differential of the function $\varphi(x, y, z) = f(u)$ if $u = xyz$.

$$f_x = f'(u)yz \,;\; f_y = f'(u)xz \,;\; f_z = f'(u)xy$$

$$f_{xx} = f''(u)y^2z^2 \,;\; f_{yy} = f''(u)x^2z^2 \,;\; f_{zz} = f''(u)x^2y^2$$

$$f_{xy} = f''(u)xyz^2 + f'(u)z \,;\; f''(u)xy^2z + f'(u)y \,;$$

$$f_{yz} = f''(u)x^2yz + f'(u)x$$

$$d^2\varphi = f''(u)y^2z^2dx^2 + f''(u)x^2z^2dy^2 + f''(u)x^2y^2dz^2 + 2(f''(u)xyz^2 + f'(u)z)dxdy +$$

$$+ 2(f''(u)xy^2z + f'(u)y)dxdz + 2(f''(u)x^2yz + f'(u)x)dydz$$

10. Find $d^3 f$ if

(a) $f = \sin(x^2 + y^2)$; (b) (HW) $f = e^{x^2 y}$ at $(0,1) = C$

$$d^3 f = (f_x + f_y)^3 f = (f_{xxx} + f_{yyy} + 3f_{xyy} + 3f_{xxy})$$

$$f_x = 2xye^{x^2y}\,dx \qquad f_{xx} = e^{x^2y}(2y + 4x^2y^2)\,dx^2$$

$$f_y = x^2 e^{x^2y}\,dy \qquad f_{yy} = x^4 e^{x^2y}\,dy^2$$

$$f_{xy} = f_{yx} = e^{x^2y}(2x + 2x^3)\,dx\,dy$$

$$f_{xxx} = e^{x^2y}(12xy + 8x^3y^3)\,dx^3$$

$$f_{yyy} = x^6 e^{x^2y}$$

$$f_{xyy} = f_{yxy} = f_{yyx} = e^{x^2y}(2x^3 + 2x^5)\,dx\,dy^2$$

$$f_{xxy} = f_{xyx} = f_{yxx} = e^{x^2y}(4x^2y + 4x^4y + 6x^2 + 2)\,dx^2dy$$

$$f_{yyy}(c) = 0 \quad f_{xxx}(c) = 0 \quad f_{xyy}(c) = 0 \quad f_{xxy}(c) = 2dx^2dy$$

Hence $d^3 f = 4ye^{x^2y}(3xy + 2x^3y^2)\,dx^3 + 6e^{x^2y}(1 + 6x^2y + 4x^4y^2)\,dx^2dy +$

$+ 6x^3e^{x^2y}(2 + x^2)\,dy^2dx + x^6e^{x^2y}\,dy^3$

Thus at $(0,1)$ $4e^0(0+0)dx^3 + 6e^0(1+0+0)\,dx^2dy + 6\cdot 0\cdot e^0(2+0)dy^2dx + 0\cdot e^0 dy^3 = 6\,dx^2dy$

11. Find $d^4 f$ if

(a) $f = \cos(x+y)$; (b) (HW) $f = \ln(x^x y^y z^z)$, $x > 0,\ y > 0,\ z > 0$.

$$f_x = \ln(x^x y^y z^z)'_x = (x\ln(x) + \ln(y^y z^z))'_x = \ln(x) + 1$$

$$f_y = \ln(y) + 1 \ ; \ f_z = \ln(z) + 1$$

Thus $df = (\ln(x)+1)dx + (\ln(y)+1)dy + (\ln(z)+1)dz$

$$d^2 f = \frac{1}{x}dx^2 + \frac{1}{y}dy^2 + \frac{1}{z}dz^2$$

$$d^3 f = -\frac{1}{x^2}dx^3 - \frac{1}{y^2}dy^3 - \frac{1}{z^2}dz^3$$

$$d^4 f = \frac{2}{x^3}dx^4 + \frac{2}{y^3}dy^4 + \frac{2}{z^3}dz^4$$

(b) (HW) $f = \ln(x+y+z)$.

$$df = \frac{dx+dy+dz}{x+y+z}$$

$$d^2 f = -\frac{(dx+dy+dz)^2}{(x+y+z)^2}$$

$$d^3 f = \frac{2(dx+dy+dz)^3}{(x+y+z)^3}$$

$$d^4 f = -6\left(\frac{dx+dy+dz}{x+y+z}\right)^4$$

$\vdots \qquad\qquad \vdots$

$$d^n = (-1)^{n-1}\cdot(n-1)!\left(\frac{dx+dy+dz}{x+y+z}\right)^n$$