Local Vulnerabilities and Global Robustness of Coupled Dynamical Systems on Complex Networks

PhD Defense

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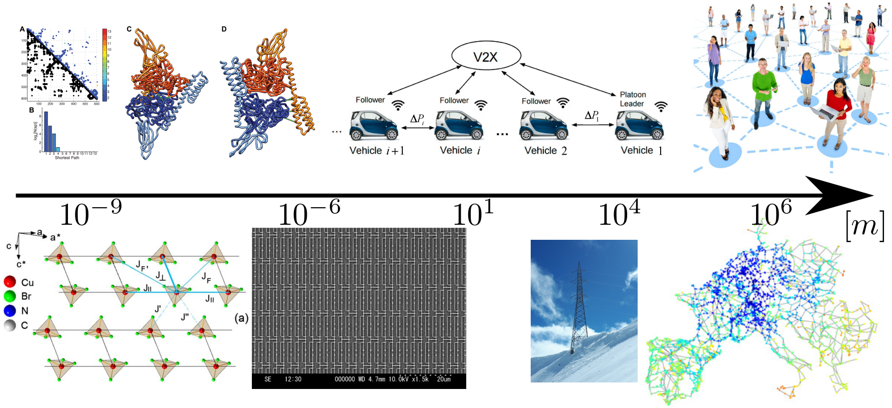
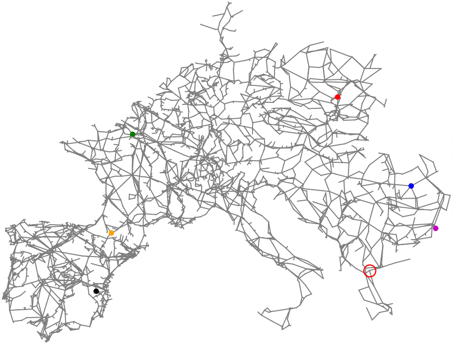
December 16, 2019

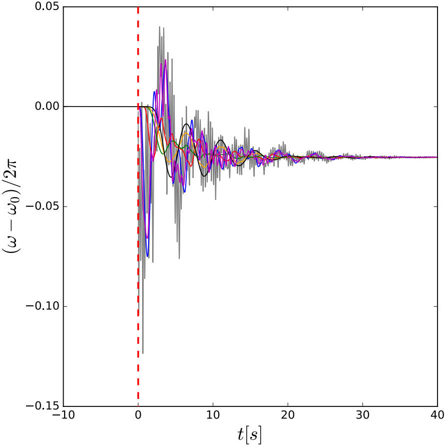
Introduction – Coupled Dynamical Systems



Sources: Phys. Rev. B **80**, 094411 (2009)
PLoS Comput Biol 11(6), e1004262 (2015)
Phys. Rev. Lett. **85**, 1974 (2000).
Transportation Research Part C: Emerging Technologies Volume 84, November 2017, Pages 21-47.
https://etranselec.ch/2019/10/07/Swissgri.html
https://researchoutreach.org/articles/opinion-dynamics-consensus-social-networks/

Transient dynamics → small-signal response





Multistability $\rightarrow$ transitions between fixed points

.. Silk, J. & Vilenkin, A. *Phys. Rev. Lett.* **53,** 1700 (1984).
9. Hogan, C. J. *Phys. Lett.* **143B,** 87 (1984).
10. Vilenkin, A. & Field, G. *Nature* (in the press).

Flux quantization in a high-T_c superconductor

C. E. Gough*, M. S. Colclough*, E. M. Forgan*, R. G. Jordan*, M. Keene*, C. M. Muirhead*, A. I. M. Rae*, N. Thomas*, J. S. Abell† & S. Sutton†

* Department of Physics, University of Birmingham, Birmingham B15 2TT, UK
† Department of Metallurgy and Materials, University of Birmingham, Birmingham B15 2TT, UK

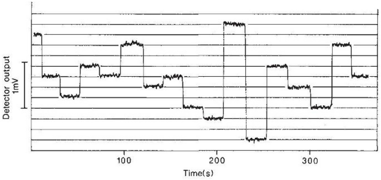
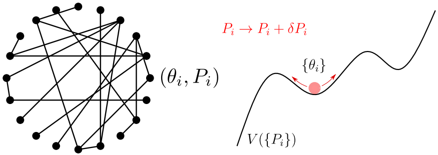


Fig. 2 Output of the r.f.-SQUID magnetometer showing small integral numbers of flux quanta jumping in and out of the ring.

way we deduce that the magnitude of the flux jumps shown in Fig. 1 is typically 100 ($h/2e$).

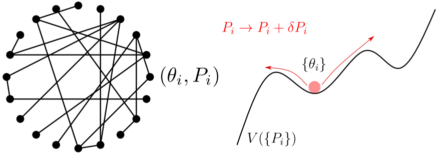
Small-signal response

- Complex network metric & response to external perturbations
- Local vulnerabilities & global robustness



Transitions

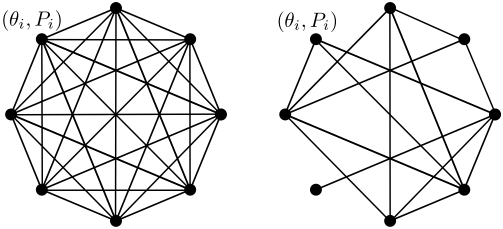
- Estimate for survival probability & shape of the basin of attraction



Kuramoto model

$$\dot{\theta}_i = P_i - \sum_j a_{ij} \sin(\theta_i - \theta_j) \ , \ i = 1, ..., n.$$

$a_{ij} = a_{ji} \geq 0$.
P_i : natural frequencies.



Y. Kuramoto, *Lecture Notes in Physics* **39**, *International Symposium on Mathematical Problems in Theoretical Physics* (1975).

Coupled Oscillators

Second order Kuramoto model

$$m_i\ddot{\theta}_i + d_i\dot{\theta}_i = P_i - \sum_j a_{ij}\sin(\theta_i - \theta_j) \;\; , \;\; i = 1, ..., n.$$

$a_{ij} = a_{ji} \geq 0$.
P_i : natural frequencies.
m_i : inertia.
d_i : damping.

Electric Power Network (in the lossless line approximation)

P_i : injected/consumed power.
$m_i = 0$: loads.
$m_i \neq 0$: generators.
$a_{ij}\sin(\theta_i - \theta_j)$: power flow from i to j .

J. A. Acebrón, L. L. Bonilla, Conrad J. Pérez Vicente, F. Ritort, and R. Spigler, Rev. Mod. Phys. **77**, 137 (2005)

Second order Kuramoto model

$$m_i\ddot{\theta}_i + d_i\dot{\theta}_i = P_i - \sum_j a_{ij}\sin(\theta_i - \theta_j) \ \ , \ i = 1, ..., n.$$

$a_{ij} = a_{ji} \geq 0$.

