Markov Chain

Definition: A Markov chain is a stochastic process that evolves over time through a sequence of states, where the transition from one state to another is probabilistic and follows the Markov property. In a Markov chain, the future state depends only on the current state and is independent of the past states. This means that the probability of transitioning to the next state is determined solely by the current state and not by the sequence of events leading up to it.

Example and Explanation

- Let's illustrate this with an example:
- Suppose we have a simple weather model with three states: rainy (R), cloudy (C), and sunny (S). The transition probabilities are as follows:
- If it is rainy today, there is a 0.6 probability of rain tomorrow, 0.3 probability of cloudy weather, and 0.1 probability of sun tomorrow.
- If it is cloudy today, there is a 0.2 probability of rain tomorrow, 0.5 probability of remaining cloudy, and 0.3 probability of sun tomorrow.
- If it is sunny today, there is a 0.4 probability of rain tomorrow, 0.2 probability of becoming cloudy, and 0.4 probability of sun tomorrow.

We represent these transition probabilities using a transition matrix. Let's denote the matrix P where P_{ij} represents the probability of transitioning from state i to state j. The transition matrix for this weather model is:

possibility of the next event happening

$$P = \begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.4 & 0.2 & 0.4 \end{pmatrix}$$

current state

To iterate the transition matrix and determine the weather probabilities over multiple days, we start with an initial probability distribution vector representing the weather probabilities for the first day. For example, if on the first day, the probabilities of rainy, cloudy, and sunny weather are 0.3, 0.4, and 0.3 respectively, we represent this as:

$$\mathbf{v}_1 = \begin{pmatrix} 0.3 \\ 0.4 \\ 0.3 \end{pmatrix}$$

To determine the weather probabilities for the next day ($\mathbf{v}_2$), we multiply the initial probability distribution vector ($\mathbf{v}_1$) by the transition matrix (P):

$$\mathbf{v}_2 = \mathbf{v}_1 \cdot P$$

Similarly, for subsequent days, we continue to multiply the previous day's weather probabilities ($\mathbf{v}_n$) by the transition matrix (P) to obtain the weather probabilities for the next day ($\mathbf{v}_{n+1}$):

$$\mathbf{v}_{n+1} = \mathbf{v}_n \cdot P$$

This iterative process allows us to predict the weather probabilities for each day based on the initial probabilities and the transition dynamics described by the transition matrix. Over time, the weather probabilities will stabilize, reaching a steady-state distribution that represents the long-term behavior of the weather model.

CHAPTER 16

Markov Chains

Chapter 15 focused on decision making in the face of uncertainty about *one* future event (learning the true state of nature). However, some decisions need to take into account uncertainty about *many* future events. We now begin laying the groundwork for decision making in this broader context.

In particular, this chapter presents probability models for processes that *evolve over time* in a probabilistic manner. Such processes are called *stochastic processes.* After briefly introducing general stochastic processes in the first section, the remainder of the chapter focuses on a special kind called a *Markov chain.* Markov chains have the special property that probabilities involving how the process will evolve in the future depend only on the present state of the process, and so are independent of events in the past. Many processes fit this description, so Markov chains provide an especially important kind of probability model.

For example, you will see in the next chapter that *continuous-time Markov chains* (described in Sec. 16.8) are used to formulate most of the basic models of *queueing theory*. Markov chains also provide the foundation for the study of *Markov decision models* in Chapter 19. There are a wide variety of other applications of Markov chains as well. A considerable number of books and articles present some of these applications. One is Selected Reference 4, which describes applications in such diverse areas as the classification of customers, DNA sequencing, the analysis of genetic networks, the estimation of sales demand over time, and credit rating. You also will see an application vignette in Sec. 16.2 that involves credit rating, as well as an application vignette in Sec. 16.8 that involves machine maintenance. Selected Reference 6 focuses on applications in finance and Selected Reference 3 describes applications for analyzing baseball strategy. The list goes on and on, but let us turn now to a description of stochastic processes in general and Markov chains in particular.

16.1 STOCHASTIC PROCESSES

A **stochastic process** is defined to be an indexed collection of random variables $\{X_t\}$, where the index t runs through a given set T. Often T is taken to be the set of nonnegative integers, and X_t represents a measurable characteristic of interest at time t. For example, X_t might represent the inventory level of a particular product at the end of week t.

Stochastic processes are of interest for describing the behavior of a system operating over some period of time. A stochastic process often has the following structure.

> The current status of the system can fall into any one of $M + 1$ mutually exclusive categories called **states.** For notational convenience, these states are labeled $0, 1, \ldots, M$. The random variable X_t represents the *state of the system* at time t, so its only possible values are $0, 1, \ldots, M$. The system is observed at particular points of time, labeled $t = 0, 1, 2, \ldots$. Thus, the stochastic process $\{X_t\} = \{X_0, X_1, X_2, \ldots\}$ provides a mathematical representation of how the status of the physical system evolves over time.

This kind of process is referred to as being a *discrete time* stochastic process with a *finite state space.* Except for Sec. 16.8, this will be the only kind of stochastic process considered in this chapter. (Section 16.8 describes a certain *continuous time* stochastic process.)

A Weather Example

The weather in the town of Centerville can change rather quickly from day to day. However, the chances of being dry (no rain) tomorrow are somewhat larger if it is dry today than if it rains today. In particular, the probability of being dry tomorrow is **0.8** if it is dry today, but is only **0.6** if it rains today. These probabilities do not change if information about the weather before today is also taken into account.

