1. (1 point per item) For each of the following linear operators φ on a vector space $\mathbb{V}$, test φ for diagonalizability, and if φ is diagonalizable, find an ordered basis B for $\mathbb{V}$ such that $T(\varphi, B)$ is a diagonal matrix and write down $T(\varphi, B)$.

   (a) $\mathbb{V} = \mathbb{R}^3$ and φ is given by its coordinate matrix (with respect to a some ordered basis A of $\mathbb{V}$):

   $$T(\varphi, A) = \begin{bmatrix} -11 & 27 & -9 \\ -4 & 7 & -6 \\ 3 & -9 & 1 \end{bmatrix};$$

   [**hint:** see Problem 1 from Seminar 23]

   (b) $\mathbb{V} = \mathbb{R}^3$ and φ is given by its coordinate matrix (with respect to a some ordered basis A of $\mathbb{V}$):

   $$T(\varphi, A) = \begin{bmatrix} -14 & 6 & 6 \\ -18 & 7 & 9 \\ -12 & 6 & 4 \end{bmatrix};$$

   (c) $\mathbb{V} = \mathbb{R}[x; n]$ (that is, $\mathbb{V}$ is the vector space of all polynomials of degree at most n with real coefficients) and φ is given by the following formula

   $$\varphi(p(x)) = p(x+1), \quad \text{for every } p(x) \in \mathbb{R}[x; n].$$

   (for example, $\varphi(5x^3 - 2x + 7) = 5(x+1)^3 - 2(x+1) + 7$)

   [**hint:** consider $T(\varphi, A)$ for some "simple" ordered basis A of $\mathbb{R}[x; n]$; see Item c) of Problem 1 from Seminar 23]

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a) find χ_φ :

$$\begin{vmatrix} -11-x & 27 & -9 \\ -4 & 7-x & -6 \\ 3 & -9 & 1-x \end{vmatrix} = -(x-1)(x+2)^2$$

$a.m(-2) = 2$ since $1 \le g.m(\lambda) \le a.m(\lambda)$

$\Rightarrow Spec(\varphi) = \{1, -2\} \wedge a.m(1) = 1 \Rightarrow g.m(1) = 1$

$\Rightarrow a.m.(1) + a.m(-2) = \dim(\mathbb{R}^3) = 3$ ✓

then find g.m.(-2):

$g.m.(-2) = \dim(\ker(T(\varphi, A)) + 2I))$; $\ker(T(\varphi,A)) + 2I) = \ker\left(\begin{bmatrix} -9 & 27 & -9 \\ -4 & 9 & -6 \\ 3 & -9 & 3 \end{bmatrix}\right) = \left(\begin{bmatrix} -3 \\ -2/3 \\ 1 \end{bmatrix}\right)$

hence $g.m(-2) = 1 \Rightarrow \varphi$ isn't diagonaziable, since $g.m(\lambda) < a.m(\lambda)$

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b) find χ_φ: $\det\left(\begin{bmatrix} -14 & 6 & 6 \\ -18 & 7 & 9 \\ -12 & 6 & 4 \end{bmatrix} - xI_3\right) = -(x-1)(x+2)^2 \Rightarrow Spec(\varphi) = \{1, -2\}$

also since $1 \le g.m(\lambda) \le a.m(\lambda)$ $a.m(1) = g.m(1) = 1$; $a.m(-2) = 2$; $a.m(1) + a.m(-2) = \dim(\mathbb{V}) = 3$.

find $g.m(-2)$: $\dim(\ker(T(\varphi, A) + 2I_3)) = 2$, since $\ker(T(\varphi,A) + 2I_3) = \ker\left(\begin{bmatrix} -12 & 6 & 6 \\ -18 & 9 & 9 \\ -12 & 6 & 6 \end{bmatrix}\right) = \left(\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}\right)$

