Analysis I Solutions

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Introduction

1.1 What is analysis?

No exercises in this section.

1.2 Why do analysis?

No exercises in this section.

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Starting at the beginning: the natural numbers

2.1 The Peano axioms

No exercises in this section.

2.2 Addition

Ex. 2.2.1. Fix b and c and induct on a. For $a = 0$ we have $(0 + b) + c = b + c = b+(c+0)$, proving the base case. Now suppose $(a+b)+c = a+(b+c)$. Then,

$$\begin{aligned}(a{+\!+}\, b) + c &= (a + b){+\!+}\ + c \\ &= ((a + b) + c){+\!+} \\ &= (a + (b + c)){+\!+} \\ &= a{+\!+}\ + (b + c).\end{aligned}$$

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Ex. 2.2.2. First we show existence by induction, starting at $a = 1$. We have $1 = 0{+\!+}$ by definition. Now suppose there exists a b such that $a = b{+\!+}$. Then $a{+\!+} = (b{+\!+}\ ){+\!+}$ will do. Now for uniqueness. If $a = b{+\!+}$ and $a = b'{+\!+}$ then by Axiom 2.4 we have $b = b'$.

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Ex. 2.2.3. (a) Since $a = a + 0$ we have $a \geq a$.

(b) Since $a \geq b$ there exists a natural number m such that $a = b+m$. Similarly, since $b \geq c$ there exists a natural number n such that $b = c + n$. Thus, $a = b + m = c + (m + n)$ whence $a \geq c$.

(c) Since $a \geq b$ there exists a natural number m such that $a = b+m$. Similarly, since $b \geq a$ there exists a natural number n such that $b = a + n$. Thus, $a = b + m = a + m + n$ whence $m + n = 0$ by the Cancellation law, and so $m = n = 0$ by Corollary 2.2.9.

(d) If $a \geq b$ then $a = b + n$ for some natural number n. Thus $a + c = b + c + n$ whence $a + c \geq b + c$. Conversely, if $a + c \geq b + c$ then $a + c = b + c + n$ for some natural number n. Applying the Cancellation law yields $a = b + n$ whence $a \geq b$.

(e) If $a < b$ then $b = a + n$ for some natural number n. Note that $n \neq 0$ since then we would have $a = b$, which by definition is not the case. Since $n \neq 0$ we can, by Lemma 2.2.10. write $n = m++$ for some unique natural number m. Thus $b = a+m{++} = a{++} +m$ whence $a{++} \leq b$.

Conversely, if $a{++} \leq b$ then $b = a{++} + n$, so $b = a + n{++}$. The point being that $n \neq 0$ by Axiom 2.3 whence $a \neq b$, or in other words, $a < b$.

(f) If $a < b$ then $a{++} \leq b$ so that $b = a{++} + n = a + n{++}$. By Axioms 2.3 $n{++}$ is not zero, and so is positive.

Conversely, if $b = a+d$ for some $d > 0$ then by Lemma 2.2.10 there exists a unique c such that $c{++} = d$. Thus $b = a+c{++} = a{++} +c$, whence $a{++} \leq b$ and by the above exercise $a < b$.

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Ex. 2.2.4. We have $0 \leq b$ for all b since $b = b+0$. If $a > b$ then since $a{++} > a$ we have $a{++} > b$ by transitivity. Finally, if $a = b$ then $a{++} = b{++} > b$ since for all a we have, again, $a{++} > a$.

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Ex. 2.2.5. Define $Q(n)$ to be the property that $P(m)$ is true for all $m_0 \leq m < n$. We shall induct on n. We have that in the base case $n = 0$ the statement is true vacously. In fact, it is true vacously for all $n \leq m_0$. Now suppose $Q(n)$ holds for n, or in other words, $P(m)$ is true for all $m_0 \leq m < n$. By assumption this implies $P(m + 1)$ is true also, whence $Q(n+1)$ is true. Thus, by the induction principle $Q(n)$ holds for all n, whence $P(m)$ hold for all natural numbers $m \geq m_0$.

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Ex. 2.2.6. We shall induct on n. If $n = 0$ then $P(0)$ is true by assumption and so $P(m)$ is true for all $m \leq 0$. Suppose now that $P(n{++}\ )$ is true. Then by assumption $P(n)$ is true, and so by the induction hypothesis

$P(m)$ is true for all $m \leq n$. But $P(n{+}{+})$ was true also, so $P(m)$ is true for all $m \leq n{+}{+}$.

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2.3 Multiplication

Ex. 2.3.1. First we show $m \times 0 = 0$. By induction on m we have for $m = 0$ that $0 \times 0 = 0$ by the definition of multiplication. If $m \times 0 = 0$ then $m{+}{+} \times 0 = (m \times 0) + 0 = 0$ by the induction hypothesis. Hence, $m \times 0 = 0$ for all m.

Next we show $m \times n{+}{+} = (m \times n) + m$ by induction on m. In the base case $m = 0$ we already have $0 \times n{+}{+} = 0 = (m \times 0) + 0$ from the paragraph above. Now suppose $m \times n{+}{+} = (m \times n) + m$. Then $m{+}{+} \times n{+}{+} = (m + n{+}{+}) + m$ by definition of multiplication, and by the induction hypothesis this equals $(m + n) + m + m$ which in turn is equal to $(m{+}{+} + n) + m{+}{+}$.

Finally we show multiplication is commutative. We induct on n. In the base case $m \times 0 = 0 \times m$. Suppose $m \times n = n \times m$. Then $m \times n{+}{+} = (m \times n) + m = (n \times m) + m = (n{+}{+} \times m)$.

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Ex. 2.3.2. If $n = 0$ or $m = 0$ then $m \times 0 = 0 \times n = 0$, respectively, by the above exercise.

Conversely, suppose $m > 0$ and $n > 0$. We induct on m. In the base case we have $m = 1$ and $1 \times n = n > 0$. Now suppose $m \times n > 0$. Then $m{+}{+} \times n = (m \times n) + m > 0$, by the induction principle and Proposition 2.2.8.

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Ex. 2.3.3. We fix b and c and induct on a. In the base case $a = 0$ and $(0 \times b) \times c = 0 = 0 \times (b \times c)$. Now suppose abc is unambiguous. Then $((a+1) \times b) \times c = abc+bc$ by the Distributive law. But also, $(a+1) \times (b \times c) = abc+bc$ by the Distributive law.

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Ex. 2.3.4. We have $(a+b)^2 = (a+b)(a+b) = a(a+b)+b(a+b) = a^2+2ab+b^2$.

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Ex. 2.3.5. We fix q and induct on n. In the base case $n = 0$ we have $m = 0$ and $r = 0$. Then $n = 0 = 0q + 0$ and $0 \leq r < q$ since q is positive.

Now suppose $n = mq+r$ and consider $n+1$. Then $n+1 = mq+r+1$. If $r + 1 < q$ we are done. Otherwise $r = q$ and so $n = (m + 1)q + 0$.

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