Chapter - 4

Arrays, Records and Pointers

4.1 INTRODUCTION

Data structures are classified as either linear or nonlinear. A data structure is said to be linear if its elements form a sequence, or, in other words, a linear list. There are two basic ways of representing such linear structures in memory. One way is to have the linear relationship between the elements represented by means of sequential memory locations. These linear structures are called *arrays* and form the main subject matter of this chapter. The other way is to have the linear relationship between the elements represented by means of pointers or links. These linear structures are called *linked lists*; they form the main content of Chapter 5. Nonlinear structures such as trees and graphs are treated in later chapters.

The operations one normally performs on any linear structure, whether it be an array or a linked list, include the following:

(a) *Traversal.* Processing each element in the list.
(b) *Search.* Finding the location of the element with a given value or the record with a given key.
(c) *Insertion.* Adding a new element to the list.
(d) *Deletion.* Removing an element from the list.
(e) *Sorting.* Arranging the elements in some type of order.
(f) *Merging.* Combining two lists into a single list.

The particular linear structure that one chooses for a given situation depends on the relative frequency with which one performs these different operations on the structure.

This chapter discusses a very common linear structure called an array. Since arrays are usually easy to traverse, search and sort, they are frequently used to store relatively permanent collections of data. On the other hand, if the size of the structure and the data in the structure are constantly changing, then the array may not be as useful a structure as the linked list, discussed in Chapter 5.

4.2 LINEAR ARRAYS

A *linear array* is a list of a finite number n of *homogeneous* data elements (i.e., data elements of the same type) such that:

(a) The elements of the array are referenced respectively by an *index set* consisting of n consecutive numbers.

(b) The elements of the array are stored respectively in successive memory locations.

The number n of elements is called the *length* or *size* of the array. If not explicitly stated, we will assume the index set consists of the integers 1, 2, ..., n. In general, the length or the number of data elements of the array can be obtained from the index set by the formula

$$\text{Length} = \text{UB} - \text{LB} + 1 \tag{4.1}$$

where UB is the largest index, called the *upper bound*, and LB is the smallest index, called the *lower bound*, of the array. Note that length = UB when LB = 1.

The elements of an array A may be denoted by the subscript notation

$$A_1, A_2, A_3, \ldots, A_n$$

or by the bracket notation (used in C)

$$\text{A[1], A[2], A[3], ..., A[N]}$$

We will usually use the subscript notation or the bracket notation. Regardless of the notation, the number K in A[K] is called a *subscript* or an *index* and A[K] is called a *subscripted variable*. Note that subscripts allow any element of A to be referenced by its relative position in A.

Example 4.1

(a) Let DATA be a 6-element linear array of integers such that
DATA[1] = 247 DATA[2] = 56 DATA[3] = 429 DATA[4] = 135 DATA[5] = 87 DATA[6] = 156.
Sometimes we will denote such an array by simply writing

DATA: 247, 56, 429, 135, 87, 156

The array DATA is frequently pictured as in Fig. 4.1(a) or Fig. 4.1(b).

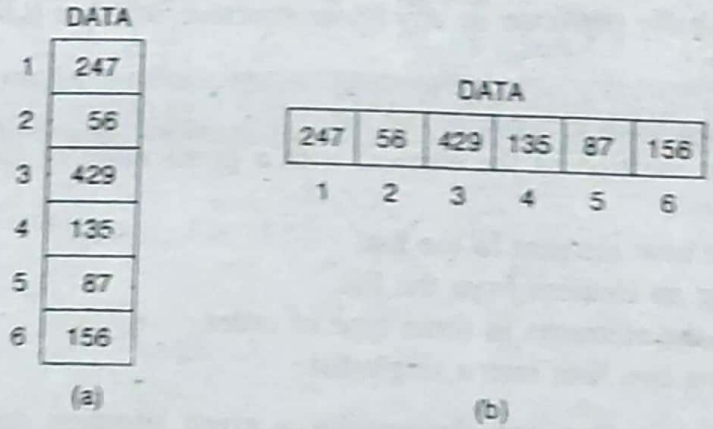


Fig. 4.1

(b) An automobile company uses an array AUTO to record the number of automobiles sold each year from 1932 through 1984. Rather than beginning the index set with 1, it is more useful

to begin the index set with 1932 so that

AUTO[K] = number of automobiles sold in the year K

Then LB = 1932 is the lower bound and UB = 1984 is the upper bound of AUTO. By Eq. (4.1),

Length = UB – LB + 1 = 1984 – 1930 + 1 = 55

That is, AUTO contains 55 elements and its index set consists of all integers from 1932 through 1984.

Program 4.1

```
/*
Defining Arrays in C*/
#include <stdio.h>
main()
{
  int a[10]; //1
  for(int i = 0;i<10;i++)
  {
    a[i]=i;
  }
  printaray(a);
}
void printaray(int a[])
{
  for(int i = 0;i<10;i++)
  {
    printf("Value in the array %d\n",a[i]);
  }
}
```

The above program helps to define an array. Statement 1 defines an array of integers of the size 10, which means you can store 10 integers. When we define the array, the size should be known. Subscripts are used to refer the elements of the array where 0 is considered to be the lowest subscript always and the highest subscript is (size –1), which is 9 in this case. We can refer to any element as a[0], a[1], a[2], etc.

An array can also be processed using a for loop. The consecutive memory locations of the array are allocated and the element size is the same. We should always keep in mind that among all the operators used the subscript of this array must have the highest precedence.

Each programming language has its own rules for declaring arrays. Each such declaration must give, implicitly or explicitly, three items of information: (1) the name of the array, (2) the data type of the array and (3) the index set of the array.

Example 4.2

(a) Suppose DATA is a 6-element linear array containing real values. C language declares such an array as follows:

```
float DATA[6];
```

We will declare such an array, when necessary, by writing DATA(6). (The context will usually indicate the data type, so it will not be explicitly declared.)

(b) Consider the integer array AUTO with the lower bound LB = 1932 and upper bound UB = 1984. In C, the lower bound of an array is always 0; it cannot be customized to some other value. Thus, to implement a scenario where LB = 1932, we need to use an offset value that logically represents the LB index value as 1932. Here's a C program depicting such a case.

Program 4.2

```
#include <stdio.h>
#include <conio.h>

void main()
{
  int AUTO[60];
  int i, offset = 1932;
  clrscr();

  printf("Array AUTO goes like this:\n\n");
  for(i=0;i<5;i++)
  printf("AUTO[%d], ",i+offset);

  printf("...and so on!");
  getch();
}
```

Output:

```
Array AUTO goes like this:

AUTO[1932], AUTO[1933], AUTO[1934], AUTO[1935], AUTO[1936], ...and
so on!
```

Some programming languages (e.g. FORTRAN and Pascal) allocate memory space for arrays statically, i.e., during program compilation; hence the size of the array is fixed during program execution. On the other hand, some programming languages (e.g. C, etc.) allow one to read an integer n and then declare an array with n elements; such programming languages are said to allocate memory *dynamically*.

4.3 ARRAYS AS ADT

An array is a fundamental Abstract Data Type (ADT). Due to its' versatility, it can be changed as per requirement. An array is a pre-defined set which has a sequence of cells where we can store different types of elements.

Consider the following example: object (A,N) Here an array A is created which can store N number of items in it.

A[*i*] is an item in the array A stored in its i^{th} position. A[*i*] is also used to return the value which is stored in that location.

An array can also be considered a vector; in this case we need to stress more on the function which creates and accesses the array. We can label the arrays as we like However, to make it simple we assume they are labeled A[0]...., A[N – 1]. as is the convention. The arrays are named as A[1]...., A[N]. We should also make sure that we stick to any one of these conventions, If A[*i*] is accessed without being assigned any value then it gives an error.

In computers too, data is mostly stored in the form of arrays. When we bring this to practice the same can be executed for almost all the other ADT's. Here is an example:

Example 4.3

Consider an array of length L having different color flags. The flags are red, yellow or blue in color. Now let us rearrange the flags in such a way that all the blue flags appear first followed by the yellow ones and then the red ones. The tools available are:

- Predicate B(*i*) returns true if the flag is blue, Y(*i*) is true if the flag is yellow and R(*i*) is true if the flag is red respectively; and
- On doing swap(*i*, *j*) the flags in the positions *i* and *j* are interchanged without excluding the case *i* = *j*.

All the predicates are calculated only once for each flag. We should aim to bring the arrangement of flags in the required order using the swap operation with the least number of calls.

In this example, our aim is to sort the flags so that the blue flag B comes before the yellow flag Y and the red flag R comes in the end. We also have a common area X having flags of unknown color. This X is followed by Y and precedes R. In the beginning B, Y and R are empty and all the items in the array are in X. Three variables are used to track the current state of the array. The variables *b* and *y* point to the flags which come immediately after the blue and yellow areas and *r* points to the token that immediately precedes the red area.

Fig. 4.2 Sorting the array of flags

Figure 4.2 shows a stage in the sorting of arrays. Let us now see the following algorithm:

Let *b* = 1, *y* = 1, *r* = N;

1. While (*y* < *r* + 1) begin
   (a) if (Y(*y*)) *y* = *y* + 1 [it was a yellow flag]
   else if (B(*y*)) begin [it was a blue flag]
   swap(*y*, *b*); *b* = *b* + 1; *y* = *y* + 1
   end
   (b) else begin [it was a red flag]
   swap(*r*, *y*); *r* = *r* – 1
   end
   [End of While structure]

In each stage we can see that either r is decreased or y is increased, thus showing the algorithm is finite and the length $|X|$ of the region X, which is $r + 1 - y$ decreases each time, and the 'sorted' condition

$B < Y < X < R$

remains true. When $X = 0$, the algorithm terminates, and thus we get our sorted list. The loop executed N number of times there are $|B| + |R|$ calls to the routine swap.

4.4 REPRESENTATION OF LINEAR ARRAYS IN MEMORY

Let LA be a linear array in the memory of the computer. Recall that the memory of the computer is simply a sequence of addressed locations as pictured in Fig. 4.3. Let us use the notation

LOC(LA[K]) = address of the element LA[K] of the array LA

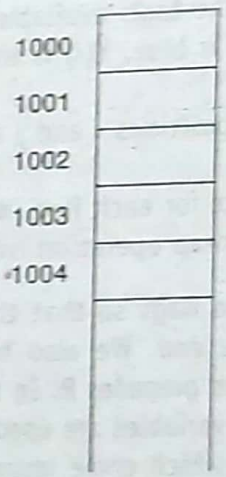


Fig. 4.3 Computer Memory

As previously noted, the elements of LA are stored in successive memory cells. Accordingly, the computer does not need to keep track of the address of every element of LA, but needs to keep track only of the address of the first element of LA, denoted by

Base(LA)

and called the *base address* of LA. Using this address *Base*(LA), the computer calculates the address of any element of LA by the following formula:

$$LOC(LA[K]) = Base(LA) + w(K - \text{lower bound}) \quad (4.2)$$

where w is the number of words per memory cell for the array LA. Observe that the time to calculate LOC(LA[K]) is essentially the same for any value of K. Furthermore, given any subscript K, one can locate and access the content of LA[K] without scanning any other element of LA.

Example 4.4

Consider the array AUTO in Example 4.1(b), which records the number of automobiles sold each year from 1932 through 1984. Suppose AUTO appears in memory as pictured in Fig. 4.4. That is, Base(AUTO) = 200, and $w = 4$ words per memory cell for AUTO. Then

LOC(AUTO[1932]) = 200, LOC(AUTO[1933]) = 204, LOC(AUTO[1934]) = 208, ...

The address of the array element for the year K = 1965 can be obtained by using Eq. (4.2):

$$\begin{aligned} LOC(AUTO[1965]) &= Base(AUTO) + w(1965 - \text{lower bound}) \\ &= 200 + 4(1965 - 1932) = 332 \end{aligned}$$

Again we emphasize that the contents of this element can be obtained without scanning any other element in array AUTO.

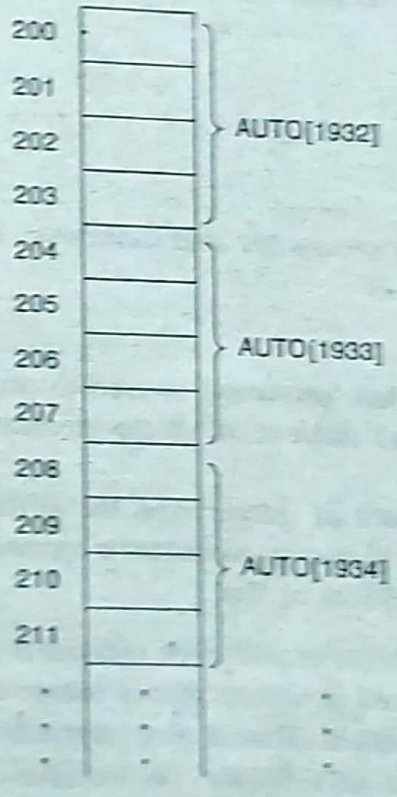


Fig. 4.4

Program 4.3

There is a memory address for all the elements of an array. In the following program, we can see how an array limit value and an array element address are printed.

```
#include <stdio.h>
void printaray(int x[]);
main()
```

```
{
  int x[15];
  for(int i = 0;i<15;i++)
  {
    x[i]=i;
  }
  printaray(x);
}
void printaray(int x[])
{
  for(int i = 0;i<15;i++)
  {
    printf("Value in the array %d\n",x[i]);
  }
}
void printdetail(int x[])
{
  for(int i = 0;i<15;i++)
  {
    printf("Value in the array %d and address is %16lu\n",x[i],&x[i]);
  }
}
```

In the above program, the function 'printaray' is used to print values of each element in the array 'aray'. To print the value and address of all the elements and its address the printdetail function is used.

Now that we know all the elements are integer type, the difference between address is 2.

All the elements in the array are placed in consecutive memory spaces. The memory addresses can be printed using place holders %16lu or %p.

Remark: A collection A of data elements is said to be *indexed* if any element of A, which we shall call A_K, can be located and processed in a time that is independent of K. The above discussion indicates that linear arrays can be indexed. This is very important property of linear arrays. In fact, linked lists, which are covered in the next chapter, do not have this property.

4.5 TRAVERSING LINEAR ARRAYS

Let A be a collection of data elements stored in the memory of the computer. Suppose we want to print the contents of each element of A, or suppose we want to count the number of elements of A with a given property. This can be accomplished by *traversing* A, that is, by accessing and processing (frequently called *visiting*) each element of A exactly once.

The following algorithm traverses a linear array LA. The simplicity of the algorithm comes from the fact that LA is a linear structure. Other linear structures, such as linked lists, can also be easily traversed. On the other hand, the traversal of nonlinear structures, such as trees and graphs, is considerably more complicated.

Algorithm 4.1: (Traversing a Linear Array) Here LA is a linear array with lower bound LB and upper bound UB. This algorithm traverses LA applying an operation PROCESS to each element of LA.

1. [Initialize counter.] Set K := LB.
2. Repeat Steps 3 and 4 while K ≤ UB.
3. [Visit element.] Apply PROCESS to LA[K].
4. [Increase counter.] Set K := K + 1.
   [End of Step 2 loop.]
5. Exit.

