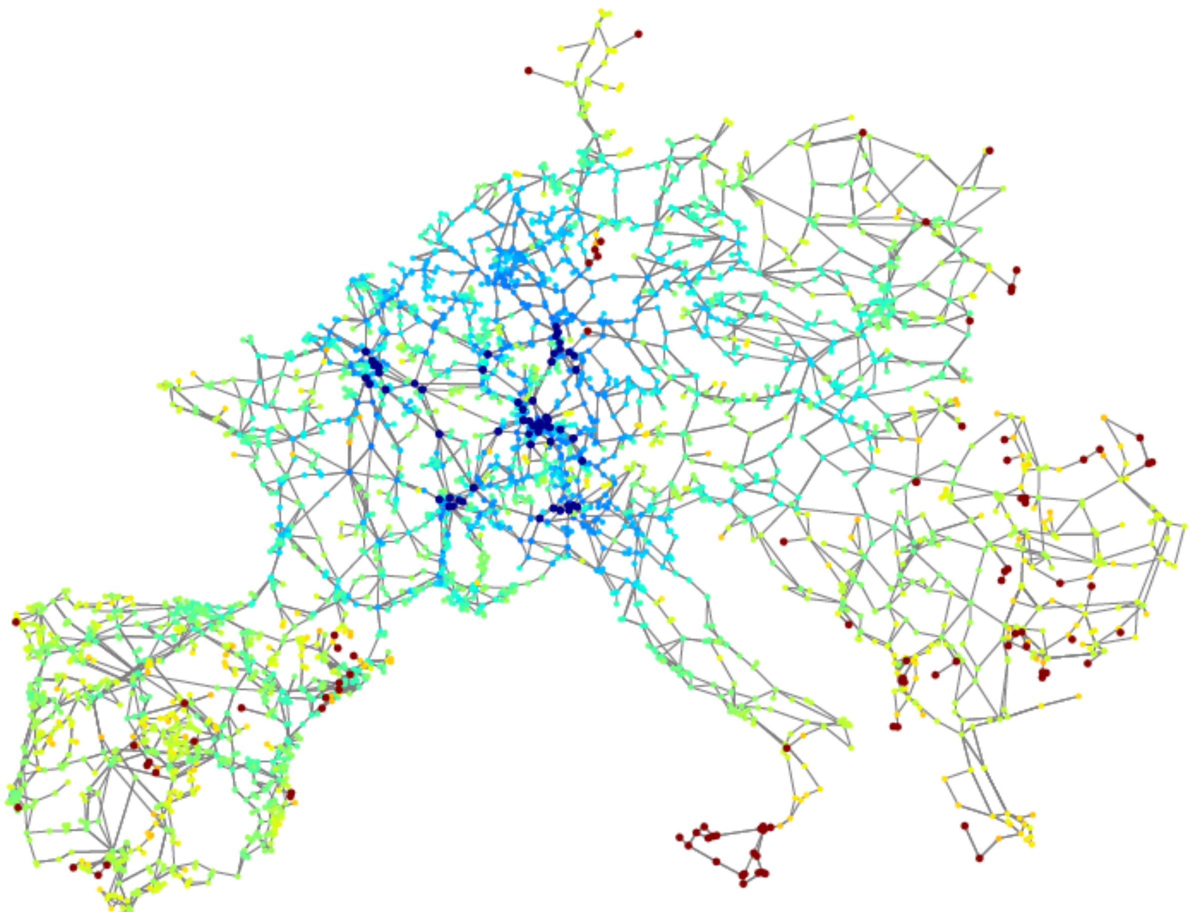
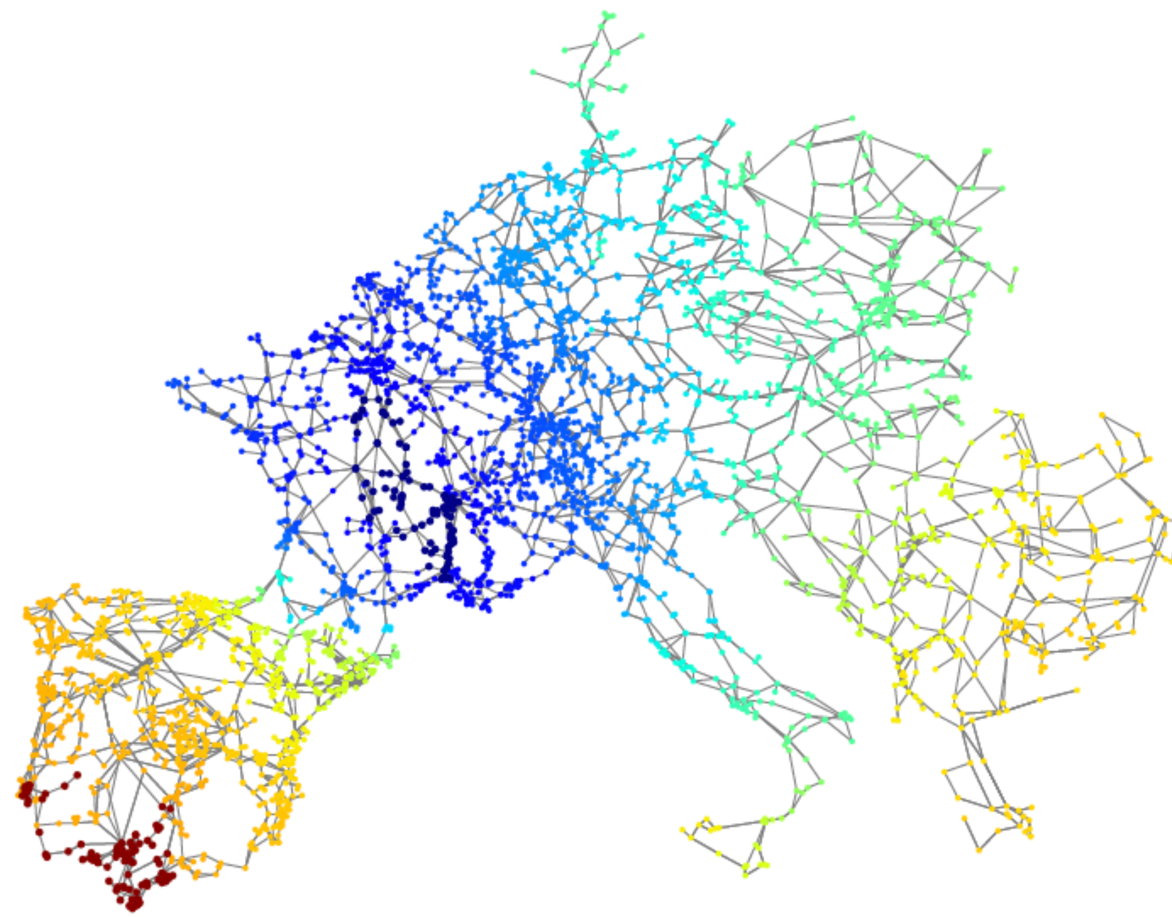
Spectral geometry
for
Dynamical Systems on Complex Graphs
Part I





Philippe Jacquod - GeoCow2020

UNIVERSITÉ DE GENÈVE
FACULTÉ DES SCIENCES
Hes·so VALAIS WALLIS
School of Engineering
FNSNF
swissgrid

“It is nice to know that the computer understands the problem

…

-E.P. Wigner

"It is nice to know that the computer understands the problem

...

but I would like to understand it too."

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"Truth (...) is much too complicated to allow anything but approximations."

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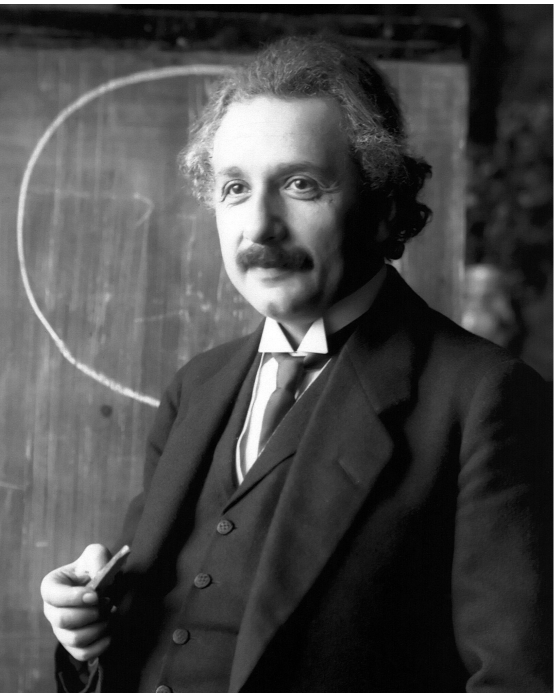
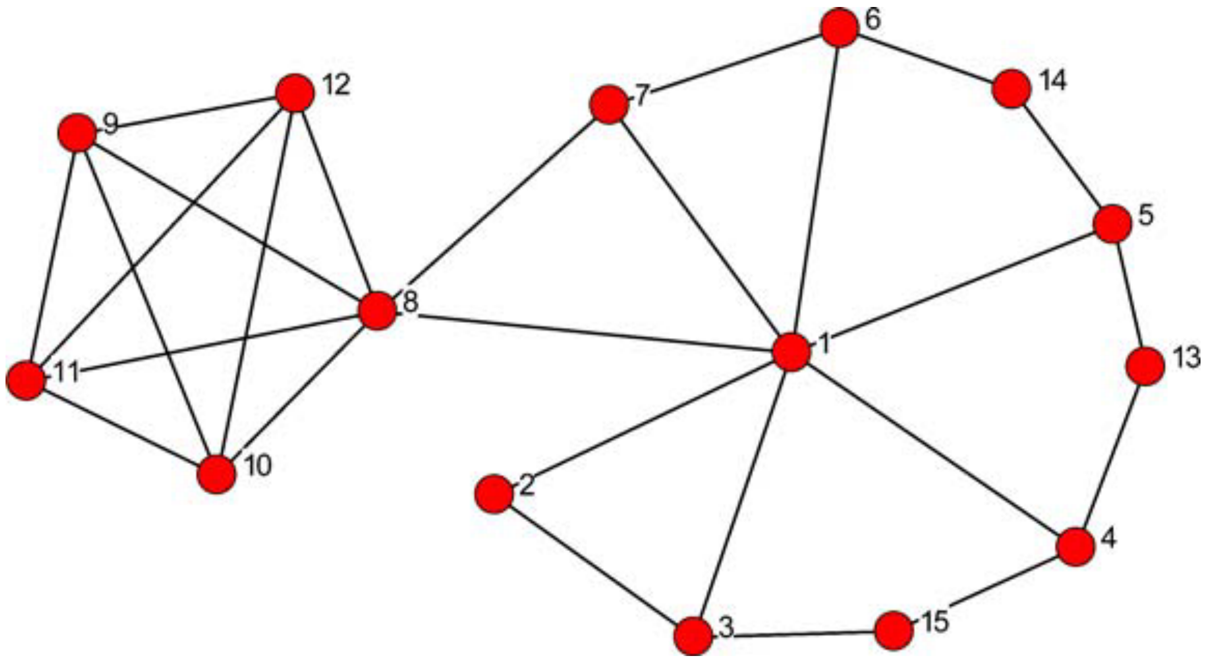


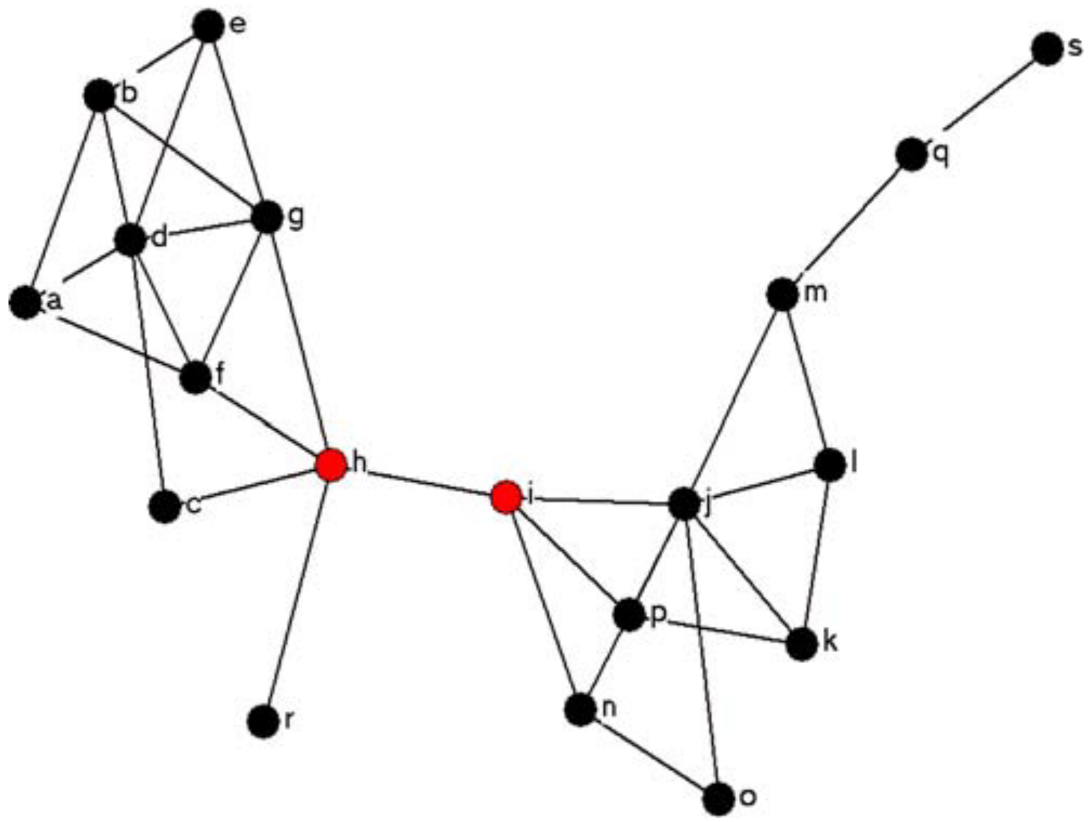
“Everything should be made as simple as possible, but not simpler.”

-A. Einstein

The key player problem

"Given a network/graph, find the agent/node which, if removed, would maximally disrupt communication among the remaining nodes."

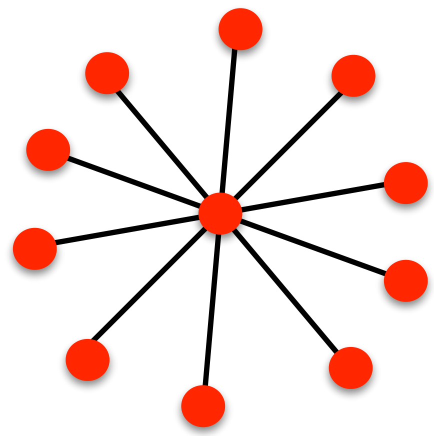




Borgatti : "The key player problem" (2003); Comput Math Organiz Theor (2006)

The key player problem : an answer

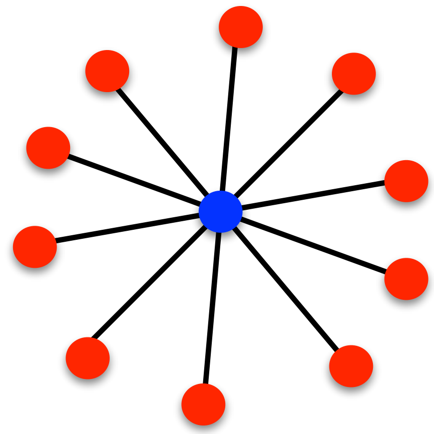
Star graph :



Which node should one remove to maximally disrupt communication among the remaining nodes ?

The key player problem : an answer

Star graph :



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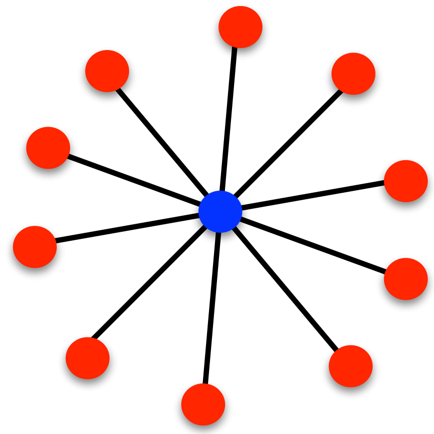

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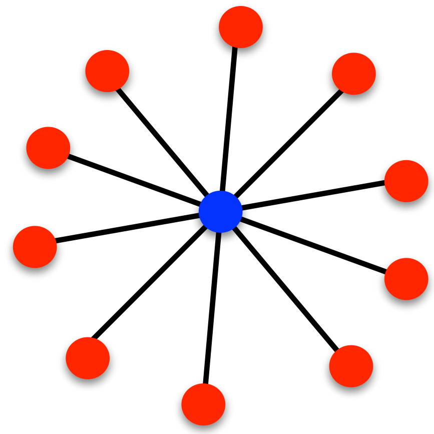

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* with largest degree

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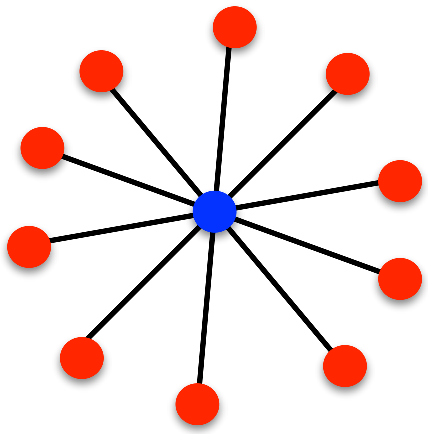

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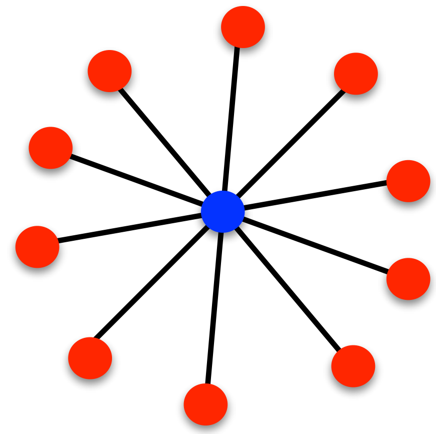
...but then the central node is the node

* with largest degree
* that is closest to all other nodes

The key player problem : an answer

Star graph :



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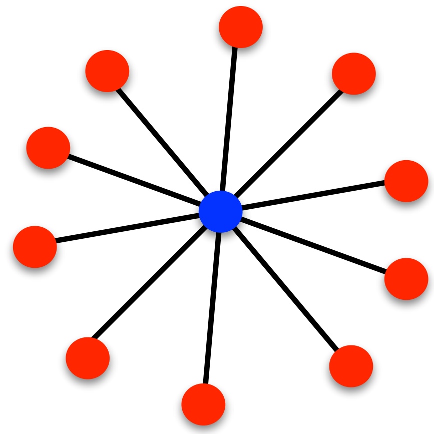

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* that maximizes the dominant eigenvector of the adjacency matrix

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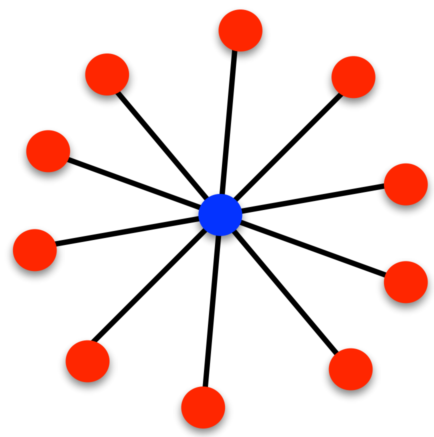

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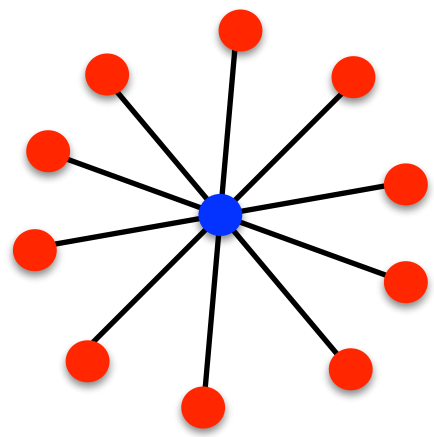

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Which of these property makes the central node the key player ?

The key player problem : an answer

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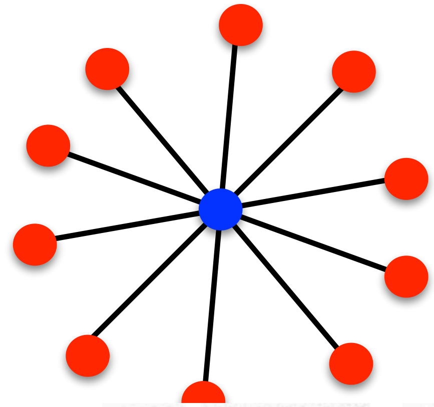

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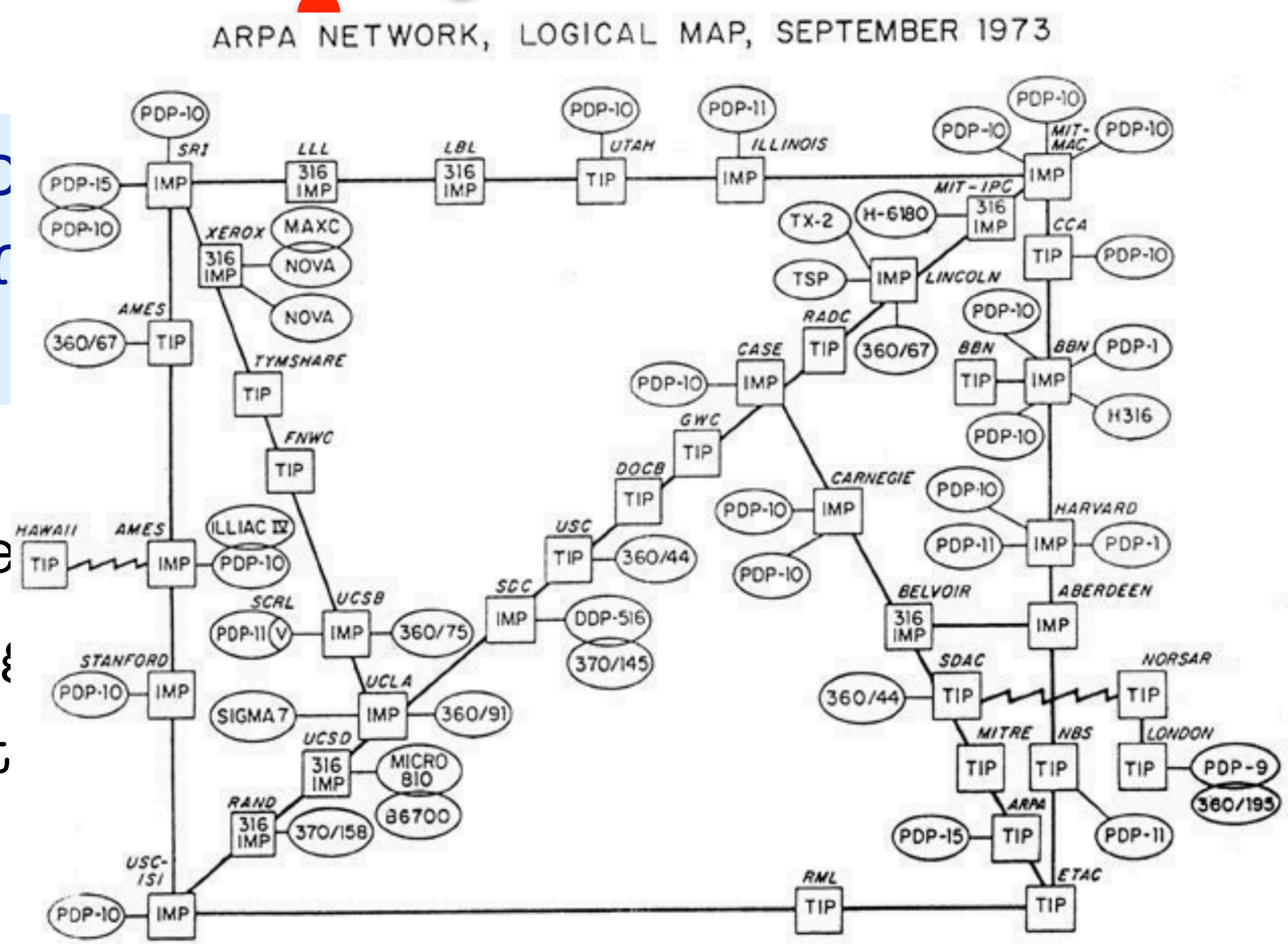

The whole internet, 1973

...but then the ce

* with largest de

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* with the highe
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The key player problem : an answer

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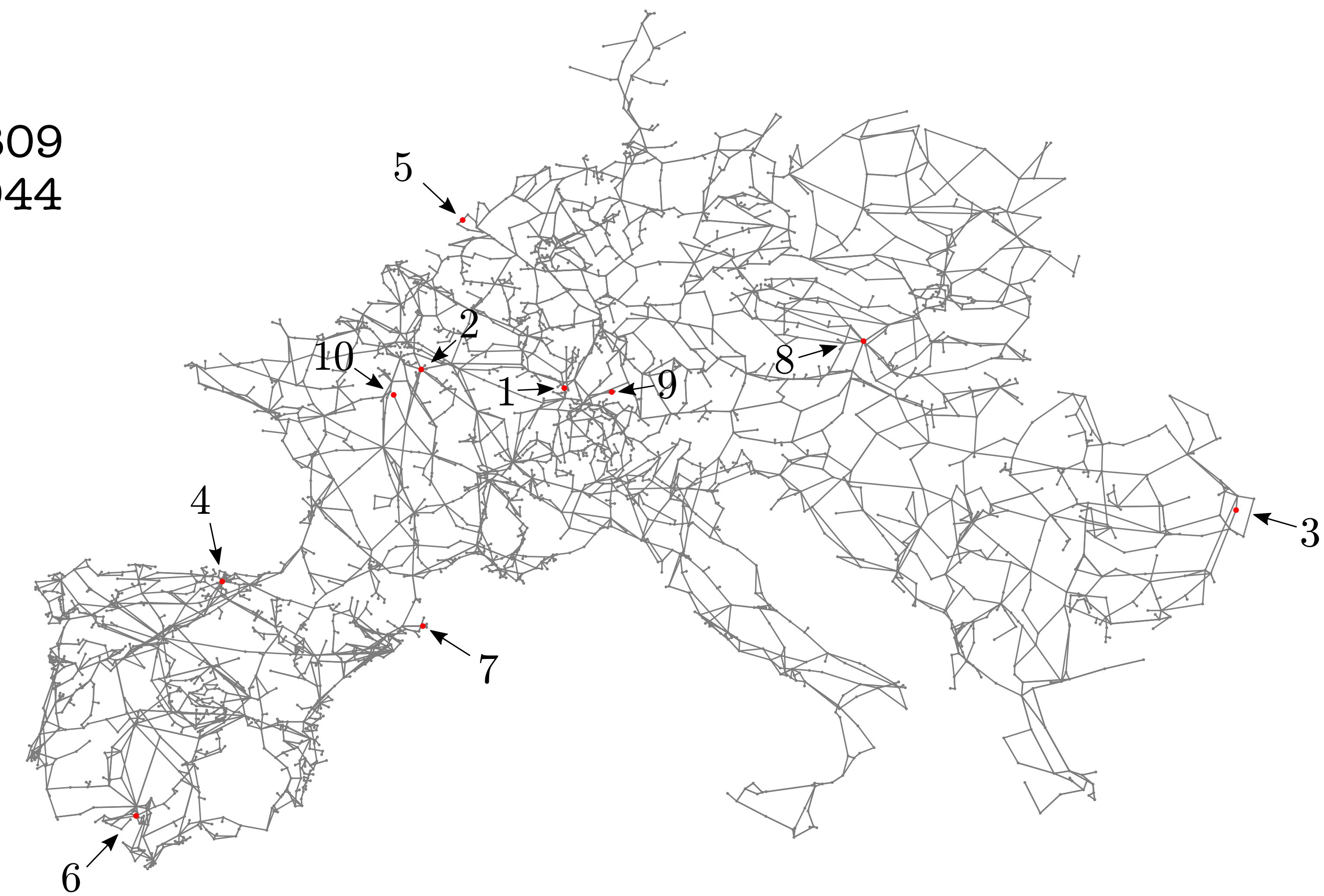
#nodes : 3809
#edges : 4944



Power grid of continental Europe

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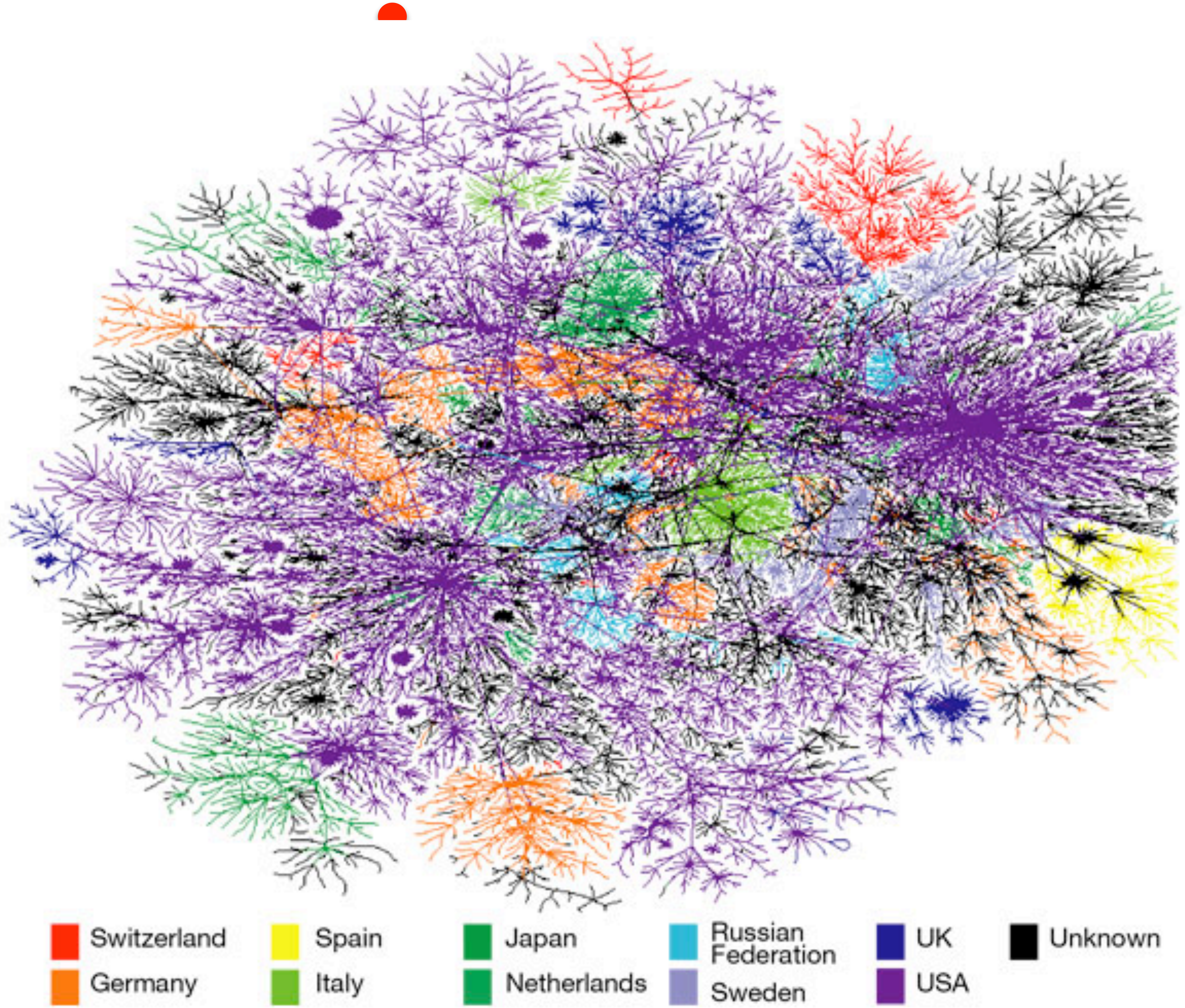

The whole internet, 2013

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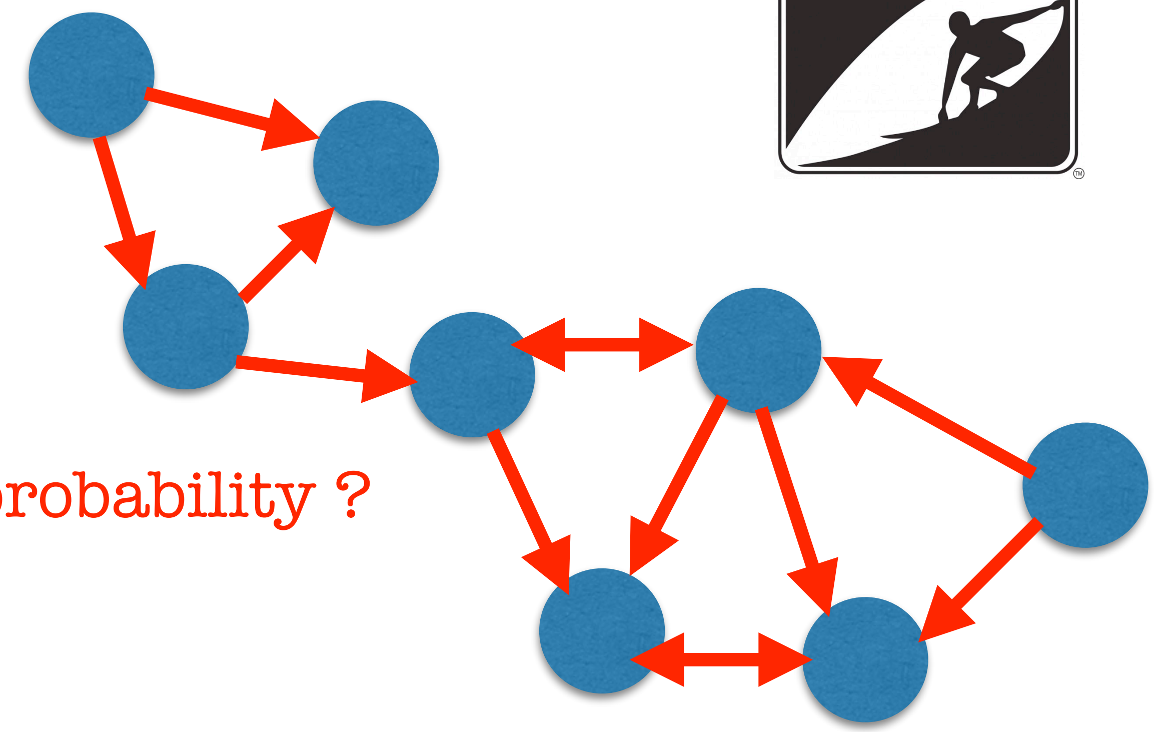
The key player problem : Google's answer

From Markov to PageRank:

“Internet surfing as random walk
on a directed network”

? Where do you end up with the highest probability ?



Brin and Page : Comput. Netw. ISDN Syst. (1998)

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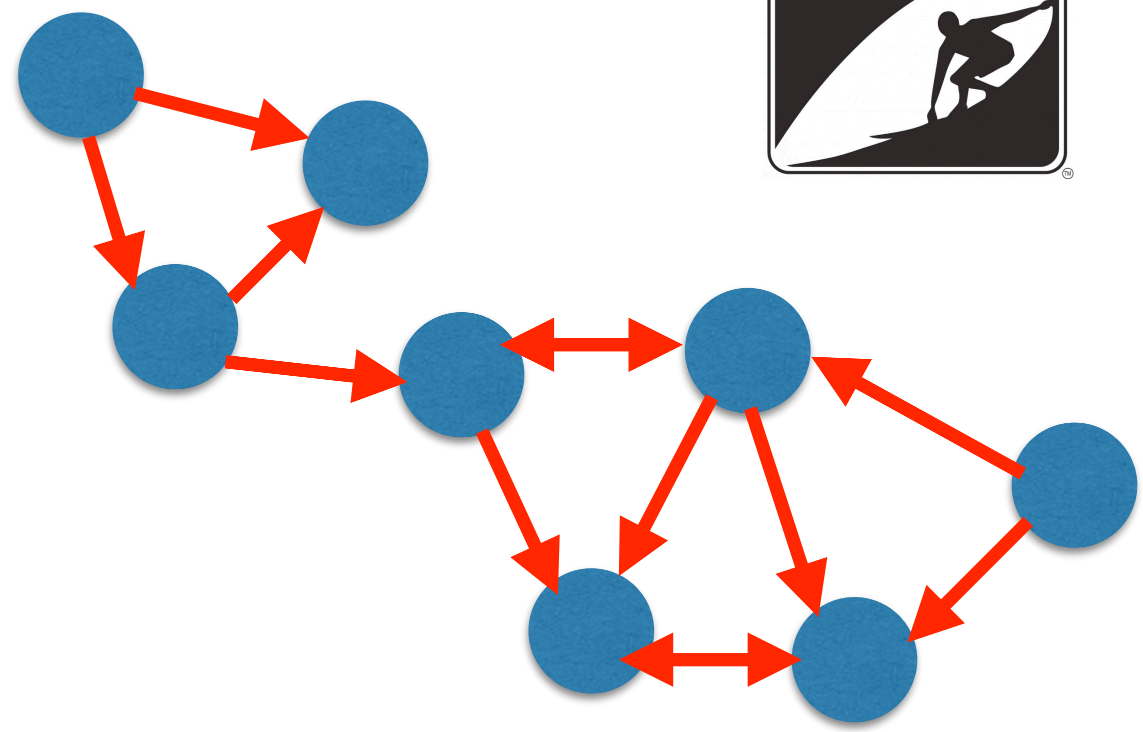

From Markov to PageRank:

"Internet surfing as random walk on a directed network"

(i) define the adjacency matrix

$$A_{ij} = \begin{cases} 1 \text{ if } j \to i \\ 0 \text{ if } j \nrightarrow i \end{cases}$$

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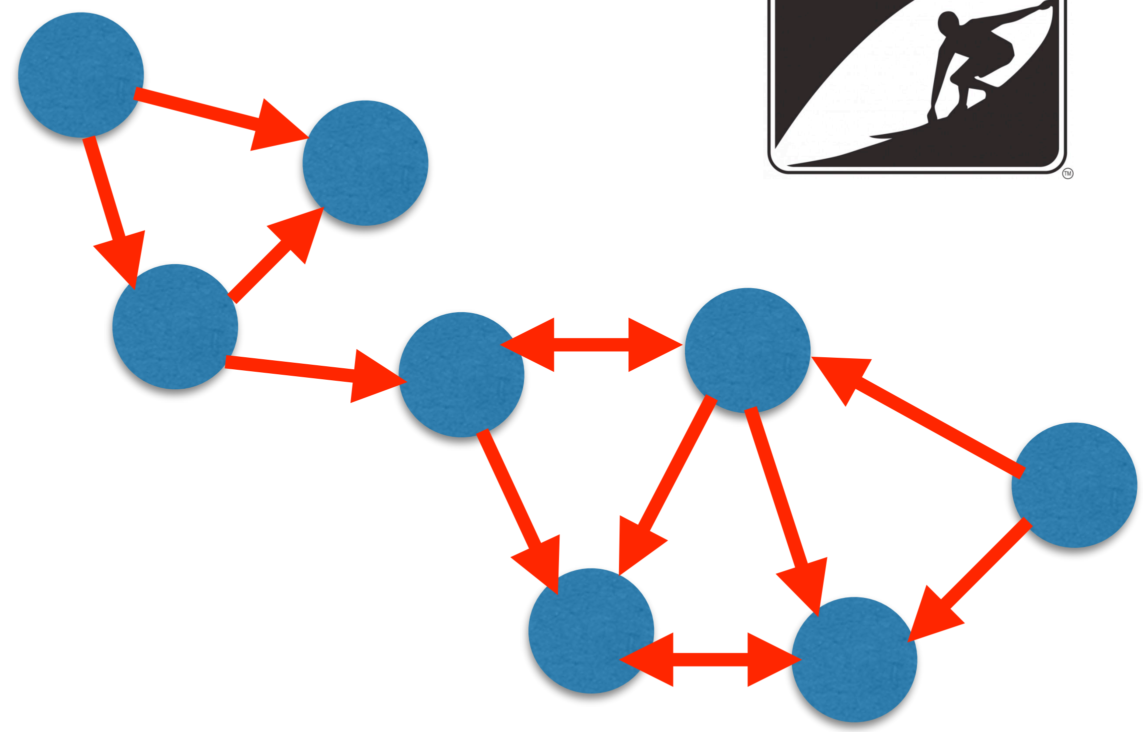

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$$A_{ij} = \begin{cases} 1 \ \text{ if } \ j \to i \\ 0 \ \text{ if } \ j \nrightarrow i \end{cases}$$

(ii) define the stochastic (Markov process) matrix

$$S_{ij} = \begin{cases} A_{ij} / \sum_k A_{kj} \ \text{ if } \ \sum_k A_{kj} \neq 0 \\ 1/N \ \text{ otherwise} \end{cases}$$

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & 1/8 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/8 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 1/8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1/8 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 1/8 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/8 & 1/2 & 1/3 & 0 & 0 & 1 \\ 0 & 0 & 1/8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/8 & 0 & 1/3 & 1 & 1/2 & 0 \end{pmatrix}$$

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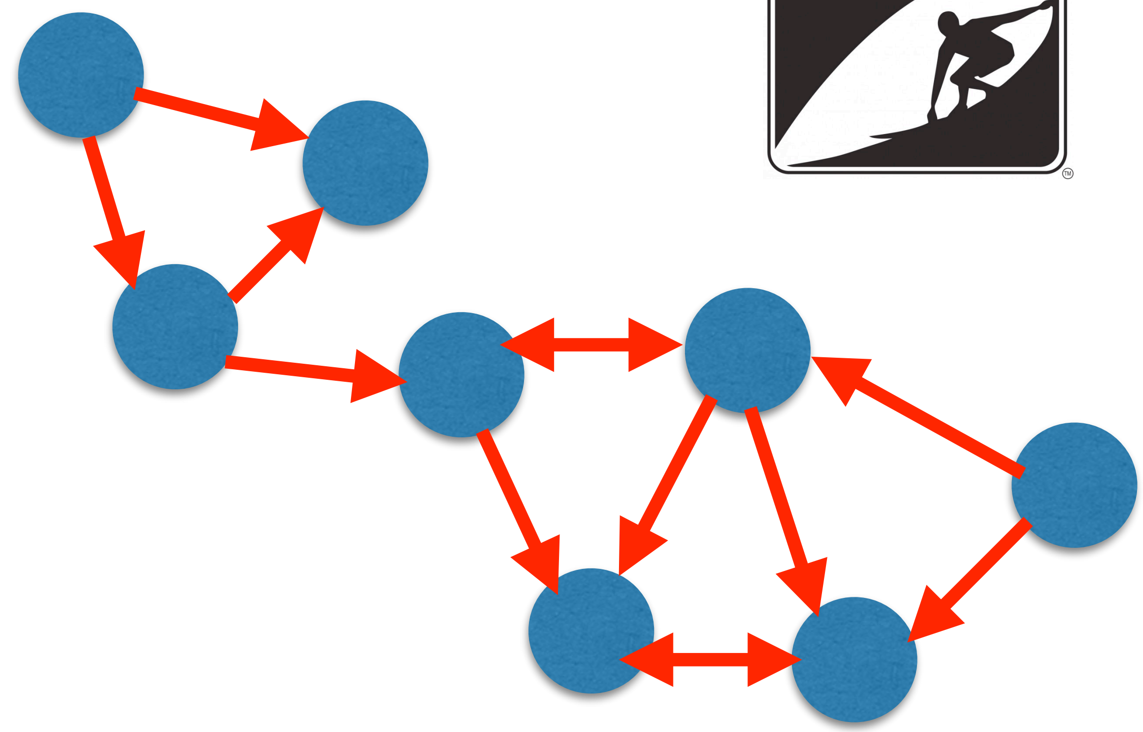

