2. (HW) Find the length of the indicated arc of the given curve:

(a) $y = x^2$, $1 \leqslant x \leqslant 2$; (b) $y = \dfrac{x^2}{2} - \dfrac{\ln x}{4}$, $1 \leqslant x \leqslant 3$.

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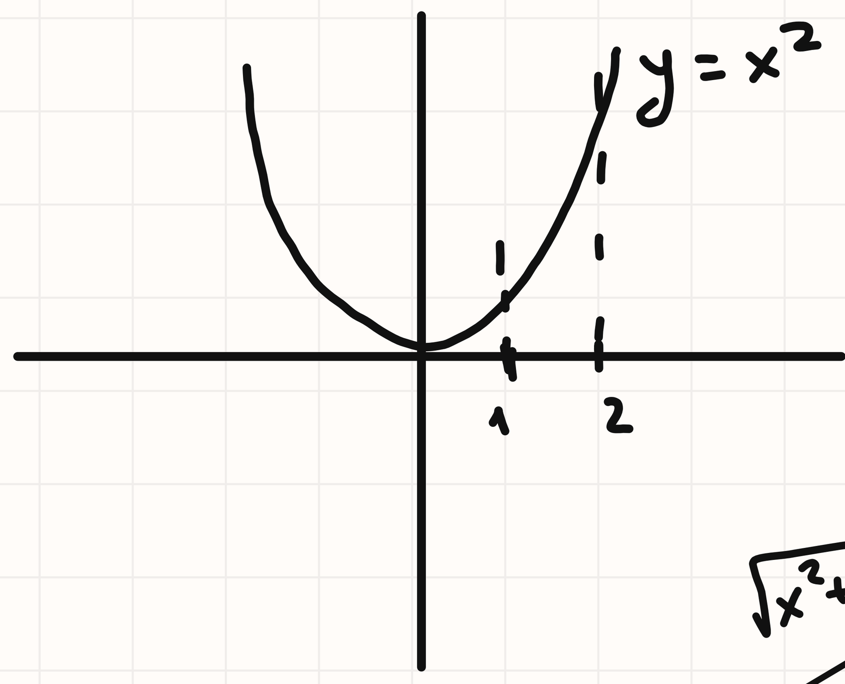
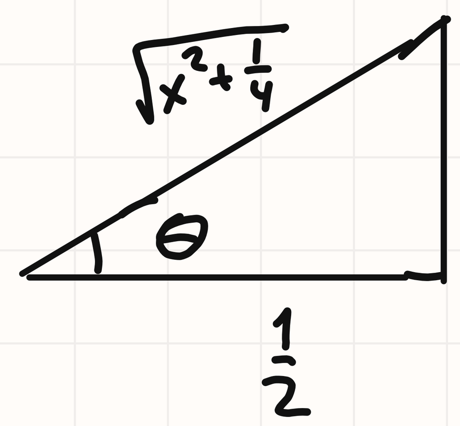


a) $y = x^2$; $x = x$; $1 \le x \le 2$;

$$\int_1^2 \sqrt{\dot{x}^2 + \dot{y}^2}\, dx = \int_1^2 \sqrt{1+4x^2}\, dx =$$

$$= 2\int_1^2 \sqrt{\frac{1}{4} + x^2}\, dx = \left\{ \begin{aligned} x &= \frac{\tan(\theta)}{2} \\ dx &= \frac{1}{2}\sec^2(\theta) \end{aligned} \right\} =$$



$$\Rightarrow \left\{ \begin{aligned} \sec(x) &= 2\sqrt{x^2+\tfrac{1}{4}} \\ \theta &= \arctan(2x) \\ \tan(x) &= 2x \end{aligned} \right.$$

$$\Rightarrow 2\int \frac{1}{2}\sec^2(\theta)\sqrt{\frac{1}{4} + \frac{\tan^2(\theta)}{4}}\, d\theta = \int \frac{1}{2}\sec^2(\theta)\sqrt{1+\tan^2(\theta)}\, d\theta = \int \frac{1}{2}\sec^2(\theta)\sqrt{\sec^2(\theta)}\, d\theta = \frac{1}{2}\int \sec^2(\theta)\sec(\theta)\, d\theta =$$

$$\frac{1}{4}\tan(\theta)\sec(\theta) + \frac{1}{4}\int \sec(\theta)\, d\theta = \frac{1}{4}\tan(\theta)\sec(\theta) + \frac{1}{4}\ln(|\tan(\theta) + \sec(\theta)|) + c \circleddash$$