Program 4.4

```
/* C Implementation of Algorithm 4.1*/
#include<stdio.h>
#define N 7          /* lower bound LB of the array is 1 and
                        upper bound UB of the array is N*/
int main(void)
{ int k,i,           LA[N]={23, 45, 56, 1, -9, -12, 123};
  k=0;                  /* Initialize counter k*/
  while(k<=N)        /*PROCESS in the algorithm is implemented
  {                    as scaling the array elements by 2*/
    k++;             /* Increase counter*/
  }
  for( i=0;i<N;i++)
  {printf("\n%d", LA[i]);   /* print array elements after PROCESS*/
  }
  return 0;
}
```

We also state an alternative form of the algorithm which uses a repeat-for loop instead of the repeat-while loop.

Algorithm 4.1′: (Traversing a Linear Array) This algorithm traverses a linear array LA with lower bound LB and upper bound UB.

1. Repeat for K = LB to UB:
   Apply PROCESS to LA[K].
   [End of loop.]
2. Exit.

Caution: The operation PROCESS in the traversal algorithm may use certain variables which must be initialized before PROCESS is applied to any of the elements in the array. Accordingly, the algorithm may need to be preceded by such an initialization step.

Algorithm 4.1 illustrates traversal of a linear array undertaking a process PROCESS on each of the elements of the array. The C implementation of PROCESS is carried out by scaling each of the elements of the array by 2.

Example 4.5

Consider the array AUTO in Example 4.1(b), which records the number of automobiles sold each year from 1932 through 1984. Each of the following modules, which carry out the given operation, involves traversing AUTO.

(a) Find the number NUM of years during which more than 300 automobiles were sold.

```
1. [Initialization step.] Set NUM := 0.
2. Repeat for K = 1932 to 1984:
       If AUTO[K] > 300, then: Set NUM := NUM + 1.
   [End of loop.]
3. Return.
```

(b) Print each year and the number of automobiles sold in that year.

```
1. Repeat for K = 1932 to 1984:
       Write: K, AUTO[K].
   [End of loop.]
2. Return.
```

(Observe that (a) requires an initialization step for the variable NUM before traversing the array AUTO.)

4.6 INSERTING AND DELETING

Let A be a collection of data elements in the memory of the computer. "Inserting" refers to the operation of adding another element to the collection A, and "deleting" refers to the operation of removing one of the elements from A. This section discusses inserting and deleting when A is a linear array.

Inserting an element at the "end" of a linear array can be easily done provided the memory space allocated for the array is large enough to accommodate the additional element. On the other hand, suppose we need to insert an element in the middle of the array. Then, on the average, half of the elements must be moved downward to new locations to accommodate the new element and keep the order of the other elements.

Similarly, deleting an element at the "end" of an array presents no difficulties, but deleting an element somewhere in the middle of the array would require that each subsequent element be moved one location upward in order to "fill up" the array.

Remark: Since linear arrays are usually pictured extending downward, as in Fig. 4.1, the term "downward" refers to locations with larger subscripts, and the term "upward" refers to locations with smaller subscripts.

Example 4.6

Suppose TEST has been declared to be a 5-element array but data have been recorded only for TEST[1], TEST[2] and TEST[3]. If X is the value of the next test, then one simply assigns

```
TEST[4] := X
```

to add X to the list. Similarly, if Y is the value of the subsequent test, then we simply assign

```
TEST[5] := Y
```

to add Y to the list. Now, however, we cannot add any new test scores to the list.

Example 4.7

Suppose NAME is an 8-element linear array, and suppose five names are in the array, as in Fig. 4.5(a). (Observe that the names are listed alphabetically, and suppose we want to keep the array names alphabetical at all times. Suppose Ford is added to the array. Then Johnson, Smith and Wagner must each be moved downward one location, as in Fig. 4.5(b). Next suppose Taylor is added to the array; then Wagner must be moved, as in Fig. 4.5(c). Last, suppose Davis is removed from the array. Then the five names Ford, Johnson, Smith, Taylor and Wagner must each be moved upward one location, as in Fig. 4.5(d). Clearly such movement of data would be very expensive if thousands of names were in the array.

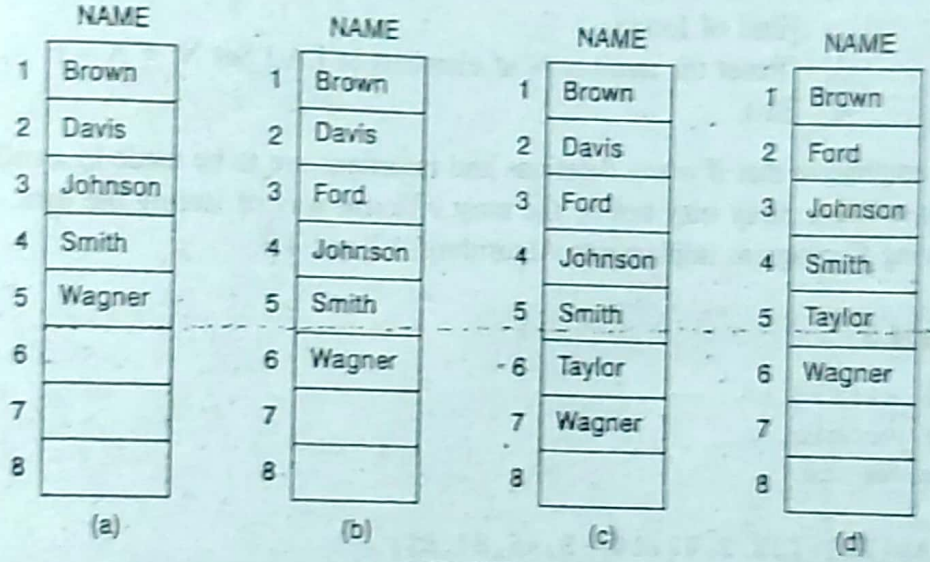


Fig. 4.5

The following algorithm inserts a data element ITEM into the Kth position in a linear array LA with N elements. The first four steps create space in LA by moving downward one location each element from the Kth position on. We emphasize that these elements are moved in reverse order—i.e., first LA[N], then LA[N – 1], ..., and last LA[K]; otherwise data might be erased. (See Solved Problem 4.3) In more detail, we first set J := N and then, using J as a counter, decrease J each time the loop is executed until J reaches K. The next step, Step 5, inserts ITEM into the array in the space just created. Before the exit from the algorithm, the number N of elements in LA is increased by 1 to account for the new element.

Algorithm 4.2: (Inserting into a Linear Array) INSERT (LA, N, K, ITEM)
Here LA is a linear array with N elements and K is a positive integer such that $K \leq N$. This algorithm inserts an element ITEM into the Kth position in LA.

1. [Initialize counter.] Set J := N.
2. Repeat Steps 3 and 4 while $J \geq K$.
3. [Move Jth element downward.] Set LA[J + 1] := LA[J].
4. [Decrease counter.] Set J := J – 1.
   [End of Step 2 loop.]
5. [Insert element.] Set LA[K] := ITEM.
6. [Reset N.] Set N := N + 1.
7. Exit.

The following algorithm deletes the Kth element from a linear array LA and assigns it to a variable ITEM.

Algorithm 4.3: (Deleting from a Linear Array) DELETE(LA, N, K, ITEM)
Here LA is a linear array with N elements and K is a positive integer such that $K \leq N$. This algorithm deletes the Kth element from LA.

1. Set ITEM := LA[K].
2. Repeat for J = K to N – 1:
   [Move J + 1st element upward.] Set LA[J] := LA[J + 1].
   [End of loop.]
3. [Reset the number N of elements in LA.] Set N: = N – 1.
4. Exit.

Remark: We emphasize that if many deletions and insertions are to be made in a collection of data elements, then a linear array may not be the most efficient way of storing the data.

The following C program implements Algorithms 4.2 and 4.3:

Program 4.5

```
#include <stdio.h>
#include <conio.h>
# define UB 10

int array[UB]={21,2,43,14,-5,46,87,8};
int insert_item(int LA[], int N, int k, int item);
int delete_item(int LA[], int N, int k);

void main()
{
int ITEM, LOC;
int i, size=8;
int choice;
clrscr();

printf("Array: ");
for(i=0;i<size;i++)
printf("%d ",array[i]);

printf("\n\nEnter your choice: \n\n1. Insert an element\n2. Delete an
element\n   ");
scanf("%d",&choice);

if(choice!=1 && choice !=2)
{
 printf("\nInvalid Choice");
 getch();
```

```
 exit();
}
else if(choice==1)
{
printf("\n\nEnter the element to be inserted in the array: ");
scanf("%d",&ITEM);

printf("\nEnter the location where element %d is to be inserted: ",
ITEM);
scanf("%d",&LOC);
if(LOC<=size)
 size=insert_item(array,size,LOC,ITEM);
else
{
 printf("\nThe entered location is out of bound");
 getch();
 exit();
}
printf("\nModified array: ");
for(i=0;i<size;i++)
printf("%d ",array[i]);

getch();
}

else
{
printf("\nEnter the location from where element is to be deleted: ");
scanf("%d",&LOC);
if(LOC<=size)
 size=delete_item(array,size,LOC);
else
{
 printf("\nThe entered location is out of bound");
 getch();
 exit();
}
printf("\nModified array: ");
for(i=0;i<size;i++)
printf("%d ",array[i]);
getch();
}
}

int insert_item(int LA[], int N, int k, int item)
{
```

```
    int j=N;
    while(j>=k-1)
    {
     LA[j+1]=LA[j];
     j--;
    }
    LA[k-1]=item;
    return(N+1);
}

int delete_item(int LA[], int N, int k)
{
    int j, item;
    item=LA[k-1];
    printf("\nItem %d deleted from location %d\n",item,k);
    for(j=k-1;j<N-1;j++)
     LA[j]=LA[j+1];
    return(N-1);
}
```

Output (Insertion):

```
Array: 21 2 43 14 -5 46 87 8
Enter your choice:
1. Insert an element
2. Delete an element
   1

Enter the element to be inserted in the array: 99
Enter the location where element 99 is to be inserted: 4
Modified array: 21 2 43 99 14 -5 46 87 8
```

Output (Deletion)

```
Array: 21 2 43 14 -5 46 87 8
Enter your choice:
1.  Insert an element
2.  Delete an element
    2

Enter the location from where element is to be deleted: 6
Item 46 deleted from location 6
Modified array: 21 2 43 14 -5 87 8
```

4.7 SORTING; BUBBLE SORT

Let A be a list of n numbers. *Sorting* A refers to the operation of rearranging the elements of A so they are in increasing order, i.e., so that,

$$A[1] < A[2] < A[3] < \ldots < A[N]$$

For example, suppose A originally is the list

8, 4, 19, 2, 7, 13, 5, 16

After sorting, A is the list

2, 4, 5, 7, 8, 13, 16, 19

Sorting may seem to be a trivial task. Actually, sorting efficiently may be quite complicated. In fact, there are many, many different sorting algorithms; some of these algorithms are discussed in Chapter 9. Here we present and discuss a very simple sorting algorithm known as the *bubble sort*.

Remark: The above definition of sorting refers to arranging numerical data in increasing order; this restriction is only for notational convenience. Clearly, sorting may also mean arranging numerical data in decreasing order or arranging non-numerical data in alphabetical order. Actually, A is frequently a file of records, and sorting A refers to rearranging the records of A so that the values of a given key are ordered.

Bubble Sort

Suppose the list of numbers A[1], A[2], ..., A[N] is in memory. The bubble sort algorithm works as follows:

Step 1. Compare A[1] and A[2] and arrange them in the desired order, so that A[1] < A[2]. Then compare A[2] and A[3] and arrange them so that A[2] < A[3]. Then compare A[3] and A[4] and arrange them so that A[3] < A[4]. Continue until we compare A[N – 1] with A[N] and arrange them so that A[N – 1] < A[N].

Observe that Step 1 involves $n - 1$ comparisons. (During Step 1, the largest element is "bubbled up" to the nth position or "sinks" to the nth position.) When Step 1 is completed, A[N] will contain the largest element.

Step 2. Repeat Step 1 with one less comparison; that is, now we stop after we compare and possibly rearrange A[N – 2] and A[N – 1]. (Step 2 involves N – 2 comparisons and, when Step 2 is completed, the second largest element will occupy A[N – 1].)

Step 3. Repeat Step 1 with two fewer comparisons; that is, we stop after we compare and possibly rearrange A[N – 3] and A[N – 2].

. . .

Step N – 1. Compare A[1] with A[2] and arrange them so that A[1] < A[2].

After $n - 1$ steps, the list will be sorted in increasing order.

The process of sequentially traversing through all or part of a list is frequently called a "pass," so each of the above steps is called a pass. Accordingly, the bubble sort algorithm requires $n - 1$ passes, where n is the number of input items.

Example 4.8

Suppose the following numbers are stored in an array A:

32, 51, 27, 85, 66, 23, 13, 57

We apply the bubble sort to the array A. We discuss each pass separately.

Pass 1. We have the following comparisons:

(a) Compare A_1 and A_2. Since 32 < 51, the list is not altered.

(b) Compare A_2 and A_3. Since 51 > 27, interchange 51 and 27 as follows:

32, (27), (51), 85, 66, 23, 13, 57

(c) Compare A_3 and A_4. Since 51 < 85, the list is not altered.

(d) Compare A_4 and A_5. Since 85 > 66, interchange 85 and 86 as follows:

32, 27, 51, (66), (85), 23, 13, 57

(e) Compare A_5 and A_6. Since 85 > 23, interchange 85 and 23 as follows:

32, 27, 51, 66, (23), (85), 13, 57

(f) Compare A_6 and A_7. Since 85 > 13, interchange 85 and 13 to yield:

32, 27, 51, 66, 23, (13), (85), 57

(g) Compare A_7 and A_8. Since 85 > 57, interchange 85 and 51 to yield:

32, 27, 51, 66, 23, 13, (57), (85)

At the end of this first pass, the largest number, 85, has moved to the last position. However, the rest of the numbers are not sorted, even though some of them have changed their position.

For the remainder of the passes, we show only the interchanges.

Pass 2. (27), (33), 51, 66, 23, 13, 57, 85

27, 33, 51, (23), (66), 13, 57, 85

27, 33, 51, 23, (13), (66), 57, 85

27, 33, 51, 23, 13, (57), (66), 85

At the end of Pass 2, the second largest number, 66, has moved its way down to the next-to-last position.

Pass 3. 27, 33, (23), (51), 13, 57, 66, 85

27, 33, 23, (13), (51), 57, 66, 85

Pass 4. 27, (23), (33), 13, 51, 57, 66, 85

27, 23, (13), (33), 51, 57, 66, 85

Pass 5. (23), (27), 13, 33, 51, 57, 66, 85

23, (13), (27), 33, 51, 57, 66, 85

Pass 6. (13), (23), 27, 33, 51, 57, 66, 85

Pass 6 actually has two comparisons, A_1 with A_2 and A_2 and A_3. The second comparison does not involve an interchange.

Pass 7. Finally, A_1 is compared with A_2. Since 13 < 23, no interchange takes place.

Since the list has 8 elements; it is sorted after the seventh pass. (Observe that in this example, the list was actually sorted after the sixth pass. This condition is discussed at the end of the section.)

We now formally state the bubble sort algorithm.

Algorithm 4.4: (Bubble Sort) BUBBLE(DATA, N)
Here DATA is an array with N elements. This algorithm sorts the elements in DATA.

