Near Equilibirum Dynamics and Transitions in Complex Network-Coupled Systems

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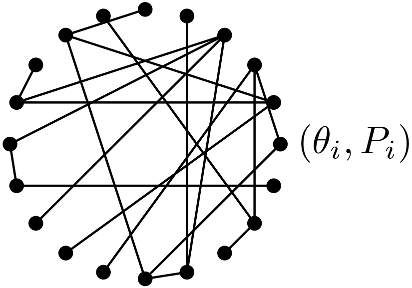
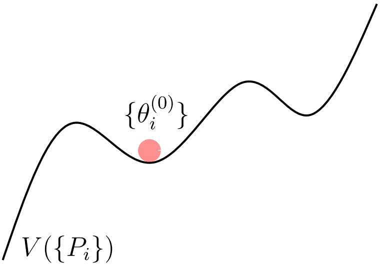
October 17, 2019

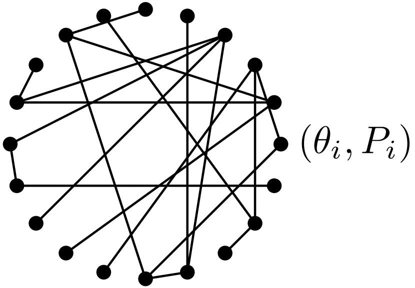
MT,Coletta and Jacquod, Phys. Rev. Lett. **120**, 084101 (2018).
MT,Pagnier and Jacquod to appear in *Science Advances* (2019), arXiv:1810.09694.
MT, Delabays and Jacquod, Phys. Rev. E **99**, 062213 (2019).
MT and Jacquod, Phys. Rev. E **100**, 032303 (2019).

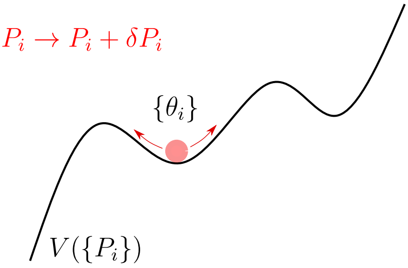
Introduction



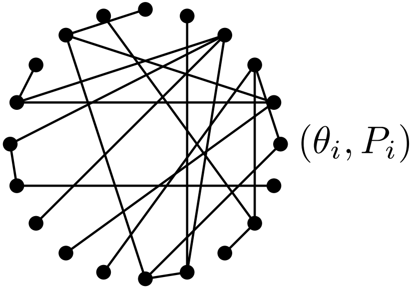


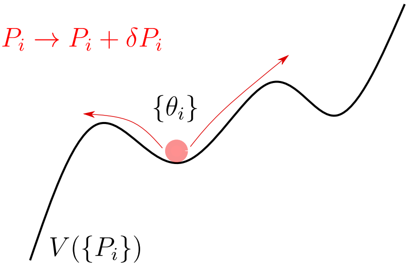
Introduction – Near Equilibrium Dynamics





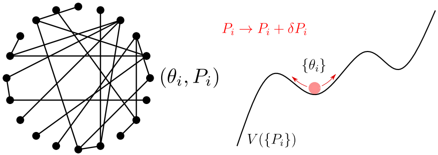
Introduction – Transitions





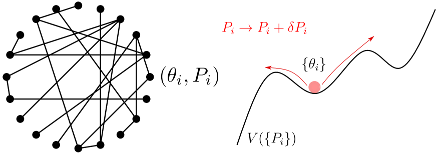
Near equilibrium dynamics

- Complex network metric vs. response to external perturbations
- Local vulnerabilities vs. global robustness

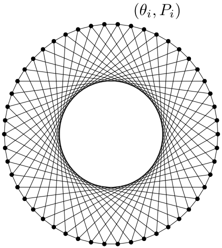


Transitions

- Estimate survival probability vs. shape of the basin of attraction

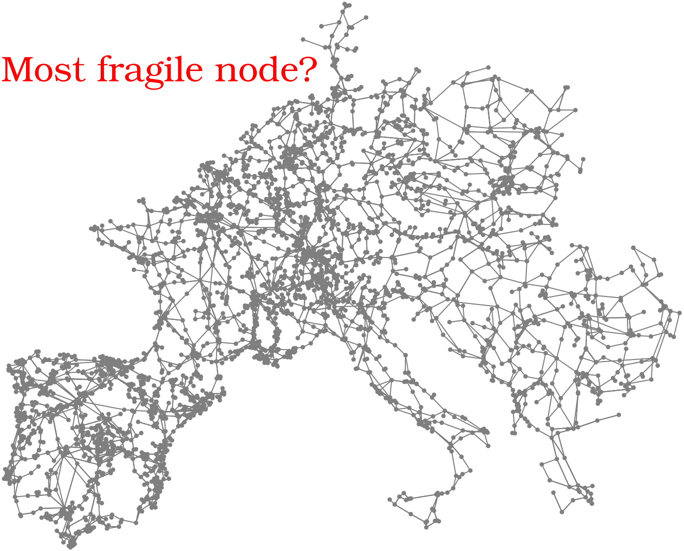


Most fragile network?



MT,Coletta and Jacquod, Phys. Rev. Lett. **120** 084101 (2018).

Near Equilibrium Dynamics – Question 2



MT,Pagnier and Jacquod to appear in *Science Advances*, arXiv:1810.09694.

Complex Networks Metrics

Centrality Measures (Local)

- Geodesic Distance
- PageRank
- Katz centrality
- Harmonic
- Degree
- Communicability
- Betweenness Centrality
- ...

Graph Measures (Global)

- Average Geodesic Distance
- Degree Heterogeneity
- Average Degree
- Degree Distribution: Scale-Free...
- Clustering Coefficient
- A,Di-ssortativity
- ...

P. Boldi and S. Vigna, Internet Mathematics **10**, 222 (2014).

Complex Networks Metrics $\leftrightarrow$ Coupled Dynamical System

Centrality Measures (Local)

- Geodesic Distance
- PageRank $\rightarrow$ RWs on complex networks.
- Katz centrality
- Harmonic/Closeness
- Degree
- Communicability $\rightarrow$ R diff. processes on complex networks.
- Betweenness Centrality
- ...

Graph Measures (Global)

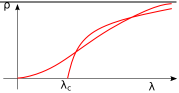
- Average Geodesic Distance
- Degree Heterogeneity
- Average Degree
- Degree Distribution: Scale-Free... $\tilde{\rightarrow}$ SIS transition.
- Clustering Coefficient
- A,Di-ssortativity
- ...

P. Boldi and S. Vigna, Internet Mathematics **10**, 222 (2014).

Complex Networks Metrics $\leftrightarrow$ Coupled Dynamical System

Degree Distribution: Scale-Free... $\tilde{\rightarrow}$ SIS transition.

VOLUME 86, NUMBER 14 PHYSICAL REVIEW LETTERS 2 APRIL 2001



Epidemic Spreading in Scale-Free Networks

Romualdo Pastor-Satorras[1] and Alessandro Vespignani[2]

[1]*Departament de Física i Enginyeria Nuclear, Universitat Politècnica de Catalunya, Campus Nord, Mòdul B4, 08034 Barcelona, Spain*

[2]*The Abdus Salam International Centre for Theoretical Physics (ICTP), P.O. Box 586, 34100 Trieste, Italy*

(Received 20 October 2000)

Communicability $\rightarrow$ R diff. processes on complex networks.

