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✓ **2.** (HW) Find the indefinite integral using basic integration rules. Check the result by differentiation.

(a) $\int x^{7/8}\left(2x^{3/5}-3x^{-2}+\sqrt{x}\right)dx$; (b) $\int \frac{\sqrt{x}-x^3e^x+x^2}{x^3}\,dx$; (c) $\int \frac{x\sin 2x+\sqrt[5]{x^2}\cos x}{x\cos x}\,dx$;

(d) $\int \frac{\sqrt{3+x^2}-\sqrt{3-x^2}}{\sqrt{9-x^4}}\,dx$; (e) $\int (\tan x+\cot x)^2\,dx$; (f) $\int \frac{1+2x^2}{x^2(1+x^2)}\,dx$.

✓ a) $\int x^{7/8}\left(2x^{3/5}-3x^{-2}+\sqrt{x}\right)dx = \int\left[2x^{59/40}-3x^{-9/8}+x^{11/8}\right]dx = \frac{80}{99}x^{99/40}+24x^{-1/8}+\frac{8}{19}x^{19/8}+C \quad \Big| \frac{d}{dx} = \frac{80}{99}x^{59/40}-3x^{-9/8}+x^{11/8}$

✓ b) $\int \frac{\sqrt{x}-x^3e^x+x^2}{x^3}\,dx = \int\left[x^{-5/2}-e^x+x^{-1}\right]dx = -\frac{2}{3}x^{-3/2}-e^x+\ln(|x|)+C \quad \Big| \frac{d}{dx} = -\frac{2}{3}\cdot\left(-\frac{3}{2}\right)x^{-5/2}-e^x+\frac{1}{\sqrt{x^2}}\cdot\frac{1}{2\sqrt{x^2}}\cdot 2x+0 = x^{-5/2}-e^x+\frac{1}{x} = \frac{1+x\sqrt{x}-e^x x^2\sqrt{x}}{x^2\sqrt{x}}\Big|\begin{matrix}\cdot\sqrt{x}\\ \cdot\sqrt{x}\end{matrix} = \frac{\sqrt{x}+x^2-x^3e^x}{x^3}$

✓ c) $\int \frac{x\sin(2x)+x^{2/5}\cos(x)}{x\cos(x)}\,dx = \int\frac{\sin(2x)}{\cos(x)}dx+\int x^{-3/5}dx = 2\int\sin(x)\,dx+\int x^{-3/5}dx = -2\cos(x)+\frac{5}{2}x^{2/5}+C \quad \Big| \frac{d}{dx} = 2\sin(x)+x^{-3/5} = \frac{x2\sin(x)\cos(x)+x^{2/5}\cos(x)}{x\cos(x)}$

$\int\frac{1}{\sqrt{x^2+a^2}}dx = \ln\left(\left|x+\sqrt{x^2+a^2}\right|\right)+C$

✓ d) $\int \frac{\sqrt{3+x^2}-\sqrt{3-x^2}}{\sqrt{9-x^4}}\,dx = \int\frac{\sqrt{3+x^2}-\sqrt{3-x^2}}{\sqrt{(3+x^2)(3-x^2)}}dx = \int\frac{1}{\sqrt{3-x^2}}dx-\int\frac{1}{\sqrt{3+x^2}}dx = \arcsin\left(\frac{x}{\sqrt{3}}\right)-\ln\left(\left|x+\sqrt{x^2+3}\right|\right)+C \quad \Big| \frac{1}{\sqrt{3-x^2}}-\frac{1}{\sqrt{3+x^2}} = \frac{\sqrt{3+x^2}-\sqrt{3-x^2}}{\sqrt{3+x^2}\sqrt{3-x^2}} = \frac{\sqrt{3+x^2}-\sqrt{3-x^2}}{\sqrt{9-x^4}}$

straight forward

e) $\int(\tan(x)+\cot(x))^2dx = \int\tan^2(x)+2\int\tan(x)\cot(x)\,dx+\int\cot^2(x)\,dx = \underbrace{\int\tan^2(x)\,dx}_{(1)}+2\overbrace{\int dx}^{2x+C}+\underbrace{\int\cot^2(x)\,dx}_{(2)} = 2x+\sin^3(x)\sec(x)-x+\frac{1}{2}\sin(2x)-\cos^3(x)\csc(x)-x-\frac{1}{2}\sin(2x)+C \circledcirc$

