1)* Suppose $A \cap B = \emptyset$. Then $C^{A \cup B} \sim C^{A} \times C^{B}$ Novosad Ivan

note that $X^{Y} \Leftrightarrow \{ f \in P(Y \times X) \mid f \text{ is a function from } A \text{ to } B \}$;

if A has n elements, then $P(A)$ hase 2^{n} elements;

if A has a n elements and B has m elements, then $A \times B$ has $n \cdot m$ elements.

then obviously $A^{B} = 2^{n \cdot m}$ (since $A \times B$ has $n \cdot m$ elements, then $P(A \times B)$ has $2^{n \cdot m}$ element s)

then $A \cup B$ has $n + m$ elements ($|A| = n$ $|B| = m$ $\Rightarrow |A \cup B| = |A| + |B| - |\overset{=0}{A \cap B}| \Rightarrow |A \cup B| = n + m$)

suppose C has k elements ($|C| = k$) then $|C^{A \cup B}| = |P((A \cup B) \times (C))| = 2^{(n+m)k}$

then $C^{A \cup B}$ has $(n+m)k$ elements.

at the other hand $|C^{A}| = |P(A \times C)| = 2^{nk}$; $|C^{B}| = |P(B \times C)| = 2^{mk}$

then $C^{A} \times C^{B}$ has $2^{nk} \cdot 2^{mk} = 2^{(n+m)k}$ elements

$|C^{A \cup B}| = |C^{A} \times C^{B}| \Rightarrow C^{A \cup B} \sim C^{A} \times C^{B}$ ∎

this coll. is obvious. (that was on the lecture.)

2. Let $\mathcal{P}_1(A)$ be the set of all subsets of A of the form $\{x\}$. Prove that $\mathcal{P}_1(A) \sim A$ for any A.

3. Using indicator functions, prove the following statements for arbitrary A, B, C:

a) $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$;

b) $(A \setminus B) \cup B = A \iff B \subseteq A$.

2) Let A have n many elements. Then $P_1(A)$ has n many singletons of each element from A. Hence we can create a function $f: A \to P_1(A)$, which maps every elements from a to the couispanding elements in $P_1(A)$

Hence $P_1(A) \sim A$

3 $I_{(A \cup B) \setminus C} = I_{\bar{C}} = 1 - I_C$

$I_{(A \setminus C) \cup (B \setminus C)} = I_{(\bar{C} \cap B) \cup (\bar{C} \cap A)} =$

$= (1 - I_C) I_B + (1 - I_C) I_A - (1 - I_C) I_B (1 - I_C) I_B =$

$= I_B - I_C I_B + I_A - I_C I_A - (I_B - I_C I_B)(I_A - I_C I_A) =$

$= I_B - I_C I_B + I_A - I_C I_A - (I_B I_A - I_C I_A I_B - I_A I_B I_C + I_C^2 I_A I_B) =$

$= I_B - I_C I_B + I_A - I_C I_A - I_B I_A - I_C I_A I_B = (A \cup B) \setminus C$

b) $(A \setminus B) \cup B = I_A \cdot I_{\bar{B}} + I_B - \underbrace{I_A \cdot I_{\bar{B}} \cdot I_B}_{= 0} = I_A \cdot I_{\bar{B}} + I_B =$

$= I_A (1 - I_B) + I_B = I_A - I_B I_A + I_B = I_A + I_B - I_A I_B = A \cup B = A \Rightarrow B \subseteq A.$

4. Applying Cantor–Bernstein–Schröder theorem (if you need), prove that:

a) $\mathbb{N}^{\mathbb{N}\times\mathbb{Q}} \times \mathbb{N} \sim \mathbb{R}^{\mathbb{Q}}$;

b) $\underline{5}^{\mathbb{N}} \sim \underline{3}^{\mathbb{N}}$;

c) any square (with the interior) and disc (the interior of a circle) in the plane are equivalent to each other;
(*Hint: think of the motions of the plane.*)

d) the set of all possible triangles in the plane is equivalent to $\mathbb{R}$.

a) $\mathbb{N}^{\mathbb{N}\times\mathbb{Q}} \times \mathbb{N} \sim \mathbb{R}^{\mathbb{Q}} \times \mathbb{N} \precsim \mathbb{R}^{\mathbb{R}} \times \mathbb{R} \sim \mathbb{R}^{3} \sim \mathbb{R} \precsim \mathbb{R}^{\mathbb{Q}} \precsim \mathbb{R}^{2} \sim \mathbb{R} \precsim \mathbb{R}^{\mathbb{Q}}$

Hence, $\mathbb{N}^{\mathbb{N}\times\mathbb{Q}} \times \mathbb{N} \sim \mathbb{R}^{\mathbb{Q}}$

b) $5^{\mathbb{N}} \precsim \mathbb{N}^{\mathbb{N}} \sim \mathbb{R} \sim \underline{2}^{\mathbb{N}} \precsim \underline{3}^{\mathbb{N}} \precsim 5^{\mathbb{N}}$

