Lec-01

Laplace and Fourier Transformation:

$$\mathcal{L}\{F(t)\} = f(s) = \boxed{\int_0^{\infty} e^{-st} F(t)\, dt} \rightarrow \text{parameter}$$

$f(s)$ → Algebraic format

1. (i) $F(t) = 1$

$$\mathcal{L}\{F(t)\} = \int_0^{\infty} e^{-st} F(t)\, dt$$

$$\mathcal{L}\{1\} = \int_0^{\infty} e^{-st} \cdot 1\, dt = \left[\frac{e^{-st}}{-s}\right]_0^{\infty} = \frac{1}{-s}[0-1] = \frac{1}{s}$$

(ii) $F(t) = t^2$

$$\mathcal{L}\{F(t)\} = \mathcal{L}\{t^2\} = \int_0^{\infty} e^{-st} t^2\, dt = \left[t^2 \frac{e^{-st}}{-s} - \int 2t \frac{e^{-st}}{-s} dt\right]_0^{\infty}$$

$$= \frac{t^2 \cdot e^{-st}}{-s} + \frac{2}{s}\left[t \frac{e^{-st}}{-s} - \int 1 \cdot \frac{e^{-st}}{-s} dt\right]$$

$$= 0 + \frac{2}{-s^2}\left[t e^{-st} + \frac{e^{-st}}{s}\right]_0^{\infty}$$

$$= \frac{2}{-s^2}\left[0 - \frac{1}{s}\right]$$

$$= \boxed{\frac{2}{s^3}}$$

Formulla

$$\mathcal{L}\{t^3\} = \frac{3!}{s^4}$$

$\vdots$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\{t\} = \frac{1!}{s^2}$$

$$= \frac{1}{s^2}$$

(2) $\mathcal{L}\{f(t) = e^{at}\} = \int_0^\infty e^{-st} e^{at} \, dt$

$$= \cancel{\int_0^\infty e^{-t(a-s)} \, dt} = \int_0^\infty e^{-t(s-a)} \, dt$$

$$\cancel{= \left[\frac{e^{-t(a-s)}}{-(a-s)} \right]_0^\infty} = \left[\frac{e^{-t(s-a)}}{-(s-a)} \right]_0^\infty$$

$$= \frac{1}{-(s-a)} [0 - 1]$$

$$= \frac{1}{s-a}$$

$$\boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a}}$$

$$\int uv = u\int v\,dx - \int \frac{d}{dx}(u)\int v\,dx\,dx$$

$$\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st} \sin at\,dt$$? Ans:

$$\int_0^\infty e^{\alpha t} \sin\beta t\,dt$$

$$\int_0^\infty e^{\alpha t} \cos\beta t\,dt$$

$$\int_0^\infty e^{\alpha t} \cos\beta t\,dt$$

$$e^{\alpha t}\int_0^\infty \cos\beta t\,dt - \int_0^\infty \left(\frac{d}{dx}(e^{\alpha t})\int_0^\infty \cos\beta t\,dt\right)dt$$

$$= e^{\alpha t}\left[\frac{\sin\beta t}{\beta}\right]_0^\infty - \int_0^\infty \frac{e^{\alpha t}}{\alpha}$$

$$\int e^{at}\cos\beta t\,dt\left(1+\frac{\beta^2}{a^2}\right)$$

derivation আসবে

$I = \int e^{\alpha t} \cos\beta t \, dt$

$= \cos\beta t \frac{e^{\alpha t}}{\alpha} - \int -\sin\beta t \cdot \beta \cdot \frac{e^{\alpha t}}{\alpha} dt$

$= \frac{e^{\alpha t}\cos\beta t}{\alpha} + \frac{\beta}{\alpha}\int \sin\beta t \, e^{\alpha t} dt$

$= \frac{e^{\alpha t}\cos\beta t}{\alpha} + \frac{\beta}{\alpha}\left[\sin\beta t \frac{e^{\alpha t}}{\alpha} - \int \cos\beta t \cdot \beta \frac{e^{\alpha t}}{\alpha} dt\right]$

$I = \frac{e^{\alpha t}\cos\beta t}{\alpha} + \frac{\beta \sin\beta t \, e^{\alpha t}}{\alpha^2} - \frac{\beta^2}{\alpha^2}\int \cos\beta t \, e^{\alpha t} dt$

$\Rightarrow \int \cos\beta t \, e^{\alpha t} dt \left(1 + \frac{\beta^2}{\alpha^2}\right) = \frac{\alpha\left(e^{\alpha t}\cos\beta t + \beta \sin\beta t \, e^{\alpha t}\right)}{\alpha^2}$

$$\boxed{\Rightarrow \int \cos\beta t \, e^{\alpha t} dt = \frac{e^{\alpha t}(\alpha\cos\beta t + \beta\sin\beta t)}{(\alpha^2+\beta^2)}}$$

