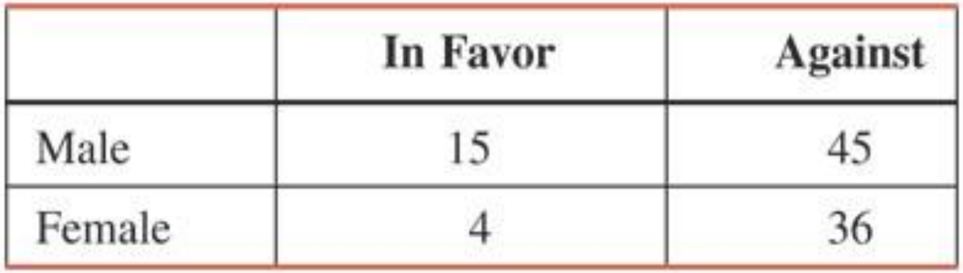
4.4 Marginal and Conditional Probabilities

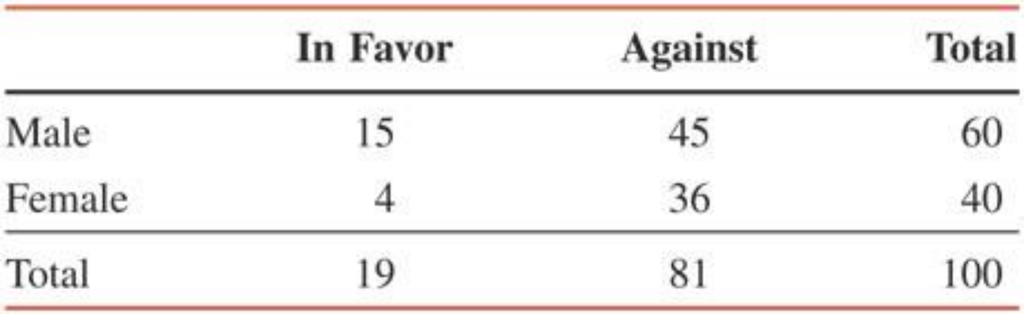
Suppose all 100 employees of a company were asked whether they are in favor of or against paying high salaries to CEOs of U.S. companies. Table 4.3 gives a two way classification of the responses of these 100 employees.

Table 4.3 Two-Way Classification of Employee Responses



	In Favor	Against
Male	15	45
Female	4	36

Table 4.4 Two-Way Classification of Employee Responses with Totals



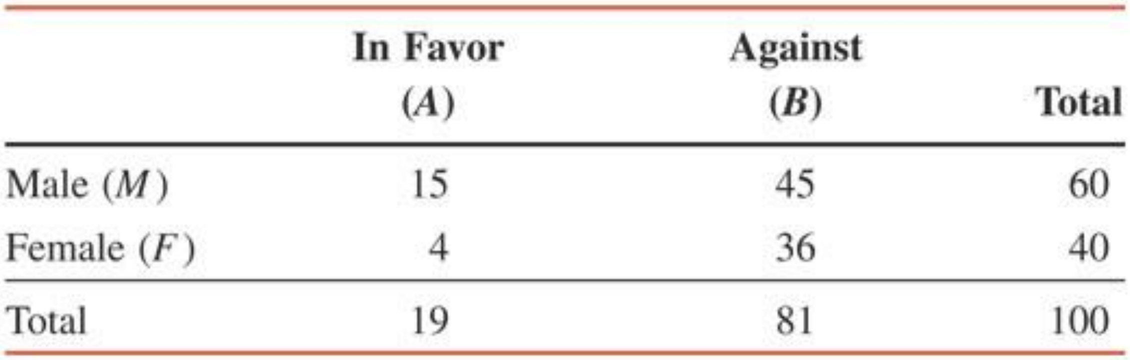
	In Favor	Against	Total
Male	15	45	60
Female	4	36	40
Total	19	81	100

MARGINAL AND CONDITIONAL PROBABILITIES

Definition

Marginal probability is the probability of a single event without consideration of any other event. Marginal probability is also called ***simple probability***.

Table 4.5 Listing the Marginal Probabilities



	In Favor (A)	Against (B)	Total
Male (M)	15	45	60
Female (F)	4	36	40
Total	19	81	100

$P(M) = 60/100 = .60$

$P(F) = 40/100 = .40$

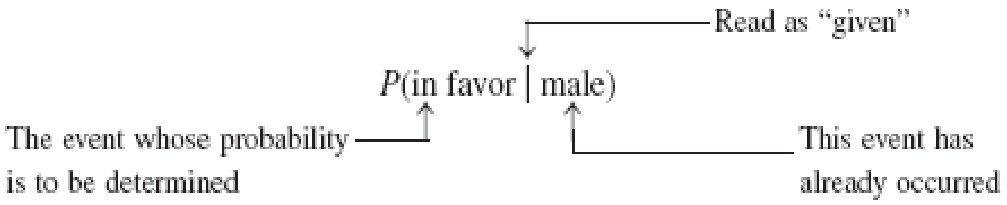
$P(A) = 19/100$

$= .19$

$P(B) = 81/100$

$= .81$

MARGINAL AND CONDITIONAL PROBABILITIES



$P(\text{in favor} \mid \text{male})$

Read as "given"

The event whose probability is to be determined

This event has already occurred

MARGINAL AND CONDITIONAL PROBABILITIES

Definition

Conditional probability is the probability that an event will occur given that another has already occurred. If *A* and *B* are two events, then the conditional probability *A* given *B* is written as

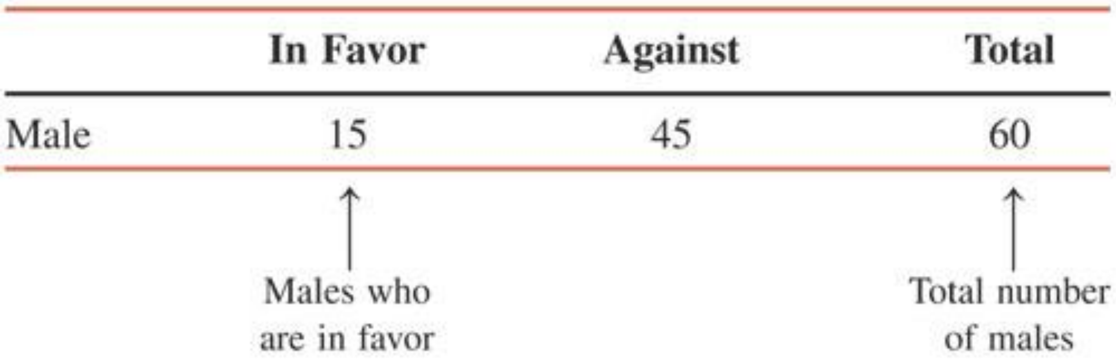
$$P(A \mid B)$$

and read as “the probability of *A* given that *B* has already occurred.”

Example 4-15

Compute the conditional probability *P* (in favor | male) for the data on 100 employees given in Table 4.4.

