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2. (HW) Let $f(x,y) = \arctan(y/x)$, $x = \cos t$, $y = \sin t$ and $F(t) = f(x(t), y(t))$. Compute $F'(t)$ in two ways: (i) by using formula for the derivative of a composition; (ii) by first substituting x and y into $f(x,y)$ and then differentiating the resulting function of t.

$$\text{I)}\ F'(t) = \frac{df}{dx}x' + \frac{df}{dy}y' = \left(-\frac{y}{y^2+x^2}\right)(-\sin(t)) + \left(\frac{x}{x^2+y^2}\right)(\cos(t)) = \frac{y\sin(t) + x\cos(t)}{y^2+x^2} = \frac{\sin^2(t)+\cos^2(t)}{\sin^2(t)+\cos^2(t)} = 1$$

$$\text{II)}\ F(t) = \arctan(\sin(t)/\cos(t)) = \arctan(\tan(t)) = t \Rightarrow F'(t) = 1$$

4. (HW) Compute the derivative of the function $f(x,y,z) = xyz + x^2 + y^2 - z^2$ with respect to the curve $x = 2t+1$, $y = \sin t$, $z = e^t$ at the point $t = 0$.

$$F'(t) = \frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z' :$$

$$\frac{df}{dx} = yz + 2x \ ; \ \frac{dx}{dt} = 2 \Rightarrow \frac{df}{dx}x' = 2yz + 4x = 2\sin(t)e^t + 4t + 4$$

$$\frac{df}{dy} = xz + 2y \ ; \ \frac{dy}{dt} = \cos(t) \Rightarrow \frac{df}{dy}y' = \cos(t)(xz + 2y) = \cos(t)(2e^t t + e^t + \cos(t))$$

$$\frac{df}{dz} = xy - 2z \ ; \ \frac{dz}{dt} = e^t \Rightarrow \frac{df}{dz}z' = e^t(xy - 2z) = e^t(2t\sin(t) + \sin(t) - 2e^t)$$

$$F'(t) = 2\sin(t)e^t + 4t + 4 + \cos(t)(2e^t t + e^t + \cos(t)) + e^t(2t\sin(t) + \sin(t) - 2e^t)$$

6. (HW) Let $f(x,y,z)$ have continuous derivatives and let $x = u+v$, $y = u-v$, $z = 2u+v$. $f_u^2 + f_v^2$.

$$f_u = f_x \cdot x_u' + f_y \cdot y_u' + f_z \cdot z_u' = f_x + f_y + 2f_z$$

$$f_v = f_x \cdot x_v' + f_y \cdot y_v' + f_z \cdot z_v' = f_x - f_y + f_z$$

$$f_u^2 = f_x^2 + f_y^2 + 4f_z^2 + 2f_xf_y + 4f_xf_z + 4f_yf_z$$

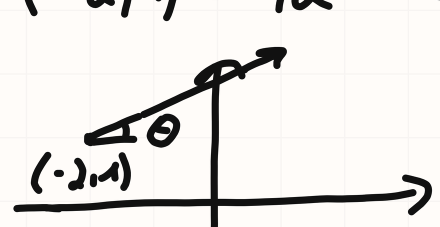
$$f_v^2 = f_x^2 + f_y^2 + f_z^2 - 2f_xf_y + 2f_xf_z - 2f_yf_z$$

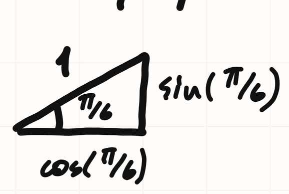
$$f_u^2 + f_v^2 = 2f_x^2 + 2f_y^2 + 5f_z^2 + 6f_xf_z + 2f_yf_z$$

8. Find the directional derivative of f at P in the direction of $\mathbf{u}$ that makes an angle of θ with the positive x-axis if:
(a) $f(x,y) = e^{xy}$, $\theta = \pi/3$, and $P(-2,0)$; (b) (HW) $f(x,y) = -3x^2 - 8y^2$, $\theta = \pi/6$, and $P(-2,1)$.