PHYSICAL REVIEW E **77**, 036111 (2008)

Communicability in complex networks

$$G_{pq} = \sum_{k=0}^{\infty} \frac{(\mathbf{A}^k)_{pq}}{k!} = (e^{\mathbf{A}})_{pq}.$$

Ernesto Estrada[1,*] and Naomichi Hatano[2]

[1]*Complex Systems Research Group, X-rays Unit, RIAIDT, Edificio CACTUS, University of Santiago de Compostela, 15076 Santiago de Compostela, Spain*

[2]*Institute of Industrial Science, University of Tokyo, 4-6-1 Komaba, Meguro, Tokyo 153-8505, Japan*

(Received 21 August 2007; published 11 March 2008)

PageRank $\rightarrow$ RWs on complex networks.

The PageRank Citation Ranking: Bringing Order to the Web

L. Page, S. Brin, R. Motwani and T. Winograd

January 29, 1998

Complex Networks Metrics $\leftrightarrow$ Coupled Dynamical System

Centrality Measures (Local)

- Geodesic Distance
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Graph Measures (Global)

- Average Geodesic Distance
- Degree Heterogeneity
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- Degree Distribution: Scale-Free...
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- A,Di-ssortativity
- ...

Not related to particular network dynamics... can be misleading.

P. Boldi and S. Vigna, Internet Mathematics **10**, 222 (2014).
P. Hines, E. Cortilla-Sanchez and S. Blumsack, Chaos **22**, 033122 (2010).

Consensus Algorithms

First-Order Consensus

$$\dot{x}_i = -\sum_j b_{ij}(x_i - x_j) \ ,$$

Second-Order Consensus

$$\ddot{x}_i + \gamma\dot{x}_i = -\sum_j b_{ij}(x_i - x_j) - \sum_j b'_{ij}(\dot{x}_i - \dot{x}_j) \ ,$$

$$\ddot{\mathbf{x}} + \gamma\dot{\mathbf{x}} = -\mathbb{L}\,\mathbf{x} - \mathbb{L}'\,\dot{\mathbf{x}} \ ,$$

$b_{ij} = b_{ji}$ and $b_{ij} \geq 0\,.$

H. G. Oral, E. Mallada and D. F. Gayme, IEEE 56th CDC (2017)

Leaders-Followers Dynamics

Opinion Dynamics

$$\dot{x}_i = -\sum_j b_{ij}(x_i - x_j) - \kappa(x_i - P_i)\,, \quad i \in V_l\,,$$
$$\dot{x}_i = -\sum_j b_{ij}(x_i - x_j)\,, \quad i \in V_f\,,$$

$b_{ij} = b_{ji}$ and $b_{ij} \geq 0$.
V_l : set of leaders.
V_f : set of followers.

$$\dot{\mathbf{x}} = -(\mathbb{L} + K)\,\mathbf{x} + \mathbf{P}^{\kappa}\,.$$

S. Patterson, Y. Yi, and Z. Zhang, IEEE TCNS (2018).
F. Baumann, I. M. Sokolov, and MT, arXiv:1910.01897 (2019).

Second-order Kuramoto (swing equations in the lossless line limit)

$$m_i\ddot{\theta}_i + d_i\dot{\theta}_i = P_i - \sum_j b_{ij}\sin(\theta_i - \theta_j) \;\;,\;\; i = 1, ..., n.$$

$b_{ij} = b_{ji} \geq 0$.
P_i : natural frequencies.
m_i : inertia.
d_i : damping.

J. A. Acebrón, L. L. Bonilla, Conrad J. Pérez Vicente, F. Ritort, and R. Spigler, Rev. Mod. Phys. **77**, 137 (2005)

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$b_{ij} = b_{ji} \geq 0$.

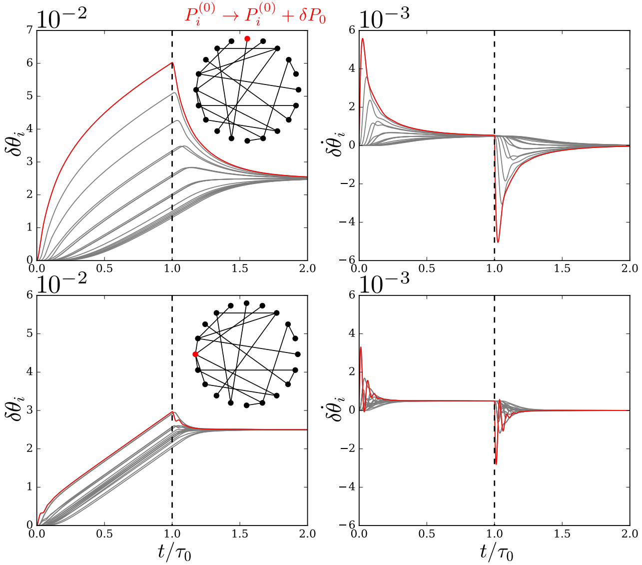
Steady-state solutions Synchronous state $\{\theta_i^{(0)}\}$ such that:

$$P_i = \sum_j b_{ij}\sin(\theta_i^{(0)} - \theta_j^{(0)}) \;\;,\;\; i = 1, ..., n.$$

$\sum_i P_i = 0.$

Perturbations $P_i \to P_i^{(0)} + \delta P_i(t)$.

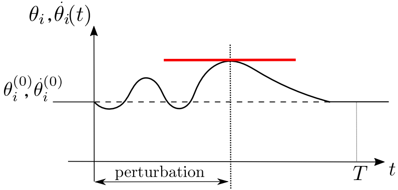
Coupled Dynamical Systems: Example



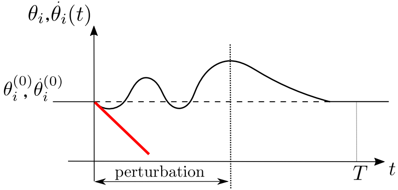
MT and Jacquod, Phys. Rev. E **100**, 032303 (2019).

Quantifying Robustness

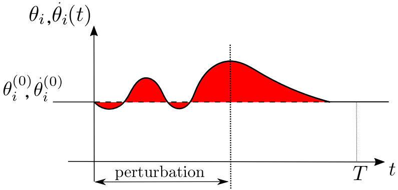
- Maximum of the response, $\max_t(\theta_i)$.



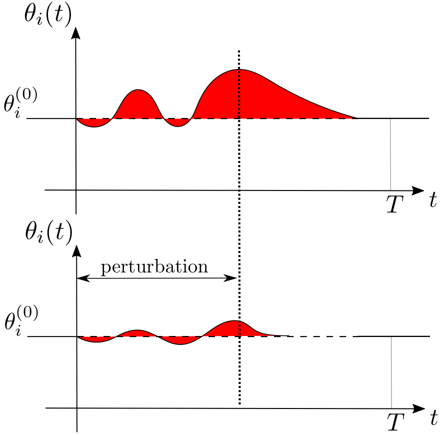
- Rate of change of frequency (RoCoF), $\ddot{\theta}_i$.



