**LAaG** | **Homework #17** | **Group: 231 (M+P+)**

**Release: 25.01.2024**
**Deadline: 04.02.2024**

*In this HW, you can transform any matrix into REF/RREF by using a machine.*

1. (2 points) Let

$$A(\lambda) = \begin{bmatrix} 3 & 1 & 1 & 4 \\ \lambda & 4 & 10 & 1 \\ 1 & 7 & 17 & 3 \\ 2 & 2 & 4 & 3 \end{bmatrix} \in \mathrm{Mat}_4(\mathbb{R}).$$

Then, for every $\lambda \in \mathbb{R}$, find $\mathrm{rk}(A(\lambda))$.

[**hint:** since $\mathrm{rk}_c(A) = \mathrm{rk}_r(A)$ for every matrix $A$ (see Theorem 17.1), due to Definitions 16.5 and 16.6, permutations of the rows/columns of a matrix do not change its rank; try to "move" $\lambda$ in the right-bottom corner of the matrix; then transform the new matrix into a matrix in row echelon form]

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1)
$$\begin{bmatrix} 3 & 1 & 1 & 4 \\ \lambda & 4 & 10 & 1 \\ 1 & 7 & 17 & 3 \\ 2 & 2 & 4 & 3 \end{bmatrix} \to \begin{bmatrix} 4 & 1 & 1 & 3 \\ 3 & 2 & 4 & 2 \\ 3 & 7 & 17 & 1 \\ 1 & 4 & 10 & \lambda \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2/5 & 4/5 \\ 0 & 1 & 13/5 & -1/5 \\ 0 & 0 & 0 & 0 \\ 1 & 4 & 10 & \lambda \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2/5 & 4/5 \\ 0 & 1 & 13/5 & -1/5 \\ 0 & 4 & 52/5 & \lambda - 4/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & -2/5 & 4/5 \\ 0 & 1 & 13/5 & -1/5 \\ 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

hence if $\lambda = 0 \Rightarrow \mathrm{rk}(A) = 2$

otherwise $\mathrm{rk}(A) = 3 \leftarrow$

2. (2 points per item) Let

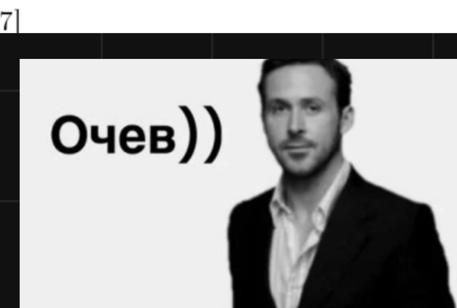
$$A = \begin{bmatrix} 1 & -2 & 2 & 7 \\ 2 & -4 & 1 & 8 \\ -1 & 2 & 1 & -1 \\ 1 & -2 & 0 & 3 \end{bmatrix} \in \mathrm{Mat}_4(\mathbb{R}).$$

Then

(a) find the smallest $k \in \mathbb{N}$ and matrices $C \in \mathrm{Mat}(4, k, \mathbb{R})$ and $F \in \mathrm{Mat}(k, 4, \mathbb{R})$ such that the equality $A = CF$ holds true (note that $C$ and $F$ are *not* unique; such $k$ is called the *factorization rank* of $A$);
[**hint**: see Problem 3 from Seminar 17; also, an algorithm on page 17.4 may be useful; it is not a part of this problem, but it is highly advisable to verify that your matrices $C$ and $F$ indeed satisfy the equality $A = CF$]

(b) find the smallest $k \in \mathbb{N}$ and matrices $A_1, A_2, \ldots, A_k$ such that $\mathrm{rk}(A_i) = 1$, for every $i \in \{1, \ldots, k\}$, and the equality $A = A_1 + A_2 + \cdots + A_k$ holds true (note that $A_1, A_2, \ldots, A_k$ are *not* unique; such $k$ is called the *decomposition rank* of $A$).
[**hint**: see Problem 4 from Seminar 17]