Example 4-15: Solution



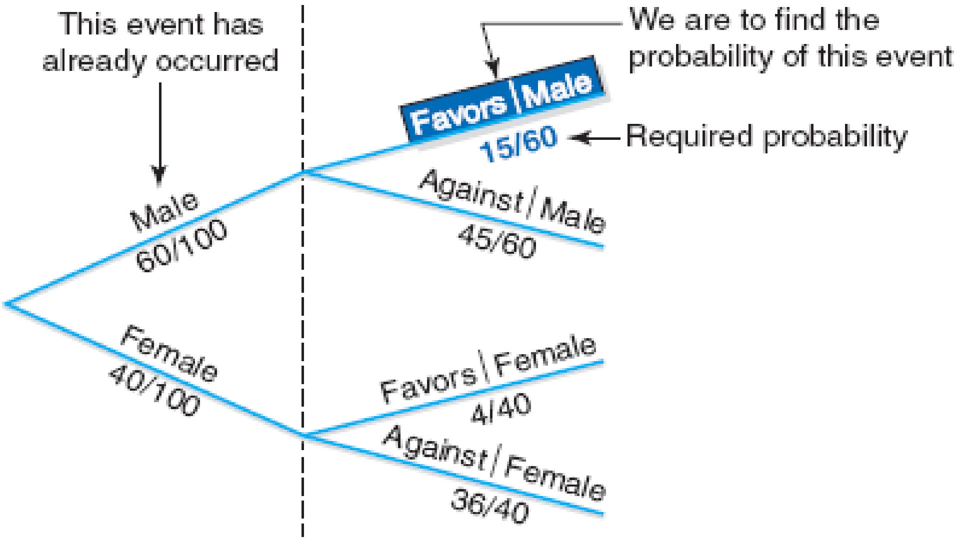
	In Favor	Against	Total
Male	15	45	60

Males who are in favor

Total number of males

$$P(\text{in favor} \mid \text{male}) = \frac{\text{Number of males who are in favor}}{\text{Total number of males}} = \frac{15}{60} = .25$$

Figure 4.6 Tree Diagram.

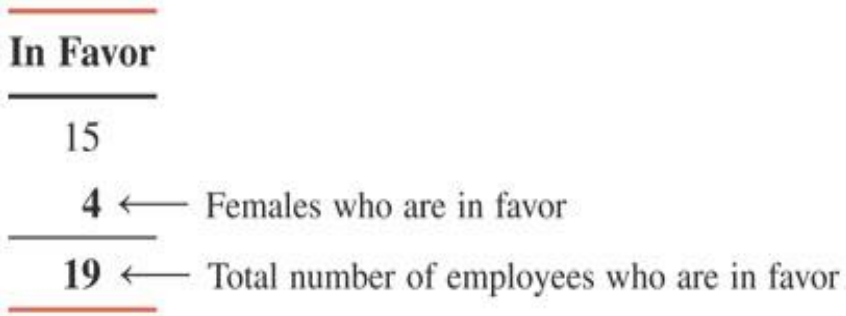


Prem Mann, *Introductory Statistics*, *7/E*

Example 4-16

For the data of Table 4.4, calculate the conditional probability that a randomly selected employee is a female given that this employee is in favor of paying high salaries to CEOs.

Example 4-16: Solution

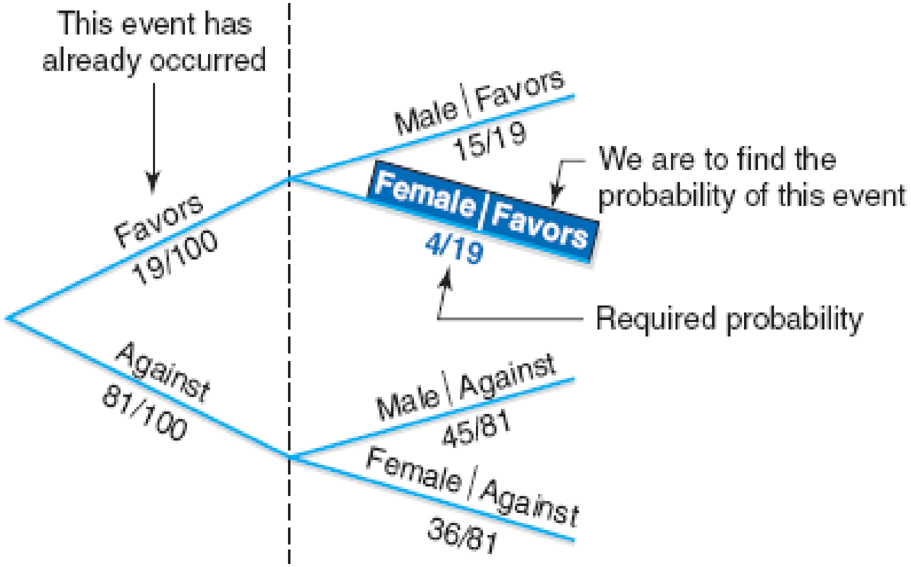


In Favor	
15	
4	← Females who are in favor
19	← Total number of employees who are in favor

$$P(\text{female} \mid \text{in favor}) = \frac{\text{Number of females who are in favor}}{\text{Total number of employees who are in favor}}$$

$$= \frac{4}{19} = .2105$$

Figure 4.7 Tree diagram.



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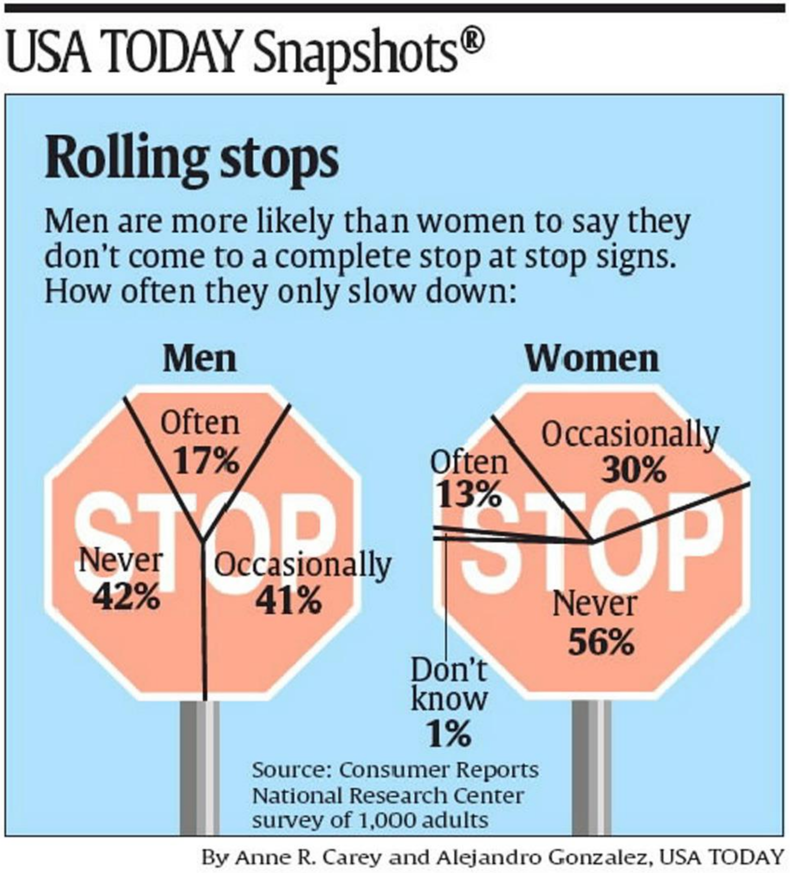
Case Study 4-1 Rolling Stops



USA TODAY Snapshots®

Rolling stops

Men are more likely than women to say they don't come to a complete stop at stop signs. How often they only slow down:

Source: Consumer Reports National Research Center survey of 1,000 adults

By Anne R. Carey and Alejandro Gonzalez, USA TODAY

4.5 Mutually Exclusive Events

Definition

Events that cannot occur together are said to be ***mutually exclusive events***.

Example 4-17

Consider the following events for one roll of a die:

A= an even number is observed= {2, 4, 6}

B= an odd number is observed= {1, 3, 5}

C= a number less than 5 is observed= {1, 2, 3, 4}

Are events *A* and B mutually exclusive? Are events *A* and *C* mutually exclusive?