- Performance measure (quadratic integrals over the transient).



Performance measures



$$\mathcal{P}_1(T) = \sum_i \int_0^T |\theta_i(t) - \theta_i^{(0)}|^2 \mathrm{d}t \,,$$

$$\mathcal{P}_2(T) = \sum_i \int_0^T |\dot\theta_i(t) - \dot\theta_i^{(0)}|^2 \mathrm{d}t \,.$$

$$\mathcal{P}_{1,2}^{\infty} = \mathcal{P}_{1,2}(T \to \infty)\,.$$

Noisy disturbances $\to$ divide by T.

Perturbations : $P_i \to P_i^{(0)} + \delta P_i(t)$.

Response to Perturbations: Linearization

Linear response Perturbation of the natural frequencies.

- $P_i(t) = P_i^{(0)} + \delta P_i(t) \to \theta_i(t) = \theta_i^{(0)} + \delta\theta_i(t)$:

$$m\delta\ddot{\boldsymbol{\theta}}(t) + d\delta\dot{\boldsymbol{\theta}}(t) = \delta\boldsymbol{P}(t) - \mathbb{L}(\{\theta_i^{(0)}\})\delta\boldsymbol{\theta}(t) \ ,$$

$\mathbb{L}(\{\theta_i^{(0)}\})$: the weighted Laplacian matrix,

$$\mathbb{L}_{ij}(\{\theta_i^{(0)}\}) = \begin{cases} -b_{ij}\cos(\theta_i^{(0)} - \theta_j^{(0)}) \, , & i \neq j \, , \\ \sum_k b_{ik}\cos(\theta_i^{(0)} - \theta_k^{(0)}) \, , & i = j \, . \end{cases}$$

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Topology $\rightarrow b_{ij}$.

Steady state $\rightarrow \{\theta_i^{(0)}\}$.

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Topology $\to b_{ij}$.
Steady state $\to \{\theta_i^{(0)}\}$.
Linear response $\to$ **Can be applied to linear consensus, opinion dynamics models.**

Response to Perturbations: Linearization

Linear response

$$m\delta\ddot{\boldsymbol{\theta}}(t) + d\delta\dot{\boldsymbol{\theta}}(t) = \delta\boldsymbol{P}(t) - \mathbb{L}(\{\theta_i^{(0)}\})\delta\boldsymbol{\theta}(t) \ ,$$

Expanding on the eigenvectors $\mathbf{u}_\alpha$ of $\mathbb{L}$, we have $\delta\boldsymbol{\theta}(t) = \sum_\alpha c_\alpha(t)\mathbf{u}_\alpha$.

$$c_\alpha(t) = m^{-1}\, e^{-(\gamma+\Gamma_\alpha)t/2} \int_0^t e^{\Gamma_\alpha t_1} \int_0^{t_1} \delta\boldsymbol{P}(t_2)\cdot\mathbf{u}_\alpha\, e^{(\gamma-\Gamma_\alpha)t_2/2}\,\mathrm{d}t_2\mathrm{d}t_1 \ ,$$

$\gamma = d/m$ and $\Gamma_\alpha = \sqrt{\gamma^2 - 4\lambda_\alpha/m}$. $\rightarrow \mathcal{P}_1$, $\mathcal{P}_2$ for specific perturbations,

$$\begin{aligned} \mathcal{P}_1^\infty &= \sum_{\alpha\geq 2}\int_0^\infty c_\alpha^2(t)\mathrm{d}t \ , \\ \mathcal{P}_2^\infty &= \sum_{\alpha\geq 2}\int_0^\infty \dot{c}_\alpha^2(t)\mathrm{d}t \ . \end{aligned}$$

Response to Perturbations: Time Scales

Intrinsic Time Scales

- Individual elements: m/d.
- Network relaxation: d/λ_α with $\{\lambda_\alpha\}$ the eigenvalues of $\mathbb{L}$.

Perturbation Time Scale

- Correlation time of the external perturbation $\delta \boldsymbol{P}(t)$.

Quench perturbations

- $\delta P_i(t) = \delta P_{0i} \Theta(t) \Theta(\tau_0 - t)$.

Quench duration $\rightarrow \tau_0$.

Noisy time correlated perturbations

- $\langle \delta P_i(t) P_j(t') \rangle = \delta P_{0i}^2 \delta_{ij} \exp[-|t - t'|/\tau_0]$.

Correlation time $\rightarrow \tau_0$.

MT,Coletta and Jacquod, Phys. Rev. Lett. **120**, 084101 (2018).
MT,Pagnier and Jacquod to appear in *Science Advances*, arXiv:1810.09694.
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MT and Jacquod, Phys. Rev. E **100**, 032303 (2019).

Performance Measures

Performance measures for quench perturbations

$$
\begin{aligned}
\mathcal{P}_1^\infty &= \frac{m}{8\gamma}\sum_{\alpha\geq 2}\frac{(\delta \boldsymbol{P}_0\cdot\mathbf{u}_\alpha)^2}{\Gamma_\alpha\lambda_\alpha^3}\left[2\Gamma_\alpha(4\gamma\tau_0\lambda_\alpha/m-3\gamma^2-\Gamma_\alpha^2)\right.\\
&+ \left.(\gamma+\Gamma_\alpha)^3e^{-\tau_0\frac{(\gamma-\Gamma_\alpha)}{2}}-(\gamma-\Gamma_\alpha)^3e^{-\tau_0\frac{(\gamma+\Gamma_\alpha)}{2}}\right]\,,\\
\mathcal{P}_2^\infty &= \frac{1}{2d}\sum_{\alpha\geq 2}\frac{(\delta \boldsymbol{P}_0\cdot\mathbf{u}_\alpha)^2}{\Gamma_\alpha\lambda_\alpha}\left[2\Gamma_\alpha-(\gamma+\Gamma_\alpha)e^{-\frac{\tau_0(\gamma-\Gamma_\alpha)}{2}}\right.\\
&+ \left.(\gamma-\Gamma_\alpha)e^{-\frac{\tau_0(\gamma+\Gamma_\alpha)}{2}}\right]\,.
\end{aligned}
$$

MT and Jacquod, Phys. Rev. E **100**, 032303 (2019).