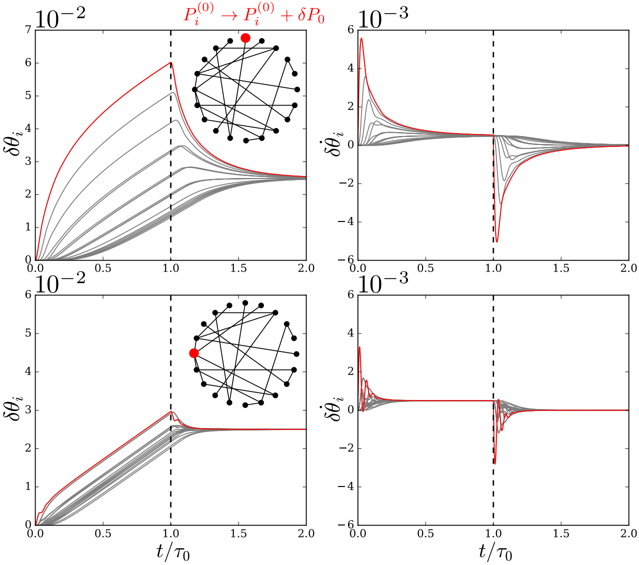
Steady-state solutions Synchronous state $\{\theta_i^{(0)}\}$ such that:

$$P_i = \sum_j a_{ij}\sin(\theta_i^{(0)} - \theta_j^{(0)}) \ \ , \ i = 1, ..., n.$$

$\sum_i P_i = 0 \ [\theta_i(t) \to \theta_i(t) + \Omega t]$.

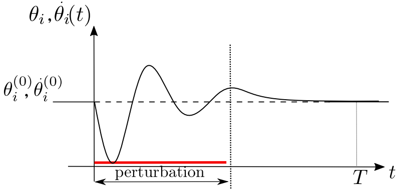
Perturbations $P_i \to P_i^{(0)} + \delta P_i(t)$.

Coupled Dynamical Systems: Example

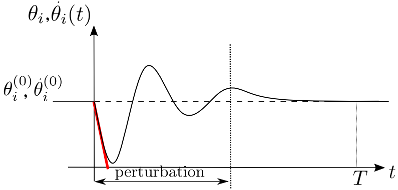


Quantifying Robustness

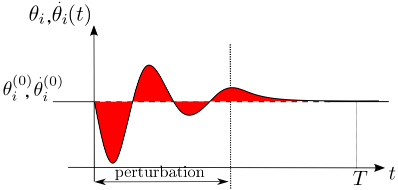
- Maximum of the response, $\max_t(\theta_i)$.



- Rate of change of frequency (RoCoF), $\ddot{\theta}_i$.

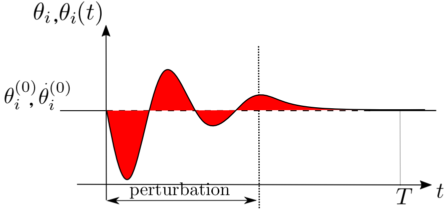


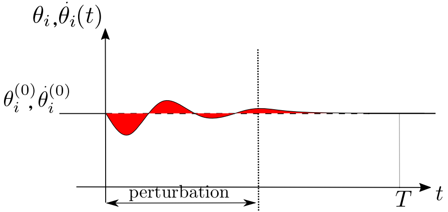
- Performance measure (quadratic integrals over the transient).



Quantifying Robustness

Performance measures





$$\mathcal{P}_1(T) = \sum_i \int_0^T |\theta_i(t) - \theta_i^{(0)}|^2 \mathrm{d}t\,,$$

$$\mathcal{P}_2(T) = \sum_i \int_0^T |\dot\theta_i(t) - \dot\theta_i^{(0)}|^2 \mathrm{d}t\,.$$

$$\mathcal{P}_{1,2}^{\infty} = \mathcal{P}_{1,2}(T \to \infty)\,.$$

Noisy disturbances $\rightarrow$ divide by T.

Perturbations : $P_i \rightarrow P_i^{(0)} + \delta P_i(t)$.

Response to Perturbations: Linearization

Linear response Perturbation of the natural frequencies.

- $P_i(t) = P_i^{(0)} + \delta P_i(t) \rightarrow \theta_i(t) = \theta_i^{(0)} + \delta\theta_i(t)$:

$$\delta\dot{\boldsymbol{\theta}}(t) = \delta\boldsymbol{P}(t) - \mathbb{L}(\{\theta_i^{(0)}\})\delta\boldsymbol{\theta}(t) \ ,$$

$\mathbb{L}(\{\theta_i^{(0)}\})$: the weighted Laplacian matrix,

$$\mathbb{L}_{ij}(\{\theta_i^{(0)}\}) = \begin{cases} -a_{ij}\cos(\theta_i^{(0)} - \theta_j^{(0)}) \, , & i \neq j \, , \\ \sum_k a_{ik}\cos(\theta_i^{(0)} - \theta_k^{(0)}) \, , & i = j \, . \end{cases}$$

Topology $\rightarrow a_{ij}$.
Steady state $\rightarrow \{\theta_i^{(0)}\}$.

Response to Perturbations: Linearization

Linear response Perturbation of the natural frequencies.

- $P_i(t) = P_i^{(0)} + \delta P_i(t) \rightarrow \theta_i(t) = \theta_i^{(0)} + \delta\theta_i(t)$:

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Topology $\rightarrow a_{ij}$.
Steady state $\rightarrow \{\theta_i^{(0)}\}$.
Linear response $\rightarrow$ **Can be applied to linear consensus, opinion dynamics models.**

Response to Perturbations: Linearization

Linear response

$$\delta\dot{\boldsymbol{\theta}}(t) = \delta\boldsymbol{P}(t) - \mathbb{L}(\{\theta_i^{(0)}\})\delta\boldsymbol{\theta}(t) \ ,$$

Expanding on the eigenvectors $\mathbf{u}_\alpha$ of $\mathbb{L}$, we have $\delta\boldsymbol{\theta}(t) = \sum_\alpha c_\alpha(t)\mathbf{u}_\alpha$.

$$\dot{c}_\alpha = \delta\boldsymbol{P}(t)\cdot\mathbf{u}_\alpha - \lambda_\alpha c_\alpha \ , \quad \alpha = 1, ..., n \, .$$

$$c_\alpha(t) = e^{-\lambda_\alpha t}\int_0^t e^{\lambda_\alpha t'}\delta\boldsymbol{P}(t')\cdot\mathbf{u}_\alpha \,\mathrm{d}t' \ .$$

Performance Measures

$$\mathcal{P}_1^\infty = \sum_i \int_0^\infty |\delta\theta_i(t) - \Delta(t)|^2 \mathrm{d}t = \sum_{\alpha\geq 2}\int_0^\infty c_\alpha^2(t)\mathrm{d}t \ ,$$

$$\mathcal{P}_2^\infty = \sum_i \int_0^\infty |\delta\dot{\theta}_i(t) - \dot{\Delta}(t)|^2 \mathrm{d}t = \sum_{\alpha\geq 2}\int_0^\infty \dot{c}_\alpha^2(t)\mathrm{d}t \ ,$$

Response to Perturbations: Time Scales

Intrinsic Time Scales

- Network relaxation: $1/\lambda_\alpha$ with $\{\lambda_\alpha\}$ the eigenvalues of $\mathbb{L}$.
- Individual elements: m/d.

Perturbation Time Scale

- Correlation time of the external perturbation $\delta\boldsymbol{P}(t)$.

Quench perturbations

- $\delta P_i(t) = \delta P_{0i}\Theta(t)\Theta(\tau_0 - t)$.

Quench duration $\rightarrow \tau_0$.

Noisy time correlated perturbations

- $\overline{\delta P_i(t)P_j(t')} = \delta P_{0i}^2\delta_{ij}\exp[-|t-t'|/\tau_0]$.

