1. (1 point) Let

$$f(x) = -2ix^4 + (2+2i)x^3 + (2-i)x^2 + (1+2i)x + 3 - i,$$
$$g(x) = -x^2 + (1-i)x - i$$

be two polynomials in $\mathbb{C}[x]$. Then, find (uniquely defined) polynomials $q(x)$, $r(x) \in \mathbb{C}[x]$ such that

$$f(x) = q(x)g(x) + r(x) \quad \text{and} \quad \deg\big(r(x)\big) < \deg\big(g(x)\big).$$

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$$\begin{array}{l|l} -2ix^4+(2+2i)x^3+(2-i)x^2+(1+2i)x+3-i & -x^2+(1-i)x-i \\ \hline -\;\; -2ix^4+x^3(2+2i)\;+2x^2 & 2ix^2+i \\ \hline \quad -ix^2+(1+2i)x+3-i & \\ \quad -\;\; -ix^2+(1+i)x+1 & \\ \hline \quad\quad xi+2-i & \end{array}$$

then $f(x) = (2ix^2+i)(-x^2+(1-i)x-i) + xi+2-i$

$$\big(q(x) = (2ix^2+i) \wedge r(x) = xi+2-i\big)$$

2. (1 point) Find the interpolation polynomial in the Lagrange form, $p(x)$, such that:

$$p(-1) = 6, \quad p(0) = 5, \quad p(1) = 0, \quad p(2) = 3.$$

$a_0 b_0 \ (-1;6) \qquad a_1 b_1 = (0;5) \qquad a_2 b_2 \ (1;0) \qquad a_3 b_3 \ (2;3)$

then $\quad p(x) = \sum_{j=0}^{3} b_j \, l_j(x)$

$$l_0 = \frac{(x-0)(x-1)(x-2)}{(-1-0)(-1-1)(-1-2)} \qquad l_1 = \frac{(x+1)(x-1)(x-2)}{(0+1)(0-1)(0-2)}$$

$$l_2 = \frac{(x+1)(x-0)(x-2)}{(1+1)(1-0)(1-2)} \qquad l_3 = \frac{(x+1)(x-0)(x-1)}{(2+1)(2-0)(2-1)}$$

then $\quad p(x) = 6 l_0 + 5 l_1 + 0 l_2 + 3 l_3$

3. (2 points) Factor the polynomial $x^6 + 1 \in \mathbb{R}[x]$ into *irreducible* polynomials in $\mathbb{R}[x]$.

[**hint**: use the same approach as we did for $x^4 + 1$ is Seminar 9]

$$x^6 + 1 \iff x^6 + 3x^4 + 3x^2 + 1 - 3x^4 - 3x^2 \iff$$

$$\iff (x^2+1)^3 - 3x^2(x^2+1) \iff (x^2+1)\left((x^2+1)^2 - 3x^2\right) \iff$$

$$\iff \underline{(x^2+1)(x^2+1-\sqrt{3}x)(x^2+1+\sqrt{3}x)} \quad \text{Answer}$$

$$\Updownarrow$$

$$(x-i)(x+i)\left(x+\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\left(x+\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\left(x-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)\left(x+\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)$$

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extended solution, but, obviously not a answer to the task.

5. (2 points) Let a, b, and c be the complex roots of the polynomial $p(x) = x^3 + 2x^2 - x + 1$. Then, find the value of $a^4 + b^4 + c^4$.

[**hint**: any symmetric polynomial in variables a, b, c (for example, $a^4+b^4+c^4$) can be expressed as a *polynomial* in variables p, q, r, where $p = a+b+c$, $q = ab+bc+ac$, $r = abc$. For example, $a^3+b^3+c^3 = p^3 - 3pq + 3r$; find a similar expression for $a^4+b^4+c^4$ (this may take more than one try) and use The Vieta Theorem for cubic polynomials]

$$\begin{cases} a+b+c = -2 = p \\ ab+ac+bc = -1 = q \\ abc = -1 = r \end{cases}$$

$$(a+b+c)^4 = a^4 + 4a^3b + 4a^3c + 6a^2b^2 + 12a^2bc + 6a^2c^2 + 4ab^3 +$$

$$+ 12ab^2c + 12abc^2 + 4ac^3 + b^4 + 4b^3c + 6b^2c^2 + 4bc^3 + c^4$$

$$\cancel{(ab+ac+bc)(a+b+c) = a^2b + a^2c + ab^2 + 3abc + ac^2 + b^2c + bc^2}$$

$$p^4 - (ab+ac+bc)(a+b+c)^2 = a^3b + a^3c + 2a^2b^2 + 5a^2bc + 2a^2c^2 +$$

$$+ ab^3 + 5abc^2 + ac^3 + b^3c + 2b^2 + c^2 + bc^3$$

then $(a+b+c)^4 - (ab+ac+bc)(a+b+c)^2 4 \Leftrightarrow$

$$\Leftrightarrow a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2a^2c^2 - 8(a^2bc + ab^2c + abc^2)$$

then $(a+b+c)^4 - 4(a+b+c)^2(ab+ac+bc) + 8(abc)(a+b+c) =$

$$= a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2$$

then $(a+b+c)^4 - 4(a+b+c)^2(ab+ac+bc) + 8(abc)(a+b+c) + 2(ab+ac+bc)^2$

$= a^4 + b^4 + c^4 + 4(a^2bc + ab^2c + abc^2)$

then $a^4 + b^4 + c^4 =$

$(a+b+c)^4 - 4(a+b+c)^2(ab+ac+bc) + 8(abc)(a+b+c) + 2(ab+ac+bc)^2 - 4(a+b+c)abc$

$= a^4 + b^4 + c^4$

then

$a^4 + b^4 + c^4 = p^4 - 4p^2q + 8rp + 2q^2 - 4pr$

$-2; -1; -1$

then $a^4 + b^4 + c^4 = (-2)^4 - 4(-2)^2(-1) + 8(-1)(-2) + 2(-1)^2 - 4(-2)(-1) =$

$= 42$

4. (2 points) Find the multiplicity of the root $\lambda = 1$ of the polynomial $p(x) = x^{2n} - nx^{n+1} + nx^{n-1} - 1$, where $n \in \mathbb{N}$.