(1) $\int\tan^2(x)\,dx = \int\frac{\sin^2(x)}{\cos^2(x)}dx = \left|\begin{matrix}u=\sin^2(x) & dw=\sec^2(x)\\ du=\sin(2x) & w=\tan(x)\end{matrix}\right| = \sec(x)\sin^3(x)-\int\frac{2\sin^2(x)\cos(x)}{\cos(x)}dx = \sin^3(x)\sec(x)-2\int\sin^2(x)\,dx = \sin^3(x)\sec(x)-2\int\frac{1}{2}dx+2\int\frac{\cos(2x)}{2}dx = \sin^3(x)\sec(x)-x+\frac{1}{2}\sin(2x)+C$

$\left(\sin^2(x)=\frac{1-\cos(2x)}{2}\right)$

(2) $\int\cot^2(x)\,dx = \int\cos^2(x)\csc^2(x)\,dx = \left|\begin{matrix}u=\cos^2(x) & dw=\csc^2(x)\\ du=-\sin(2x)dx & w=-\cot(x)\end{matrix}\right| = -\cos^3(x)\csc(x)-2\int\frac{\sin(x)\cos^2(x)}{\sin(x)}dx = -\cos^3(x)\csc(x)-2\int\cos^2(x) = -\cos^3(x)\csc(x)-2\int\frac{1+\cos(2x)}{2}dx = -\cos^3(x)\csc(x)-x-\frac{1}{2}\sin(2x)+C$

$\circledcirc \ \sin^3(x)\sec(x)-\cos^3(x)\csc(x)+C = \frac{\sin^3(x)}{\cos(x)}-\frac{\cos^3(x)}{\sin(x)}+C = \frac{\sin^4(x)-\cos^4(x)}{\sin(x)\cos(x)}+C = \frac{-(\cos^4(x)-\sin^4(x))}{\frac{1}{2}\sin(2x)}+C = -2\frac{(\cos^2(x)-\sin^2(x))\overbrace{(\cos^2(x)+\sin^2(x))}^{1}}{\sin(2x)}+C = -2\frac{\cos^2(x)-\sin^2(x)}{\sin(2x)}+C = -2\frac{\cos(2x)}{\sin(2x)}+C = -2\cot(2x)+C$

check: $-2\cot(2x)\frac{d}{dx} = -2(-\csc^2(2x))2 = 4\csc^2(2x) \overset{⊛}{=} (2\csc(2x))^2 = \left(\frac{2}{2\sin(x)\cos(x)}\right)^2 \overset{⊛}{=} \left(\frac{1}{\sin(x)\cos(x)}\right)^2 \overset{⊛}{=} \left(\frac{\sin^2(x)+\cos^2(x)}{\sin(x)\cos(x)}\right)^2 \overset{⊛}{=} \left(\frac{\sin^2(x)}{\sin(x)\cos(x)}+\frac{\cos^2(x)}{\sin(x)\cos(x)}\right)^2 \overset{⊛}{=} (\tan(x)+\cot(x))^2$ ∎

✓ e) From ⊛: $\int(\tan(x)+\cot(x))^2dx \overset{*}{=} \int 4\csc^2(2x) = 4\int\csc^2(2x) = 4\frac{1}{2}(-\cot(2x))+C = -2\cot(2x)$

f) $\int\frac{1+2x^2}{x^2(1+x^2)}dx = \int\left[\frac{1}{x^2}+\frac{1}{1+x^2}\right]dx = \int x^{-2}dx+\int\frac{1}{1+x^2}dx = -\frac{1}{x}+\arctan(x)+C \quad \Big| \frac{d}{dx} = \frac{1}{x^2}+\frac{1}{1+x^2} = \frac{1+x^2+x^2}{x^2(1+x^2)} = \frac{2x^2+1}{x^2(1+x^2)}$ ∎

3. Find the indefinite integral by the method of substitution.

(a) $\displaystyle\int \frac{\ln x}{x}\,dx$; (b) $\displaystyle\int \frac{2x}{x^2+1}\,dx$; (c) $\displaystyle\int \frac{\sin\sqrt{x}}{2\sqrt{x}}\,dx$; (d) $\displaystyle\int \frac{dx}{x\ln^3 x}$;

(e) $\displaystyle\int \frac{e^x+e^{-x}}{e^x-e^{-x}}\,dx$; (f) $\displaystyle\int -\frac{dx}{\arccos^2 x\cdot\sqrt{1-x^2}}$; (g) $\displaystyle\int (10x+1)\sqrt[3]{5x^2+x+5}\,dx$.