Correlation time $\rightarrow \tau_0$.

Response to Perturbations: Specific $\delta \boldsymbol{P}_0$

Quench perturbations

$$\mathcal{P}_1^\infty = \sum_{\alpha\geq 2} \frac{(\delta \boldsymbol{P}_0 \cdot \mathbf{u}_\alpha)^2}{\lambda_\alpha^3}(\lambda_\alpha \tau_0 - 1 + e^{-\lambda_\alpha \tau_0}) \, .$$

Noisy time correlated perturbations

$$\overline{\mathcal{P}_1} = \sum_{\alpha} \frac{\sum_{i\in N_{\rm n}} \delta P_{0i}^2 \, u_{\alpha,i}^2}{\lambda_\alpha(\lambda_\alpha + \tau_0^{-1})} \, .$$

N_n: noisy nodes.

Averaged perturbations

$$\langle \delta P_{0,i} \delta P_{0,j} \rangle = \delta_{ij} \langle \delta P_0^2 \rangle$$

Quench perturbations

$$\langle \mathcal{P}_1^\infty \rangle \simeq \begin{cases} \langle \delta P_0^2 \rangle \, \tau_0^2 \, \sum_{\alpha \geq 2} \lambda_\alpha^{-1} / 2 \,, & \lambda_\alpha \tau_0 \ll 1 \,, \forall \alpha \,, \\ \langle \delta P_0^2 \rangle \, \tau_0 \, \sum_{\alpha \geq 2} \lambda_\alpha^{-2} \,, & \lambda_\alpha \tau_0 \gg 1 \,, \forall \alpha \,. \end{cases}$$

Noisy time correlated perturbations

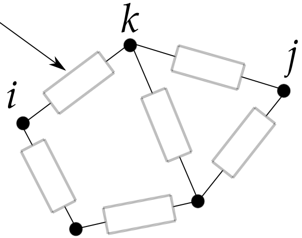
$$\overline{\mathcal{P}_1^\infty} \simeq \begin{cases} \langle \delta P_0^2 \rangle \, \tau_0 \, \sum_{\alpha \geq 2} \lambda_\alpha^{-1} / 2 \,, & \lambda_\alpha \tau_0 \ll 1 \,, \forall \alpha \,, \\ \langle \delta P_0^2 \rangle \, \sum_{\alpha \geq 2} \lambda_\alpha^{-2} \,, & \lambda_\alpha \tau_0 \gg 1 \,, \forall \alpha \,. \end{cases}$$

Resistance Distance

$$\Omega_{ij}^{(1)} = \mathbb{L}_{ii}^\dagger + \mathbb{L}_{jj}^\dagger - \mathbb{L}_{ij}^\dagger - \mathbb{L}_{ji}^\dagger = \sum_{\alpha \geq 2} \frac{(u_{\alpha,i} - u_{\alpha,j})^2}{\lambda_\alpha} .$$

$\mathbb{L}^\dagger$: pseudo inverse of $\mathbb{L}$ (because of $\lambda_1 = 0$).

$$R_{ik} = [a_{ik} \cos(\theta_i^{(0)} - \theta_k^{(0)})]^{-1}$$

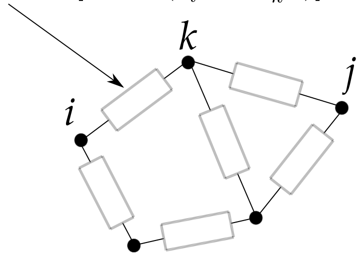


Klein and Randić, *J. Math. Chem.* **12**, 81 (1993).

Kirchhoff Index

$$Kf_1 = \sum_{i<j} \Omega_{ij}^{(1)} = n \sum_{\alpha \geq 2} \lambda_\alpha^{-1} \,.$$

$$R_{ik} = [a_{ik} \cos(\theta_i^{(0)} - \theta_k^{(0)})]^{-1}$$



Generalized Resistance Distances

$$\begin{aligned}\Omega_{ij}^{(p)} &= \mathbb{L}'^{\dagger}_{ii} + \mathbb{L}'^{\dagger}_{jj} - \mathbb{L}'^{\dagger}_{ij} - \mathbb{L}'^{\dagger}_{ji} \\ &= \sum_{\alpha \geq 2} \frac{(u_{\alpha,i} - u_{\alpha,j})^2}{\lambda_\alpha^p}\,, \\ \mathbb{L}' &= \mathbb{L}^p\,.\end{aligned}$$

Generalized Kirchhoff Indices

$$Kf_p = \sum_{i<j} \Omega_{ij}^{(p)} = n \sum_{\alpha \geq 2} \lambda_\alpha^{-p}\,.$$

Response to Perturbations: Global Robustness

Averaged perturbations

$$\langle \delta P_{0,i} \delta P_{0,j} \rangle = \delta_{ij} \langle \delta P_0^2 \rangle$$

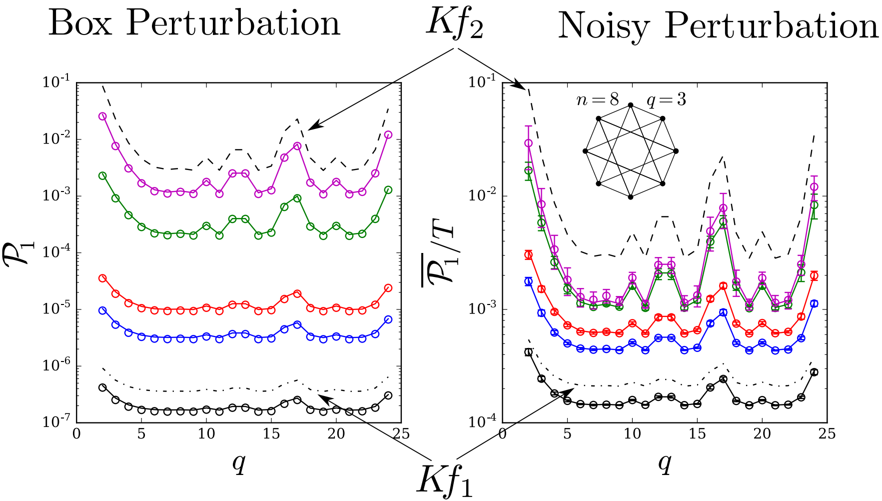
Quench perturbations

$$\langle \mathcal{P}_1^\infty \rangle \simeq \begin{cases} \langle \delta P_0^2 \rangle \, \tau_0^2 \, K f_1 / 2n \,, & \lambda_\alpha \tau_0 \ll 1 \,, \forall \alpha \,, \\ \langle \delta P_0^2 \rangle \, \tau_0 \, K f_2 / n \,, & \lambda_\alpha \tau_0 \gg 1 \,, \forall \alpha \,. \end{cases}$$

Noisy time correlated perturbations

$$\overline{\mathcal{P}_1^\infty} \simeq \begin{cases} \langle \delta P_0^2 \rangle \, \tau_0 \, K f_1 / n \,, & \lambda_\alpha \tau_0 \ll 1 \,, \forall \alpha \,, \\ \langle \delta P_0^2 \rangle \, K f_2 / n \,, & \lambda_\alpha \tau_0 \gg 1 \,, \forall \alpha \,. \end{cases}$$