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(iii) define the Google matrix

$$G_{ij} = \alpha S_{ij} + (1-\alpha)/N$$

$$\mathbf{G} = \begin{pmatrix} 1/40 & 1/40 & 1/8 & 1/40 & 1/40 & 1/40 & 1/40 & 1/40 \\ 17/40 & 1/40 & 1/8 & 1/40 & 1/40 & 1/40 & 1/40 & 1/40 \\ 17/40 & 17/40 & 1/8 & 1/40 & 1/40 & 1/40 & 1/40 & 1/40 \\ 1/40 & 17/40 & 1/8 & 1/40 & 7/24 & 1/40 & 1/40 & 1/40 \\ 1/40 & 1/40 & 1/8 & 17/40 & 1/40 & 1/40 & 17/40 & 1/40 \\ 1/40 & 1/40 & 1/8 & 17/40 & 7/24 & 1/40 & 1/40 & 33/40 \\ 1/40 & 1/40 & 1/8 & 1/40 & 1/40 & 1/40 & 1/40 & 1/40 \\ 1/40 & 1/40 & 1/8 & 1/40 & 7/24 & 33/40 & 17/40 & 1/40 \end{pmatrix}$$

$\alpha = 0.8$

The key player problem : Google's answer

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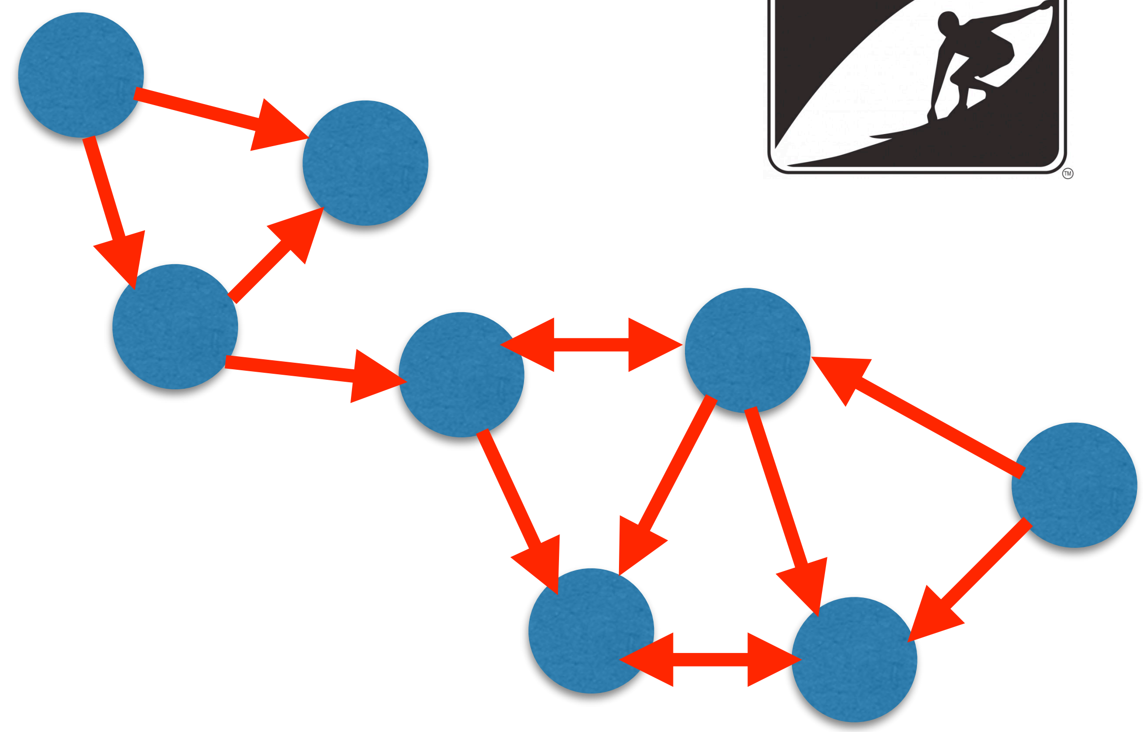

Properties of the Google matrix

$$G_{ij} = \alpha S_{ij} + (1-\alpha)N$$

Only real, positive coefficients
-> one of its eigenvectors has (Perron-Frobenius)
(1) a real, positive eigenvalue with largest absolute value
(2) all its components real and positive

The key player problem : Google's answer

From Markov to PageRank:

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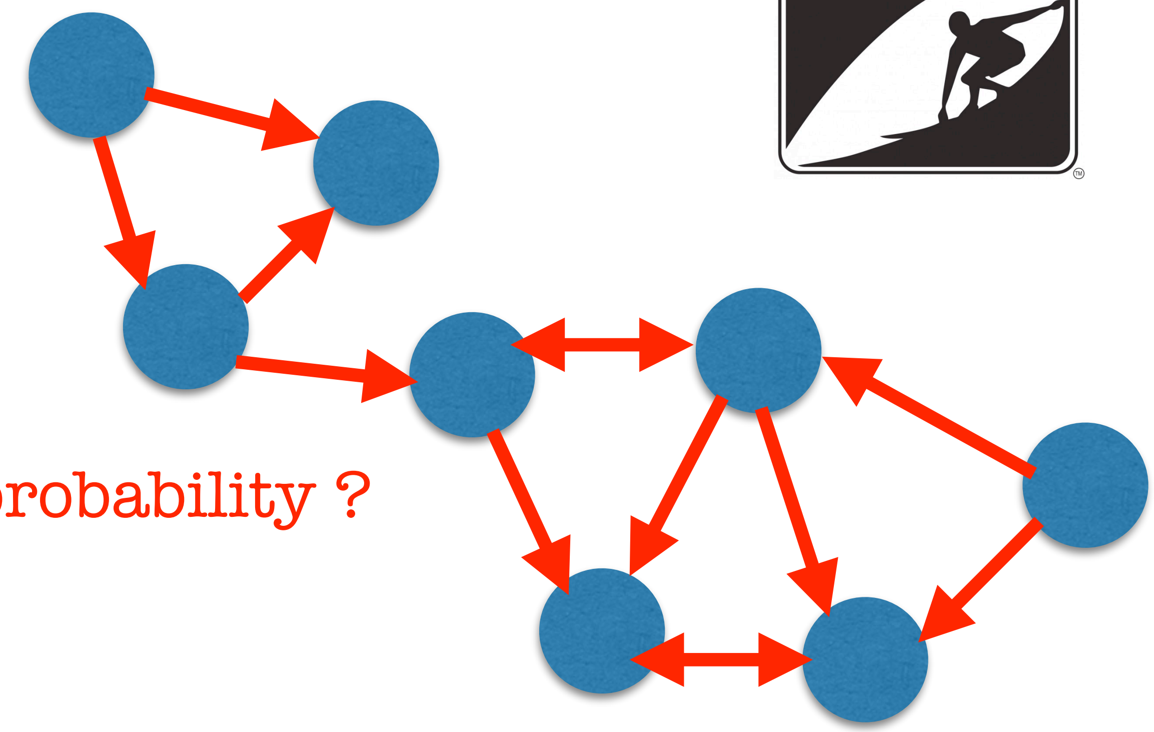

A: on the node with largest component of the Perron-Frobenius mode !

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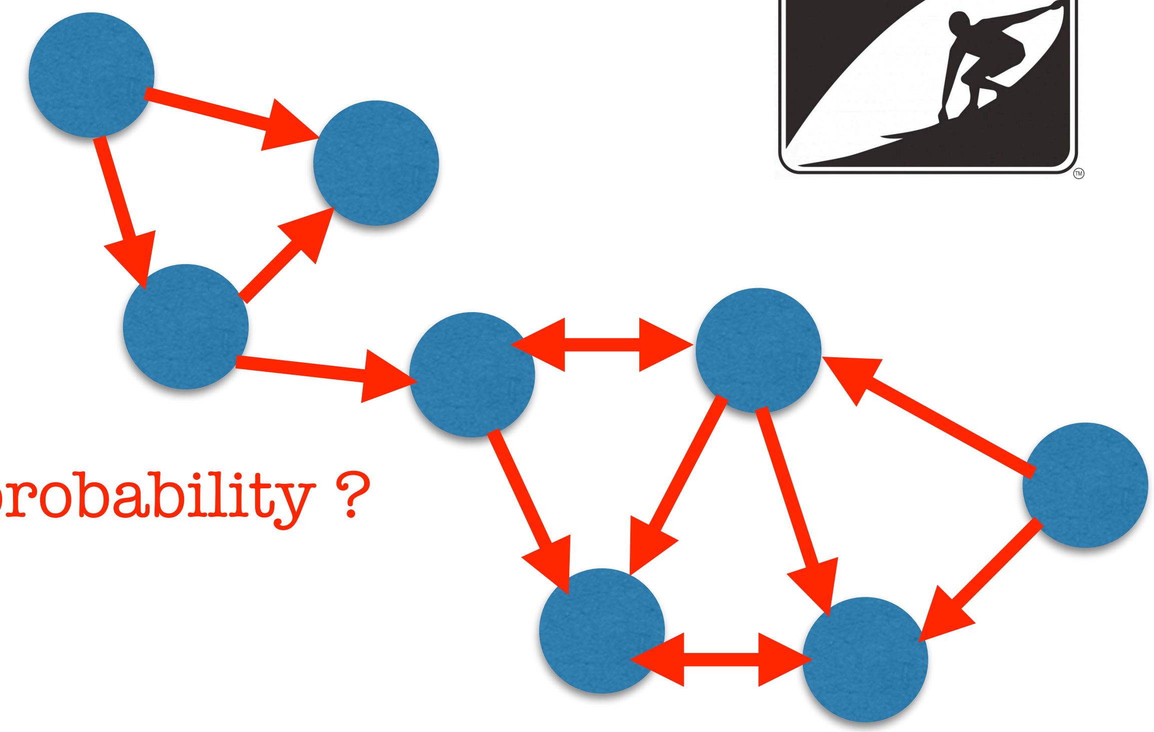

A: on the node with largest component of the Perron-Frobenius mode !

-> PageRank = ranking nodes on a network with the components of the Perron-Frobenius mode

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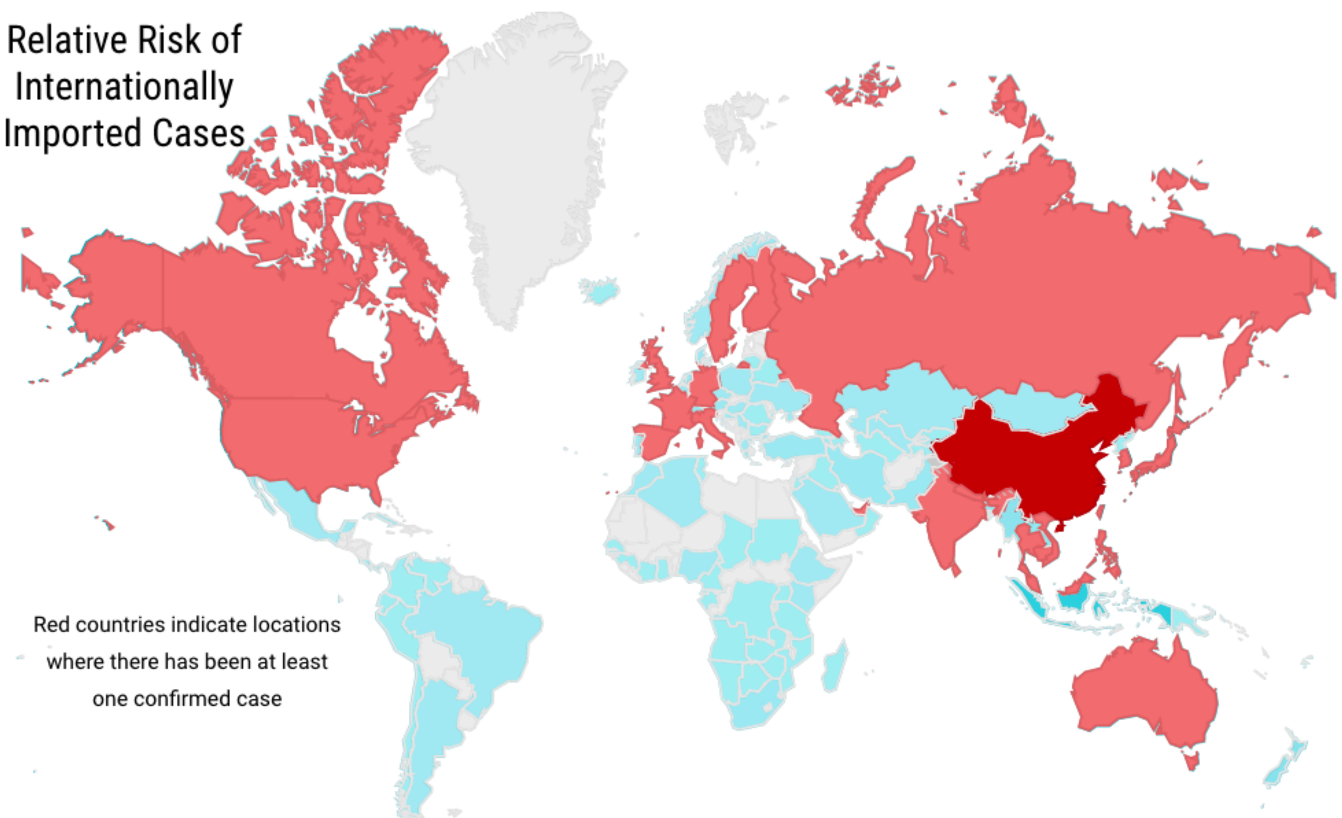
Two classes of networked problems in sciences

1. Random/probabilistic processes on networks: web surfing, epidemia, rumour propagation etc.

Relative Risk of Importing a case of 2019-nCoV

Relative Risk of Internationally Imported Cases



Source: 2019-nCoV Global Cases by John Hopkins CSSE

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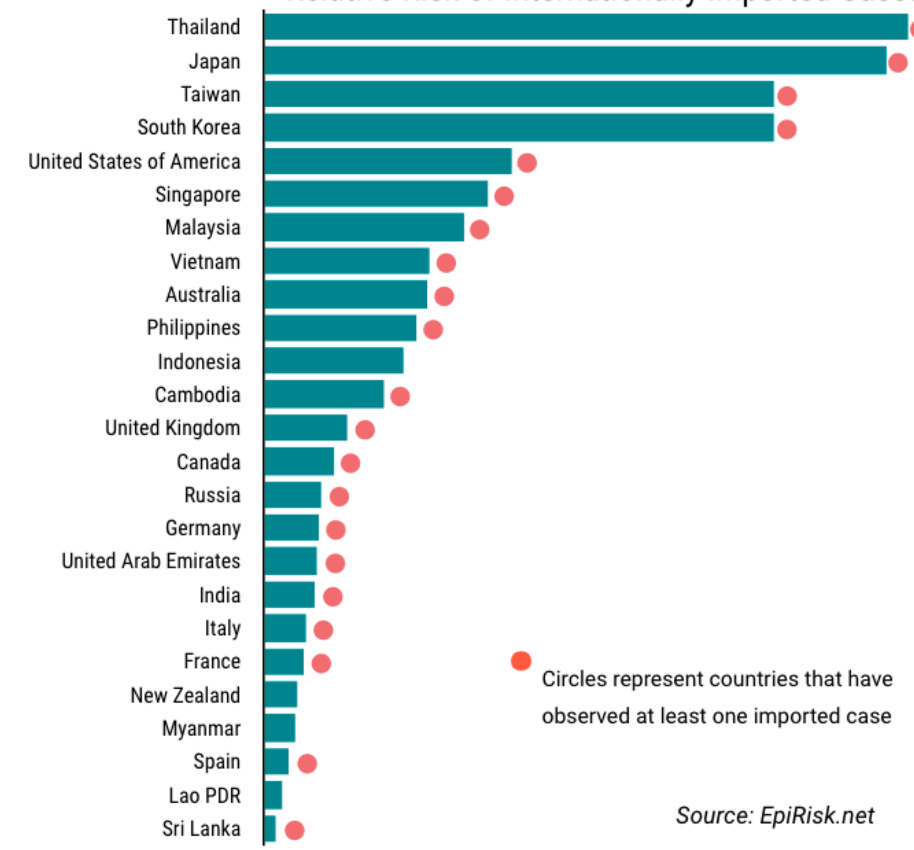


Source: EpiRisk.net

2019-nCoV Cases within China

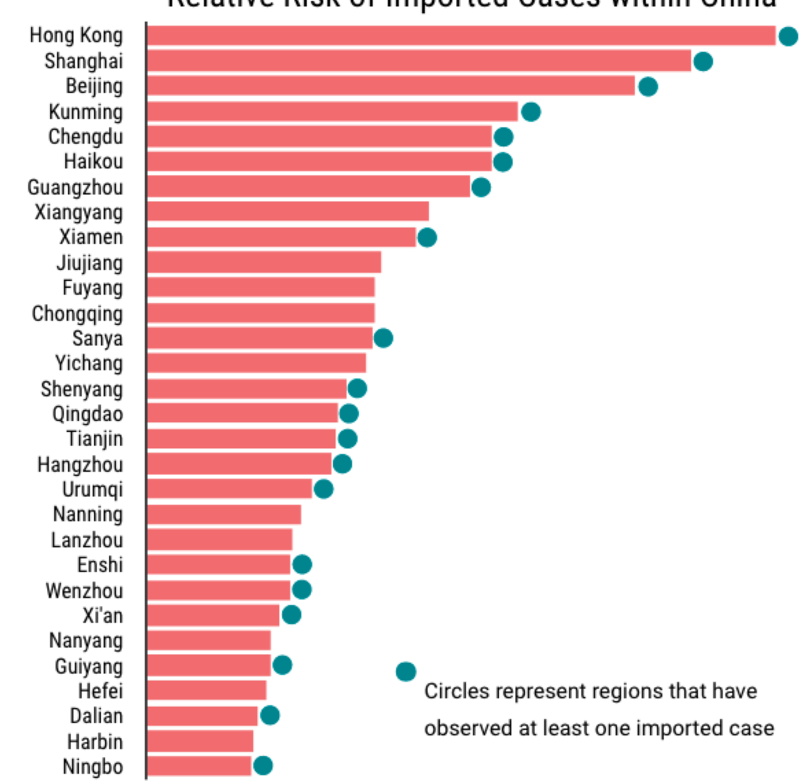
Province	Confirmed ▾	Suspected
Total	17,492	21,558
Hubei	11,177	-
Guangdong	725	-
Zhejiang	724	-
Henan	566	-
Hunan	521	-
Anhui	408	-
Jiangxi	391	-
Chongqing	312	-
Jiangsu	271	-

1 - 10 / 55 ‹ ›

Source: DXY.cn (translated by Kaiyun Sun)

Locations of 2019-nCoV Cases within China

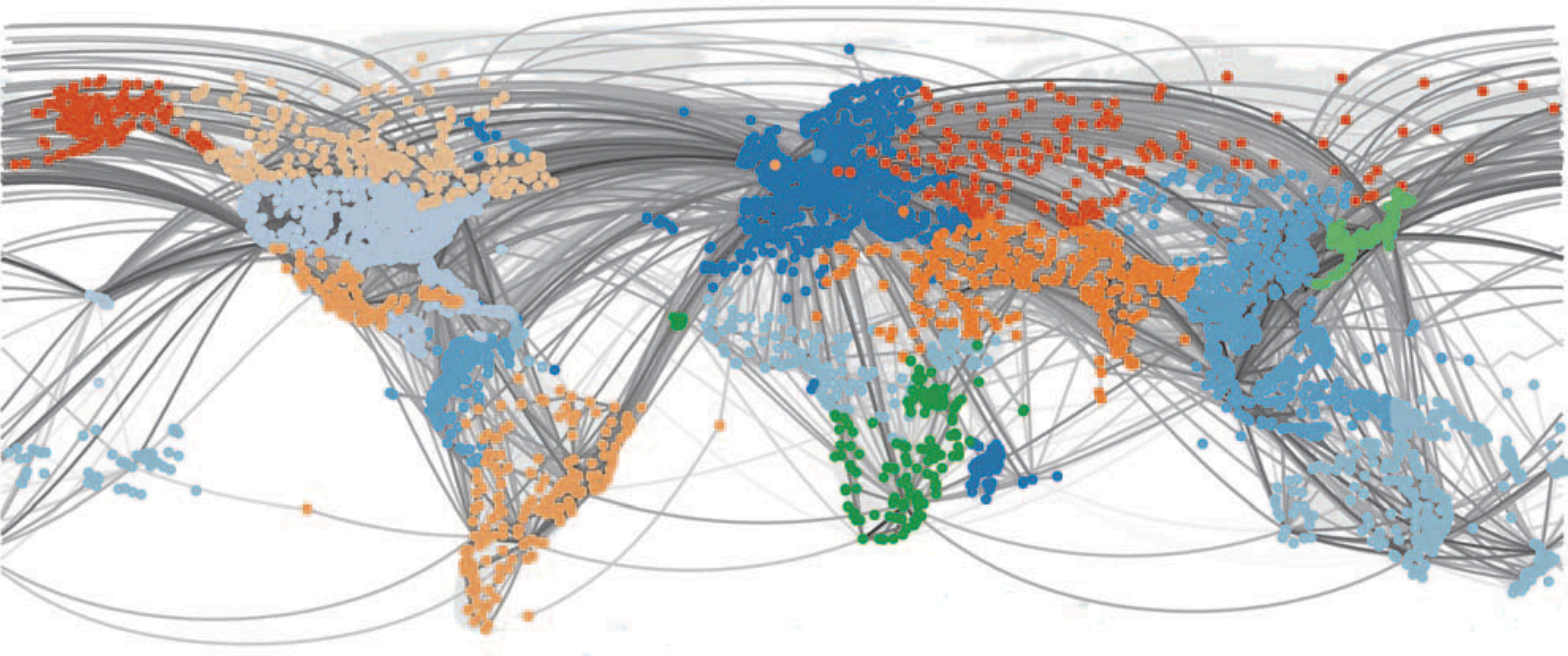
Relative Risk of Imported Cases within China



FONDAZIONE BRUNO KESSLER · NIH Fogarty International Center · Institute for Scientific Interchange Foundation · UF Department of Biostatistics, College of Public Health and Health Professions & College of Medicine · School of Public Health, University of Washington · Fred Hutch Cures Start Here · MOBS LAB · Laboratory for the Modeling of Biological + Socio-technical Systems · netsi · Northeastern Universi Network Science Institute

https://datastudio.google.com/u/0/reporting/3ffd36c3-0272-4510-a140-39e288a9f15c/page/U5lCB

Global mobility network (airlines)



Brockman & Helbing, Science 2013

Global mobility network (airlines)

S(ane) I(nfected) R(ecovered) model :

$$\partial_t S_n = -\alpha I_n S_n / N_n,$$

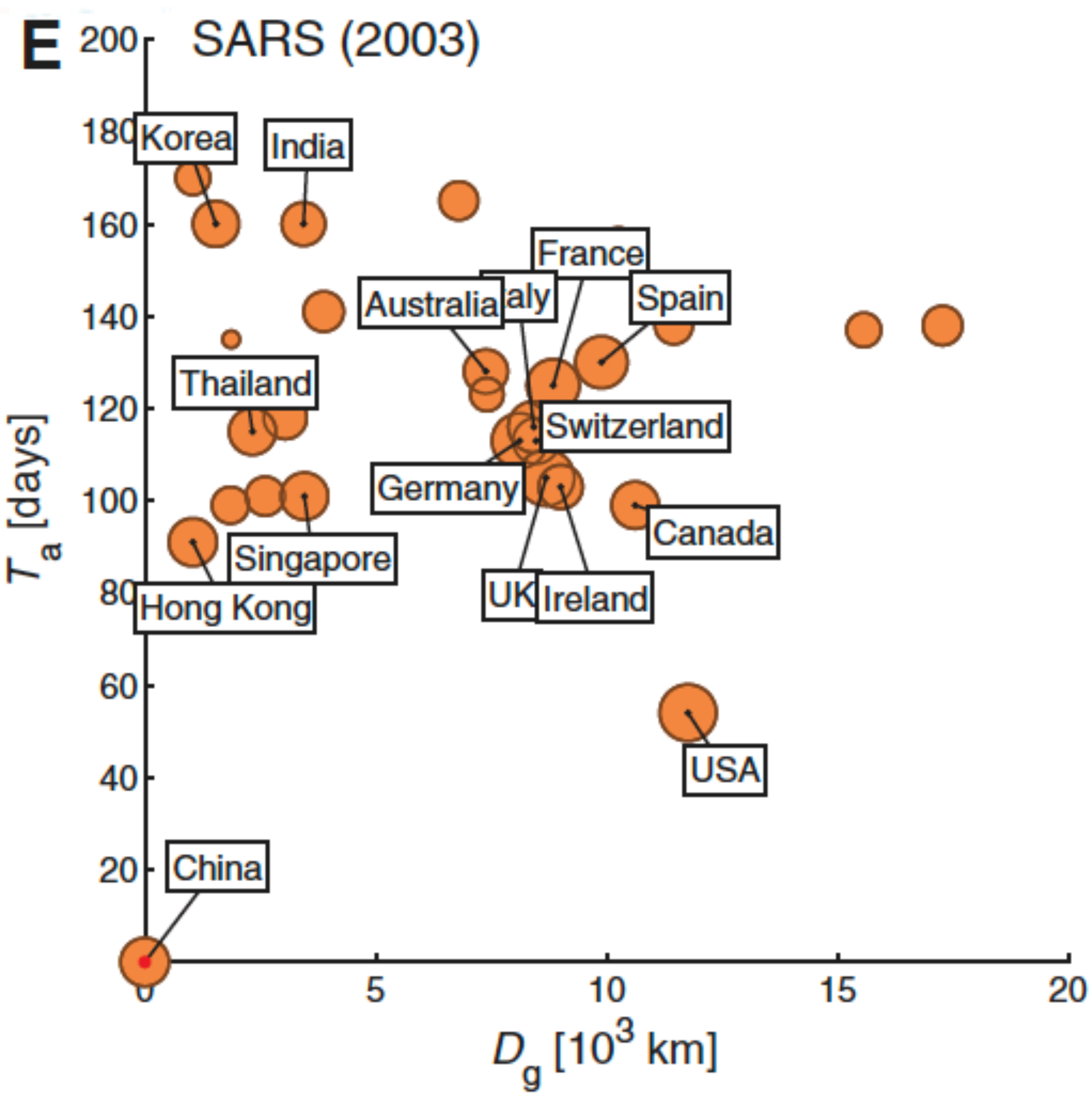
$$\partial_t I_n = \alpha I_n S_n / N_n - \beta I_n \quad n = 1, \ldots, M$$

$$R_n = N_n - S_n - I_n$$

Interactions between different populations n

$$\partial_t U_n = \sum_{m \neq n} w_{nm} U_m - w_{mn} U_n$$

U=S,I,R



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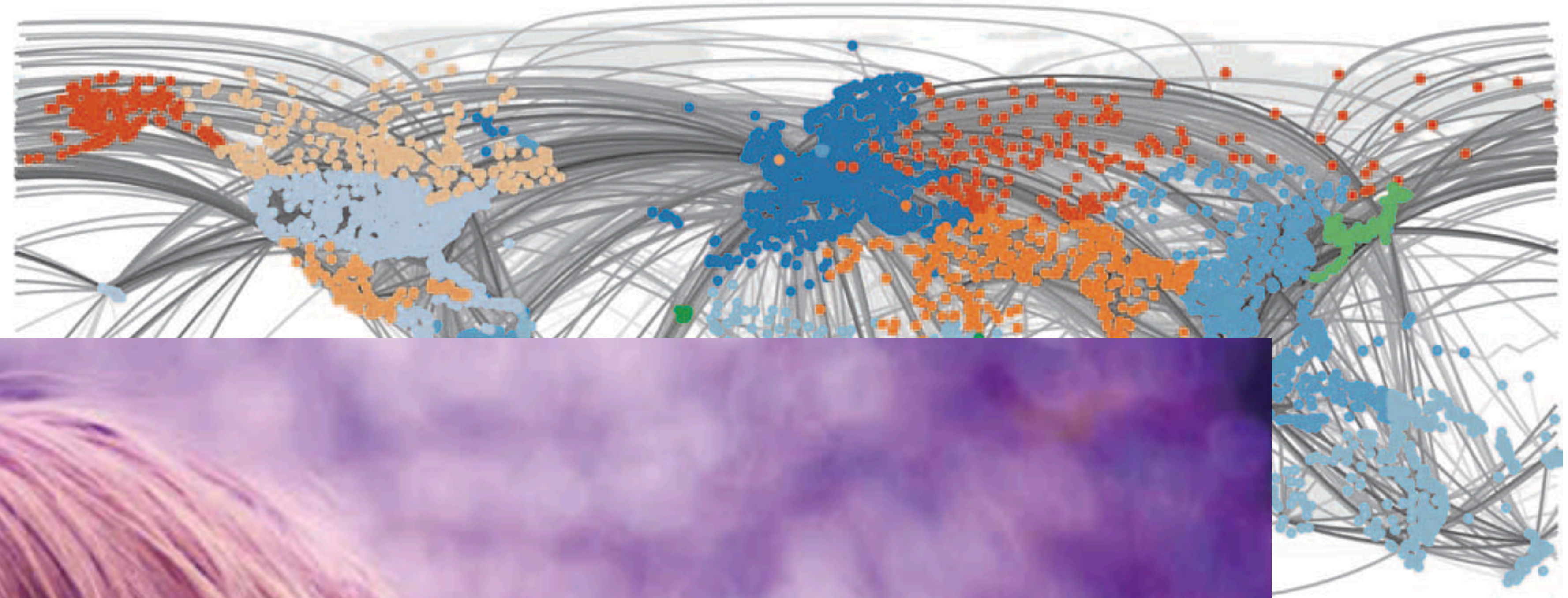

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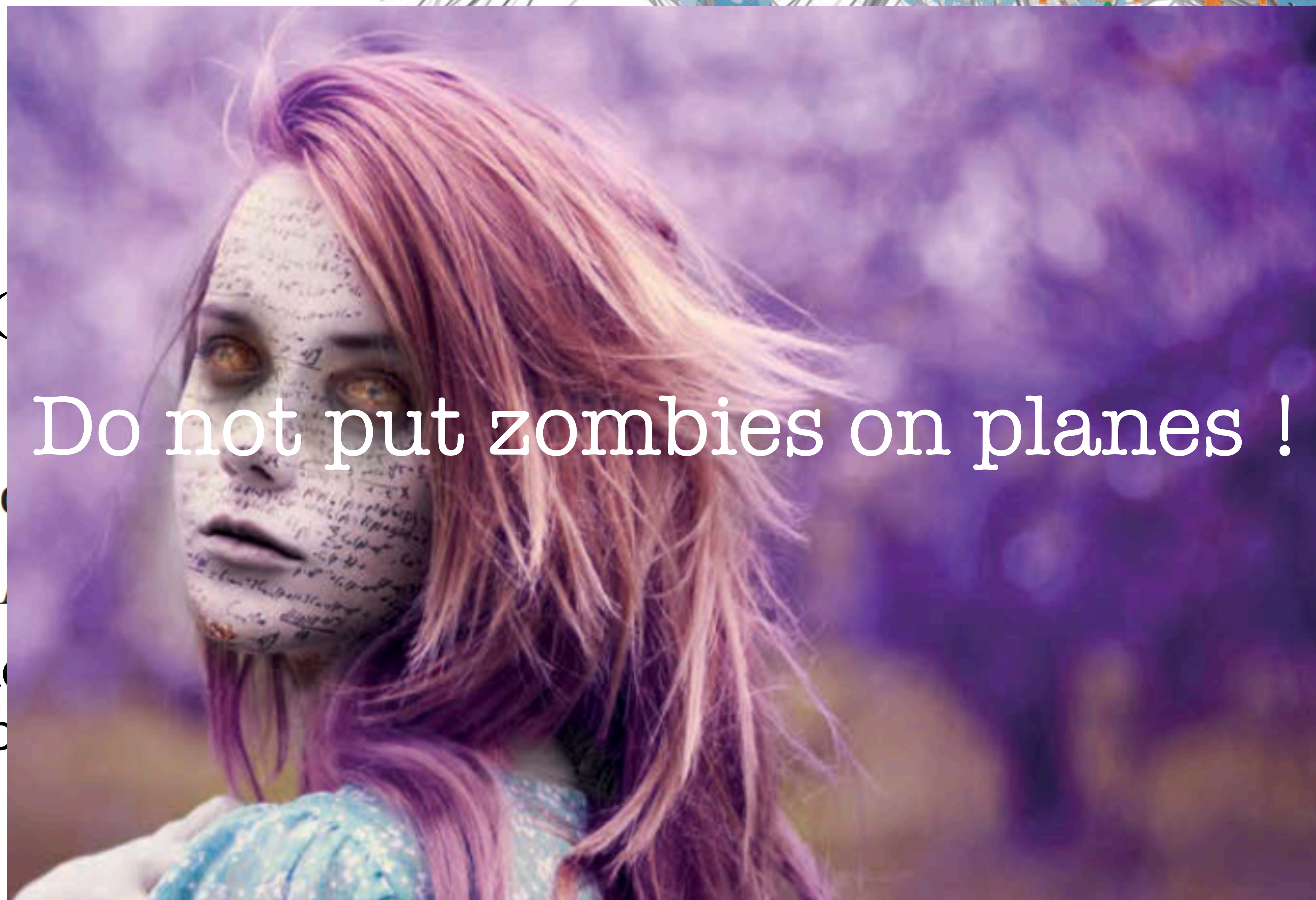

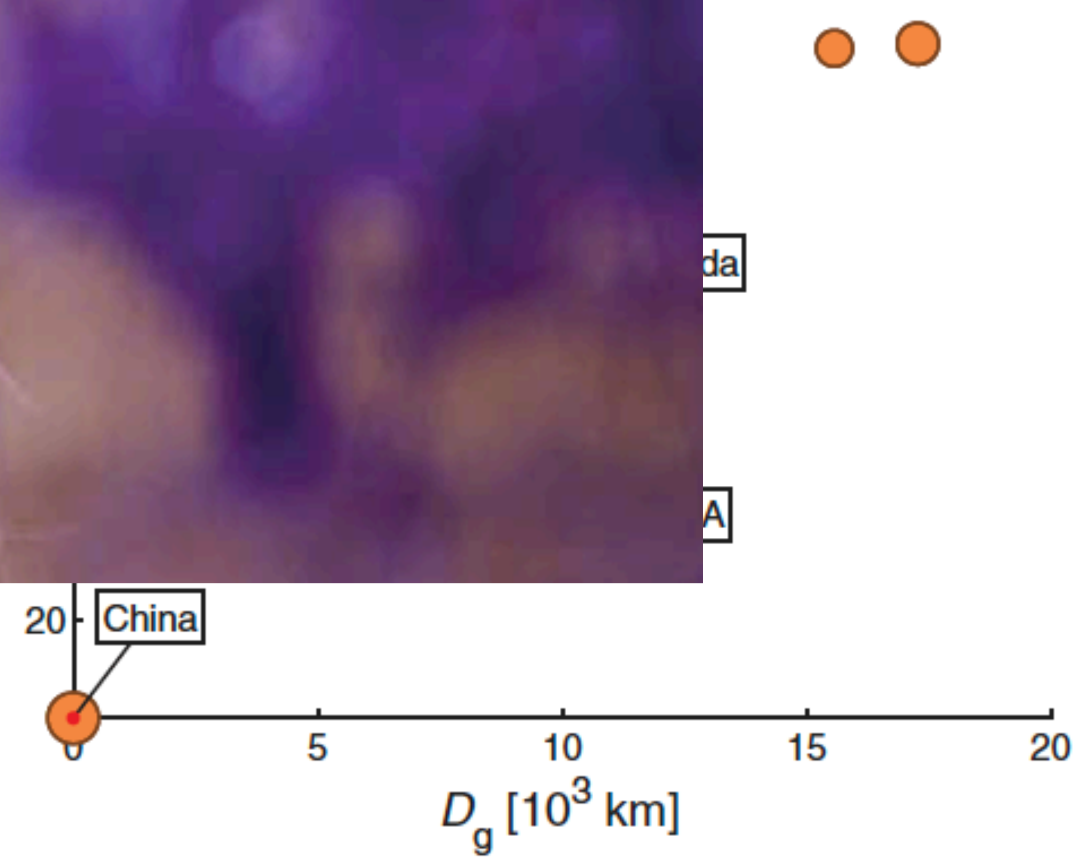

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Brockman & Helbing, Science 2013

Note : why did we start to care with only few hundred deaths in China ?

Answer : exponential increase of I above epidemic threshold

$$\frac{\alpha}{\beta} \simeq \lambda_n^{-1}$$

i.e. ratio of infection with recovery rate vs. Inverse of largest eigenvalue of adjacency matrix

Two classes of networked problems in sciences

1. Random/probabilistic processes on networks: web surfing, epidemia, rumour propagation etc.

2. Physical processes with determinism, conservation laws etc.
   -power grids, neurons, biological systems, Josephson arrays etc.

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!!! They are not the same !!!

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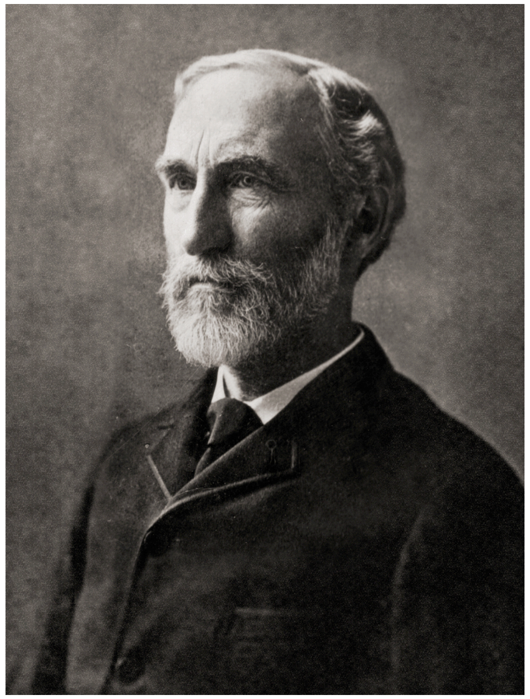

"A mathematician may say anything he pleases, but a physicist must be at least partially sane."

- J. Willard Gibbs

Laplacian matrices, resistance distance and all that jazz

Laplacian matrix, e-vectors and -values $\mathbb{L}\vec{\phi}_\alpha = \lambda_\alpha \vec{\phi}_\alpha$

Klein and Randic, J. Math. Chem. '93

Laplacian matrices, resistance distance and all that jazz

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Undirected graph $\lambda_1 = 0 \qquad \phi_{1,i} = n^{-1/2}$

$$\sum_i \phi_{\alpha,i} = 0 \quad \forall \alpha \geq 2$$

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Resistance distance
(pseudo-inverse of L)

$$\boxed{\Omega_{ij} = \mathbb{L}^\dagger_{ii} + \mathbb{L}^\dagger_{jj} - \mathbb{L}^\dagger_{ij} - \mathbb{L}^\dagger_{ji}}$$

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~graph-theoretic distance metric (zero from i to i; positive; triangle inequality)

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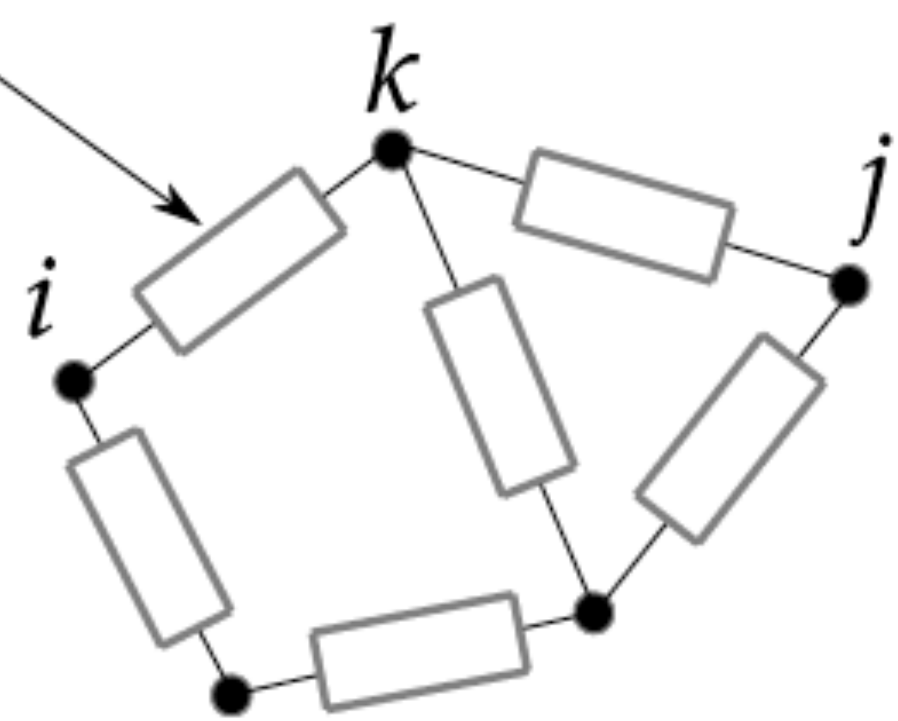
$$R_{ik} = 1/\mathbb{L}_{ik}$$



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Undirected graph $\quad \lambda_1 = 0 \qquad \phi_{1,i} = n^{-1/2}$

$$\sum_i \phi_{\alpha,i} = 0 \quad \forall \alpha \geq 2$$

Resistance distance

$$\boxed{\Omega_{ij} = \mathbb{L}^\dagger_{ii} + \mathbb{L}^\dagger_{jj} - \mathbb{L}^\dagger_{ij} - \mathbb{L}^\dagger_{ji} = \sum_{\lambda \geq 2} \frac{(\phi_{\alpha,i} - \phi_{\alpha,j})^2}{\lambda_\alpha}}$$

Klein and Randic, J. Math. Chem. '93; Tyloo, Pagnier, PJ, Sc. Adv. '19

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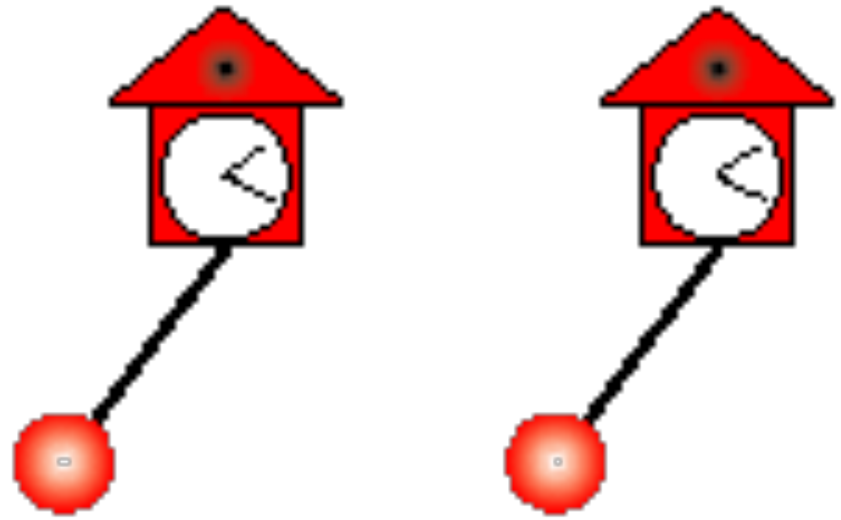
Graph efficiency
A.k.a. "Kirchhoff index"

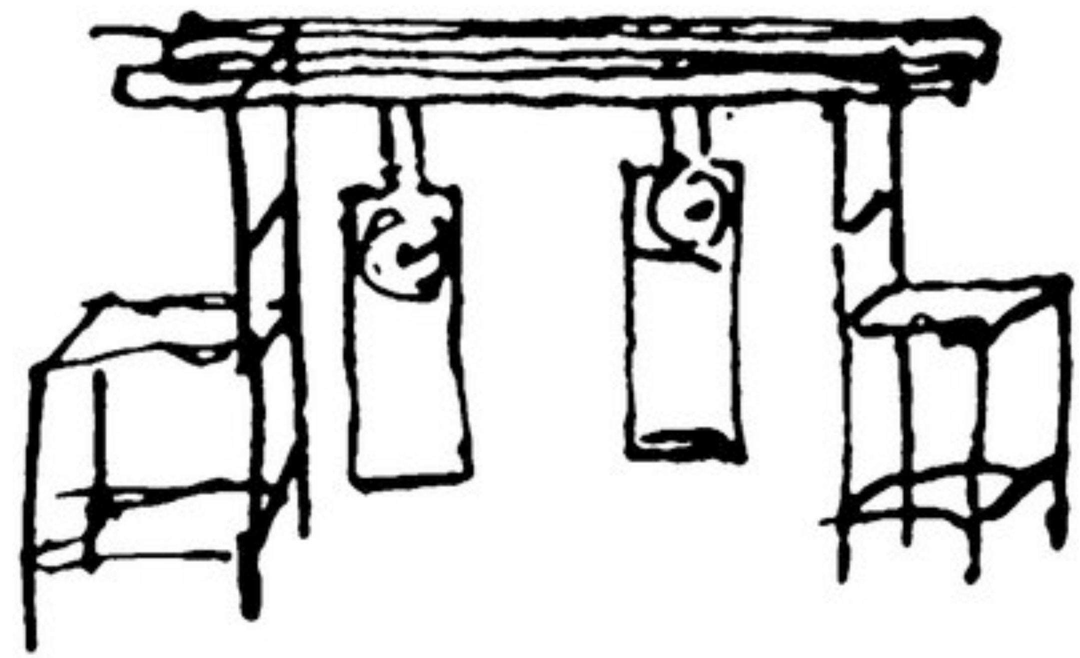
$$Kf_1 = \sum_{i<j} \Omega_{ij} = n \sum_{\alpha \geq 2} \lambda_\alpha^{-1}$$

Klein and Randic, J. Math. Chem. '93; Tyloo, Pagnier, PJ, Sc. Adv. '19

Synchrony and how to understand it

C Huygens





"... when we suspended two clocks (...), the motions of each pendulum in opposite swings were so much in agreement that (...) the sound of each was always heard simultaneously. Further, if this agreement was disturbed by some interference, it reestablished itself in a short time. (...) After a careful examination I finally found that the cause of this is due to the motion of the beam, even though this is hardly perceptible."