✓ a) $\displaystyle\int \frac{\ln(x)}{x}\,dx = \left|\begin{array}{l} u=\ln(x) \\ du = \frac{1}{x}dx \end{array}\right| = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\ln^2(x) + C$

✓ b) $\displaystyle\int \frac{2x}{x^2+1}\,dx = \left|\begin{array}{l} u=x^2+1 \\ du = 2x\,dx \end{array}\right| = \int \frac{1}{u}\,du = \ln(u) + C = \ln(x^2+1) + C$

✓ c) $\displaystyle\int \frac{\sin(\sqrt{x})}{2\sqrt{x}}\,dx = \left|\begin{array}{l} u=\sqrt{x} \\ du = (2\sqrt{x})^{-1}dx \end{array}\right| = \int \sin(u)\,du = -\cos(u) + C = -\cos(\sqrt{x}) + C$

✓ d) $\displaystyle\int \frac{dx}{x\ln^3(x)} = \left|\begin{array}{l} \ln(x)=u \\ du = \frac{1}{x}dx \end{array}\right| = \int \frac{1}{u^3}\,du = -\frac{1}{2}u^{-2} + C = -\frac{1}{2\ln^2(x)} + C$

✓ e) $\displaystyle\int \frac{e^x+e^{-x}}{e^x-e^{-x}} \left|\begin{array}{l} u=e^x-e^{-x} \\ du = (e^x+e^{-x})dx \end{array}\right| = \int \frac{1}{u}\,du = \ln(|u|) + C = \ln(|e^x-e^{-x}|) + C$

f) $\displaystyle\int -\frac{dx}{\arccos^2(x)\sqrt{1-x^2}} = \left|\begin{array}{l} u=\arccos(x) \\ du = -\frac{1}{\sqrt{1-x^2}}dx \end{array}\right| = \int \frac{du}{u^2} = -u^{-1} + C = -\frac{1}{\arccos(x)} + C$

g) $\displaystyle\int (10x+1)\sqrt[3]{5x^2+x+5}\,dx = \left|\begin{array}{l} u=5x^2+x+5 \\ du = [10x+1]dx \end{array}\right| = \int u^{1/3}\,du = \frac{3}{4}u^{4/3} + C = \frac{3}{4}|5x^2+x+5|\sqrt[3]{5x^2+x+5} + C$

4. (HW) Find the indefinite integral by the method of substitution.

$$\text{(a)} \int \frac{4x+5}{2x^2+5x-6}\,dx; \qquad \text{(b)} \int \frac{\cos 2x}{\sin x \cdot \cos x}\,dx; \qquad \text{(c)} \int \frac{x^3}{\sqrt{1-x^8}}\,dx;$$

$$\text{(d)} \int \frac{\arctan x}{1+x^2}\,dx; \qquad \text{(e)} \int e^{\cos x} \cdot \sin x\,dx; \qquad \text{(f)} \int \frac{\cos(\ln x)}{x}\,dx.$$

$$\checkmark\ a) \int \frac{4x+5}{2x^2+5x-6}\,dx \left| \begin{array}{l} u = 2x^2+5x-6 \\ 4x+5\,dx = du \end{array} \right| = \int \frac{1}{u}\,du = \ln\left(|2x^2+5x-6|\right)+c$$

note: $\cos^2(x)-\sin^2(x) = \cos(2x)$

$$\checkmark\ b) \int \frac{\cos(2x)}{\sin(x)\cos(x)}\,dx = \int \frac{2\cos(2x)}{\sin(2x)}\,dx = 2\int \frac{\cos(2x)}{\sin(2x)}\,dx = \left| \begin{array}{l} t = \sin(2x) \\ dt = 2\cos(2x) \\ 2\cos(2x)\,dx = dt \end{array} \right| = \int \frac{1}{t}\,dt = \ln(|t|)+c = \ln(|\sin(2x)|)+c$$

$$\checkmark\ c) \int \frac{x^3}{\sqrt{1-x^8}}\,dx = \left| \begin{array}{l} t = x^4 \\ dt = 4x^3 \\ x^3 dx = \frac{1}{4}\,dt \end{array} \right| = \frac{1}{4}\int \frac{dt}{\sqrt{1-t^2}} = \frac{1}{4}\arcsin(t)+c \Rightarrow \frac{1}{4}\arcsin(x^4)+c$$

$$\checkmark\ d) \int \frac{\arctan(x)}{1+x^2}\,dx = \left| \begin{array}{l} u = \arctan(x) \\ du = \frac{1}{1+x^2} \\ \frac{1}{1+x^2}\,dx = dt \end{array} \right| = \int u\,du = \frac{1}{2}u^2+c = \frac{1}{2}\arcsin^2(x)+c$$

