2. (HW) Find $f_x(x,y)$ and $f_y(x,y)$ for the function $f(x,y) = xe^{x^3y}$, and evaluate each at the point $(1, \ln 2)$.

$$f_x(x,y) = e^{x^3y} + x \cdot e^{x^3y} \cdot x^2y \cdot 3 = e^{x^3y}(1 + 3x^3y)$$

$$f_y(x,y) = x^4 e^{x^3y}$$

So $f_x(1, \ln(2)) = e^{\ln(2)}(1 + 3\ln(2)) = 2 + 6\ln(2)$

$f_y(1, \ln(2)) = e^{\ln(2)} = 2$

4. (HW) Find $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$, and $\dfrac{\partial f}{\partial z}$ if $f(x,y,z) = z\ln(xy^2 - 2x^2\sin z)$.

$$\frac{df}{dx} = \frac{z(y^2 - 4x\sin(z))}{xy^2 - 2x^2\sin(z)} \qquad \frac{df}{dy} = \frac{2xyz}{x^2y - 2x^2\sin(z)}$$

$$\frac{df}{dz} = \ln(xy^2 - 2x^2\sin(z)) + \frac{z(-2x^2\cos(z))}{xy^2 - 2x^2\sin(z)}$$

6. (HW) Show that $f_x(0,0)$ and $f_y(0,0)$ both exist, but f is not differentiable at $(0,0)$ where

$$f(x,y) = \begin{cases} \frac{(x+y)^2}{x^2+y^2}, & \text{if } x^2+y^2 \neq 0, \\ 1, & \text{if } x^2+y^2 = 0, \end{cases}$$

by definition:

$$f_x(0,0) = f'(x,0) \Rightarrow \frac{(x+0)^2}{x^2+0^2} = \frac{x^2}{x^2} = 1 \Rightarrow f'(x,0) = 0 \Rightarrow f_x(0,0) = 0$$

$$f_y(0,0) = f'(0,y) \Rightarrow \frac{y^2}{y^2} = 1 \Rightarrow f'(0,y) = f_y(0,0) = 0$$

to show that $f(x,y)$ has discontin. at $(0,0)$:

consider a seq. $\begin{cases} x_n = 1/n \\ y_n = -1/n \end{cases} \Rightarrow f_n = \dfrac{1/n - 1/n}{1/n^2 + 1/n^2} = 0 \xrightarrow[n\to\infty]{} 0$

Hence $f(x,y)$ isn't continuous $\Rightarrow$ not differentiable.

10. Show that the following functions are differentiable at $(0,0)$:

(a) (HW) $f(x,y) = |y|\sin x$, (b) $f(x,y) = \cosh\sqrt{|xy|}$, (c) (HW) $f(x,y) = (\sin x + \sqrt[3]{xy})^2$.

a) by definition: since $f(0,y)=0$ and $f(x,0)=0$

$$f_x'(0,0) = \lim_{h\to 0}\left(\frac{f(h,0)-f(0,0)}{h}\right) = \lim_{h\to 0}\left(\frac{0}{h}\right) = 0$$

$$f_y'(0,0) = \lim_{h\to 0}\left(\frac{f(0,h)-f(0,0)}{h}\right) = \lim_{h\to 0}\left(\frac{0}{h}\right) = 0$$

For differentiability:

$$\frac{\Delta f(x,y) - f_x'\cdot x - f_y'\cdot y}{\sqrt{x^2+y^2}} = \frac{|y|\sin(x) - 0(x+y)}{\sqrt{x^2+y^2}} \to 0$$

So $\forall\varepsilon\ \exists\delta=\varepsilon$ s.t. $0<\rho=\sqrt{x^2+y^2}<\delta$:

$$\left|\frac{y\sin(x)}{\sqrt{x^2+y^2}}\right| \leq \left|\frac{yx}{\sqrt{x^2+y^2}}\right| \leq \left|\frac{y\cdot x}{\sqrt{x^2+0}}\right| \leq |y| < \delta$$

Hence $\lim \to 0$, hence $f(x,y)$ is diff. at $(0,0)$

c) by definition: $x=0$: $\sin(0)+\sqrt{0\cdot y}$

$$f_y'(0,0) = \lim_{h\to 0}\left(\frac{f(0,h)-f(0,0)}{h}\right) = \lim_{h\to 0}\left(\frac{0}{h}\right) = 0$$

$$f_x'(0,0) = \lim_{h\to 0}\left(\frac{f(h,0)-f(0,0)}{h}\right) = \lim_{h\to 0}\left(\frac{\sin(h)+0-0}{h}\right) = 1$$

For differentiability:

$$\frac{\sin(x)+(xy)^{2/3} - 1\cdot x - 0\cdot y}{\sqrt{x^2+y^2}} \to 0\text{, and it is, cuz}$$

$\forall\varepsilon>0\ \exists\delta=$ s.t. $0<\rho=\sqrt{x^2+y^2}<\delta \Rightarrow$

$$\left|\frac{\sin(x)+(xy)^{2/3}-x}{\sqrt{x^2+y^2}}\right| \leq \left|\frac{x+(xy)^{2/3}-x}{\sqrt{x^2+y^2}}\right| \leq \left|\frac{(xy)^{2/3}}{\sqrt{x^2+y^2}}\right| \leq \left|\frac{xy}{\sqrt{x^2+0}}\right| \leq |y| < \delta$$

Hence $\lim \to 0$, hence $f(x,y)$ is diff. at $(0,0)$

12. (HW) Prove that the following functions are not differentiable at $(0,0)$:

(a) $f(x,y)=\sqrt{x^2+y^2}$, (b) $f(x,y)=\sqrt[5]{x^5-y^5}$.