Example 4-17: Solution

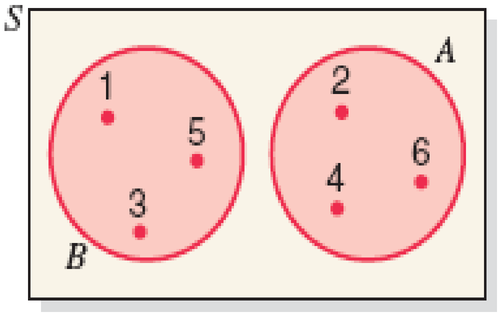


Figure 4.8 Mutually exclusive events *A* and *B*.

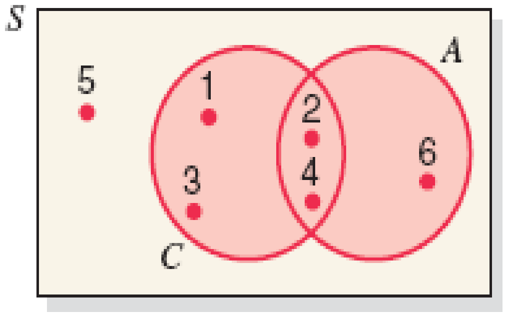


Figure 4.9 Mutually nonexclusive events *A* and *C*.

Example 4-18

Consider the following two events for a randomly selected adult:

Y = this adult has shopped on the Internet at least once

N = this adult has never shopped on the Internet

Are events Y and N mutually exclusive?

Example 4-18: Solution

As we can observe from the definitions of events Y and N and from Figure 4.10, events Y and N have no common outcome. Hence, these two events are mutually exclusive.

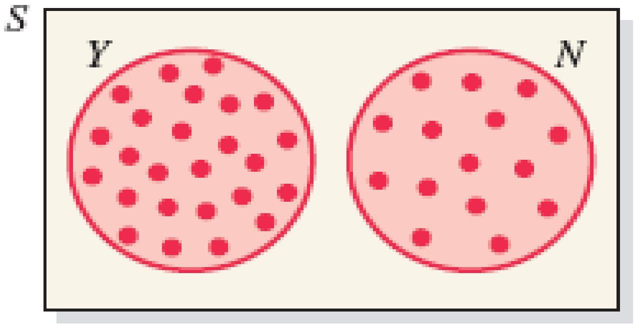


Figure 4.10 Mutually exclusive events *Y* and *N*.

4.6 Independent Versus Dependent Events

Definition

Two events are said to be ***independent*** if the occurrence of one does not affect the probability of the occurrence of the other. In other words, A and B are ***independent events*** if

either $P(A \mid B) = P(A)$ or $P(B \mid A) = P(B)$

Example 4-19

Refer to the information on 100 employees given in Table 4.4 in Section 4.4. Are events “female (*F*)” and “in favor (*A*)” independent?

Example 4-19: Solution

Events *F* and *A* will be independent if

$$P\,(F) = P\,(F \mid A)$$

Otherwise they will be dependent.

Using the information given in Table 4.4, compute

$P\,(F) = 40/100 = .40$ and
$P\,(F \mid A) = 4/19 = .2105$

Because these two probabilities are not equal, the two events are dependent.

Example 4-20

A box contains a total of 100 CDs that were manufactured on two machines. Of them, 60 were manufactured on Machine I. Of the total CDs, 15 are defective. Of the 60 CDs that were manufactured on Machine I, 9 are defective.
Let *D* be the event that a randomly selected CD is defective, and let *A* be the event that a randomly selected CD was manufactured on Machine I. Are events *D* and *A* independent?

Example 4-20: Solution

From the given information,

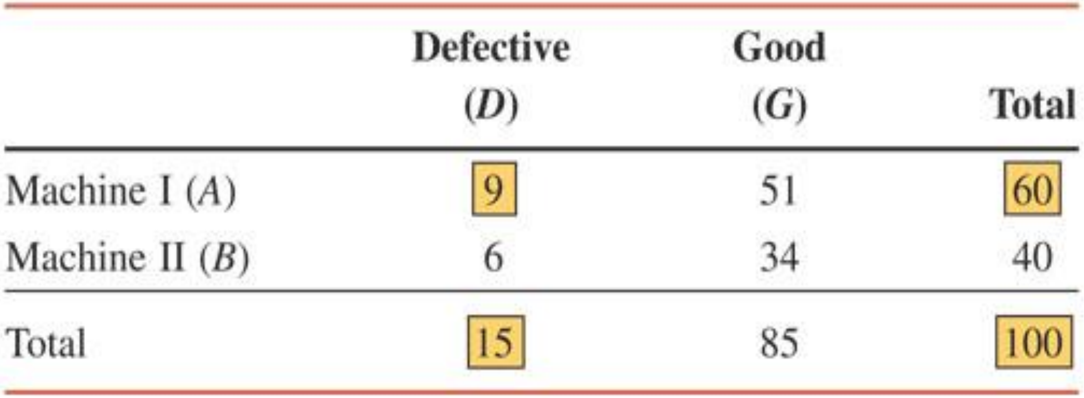
$$P\ (D) = 15/100 = .15 \quad \text{and}$$
$$P\ (D \mid A) = 9/60 = .15$$

Hence,

$$P\ (D) = P\ (D \mid A)$$

Consequently, the two events, D and A, are independent.

Table 4.6 Two-Way Classification Table



	Defective (*D*)	Good (*G*)	Total
Machine I (*A*)	9	51	60
Machine II (*B*)	6	34	40
Total	15	85	100

4.7 Complementary Events

Definition

The ***complement of event A***, denoted by $\bar{A}$ and is read as “*A* bar” or “*A* complement,” is the event that includes all the outcomes for an experiment that are not in *A*.

Figure 4.11 Venn diagram of two complementary events.

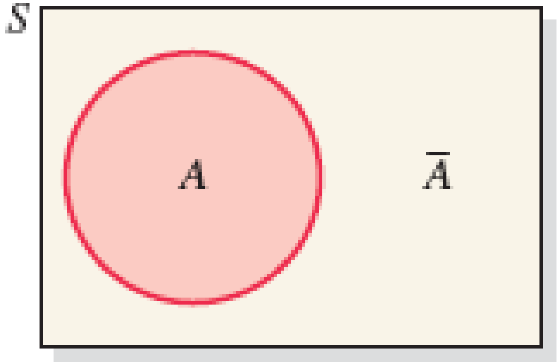


Figure 4.11 Venn diagram of two complementary events.

Example 4-21

In a group of 2000 taxpayers, 400 have been audited by the IRS at least once. If one taxpayer is randomly selected from this group, what are the two complementary events for this experiment, and what are their probabilities?

Example 4-21: Solution

The two complementary events for this experiment are

- A = the selected taxpayer has been audited by the IRS at least once
- $\bar{A}$ = the selected taxpayer has never been audited by the IRS

The probabilities of the complementary events are

$$P(A) = 400/2000 = .20 \quad \text{and}$$

$$P(\bar{A}) = 1600/2000 = .80$$

Figure 4.12 Venn diagram.

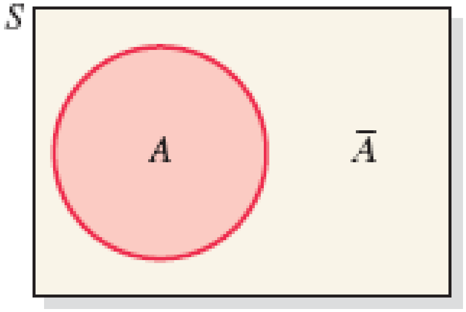


Figure 4.12 Venn diagram.

Example 4-22

In a group of 5000 adults, 3500 are in favor of stricter gun control laws, 1200 are against such laws, and 300 have no opinion. One adult is randomly selected from this group. Let A be the event that this adult is in favor of stricter gun control laws. What is the complementary event of A? What are the probabilities of the two events?