A. Pikovsky and M. Rosenblum
scholarpedia.org/article/Synchronization

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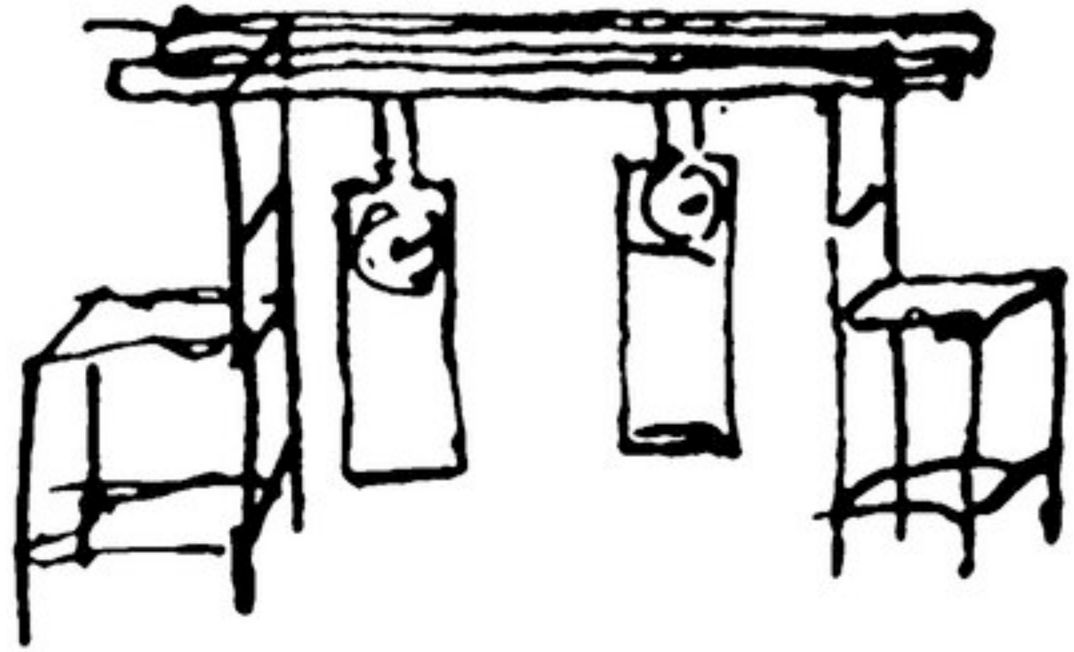

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Jacobian (giving coupling between i and j) must be negative (semi)definite

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Jacobian (giving coupling between i and j) must be negative (semi)definite

- Take minus a Laplacian matrix $\mathbb{A} \to -\mathbb{L}$
- Consensus dynamics ~opinion formation (computer sciences, social networks, vehicle platoons etc.)

$$\dot{\mathbf{x}} = -\mathbb{L}\mathbf{x}$$

Consensus algorithm:
JK French 1956, F Harary, 1959, MH DeGroot, 1974

Synchrony and how to understand it

- Synchronization between oscillators :
  i.e. compact variables ~periodicity

$$\mathbf{x} \to \boldsymbol{\theta}$$

$$\theta_i \in [-\pi, \pi[$$

- Oscillators have natural frequencies

$$\theta_i(t) = \theta_i(0) + P_i t$$

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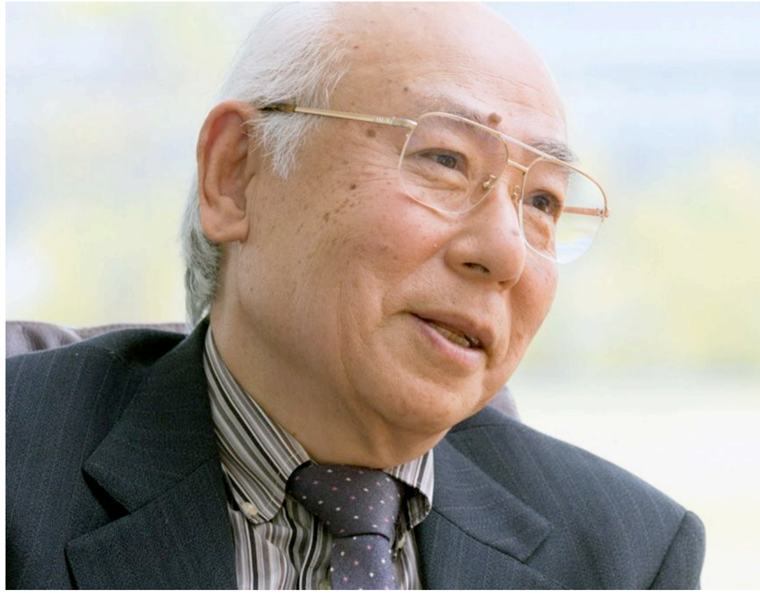

Yoshiki Kuramoto

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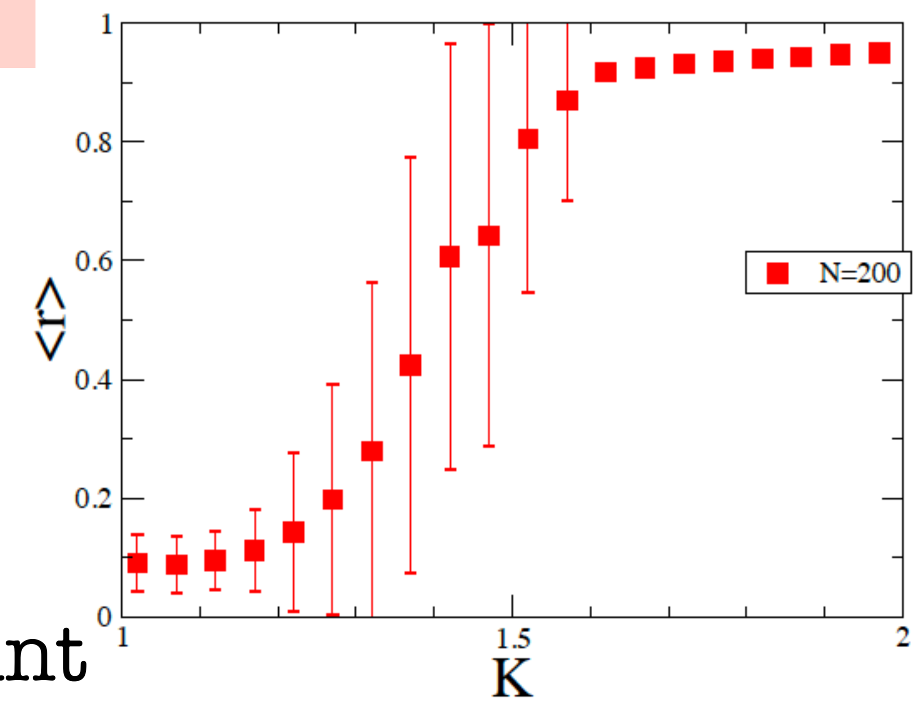
- Winfree '67
- Kuramoto '84 : all-to-all coupling
  mfa + introduce order parameter

$$r(t) = \left| \frac{1}{N} \sum_j \exp[i\theta_j(t)] \right| = |r(t)| \exp[i\Psi(t)]$$

Bifurcation at $K_c=4/\pi$

$K>K_c$: there is a stable synchronous fixed point



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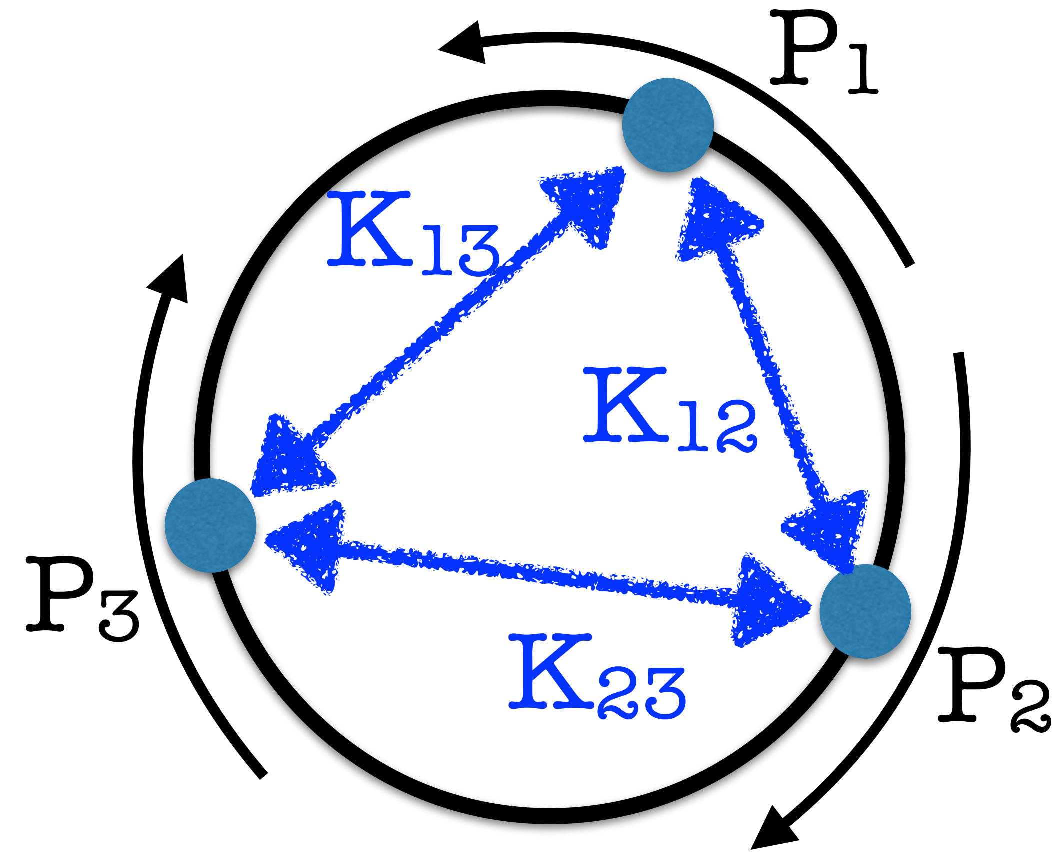
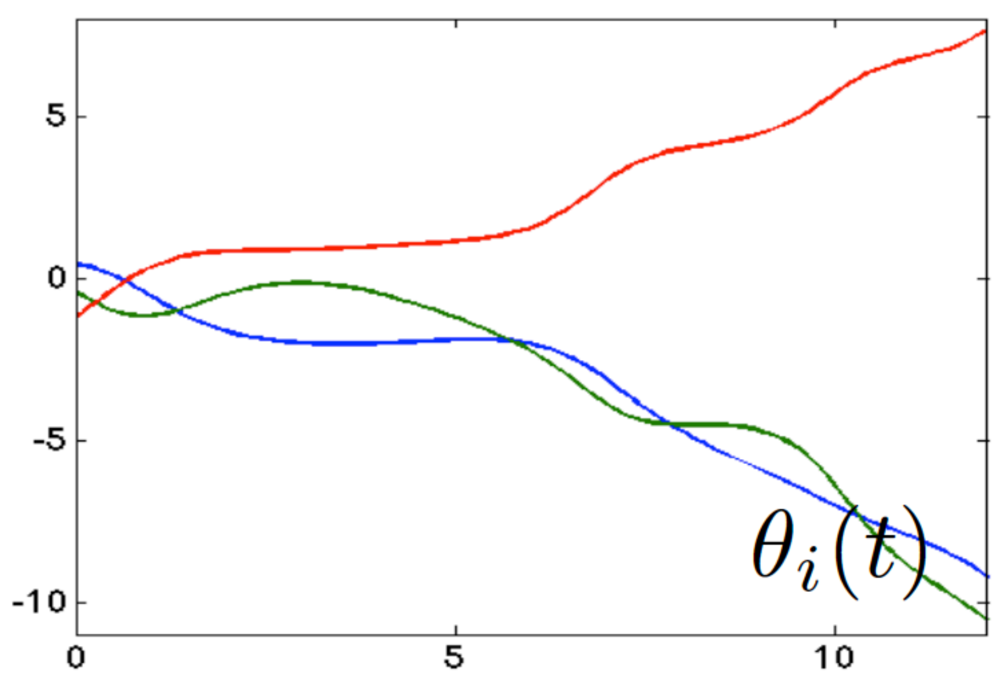
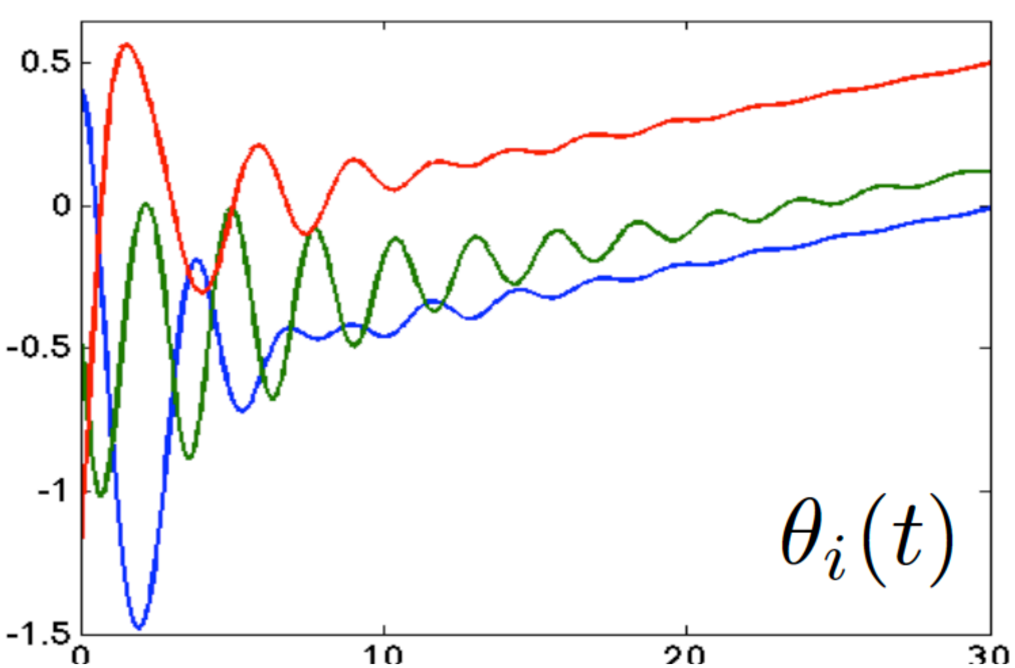


Weak coupling



Strong coupling



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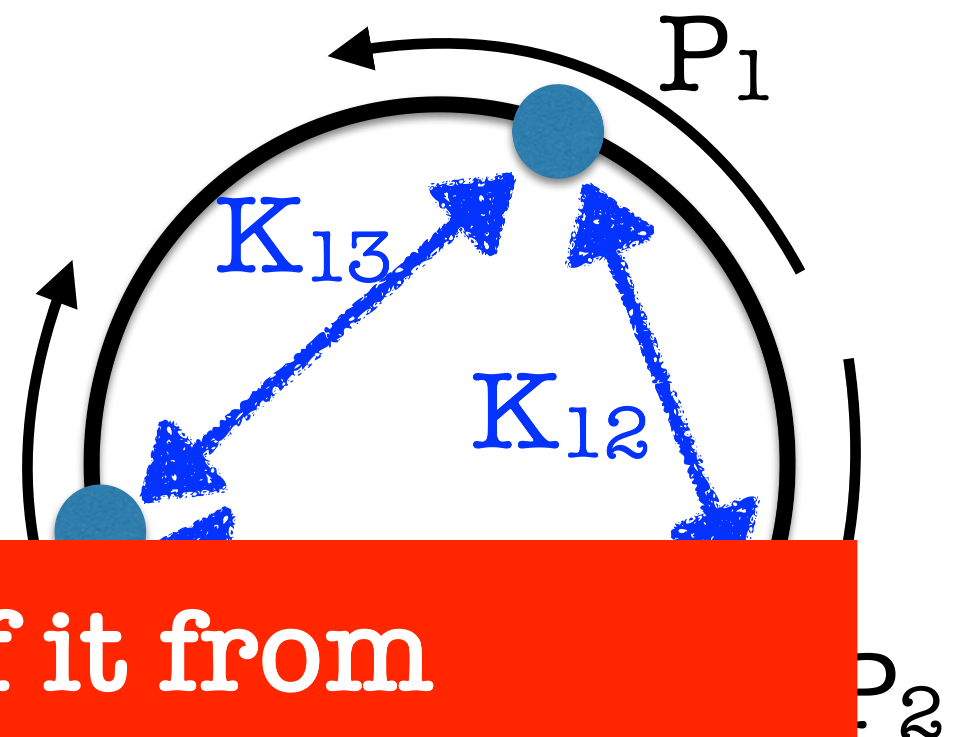
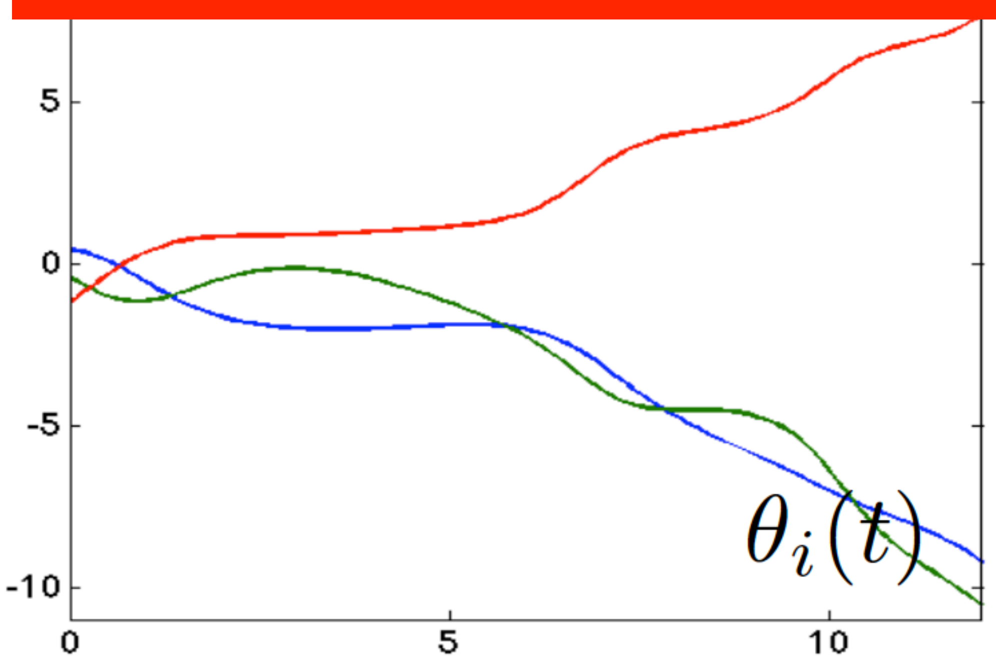
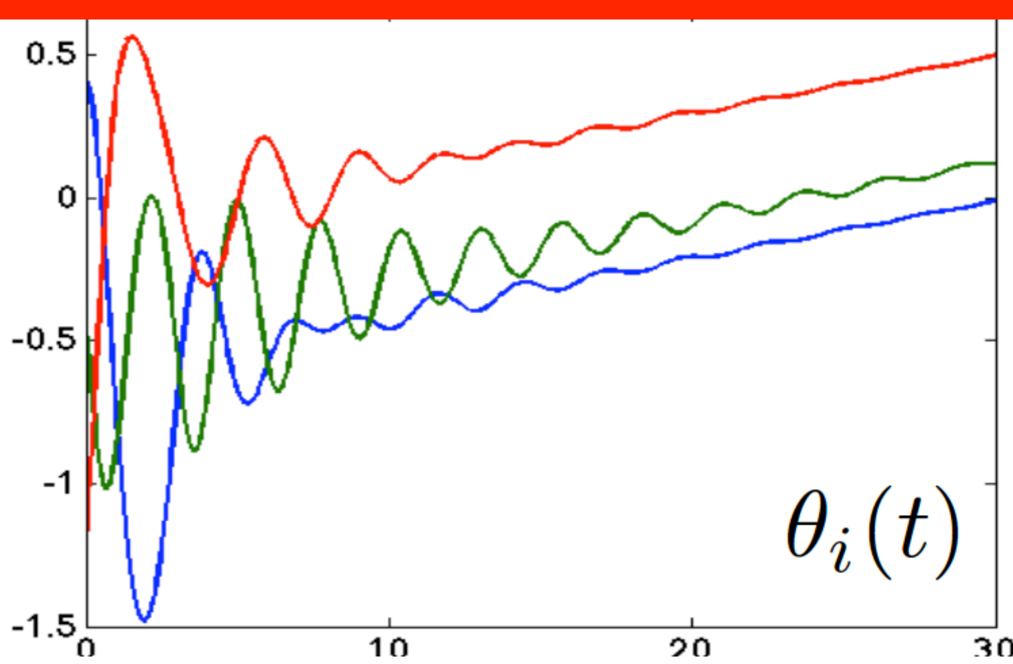


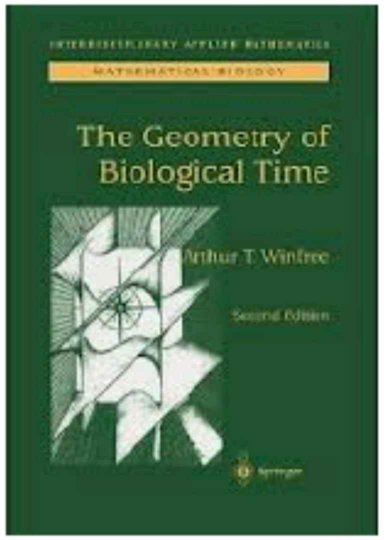
Synchrony/absence of it from competition between natural dynamics and coupling between individual entities

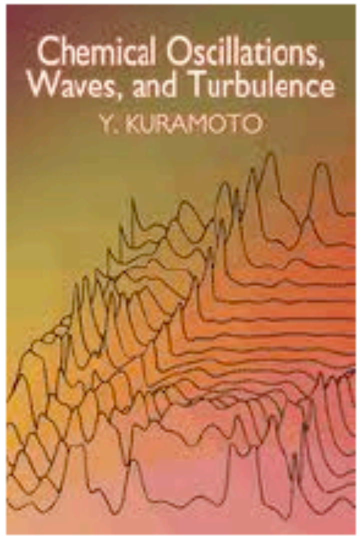


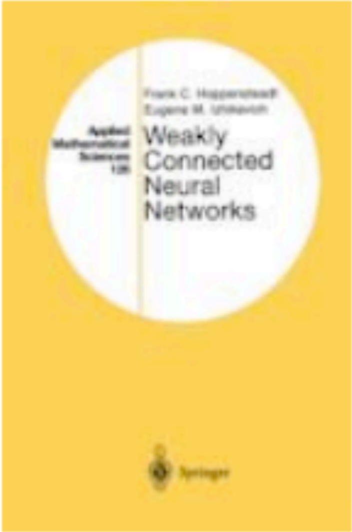


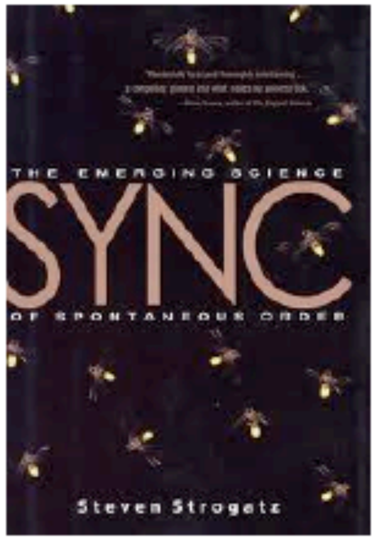
Synchrony is ubiquitous

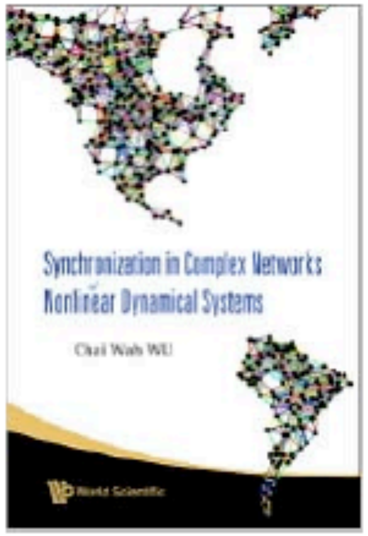
- Sync in mathematical biology [Winfree ’80, Strogatz ’03 …]
- Sync in natural sciences [Kuramoto ’83, Mézard ’87 …]
- Sync in neural networks [Hoppensteadt and Izhikevich ’00 …]
- Sync in complex networks [Wu ’07, Bocaletti ’08 …]
- Today : sync and electric power grids

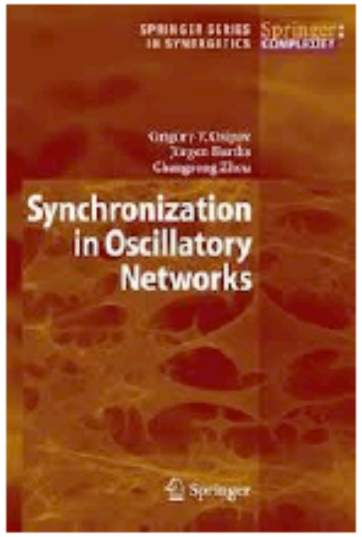


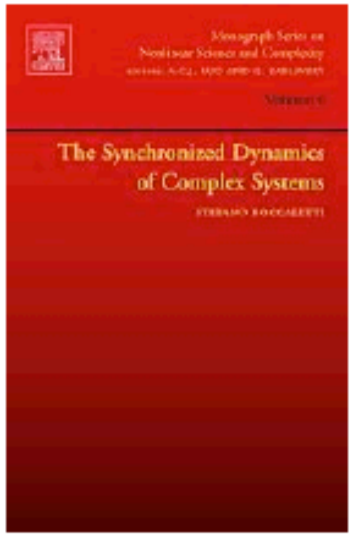












The Kuramoto model : fixed points

$$\frac{d\theta_i}{dt} = P_i - \sum_j K_{ij} \sin(\theta_j - \theta_i)$$

It's a gradient flow : $$-\frac{\partial \mathcal{V}}{\partial \theta_i} = \dot{\theta}_i$$

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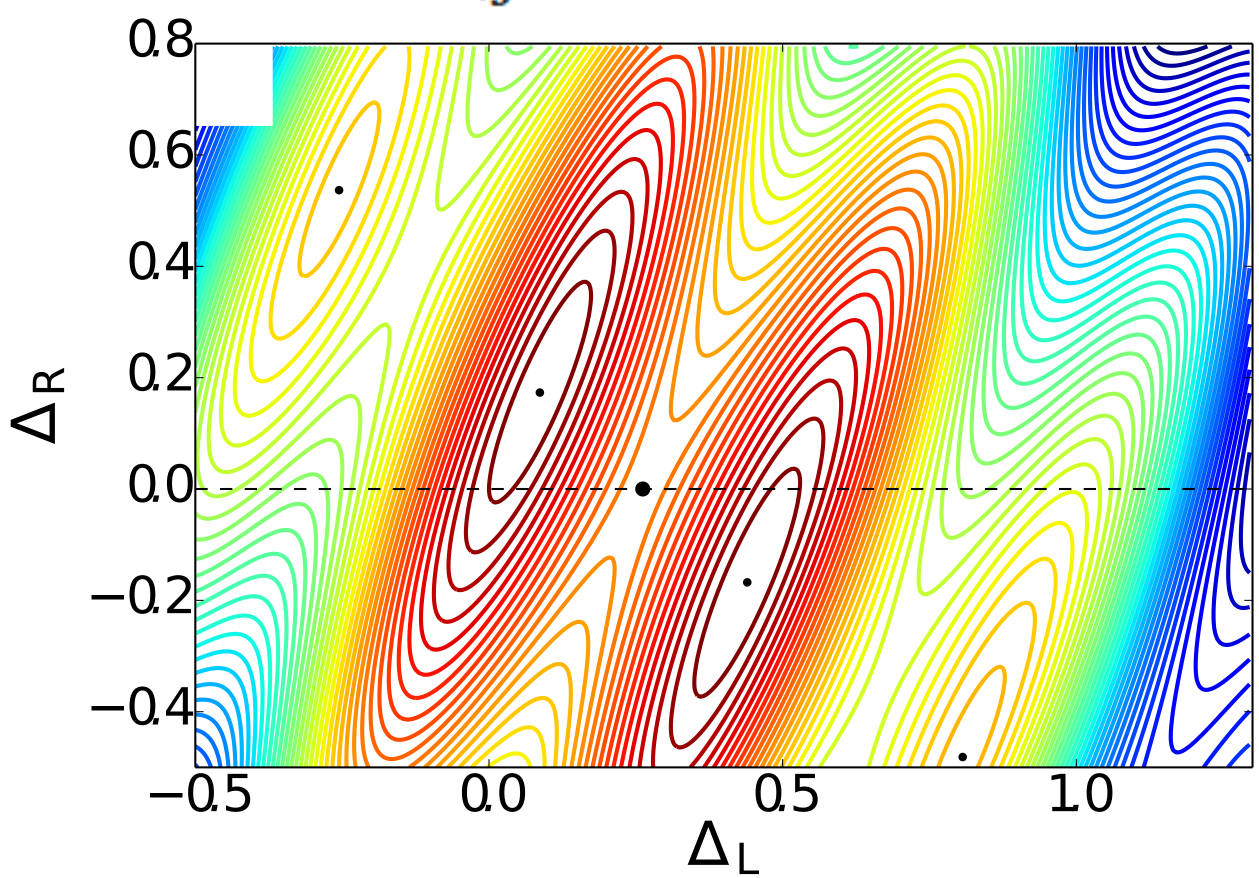

Coletta, Delabays, Adagideli and PJ, New J Phys '16

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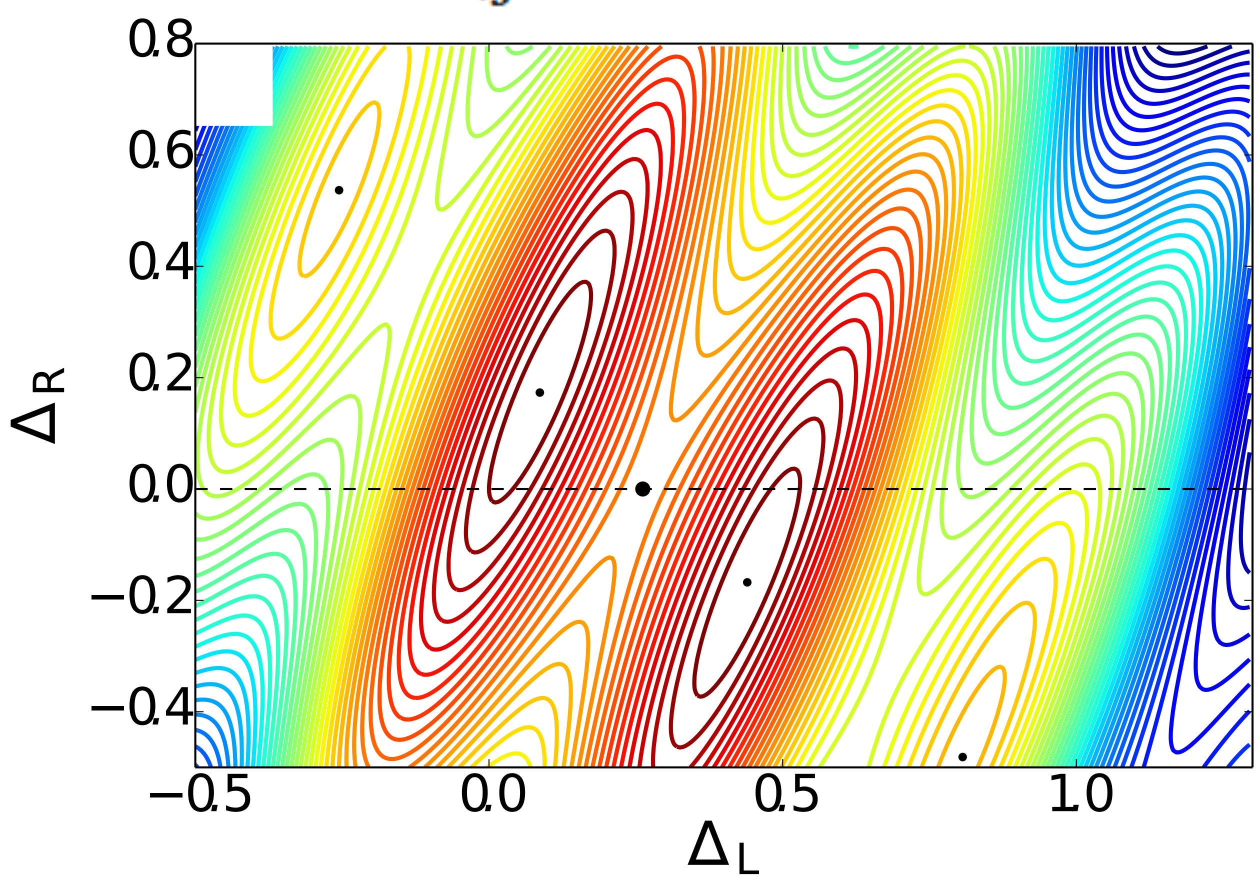

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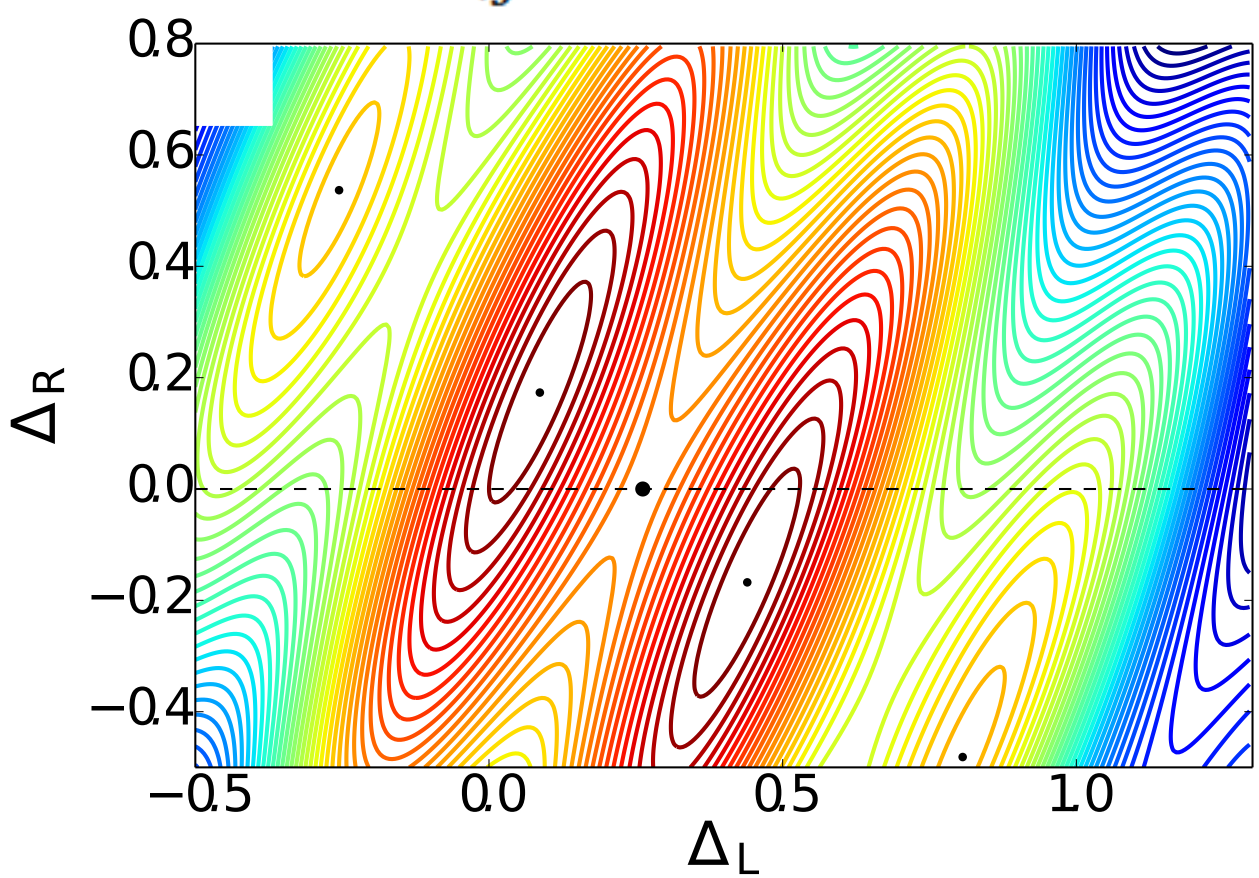


Coletta, Delabays, Adagideli and PJ, New J Phys '16

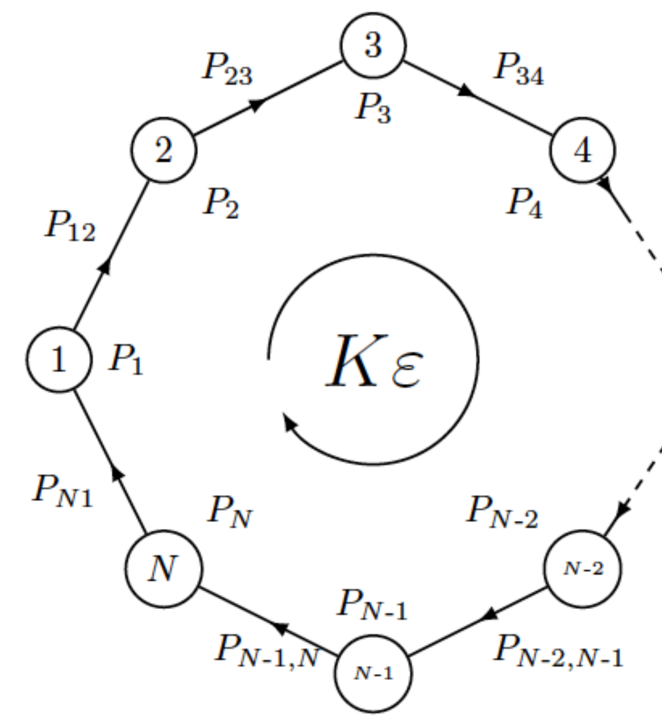
Circulating loop flows

*Thm: Different solutions to the following power-flow problem

$$P_i = \sum_j B_{ij} \sin(\theta_i - \theta_j)$$

may differ only by circulating loop current(s) in any network

Dörfler, Chertkov, Bullo, PNAS '13; Delabays, Coletta and PJ, JMP '16



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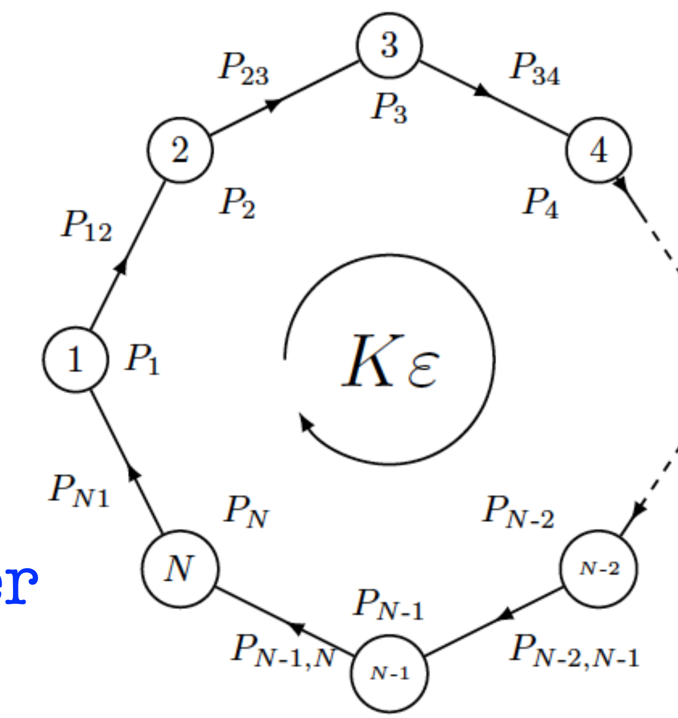

* Angle uniquely defined

→ $q = \sum_i |\theta_{i+1} - \theta_i|_{2\pi} / 2\pi \quad \in \mathbb{Z}$ ~topological winding number