Averaged Global Robustness and Kf_p's: Numerics



Response to Perturbations: Local Vulnerabilities

Local perturbations

$$\delta P_{0,i} = \delta_{ik}\delta P_0^2$$

Quench perturbations

$$\mathcal{P}_1^\infty(k) \simeq \begin{cases} \frac{\delta P_0^2\,\tau_0^2}{2}\sum_{\alpha\geq 2}\frac{u_{\alpha,k}^2}{\lambda_\alpha}\,, & \lambda_\alpha\tau_0 \ll 1\,,\forall\alpha\,, \\ \delta P_0^2\,\tau_0\,\sum_{\alpha\geq 2}\frac{u_{\alpha,k}^2}{\lambda_\alpha^2}\,, & \lambda_\alpha\tau_0 \gg 1\,,\forall\alpha\,. \end{cases}$$

Noisy time correlated perturbations

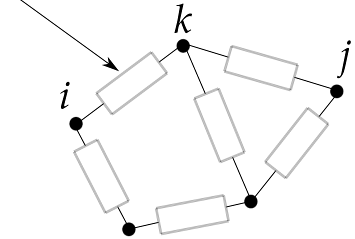
$$\overline{\mathcal{P}_1^\infty}(k) \simeq \begin{cases} \delta P_0^2\,\tau_0\,\sum_{\alpha\geq 2}\frac{u_{\alpha,k}^2}{\lambda_\alpha}\,, & \lambda_\alpha\tau_0 \ll 1\,,\forall\alpha\,, \\ \delta P_0^2\,\sum_{\alpha\geq 2}\frac{u_{\alpha,k}^2}{\lambda_\alpha^2}\,, & \lambda_\alpha\tau_0 \gg 1\,,\forall\alpha\,. \end{cases}$$

Resistance Distances, $Kf_p's$ and C_p's

Generalized Resistance Centralities

$$C_p(k) = \left[n^{-1}\sum_j \Omega_{kj}^{(p)}\right]^{-1} = \left[\sum_{\alpha\geq 2}\frac{u_{\alpha,k}^2}{\lambda_\alpha^p} + n^{-2}Kf_p\right]^{-1}.$$

$$R_{ik} = [a_{ik}\cos(\theta_i^{(0)} - \theta_k^{(0)})]^{-1}$$



Response to Perturbations: Local Vulnerabilities

Local perturbations

$$\delta P_{0,i} = \delta_{ik}\delta P_0^2$$

Quench perturbations

$$\mathcal{P}_1^{\infty}(k) \simeq \begin{cases} \frac{\delta P_0^2\,\tau_0^2}{2}[C_1^{-1}(k) - n^{-2}Kf_1]\,, & \lambda_\alpha\tau_0 \ll 1\,, \forall\alpha\,, \\ \delta P_0^2\,\tau_0\,[C_2^{-1}(k) - n^{-2}Kf_2], & \lambda_\alpha\tau_0 \gg 1\,, \forall\alpha\,. \end{cases}$$

Noisy time correlated perturbations

$$\overline{\mathcal{P}_1^{\infty}}(k) \simeq \begin{cases} \delta P_0^2\,\tau_0\,[C_1^{-1}(k) - n^{-2}Kf_1]\,, & \lambda_\alpha\tau_0 \ll 1\,, \forall\alpha\,, \\ \delta P_0^2\,[C_2^{-1}(k) - n^{-2}Kf_2]\,, & \lambda_\alpha\tau_0 \gg 1\,, \forall\alpha\,. \end{cases}$$

Global Robustness $\rightarrow$ Kf_p's Local Vulnerabilities $\rightarrow$ C_p's

Local Vulnerability

Perturbing a specific node k i.e. $\delta P_{0i} = \delta_{ik}\delta P_0$,

$\tau_0 \ll m/d,\, (\gamma \pm \Gamma_\alpha)^{-1}$

$$\mathcal{P}_1^\infty(k) = \frac{\delta P_0^2 \tau_0^2}{2d}[C_1^{-1}(k) - n^{-2} Kf_1]\,,$$
$$\mathcal{P}_2^\infty(k) = \frac{\delta P_0^2 \tau_0^2}{2md}\frac{(n-1)}{n}\,,$$

$\tau_0 \gg m/d,\, (\gamma \pm \Gamma_\alpha)^{-1}$

$$\mathcal{P}_1^\infty(k) = \delta P_0^2 \tau_0 [C_2^{-1}(k) - n^{-2} Kf_2]\,,$$
$$\mathcal{P}_2^\infty(k) = \frac{\delta P_0^2}{d}[C_1^{-1}(k) - n^{-2} Kf_1]\,.$$

Global Robustness

Averaging over an ergodic ensemble of perturbation vectors,

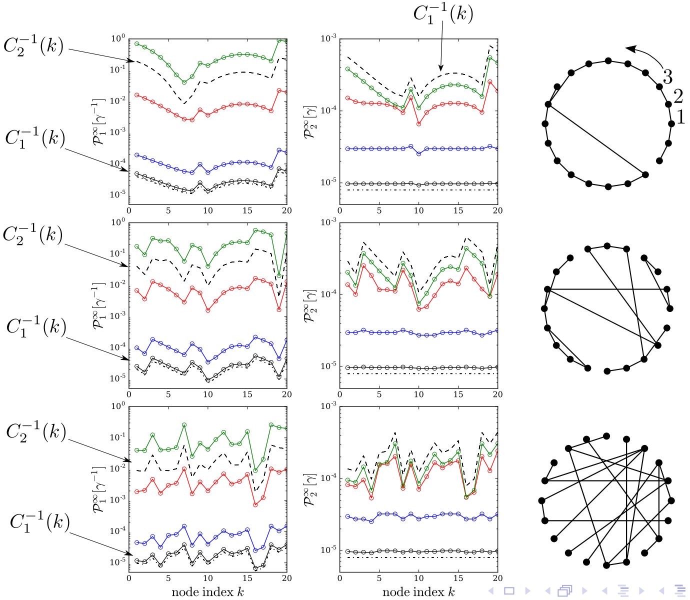
$\tau_0 \ll m/d,\, (\gamma \pm \Gamma_\alpha)^{-1}$

$$\langle\mathcal{P}_1^\infty\rangle = \frac{\langle\delta P_0^2\rangle \tau_0^2}{2nd} Kf_1\,,$$
$$\langle\mathcal{P}_2^\infty\rangle = \frac{\langle\delta P_0^2\rangle \tau_0^2}{2md}\frac{n-1}{n}\,.$$

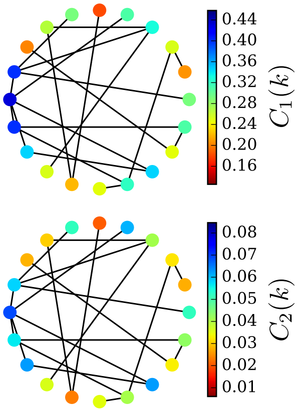
$\tau_0 \gg m/d,\, (\gamma \pm \Gamma_\alpha)^{-1}$

