Multigrid Solver for Helmholtz/Poission Equation using PETSc

Yu-Hsuan Shih
yhs264

PETSc - Portable, Extensible Toolkit for Scientific Computation

Time dependent Stokes Equation

$$u_t(x,t) + \nabla p(x,t) = \mu \nabla^2 u(x,t) + f(x,t)$$
$$\nabla \cdot u(x,t) = 0$$

Temporal Discretization
Crank-Nicholson for Diffusion + Midpoint Rule for pressure and source

$$\frac{u^{n+1} - u^n}{\Delta t} + \nabla p^{n+\frac{1}{2}} = \mu \nabla^2 \left(\frac{u^{n+1} + u^n}{2} \right) + f^{n+\frac{1}{2}}$$
$$\nabla \cdot u^{n+1} = 0$$

=> Linear System to Solve KSP Matrix A Solver Preconditioner P

$$\begin{bmatrix} I - \frac{\mu \Delta t}{2} L^x & 0 & \Delta t G^x \\ 0 & I - \frac{\mu \Delta t}{2} L^y & \Delta t G^y \\ \Delta t D^x & \Delta t D^y & 0 \end{bmatrix} \begin{bmatrix} u^{n+1} \\ v^{n+1} \\ p^{n+\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} \left(I + \frac{\mu \Delta t}{2} L^x \right) u^n + \Delta t f_u^{n+\frac{1}{2}} \\ \left(I + \frac{\mu \Delta t}{2} L^y \right) v^n + \Delta t f_v^{n+\frac{1}{2}} \\ 0 \end{bmatrix}$$

Preconditioner - Pressure-Free Projection Method

PCSetType(pc, PCSHELL), ShellPCApply

Step 1: Solve an analog of equation for an intermediate quantity (u^*, v^*)

$$\left(I - \frac{\mu\Delta t}{2} L^x\right) u^* = \left(I + \frac{\mu\Delta t}{2} L^x\right) u^n + \Delta t f_u$$

$$\left(I - \frac{\mu\Delta t}{2} L^y\right) v^* = \left(I + \frac{\mu\Delta t}{2} L^y\right) v^n + \Delta t f_v$$

Step 2: Project (u^*, v^*) **onto the space of divergence-free fields** (u^{n+1}, v^{n+1})

$$-DG\phi = -L^c\phi = -\frac{1}{\Delta t} Du^*$$

$$\mathbf{u}^{n+1} = \mathbf{u}^* - \Delta t G\phi$$

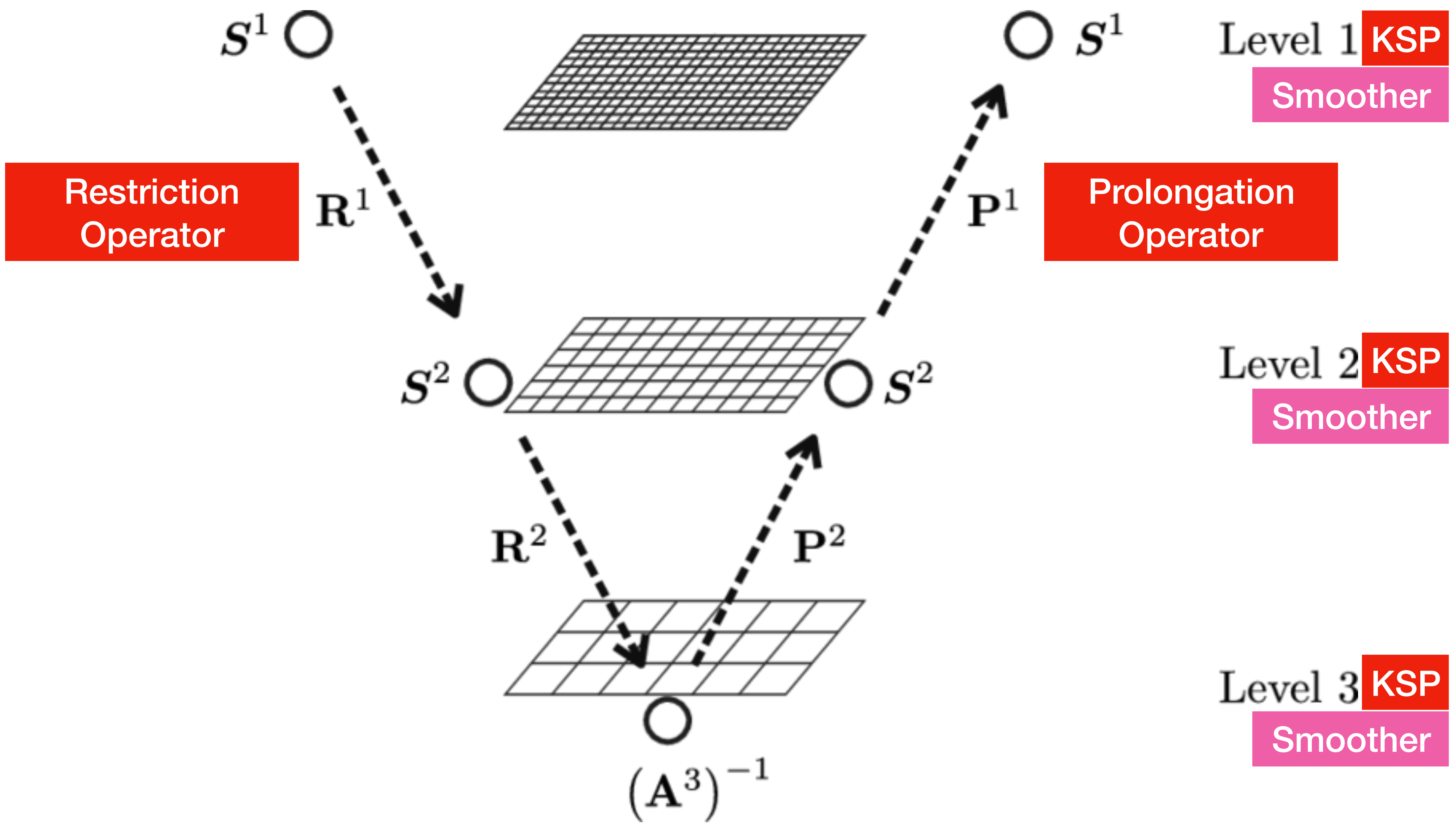
Step 3: Correct the pressure term

$$\mathbf{LG} = \mathbf{G}L^c$$

$$p^{n+\frac{1}{2}} = \left(I - \frac{\mu\Delta t}{2} L^c\right)\phi$$

Reference: An accurate and efficient method for the incompressible Navier–Stokes equations using the projection method as a preconditioner