Example 4-22: Solution

The two complementary events for this experiment are

- A = the selected adult is in favor of stricter gun control laws
- $\bar{A}$ = the selected adult either is against such laws or has no opinion

The probabilities of the complementary events are

$$P(A) = 3500/5000 = .70 \quad \text{and}$$

$$P(\bar{A}) = 1500/5000 = .30$$

Figure 4.13 Venn diagram.

Figure 4.13 Venn diagram.

INTERSECTION OF EVENTS AND THE MULTIPLICATION RULE

Intersection of Events

Definition

Let *A* and *B* be two events defined in a sample space. The ***intersection*** of *A* and *B* represents the collection of all outcomes that are common to both *A* and *B* and is denoted by

A and *B*

Figure 4.14 Intersection of events *A* and *B*.

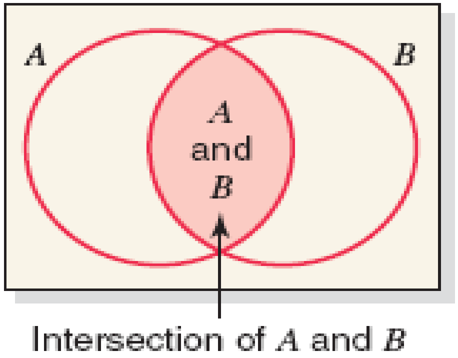


Figure 4.14 Intersection of events *A* and *B*.

INTERSECTION OF EVENTS AND THE MULTIPLICATION RULE

Multiplication Rule

Definition

The probability of the intersection of two events is called their ***joint probability***. It is written as

$$P(A \text{ and } B)$$

INTERSECTION OF EVENTS AND THE MULTIPLICATION RULE

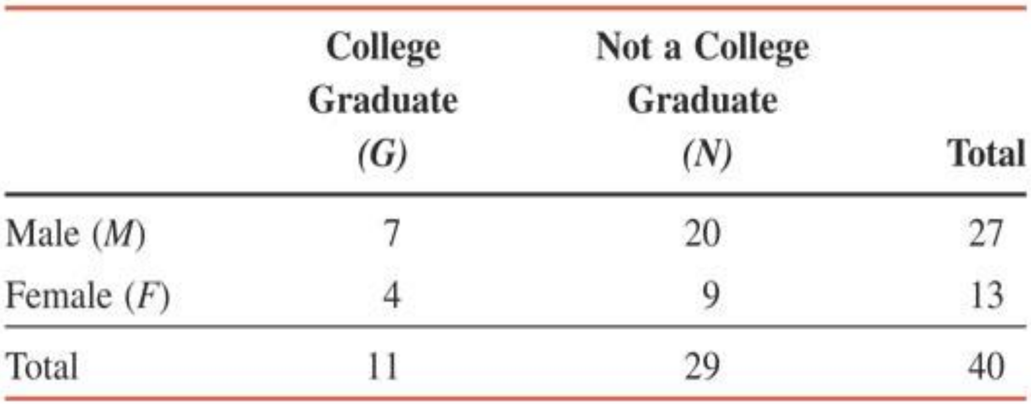
Multiplication Rule to Find Joint Probability

The probability of the intersection of two events A and B is

$$P(A \text{ and } B) = P(A)\, P(B\,|A)$$

Example 4-23

Table 4.7 gives the classification of all employees of a company given by gender and college degree.



	College Graduate (G)	Not a College Graduate (N)	Total
Male (M)	7	20	27
Female (F)	4	9	13
Total	11	29	40

Example 4-23

If one of these employees is selected at random for membership on the employee-management committee, what is the probability that this employee is a female and a college graduate?

Example 4-23: Solution

We are to calculate the probability of the intersection of the events *F and G.*

$P(F \text{ and } G) = P(F)\, P(G\,|F)$

$P(F) = 13/40$

$P(G\,|F) = 4/13$

$P(F \text{ and } G) = P(F)\, P(G\,|F)$

$= (13/40)(4/13) = .100$

Figure 4.15 Intersection of events *F* and *G*.

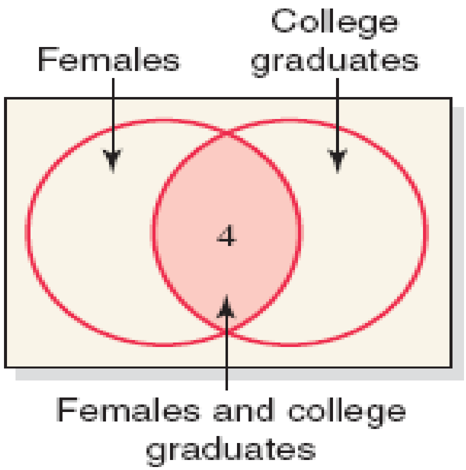
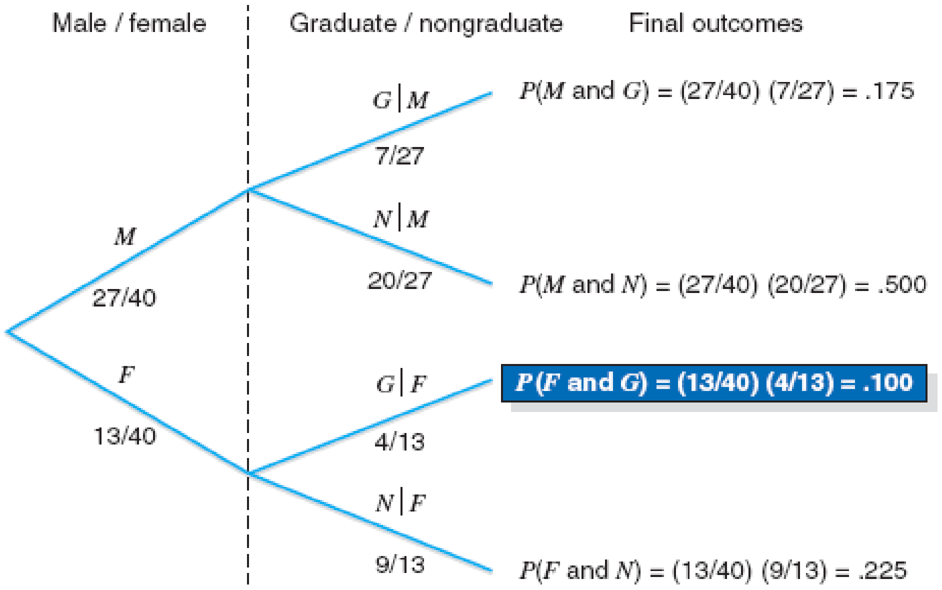


Figure 4.15 Intersection of events *F* and *G*.

Figure 4.16 Tree diagram for joint probabilities.



Example 4-24

A box contains 20 DVDs, 4 of which are defective. If two DVDs are selected at random (without replacement) from this box, what is the probability that both are defective?