The evolution of the weather from day to day in Centerville is a stochastic process. Starting on some initial day (labeled as day 0), the weather is observed on each day t, for $t = 0, 1, 2, \ldots$. The state of the system on day t can be either

State 0 = Day t is dry

or

State 1 = Day t has rain.

Thus, for $t = 0, 1, 2, \ldots$, the random variable X_t takes on the values,

$$X_t = \begin{cases} 0 & \text{if day } t \text{ is dry} \\ 1 & \text{if day } t \text{ has rain.} \end{cases}$$

The stochastic process $\{X_t\} = \{X_0, X_1, X_2, \ldots\}$ provides a mathematical representation of how the status of the weather in Centerville evolves over time.

An Inventory Example

Dave's Photography Store has the following inventory problem. The store stocks a particular model camera that can be ordered weekly. Let $D_1, D_2, \ldots$ represent the *demand* for this camera (the number of units that would be sold if the inventory is not depleted) during the first week, second week, . . . , respectively, so the random variable D_t (for $t = 1, 2, \ldots$) is

D_t = number of cameras that would be sold in week t if the inventory is not depleted. (This number includes lost sales when the inventory is depleted.)

It is assumed that the D_t are independent and identically distributed random variables having a *Poisson distribution* with a mean of 1. Let X_0 represent the number of cameras on hand at the outset, X_1 the number of cameras on hand at the end of week 1, X_2 the number of cameras on hand at the end of week 2, and so on, so the random variable X_t (for $t = 0, 1, 2, \ldots$) is

X_t = number of cameras on hand at the end of week t.

Assume that $X_0 = 3$, so that week 1 begins with three cameras on hand.

$$\{X_t\} = \{X_0, X_1, X_2, \ldots\}$$

is a stochastic process where the random variable X_t represents the state of the system at time t, namely,

State at time t = number of cameras on hand at the end of week t.

As the owner of the store, Dave would like to learn more about how the status of this stochastic process evolves over time while using the current ordering policy described below.

At the end of each week t (Saturday night), the store places an order that is delivered in time for the next opening of the store on Monday. The store uses the following order policy:

If $X_t = 0$, order 3 cameras.
If $X_t > 0$, do not order any cameras.

Thus, the inventory level fluctuates between a minimum of zero cameras and a maximum of three cameras, so the possible states of the system at time t (the end of week t) are

Possible states = 0, 1, 2, or 3 cameras on hand.

Since each random variable X_t ($t = 0, 1, 2, \ldots$) represents the state of the system at the end of week t, its only possible values are 0, 1, 2, or 3. The random variables X_t are dependent and may be evaluated iteratively by the expression

$$X_{t+1} = \begin{cases} \max\{3 - D_{t+1}, 0\} & \text{if} \quad X_t = 0 \\ \max\{X_t - D_{t+1}, 0\} & \text{if} \quad X_t \geq 1, \end{cases}$$

for $t = 0, 1, 2, \ldots$.

These examples are used for illustrative purposes throughout many of the following sections. Section 16.2 further defines the particular type of stochastic process considered in this chapter.

16.2 MARKOV CHAINS

Assumptions regarding the joint distribution of $X_0, X_1, \ldots$ are necessary to obtain analytical results. One assumption that leads to analytical tractability is that the stochastic process is a Markov chain, which has the following key property:

A stochastic process $\{X_t\}$ is said to have the **Markovian property** if $P\{X_{t+1} = j \mid X_0 = k_0, X_1 = k_1, \ldots, X_{t-1} = k_{t-1}, X_t = i\} = P\{X_{t+1} = j \mid X_t = i\}$, for $t = 0, 1, \ldots$ and every sequence $i, j, k_0, k_1, \ldots, k_{t-1}$.

In words, this Markovian property says that the conditional probability of any future "event," given any past "events" and the present state $X_t = i$, is *independent* of the past events and depends only upon the present state.

A stochastic process $\{X_t\}$ ($t = 0, 1, \ldots$) is a **Markov chain** if it has the *Markovian property.*

The conditional probabilities $P\{X_{t+1} = j \mid X_t = i\}$ for a Markov chain are called (one-step) **transition probabilities.** If, for each i and j,

$$P\{X_{t+1} = j \mid X_t = i\} = P\{X_1 = j \mid X_0 = i\}, \qquad \text{for all } t = 1, 2, \ldots,$$

then the (one-step) transition probabilities are said to be *stationary.* Thus, having **stationary transition probabilities** implies that the transition probabilities do not change over time. The existence of stationary (one-step) transition probabilities also implies that, for each i, j, and n ($n = 0, 1, 2, \ldots$),

$$P\{X_{t+n} = j \mid X_t = i\} = P\{X_n = j \mid X_0 = i\}$$

for all $t = 0, 1, \ldots$. These conditional probabilities are called ***n*-step transition probabilities.**

To simplify notation with stationary transition probabilities, let

$$p_{ij} = P\{X_{t+1} = j \mid X_t = i\},$$
$$p_{ij}^{(n)} = P\{X_{t+n} = j \mid X_t = i\}.$$

Thus, the n-step transition probability $p_{ij}^{(n)}$ is just the conditional probability that the system will be in state j after exactly n steps (time units), given that it starts in state i at any time t. When $n = 1$, note that $p_{ij}^{(1)} = p_{ij}$.[1]

Because the $p_{ij}^{(n)}$ are conditional probabilities, they must be nonnegative, and since the process must make a transition into some state, they must satisfy the properties

$$p_{ij}^{(n)} \geq 0, \qquad \text{for all } i \text{ and } j;\ n = 0, 1, 2, \ldots,$$

and

$$\sum_{j=0}^{M} p_{ij}^{(n)} = 1 \qquad \text{for all } i;\ n = 0, 1, 2, \ldots.$$