$\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st}\cos at \, dt$

$= \left[\frac{e^{-st}(-s\cos at + a\sin at)}{s^2+a^2}\right]_0^\infty$

$= \frac{1}{s^2+a^2}[0+s] = \frac{s}{s^2+a^2}$

$\int e^{ax}\sin bx \, dx$

$\frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

$\int e^{ax}\cos bx \, dx$

e^{ax}

H.task

$$\boxed{\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}}$$

$$\boxed{\cosh x = \frac{1}{2}(e^{x}+e^{-x})}$$

$$\boxed{\sinh x = \frac{1}{2}(e^{x}-e^{-x})}$$

$$\mathcal{L}\{\cosh at\} = \int_0^\infty e^{-st} \cosh at \, dt$$

Properties of laplace transformation:

property: $\mathcal{L}\{F_1(t)\} = f_1(s)$

$\mathcal{L}\{F_2(t)\} = f_2(s)$

$$\mathcal{L}\{c_1 F_1(t) + c_2 F_2(t)\} = c_1 f_1(s) + c_2 f_2(s)$$

constant

$$\mathcal{L}\{4t^2 - 3\cos 2t + 5e^{-t}\} = \mathcal{L}\{4t^2\} - \mathcal{L}\{3\cos 2t\} + \mathcal{L}\{5e^{-t}\}$$

$$= 4 \cdot \frac{2!}{s^3} - 3 \cdot \frac{s}{s^2+4} + 5 \cdot \frac{1}{s+1}$$

$$= \frac{8}{s^3} - \frac{3s}{s^2+4} + \frac{5}{s+1}$$

Ans.

First Translation or shifting property: prove করতে পারে

If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{e^{at} F(t)\} = f(s-a)$

proof: $\mathcal{L}\{e^{at} F(t)\} = \int_0^\infty e^{-st} e^{at} F(t)\, dt$

$$= \int_0^\infty e^{-(s-a)t} F(t)\, dt = f(s-a) \text{ (proved)}$$

Q: $\mathcal{L}\{e^{-t} \cos 2t\}$

let

$F(t) = \cos 2t$

$\mathcal{L}\{F(t)\} = \mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4}$

$\therefore \mathcal{L}\{e^{-t} \cos 2t\} = \underline{f(s-a)} = f(s+1)$

$$= \frac{s+1}{(s+1)^2+4} \text{ Ans.}$$

2nd translation or shifting:

If $\mathcal{L}\{F(t)\} = f(s)$ and $G(t) = \begin{cases} F(t-a) & t > a \\ 0 & t < a \end{cases}$, then

$$\mathcal{L}\{G(t)\} = e^{-as} f(s)$$

Find the laplace transformation of $G(t) = \begin{cases} (t-2)^3 & t > 2 \\ 0 & t < 2 \end{cases}$

Solⁿ: let, $F(t) = t^3$

$$\Rightarrow \mathcal{L}\{F(t)\} = \frac{3!}{s^4}$$

$$\therefore \mathcal{L}\{G(t)\} = e^{-2s} \cdot \frac{6}{s^4} = \frac{6e^{-2s}}{s^4} \text{ Ans:}$$

derivatative transformation:

If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{F'(t)\} = sf(s) - F(0)$

$$\boxed{\mathcal{L}\{F^n(t)\} = s^n F(s) - s^{n-1} F(0) - s^{n-2} F'(0) - s^{n-3} F''(0) \cdots s^{n-n} F^{n-1}(0)}$$

proof: $\mathcal{L}\{F(t)\} = f(s)$

$$\mathcal{L}\{F'(t)\} = \int_0^\infty e^{-st} \cdot F'(t)\, dt$$

$$= \left[e^{-st} \int F'(t)\, dt\right]_0^\infty - \int_0^\infty \left[\frac{d}{dt}(e^{-st}) \int F'(t)\, dt\right] dt$$

$$= \left[e^{-st} F(t)\right]_0^\infty - \int_0^\infty e^{-st}(-s)\, F(t)\, dt$$

$$= 0 - F(0) + \boxed{s\int_0^\infty e^{-st} F(t)\, dt}$$

$$= -F(0) + \cancel{sf(s)}\ sf(s)$$

$$= \boxed{sf(s) - F(0)}$$

If $F(t) = \cos 3t$ then find $\{F'(t)\}$

Sol^n: $\mathcal{L}\{F'(t)\} = sf(s) - F(0)$ — (1)

