Two Wheeled Self Balancing Robot

(3DOF Dynamic Modelling & Control Design)

Design Linear (LQR) and Nonlinear Control (Partial Feedback Linearization) Techniques for Balancing & Trajectory Tracking

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1 Introduction: Two wheeled Self Balancing Robot has three degree-of-freedom motion including forward motion in robot's x-direction, pitch and yaw. This robot consists of inverted pendulum that is driven by only two wheels. A robot with an unstable mobile platform belongs to a nonlinear system and underactuated system such as only two input torque exerted at wheels.

1.1 Advantages

- It can pass through narrow spaces unlike three wheelers or four wheelers.
- Compact design and versatility in congested urban areas.
- It has a mechanically small footprint and does not have any caster wheel.

2 Mathematical Modelling: In this section, the motion equation of Self Balancing Robot for mobile control are derived.

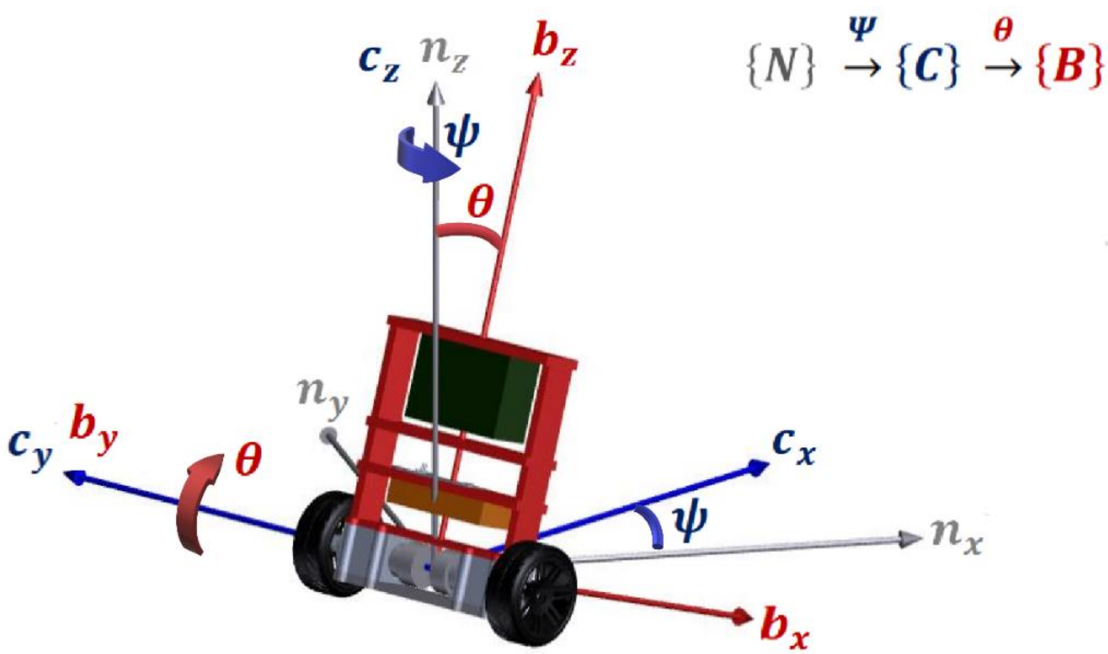

Figure 2.1 Schematic of the Two wheel Self Balancing Robot

2.1 Model Parameters of SBR

Symbol	Description
P_n	Position vector of body COG in the initial frame
V_n	Velocity vector of body COG in the initial frame
$x_c,\ y_c$	Origin of vehicle fixed coordinate / point attached to the centre of wheel axis w.r.t the initial frame
$x_b,\ y_b$	Point attached to the centre of wheel axis w.r.t the body fixed frame
θ	Inclination angle of body COG /Pitch
ψ	Steering angle of the vehicle /Yaw
θ_R, θ_L	Rotational angle of Right & Left wheel
$\dot{x}$	Forward velocity of robot in body frame
T_L, T_R	Wheel torques exerted at left & right
Ω	Angular velocity vector of body COG in body frame
Ω_L, Ω_R	Angular velocity vector of left & right wheel in C frame
${}^{A}R_B(\theta)$	Rotation matrix such as transformation from B frame to A frame with θ