[**hint**: recall that we proved the fact: if λ ia a root of multiplicity k for a polynomial $p(x)$, then, λ ia a root of multiplicity $k-1$ for the derivative $p'(x)$; what is the multiplicity of λ for $p''(x)$?; do not forget about the

$$f(x) = x^{2n} - n x^{n+1} + n x^{n-1} - 1 \qquad n \in N \Rightarrow f(1) = 0$$

$$f'(x) = 2n x^{2n-1} - n(n+1) x^{n} + (n+1) n x^{n-2} \qquad n \in N \Rightarrow f'(1) = 0$$

$$f''(x) = (4n^2 - 2n) x^{2n-2} - (n^3 + n^2) x^{n-1} + (n^2 - n - 2) x^{n-3} \qquad n \in N$$

$$\Rightarrow f''(1) = 0$$

$$f'''(x) = (2n-2)(4n-2) n x^{2n-2} - (n+1) n^2 x^{n-1} + (n^2 - n - 2)(n-3) x^{n-4}$$

$$\Rightarrow f'''(1) = (2n-2)(4n-2) n - n^3 + n^2 + (n^2 - n - 2)(n-3) =$$

$$= 8n^3 - 12n^2 + 4n - n^3 + n^2 + n^3 - 4n^2 + n + 6 \neq 0 \Rightarrow$$

$\Rightarrow$ multiplicity equal 3 $\Rightarrow \forall n \in \mathbb{N} \wedge n \geq 4$

for $n < 4$

$n = 1$

$p(x) = k^2 - k^2 + 1 - 1 = 0 \Rightarrow k = 0$

$n = 2$

$p(x) = x^4 - 2x^3 + 2x - 1 = x^4 - x^3 - (x^3 - x^2) - (x^2 - x) + x - 1 =$

$= (x-1)(x^3 - x^2 - x + 1) = (x-1)^3(x+1) \Rightarrow k = 3$

$n = 3$

$p(x) = x^6 - 3x^4 + 3x^2 - 1 \Leftrightarrow (x^2 - 1)^3 = 0 \Rightarrow k = 3$

$$\Rightarrow K = \begin{cases} 3 & n \geq 2 \\ 0 & n = 1 \end{cases}$$

//—//

6. (2 points) Let

$$p(x) = (x^2 + x + 1)^{2077} + x + 1 \in \mathbb{C}[x].$$ [1]

Note that, since (by Theorem 11.2) the field $\mathbb{C}$ is algebraically closed and $\deg(p(x)) = 4154$, due to Lemma 11.1, $p(x)$ has 4154 many complex roots, say $\lambda_1, \lambda_2, \ldots, \lambda_{4154} \in \mathbb{C}$.

Following the instructions, find the values of the sum:

$$S = \sum_{i=1}^{4154} \frac{1}{1 - \lambda_i}.$$

Instructions:

(a) Note that, due to Lemma 11.1, we have $p(x) = 1 \cdot (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_{4154})$. Then, by the well-known derivative formula $(uv)' = u'v + uv'$, we have

$$p'(x) = \sum_{j=1}^{4154} h_j(x), \quad \text{where } h_j(x) = (x - \lambda_1) \cdots (x - \lambda_{j-1}) \cdot (x - \lambda_{j+1}) \cdots (x - \lambda_{4154}),\ j \in \{1, \ldots, 4154\}.$$

[1] Since $\mathbb{R}$ is a subfield of $\mathbb{C}$, every polynomial with real coefficients is also a polynomial with complex coefficients.

(b) Taking into account the above expression for $p(x)$ and $p'(x)$, consider the rational function $f(x) = \frac{p'(x)}{p(x)}$.

$f(x)$ is very similar to lagrange polynomial, then:

$$p'(x) = \sum_{j=1}^{4154} h_j(x) \quad , \text{ where } h_j(x) = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_{j-1})(x-\lambda_{j+1})\cdots(x-\lambda_{4154})$$

$$p(x) = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_{4154})$$

then $$f(x) = \frac{p'(x)}{p(x)} = \sum_{j=1}^{4154} \frac{(x-\lambda_1)\cdots(x-\lambda_{j-1})(x-\lambda_{j+1})\cdots(x-\lambda_{4154})}{(x-\lambda_1)\cdots(x-\lambda_{j-1})(x-\lambda_{j+1})\cdots(x-\lambda_{4154})} \cdot \frac{1}{x-\lambda_j}$$

$$= \sum_{j=1}^{4154} \frac{1}{x-\lambda_j} \quad , \text{ then } f(x) = S$$

We obtain $x = 1$, due to $S = \sum_{j=1}^{4154} \frac{1}{1-\lambda_j}$,

then we need to find $f(1)$:

$$p(1) = 3^{2077} + 2$$

$$p(x)' = 2077\left(x^2 + x + 1\right)^{2076}(2x+1) + x + 1$$

$$p'(1) = 2077(3)^{2076}(3) + 1 + 1 = 2077 \cdot 3^{2077} + 2$$

then $S = \dfrac{3^{2077} + 2}{2077 \cdot 3^{2077} + 2}$ $\square$

Tnx for checking!