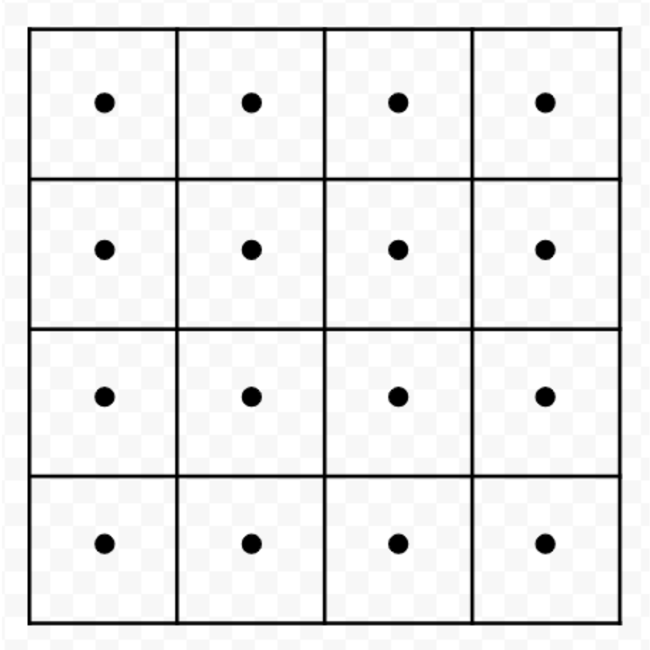
Multigrid (V-cycle)



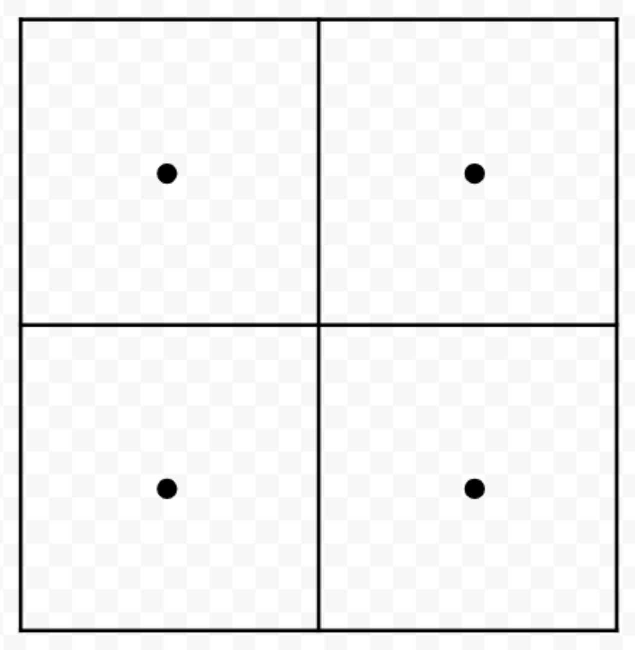
Picture from: https://www.researchgate.net/figure/Illustration-of-a-multigrid-V-cycle-with-3-levels_fig1_301896254

Reference: EFFICIENT VARIABLE-COEFFICIENT FINITE-VOLUME STOKES SOLVERS

Cell-Centered



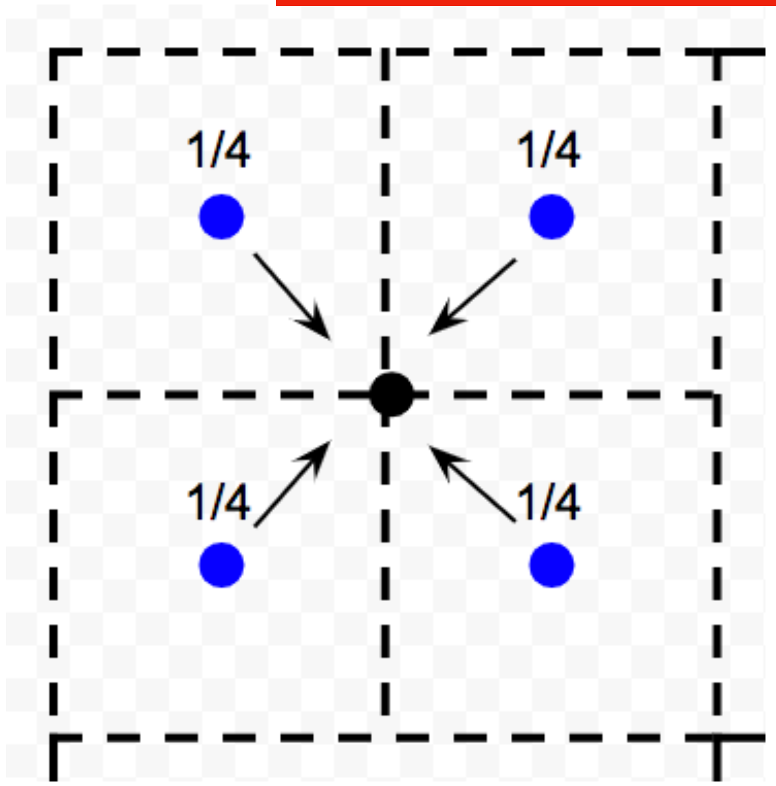
Fine



Coarse

Restriction (Fine to Coarse)