Example 4-24: Solution

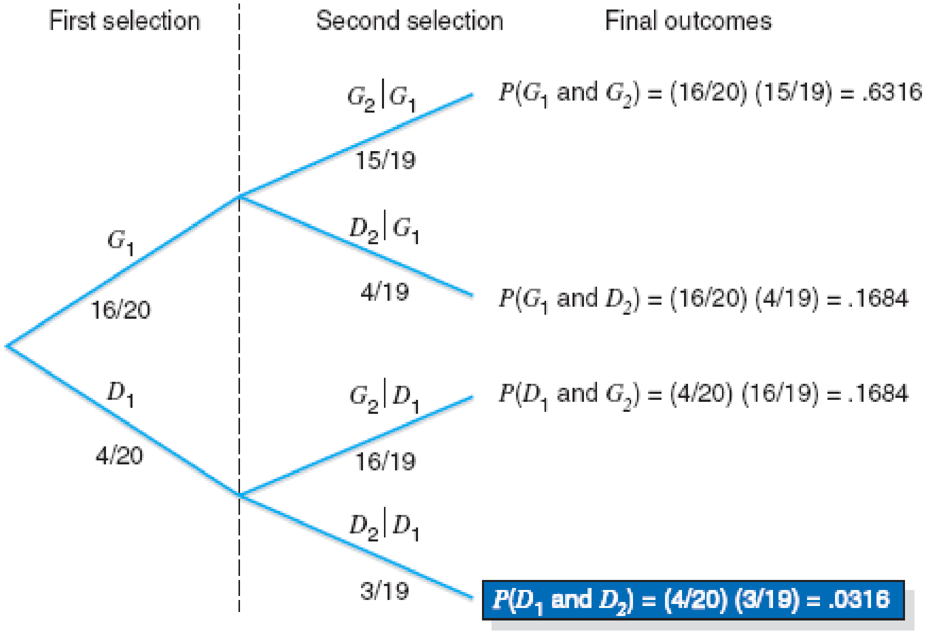
Let us define the following events for this experiment:

G_1 = event that the first DVD selected is good
D_1 = event that the first DVD selected is defective
G_2 = event that the second DVD selected is good
D_2 = event that the second DVD selected is defective

We are to calculate the joint probability of D_1 *and* D_2,

$P(D_1 \text{ and } D_2) = P(D_1)\, P(D_2 \mid D_1)$
$P(D_1) = 4/20$
$P(D_2 \mid D_1) = 3/19$
$P(D_1 \text{ and } D_2) = (4/20)(3/19) = .0316$

Figure 4.17 Selecting two DVDs.



Prem Mann, *Introductory Statistics*, *7/E*

INTERSECTION OF EVENTS AND THE MULTIPLICATION RULE

Calculating Conditional Probability

If *A* and *B* are two events, then,

$$P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)} \quad \text{and} \quad P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

given that $P(A) \neq 0$ and $P(B) \neq 0$.

Example 4-25

The probability that a randomly selected student from a college is a senior is .20, and the joint probability that the student is a computer science major and a senior is .03. Find the conditional probability that a student selected at random is a computer science major given that the student is a senior.

Example 4-25: Solution

Let us define the following two events:

- A = the student selected is a senior
- B = the student selected is a computer science major

From the given information,

$$P(A) = .20 \quad \text{and} \quad P(A \text{ and } B) = .03$$

Hence,

$$P\,(B \mid A) = P(A \text{ and } B)/P(A) = .03/.20 = .15$$

Multiplication Rule for Independent Events

Multiplication Rule to Calculate the Probability of Independent Events

The probability of the intersection of two independent events *A* and *B* is

$$P(A \text{ and } B) = P(A)\, P(B)$$

Example 4-26

An office building has two fire detectors. The probability is .02 that any fire detector of this type will fail to go off during a fire. Find the probability that both of these fire detectors will fail to go off in case of a fire.

Example 4-26: Solution

We define the following two events:

A = the first fire detector fails to go off during a fire

B = the second fire detector fails to go off during a fire

Then, the joint probability of A and B is

$$P(A \text{ and } B) = P(A)\,P(B) = (.02)(.02) = .0004$$

Example 4-27

The probability that a patient is allergic to penicillin is .20. Suppose this drug is administered to three patients.

a) Find the probability that all three of them are allergic to it.
b) Find the probability that at least one of the them is not allergic to it.

Example 4-27: Solution

a) Let *A*, *B*, and *C* denote the events the first, second and third patients, respectively, are allergic to penicillin. Hence,

$$P\ (A \text{ and } B \text{ and } C) = P(A)\ P(B)\ P(C)$$
$$= (.20)\ (.20)\ (.20) = .008$$

Example 4-27: Solution

b) Let us define the following events:

G = all three patients are allergic

H = at least one patient is not allergic

- $P(G) = P(A \text{ and } B \text{ and } C) = .008$
- Therefore, using the complementary event rule, we obtain
- $P(H) = 1 – P(G) = 1 - .008 = .992$

Figure 4.18 Tree diagram for joint probabilities.

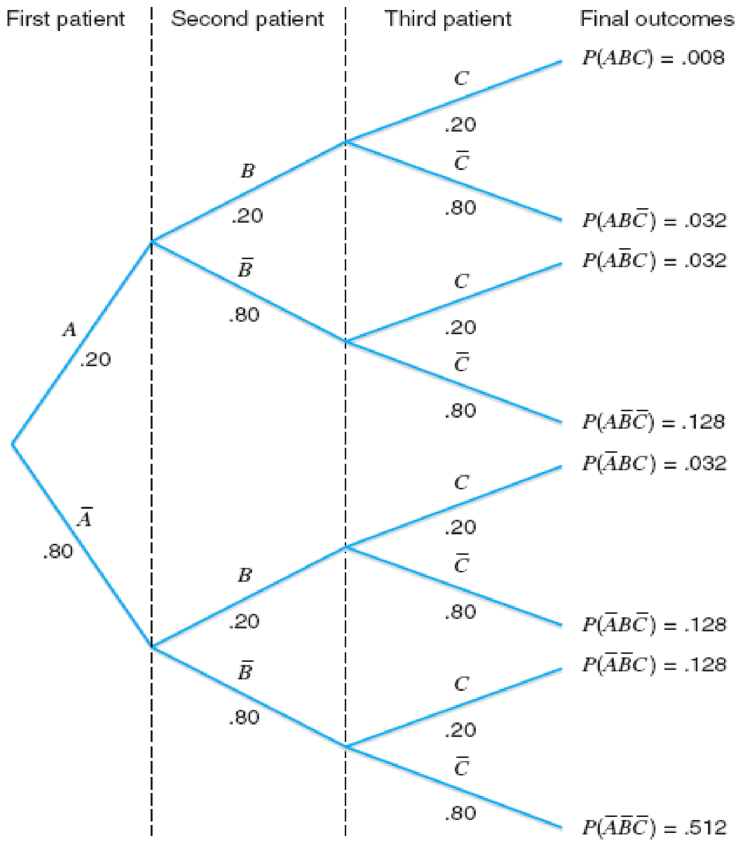


Figure 4.18 Tree diagram for joint probabilities.

Multiplication Rule for Independent Events

Joint Probability of Mutually Exclusive Events

The joint probability of two mutually exclusive events is always zero. If *A* and *B* are two mutually exclusive events, then

$$P(A \text{ and } B) = 0$$

Example 4-28

Consider the following two events for an application filed by a person to obtain a car loan:

A = event that the loan application is approved

R = event that the loan application is rejected

What is the joint probability of A and R?

Example 4-28: Solution

The two events A and R are mutually exclusive. Either the loan application will be approved or it will be rejected. Hence,

$$P(A \text{ and } R) = \mathbf{0}$$

4.9 Union of Events and the Addition Rule

Definition

Let *A* and *B* be two events defined in a sample space. The ***union of events*** *A* and *B* is the collection of all outcomes that belong to either *A* or *B* or to both *A* and *B* and is denoted by

$$A \text{ or } B$$

Example 4-29

A senior citizen center has 300 members. Of them, 140 are male, 210 take at least one medicine on a permanent basis, and 95 are male *and* take at least one medicine on a permanent basis. Describe the union of the events “male” and “take at least one medicine on a permanent basis.”

