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In this HW, you can perform all arithmetic operations on matrices (e.g. multiplication, transforming into RREF, finding the inverse, etc) by a machine.

1. (1 point per item) Let $\mathbb{V}$ and $\mathbb{W}$ be two vector spaces; let $\mathcal{A}$ and $\mathcal{B}$ be an ordered basis for $\mathbb{V}$ and $\mathbb{W}$, respectively; let $\varphi\colon \mathbb{V} \to \mathbb{W}$ be a linear transformation (that is, $\varphi \in \mathcal{L}(\mathbb{V}, \mathbb{W})$). Then, find the coordinate matrix of φ with respect to $\mathcal{A}$ and $\mathcal{B}$ (that is, find $T(\varphi, \mathcal{A}, \mathcal{B})$) if:

   (a) $\mathbb{V} = \mathbb{R}[x;3]$ [1], $\mathbb{W} = \mathbb{R}[x;2]$, $\mathcal{A} = (1+x^3,\ x,\ 1+x+x^2,\ 2+x)$, $\mathcal{B} = (1,\ x^2,\ x)$, and

   $$\varphi(ax^3+bx^2+cx+d) = 3ax^2+2bx+c, \quad \text{for every } ax^3+bx^2+cx+d \in \mathbb{R}[x,3];$$

   [**hint:** since $(1,\ x^2,\ x) \neq (1,\ x,\ x^2)$, be careful about the order of coefficients]

   (b) $\mathbb{V} = \mathbb{W} = \mathbb{R}[x;2]$, $\mathcal{A} = (1+x^2,\ 1-x,\ 1-x+x^2)$, $\mathcal{B} = (2+x,\ x^2,\ 1+x+x^2)$, and

   $$\varphi(ax^2+bx+c) = bx^2+cx, \quad \text{for every } ax^2+bx+c \in \mathbb{R}[x,2];$$

   [**remark:** it is not a part of this problem, but it is advisable to verify that φ is indeed a linear transformation]

   (c) $\mathbb{V} = \mathbb{W} = \mathrm{Mat}_2(\mathbb{R})$, $\mathcal{A} = \mathcal{B} = \left(\begin{bmatrix}1&1\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\1&1\end{bmatrix}\right)$, and $\varphi\colon A \mapsto A^{\mathrm{T}}$, for every $A \in \mathrm{Mat}_2(\mathbb{R})$.

$T(\varphi,A,B)$ a)

$$\frac{d}{dx}(1+x^3) = 3x^2 \qquad T[1+x^3]_A = T(e_1)_A = [3x^2]_B = \begin{bmatrix}0\\3\\0\end{bmatrix}$$

$$\frac{d}{dx}(x) = 1 \qquad T[x]_A = T(e_2)_A = [1]_B = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$\frac{d}{dx}(1+x+x^2) = 1+2x \qquad T[1+x+x^2]_A = T(e_3)_A = [1+2x]_B = \begin{bmatrix}1\\0\\2\end{bmatrix}$$

$$\frac{d}{dx}(2+x) = 1 \qquad T[2+x]_A = T(e_4)_A = [1]_B = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix}0&1&1&1\\3&0&0&0\\0&0&2&0\end{bmatrix} = T(\varphi, A, B)$$

$(2+x \quad x^2 \quad 1+x+x^2)$

b)

$$T[1+x^2]_A = Te_1 = [x]_B = \begin{bmatrix}-1\\-3\\2\end{bmatrix}$$

$T = T(\varphi,A,B)$

$$T[1-x]_A = [-x^2+x]_B = Te_2 = \begin{bmatrix}-1\\-3\\2\end{bmatrix}$$

$$T[1-x+x^2]_A = [-x^2+x]_B = Te_3 = \begin{bmatrix}-1\\-3\\2\end{bmatrix}$$

$$\Downarrow$$

$$T(\varphi, A, B) = \begin{bmatrix}-1&-1&-1\\-2&-3&-3\\2&2&2\end{bmatrix}$$

$T(\varphi,A,B)$ c)