A convenient way of showing all the n-step transition probabilities is the *n-step transition matrix*

$$\mathbf{P}^{(n)} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ \vdots \\ M \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & \cdots & M \end{array} \\ \begin{bmatrix} p_{00}^{(n)} & p_{01}^{(n)} & \cdots & p_{0M}^{(n)} \\ p_{10}^{(n)} & p_{11}^{(n)} & \cdots & p_{1M}^{(n)} \\ \cdots & \cdots & \cdots & \cdots \\ p_{M0}^{(n)} & p_{M1}^{(n)} & \cdots & p_{MM}^{(n)} \end{bmatrix} \end{array}$$

Note that the transition probability in a particular row and column is for the transition *from* the row state *to* the column state. When $n = 1$, we drop the superscript n and simply refer to this as the *transition matrix.*

The Markov chains to be considered in this chapter have the following properties:

1. A finite number of states.
2. Stationary transition probabilities.

We also will assume that we know the initial probabilities $P\{X_0 = i\}$ for all i.

Formulating the Weather Example as a Markov Chain

For the weather example introduced in the preceding section, recall that the evolution of the weather in Centerville from day to day has been formulated as a stochastic process $\{X_t\}$ ($t = 0, 1, 2, \ldots$) where

$$X_t = \begin{cases} 0 & \text{if day } t \text{ is dry} \\ 1 & \text{if day } t \text{ has rain.} \end{cases}$$

[1]For $n = 0$, $p_{ij}^{(0)}$ is just $P\{X_0 = j \mid X_0 = i\}$ and hence is 1 when $i = j$ and is 0 when $i \neq j$.

An Application Vignette

Merrill Lynch is a leading full-service financial service firm. It provides brokerage, investment, and banking services to individual retail clients and small businesses while also helping major corporations and institutions around the world raise capital. One of Merrill Lynch's affiliates, *Merrill Lynch (ML) Bank USA*, has assets of over $60 billion obtained by accepting deposits from Merrill Lynch retail customers and using these deposits to fund loans and make investments.

In 2000, ML Bank USA began to establish revolving credit lines for client companies. Within a few years, the bank had developed a portfolio of about $13 billion in credit-line commitments with over 100 institutions. Long before this point was reached, Merrill Lynch's outstanding OR group was asked to guide the management of this growing portfolio by using OR techniques to assess the *liquidity risk* (the bank's potential inability to meet its cash obligations) associated with its current and prospective credit-line commitments.

The OR group developed a *simulation model* (the topic of Chap. 20) for this purpose. However, the most important input to this model is a *Markov chain* that describes the evolution of each customer's credit rating over time. The states of the Markov chain are the various possible credit ratings (ranging from *highest investment grade to default*) that are assigned to major companies by such credit-rating agencies as Standard and Poor's and Moody's. The transition probability from state i to state j in the transition matrix for a given company is the probability that the credit-rating agency will shift its rating of the company from state i to state j from one month to the next, based on historical patterns for similar companies.

This application of operations research, including Markov chains, enabled ML Bank USA to *free up about* **$4 billion** *of liquidity for other use*, as well as to expand its portfolio of credit-line commitments by over 60 percent in less than two years. Other benefits include the ability to evaluate extreme-risk scenarios and to perform long-range planning. This outstanding work led to Merrill Lynch winning the prestigious *Wagner Prize for Excellence in Operations Research Practice* for 2004.

Source: Duffy, T., M. Hatzakis, W. Hsu, R. Labe, B. Liao, X. Luo, J. Oh, A. Setya, and L. Yang: "Merrill Lynch Improves Liquidity Risk Management for Revolving Credit Lines," *Interfaces*, **35**(5): 353–369, Sept.–Oct. 2005. (A link to this article is provided on our website, www.mhhe.com/hillier.)

$$P\{X_{t+1} = 0 \mid X_t = 0\} = 0.8,$$
$$P\{X_{t+1} = 0 \mid X_t = 1\} = 0.6.$$

Furthermore, because these probabilities do not change if information about the weather before today (day t) is also taken into account,

$$P\{X_{t+1} = 0 \mid X_0 = k_0, X_1 = k_1, \ldots, X_{t-1} = k_{t-1}, X_t = 0\} = P\{X_{t+1} = 0 \mid X_t = 0\}$$
$$P\{X_{t+1} = 0 \mid X_0 = k_0, X_1 = k_1, \ldots, X_{t-1} = k_{t-1}, X_t = 1\} = P\{X_{t+1} = 0 \mid X_t = 1\}$$

for $t = 0, 1, \ldots$ and every sequence $k_0, k_1, \ldots, k_{t-1}$. These equations also must hold if $X_{t+1} = 0$ is replaced by $X_{t+1} = 1$. (The reason is that states 0 and 1 are mutually exclusive and the only possible states, so the probabilities of the two states must sum to 1.) Therefore, the stochastic process has the *Markovian property,* so the process is a Markov chain.