by def:

$$f'_x(0,0)=\lim_{h\to 0}\left(\frac{f(h,0)-f(0,0)}{h}\right)=\lim_{h\to 0}\left(\frac{0}{h}\right)=0$$

$$f'_y(0,0)=\lim_{h\to 0}\left(\frac{f(0,h)-f(0,0)}{h}\right)=\lim_{h\to 0}\left(\frac{0}{h}\right)=0$$

So we need to prove, that $\lim \not\to 0$

$$\frac{\Delta f(x,y)-f'_x(0,0)\cdot x-f_y(0,0)\cdot y}{\sqrt{x^2+y^2}}=\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}=1 \quad \forall x\,\forall y,\ (x,y)\neq(0,0)$$

b) by def:

$$f'_x(0,0)=\lim_{h\to 0}\left(\frac{f(h,0)-f(0,0)}{h}\right)=\lim_{h\to 0}\left(\frac{0}{h}\right)=0$$

$$f'_y(0,0)=\lim_{h\to 0}\left(\frac{f(0,h)-f(0,0)}{h}\right)=\lim_{h\to 0}\left(\frac{0}{h}\right)=0$$

So we need to prove, that $\lim \not\to 0$

$$\frac{\Delta f(x,y)-f'_x(0,0)\cdot x-f_y(0,0)\cdot y}{\sqrt{x^2+y^2}}=\frac{(x^5-y^5)^{1/5}}{(x^2+y^2)^{1/2}}\to 0$$

but it's not, since: $\begin{cases} x_n=1/n \\ y_n=-1/n \end{cases} \Rightarrow f_n(x_n,y_n)=\dfrac{(1/n^5-1/n^5)^{1/5}}{(1/n^2+1/n^2)^{1/2}}=\dfrac{\sqrt{2}/n}{1/|n|}=\sqrt{2}\cdot\mathrm{sgn}(n)\xrightarrow[n\to\infty]{}\sqrt{2}$

13*. (HW) Find the values of α for which the following function is continuous and differentiable at $(0,0)$:

$$f(x,y)=\begin{cases}|x|^\alpha \ln(1+y^2), & \text{if } x\neq 0,\\ 0, & \text{if } x=0.\end{cases}$$

$$\lim_{(x,y)\to 0}\left(|x|^\alpha \ln(1+y^2)\right)$$

cont. $\longrightarrow$ 1) $\alpha \geq 0$: $\lim\limits_{(x,y)\to(0,0)}\left(|x|^\alpha \ln(1+y^2)\right)=0$

$\forall x \leq 1$ near $(0,0)$

$\forall \varepsilon>0\ \exists \delta=\frac{\sqrt{\varepsilon}}{10}\quad 0<\sqrt{x^2+y^2}<\delta \Rightarrow \left||x|^\alpha \ln(1+y^2)\right| \leq \left||x|^\alpha |y|\right| \leq |x^\alpha y| \leq |x||y|<\delta^2$

Hence function is continuous $\forall \alpha \geq 0$ since $\ln(1+y^2)\leq |y|\ \forall y\in\mathbb{R}$

dis cont. $\longrightarrow$ not diff. 2) $\alpha<0$: $\lim\limits_{(x,y)\to(0,0)}\left(\frac{\ln(1+y^2)}{|x|^\alpha}\right)$ DNE

since logarithmic growth is slower than pol. growth.

or let just consider $\begin{cases}x_n = 1/n\\ y=1/n\end{cases} \Rightarrow f_n(x_n,y_n)=\frac{\ln(1+1/n^2)}{|1/n|^\alpha}=|n|^\alpha \ln\left(\frac{n^2+1}{n^2}\right)\xrightarrow[n\to\infty]{}+\infty$

since $\frac{n^2+1}{n^2}>1\ \forall n$ or even $\neq 0$, that's enough

Hence $f(x,y)$ isn't continuous at $(0,0) \Rightarrow$ isn't diff.

Step 2: by def: $f'_x(0,0)=\lim\limits_{h\to 0}\left(\frac{f(h,0)-f(0,0)}{h}\right)=\lim\limits_{h\to 0}\left(\frac{0}{h}\right)=0$

$f'_y(0,0)=\lim\limits_{h\to 0}\left(\frac{f(0,h)-f(0,0)}{h}\right)=\lim\limits_{h\to 0}\left(\frac{0}{h}\right)=0$

Thus we need to show, that $\frac{\Delta f(x,y)-f'_x(0,0)\cdot x - f'_y(0,0)\cdot y}{\sqrt{x^2+y^2}}\longrightarrow 0$

only for $\alpha\geq 0$ ofc.

$$\left|\frac{|x|^\alpha \ln(1+y^2)}{\sqrt{x^2+y^2}}\right| \leq \left|\frac{x^\alpha\cdot y}{\sqrt{x^2+y^2}}\right| \leq \left|\frac{x^\alpha \cdot y}{|y|}\right|=|x|^\alpha$$

now for $0\leq\alpha<1$: $|x|^\alpha>|x|\quad \forall x\ |x|<1 \longrightarrow$ DNE

for $\alpha\geq 1$: $|x|^\alpha\leq |x|$, near 0, for $\forall x\ |x|\leq 1 \Rightarrow |x|^\alpha \leq |x|<\delta$ (1)

(1) $\alpha\geq 1\ \forall\varepsilon>0\ \exists\delta=\varepsilon$ s.t. $0<\sqrt{x^2+y^2}<\delta \Rightarrow |f^*(x,y)|<\varepsilon$, where f^* is $f(x,y)\wedge \alpha\geq 1$.

Thus $f(x,y)$ is continuous for $\alpha\geq 0$ and differentiable for $\alpha \geq 1$

discontinuous for $\alpha<0$ not differentiable for $\alpha<1$

Tnx for checking ♡ - Novosad Ivan