$\Rightarrow T(\varphi, A)$ is diagonaziable, since $a.m.(\lambda) = g.m(\lambda) \wedge \sum a.m(\lambda) = \dim(\mathbb{V})$ (lazy notation)

$$\Rightarrow T(\varphi, A) = \begin{bmatrix} 1/2 & 1/2 & 1 \\ 1 & 0 & 3/2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 1 \\ 1 & 0 & 3/2 \\ 0 & 1 & 1 \end{bmatrix}^{-1}$$

$\Rightarrow$ ordered basis B for $\mathbb{V}$ = $\Bigg(\underbrace{\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}}_{E_\varphi(-2)}, \underbrace{\begin{bmatrix} 1 \\ 3/2 \\ 1 \end{bmatrix}}_{E_\varphi(1)}\Bigg)$, and $T(\varphi, B) = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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c) let $\underline{A}$ be an "classic" basis for $\mathbb{R}[x, n]$, s.t. $\langle 1, x, x^2, x^3, \dots, x^n \rangle \Leftrightarrow \left(\underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{e_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{e_2}, \dots \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{e_n}\right)$

then $T(\varphi, A) = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & 3 & \cdots & n \\ 0 & 0 & 1 & 3 & \cdots & * \\ 0 & 0 & 0 & 1 & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & n \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$, we easily obtain it using pascal triangular

so, consider $\chi_\varphi(x) = \det(T(\varphi, A) - xI) = \begin{vmatrix} 1-x & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1-x & 2 & 3 & \cdots & n \\ 0 & 0 & 1-x & 3 & \cdots & * \\ 0 & 0 & 0 & 1-x & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & n \\ 0 & 0 & 0 & 0 & \cdots & 1-x \end{vmatrix} = (-1)^n (1-x)^{n+1}$, since matrix in upper triangular form.

so, $Spec(\varphi) = \{1\}$ and $a.m(1) = n+1$

now, consider $\dim(\ker(A - I)) = \dim\left(\ker\left(\begin{bmatrix} 0 & 1 & & * \\ & 0 & 1 & \\ & & 0 & \ddots \\ 0 & & & 0 \end{bmatrix}\right)\right) \xrightarrow{RREF} = \dim\left(\ker\left(\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ & & & 1 & \\ 0 & & & & \ddots & \\ & & & & & 1 \\ & & & & & & 0 \end{bmatrix}\right)\right) = 1$ (since there is only 1 free el.)

$\Rightarrow a.m(1) = n+1 \wedge g.m(1) = 1 \Rightarrow \begin{cases} \varphi \text{ is diagonaziable for } n = 0 \\ \varphi \text{ isn't diagonaziable for } n > 0 \end{cases}$ $n \in \mathbb{N} + \{0\}$

in fact for $n = 0$ $T(\varphi, A) = [1]$

diagonal mat.

2. (1 point) Let $\mathbb{V}$ be a finite dimensional vector space and let φ be a *diagonalizable* linear operator on $\mathbb{V}$ such that $\mathrm{Spec}(\varphi) = \{0, 1\}$. Then, for every positive integer k, find $\varphi^k \stackrel{\text{def}}{=} \underbrace{\varphi \circ \varphi \circ \cdots \circ \varphi}_{k \text{ times}}$.

[**hint:** if B is an ordered basis for $\mathbb{V}$ such that $T(\varphi, B)$ is diagonal, then, by Theorem 19.2, $T(\varphi^k, B) = ?$]

Since φ is diagonaziable, exist an ordered basis B,

s.t. $b_1 \oplus b_2 \oplus b_3 \oplus \dots \oplus b_n = \mathbb{V}$, where b_i is i-th vector of ordered basis B.

