Analysis I Solutions

Hannes Thorsell

August 13, 2022

1

Introduction

1.1 What is analysis?

No exercises in this section.

1.2 Why do analysis?

No exercises in this section.

2

Starting at the beginning: the natural numbers

2.1 The Peano axioms

No exercises in this section.

2.2 Addition

Ex. 2.2.1. Fix b and c and induct on a. For $a = 0$ we have $(0 + b) + c = b + c = b + (c + 0)$, proving the base case. Now suppose $(a+b)+c = a+(b+c)$. Then,

$$\begin{aligned}
(a{+\!+}\, b) + c &= (a + b){+\!+} \ + c \\
&= ((a + b) + c){+\!+} \\
&= (a + (b + c)){+\!+} \\
&= a{+\!+} \ + (b + c).
\end{aligned}$$

■

Ex. 2.2.2. First we show existence by induction, starting at $a = 1$. We have $1 = 0{+\!+}$ by definition. Now suppose there exists a b such that $a = b{+\!+}$. Then $a{+\!+} = (b{+\!+})\,{+\!+}$ will do. Now for uniqueness. If $a = b{+\!+}$ and $a = b'{+\!+}$ then by Axiom 2.4 we have $b = b'$.

■

Ex. 2.2.3. (a) Since $a = a + 0$ we have $a \geq a$.

(b) Since $a \geq b$ there exists a natural number m such that $a = b+m$. Similarly, since $b \geq c$ there exists a natural number n such that $b = c + n$. Thus, $a = b + m = c + (m + n)$ whence $a \geq c$.

(c) Since $a \geq b$ there exists a natural number m such that $a = b+m$. Similarly, since $b \geq a$ there exists a natural number n such that $b = a + n$. Thus, $a = b + m = a + m + n$ whence $m + n = 0$ by the Cancellation law, and so $m = n = 0$ by Corollary 2.2.9.

(d) If $a \geq b$ then $a = b + n$ for some natural number n. Thus $a + c = b + c + n$ whence $a + c \geq b + c$. Conversely, if $a + c \geq b + c$ then $a + c = b + c + n$ for some natural number n. Applying the Cancellation law yields $a = b + n$ whence $a \geq b$.

(e) If $a < b$ then $b = a + n$ for some natural number n. Note that $n \neq 0$ since then we would have $a = b$, which by definition is not the case. Since $n \neq 0$ we can, by Lemma 2.2.10. write $n = m++$ for some unique natural number m. Thus $b = a+m{++} = a{++} + m$ whence $a{++} \leq b$.

Conversely, if $a{++} \leq b$ then $b = a{++} + n$, so $b = a + n{++}$. The point being that $n \neq 0$ by Axiom 2.3 whence $a \neq b$, or in other words, $a < b$.

(f) If $a < b$ then $a{++} \leq b$ so that $b = a{++} + n = a + n{++}$. By Axioms 2.3 $n{++}$ is not zero, and so is positive.

Conversely, if $b = a+d$ for some $d > 0$ then by Lemma 2.2.10 there exists a unique c such that $c{++} = d$. Thus $b = a+c{++} = a{++} + c$, whence $a{++} \leq b$ and by the above exercise $a < b$.

■

Ex. 2.2.4. We have $0 \leq b$ for all b since $b = b+0$. If $a > b$ then since $a{++} > a$ we have $a{++} > b$ by transitivity. Finally, if $a = b$ then $a{++} = b{++} > b$ since for all a we have, again, $a{++} > a$.

■

Ex. 2.2.5. Define $Q(n)$ to be the property that $P(m)$ is true for all $m_0 \leq m < n$. We shall induct on n. We have that in the base case $n = 0$ the statement is true vacously. In fact, it is true vacously for all $n \leq m_0$. Now suppose $Q(n)$ holds for n, or in other words, $P(m)$ is true for all $m_0 \leq m < n$. By assumption this implies $P(m + 1)$ is true also, whence $Q(n+1)$ is true. Thus, by the induction principle $Q(n)$ holds for all n, whence $P(m)$ hold for all natural numbers $m \geq m_0$.

■

Ex. 2.2.6. We shall induct on n. If $n = 0$ then $P(0)$ is true by assumption and so $P(m)$ is true for all $m \leq 0$. Suppose now that $P(n{++})$ is true. Then by assumption $P(n)$ is true, and so by the induction hypothesis

$P(m)$ is true for all $m \leq n$. But $P(n{+}{+})$ was true also, so $P(m)$ is true for all $m \leq n{+}{+}$.