$$\text{a)}\ f_x' = -6x \ ; \ f_y' = -16y \Rightarrow \operatorname{grad}(f) = (-6x, -16y)$$

$$f_x'(-2,1) = 12 \quad f_y'(-2,1) = -16 \Rightarrow \operatorname{grad}(f(-2,1)) = (12, -16)$$





$$\Rightarrow \bar{u} = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

$$\partial f_{\bar{u}}(-2,0) = (\operatorname{grad}(f(-2,0)), \bar{u}) = \frac{12\cdot\sqrt{3}}{2} + \frac{1}{2}\cdot(-16) = 6\sqrt{3} - 8$$

10. (HW) Compute the derivative of the function f in the direction $\mathbf{v}$ at the point P if
(a) $f = \dfrac{1+x^2}{1+y^2}$, $\mathbf{v} = (3,4)$, and $P = (-1,1)$;
(b) $f = e^x \cos y + e^z \sin y$, $\mathbf{v} = (-3,4,5)$, and $P = (0, \pi/2, 0)$.

$$\text{a)}\ \frac{\partial f}{\partial x} = \frac{2x}{1+y^2} \quad \frac{\partial f}{\partial y} = -\frac{2y + 2x^2y}{(1+y^2)^2} \Rightarrow \operatorname{grad}(f) = \left(\frac{2x}{1+y^2}, -\frac{2y+2x^2y}{(1+y^2)^2}\right)$$

$$\frac{\partial f}{\partial x}(-1,1) = \frac{-1}{1+1} = -1/2 \quad \frac{\partial f}{\partial y} = -\frac{2+2}{(1+1)^2} = -1 \Rightarrow \operatorname{grad} f(-1,1) = (-1/2, -1)$$

$$\partial f_v(-1,1) = 3\cdot(-1/2) + 4(-1) = -11/2$$

$$\text{b)}\ \frac{\partial f}{\partial x} = \cos(y)e^x \ ; \ \frac{\partial f}{\partial y} = -e^x\sin(y) + e^z\cos(y) \ ; \ \frac{\partial f}{\partial z} = \sin(y)e^z$$

$$\left.\begin{matrix} \dfrac{\partial f}{\partial x}(0,\tfrac{\pi}{2},0) = 0 & \dfrac{\partial f}{\partial y}(0,\tfrac{\pi}{2},0) = -1 & \dfrac{\partial f}{\partial z}(0,\tfrac{\pi}{2},0) = 1 \end{matrix}\right\} \Rightarrow \operatorname{grad} f(0,\pi/2,0) = (0,-1,1)$$

$$\text{Hence } \partial f_v(0,\pi/2,0) = -3\cdot 0 + 4(-1) + 5(1) = 1$$

2. (HW) (a) Find the directional derivative of $f = 2x^3y - 3y^2z$ at $P(1,2,-1)$ in a direction toward $Q(3,-1,5)$. (b) In what direction from P is the directional derivative a maximum? (c) What is the magnitude of the maximum directional derivative?

$$\text{a)}\ \frac{\partial f}{\partial x} = 6x^2y \ ; \ \frac{\partial f}{\partial y} = 2x^3 - 6yz \ ; \ \frac{\partial f}{\partial z} = -3y^2$$

$$\operatorname{grad}(f(1,2,-1)) = (12, 14, -12)$$

$$\partial f_Q(3,-1,5) = 36 - 14 - 60 = 22 - 60 = -38$$

rate of increase

$$\text{b)}\ \sqrt{12^2 + 14^2 + (-12)^2} = 22 \leftarrow \text{magnitude of } \nabla f$$

