Homework

1. List all cosets of the cyclic subgroup $\langle 8 \rangle$ of $(\mathbb{Z}_9^*, \cdot)$. Count the number of cosets.

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$\mathbb{Z}_9^* = \{1,2,4,5,7,8\}$, $3,6 \notin \mathbb{Z}_9^*$, since $\gcd(3,9) \neq 1$ as well as $\gcd(6,9)$

$\langle 8 \rangle$ generates : $\{1,8\}$ (since $8^1 \equiv 8$; $8^2 = 81 \equiv 1$)

list of all cosets:

$1 \cdot \langle 8 \rangle = \{1, 8\}$ (1)

$2 \cdot \langle 8 \rangle = \{2, 16 \equiv 7(9)\} = \{2, 7\}$ (2)

$4 \cdot \langle 8 \rangle = \{4, 32 \equiv 5(9)\} = \{4, 5\}$ (3)

$5 \cdot \langle 8 \rangle = \{5, 40 \equiv 4(9)\} = \{5, 4\}$ (3)

$7 \cdot \langle 8 \rangle = \{7, 56 \equiv 2(9)\} = \{7, 2\}$ (2)

$8 \cdot \langle 8 \rangle = \{8, 64 \equiv 1(9)\} = \{8, 1\}$ (1)

Hence, 3 unique cosets.

Moreover: $H \sqcup 2H \sqcup 4H = \mathbb{Z}_9^*$

Since $\mathbb{Z}_9^*$ is commutative $gH = Hg \Rightarrow \langle 8 \rangle$ is a normal subgroup.

2. List all subgroups of $(\mathbb{Z}_8^*, \cdot)$. Count the number of subgroups.

$\mathbb{Z}_8^* = \{1, 3, 5, 7\} = G$, 2, 4, 6 not in G, since they are not co-prime with 8.

$\langle 1 \rangle = \{1\}$

$\langle 3 \rangle = \{1, 3\}$

$\langle 5 \rangle = \{1, 5\}$

$\langle 7 \rangle = \{1, 7\}$

and $\mathbb{Z}_8^*$ is also subgroup (since any group is a subgroup)

Hence there are 5 cyclic subgroups.

3. Let G be a group and $H, N \subseteq G$ be its subgroups such that $|H| = 471$ and $|N| = 226$. Find $|H \cap N|$ (make sure to prove the answer is correct).

Since $H \cap N$ is also a subgroup of G, it's cardinality must divide both $|H|$ and $|N|$.

- Lagrange.

$|N|$ is divisiable by $1, 2, 113, 226$

$|H|$ is divisiable by $1, 3, 157, 471$

Hence $|H \cap N|$ is 1, no other Natural number can divide both $|H|$ and $|N|$

0 can not be the cardinality of $|H \cap N|$, since it's a subgroup, thus, it must contain at least the neutral element.

4. Let G be the group of non-degenerate uppertriangular 2 by 2 matrices with real coefficients. Show that the subset $H \subseteq G$ consisting of the matrices of the form $\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)$ is a normal subgroup of G.

First of all, let's prove that H is a subgroup:

1) clearly, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ is a neutral element, since

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

2) Since G consist of non-degenerative matrices, H also does, hence $\forall A \in H$, $\det(A) \neq 0 \Rightarrow \exists! A^{-1}$ s.t. $A \cdot A^{-1} = A^{-1} A = I$

3) $$\underbrace{\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}}_{\in H}\underbrace{\begin{bmatrix} a' & b' \\ 0 & 1 \end{bmatrix}}_{\in H} = \begin{bmatrix} aa' & ab'+b \\ 0 & 1 \end{bmatrix} \in H$$, closed under multiplication

II) Normal subgroup $\Leftrightarrow ghg^{-1} \subseteq H$, $\forall g \in G, \forall h \in H$

$$ghg^{-1} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \cdot \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}^{-1} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/a & -b/ac \\ 0 & 1/c \end{bmatrix} = \begin{bmatrix} x & \frac{-bx+ay+b}{c} \\ 0 & 1 \end{bmatrix} \in H$$

since $ghg^{-1} \subseteq H \Rightarrow H \trianglelefteq G$