1. Repeat Steps 2 and 3 for K = 1 to N – 1.
2. Set PTR := 1. [Initializes pass pointer PTR.]
3. Repeat while PTR ≤ N – K: [Executes pass.]
   (a) If DATA[PTR] > DATA[PTR + 1], then:
   Interchange DATA[PTR] and DATA[PTR + 1].
   [End of If structure.]
   (b) Set PTR := PTR + 1.
   [End of inner loop.]
   [End of Step 1 outer loop.]
4. Exit.

Observe that there is an inner loop which is controlled by the variable PTR, and the loop is contained in an outer loop which is controlled by an index K. Also observe that PTR is used as a subscript but K is not used as a subscript, but rather as a counter.

Here's is a C implementation of the above algorithm:

Program 4.6

```
#include <stdio.h>
#include <conio.h>

int DATA[10]={22,65,1,99,32,17,74,49,33,2};
int N=10;
void BUBBLE(void);

void main()
{
int i;
clrscr();

printf("Values contained in DATA[10] =");
for(i=0;i<10;i++)
```

```
  printf(" %d",DATA[i]);

BUBBLE();

printf("\n\nValues contained in DATA[10] after sorting =");
for(i=0;i<10;i++)
 printf(" %d",DATA[i]);

getch();
}

void BUBBLE(void)
{
int K,PTR,TEMP;

for(K=0;K<=(N-1-1);K++)
{
 PTR=0;
 while(PTR<=(N-K-1-1))
 {
  if(DATA[PTR]>DATA[PTR+1])
  {
   TEMP=DATA[PTR];
   DATA[PTR]=DATA[PTR+1];
   DATA[PTR+1]=TEMP;
  }
 PTR=PTR+1;
 }
}
}
```

Output:

```
Values contained in DATA[10] = 22 65 1 99 32 17 74 49 33 2
Values contained in DATA[10] after sorting = 1 2 17 22 32 33 49 65
74 99
```

Complexity of the Bubble Sort Algorithm

Traditionally, the time for a sorting algorithm is measured in terms of the number of comparisons. The number $f(n)$ of comparisons in the bubble sort is easily computed. Specifically, there are $n-1$ comparisons during the first pass, which places the largest element in the last position; there are $n-2$ comparisons in the second step, which places the second largest element in the next-to-last position; and so on. Thus

$$f(n) = (n-1) + (n-2) + \ldots + 2 + 1 = \frac{n(n-1)}{2} = \frac{n^2}{2} + O(n) = O(n^2)$$

In other words, the time required to execute the bubble sort algorithm is proportional to n^2, where n is the number of input items.

Remark: Some programmers use a bubble sort algorithm that contains a 1-bit variable FLAG (or a logical variable FLAG) to signal when no interchange takes place during a pass. If FLAG = 0 after any pass, then the list is already sorted and there is no need to continue. This may cut down on the number of passes. However, when using such a flag, one must initialize, change and test the variable FLAG during each pass. Hence the use of the flag is efficient only when the list originally is "almost" in sorted order.

4.8 SEARCHING; LINEAR SEARCH

Let DATA be a collection of data elements in memory, and suppose a specific ITEM of information is given. *Searching* refers to the operation of finding the location LOC of ITEM in DATA, or printing some message that ITEM does not appear there. The search is said to be *successful* if ITEM does appear in DATA and *unsuccessful* otherwise.

Frequently, one may want to add the element ITEM to DATA after an unsuccessful search for ITEM in DATA. One then uses a *search and insertion* algorithm, rather than simply a *search* algorithm; such search and insertion algorithms are discussed in the problem sections.

There are many different searching algorithms. The algorithm that one chooses generally depends on the way the information in DATA is organized. Searching is discussed in detail in Chapter 9. This section discusses a simple algorithm called *linear search*, and the next section discusses the well-known algorithm called *binary search*.

The complexity of searching algorithms is measured in terms of the number $f(n)$ of comparisons required to find ITEM in DATA where DATA contains n elements. We shall show that linear search is a linear time algorithm, but that binary search is a much more efficient algorithm, proportional in time to $\log_2 n$. On the other hand, we also discuss the drawback of relying only on the binary search algorithm.

Linear Search

Suppose DATA is a linear array with n elements. Given no other information about DATA, the most intuitive way to search for a given ITEM in DATA is to compare ITEM with each element of DATA one by one. That is, first we test whether DATA[1] = ITEM, and then we test whether DATA[2] = ITEM, and so on. This method, which traverses DATA sequentially to locate ITEM, is called *linear search* or *sequential search*.

To simplify the matter, we first assign ITEM to DATA[N + 1], the position following the last element of DATA. Then the outcome

$$LOC = N + 1$$

where LOC denotes the location where ITEM first occurs in DATA, signifies the search is unsuccessful. The purpose of this initial assignment is to avoid repeatedly testing whether or not we have reached the end of the array DATA. This way, the search must eventually "succeed."

A formal presentation of linear search is shown in Algorithm 4.5.

Observe that Step 1 guarantees that the loop in Step 3 must terminate. Without Step 1 (see Algorithm 2.4), the Repeat statement in Step 3 must be replaced by the following statement, which involves two comparisons, not one:

Repeat while LOC ≤ N and DATA[LOC] ≠ ITEM:

On the other hand, in order to use Step 1, one must guarantee that there is an unused memory

location at the end of the array DATA; otherwise, one must use the linear search algorithm discussed in Algorithm 2.4.

Algorithm 4.5: (Linear Search) LINEAR(DATA, N, ITEM, LOC)
Here DATA is a linear array with N elements, and ITEM is a given item of information. This algorithm finds the location LOC of ITEM in DATA, or sets LOC := 0 if the search is unsuccessful.

1. [Insert ITEM at the end of DATA.] Set DATA[N + 1] := ITEM.
2. [Initialize counter.] Set LOC := 1.
3. [Search for ITEM.]
   Repeat while DATA[LOC] ≠ ITEM:
   Set LOC := LOC + 1.
   [End of loop.]
4. [Successful?] If LOC = N + 1, then: Set LOC := 0.
5. Exit.

Here's a C implementation for Algorithm 4.5:

Program 4.7

```
#include <stdio.h>
#include <conio.h>

int DATA[20]={22,65,1,99,32,17,74,49,33,2};
int N=10;
int LINEAR(int);

void main()
{
int i,ITEM,LOC;
clrscr();

printf("Values contained in DATA[10] =");
for(i=0;i<10;i++)
 printf(" %d",DATA[i]);

printf("\n\n\nEnter the ITEM to be searched - ");
scanf("%d",&ITEM);

LOC=LINEAR(ITEM);

if(LOC==-1)
 printf("\n\nITEM is not present in DATA[10]");
else
 printf("\n\nThe location of ITEM in DATA[10] is = %d",LOC);

getch();
```

```
}

int LINEAR(int I)
{
int L;
DATA[N]=I;
L=0;

while(DATA[L]!=I)
  L=L+1;
if(L==N)
L=-1;

return(L);
}
```

Output:

```
Values contained in DATA[10] = 22 65 1 99 32 17 74 49 33 2
Enter the ITEM to be searched = 99
The location of ITEM in DATA[10] is = 3
```

Example 4.9

Consider the array NAME in Fig. 4.6(a), where $n = 6$.

(a) Suppose we want to know whether Paula appears in the array and, if so, where. Our algorithm temporarily places Paula at the end of the array, as pictured in Fig. 4.6(b), by setting NAME[7] = Paula. Then the algorithm searches the array from top to bottom. Since Paula first appears in NAME[N + 1], Paula is not in the original array.

(b) Suppose we want to know whether Susan appears in the array and, if so, where. Our algorithm temporarily places Susan at the end of the array, as pictured in Fig. 4.6(c), by setting NAME[7] = Susan. Then the algorithm searches the array from top to bottom. Since Susan first appears in NAME[4] (where $4 \leq n$), we know that Susan is in the original array.

	NAME (a)	NAME (b)	NAME (c)
1	Mary	Mary	Mary
2	Jane	Jane	Jane
3	Diane	Diane	Diane
4	Susan	Susan	Susan
5	Karen	Karen	Karen
6	Edith	Edith	Edith
7		Paula	Susan
8			

Fig. 4.6

Complexity of the Linear Search Algorithm

As noted above, the complexity of our search algorithm is measured by the number $f(n)$ of comparisons required to find ITEM in DATA where DATA contains n elements. Two important cases to consider are the average case and the worst case.

Clearly, the worst case occurs when one must search through the entire array DATA, i.e., when ITEM does not appear in DATA. In this case, the algorithm requires

$$f(n) = n + 1$$

comparisons. Thus, in the worst case, the running time is proportional to n.

The running time of the average case uses the probabilistic notion of expectation. (See Sec. 2.5.) Suppose p_k is the probability that ITEM appears in DATA[K], and suppose q is the probability that ITEM does not appear in DATA. (Then $p_1 + p_2 + \ldots + p_n + q = 1$.) Since the algorithm uses k comparisons when ITEM appears in DATA[K], the average number of comparisons is given by

$$f(n) = 1 \cdot p_1 + 2 \cdot p_2 + \ldots + n \cdot p_n + (n + 1) \cdot q$$

In particular, suppose q is very small and ITEM appears with equal probability in each element of DATA. Then $q \approx 0$ and each $p_i = 1/n$. Accordingly,

$$f(n) = 1 \cdot \frac{1}{n} + 2 \cdot \frac{1}{n} + \ldots + n \cdot \frac{1}{n} + (n+1) \cdot 0 = (1 + 2 + \ldots + n) \cdot \frac{1}{n}$$

$$= \frac{n(n+1)}{2} \cdot \frac{1}{n} = \frac{n+1}{2}$$

That is, in this special case, the average number of comparisons required to find the location of ITEM is approximately equal to half the number of elements in the array.

4.9 BINARY SEARCH

Suppose DATA is an array which is sorted in increasing numerical order or, equivalently, alphabetically. Then there is an extremely efficient searching algorithm, called *binary search*, which can be used to find the location LOC of a given ITEM of information in DATA. Before formally discussing the algorithm, we indicate the general idea of this algorithm by means of an idealized version of a familiar everyday example.

Suppose one wants to find the location of some name in a telephone directory (or some word in a dictionary). Obviously, one does not perform a linear search. Rather, one opens the directory in the middle to determine which half contains the name being sought. Then one opens that half in the middle to determine which quarter of the directory contains the name. Then one opens that quarter in the middle to determine which eighth of the directory contains the name. And so on. Eventually, one finds the location of the name, since one is reducing (very quickly) the number of possible locations for it in the directory.

The binary search algorithm applied to our array DATA works as follows. During each stage of our algorithm, our search for ITEM is reduced to a *segment* of elements of DATA:

DATA[BEG], DATA[BEG + 1], DATA[BEG + 2], ..., DATA[END]

Note that the variables BEG and END denote, respectively, the beginning and end locations of

the segment under consideration. The algorithm compares ITEM with the middle element DATA[MID] of the segment, where MID is obtained by

$$MID = INT((BEG + END)/2)$$

(We use INT(A) for the integer value of A.) If DATA[MID] = ITEM, then the search is successful and we set LOC := MID. Otherwise a new segment of DATA is obtained as follows:

(a) If ITEM < DATA[MID], then ITEM can appear only in the left half of the segment:

DATA[BEG], DATA[BEG + 1], ..., DATA[MID – 1]

So we reset END := MID – 1 and begin searching again.

(b) If ITEM > DATA[MID], then ITEM can appear only in the right half of the segment:

DATA[MID + 1], DATA[MID + 2], ..., DATA[END]

So we reset BEG := MID + 1 and begin searching again.

Initially, we begin with the entire array DATA; i.e., we begin with BEG = 1 and END = n, or, more generally, with BEG = LB and END = UB.

If ITEM is not in DATA, then eventually we obtain

$$END < BEG$$

This condition signals that the search is unsuccessful, and in such a case we assign LOC := NULL. Here NULL is a value that lies outside the set of indices of DATA. (In most cases, we can choose NULL = 0.)

We state the binary search algorithm formally.

Algorithm 4.6: (Binary Search) BINARY(DATA, LB, UB, ITEM, LOC)
Here DATA is a sorted array with lower bound LB and upper bound UB, and ITEM is a given item of information. The variables BEG, END and MID denote, respectively, the beginning, end and middle locations of a segment of elements of DATA. This algorithm finds the location LOC of ITEM in DATA or sets LOC = NULL.

1. [Initialize segment variables.]
   Set BEG := LB, END := UB and MID = INT((BEG + END)/2).
2. Repeat Steps 3 and 4 while BEG ≤ END and DATA[MID] ≠ ITEM.
3. If ITEM < DATA[MID], then:
   Set END := MID – 1.
   Else:
   Set BEG := MID + 1.
   [End of If structure.]
4. Set MID := INT((BEG + END)/2).
   [End of Step 2 loop.]
5. If DATA[MID] = ITEM, then:
   Set LOC := MID.
   Else:
   Set LOC := NULL.
   [End of If structure.]
6. Exit.

```
}

if(DATA[MID]==I)
  L=MID;
else
  L=-1;
return(L);
}
```

Output:

```
Values contained in DATA[10] = 1 2 17 22 32 33 49 65 74 99

Enter the ITEM to be searched = 33

The location of ITEM in DATA[10] is =  5
```

Remark: Whenever ITEM does not appear in DATA, the algorithm eventually arrives at the stage that BEG = END = MID. Then the next step yields END < BEG, and control transfers to Step 5 of the algorithm. This occurs in part (b) of the next example.

Example 4.10

Let DATA be the following sorted 13-element array:

DATA: 11, 22, 30, 33, 40, 44, 55, 60, 66, 77, 80, 88, 99

We apply the binary search to DATA for different values of ITEM.

(a) Suppose ITEM = 40. The search for ITEM in the array DATA is pictured in Fig. 4.7, where the values of DATA[BEG] and DATA[END] in each stage of the algorithm are indicated by circles and the value of DATA[MID] by a square. Specifically, BEG, END and MID will have the following successive values:

1. Initially, BEG = 1 and END = 13. Hence

   MID = INT[(1 + 13)/2] = 7 and so DATA[MID] = 55

2. Since 40 < 55, END has its value changed by END = MID − 1 = 6. Hence

   MID = INT[(1 + 6)/2] = 3 and so DATA[MID] = 30

3. Since 40 > 30, BEG has its value changed by BEG = MID + 1 = 4. Hence

   MID = INT[(4 + 6)/2] = 5 and so DATA[MID] = 40

We have found ITEM in location LOC = MID = 5.

(1) (11, 22, 30, 33, 40, 44, [55], 60, 66, 77, 80, 88, (99))
(2) (11, 22, [30], 33, 40, (44), 55, 60, 66, 77, 80, 88, 99)
(3) 11, 22, 30, (33), [40], (44) 55, 60, 66, 77, 80, 88, 99 [Successful]

Fig. 4.7 Binary Search for ITEM = 40

(b) Suppose ITEM = 85. The binary search for ITEM is pictured in Fig. 4.8. Here BEG, END and MID will have the following successive values:

1. Again initially, BEG = 1, END = 13, MID = 7 and DATA[MID] = 55.
2. Since 85 > 55, BEG has its value changed by BEG = MID + 1 = 8. Hence
   MID = INT[(8 + 13)/2] = 10 and so DATA[MID] = 77
3. Since 85 > 77, BEG has its value changed by BEG = MID + 1 = 11. Hence
   MID = INT[(11 + 13)/2] = 12 and so DATA[MID] = 88
4. Since 85 < 88, END has its value changed by END = MID − 1 = 11. Hence
   MID = INT[(11 + 11)/2] = 11 and so DATA[MID] = 80

(Observe that now BEG = END = MID = 11.)

