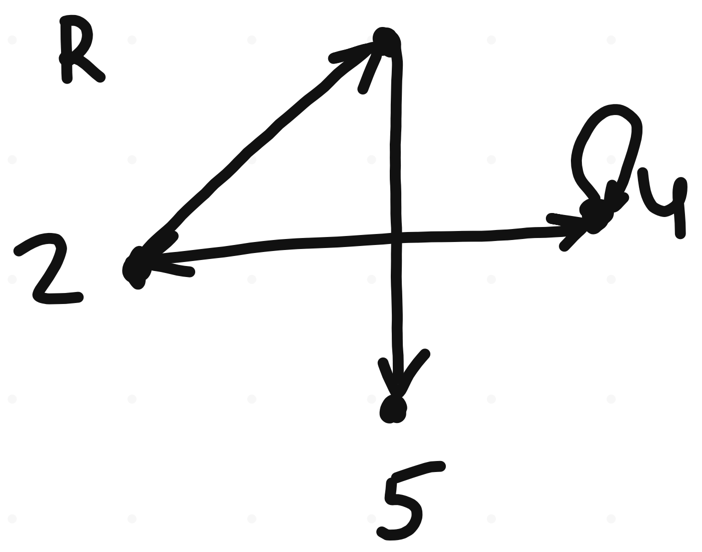
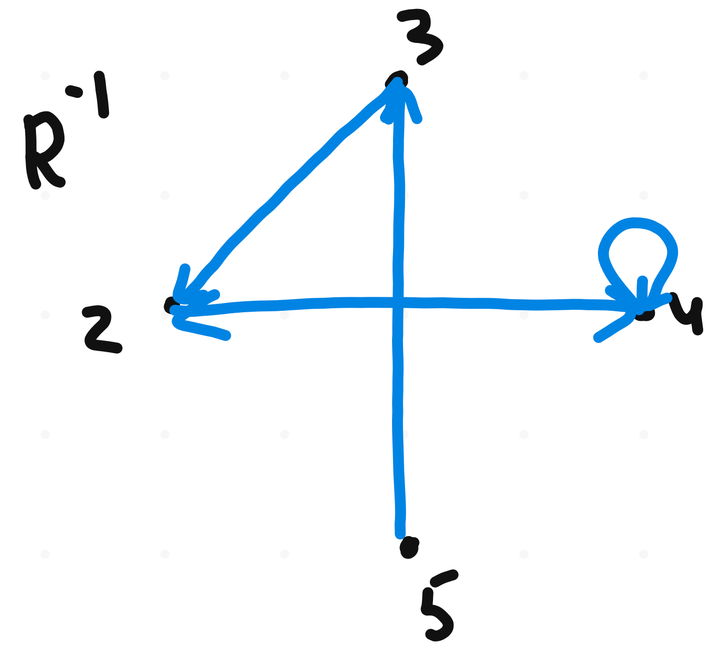
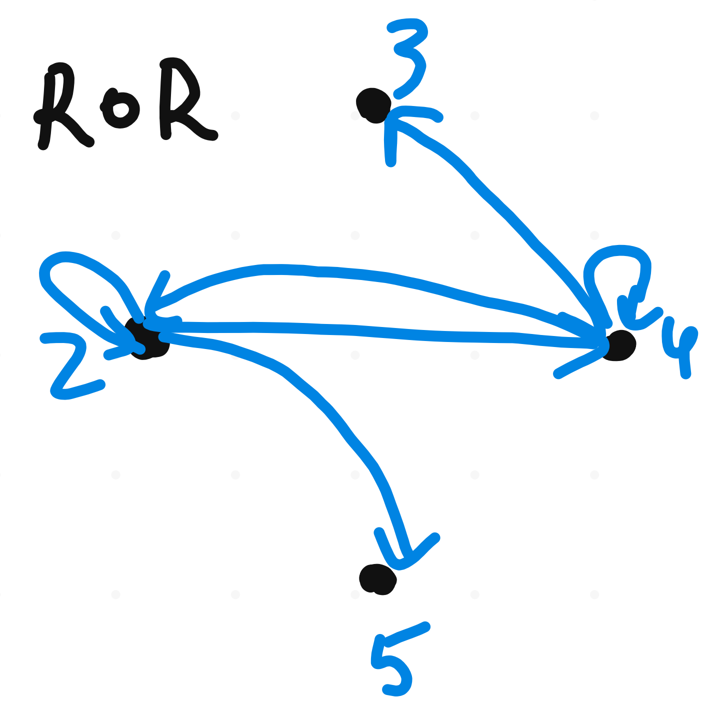
1. Let $R = \{(2,3),(3,5),(2,4),(4,4),(4,2)\}$. Draw 'arrow diagrams' for the relations R, R^{-1} and $R \circ R$. What are the sets $\mathrm{dom}(R \circ R \circ R)$ and $\mathrm{rng}(R \circ R \circ R)$? (Name every element of these sets.)