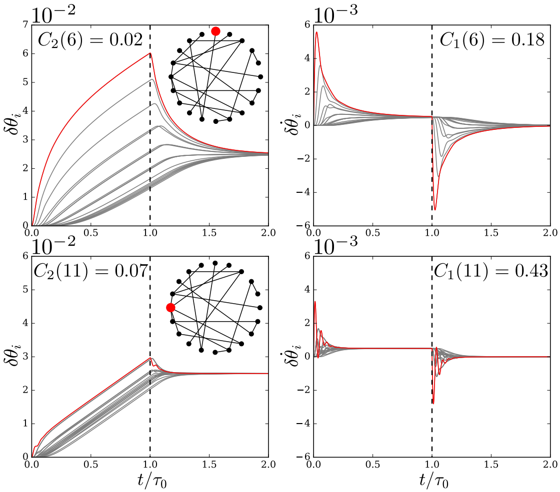
$$\langle\mathcal{P}_1^\infty\rangle = \frac{\langle\delta P_0^2\rangle \tau_0}{n} Kf_2\,,$$
$$\langle\mathcal{P}_2^\infty\rangle = \frac{\langle\delta P_0^2\rangle}{nd} Kf_1\,.$$

Specific Local Vulnerabilities and C_p's: Numerics

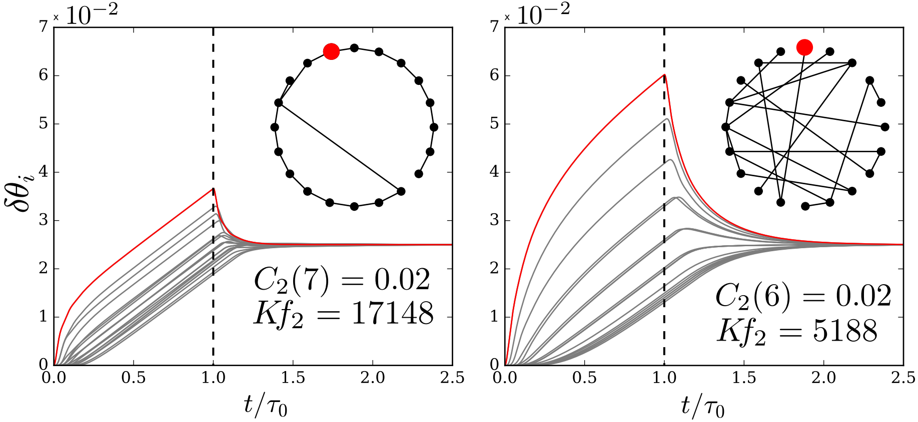


Specific Local Vulnerabilities and C_p's: Numerics



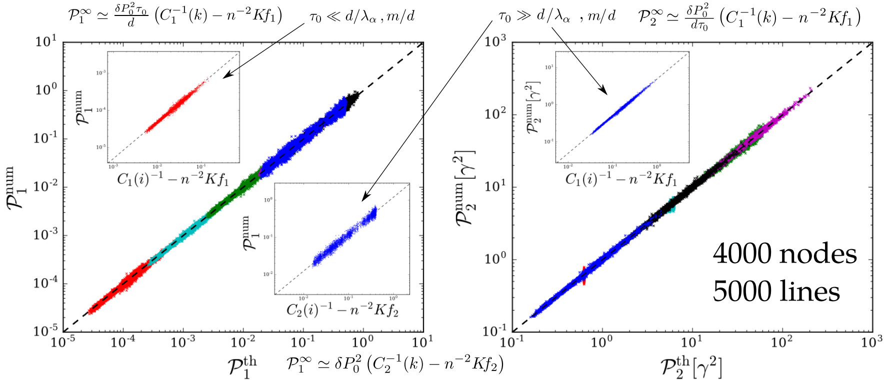


Global Robustness vs. Local Vulnerabilities



$$\delta P_k(t) = \delta P_0 \Theta(t)\Theta(\tau_0 - t) \ ,$$
$$\mathcal{P}_1^{\infty}(k) = \delta P_0^2 \tau_0 [C_2^{-1}(k) - n^{-2} K f_2] \, .$$

Specific Local Vulnerabilities and C_p's: Numerics



$\tau_0 \ll d/\lambda_\alpha \, , m/d$

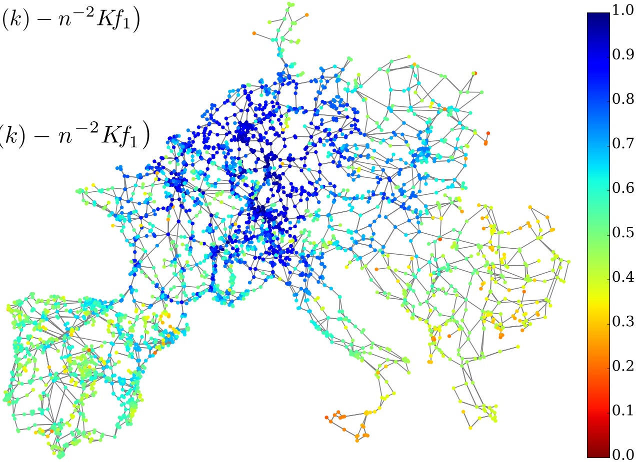
$\mathcal{P}_1^\infty \simeq \frac{\delta P_0^2 \tau_0}{d} \left(C_1^{-1}(k) - n^{-2} K f_1 \right)$

$\mathcal{P}_2^\infty \simeq \frac{\delta P_0^2 \tau_0}{dm} \frac{(n-1)}{n}$

$\tau_0 \gg d/\lambda_\alpha \;\; , m/d$

$\mathcal{P}_2^\infty \simeq \frac{\delta P_0^2}{d\tau_0} \left(C_1^{-1}(k) - n^{-2} K f_1 \right)$

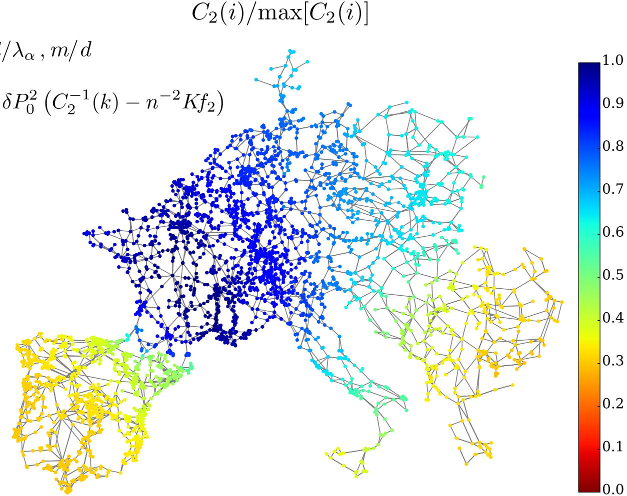
$C_1(i)/\max[C_1(i)]$



Physical Realization : European Electrical Grid

$$\tau_0 \gg d/\lambda_\alpha\,,\, m/d$$

$$\mathcal{P}_1^\infty \simeq \delta P_0^2 \left(C_2^{-1}(k) - n^{-2} K f_2\right)$$