■

2.3 Multiplication

Ex. 2.3.1. First we show $m \times 0 = 0$. By induction on m we have for $m = 0$ that $0 \times 0 = 0$ by the definition of multiplication. If $m \times 0 = 0$ then $m{+}{+} \times 0 = (m \times 0) + 0 = 0$ by the induction hypothesis. Hence, $m \times 0 = 0$ for all m.

Next we show $m \times n{+}{+} = (m \times n) + m$ by induction on m. In the base case $m = 0$ we already have $0 \times n{+}{+} = 0 = (m \times 0) + 0$ from the paragraph above. Now suppose $m \times n{+}{+} = (m \times n) + m$. Then $m{+}{+} \times n{+}{+} = (m + n{+}{+}) + m$ by definition of multiplication, and by the induction hypothesis this equals $(m + n) + m + m$ which in turn is equal to $(m{+}{+} + n) + m{+}{+}$.

Finally we show multiplication is commutative. We induct on n. In the base case $m \times 0 = 0 \times m$. Suppose $m \times n = n \times m$. Then $m \times n{+}{+} = (m \times n) + m = (n \times m) + m = (n{+}{+} \times m)$.

■

Ex. 2.3.2. If $n = 0$ or $m = 0$ then $m \times 0 = 0 \times n = 0$, respectively, by the above exercise.

Conversely, suppose $m > 0$ and $n > 0$. We induct on m. In the base case we have $m = 1$ and $1 \times n = n > 0$. Now suppose $m \times n > 0$. Then $m{+}{+} \times n = (m \times n) + m > 0$, by the induction principle and Proposition 2.2.8.

■

Ex. 2.3.3. We fix b and c and induct on a. In the base case $a = 0$ and $(0 \times b) \times c = 0 = 0 \times (b \times c)$. Now suppose abc is unambiguous. Then $((a+1) \times b) \times c = abc + bc$ by the Distributive law. But also, $(a+1) \times (b \times c) = abc + bc$ by the Distributive law.

■

Ex. 2.3.4. We have $(a+b)^2 = (a+b)(a+b) = a(a+b) + b(a+b) = a^2 + 2ab + b^2$.

■

Ex. 2.3.5. We fix q and induct on n. In the base case $n = 0$ we have $m = 0$ and $r = 0$. Then $n = 0 = 0q + 0$ and $0 \leq r < q$ since q is positive.

Now suppose $n = mq + r$ and consider $n + 1$. Then $n + 1 = mq + r + 1$. If $r + 1 < q$ we are done. Otherwise $r = q$ and so $n = (m + 1)q + 0$.

■

3

Set theory

3.1 Fundamentals

Ex. 3.1.1. Let A, B, and C be sets. First, $A = A$ for all A since every element of A is an element of A. Second, if $A = B$ then A and B contain the exact same elements, and so $B = A$. Finally, if $A = B$ and $B = C$ then every element of A belong to B, and every element of B belongs to C. Hence every element of A belongs to C, and vice versa.

■

Ex. 3.1.2. We have $\varnothing \neq \{ \varnothing \}$ since the empty set has no elements, but $\{ \varnothing \}$ has one element, namely $\varnothing$. These are both distinct from the set $\{ \{ \varnothing \} \}$ since it contains the element $\{ \varnothing \}$ which both of the other sets lack. All three are distinct from $\{ \varnothing, \{ \varnothing \} \}$ since the latter has the element $\{ \varnothing \}$ which the first two sets lack, and the element $\varnothing$ which the third set lacks.

■

Ex. 3.1.3. First $\{ a, b \} = \{ a \} \cup \{ b \}$ immedtiately by Axiom 3.4. Next, $A \cup A = A \cup \varnothing = \varnothing \cup A = A$ by Axiom 3.4 again. Finally, commutativity follows from Axiom 3.4 and commutativity of the logical OR operation.

■

Ex. 3.1.4. If $A \subseteq B$ and $B \subseteq A$ then if $x \in A$ then $x \in B$, and if $x \in B$ then $x \in A$. This is exactly Definition 3.1.4. For the final statement we already know $A \subseteq C$, given $A \subsetneq B$ and $B \subsetneq C$. There is some element x in B that is not an element of A. Since $B \subseteq C$ (note carefully the symbol used!) we have $x \in C$ and so $A \subsetneq C$.