Since 85 > 80, BEG has its value changed by BEG = MID + 1 = 12. But now BEG > END. Hence ITEM does not belong to DATA.

(1) (11), 22, 30, 33, 40, 44, [55], 60, 66, 77, 80, 88, (99)

(2) 11, 22, 30, 33, 40, 44, 55, (60), 66, [77], 80, 88, (99)

(3) 11, 22, 30, 33, 40, 44, 55, 60, 66, 77, (80), [88], (99)

(4) 11, 22, 30, 33, 40, 44, 55, 60, 66, 77, (80), 88, 99 [Unsuccessful]

Fig. 4.8 Binary Search for ITEM = 85

Complexity of the Binary Search Algorithm

The complexity is measured by the number $f(n)$ of comparisons to locate ITEM in DATA where DATA contains n elements. Observe that each comparison reduces the sample size in half. Hence we require at most $f(n)$ comparisons to locate ITEM where

$$2^{f(n)} > n \quad \text{or equivalently} \quad f(n) = \lfloor \log_2 n \rfloor + 1$$

That is, the running time for the worst case is approximately equal to $\log_2 n$. One can also show that the running time for the average case is approximately equal to the running time for the worst case.

Example 4.11

Suppose DATA contains 1 000 000 elements. Observe that

$$2^{10} = 1024 > 1000 \quad \text{and hence} \quad 2^{20} > 1000^2 = 1\,000\,000$$

Accordingly, using the binary search algorithm, one requires only about 20 comparisons to find the location of an item in a data array with 1 000 000 elements.

Limitations of the Binary Search Algorithm

Since the binary search algorithm is very efficient (e.g., it requires only about 20 comparisons to find an initial list of 1 000 000 elements), why would one want to use any other search algorithm?

Observe that the algorithm requires two conditions: (1) the list must be sorted and (2) one must have direct access to the middle element in any sublist. This means that one must essentially use a sorted array to hold the data. But keeping data in a sorted array is normally very expensive when there are many insertions and deletions. Accordingly, in such situations, one may use a different data structure, such as a linked list or a binary search tree, to store the data.

4.10 MULTIDIMENSIONAL ARRAYS

The linear arrays discussed so far are also called *one-dimensional arrays*, since each element in the array is referenced by a single subscript. Most programming languages allow two-dimensional and three-dimensional arrays, i.e., arrays where elements are referenced, respectively, by two and three subscripts. In fact, some programming languages allow the number of dimensions for an array to be as high as 7. This section discusses these multidimensional arrays.

Two-Dimensional Arrays

A two-dimensional $m \times n$ array A is a collection of $m \cdot n$ data elements such that each element is specified by a pair of integers (such as J, K), called *subscripts*, with the property that

$$1 \le J \le m \qquad \text{and} \qquad 1 \le K \le n$$

The element of A with first subscript j and second subscript k will be denoted by

$$A_{J,K} \quad \text{or} \quad A[J, K]$$

Two-dimensional arrays are called *matrices* in mathematics and *tables* in business applications; hence two-dimensional arrays are sometimes called *matrix arrays*.

There is a standard way of drawing a two-dimensional $m \times n$ array A where the elements of A form a rectangular array with m rows and n columns and where the element A[J, K] appears in row J and column K. (A *row* is a horizontal list of elements, and a *column* is a vertical list of elements.) Figure 4.9 shows the case where A has 3 rows and 4 columns. We emphasize that each row contains those elements with the same first subscript, and each column contains those elements with the same second subscript.

Columns

Rows	1	2	3	4
1	A[1, 1]	A[1, 2]	A[1, 3]	A[1, 4]
2	A[2, 1]	A[2, 2]	A[2, 3]	A[2, 4]
3	A[3, 1]	A[3, 2]	A[3, 3]	A[3, 4]

Fig. 4.9 Two-Dimensional 3 × 4 Array A

Example 4.12

Suppose each student in a class of 25 students is given 4 tests. Assuming the students are numbered from 1 to 25, the test scores can be assigned to a 25 × 4 matrix array SCORE as pictured

in Fig. 4.10. Thus SCORE[K, L] contains the Kth student's score on the Lth test. In particular, the second row of the array,

SCORE[2, 1], SCORE[2, 2], SCORE[2,3], SCORE[2, 4]

contains the four test scores of the second student.

Student	Test 1	Test 2	Test 3	Test 4
1	84	73	88	81
2	95	100	88	96
3	72	66	77	72
:	:	:	:	:
25	78	82	70	85

Fig. 4.10 Array SCORE

Suppose A is a two-dimensional $m \times n$ array. The first dimension of A contains the *index set* 1, ..., m, with *lower bound* 1 and *upper bound* m; and the second dimension of A contains the *index set* 1, 2, ..., n, with *lower bound* 1 and *upper bound* n. The *length* of a dimension is the number of integers in its index set. The pair of lengths $m \times n$ (read "m by n") is called the *size* of the array.

Some programming languages allow one to define multidimensional arrays in which the lower bounds are not 1. (Such arrays are sometimes called *nonregular*.) However, the index set for each dimension still consists of the consecutive integers from the lower bound to the upper bound of the dimension. The length of a given dimension (i.e., the number of integers in its index set) can be obtained from the formula

$$\text{Length} = \text{upper bound} - \text{lower bound} + 1 \tag{4.3}$$

(Note that this formula is the same as Eq. (4.1), which was used for linear arrays.) Generally speaking, unless otherwise stated, we will always assume that our arrays are *regular*, that is, that the lower bound of any dimension of an array is equal to 1.

Each programming language has its own rules for declaring multidimensional arrays. (As is the case with linear arrays, all element in such arrays must be of the same data type.) Suppose, for example, that DATA is a two-dimensional 4×8 array with elements of the *real* type. C language would declare such an array as follows:

```
float DATA[4][8];
```

Storage representations

Row-major Representation

We can consider a two-dimensional array as a one-dimensional array since it has elements with a single dimension. As a result of this a two-dimensional array can be assumed as a single column with many rows and mapped sequentially as shown in Fig. 4.11. Such a representation is called a row-major representation.

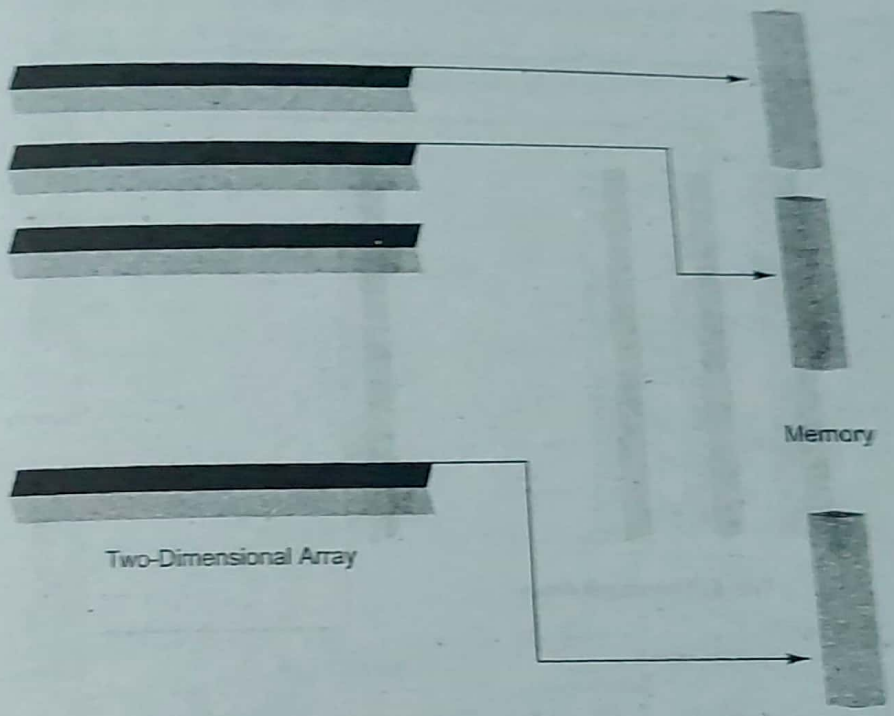


Fig. 4.11 Row Major Representation of a two-dimensional array

We can calculate the address of the element of the m^{th} row and the n^{th} column in a two-dimensional array using the formula shown below:

addr(a[m, n]) = (total number of rows present before the m^{th} row × size of a row) + (total number of elements present before the n^{th} element in the m^{th} row × size of element)

In the above equation:

The total number of rows present before m^{th} row = $(m - lb1)$ and

$lb1$ is a first dimensional lower bound.

Size of a row = total number of elements present in a row × size of an element.

Total number of elements in a row is calculated using $(ub2 - lb2 + 1)$ and

$ub2$ and $lb2$ are the second dimensional upper and lower bounds.

Now that we have a clear idea about all the variables present the above equation can be written in a simple form as:

$addr(a[m, n]) = ((m - lb1) \times (ub2 - lb2 + 1) \times size) + ((n - lb2) \times size)$

Column-major Representation

We can also represent a two-dimensional array as one single row of columns and map it sequentially as shown in Fig. 4.12. Such a representation is called a column-major representation.

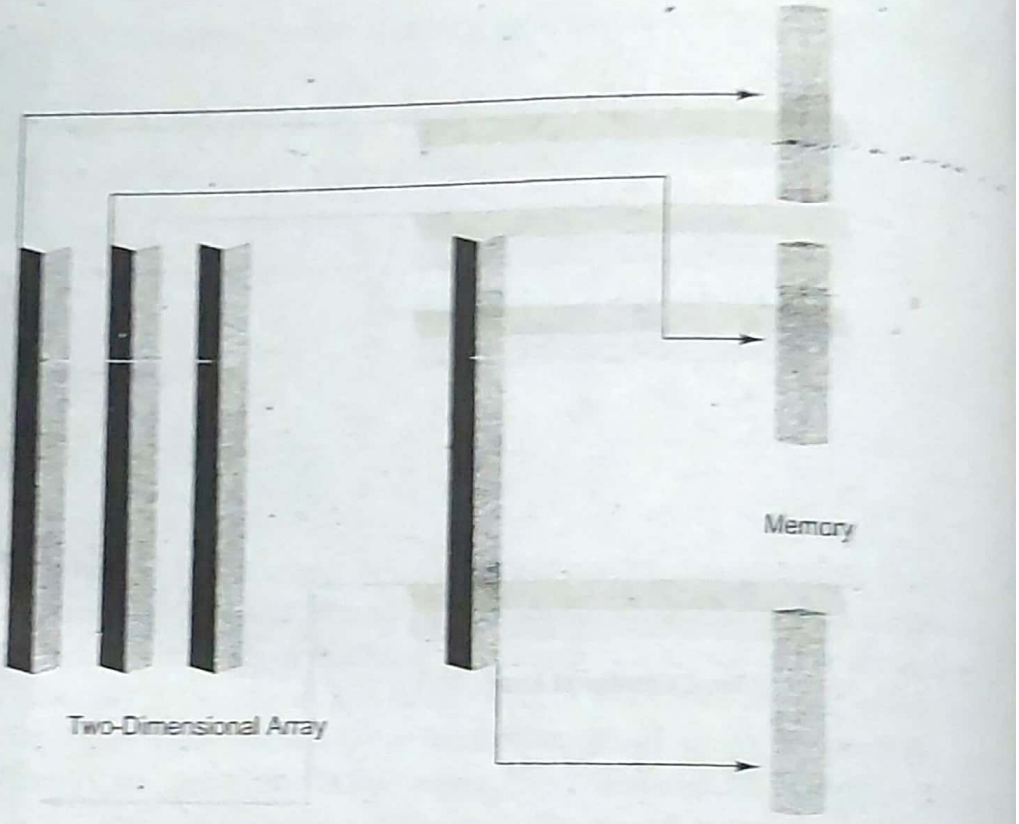


Fig. 4.12 Column-major Representation of a two-dimensional array

We can calculate the address of the element of the m^{th} row and the n^{th} column as follows:

addr(a[m, n]) = (Total number of columns present before n^{th} column × column size) + (Total number of elements present before the m^{th} element of the n^{th} column × each elements size)

Columns which are placed before n^{th} column = $(n - lb2)$ where $lb2$ is a second dimension lower bound.

Column size = Total number of elements present in a column × Element size

Number of elements in a column = $(ub1 - lb1 + 1)$,

here $ub1$ is first dimensional upper bound and $lb1$ first dimensional lower bound.

Therefore:

$addr(a[m, n]) = ((n - lb2) \times (ub1 - lb1 + 1) \times size) + ((m - lb1) \times size)$

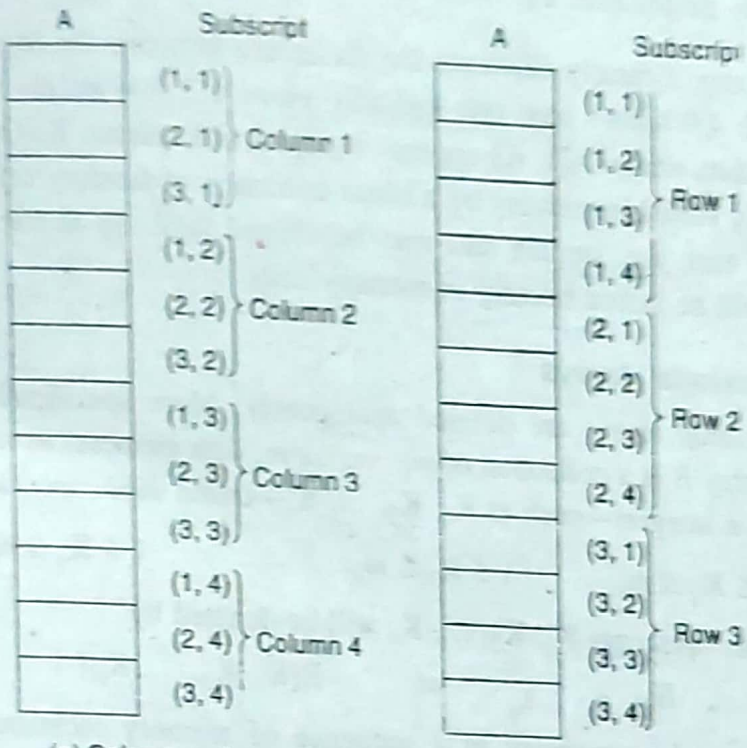
Representation of Two-Dimensional Arrays in Memory

Let A be a two-dimensional $m \times n$ array. Although A is pictured as a rectangular array of elements with m rows and n columns, the array will be represented in memory by a block of $m \cdot n$ sequential memory locations. Specifically, the programming language will store the array A either (1) column by column, which is what is called *column-major order*, or (2) row by row, in *row-major order*. Figure 4.13 shows these two ways when A is a two-dimensional 3×4 array. We emphasize that the particular representation used depends upon the programming language, not the user.