(well-know formula $\int \sec^m(x)\, dx = \frac{1}{m-1}\sin(x)\sec^{m-1}(x)\ \frac{m-2}{m-1}\int \sec^{m-2}(x)\, dx$

$$\int \sec(x) = \int \frac{\sec^2 x + \sec(x)\tan(x)}{\sec(x)+\tan(x)}\, dx = \left\{ \begin{aligned} u &= \sec(x) + \tan(x) \\ du &= \sec^2(x) + \sec(x)\tan(x) \end{aligned} \right\} = \ln(|u|) + c ..\}$$

$$\circleddash \frac{1}{2}x\sqrt{1+4x^2} + \frac{1}{4}\ln(|2x + \sqrt{1+4x^2}|) + c \Rightarrow$$

$$\Rightarrow \int_1^2 \sqrt{1+4x^2}\, dx = \frac{x}{2}\sqrt{1+4x^2} + \frac{1}{4}\ln(|2x+\sqrt{1+4x^2}|)\Big|_1^2 = \frac{1}{4}\ln\left(\left|\frac{4+\sqrt{17}}{2+\sqrt{5}}\right|\right) + \sqrt{17} - \frac{\sqrt{5}}{2}$$

b) $y = \frac{x^2}{2} - \frac{\ln(x)}{4}$, $1 \le x \le 3$ $\Rightarrow$ $y' = \dfrac{4x^2 - 1}{4x}$

$$\text{Arc lenght} = \int_1^3 \sqrt{\dot{x}^2 + \dot{y}^2}\, dx = \int_1^3 \sqrt{1 + \left(\frac{4x^2-1}{4x}\right)^2}\, dx = \int_1^3 \sqrt{1 + \frac{16x^4 - 8x^2 + 1}{16x^2}}\, dx$$

$$= \int_1^3 \left(\frac{16x^4 + 8x^2 + 1}{16x^2}\right)^{1/2} dx = \int_1^3 \left(\frac{(4x^2+1)^2}{(4x)^2}\right)^{1/2} dx = \int_1^3 \frac{4x^2+1}{4x}\, dx = \frac{x^2}{2} + \frac{1}{4}\ln(x)\Big|_1^3 =$$

since $x > 0$

$$= \frac{9}{2} + \frac{1}{4}\ln(3) - \frac{1}{2} = 4 + \frac{1}{4}\ln(3)$$

4. (HW) Find the arc length of the curve

$$x(t) = \sin^3(e^t), \quad y(t) = \cos^3(e^t), \quad \ln\frac{\pi}{4} \leqslant t \leqslant \ln\frac{\pi}{2}.$$

$a = \ln(\pi/4)$ $b = \ln(\pi/2)$ since I'm lazy.

$x(t)' = 3\sin^2(e^t)\cos(e^t)e^t$

$y(t) = 3\cos^2(e^t)\sin(e^t)e^t$

arc lenght $= \int_a^b \sqrt{9\sin^4(e^t)\cos^2(e^t)e^{2t} + 9\cos^4(e^t)\sin^2(e^t)e^{2t}}\, dt =$

$= \int_a^b 3e^t\sqrt{\sin^4(e^t)\cos^2(e^t)+\cos^4(e^t)\sin^2(e^t)}\, dt = \left\{ \begin{array}{l} k = e^t \\ dk = e^t dt \end{array} \right\} =$

$= 3\int_{e^a}^{e^b} \left(\sin^2(k)\cos^2(k)\left(\sin^2(k) + \cos^2(k)\right)\right)^{1/2} dk = 3\int_{\frac{\pi}{4}}^{\pi/2} \sin(k)\cos(k)\, dk =$

$= \frac{3}{2}\int_{\frac{\pi}{4}}^{\pi/2} \sin(2k)\, dk = -\frac{3}{4}\cos(2k)\Big|_{\pi/4}^{\pi/2} = -\frac{3}{4}\left(\cos(\pi) - \cos(\pi/2)\right) = -\frac{3}{4}(-1-0) = \frac{3}{4}$

6. (HW) Find the arc length of the curve

$$x(t) = \sin^2 t, \quad y(t) = \sin t\cos t, \quad z(t) = \frac{t^2}{2}, \quad 0 \leqslant t \leqslant 1.$$

