LAaG **Homework #24** *Novosad Ivan*

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In this HW, you can perform all arithmetic operations on matrices (e.g. multiplication, transforming into RREF, finding the inverse, etc) by a machine.
In this HW, you can calculate every characteristic polynomial by a machine.

1. (0.5s point per item) Which of the following matrices are Jordan matrices and which are not (see Definition 24.2)? For those that are, denote them by a notation of the form $J_{n_1,\dots,n_k}(\lambda_1,\dots,\lambda_k)$.

✓ a) $\begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&1\end{bmatrix}$; ✗ b) $\begin{bmatrix} 2&-1&0&0\\0&2&0&0\\0&0&7&0\\0&0&0&8\end{bmatrix}$; ✗ c) $\begin{bmatrix} 1&1&0&0&0\\0&-1&0&0&0\\0&0&2&1&0\\0&0&0&2&1\\0&0&0&0&2\end{bmatrix}$ can't; ✓ d) $\begin{bmatrix} 0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&2&0\\0&0&0&0&3\end{bmatrix}$.

a) $J_{1,1,2}(1,1,1)$ d) $J_{3,1,1}(0,2,3)$

(b) $A = J_k(a)^3$, where $a \neq 0$ and $J_k(\lambda)$ is the Jordan block of size k and value λ (see Definition 24.1).

$$J_k(a)^3 = \underbrace{\begin{bmatrix} a & 1 & & 0 \\ & a & 1 & \\ & & a & \ddots \\ 0 & & & \ddots \; 1 \\ & & & a \end{bmatrix}^3}_{k} k, \quad a \neq 0$$

$$J_k(a)^3 = \begin{bmatrix} a^3 & 3a^2 & 3a & 1 & 0 & \cdots & 0 \\ & a^3 & 3a^2 & 3a & 1 & \cdots & \vdots \\ & & a^3 & 3a^2 & 3a & \cdots & 0 \\ & & & a^3 & 3a^2 & & 1 \\ & & & & a^3 & \ddots & 3a \\ & 0 & & & & \ddots & 3a^2 \\ & & & & & & a^3 \end{bmatrix}$$

consider φ s.t. $T(\varphi, B) = J_k(a)^3 = A$, B - basis of F^k

So $\chi_\varphi(x) = \det(A - xI_k) = (-1)^k (a^3 - x)^k$

Thus $Spec(\varphi) = \{a^3\} \wedge a.m(a^3) = k$

let's find $g.m(a^3) = \dim(\ker(A - a^3 I))$

$$\dim\left(\ker\left(\begin{bmatrix} 0 & 3a^2 & & * \\ & 0 & 3a^2 & \\ & & 0 & 3a^2 \\ & & & 0 & \ddots \\ 0 & & & & \ddots & 3a^2 \\ & & & & & 0 \end{bmatrix}\right)\right) \xrightarrow[\text{since } a \neq 0]{RREF} = \dim\left(\ker\left(\begin{bmatrix} I_{k-1} & \begin{matrix} 0 \\ 0 \\ \vdots \end{matrix} \\ 0\;0\cdots & 0 \end{bmatrix}\right)\right)$$

$= 1$ (since there is only one free el.)

So $a.m(a^3) = k \wedge g.m(a^3) = 1 \Rightarrow$ JNF of A is $\begin{bmatrix} a^3 & 1 & & 0 \\ & a^3 & 1 & \\ & & a^3 & \ddots \\ 0 & & & \ddots\; 1 \\ & & & a^3 \end{bmatrix} = J_k(a^3)$

2. (2 points per item) Find the Jordan normal form (see Definition 24.2) of the following matrices:

(a) $A = \begin{bmatrix} 3 & -1 & 0 & -2 & 0 \\ -3 & -4 & -2 & 1 & 3 \\ 0 & -7 & 1 & -5 & 2 \\ 3 & 4 & 1 & 1 & -2 \\ -6 & -19 & -5 & -3 & 10 \end{bmatrix}$;

[**hint:** use the algorithm from Seminar 24 (see pp. 24.4-24.7; also see Problem 2); *do not even try* to perform calculations by hands, use a machine]

I don't even thought to perform any calc. by hands :)

let A be coordinate matrix of φ;

$\chi_\varphi(x) = \det(A - xI_5) = -(x-3)(x-2)^4$, hence $Spec(\varphi) = \{3, 2\}$

and $a.m(3) = 1 \Rightarrow g.m(3) = 1$; $a.m(2) = 4$, lets find $g.m(2)$

$g.m(2) = \dim(\ker(A - 2I_5)) = \dim\left(\left(\begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/5 \\ 1/5 \\ 3/5 \\ 0 \\ 1 \end{bmatrix}\right)\right) = 2$

Since we have two possibilities now $(J_{1,3,1}(3,2,2) \vee J_{1,2,2}(3,2,2))$
which are different not just up to order

We need to calculate generelised eigenspaces:

$\dim(\ker(A - 2I_5)^2) = \dim\left(\left(\begin{bmatrix} -3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)\right) = 4$

since $(A - 2I_5)^2 = (A - 2I_5)^3 \Rightarrow \dim(\ker(A - 2I_5)^3) = 4$,
so, that's it.