Recall that, for a linear array LA, the computer does not keep track of the address LOC(LA[K])

of every element LA[K] of LA, but does keep track of *Base*(LA), the address of the first element of LA. The computer uses the formula

$$LOC(LA[K]) = Base(LA) + w(K - 1)$$



(a) Column-major order (b) Row-major order

Fig. 4.13

to find the address of LA[K] in time independent of K. (Here w is the number of words per memory cell for the array LA, and 1 is the lower bound of the index set of LA.)

A similar situation also holds for any two-dimensional $m \times n$ array A. That is, the computer keeps track of *Base*(A)—the address of the first element A[1, 1] of A—and computes the address LOC(A[J, K]) of A[J, K] using the formula

$$\text{(Column-major order)} \quad LOC(A[J, K]) = Base(A) + w[M(K - 1) + (J - 1)] \tag{4.4}$$

or the formula

$$\text{(Row-major order)} \quad LOC(A[J, K]) = Base(A) + w[N(J - 1) + (K - 1)] \tag{4.5}$$

Again, w denotes the number of words per memory location for the array A. Note that the formulas are linear in J and K, and that one can find the address LOC(A[J, K]) in time independent of J and K.

Example 4.13

Consider the 25 × 4 matrix array SCORE in Example 4.12. Suppose *Base*(SCORE) = 200 and there are w = 4 words per memory cell. Furthermore, suppose the programming language stores two-

dimensional arrays using row-major order. Then the address of SCORE[12, 3], the third test of the twelfth student, follows:

$$LOC(SCORE[12, 3]) = 200 + 4[4(12 - 1) + (3 - 1)] = 200 + 4[46] = 384$$

Observe that we have simply used Eq. (4.5).

Multidimensional arrays clearly illustrate the difference between the logical and the physical views of data. Figure 4.9 shows how one logically views a 3 × 4 matrix array A, that is, as a rectangular array of data where A[J, K] appears in row J and column K. On the other hand, the data will be physically stored in memory by a linear collection of memory cells. This situation will occur throughout the text; e.g., certain data may be viewed logically as trees or graphs although physically the data will be stored linearly in memory cells.

General Multidimensional Arrays

General multidimensional arrays are defined analogously. More specifically, an n-dimensional $m_1 \times m_2 \times \ldots \times m_n$ array B is a collection of $m_1 \cdot m_2 \ldots m_n$ data elements in which each element is specified by a list of n integers—such as $K_1, K_2, \ldots, K_n$—called *subscripts*, with the property that

$$1 \le K_1 \le m_1, \qquad 1 \le K_2 \le m_2, \qquad \ldots, \qquad 1 \le K_n \le m_n$$

The element of B with subscripts $K_1, K_2, \ldots, K_n$ will be denoted by

$$B_{K_1, K_2, \ldots K_n} \qquad \text{or} \qquad B[K_1, K_2, \ldots, K_N]$$

The array will be stored in memory in a sequence of memory locations. Specifically, the programming language will store the array B either in row-major order or in column-major order. By *row-major order*, we mean that the elements are listed so that the subscripts vary like the automobile odometer, i.e., so that the last subscript varies first (most rapidly), the next-to-last subscript varies second (less rapidly), and so on. By *column-major order*, we mean that the elements are listed so that the first subscript varies first (most rapidly), the second subscript second (less rapidly), and so on.

Example 4.14

Suppose B is a three-dimensional 2 × 4 × 3 array. Then B contains 2 · 4 · 3 = 24 elements. These elements of B are usually pictured as in Fig. 4.14; i.e., they appear in three layers, called *pages*, where each page consists of the 2 × 4 rectangular array of elements with the same third subscript. (Thus the three subscripts of an element in a three-dimensional array are called, respectively, the *row*, *column* and *page* of the element.) The two ways of storing B in memory appear in Fig. 4.15. Observe that the arrows in Fig. 4.14 indicate the column-major order of the elements.

The definition of general multidimensional arrays also permits lower bounds other than 1. Let C be such an n-dimensional array. As before, the index set for each dimension of C consists of the consecutive integers from the lower bound to the upper bound of the dimension. The length L_i of dimension i of C is the number of elements in the index set, and L_i can be calculated, as before, from

$$L_j = \text{upper bound} - \text{lower bound} + 1 \tag{4.6}$$

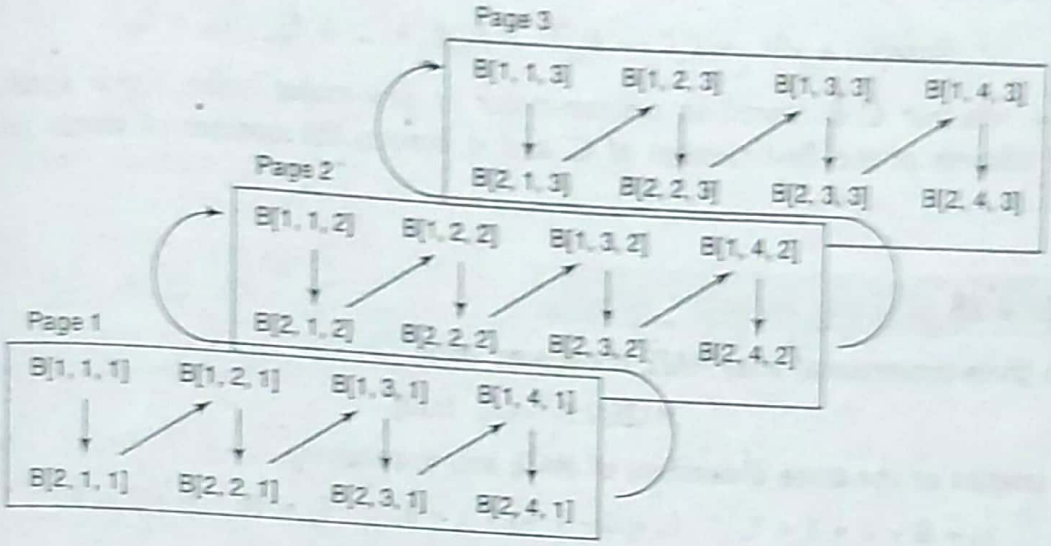
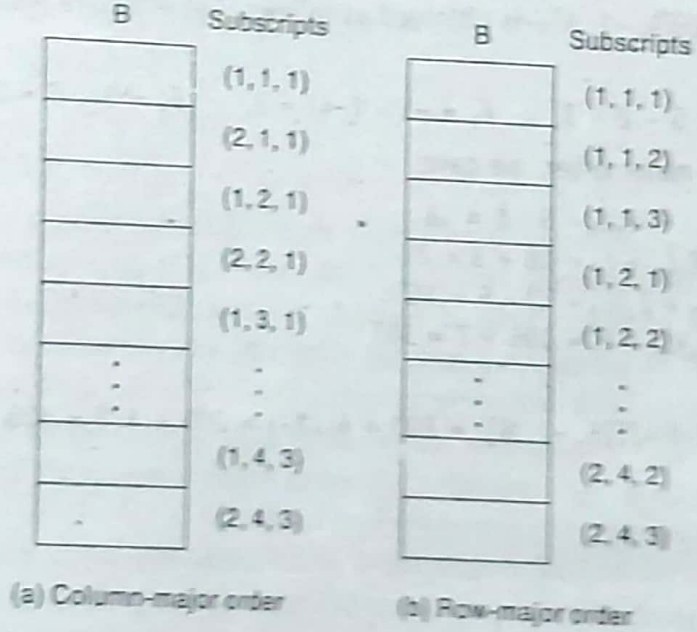


Fig. 4.14



(a) Column-major order (b) Row-major order

Fig. 4.15

For a given subscript K_j, the effective index E_j of L_j is the number of indices preceding K_j in the index set, and E_j can be calculated from

$$E_j = K_j - \text{lower bound} \tag{4.7}$$

Then the address $\text{LOC}(C[K_1, K_2, \ldots, K_N])$ of an arbitrary element of C can be obtained from the formula

$$Base(C) + w[(((\ldots(E_N L_{N-1} + E_{N-1})L_{N-2}) + \ldots + E_3)L_2 + E_2)L_1 + E_1] \quad (4.8)$$

or from the formula

$$Base(C) + w[(\ldots((E_1 L_2 + E_2)L_3 + E_3)L_4 + \ldots + E_{N-1})L_N + E_N] \quad (4.9)$$

according to whether C is stored in column-major or row-major order. Once again, $Base(C)$ denotes the address of the first element of C, and w denotes the number of words per memory location.

Example 4.15

Suppose a three-dimensional array MAZE is declared using

MAZE(2:8, −4:1, 6:10)

Then the lengths of the three dimensions of MAZE are, respectively,

$$L_1 = 8 - 2 + 1 = 7, \qquad L_2 = 1 - (-4) + 1 = 6, \qquad L_3 = 10 - 6 + 1 = 5$$

Accordingly, MAZE contains $L_1 \cdot L_2 \cdot L_3 = 7 \cdot 6 \cdot 5 = 210$ elements.

Suppose the programming language stores MAZE in memory in row-major order, and suppose $Base(MAZE) = 200$ and there are $w = 4$ words per memory cell. The address of an element of MAZE—for example, MAZE[5, −1, 8]—is obtained as follows. The effective indices of the subscripts are, respectively,

$$E_1 = 5 - 2 = 3, \qquad E_2 = -1 - (-4) = 3, \qquad E_3 = 8 - 6 = 2$$

Using Eq. (4.9) for row-major order, we have:

$$E_1 L_2 = 3 \cdot 6 = 18$$
$$E_1 L_2 + E_2 = 18 + 3 = 21$$
$$(E_1 L_2 + E_2)L_3 = 21 \cdot 5 = 105$$
$$(E_1 L_2 + E_3)L_3 + E_3 = 105 + 2 = 107$$

Therefore,

$$LOC(MAZE[5, -1, 8]) = 200 + 4(107) = 200 + 428 = 628$$

Program 4.9

```
#include <stdio.h>
main()
{
int a[2][3][4]; \\ 1
int b[3][4];    \\ 2
int c[4];       \\ 3
int count=0;

for(int i=0;i<2;i++)
for(int j=0;j<3;j++)
```

```
for(int k=0;k<4;k++)
{
a[i][j][k] = count;
count;
}
}
void print_oneDim(int a[]) \\ 4
{
for(int i=0;i<4;i++)
printf("%d",a[i]);
}
void print_twoDim(int a[][4]) \\ 5
{
for(int j=0;j<3;j++)
print_oneDim(a[j]);
printf("\n");
}
void print_threeDim(int a[][3][4]) \\ 6
{
printf("Every two dimensional matrix\n");
for(int j=0;j<2;j++)
print_twoDim(a[j]);;
}
```

Consider the above program where the three-dimensional array has two arrays of the size 3 x 4 the two arrays are called as a[0], a[1]. Hence there are 12 elements in a[0] and 12 elements in a[1] respectively. Every two-dimensional array is assumed as three arrays of size 4 each.

We use a function 'oneDim', which is a print function for printing all single-dimensional arrays and function 'twoDim' which is a print function to print two-dimensional array. It also calls a function which prints a single-dimensional value. In the same manner the function 'threeDim' is a print function to print a three-dimensional array and calls the respective function to print a three-dimensional value.

Dimension two which is sounds like a name of an array is called the outermost dimension and dimension four which is further away from the array name declaration, is called the innermost dimension.

When we pass an array to the function, we need to specify the inner dimension (which is far from the array name declaration). For example, to print 'twoDim' we have to specify the inner dimension, i.e., 4, and for print 'threeDim', we need to pass 3 and 4 as inner dimensions.

On passing a one-dimension array, there is no need for passing a dimension as the the function which is the best address for the array. Whereas, in two-dimensional array the inner dimension needs to be passed since only then does the function know the base address of each array. For example, if the declaration is

int a[3][4] has three arrays of size 4. Here the base address of a[0] is 'a'. The base address of a[1] is 'a+8' as it has 4 elements in the first row and its size is 2 bytes each. The base address of a[1] can

be calculated in a similarly fashion. In the same manner the base address of "a[2]" is "a+16". Hence we can conclude that we must know the inner dimension 4 for calculating the base address.

4.11 REPRESENTATION OF POLYNOMIALS USING ARRAYS

Sometimes we will need to evaluate several polynomial expressions and perform basic arithmetic operations like addition, multiplication, etc. on them. Therefore, we need a method to represent a polynomial. The easiest way to represent a polynomial of degree 'n' is by storing the coefficients of $(n + 1)$ terms of a polynomial in an array. An expression $x + 4x^2 - 7x^5$ is a polynomial of degree 5.

All the elements of an array has two values, namely coefficient and exponent. We may also assume that exponent of each successive term is less than that of the previous term. Once we build an array for polynomials, we can use it to perform various operations including addition and multiplication.

A single-dimensional array is used for representing a single variable polynomial. The index of such an array can be considered as an exponent and the coefficient can be stored at that particular index.

Example 4.16

The polynomial expression $3x^4 + 5x^3 + 6x^2 + 10x - 14$ can be stored in a single-dimensional array as shown in Fig. 4.16.

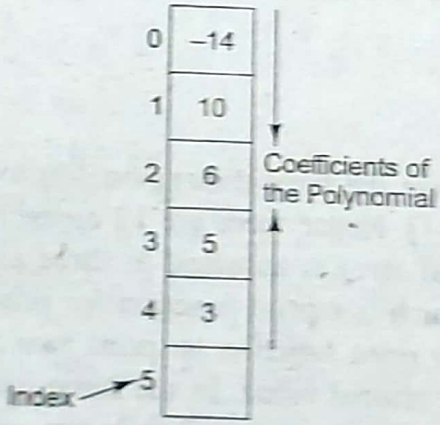


Fig. 4.16 Polynomial Representation Using an Array

This representation is definitely easy to handle. However, it has its own drawbacks. Suppose the exponent is too large, then the size of the array will become more. For instance, if we have something like $4x^{999}$. In that case, we will have to store the coefficient 4 at index 999 in the array, and the array size will have to be 1000. Imagine scanning such a large array—it will be time consuming.

Another issue faced is the wastage of space. Suppose you have a polynomial $6x^{99} - 10$, then only two elements will be stored in the array of size 100. The remaining space will be empty and therefore not utilized.

A third problem is faced when we cannot decide the size of the array. Suppose we declare an array of size 15, and the exponent value of the polynomial is 50, then we cannot store the value and we will have to change the array size.

