LAaG **List of Examples for EX 2**

1. For a given linear operator, find its spectrum.
   For example:
   Let $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ where
   $$\varphi\colon \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \underbrace{\begin{bmatrix} -9 & 7 & -8 \\ 16 & -12 & 14 \\ 23 & -17 & 20 \end{bmatrix}}_{A} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad \text{for any } [x,\ y,\ z]^{\mathrm{T}} \in \mathbb{R}^3.$$
   Then, find $\operatorname{Spec}(\varphi)$.

$$1)\ \chi_\varphi(x) = \det(A - xI);$$

$$\det\left(\begin{bmatrix} -9-x & 7 & -8 \\ 16 & -12-x & 14 \\ 23 & -17 & 20-x \end{bmatrix}\right) = (23)(12+x)(8) + (17)(9+x)(14) + (16)(7)(20-x)$$
$$-(9+x)(12+x)(20-x) - (7)(14)(23) - (8)(16)(17)$$
$$= 2x - x^2 - x^3$$
$$x(x^2+x-2) \Rightarrow x(x-1)(x+2) \Rightarrow \operatorname{Spec}(\varphi) = \{0, 1, -2\}$$

2. For any element of the spectrum of a given linear operator, find a basis for the corresponding eigenspace.
   For example:
   Let $\varphi\colon \mathbb{R}[x;5] \to \mathbb{R}[x;5]$ where
   $$\varphi\colon p(x) \mapsto x^2 p(x)'', \qquad \text{for any } p(x) \in \mathbb{R}[x;5].$$
   Then, for any $\lambda \in \operatorname{Spec}(\varphi)$, find a basis for $\mathrm{E}_\lambda(\varphi)$.

$$1 \to x^2 \cdot 0 \Rightarrow 0$$
$$x \to x^2 \cdot 0 \Rightarrow 0$$
$$x^2 \to x^2 \cdot 2 \Rightarrow 2x^2$$
$$x^3 \to x^2 \cdot 6x = 6x^3$$
$$x^4 \to x^2 \cdot 12x^2 = 12x^4$$
$$x^5 \to 20x^5$$

$$\underbrace{\begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&2&0&0&0 \\ 0&0&0&6&0&0 \\ 0&0&0&0&12&0 \\ 0&0&0&0&0&20 \end{bmatrix}}_{A} \Rightarrow \operatorname{Spec}(\varphi) = \{0, 2, 6, 12, 20\}$$

$$E_\varphi(0) = \ker(A) =$$

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \end{matrix} \\ \left[\begin{array}{cccccc|c} 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&2&0&0&0&0 \\ 0&0&0&6&0&0&0 \\ 0&0&0&0&12&0&0 \\ 0&0&0&0&0&20&0 \end{array}\right] \end{array} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow E_\varphi(0) = \left\langle \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$$

basis

$$\left[\begin{array}{cccccc|c} -2&0&0&0&0&0&0 \\ 0&-2&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&4&0&0&0 \\ 0&0&0&0&10&0&0 \\ 0&0&0&0&0&18&0 \end{array}\right] \to \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \alpha \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow E_\varphi(2) = \left\langle \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\rangle$$

Thus $E_\varphi(6) = \left\langle \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle$ $E_\varphi(12) = \left\langle \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\rangle$ $E_\varphi(20) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

3. For any element of the spectrum of a given linear operator, find its algebraic and geometric multiplicities.
   For example:
   Let $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ where
   $$\varphi\colon \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} -3 & 8 & -19 \\ 5 & -4 & 13 \\ 3 & -4 & 11 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad \text{for any } [x,\ y,\ z]^{\mathrm{T}} \in \mathbb{R}^3.$$
   Then, for any $\lambda \in \operatorname{Spec}(\varphi)$, find the algebraic and geometric multiplicity of λ.