Performance Measures Asymptotics

Short duration $\tau_0 \ll m/d,\ (\gamma \pm \Gamma_\alpha)^{-1}$

$$\mathcal{P}_1^\infty = \frac{\tau_0^2}{2d} \sum_{\alpha \geq 2} \frac{(\delta \boldsymbol{P}_0 \cdot \mathbf{u}_\alpha)^2}{\lambda_\alpha}\,,$$

$$\mathcal{P}_2^\infty = \frac{\tau_0^2}{2md} \sum_{\alpha \geq 2} (\delta \boldsymbol{P}_0 \cdot \mathbf{u}_\alpha)^2\,.$$

Long duration $\tau_0 \gg m/d,\ (\gamma \pm \Gamma_\alpha)^{-1}$

$$\mathcal{P}_1^\infty = \tau_0 \sum_{\alpha \geq 2} \frac{(\delta \boldsymbol{P}_0 \cdot \mathbf{u}_\alpha)^2}{\lambda_\alpha^2}\,,$$

$$\mathcal{P}_2^\infty = d^{-1} \sum_{\alpha \geq 2} \frac{(\delta \boldsymbol{P}_0 \cdot \mathbf{u}_\alpha)^2}{\lambda_\alpha}.$$

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Local Vulnerability
Perturbing a specific node k i.e. $\delta P_{0i} = \delta_{ik}\delta P_0$,
$\tau_0 \ll m/d, (\gamma \pm \Gamma_\alpha)^{-1}$

$$\mathcal{P}_1^\infty(k) = \frac{\delta P_0^2 \tau_0^2}{2d} \sum_{\alpha\geq 2} \frac{u_{\alpha,k}^2}{\lambda_\alpha},$$
$$\mathcal{P}_2^\infty(k) = \frac{\delta P_0^2 \tau_0^2}{2md} \sum_{\alpha\geq 2} u_{\alpha,k}^2,$$

$\tau_0 \gg m/d, (\gamma \pm \Gamma_\alpha)^{-1}$

$$\mathcal{P}_1^\infty(k) = \delta P_0^2 \tau_0 \textstyle\sum_{\alpha\geq 2} \frac{u_{\alpha,k}^2}{\lambda_\alpha^2},$$
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Global Robustness

Averaging over an ergodic ensemble of perturbation vectors, $\tau_0 \ll m/d, \, (\gamma \pm \Gamma_\alpha)^{-1}$

$$\begin{aligned}
\langle \mathcal{P}_1^\infty \rangle &= \frac{\langle \delta P_0^2 \rangle \tau_0^2}{2d} \sum_{\alpha \geq 2} \lambda_\alpha^{-1} \, , \\
\langle \mathcal{P}_2^\infty \rangle &= \frac{\langle \delta P_0^2 \rangle \tau_0^2}{2md} \frac{n-1}{n} \, .
\end{aligned}$$

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$\tau_0 \gg m/d,\ (\gamma \pm \Gamma_\alpha)^{-1}$

$$\langle \mathcal{P}_1^\infty \rangle = \langle \delta P_0^2 \rangle \tau_0 \sum_{\alpha\geq 2} \lambda_\alpha^{-2}\,,$$

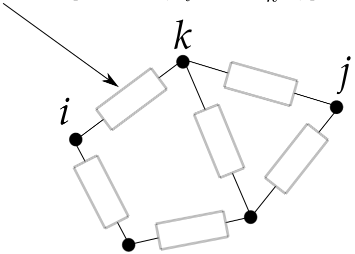
$$\langle \mathcal{P}_2^\infty \rangle = \frac{\langle \delta P_0^2 \rangle}{d} \sum_{\alpha\geq 2} \lambda_\alpha^{-1}\,.$$

Resistance Distance

$$\Omega_{ij} \;=\; \mathbb{L}^{\dagger}_{ii} + \mathbb{L}^{\dagger}_{jj} - \mathbb{L}^{\dagger}_{ij} - \mathbb{L}^{\dagger}_{ji} = \sum_{\alpha \geq 2} \frac{(u_{\alpha,i} - u_{\alpha,j})^2}{\lambda_\alpha} \,.$$

$\mathbb{L}^{\dagger}$: pseudo inverse of $\mathbb{L}$ (because of $\lambda_1 = 0$).

$$R_{ik} = [b_{ik} \cos(\theta_i^{(0)} - \theta_k^{(0)})]^{-1}$$

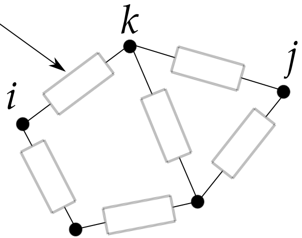


Klein and Randić, *J. Math. Chem.* **12**, 81 (1993).

Kirchhoff Index

$$Kf_1 = \sum_{i<j} \Omega_{ij} = n \sum_{\alpha \geq 2} \lambda_\alpha^{-1} \,.$$

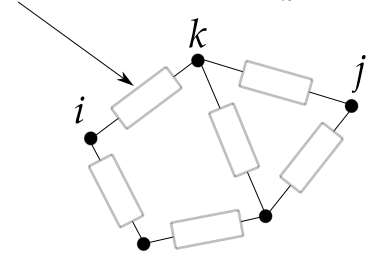
$$R_{ik} = [b_{ik} \cos(\theta_i^{(0)} - \theta_k^{(0)})]^{-1}$$



Resistance Centrality

$$C_1(k) = \left[n^{-1}\sum_j \Omega_{kj}\right]^{-1} = \left[\sum_{\alpha\geq 2}\frac{u_{\alpha,k}^2}{\lambda_\alpha} + n^{-2}Kf_1\right]^{-1}.$$

$$R_{ik} = [b_{ik}\cos(\theta_i^{(0)} - \theta_k^{(0)})]^{-1}$$



Generalized Resistance Distances

$$\begin{aligned}\Omega_{ij}^{(p)} &= \mathbb{L}'^{\dagger}_{ii} + \mathbb{L}'^{\dagger}_{jj} - \mathbb{L}'^{\dagger}_{ij} - \mathbb{L}'^{\dagger}_{ji} \\ &= \sum_{\alpha\geq 2} \frac{(u_{\alpha,i} - u_{\alpha,j})^2}{\lambda_\alpha^p} \,, \\ \mathbb{L}' &= \mathbb{L}^p \,.\end{aligned}$$

Generalized Kirchhoff Indices

$$Kf_p = \sum_{i<j} \Omega_{ij}^{(p)} = n \sum_{\alpha\geq 2} \lambda_\alpha^{-p} \,.$$

Generalized Resistance Centralities

$$C_p(k) = \left[n^{-1} \sum_j \Omega_{kj}^{(p)}\right]^{-1} = \left[\sum_{\alpha\geq 2} \frac{u_{\alpha,k}^2}{\lambda_\alpha^p} + n^{-2} Kf_p\right]^{-1} \,.$$

MT,Coletta and Jacquod, Phys. Rev. Lett. **120**, 084101 (2018).
MT,Pagnier and Jacquod to appear in *Science Advances*, arXiv:1810.09694.

Global Robustness $\rightarrow$ Kf_p's Local Vulnerabilities $\rightarrow$ C_p's

Local Vulnerability

Perturbing a specific node k i.e.
$\delta P_{0i} = \delta_{ik}\delta P_0$,
$\tau_0 \ll m/d,\, (\gamma \pm \Gamma_\alpha)^{-1}$

$$\mathcal{P}_1^\infty(k) = \frac{\delta P_0^2 \tau_0^2}{2d}[C_1^{-1}(k) - n^{-2}Kf_1]\,,$$
$$\mathcal{P}_2^\infty(k) = \frac{\delta P_0^2 \tau_0^2}{2md}\frac{(n-1)}{n}\,,$$

$\tau_0 \gg m/d,\, (\gamma \pm \Gamma_\alpha)^{-1}$

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MT and Jacquod, Phys. Rev. E **100**, 032303 (2019).

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MT and Jacquod, Phys. Rev. E **100**, 032303 (2019).

