1. (1 point) Let $\mathbb{V}$ be a 3-dimensional vector space over the field of reals; let $\mathcal{A}$ be an ordered basis for $\mathbb{V}$; let

$$\beta(\mathbf{x},\mathbf{y}) = 3x_1y_1 - x_1y_2 - 2x_1y_3 + 3x_2y_1 - 5x_3y_1 + 2x_3y_2 - x_3y_3,$$

where $[\mathbf{x}]_{\mathcal{A}} = [x_1\ x_2\ x_3]^{\mathrm{T}}$ and $[\mathbf{y}]_{\mathcal{A}} = [y_1\ y_2\ y_3]^{\mathrm{T}}$, be a bilinear form on $\mathbb{V}$. Then, find q_β.

[**hint**: you can either use Definition 26.1 or Item 1 of Theorem 26.2]

$$\beta(x,y) = [x_1\ x_2\ x_3]\begin{bmatrix} 3 & -1 & -2 \\ 3 & 0 & 0 \\ -5 & 2 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\text{so } q_\beta : [x_1\ x_2\ x_3]\begin{bmatrix} 3 & -1 & -2 \\ 3 & 0 & 0 \\ -5 & 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [3x_1^2 + 2x_1x_2 - 7x_1x_3 + 2x_2x_3 - x_3^2]$$

2. Let $\mathbb{V}$ be the vector space of all symmetric matrices of size 2 over the field of reals; let $\mathcal{A} = \left(\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{e_1}, \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{e_2}, \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}}_{e_3}\right)$ be an ordered basis for $\mathbb{V}$; let $q(A) = \mathrm{tr}(A^2)$, for $A \in \mathbb{V}$, be a quadratic form on $\mathbb{V}$. Then:

(a) (0,5 points) find a mistake in the following statement: since $\beta_1(A,B) = \mathrm{tr}(A \cdot B)$ and $\beta_2(A;B) = \mathrm{tr}(A \cdot B^{\mathrm{T}})$ are distinct symmetric bilinear forms such that $\beta_1(A,A) = \beta_2(A,A) = q(A)$, the uniqueness part of Statement 26.1 is not correct;

not

(b) (1 points) find $H(q;\mathcal{A})$;

[**hint**: for example, you can use Item (a) and Theorem 26.2]

(c) (1 point) using Equality (25.5), find $H(q;\mathcal{B})$, where $\mathcal{B} = \left(\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}}_{f_1}, \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{f_2}, \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{f_3}\right)$ is another ordered basis for $\mathbb{V}$.

a) since $\mathrm{tr}(A) = \mathrm{tr}(A^T) \Rightarrow \mathrm{tr}(AB) = \mathrm{tr}(AB^T)$

$\beta_1 = \beta_2$ (in fact it's not distingue BFs).

b) $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ (sym. mat 2x2) can be assos. with $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$\text{so } [a, b, c]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a^2 + 2b^2 + c^2$$

$$\begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}^2 = \begin{bmatrix} a_1^2 + a_2^2 & a_1a_2 + a_2a_3 \\ a_1a_2 + a_2a_3 & a_2^2 + a_3^2 \end{bmatrix} \to \mathrm{tr} = a_1^2 + 2a_2^2 + a_3^2$$

$$\begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_2+b_2 & a_3+b_3 \end{bmatrix}^2 \to \mathrm{tr} = \underline{a_1^2} + \underline{2a_1b_1} + \underline{b_1^2} + \underline{a_2^2} + \underline{2a_2b_2} + \underline{b_2^2} + \underline{a_2^2} + \underline{2a_2b_2} + \underline{b_2^2} + a_3^2 + 2a_3b_3 + b_3^2 =$$

$$= a_1^2 + 2a_1b_1 + 2a_2^2 + b_1^2 + 4a_2b_2 + 2b_2^2 + a_3^2 + 2a_3b_3 + b_3^2$$

$$\beta(A,B) = \frac{1}{2}\left(q(A+B) - q(A) - q(B)\right) = \frac{1}{2}\left(a_1^2 + 2a_1b_1 + 2a_2^2 + b_1^2 + 4a_2b_2 + 2b_2^2 + a_3^2 + 2a_3b_3 + b_3^2 - a_1^2 - 2a_2^2 - a_3^2 - b_1^2 - 2b_2^2 - b_3^2\right) =$$

$$= a_1b_1 + 2a_2b_2 + a_3b_3$$

and Matrix rep of $\beta(A,B) = [a_1\ a_2\ a_3]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1b_1 + 2a_2b_2 + a_3b_3$

$$\text{So } \beta(e_1, e_1) = [0\ 1\ 0]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 2$$

$$\beta(e_1, e_2) = [0\ 1\ 0]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0$$