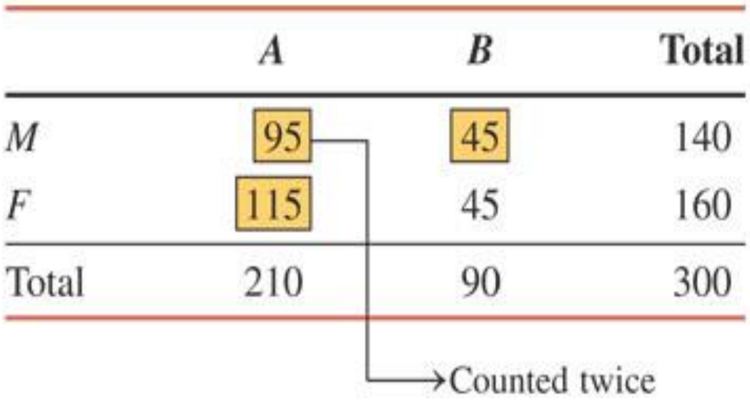
Example 4-29: Solution

- Let us define the following events:
  M = a senior citizen is a male
  F = a senior citizen is a female
  A = a senior citizen takes at least one medicine
  B = a senior citizen does not take any medicine

- The union of the events “male” and “take at least one medicine” includes those senior citizens who are either male or take at least one medicine or both. The number of such senior citizen is

$$140 + 210 – 95 = 255$$

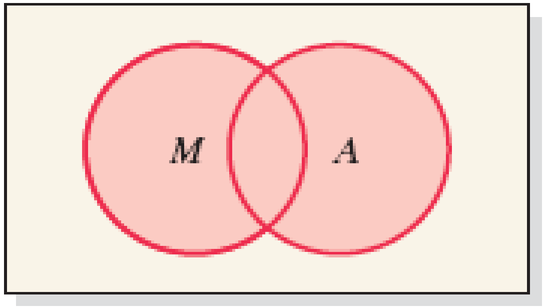
Table 4.8



	A	*B*	**Total**
M	95	45	140
F	115	45	160
Total	210	90	300

Counted twice

Figure 4.19 Union of events *M* and *A*.



Area shaded in red gives the union of events *M* and *A*, and includes 255 senior citizens

UNION OF EVENTS AND THE ADDITION RULE

Addition Rule

Addition Rule to Find the Probability of Union of Events

The portability of the union of two events A and B is

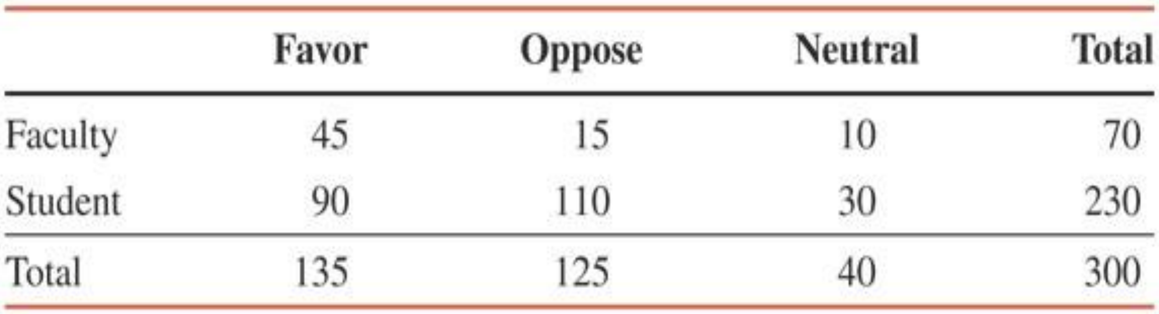
$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Example 4-30

A university president has proposed that all students must take a course in ethics as a requirement for graduation. Three hundred faculty members and students from this university were asked about their opinion on this issue. Table 4.9 gives a two-way classification of the responses of these faculty members and students.

Find the probability that one person selected at random from these 300 persons is a faculty member or is in favor of this proposal.

Table 4.9 Two-Way Classification of Responses



	Favor	Oppose	Neutral	Total
Faculty	45	15	10	70
Student	90	110	30	230
Total	135	125	40	300

Example 4-30: Solution

Let us define the following events:

A = the person selected is a faculty member
B = the person selected is in favor of the proposal

From the information in the Table 4.9,

$P(A) = 70/300 = .2333$
$P(B) = 135/300 = .4500$
$P(\text{A and B}) = P(A)\ P(B \mid A) = (70/300)(45/70) = .1500$

Using the addition rule, we obtain

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$
$$= .2333 + .4500 - .1500 = \mathbf{.5333}$$

Prem Mann, *Introductory Statistics*, *7/E*

Example 4-31

In a group of 2500 persons, 1400 are female, 600 are vegetarian, and 400 are female and vegetarian. What is the probability that a randomly selected person from this group is a male or vegetarian?

Example 4-31: Solution

Let us define the following events:

F = the randomly selected person is a female

M = the randomly selected person is a male

V = the randomly selected person is a vegetarian

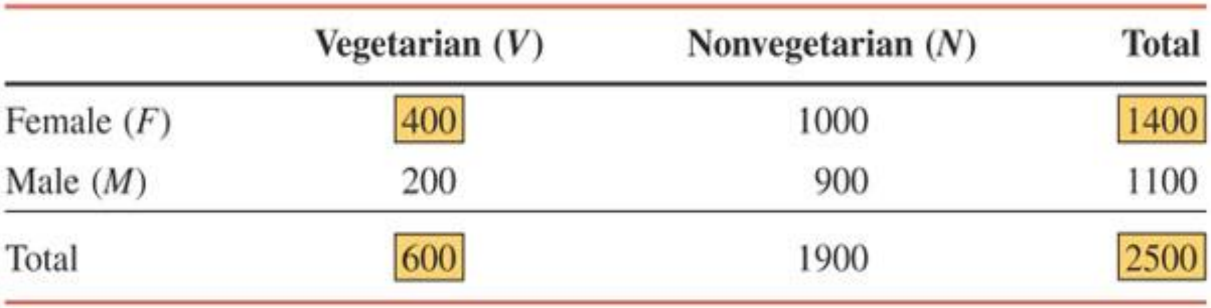
N = the randomly selected person is a non-vegetarian.

$$P(M \text{ or } V) = P(M) + P(V) - P(M \text{ and } V)$$

$$= \frac{1100}{2500} + \frac{600}{2500} - \frac{200}{2500}$$

$$= .44 + .24 - .08 = .60$$

Table 4.10 Two-Way Classification Table



	Vegetarian (V)	**Nonvegetarian** (N)	**Total**
Female (F)	400	1000	1400
Male (M)	200	900	1100
Total	600	1900	2500

Addition Rule for Mutually Exclusive Events

Addition Rule to Find the Probability of the Union of Mutually Exclusive Events

The probability of the union of two mutually exclusive events A and B is

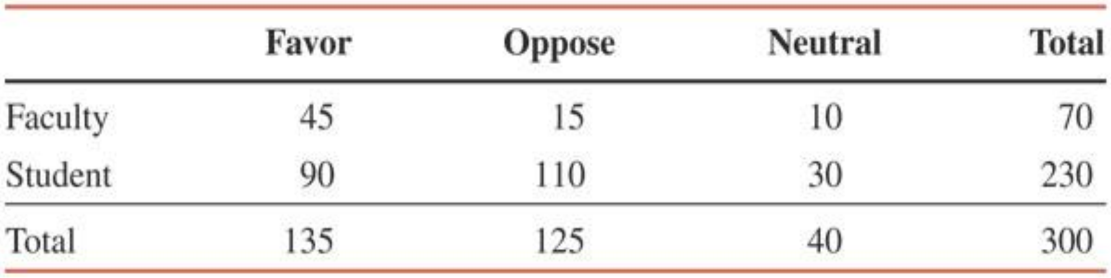
$$P(A \text{ or } B) = P(A) + P(B)$$

Example 4-32

A university president has proposed that all students must take a course in ethics as a requirement for graduation. Three hundred faculty members and students from this university were asked about their opinion on this issue. The following table, reproduced from Table 4.9 in Example 4-30, gives a two-way classification of the responses of these faculty members and students.