(c) $\mathbb{V} = \mathbb{W} = \mathrm{Mat}_2(\mathbb{R})$, $\mathcal{A} = \mathcal{B} = \left(\begin{bmatrix}1&1\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\1&1\end{bmatrix}\right)$, and $\varphi\colon A \mapsto A^{\mathrm{T}}$, for every $A \in \mathrm{Mat}_2(\mathbb{R})$.

$$T\left(\begin{bmatrix}1&1\\0&0\end{bmatrix}\right) = \left(\begin{bmatrix}1&0\\1&0\end{bmatrix}\right)_A = \left(\begin{bmatrix}1&0\\1&0\end{bmatrix}\right)_B = \begin{bmatrix}1\\-1\\1\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0&1\\0&0\end{bmatrix}\right) = \left(\begin{bmatrix}0&0\\1&0\end{bmatrix}\right)_{A/B} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0&0\\1&0\end{bmatrix}\right) = \left(\begin{bmatrix}0&1\\0&0\end{bmatrix}\right)_{A/B} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0&0\\1&1\end{bmatrix}\right) = \left(\begin{bmatrix}0&1\\0&1\end{bmatrix}\right)_{A/B} = \begin{bmatrix}0\\1\\-1\\1\end{bmatrix}$$

$$\Rightarrow T(\varphi, A, B) = \begin{bmatrix}1&0&0&0\\-1&0&1&1\\1&1&0&-1\\0&0&0&1\end{bmatrix}$$

2. (0.5 points per matrix) Let $\mathbb{V}$ be a vector space over the field of reals; let $\mathcal{A} = (\mathbf{e}_1, \mathbf{e}_2)$ and $\mathcal{B} = (\mathbf{e}_1 + \mathbf{e}_2, 2\mathbf{e}_1 + \mathbf{e}_2)$ be two ordered bases for $\mathbb{V}$; let $\varphi\colon \mathbb{V} \to \mathbb{V}$ be a linear transformation which is defined as:

$$\varphi\colon \begin{bmatrix} x \\ y \end{bmatrix}_{\mathcal{A}} \mapsto \begin{bmatrix} x+2y \\ x-y \end{bmatrix}_{\mathcal{B}}, \quad \text{for every } x \cdot \mathbf{e}_1 + y \cdot \mathbf{e}_2 \in \mathbb{V}.$$

Then, find $T(\varphi, \mathcal{A}, \mathcal{A})$, $T(\varphi, \mathcal{A}, \mathcal{B})$, $T(\varphi, \mathcal{B}, \mathcal{A})$, and $T(\varphi, \mathcal{B}, \mathcal{B})$.

[**hint:** see Problem 3 from Seminar 19]

$$A : \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \quad B : \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$$

$T(\varphi, A, B)$:

$$\varphi(e_1) = \varphi\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}_A\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_B$$
$$\varphi(e_2) = \varphi\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}_A\right) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}_B \quad \Rightarrow T(\varphi, A, B) = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

$T(\varphi, A, A)$:

$$T(\varphi, A, A) = C(A,B)\, T(\varphi, A, B) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}$$

$$C(A,B) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix};$$

$T(\varphi, B, B)$

$$(V, A) \xrightarrow{\varphi} (V, A) \Rightarrow (V, B) \xrightarrow{\varphi} (V, B)$$

thus: $A \to A$; $B \to A$; $A' \to B$; $B' \to B$; $W \to V \Rightarrow T(\varphi, A' B') = C(B' B)\, T(\varphi, A, B)\, C(A, A') \Rightarrow$

$$\Rightarrow C(B, A) \cdot T(\varphi, A, A)\, C(A, B) = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 1 \end{bmatrix}$$

$$T(\varphi, B, A) = C(A, B)\, T(\varphi, B, B)\, \overset{= I_2}{C(B, B)} =$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 3 & 5 \end{bmatrix}$$

3. (1 point per item) Let $\mathbb{V}$ and $\mathbb{W}$ be two vector spaces over the field of reals; let $\mathcal{A} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and $\mathcal{A}' = (\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3)$ be two ordered bases for $\mathbb{V}$; let $\mathcal{B} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ and $\mathcal{B}' = (\mathbf{f}'_1, \mathbf{f}'_2, \mathbf{f}'_3)$ be two ordered bases for $\mathbb{W}$.