Using the notation introduced in this section, the (one-step) transition probabilities are

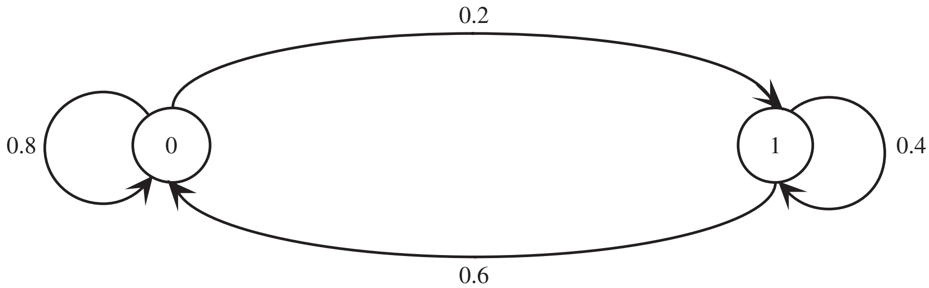
$$p_{00} = P\{X_{t+1} = 0 \mid X_t = 0\} = 0.8,$$
$$p_{10} = P\{X_{t+1} = 0 \mid X_t = 1\} = 0.6$$

for all $t = 1, 2, \ldots$, so these are *stationary* transition probabilities. Furthermore,

$$p_{00} + p_{01} = 1, \qquad \text{so} \qquad p_{01} = 1 - 0.8 = 0.2,$$
$$p_{10} + p_{11} = 1, \qquad \text{so} \qquad p_{11} = 1 - 0.6 = 0.4.$$

Therefore, the (one-step) transition matrix is

$$\mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \end{array}\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} \end{array} = \begin{array}{c} \text{State} \\ 0 \\ 1 \end{array}\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} \end{array}$$



■ **FIGURE 16.1**
The state transition diagram for the weather example.

where these transition probabilities are for the transition *from* the row state *to* the column state. Keep in mind that state 0 means that the day is dry, whereas state 1 signifies that the day has rain, so these transition probabilities give the probability of the state the weather will be in tomorrow, given the state of the weather today.

The state transition diagram in Fig. 16.1 graphically depicts the same information provided by the transition matrix. The two nodes (circle) represent the two possible states for the weather, and the arrows show the possible transitions (including back to the same state) from one day to the next. Each of the transition probabilities is given next to the corresponding arrow.

The n-step transition matrices for this example will be shown in the next section.

Formulating the Inventory Example as a Markov Chain

Returning to the inventory example developed in the preceding section, recall that X_t is the number of cameras in stock at the end of week t (before ordering any more), so X_t represents the *state of the system* at time t (the end of week t). Given that the current state is $X_t = i$, the expression at the end of Sec. 16.1 indicates that X_{t+1} depends only on D_{t+1} (the demand in week $t + 1$) and X_t. Since X_{t+1} is independent of any past history of the inventory system prior to time t, the stochastic process $\{X_t\}$ ($t = 0, 1, \ldots$) has the *Markovian property* and so is a Markov chain.

Now consider how to obtain the (one-step) transition probabilities, i.e., the elements of the (one-step) *transition matrix*

$$\mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & \;\;1 & \;\;2 & \;\;3 \end{array} \\ \begin{bmatrix} p_{00} & p_{01} & p_{02} & p_{03} \\ p_{10} & p_{11} & p_{12} & p_{13} \\ p_{20} & p_{21} & p_{22} & p_{23} \\ p_{30} & p_{31} & p_{32} & p_{33} \end{bmatrix} \end{array}$$

given that D_{t+1} has a Poisson distribution with a mean of 1. Thus,

$$P\{D_{t+1} = n\} = \frac{(1)^n e^{-1}}{n!}, \qquad \text{for } n = 0, 1, \ldots,$$

so (to three significant digits)

$$P\{D_{t+1} = 0\} = e^{-1} = 0.368,$$
$$P\{D_{t+1} = 1\} = e^{-1} = 0.368,$$
$$P\{D_{t+1} = 2\} = \frac{1}{2}e^{-1} = 0.184,$$
$$P\{D_{t+1} \geq 3\} = 1 - P\{D_{t+1} \leq 2\} = 1 - (0.368 + 0.368 + 0.184) = 0.080.$$

For the first row of **P,** we are dealing with a transition from state $X_t = 0$ to some state X_{t+1}. As indicated at the end of Sec. 16.1,

$$X_{t+1} = \max\{3 - D_{t+1}, 0\} \qquad \text{if} \qquad X_t = 0.$$

Therefore, for the transition to $X_{t+1} = 3$ or $X_{t+1} = 2$ or $X_{t+1} = 1$,

$$p_{03} = P\{D_{t+1} = 0\} = 0.368,$$
$$p_{02} = P\{D_{t+1} = 1\} = 0.368,$$
$$p_{01} = P\{D_{t+1} = 2\} = 0.184.$$

A transition from $X_t = 0$ to $X_{t+1} = 0$ implies that the demand for cameras in week $t + 1$ is 3 or more after 3 cameras are added to the depleted inventory at the beginning of the week, so

$$p_{00} = P\{D_{t+1} \geq 3\} = 0.080.$$

For the other rows of **P,** the formula at the end of Sec. 16.1 for the next state is

$$X_{t+1} = \max\ \{X_t - D_{t+1}, 0\} \qquad \text{if} \qquad X_t \geq 1.$$

This implies that $X_{t+1} \leq X_t$, so $p_{12} = 0$, $p_{13} = 0$, and $p_{23} = 0$. For the other transitions,