Table 2.1 Model parameters of Two wheel SBR

2.2 Nomenclature:

Symbol	Description	Value & Unit
m_B	Mass of pendulum body (except wheels)	1.15 kg
m_W	Mass of each wheel	0.053 kg

I_x, I_y, I_z	MOI of pendulum body w.r.t. body	0.007402 $kg.m^2$, 0.004295 $kg.m^2$, 0.003350 $kg.m^2$
I_{Wd}	Inertia of wheel about its diameter/radial	0.0000183 $kg.m^2$
I_{Wa}	Inertia of wheel axis/axial	0.0000365 $kg.m^2$
d	Distance between two wheels	0.23 m
l	distance between the origin of vehicle fixed coordinates and body COG	0.045 m
r	Radius of wheel	0.0425 m
g	Gravitational Constant	9.81 m/sec^2
C_α	Coefficient of viscous friction on the wheel axis	0 $N/(m/sec)$

Table 2.2 Model parameters of Two wheel SBR

2.3 Differential motion constraints

There are three constraints in the robot,

Two Nonholonomic constraints as shown in Fig, such as there is no side motion, or body can not move in y_b direction, Therefore, $\boldsymbol{\dot{y}_b = 0}$

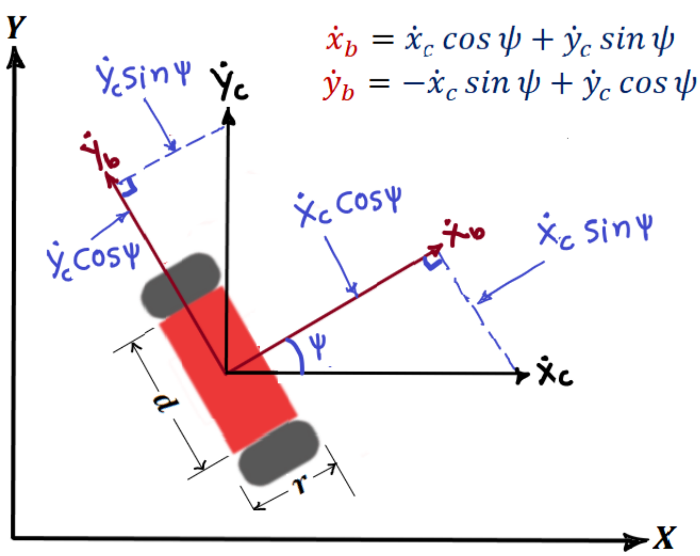

Figure 2.2 Differential motion constraints

$$\dot{x}_b = \dot{x}_c \cos\psi + \dot{y}_c \sin\psi \tag{2.1}$$

$$\dot{y}_b = -\dot{x}_c \sin\psi + \dot{y}_c \cos\psi \tag{2.2}$$

Forward velocity of robot in body frame is

$$\dot{x}_b = \dot{x} \tag{2.3}$$

There is no side motion (Nonholonomic constraint)

$$\dot{y}_b = 0 \tag{2.4}$$

Therefore, by putting Eq^n(2.3) & (2.4) into Eq^n(2.1) & (2.2), we get,

$$\dot{x}_c \cos\psi + \dot{y}_c \sin\psi = \dot{x} \tag{2.5}$$

$$-\dot{x}_c \sin\psi + \dot{y}_c \cos\psi = 0 \tag{2.6}$$

2.4 Kinematics of Self Balancing Robot

$\{\boldsymbol{N}\} \xrightarrow{\boldsymbol{\Psi}} \{\boldsymbol{C}\} \xrightarrow{\boldsymbol{\theta}} \{\boldsymbol{B}\}$

Where:

$\{N\}$ denotes the Newtonian/ Initial/ Global frame fixed in space,

$\{C\}$ denotes the chassis frame attached at the centre of the wheel axis,

$\{B\}$ denotes the body frame whose origin is located at the COG of SBR.