Suppose that:

$$[\mathbf{e}'_1]_{\mathcal{A}} = \begin{bmatrix}1\\1\\1\end{bmatrix},\quad [\mathbf{e}'_2]_{\mathcal{A}} = \begin{bmatrix}1\\2\\3\end{bmatrix},\quad [\mathbf{e}'_3]_{\mathcal{A}} = \begin{bmatrix}-2\\-1\\1\end{bmatrix},\quad [\mathbf{f}'_1]_{\mathcal{B}} = \begin{bmatrix}1\\2\\1\end{bmatrix},\quad [\mathbf{f}'_2]_{\mathcal{B}} = \begin{bmatrix}-3\\-3\\-2\end{bmatrix},\quad [\mathbf{f}'_3]_{\mathcal{B}} = \begin{bmatrix}3\\1\\1\end{bmatrix},$$

and a linear transformation $\varphi \in \mathcal{L}(\mathbb{V}, \mathbb{W})$ is given by its coordinate matrix:

$$T(\varphi, \mathcal{A}, \mathcal{B}) = \begin{bmatrix}-13 & 19 & -7\\-20 & 29 & -11\\-11 & 16 & -6\end{bmatrix}.$$

Then:

(a) find $T(\varphi, \mathcal{A}', \mathcal{B}')$;
[**hint:** use Theorem 19.3]

a) $C(A\ A') = \begin{bmatrix}1 & 1 & -2\\1 & 2 & -1\\1 & 3 & 1\end{bmatrix} \quad C(B\ B') = \begin{bmatrix}1 & -3 & 3\\2 & -3 & 1\\1 & -2 & 1\end{bmatrix}$

$$T(\varphi, A, B): \quad (\mathbb{V}, A) \xrightarrow{\varphi} (\mathbb{W}, B)$$
$$\updownarrow$$
$$(\mathbb{V}, A') \xrightarrow{\varphi} (\mathbb{W}, B')$$

$$T(\varphi, A', B') = C(B'\ B)\, T(\varphi, A, B)\, C(A, A')$$

$$T(\varphi, A', B') = \begin{bmatrix}1 & -3 & 3\\2 & -3 & 1\\1 & -2 & 1\end{bmatrix}^{-1} \begin{bmatrix}-13 & 19 & -7\\-20 & 29 & -11\\-11 & 16 & -6\end{bmatrix} \begin{bmatrix}1 & 1 & -2\\1 & 2 & -1\\1 & 3 & 1\end{bmatrix} =$$

$$= \begin{bmatrix}1 & 3 & -6\\1 & 2 & -5\\1 & 1 & -3\end{bmatrix} \begin{bmatrix}-1 & 4 & 0\\-2 & 5 & 0\\-1 & 3 & 0\end{bmatrix} = \begin{bmatrix}-1 & 1 & 0\\0 & -1 & 0\\0 & 0 & 0\end{bmatrix}$$

(b) find $[\varphi(2\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3)]_{\mathcal{B}'}$;
[**hint:** use Item 4) of Theorem 15.1 and Item 1 of Theorem 19.1]

$$[\varphi(2e_1 - e_2 + e_3)]_{B'}$$
$$\parallel$$
$$T(\varphi, A, B') \cdot [2e_1 - e_2 + e_3]_A$$

$$T(\varphi, A, B): \quad (\mathbb{V}, A) \xrightarrow{\varphi} (\mathbb{V}, B)$$
$$(\mathbb{V}, A) \to (\mathbb{V}, B')$$