here, $F(t) = \cos 3t$

$\Rightarrow F(0) = 1$

$\mathcal{L}\{F(t)\} = \mathcal{L}\{\cos 3t\} = \frac{s}{s^2+9} = f(s)$

(1) $\Rightarrow \mathcal{L}\{F'(t)\} = s\left(\frac{s}{s^2+9}\right) - 1$

$= \frac{s^2 - s^2 - 9}{s^2+9} = \frac{-9}{s^2+9}$

$$\boxed{\mathcal{L}\{F''(t)\} = s^2 f(s) - sF(0) - F'(0)}$$

$\mathcal{L}\{F'(t)\}$

* $\mathcal{L}\{F'(t)\} = sf(s) - F(0)$

$\mathcal{L}\{F''(t)\} = \int_0^\infty e^{-st} F''(t)\,dt$

$= \left[e^{-st} \int F''(t)\,dt\right]_0^\infty - \int_0^\infty \left[\frac{d}{dt}(e^{-st}) \int F''(t)\,dt\right] dt$

$= \left[e^{-st} F'(t)\right]_0^\infty - \int_0^\infty e^{-st}(-s) F'(t)\,dt$

$$= 0 - F(0) + s\int_0^\infty sF' \quad - F(0)\}$$

$$= s^2 F(s) - sF(0) - F(0)$$

Lec-9

multiplication by the power of t

If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s) = (-1)^n f^n(s)$

* Find $\mathcal{L}\{t^2 \cos at\}$

by the def^n: $\mathcal{L}\{t^2 \cos at\} = (-1)^2 f^2(s)$ —— (1) no

let, $F(t) = \cos at$

$$\mathcal{L}\{F(t)\} = \mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2} = f(s)$$

$$f'(s) = \frac{(s^2+a^2).1 - s.2s}{(s^2+a^2)^2} = \frac{a^2-s^2}{(s^2+a^2)^2}$$

Again:

$$f^2(s) = \frac{2\cancel{(s^2+a^2)}.2s -}{\cancel{(s^2+a^2)^4}}$$

$$= \frac{(s^2+a^2)^2(-2s) - (a^2-s^2).2(s^2+a^2).2s}{(s^2+a^2)^4}$$

$$= \frac{(s^2+a^2) \cdot (-2s^3 - 2a^2 s - 4a^2 s + 4s^3)}{(s^2+a^2)^4}$$

$$= \frac{(2s^3 - 6a^2 s)}{(s^2+a^2)^3}$$

$$\mathcal{L}\{t^2 \cos at\} = \frac{(2s^3 - 6a^2 s)}{(s^2+a^2)^3}$$ Ans:

Division by t:

If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)\,du$

$\boxed{\text{Proof: Do it yourself}}$

* show that $\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1} = f(s)$$

$$f(u) = \frac{1}{u^2+1}$$

$$\therefore \int_s^\infty \frac{\sin t}{t}\,dt = \int_0^\infty \frac{1}{u^2+1}\,du = \left[\tan^{-1}(u)\right]_0^\infty = \frac{\pi}{2}$$

Inverse laplace transformation:

$f(s)$	$\mathcal{L}^{-1}\{f(s)\} = F(t)$
$\frac{1}{s}$	1
$\frac{1}{s^2}$	t
$\frac{1}{s^{n+1}}$	$\frac{t^n}{n!}$
$\frac{1}{s-a}$	e^{at}
$\frac{1}{s^2+a^2}$	$\frac{\sin at}{a}$
$\frac{s}{s^2+a^2}$	$\cos at$
$\frac{1}{s^2-a^2}$	$\sinh(at)$
$\frac{s}{s^2-a^2}$	$\cosh(at)$

(i) Linear property:

$$\mathcal{L}^{-1}\{c_1 f_1(s) + c_2 f_2(s)\} = c_1 F_1(t) + c_2 F_2(t)$$

(ii) First translation: If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ then

$$\mathcal{L}^{-1}\{f(s-a)\} = e^{at} F(t)$$

ex: (1) $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$

$= \frac{1}{2} \cdot \sin 2t$

(ii) $\mathcal{L}^{-1}\left\{\frac{1}{s^2-2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2-2.s.1+1^2+4}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+2^2}\right\}$ —(i) no

we know, $\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2} = f(s)$

$\therefore f(s-a) = \frac{b}{(s-a)^2+b^2}$

$\mathcal{L}^{-1}\{f(s-a)\} = e^{at} F(t)$

(i) no $\Rightarrow$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2-2s+5}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2+2^2}\right\}$