a) $A = CF$



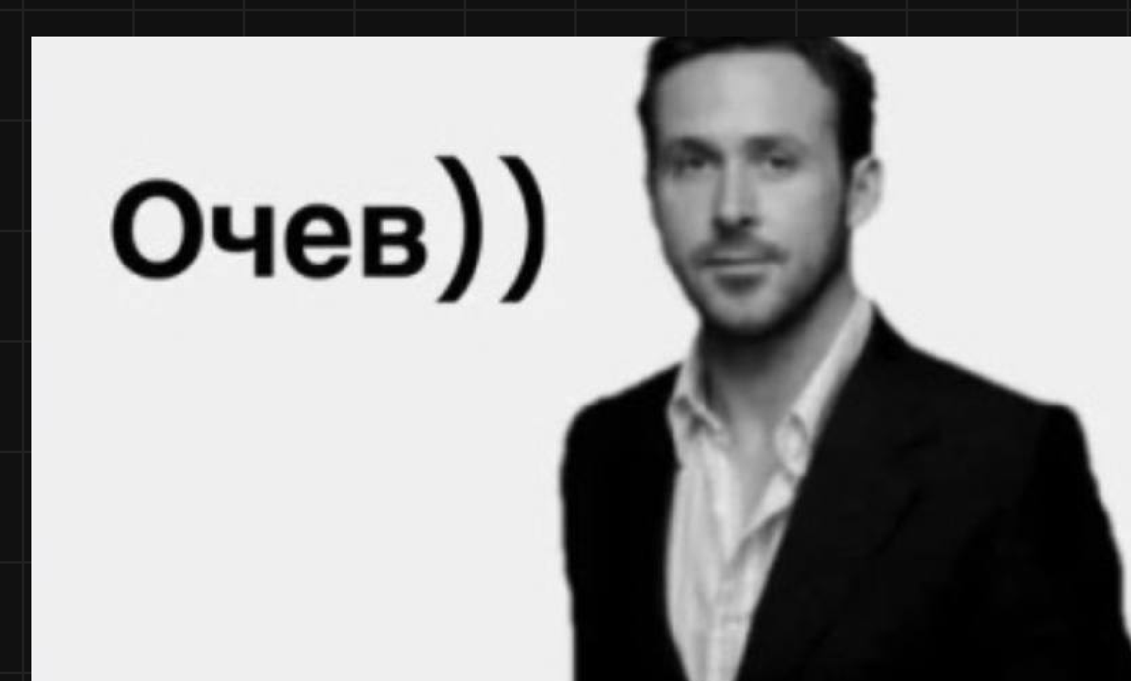

$$C = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 1 \\ 1 & 0 \end{bmatrix} \quad F = \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$A = \hat{A}_1 + \hat{A}_2 \wedge \mathrm{rk}(\tilde{A}_1) = \mathrm{rk}(\hat{A}_2) = 1$

$\Uparrow$

b)
$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 & 3 \\ 2 & -4 & 0 & 6 \\ -1 & 2 & 0 & -3 \\ 1 & -2 & 0 & 3 \end{bmatrix} \tilde{A}_1$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tilde{A}_2$$



3. Let $A$ be a square matrix of size $n$ over a field of reals. Then, following the instructions, find $\operatorname{rk}(\operatorname{adj}(A))$, where $\operatorname{adj}(A)$ is the adjugate of $A$ (see Definition 8.2).

**Instructions:**

(a) consider three cases: $\operatorname{rk}(A) = n$, $\operatorname{rk}(A) = n-1$, and $\operatorname{rk}(A) \leqslant n-2$;

(b) (1 point) if $\operatorname{rk}(A) = n$ then use Item 7 of Theorem 17.2, Statement 8.1, and Theorem 7.3;

(c) (1 point) if $\operatorname{rk}(A) \leqslant n-2$ then use Theorem I (see below) and Definition 8.2;

(d) (2 points) if $\operatorname{rk}(A) = n-1$ then: 1) use Item 7 of Theorem 17.2 and Statement 8.1 to find $B$ in the equality $A \cdot \operatorname{adj}(A) = B$; 2) use Item 9 of Theorem 17.2, the result of the previous step, and the fact that $\operatorname{rk}(A) = n-1$ to obtain an inequality of the form $\operatorname{rk}(\operatorname{adj}(A)) \leqslant ?$; 3) use Theorem I, Definition 8.2, and the fact that $\operatorname{rk}(A) = n-1$ to obtain an inequality of the form $\operatorname{rk}(\operatorname{adj}(A)) \geqslant ?$.

b) if $rk(A)=n$: from Th 17.2: Let $adj(A)=B$

if $rk(A)=n \Rightarrow det(A)\neq 0$. also note that $AB=det(A)I_n$;

since $det(A)\neq 0$ we obtain that B has full rank $(n)$.

c) If $rk(A)\leq n-2 \Rightarrow det(A)=0 \Rightarrow B=O_n$

since in general $I_n \cdot det(A) = A\cdot adj(A) \to I_n \cdot 0 = A\cdot adj(A) \Leftrightarrow A\cdot adj(A)=O_n \Rightarrow$

$\Rightarrow \left[\begin{array}{l} A=O_n \\ adj(A)=O_n \end{array}\right. \Rightarrow adj(A)=O_n$

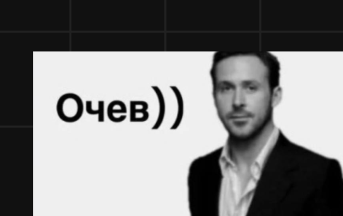
and, as follows $rk(B)=0$, since $rk(O_n)=0$; and even more $rk(B)=0 \Leftrightarrow B=O_n$

(since if $A=O_n$ it's clear that $adj(A)=O_n$ as well.

d) If $rk(A)=n-1 \Rightarrow det(A)=0$

then using the identity $A\cdot B = det(A)\cdot I_n$ we obtain that $A\cdot B=O_n$ (since $det(A)=0$)

and from it, it follows that all collumns of B is belong to the $ker(A)$, which is one-dimensional $\Rightarrow rk(B)=1$.

( Just in case, $dim(ker(A))=1$ follows from nature of kernel, if we "lost" one dim the $ker(A)=1$ and so on, it's 

, if not, call me to defend my HW.

(or if something else is not obv.)

Another way to prove b):

if $rk(A)=n$ then $A$ is invertable and $adj(A)\cdot A = det(A)\cdot I_n$ shows that $adj(A)=\frac{1}{det(A)}A^{-1}$ is also invertable $\Rightarrow rk(adj(A))=n$.