Rankings of Vulnerabilities

Ranking of the nodes

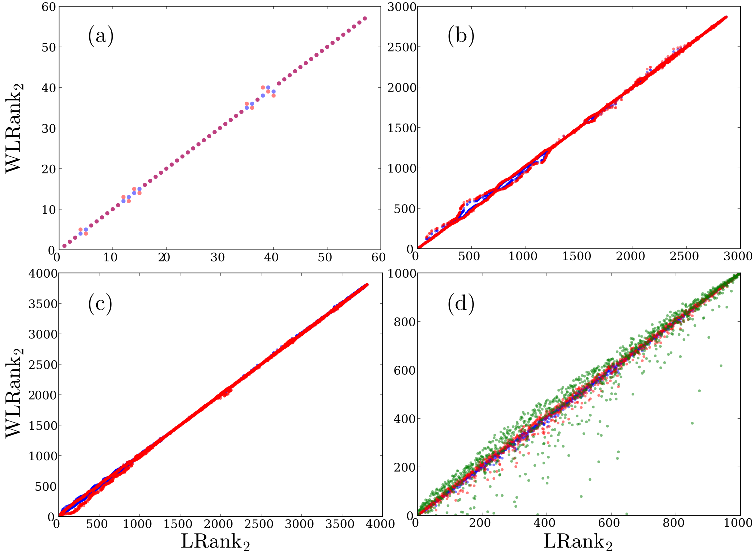
$\mathrm{WLRank}_{1,2}$: Based on $C_{1,2}$ related to

$$\mathbb{L}_{ij}(\{\theta_i^{(0)}\}) = \begin{cases} -a_{ij}\cos(\theta_i^{(0)} - \theta_j^{(0)})\,, & i \neq j\,, \\ \sum_k a_{ik}\cos(\theta_i^{(0)} - \theta_k^{(0)})\,, & i = j\,. \end{cases}$$

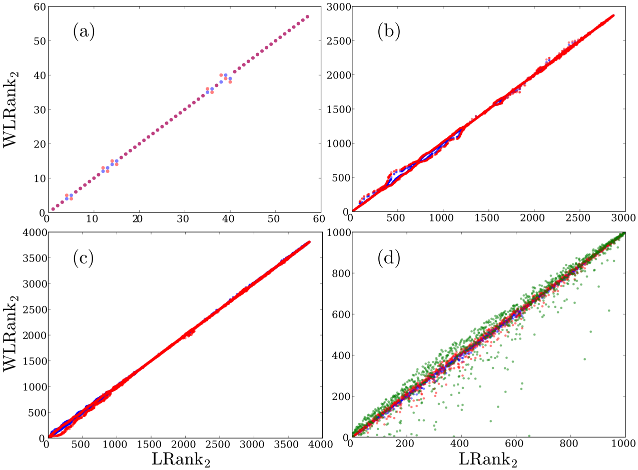
$\mathrm{LRank}_{1,2}$: Based on $C_{1,2}$ related to

$$\mathbb{L}_{ij} = \begin{cases} -a_{ij}\,, & i \neq j\,, \\ \sum_k a_{ik}\,, & i = j\,. \end{cases}$$

Rankings of Vulnerabilities



Rankings of Vulnerabilities



- Rank the nodes: independent of the operational/synchronous state if $|\Delta\theta| < 30^\circ \to \mathrm{LRank} \cong \mathrm{WLRank}$.

Small-Signal Response – Conclusion

Global Robustness

- Generalized Kirchhoff Indices, Kf_p's.

Local Vulnerabilities

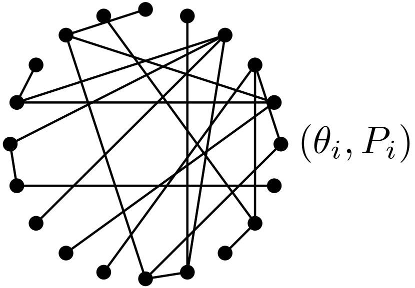
- Generalized Resistance Centralities, C_p's.
- Establish a ranking of the nodes.

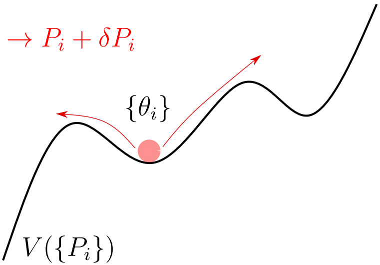
$\rightarrow$ p depends on which performance measures you are interested in and on the correlation time of the perturbation.

Inertia

- No effect on performance measures in both asymptotics in τ_0 except for frequencies and short τ_0.

Introduction – Transitions





Second-order Kuramoto

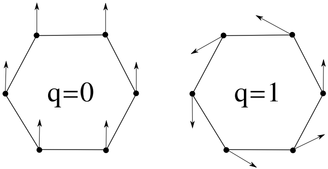
$$m_i\ddot{\theta}_i + d_i\dot{\theta}_i = P_i - \sum_j a_{ij}\sin(\theta_i - \theta_j)\ \ ,\ i = 1, ..., n.$$

$a_{ij} = a_{ji} \geq 0$.

Steady-state solutions Synchronous state $\{\theta_i^{(0)}\}$ such that:

$$P_i = \sum_j a_{ij}\sin(\theta_i^{(0)} - \theta_j^{(0)})\ \ ,\ i = 1, ..., n.$$

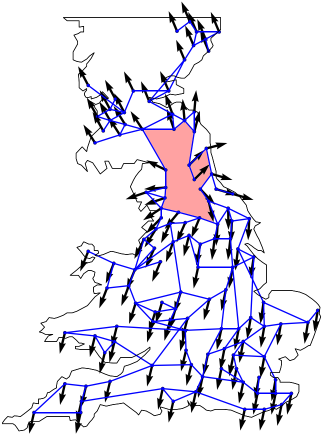
Winding number $q = (2\pi)^{-1}\sum_{i\in c}|\theta_{i+1} - \theta_i|_{[-\pi,\pi]}$.



Delabays,Coletta and Jacquod, Journal of Mathematical Physics **58**, 032703 (2017).

Transitions – Steady State Identification

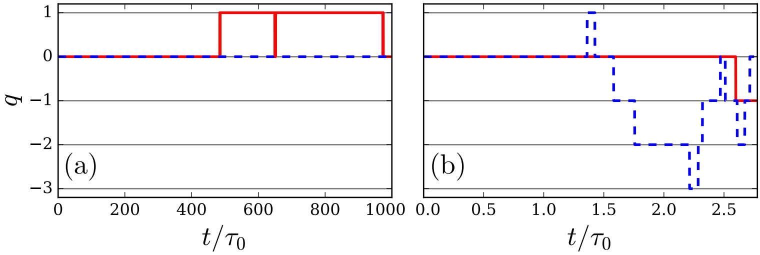
Meshed network



Delabays, MT and Jacquod, Chaos **27**, 103109 (2017).

Noisy natural frequencies

- $\langle \delta P_i(t) \rangle = 0$,
- $\langle \delta P_i(t) \delta P_j(t') \rangle = \delta_{ij}\, \delta P_0^2 \, \exp[-|t - t'|/\tau_0]$.



