1. Using the definition, show that

(a) $\lim_{(x,y)\to(a,b)} (2x+y) = 2a+b$; (b) (HW) $\lim_{(x,y)\to(a,b)} (x-3y) = a-3b$.

$0 < \sqrt{(x-a)^3+(y-b)^2} < \delta \Rightarrow |x-a| < \delta \wedge |y-b| < \delta$

So $|x-3y-(a-3b)| = |(x-a)-3(y-b)| \le |x-a| + 3|y-b| < 4\delta$

Hence chosing $\delta = \varepsilon/4$:

it follows that $|x-3y-(a-3b)| < \varepsilon$ when $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$

6. (HW) Prove by definition that the following limits exist.

(a) $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{\sqrt{x^2+y^2}}$; (b) $\lim_{(x,y)\to(0,0)} \frac{y^2 \sin x}{x^2+3y^2}$.

a) $\lim_{(x,y)\to(0,0)} \left(\frac{x^2-y^2}{\sqrt{x^2+y^2}} \right) = 0$

note, that $|y| \le \sqrt{x^2+y^2}$

So if $0 < \sqrt{x^2+y^2} < \delta$, $(x,y) \ne (0,0)$

$|f(x,y) - 0| = \left| \frac{x^2-y^2}{\sqrt{x^2+y^2}} \right| \le \left| \frac{x^2+y^2}{\sqrt{x^2+y^2}} \right| = \left|\sqrt{x^2+y^2}\right| = \sqrt{x^2+y^2} < \delta$

So, we can choose $\delta = \varepsilon$ and conclude that $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{\sqrt{x^2+y^2}} = 0$

b) $\lim_{(x,y)\to(0,0)} \left(\frac{y^2 \sin(x)}{x^2+3y^2} \right) = 0$ let $0 < \sqrt{x^2+y^2} < \delta$

since $\sin(x) \le x$: $\left| \frac{y^2 \sin(x)}{x^2+3y^2} \right| \le \frac{y^2|x|}{x^2+y^2} \le \frac{y^2|x|}{y^2} = |x| \le \sqrt{x^2+y^2} < \delta = \varepsilon$

$\delta = \varepsilon$

So, if we choose $\delta = \varepsilon$, we can conclude, that $\lim = 0$

7. (HW) If the following limit exists, prove it by definition. If the limit does not exist, explain why.

(a) $\lim_{(x,y)\to(0,0)} \frac{x+5y}{x^2+3y^2}$; (b) $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$; (c) $\lim_{(x,y)\to(0,0)} \frac{3xy^2-x^2y}{x^2+y^6}$.

a) $\lim_{(x,y)\to(0,0)} \frac{x+5y}{x^2+3y^2}$, by Heine: consider two sets (approaching)

then if lim exist, then for any series/sets of points tends to $(x_0\ y_0)$ $\lim(f(x_n\ y_n))$ is equal to $\lim_{(x,y)\to(x_0,y_0)}(f(x,y))$

$$\begin{cases} x_n = \frac{1}{n} \\ y_n = 0 \end{cases} \Rightarrow f(x_n, y_n) = \frac{1/n}{1/n^2} = n \xrightarrow[n\to\infty]{} \infty$$

$$\begin{cases} \tilde{x}_n = 1/n \\ \tilde{y}_n - 1/5n \end{cases} \Rightarrow f(\tilde{x}_n\ \tilde{y}_n) = \frac{1/n - 1/n}{1/n^2 + 3/n^2} = \frac{0}{4/n^2} = 0 \xrightarrow[n\to\infty]{} 0$$

So, since for diffrent appoaching limits are different, limit does not exist.

b) $\lim_{(x,y)\to(0,0)} \left(\frac{x^3+y^3}{x^2+y^2}\right) = 0$; let $0 < \sqrt{x^2+y^2} < \delta \Rightarrow |x| < \delta \wedge |y| < \delta$

since $|x| \leq \sqrt{x^2+y^2}$
$|y| \leq \sqrt{x^2+y^2}$

then $\left|\frac{x^3+y^3}{x^2+y^2}\right| \leq \left|\frac{x^3}{x^2+y^2}\right| + \left|\frac{y^3}{x^2+y^2}\right| \leq \left|\frac{x^3}{x^2}\right| + \left|\frac{y^3}{y^2}\right| = |x| + |y| < 2\delta = \varepsilon$

So if we choose $\delta = \frac{1}{2}\varepsilon$, we can conclude that lim exist and equal 0.