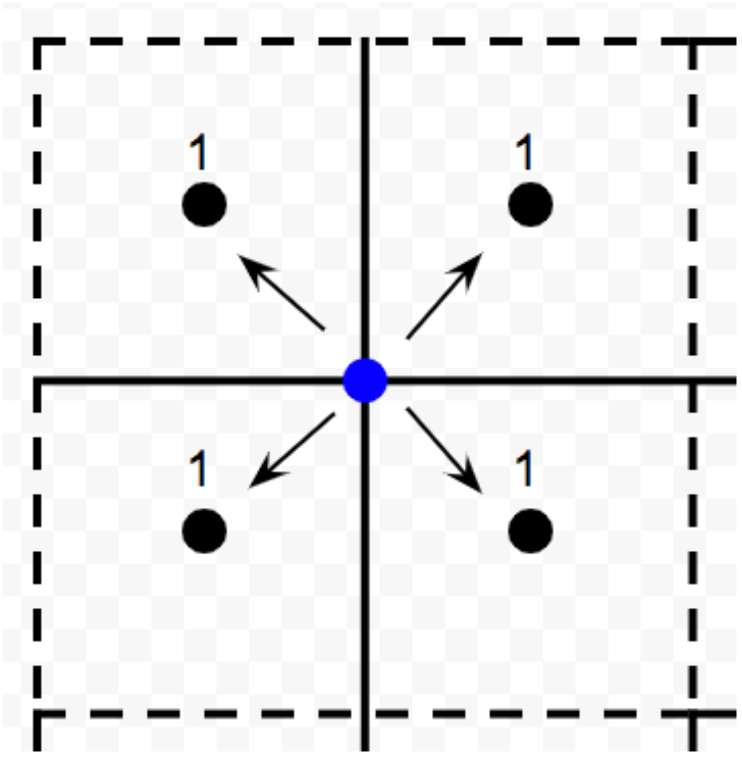
PCMGSetRestriction



R

Prolongation (Coarse to Fine)