Hence $\underline{5}^{\mathbb{N}} \sim \underline{3}^{\mathbb{N}}$

c) square on a plane has 4 points $x_1\ x_2\ y_1\ y_2$

Hence, it's $\mathbb{R}^4$

disk on a plane has 3 points x, y, r

Hence it's $\mathbb{R}^3$

Since $\mathbb{R}^4 \sim \mathbb{R} \sim \mathbb{R}^3 \Rightarrow \mathbb{R}^4 \sim \mathbb{R}^3$

d) triangle on a plane has 3 points $x\ y\ z$

Hence it's $\mathbb{R}^3$;

Well know fact $\mathbb{R}^3 \sim \mathbb{R}$

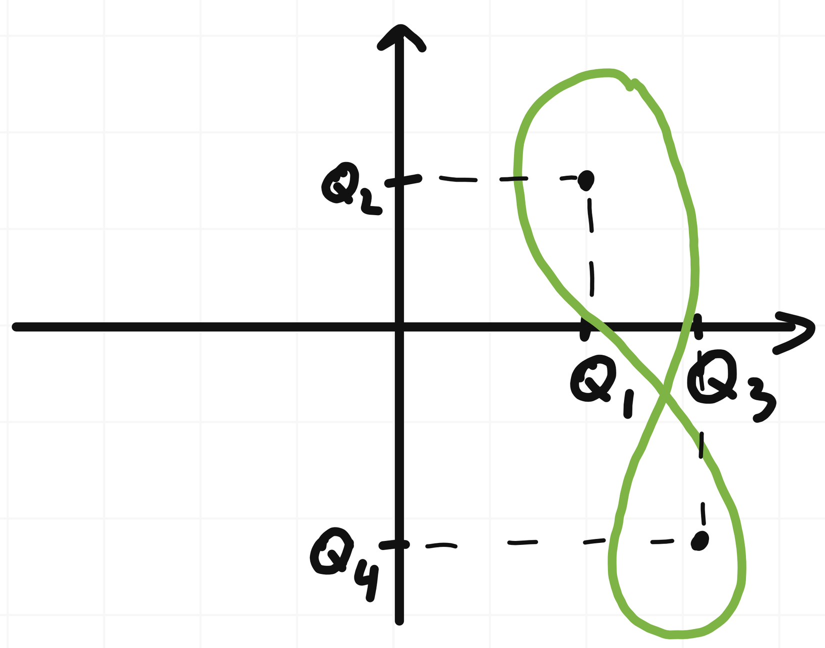
5*. Let X be some set of pairwise disjoint figures-of-eight in the plane. Prove that $X \lesssim \mathbb{N}$, that is, there are no more than countably many such figures in X.

6*. Prove that there exists a set $S \subseteq \mathcal{P}(\mathbb{R})$ such that all of the following hold: (a) $S \sim \mathbb{R}$; (b) if $X, Y \in S$ and $X \neq Y$, then $X \cap Y = \varnothing$; (c) if $X \in S$, then $X \sim \mathbb{R}$. (*Hint: try to find a similar set $S' \subseteq \mathcal{P}(\mathbb{R}^2)$; then apply a bijection.*)

7*. Let $C = \{f \in \mathbb{R}^{\mathbb{R}} \mid \text{the function } f \text{ is continuous}\}$. Prove that $C \sim \mathbb{R}$. (*Heine's (sequential) definition of continuity might be helpful.*)

(5) let $8 \in X$, then $f: 8 \to Q^2 \times Q^2$, there Q^2 is a point inside the loop of 8.

i.e.



If any two figures of eight intersect, they must share an ordered pair $Q^2 \times Q^2 \Rightarrow \bot$

Hence $X \lesssim Q^2 \times Q^2 \sim Q^4 \sim \mathbb{N}^4 \sim \mathbb{N}$

(6) Let f be a bijetion : $\mathbb{R}^2 \to \mathbb{R}$ and $S = \{ f(\{x\} \times \mathbb{R}) \mid x \in \mathbb{R} \} \Rightarrow$ b. holds

Let $f(\{c\} \times \mathbb{R}) \in S$, then $g: \mathbb{R} \to f(\{c\} \times \mathbb{R})$

$g(x) = f(c, x)$ is a bijection, hence $f(\{c\} \times \mathbb{R}) \sim \mathbb{R} \Rightarrow$ c. holds.

Let $h: \mathbb{R} \to S$ as $h(x) = f(\{x\} \times \mathbb{R})$ is a bijetion and $S \sim \mathbb{R} \Rightarrow$ a. holds.

(7) From Heine's definition:

$\forall x \in \mathbb{R}$ exist a sequence q_n $n \in \mathbb{N}$ of rationals converging to x.

since f is continuous $\Rightarrow f(x) = \lim_{n \to \infty} (f(q_n))$

So, for any value of x there's a sequence, hence, $\mathbb{R}^{\mathbb{R}} \sim Q^{\mathbb{N}} \sim \mathbb{R}$

tnx for checking!