Novosad Ivan 231 Discrete math 2b.

9. Suppose a number $a > 1$ is divisible by 2 but not by 4. Then a has as many positive *even* divisors as it has positive *odd* divisors.

∇ Assume $a>1$ have n odd divisors $\wedge$ $2|a$ $\wedge$ $\neg(4|a) \to$ a have n even divisors.
(a have n odd divisors: $x_1\ x_2\ x_3\ x_4 \cdots x_n$ $\to$ a have n even divisors : $2x_1\ 2x_2\ 2x_3\ 2x_4 \ldots 2x_n$)
Using MIP: base: number 2 have 1 odd divisor : $\underset{x_1}{1}$ and 1 even divisor: $\underset{x_1 \cdot 2}{2}$
$\varphi(n)$ Assume number $a>1$ have n odd divisors $\wedge$ $2|a$ $\wedge \neg(4|a)$ $\wedge$ a have n even divisors.
$\varphi(n+1)$ then a have $n+1$ odd divisors $\wedge$ $2|a$ $\wedge$ $\neg(4|a)$ $\wedge a>1$
since a have $(n+1)$ odd divisors: $x_1\ x_2\ x_3 \ldots x_n\ x_{n+1}$ and $2|a \Rightarrow a$ have $n+1$ even divisors: $2x_1\ 2x_2\ 2x_3\ 2x_4 \ldots 2x_n\ 2x_{n+1} \Rightarrow \varphi(n)$ Δ

10. Suppose that each of the digits 0, 1, and 2 has exactly 100 occurrences in the decimal notation of a certain integer x. No other digit occurs there. Prove there is no such integer y that $x = y^2$.

1) Since any power of 10 is congruent to $1 \pmod 3$ $\overline{a_1 a_2 a_3 \ldots a_n} \vdots 3 = \sum_{i=1}^{n} a_i$
then it ought to be clear, that $3|x$ $(100(1+0+2) = 300 = 0 \bmod (3))$
2) Since any power of 10 is congruent to $1 \bmod (3)$ $\overline{a_1 a_2 a_3 \ldots a_n} \vdots 3 = \sum_{i=1}^{n} a_i$
then it ought to be clear, that $\neg 9|x$ $(100(1+0+2) = 300 = 3 \bmod 9)$
3) then note that if $3|x \to x = 3 \cdot p_1^{x_1} \cdot p_2^{x_2} \cdot p_3^{x_3} \ldots p_n^{x_n}$ by FTA $\Rightarrow y = x \cdot x \Leftrightarrow$
$y = 3 \cdot p_1^{x_1} p_2^{x_2} \ldots p_n^{x_n} \cdot 3 \cdot p_1^{x_1} \cdot p_2^{x_2} \ldots p_n^{x_n} \Leftrightarrow y = 3^2 \cdot p_1^{x_1} \cdot p_2^{x_2} \ldots p_n^{x_n}$ $\perp$ ($\perp$ because in our case $3|x \wedge \neg 9|x$)

ertain integer x. No other digit occurs there. Prove there is no such inte

11. Prove that there are infinitely many primes of the form $6k + 5$.

∇ Assume there are finally many of them.
$\{p_1 p_2 \ldots p_s\}$ are all the primes of the form $(6k+5)$
Consider $N = 6p_1 \cdot p_2 \cdots p_{s} - 1 = 5\ (6)$ (It's clear that $N > 1$)
$p_i | N \Rightarrow p_i | 4 p_1 \cdots p_i \cdots p_s - 1 \Rightarrow p_i | 1 \Rightarrow \perp$ $(\forall i\ p_i \nmid N)$
by FTA, let's factorize N:
$N = 2^a \cdot q_1^{b_1} \cdots q_t^{l_t}$; where $\forall i\ q_i = 1(6)$
then $2|N \Rightarrow 2|6 p_1 \cdots p_s - 1 \Rightarrow 2|1 \Rightarrow \perp$
then $a = 0$, then $N = q_1^{b_1} \cdots q_t^{b_t} = 1^{b_1} \cdot 1^{b_2} \ldots 1^{b_n} \equiv 1(6) \Rightarrow 1 \equiv 3(4) \Rightarrow \perp$
Then we have infinity many primes of the form $6k+5$. Δ

any positive numbers x, y, z.