Hence $\left(\frac{12}{22}, \frac{14}{22}, \frac{-12}{22}\right)$ is the direction from P, in which the func. increase most rapidly

$$\text{c) magnitude of } \left(\frac{6}{11}, \frac{7}{11}, \frac{-6}{11}\right) = \sqrt{\left(\frac{6}{11}\right)^2 + \left(\frac{7}{11}\right)^2 + \left(\frac{(-6)}{11}\right)^2} = 1$$

'also the direction is always unit-vector. $\|\text{direction vector}\| = 1$

13. (HW) Compute the length and the direction of the gradient of the function $u = \frac{1}{r}$ where $r = \sqrt{x^2+y^2+z^2}$ at a point $M(x_0, y_0, z_0)$. (*Remark: the direction should be described by the unit vector having the same direction as* ∇f)

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{-x}{(x^2+y^2+z^2)^{3/2}} \\ \frac{\partial u}{\partial y} &= \frac{-y}{(x^2+y^2+z^2)^{3/2}} \\ \frac{\partial u}{\partial z} &= \frac{-z}{(x^2+y^2+z^2)^{3/2}} \end{aligned}\right\} \nabla u_M = \begin{pmatrix} \dfrac{-x_0}{(x_0^2+y_0^2+z_0^2)^{3/2}} \\ \dfrac{-y_0}{(x_0^2+y_0^2+z_0^2)^{3/2}} \\ \dfrac{-z_0}{(x_0^2+y_0^2+z_0^2)^{3/2}} \end{pmatrix}$$

$$\|\nabla u_M\| = \sqrt{\frac{x_0^2+y_0^2+z_0^2}{(x_0^2+y_0^2+z_0^2)^3}} = \frac{1}{x_0^2+y_0^2+z_0^2} \leftarrow \text{len. of } \nabla$$

Hence, since the function increases most rapidly in the direction of it's ∇:

$$\begin{pmatrix} -x_0(x_0^2+y_0^2+z_0^2)^{-1/2} \\ -y_0(x_0^2+y_0^2+z_0^2)^{-1/2} \\ -z_0(x_0^2+y_0^2+z_0^2)^{-1/2} \end{pmatrix} = \bar{v} \text{ is the direction of } \nabla \text{, so } \|\bar{v}\| = 1$$

14*. A function $f(x) = f(x_1, x_2, \ldots, x_n)$ is said to be *homogeneous of degree* k if $f(tx_1, tx_2, \ldots, tx_n) = t^k f(x_1, x_2, \ldots, x_n)$ for any positive number t. Prove that a function $f(x)$ having continuous first derivatives in $\mathbb{R}^n$ is homogeneous of degree k if, and only if, it satisfies Euler's equation

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = kf.$$

Hint: Let $F(x,t) = t^{-k} f(tx)$. Prove that $t^{k+1}\frac{\partial F}{\partial t} = \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} - kf$.

1) f is homogeneous $\Leftarrow \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = kf$

$$\frac{\partial F}{\partial t} = -kt^{-k-1} f(tx) + t^{-k}\left(\frac{f_{x_1}}{t} \cdot x_1 + \frac{f_{x_2}}{t} x_2 + \ldots + \frac{f_{x_n}}{t} x_n\right) =$$

$$= -k t^{k-1} \cdot f(tx) + t^{-k} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot \frac{x_i}{t}$$

$$t^{k+1} \frac{\partial F}{\partial t} = -k \cdot f(tx) + t \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot \frac{x_i}{t}$$

$$t^{k+1} \frac{\partial F}{\partial t} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} x_i - k \cdot f(tx)$$

$$F(x,t) = t^{-k} \cdot f(tx) = t^{-k} \cdot t^{k} \cdot f(x) = f(x)$$

Thus $F(x,t)$ does not dep. on t

2) f is homogeneous $\Rightarrow \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = kf$

since f is hom. $0 = \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} - kf$, then

$t^{-k} f(tx) = f(x)$, so $f(tx) = t^k f(x)$.

Tnx for checking