The whole body of the robot moves together by straight motion and yaw rotation and the inverted pendulum has additional pitch motion. The transformation matrix between the frames can be written as

Rotation from C frame to N frame is given by

$${}^{N}R_C(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.7}$$

Rotation from B frame to C frame is given by

$${}^{C}R_B(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{2.8}$$

C is the intermediate frame, therefore Rotation from B frame to N frame is given by

$$^{N}R_{B}(\psi,\theta) = \ ^{N}R_{C}(\psi).\ ^{C}R_{B}(\theta) \tag{2.9}$$

Therefore, solve Eq^{n}(2.7) to (2.9), we get,

$$^{N}R_{B}(\psi,\theta) = \begin{bmatrix} cos\,\psi\, cos\,\theta & -sin\,\psi & sin\,\theta\, cos\,\psi \\ sin\,\psi\, cos\,\theta & cos\,\psi & sin\,\psi\, sin\,\theta \\ -sin\,\theta & 0 & cos\,\theta \end{bmatrix} \tag{2.10}$$

The position of COG of the body P_b, in body frame

$$P_b = \begin{bmatrix} 0 \\ 0 \\ l \end{bmatrix} \tag{2.11}$$

The position of COG of the body, P_n with respect to the global coordinates is given as

$$P_n = \begin{bmatrix} x_C \\ y_C \\ 0 \end{bmatrix} + \ ^{N}R_{B} \cdot P_b \tag{2.12}$$

Therefore, solve Eq^{n}(2.10) to (2.12), we get,

$$P_n = \begin{bmatrix} x_C \\ y_C \\ 0 \end{bmatrix} + \begin{bmatrix} cos\,\psi\, cos\,\theta & -sin\,\psi & sin\,\theta\, cos\,\psi \\ sin\,\psi\, cos\,\theta & cos\,\psi & sin\,\psi\, sin\,\theta \\ -sin\,\theta & 0 & cos\,\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ l \end{bmatrix}$$

$$\Rightarrow\ P_n = \begin{bmatrix} x_C \\ y_C \\ 0 \end{bmatrix} + \begin{bmatrix} sin\,\theta\, cos\,\psi \\ l\, sin\,\psi\, sin\,\theta \\ l\, cos\,\theta \end{bmatrix}$$

$$\Rightarrow\ P_n = \begin{bmatrix} x_c + l\, sin\,\theta\, cos\,\psi \\ y_c + l\, sin\,\psi\, sin\,\theta \\ l\, cos\,\theta \end{bmatrix} \tag{2.13}$$

And velocity, v_n with respect to the global coordinates is given as

$$v_n = \dot{P}_n \tag{2.14}$$

Therefore, solve Eq^{n}(2.13) & (2.14), we get,

$$\Rightarrow\ v_n = \begin{bmatrix} \dot{x}_C + l\cos\psi\cos\theta\dot{\theta} - l\sin\psi\sin\theta\dot{\psi} \\ \dot{y}_C + l\sin\theta\cos\psi\dot{\psi} + l\sin\psi\cos\theta\dot{\theta} \\ -l\sin\theta\dot{\theta} \end{bmatrix} \tag{2.15}$$

Let say :

$$^{N}R_{B}(\psi,\theta) = R$$

Then angular velocity vector of COG with respect to body fixed frame is given by Ω

$$\Omega = R^T \dot{R} = \begin{bmatrix} 0 & -\cos\theta\dot{\psi} & \dot{\theta} \\ \cos\theta\dot{\psi} & 0 & \sin\theta\dot{\psi} \\ -\dot{\theta} & -\sin\theta\dot{\psi} & 0 \end{bmatrix}$$

$$\Rightarrow \ \Omega = \begin{bmatrix} -\sin\theta\,\dot{\psi} \\ \dot{\theta} \\ \cos\theta\dot{\psi} \end{bmatrix} \tag{2.16}$$

The angular velocity of left & right wheel in C frame is given by Ω_L and Ω_R respectively

$$\Omega_L = \begin{bmatrix} 0 \\ \dot{\theta}_L \\ \dot{\psi} \end{bmatrix} \tag{2.17}$$

$$\Omega_R = \begin{bmatrix} 0 \\ \dot{\theta}_R \\ \dot{\psi} \end{bmatrix} \tag{2.18}$$