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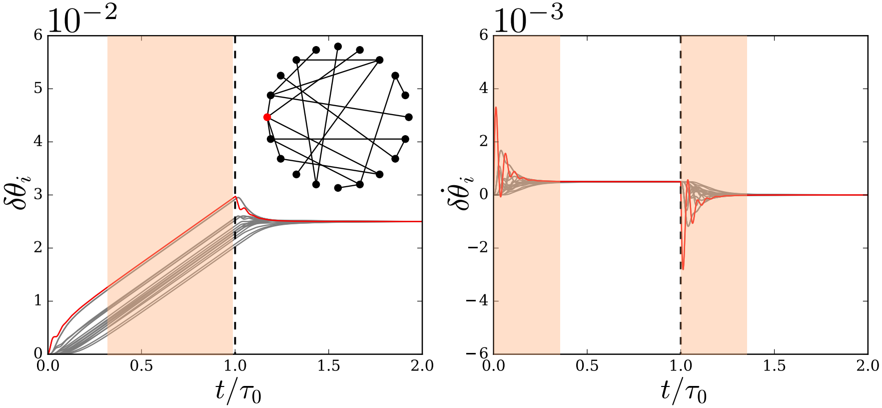
MT and Jacquod, Phys. Rev. E **100**, 032303 (2019).

Global Robustness → Kf_p's Local Vulnerabilities → C_p's



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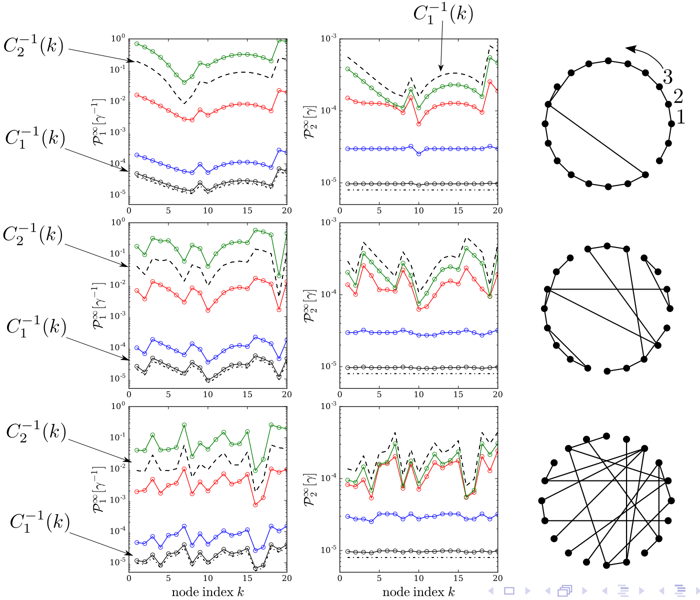
MT and Jacquod, Phys. Rev. E **100**, 032303 (2019).

Specific Local Vulnerabilities and C_p's: Numerics



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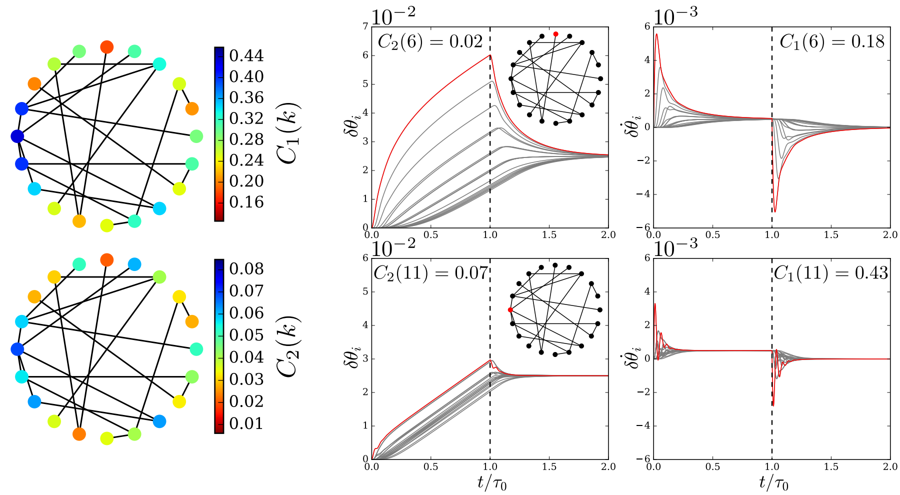

MT and Jacquod, Phys. Rev. E **100**, 032303 (2019).

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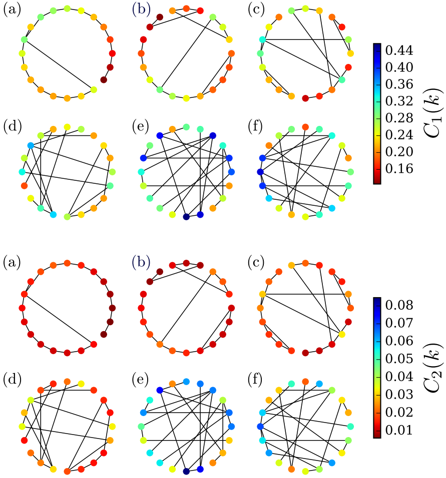

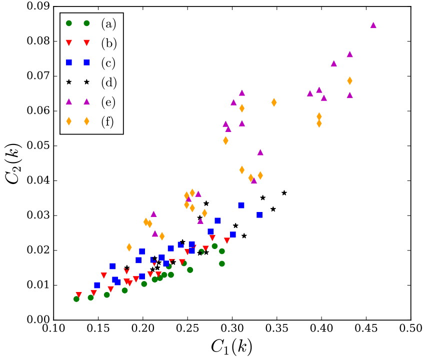

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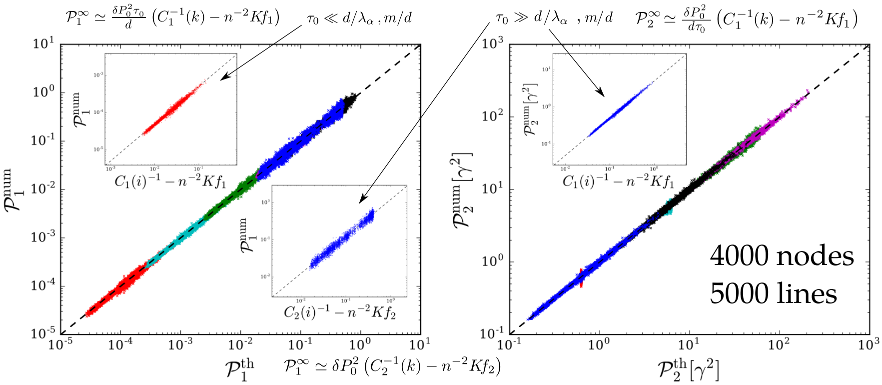

MT,Pagnier and Jacquod to appear in *Science Advances*, arXiv:1810.09694.