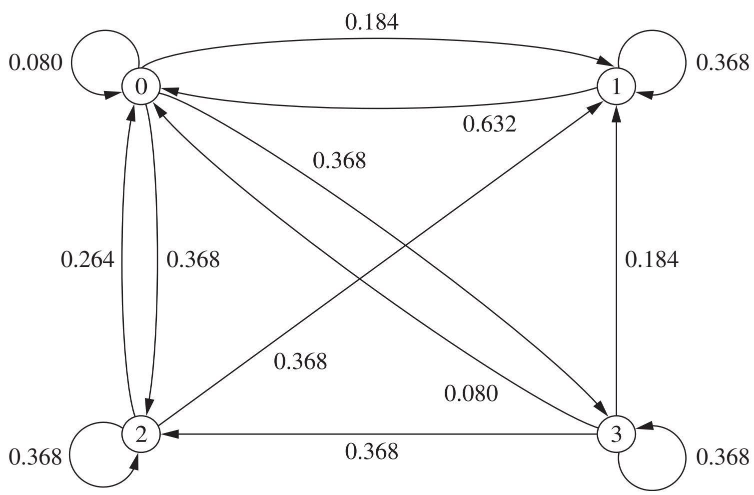
$$p_{11} = P\{D_{t+1} = 0\} = 0.368,$$
$$p_{10} = P\{D_{t+1} \geq 1) = 1 - P\{D_{t+1} = 0\} = 0.632,$$
$$p_{22} = P\{D_{t+1} = 0\} = 0.368,$$
$$p_{21} = P\{D_{t+1} = 1\} = 0.368,$$
$$p_{20} = P\{D_{t+1} \geq 2\} = 1 - P\{D_{t+1} \leq 1\} = 1 - (0.368 + 0.368) = 0.264.$$

For the last row of **P,** week $t + 1$ begins with 3 cameras in inventory, so the calculations for the transition probabilities are exactly the same as for the first row. Consequently, the complete transition matrix (to three significant digits) is

$$\mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} \;\;0\;\; & \;\;\;1\;\;\; & \;\;\;2\;\;\; & \;\;\;3\;\; \end{array} \\ \begin{bmatrix} 0.080 & 0.184 & 0.368 & 0.368 \\ 0.632 & 0.368 & 0 & 0 \\ 0.264 & 0.368 & 0.368 & 0 \\ 0.080 & 0.184 & 0.368 & 0.368 \end{bmatrix} \end{array}$$

The information given by this transition matrix can also be depicted graphically with the state transition diagram in Fig. 16.2. The four possible states for the number of cameras

FIGURE 16.2
The state transition diagram for the inventory example.



on hand at the end of a week are represented by the four nodes (circles) in the diagram. The arrows show the possible transitions from one state to another, or sometimes from a state back to itself, when the camera store goes from the end of one week to the end of the next week. The number next to each arrow gives the probability of that particular transition occurring next when the camera store is in the state at the base of the arrow.

Additional Examples of Markov Chains

A Stock Example. Consider the following model for the value of a stock. At the end of a given day, the price is recorded. If the stock has gone up, the probability that it will go up tomorrow is **0.7.** If the stock has gone down, the probability that it will go up tomorrow is only **0.5.** (For simplicity, we will count the stock staying the same as a decrease.) This is a Markov chain, where the possible states for each day are as follows:

State 0: The stock increased on this day.
State 1: The stock decreased on this day.

The transition matrix that shows each probability of going from a particular state today to a particular state tomorrow is given by

$$\mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{bmatrix} \end{array}$$

The form of the state transition diagram for this example is exactly the same as for the weather example shown in Fig. 16.1, so we will not repeat it here. The only difference is that the transition probabilities in the diagram are slightly different (0.7 replaces 0.8, 0.3 replaces 0.2, and 0.5 replaces both 0.6 and 0.4 in Fig. 16.1).

A Second Stock Example. Suppose now that the stock market model is changed so that the stock's going up tomorrow depends upon whether it increased today *and* yesterday. In particular, if the stock has increased for the past two days, it will increase tomorrow with probability **0.9.** If the stock increased today but decreased yesterday, then it will increase tomorrow with probability **0.6.** If the stock decreased today but increased yesterday, then it will increase tomorrow with probability **0.5.** Finally, if the stock decreased for the past two days, then it will increase tomorrow with probability **0.3.** If we define the state as representing whether the stock goes up or down today, the system is no longer a Markov chain. However, we can transform the system to a Markov chain by defining the states as follows:[2]

State 0: The stock increased both today and yesterday.
State 1: The stock increased today and decreased yesterday.
State 2: The stock decreased today and increased yesterday.
State 3: The stock decreased both today and yesterday.

This leads to a four-state Markov chain with the following transition matrix:

$$\mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.9 & 0 & 0.1 & 0 \\ 0.6 & 0 & 0.4 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0.3 & 0 & 0.7 \end{bmatrix} \end{array}$$

[2]We again are counting the stock staying the same as a decrease. This example demonstrates that Markov chains are able to incorporate arbitrary amounts of history, but at the cost of significantly increasing the number of states.

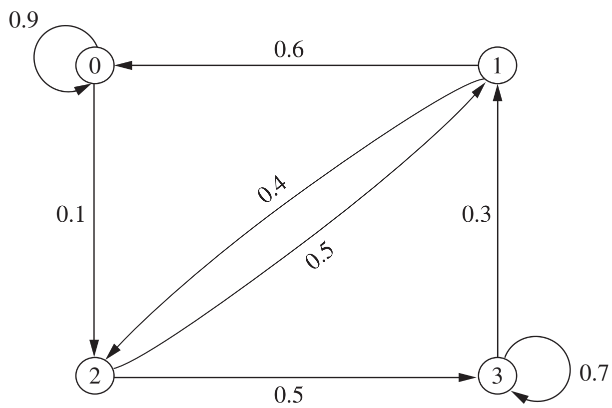
Figure 16.3 shows the state transition diagram for this example. An interesting feature of the example revealed by both this diagram and all the values of 0 in the transition matrix is that so many of the transitions from state i to state j are impossible in one step. In other words, $p_{ij} = 0$ for 8 of the 16 entries in the transition matrix. However, check out how it always is possible to go from any state i to any state j (including $j = i$) in two steps. The same holds true for three steps, four steps, and so forth. Thus, $p_{ij}^{(n)} > 0$ for n = 2, 3, . . . for all i and j.