and B is eigenbasis of $\mathbb{V}$ (that is, $b_1, b_2, \dots b_m \in E_\varphi(1)$, where m is $\underset{\text{g.m.}(1)}{\overset{\text{a.m}(1)}{\shortparallel}}$ and $b_{m+1}, b_{m+2}, \dots b_n \in E_\varphi(0)$, where $n = \dim(\mathbb{V})$)

then $T(\varphi, B)$ is diagonal matrix

hence $\varphi^k = T(\varphi, B)^k = \underbrace{\begin{bmatrix} 1 & & & & & \\ & 1 & & & 0 & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & 0 & \\ & 0 & & & & \ddots \;\; 0 \end{bmatrix}}_{m \text{ times} \quad n-m \text{ times}}^{k} = \underbrace{\begin{bmatrix} 1 & & & & & \\ & 1 & & & 0 & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & 0 & \\ & 0 & & & & \ddots \;\; 0 \end{bmatrix}}_{m \text{ times} \quad n-m \text{ times}} = T(\varphi, B) = \varphi$

3. (1 point) Let $\mathbb{V}$ be an n-dimensional vector space, let φ be a *diagonalizable* linear operator on $\mathbb{V}$, and let $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n$ be the eigenvalues of φ (note that $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n$ *may not* be pairwise distinct). Then, for every positive integer k, find the trace of φ^k.

[**hint:** see Definition 21.5, Proposition 21.1 and use the fact that φ is diagonalizable]

Theorem (Cayley–Hamilton Theorem)**.** *Let A be a square matrix of size n and*

$$\chi(x) \overset{\text{def}}{=} \det(A - xI_n) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$$

be the characteristic polynomial of A. Then,

$$\chi(A) = c_n A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I_n = 0_n, \tag{1}$$

Since $tr(\varphi)$ do not depens on choice of basis, we can choose eigen basis A, then $tr(\varphi) = tr(T(\varphi, A))$

$$T(\varphi, A) = \begin{bmatrix} \lambda_1 & & & & 0 \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ 0 & & & & \lambda_n \end{bmatrix} \Rightarrow \sum_{i=1}^{n} \lambda_i = tr(\varphi) \quad \text{and since } tr(\varphi) = tr(\varphi^k, A)$$

$$tr(\varphi^k) = tr\begin{bmatrix} \lambda_1 & & & & 0 \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ 0 & & & & \lambda_n \end{bmatrix}^k = tr\begin{bmatrix} \lambda_1^k & & & & 0 \\ & \lambda_2^k & & & \\ & & \lambda_3^k & & \\ & & & \ddots & \\ 0 & & & & \lambda_n^k \end{bmatrix} = \sum_{i=1}^{n} \lambda_i^k \quad (\text{where } \lambda_i \in Spec(\varphi))$$

4. (1 point per item) Let

$$A = \begin{bmatrix} 1 & 5 & 4 \\ -1 & 2 & -1 \\ 1 & -5 & -2 \end{bmatrix}.$$

Then:

(a) find the characteristic polynomial of A;

(b) using Equality (1), find real numbers α, β, γ such that

$$A^{-1} = \alpha A^2 + \beta A + \gamma I_3;$$

Remark: after solving this item, you should understand that if A is an invertible square matrix of size n, then

$$A^{-1} \in \langle A^{n-1}, A^{n-2}, \dots, A, I_n \rangle.$$

(c) for every positive integer k, find A^k.

[**hint:** see Problem 2 from Seminar 23]

a) characteristic polynomial of $A = \det(A - xI_3) =$

$$= \begin{vmatrix} 1-x & 5 & 4 \\ -1 & 2-x & -1 \\ 1 & -5 & -2-x \end{vmatrix} = -(x-2)^2(x+3) = -x^3 + x^2 + 8x - 12$$

(since we're able to use calc. for that)

b)
$$O_3 = -A^3 + A^2 + 8A - 12I$$
$$\Updownarrow$$
$$12I_3 = -A^3 + A^2 + 8A$$
$$\Updownarrow$$
$$12I_3 = A(-A^2 + A + 8I_3)$$

$$\Rightarrow 12A^{-1} = -A^2 + A + 8I_2$$

$$A^{-1} = \frac{1}{12}(-A^2 + A + 8I_2)$$

$$A^{-1} = \frac{1}{12}\left(-\begin{bmatrix} 1 & 5 & 4 \\ -1 & 2 & -1 \\ 1 & -5 & -2 \end{bmatrix}^2 + \begin{bmatrix} 1 & 5 & 4 \\ -1 & 2 & -1 \\ 1 & -5 & -2 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}\right)$$