Physical Realization : European Electrical Grid

$\tau_0 \ll d/\lambda_\alpha \, , m/d$

$\mathcal{P}_1^\infty \simeq \frac{\delta P_0^2 \tau_0}{d} \left(C_1^{-1}(k) - n^{-2} K f_1 \right)$

$\mathcal{P}_2^\infty \simeq \frac{\delta P_0^2 \tau_0}{dm} \frac{(n-1)}{n}$

$\tau_0 \gg d/\lambda_\alpha \quad , m/d$

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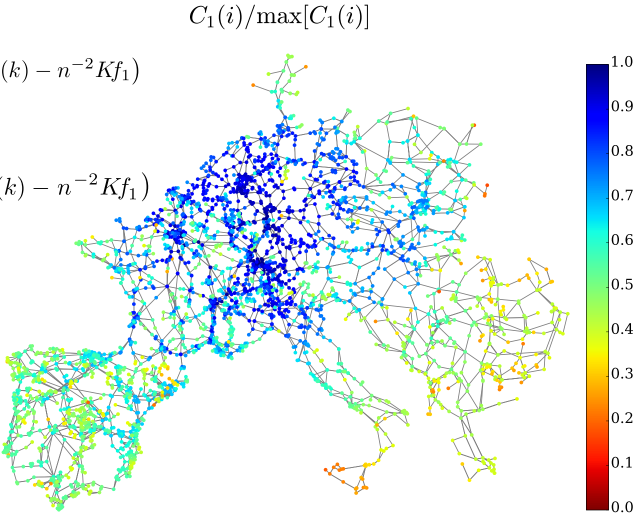

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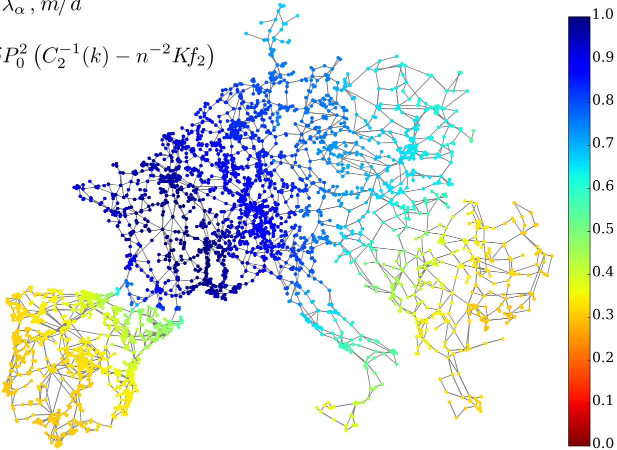
Physical Realization : European Electrical Grid

$$C_2(i)/\max[C_2(i)]$$

$\tau_0 \gg d/\lambda_\alpha \,, m/d$

$\mathcal{P}_1^\infty \simeq \delta P_0^2 \left(C_2^{-1}(k) - n^{-2} K f_2 \right)$



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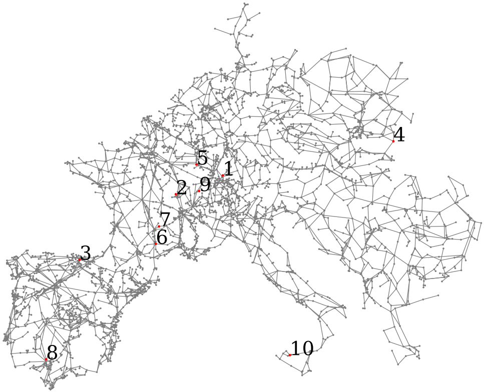

node #	$C_{\rm geo}$	Degree	PageRank	C_1	C_2	$\mathcal{P}_1^{\rm num}$	$\mathcal{P}_2^{\rm num}$ $[\gamma^2]$
1	7.84	4	3024	31.86	5.18	0.047	0.035
2	6.8	1	2716	22.45	5.68	0.021	0.118
3	5.56	10	896	22.45	2.33	0.32	0.116
4	4.79	3	1597	21.74	3.79	0.126	0.127
5	7.08	1	1462	21.74	5.34	0.026	0.125
6	4.38	6	2945	21.69	5.65	0.023	0.129
7	5.11	2	16	19.4	5.89	0.016	0.164
8	4.15	6	756	19.38	1.83	0.453	0.172
9	5.06	1	1715	10.2	5.2	0.047	0.449
10	2.72	4	167	7.49	2.17	0.335	0.64

MT,Pagnier and Jacquod to appear in *Science Advances*, arXiv:1810.09694.

Ranking of the nodes

$\mathrm{WLRank}_{1,2}$: Based on $C_{1,2}$ related to

$$\mathbb{L}_{ij}(\{\theta_i^{(0)}\}) = \begin{cases} -b_{ij}\cos(\theta_i^{(0)} - \theta_j^{(0)})\,, & i \neq j\,, \\ \sum_k b_{ik}\cos(\theta_i^{(0)} - \theta_k^{(0)})\,, & i = j\,. \end{cases}$$

$\mathrm{LRank}_{1,2}$: Based on $C_{1,2}$ related to

$$\mathbb{L}_{ij} = \begin{cases} -b_{ij}\,, & i \neq j\,, \\ \sum_k b_{ik}\,, & i = j\,. \end{cases}$$

MT,Pagnier and Jacquod to appear in *Science Advances*, arXiv:1810.09694.

Summary: Local Vulnerabilities

- Rank the nodes: independent of the operational/synchronous state if $|\Delta\theta| < 30^\circ \rightarrow \mathrm{LRank} \cong \mathrm{WLRank}$.
  More details $\rightarrow$ arXiv:1810.09694 (2018) (or in Science Advances soon).
- Most vulnerable nodes $\rightarrow$ C_1 or C_2 depending on τ_0 and on the dynamical variable of interest.

MT,Pagnier and Jacquod to appear in *Science Advances*, arXiv:1810.09694.

Global Robustness $\rightarrow$ Kf_p's Local Vulnerabilities $\rightarrow$ C_p's

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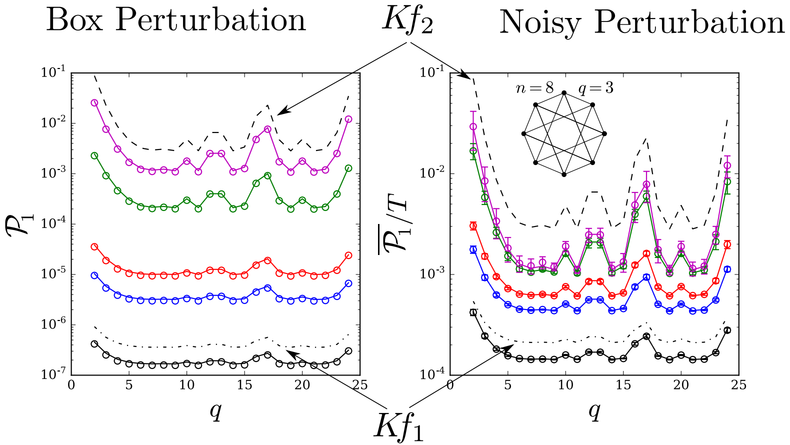
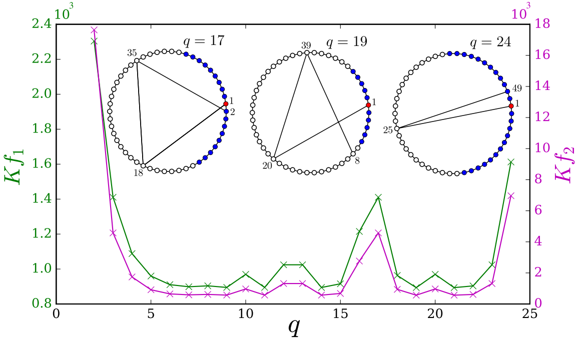
MT and Jacquod, Phys. Rev. E **100**, 032303 (2019).