A Gambling Example. Another example involves gambling. Suppose that a player has \$1 and with each play of the game wins \$1 with probability $p > 0$ or loses \$1 with probability $1 - p > 0$. The game ends when the player either accumulates \$3 or goes broke. This game is a Markov chain with the states representing the player's current holding of money, that is, 0, \$1, \$2, or \$3, and with the transition matrix given by

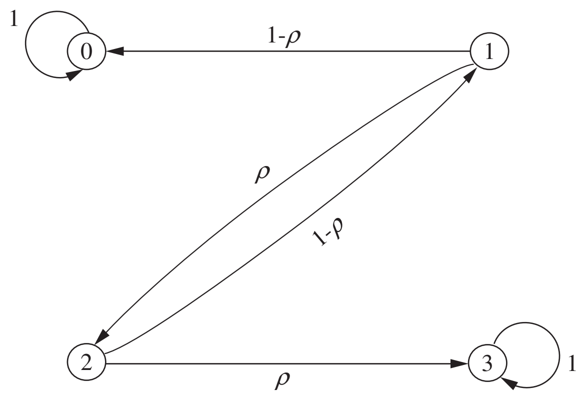
$$\mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The state transition diagram for this example is shown in Fig. 16.4. This diagram demonstrates that once the process enters either state 0 or state 3, it will stay in that state

■ **FIGURE 16.3**
The state transition diagram for the second stock example.



■ **FIGURE 16.4**
The state transition diagram for the gambling example.



forever after, since $p_{00} = 1$ and $p_{33} = 1$. States 0 and 3 are examples of what are called an **absorbing state** (a state that is never left once the process enters that state). We will focus on analyzing absorbing states in Sec. 16.7.

Note that in both the inventory and gambling examples, the numeric labeling of the states that the process reaches coincides with the physical expression of the system—i.e., actual inventory levels and the player's holding of money, respectively—whereas the numeric labeling of the states in the weather and stock examples has no physical significance.

16.3 CHAPMAN-KOLMOGOROV EQUATIONS

Section 16.2 introduced the n-step transition probability $p_{ij}^{(n)}$. The following *Chapman-Kolmogorov equations* provide a method for computing these n-step transition probabilities:

$$p_{ij}^{(n)} = \sum_{k=0}^{M} p_{ik}^{(m)} p_{kj}^{(n-m)}, \qquad \begin{aligned} \text{for all } i &= 0, 1, \ldots, M, \\ j &= 0, 1, \ldots, M, \\ \text{and any } m &= 1, 2, \ldots, n-1, \\ n &= m+1, m+2, \ldots .^{3} \end{aligned}$$

These equations point out that in going from state i to state j in n steps, the process will be in some state k after exactly m (less than n) steps. Thus, $p_{ik}^{(m)}\, p_{kj}^{(n-m)}$ is just the conditional probability that, given a starting point of state i, the process goes to state k after m steps and then to state j in $n - m$ steps. Therefore, summing these conditional probabilities over all possible k must yield $p_{ij}^{(n)}$. The special cases of $m = 1$ and $m = n - 1$ lead to the expressions

$$p_{ij}^{(n)} = \sum_{k=0}^{M} p_{ik} p_{kj}^{(n-1)}$$

and

$$p_{ij}^{(n)} = \sum_{k=0}^{M} p_{ik}^{(n-1)} p_{kj},$$

for all states i and j. These expressions enable the n-step transition probabilities to be obtained from the one-step transition probabilities recursively. This recursive relationship is best explained in matrix notation (see Appendix 4). For $n = 2$, these expressions become

$$p_{ij}^{(2)} = \sum_{k=0}^{M} p_{ik} p_{kj}, \qquad \text{for all states } i \text{ and } j,$$

where the $p_{ij}^{(2)}$ are the elements of a matrix $\mathbf{P}^{(2)}$. Also note that these elements are obtained by multiplying the matrix of one-step transition probabilities by itself; i.e.,

$$\mathbf{P}^{(2)} = \mathbf{P} \cdot \mathbf{P} = \mathbf{P}^2.$$

In the same manner, the above expressions for $p_{ij}^{(n)}$ when $m = 1$ and $m = n - 1$ indicate that the matrix of n-step transition probabilities is

$$\begin{aligned} \mathbf{P}^{(n)} &= \mathbf{P}\mathbf{P}^{(n-1)} = \mathbf{P}^{(n-1)}\mathbf{P} \\ &= \mathbf{P}\mathbf{P}^{n-1} = \mathbf{P}^{n-1}\mathbf{P} \\ &= \mathbf{P}^n. \end{aligned}$$

[3]These equations also hold in a trivial sense when $m = 0$ or $m = n$, but $m = 1, 2, \ldots, n - 1$ are the only interesting cases.

Thus, the n-step transition probability matrix $\mathbf{P}^n$ can be obtained by computing the nth power of the one-step transition matrix **P.**

n-Step Transition Matrices for the Weather Example

For the weather example introduced in Sec. 16.1, we now will use the above formulas to calculate various n-step transition matrices from the (one-step) transition matrix **P** that was obtained in Sec. 16.2. To start, the two-step transition matrix is

$$\mathbf{P}^{(2)} = \mathbf{P} \cdot \mathbf{P} = \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.76 & 0.24 \\ 0.72 & 0.28 \end{bmatrix}.$$

Thus, if the weather is in state 0 (dry) on a particular day, the probability of being in state 0 two days later is 0.76 and the probability of being in state 1 (rain) then is 0.24. Similarly, if the weather is in state 1 now, the probability of being in state 0 two days later is 0.72 whereas the probability of being in state 1 then is 0.28.