→ discretization of these loop currents ~vortex flows

Janssens and Kamagate '03

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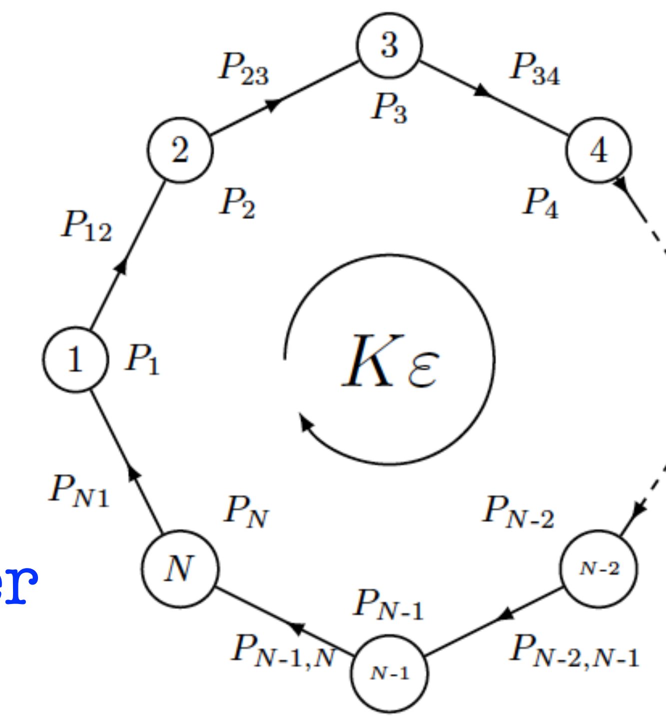

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Delabays, Coletta and PJ, JMP '16, JMP '17; Coletta, Delabays, Adagideli and PJ, NJP '16

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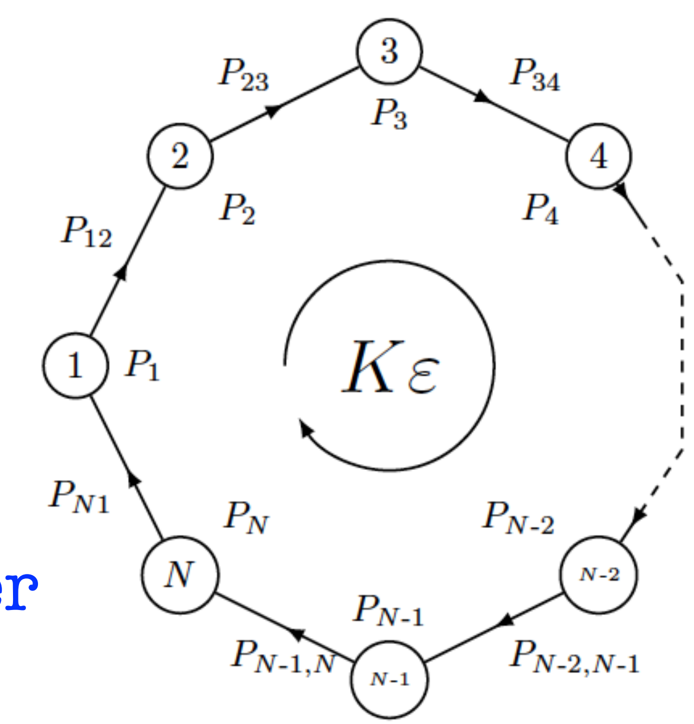

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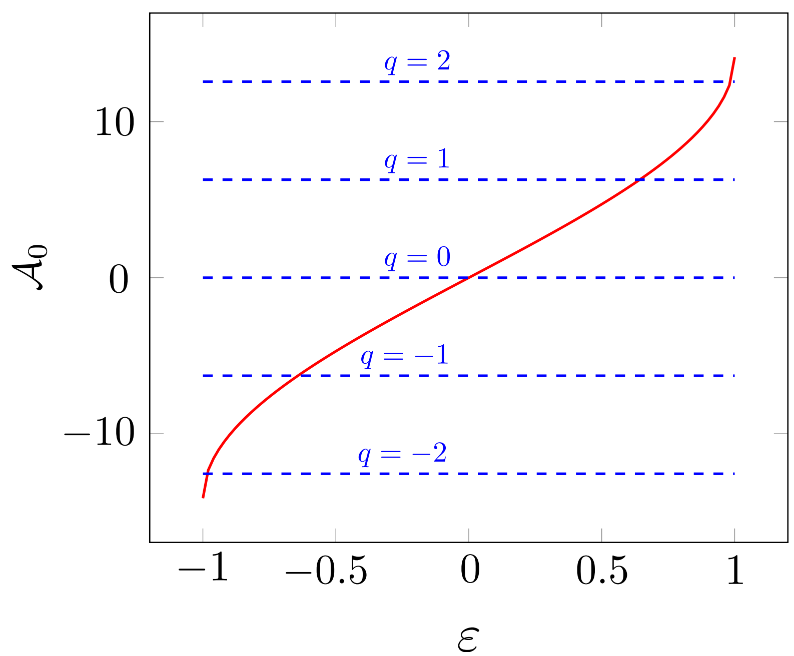

Single cycle

$$\mathcal{N} \leq 2\,\mathrm{Int}\,[n/4] + 1$$

Multi-cycle planar graph (conjecture)

$$\mathcal{N} \leq \prod_{k=1}^{c} \left[2 \cdot \mathrm{Int}\left((n_k + n'_k)/4\right) + 1\right]$$

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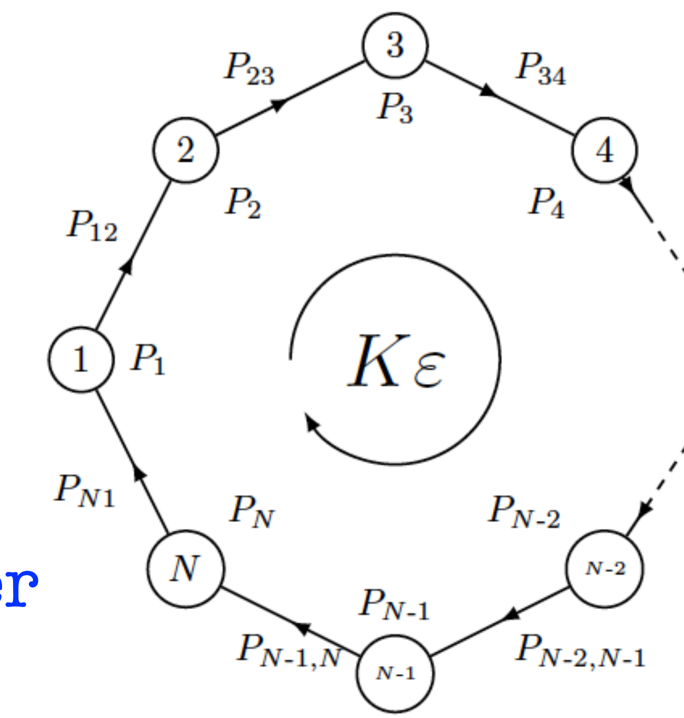

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Janssens and Kamagate '03

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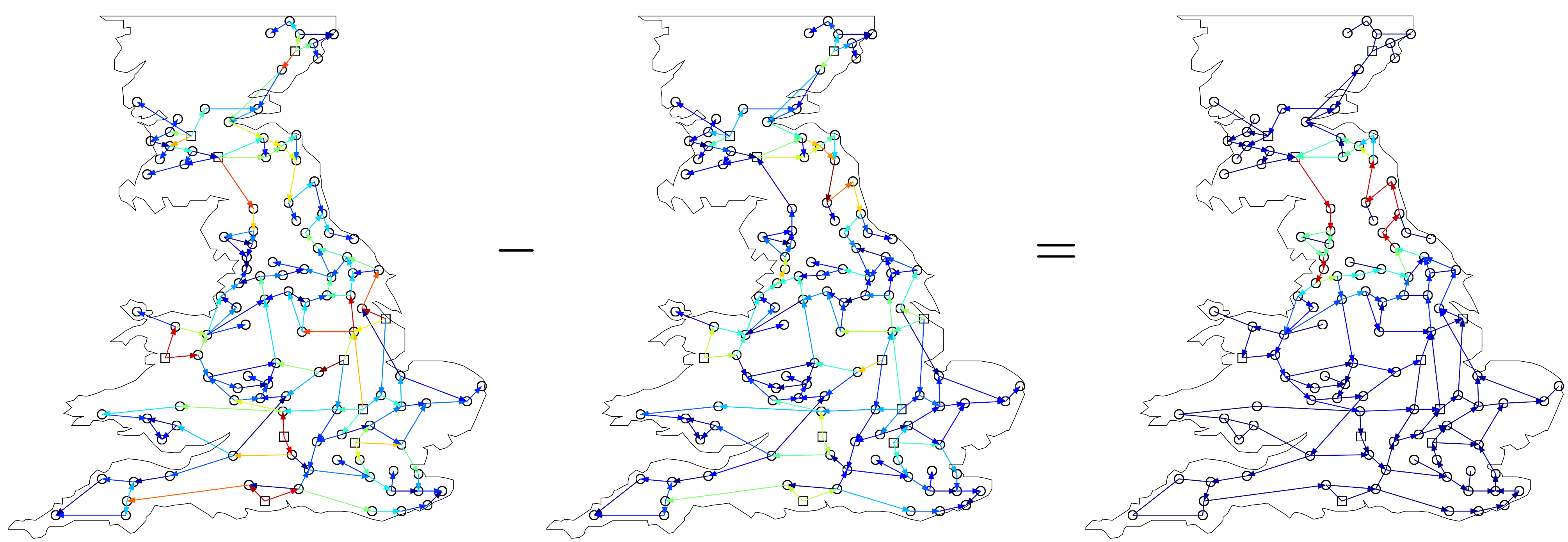

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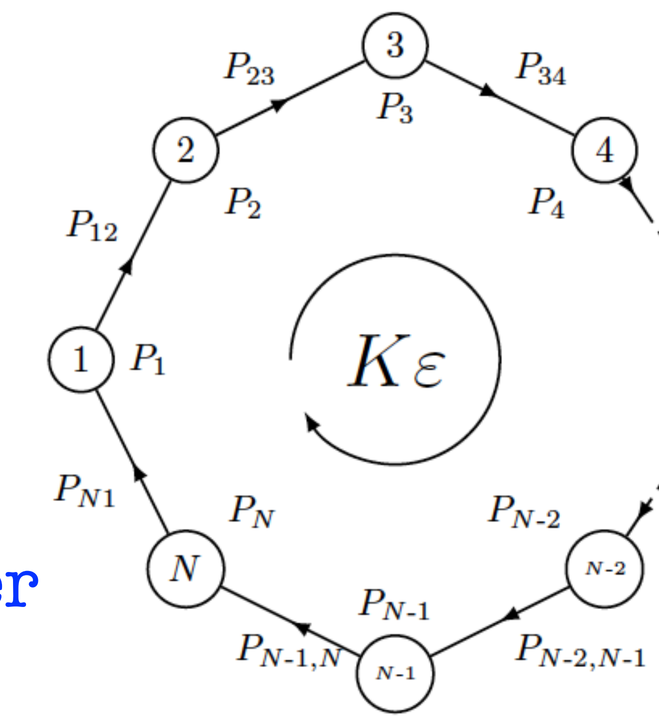

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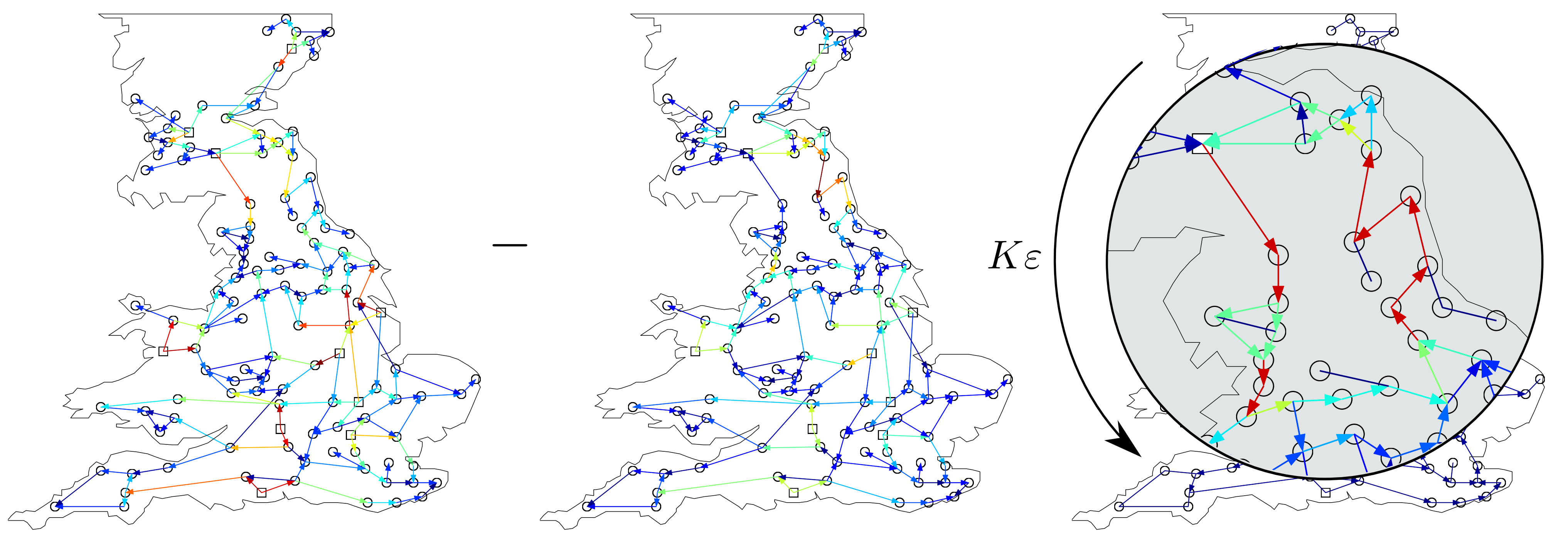

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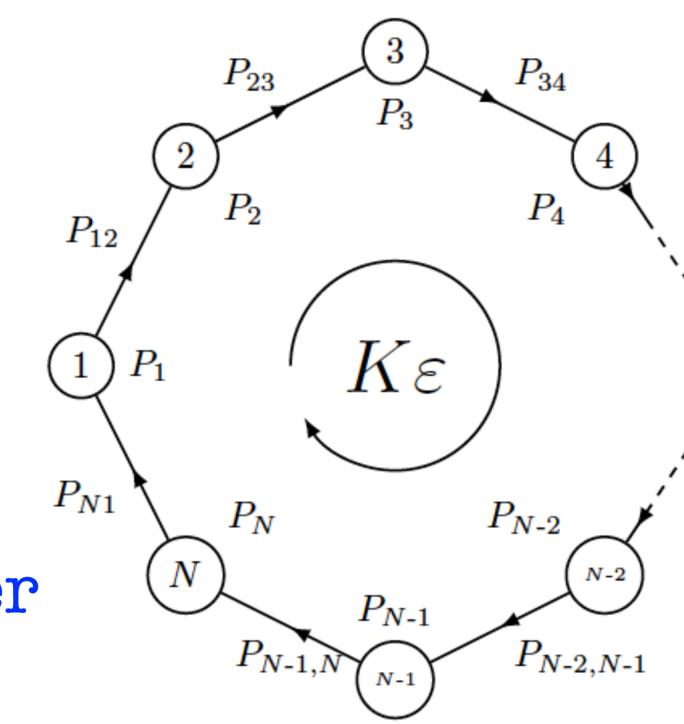

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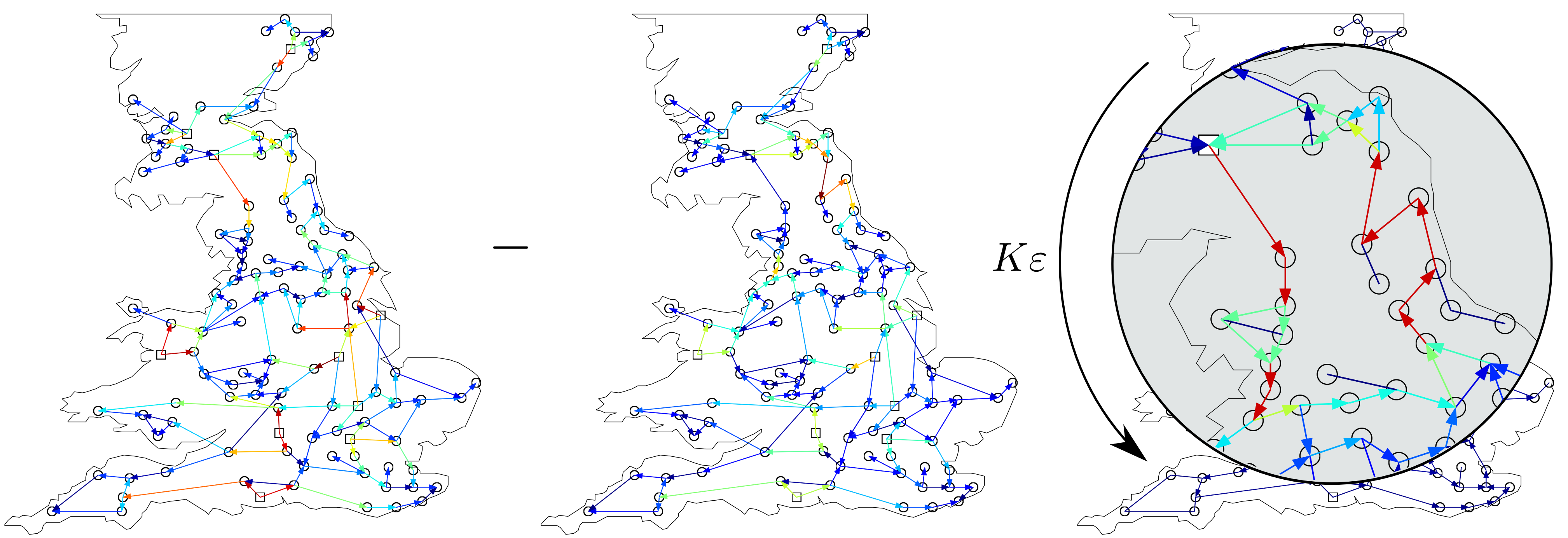

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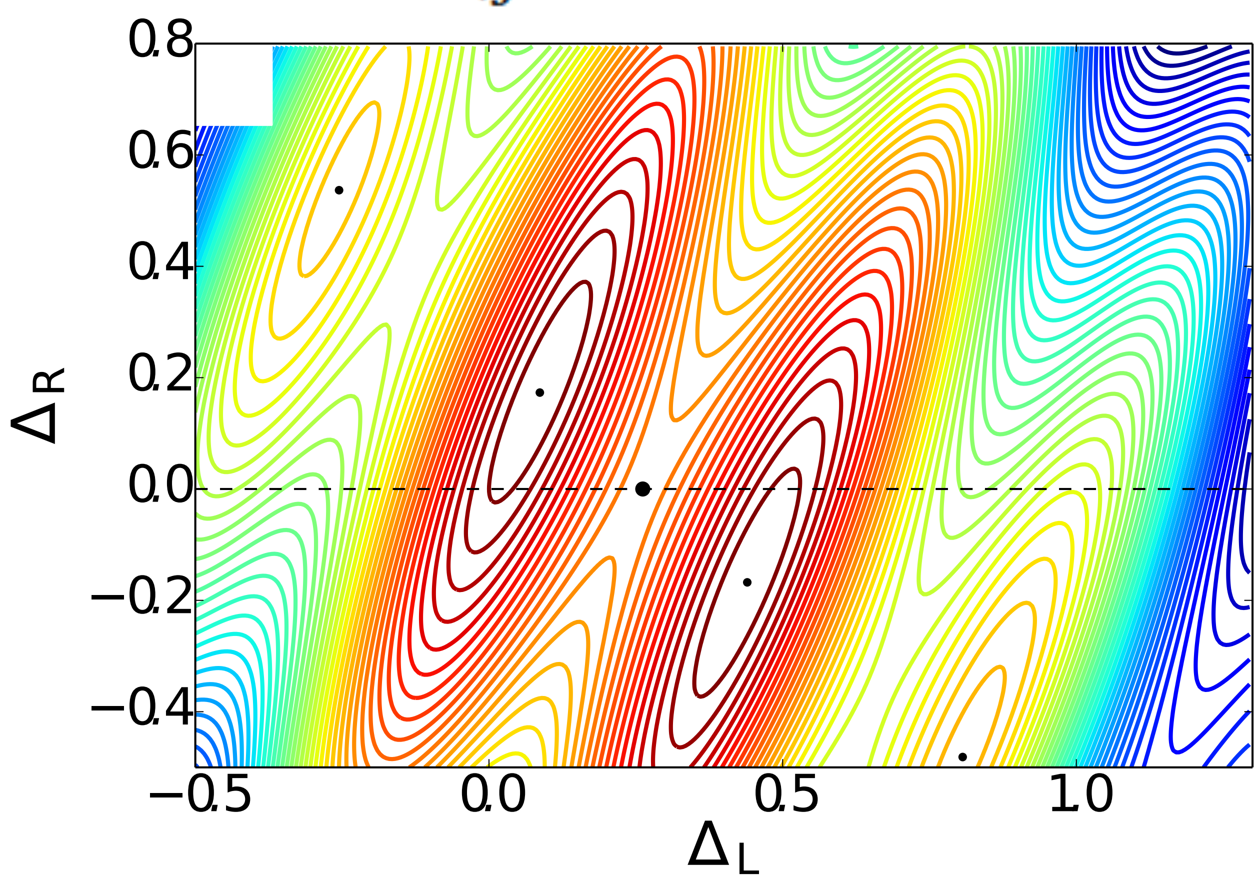
Fixed-point characterized by **topological indices / winding #**

Boundary of basin have **saddle points**



Coletta, Delabays, Adagideli and PJ, New J Phys '16

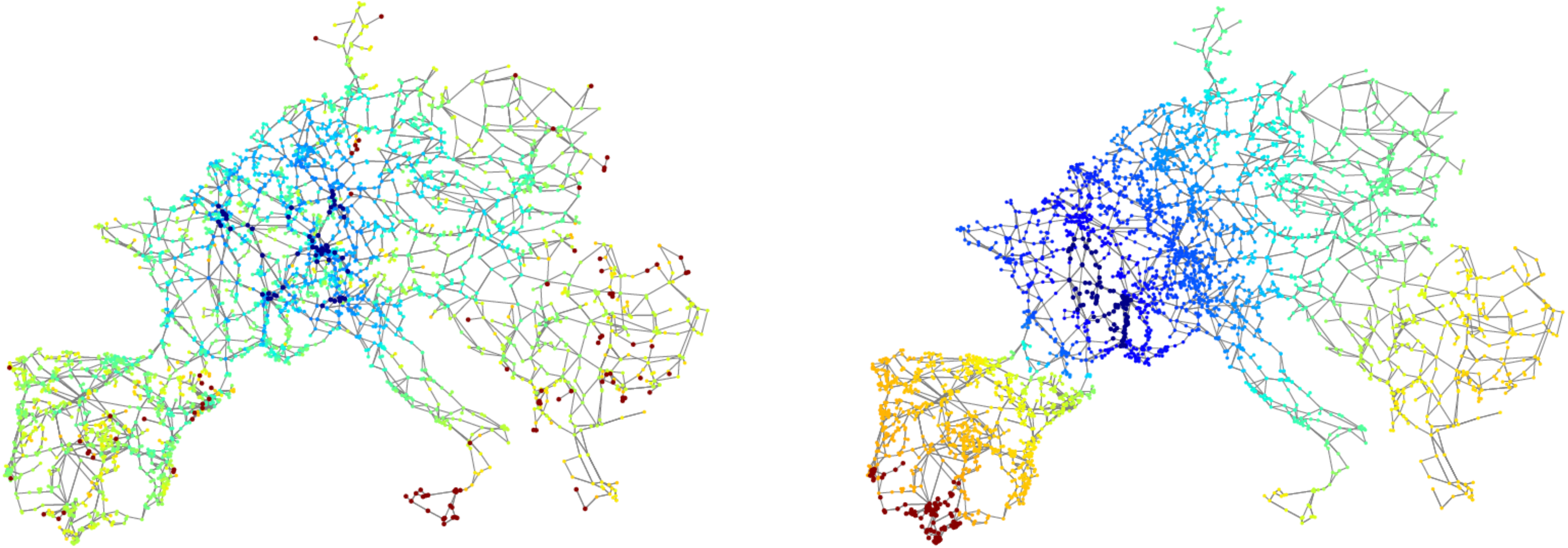
“Young man, in mathematics you don't understand things. You just get used to them.”

-J. Von Neumann

“The best that most of us can hope to achieve in physics is simply to misunderstand at a deeper level.”

-W. Pauli

Spectral geometry
for
Dynamical Systems on Complex Graphs
Part II

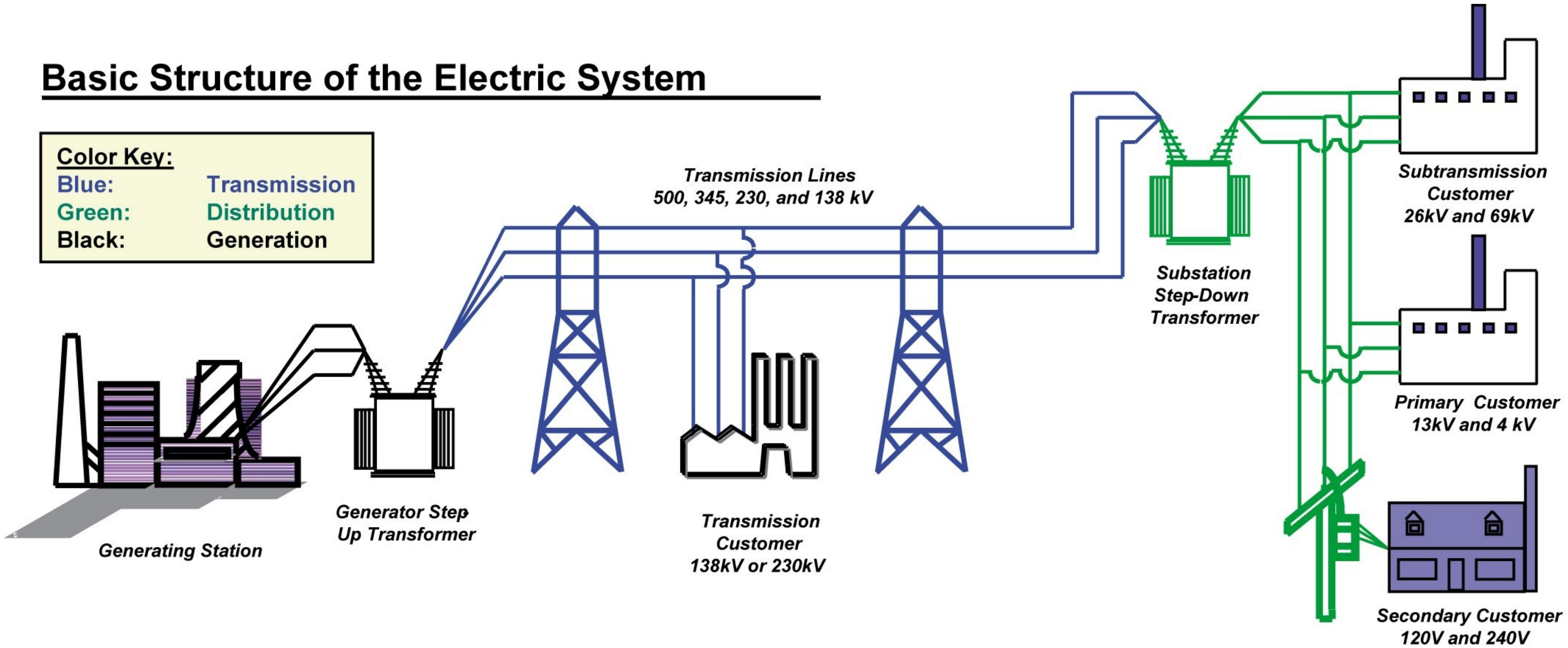


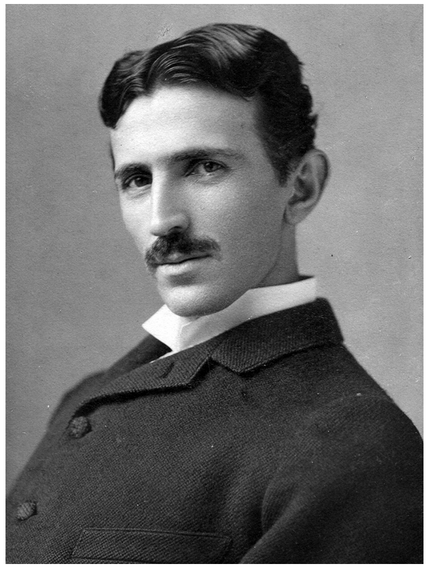
Philippe Jacquod - GeoCow2020

Content

- From Kuramoto to electric power grids
- Small signal perturbations
- Larger perturbations and transient dynamics

What are electric power systems ?





N Tesla 1856-1943

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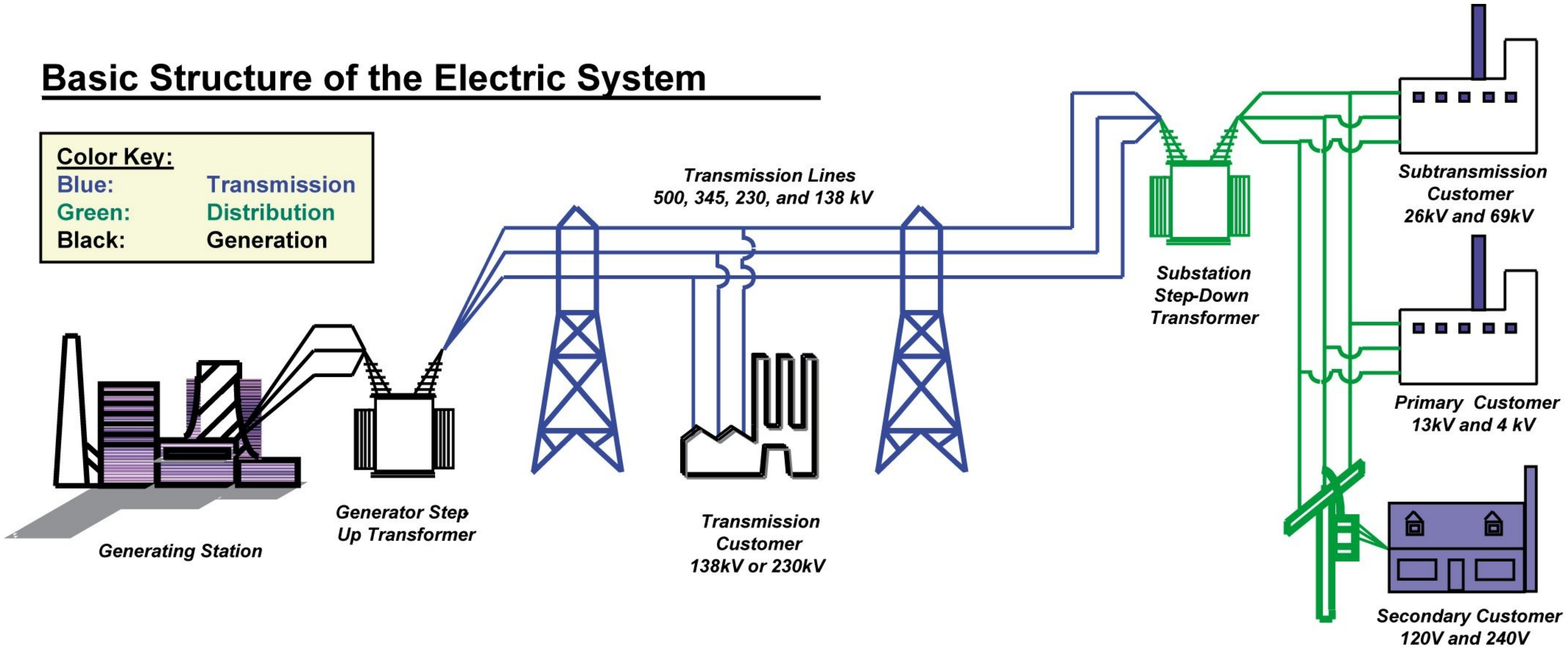
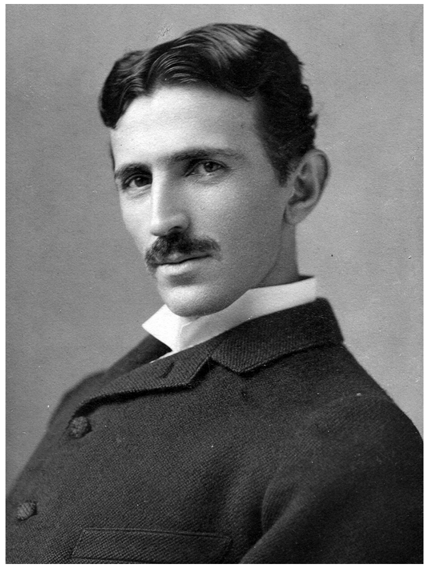
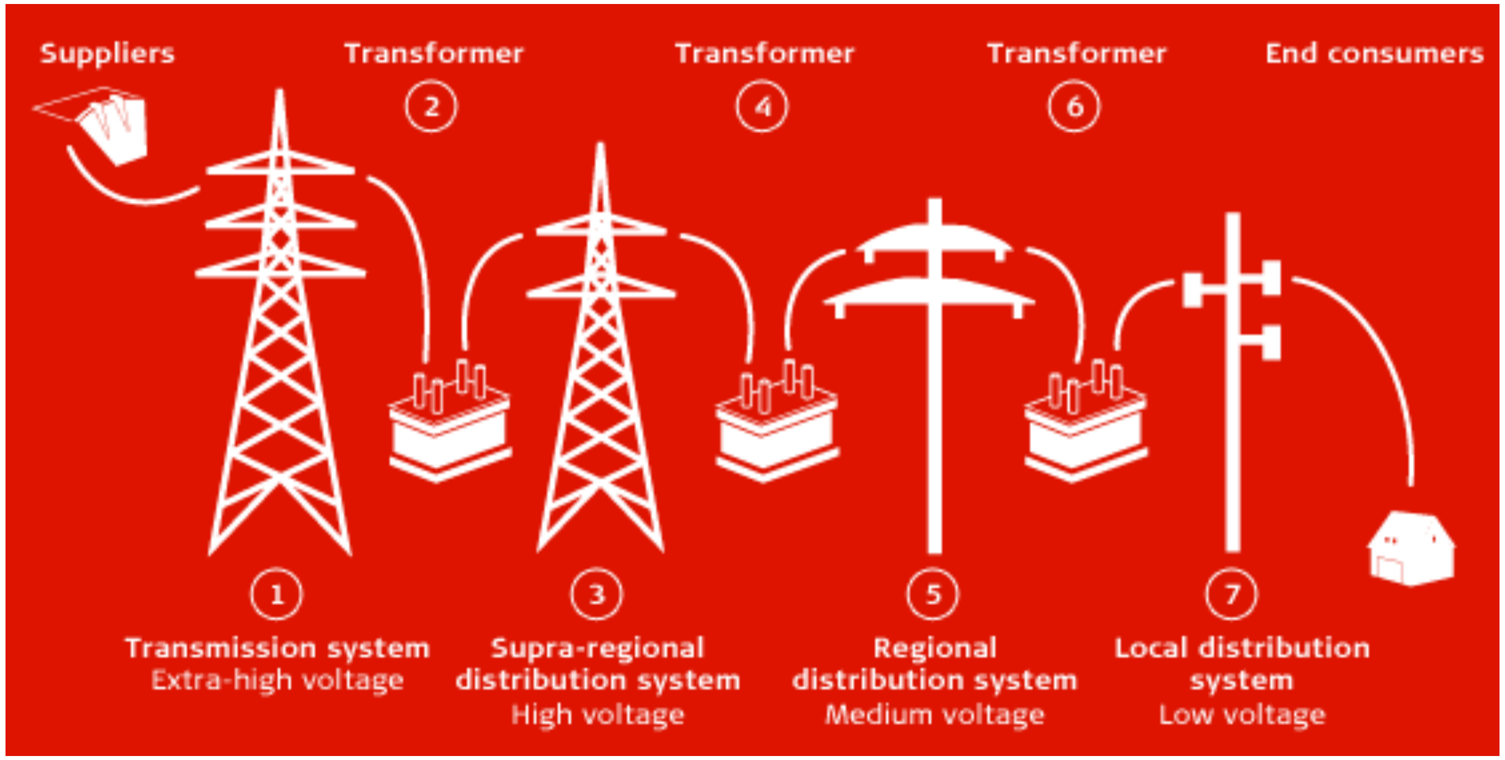




N Tesla 1856-1943

Seven levels of electric power systems



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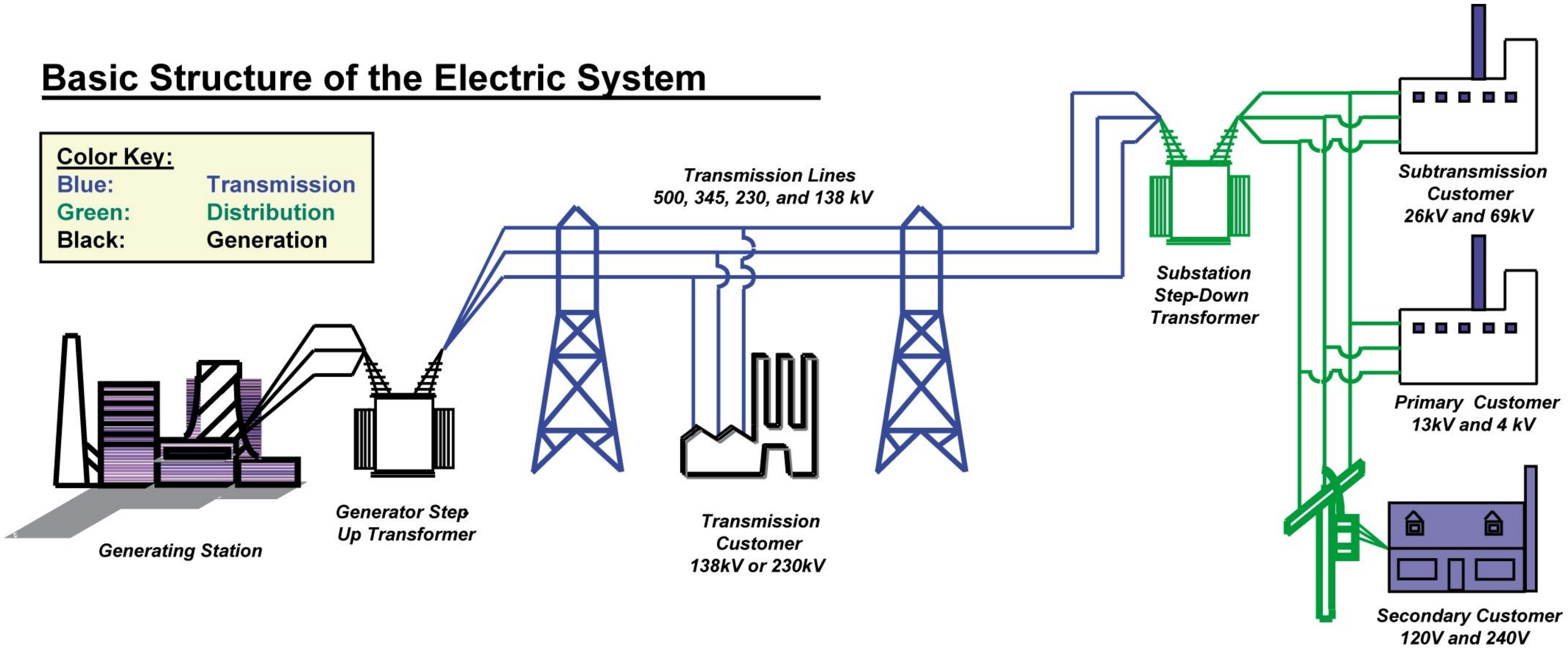