Averaged Global Robustness and Kf_p's: Numerics



MT,Coletta and Jacquod, Phys. Rev. Lett. **120**, 084101 (2018).

Averaged Global Robustness and Kf_p's: Regular Networks

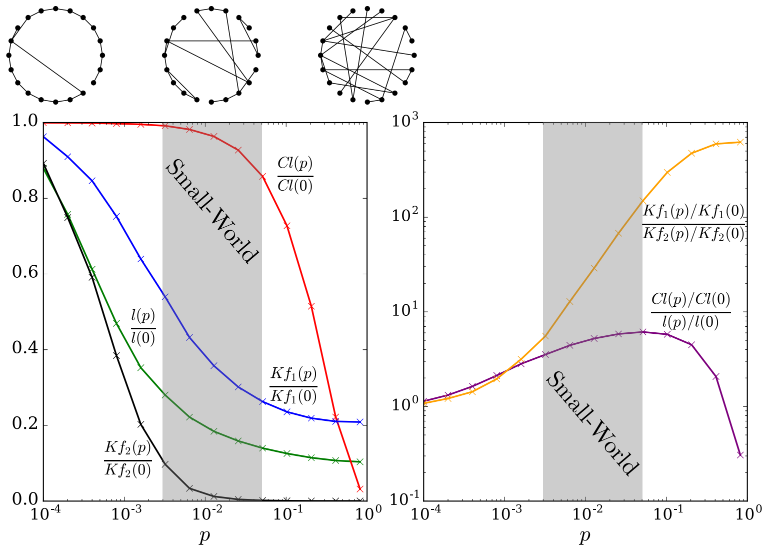


$$Kf_p = \sum_{\alpha \geq 2} [4 - 2\cos(k_\alpha) - 2\cos(qk_\alpha)]^{-p}$$

$$k_\alpha = \frac{2\pi(\alpha - 1)}{n}.$$

MT,Coletta and Jacquod, Phys. Rev. Lett. **120**, 084101 (2018).
MT and Jacquod, Phys. Rev. E **100**, 032303 (2019).

Averaged Global Robustness and Kf_p's: Small-World



MT and Jacquod, Phys. Rev. E **100**, 032303 (2019).

Near Equilibrium Dynamics – Conclusion

Global Robustness

- Generalized Kirchhoff Indices, Kf_p's.

Local Vulnerabilities

- Generalized Resistance Centralities, C_p's.
- Establish a ranking of the nodes.

$\rightarrow$ p depends on which performance measures you are interested in and on the correlation time of the perturbation.

Inertia

- No effect on performance measures in both asymptotics in τ_0 except for frequencies and short τ_0.

MT,Coletta and Jacquod, Phys. Rev. Lett. **120**, 084101 (2018).
MT,Pagnier and Jacquod to appear in *Science Advances*, arXiv:1810.09694.
MT and Jacquod, Phys. Rev. E **100**, 032303 (2019).

Transitions – Steady State Identification

Swing Equations in the lossless line limit (second-order Kuramoto)

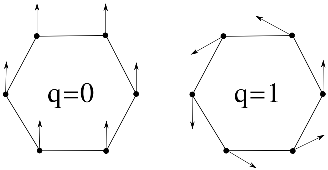
$$m_i\ddot{\theta}_i + d_i\dot{\theta}_i = P_i - \sum_j b_{ij}\sin(\theta_i - \theta_j) \ \ , \ i = 1, ..., n.$$

$b_{ij} = b_{ji} \geq 0$.

Steady-state solutions Synchronous state $\{\theta_i^{(0)}\}$ such that:

$$P_i = \sum_j b_{ij}\sin(\theta_i^{(0)} - \theta_j^{(0)}) \ \ , \ i = 1, ..., n.$$

Winding number $q = (2\pi)^{-1}\sum_{i\in c}|\theta_{i+1} - \theta_i|_{[-\pi,\pi]}$.

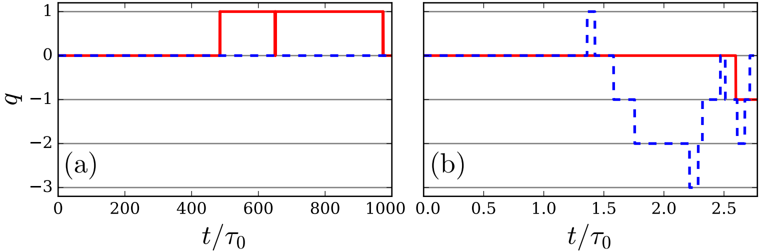


Delabays,Coletta and Jacquod, Journal of Mathematical Physics **58**, 032703 (2017).

Noisy natural frequencies

$\langle \delta P_i(t) \rangle = 0$

$\langle \delta P_i(t) \delta P_j(t') \rangle = \delta_{ij}\, \delta P_0^2 \, \exp[-|t - t'|/\tau_0]$



MT, Delabays and Jacquod, Phys. Rev. E **99**, 062213 (2019).

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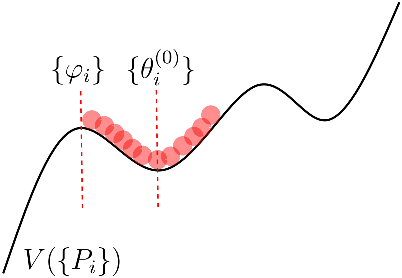


Escape criterion $\langle \delta \boldsymbol{\theta}^2 \rangle \cong \|\boldsymbol{\theta}^{(0)} - \boldsymbol{\varphi}\|_2^2$

MT, Delabays and Jacquod, Phys. Rev. E **99**, 062213 (2019).

Transitions – Linear Response

Noisy natural frequencies

- $\langle \delta P_i(t) \rangle = 0$,
- $\langle \delta P_i(t) \delta P_j(t') \rangle = \delta_{ij}\, \delta P_0^2 \, \exp[-|t-t'|/\tau_0]$.

$$\lim_{t\to\infty} \langle \boldsymbol{\delta\theta}^2(t) \rangle = \delta P_0^2 \sum_{\alpha \geq 2} \frac{\tau_0 + m/d}{\lambda_\alpha(\lambda_\alpha \tau_0 + d + m/\tau_0)} \, . \tag{3}$$

In the two limits of long and short τ_0, one has

$$\lim_{t\to\infty} \langle \boldsymbol{\delta\theta}^2(t) \rangle \simeq \begin{cases} \dfrac{\delta P_0^2 \, \tau_0}{nd} K f_1 \, , & \tau_0 \ll \frac{d}{\lambda_\alpha} \, , \frac{m}{d} \, , \\ \dfrac{\delta P_0^2}{n} K f_2 \, , & \tau_0 \gg \frac{d}{\lambda_\alpha} \, , \frac{m}{d} \, . \end{cases} \tag{4}$$

MT, Delabays and Jacquod, Phys. Rev. E **99**, 062213 (2019).