The probabilities of the state of the weather three, four, or five days into the future also can be read in the same way from the three-step, four-step, and five-step transition matrices calculated to three significant digits below.

$$\mathbf{P}^{(3)} = \mathbf{P}^3 = \mathbf{P} \cdot \mathbf{P}^2 = \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} \begin{bmatrix} 0.76 & 0.24 \\ 0.72 & 0.28 \end{bmatrix} = \begin{bmatrix} 0.752 & 0.248 \\ 0.744 & 0.256 \end{bmatrix}$$

$$\mathbf{P}^{(4)} = \mathbf{P}^4 = \mathbf{P} \cdot \mathbf{P}^3 = \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} \begin{bmatrix} 0.752 & 0.248 \\ 0.744 & 0.256 \end{bmatrix} = \begin{bmatrix} 0.75 & 0.25 \\ 0.749 & 0.251 \end{bmatrix}$$

$$\mathbf{P}^{(5)} = \mathbf{P}^5 = \mathbf{P} \cdot \mathbf{P}^4 = \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} \begin{bmatrix} 0.75 & 0.25 \\ 0.749 & 0.251 \end{bmatrix} = \begin{bmatrix} 0.75 & 0.25 \\ 0.75 & 0.25 \end{bmatrix}$$

Note that the five-step transition matrix has the interesting feature that the two rows have identical entries (after rounding to three significant digits). This reflects the fact that the probability of the weather being in a particular state is essentially independent of the state of the weather five days before. Thus, the probabilities in either row of this five-step transition matrix are referred to as the *steady-state probabilities* of this Markov chain.

We will expand further on the subject of the steady-state probabilities of a Markov chain, including how to derive them more directly, at the beginning of Sec. 16.5.

n-Step Transition Matrices for the Inventory Example

Returning to the inventory example included in Sec. 16.1, we now will calculate its n-step transition matrices to three decimal places for $n = 2$, 4, and 8. To start, its one-step transition matrix **P** obtained in Sec. 16.2 can be used to calculate the two-step transition matrix $\mathbf{P}^{(2)}$ as follows:

$$\mathbf{P}^{(2)} = \mathbf{P}^2 = \begin{bmatrix} 0.080 & 0.184 & 0.368 & 0.368 \\ 0.632 & 0.368 & 0 & 0 \\ 0.264 & 0.368 & 0.368 & 0 \\ 0.080 & 0.184 & 0.368 & 0.368 \end{bmatrix} \begin{bmatrix} 0.080 & 0.184 & 0.368 & 0.368 \\ 0.632 & 0.368 & 0 & 0 \\ 0.264 & 0.368 & 0.368 & 0 \\ 0.080 & 0.184 & 0.368 & 0.368 \end{bmatrix}$$

$$= \begin{bmatrix} 0.249 & 0.286 & 0.300 & 0.165 \\ 0.283 & 0.252 & 0.233 & 0.233 \\ 0.351 & 0.319 & 0.233 & 0.097 \\ 0.249 & 0.286 & 0.300 & 0.165 \end{bmatrix}.$$

For example, given that there is one camera left in stock at the end of a week, the probability is 0.283 that there will be no cameras in stock 2 weeks later, that is, $p_{10}^{(2)} = 0.283$. Similarly, given that there are two cameras left in stock at the end of a week, the probability is 0.097 that there will be three cameras in stock 2 weeks later, that is, $p_{23}^{(2)} = 0.097$.

The four-step transition matrix can also be obtained as follows:

$$\mathbf{P}^{(4)} = \mathbf{P}^4 = \mathbf{P}^{(2)} \cdot \mathbf{P}^{(2)}$$

$$= \begin{bmatrix} 0.249 & 0.286 & 0.300 & 0.165 \\ 0.283 & 0.252 & 0.233 & 0.233 \\ 0.351 & 0.319 & 0.233 & 0.097 \\ 0.249 & 0.286 & 0.300 & 0.165 \end{bmatrix} \begin{bmatrix} 0.249 & 0.286 & 0.300 & 0.165 \\ 0.283 & 0.252 & 0.233 & 0.233 \\ 0.351 & 0.319 & 0.233 & 0.097 \\ 0.249 & 0.286 & 0.300 & 0.165 \end{bmatrix}$$

$$= \begin{bmatrix} 0.289 & 0.286 & 0.261 & 0.164 \\ 0.282 & 0.285 & 0.268 & 0.166 \\ 0.284 & 0.283 & 0.263 & 0.171 \\ 0.289 & 0.286 & 0.261 & 0.164 \end{bmatrix}.$$

For example, given that there is one camera left in stock at the end of a week, the probability is 0.282 that there will be no cameras in stock 4 weeks later, that is, $p_{10}^{(4)} = 0.282$. Similarly, given that there are two cameras left in stock at the end of a week, the probability is 0.171 that there will be three cameras in stock 4 weeks later, that is, $p_{23}^{(4)} = 0.171$.