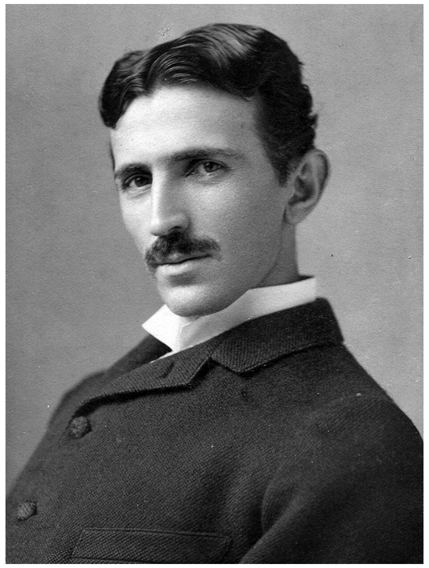

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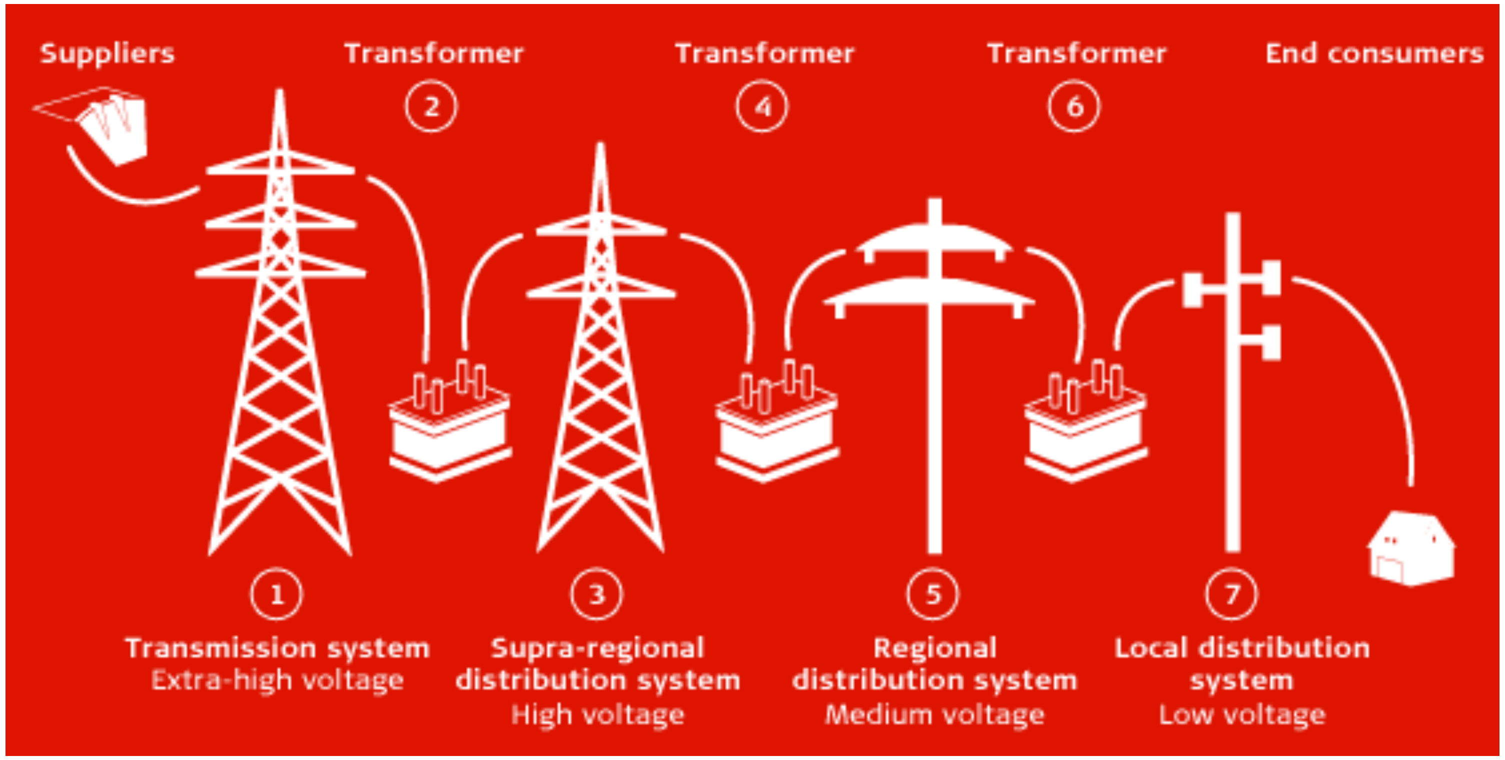
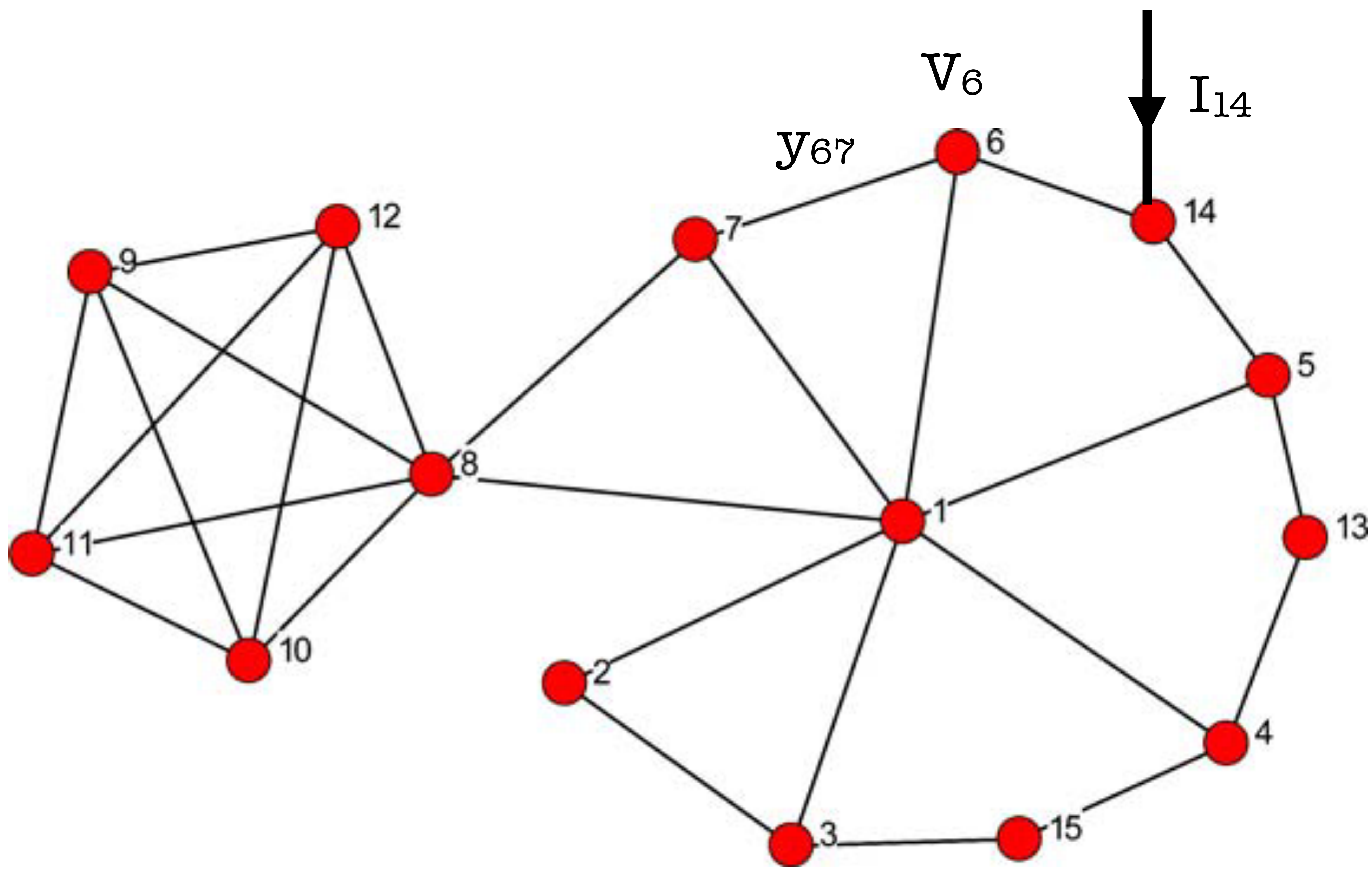


Power :
* conserved (~)
* control parameter
"write Eq. for power"

Operating steady-state of AC power grids

- Ohm's law $\boxed{I_i = \sum_j Y_{ij} V_j}$

I_i : current injected/collected at node i
V_i : voltage at node i
Y_{ij} : admittance matrix



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- Active power vs. reactive power

Active power	vs.	reactive power
P=Re(S)		Q=Im(S)
finite time-average		zero time-average
"truly transmitted"		"oscillating in the circuit"
(injected and consumed)		

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Power flow equations

$$P_i = \sum_j |V_i V_j| \left[G_{ij} \cos(\theta_i - \theta_j) + B_{ij} \sin(\theta_i - \theta_j)\right]$$

$$Q_i = \sum_j |V_i V_j| \left[G_{ij} \sin(\theta_i - \theta_j) - B_{ij} \cos(\theta_i - \theta_j)\right]$$

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Voltages at buses i and j

Conductance matrix

Voltage angles at buses i and j

Susceptance matrix

"It is nice to know that the computer understands the problem

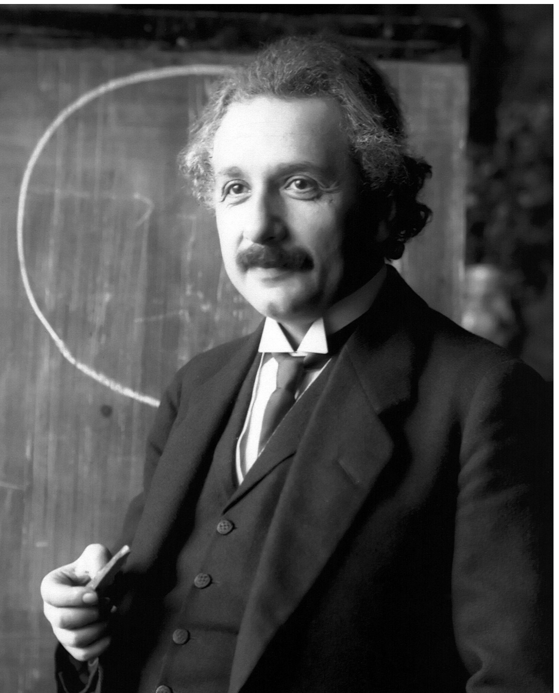
...

but I would like to understand it too."

-E.P. Wigner

"Truth (...) is much too complicated to allow anything but approximations."

-J. Von Neumann



"Everything should be made as simple as possible, but not simpler."

-A. Einstein

Approximated power flow equations : very high voltage

- Admittance dominated by its imaginary part for large conductors ~ high voltage G/B < 0.1 for 200kV and more

neglect conductance

$$P_i \simeq \sum_j |V_i V_j| B_{ij} \sin(\theta_i - \theta_j)$$

$$Q_i \simeq -\sum_j |V_i V_j| B_{ij} \cos(\theta_i - \theta_j)$$

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neglect conductance

- No conductance ~no voltage drop

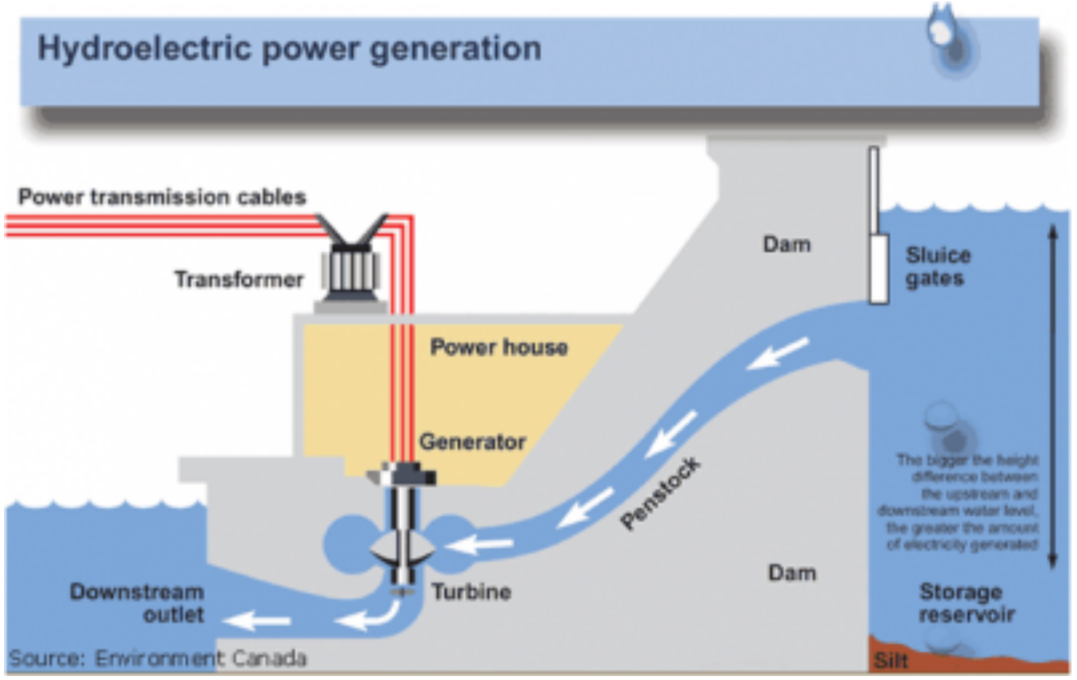
consider constant voltage

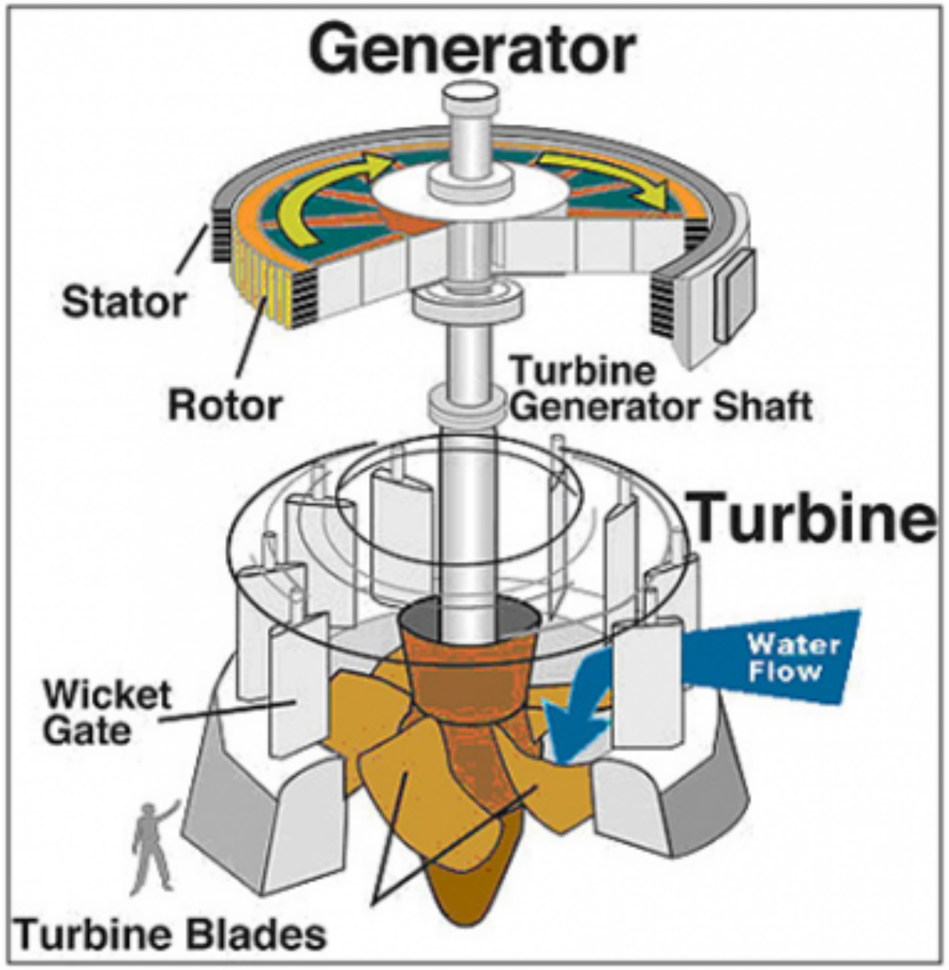
✳decoupling between P and Q
✳consider P only
(a.k.a. lossless line approx.)

$$P_i = \sum_j B_{ij} \sin(\theta_i - \theta_j)$$

A bit of electric power engineering

- Electricity production with rotating machines
- Potential, chemical, nuclear or thermal energy converted into mechanical energy (rotation)
- Mechanical energy converted into electric energy
- Time-dependence : think power instead of energy
- Balance between power in, power out and energy change in rotator : **SWING EQUATIONS**





Time-evolution of frequency : swing equations

- Power balance

Change in KE of rotator

$$\frac{dW_i}{dt} + P_i^{(d)} = P_i^{(m)} - P_i^{(g)}$$

Damping/control power

Power input

Electric power output

Time-evolution of frequency : swing equations

- Power balance

Change in KE of rotator

$$\frac{dW_i}{dt} + P_i^{(d)} = P_i^{(m)} - P_i^{(g)}$$

Electric power output

Damping/control power

Power input

- Swing equation for angles (in rotating frame @ 50Hz)

$$M\frac{d^2\theta_i}{dt^2} + D\frac{d\theta_i}{dt} = P_i - \sum_j B_{ij}\sin(\theta_i - \theta_j)$$

Inertia of rotator

Damping / control

Time-evolution of frequency : swing equations

- Power balance

$$\frac{dW_i}{dt} + P_i^{(d)} = P_i^{(m)} - P_i^{(g)}$$

Change in KE of rotator

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Power input

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Inertia of rotator

Damping / control

- In standard operation, power grids are in a steady-state solutions of the swing eqs. ~synchronous solution

Dynamics of AC power grids

Dynamics: Swing Equations.

- from few AC cycles to ~30 secs.
- In rotating frame @ 50/60 Hz

$$I_i\ddot{\theta} + D_i\dot{\theta}_i = P_i - \sum B_{ij}\sin(\theta_i - \theta_j)$$

Bergen and Hill, 1981

- I : inertia ~ rot. kinetic energy
- D : damping from primary control + frequency-dependence of loads

- Coupling between power balance and frequency

Change in kinetic energy = balance of power

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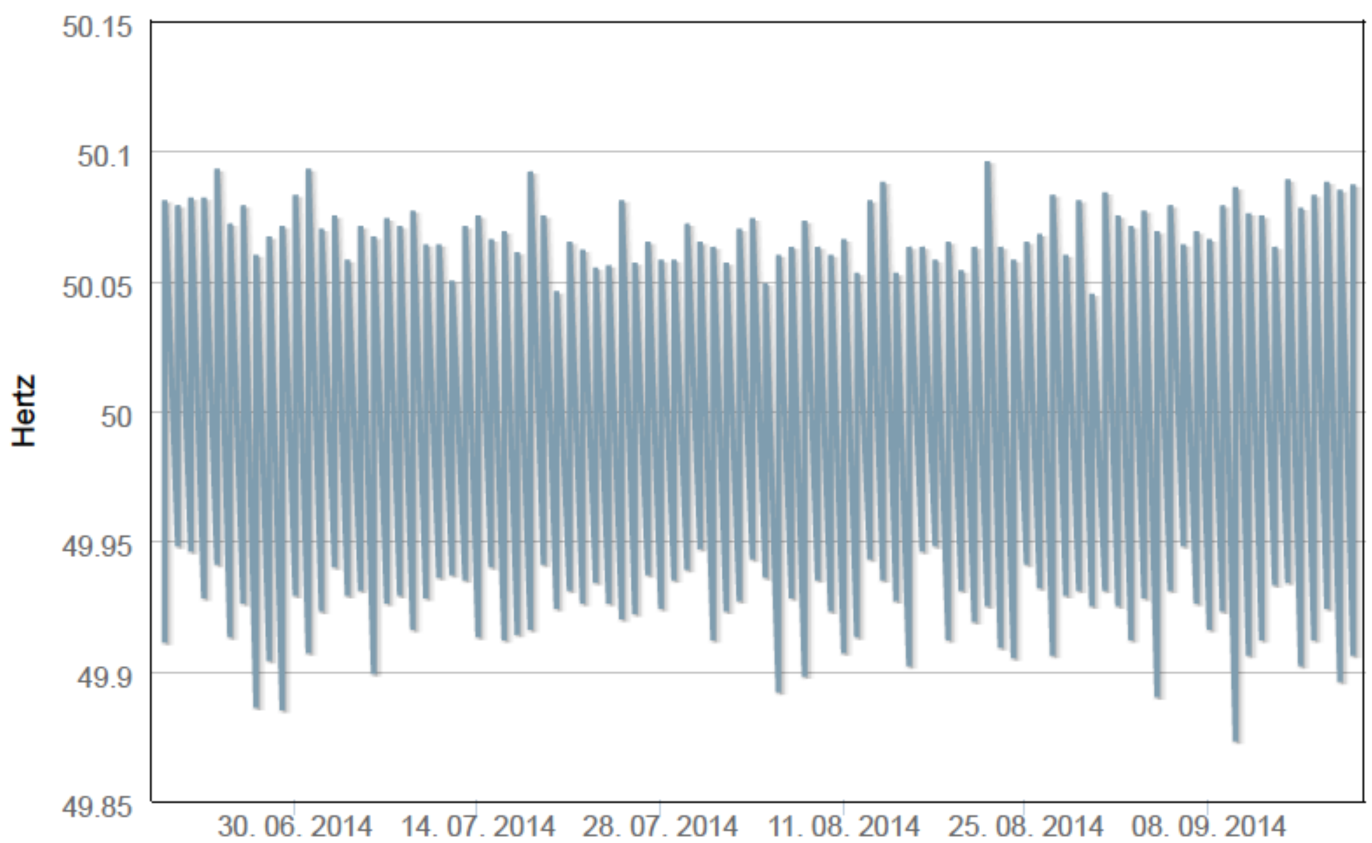
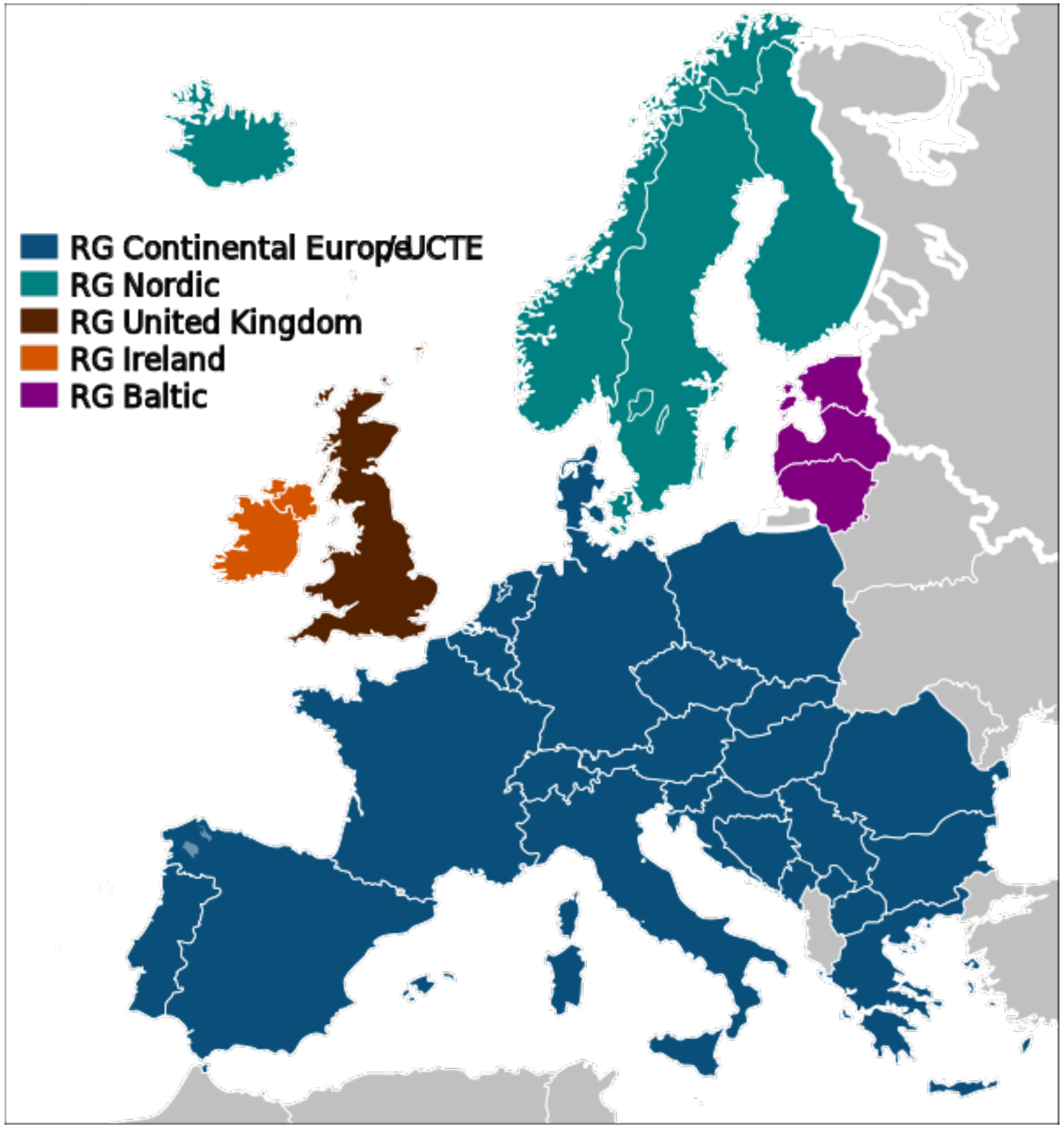
Steady-state operation : synchrony over 1000's of kms



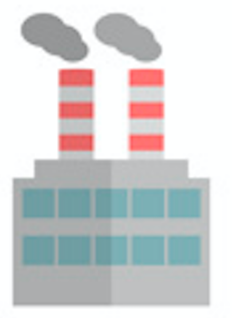


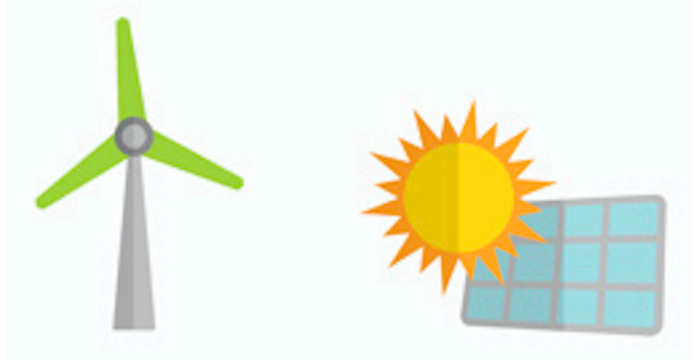
Loss of synchrony : blackouts

Italy blackout, sep 28 2003

Northeast blackout, aug 14 2003

Today’s vs. tomorrow’s power grids





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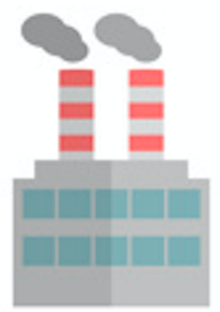
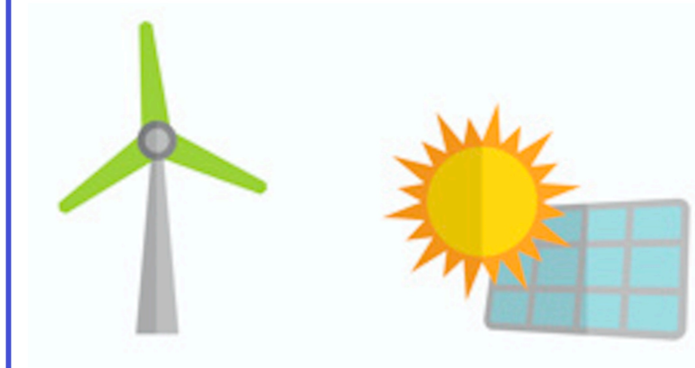
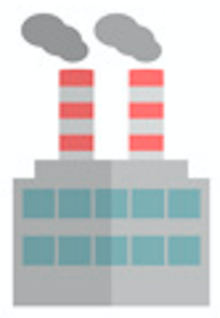


- Centralized



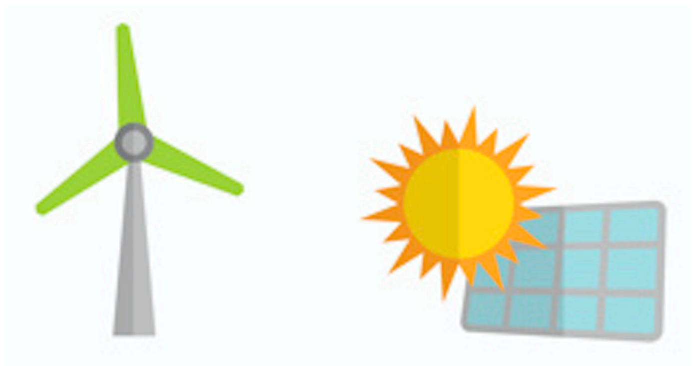
- Decentralized
  - -issues at distribution level
  - -easier at transmission level (=reduction in load)

Today's vs. tomorrow's power grids



- Centralized

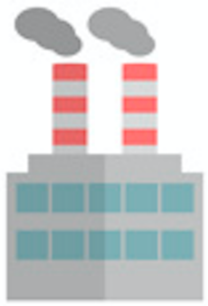
- Dispatchable/predictable



- Decentralized
  - -issues at distribution level
  - -easier at transmission level (=reduction in load)

- Uncertain
  - -power / energy reserve

Today's s



Aaron Rupar
@atrupar

Suivre

TRUMP: "If Hillary got in... you'd be doing wind. Windmills. Weeeee. And if it doesn't blow, you can forget about television for that night. 'Darling, I want to watch television.' 'I'm sorry! The wind isn't blowing.' I know a lot about wind."

Traduire le Tweet

- Centr
- Dispa

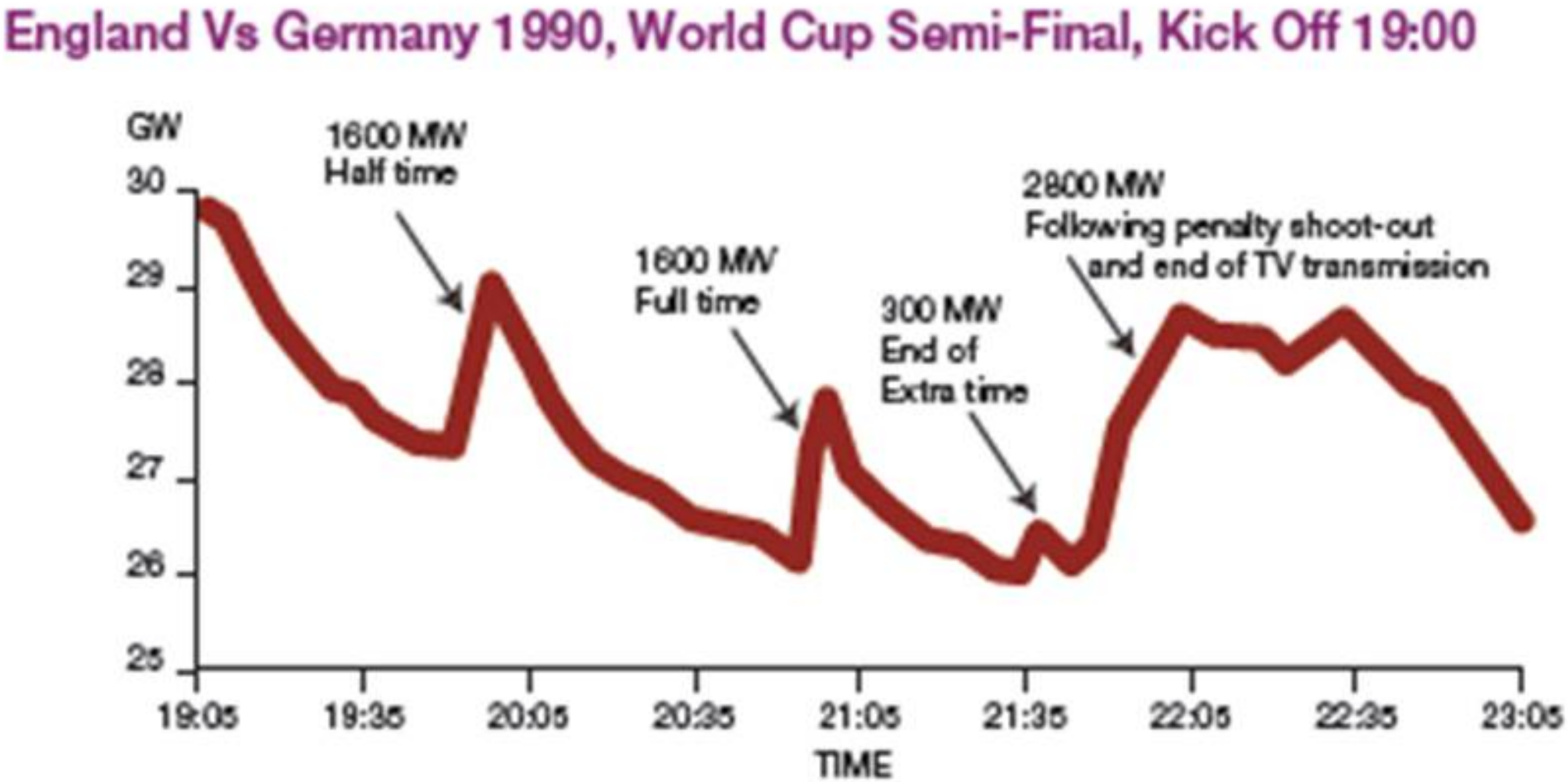
n level
on level

erve

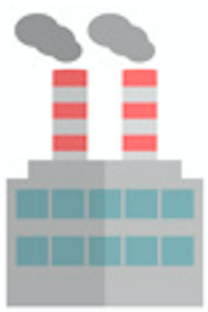
Hes·so // VALAIS WALLIS
School of Engineering

UNIVERSITÉ DE GENÈVE
FACULTÉ DES SCIENCES

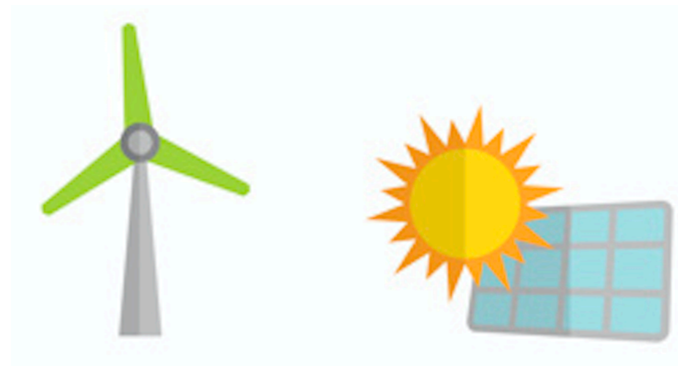
Variability and uncertainty are nothing new for grid operators



Old vs. new productions



- Rotating machine with inertia
  - -frequency definition
  - -grid stability :
    - * energy reserve
    - * droop control



- No or much less inertia (inverter-connected)

The questions of interest

$$I_i\ddot{\theta} + D_i\dot{\theta}_i = P_i - \sum B_{ij}\sin(\theta_i - \theta_j)$$

1. Start from stable fixed point

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$$I_i\ddot{\theta} + D_i\dot{\theta}_i = P_i - \sum B_{ij}\sin(\theta_i - \theta_j)$$

1. Start from stable fixed point

2. Perturb it

    power outage $P_i \to 0$

    power fluctuations $\langle P_i(t)P_j(t')\rangle = \delta_{ij}\, P_0^2\, e^{-|t-t'|/\tau_0}$

    line fault $B_{ij} \to 0$

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   2a. Weakly
   2b. More strongly

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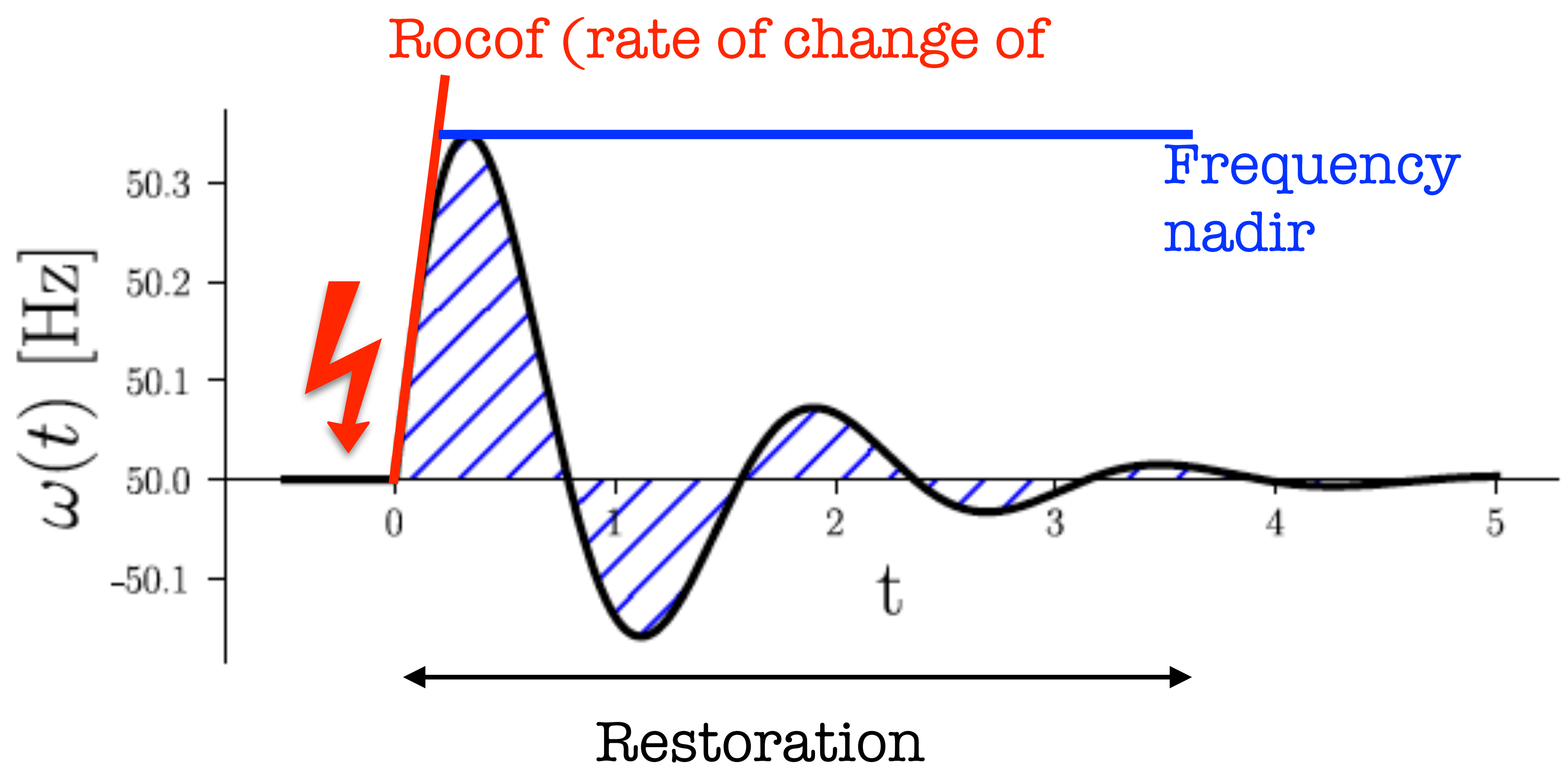
   2a. Weakly
   2b. More strongly

3. Look at dynamical response

The questions of interest : weak perturbation

$$I_i\ddot{\theta} + D_i\dot{\theta}_i = P_i - \sum B_{ij}\sin(\theta_i - \theta_j)$$



Weak pert. ~linearize DE ~damped wave equation
-> look at pert. propagation

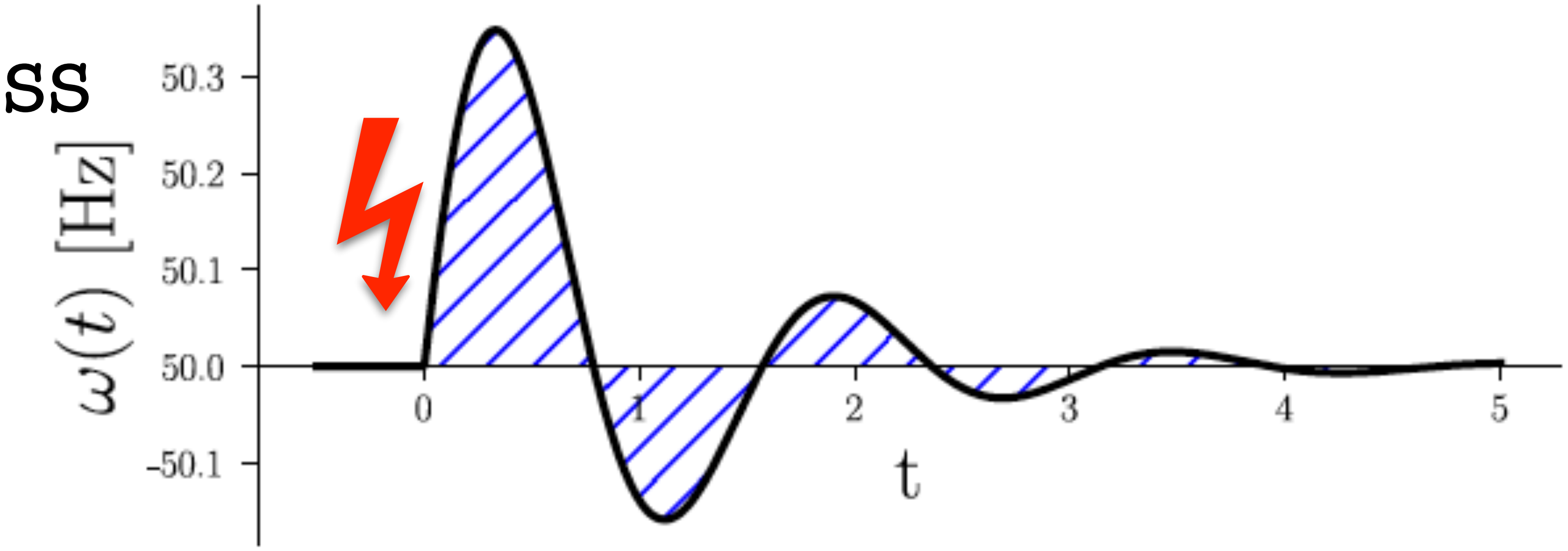
Evaluate total response (~shaded area) as measure of
-vulnerability under local perturbation
-robustness under average perturbation