$$T(\varphi, A, B') = C(B', B)\, T(\varphi, A, B)$$

$$T(\varphi, A, B') = \begin{bmatrix}1 & -3 & 3\\2 & -3 & 1\\1 & -2 & 1\end{bmatrix}^{-1} \begin{bmatrix}-13 & 19 & -7\\-20 & 29 & -11\\-11 & 16 & -6\end{bmatrix} =$$

$$= \begin{bmatrix}-7 & 10 & -4\\2 & -3 & 1\\0 & 0 & 0\end{bmatrix};$$

$$[2e_1 - e_2 + e_3]_A = \begin{bmatrix}2\\-1\\1\end{bmatrix}$$

$$[\varphi([2e_1 - e_2 + e_3])]_{B'} = \begin{bmatrix}-7 & 10 & -4\\2 & -3 & 1\\0 & 0 & 0\end{bmatrix} \begin{bmatrix}2\\-1\\1\end{bmatrix} = \begin{bmatrix}-28\\8\\0\end{bmatrix}$$

$$T(\varphi, A, B): \quad (\mathbb{V}, A) \to (\mathbb{W}, B)$$
$$(\mathbb{V}, A'') \to (\mathbb{W}, B'')$$

$$T(\varphi, A'', B'') = C(B'', B)\, T(\varphi, A, B)\, C(A, A'')$$
$$T(\varphi, A'', B'') = C(B'', B')\, T(\varphi, A', B')\, C(A', A'')$$

$$\begin{bmatrix}-13 & 19 & -7\\-20 & 29 & -11\\-11 & 16 & -6\end{bmatrix} \xrightarrow{RREF} \begin{bmatrix}1 & 0 & 2\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix} \Rightarrow \begin{matrix}\dim(Im(\varphi)) = 2\\ \dim(Ker(\varphi)) = 1\end{matrix} \Rightarrow$$

$\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\right)$ - basis for $Im(\varphi)$

$\left(\begin{bmatrix}-2\\-1\\1\end{bmatrix}\right)$ - basis for $Ker(\varphi)$

hence

$$\to \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-2a\\-1a\\a\end{bmatrix}$$

hence $A'' = \left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\-1\\1\end{bmatrix}\right)$

$$\downarrow\varphi \quad \downarrow\varphi \quad \xrightarrow{\varphi} 0$$

hence $B'' = \left(\begin{bmatrix}-13\\-20\\-11\end{bmatrix}, \begin{bmatrix}19\\29\\16\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right)$

Let's prove it: $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix} = C(B'', B)\, T(\varphi, A, B)\, C(A, A'')$

$$\begin{bmatrix}-13 & 19 & 0\\-20 & 29 & 0\\-11 & 16 & 1\end{bmatrix}^{-1} \begin{bmatrix}-13 & 19 & -7\\-20 & 29 & -11\\-11 & 16 & -6\end{bmatrix} \begin{bmatrix}1 & 0 & -2\\0 & 1 & -1\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}29/3 & -19/3 & 0\\20/3 & -13/3 & 0\\-1/3 & -1/3 & 1\end{bmatrix} \begin{bmatrix}-13 & 19 & 0\\-20 & 29 & 0\\-11 & 16 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$$

4. (2 points) Let $\mathcal{A} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4)$ be an ordered basis for a vector space $\mathbb{V}$, $\mathcal{B} = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ be an ordered basis for a vector space $\mathbb{W}$, and $\mathcal{C} = (\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4, \mathbf{c}_5)$ be an ordered basis for a vector space $\mathbb{Z}$. Then, it can be easily verified that $\mathcal{E} = (3\mathbf{a}_1 + 2\mathbf{a}_3 - \mathbf{a}_4,\ 4\mathbf{a}_2 + 3\mathbf{a}_3 - \mathbf{a}_4,\ \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3,\ \mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3 + \mathbf{a}_4)$ is an ordered basis for $\mathbb{V}$, both $\mathcal{F} = (3\mathbf{b}_1 + \mathbf{b}_3,\ -\mathbf{b}_1 + \mathbf{b}_2,\ \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3)$ and $\mathcal{G} = (2\mathbf{b}_1 - 2\mathbf{b}_2 + \mathbf{b}_3,\ \mathbf{b}_1 + \mathbf{b}_2,\ 2\mathbf{b}_1 - \mathbf{b}_2 + \mathbf{b}_3)$ are ordered bases for $\mathbb{W}$, and $\mathcal{H} = (\mathbf{c}_1 + \mathbf{c}_2,\ \mathbf{c}_2 + \mathbf{c}_3,\ \mathbf{c}_3,\ \mathbf{c}_4 + \mathbf{c}_5,\ \mathbf{c}_5)$ is an ordered basis for $\mathbb{Z}$. Suppose that linear transformations $\varphi \in \mathcal{L}(\mathbb{V}, \mathbb{W})$ and $\psi \in \mathcal{L}(\mathbb{W}, \mathbb{Z})$ are given by the coordinate matrices:

$$T(\varphi, \mathcal{E}, \mathcal{F}) = \begin{bmatrix} 1 & 1 & -7 & 1 \\ 0 & -2 & 3 & -2 \\ 2 & 0 & -3 & 2 \end{bmatrix} \quad \text{and} \quad T(\psi, \mathcal{G}, \mathcal{H}) = \begin{bmatrix} 1 & 1 & -7 \\ 0 & -2 & 3 \\ 1 & 1 & -3 \\ -1 & 0 & 1 \\ 2 & -1 & -2 \end{bmatrix}.$$

Then, find $T(\psi \circ \varphi, \mathcal{A}, \mathcal{C})$.

[**hint:** see a similar problem from Seminar 19]

Novosad Ivan !

$A = (\bar{a}_1; \bar{a}_2; \bar{a}_3; \bar{a}_4)$ for $\mathbb{V}$

$\mathcal{E} = (3\bar{a}_1 + 2\bar{a}_3 - \bar{a}_4;\ 4\bar{a}_2 + 3\bar{a}_3 - \bar{a}_4;\ \bar{a}_1 + \bar{a}_2 + \bar{a}_3;\ \bar{a}_1 - \bar{a}_2 - \bar{a}_3 + \bar{a}_4)$ for $\mathbb{V}$

$B = (\bar{b}_1; \bar{b}_2; \bar{b}_3)$ for $\mathbb{W}$

$F = (3\bar{b}_1 + \bar{b}_3;\ -\bar{b}_1 + \bar{b}_2;\ \bar{b}_1 + \bar{b}_2 + \bar{b}_3)$ for $\mathbb{W}$

$G = (2\bar{b}_1 - 2\bar{b}_2 + \bar{b}_3;\ \bar{b}_1 + \bar{b}_2;\ 2\bar{b}_1 - \bar{b}_2 + \bar{b}_3)$ for $\mathbb{W}$

$C = (\bar{c}_1; \bar{c}_2; \bar{c}_3; \bar{c}_4; \bar{c}_5)$ for $\mathbb{Z}$

$H = (\bar{c}_1 + \bar{c}_2;\ \bar{c}_2 + \bar{c}_3;\ \bar{c}_3;\ \bar{c}_4 + \bar{c}_5;\ \bar{c}_5)$ for $\mathbb{Z}$

$$T(\psi\ G\ H):\quad \begin{array}{ccc} (\mathbb{W}, G) & \xrightarrow{\psi} & (\mathbb{Z}, H) \\ & \downarrow & \\ (\mathbb{W}, B) & \xrightarrow{\psi} & (\mathbb{Z}, C) \end{array}$$

req: $T(\psi \circ \varphi, A, C) = T(\psi, B, C) \cdot T(\varphi\ A, B)$

$T(\psi, B, C) = C(C, H)\, T(\psi, G, H)\, C(G, B)$

$H = (\bar{c}_1 + \bar{c}_2;\ \bar{c}_2 + \bar{c}_3;\ \bar{c}_3;\ \bar{c}_4 + \bar{c}_5;\ \bar{c}_5)$ $G = (2\bar{b}_1 - 2\bar{b}_2 + \bar{b}_3;\ \bar{b}_1 + \bar{b}_2;\ 2\bar{b}_1 - \bar{b}_2 + \bar{b}_3)$

$$C(C, H) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \quad C(B, G) = \begin{bmatrix} 2 & 1 & 2 \\ -2 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$T(\varphi, \mathcal{E}, F):\quad \begin{array}{ccc} (\mathbb{V}, \mathcal{E}) & \xrightarrow{\varphi} & (\mathbb{W}, F) \\ & \Downarrow & \\ (\mathbb{V}, A) & \xrightarrow{\varphi} & (\mathbb{W}, B) \end{array}$$