$$A^{-1} = \frac{1}{12}\begin{bmatrix} 9 & 10 & 13 \\ 3 & 6 & 3 \\ -3 & -10 & -7 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 3/4 & 5/6 & 13/6 \\ 1/4 & 1/2 & 1/4 \\ -1/4 & -5/6 & -7/12 \end{bmatrix}$$

c) $x^n = q(x)\chi_A(x) + r(x)$, where $\deg(r(x)) < \deg(q(x))$

since $\chi_A(A) = O_3$, we have $A^n = r(A)$

since $\deg(\chi_A(x)) = 3$, $\deg(r(x)) \leq 2 \iff r(x) = ax^2 + bx + c$, $a, b, c \in \mathbb{R}$

then by substitution x with eigenvalues of A:

$2^n = q(2)\chi_A(2) + 4a + 2b + c = 4a + 2b + c$

$(-3)^n = 9a - 3b + c$

$(x^n)' = (q(x)\chi_A(x) + r(x))' \iff nx^{n-1} = q'(x)\chi_A(x) + q(x)\chi_A'(x) + r'(x)$

note that if 2 has a.m. $= 2$ for $\chi_A(x)$, then it has a.m. $= 1$ for $\chi_A'(x)$,

that is $\chi_A(2) = \chi_A'(2) = 0 \Rightarrow nx^{n-1} = r'(x) = 2ax + b$

So $n \cdot 2^{n-1} = 4a + b$

Thus, we get the following system:
$$\begin{cases} 2^n = 4a + 2b + c \\ (-3)^n = 9a - 3b + c \\ n \cdot 2^{n-1} = 4a + b \end{cases} \Rightarrow \begin{cases} a = \dfrac{(-3)^n}{25} + \dfrac{2^n n}{10} - \dfrac{2^n}{25} \\ b = \dfrac{-8(-3)^n + 5 \cdot 2^n \cdot n + 8 \cdot 2^n}{50} \\ c = -\dfrac{-4(-3)^n + 15 \cdot 2^n \cdot n - 21 \cdot 2^n}{25} \end{cases}$$

hence
$$A^n = \left(\frac{(-3)^n}{25} + \frac{2^n n}{10} - \frac{2^n}{25}\right)A^2 + \left(\frac{-8(-3)^n + 5 \cdot 2^n \cdot n + 8 \cdot 2^n}{50}\right)A - \left(\frac{-4(-3)^n + 15 \cdot 2^n \cdot n - 21 \cdot 2^n}{25}\right)I_3$$

check:

$n = 0$ ✓

$n = 1$ ✓ $\quad n = 3$ ✗

$n = 2$ ✓ $\quad n = 4$ ✗

5. (2 points) Suppose that there are three types of pokemons[1]: blue, red, and orange pokemons.

It is known that in one day time:

(a) a blue pokemon evolves into one red pokemon;

(b) a red pokemon evolves into two orange pokemons;

(c) an orange pokemon evolves into two blue and one red pokemons.

For example: if you start with one blue and one orange pokemons, then, *in one day* you will have two blue and two red pokemons ($B \to R$ and $O \to 2B + R$) and *in two days* you will have two red and four orange pokemons ($2B \to 2R$ and $2R \to 4O$).

Suppose you start with five blue pokemons, then, how many blue, red, and orange pokemons will you have in 60 days?

как бы векторным пространством может быть что угодно, например его можно заспанить тремя покемонами

15:19

we can solve it using numerical theory, but instead we will use VS.

Consider an ordered basis $\left(\underset{e_1}{\begin{bmatrix}1\\0\\0\end{bmatrix}},\underset{e_2}{\begin{bmatrix}0\\1\\0\end{bmatrix}},\underset{e_3}{\begin{bmatrix}0\\0\\1\end{bmatrix}}\right) =: A$ which related to amount of pokemons.

s.t. $\begin{bmatrix}3\\1\\2\end{bmatrix}$ stands for 3 blue pokemons, 1 red pokemons and 2 orange pokemons.