$$e) \int e^{\cos x}\cdot\sin(x)\,dx \xrightarrow{\text{by mind tecnique}} -e^{\cos(x)}+c$$

$$\int e^{\cos(x)}\sin(x)\,dx = \left| \begin{array}{l} u = \cos(x) \\ \sin(x)\,dx = -du \end{array} \right| = -\int e^u\,du = -e^u+c = -e^{\cos(x)}+c$$

$$f) \int \frac{\cos(\ln(x))}{x}\,dx = \left| \begin{array}{l} u = \ln(x) \\ du = \frac{1}{x} \\ \frac{1}{x}dx = du \end{array} \right| = \int \cos(u)\,du = \sin(u)+c = \sin(\ln(x))+c$$

$$\int \frac{\cos(2x)}{\sin(x)\cos(x)}\,dx = \left| \begin{array}{l} u = \sin(x)\cos(x) \\ du = \cos(2x)\,dx \\ \cos(2x)\,dx = du \end{array} \right| = \int \frac{1}{u}\,du = \ln(|u|)+c = \ln(|\sin(x)\cos(x)|)+c$$

5. Find the indefinite integral by making a linear substitution.

(a) $\int \cos(2x+3)\,dx$; (b) $\int \frac{dx}{3-5x}$; (c) $\int \frac{dx}{x^2+4x+5}$; (d) $\int \frac{dx}{\sqrt{-x^2+2x+8}}$.

√ a) $\int \cos(2x+3)dx = \frac{1}{2}\sin(2x+3)+C$ (by mind-integration tecnique)

√ b) $\int \frac{dx}{3-5x} = -\frac{1}{5}\ln(3-5x)+C$ (by mind-integration tecnique)

c) $\int \frac{dx}{x^2+4x+5} = \int \frac{dx}{(x+2)^2+1} = \left| \begin{matrix} u = x+2 \\ du = dx \end{matrix} \right| = \int \frac{du}{u^2+1} = \arctan(u)+C = \arctan(x+2)+C$

d) $\int \frac{dx}{\sqrt{-x^2+2x+8}} = -\int \frac{dx}{\sqrt{x^2-2x-8}} = -\int \frac{dx}{\sqrt{(x-1)^2-9}} = \left| \begin{matrix} t = x-1 \\ dt = dx \end{matrix} \right| = -\int \frac{dt}{\sqrt{t^2-9}} = -\ln\left(\left|t+\sqrt{t^2-9}\right|\right)^{+C} = -\ln\left(\left|x-1+\sqrt{x^2-2x-8}\right|\right)+C$

6. (HW) Find the indefinite integral by making a linear substitution.

$$\text{(a)} \int e^{7x-2}\,dx; \qquad \text{(b)} \int (4-3x)^{7/2}\,dx; \qquad \text{(c)} \int \frac{dx}{3+(2x+5)^2}.$$

✓ a) $\displaystyle\int e^{7x-2}\,dx = \left| \begin{array}{l} u = 7x-2 \\ du = 7 \\ dx = \frac{1}{7}\,du \end{array} \right| = \frac{1}{7}\int e^{u}\,du = \frac{1}{7}e^{u} + c = \frac{1}{7}e^{7x-2} + c$

✓ b) $\displaystyle\int (4-3x)^{7/2}\,dx = \left| \begin{array}{l} t = 4-3x \\ dt = -3 \\ dx = -\frac{1}{3}dt \end{array} \right| = -\frac{1}{3}\int t^{7/2}\,dt = -\frac{1}{3}\cdot\frac{2}{9}\cdot t^{9/2} + c = -\frac{2}{27}(4-3x)^{9/2} + c$

✓ c) $\displaystyle\int \frac{dx}{3+(2x+5)^2} \quad \left| \begin{array}{l} t = 2x+5 \\ dt = 2 \\ dx = \frac{1}{2}\,dt \end{array} \right| = \frac{1}{2}\int \frac{dt}{3+t^2} = \frac{1}{2\sqrt{3}}\arctan\left(\frac{t}{\sqrt{3}}\right) + c \quad f(x) = \frac{\sqrt{3}\arctan\left(\frac{(2x+5)}{\sqrt{3}}\right)}{6} + c$

7. Find following indefinite integrals:

(a) $\int \frac{2x+7}{x^2+6x+5}\,dx$; (b) $\int \frac{3x+2}{\sqrt{x^2+2x+2}}\,dx$; (c) (HW) $\int \frac{2x+11}{x^2-6x+5}\,dx$;
(d) (HW) $\int \frac{3x-7}{x^2+8x+19}\,dx$; (e) (HW) $\int \frac{5x+1}{\sqrt{1+2x-x^2}}\,dx$.

