In this HW, you can transform any matrix into REF/RREF by using a machine.

1. (2 points) Let $V = \mathbb{R}^2$ and operations $\oplus \colon V \times V \to V$, $\odot \colon \mathbb{R} \times V \to V$ be defined as

$$(u, v) \oplus (x, y) = (u + x + 1, v + y - 3);$$
$$\lambda \odot (x, y) = (\lambda x + \lambda - 1, \lambda y - 3\lambda + 3),$$

for every $(u, v), (x, y) \in V$ and $\lambda \in \mathbb{R}$. Then, using Definition 12.1, prove that $(V, \oplus, \odot)$ is a vector space over the field of real numbers.

[**hint**: since $(0, 0) \oplus (x, y) = (0 + x + 1, 0 + y - 3) = (x + 1, y - 3) \neq (x, y)$, it ought to be clear that $(0, 0)$ is not the zero vector]

1) $a + b = b + a \quad (a = (u,v); b = (x,y))$

$(u,v) + (x,y) = (x,y) + (u,v)$

$(u+x+1, v+y-3) = (x+u+1, y+v-3)$

2) $(a+b)+c = a+(b+c) \quad (a = (u,v); b = (x,y); c = (t,m))$

$((u,v) + (x,y)) + (t,m) = (u,v) + ((x,y) + (t,m))$

$(u+x+1, v+y-3) + (t,m) = (u,v) + ((x+t+1, y+m-3))$

$(u+x+t+2, v+y+m-6) = (x+t+u+2, y+m+v-6)$

3) $a + \bar{o} = a \quad (a = (x,y))$

$(x,y) + (-1;3) = (x+1-1, y+3-3) = (x,y)$

4) $a + (-a) = 0 \quad (\text{where } a = (x,y), \text{ then } -a = (-x-1, -y+3))$

$(x,y) + (-x-1; -y+3) = (x-x+1-1, y-y+3-3) = (0,0)$

5) $\lambda(a+b) = \lambda a + \lambda b \quad (\text{where } \lambda = 2,\ a = (x,y),\ b = (t,m))$

$2((x,y) + (t,m)) = 2(x,y) + 2(t,m)$

$2((x+t+1, y+m-3)) = (2x+1, 2y-3) + (2t+1, 2m-3)$

$(2x+2t+2, 2y+2m-6) = (2x+2t+2, 2y+2m-6)$

6) $(\lambda_1 + \lambda_2)a = \lambda_1 a + \lambda_2 a \quad (\text{where } \lambda_1 = 2,\ \lambda_2 = 3,\ a = (x,y))$

$(2+3)(x,y) = 2(x,y) + 3(x,y)$

$5(x,y) = (2x+2-1, 2y-6+3) + (3x+3-1, 3y-9+3)$

$(5x+5-1, 5y-15+3) = (2x+1, 2y-3) + (3x+2, 3y-6)$

$(5x+4, 5y-12) = (2x+1+3x+2+1, 2y-3+3y-6-3)$

$(5x+4, 5y-12) = (5x+4, 5y-12)$

7) $(\lambda_1 \cdot \lambda_2)a = \lambda_1(\lambda_2 a) \quad (\text{where } \lambda_1 = 2,\ \lambda_2 = 3,\ a = (x,y))$

$(2 \cdot 3)(x,y) = 2(3(x,y))$

$6(x,y) = 2(3x+3-1, 3y-9+3)$

$(6x+6-1, 6y-18+3) = 2(3x+2, 3y-6)$

$(6x+5, 6y-15) = (6x+2+4-1, 6y-12-6+3)$

$(6x+5, 6y-15) = (6x+5, 6y-15)$

8) $1 \cdot a = a \quad (a = (x,y))$

$1(x,y) = (x+1-1, y-3+3) = (x,y)$

then it's a vector space

2. (0.5 points per item) Using Definition 12.3, determine if a set S is a subspace of a vector space $\mathbb{V}$ if

(a) $S = \{A \in \mathrm{Mat}_n(\mathbb{R}) \,|\, \mathrm{tr}(A) = 0\}$ and $\mathbb{V} = \mathrm{Mat}_n(\mathbb{R})$ (that is, $\mathbb{V}$ is the vector space of all square matrices of size n with real coefficients over the field of real numbers with standard operations of matrix addition and matrix scalar multiplication);

(b) $S = \{A \in \mathrm{Mat}_n(\mathbb{R}) \,|\, A \text{ is invertible}\}$ and $\mathbb{V} = \mathrm{Mat}_n(\mathbb{R})$;

(c) $S = \{p(x) \in \mathbb{R}[x] \,|\, p(-3) = 0\}$ and $\mathbb{V} = \mathbb{R}[x]$ (that is, $\mathbb{V}$ is the vector space of all polynomials with real coefficients over the field of real numbers with standard operations of addition and scalar multiplication);