$x(t)' = 2\sin(t)\cos(t) \quad \Rightarrow \quad \dot{x}^2 = \sin(2t)^2$

$y(t)' = \frac{1}{2}(\sin(2t))' = \cos(2t) \quad \dot{y}^2 = \cos^2(2t)$

$z(t)' = t \quad \Rightarrow \dot{z} = t^2$

arc lenght $= \int_0^1 (\sin^2(2t) + \cos^2(2t) + t^2)^{1/2} dt = \int_0^1 \sqrt{1+t^2}\, dt \Rightarrow$

$\cdot(\sec(\theta)+\tan(\theta))$ $\quad u = \cos(\theta)$

$\left\{ \begin{array}{l} t = \tan(\theta) \\ dt = \sec^2(\theta) \end{array} \right\} \Rightarrow \int_0^{\pi/4} \sec^3(\theta)\, d\theta = \int_0^{\pi/4} \sec(\theta)\, d\theta + \int_0^{\pi/4} \sec(\theta)\tan(\theta)\, d\theta =$

$= \frac{1}{2}\tan(\theta)\sec(\theta) + \frac{1}{2}\ln(|\sec(\theta)+\tan(\theta)|)\Big|_0^{\pi/4} = \frac{1}{2}\left(1\cdot\frac{2}{\sqrt{2}} + \ln\left(\left|\frac{\frac{2}{\sqrt{2}}+1}{1+0}\right|\right)\right) = \frac{\sqrt{2}}{2} + \frac{1}{2}\ln(\sqrt{2}+1)$

8. (HW) Find the area of the surface generated by revolving the arc of $x = y^3$ from $y = 0$ to $y = 1$ about the y-axis.

$$S = 2\pi \int_0^1 y^3 \sqrt{1+9y^4}\, dy \rightarrow \left\{ \begin{array}{l} \theta = 1 + 9y^4 \\ d\theta = 36 y^3 dy \end{array} \right\} \Rightarrow \frac{2\pi}{36} \int \sqrt{\theta}\, d\theta = \frac{\pi}{27} (\theta)^{3/2} + C \Rightarrow$$

$$\Rightarrow S = \frac{\pi}{27} (1+9y^4)^{3/2} \Big|_0^1 = \frac{\pi}{27} \left(10\sqrt{10} - 1 \right)$$

10. (HW) Find the area of the surface of solid obtained by revolving the arc of

$$x(t) = e^t \cos t, \quad y(t) = e^t \sin t$$

from $t = 0$ to $t = \pi$ about the x-axis.

$$x(t)' = e^t \cos(t) - e^t \sin(t) = e^t (\cos(t) - \sin(t))$$

$$y(t)' = e^t \sin(t) + e^t \cos(t) = e^t (\cos(t) + \sin(t))$$

$$S = 2\pi \int_0^\pi e^t \sin(t) \left(e^{2t} (\cos^2(t) - \sin^2(2t) + \sin^2(t)) + e^{2t} (\cos^2(t) + \sin(2t) + \sin^2(t)) \right)^{1/2} dt =$$

$$= 2\pi \int_0^\pi e^t \sin(t) \left(e^{2t} (1 - \sin^2(2t) + 1 + \sin^2(2t)) \right)^{1/2} dt =$$

$$= 2\pi \int_0^\pi e^{2t} \sin(t) \sqrt{2}\, dt = \frac{4\sqrt{2}\pi}{5} \left(e^{2t} \left(\sin(t) - \tfrac{1}{2} \cos(t) \right) \right) \Big|_0^\pi = \frac{4\sqrt{2}}{5} \pi \left(\frac{e^{2\pi}}{2} + \frac{1}{2} \right)$$

$$\int e^{2t} \sin(t) dt = \left\{ \begin{array}{ll} u = \sin(t) & dw = e^{2t} dt \\ du = \cos(t) dt & w = \frac{1}{2} e^{2t} \end{array} \right\} \checkmark = \frac{1}{2} \sin(t) e^{2t} - \frac{1}{2} \int e^{2t} \cos(t) dt =$$