■

Ex. 3.1.5. If $A \subseteq B$ then if $x \in A$ also $x \in B$ and so $A \subseteq A \cap B$, whence $A = A \cap B$ for $A \cap B \subseteq A$ by definition.

If $A \cap B = A$ then if $x \in A \cup B$ then either $x \in B$ and $A \cup B \subseteq B$ or $x \in A = A \cap B$ and so $x \in B$ and again $A \cup B \subseteq B$. So $A \cup B = B$ since $B \subseteq A \cup B$ by definition.

Finally, if $A \cup B = B$ we have for any $x \in A$ also $x \in A \cup B = B$ and so $A \subseteq B$.

■

Ex. 3.1.6. (a) By Lemma 3.1.13 and definition of set intersection.

(b) By Exercise 3.1.5.

(c) By part (b) replacing X by A and noting $A \subseteq A$ for all A.

(d) By Exercise 3.3.3. as well as commutativity of AND operation combined wit the definition of set intersection.

(e) Associativity of unions and intersections follow from associativity of OR and AND, respectively, combined with the definitions of set union and set intersection.

(f) We show the first equality. The second equality is similar. If $x \in A \cap (B \cup C)$ then $x \in A$ and $x \in B \cup C$. If $x \in B$ then $x \in A \cap B$. If $x \in C$ then $x \in A \cap C$. In either case, $x \in (A \cap B) \cup (A \cap C)$. Thus, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. Conversely, if $x \in (A \cap B) \cup (A \cap C)$ then if $x \in A \cap B$ we have $x \in A \cap (B \cup C)$ for $x \in A$ and $x \in B$. If $x \in A \cap C$ then also $x \in A \cap (B \cup C)$ for $x \in A$ and $x \in C$. Regardless, $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. All in all, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(g) Clearly $A \cup (X - A) \subseteq X$ so we only need show $X \subseteq A \cup (X - A)$. But this is true, since if $x \in X$ then either $x \in A$ or $x \notin A$. In either case, $x \in A \cup (X - A)$.
Similarly, clearly $\varnothing \subseteq A \cap (X - A)$, and we need only show $A \cap (X - A)$ is empty. If $x \in A \cap (X - A)$ then $x \in A$ and $x \notin A$, which is impossible.

(h) We show the first law. The second one is analogous. If $x \in X - (A \cup B)$ then $x \in X$ and x is neither in A nor in B. Thus $x \in X - A$ and $x \in X - B$. Thus $x \in (X - A) \cap (X - B)$. Conversely, if $x \in (X - A) \cap (X - B)$ we have $x \in X$ regardless, but also $x \notin A$ and $x \notin B$, and so $x \in X - (A \cup B)$.

■

Ex. 3.1.7. If $x \in A \cap B$ then $x \in A$ and so $A \cap B \subseteq B$. Also, $x \in B$ and so $A \cap B \subseteq B$.

If $C \subseteq A$ and $C \subseteq B$ we have that if $x \in C$ then $x \in A$ and $x \in B$ and so $x \in A \cap B$. Thus, $C \subseteq A \cap B$. Conversely, if $C \subseteq A \cap B$ then if $x \in C$ then $x \in A$ and $x \in B$, and so $C \subseteq A$ and $C \subseteq B$.

The other case is done analogously.

■

Ex. 3.1.8. By the distributive property, $A \cap (A \cup B) = (A \cap A) \cup (A \cap B) = A \cup (A \cap B)$. Clearly $A \subseteq A \cup (A \cap B)$. But also, if $x \in A \cup (A \cap B)$ either $x \in A$ and we are done, or $x \in A \cap B$ and so $x \in A$ and we are done, for in both cases $x \in A$ and so $A \cup (A \cap B) \subseteq A$. All in all, $A = A \cup (A \cap B)$.

■

Ex. 3.1.9. Suppose $x \in A$. Then $x \notin B$ and so $x \in X$ since $X = A \cup B$. Thus $A \subseteq X - B$. Conversely, if $x \in X - B$ then $x \in X$ and $x \notin B$. Thus $x \in A$ since $X = A \cup B$. Hence, $X - B \subseteq A$. All in all, $A = X - B$. The other case is analogous.