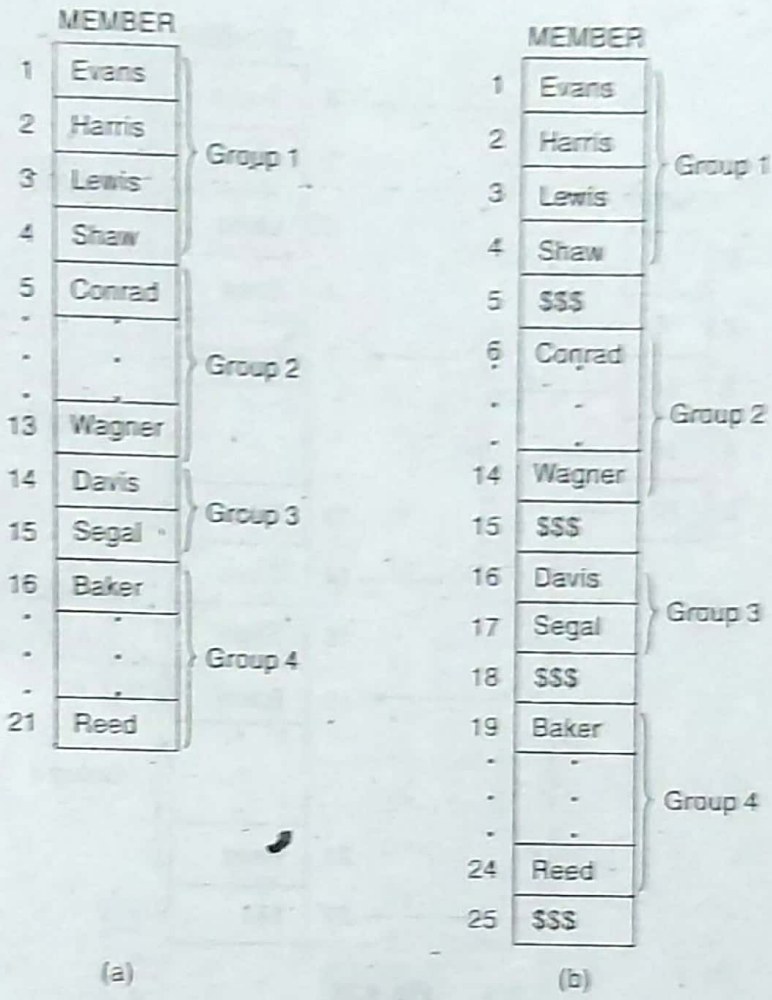


Fig. 4.20

Example 4.18

Suppose one wants to print only the names in the Lth group in Fig. 4.21, where the value of L is part of the input. Since GROUP[L] and GROUP[L + 1] – 1 contain, respectively, the locations of the first and last name in the Lth group, the following module accomplishes our task:

1. Set FIRST := GROUP[L] and LAST := GROUP[L + 1] – 1.
2. Repeat for K = FIRST to LAST:
   Write: MEMBER[K].
   [End of loop.]
3. Return.

The simplicity of the module comes from the fact that the pointer array GROUP indexes the Lth group. The variables FIRST and LAST are used mainly for notational convenience.

A slight variation of the data structure in Fig. 4.20 is pictured in Fig. 4.21, where unused memory cells are indicated by the shading. Observe that now there are some empty cells between the groups. Accordingly, a new element may be inserted in a group without necessarily moving the elements in any other group. Using this data structure, one requires an array NUMB which gives

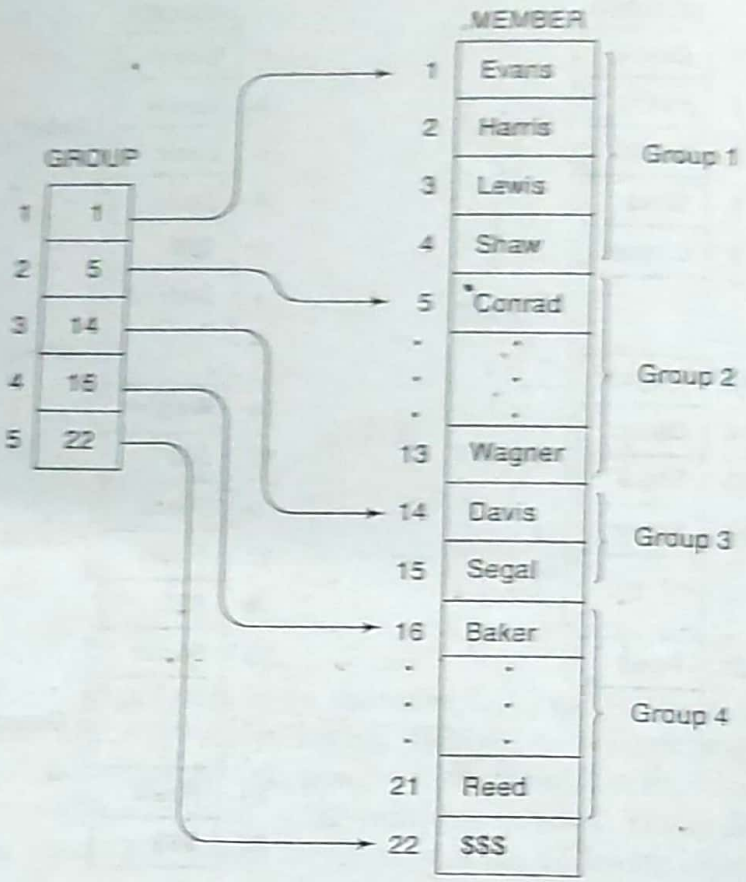


Fig. 4.21

the number of elements in each group. Observe that GROUP[K + 1] – GROUP[K] is the total amount of space available for Group K; hence

$$FREE[K] = GROUP[K + 1] - GROUP[K] - NUMB[K]$$

is the number of empty cells following GROUP K. Sometimes it is convenient to explicitly define the extra array FREE.

Example 4.19

Suppose, again, one wants to print only the names in the Lth group, where L is part of the input, but now the groups are stored as in Fig. 4.22. Observe that

$$GROUP[L] \quad \text{and} \quad GROUP[L] + NUMB[L] - 1$$

contain, respectively, the locations of the first and last names in the Lth group. Thus the following module accomplishes our task:

1. Set FIRST := GROUP[L] and LAST := GROUP[L] + NUMB[L] – 1.
2. Repeat for K = FIRST to LAST:
   Write: MEMBER[K].
   [End of loop.]
3. Return.

The variables FIRST and LAST are mainly used for notational convenience.

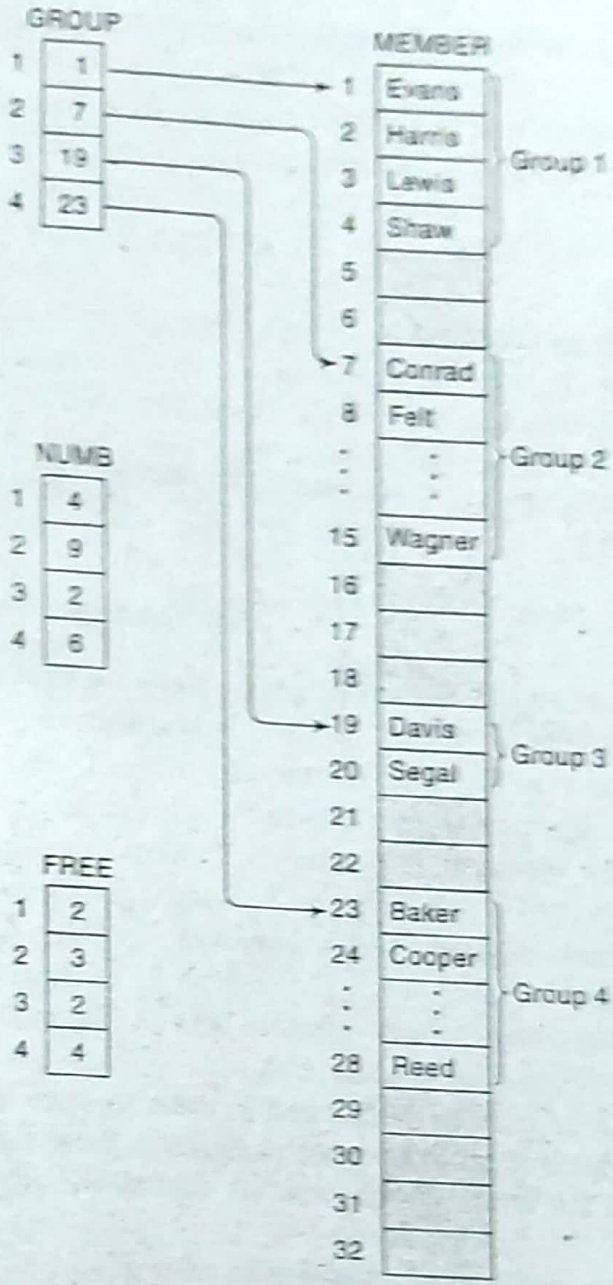


Fig. 4.22

Program 4.11

In this program, we see how to define pointer arrays.

```
#include <stdio.h>
void printArray(int *x[]);
void printArray_usingptr(int *x[]);
int *x[6];
main()
{
```

[3]	5	0	-15
[4]	1	1	11
[5]	2	1	3
[6]	3	2	-6
[7]	0	4	91
[8]	2	5	28

SOLVED PROBLEMS

Linear Arrays

4.1 Consider the linear arrays AAA(5:50), BBB(–5:10) and CCC(18).
(a) Find the number of elements in each array.
(b) Suppose *Base*(AAA) = 300 and w = 4 words per memory cell for AAA. Find the address of AAA[15], AAA[35] and AAA[55].

(a) The number of elements is equal to the length, hence use the formula

$$\text{Length} = \text{UB} - \text{LB} + 1$$

Accordingly,
$$\text{Length(AAA)} = 50 - 5 + 1 = 46$$
$$\text{Length(BBB)} = 10 - (-5) + 1 = 16$$
$$\text{Length(CCC)} = 18 - 1 + 1 = 18$$

Note that Length(CCC) = UB, since LB = 1.

(b) Use the formula

$$\text{LOC(AAA[K])} = \textit{Base}(\text{AAA}) + w(\text{K} - \text{LB})$$

Hence:
$$\text{LOC(AAA[15])} = 300 + 4(15 - 5) = 340$$
$$\text{LOC(AAA[35])} = 300 + 4(35 - 5) = 420$$

AAA[55] is not an element of AAA, since 55 exceeds UB = 50.

4.2 Suppose a company keeps a linear array YEAR(1920:1970) such that YEAR[K] contains the number of employees born in year K. Write a module for each of the following tasks.
(a) To print each of the years in which no employee was born.
(b) To find the number NNN of years in which no employee was born.
(c) To find the number N50 of employees who will be at least 50 years old at the end of the year. (Assume 1984 is the current year.)
(d) To find the number NL of employees who will be at least L years old at the end of the year. (Assume 1984 is the current year.)

Each module traverses the array.

(a)
1. Repeat for K = 1920 to 1970:
   If YEAR[K] = 0, then: Write: K.
   [End of loop.]
2. Return.

(b) 1. Set NNN := 0.
2. Repeat for K = 1920 to 1970:
If YEAR[K] = 0, then: Set NNN := NNN + 1.
[End of loop.]
3. Return.

(c) We want the number of employees born in 1934 or earlier.
1. Set N50 := 0.
2. Repeat for K = 1920 to 1934:
Set N50 := N50 + YEAR[K].
[End of loop.]
3. Return.

(d) We want the number of employees born in year 1984 – L or earlier.
1. Set NL := 0 and LLL := 1984 – L
2. Repeat for K = 1920 to LLL:
Set NL := NL + YEAR[K].
[End of loop.]
3. Return.

4.3 Suppose a 10-element array A contains the values $a_1, a_2, \ldots, a_{10}$. Find the values in A after each loop.

(a) Repeat for K = 1 to 9:
Set A[K + 1] := A[K].
[End of loop.]

(b) Repeat for K = 9 to 1 by –1:
Set A[K + 1] := A[9].
[End of loop.]

Note that the index K runs from 1 to 9 in part (a) but in reverse order from 9 back to 1 in part (b).

(a) First A[2] := A[1] sets A[2] = a_1, the value of A[1].
Then A[3] := A[2] sets A[3] = a_1, the current value of A[2].
Then A[4] := A[3] sets A[4] = a_1, the current value of A[3]. And so on.
Thus every element of A will have the value x_1, the original value of A[1].

(b) First A[10] : = A[9] sets A[10] = a_9.
Then A[9]:= A[8] sets A[9] = a_8.
Then A[8]: = A[7] sets A[8] = a_7. And so on.
Thus every value in A will move to the next location. At the end of the loop, we still have A[1] = x_1.

Remark: This example illustrates the reason that, in the insertion algorithm, Algorithm 4.4, the elements are moved downward in reverse order, as in loop (b) above.

4.4 Consider the alphabetized linear array NAME in Fig. 4.30.

	NAME
1	Allen
2	Clark
3	Dickens
4	Edwards
5	Goodman
6	Hobbs
7	Irwin
8	Klein
9	Lewis
10	Morgan
11	Richards
12	Scott
13	Tucker
14	Walton

Fig. 4.30

(a) Find the number of elements that must be moved if Brown, Johnson and Peters are inserted into NAME at three different times.
(b) How many elements are moved if the three names are inserted at the same time?
(c) How does the telephone company handle insertions in a telephone directory?

(a) Inserting Brown requires 13 elements to be moved, inserting Johnson requires 7 elements to be moved and inserting Peters requires 4 elements to be moved. Hence 24 elements are moved.
(b) If the elements are inserted at the same time, then 13 elements need be moved, each only once (with the obvious algorithm).
(c) The telephone company keeps a running list of new numbers and then updates the telephone directory once a year.

Searching, Sorting

4.5 Consider the alphabetized linear array NAME in Fig. 4.30.

(a) Using the linear search algorithm, Algorithm 4.5, how many comparisons C are used to locate Hobbs, Morgan and Fisher?
(b) Indicate how the algorithm may be changed for such a sorted array to make an unsuccessful search more efficient. How does this affect part (a)?

(a) C(Hobbs) = 6, since Hobbs is compared with each name, beginning with Allen, until Hobbs is found in NAME[6].
C(Morgan) = 10, since Morgan appears in NAME[10].
C(Fisher) = 15, since Fisher is initially placed in NAME[15] and then Fisher is compared with every name until it is found in NAME[15]. Hence the search is unsuccessful.

(b) Observe that NAME is alphabetized. Accordingly, the linear search can stop after a given name XXX is compared with a name YYY such that XXX < YYY (i.e., such that alphabetically, XXX comes before YYY). With this algorithm, C(Fisher) = 5, since the search can stop after Fisher is compared with Goodman in NAME[5].

4.6 Suppose the binary search algorithm, Algorithm 4.6, is applied to the array NAME in Fig. 4.30 to find the location of Goodman. Find the ends BEG and END and the middle MID for the test segment in each step of the algorithm.

Recall that MID = INT((BEG + END)/2), where INT means integer value.
Step 1. Here BEG = 1 [Allen] and END = 14 [Walton], so MID = 7 [Irwin].
Step 2. Since Goodman < Irwin, reset END = 6. Hence MID = 3 [Dickens].
Step 3. Since Goodman > Dickens, reset BEG = 4. Hence MID = 5 [Goodman].
We have found the location LOC = 5 of Goodman in the array. Observe that, there were C = 3 comparisons.

4.7 Modify the binary search algorithm, Algorithm 4.6, so that it becomes a search and insertion algorithm.

There is no change in the first four steps of the algorithm. The algorithm transfers control to Step 5 only when ITEM does not appear in DATA. In such a case, ITEM is inserted before or after DATA[MID] according to whether ITEM < DATA[MID] or ITEM > DATA[MID]. The algorithm follows.

Algorithm P4.7: (Binary Search and Insertion) DATA is a sorted array with N elements, and ITEM is a given item of information. This algorithm finds the location LOC of ITEM in DATA or inserts ITEM in its proper place in DATA.

Steps 1 through 4. Same as in Algorithm 4.6.
5. If ITEM < DATA[MID], then:
Set LOC := MID
Else:
Set LOC := MID +
[End of If structure.]
6. Insert ITEM into DATA[LOC] using Algorithm 4.2.
7. Exit.