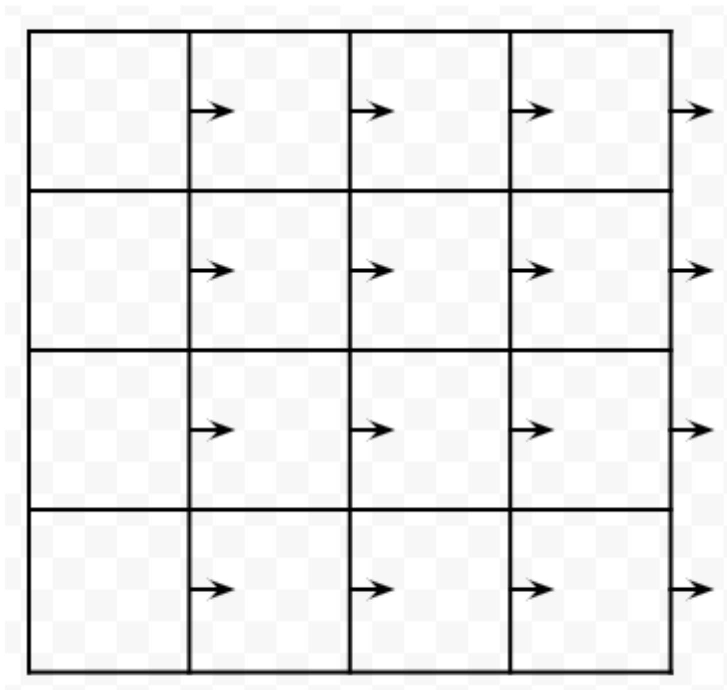
PCMGSetInterpolation



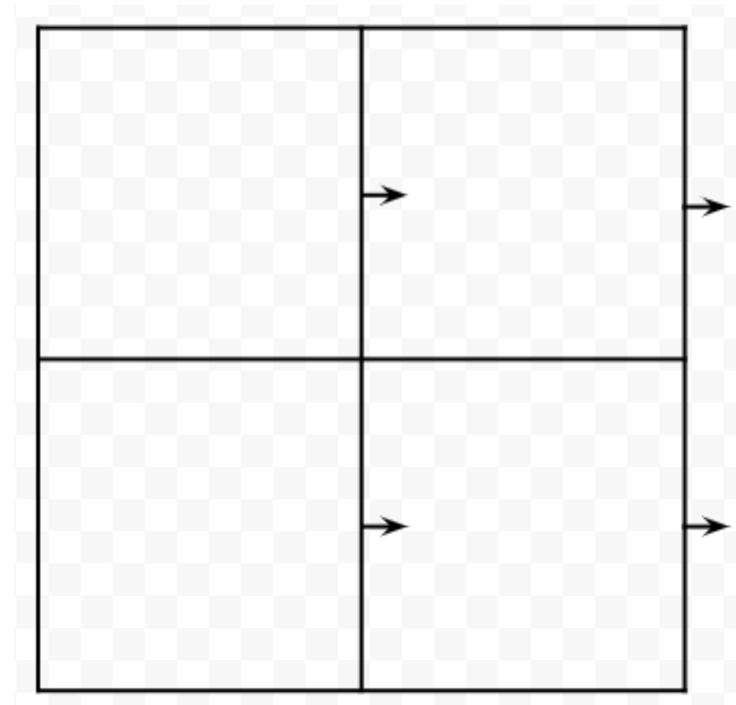
P

Reference: EFFICIENT VARIABLE-COEFFICIENT FINITE-VOLUME STOKES SOLVERS

Cell-Faced



Fine

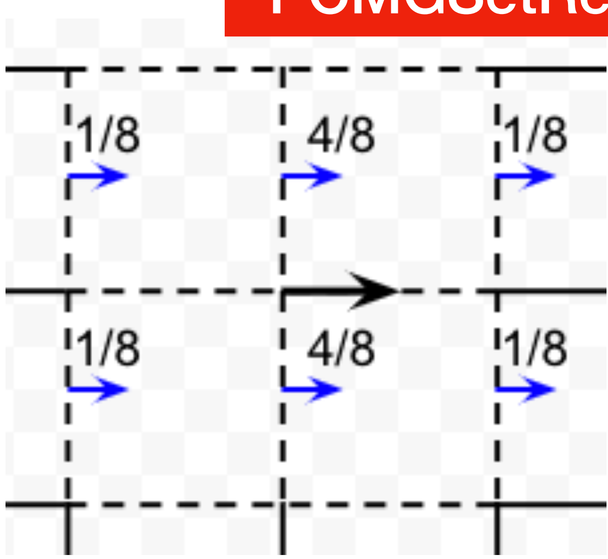


Coarse

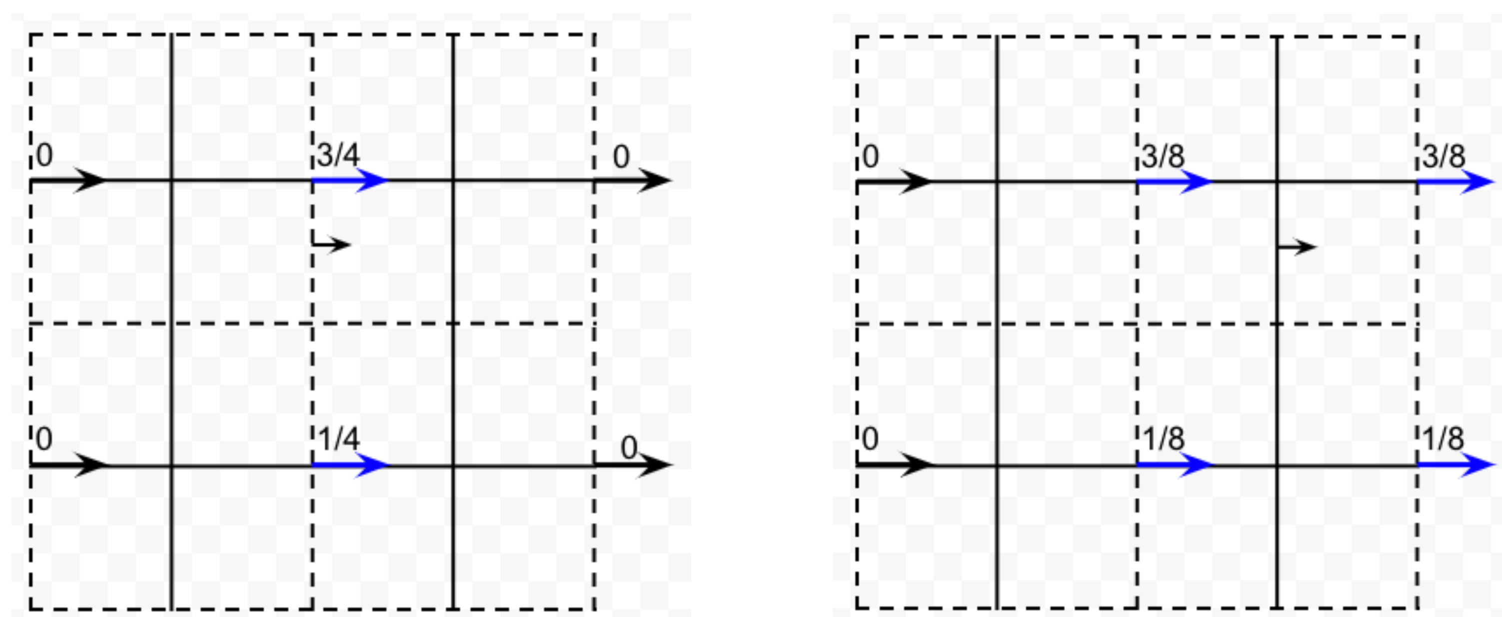
PCMGSetInterpolation

Restriction (Fine to Coarse)

PCMGSetRestriction



Prolongation (Coarse to Fine)



Code Implementation

KSP (Stokes Equation)

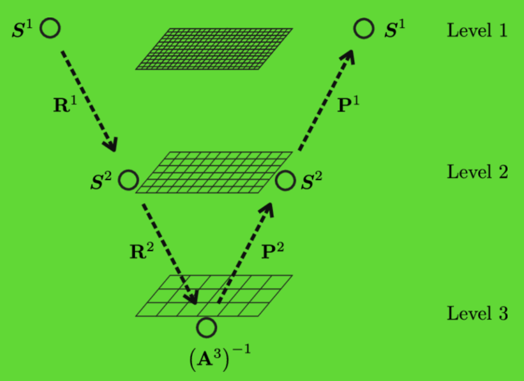
$$\begin{bmatrix} I-\frac{\mu\Delta t}{2}L^x & 0 & \Delta t G^x \\ 0 & I-\frac{\mu\Delta t}{2}L^y & \Delta t G^y \\ \Delta t D^x & \Delta t D^y & 0 \end{bmatrix} \begin{bmatrix} u^{n+1} \\ v^{n+1} \\ p^{n+\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} \left(I+\frac{\mu\Delta t}{2}L^x\right)u^n + \Delta t f_u^{n+\frac{1}{2}} \\ \left(I+\frac{\mu\Delta t}{2}L^y\right)v^n + \Delta t f_v^{n+\frac{1}{2}} \\ 0 \end{bmatrix}$$

PCShell (Projection Method)