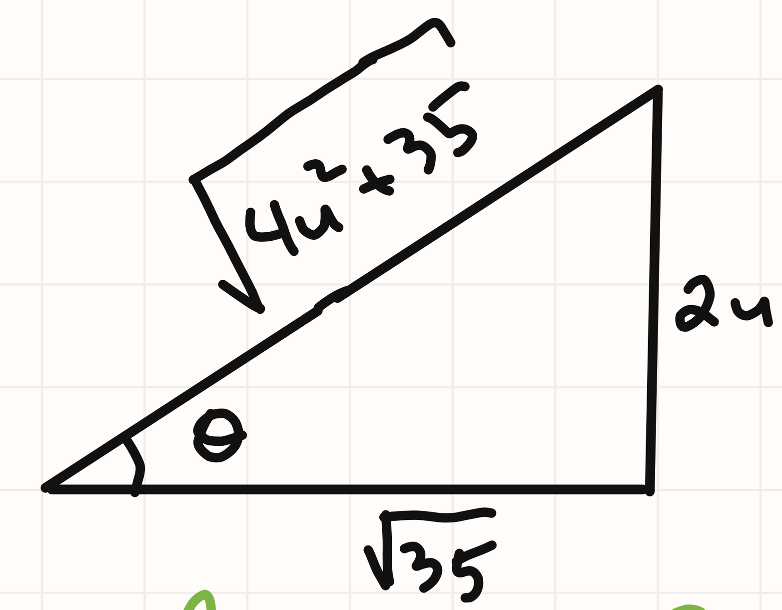
$$= \left\{ \begin{array}{ll} u = \cos(t) & dw = e^{2t} dt \\ du = -\sin(t) dt & w = \frac{1}{2} e^{2t} \end{array} \right\} = \frac{1}{2} \sin(t) e^{2t} - \frac{1}{4} e^{2t} \cos(t) - \frac{1}{4} \int e^{2t} \sin(t) dt \Rightarrow$$

$$\int e^{2t} \sin(t) dt = \frac{2}{5} e^{2t} \left(\sin(t) - \frac{1}{2} \cos(t) \right)$$

11*. (HW) Find the arc length of the curve

$$x(t) = 3t, \quad y(t) = \frac{2}{3}t^{3/2}, \quad z(t) = 0.5t^2, \quad 0 \leqslant t \leqslant 1.$$

$\dot{x}(t) = 3 \qquad \dot{y}(t) = \sqrt{t} \qquad \dot{z}(t) = t$



$$\int_0^1 \sqrt{9+t+t^2}\,dt = \int_0^1 \sqrt{\left(t+\tfrac{1}{2}\right)^2+\tfrac{35}{4}}\,dt = \left\{ \begin{array}{l} u = t+\frac{1}{2} \\ du = dt \\ 1 \to \frac{3}{2} \quad 0 \to \frac{1}{2} \end{array} \right\}$$

$$= \int_{1/2}^{3/2} \sqrt{u^2+\tfrac{35}{4}}\,du = \left\{ \begin{array}{ll} u = \frac{\sqrt{35}}{2}\tan(\theta) & 3/2 \to \overbrace{\arctan\left(\frac{3}{\sqrt{35}}\right)}^{b} \\ du = \frac{\sqrt{35}}{2}\sec^2(\theta)\,d\theta & 1/2 \to \underbrace{\arctan\left(\frac{1}{\sqrt{35}}\right)}_{=a} \end{array} \right\} = \left\{ \begin{array}{l} \tan(\theta) = \frac{2u}{\sqrt{35}} \\ \sec(\theta) = \sqrt{4u^2+35}/\sqrt{35} \end{array} \right\}$$

$$= \frac{35}{4}\int_a^b \sec^3(\theta)\,d\theta = \frac{35}{8}\sec(\theta)\tan(\theta) + \frac{35}{8}\ln\left(\left|\sec(\theta)+\tan(\theta)\right|\right)\Bigg|_a^b =$$

$$\frac{2u\sqrt{4u^2+35} + 35\ln\left(\left|2u+\sqrt{4u^2+35}\right|\right) - \ln\left(\sqrt{35}\right)}{8}\Bigg|_{1/2}^{3/4} =$$

$$\frac{u}{4}\sqrt{4u^2+35} + \frac{35}{8}\ln\left(2u+\sqrt{4u^2+35}\right)\Bigg|_{1/2}^{3/2} =$$

$$\frac{3}{8}\sqrt{44} + \frac{35}{8}\ln\left(\left|\frac{3+\sqrt{44}}{7}\right|\right) - \frac{6}{8} = \frac{3(\sqrt{11}-1)}{4} + \frac{35}{8}\ln\left(\left|\frac{3+2\sqrt{11}}{7}\right|\right)$$

Tnx for checking

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