$\beta(e_1, e_3) = 2 \quad \beta(e_2\, e_2) = 1 \quad \beta(e_2\, e_3) = 0 \quad \beta(e_3\, e_3) = 3$

Hence $H(q, A) = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix}$

$$\text{c) } \beta(f_1\, f_1) = [0\ 1\ 1]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 3$$

$$\beta(f_1\, f_2) = [0\ 1\ 1]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1$$

Similarly $\beta(f_1\, f_3) = 2$

$\beta(f_2\, f_2) = 2 \quad \beta(f_2\, f_3) = 1 \quad \beta(f_3\, f_3) = 3$

Hence $H(q, B) = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

3. Let $\mathbb{V}$ be a 3-dimensional vector space over the field of reals; let $\mathcal{A}$ be an ordered basis for $\mathbb{V}$; let

$$q(\mathbf{x}) = 2x_1^2 - x_2^2 + 3x_3^2 - 4x_1x_2 - 6x_1x_3 + 2x_2x_3,$$

where $[\mathbf{x}]_{\mathcal{A}} = [x_1\ x_2\ x_3]^{\mathrm{T}}$, be a quadratic form on $\mathbb{V}$. Then:

(a) (1 point) find the polynomial representation[1] of β_q;

(b) (1.5 points) find two distinct bilinear forms β_1 and β_2 on $\mathbb{V}$ such that $\beta_1(\mathbf{x},\mathbf{x}) = \beta_2(\mathbf{x},\mathbf{x}) = q(\mathbf{x})$, for every $\mathbf{x} \in \mathbb{V}$;

[**hint**: see Problem 2 from Seminar 26; note that we did not discuss this problem at the seminar, that is, I leave it for self-study]

(c) (2 points) using Lagrange's method for quadratic forms, find a canonical form of q and a canonical basis of q (see Definition 26.3).

[**hint**: see Problem 3 from Seminar 26; "Lagrange's method for quadratic forms" is the name of the algorithm we used to solve Problem 3; it is not a part of this problem but it is highly advisable to verify your calculations by matrix multiplication (see Page 26.9 of Seminar 26)]

a) $q(x) : [x_1\ x_2\ x_3]\begin{bmatrix}2&-2&-3\\-2&-1&1\\-3&1&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

So $B_q(x,y): [x_1\ x_2\ x_3]\begin{bmatrix}2&-2&-3\\-2&-1&1\\-3&1&3\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$

$B_q: 2x_1y_1 - 2x_2y_1 - 3x_2y - 2x_1y_1 - x_2y_2 - x_3y_2 - 3x_1y_3 + x_2y_3 + 3x_3y_3$

b) Since Matrix rep. of $q(x)$ is not unique, i.e. $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}^T\begin{bmatrix}2&-2&-3\\-2&-1&1\\-3&1&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}^T\begin{bmatrix}2&-4&-6\\0&-1&2\\0&0&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

So, we will abuse this fact. then $B_1(\bar{x},\bar{y}) = [x_1\ x_2\ x_3]\begin{bmatrix}2&-2&-3\\-2&-1&1\\-3&1&3\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$

Thus clearly $B_1 \neq B_2$ $\quad B_2(\bar{x},\bar{y}) = [x_1\ x_2\ x_3]\begin{bmatrix}2&-4&-6\\0&-1&2\\0&0&3\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$

but $\beta_1(x,x) = \beta_2(x,x) = q(x) = 2x_1^2 - x_2^2 + 3x_3^2 - 4x_1x_2 - 6x_1x_3 + 2x_2x_3$

c) $2(x_1^2 - 2x_1x_2 - 3x_1x_3) - x_2^2 + 3x_3^2 + 2x_2x_3 = 2(x_1^2 - 2x_1(x_2 + \frac{3}{2}x_3)) - x_2^2 + 3x_3^2 + 2x_2x_3 =$

$= 2(x_1 - x_2 - \frac{3}{2}x_3)^2 - 2(x_2 + \frac{3}{2}x_3)^2 - x_2 + 3x_3^2 + 2x_2x_3 = 2(x_1 - x_2 - \frac{3}{2}x_3)^2 - 2x_2^2 - 6x_2x_3 - \frac{9}{2}x_3^2 - x_2 + 3x_3^2 + 2x_2x_3 =$