KSP (Helmholtz Equation)

$$\left(I-\frac{\mu\Delta t}{2}L^x\right)u^* = \left(I+\frac{\mu\Delta t}{2}L^x\right)u^n + \Delta t f_u$$

PC (MultiGrid)



KSP

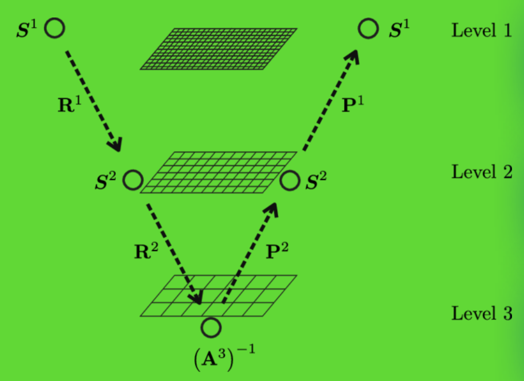
KSP

KSP

KSP (Poission Equation)

$$-DG\phi = -L^c\phi = -\frac{1}{\Delta t}Du^*$$

PC (MultiGrid)



KSP

KSP

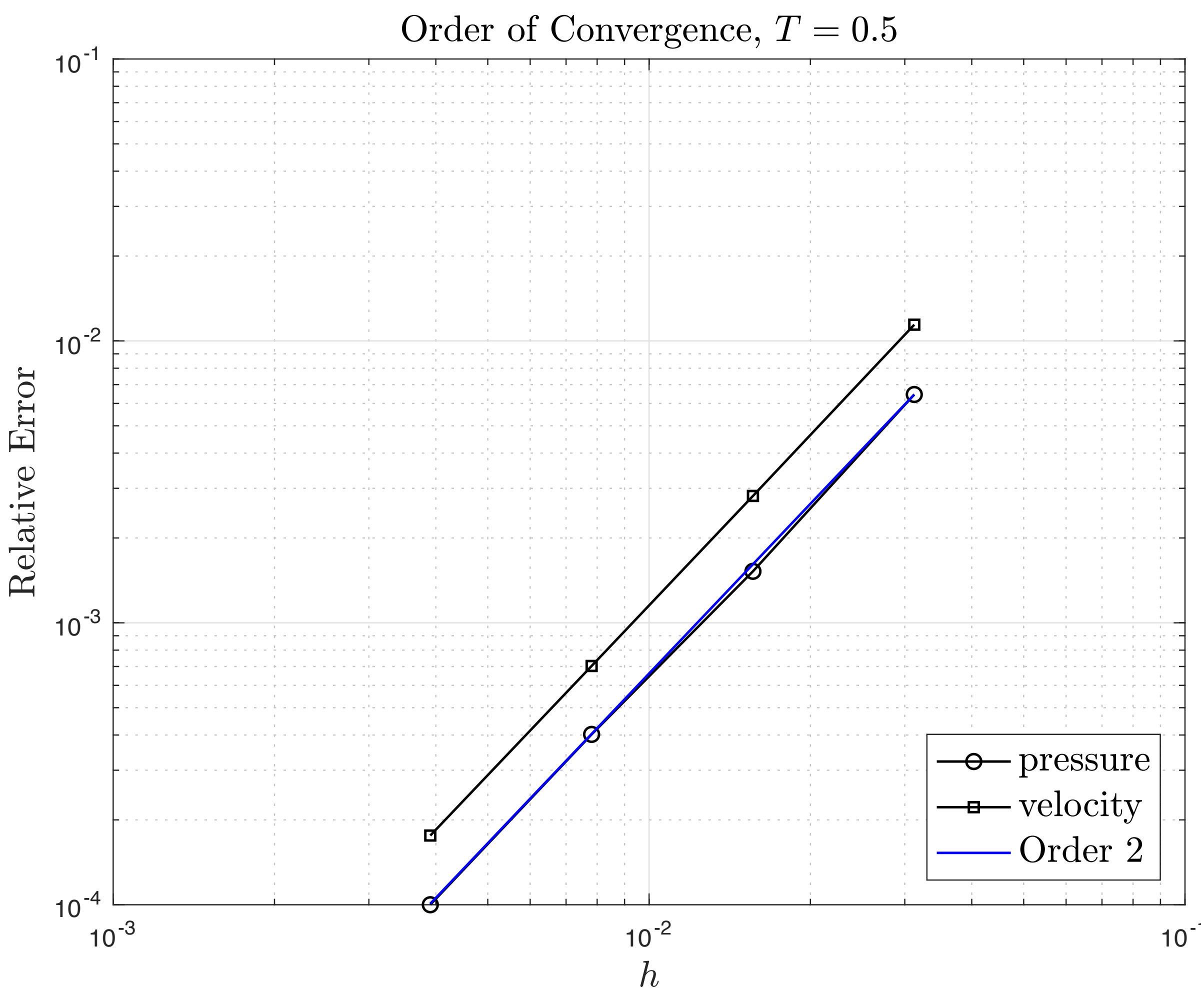
KSP

Code Validation

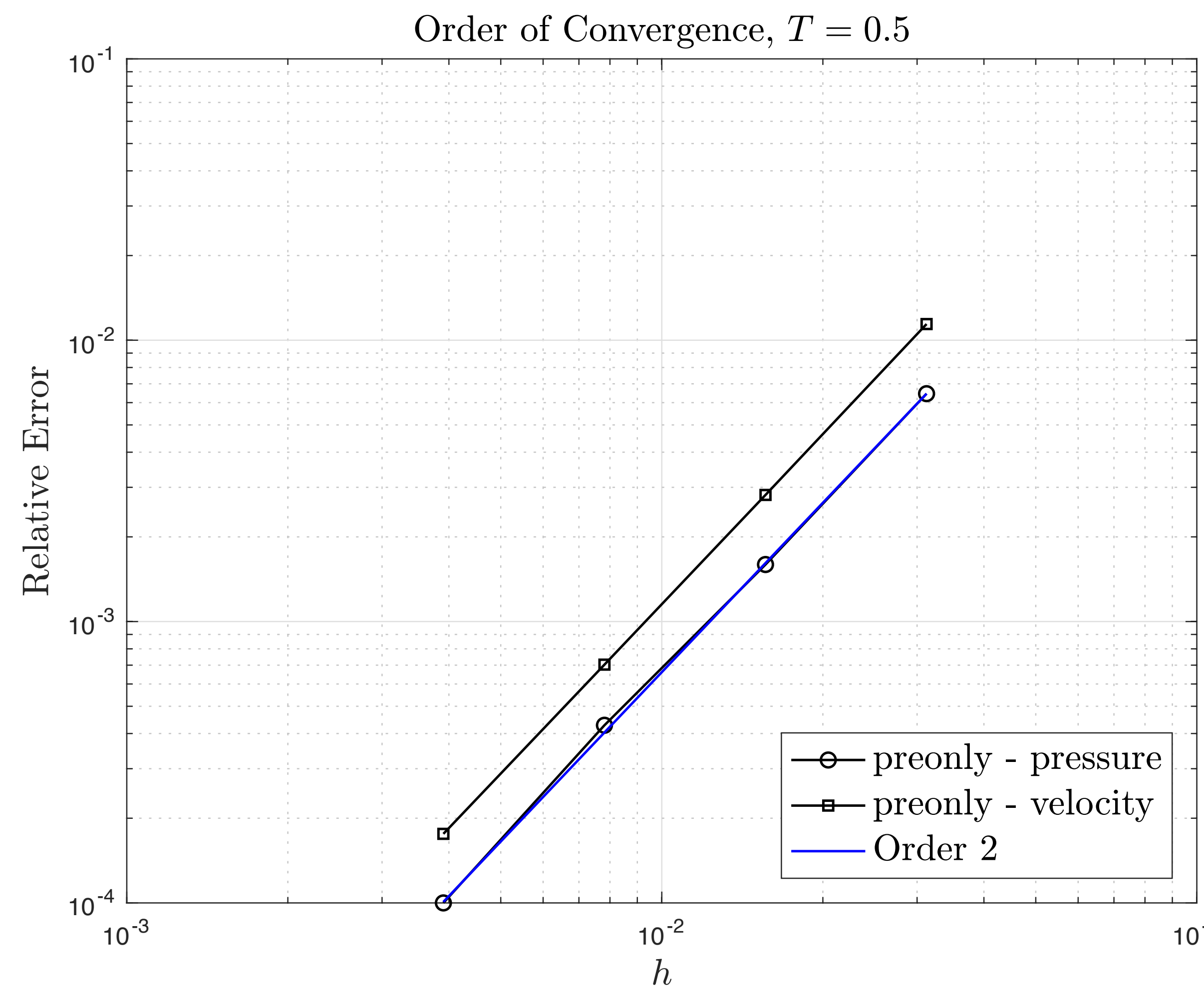
Manufactured Solution

$$
\begin{aligned}
u(x,t) &= 1 - 2\cos(2\pi(x-t))\sin(2\pi(y-t)) \\