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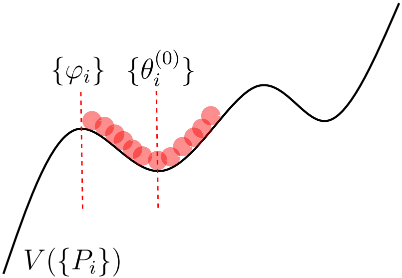


Escape criterion $\langle \delta \boldsymbol{\theta}^2 \rangle \cong \|\boldsymbol{\theta}^{(0)} - \boldsymbol{\varphi}\|_2^2$

Transitions – Linear Response

Escape criterion $\langle \delta\boldsymbol{\theta}^2 \rangle \cong \|\boldsymbol{\theta}^{(0)} - \boldsymbol{\varphi}\|_2^2$

Noisy natural frequencies

- $\langle \delta P_i(t) \rangle = 0$,
- $\langle \delta P_i(t) \delta P_j(t') \rangle = \delta_{ij}\, \delta P_0^2 \, \exp[-|t-t'|/\tau_0]$.

$$\lim_{t\to\infty} \langle \boldsymbol{\delta\theta}^2(t) \rangle = \delta P_0^2 \sum_{\alpha\geq 2} \frac{\tau_0 + m/d}{\lambda_\alpha(\lambda_\alpha \tau_0 + d + m/\tau_0)} \, .$$

Transitions – Distance to Closest Saddle

Escape criterion $\langle \delta\boldsymbol{\theta}^2 \rangle \cong \|\boldsymbol{\theta}^{(0)} - \boldsymbol{\varphi}\|_2^2$

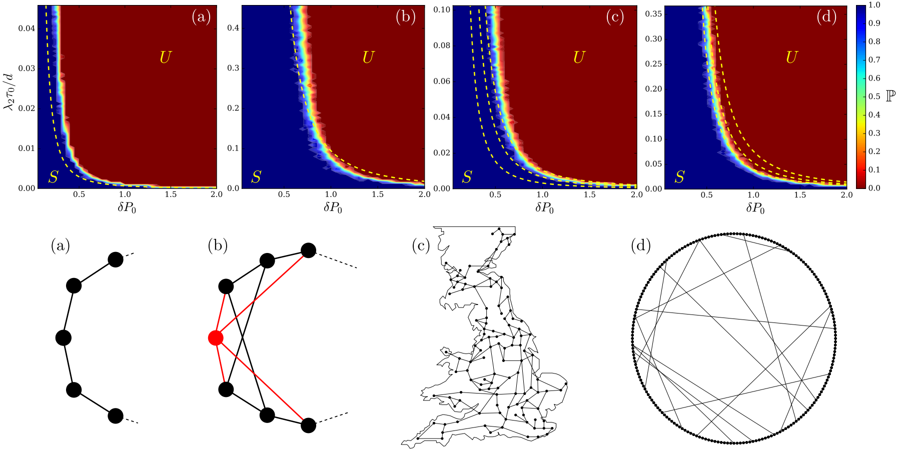
Single cycle
For identical natural frequencies:

$$\Delta^2 = \left\|\boldsymbol{\theta}^{(0)} - \boldsymbol{\varphi}\right\|_2^2 = \frac{n(n^2-1)}{12(n-2)^2}\pi^2\,.$$

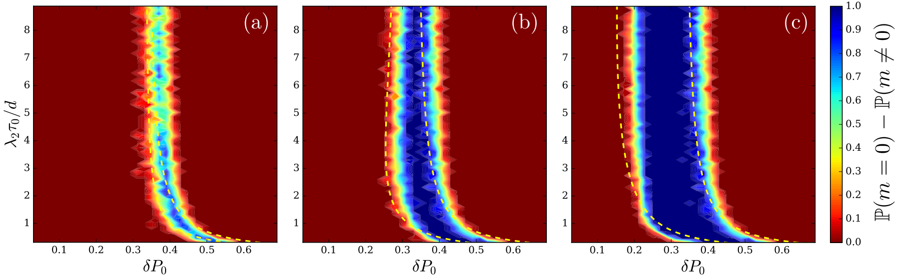
Complex networks
Find saddles numerically, e.g. Newton-Raphson.

Delabays, MT and Jacquod, Chaos **27**, 103109 (2017).

Transitions – Prediction



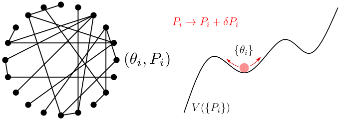
Effect of inertia



Overall Conclusion

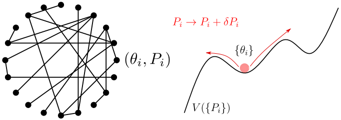
Small-signal response

- Response of coupled oscillators to external perturbations.
- Global network robustness $\rightarrow$ Kf_p .
- Local network vulnerabilities $\rightarrow$ C_p, Kf_p .



Transitions

- Estimate for escape probability based on distances between stable and unstable fixed points.
- Estimate for first escape time.



Extensions and related research directions

- Ranking of edges based on Kf_p's $\rightarrow Kf_1^{new} = Kf_1 + \frac{a_{\alpha\beta}\Omega_{\alpha\beta}^{(2)}}{1-a_{\alpha\beta}\Omega_{\alpha\beta}}$.
- Consider other robustness quantifiers: RoCoF for line contingencies

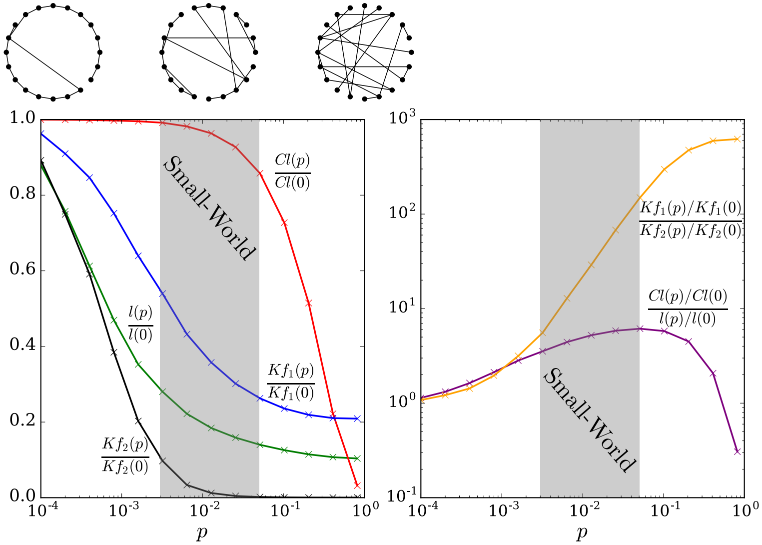
$$\rightarrow \ddot{\theta}_k = \dot{\omega}_k = (\delta_{ik} - \delta_{jk})\frac{a_{ij}[\theta_i(0) - \theta_j(0)]}{m_k}\,,$$

  Delabays, MT, Jacquod Chaos **29**, 103130 (2019).
- Consider non homogeneous inertia and damping parameters:

$$\begin{aligned} m\,\delta\ddot{\boldsymbol{\theta}} + d\,\delta\dot{\boldsymbol{\theta}} &= \delta\boldsymbol{P} - \mathbb{L}(\{\theta_i^{(0)}\})\delta\boldsymbol{\theta}(t)\,, \\ \rightarrow \gamma^{-1}\delta\ddot{\boldsymbol{\varphi}} + \delta\dot{\boldsymbol{\varphi}} &= \boldsymbol{D}^{-1/2}\delta\boldsymbol{P} - \boldsymbol{D}^{-1/2}\mathbb{L}\boldsymbol{D}^{-1/2}\,\delta\boldsymbol{\varphi}\,. \end{aligned}$$

  with $\delta\boldsymbol{\varphi} = \boldsymbol{D}^{1/2}\delta\boldsymbol{\theta}$ and $\gamma = d_i/m_i$ (under preparation, almost submitted).
- Network inference: $\mathcal{P}_{1,2} \rightarrow \mathbb{L}$.
- Opinion dynamics.

