4. (HW) Find the limit using Taylor expansion:

$$\text{(a)}\ \lim_{x\to0}\frac{e^x-1-x}{x^2};\qquad \text{(b)}\ \lim_{x\to0}\frac{\cos x-1+x^2/2}{x^4}.$$

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a) $$\lim_{x\to0}\frac{e^x-1-x}{x^2}=\lim_{x\to0}\left(\frac{1+x+\frac{x^2}{2}-1-x+o(x^2)}{x^2}\right)=\lim_{x\to0}\left(\frac{\frac{x^2}{2}+o(x^2)}{x^2}\right)=\lim_{x\to0}\left(\frac{\frac{1}{2}+\underbrace{\frac{o(x^2)}{x^2}}_{\to 0}}{1}\right)=\frac{1}{2}$$

since $e^x=\sum_{k=0}^{n}\frac{x^k}{k!}+o(x^n)$

b) $$\lim_{x\to0}\left(\frac{\cos(x)-1+x^2/2}{x^4}\right)=\lim_{x\to0}\left(\frac{1-\frac{x^2}{2}+\frac{x^4}{24}-1+\frac{x^2}{2}+o(x^5)}{x^4}\right)=\lim_{x\to0}\left(\frac{\frac{x^4}{24}+o(x^5)}{x^4}\right)=\lim_{x\to0}\left(\frac{\frac{1}{24}+\underbrace{\frac{o(x^5)}{x^4}}_{\to 0}}{1}\right)=\frac{1}{24}$$

since $\cos(x)=1-\frac{x^2}{2}+\frac{x^4}{24}+o(x^5)$

5. (HW) Find the limit using Taylor expansion:

$$\text{(a)}\ \lim_{x\to0}\frac{\cosh 3x+\cos 3x-2}{x^4};\qquad \text{(b)}\ \lim_{x\to0}\frac{\sinh 2x-2\sinh x}{x^3}.$$

a) $$\lim_{x\to0}\left(\frac{\cosh(3x)+\cos(3x)-2}{x^4}\right)=\lim_{x\to0}\left(\frac{1+\frac{9x^2}{2}+\frac{81x^4}{24}+o(3^5x^5)+1-\frac{9x^2}{2}+\frac{81x^4}{24}+o(3^5x^5)-2}{x^4}\right)\ (=)$$

since $\cosh(3x)=1+\frac{9x^2}{2}+\frac{81x^4}{24}+o((3x)^5)$

$\cos(3x)=1-\frac{9x^2}{2}+\frac{81x^4}{24}+o((3x)^5)$

$$(=)\ \lim_{x\to0}\left(\frac{\frac{81x^4}{12}+2o(x^5 3^5)}{x^4}\right)=\lim_{x\to0}\left(\frac{\frac{81}{12}+\underbrace{\frac{2o(x^5 3^5)}{x^4}}_{\to 0}}{1}\right)\quad \frac{81}{12}=\frac{27}{4}$$

b) $$\lim_{x\to0}\left(\frac{\sinh(2x)-2\sinh(x)}{x^3}\right)=\lim_{x\to0}\left(\frac{2x+\frac{8x^3}{6}+o(x^4)-2x-\frac{2x^3}{6}-2o(x^4)}{x^3}\right)=\lim_{x\to0}\left(\frac{x^3-2o(x^4)}{x^3}\right)=1$$

Since: $\sinh(2x)=2x+\frac{8x^3}{6}+o(x^4)$

$2\sinh(x)=2x+\frac{2x^3}{6}+2o(x^4)$

6. (HW) Find the limit using Taylor expansion:

$$\text{(a)}\ \lim_{x\to0}\frac{e^x-\sqrt{1+2x}}{\ln\cos x};\qquad \text{(b)}\ \lim_{x\to0}\frac{3\cos x+\arcsin x-3\sqrt[3]{1+x}}{\ln(1-x^2)}.$$

a) $$\lim_{x\to0}\left(\frac{e^x-\sqrt{1+2x}}{\ln(\cos(x))}\right)\ (=)$$

since: $f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}x^k$

e^x

$f^0(x)=e^x \wedge f^0(0)=1$

$f'(x)=e^x \wedge f'(0)=1$

$f^2(x)=e^x \wedge f^2(0)=1$

$f^3(x)=e^x \wedge f^3(0)=1$

then $e^x=\frac{1}{0!}x^0+\frac{1}{1!}x^1+\frac{1}{2!}x^2+\frac{1}{3!}x^3+o(x^3)$

$=1+x+\frac{x^2}{2}+\frac{x^3}{6}+o(x^3)$

$-\sqrt{2x+1}\approx\frac{-1}{0!}x^0+\frac{-1}{1}x^1+\frac{1}{2!}x^2+\frac{-3!}{3!}x^3\ (\approx)$