v(x,t) &= 1 + 2\sin(2\pi(x-t))\cos(2\pi(y-t)) \\
p(x,t) &= -(\cos(4\pi(x-t)) + \cos(4\pi(y-t)))
\end{aligned}
$$

Using GMRES + Projection Method PC



Projection Method as a direct solver



MultiGrid Convergence

$$||\text{Residual after 1 V-Cycle}||_2 \leq \rho ||\text{Original Residual}||_2$$

$$\rho \text{ independent of grid spacing } h = \frac{1}{N}$$

Multigrid Method for Poission Equation

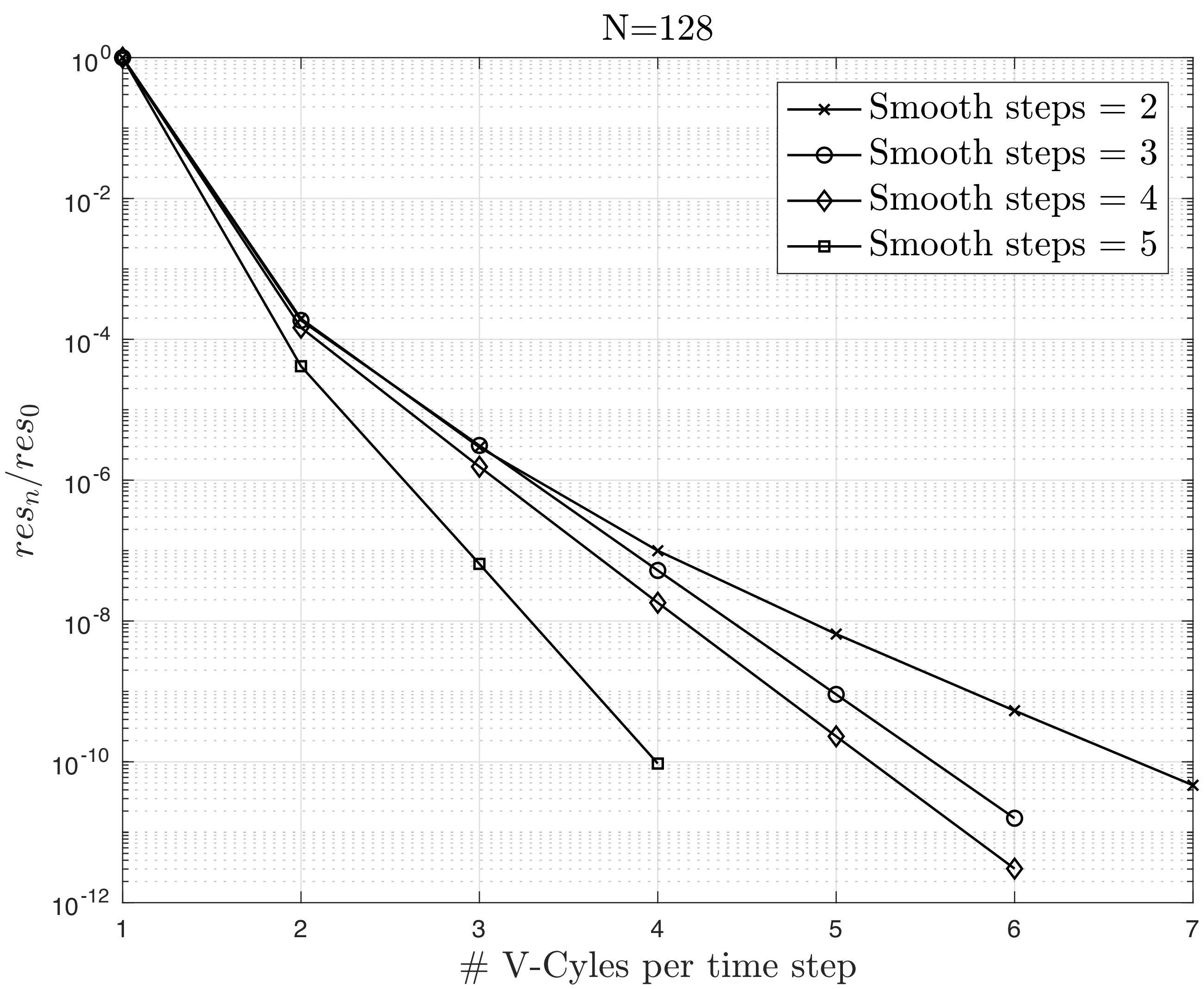
Smoother - Weighted Jacobi
2 smoothing sweeps/level