$= 2(x_1 - x_2 - \frac{3}{2}x_3)^2 - 3x_2^2 - 4x_2x_3 - \frac{3}{2}x_3^2$

$2(x_1 - x_2 - \frac{3}{2}x_3)^2 - 3(x_2 + \frac{2}{3}x_3)^2 - \frac{1}{6}x_3^2$

So $\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix} = \begin{bmatrix}x_1 - x_2 - \frac{3}{2}x_3\\x_2 + \frac{2}{3}x_3\\x_3\end{bmatrix}$

So $M(q,A) = \begin{bmatrix}2&0&0\\0&-3&0\\0&0&-\frac{1}{6}\end{bmatrix} \Rightarrow \begin{bmatrix}x_1 - x_2 - \frac{3}{2}x_3\\x_2 + \frac{2}{3}x_3\\x_3\end{bmatrix}^T\begin{bmatrix}2&0&0\\0&-3&0\\0&0&-\frac{1}{6}\end{bmatrix}\begin{bmatrix}x_1 - x_2 - \frac{3}{2}x_3\\x_2 + \frac{2}{3}x_3\\x_3\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}^T\begin{bmatrix}2&-2&-3\\-2&-1&1\\-3&1&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

$(C\bar{x})^T\begin{bmatrix}2&0&0\\0&-3&0\\0&0&-\frac{1}{6}\end{bmatrix}(C\bar{x}) = \bar{x}^T\begin{bmatrix}2&-2&-3\\-2&-1&1\\-3&1&3\end{bmatrix}\bar{x}$

$\bar{x}^T C^T\begin{bmatrix}2&0&0\\0&-3&0\\0&0&-\frac{1}{6}\end{bmatrix}C\bar{x} = \bar{x}^T\begin{bmatrix}2&-2&-3\\-2&-1&1\\-3&1&3\end{bmatrix}\bar{x}$

$C^T\begin{bmatrix}2&0&0\\0&-3&0\\0&0&-\frac{1}{6}\end{bmatrix}C = \begin{bmatrix}2&-2&-3\\-2&-1&1\\-3&1&3\end{bmatrix}$ $\quad$ So $\left(\begin{bmatrix}2\\0\\0\end{bmatrix},\begin{bmatrix}0\\-3\\0\end{bmatrix},\begin{bmatrix}0\\0\\-\frac{1}{6}\end{bmatrix}\right) \leftarrow$ canonical basis

$q(\bar{y}) = 2y_1^2 - 3y_2^2 - \frac{1}{6}y_3^2$ $\leftarrow$ canonical form

4. (2 points) Let

$$A = \begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix} \in \mathrm{Mat}_3(\mathbb{R}).$$

Then, find a *lower triangular* matrix L such that $A = LL^T$.

Using sym.-gauss:

$$\begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix} \xrightarrow[\hat{\ell}_{1,3,4}]{(1)\ \ell_{1,3,4}} \begin{bmatrix} 4 & 12 & 0 \\ 12 & 37 & 5 \\ 0 & 5 & 34 \end{bmatrix} \xrightarrow[\hat{\ell}_{1,2,-3}]{(2)\ \ell_{1,2,-3}} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 5 & 34 \end{bmatrix} \xrightarrow[\hat{\ell}_{2,3,-5}]{(3)\ \ell_{2,3,-5}} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix} \xrightarrow[\hat{d}_{1,1/2}]{(4)\ d_{1,1/2}}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix} \xrightarrow[\hat{d}_{3,1/3}]{(5)\ d_{3,1/3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{(1)\ \ell_{1,3,4}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \xrightarrow{(2)\ \ell_{1,2,-3}} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \xrightarrow{(3)\ \ell_{2,3,-5}} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 19 & -5 & 1 \end{bmatrix} \xrightarrow[d_{3,1/3}\ (6)]{(5)\ d_{1,1/2}} \begin{bmatrix} 1/2 & 0 & 0 \\ -3 & 1 & 0 \\ 19/3 & -5/3 & 1/3 \end{bmatrix} = L^{-1}$$

$$\begin{bmatrix} 1/2 & 0 & 0 \\ -3 & 1 & 0 \\ 19/3 & -5/3 & 1/3 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1/2 & -3 & 19/3 \\ 0 & 1 & -5/3 \\ 0 & 0 & 1/3 \end{bmatrix}^{-1} = A, \text{ where } L = \begin{bmatrix} 2 & 6 & -8 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

Tnx for checking! Have a nice day.