$(\approx)\ -1-x+\frac{x^2}{2}-\frac{x^3}{2}$

$\ln(\cos(x))$

$\to f^0(x)=\ln(\cos(x)) \wedge f^0(0)=0$

$f^{(1)}(x)=-\tan(x) \wedge f^{(1)}(0)=0$

$f^{(2)}(x)=-\tan^2(x)-1 \quad f^{(2)}(0)=-1$

$f^{(3)}(x)=-2\left(\tan^3(x)+\tan(x)\right) \quad f^{(3)}(0)=0$

$-\sqrt{2x+1}$

$f^0(x)=-\sqrt{2x+1} \wedge f^0(0)=-1$

$f'(x)=-\frac{1}{\sqrt{2x+1}} \wedge f'(0)=-1$

$f^2(x)=\frac{1}{(2x+1)^{3/2}} \wedge f^2(0)=1$

$f^3(x)=-\frac{3}{(2x+1)^{5/2}} \wedge f^3(0)=-3$

$\ln(\cos(x))\approx\frac{0}{0!}x^0+\frac{0}{1!}x^1+\frac{-1}{2!}x^2+\frac{0}{3!}x^3$

$\ln(\cos(x))\approx-\frac{1}{2}x^2$

$$(=)\ \lim_{x\to0}\left(\frac{1+x+\frac{x^2}{2}+\cancel{\frac{x^3}{6}}+o(x^3)-1-x+\frac{x^2}{2}-\cancel{\frac{x^3}{2}}+o(x^3)}{-\frac{1}{2}x^2+o(x^3)}\right)=\lim_{x\to0}\left(\frac{x^2+2o(x^2)}{-\frac{1}{2}x^2+o(x^3)}\right)=-2$$

2) $$\lim_{x\to0}\left(\frac{3\cos(x)+\arcsin(x)-3\sqrt[3]{1+x}}{\ln(1-x^2)}\right)=\lim_{x\to0}\left(\frac{3-\frac{3x^2}{2}+o(x^2)+x+o(x^2)-3-x+\frac{1}{3}x^2+o(x^2)}{-x^2+o(x^2)}\right)=\lim_{x\to0}\left(\frac{-\frac{7}{6}x^2+o(x^2)}{-x^2+o(x^2)}\right)=\frac{7}{6}$$

$\cos(x)=1-\frac{x^2}{2}+o(x^2)$

$\arcsin(x)=x+o(x^2)$

$\ln(1-x^2)=-x^2+o(x^2)$

$\sqrt[3]{1+x}=f(x)$

$f(0)=1$

$f'(x)=\frac{1}{3\sqrt[3]{(1+x)^2}} \wedge f'(0)=\frac{1}{3}$

$f^2(x)=-\frac{2}{9(1+x)^{\frac{5}{3}}} \wedge f^2(0)=-\frac{2}{9}$

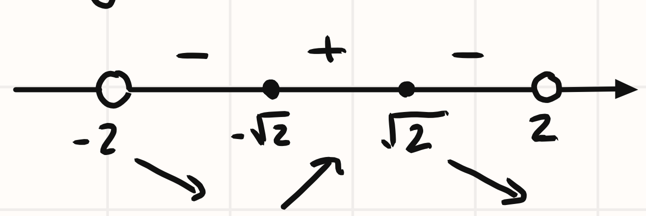
$3f(x)=3+x-\frac{1}{3}x^2+o(x^2)$

9. (HW) Determine the intervals on which $f(x)$ is increasing or decreasing. Find critical points and determine whether they are local maximum point, local minimum points or neither.

$$\text{(a) } y = x\sqrt{4-x^2}; \qquad \text{(b) } y = \frac{2x^2-1}{x^4}.$$

a) $y = x\sqrt{4-x^2}$ $\qquad \mathcal{D}(y) = [-2;2]$

$f'(x) = \dfrac{4-2x^2}{\sqrt{4-x^2}}$



Hence: from $(-2;-\sqrt{2}] \cup [\sqrt{2},2)$ decreasing

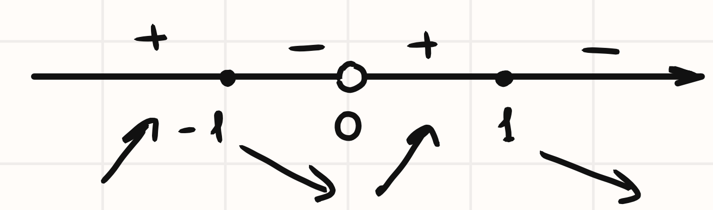
$[-\sqrt{2};\sqrt{2}]$ - increasing

$x=\sqrt{2}$ is local maximum $\qquad x = \pm 2$ - critical points

$x=-\sqrt{2}$ is local minimum $\qquad x = \pm\sqrt{2}$ stationary points

b) $y = \dfrac{2x^2-1}{x^4}$ $\qquad x \neq 0 \qquad \mathcal{D}(y) = (-\infty,0)\cup(0;+\infty)$

$f'(x) = \dfrac{-4x^2+4}{x^5}$



$x=0$ - critical point

$x=\pm 1$ - critical point $\qquad x = \pm 1$ - stationary point

$(-\infty;-1] \cup (0;1]$ - increasing

$[-1;0) \cup [1;+\infty)$ - decreasing

$x = \pm 1$ – local maximum