$T(\varphi, A, B) = C(B, F) \cdot T(\varphi, \mathcal{E}, F) \cdot C(\mathcal{E}, A)$

$B = (\bar{b}_1; \bar{b}_2; \bar{b}_3)$ $F = (3\bar{b}_1 + \bar{b}_3;\ -\bar{b}_1 + \bar{b}_2;\ \bar{b}_1 + \bar{b}_2 + \bar{b}_3)$

$\mathcal{E} = (3\bar{a}_1 + 2\bar{a}_3 - \bar{a}_4;\ 4\bar{a}_2 + 3\bar{a}_3 - \bar{a}_4;\ \bar{a}_1 + \bar{a}_2 + \bar{a}_3;\ \bar{a}_1 - \bar{a}_2 - \bar{a}_3 + \bar{a}_4)$ $A = (\bar{a}_1; \bar{a}_2; \bar{a}_3; \bar{a}_4)$ for $\mathbb{V}$

$$C(B, F) = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad C(A, \mathcal{E}) = \begin{bmatrix} 3 & 0 & 1 & 1 \\ 0 & 4 & 1 & -1 \\ 2 & 3 & 1 & -1 \\ -1 & -1 & 0 & 1 \end{bmatrix}$$

$$T(\psi, B, C) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -7 \\ 0 & -2 & 3 \\ 1 & 1 & -3 \\ -1 & 0 & 1 \\ 2 & -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ -2 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}^{-1} =$$

$$= \begin{bmatrix} 1 & 1 & -7 \\ 1 & -1 & -4 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -3 \\ 1 & 0 & -2 \\ -1 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 9 & -8 & -33 \\ 4 & -5 & -17 \\ 0 & -1 & -1 \\ -2 & 2 & 7 \\ 1 & -2 & -5 \end{bmatrix}$$

$$T(\varphi, A, B) = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -7 & 1 \\ 0 & -2 & 3 & -2 \\ 2 & 0 & -3 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 & 1 \\ 0 & 4 & 1 & -1 \\ 2 & 3 & 1 & -1 \\ -1 & -1 & 0 & 1 \end{bmatrix}^{-1} =$$

$$= \begin{bmatrix} 5 & 5 & -27 & 7 \\ 2 & -2 & 0 & 0 \\ 3 & 1 & -10 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 & -2 \\ 2 & 3 & -5 & -4 \\ -5 & -7 & 13 & 11 \\ 3 & 4 & -7 & -5 \end{bmatrix} = \begin{bmatrix} 171 & 237 & -435 & -362 \\ -2 & -4 & 6 & 4 \\ 64 & 88 & -162 & -135 \end{bmatrix}$$

$$T(\psi \circ \varphi, A, C) = \begin{bmatrix} 9 & -8 & -33 \\ 4 & -5 & -17 \\ 0 & -1 & -1 \\ -2 & 2 & 7 \\ 1 & -2 & -5 \end{bmatrix} \begin{bmatrix} 171 & 237 & -435 & -362 \\ -2 & -4 & 6 & 4 \\ 64 & 88 & -162 & -135 \end{bmatrix}$$

$$= \begin{bmatrix} -557 & -739 & 1383 & 1165 \\ -394 & -528 & 984 & 827 \\ -62 & -84 & 156 & 131 \\ 102 & 134 & -252 & -213 \\ -145 & -195 & 363 & 305 \end{bmatrix}$$

Ans

thx for checking