COG Mass Moment of Inertia is given by

$$I_B = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \tag{2.19}$$

Wheel Mass Moment of Inertia is given by

$$I_W = \begin{bmatrix} I_{Wd} & 0 & 0 \\ 0 & I_{Wa} & 0 \\ 0 & 0 & I_{Wd} \end{bmatrix} \tag{2.20}$$

2.5 Lagrangian equations of motion

Translation kinetic energy of body COG is given by

$$T_{t_b} = \frac{1}{2} m_B (v_n^T \cdot v_n) \tag{2.21}$$

Translation kinetic energy of both left & right wheel is given by

$$T_{t_w} = \frac{1}{2} m_W r^2 \left(\dot{\theta}_L^2 + \dot{\theta}_R^2\right) \tag{2.22}$$

Rotation kinetic energy of body COG is given by

$$T_{r_b} = \frac{1}{2}\Omega^T I_B \Omega \tag{2.23}$$

Rotational kinetic energy of both left & right wheel is given by

$$T_{r_w} = \frac{1}{2}\Omega_L^T I_W \Omega_L + \frac{1}{2}\Omega_R^T I_W \Omega_R \tag{2.24}$$

Total Translation kinetic energy is given by

$$T_t = T_{t_b} + T_{t_w} \tag{2.25}$$

Total Rotational kinetic energy is given by

$$T_r = T_{r_b} + T_{r_w} \tag{2.26}$$

Total Kinetic energy is given by

$$T = T_t + T_r \tag{2.27}$$

Solving Eq^n (2.14) to (2.27), we get total kinetic energy is:

$$\begin{aligned} T \quad = \quad & \left(\frac{I_{Wa} + m_W r^2}{2}\right)\left(\dot{\theta}_L^2 + \dot{\theta}_R^2\right) + \frac{m_B}{2}\left(\dot{x}_C^2 + \dot{y}_C^2\right) \\ & + \left(\frac{I_y + m_B l^2}{2}\right)\dot{\theta}^2 + \left(Iw_d + \frac{I_x \sin^2\theta}{2} - \frac{m_B l^2 \cos^2\theta}{2}\right)\dot{\psi}^2 \\ & + m_B l^2 \cos\psi \cos\theta \dot{x}_c \dot{\theta} + m_B l \sin\theta \cos\psi \dot{\psi} \dot{y}_C \\ & + m_B l \sin\psi \cos\theta \dot{y}_C \dot{\theta} - m_B l \sin\psi \sin\theta \dot{\psi} \dot{x}_C \end{aligned} \tag{2.28}$$

Total Potential energy is given by

$$V = m_B g l \cos\theta \tag{2.29}$$

The Lagrangian is given by

$$\mathcal{L} = T - V \tag{2.30}$$

Solve Eq^{n}(2.28) to (2.30), we get Lagrangian:

$$\begin{aligned}\mathcal{L} = & \left(\frac{I_{Wa} + m_W r^2}{2}\right)\left(\dot{\theta}_L^2 + \dot{\theta}_R^2\right) + \frac{m_B}{2}\left(\dot{x}_C^2 + \dot{y}_C^2\right) + \left(\frac{I_y + m_B l^2}{2}\right)\dot{\theta}^2 \\ & + \left(Iw_d + \frac{I_x \sin^2\theta}{2} - \frac{m_B l^2 \cos^2\theta}{2}\right)\dot{\psi}^2 + m_B l^2 \cos\psi \cos\theta \dot{x}_c \dot{\theta} \\ & + m_B l \sin\theta \cos\psi \psi \dot{\psi} \dot{y}_C + m_B l \sin\psi \cos\theta \dot{y}_C \dot{\theta} - m_B l \sin\psi \sin\theta \dot{\psi} \dot{x}_C \\ & - m_B g l\, cos\,\theta \end{aligned} \tag{2.31}$$

This Documentation is under progress, I am working on this