Now, since I miss all classes due to illness, I'll use tecnique from „LA Done right" book:

2 - level • •

1 - level • •

$\Rightarrow$ we have two 2x2 Jordan boxes for (2)

hence Jordan normal form of A is $\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} = J_{1,2,2}(3,2,2) = J$

3. (2 points) For every positive integer n, find a matrix B_n such that

$$(B_n)^n = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}.$$

Remark 1 B_n is (odiously) a square matrix os size 2, index n in the notation B_n shows that the matrix depends on n.

Remark 2 You can say that B_n is an n-th root of $\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$;

Remark 3 It can be proved (and, actually, you can do it!) that, for every positive integer n and every square matrix A such that its characteristic polynomial splits into linear factors, there is a matrix B such that $B^n = A$ (that is, a matrix equation $X^n = A$ has a solution).

Instructions: (It would be 3 times as interesting without the instructions)

✓(a) find the Jordan normal form of $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$, say it is J;

✓(b) find a matrix C such that $C^{-1}JC = A$ (~~for this, rewrite the last matrix equality as $JC = CA$, assume that $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, for some unknowns a, b, c, and d, find a solution to the system of linear equations $JC = CA$ such that C is invertible (note that there are infinitely many such solutions, but any will do)~~);

✓(c) find a matrix D_n such that $D_n^n = J$ (for this, using approach similar to (b), for every positive integer n and every $\lambda \neq 0$, find a matrix E such that

$$E^{-1} \cdot \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^n \cdot E = \begin{bmatrix} \lambda^n & 1 \\ 0 & \lambda^n \end{bmatrix};$$

find λ such that $\begin{bmatrix} \lambda^n & 1 \\ 0 & \lambda^n \end{bmatrix} = J$; note that, for every square matrix F of size 2, we have $(E^{-1}FE)^n = E^{-1}F^nE$; now it ought to be clear that $D_n = ?$);

✓(d) using Items (b) and (c), find B_n.

a) (let $T(\varphi_A) = A$) so $\chi_\varphi(x) = \det(A - xI) = (x-2)^2$

$\dim(\ker(A - 2I)) = 1$, so $J = J_2(2) = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

b) A has only one eigen-vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $(E_\varphi(2))$

and $\dim(\ker(A - 2I)^2) = \dim(\ker(O_2)) = 2$, so

generelised eigenvectors of level 2: $\langle(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix})\rangle$, but

since we are free to choose, we will chose $\langle(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix})\rangle$

Thus: 2^{nd}-level: $\bullet\ \bar{w}_2$

$\downarrow$

1^{st}-level: $\bullet\ \bar{w}_1$, formally $\bar{w}_2 \in \ker(A - 2I)^2 \wedge$

$\wedge\ \bar{w}_2 \notin \ker(A - 2I)$

so, since $w_{k-1} = (A - \lambda I)w_k$, $\bar{w}_1 = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

so a Jordan basis of φ is $(\underbrace{\begin{bmatrix} -1 \\ -1 \end{bmatrix}}_{\bar{j}_1}, \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\bar{j}_2})$

so $J = C(A,J)\, A\, C(J,A) = \begin{bmatrix} 1 & -3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

also $A = \underbrace{\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}}_{= M}\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\underbrace{\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}^{-1}}_{= M^{-1}} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$

c) $D_n^n = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \iff D_n = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}^{1/n}$

Thus $D_n = \begin{bmatrix} \sqrt[n]{2} & \frac{2^{\frac{1-n}{n}}}{n} \\ 0 & \sqrt[n]{2} \end{bmatrix} = \begin{bmatrix} \sqrt[n]{2} & \frac{\sqrt[n]{2}}{2n} \\ 0 & \sqrt[n]{2} \end{bmatrix}$

since $J^n = 2^{n-1}\begin{bmatrix} 2 & n \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2^n & \frac{2^{n-1}}{n} \\ 0 & 2^n \end{bmatrix}$

So it's easy to find n-th root of J just by sub. n with $1/n$!