$$\left[\begin{array}{ccc|cc} -3 & 8 & -19 & -3 & 8 \\ 5 & -4 & 13 & 5 & -4 \\ 3 & -4 & 11 & 3 & -4 \end{array}\right] \Rightarrow \chi_\varphi(x) = x(x^2-4x+4) = x(x-2)^2$$

$$(3+x)(4+x)(11-x) + 24 \cdot 13 + 380$$
$$-(57)(4+x) - 4 \cdot 13(3+x) + 40(11-x)$$
$$12 + 4x + 3x + x^2$$
$$x^2 + 7x + 12 + 312 + 380$$
$$-228 - 57x - 157 - 52x + 440 - 40x$$
$$x^2 - 142x + 707 = 0$$

$$\Rightarrow \operatorname{Spec}(\varphi) = \{0, 2\}$$
$$a.m(0) = 1$$
$$a.m(2) = 2$$
$$g.m(0) = 1$$
$$g.m(2) = ?\ 1$$

$$\begin{bmatrix} -5 & 8 & -19 \\ 5 & -6 & 13 \\ 3 & -4 & 9 \end{bmatrix} \to \begin{bmatrix} 5 & -8 & 19 \\ 0 & 2 & -6 \\ 3 & -4 & 9 \end{bmatrix} \to \begin{bmatrix} 5 & 0 & -5 \\ 0 & 2 & -6 \\ 3 & 0 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 1 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ 3\alpha \\ \alpha \end{bmatrix} \Rightarrow E_\varphi(2) = \left\langle \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\rangle$$

4. Find the n-th power of a given matrix (application of the Cayley-Hamilton Theorem is strongly recommended but not mandatory).
   For example:
   For any positive integer n, find
   $$A_n = \begin{bmatrix} 1 & -3 & -4 \\ -5 & 8 & 13 \\ 4 & -7 & -11 \end{bmatrix}^n.$$

$$\begin{bmatrix} 1 & -3 & -4 \\ -5 & 8 & 13 \\ 4 & -7 & -11 \end{bmatrix}\begin{bmatrix} 1 & -3 & -4 \\ -5 & 8 & 13 \\ 4 & -7 & -11 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 7 & -12 & -19 \\ -5 & 9 & 14 \end{bmatrix}$$

$$\chi_\varphi(x) = \left[\begin{array}{ccc|cc} 1-x & -3 & -4 & 1-x & 3 \\ -5 & 8-x & 13 & -5 & 8-x \\ 4 & -7 & -11-x & 4 & -7 \end{array}\right]$$

$$\chi_\varphi(x) = (1-x)(8-x)(-11-x) - 12 \cdot 13 - 28 \cdot 5$$
$$+16(8-x) + (7-7x)13 - 15(11+x)$$
$$= x^3 + 2x^2 + x$$
$$\Rightarrow \operatorname{Spec}(\varphi) = \{0, 1\} \wedge a.m(1) = 2$$

$$\begin{cases} c = 0 \\ a + b = 1 \\ 2a + b = n1^{n-1} \end{cases} \Rightarrow \begin{cases} c = 0 \\ a = 1 - b \\ a = \frac{n-b}{2} \end{cases} \Rightarrow \begin{matrix} c = 0 \\ a = 1 - b \\ 1 - b = \frac{n-b}{2} \end{matrix} \Rightarrow \begin{matrix} 2 - 2b = n - b \\ -b = n - 2 \\ b = 2 - n \end{matrix}$$
$$a = 1 - 2 + n$$
$$a = n - 1$$

$$0^n = a(0)^2 + b(0) + c \Rightarrow c = 0$$
$$1^n = a(1)^n + b(1)^n + 0 \Rightarrow 1 = a + b$$
$$(n)1^{n-1} = (ax^2 + bx + c)' = 2ax + b = 2a + b = 1$$
$$\Rightarrow (a, b, c) = (n-1,\ 2-n,\ 0)$$

Thus $r(x) = (n-1)x^2 + (2-n)x$
$$A^n = r(A) \Rightarrow (n-1)A^2 + (2-n)A + 0I$$
$$A^n = (n-1)\begin{bmatrix} 0 & 1 & 1 \\ 7 & -12 & -19 \\ -5 & 9 & 14 \end{bmatrix} + (2-n)\begin{bmatrix} 1 & -3 & -4 \\ -5 & 8 & 13 \\ 4 & -7 & -11 \end{bmatrix}$$

$$A^n = \begin{bmatrix} 2-n & 4n-7 & 5n-9 \\ 12n-17 & 28-20n & 45-32n \\ -9n+13 & 16n-23 & 25n-36 \end{bmatrix}$$

False, but why?