(d) $S = \{p(x) \in \mathbb{R}[x] \,|\, \deg(p(x)) = 3\}$ and $\mathbb{V} = \mathbb{R}[x]$;

(e) $S = \{(x, 3) \in \mathbb{R}^2\}$ and $\mathbb{V}$ is the vector space from Problem 1;

a) Yes, since if $X, Y \in A$, then $X + Y \in A$ $(tr(A+B) = tr(A) + tr(B))$

$\wedge\ \lambda X \in A$ $(tr(\lambda A)) = \lambda \cdot tr(A))$

b) No, since if $X, Y \in A$, then $X + Y$ can still be not invertable

c) Yes, since if $f(x), g(x) \in A$, then $f(x) + g(x) = A$ $(f(-3) + g(-3) = 0)$

$f(x) \cdot g(x) = 0$

d) No, since if $f(x), g(x) \in A$, $\deg(f(x) + g(x))$ can be less than 3.

e) Yes, since if $(x,3) \wedge (y,3) \in A$, $(x,3) + (y,3) \in A \wedge \lambda(x,3) \in A$

$\Big((x,3) + (y,3) = (x + y + 1, 3)$

$\lambda \cdot (x,3) = (\lambda x + \lambda - 1, \cancel{3\lambda} - \cancel{3\lambda} + 3)\Big)$

3. (1 point) Find all $\lambda \in \mathbb{R}$ such that the set of vectors

$$S = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ \lambda \end{bmatrix} \right\}$$

is linearly dependent.

$$a \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{c} \begin{matrix} a & b & c & \end{matrix} \\ \left[\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 2 & -3 & 0 & 0 \\ 1 & 2 & \lambda & 0 \end{array}\right] \end{array} \xrightarrow[l_{2,1,2}]{l_{3,1,1}} \left[\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 4 & \lambda+1 & 0 \end{array}\right] \xrightarrow[\substack{l_{3,2,-4} \\ d_{1,-1}}]{l_{1,2,-2}} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & \lambda-7 & 0 \end{array}\right]$$

Now we obtain 2 cases:

1) $\lambda \neq 7$, then $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & \lambda-7 & 0 \end{array}\right] \xrightarrow[\substack{l_{1,3,-1} \\ l_{2,3,-3}}]{d_{3,(\lambda-7)^{-1}}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ and then vectors are LI

2) $\lambda = 7$, then $\begin{array}{c} \begin{matrix} a & b & c & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -3c \\ -2c \\ \lambda \end{bmatrix}$, where $\lambda \in \mathbb{R}$

, then vectors are LD

4. (1 point) Find all $\lambda \in \mathbb{R}$ such that the vector $p(x) = x^2 + \lambda x - 3$ belongs to the linear span

$$\langle x^2 - 1,\ x + 1,\ x^2 + 3x + 2 \rangle.$$

$$a(x^2-1) + b(x+1) + c(x^2+3x+2) = x^2 + \lambda x - 3 \quad \Big|\ a,b,c \in \mathbb{R}$$

$$ax^2 - a + bx + b + cx^2 + 3xc + 2c = x^2 + \lambda x - 3$$

$$x^2(a+c) + x(3c+b) + (2c+b-a) = x^2 + \lambda x - 3$$

$$\Updownarrow$$

$$\begin{cases} a + c = 1 \\ 3c + b = \lambda \\ 2c - a + b = -3 \end{cases} \iff \begin{cases} a = 1 - c \\ b = -2 - 3c \\ \lambda = -2 \end{cases} \text{, where } c \in \mathbb{R}$$

5. (0,5 points per item) Which of the following statements are true and which are false (if a statement is true, then, prove if; if a statement is false, then, provide a counterexample)?

   (a) if S and A are non-empty subsets of a vector space $\mathbb{V}$, S is linearly dependent, and $A \subseteq S$, then, A is also linearly dependent;

   (b) if S and A are non-empty subsets of a vector space $\mathbb{V}$, S is linearly independent, and $A \subseteq S$, then, A is also linearly independent;

   (c) if S and B are non-empty subsets of a vector space $\mathbb{V}$, S is linearly dependent, and $S \subseteq B$, then, B is also linearly dependent;

a) No, if $S = \{a, b, c\}$, $a \cdot 0 + b + 2c = 0$; S is linearly dependent

b) Yes, since if A wouldn't be lineary independent, it would be possible to construct.

$$\sum_{i=a}^{n} a_i \lambda_i + \sum_{j=1}^{n} s_j \lambda_i = 0 \text{ , s.t. } a_i = A; s_j \in S, s_j \notin A. \ \exists i \ \lambda_i \neq 0 \Rightarrow S \text{ would be LI}$$

c) Yes, check point b

d) No, check point a

e) Yes, since there can be solutions, where at least one λ is not zero

g) No, for example $A = \{0\}$ $B = \{x^2+1\}$ $\Rightarrow 0 \in \langle B \rangle$