d) since $A = MJM^{-1}$

and $J = D_n^n$

$A = M D_n^n M^{-1}$

so $M D_n^n M^{-1} = B_n^n$

$B_n^n = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \sqrt[n]{2} & \frac{\sqrt[n]{2}}{2n} \\ 0 & \sqrt[n]{2} \end{bmatrix}^n\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}^{-1}$

$B_n = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \sqrt[n]{2} & \frac{\sqrt[n]{2}}{2n} \\ 0 & \sqrt[n]{2} \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}^{-1} = \frac{\sqrt[n]{2}}{n} \cdot \begin{bmatrix} 2n-1 & 1 \\ -1 & 2n+1 \end{bmatrix}$

check: $B_n^n = \left(\frac{\sqrt[n]{2}}{n}\right)^n \begin{bmatrix} 2n-1 & 1 \\ -1 & 2n+1 \end{bmatrix}^n =$

$= \left(\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \sqrt[n]{2} & \frac{\sqrt[n]{2}}{2n} \\ 0 & \sqrt[n]{2} \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}^{-1}\right)^n =$

$\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \sqrt[n]{2} & \frac{\sqrt[n]{2}}{2n} \\ 0 & \sqrt[n]{2} \end{bmatrix}^n\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}^{-1} = A$

4. (2 points) Let

$$A = \begin{bmatrix} 16 & 18 & -6 & 0 & 15 \\ -19 & -20 & 8 & 3 & -17 \\ -25 & -24 & 7 & 3 & -18 \\ 18 & 18 & -6 & -2 & 15 \\ -7 & -6 & 3 & 3 & -5 \end{bmatrix}.$$

Then, find a matrix C such that $C^{-1}AC$ is a Jordan matrix.

Consid φ s.t. $T(\varphi, \mathcal{B}) = A \ldots)$

$\chi_\varphi(X) = \det(A - xI) = -(x-1)^2(x+2)^3$

So $\mathrm{Spec}(\varphi) = \{1, -2\}$, a.m.$(1) = 2$ a.m$(-2) = 3$

Let N_λ be $(A - \lambda I)$

$$\ker(N_1) = \left(\underbrace{\begin{bmatrix} -3 \\ 3 \\ 4 \\ -3 \\ 1 \end{bmatrix}}_{\bar{w}_1} \right) \qquad \ker(N_{-2}) = \left(\underbrace{\begin{bmatrix} 3 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\bar{u}_1}, \begin{bmatrix} 6 \\ -11/2 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right)$$

g.m$(1) = 1$ g.m$(-2) = 2$

$$\ker(N_1^2) = \left(\underbrace{\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\bar{w}_2}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right) \qquad \ker(N_{-2}^2) = \left(\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \underbrace{\begin{bmatrix} 3 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\bar{u}_1}, \underbrace{\begin{bmatrix} 4 \\ -3 \\ 0 \\ 0 \\ 2 \end{bmatrix}}_{\bar{m}_2} \right)$$

$\ker(N_1^3) = \ker(N_1^2)$ $\qquad$ $\ker(N_{-2}^3) = \ker(N_{-2}^2)$

2nd - level : $\bar{w}_2$ • $\downarrow$ 1 box
1st - level : $\bar{w}_1$ • size 2

2nd - level : $\bar{m}_2$ • $\downarrow$ 2 boxes
1st - level : $\bar{m}_1$ • $\bar{u}_1$ • size 2 ^ 1

$$\text{Thus } C = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Since $m_{k-1} = N_\lambda \cdot m_k$: $\bar{m}_1 = N_{-2} \cdot \bar{m}_2 = \begin{bmatrix} 48 \\ -56 \\ -64 \\ 48 \\ -16 \end{bmatrix}$

$\bar{w}_1 = N_1 \bar{w}_2$

Hence $(\bar{u}_1; \bar{m}_1; \bar{m}_2; \bar{w}_1; \bar{w}_2)$ is Jordan basis

$$\text{So, } A = \begin{bmatrix} 3 & 48 & 4 & -3 & -1 \\ -3 & -56 & -3 & 3 & 1 \\ 0 & -64 & 0 & 4 & 1 \\ 1 & 48 & 0 & -3 & -1 \\ 0 & -16 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 48 & 4 & -3 & -1 \\ -3 & -56 & -3 & 3 & 1 \\ 0 & -64 & 0 & 4 & 1 \\ 1 & 48 & 0 & -3 & -1 \\ 0 & -16 & 2 & 1 & 0 \end{bmatrix}^{-1}$$

5. (*) (2 points) Find the Jordan normal form of $J_k(0)^3$, where $J_k(0)$ the Jordan block of size k and value 0.

Note 1: this problem is somewhat surprisingly more difficult then Item (b) of Problem 2.