What is the probability that a randomly selected person from these 300 faculty members and students is in favor of the proposal or is neutral?

Example 4-32: Solution



	Favor	Oppose	Neutral	Total
Faculty	45	15	10	70
Student	90	110	30	230
Total	135	125	40	300

Example 4-32: Solution

Let us define the following events:

F = the person selected is in favor of the proposal

N = the person selected is neutral

From the given information,

$P(F) = 135/300 = .4500$

$P(N) = 40/300 = .1333$

Hence,

$P(F \text{ or } N) = P(F) + P(N) = .4500 + .1333 = \mathbf{.5833}$

Figure 4.20 Venn diagram of mutually exclusive events.

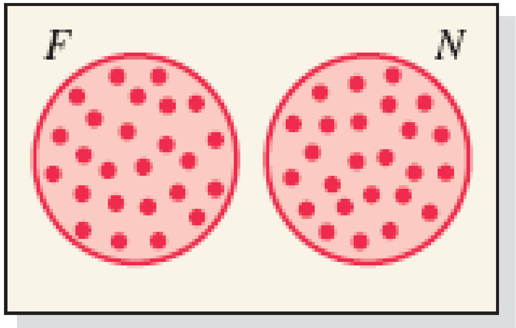


Figure 4.20 Venn diagram of mutually exclusive events.

Example 4-33

Consider the experiment of rolling a die twice. Find the probability that the sum of the numbers obtained on two rolls is 5, 7, or 10.

Table 4.11 Two Rolls of a Die

		Second Roll of the Die					
		1	**2**	**3**	**4**	**5**	**6**
First Roll of the Die	1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
	2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
	3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
	4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
	5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
	6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Example 4-33: Solution

P(sum is 5 or 7 or 10)

= P(sum is 5) + P(sum is 7) + P(sum is 10)

= 4/36 + 6/36 + 3/36 = 13/36 = **.3611**

Example 4-34

The probability that a person is in favor of genetic engineering is .55 and that a person is against it is .45. Two persons are randomly selected, and it is observed whether they favor or oppose genetic engineering.

a) Draw a tree diagram for this experiment
b) Find the probability that at least one of the two persons favors genetic engineering.

Example 4-34: Solution

a) Let

F = a person is in favor of genetic engineering

A = a person is against genetic engineering

This experiment has four outcomes. The tree diagram in Figure 4.21 shows these four outcomes and their probabilities.

Figure 4.21 Tree diagram.

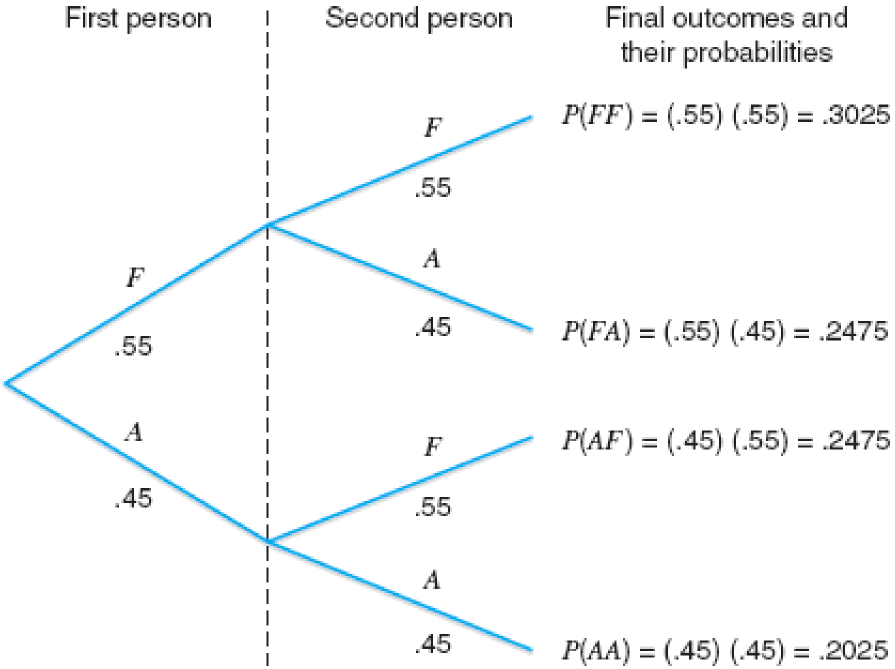


Figure 4.21 Tree diagram.

Example 4-34: Solution

b) $P(\text{at least one person favors})$

$= P(FF \text{ or } FA \text{ or } AF)$

$= P(FF) + P(FA) + P(AF)$

$= .3025 + .2475 + .2475 =$ **.7975**

2.7 Bayes' Rule

Bayesian statistics is a collection of tools that is used in a special form of statistical inference which applies in the analysis of experimental data in many practical situations in science and engineering. Bayes' rule is one of the most important rules in probability theory. It is the foundation of Bayesian inference, which will be discussed in Chapter 18.

Total Probability

Let us now return to the illustration of Section 2.6, where an individual is being selected at random from the adults of a small town to tour the country and publicize the advantages of establishing new industries in the town. Suppose that we are now given the additional information that 36 of those employed and 12 of those unemployed are members of the Rotary Club. We wish to find the probability of the event A that the individual selected is a member of the Rotary Club. Referring to Figure 2.12, we can write A as the union of the two mutually exclusive events $E \cap A$ and $E' \cap A$. Hence, $A = (E \cap A) \cup (E' \cap A)$, and by Corollary 2.1 of Theorem 2.7, and then Theorem 2.10, we can write

$$\begin{aligned} P(A) &= P[(E \cap A) \cup (E' \cap A)] = P(E \cap A) + P(E' \cap A) \\ &= P(E)P(A|E) + P(E')P(A|E'). \end{aligned}$$

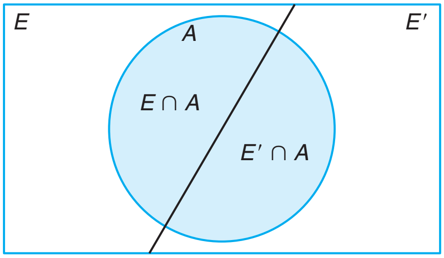


Figure 2.12: Venn diagram for the events A, E, and E'.

The data of Section 2.6, together with the additional data given above for the set A, enable us to compute

$$P(E) = \frac{600}{900} = \frac{2}{3}, \quad P(A|E) = \frac{36}{600} = \frac{3}{50},$$

and

$$P(E') = \frac{1}{3}, \quad P(A|E') = \frac{12}{300} = \frac{1}{25}.$$

If we display these probabilities by means of the tree diagram of Figure 2.13, where the first branch yields the probability $P(E)P(A|E)$ and the second branch yields

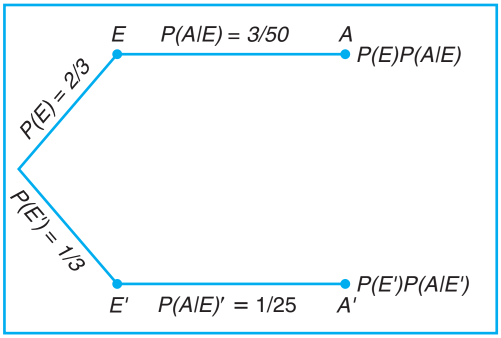


Figure 2.13: Tree diagram for the data on page 63, using additional information on page 72.

the probability $P(E')P(A|E')$, it follows that

$$P(A) = \left(\frac{2}{3}\right)\left(\frac{3}{50}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{25}\right) = \frac{4}{75}.$$

A generalization of the foregoing illustration to the case where the sample space is partitioned into k subsets is covered by the following theorem, sometimes called the **theorem of total probability** or the **rule of elimination**.