✓ c) $\int \frac{2x+11}{x^2-6x+5}\,dx = \int\left[-\frac{13}{4(x-1)} + \frac{21}{4(x-5)}\right]dx = \frac{21}{4}\int\frac{dx}{x-5} - \frac{13}{4}\int\frac{dx}{x-1} = \frac{21}{4}\ln(|x-5|) - \frac{13}{4}\ln(|x-1|) + c$

✓ d) $\int \frac{3x-7}{x^2+8x+19}\,dx = \int \frac{3/2\,(2x+8)-19}{x^2+8x+19}\,dx = \frac{3}{2}\underbrace{\int\frac{2x+8}{x^2+8x+19}\,dx}_{(1)} - \underbrace{19\int\frac{dx}{x^2+8x+19}}_{(2)} = \frac{1}{3}\ln(|x^2+8x+19|) - \frac{19}{\sqrt{3}}\arctan\left(\frac{x+4}{\sqrt{3}}\right) + c$

(1) $\frac{3}{2}\int\frac{2x+8}{x^2+8x+19}\,dx = \left|\begin{array}{l} t = x^2+8x+19 \\ dt = 2x+8 \\ (2x+8)dx = dt \end{array}\right| = \frac{3}{2}\int\frac{1}{t}\,dt = \frac{3}{2}\ln(|t|)+c = \frac{1}{3}\ln(|x^2+8x+19|)+c$

(2) $19\int\frac{dx}{x^2+8x+19} = 19\int\frac{dx}{x^2+8x+16+3} = 19\int\frac{dx}{(x+4)^2+3} = \left|\begin{array}{l} t=(x+4) \\ dt = 1 \\ dt = dx \end{array}\right| = 19\int\frac{1}{t^2+3} = \frac{19}{\sqrt{3}}\arctan\left(\frac{t}{\sqrt{3}}\right)+c = \frac{19}{\sqrt{3}}\arctan\left(\frac{x+4}{\sqrt{3}}\right)+c$

e) $\int\frac{5x+1}{\sqrt{1+2x-x^2}}\,dx = \int\frac{5x+1}{\sqrt{2-(x-1)^2}}\,dx \left|\begin{array}{l} t = x-1 \\ dt = dx \\ 5x+1 = 5t+6 \end{array}\right| = \int\frac{5t+6}{\sqrt{2-t^2}}\,dt \overset{\text{trivial}}{=} \underbrace{\int\frac{5t}{\sqrt{2-t^2}}\,dt}_{(1)} + \underbrace{\int\frac{6}{\sqrt{2-t^2}}\,dt}_{(2)} \overset{\text{checked}}{=}$

checked (1) $\int\frac{5t}{\sqrt{2-t^2}}\,dt = \left|\begin{array}{l} w = 2-t^2 \\ dw = -2t\,dt \\ dt = -\frac{1}{2t}dw \end{array}\right| = -\frac{5}{2}\int\frac{1}{\sqrt{w}}\,dw = -\frac{5}{2}\cdot 2\sqrt{w}+c = -5\sqrt{w}+c = -5\sqrt{2-t^2}+c$

checked (2) $\int\frac{6}{\sqrt{2-t^2}}\,dt = 6\int\frac{dt}{\sqrt{2-t^2}} = 6\arcsin\left(\frac{t}{\sqrt{2}}\right)+c$

$\overset{\text{checked}}{=} -5\sqrt{2-t^2} + 6\arcsin\left(\frac{t}{\sqrt{2}}\right)+c \overset{\text{checked}}{=} -5\sqrt{2-(x-1)^2} + 6\arcsin\left(\frac{x-1}{\sqrt{2}}\right)+c \overset{\text{checked}}{=} -5\sqrt{1+2x-x^2} + 6\arcsin\left(\frac{x-1}{\sqrt{2}}\right)+c$

$f(x) = -5\sqrt{1-x^2+2x} + 6\arcsin\left(\frac{x-1}{\sqrt{2}}\right) \xrightarrow[\text{photomath}]{\text{by}} f'(x) = \frac{5x+1}{\sqrt{1+2x-x^2}}$

Thx for your work! Have a nice day! ❤