■

Ex. 3.1.10. The sets $A - B$ and $A \cap B$ are disjoint for the first contains no elements from B (or is empty) and the second contains element of B (or is empty). The sets $A - B$ and $B - A$ are disjoint for the first contains elements of A (or is empty) and the second contains no elements from A (or is empty). The sets $A \cap B$ and $B - A$ are disjoint for the second contains elements from A (or is empty) and the first contains no elements from A (or is empty).

Now, we have $(A - B) \cup (A \cap B) \cup (B - A) = A \cup B$ since any element of the LHS is an element of exactly one of the three sets, and in either case it must belong to either A, B, or both. Similarly any element of $A \cup B$ belong either to A, B, or both, and is thus an element of exactly one of the three sets on the LHS.

■

Ex. 3.1.11. Let A be a set, and for each $x \in A$ let $P(x)$ be a property pertaining to x. Let us form the statement $Q(x, y)$ which means $P(x)$ is true and $x = y$. Then by the Axiom of Replacement we replace each x by y (which is equal to x) such that $P(x)$ is true. This is nothing but the Axiom of Specification.

■

3.2 Russel's paradox (Optional)

Ex. 3.2.1 For each axiom we need only specify predicate P. So for Axiom 3.2 let $P(x)$ be "x is a dog and x is not a dog.". h(z) = For Axiom 3.3 let P be $P(x) = y$ for the singleton set $\{ y \}$. For Axiom 3.4 let $P(x)$ be $x \in A$ or $x \in B$. Axiom 3.5 is already selected based on a predicate, we need only add $P(x)$ to be $x \in A$ and x satisfies whatever predicate Q that Axiom 3.5 uses. For Axiom 3.6 let $P(x)$ mean just as in the previous case, but add again that $x \in A$.

■

Ex. 3.2.2 Let A be a set. If $A \in A$ but A is a set, then we can form the singleton set $\{ A \}$. But by the regularity axiom we would require that one element of $\{ A \}$ either not be a set, which isn't the case, or disjoint from $\{ A \}$, which is not the case.

■

Ex. 3.2.3 The existence of the universal set is given by a vacuous predicate P. Now, suppose we can form a universal set Ω. Then by the axiom of specification the predicate P would then be exactly the predicate described in the axiom of universal specification, operating on this universal set Ω.

■

3.3 Functions

Ex. 3.3.1. The definition is reflexive, for $f = f$ for all f, since f's domain it equal to f's domain, and $f(x) = f(x)$ for all $x \in X$. Symmetry follows from symmetry of set equality and equality of objects in the image of f. If $f = g$ and $g = h$, then f, g, and h all have the same domain and range from transitivity of set equality, and $f(x) = g(x) = h(x)$ from transitivity of the objects in the image of f (or g or h).

Finally, for substitution, the domain of $g \circ f$ is the same as $\tilde{g} \circ \tilde{f}$ by set substitution, and the same is true of the domain. Moreover, if $f = \tilde{f}$ and $g = \tilde{g}$ then by substitution in the range the function compositions must be equal.

Note: All of the above relies on equality defined on the range of f, g, $\tilde{f}$, and $\tilde{g}$ being well-defined, i.e. satisfying the substitution axiom.

■

Ex. 3.3.2. If $x \neq x'$ then $f(x) \neq f(x')$ since f is injective. Thus, since g is also injective, $g(f(x)) \neq g(f(x'))$. The case is similar for surjection.

■

Ex. 3.3.3. The empty function is vacuously injective. It is surjective (and bijective) if and only if its range is specified as the empty set.

■

Ex. 3.3.4. Suppose $g \circ f = g \circ \tilde{f}$. The functions f and $\tilde{f}$ have the same domain and range. Since g is injective, $g \circ f(x) = g \circ \tilde{f}(x)$ implies $f(x) = \tilde{f}(x)$ for all $x \in X$. It does not hold if we drop the assumption that g is injective, for then f and $\tilde{f}$ may map some x to different elements which g in turn maps to the same element.

The other case is analogous, with the requirement that f being surjective being mandatory as well.

■

Ex. 3.3.5. If f is not injective, $f(x) = f(x')$ for some $x \neq x'$. But then $g \circ f(x) = g \circ f(x')$ as well, so it is not injective. It is not necessarily true that g also is injective.

If g is not surjective then immediately $g \circ f$ is not surjective. It is not necessarily true that f also is surjective.

