HW 19 Calculus; Novosad Ivan

✓ 1) Evaluate define integral

✓ a) $\int_0^3 \frac{x+2}{\sqrt{x+1}}\,dx = \left| \begin{matrix} t = x+1 \\ dt = dx \end{matrix} \right| = \int_1^4 \frac{t+1}{\sqrt{t}}\,dt = \int_1^4 [t^{1/2} + t^{-1/2}]\,dt = \frac{2}{3} t^{3/2} + 2t^{1/2} \Big|_1^4 = \frac{16}{3} + 4 - \frac{2}{3} - 2 = \frac{14}{3} + 2 = \frac{14+6}{3} = \frac{20}{3}$

✓ b) $\int_0^2 x^2\sqrt{4-x^2}\,dx \sim \left| \begin{matrix} x = 2\sin(\theta) \\ dx = 2\cos(\theta) \end{matrix} \right| \sim \int 16\sin^2(\theta)\cos^2(\theta)\,d\theta = 4\int \sin(2\theta)^2\,d\theta = \{u = 2\theta\} = 2\int \sin^2(u)\,du = \int [1 - \cos(2u)]\,du = u - \frac{\sin(2u)}{2} + C =$

✓ $= 2\theta - \frac{\sin(4\theta)}{2} \overset{✓}{=} 2\arcsin\left(\frac{x}{2}\right) - \frac{\sin\left(4\arcsin\left(\frac{x}{2}\right)\right)}{2} = \Big|_0^2 = \underbrace{2\arcsin(1)}_{2\cdot\frac{\pi}{2} = \pi} - \underbrace{\frac{\sin(4\arcsin(1))}{2}}_{\frac{\sin(2\pi)}{2} = 0} - 2\overset{=0}{\arcsin(0)} + \overset{=0}{\frac{\sin(4\arcsin(0))}{2}} = \pi$

$\sin(\theta) = \frac{x}{2} \Rightarrow \theta = \arcsin\left(\frac{x}{2}\right)$

✓ c) $\int_1^3 \arctan(\sqrt{x})\,dx = \left| \begin{matrix} t = \sqrt{x} \\ dt = \frac{1}{2}\sqrt{x}^{\,-1} \\ dx = 2\sqrt{x} \end{matrix} \right| = \int_1^{\sqrt{3}} 2t\arctan(t)\,dt = \left\{ \begin{matrix} m = \arctan(t) & dw = t\,dt \\ dm = \frac{dt}{1+t^2} & w = \frac{1}{2}t^2 \end{matrix} \right\} = t^2\arctan(t)\Big|_1^{\sqrt{3}} - \int_1^{\sqrt{3}} \frac{t^2}{1+t^2}\,dt = t^2\arctan(t)\Big|_1^{\sqrt{3}} - \int_1^{\sqrt{3}} \left(1 - \frac{1}{1+t^2}\right)dt =$

$= t^2\arctan(t) - t + \arctan(t)\Big|_1^{\sqrt{3}} = \pi - \sqrt{3} + \frac{\pi}{3} - \frac{\pi}{4} + 1 - \frac{\pi}{4} = \frac{5\pi}{6} - \sqrt{3} + 1$

4. (HW) Prove that if $f(x)$ is integrable on $[a,b]$ and $\int_a^b f(x)\,dx > 1$, then there exists a point c in (a,b) such that $f(c) > \frac{1}{b-a}$.

if $f(x)$ is integrable $\Rightarrow$ it has max and min value : M, m;

s.t. $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$;

note that m and M are just min and max values of f on (a,b); then suppose $M = f(c) : c \in (a,b)$

$$\int_a^b f(x)\,dx \leq f(c)(b-a) \iff \frac{\int_a^b f(x)\,dx}{b-a} \leq f(c)$$ (if $(b-a) \geq 0$)

we use $[a,b]$ notation as $a \leq b$ or $a < b$ but not $b \leq a$;

$$\Downarrow$$

$$f(c) > \frac{1}{b-a} \Leftarrow f(c) \geq \frac{\int_a^b f(x)\,dx}{b-a}$$ > 1

since $\int > 1$ (given) ∎

5. (HW) Prove that if $f(x)$ is integrable and continuous over $[a, b]$ and if $\int_\alpha^\beta f(x)\,dx \geqslant 0$ for any subinterval $[\alpha, \beta]$ of (a, b), then $f(x) \geqslant 0$ in $[a, b]$.

suppose it's not: $\exists c$ s.t. $f(c) < 0$;

since $f(x)$ is continuous: $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

then we can denote α as $c - \delta/2$ and β as $c + \delta/2$; then $\forall x \in [c - \delta/2; c + \delta/2]$ $f(x) < 0 \Rightarrow$ $\forall x \in [\alpha, \beta]$

$\int_\alpha^\beta f(x)\,dx < 0$ but $\int_\alpha^\beta f(x)\,dx \geqslant 0$ ($\forall \alpha\ \forall \beta$ s.t. $\alpha < \beta \wedge \alpha \geqslant a \wedge \beta \leqslant b$) $\Rightarrow$ (!), then $f(x) \geqslant 0$ on $[a, b]$; ■