Risk of Rain $R \circ R = \{(2,5)(2,4)(2,2)(4,4)(4,2)(4,3)\}$

$R \circ R \circ R \{(2,4)(2,2)(2,3)(4,5)(4,4)(4,2)(4,3)\}$

$\mathrm{dom}(R \circ R \circ R) = \{2,4\}$

$\mathrm{rng}(R \circ R \circ R) = \{2,3,4,5\}$

2. Let $A = \{1,2,3\}$. What is the relation $\subseteq \circ \subseteq$ on $\mathcal{P}(A)$? (Name every element of this relation.)

$\mathcal{P}(A) = \{\{1\}\{2\}\{3\}\{1,2\}\{1,3\}\{2,3\}\{1,2,3\}\{\emptyset\}\}$

$\subseteq \circ \subseteq \{(\{\emptyset\},\{\emptyset\})(\{\emptyset\},\{1\})(\{\emptyset\}\{2\})(\{\emptyset\}\{3\})(\{\emptyset\}\{1,2\})(\{\emptyset\}\{1,3\})(\{\emptyset\}\{2,3\})(\{\emptyset\}\{1,2,3\})$
$(\{1\},\{1\})(\{2\}\{2\})(\{3,2\}\{2,3\})(\{1\}\{1,2\})(\{1\}\{1,3\})(\{2\}\{1,2\})(\{1\}\{1,2,3\})$
$(\{1\ 2\},\{1,2\})(\{1,3\}\{1,3\})(\{1,3\}\{1,2,3\})(\{1,2\}\{1,2,3\})(\{2\}\{2,3\})(\{2\}\{1,2,3\})$
$(\{3,2\}\{1,2,3\})(\{1,2,3\}\{1,2,3\})(\{3\}\{1,3\})(\{3\}\{2,3\})(\{3\}\{1,2,3\})\}$

3. What is the set $R^{-1}[\{12,15,42\}]$, where R is the divisibility relation $\mid$ on the set $\mathbb{Z}$?

$R = \{(\pm1,12)(\pm2,12)(\pm3,12)(\pm4,12)(\pm6,12)(\pm12,12)$
$(\pm1,15)(\pm3,15)(\pm5,15)(\pm15,15)(\pm1,42)(\pm2,42)(\pm3,42)$
$(\pm6,42)(\pm7,42)(\pm14,42)(\pm21,42)(\pm42,42)\}$

$(a,42)$
$(-a,42)$

$(\pm a,42)$

4. Prove that $R \circ (P \cup Q) = (R \circ P) \cup (R \circ Q)$ for every relations P, Q, R.

By Lemma 9.22 and Theorem 9.19, it holds that

$$\begin{aligned}(R \circ (P \cup Q))^{-1} &= (P \cup Q)^{-1} \circ R^{-1} = \\ &= (P^{-1} \cup Q^{-1}) \circ R^{-1} = (P^{-1} \circ R^{-1}) \cup (Q^{-1} \circ R^{-1}) = \\ &= (R \circ P)^{-1} \cup (R \circ Q)^{-1} = ((R \circ P) \cup (R \circ Q))^{-1}.\end{aligned}$$

Hence,

$$R \circ (P \cup Q) = ((R \circ (P \cup Q))^{-1})^{-1} = ((R \circ P) \cup (R \circ Q))^{-1})^{-1} = (R \circ P) \cup (R \circ Q).$$

Lemma 9.22. *Let R, P, Q be arbitrary binary relations. Then*

1. $(P \cup Q) \circ R = (P \circ R) \cup (Q \circ R)$;
2. $(P \cap Q) \circ R \subseteq (P \circ R) \cap (Q \circ R)$.

Proof. For any pair (a, c),

$$\begin{aligned}(a,c) \in (P \cup Q) \circ R \quad &\Longleftrightarrow \quad \exists b\,(aRb \wedge (b,c) \in P \cup Q) \\ &\Longleftrightarrow \quad \exists b\,(aRb \wedge (bPc \text{ or } bQc)) \\ &\Longleftrightarrow \quad \exists b\,((aRb \wedge bPc) \text{ or } (aRb \wedge bQc)) \\ &\Longleftrightarrow \quad \exists b\,(aRb \wedge bPc) \text{ or } \exists b\,(aRb \wedge bQc) \\ &\Longleftrightarrow \quad (a,c) \in P \circ R \text{ or } (a,c) \in Q \circ R \\ &\Longleftrightarrow \quad (a,c) \in (P \circ R) \cup (Q \circ R).\end{aligned}$$

Here we have applied a certain law of logic. Indeed, $\exists x\,(A \vee B)$ ("there exists a unicorn either black or tame") iff $\exists x\,A \vee \exists x\,B$ ("there exists a black unicorn or there exists a tame unicorn").