■

Ex. 3.3.6. Let $f(x) = y$. Then $f^{-1}(y) = x$ by definition. Thus, $f^{-1}(y) = f^{-1}(f(x)) = x$. Similarly $f(f^{-1}(y)) = y$. We conclude by letting $h = (f^{-1})^{-1}$ and noting that $f^{-1} \circ h(x) = f^{-1} \circ (f^{-1})^{-1}(x) = x$ so that $h = f$. (Here we used the uniqueness of the inverse, but this can easily be shown).

■

Ex. 3.3.7. The function $g \circ f$ is bijective from Exercise 3.3.2. We also have $g \circ f)^{-1} \circ (g \circ f) = I$ where I is the identity function. Thus, by cancelling on the right, we get $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

■

Ex. 3.3.8. (a) Suppose $X \subseteq Y \subseteq Z$. Then $\imath_{Y\to Z}(\imath_{X\to Y}(x)) = \imath_{Y\to Z}(x) = x$.

(b) The domains and ranges being equal is immediate. Now, let $x \in A$. Then $f(\imath_{A\to A}(x)) = f(x) = \imath_{B\to B}(f(x))$.

(c) This is just Exercise 3.3.6.

(d) Define

$$h(z) = \begin{cases} f(z) & z \in A, \\ g(z) & z \in B. \end{cases}$$

This is well-defined since X and Y are disjoint. The range of h is indeed Z since the range of both f and g is Z.

■

3.4 Images and inverse images

Ex. 3.4.1. To avoid confusion let $f^{-1}(V)$ denote the image of V under the inverse of f, whilst letting g denote the inverse image of V under f.

If $x \in f^{-1}(V)$ then there exists a $y \in V$ such that $f^{-1}(y) = x$, or equivalently $y = f(x)$. But then $x \in g(V)$, by definition, and vice versa.

■

Ex. 3.4.2. We have $S \subseteq f^{-1}(f(S))$ since if $x \in S$ then $f(x) \in f(S)$, and so $x \in f^{-1}(f(S))$. We also have $f(f^{-1}(U)) \subseteq U$ since if $y \in f(f^{-1}(U))$ then there exists an $x \in f^{-1}(U)$ such that $y = f(x)$, and so $y \in U$.

■

Ex. 3.4.3. First we show $f(A \cap B) \subseteq f(A) \cap f(B)$. Suppose $f(x) \in f(A \cap B)$. Then $x \in A$ and $x \in B$. Thus $f(x) \in f(A)$ and $f(x) \in f(B)$, so $f(x) \in f(A) \cap f(B)$. This cannot be strengthened to an equality. To see why, consider $A = \{ 0 \}$ and $B = \{ 1 \}$, and define $f : \{ 0, 1 \} \to \{ 0 \}$ where $f : 0 \mapsto 0$ and $f : 1 \mapsto 0$. Then $f(A \cap B) = f(\varnothing) = \varnothing \neq \{ 0 \} = \{ 0 \} \cap \{ 0 \} = f(A) \cap f(B)$.

Second we show that $f(A) - f(B) \subseteq f(A - B)$. Suppose $f(x) \in f(A) - f(B)$. Then $x \in A$ but $x \notin B$. Thus $f(x) \in f(A - B)$. This cannot be strengthened to eqaulity. Consider the above example again, from which we get $\varnothing \neq \{ 0 \}$.

Finally, we show that $f(A \cup B) = f(A) \cup f(B)$. Suppose $f(x) \in f(A \cup B)$, and let $x \in A$, say. Then $f(x) \in f(A) \subseteq f(A) \cup f(B)$. Conversely, suppose $f(x) \in f(A)$, say. Then $x \in A \subseteq A \cup B$ and so $f(x) \in f(A \cup B)$.

■

Ex. 3.4.4. We show only the first claim; the others are analogous.

Suppose $x \in f^{-1}(U \cup V)$. Then $f(x) = y$ for some $y \in U \cup V$, say $y \in U$. Consequently, $x \in f^{-1}(U)$, by definition, and so $x \in f^{-1}(U) \cup f^{-1}(V)$.

Conversely, if $x \in f^{-1}(U) \cup f^{-1}(V)$, say $x \in f^{-1}(U)$, then there exists a $y \in U \subseteq U \cup V$ such that $f(x) = y$. Thus, by definition, $x \in f^{-1}(U \cup V)$.

■

Ex. 3.4.5. By Exercise 3.4.2 we need only show $f^{-1}(f(S)) \subseteq S$ if f is injective, and $S \subseteq f(f^{-1}(S))$ if f is surjective.