c) $\lim_{(x,y)\to(0,0)} \left(\frac{3xy^2 - x^2y}{x^2+y^6}\right)$ DNE

consider two approaching to $(0,0)$:

$$\begin{cases} \bar{x}_n = 0 \\ \bar{y}_n = 1/n \end{cases} \Rightarrow f(\bar{x}_n\ \bar{y}_n) = \frac{3\cdot 0\cdot n^{-2} - 0^2\cdot n^{-1}}{0^2 + n^{-6}} = 0 \xrightarrow[n\to\infty]{} 0$$

$$\begin{cases} \hat{x}_n = 1/n \\ \hat{y}_n = 1 \end{cases} \Rightarrow f(\hat{x}_n\ \hat{y}_n) = \frac{3/n - 1/n^2}{1/n^2 + 1} = \frac{1/n(3 - 1/n)}{1/n^2 + 1} \xrightarrow[n\to\infty]{} 3$$

So, Limit doesn't exist.

8. (HW) Is the function

$$f(x,y) = \begin{cases} \frac{6x^2+5xy^2+3y^2}{2x^2+y^2}, & (x,y) \neq (0,0) \\ 3, & (x,y) = (0,0) \end{cases}$$

, continuous at $(0,0)$?

Yes, it is, since $\lim_{(x,y)\to(0,0)} \left(\frac{6x^2+5xy^2+3y^2}{2x^2+y^2} \right) = 3$

$0 < \sqrt{x^2+y^2} < \delta$

$$\left| \frac{6x^2+5xy^2+3y^2}{2x^2+y^2} - 3 \right| = \left| \frac{6x^2-6x^2+5xy^2+3y^2-3y^2}{2x^2+y^2} \right| = \left| \frac{5xy^2}{2x^2+y^2} \right| \leq$$

$$\leq \left| \frac{5xy^2}{x^2+y^2} \right| \leq \left| \frac{5xy^2}{y^2} \right| \leq 5|x| < 5\delta = \varepsilon$$

So we can choose $\delta = \frac{1}{5}\varepsilon$, and conclude that $f(x,y)$ is continuous at $(0,0)$

10*. (HW) Find the points at which the function

$$f(x,y) = xy \cdot D(x) \cdot D(y) = \begin{cases} xy, & x \in \mathbb{Q} \text{ and } y \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \text{ or } y \notin \mathbb{Q} \end{cases}$$

is continuous.

10) a) Consider point that lies on axises, i.e. (x_0, y_0) s.t. $x_0 y_0 = 0$

So $x_0 = 0 \vee y_0 = 0$. Without loss of generality let $x_0 = 0$, then

for $(0, y_0)$ s.t. $|y_0| < 1 \Rightarrow |y| < |y_0| + 1$ and $|y D(x) D(y)| < |y_0| + 1$

Thus $f(x,y)$ is a product of infinitesimal on bounded. Hence

$\lim\limits_{(x,y)\to(0\, y_0)} f(x,y) = 0 = f(0, y_0)$ so $f(x,y)$ is continuous at $(x_0\, y_0)$

b) Consider point wich isn't lie on a axis ($x_0 y_0 \neq 0$)

So if lim exist, and $\lim\limits_{(x,y)\to(x_0\, y_0)} (f(x,y)) = c$, then by Heinne, limits of all seq.

$\left(\lim\limits_{k\to\infty} (f(x_k\, y_k)) = (x_0, y_0) = (0,0) \wedge (x_k\, y_k) \neq (x_0\, y_0)\right)$ their limits are the same.

But for any Real $x_0 \wedge y_0$ exist seq. of Rational numbers x'_k and y'_k

s.t. $\lim\limits_{k\to\infty} (x'_k) = x_0$ and $\lim\limits_{k\to\infty} (y'_k) = y_0$ and exist seq. of IReal numbers $\tilde{x}_k$ and $\tilde{y}_k$

s.t. $\lim\limits_{k\to\infty} (\tilde{x}_k) = x_0$ and $\lim\limits_{k\to\infty} (\tilde{y}_k) = y_0$ and $x'_k \neq x_0$, $y'_k \neq y_0$ also $\tilde{x}_k \neq x_0$, $\tilde{y}_k \neq y_0$

So $f(x'_k\, y'_k) = x'_k \cdot y'_k$ $\lim\limits_{k\to\infty} (f(x'_k\, y'_k)) = x_0 y_0$, $f(\tilde{x}_k\, \tilde{y}_k) = 0$ So $c = x_0 y_0 = 0$

Since $x_0 y_0 \neq 0$ in that case we obtain contradiction and conclude that $f(x,y)$ is discon. at $(x_0\, y_0)$

So $f(x,y)$ is continuous if $x \cdot y = 0$ (that is $x = 0$ or/and $y = 0$)

and $f(x,y)$ is discontinuous at $xy \neq 0$ (that is otherwise)

Tnx for checking -Novosad