Note 2: it is possible to find the Jordan normal form of $J_k(0)^m$, for every $k, m \in \mathbb{N}$.

1) $J_k(0)$ is a nilpotent matrix of index k

Hence $k \le 3$ NJF $J_k(0)^3 = 0_k$

also $Spec(J_k(0)^3) = \{0\}$, so let's consider generalized eigen-spaces of $J_k(0)^3$:

1) case $3|k$ $(k \neq 0)$

$\dim(\ker(J_k(0)^3)) = 3$ $\quad$ $\dim(\ker(J_k(0)^n)) = n$ $\forall n \le k$

$\dim(\ker(J_k(0)^4)) = 4$

so 1^{st}-level: $\dim(\ker(J_k(0)^3)) = 3$

2^{nd}-level: $\dim(\ker(J_k(0)^{3\cdot 2})) = 6$

n-th level: $\dim(\ker(J_k(0)^{3\cdot n})) = 3n$ $\quad \forall n \le \frac{k}{3}, \forall n > \frac{k}{3}: \dim = k$

So Jordan boxex are in the form:

$$\begin{array}{llll}
n\text{-th level}: & \bullet \bar w_n & \bullet \bar u_n & \bullet \bar m_n \\
 & \downarrow & \downarrow & \downarrow \\
(n-1)\text{-th level}: & \bullet \bar w_{n-1} & \bullet \bar u_{n-1} & \bullet \bar m_{n-1} \\
\vdots & \downarrow \vdots & \downarrow \vdots & \downarrow \vdots \\
1\text{-th level}: & \bullet \bar w_1 & \bullet \bar u_1 & \bullet \bar m_1
\end{array}$$

where $\bar w_n, \bar u_n, \bar m_n \in \ker(J_k(0)^{3n})$
$\wedge$ $\bar w_n, \bar u_n, \bar m_n \notin \ker(J_k(0)^{3(n-1)})$

also $\bar w_n$ $\bar u_n$ $\bar m_n$ are generalised eigenvectors of level n

So Jordans boxes are $n \times n$ $\quad n \times n$ $\quad n \times n$;
$\forall k$ s.t. $k = 3 \cdot m, \forall m \in \mathbb{N}$

JNF of $J_k(0)^3 = J_{\frac{k}{3},\frac{k}{3},\frac{k}{3}}(0) = J_{n,n,n}(0)$

2) case k is in the form $3n+1$: $(\forall n \in \mathbb{N} \wedge n \neq 0)$

so 1^{st}-level: $\dim(\ker(J_k(0)^3)) = 3$

2^{nd}-level: $\dim(\ker(J_k(0)^{3\cdot 2})) = 6$

n-th level: $\dim(\ker(J_k(0)^{3\cdot n})) = 3n$

$(n+1)$-th level: $\dim(\ker(J_k(0)^{3n+3})) = 3n+1$

So Jordan boxex are in the form:

$$\begin{array}{llll}
(n+1)\text{-th level}: & \bullet \bar w_{n+1} & & \\
 & \downarrow & & \\
n\text{-th level}: & \bullet \bar w_n & \bullet \bar u_n & \bullet \bar m_n \\
 & \downarrow & \downarrow & \downarrow \\
(n-1)\text{-th level}: & \bullet \bar w_{n-1} & \bullet \bar u_{n-1} & \bullet \bar m_{n-1} \\
\vdots & \downarrow \vdots & \downarrow \vdots & \downarrow \vdots \\
1\text{-th level}: & \bullet \bar w_1 & \bullet \bar u_1 & \bullet \bar m_1
\end{array}$$

(I mean for arb. n)

where $\bar w_n, \bar u_n, \bar m_n \in \ker(J_k(0)^{3n})$
$\wedge$ $\bar w_n, \bar u_n, \bar m_n \notin \ker(J_k(0)^{3(n-1)})$

also $\bar w_m$ $\bar u_m$ $\bar m_m$ are generalised eigenvectors of level m

So Jordans boxes are $(n+1)\times(n+1)$ $\quad n \times n$ $\quad n \times n$;
$\forall k$ s.t. $k = 3 \cdot m, \forall m \in \mathbb{N}$

JNF of $J_k(0)^3 = J_{n+1,n,n}(0) = J_{\frac{(k-1)}{3}+1, \frac{(k-1)}{3}, \frac{(k-1)}{3}}(0)$

case $k = 3n+2$: $J_k(0)^3 = J_{(n+1),(n+1),n}(0) = J_{\frac{(k-2)}{3}+1; \frac{(k-2)}{3}+1; \frac{k-2}{3}}(0)$
respectively.

- Novosad Ivan 231