N	64		128		256	
#V-cycle	Residual	rho	Residual	rho	Residual	rho
0	1.28E+00		1.74E+00		2.14E+00	
1	1.81E-01	**0.14**	2.64E-01	**0.15**	3.48E-01	**0.16**
2	5.25E-02	**0.29**	7.89E-02	**0.30**	1.08E-01	**0.31**
3	1.1E-02	**0.21**	1.87E-02	**0.24**	2.84E-02	**0.26**
4	3.08E-03	**0.28**	5.55E-03	**0.30**	9.09E-03	**0.32**
5	7.98E-04	**0.26**	1.59E-03	**0.29**	2.88E-03	**0.32**
6	2.35E-04	**0.30**	4.90E-04	**0.31**	9.59E-04	**0.33**
7	6.99E-05	**0.30**	1.51E-04	**0.31**	3.20E-04	**0.33**
8	2.19E-05	**0.31**	4.80E-05	**0.32**	1.08E-04	**0.34**
9	6.98E-06	**0.32**	1.53E-05	**0.32**	3.68E-05	**0.34**
10	2.28E-06	**0.33**	4.99E-06	**0.33**	1.26E-05	**0.34**
11	7.49E-07	**0.33**	1.64E-06	**0.33**	4.31E-06	**0.34**
12	2.49E-07	**0.33**	5.49E-07	**0.33**	1.48E-06	**0.34**
13	8.29E-08	**0.33**	1.86E-07	**0.34**	5.13E-07	**0.35**
14	2.77E-08	**0.34**	6.37E-08	**0.34**	1.78E-07	**0.35**

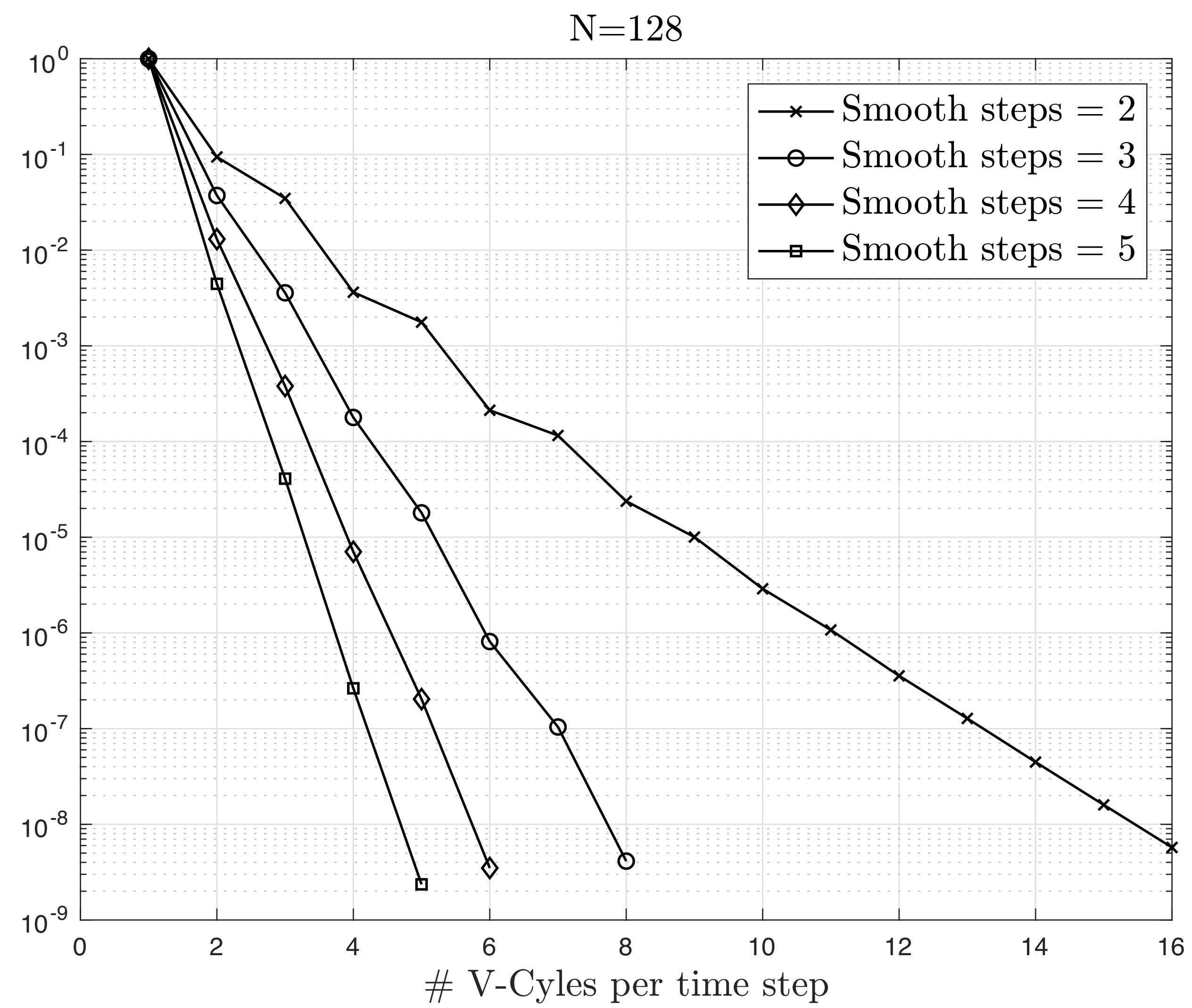
MultiGrid - Effect of number of smoothing sweeps