For the first statement, suppose $x \in f^{-1}(f(S))$. Then $f(x) \in f(S)$. Suppose now that $x \notin S$. Since f is injective there exists no $x' \in S$

such that $f(x') = f(x)$, but then we would have $f(x) \notin f(S)$. Thus we must have $x \in S$.

For the second statement, suppose $y \in S$. Then there exists an $x \in f^{-1}(S)$ such that $f(x) = y$, since f is surjective. But then $y \in f(f^{-1}(S))$.

■

Ex. 3.4.6. Simply define $A = \left\{ f^{-1}(\{ 1 \}) \,\middle|\, f \in \{ 0, 1 \}^X \right\}$. This is precisely the power set of X, as there is a one-to-one correspondence between the functions (essentially mapping a truth value whether or not an element is contained in the subset) to subsets of X. For example, $f : x \mapsto 0$ for all $x \in X$ is the empty set and $f : x \mapsto 1$ for all $x \in X$ is the entire set X.

■

Ex. 3.4.7. Simply define the set $\left\{ {Y'}^{X'} \,\middle|\, X' \in P(X), Y' \in P(Y) \right\}$, where $P(A)$ denotes the power set of A. This uses the previous exercise and the axiom of specification.

■

Ex. 3.4.8. Let A and B be sets. Form the pair $\{ A, B \}$. Then, by Axiom 3.11, $A \cup B = \bigcup \{ A, B \}$.

■

Ex. 3.4.9. If say $y \in \{ x \in A_\beta \mid x \in A_\alpha \text{ for all } \alpha \in I \}$, then particularly $y \in A_{\beta'}$, and vice versa.

The equality (3.4) holds for the same reason.

■

Ex. 3.4.10. For the first part suppose $x \in \left(\bigcup_{\alpha \in I} A_\alpha\right) \cup \left(\bigcup_{\alpha \in J} A_\alpha\right)$, say $x \in \bigcup_{\alpha \in I} A_\alpha$. Without loss of generality let $x \in A_\beta$ for some particular $\beta \in I \subseteq I \cup J$. Then $x \in \bigcup_{\alpha \in I \cup J} A_\alpha$.

Conversely, if $x \in \bigcup_{\alpha \in I \cup J} A_\alpha$ we have $x \in A_\beta$ where $\beta \in I \cup J$, say $\beta \in I$. Then $x \in \bigcup_{\alpha \in I} A_\alpha \subseteq \left(\bigcup_{\alpha \in I} A_\alpha\right) \cup \left(\bigcup_{\alpha \in J} A_\alpha\right)$.

For the second part suppose I and J are non-empty (In order to not violate ZF). Then if $x \in \left(\bigcap_{\alpha \in I} A_\alpha\right) \cap \left(\bigcap_{\alpha \in J} A_\alpha\right)$ we have $x \in A_\beta$ for all $\beta \in I$ and $x \in A_{\beta'}$ for all $\beta' \in J$. Thus $x \in A_\gamma$ for all γ in $I \cup J$ and so $x \in \bigcap_{\gamma \in I \cup J} A_\gamma$.

Conversely, if $x \in \bigcap_{\alpha \in I \cup J} A_\alpha$ then $x \in A_\gamma$ for all $\gamma \in I \cup J$, that is, for all $\alpha \in I$ and all $\beta \in J$, we have $x \in A_\alpha$ and $x \in A_\beta$. But this means that $x \in \left(\bigcap_{\alpha \in I} A_\alpha\right) \cap \left(\bigcap_{\alpha \in J} A_\alpha\right)$.

■

Ex. 3.4.11. We show only the first claim; the second is analogous.

If $x \in X - \bigcup_{\alpha \in I} A_\alpha$, then $x \in X$ but $x \notin A_\alpha$ for *any* $\alpha \in I$. Thus $x \in X - A_\alpha$ for all $\alpha \in I$, or in other words, $x \in \bigcap_{\alpha \in I}(X - A_\alpha)$.

Conversely, if $x \in \bigcap_{\alpha \in I}(X - A_\alpha)$ then $x \in X - A_\alpha$ for all $\alpha \in I$ which means $x \notin A_\alpha$ for *any* $\alpha \in I$. Thus, $x \notin \bigcup_{\alpha \in I} A_\alpha$ and so $x \in X - \bigcup_{\alpha \in I} A_\alpha$.

■

3.5 Cartesian products

3.6 Cardinality of sets