Single cycle

For identical natural frequencies:

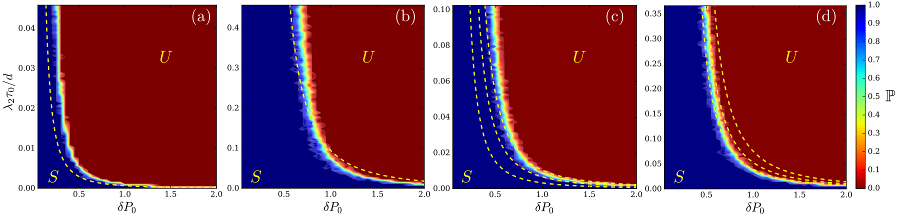
$$\Delta^2 = \left\| \boldsymbol{\theta}^{(0)} - \boldsymbol{\varphi} \right\|_2^2 = \frac{n(n^2-1)}{12(n-2)^2}\pi^2\,.$$

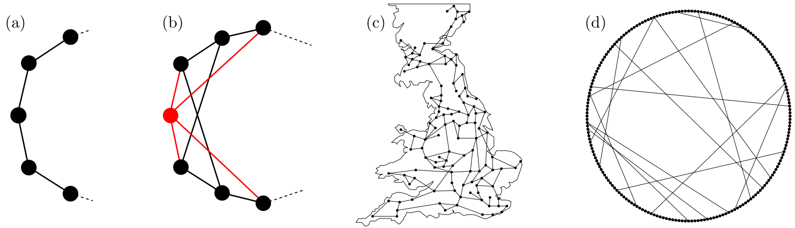
Complex networks

Find saddles numerically, e.g. Newton-Raphson.

Delabays, MT and Jacquod, Chaos **27**, 103109 (2017).
MT, Delabays and Jacquod, Phys. Rev. E **99**, 062213 (2019).

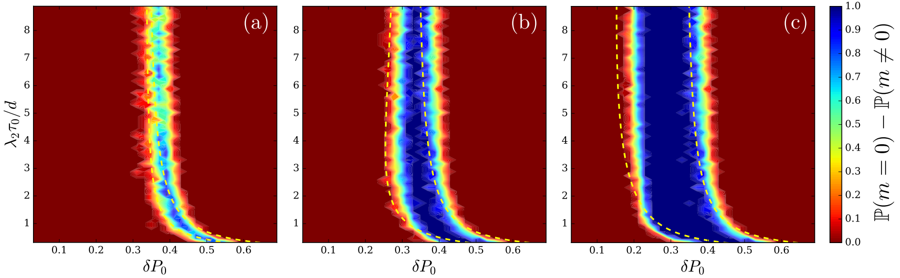
Transitions – Prediction





MT, Delabays and Jacquod, Phys. Rev. E **99**, 062213 (2019).

Transitions – Prediction



MT, Delabays and Jacquod, Phys. Rev. E **99**, 062213 (2019).

Escape

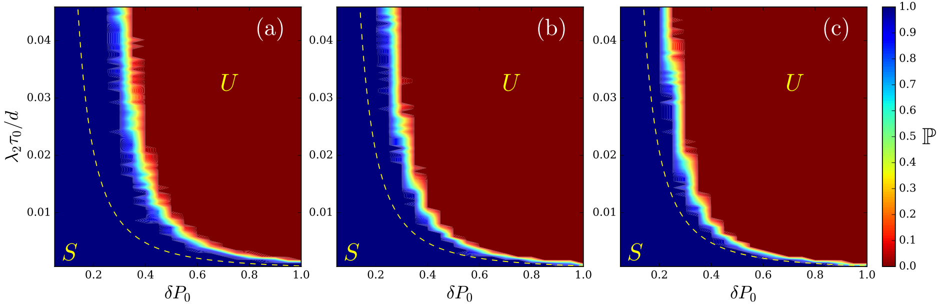
- Heuristics $\langle \delta \boldsymbol{\theta}^2 \rangle \cong \|\boldsymbol{\theta}^{(0)} - \boldsymbol{\varphi}\|_2$ + linear response $\rightarrow$ fair prediction.
- Estimate first escape time $\rightarrow$ Phys. Rev. E **99**, 062213 (2019) (or ask me).

Inertia

- Increasing inertia reduces stability region.

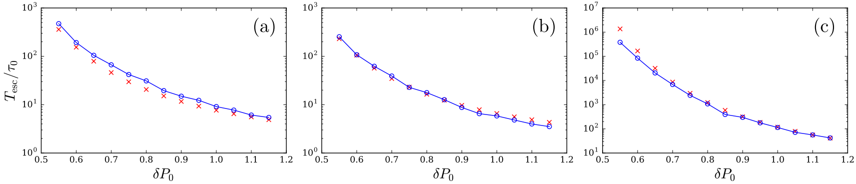
MT, Delabays and Jacquod, Phys. Rev. E **99**, 062213 (2019).

Transitions – Prediction



MT, Delabays and Jacquod, Phys. Rev. E **99**, 062213 (2019).

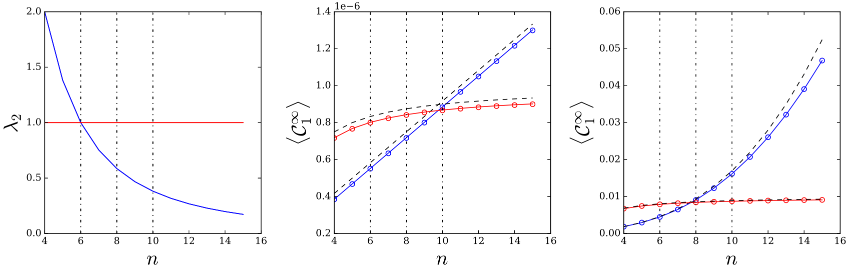
Transitions – Prediction



$$T_{\mathrm{esc}} \propto \left[2\int_{\beta\Delta}^{\infty} P(\overline{\delta\theta})\mathrm{d}(\overline{\delta\theta})\right]^{-1} \quad (5)$$

MT, Delabays and Jacquod, Phys. Rev. E **99**, 062213 (2019).

Supplemental Material

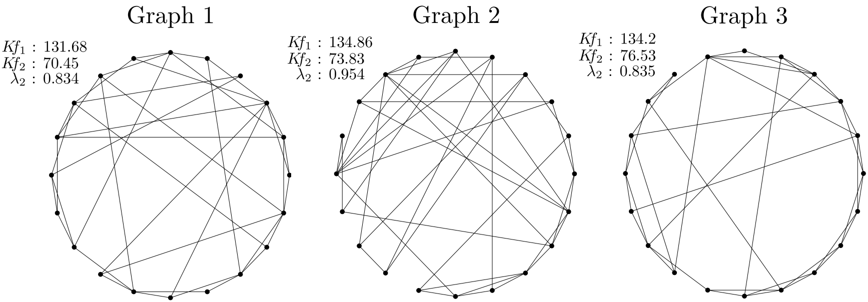


blue : cycle graph

red : star graph

MT,Coletta and Jacquod, *Phys. Rev. Lett.* **120**, 084101 (2018)

Supplemental Material



MT,Coletta and Jacquod, *Phys. Rev. Lett.* **120**, 084101 (2018)