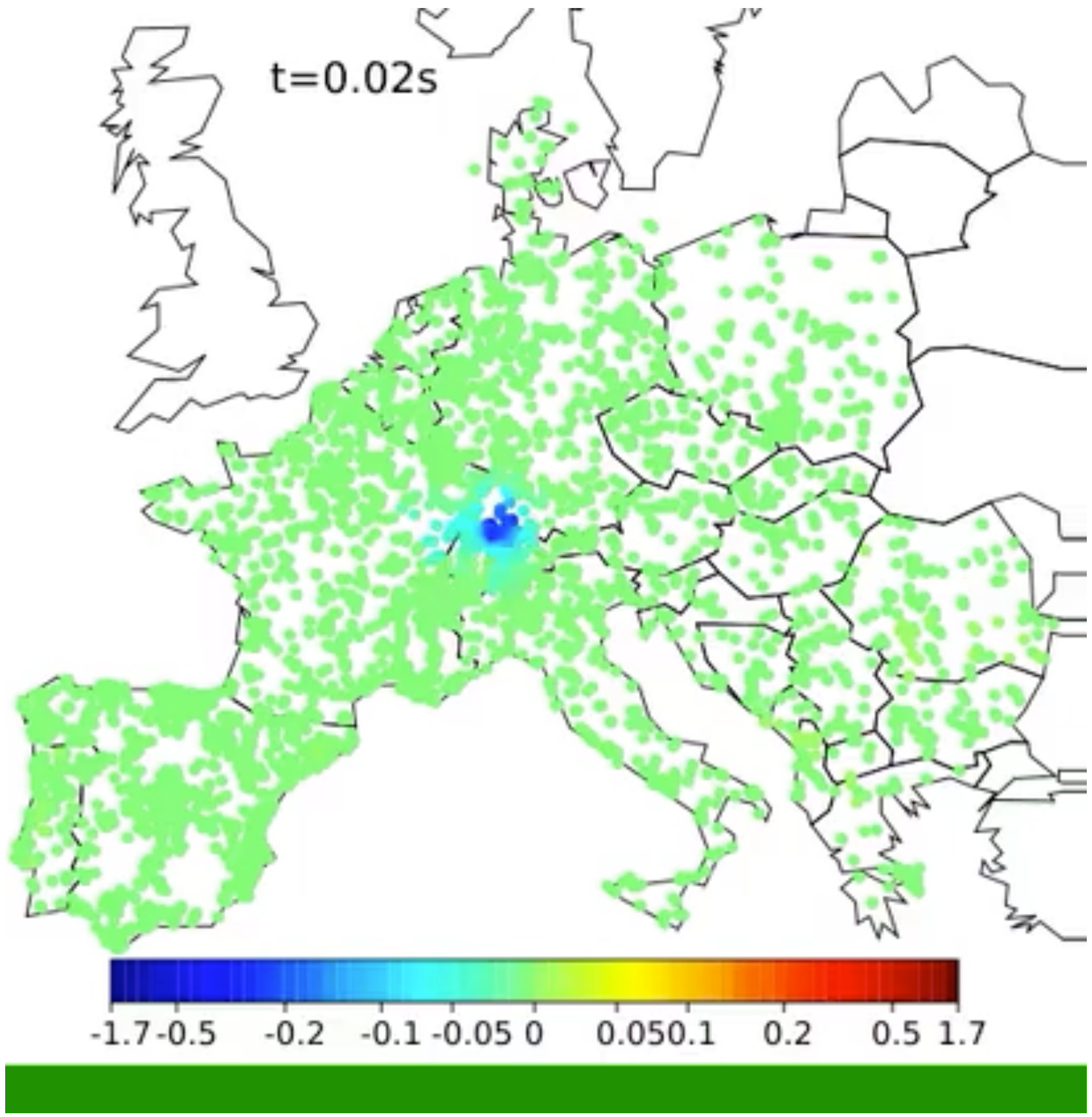
Weak perturbation : wave propagation

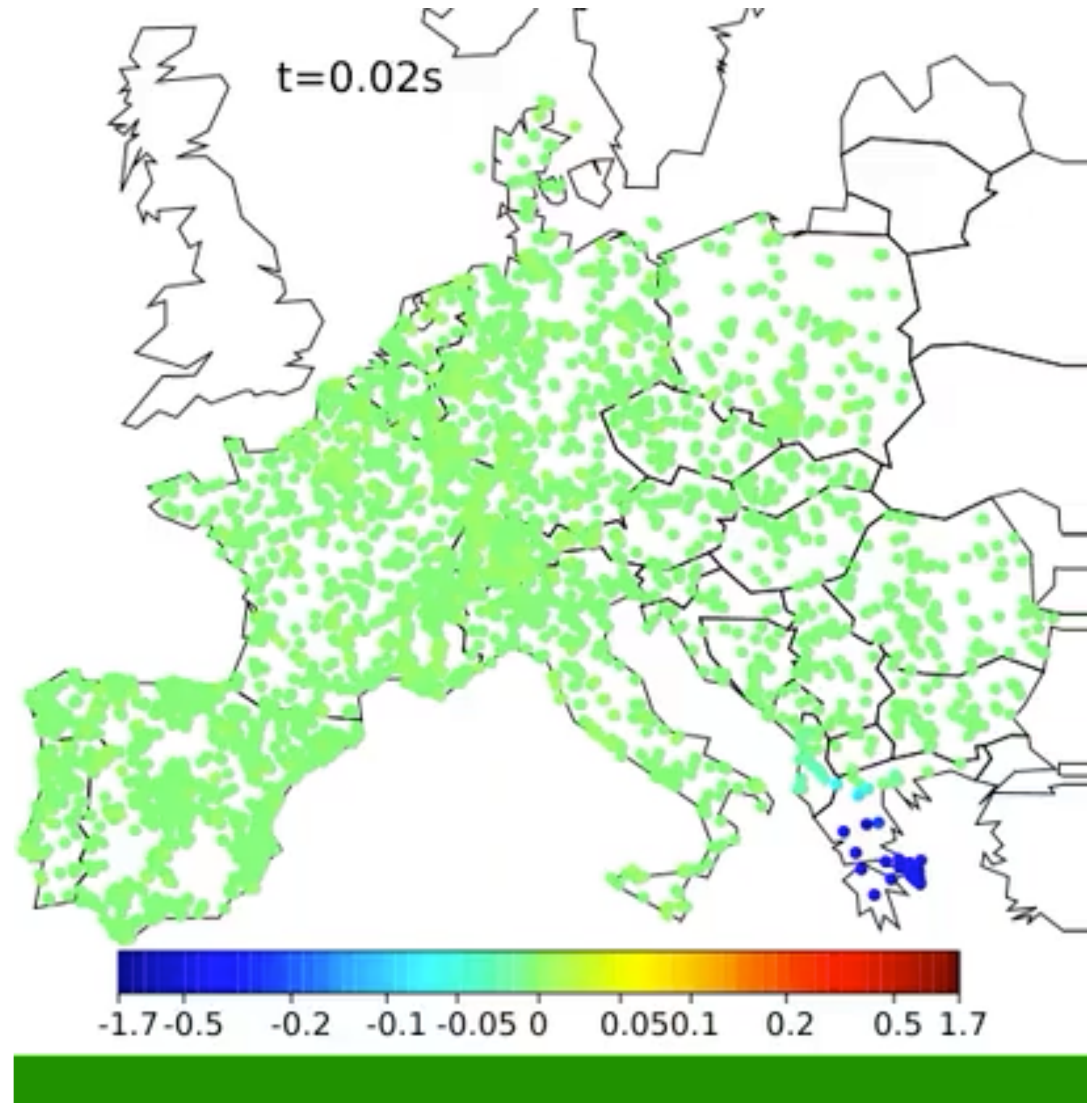
Fault : sudden power loss

$$P_b = P_b^{(0)} - \Delta P$$

$$\Delta P = 900 \text{ MW}$$







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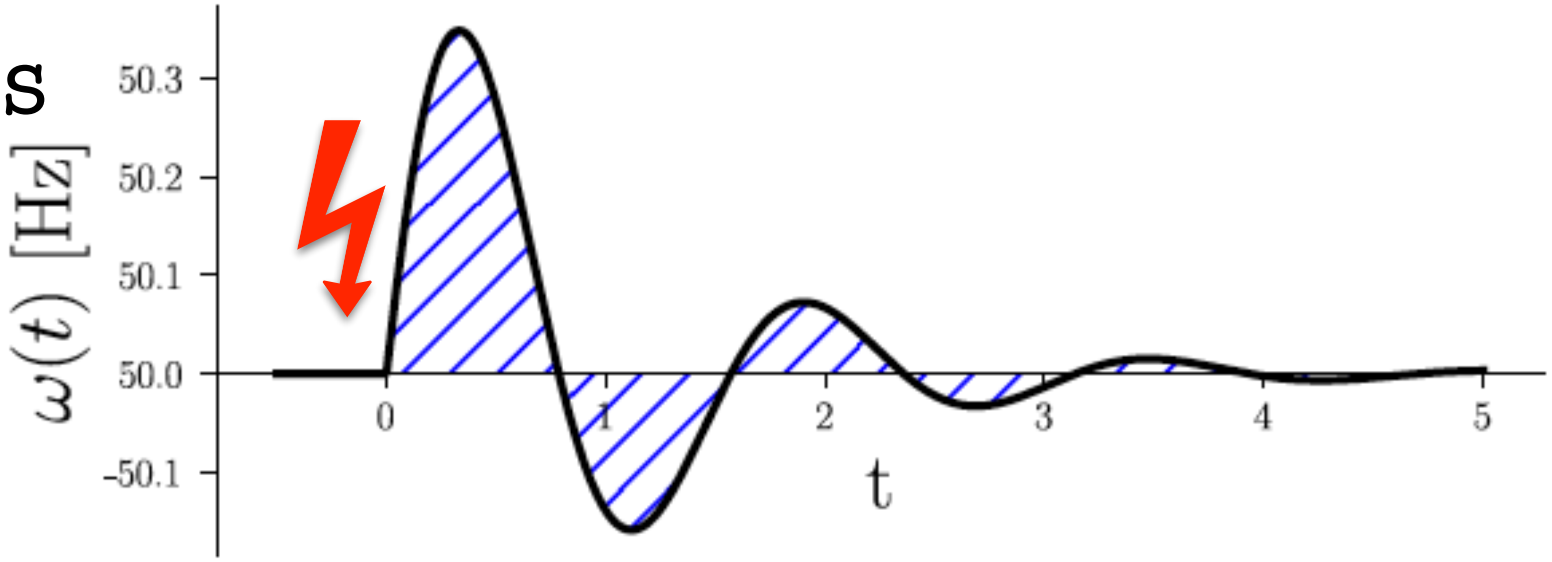

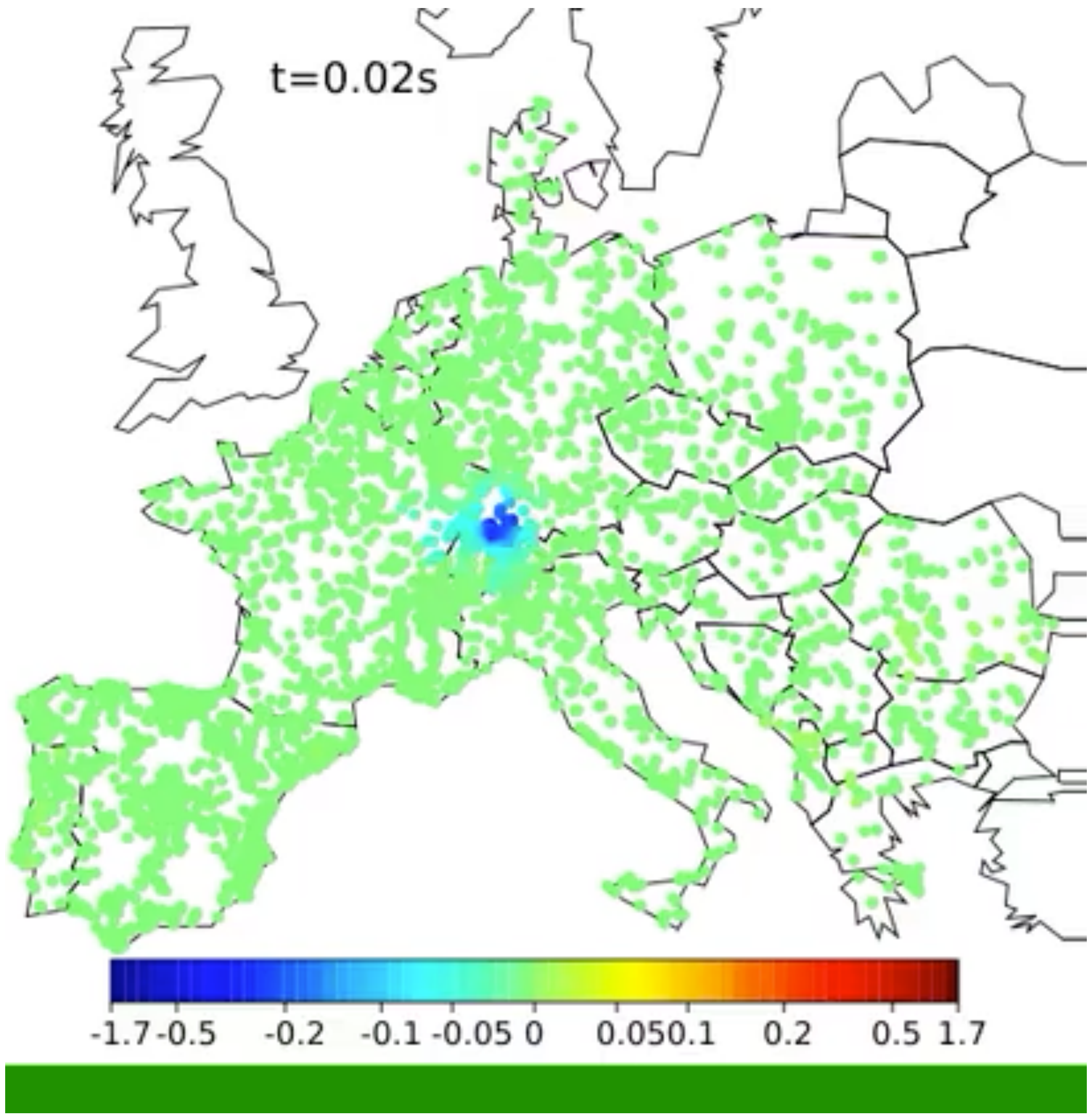

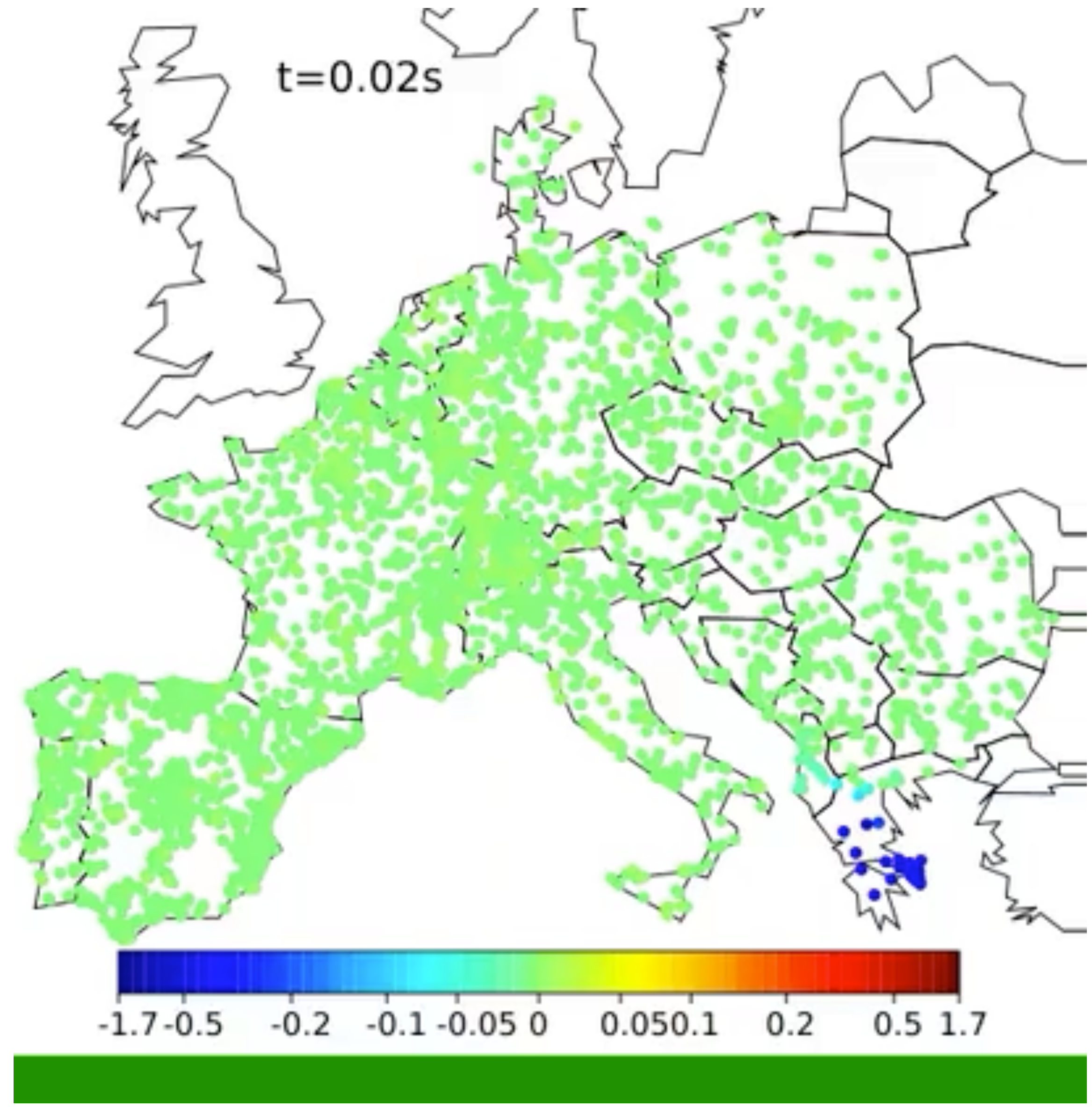

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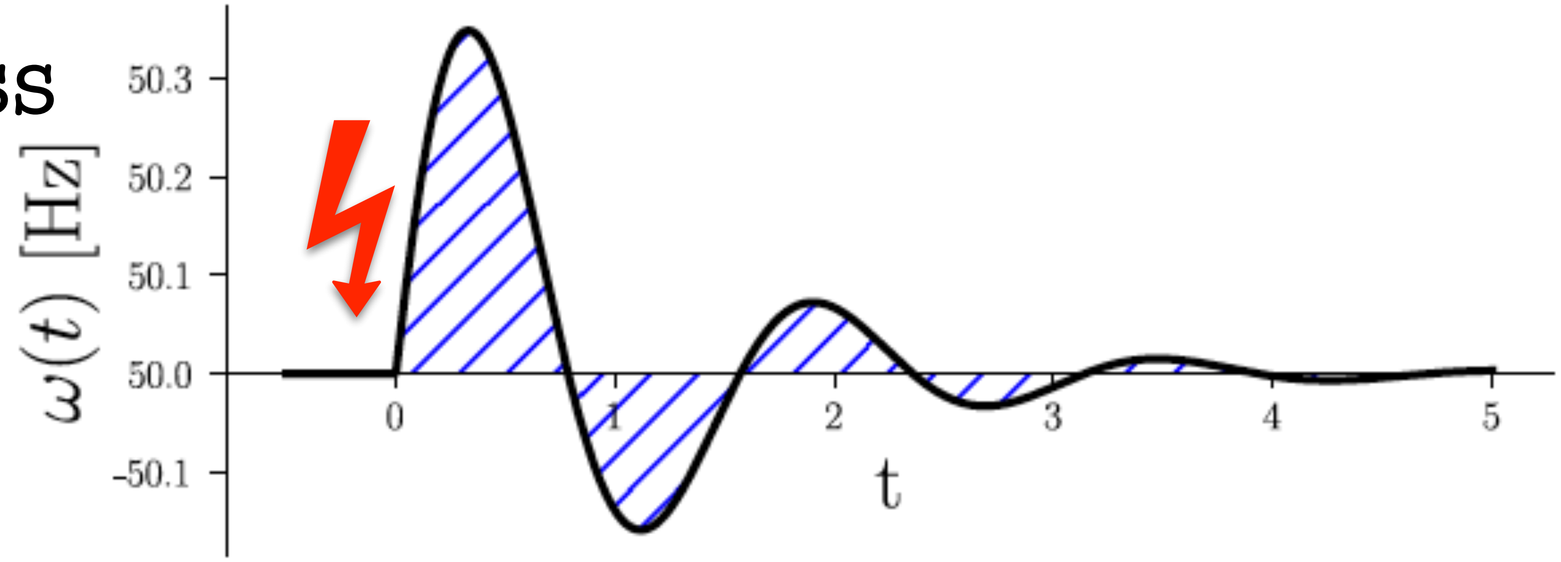
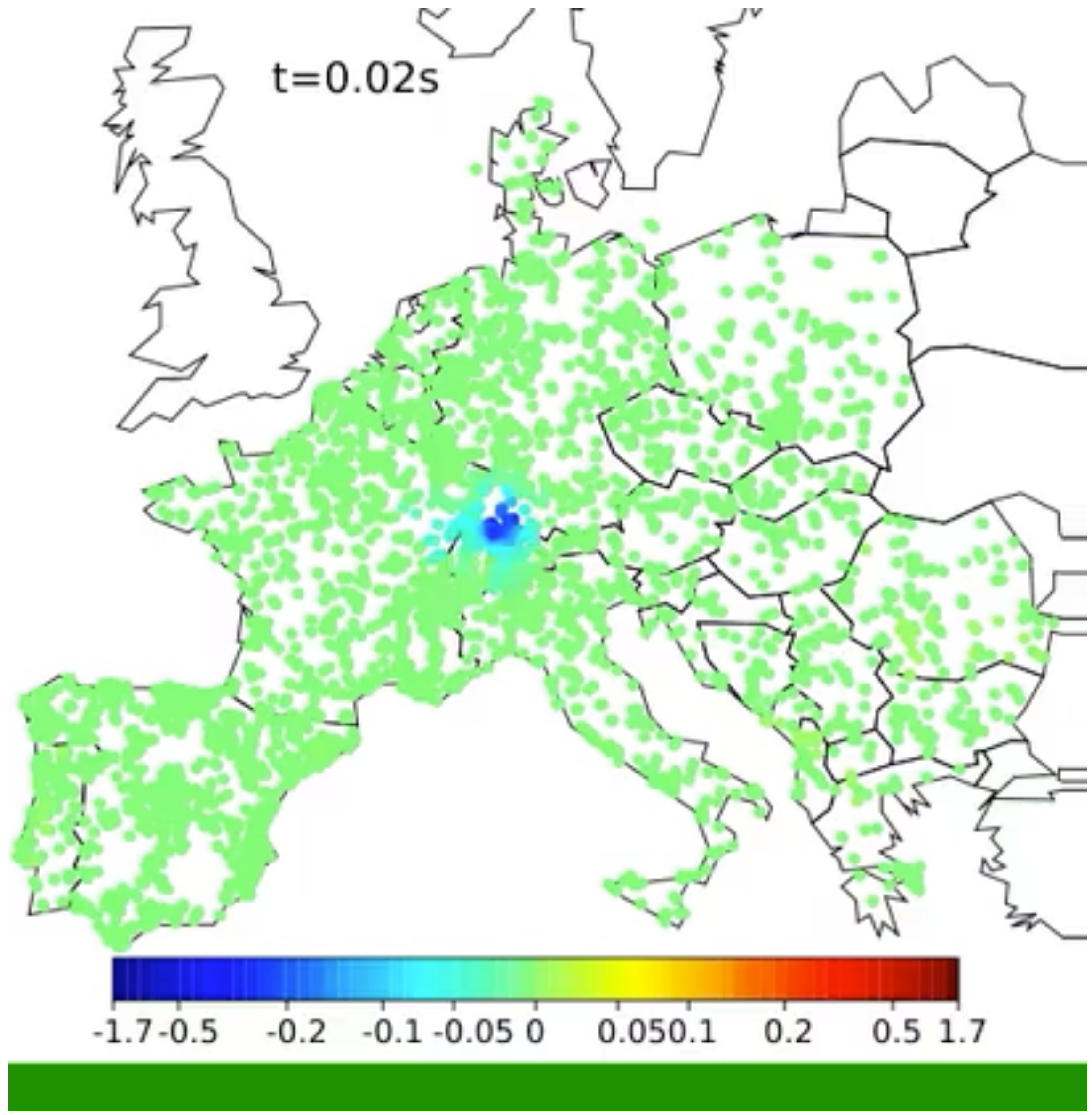
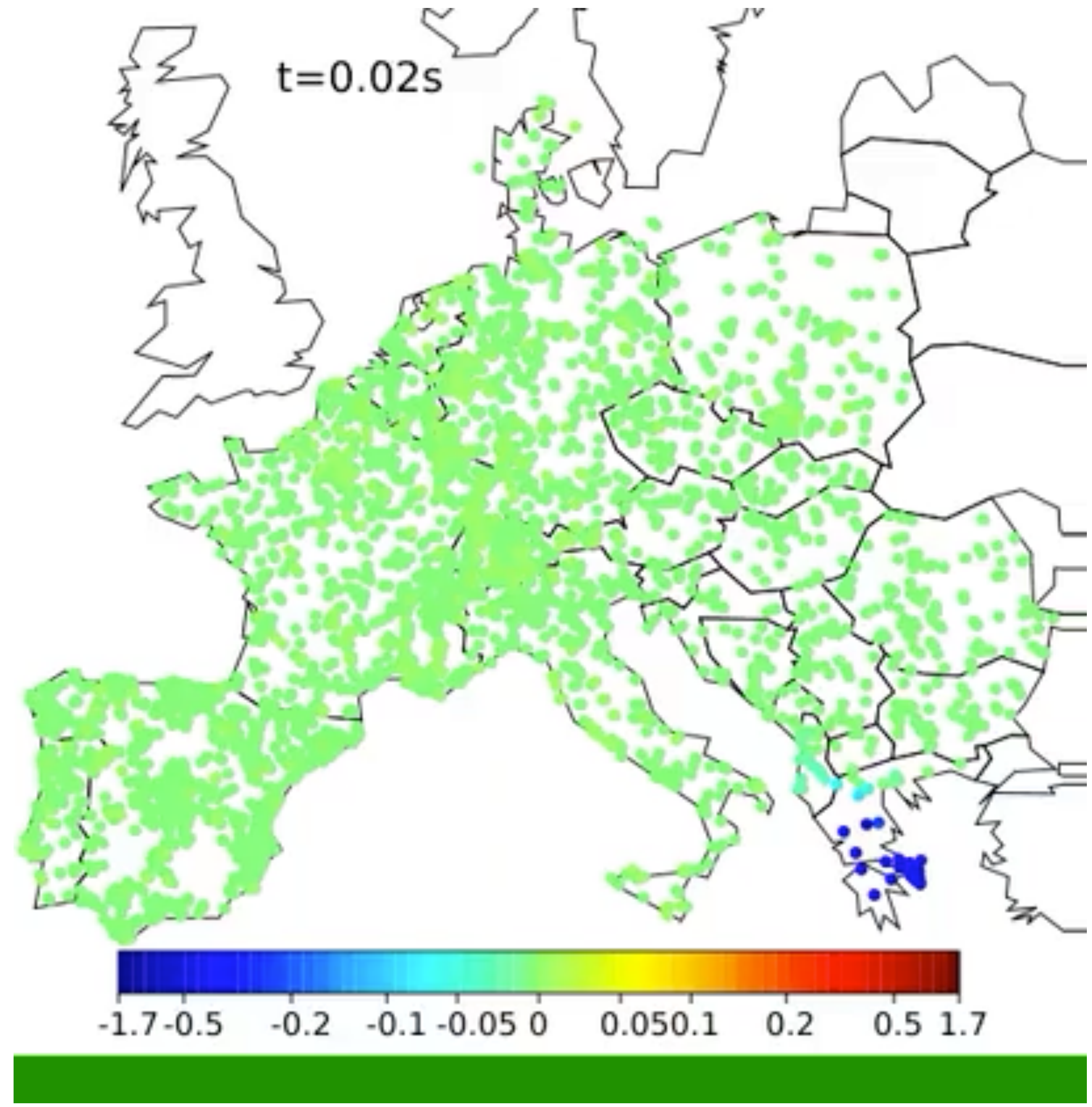
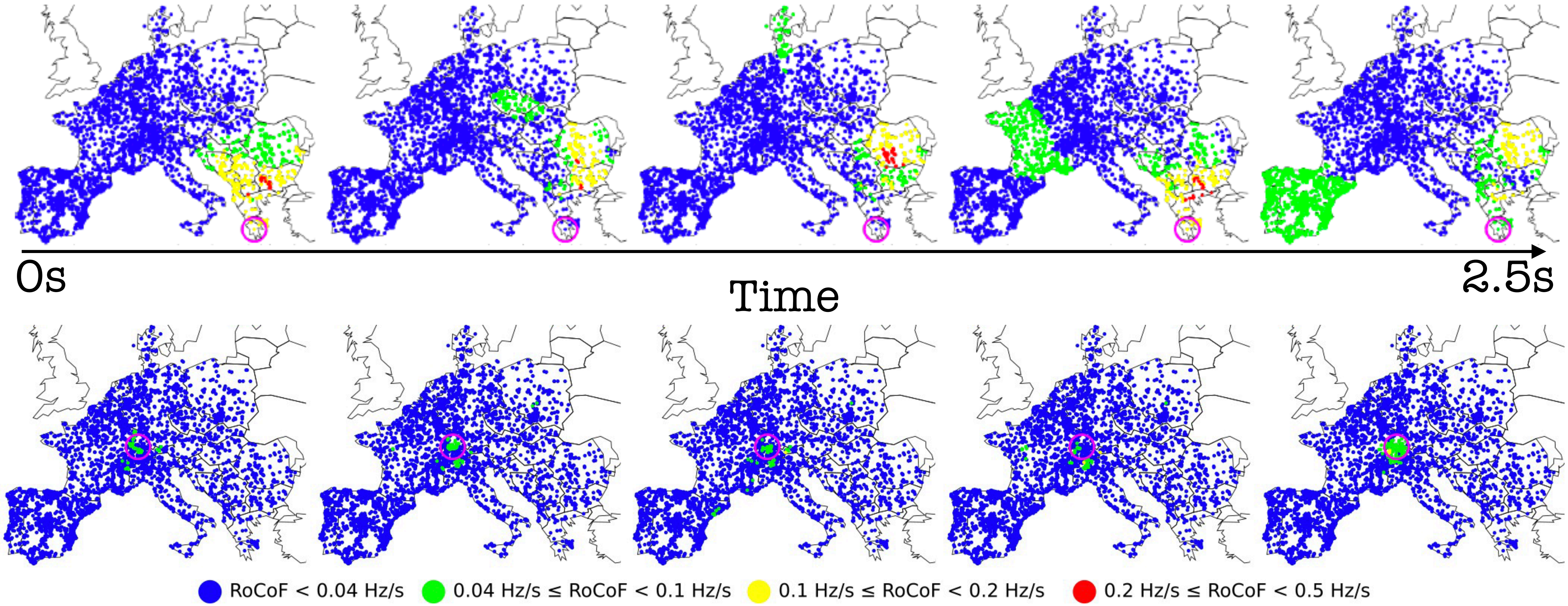






Weak perturbation : wave propagation

Evolution of RoCoF



Pagnier and PJ, PLoS ONE ‘19

Weak perturbation : wave propagation

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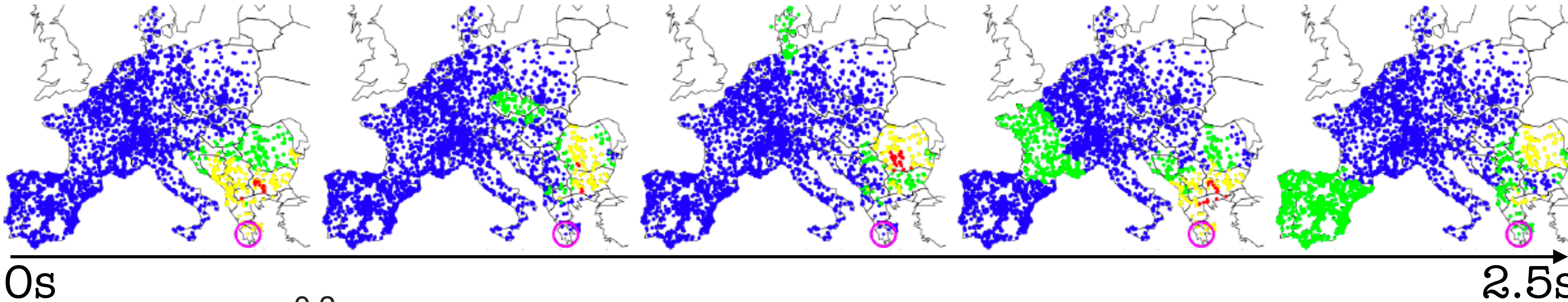

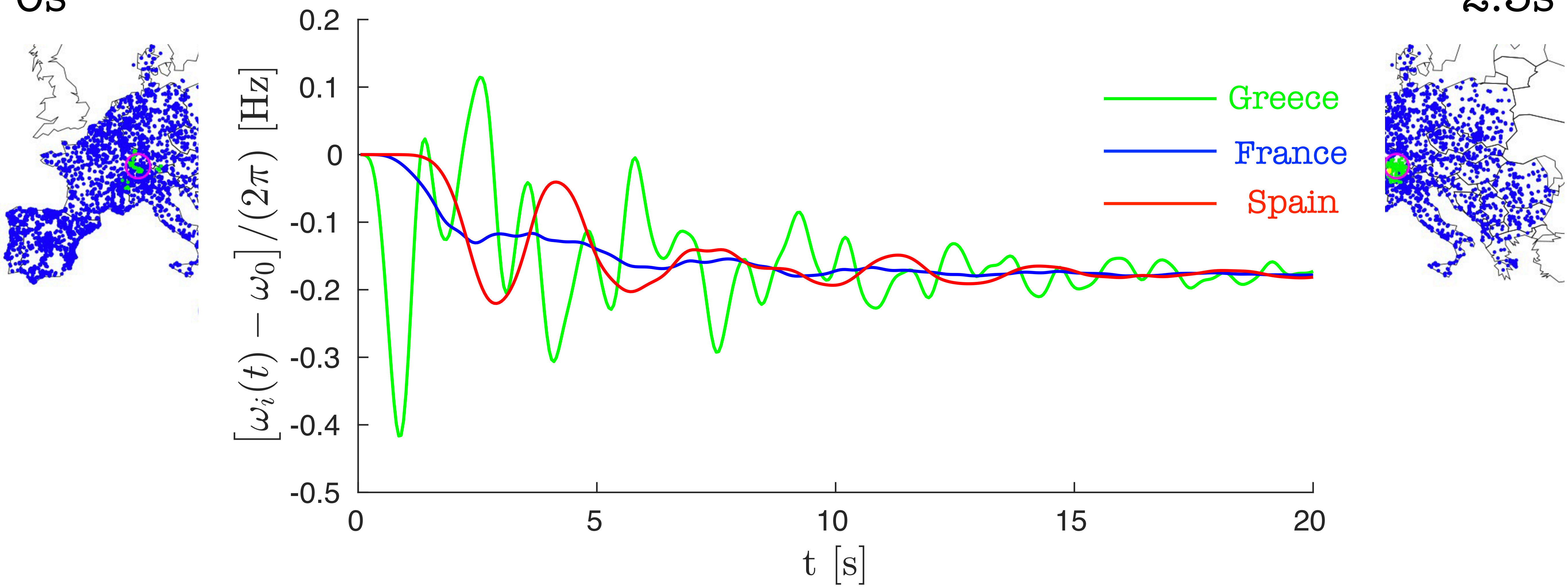

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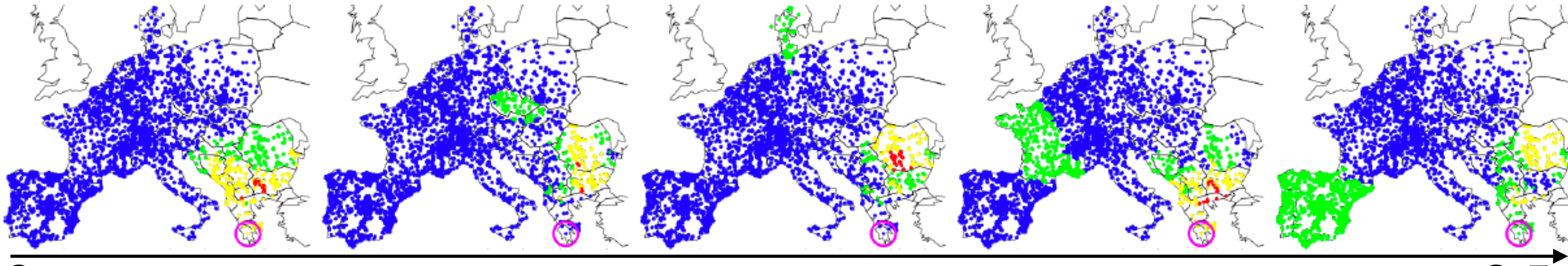
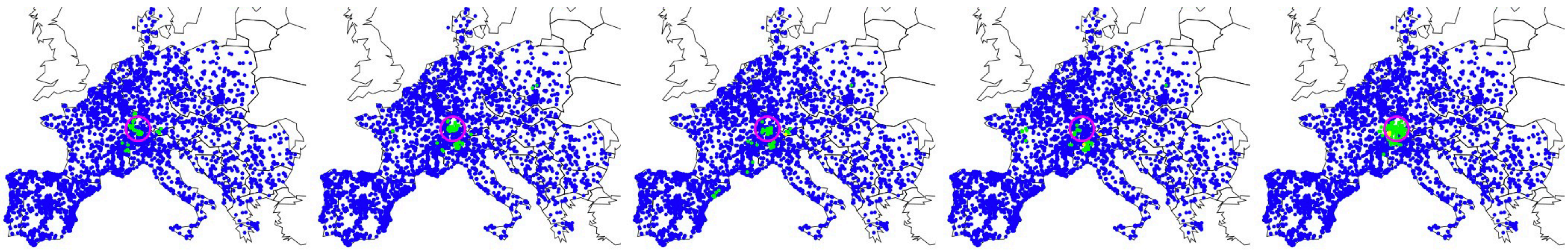


0s

2.5s

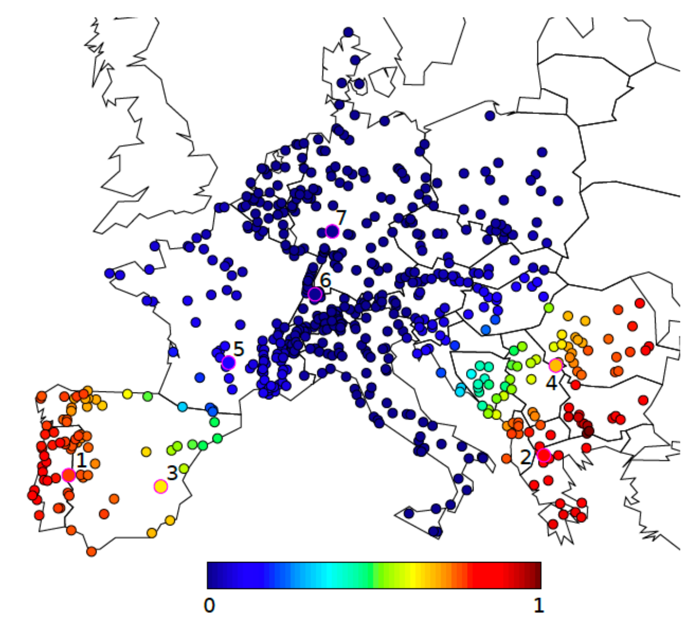
Time



RoCoF < 0.04 Hz/s   0.04 Hz/s ≤ RoCoF < 0.1 Hz/s   0.1 Hz/s ≤ RoCoF < 0.2 Hz/s   0.2 Hz/s ≤ RoCoF < 0.5 Hz/s

Propagation correlates with the
Fiedler mode (lowest nonzero mode)

~less damping



Pagnier and PJ, PLoS ONE '19

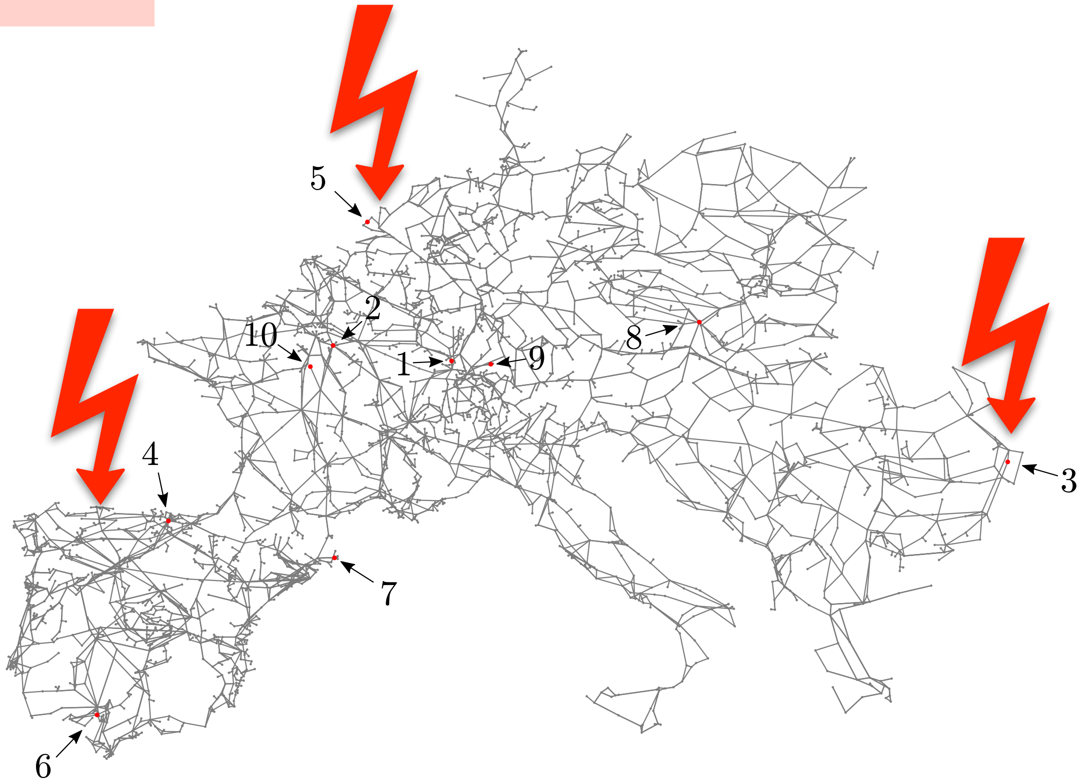
Weak perturbation :Global robustness vs. local vulnerabilities

$$I_i\ddot{\theta} + D_i\dot{\theta}_i = P_i - \sum [B_{ij}\sin(\theta_i - \theta_j)$$

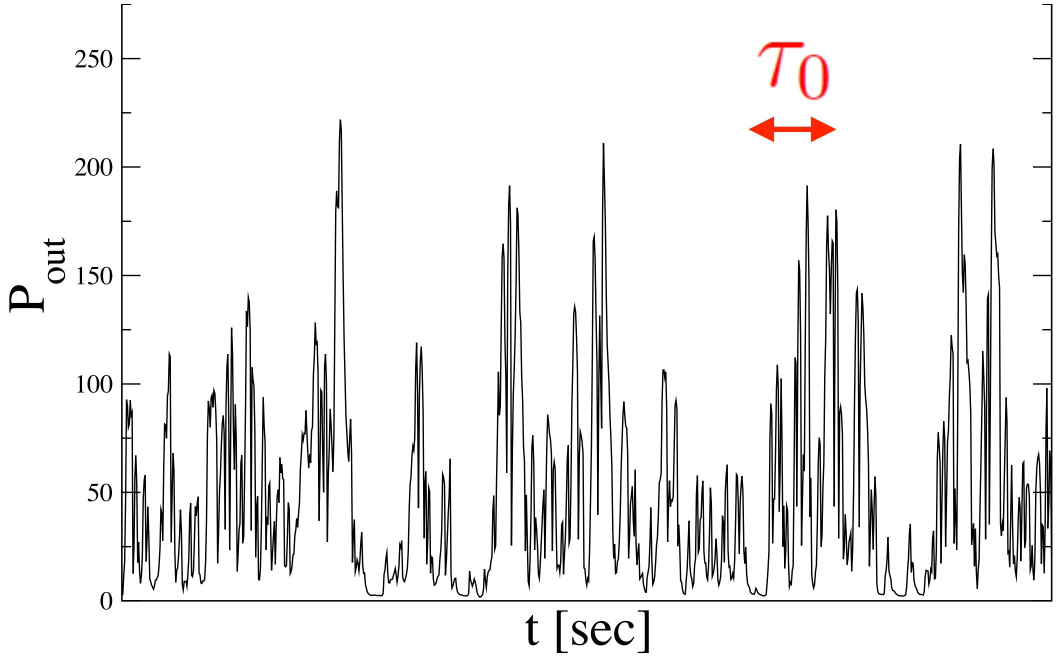
$$P_i(t) = P_i^{(0)} + \delta P_i(t)$$

$$\langle \delta P_i(t) \rangle = 0$$

$$\langle \delta P_i(t_1)\delta P_j(t_2) \rangle = \delta P_0^2 \delta_{ij} e^{-|t_1 - t_2|/\tau_0}$$



- No spatial correlation
- Characteristic time τ_0



Tyloo, Coletta and PJ, PRL '18
Tyloo, Pagnier and PJ, Sc. Adv. '19

Weak perturbation :Global robustness vs. local vulnerabilities

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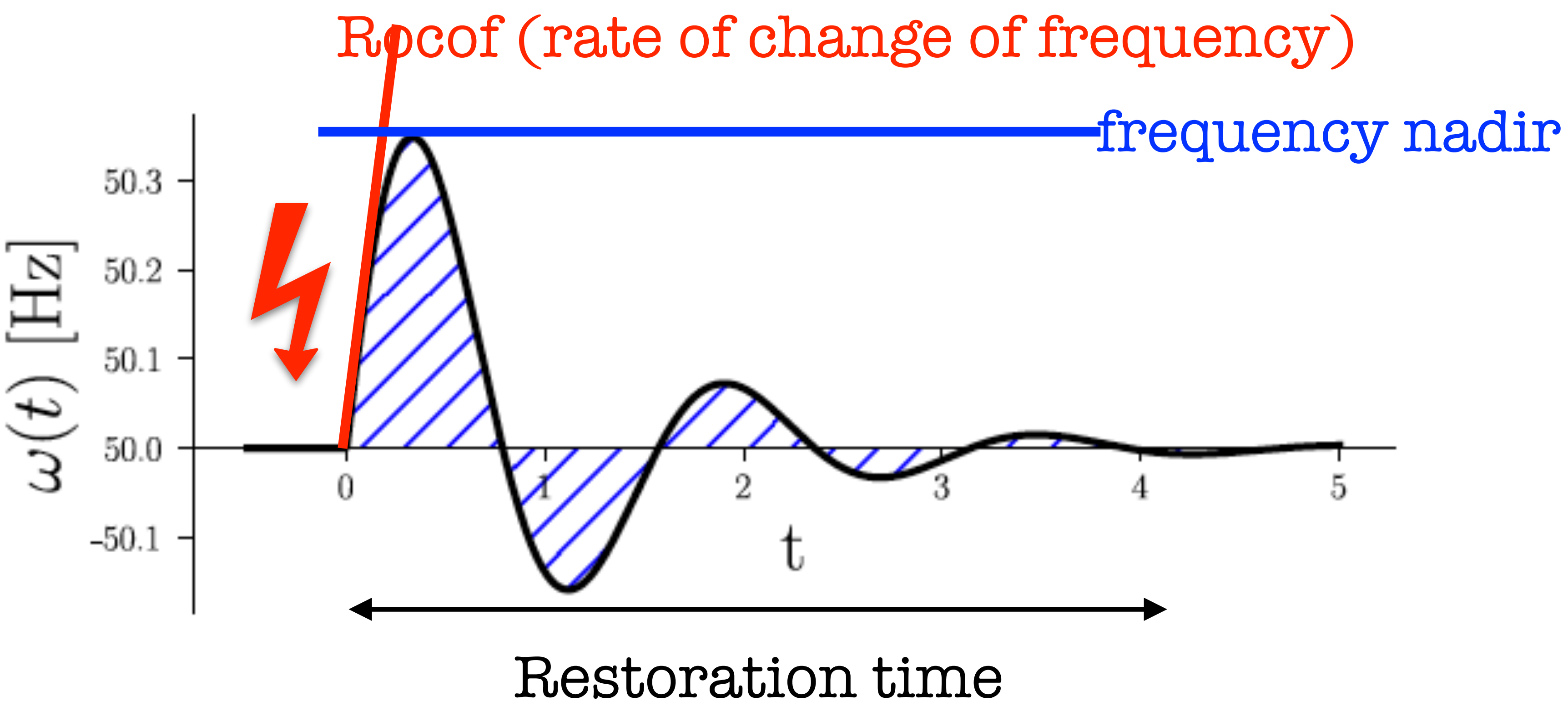