13. Let p be a prime greater than 3. Prove that $24 \,|\, (p^2 - 1)$. ∇

Let's consider all posibals remainders of primary numbers by 24:
it's ought to be clear, that remainder ought to be

odd. ('cause otherwise it won't be prime)

then we have: $1, \cancel{2}, \cancel{3}, \cancel{4}, 5, \cancel{6}, 7, \cancel{8}, \cancel{9}, \cancel{10}, 11, \cancel{12}, 13, \cancel{14}, \cancel{15}, \cancel{16}, 17, \cancel{18}, 19, \cancel{20}, \cancel{21}, \cancel{22}, 23, \cancel{0}$

let's consider the following remainders:

$24k+3$ can not be prime, 'cause $3 \mid 24k+3 \Rightarrow$
leftovers from dividing primes by 24 can not be equal 3, 9, 15, 21

then let's consider leftovers from dividing primes by 24.

$1^2 - 1 = 0 \equiv 0 \mod(24)$

$5^2 - 1 = 25 - 1 = 24 \equiv 0 \mod(24)$

$7^2 - 1 = 49 - 1 = 24 \cdot 2 \equiv 0 \mod(24)$

$11^2 - 1 = 121 - 1 = 120 = 5 \cdot 24 \equiv 0 \mod(24)$

$13^2 - 1 = 169 - 1 = 168 = 7 \cdot 24 \equiv 0 \mod(24)$

$17^2 - 1 = 289 - 1 = 288 = 12 \cdot 24 \equiv 0 \mod(24)$

$19^2 - 1 = 361 - 1 = 360 = 15 \cdot 24 \equiv 0 \mod(24)$

$23^2 - 1 = 529 - 1 = 528 = 22 \cdot 24 \equiv 0 \mod(24)$ Δ

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14. Prove that there is no arithmetic progression $\{a_k\}_{k\in\mathbb{N}}$ (whose difference is non-zero) s. t. the numbers $a_1, \ldots, a_n$ are pairwise coprime for each $n > 0$.

let $a_1 = c$ $(c \in \mathbb{N})$ $\Rightarrow a_n = c + (n-1)d$ – formula of arbitrary arifm progression

Consider: $N = c+1 \Rightarrow a_n = c + cb \Rightarrow a_n = c(b+1) \Rightarrow c \mid a_n (N = c+1)$

$\Rightarrow$ there is not any co-prime progression with non-zero step.

12*. Put $\mathrm{lcm}(a,b,c) = \mathrm{lcm}(\mathrm{lcm}(a,b),c)$ and similarly for gcd. Prove that

$$\mathrm{lcm}(x,y,z) = \frac{xyz\cdot\gcd(x,y,z)}{\gcd(x,y)\cdot\gcd(x,z)\cdot\gcd(y,z)}$$

let $x = p_1^{x_1}\cdot p_2^{x_2}\cdot p_3^{x_3}\cdots p_n^{x_n}$

let $y = p_1^{y_1}\cdot p_2^{y_2}\cdot p_3^{y_3}\cdots p_n^{y_n}$

let $z = p_1^{z_1}\cdot p_2^{z_2}\cdot p_3^{z_3}\cdots p_n^{z_n}$

then $lmc(x y z) = p_1^{\max(x_1 y_1 z_1)}\cdot p_2^{\max(x_2 y_2 z_2)}\cdots p_n^{\max(x_n y_n z_n)}\ \wedge$