4.8 Suppose A is a sorted array with 200 elements, and suppose a given element x appears with the same probability in any place in A. Find the worst-case running time $f(n)$ and the average-case running time $g(n)$ to find x in A using the binary search algorithm.

For any value of k, let n_k denote the number of those elements in A that will require k comparisons to be located in A. Then:

k:	1	2	3	4	5	6	7	8
n_k:	1	2	4	8	16	32	64	73

The 73 comes from the fact that $1 + 2 + 4 + \ldots + 64 = 127$ so there are only $200 - 127 = 73$ elements left. The worst-case running time $f(n) = 8$. The average-case running time $g(n)$ is obtained as follows:

$$g(n) = \frac{1}{n}\sum_{k=1}^{8} k \cdot n_k$$

$$= \frac{1\cdot 1 + 2\cdot 2 + 3\cdot 4 + 4\cdot 8 + 5\cdot 16 + 6\cdot 32 + 7\cdot 64 + 8\cdot 73}{200}$$

$$= \frac{1353}{200} = 6.765$$

Observe that, for the binary search, the average-case and worst-case running times are approximately equal.

4.9 Using the bubble sort algorithm, Algorithm 4.4, find the number C of comparisons and the number D of interchanges which alphabetize the $n = 6$ letters in PEOPLE.

The sequences of pairs of letters which are compared in each of the $n - 1 = 5$ passes follow: a square indicates that the pair of letters is compared and interchanged, and a circle indicates that the pair of letters is compared but not interchanged.

Pass 1. [PE]OPLE, E[PO]PLE, EO(PP)LE
EOP[PL]E EOPL[PE] EOPLEP

Pass 2. (EO)PLEP, E(OP)LEP, EO[PL]EP
EOL[PE]P, EOLEPP

Pass 3. (EO)LEPP, E[OL]EPP, EL[OE]PP
ELEOPP

Pass 4. (EL)EOPP, E[LE]OPP EELOPP

Pass 5 (EE)LOPP, EELOPP

Since $n = 6$, the number of comparisons will be $C = 5 + 4 + 3 + 2 + 1 = 15$. The number D of interchanges depends also on the data, as well as on the number n of elements. In this case $D = 9$.

4.10 Prove the following identity, which is used in the analysis of various sorting and searching algorithms.

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

Writing the sum S forward and backward, we obtain:

$$S = 1 + 2 + 3 + \ldots + (n - 1) + n$$
$$S = n + (n - 1) + (n - 2) + \ldots + 2 + 1$$

We find the sum of the two values of S by adding pairs as follows:

$$2S = (n + 1) + (n + 1) + (n + 1) + \ldots + (n + 1) + (n + 1)$$

There are n such sums, so $2S = n(n + 1)$. Dividing by 2 gives us our result.

Multidimensional Arrays; Matrices

4.11 Suppose multidimensional arrays A and B are declared using

A(–2:2, 2:22) and B(1:8, –5:5, –10:5)

(a) Find the length of each dimension and the number of elements in A and B.
(b) Consider the element B[3, 3, 3] in B. Find the effective indices E_1, E_2, E_3 and the address of the element, assuming *Base*(B) = 400 and there are $w = 4$ words per memory location.

(a) The length of a dimension is obtained by:

Length = upper bound – lower bound + 1

Hence the lengths L_i of the dimensions of A are:

$$L_1 = 2 - (-2) + 1 = 5 \quad \text{and} \quad L_2 = 22 - 2 + 1 = 21$$

Accordingly, A has $5 \cdot 21 = 105$ elements. The lengths L_i of the dimensions of B are

$$L_1 = 8 - 1 + 1 = 8 \qquad L_2 = 5 - (-5) + 1 = 11 \qquad L_3 = 5 - (-10) + 1 = 16$$

Therefore, B has $8 \cdot 11 \cdot 16 = 1408$ elements.

(b) The effective index E_i is obtained from $E_i = k_i - \text{LB}$, where k_i is the given index and LB is the lower bound. Hence

$$E_1 = 3 - 1 = 2 \qquad E_2 = 3 - (-5) = 8 \qquad E_3 = 3 - (-10) = 13$$

The address depends on whether the programming language stores B in row-major order or column-major order. Assuming B is stored in column-major order, we use Eq. (4.8):

$$E_3L_2 = 13 \cdot 11 = 143 \qquad E_3L_2 + E_2 = 143 + 8 = 151$$
$$(E_3L_2 + E_2)L_1 = 151 \cdot 8 = 1208 \qquad (E_3L_2 + E_2)L_1 + E_1 = 1208 + 2 = 1210$$

Therefore, LOC(B[3, 3, 3]) = 400 + 4(1210) = 400 + 4840 = 5240

4.12 Let A be an $n \times n$ square matrix array. Write a module which
(a) Finds the number NUM of nonzero elements in A
(b) Finds the SUM of the elements above the diagonal, i.e., elements A[I, J] where I < J
(c) Finds the product PROD of the diagonal elements $(a_{11}, a_{22}, \ldots, a_{nn})$

(a) 1. Set NUM := 0.

2. Repeat for I = 1 to N:
3. Repeat for J = 1 to N:
If A[I, J] ≠ 0, then: Set NUM := NUM + 1.
[End of inner loop.]
[End of outer loop.]
4. Return.

(b) 1. Set SUM := 0.
2. Repeat for J = 2 to N:
3. Repeat for I = 1 to J – 1:
Set SUM := SUM + A[I, J].
[End of inner Step 3 loop.]
4. Return.

(c) 1. Set PROD := 1. [This is analogous to setting SUM = 0.]
2. Repeat for K = 1 to N:
Set PROD := PROD*A[K, K].
[End of loop.]
3. Return.

4.13 Consider an n-square tridiagonal array A as shown in Fig. 4.31. Note that A has n elements on the diagonal and $n - 1$ elements above and $n - 1$ elements below the diagonal. Hence A contains at most $3n - 2$ nonzero elements. Suppose we want to store A in a linear array B as indicated by the arrows in Fig. 4.31; i.e.,

$$B[1] = a_{11}, \quad B[2] = a_{12}, \quad B[3] = a_{21}, \quad B[4] = a_{22}, \quad \ldots$$

Find the formula that will give us L in terms of J and K such that

$$B[L] = A[J, K]$$

(so that one can access the value of A[J, K] from the array B).

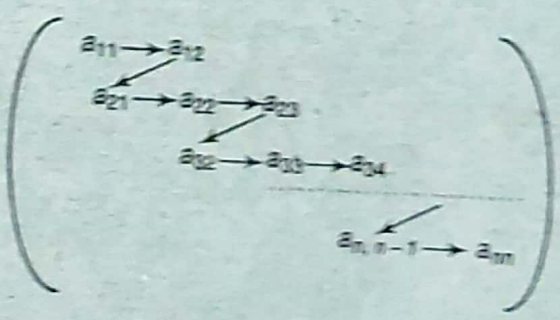


Fig. 4.31 Tridiagonal Array

Note that there are 3(J – 2) + 2 elements above A[J, K] and K – J + 1 elements to the left of A[J, K].

Hence

$$L = [3(J - 2) + 2] + [K - J + 1] + 1 = 2J + K - 2$$

The following C program demonstrates how the non-zero elements of a 5-square tridiagonal array are stored in a linear array:

Program 4.15

```
#include <stdio.h>
#include <conio.h>

void main()
{
int A[5][5];
int B[50];
int J,K,L;
int N=5;
clrscr();

for(J=0;J<N;J++)
  for(K=0;K<N;K++)
    A[J][K]=0;

for(L=0;L<50;L++)
  B[L]=0;

A[0][0]=2;
A[0][1]=22;

A[1][0]=-13;
A[1][1]=77;
A[1][2]=4;

A[2][1]=3;
A[2][2]=87;
A[2][3]=99;

A[3][2]=42;
A[3][3]=6;
A[3][4]=9;

A[4][3]=7;
A[4][4]=29;

for(J=0;J<N;J++)
  for(K=0;K<N;K++)
  {
    if(A[J][K]!=0)
    {
      L=2*(J+1)+K+1-2-1;
```

```
    B[L]=A[J][K];
   }
  }

 printf("5-Square Tridiagonal Array A:\n\n");
 printf("%d %d\n\n",A[0][0],A[0][1]);
 for(J=1,K=1;J<=3&&K<=3;J++,K++)
 {
  L=J;
  while(L!=0)
  {
   printf("\t");
   L=L-1;
  }
  printf("%d %d %d\n\n",A[J][K-1],A[J][K],A[J][K+1]);
 }
 printf("\t\t\t\t%d %d\n\n",A[4][3],A[4][4]);

 printf("5-Square Tridiagonal Array A stored in Linear Array B:\n\n");
 L=0;
 while(B[L]!=0)
 {
  printf("B[%d] = %d\n",L,B[L]);
  L=L+1;
 }

 getch();
 }

 Output:
 5-Square Tridiagonal Array A:

 2 22

         -13 77 4

                  3 87 99

                           42 6 9

                                    7 29

 5-Square Tridiagonal Array A stored in Linear Array B:

 B[0] = 2
 B[1] = 22
```

```
B[2]  = -13
B[3]  = 77
B[4]  = 4
B[5]  = 3
B[6]  = 87
B[7]  = 99
B[8]  = 42
B[9]  = 6
B[10] = 9
B[11] = 7
B[12] = 29
```

4.14 An *n*-square matrix array A is said to be *symmetric* if A[J, K] = A[K, J] for all J and K.

(a) Which of the following matrices are symmetric?

$$\begin{pmatrix} 2 & -3 & 5 \\ -3 & -2 & 4 \\ 5 & 6 & 8 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & -7 \\ 3 & 6 & -1 \\ -7 & -1 & 2 \end{pmatrix}$$

(b) Describe an efficient way of storing a symmetric matrix A in memory.

(c) Suppose A and B are two *n*-square symmetric matrices. Describe an efficient way of storing A and B in memory.

(a) The first matrix is not symmetric, since $a_{23} = 4$ but $a_{32} = 6$. The second matrix is not a square matrix so it cannot be symmetric, by definition. The third matrix is symmetric.

(b) Since A[J, K] = A[K, J], we need only store those elements of A which lie on or below the diagonal. This can be done in the same way as that for triangular matrices described in Example 4.25.

(c) First note that, for a symmetric matrix, we need store only either those elements on or below the diagonal or those on or above the diagonal. Therefore, A and B can be stored in an $n \times (n + 1)$ array C as pictured in Fig. 4.32, where C[J, K] = A[J, K] when $J \geq K$ but C[J, K] = B[J, K –1] when $J < K$.

Fig. 4.32

Pointer Arrays; Record Structures

4.15 Three lawyers, Davis, Levine and Nelson, share the same office. Each lawyer has his own clients. Figure 4.33 shows three ways of organizing the data.

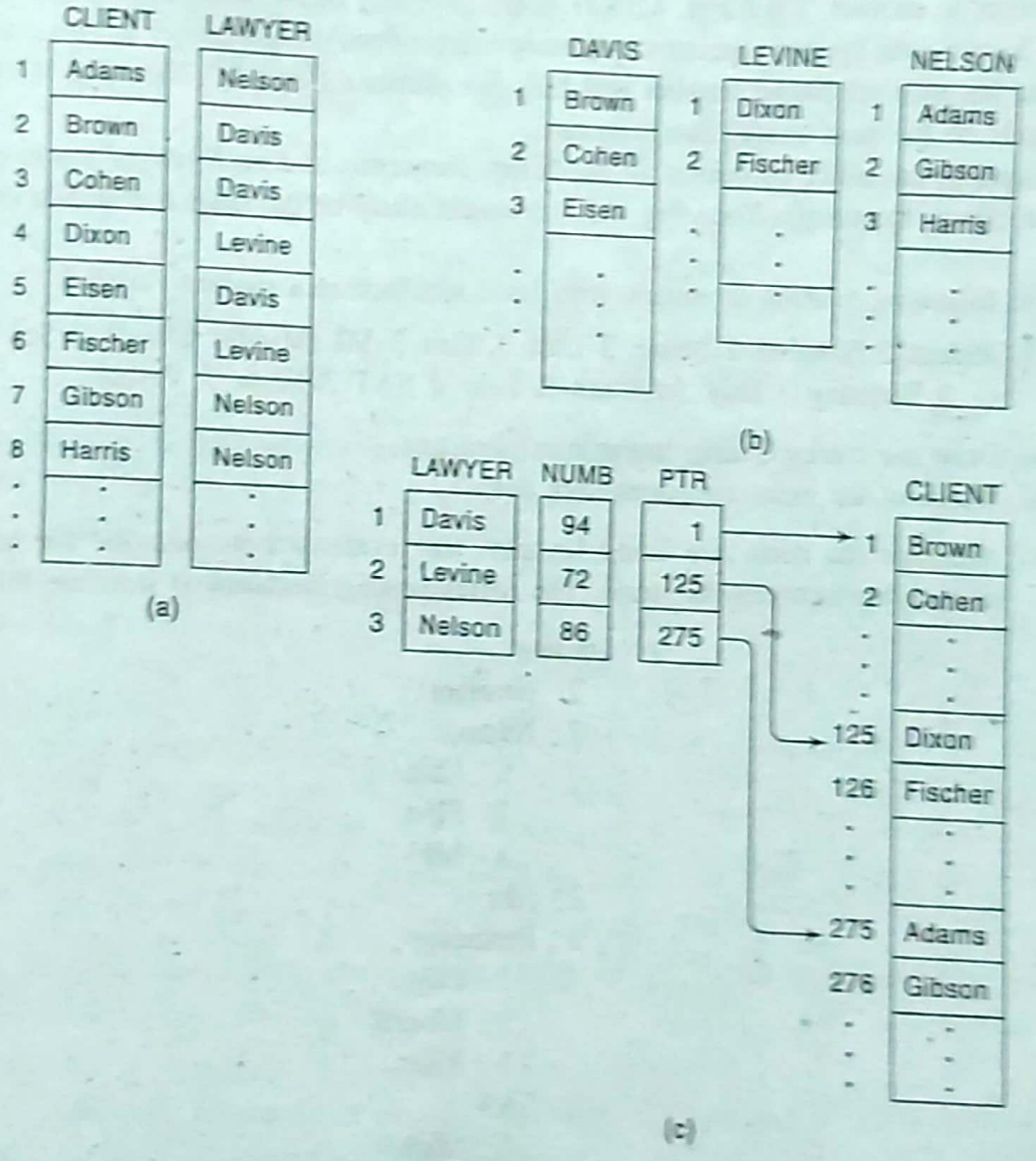


Fig. 4.33

(a) Here there is an alphabetized array CLIENT and an array LAWYER such that LAWYER[K] is the lawyer for CLIENT[K].
(b) Here there are three separate arrays, DAVIS, LEVINE and NELSON, each array containing the list of the lawyer's clients.
(c) Here there is a LAWYER array, and arrays NUMB and PTR giving, respectively, the number and location of each lawyer's alphabetized list of clients in an array CLIENT.

Which data structure is most useful? Why?

The most useful data structure depends on how the office is organized and how the clients are processed.

Suppose there are only one secretary and one telephone number, and suppose there is a single monthly billing of the clients. Also, suppose clients frequently change from one lawyer to another. Then Fig. 4.33(a) would probably be the most useful data structure.