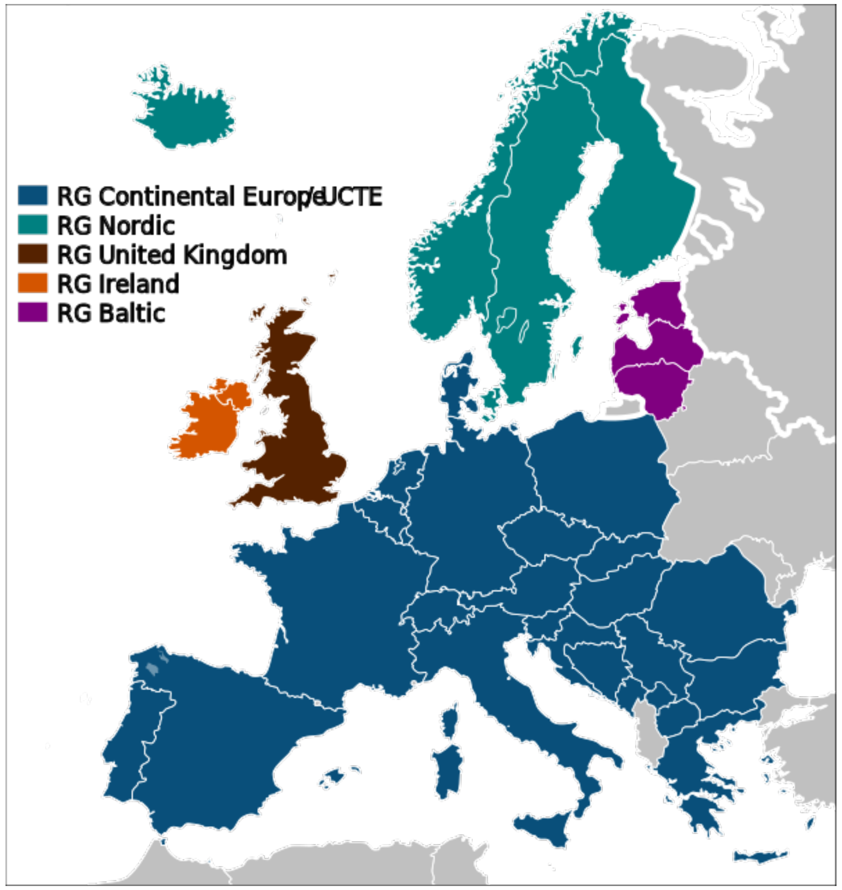

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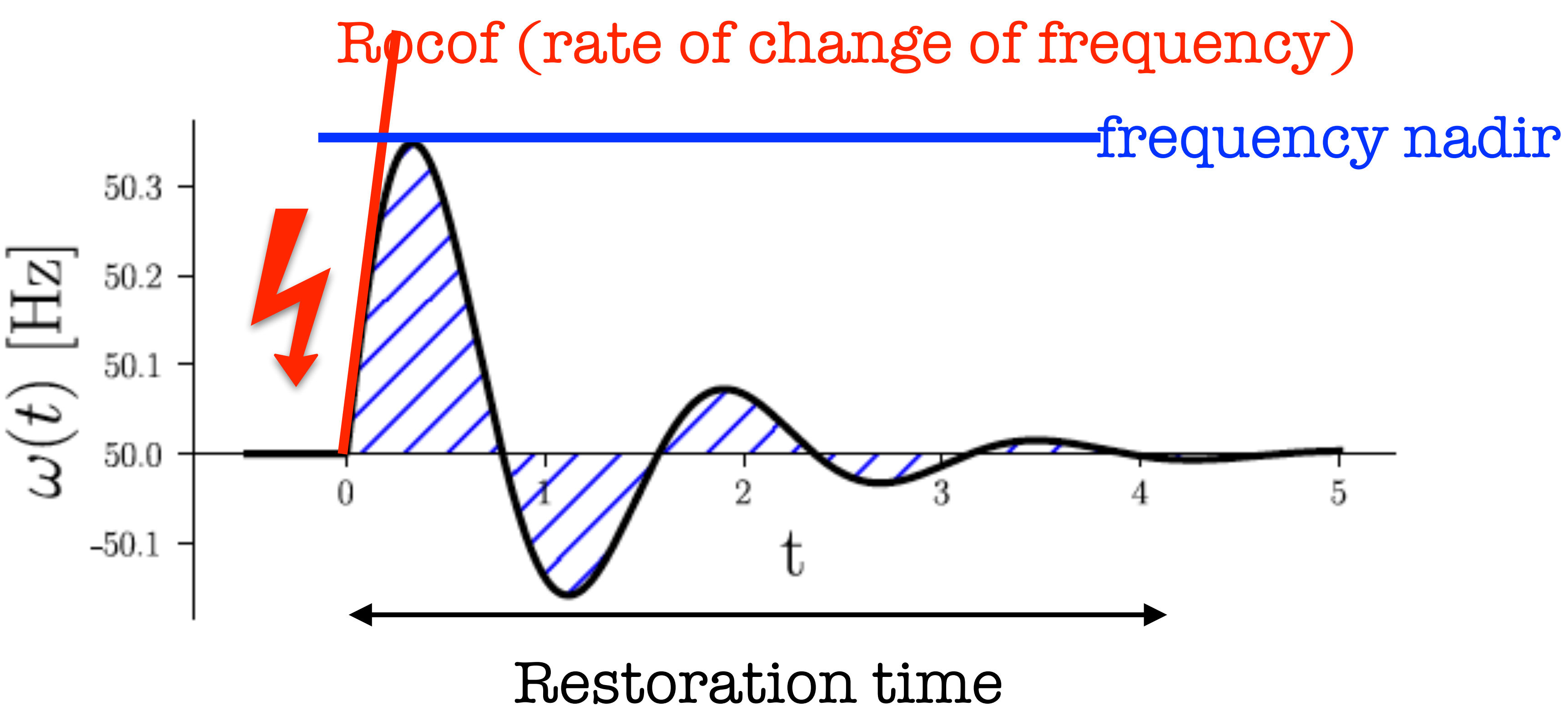
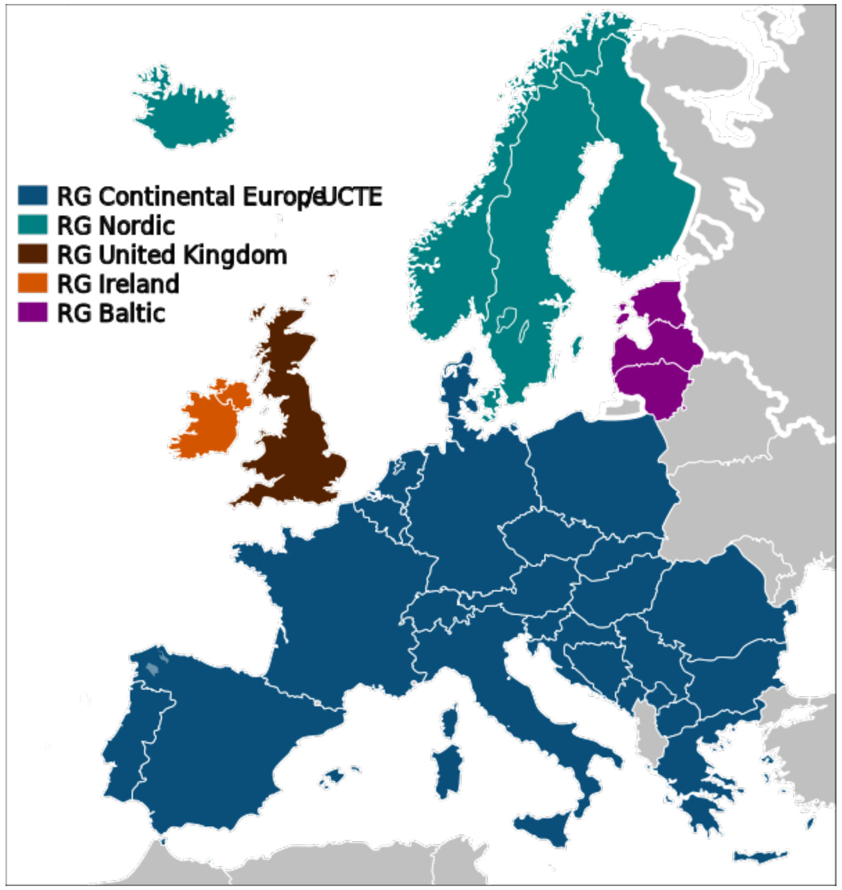




Our performance measures

$$\mathcal{P}_1(T) = \int_0^T \mathrm{d}t\,\delta\boldsymbol{\theta}^2(t)$$

$$\mathcal{P}_2(T) = \int_0^T \mathrm{d}t\,\delta\dot{\boldsymbol{\theta}}^2(t)$$

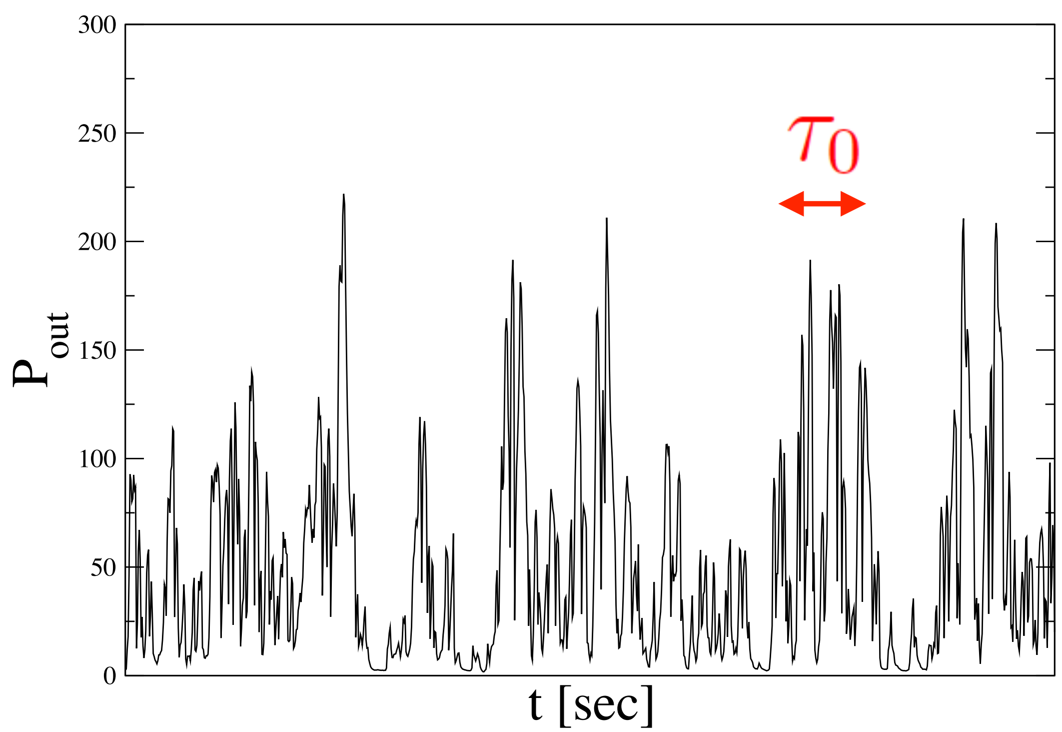
Take limit $T \to \infty$ when possible
Divide by T if needed

Weak perturbation :Global robustness vs. local vulnerabilities

Angle dynamics

$$I_i\ddot{\theta} + D_i\dot{\theta}_i = P_i - \sum[B_{ij}\sin(\theta_i - \theta_j)$$

$$P_i(t) = P_i^{(0)} + \delta P_i(t)$$
$$\theta_i(t) = \theta_i^{(0)} + \delta\theta_i(t)$$
$$P_i^{(0)} = \sum B_{ij}\sin(\theta_i^{(0)} - \theta_j^{(0)})$$



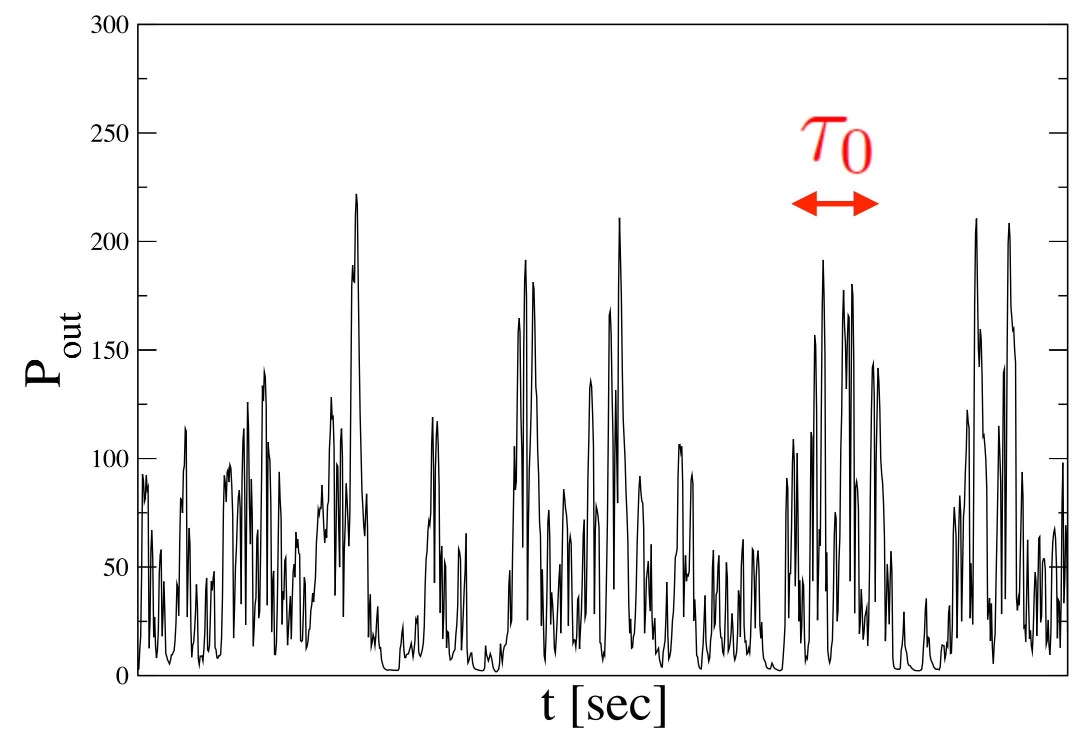
Can one characterize $\delta\theta_i(t)$ given $\delta P_i(t)$?

Weak perturbation :Global robustness vs. local vulnerabilities

Angle dynamics

$$I_i\ddot{\theta} + D_i\dot{\theta}_i = P_i - \sum [B_{ij} \sin(\theta_i - \theta_j)$$

$$\boxed{\begin{aligned} P_i(t) &= P_i^{(0)} + \delta P_i(t) \\ \theta_i(t) &= \theta_i^{(0)} + \delta\theta_i(t) \\ P_i^{(0)} &= \sum B_{ij} \sin(\theta_i^{(0)} - \theta_j^{(0)}) \end{aligned}}$$



Can one characterize $\delta\theta_i(t)$ given $\delta P_i(t)$?

A: (i) linearize the dynamics about a fixed-point solution

$$\delta\dot{\vec{\theta}} = \delta\vec{P} + \mathbb{L}(\vec{\theta}^{(0)})\, \delta\vec{\theta}$$

(ii) spectral decomposition, i.e. expand angles over eigenmodes of stability matrix

→ get equation for coefficients of expansion !

$$\delta\vec{\theta}(t) = \sum_\alpha c_\alpha(t) \vec{\phi}_\alpha$$

$$\mathbb{L}\vec{\phi}_\alpha = \lambda_\alpha \vec{\phi}_\alpha$$

Weak perturbation :Global robustness vs. local vulnerabilities

Angle dynamics

$$I_i\ddot{\theta} + D_i\dot{\theta}_i = P_i - \sum [B_{ij}\sin(\theta_i - \theta_j)$$

$$\langle \delta P_i(t) \rangle = 0$$

$$\langle \delta P_i(t_1) \delta P_j(t_2) \rangle = \delta P_0^2 \delta_{ij} e^{-|t_1 - t_2|/\tau_0}$$

Equation for coefficients of spectral expansion (no inertia)

$$\dot{c}_\alpha(t) = \lambda_\alpha c_\alpha(t) + \delta\vec{P}(t) \cdot \vec{\phi}_\alpha$$

Weak perturbation :Global robustness vs. local vulnerabilities

Angle dynamics

$$I_i\ddot{\theta} + D_i\dot{\theta}_i = P_i - \sum [B_{ij}\sin(\theta_i - \theta_j)$$

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A.k.a. Langevin equation with solution :

$$\boxed{c_\alpha(t) = e^{\lambda_\alpha t}c_\alpha(0) + e^{\lambda_\alpha t}\int_0^t e^{-\lambda_\alpha t'}\,\delta\vec{P}(t')\cdot\vec{\phi}_\alpha \mathrm{d}t'}$$

Weak perturbation :Global robustness vs. local vulnerabilities

Angle dynamics

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Performance measures to calculate

$$\mathcal{P}_1(T) = \sum_{\alpha\geq 2}\int_0^T c_\alpha^2(t)\mathrm{d}t \qquad \mathcal{P}_2(T) = \sum_{\alpha\geq 2}\int_0^T \dot{c}_\alpha^2(t)\mathrm{d}t$$

Weak perturbation :Global robustness vs. local vulnerabilities

Angle dynamics

$$I_i\ddot{\theta} + D_i\dot{\theta}_i = P_i - \sum [B_{ij}\sin(\theta_i - \theta_j)$$

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In terms of second moment of noise

$$\langle c_\alpha^2(t)\rangle = e^{2\lambda_\alpha t}\iint_0^t e^{-\lambda_\alpha(t_1+t_2)}\,\langle(\delta\vec{P}(t_1)\cdot\vec{\phi}_\alpha)\,(\delta\vec{P}(t_2)\cdot\vec{\phi}_\alpha)\rangle\mathrm{d}t_1\mathrm{d}t_2$$

$$\mathcal{P}_1(T) = \lim_{T\to\infty} T^{-1} \int_0^T \mathrm{d}t\, \delta\boldsymbol{\theta}^2(t) = \delta P_0^2 \sum_{\alpha>2} \frac{\sum_{\mathrm{noisy}\, i} |\phi_{i,\alpha}|^2 (I/D + \tau_0)}{\lambda_\alpha(\lambda_\alpha \tau_0 + D + I\tau_0^{-1})}$$

$$\mathcal{P}_2(T) = \lim_{T\to\infty} T^{-1} \int_0^T \mathrm{d}t\, \delta\dot{\boldsymbol{\theta}}^2(t) = \delta P_0^2 \sum_{\alpha\geq 2} \frac{\sum_{\mathrm{noisy}\, i} |\phi_{i,\alpha}|^2}{D(\lambda_\alpha \tau_0 + D + I\tau_0^{-1})}$$

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Local vulnerabilities

$$\boxed{\sum_{\text{noisy}i} |\phi_{i,\alpha}|^2 \to |\phi_{k,\alpha}|^2}$$

$\lambda_\alpha\tau_0 \ll D$

$$\mathcal{P}_1 \cong \frac{\delta P_0^2 \tau_0}{D} \sum_{\alpha\geq 2} \frac{\phi_{\alpha,k}^2}{\lambda_\alpha}$$

$$\mathcal{P}_2 \cong \frac{\delta P_0^2 \tau_0}{DI} \frac{(n-1)}{n}$$

$\lambda_\alpha\tau_0 \gg D$

$$\mathcal{P}_1 \cong \delta P_0^2 \sum_{\alpha\geq 2} \frac{\phi_{\alpha,k}^2}{\lambda_\alpha^2}$$

$$\mathcal{P}_2 \cong \frac{\delta P_0^2}{D\tau_0} \sum_{\alpha\geq 2} \frac{\phi_{\alpha,k}^2}{\lambda_\alpha}$$

$$\mathcal{P}_1(T) = \lim_{T\to\infty} T^{-1}\int_0^T \mathrm{d}t\,\delta\boldsymbol{\theta}^2(t) = \delta P_0^2 \sum_{\alpha>2} \frac{\sum_{\mathrm{noisy}\,i} |\phi_{i,\alpha}|^2 (I/D + \tau_0)}{\lambda_\alpha(\lambda_\alpha\tau_0 + D + I\tau_0^{-1})}$$

$$\mathcal{P}_2(T) = \lim_{T\to\infty} T^{-1}\int_0^T \mathrm{d}t\,\delta\dot{\boldsymbol{\theta}}^2(t) = \delta P_0^2 \sum_{\alpha\geq 2} \frac{\sum_{\mathrm{noisy}\,i} |\phi_{i,\alpha}|^2}{D(\lambda_\alpha\tau_0 + D + I\tau_0^{-1})}$$

Local vulnerabilities

$$\boxed{\sum_{\mathrm{noisy}i} |\phi_{i,\alpha}|^2 \rightarrow |\phi_{k,\alpha}|^2}$$

$\lambda_\alpha\tau_0 \ll D$

$$\mathcal{P}_1 \cong \frac{\delta P_0^2\tau_0}{D}\sum_{\alpha\geq 2}\frac{\phi_{\alpha,k}^2}{\lambda_\alpha}$$

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$$\mathcal{P}_2 \cong \frac{\delta P_0^2}{D\tau_0}\sum_{\alpha\geq 2}\frac{\phi_{\alpha,k}^2}{\lambda_\alpha}$$

Global robustness

$$\boxed{\sum_{\mathrm{noisy}i} |\phi_{i,\alpha}|^2 \rightarrow 1}$$

$\lambda_\alpha\tau_0 \ll D$

$$\mathcal{P}_1(T) = \frac{\delta P_0^2\tau_0}{D}\sum_{\alpha\geq 2}\frac{1}{\lambda_\alpha}$$

$$\mathcal{P}_2(T) = \frac{\delta P_0^2\tau_0}{D(D+I\tau_0^{-1})}\frac{n-1}{n}$$

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$$\mathcal{P}_2(T) = \lim_{T\to\infty} T^{-1} \int_0^T \mathrm{d}t\, \delta\dot{\boldsymbol{\theta}}^2(t) = \delta P_0^2 \sum_{\alpha\geq 2} \frac{\sum_{\mathrm{noisy}\,i} |\phi_{i,\alpha}|^2}{D(\lambda_\alpha \tau_0 + D + I\tau_0^{-1})}$$

Local vulnerabilities

$$\boxed{\sum_{\mathrm{noisy}\,i} |\phi_{i,\alpha}|^2 \to |\phi_{k,\alpha}|^2}$$

$\lambda_\alpha \tau_0 \ll D$

$$\mathcal{P}_1 \cong \frac{\delta P_0^2 \tau_0}{D} \sum_{\alpha\geq 2} \frac{\phi_{\alpha,k}^2}{\lambda_\alpha}$$

$\lambda_\alpha \tau_0 >$

They all depend on

$$\sum_{\alpha\geq 2} \frac{\phi_{\alpha,k}^2}{\lambda_\alpha^p} \quad !!!$$

$$\mathcal{P}_2 \cong \frac{\delta P_0^2}{D\tau_0} \sum_{\alpha\geq 2} \frac{\phi_{\alpha,k}^2}{\lambda_\alpha}$$

Global robustness

$$\boxed{\sum_{\mathrm{noisy}\,i} |\phi_{i,\alpha}|^2 \to 1}$$

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$$\mathcal{P}_1(T) = \frac{\delta P_0^2 \tau_0}{D} \sum_{\alpha\geq 2} \frac{1}{\lambda_\alpha}$$

$$\mathcal{P}_2(T) = \frac{\delta P_0^2 \tau_0}{D(D + I\tau_0^{-1})} \frac{n-1}{n}$$

$\lambda_\alpha \tau_0 \gg D$

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Local vulnerabilities

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$\lambda_\alpha\tau_0 \ll D$

$$\mathcal{P}_1 \cong \frac{\delta P_0^2 \tau_0}{D} \sum_{\alpha\geq 2} \frac{\phi_{\alpha,k}^2}{\lambda_\alpha}$$

$\lambda_\alpha\tau_0 >$ … $1)$ … $,k$ … 2 … λ_α … $\alpha\geq 2$

They all depend on

$$\sum_{\alpha\geq 2} \frac{\phi_{\alpha,k}^2}{\lambda_\alpha^p} \quad !!!$$

$$\mathcal{P}_2 \cong \frac{\delta P_0^2}{D\tau_0} \sum_{\alpha\geq 2} \frac{\phi_{\alpha,k}^2}{\lambda_\alpha}$$

Global robustness

$$\boxed{\sum_{\text{noisy}\, i} |\phi_{i,\alpha}|^2 \to 1}$$

$\lambda_\alpha\tau_0 \ll D$

$$\mathcal{P}_1(T) = \frac{\delta P_0^2 \tau_0}{D} \sum_{\alpha\geq 2} \frac{1}{\lambda_\alpha}$$

$\frac{n-1}{n}$

$\lambda_\alpha\tau$ … $\frac{1}{\lambda_\alpha^2}$ … ≥ 2

They all depend on

$$\sum_{\alpha\geq 2} \frac{1}{\lambda_\alpha^p} \quad !!!$$

$$\mathcal{P}_2(T) = \frac{\delta P_0^2}{D\tau_0} \sum_{\alpha\geq 2} \frac{1}{\lambda_\alpha}$$

Weak perturbation : Global robustness & Kirchhoff indices

PHYSICAL REVIEW LETTERS 120, 084101 (2018)

Robustness of Synchrony in Complex Networks and Generalized Kirchhoff Indices

M. Tyloo,[1,2] T. Coletta,[1] and Ph. Jacquod[1]
[1]*School of Engineering, University of Applied Sciences of Western Switzerland HES-SO, CH-1951 Sion, Switzerland*
[2]*Institute of Physics, EPF Lausanne, CH-1015 Lausanne, Switzerland*

Introduce "generalized Kirchhoff indices" $$Kf_p = n \sum_{\alpha \geq 2} \lambda_\alpha^{-p}$$

τ_0 is shortest time scale

$$\overline{\mathcal{P}}_1 \cong \frac{\delta P_0^2 \tau_0}{D n^2} Kf_1 \qquad \mathcal{P}_2 \cong \frac{\delta P_0^2 \tau_0}{DI} \frac{(n-1)}{n}$$

τ_0 is longest time scale

$$\overline{\mathcal{P}}_1 \cong \frac{\delta P_0^2}{n^2} Kf_2 \qquad \overline{\mathcal{P}}_2 \cong \frac{\delta P_0^2}{n^2 \tau_0} Kf_1$$

Kirchhof index Kf1 : Klein and Randic, JMC '93.

Weak perturbation : Global robustness & Kirchhoff indices

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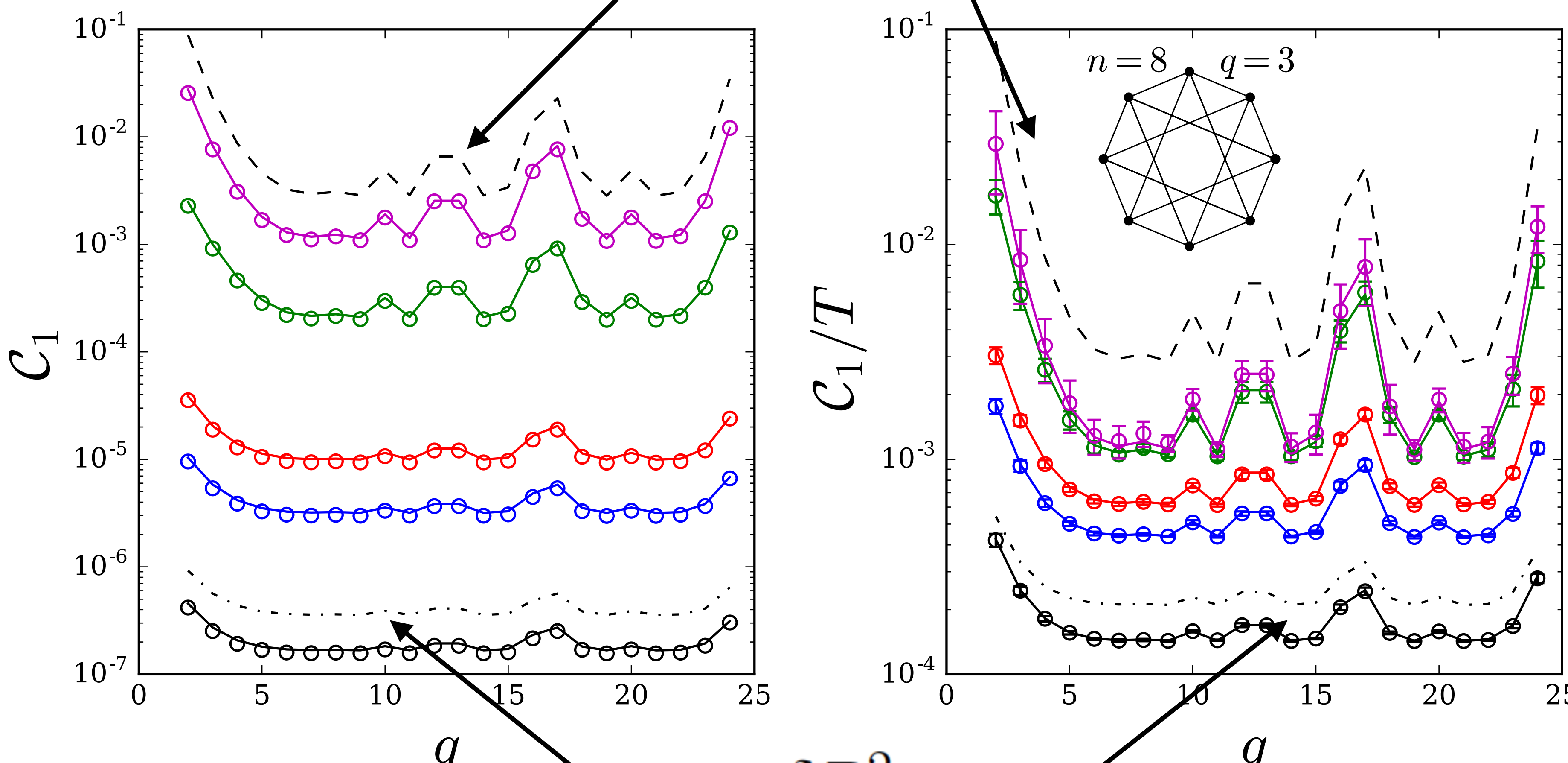
$$\overline{\mathcal{P}}_1 \cong \frac{\delta P_0^2}{n^2} K f_2$$

Box disturbance

Noise disturbance



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Weak perturbation : Global robustness & Kirchhoff indices

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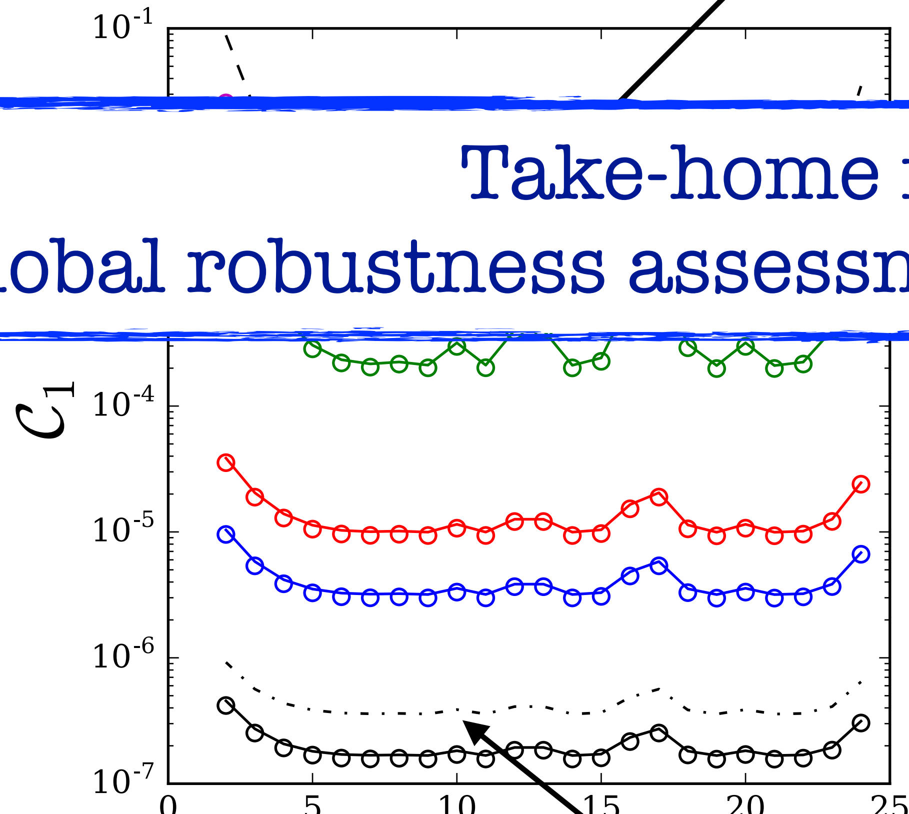

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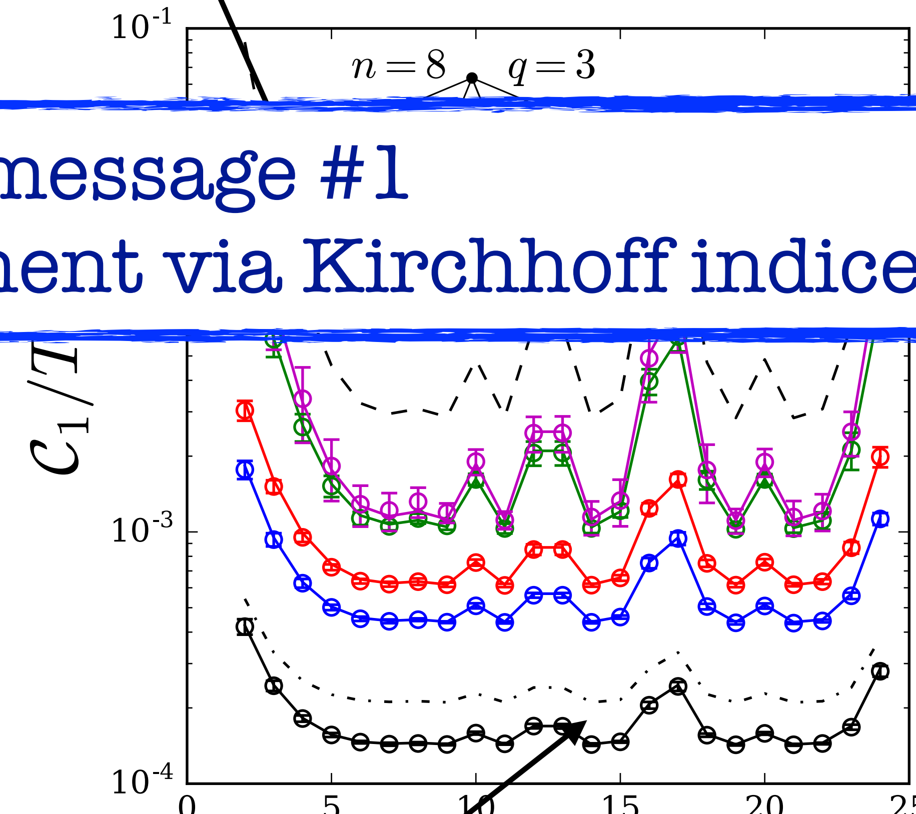

$$\overline{\mathcal{P}}_1 \cong \frac{\delta P_0^2 \tau_0}{D n^2} Kf_1$$

Take-home message #1

- Global robustness assessment via Kirchhoff indices

Weak perturbation : Local vulnerabilities & centralities

Resistance distance vs. Laplacian matrix (its pseudoinverse)

$$\Omega_{ij} = \mathbb{L}^{\dagger}_{ii} + \mathbb{L}^{\dagger}_{jj} - \mathbb{L}^{\dagger}_{ij} - \mathbb{L}^{\dagger}_{ji} = \sum_{\lambda \geq 2} \frac{(\phi_{\alpha,i} - \phi_{\alpha,j})^2}{\lambda_\alpha}$$

~effective resistance between i and k, for equivalent network of resistors

$$\rightarrow \sum_{\alpha \geq 2} \frac{\phi^2_{\alpha,k}}{\lambda_\alpha} = \sum_i \Omega_{ik} - \frac{Kf}{n^2}$$

τ_0 is shortest time scale

$$\mathcal{P}_1 = \frac{\delta P_0^2 \tau_0}{D} \left(C_k^{(1)-1} - n^{-2} K f_1 \right) \quad \mathcal{P}_2 \cong \frac{\delta P_0^2 \tau_0}{DI} \frac{(n-1)}{n}$$

τ_0 is longest time scale

$$\mathcal{P}_1 = \delta P_0^2 \left(C_k^{(2)^{-1}} - n^{-2} K f_2 \right) \quad \mathcal{P}_2 \cong \frac{\delta P_0^2}{D \tau_0} \left(C_k^{(1)-1} - n^{-2} K f_1 \right)$$

Tyloo, Pagnier and PJ, Sc. Adv. ‘19

Weak perturbation : Local vulnerabilities & centralities

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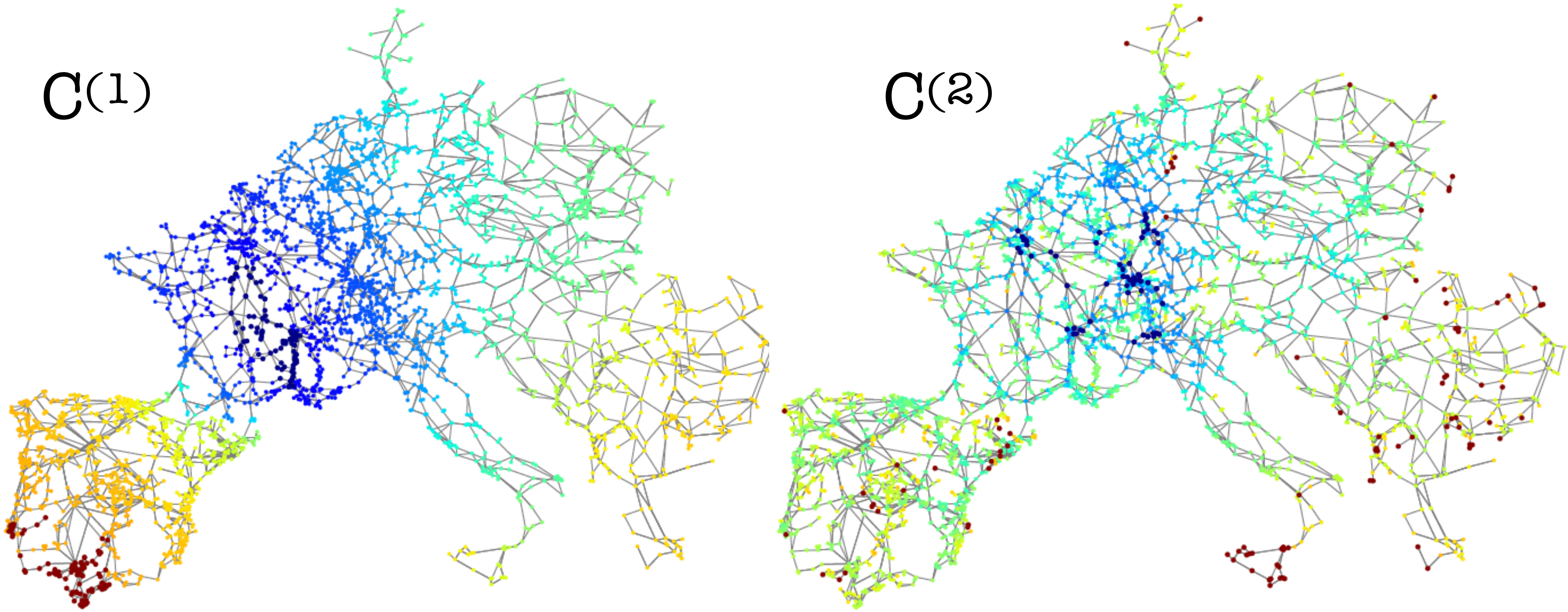


$$\mathcal{P}_1 = \delta P_0^2 \left({C_k^{(2)}}^{-1} - n^{-2} K f_2 \right) \qquad \mathcal{P}_2 \cong \frac{\delta P_0^2}{D\tau_0} \left(C_k^{(1)-1} - n^{-2} K f_1 \right)$$

Tyloo, Pagnier and PJ, Sc. Adv. '19

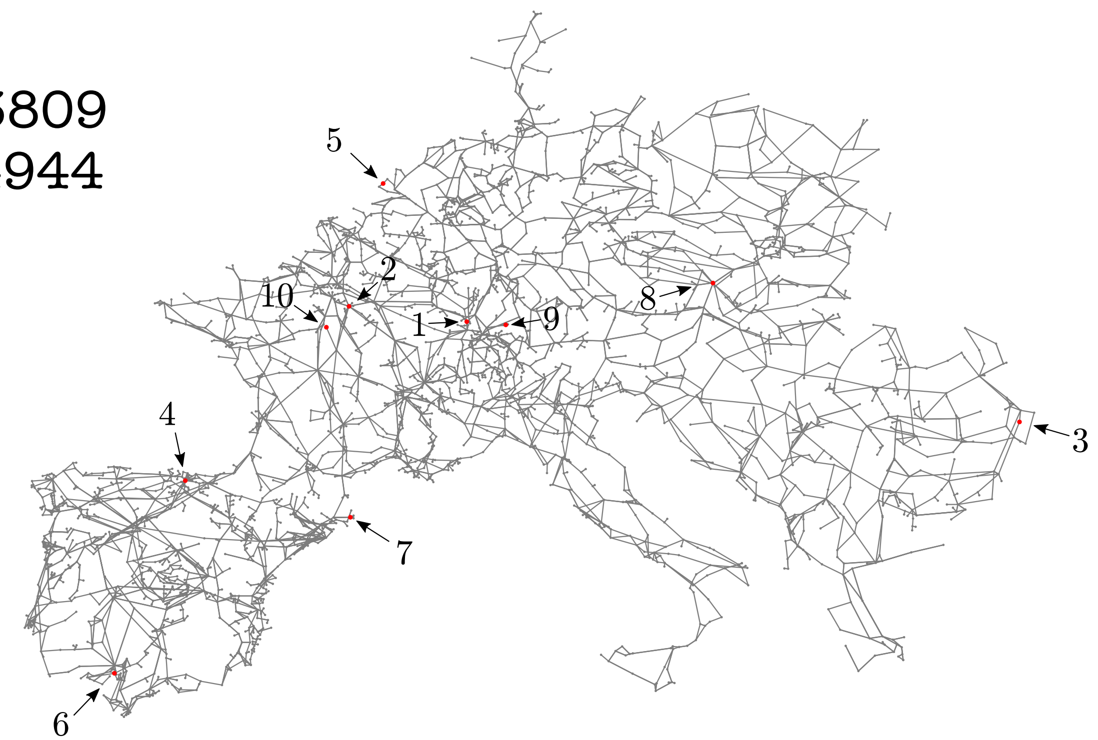
The key player problem : deterministically coupled systems