7. (HW) Prove that if $f(x)$ is integrable over $[a, b]$ and $f(x) \geqslant m > 0$, then $\sqrt{f}$ is integrable over $[a, b]$.

if $f(x)$ is integrable over $[a,b]$ $\Rightarrow$ $f(x)$ is bounded over $[a,b]$ $\Rightarrow |f(x)| \leq M$

For any partition of $[a,b]$ consider the upper and the lower Darboux sums for the function $f(x)$:

$$S = \sum_{i=1}^{n} M_i(x - x_{i-1}) \quad \text{and} \quad s = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

so, for $\sqrt{f(x)}$:

$$\bar{S} = \sum_{i=1}^{n} \sqrt{M_i}(x - x_{i-1}) \quad \text{and} \quad \bar{s} = \sum_{i=1}^{n} \sqrt{m_i}(x_i - x_{i-1})$$

by Darboux criterion, integrability of $f(x)$ implies:

$S - s \to 0$ as $\max(x_i - x_{i-1}) \to 0$. but:

$$\bar{S} - \bar{s} = \sum_{i=1}^{n} (\sqrt{M} - \sqrt{m})(x_i - x_{i-1}) \leq \sqrt{M}\,(\bar{S} - \bar{s}) \leq M(\bar{S} - \bar{s}) \leq M(S - s) \to 0$$

$= \text{iff } \sqrt{m} = 0$

which again by Darboux criterion yilds integuability of $\sqrt{f}$; ∎

8. (HW) Let $f(x)$ be a continuous function in $[a,b]$. Let $\varphi(x)$ be a function having a continuous derivative in $[a,b]$. Assume also that $a \leqslant \varphi(x) \leqslant b$. Prove that the function

$$k(x) = \int_a^{\varphi(x)} f(t)\,dt$$

is differentiable in (a,b), and $k'(x) = \varphi'(x) f(\varphi(x))$.

By the main theorem of Calculus:

$$F(x) = \int_a^x f(t)\,dt \Rightarrow \underbrace{F'(x) = f(x)}_{\text{then } F(x) \text{ is differetiable on } (a,b)} \quad | \text{ for } \forall x \in (a,b)$$

if f is continuous function on $[a,b]$

F is defined $\forall x \in [a,b]$ by:

then by abusing the notation:

$$k(x) = F(x) \;;\; X = \varphi(x) \;; \Rightarrow k(x) = \int_a^{\varphi(x)} f(t)\,dt \text{ by MTC:}$$

$k(x)$ is continuous $\Rightarrow$ $k(x)$ is differentiable on (a,b);

by chain rule (abuse the notation $x = \varphi(x)$)

$$k(x) = \int_a^{\varphi(x)} f(t)\,dt = k(\varphi(x)) = \int_a^x f(\varphi(x))\,d\varphi(x) =$$

$$\Rightarrow k(x)' = k(\varphi(x))' = \underbrace{k'(\varphi(x))}_{\substack{\shortparallel \\ f(\varphi(x)) \text{ by MTC}}} \varphi'(x) \quad \blacksquare$$

9*. (HW) Suppose

$$f(x) = \begin{cases} 0, & x = \frac{1}{n}, \quad n = 1, 2, \ldots, \\ 1, & x \neq \frac{1}{n}. \end{cases}$$

Prove that $f(x)$ is integrable over $[0,1]$ and find $\int_a^b f(x)\,dx$.

Using Riemann criterion:

Let $\varepsilon > 0$, then we chose a partition P s.t. $U(P,f) - L(P,f) < \varepsilon$; (for partion $\to 0$ $U(P,f) - L(P,f) \to 0$)

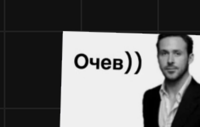
upper darboux

lower darboux

$\forall \varepsilon > 0 \ \exists P : U(P,f) - L(P,f) < \varepsilon$

Since $1/n \to 0$, $\exists N \ \forall n > N : 1/n \in [0, \varepsilon] \Rightarrow$ so only finate number of $1/n$'s lie in the interval $[\varepsilon, 1]$. Cover these finite numbers of $1/n$'s by the intervals $[x_1, x_2], [x_3, x_4], \ldots, [x_{m-1}, x_m]$ s.t. $x_i \in [\varepsilon, 1]$ $\forall i \in [\underline{m}+1]$ $\{1, 2, 3 \ldots m\}$

and the sum of the lenght of these m intervals is less than ε.

$((x_2 - x_1) + (x_4 - x_3) + \ldots (x_m - x_{m-1})) < \varepsilon$ 

since there are at least one point s.t. $< \varepsilon$ and not in form $1/n$

Consider the pantion $P = \{0; \varepsilon; x_1; x_2; x_3; \ldots; x_m\}$. It's clear that $U(P,f) - L(P,f) < 2\varepsilon$.

if not, I can spell it out to prove it, but it's obvious and the pruf is unwieldly

Hence by Riemann criterion the function is integrable. ∎

Since the lower integral is 1 and the function is integrable, $\int_0^1 f(x)dx = 1$.

I hope my explanations are clear, and I'll resive full mark, otherwise: feel free to ask me for additions v invite me to the defence;

Tnx for checking; have a nice day ❤

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