Suppose the lawyers operate completely independently: each lawyer has his own secretary and his own telephone number and bills his clients differently. Then Fig. 4.33(b) would likely be the most useful data structure.

Suppose the office processes all the clients frequently and each lawyer has to process his own clients frequently. Then Fig. 4.33(c) would likely be the most useful data structure.

4.16 The following is a list of entries, with level numbers, in a student's record:

1 Student 2 Number 2 Name 3 Last 3 First 3 MI (Middle Initial) 2 Sex
2 Birthday 3 Day 3 Month 3 Year 2 SAT 3 Math 3 Verbal

(a) Draw the corresponding hierarchical structure.
(b) Which of the items are elementary items?

(a) Although the items are listed linearly, the level numbers describe the hierarchical relationship between the items. The corresponding hierarchical structure follows:

1 Student
- 2 Number
- 2 Name
  - 3 Last
  - 3 First
  - 3 MI
- 2 Sex
- 2 Birthday
  - 3 Day
  - 3 Month
  - 3 Year
- 2 SAT
  - 3 Math
  - 3 Verbal

(b) The elementary items are the data items which do not contain subitems: Number, Last, First, MI, Sex, Day, Month, Year, Math and Verbal. Observe that an item is elementary only if it is not followed by an item with a higher level number.

4.17 A professor keeps the following data for each student in a class of 20 students:

Name (Last, First, MI), Three Tests, Final, Grade

Here Grade is a 2-character entry, for example, B+ or C or A−. Describe a C structure to store the data.

An element in a record structure may be an array itself. Instead of storing the three tests seperately, we store them in an array. Such a structure follows:

```
struct STUDENT
{
        struct NAME
        {
                char LAST[10];
                char FIRST[10];
                char MI;
        }N;
        int TEST[3];
        int FINAL;
        char GRADE[2];
}S[20];
```

4.18 A college uses the following structure for a graduating class:

```
1  Student(200)
     2  Name
          3  Last
          3  First
          3  Middle Initial
     2  Major
     2  SAT
          3  Verbal
          3  Math
     2  GPA(4)
     2  CUM
```

Here, GPA[K] refers to the grade point average during the *k*th year and CUM refers to the cumulative grade point average.

(a) How many elementary items are there in the file?
(b) How does one access (i) the major of the eighth student and (ii) the sophomore GPA of the forty-fifth student?
(c) Find each output:
  (i) Write: Name[15]
  (ii) Write: CUM
  (iii) Write: GPA[2].
  (iv) Write: GPA[I, 3].

(a) Since GPA is counted 4 times per student, there are 11 elementary items per student, so there are altogether 2200 elementary items.
(b) (i) Student.Major[8] or simply MAJOR[8]. (ii) GPA[45, 2].
(c) (i) Here Name[15] refers to the name of the fifteenth student. But Name is a group item. Hence LAST[15], First[15] and MI[15] are printed.

(ii) Here CUM refers to all the CUM values. That is,

CUM[1], CUM[2], CUM[3], ..., CUM[200]

are printed.

(iii) GPA[2] refers to the GPA array of the second student. Hence,

GPA[2, 1], GPA[2, 2], GPA[2, 3], GPA[2, 4]

are printed.

(iv) GPA[1, 3] is a single item, the GPA during the junior year of the first student. That is, only GPA[1, 3] is printed.

4.19 An automobile dealership keeps track of the serial number and price of each of its automobiles in arrays AUTO and PRICE, respectively. In addition, it uses the data structure in Fig. 4.34, which combines a record structure with pointer variables. The new Chevys, new Buicks, new Oldsmobiles, and used cars are listed together in AUTO.

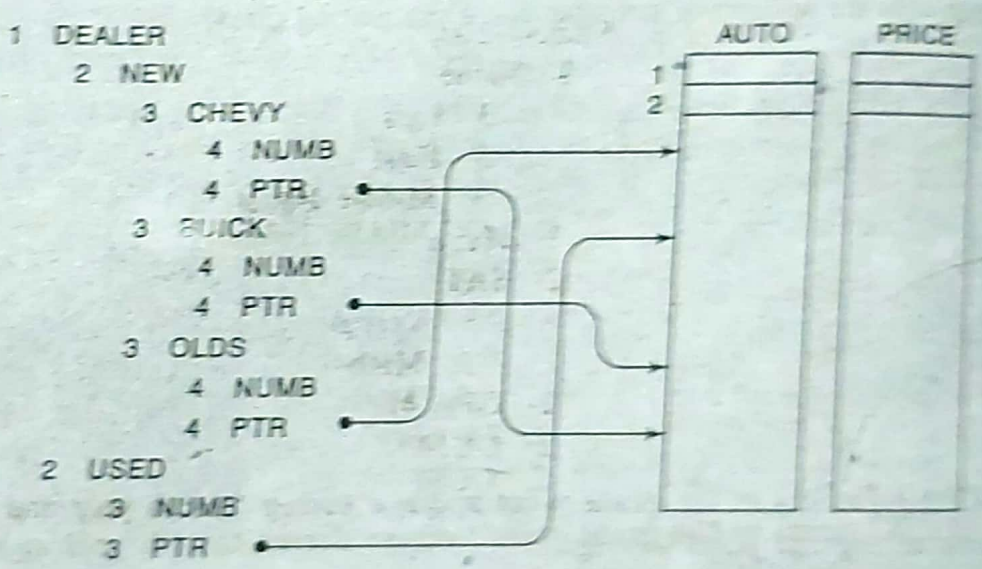


Fig. 4.34

The variables NUMB and PTR under USED give, respectively, the number and location of the list of used automobiles.

(a) How does one index the location of the list of new Buicks in AUTO?

(b) Write a procedure to print serial numbers of all new Buicks under $10 000.

(a) Since PTR appears more than once in the record structure, one must use BUICK to reference the location of the list of new Buicks in AUTO.

(b) One must traverse the list of new Buicks but print out only those Buicks whose price is less than $10 000. The procedure follows:

Procedure P4.19: The data are stored in the structure in Fig. 4.34. This procedure outputs those new Buicks whose price is less than $10 000.

1. Set FIRST := BUICK.PTR. [Location of first element in Buick list.]
2. Set LAST := FIRST + BUICK.NUMB – 1. [Location of last element in list.]
3. Repeat for K = FIRST to LAST.
   If PRICE[K] < 10 000, then:
   Write: AUTO[K], PRICE[K].
   [End of If structure.]
   [End of loop.]
4. Exit.

4.20 Suppose in Solved Problem 4.19 the dealership had also wanted to keep track of the accessories of each automobile, such as air-conditioning, radio, and rustproofing. Since this involves variable-length data, how might this be done?

This can be accomplished as in Fig. 4.35. That is, besides AUTO and PRICE, there is an array POINTER such that POINTER[K] gives the location in an array ACCESSORIES of the list of accessories (with sentinel '$$$') of AUTO[K].

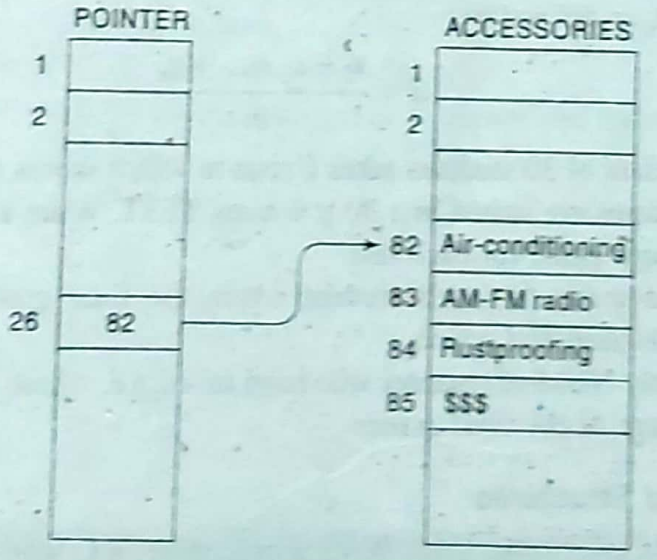


Fig. 4.35

4.21 What is an ordered list? Explain.

An *ordered list* is a list in which the order of the items is significant, and the items in it are not necessarily *sorted*. Consequently, it is possible to *change* the order of items and still have a valid ordered list.

SUPPLEMENTARY PROBLEMS

Arrays

4.1 Consider the linear arrays XXX(−10:10), YYY(1935:1985), ZZZ(35). (a) Find the number of elements in each array. (b) Suppose *Base*(YYY) = 400 and $w = 4$ words per memory cell for YYY. Find the address of YYY[1942], YYY[1977] and YYY[1988].

4.2 Consider the following multidimensional arrays:

X(−5:5, 3:33) Y(3:10, 1:15, 10:20)

(a) Find the length of each dimension and the number of elements in X and Y.

(b) Suppose *Base*(Y) = 400 and there are $w = 4$ words per memory location. Find the effective indices E_1, E_2, E_3 and the address of Y[5, 10, 15] assuming (i) Y is stored in row-major order and (ii) Y is stored in column-major order.

4.3 An array A contains 25 positive integers. Write a C program which

(a) Finds all pairs of elements whose sum is 25.

(b) Finds the number EVNUM of elements of A which are even, and the number ODNUM of elements of A which are odd.

4.4 Suppose A is a linear array with n numeric values. Write a C program

MEAN(A, N, AVE)

which finds the average AVE of the values in A. The *arithmetic mean* or *average* $\bar{x}$ of the values $x_1, x_2, \ldots, x_n$ is defined by

$$\bar{x} = \frac{x_1 + x_2 + \ldots + x_n}{n}$$

4.5 Each student in a class of 30 students takes 6 tests in which scores range between 0 and 100. Suppose the test scores are stored in a 30 × 6 array TEST. Write a C program which

(a) Finds the average grade for each test

(b) Finds the final grade for each student where the final grade is the average of the student's five highest test scores

(c) Finds the number NUM of students who have failed, i.e. whose final grade is less than 60

(d) Finds the average of the final grades

Pointer Arrays; Record Structures

4.6 Consider the data in Fig. 4-33(c). (a) Write a procedure in C which prints the list of clients belonging to LAWYER[K]. (b) Assuming CLIENT has space for 400 elements, define an array FREE such that FREE[K] contains the number of empty cells following the list of clients belonging to LAWYER[K].

4.7 The following is a list of entries, with level numbers, in a file of employee records:

1 Employee(200), 2 SSN(Social Security Number), 2 Name,
3 Last, 3 First, 3 MI (Middle Initial), 2 Address, 3 Street,
3 Area, 4 City, 4 State, 4 ZIP, 2 Age, 2 Salary, 2 Dependents

(a) Draw the corresponding hierarchical structure.
(b) Which of the items are elementary items?
(c) Describe a record structure in C to store the data.

4.8 Consider the data structure in Fig. 4.34. Write a C program to carry out each of the following:
(a) Finding the number of new Oldsmobiles selling for under $10 000.
(b) Finding the number of new automobiles selling for under $10 000.
(c) Finding the number of automobiles selling for under $10 000.
(d) Listing all automobiles selling for under $10 000.
(*Note:* Parts (c) and (d) require only the arrays AUTO and PRICE together with the number of automobiles.)

4.9 A class of student records is organized as follows:

1 Student(35), 2 Name, 3 Last, 3 First, 3 MI (Middle Initial), 2 Major
2 Test(4), 2 Final, 2 Grade

(a) How many elementary items are there?
(b) Describe a record structure in C to store the data.
(c) Describe the output of each of the following Write statements: (i) Write: Final[15], (ii) Write: Name[15] and (iii) Write: Test[4].

4.10 Consider the data structure in Solved Problem 4.18. Write a C program which
(a) Finds the average of the sophomore GPA scores
(b) Finds the number of biology majors
(c) Finds the number of CUM scores exceeding K

4.11 In a row dominated two-dimensional array, which one of the following is advantageous? Explain.

(a)
```
for(i=0;i<1000;i++)
for(j=0;j<1000;j++)
temp=temp+a[i][j];
```
(b)
```
for(j=0;j<1000;j++)
for(i=0;i<1000;i++)
temp=temp+a[i][j]
```

PROGRAMMING PROBLEMS

Arrays

Assume that the data in Table 4.1 are stored in linear arrays SSN, LAST, GIVEN, CUM and YEAR (with space for 25 students) and that a variable NUM is defined which contains the actual number of students.

4.1 Write a program for each of the following:
(a) Listing all students whose CUM is K or higher. (Test the program using K = 3.00.)
(b) Listing all students in year L. (Test the program using L = 2, or sophomore.)

MULTIPLE CHOICE QUESTIONS

4.1 _____ is a structure used to represent the linear relationship between elements by means of sequential memory locations.
(a) Linked list (b) Array
(c) Pointer (d) Stack

4.2 A _____ is a list of a finite number of homogenous data elements.
(a) Linear array (b) Pointer
(c) Linked List (d) Tree

4.3 The number of elements n is called the length or _____ of the array.
(a) Upper bound (b) Lower bound
(c) Size (d) Variable

4.4 The number K in A[K] is called the subscript or the _____.
(a) Size (b) Index
(c) Variable (d) Constant

4.5 Which of the following items are not part of the array declaration?
(a) Name of the array
(b) Data type of the array
(c) Index set of the array
(d) Length of the array

4.6 Programming languages like FORTRAN and PASCAL allocate memory space for arrays _____.
(a) Dynamically (b) Statically
(c) Successively (d) Alternatively

4.7 The process of accessing and processing each element of an array A, exactly once is called _____.
(a) Deleting (b) Inserting
(c) Traversing (d) Searching

4.8 _____ refers to the operation of rearranging the elements of an array A so that they are in increasing order.
(a) Searching (b) Sorting
(c) Traversing (d) Inserting

4.9 Two-dimensional arrays are sometimes called _____ arrays.
(a) Integer (b) Boolean
(c) Matrix (d) Real

4.10 _____ is a list in which the order of the items is significant, and the items are not necessarily sorted.
(a) Ordered list (b) Indexed list
(c) Sequential list (d) Unordered list

4.11 Representation of a two-dimensional array as one single column of rows and mapping it sequentially is called _____ representation.
(a) Row-major (b) Row
(c) Colum-major (d) Column

4.12 Matrices with relatively high proportion of zero entries are called _____ matrices.
(a) Triangular (b) Diagonal
(c) Sparse (d) Adjacency

4.13 _____ arrays are where the elements in the different arrays with the same subscript belong to the same record.
(a) One-dimensional (b) Parallel
(c) Two-dimensional (d) Static

4.14 Records can be stored in an area of memory called _____ memory.
(a) Dynamic (b) Static
(c) Simple (d) Parallel

4.15 A matrix in which nonzero entries can only occur on the diagonal or on elements immediately above or below the diagonal, is called _____ matrix.
(a) Triangular (b) Tridiagonal
(c) Sparse (d) Simple

ANSWERS TO MULTIPLE CHOICE QUESTIONS

4.1 (b) 4.2 (a) 4.3 (c) 4.4 (b) 4.5 (d) 4.6 (b) 4.7 (c)
4.8 (b) 4.9 (c) 4.10 (c) 4.11 (a) 4.12 (c) 4.13 (b) 4.14 (a)
4.15 (c)