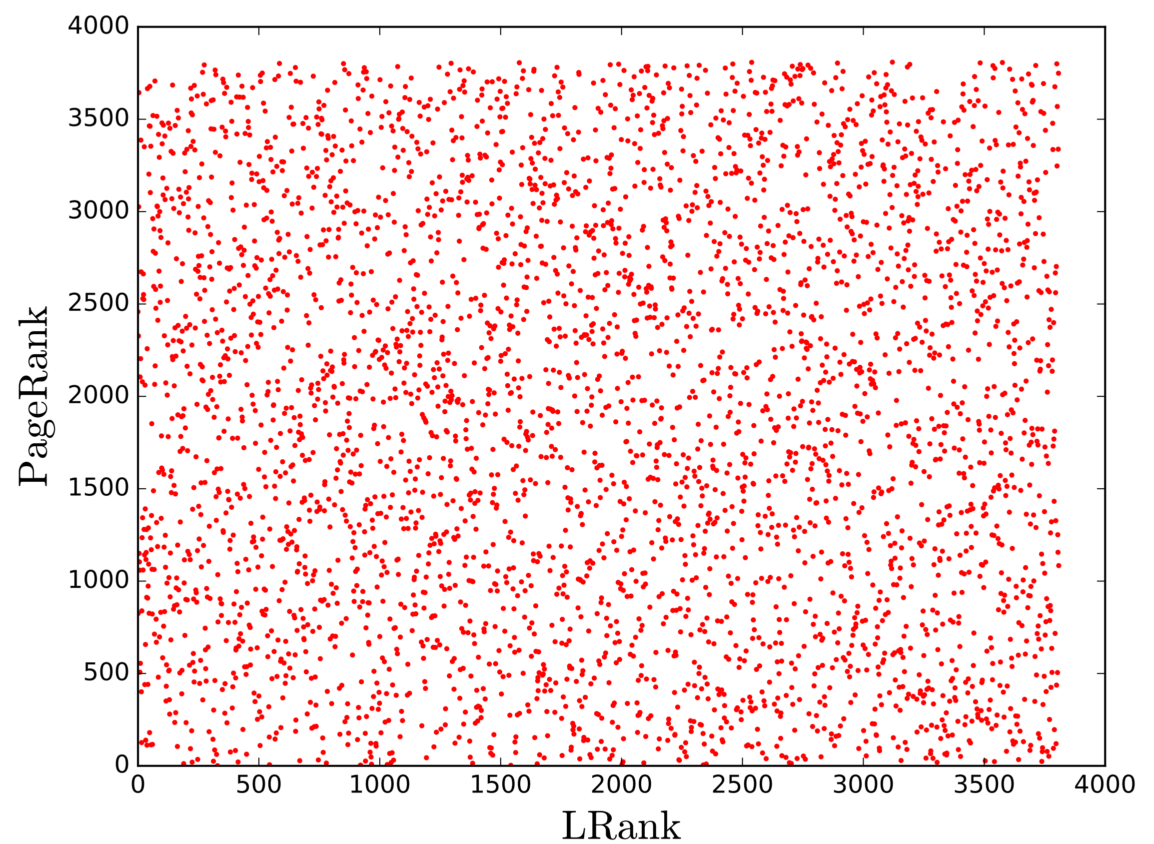
node #	$C_i^{(1)}$	C_i^{geo}	Degree	Katz	PageRank	$\mathcal{P}_2$
1	22.15	7.82	6	1.03195	1327	4.7×10^{-4}
2	22.07	7.68	5	1.03062	196	4.7×10^{-4}
3	18.59	3.54	5	1.03162	1041	6.6×10^{-4}
4	16.33	6.58	10	1.00103	1740	1.1×10^{-3}
5	16.32	5.56	2	1.03127	3470	1.1×10^{-3}
6	12.74	3.98	6	1.00067	3408	2×10^{-3}
7	10.77	6.58	3	1.00085	1076	2.5×10^{-3}
8	10.77	2.91	2	1.00016	2403	2.7×10^{-3}
9	9.66	3.53	2	1.00035	1532	3×10^{-3}
10	8.11	4.65	1	1.00001	3367	4×10^{-3}

Resistance distance centrality a.k.a. LRank

Numerically computed performance measure

#nodes : 3809
#edges : 4944





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6	12.74	3.98	6	1.00067	3408	2×10^{-3}
7	10.77	6.58	3	1.00085	1076	2.5×10^{-3}
8	1(					
9	9					
10	8					

#nodes : 3809
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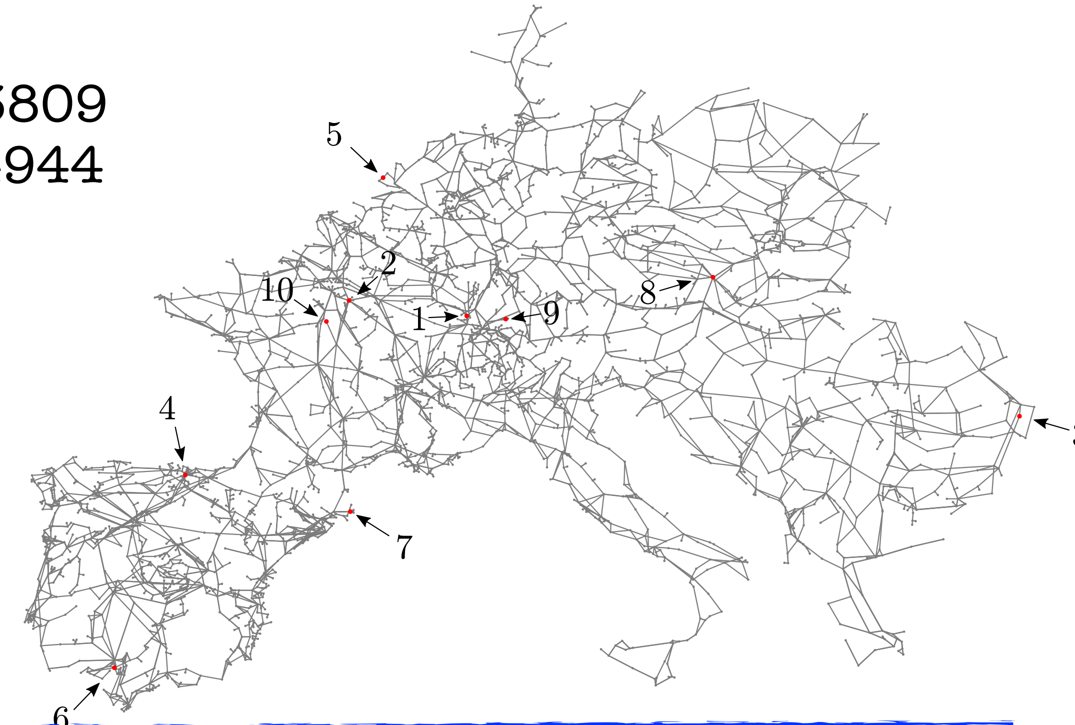
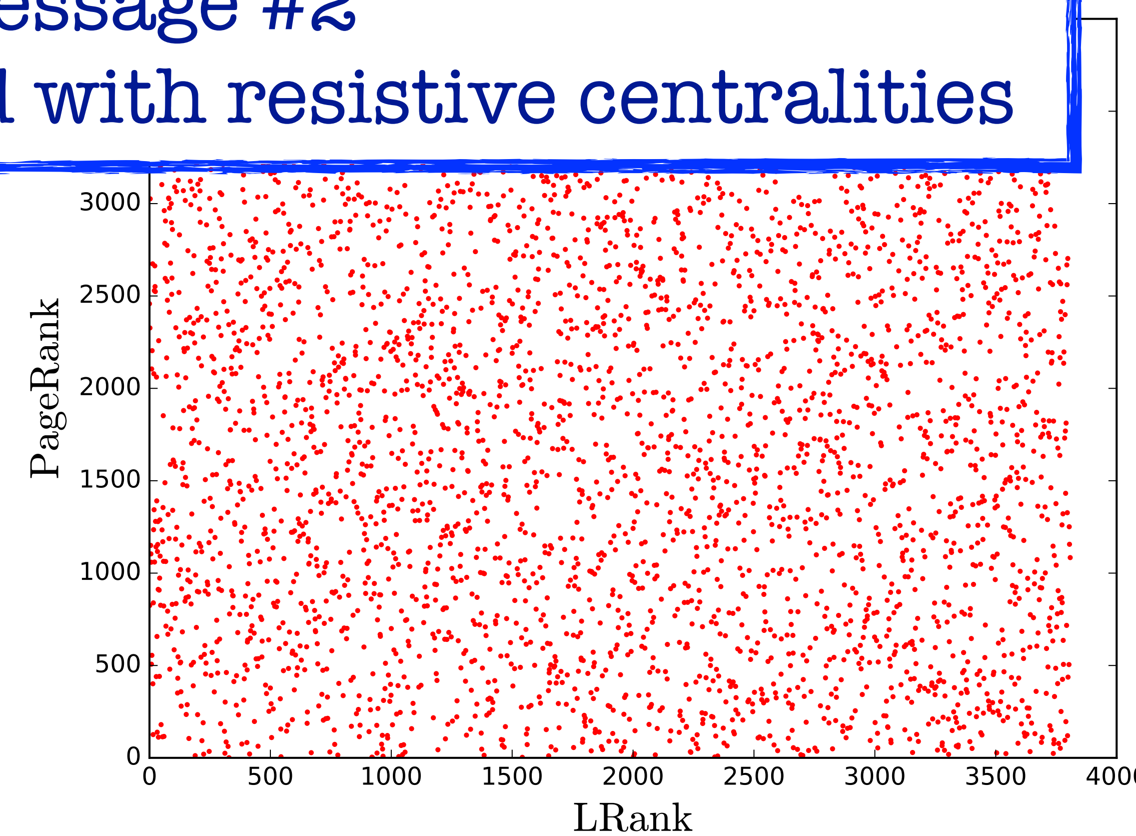


Take-home message #2

- Local vulnerabilities ranked with resistive centralities



Resistance distance centrality a.k.a. LRank

Numerically computed performance measure

Tyloo, Pagnier and PJ, Sc. Adv. '19

Strong perturbation : escape from basin of attraction

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$$\langle \delta P_i(t) \rangle = 0$$

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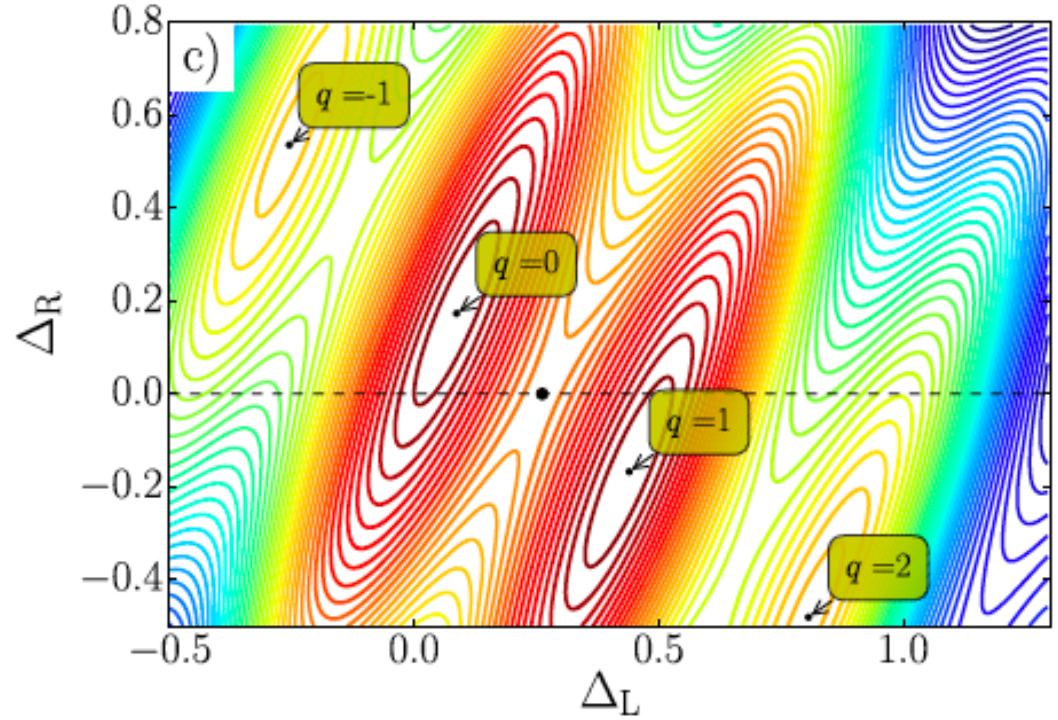

Tyloo, Delabays and PJ, Phys Rev E ‘19

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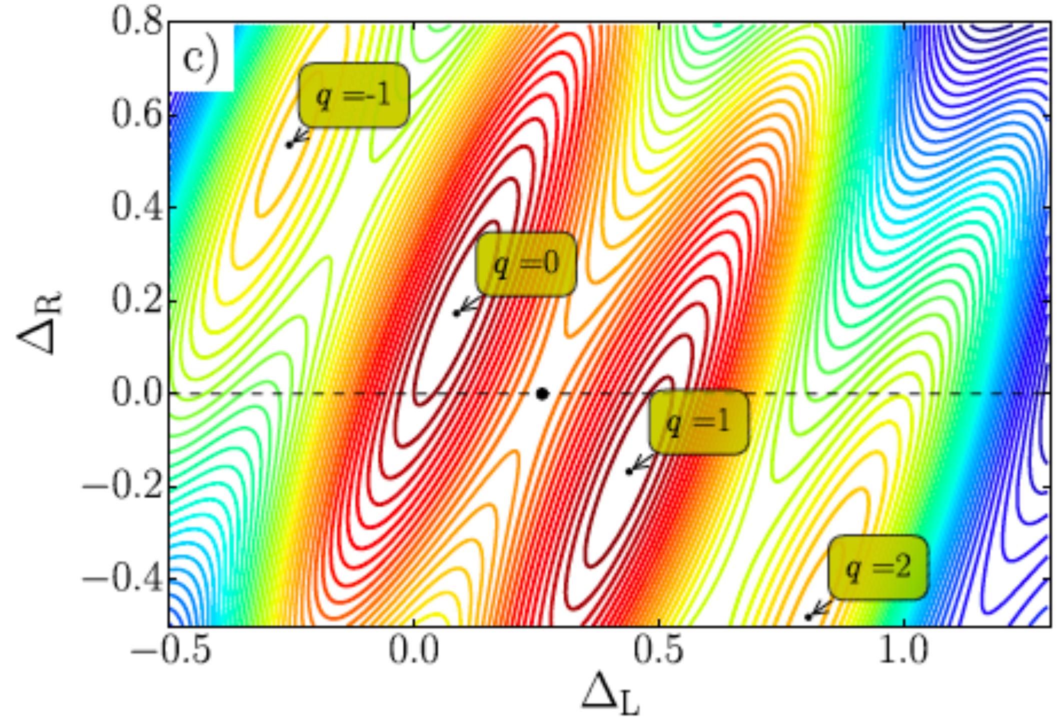

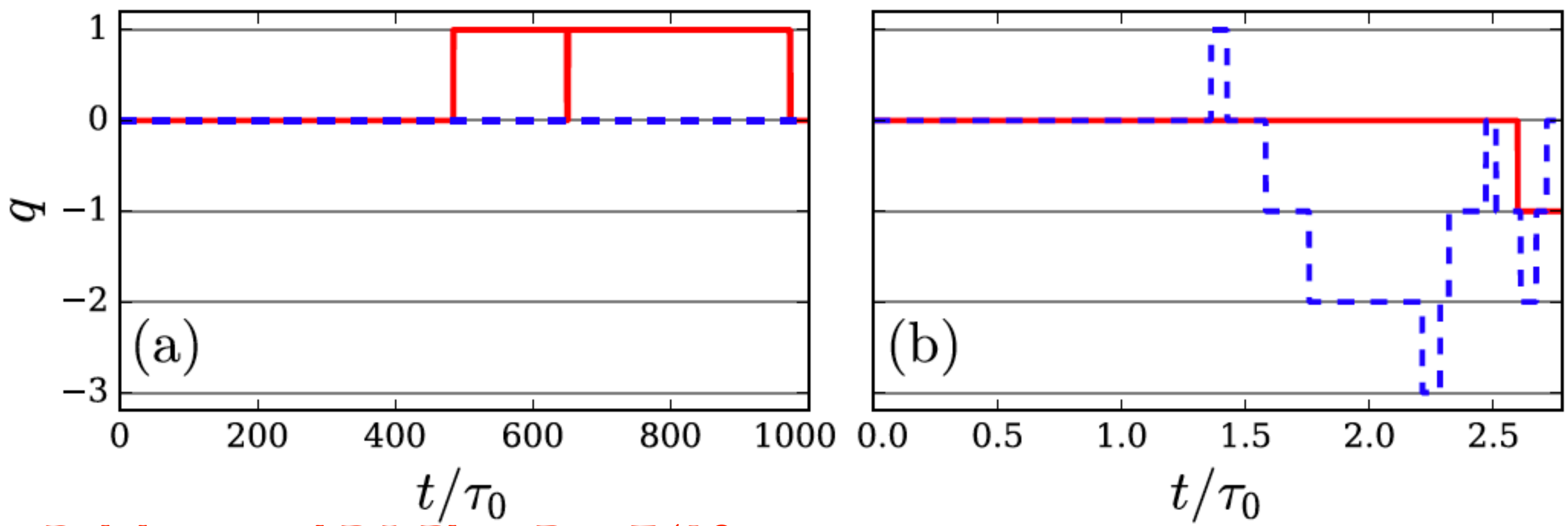

Tyloo, Delabays and PJ, Phys Rev E '19

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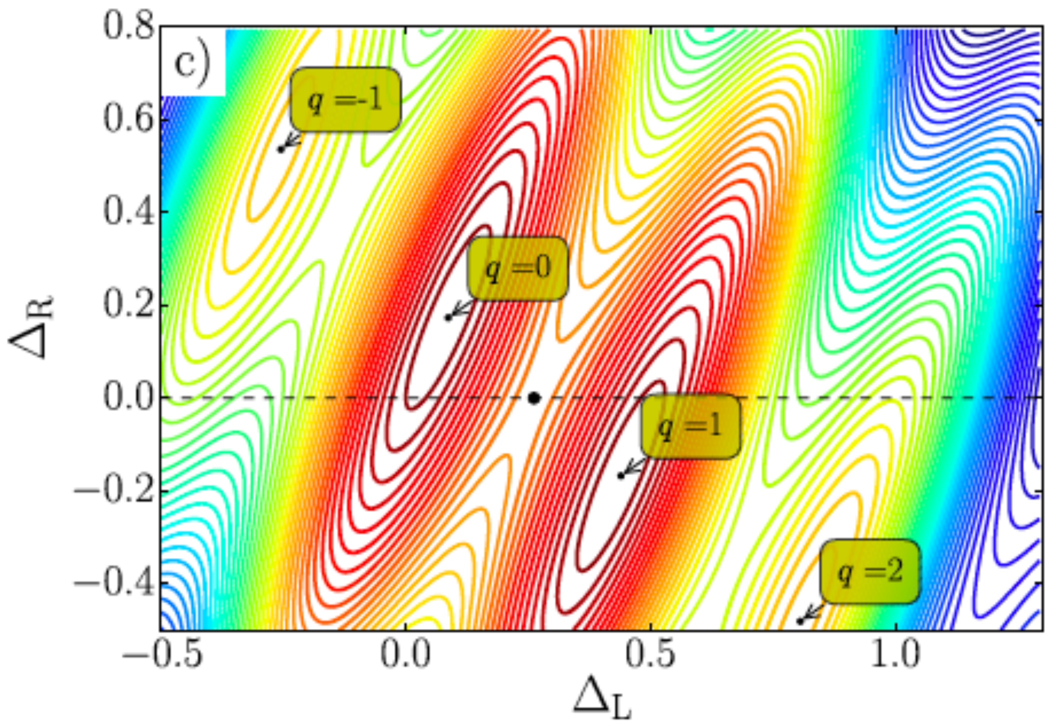

Tyloo, Delabays and PJ, Phys Rev E '19

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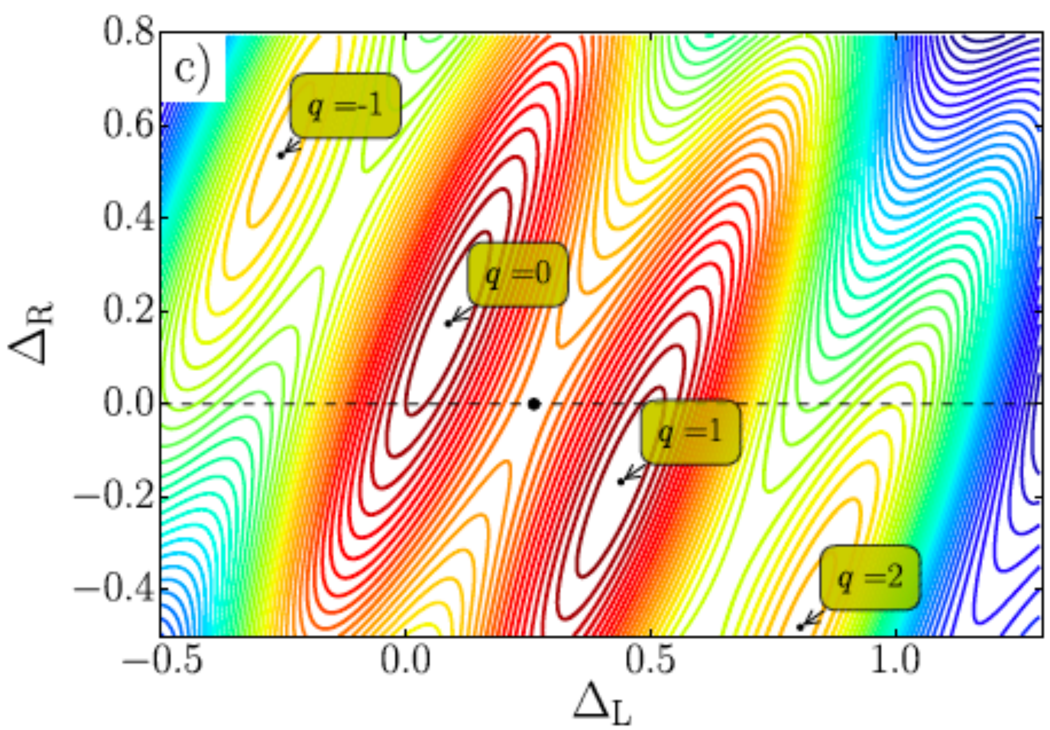

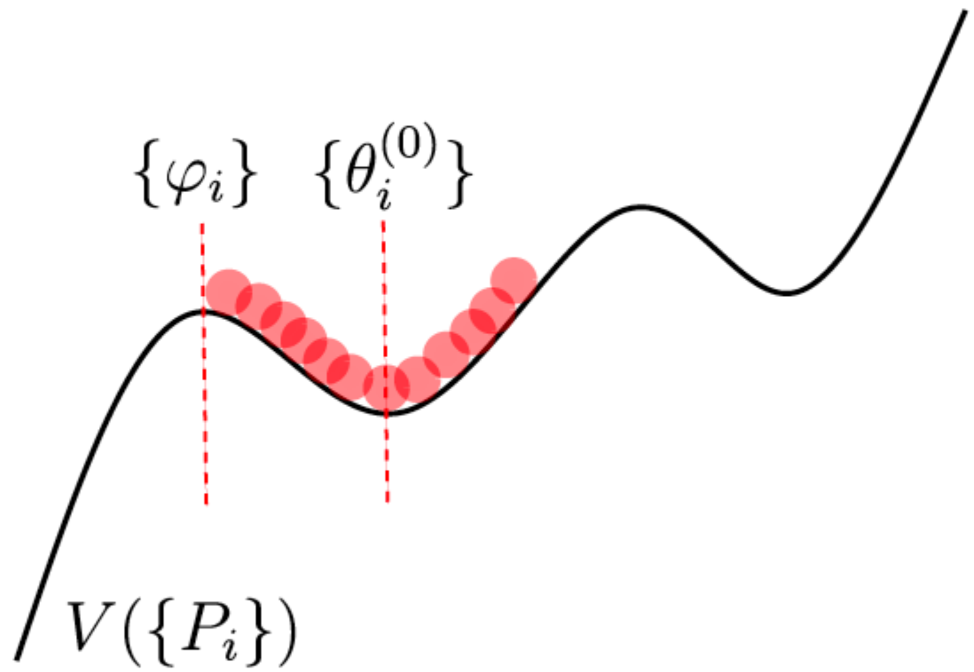

Tyloo, Delabays and PJ, Phys Rev E '19

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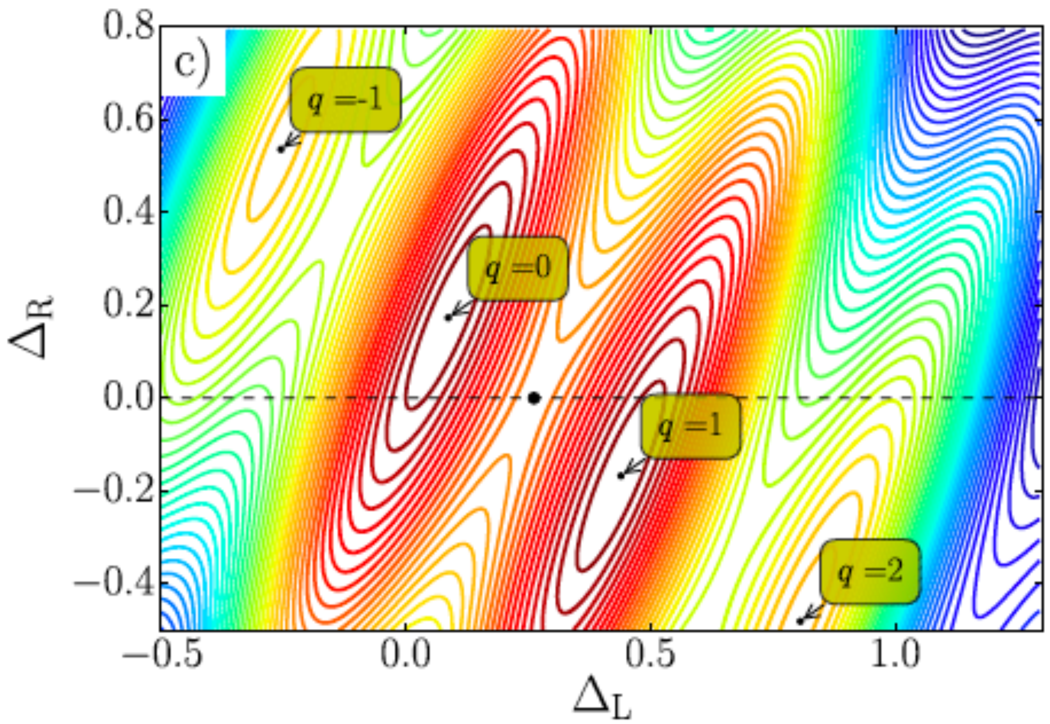

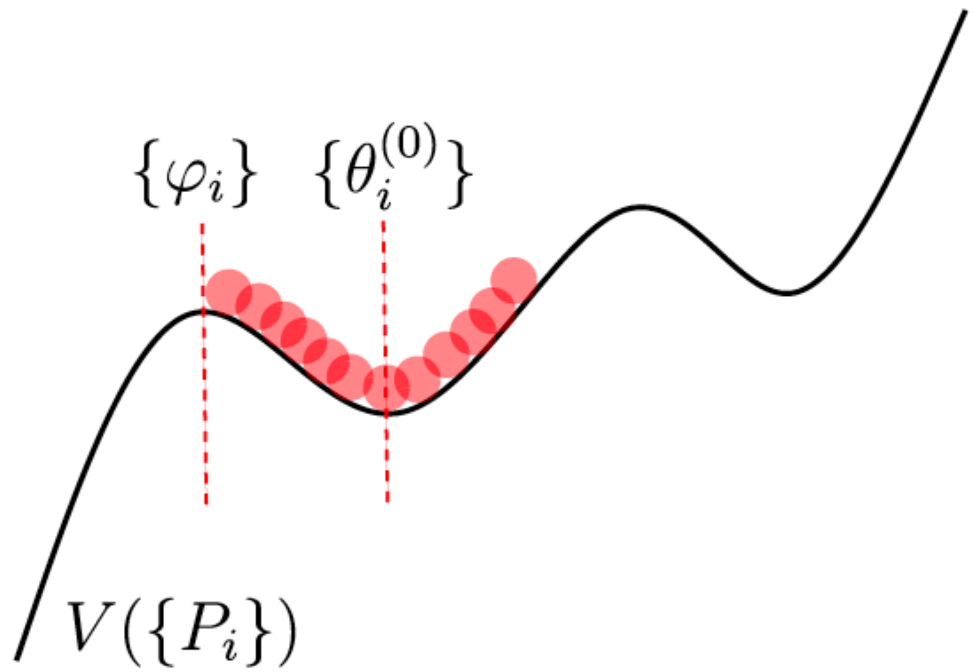

Escape criterion: $\langle \delta \boldsymbol{\theta}^2 \rangle \cong \| \boldsymbol{\theta}^{(0)} - \boldsymbol{\varphi} \|_2^2$

Tyloo, Delabays and PJ, Phys Rev E ‘19

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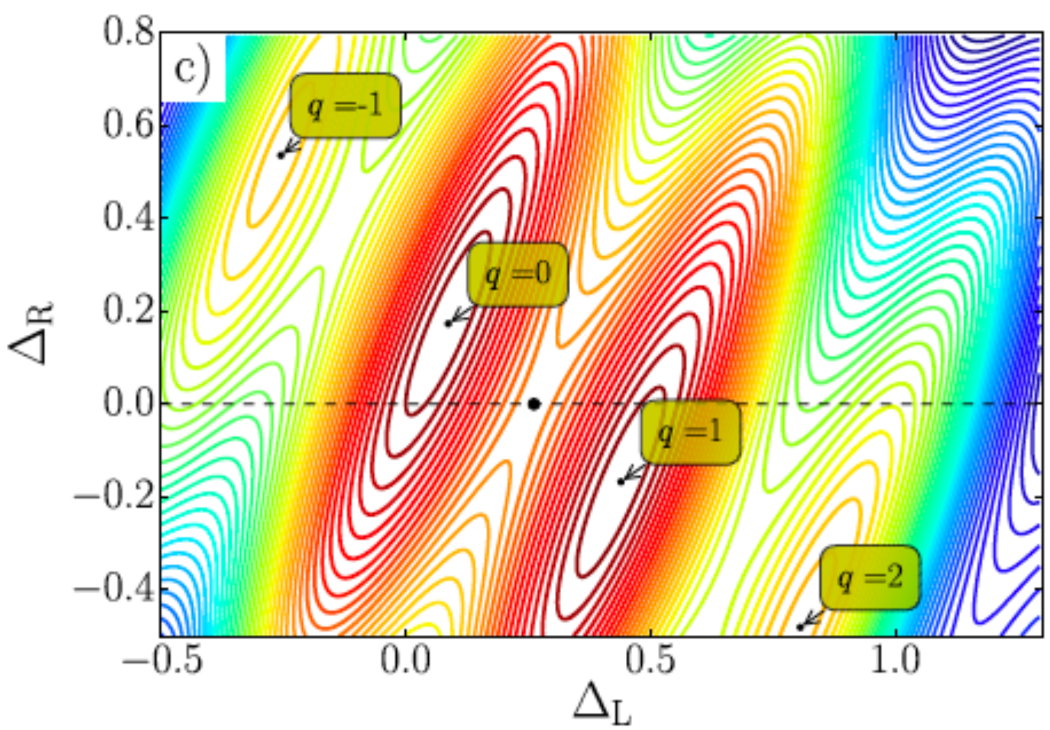

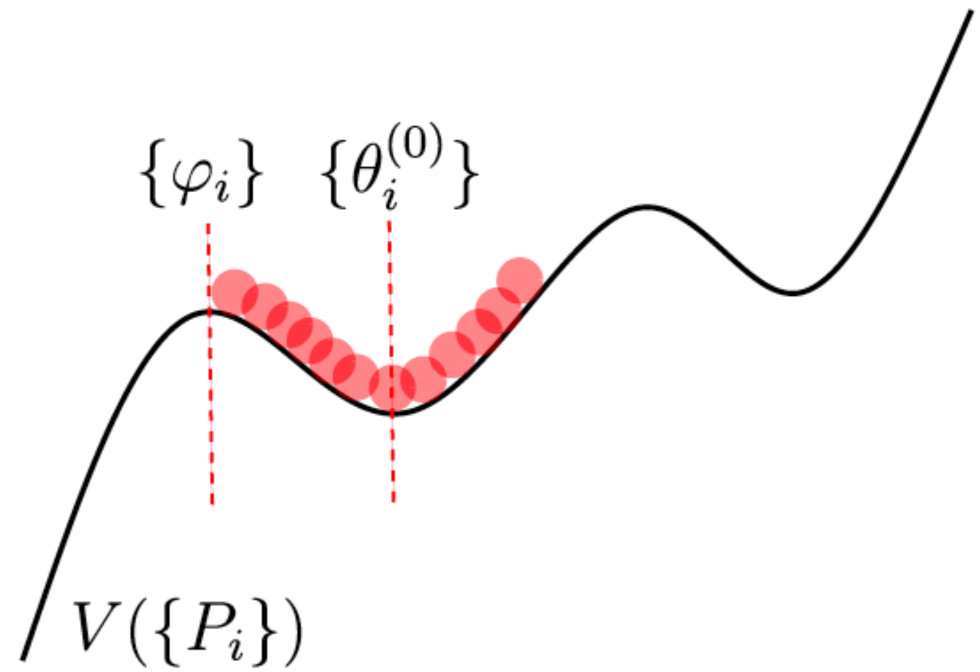

Escape criterion: $\langle \delta\boldsymbol{\theta}^2\rangle \cong \|\boldsymbol{\theta}^{(0)} - \boldsymbol{\varphi}\|_2^2$

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$$\left\|\boldsymbol{\theta}^{(0)} - \boldsymbol{\varphi}\right\|_2^2 = \frac{n(n^2-1)}{12(n-2)^2}\pi^2 \qquad \text{(Single cycle)}$$

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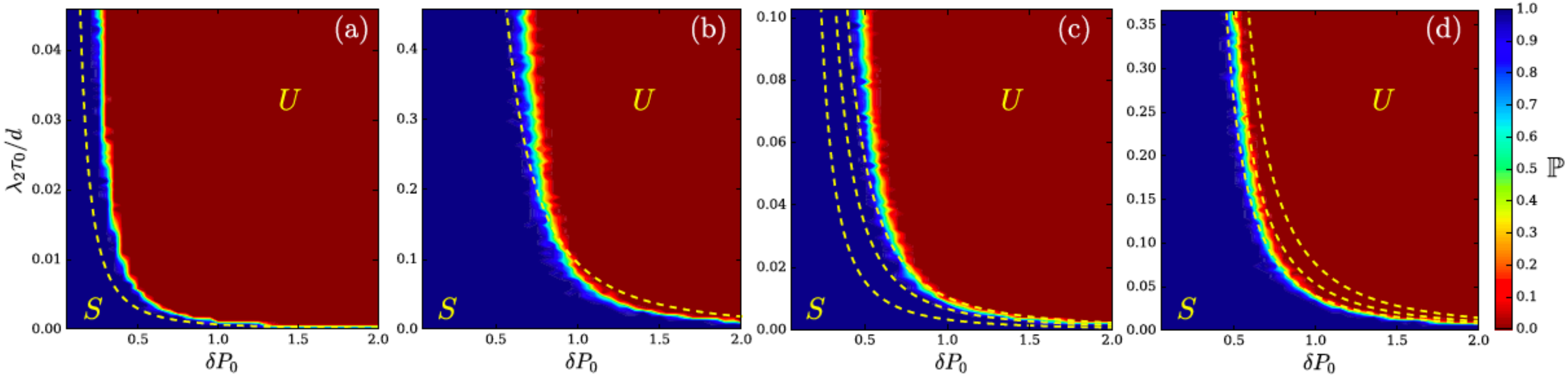
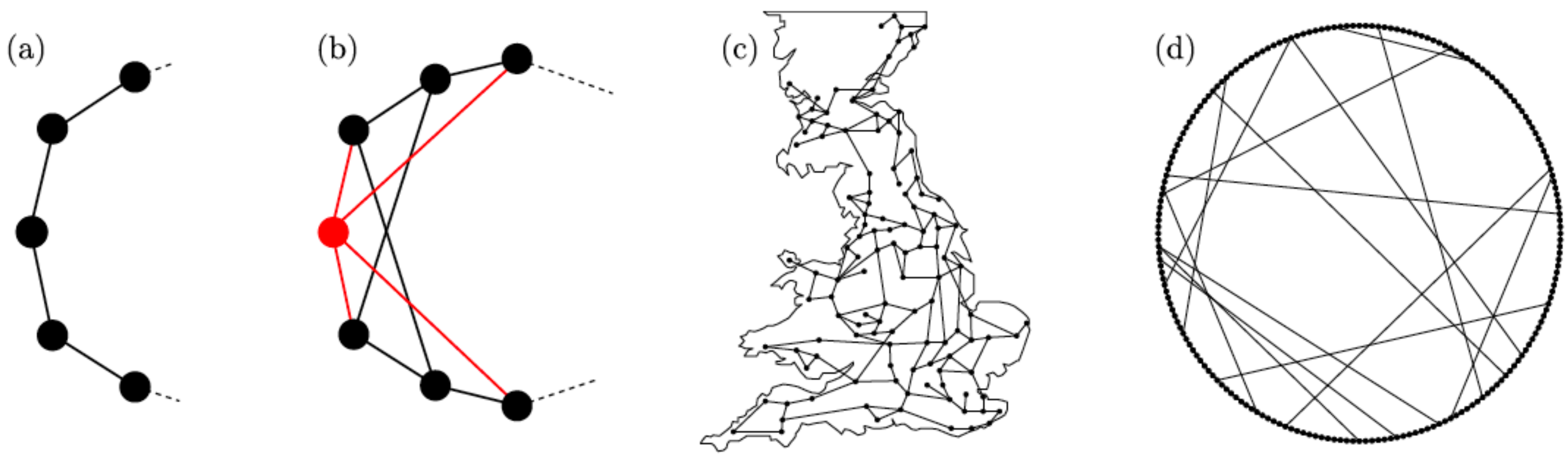
Strong perturbation : escape from basin of attraction

Escape probabilities





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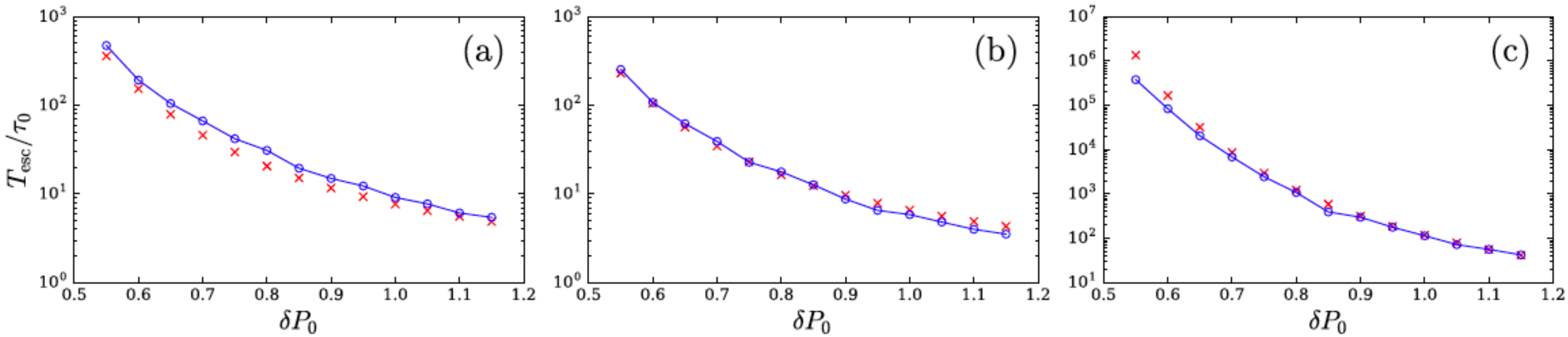
Strong perturbation : escape from basin of attraction

Escape times



$$T_{\rm esc} \propto \left[2\int_{\beta\Delta}^{\infty} P(\overline{\delta\theta})\mathrm{d}(\overline{\delta\theta})\right]^{-1}$$

Gaussian distr.
with variance

$$\lim_{t\to\infty}\langle\boldsymbol{\delta\theta}^2(t)\rangle = \delta P_0^2 \sum_{\alpha\geq 2}\frac{\tau_0 + m/d}{\lambda_\alpha(\lambda_\alpha\tau_0 + d + m/\tau_0)}$$

Tyloo, Delabays and PJ, Phys Rev E ’19
See also : Hines, PJ and Schwartz, Phys Rev E ‘19

Thank U's

Misha Chertkov @ U of A

Jason Hindes, Ira Schwartz @ Navy Res. Lab

Florian Dörfler @ ETHZ

swissgrid

FNSNF

Thank U's

Tommaso
(now with Sophia genetics)

Melvyn

Laurent
(now at U of A)

Koen

Robin

Glory

Thank U's

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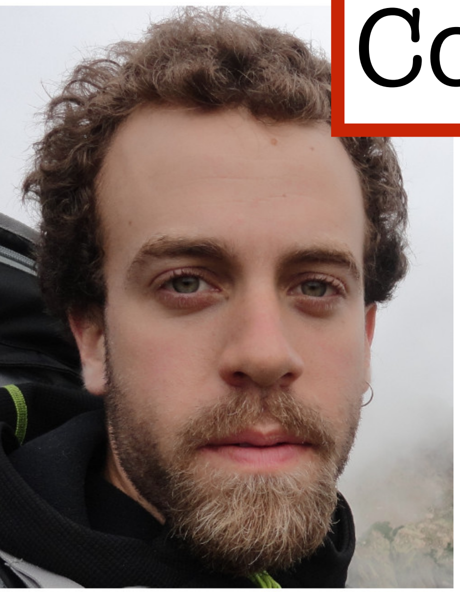
PhD position open for spring 2020
Contact me : philippe.jacquod@hevs.ch



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