The transition probabilities for the number of cameras in stock 8 weeks from now can be read in the same way from the eight-step transition matrix calculated below.

$$\mathbf{P}^{(8)} = \mathbf{P}^8 = \mathbf{P}^{(4)} \cdot \mathbf{P}^{(4)}$$

$$= \begin{bmatrix} 0.289 & 0.286 & 0.261 & 0.164 \\ 0.282 & 0.285 & 0.268 & 0.166 \\ 0.284 & 0.283 & 0.263 & 0.171 \\ 0.289 & 0.286 & 0.261 & 0.164 \end{bmatrix} \begin{bmatrix} 0.289 & 0.286 & 0.261 & 0.164 \\ 0.282 & 0.285 & 0.268 & 0.166 \\ 0.284 & 0.283 & 0.263 & 0.171 \\ 0.289 & 0.286 & 0.261 & 0.164 \end{bmatrix}$$

$$= \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} \quad 0 \quad & \quad 1 \quad & \quad 2 \quad & \quad 3 \quad \end{array} \\ \begin{bmatrix} 0.286 & 0.285 & 0.264 & 0.166 \\ 0.286 & 0.285 & 0.264 & 0.166 \\ 0.286 & 0.285 & 0.264 & 0166 \\ 0.286 & 0.285 & 0.264 & 0.166 \end{bmatrix} \end{array}$$

Like the five-step transition matrix for the weather example, this matrix has the interesting feature that its rows have identical entries (after rounding). The reason once again is that probabilities in any row are the *steady-state probabilities* for this Markov chain, i.e., the probabilities of the state of the system after enough time has elapsed that the initial state is no longer relevant.

Your IOR Tutorial includes a procedure for calculating $\mathbf{P}^{(n)} = \mathbf{P}^n$ for any positive integer $n \leq 99$.

Unconditional State Probabilities

Recall that one- or n-step transition probabilities are *conditional* probabilities; for example, $P\{X_n = j \mid X_0 = i\} = p_{ij}^{(n)}$. Assume that n is small enough that these conditional probabilities are not yet *steady-state* probabilities. In this case, if the *unconditional* probability $P\{X_n = j\}$ is desired, it is necessary to specify the probability distribution of the initial state, namely, $P\{X_0 = i\}$ for $i = 0, 1, \ldots, M$. Then

$$P\{X_n = j\} = P\{X_0 = 0\}\, p_{0j}^{(n)} + P\{X_0 = 1\} p_{1j}^{(n)} + \cdots + P\{X_0 = M\} p_{Mj}^{(n)}.$$

In the inventory example, it was assumed that initially there were 3 units in stock, that is, $X_0 = 3$. Thus, $P\{X_0 = 0\} = P\{X_0 = 1\} = P\{X_0 = 2\} = 0$ and $P\{X_0 = 3\} = 1$. Hence, the (unconditional) probability that there will be three cameras in stock 2 weeks after the inventory system began is $P\{X_2 = 3\} = (1)p_{33}^{(2)} = 0.165$.

16.4 CLASSIFICATION OF STATES OF A MARKOV CHAIN

We have just seen near the end of the preceding section that the n-step transition probabilities for the inventory example converge to steady-state probabilities after a sufficient number of steps. However, this is not true for all Markov chains. The long-run properties of a Markov chain depend greatly on the characteristics of its states and transition matrix. To further describe the properties of Markov chains, it is necessary to present some concepts and definitions concerning these states.

State j is said to be **accessible** from state i if $p_{ij}^{(n)} > 0$ for some $n \geq 0$. (Recall that $p_{ij}^{(n)}$ is just the conditional probability of being in state j after n steps, starting in state i.) Thus, state j being accessible from state i means that it is possible for the system to enter state j eventually when it starts from state i. This is clearly true for the weather example (see Fig. 16.1) since $p_{ij} > 0$ for all i and j. In the inventory example (see Fig. 16.2), $p_{ij}^{(2)} > 0$ for all i and j, so every state is accessible from every other state. In general, a sufficient condition for *all* states to be accessible is that there exists a value of n for which $p_{ij}^{(n)} > 0$ for all i and j.

In the gambling example given at the end of Sec. 16.2 (see Fig. 16.4), state 2 is not accessible from state 3. This can be deduced from the context of the game (once the player reaches state 3, the player never leaves this state), which implies that $p_{32}^{(n)} = 0$ for all $n \geq 0$. However, even though state 2 is *not* accessible from state 3, state 3 *is* accessible from state 2 since, for $n = 1$, the transition matrix given at the end of Sec. 16.2 indicates that $p_{23} = p > 0$.

If state j is accessible from state i and state i is accessible from state j, then states i and j are said to **communicate.** In both the weather and inventory examples, all states communicate. In the gambling example, states 2 and 3 do not. (The same is true of states 1 and 3, states 1 and 0, and states 2 and 0.) In general,

1. Any state communicates with itself (because $p_{ii}^{(0)} = P\{X_0 = i \mid X_0 = i\} = 1$).
2. If state i communicates with state j, then state j communicates with state i.
3. If state i communicates with state j and state j communicates with state k, then state i communicates with state k.

Properties 1 and 2 follow from the definition of states communicating, whereas property 3 follows from the Chapman-Kolmogorov equations.

As a result of these three properties of communication, the states may be partitioned into one or more separate **classes** such that those states that communicate with each other are in the same class. (A class may consist of a single state.) If there is only one class, i.e., all the states communicate, the Markov chain is said to be **irreducible.** In both the weather and inventory examples, the Markov chain is irreducible. In both of the stock examples in Sec. 16.2, the Markov chain also is irreducible. However, the gambling example contains three classes. Observe in Fig. 16.4 how state 0 forms a class, state 3 forms a class, and states 1 and 2 form a class.

Recurrent States and Transient States

It is often useful to talk about whether a process entering a state will ever return to this state. Here is one possibility.

A state is said to be a **transient** state if, upon entering this state, the process *might never return* to this state again. Therefore, state i is transient if and only if there exists a state j