Then consider a Linear operator φ s.t. maps pokemons satisfying task conditions.

then $T(\varphi, A) = \begin{bmatrix}0&0&2\\1&0&1\\0&2&0\end{bmatrix} =: M$; that's it, coordinate matrix rep. of φ

then task is $\varphi^{60}\left(\begin{bmatrix}5\\0\\0\end{bmatrix}\right) \iff \begin{bmatrix}0&0&2\\1&0&1\\0&2&0\end{bmatrix}^{60}\begin{bmatrix}5\\0\\0\end{bmatrix} = \left(\begin{bmatrix}0&0&2\\1&0&1\\0&2&0\end{bmatrix}^4\right)^{15}\begin{bmatrix}5\\0\\0\end{bmatrix} = \begin{bmatrix}0&8&8\\4&4&8\\4&8&4\end{bmatrix}^{15}\begin{bmatrix}5\\0\\0\end{bmatrix}$

since M^1, M^2, M^3 has complex eigenvalues, we will consider M^4

let's switch φ to φ^4 (M to M^4)

So let's diagonalize M^4 (denote M^4 as B)

$\chi_\varphi(x) = -(x-16)(x+4)^2 \Rightarrow \operatorname{Spec}(\varphi) = \{16, -4\}$

$\ker(B - 16I_3) = \left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) \wedge \ker(B + 4I_3) = \left(\begin{bmatrix}-2\\1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\1\end{bmatrix}\right)$

So eigenbasis for V.S. is $\left(\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}-2\\1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\1\end{bmatrix}\right)$ named it C

then $T(\varphi, C) = \begin{bmatrix}16&0&0\\0&-4&0\\0&0&-4\end{bmatrix}$ so $T(\varphi, B) = \begin{bmatrix}1&-2&-2\\1&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}16&0&0\\0&-4&0\\0&0&-4\end{bmatrix}\begin{bmatrix}1&-2&-2\\1&1&0\\1&0&1\end{bmatrix}^{-1}$

So $M^{60} = \begin{bmatrix}1&-2&-2\\1&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}16&0&0\\0&-4&0\\0&0&-4\end{bmatrix}^{15}\begin{bmatrix}1&-2&-2\\1&1&0\\1&0&1\end{bmatrix}^{-1} = \frac{1}{5}\begin{bmatrix}2^{60}-2^{32} & 2^{61}+3\cdot2^{31}-2^{32} & 2^{61}-2^{32}+3\cdot2^{31}\\ 2^{60}+2^{30} & 2^{61}-3\cdot2^{30} & 2^{61}+2^{31}\\ 2^{60}+2^{30} & 2^{61}+2^{31} & 2^{61}-3\cdot2^{30}\end{bmatrix}$

Hence after 60 days there will be $\frac{1}{5}\begin{bmatrix}2^{60}-2^{32} & 2^{61}+3\cdot2^{31}-2^{32} & 2^{61}-2^{32}+3\cdot2^{31}\\ 2^{60}+2^{30} & 2^{61}-3\cdot2^{30} & 2^{61}+2^{31}\\ 2^{60}+2^{30} & 2^{61}+2^{31} & 2^{61}-3\cdot2^{30}\end{bmatrix}\begin{bmatrix}5\\0\\0\end{bmatrix} = \begin{bmatrix}2^{60}-2^{32}\\2^{60}+2^{30}\\2^{60}+2^{30}\end{bmatrix}$ pokemons

in other words: after 60 days there will be

$2^{60}-2^{32}$ blue pokemons

$2^{60}+2^{30}$ red pokemons

$2^{60}+2^{30}$ orange pokemons

ВВ сем линал now
Бывают дети в 7 классе, которым даешь уравнение

(x-1)(x-2) = 0,

а они такие

x^2 - 3x + 2 = 0

ммм как бы это решить...

Thx for checking
~ Novosad Ivan