$\wedge\ gcd(x,y,z) = p_1^{\min(x_1 y_1 z_1)}\cdot p_2^{\min(x_2 y_2 z_2)}\cdots p_n^{\min(x_n y_n z_n)}$

then $lmc(x,y,z) = \dfrac{x,y,z\ \cdot gcd(x,y,z)}{gcd(x,y)\ gcd(x,z)\ gcd(z,y)} =$

$$= p_1^{\max(x_1,y_1,z_1)}\cdots p_n^{\max(x_n y_n z_n)} = \frac{p_1^{x_1+y_1+z_1}\cdots p_n^{x_n+y_n+z_n}\cdot p_1^{\min(x_1,y_1,z_1)}\cdots p_n^{\min(x_n y_n z_n)}}{p_1^{\min(x_1,y_1)}\cdots p_n^{\min(x_n y_n)}\ p_1^{\min(x_1,z_1)}\cdots p_n^{\min(x_n z_n)}\ p_1^{\min(z_1,y_1)}\cdots p_n^{\min(z_n y_n)}} \iff$$

$$\iff p_1^{\max(x_1,y_1,z_1)}\cdots p_n^{\max(x_n y_n z_n)} = \frac{p_1^{x_1+y_1+z_1+\min(x_1,y_1,z_1)}\cdots p_n^{x_n+y_n+z_n+\min(x_n y_n z_n)}}{p_1^{\min(x_1 y_1)+\min(x_1,z_1)+\min(z_1,y_1)}\cdots p_n^{\min(x_n y_n)+\min(x_n z_n)+\min(z_n y_n)}} \iff$$

$$\iff p_1^{\max(x_1,y_1,z_1)}\cdots p_n^{\max(x_n y_n z_n)} = p_1^{x_1+y_1+z_1+\min(x_1,y_1,z_1)-\min(x_1,y_1)-\min(x_1,z_1)-\min(z_1,y_1)}\cdots$$
$$\cdots p_n^{x_n+y_n+z_n+\min(x_n y_n z_n)-\min(x_n y_n)-\min(x_n z_n)-\min(z_n y_n)} \iff$$

$\iff \max(x_i\ y_i\ z_i) = x_i + y_i + z_i + \min(x_i\ y_i\ z_i) - \min(x_i\ y_i) - \min(x_i\ z_i) - \min(z_i\ y_i)$

assyme $x<y<z$:

$z = x+y+z+x-x-x-y \iff z=z$

assyme $y<x<z$:

$z = x+y+z+y-y-x-y \iff z=z$

assyme $x<z<y$:

$y = x+y+z+x-x-x-z \iff y=y$

assyme $z<x<y$:

$y = x+z+y+z-z-z-x \iff y=y$

assyme $y<z<x$:

$x = x+y+z+y-y-z-y \iff x=x$

assyme $z<y<x$:

$x = x+y+z-z-y-z-z \iff x=x$

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15. Prove that the fraction $\dfrac{n^2-n+1}{n^2+1}$ is irreducible for each integer $n>0$ (that is, the numerator and denominator are coprime).

$\dfrac{n^2-n+1}{n^2+1} = 1 - \dfrac{n}{n^2+1}$ now we can consider only $\dfrac{n}{n^2+1}$:

$\frac{n}{n^2+1}$ let's consider two cases.

1) if n isn't divisible by some primary number,

then $n \equiv q \pmod{p}$, where $0 \le q < P \wedge p$ is prime. $\Rightarrow$

$\Rightarrow n^2+1 \equiv q^2+1 \pmod{p}$ then $\gcd(n, n^2+1) \neq 1$

iff $q^2 = q-1$ but if q was an even number, then q^2 is even too, and q^2+1 is odd but even can not be equal. And if q was an odd number, then q^2 is odd too, and q^2+1 is even. (and they can not be equal)

the we have a contradiction $\Rightarrow$ n is co-prime to n^2+1

2) if n is divisible by some prime number,

then $n \equiv 0 \bmod (p)$, where p is prime $\Rightarrow$

$n^2+1 \equiv 1 \bmod (p) \Rightarrow$ n is a co-prime to $\underline{\underline{n^2+1}}$

Or using euclidian algorifm:

$n^2-n+1 = 1(n^2+1) - n$

$n^2+1 = n(n)+1$

$n = 1(n)+0 \Rightarrow \gcd(1,0) = 1 \Rightarrow$

$\boxed{\gcd(n^2-n+1, n^2+1) = 1}$

Tnx for check