Consider the second statement.

$$\begin{aligned}(a,c) \in (P \cap Q) \circ R \quad &\Longleftrightarrow \quad \exists b\,(aRb \wedge (b,c) \in P \cap Q) \\ &\Longleftrightarrow \quad \exists b\,(aRb \wedge (bPc \wedge bQc)) \\ &\Longleftrightarrow \quad \exists b\,((aRb \wedge bPc) \wedge (aRb \wedge bQc)) \\ &\Longrightarrow \quad \exists b\,(aRb \wedge bPc) \wedge \exists b\,(aRb \wedge bQc) \\ &\Longleftrightarrow \quad (a,c) \in P \circ R \wedge (a,c) \in Q \circ R \\ &\Longleftrightarrow \quad (a,c) \in (P \circ R) \cap (Q \circ R).\end{aligned}$$

Thus, from $(a,c) \in (P \cap Q) \circ R$, it follows that $(a,c) \in (P \circ R) \cap (Q \circ R)$. We have applied the fact that $\exists x\,(A(x) \wedge B(x))$ ("there exists a unicorn both black and tame") implies $\exists x\,A(x) \wedge \exists x\,B(x)$ ("there exists a black unicorn and there exists a tame unicorn"). The reverse implication does not hold in general. □

Theorem 9.19. *Let P, Q be arbitrary binary relations. Then $(Q \circ P)^{-1} = P^{-1} \circ Q^{-1}$.*

Proof. For any pair (a, c), obtain:

$$\begin{aligned}(a,c) \in (Q \circ P)^{-1} \quad &\Longleftrightarrow \quad (c,a) \in Q \circ P \\ &\Longleftrightarrow \quad \exists b\,(cPb \wedge bQa) \\ &\Longleftrightarrow \quad \exists b\,(aQ^{-1}b \wedge bP^{-1}c) \\ &\Longleftrightarrow \quad (a,c) \in P^{-1} \circ Q^{-1}.\end{aligned}$$

If it's illeagle plz write me, I can defend it in offline

5. Does the inclusion $(R \circ P) \cap (R \circ Q) \subseteq R \circ (P \cap Q)$ hold for every relations P, Q, R?

$R = \{(2,0)(1,0)\}$

$P = \{(0,1)\}$ $\Rightarrow$ $\{(0,0)\} \cap \{(0,0)\} \subseteq R \circ (\emptyset)$

$\Updownarrow$

$Q = \{(0,2)\}$ $\{(0,0)\} \subseteq \emptyset$ $\bot$

6. Does the inclusion $R[X] \cap R[Y] \subseteq R[X \cap Y]$ hold for every relation R and sets X and Y?

$X = \{(0,2)\}$ $Y = \{(0,0)\}$

$R = \{(2,3),(2,4)(0,4)\}$ $R[X] \cap R[Y] = \{(0,4)\}$

$R[X] = \{(0,3),(0,4)\}$ $R[X \cap Y] = R[\emptyset] = \emptyset$

$R[Y] = \{(0,4)\}$ $\Rightarrow \{(0,4)\} \subseteq \emptyset$ $\bot$

7. Does the identity $(R \cup Q)[X] = R[X] \cup Q[X]$ hold for every relations R, Q and set X?

$b \in (R \cup Q)[X] \Leftrightarrow \exists a (a \in X \wedge (a,b) \in R \cup Q) \Leftrightarrow$

$\Leftrightarrow \exists a (a \in X \wedge ((a,b) \in R \vee (a,b) \in Q)) \Leftrightarrow$

$\Leftrightarrow \exists a ((a \in X \wedge (a,b) \in R) \vee (a \in X \wedge (a,b) \in Q)) \Leftrightarrow$

$\Leftrightarrow \exists a (a \in X \wedge (a,b) \in R) \vee \exists a (a \in X \wedge (a,b) \in Q) \Leftrightarrow$

$\Leftrightarrow b \in R[x] \vee b \in Q[x] \Leftrightarrow b \in R[x] \cup Q[x]$ ■

Tnx for your work ❤

Novosad Ivan

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