-poission_mg_levels_ksp_max_it

Helmholtz Solve



Poission Solve

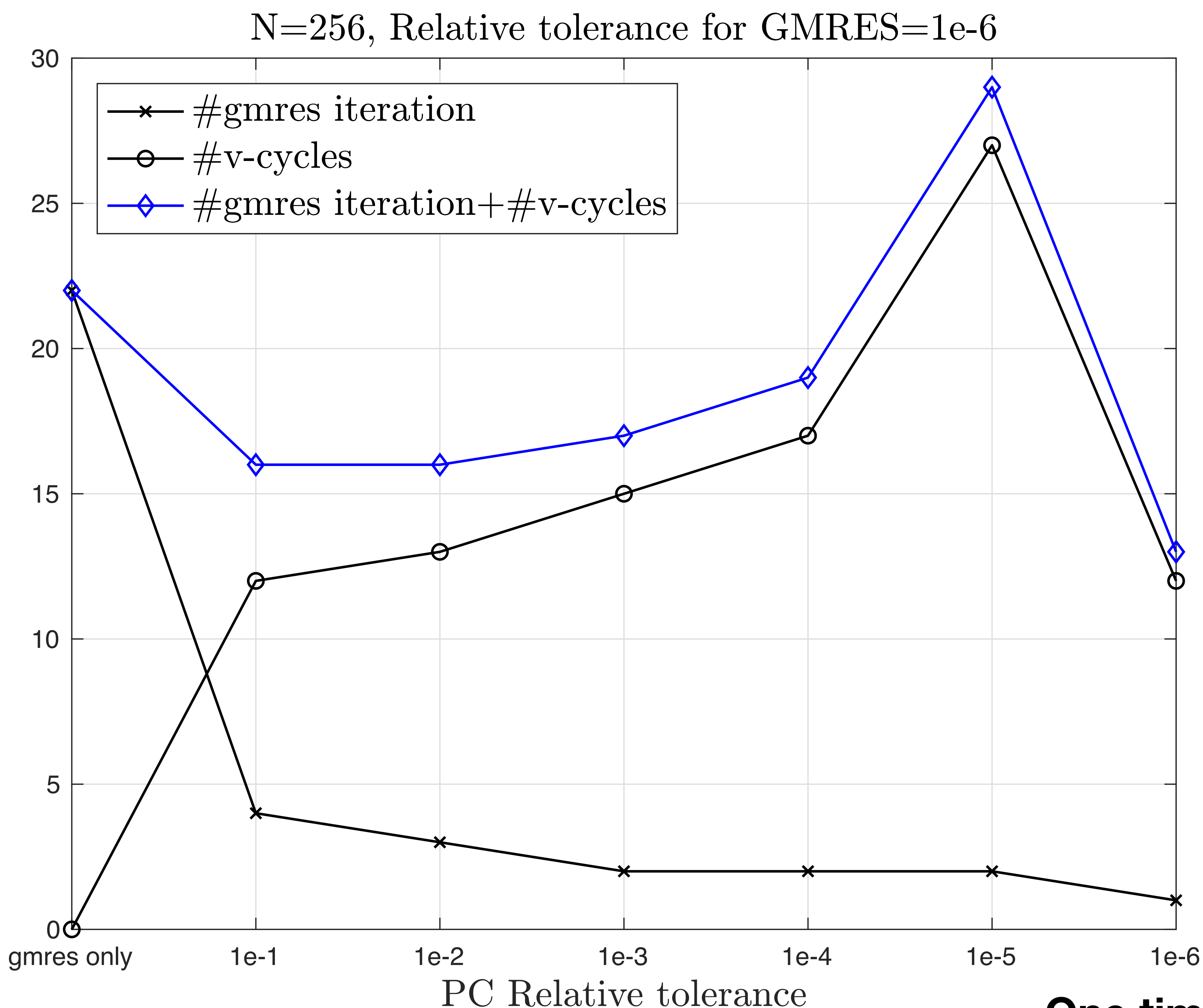


More sweeps => Better smooth effect => More reduction on Residual

More works Per V-Cycle

Effect of accuracy of the preconditioner

Number of GMRES iterations + Number of V-cyles in MultiGrid Solver $\approx$ Total amount of work



One time step

Thanks!

Have a good winter vacation :)