Theorem 2.13: If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S such that $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event A of S,

$$P(A) = \sum_{i=1}^{k} P(B_i \cap A) = \sum_{i=1}^{k} P(B_i)P(A|B_i).$$

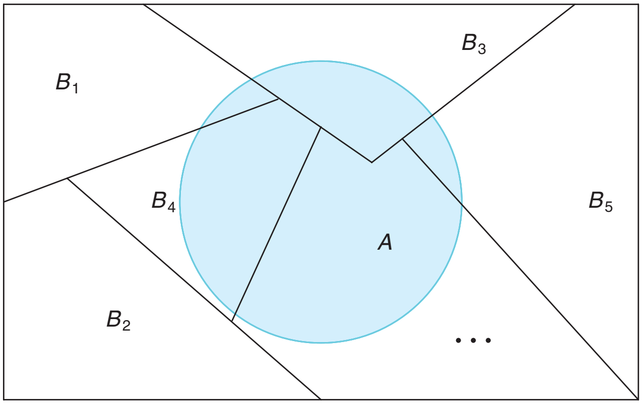


Figure 2.14: Partitioning the sample space S.

***Proof*:** Consider the Venn diagram of Figure 2.14. The event A is seen to be the union of the mutually exclusive events

$$B_1 \cap A,\ B_2 \cap A,\ \ldots,\ B_k \cap A;$$

that is,

$$A = (B_1 \cap A) \cup (B_2 \cap A) \cup \cdots \cup (B_k \cap A).$$

Using Corollary 2.2 of Theorem 2.7 and Theorem 2.10, we have

$$\begin{aligned} P(A) &= P[(B_1 \cap A) \cup (B_2 \cap A) \cup \cdots \cup (B_k \cap A)] \\ &= P(B_1 \cap A) + P(B_2 \cap A) + \cdots + P(B_k \cap A) \\ &= \sum_{i=1}^{k} P(B_i \cap A) \\ &= \sum_{i=1}^{k} P(B_i)P(A|B_i). \end{aligned}$$

Example 2.41: In a certain assembly plant, three machines, B_1, B_2, and B_3, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. Now, suppose that a finished product is randomly selected. What is the probability that it is defective?

***Solution*:** Consider the following events:

A: the product is defective,

B_1: the product is made by machine B_1,

B_2: the product is made by machine B_2,

B_3: the product is made by machine B_3.

Applying the rule of elimination, we can write

$$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3).$$

Referring to the tree diagram of Figure 2.15, we find that the three branches give the probabilities

$$\begin{aligned} P(B_1)P(A|B_1) &= (0.3)(0.02) = 0.006, \\ P(B_2)P(A|B_2) &= (0.45)(0.03) = 0.0135, \\ P(B_3)P(A|B_3) &= (0.25)(0.02) = 0.005, \end{aligned}$$

and hence

$$P(A) = 0.006 + 0.0135 + 0.005 = 0.0245.$$

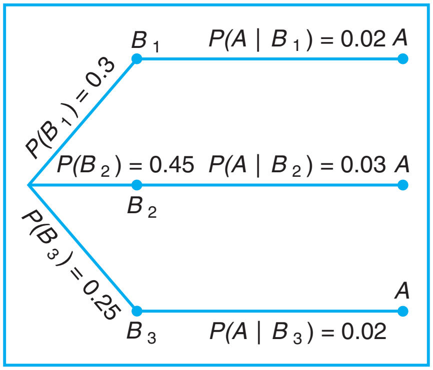


Figure 2.15: Tree diagram for Example 2.41.

Bayes' Rule

Instead of asking for $P(A)$ in Example 2.41, by the rule of elimination, suppose that we now consider the problem of finding the conditional probability $P(B_i|A)$. In other words, suppose that a product was randomly selected and it is defective. What is the probability that this product was made by machine B_i? Questions of this type can be answered by using the following theorem, called **Bayes' rule**:

Theorem 2.14: **(Bayes' Rule)** If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S such that $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event A in S such that $P(A) \neq 0$,

$$P(B_r|A) = \frac{P(B_r \cap A)}{\sum\limits_{i=1}^{k} P(B_i \cap A)} = \frac{P(B_r)P(A|B_r)}{\sum\limits_{i=1}^{k} P(B_i)P(A|B_i)} \quad \text{for } r = 1, 2, \ldots, k.$$

***Proof*:** By the definition of conditional probability,

$$P(B_r|A) = \frac{P(B_r \cap A)}{P(A)},$$

and then using Theorem 2.13 in the denominator, we have

$$P(B_r|A) = \frac{P(B_r \cap A)}{\sum\limits_{i=1}^{k} P(B_i \cap A)} = \frac{P(B_r)P(A|B_r)}{\sum\limits_{i=1}^{k} P(B_i)P(A|B_i)},$$

which completes the proof. ▟

Example 2.42: With reference to Example 2.41, if a product was chosen randomly and found to be defective, what is the probability that it was made by machine B_3?

***Solution*:** Using Bayes' rule to write

$$P(B_3|A) = \frac{P(B_3)P(A|B_3)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3)},$$

and then substituting the probabilities calculated in Example 2.41, we have

$$P(B_3|A) = \frac{0.005}{0.006 + 0.0135 + 0.005} = \frac{0.005}{0.0245} = \frac{10}{49}.$$

In view of the fact that a defective product was selected, this result suggests that it probably was not made by machine B_3. ▟

Example 2.43: A manufacturing firm employs three analytical plans for the design and development of a particular product. For cost reasons, all three are used at varying times. In fact, plans 1, 2, and 3 are used for 30%, 20%, and 50% of the products, respectively. The defect rate is different for the three procedures as follows:

$$P(D|P_1) = 0.01, \qquad P(D|P_2) = 0.03, \qquad P(D|P_3) = 0.02,$$

where $P(D|P_j)$ is the probability of a defective product, given plan j. If a random product was observed and found to be defective, which plan was most likely used and thus responsible?

***Solution*:** From the statement of the problem

$$P(P_1) = 0.30, \quad P(P_2) = 0.20, \quad \text{and} \quad P(P_3) = 0.50,$$

we must find $P(P_j|D)$ for $j = 1, 2, 3$. Bayes' rule (Theorem 2.14) shows

$$\begin{aligned} P(P_1|D) &= \frac{P(P_1)P(D|P_1)}{P(P_1)P(D|P_1) + P(P_2)P(D|P_2) + P(P_3)P(D|P_3)} \\ &= \frac{(0.30)(0.01)}{(0.3)(0.01) + (0.20)(0.03) + (0.50)(0.02)} = \frac{0.003}{0.019} = 0.158. \end{aligned}$$

Similarly,

$$P(P_2|D) = \frac{(0.03)(0.20)}{0.019} = 0.316 \text{ and } P(P_3|D) = \frac{(0.02)(0.50)}{0.019} = 0.526.$$

The conditional probability of a defect given plan 3 is the largest of the three; thus a defective for a random product is most likely the result of the use of plan 3. ▟

Using Bayes' rule, a statistical methodology called the Bayesian approach has attracted a lot of attention in applications. An introduction to the Bayesian method will be discussed in Chapter 18.

Exercises

2.95 In a certain region of the country it is known from past experience that the probability of selecting an adult over 40 years of age with cancer is 0.05. If the probability of a doctor correctly diagnosing a person with cancer as having the disease is 0.78 and the probability of incorrectly diagnosing a person without cancer as having the disease is 0.06, what is the probability that an adult over 40 years of age is diagnosed as having cancer?

2.96 Police plan to enforce speed limits by using radar traps at four different locations within the city limits. The radar traps at each of the locations L_1, L_2, L_3, and L_4 will be operated 40%, 30%, 20%, and 30% of