Averaged Global Robustness and Kf_p's: Small-World



Second order oscillators

$$m_i\ddot{\theta}_i + d_i\dot{\theta}_i = P_i - \sum_j a_{ij}\sin(\theta_i - \theta_j) \ \ , \ i = 1, ..., n.$$

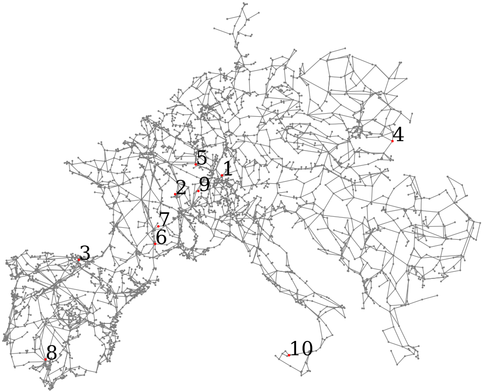
Linear response

$$m\,\delta\ddot{\boldsymbol{\theta}}(t) + d\,\delta\dot{\boldsymbol{\theta}}(t) = \delta\boldsymbol{P}(t) - \mathbb{L}(\{\theta_i^{(0)}\})\delta\boldsymbol{\theta}(t) \ .$$

$$c_\alpha(t) = m^{-1}\, e^{-(\gamma+\Gamma_\alpha)t/2}\int_0^t e^{\Gamma_\alpha t_1}\int_0^{t_1}\delta\boldsymbol{P}(t_2)\cdot\mathbf{u}_\alpha\, e^{(\gamma-\Gamma_\alpha)t_2/2}\,\mathrm{d}t_2\mathrm{d}t_1 \ ,$$

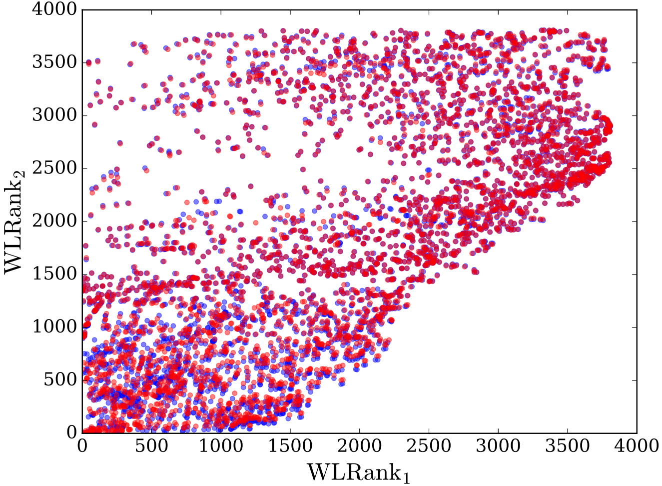
$\gamma = d/m$ and $\Gamma_\alpha = \sqrt{\gamma^2 - 4\lambda_\alpha/m}$.

Physical Realization : European Electrical Grid

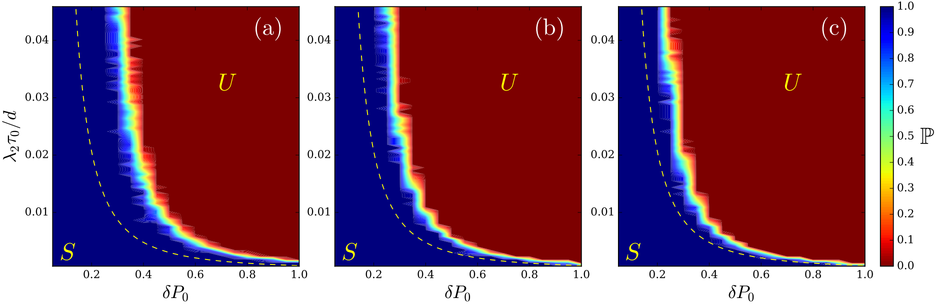


node #	C_{geo}	Degree	PageRank	C_1	C_2	$\mathcal{P}_1^{\text{num}}$	$\mathcal{P}_2^{\text{num}}$ $[\gamma^2]$
1	7.84	4	3024	31.86	5.18	0.047	0.035
2	6.8	1	2716	22.45	5.68	0.021	0.118
3	5.56	10	896	22.45	2.33	0.32	0.116
4	4.79	3	1597	21.74	3.79	0.126	0.127
5	7.08	1	1462	21.74	5.34	0.026	0.125
6	4.38	6	2945	21.69	5.65	0.023	0.129
7	5.11	2	16	19.4	5.89	0.016	0.164
8	4.15	6	756	19.38	1.83	0.453	0.172
9	5.06	1	1715	10.2	5.2	0.047	0.449
10	2.72	4	167	7.49	2.17	0.335	0.64

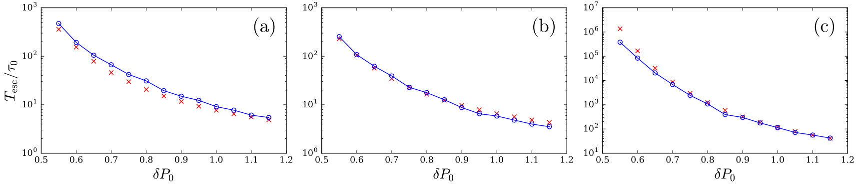
Rankings of Vulnerabilities



Transitions – Prediction

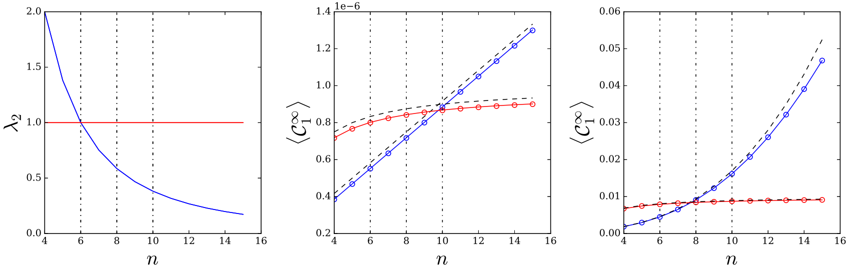


Escape Time



$$T_{\rm esc} \propto \left[2\int_{\beta\Delta}^{\infty} P(\overline{\delta\theta})\mathrm{d}(\overline{\delta\theta})\right]^{-1}$$

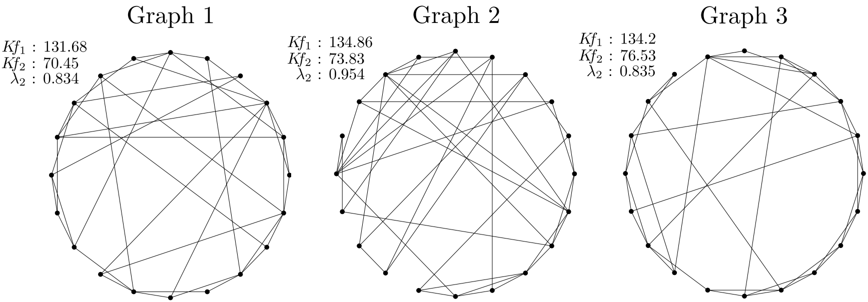
Supplemental Material



blue : cycle graph
red : star graph

MT,Coletta and Jacquod, *Phys. Rev. Lett.* **120**, 084101 (2018)

Supplemental Material



MT,Coletta and Jacquod, *Phys. Rev. Lett.* **120**, 084101 (2018)