$= \frac{1}{2} e^{t} \sin 2t$

second Translation

If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ then

$$\mathcal{L}^{-1}\{e^{-as} f(s)\} = \begin{cases} F(t-a) & t>a \\ 0 & t<a \end{cases}$$

problem: If $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$ then

$$\mathcal{L}^{-1}\left\{\frac{e^{-\frac{\pi s}{3}}}{s^2+1}\right\} = ??$$

$$= \begin{cases} \sin\left(t-\frac{\pi}{3}\right) & t>\frac{\pi}{3} \\ 0 & t<\frac{\pi}{3} \end{cases}$$

(5) Inverse laplace transform of derivatives:

If $\mathcal{L}^{-1}\{f(s)\} = f(s) = F(t)$ then

$$\mathcal{L}^{-1}\{f^n(s)\} = (-1)^n t^n F(t)$$

* Find the inverse laplace transform of $\frac{s}{(s^2+1)^2}$

Soln: $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$

We know, $\mathcal{L}^{-1}\{f(s)\} = \cancel{\frac{s}{s}}\frac{1}{s^2+1^2} = \sin t$

$$\mathcal{L}^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+1^2}\right)\right\} = \frac{(1\cdot s^2+1)\cdot 0 - 1\cdot 2s}{(s^2+1)^2}$$

$$= \frac{-2s}{(s^2+1)^2}$$

=

Now, $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{-2s}{(s^2+1)^2}\right\}$

$$= (-1)^1 \cdot t^1 \cdot \sin t \cdot \frac{1}{(-2)}$$

$$= \frac{1}{2} t \sin t$$

(6) Inverse laplace transform of integrals:

If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\left\{\int_s^\infty f(u)\,du\right\} = F(t)/t$$

problem: Find the $\mathcal{L}^{-1}\left\{\int_s^\infty \cancel{\frac{1}{s(}}\right.$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\}$$

$$= t - e^{-t}$$

$$\mathcal{L}^{-1}\left\{\int_s^\infty \left(\frac{1}{u} - \frac{1}{u+1}\right) du\right\} = \frac{t - e^{-t}}{t} \quad \text{Ans:}$$

tutorial-9

solve the following initial value problem by laplace transformation

(i) $y'' - 3y' + 2y = 4e^{2t}$ $\quad y(0) = -3,\ y'(0) = 5$

(ii) $y'' + 2y' + 5y = e^{-t}\sin t,$ $\quad y(0) = 0;\ y'(0) = 1$

note $\mathcal{L}\{y''\} = s^2 y - s y(0) - y'(0)$

(1) $\cancel{\mathcal{L}\{e^{-sy}(y'' - 3y' + 2y)}$

$y'' - 3y' + 2y = 4e^{2t}$

$\Rightarrow y'' - 3y' + 2y - 4e^{2t} = 0$

$\cancel{\mathcal{L}e^{-sy}}.$ $\mathcal{L}\{F'(y)\} = sf(s) - F(0)$

$\mathcal{L}\{F''(y)\} = s^2 F(s) - sF(0) - F'(0)$

$\mathcal{L}(2y) = 2 \cdot \frac{1}{s^2}$

$\mathcal{L}(4e^{2t}) = 4 \cdot \frac{1}{s-2}$

$\mathcal{L}\, 3y' = 3\{sf(s) - F(0)\}$

$\mathcal{L}y'' = s^2 y - sy(0) - y'(0)$

* problem:

$$y'' + 2y' + 5y = e^{-t}\sin t \quad y(0) = 0 \quad y'(0) = 1$$

$$\mathcal{L}\{f(t)\} = f(s) = \boxed{ODE} \xrightarrow{T} \boxed{\text{Algebric}}$$

solve

ILT inverse laplace trasform

Sol^n: $\mathcal{L}\{y''\} + 2\mathcal{L}(y') + 5\mathcal{L}(y) = \mathcal{L}\{e^{-t}\sin t\}$

$$\Rightarrow \{s^2 y - s y(0) + y'(0)\} + 2\{sy - y(0)\} + 5y = \frac{1}{(s+1)^2+1}$$

$$\Rightarrow s^2 y - 0 - 1 + 2sy + 5y = \frac{1}{s^2+2s+1+1} = \frac{1}{s^2+2s+2}$$

$$\Rightarrow (s^2+2s+5)y = \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$$

$$\Rightarrow y = \mathcal{L}^{-1}\left\{\frac{(s^2+2s+3)}{(s^2+2s+2)(s^2+2s+5)}\right\} \quad \text{*(1) no}$$

আংশিক ভগ্নাংশ

$$\text{let, } P = \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$$

$$= \frac{As+B}{(s^2+2s+2)} + \frac{Cs+D}{s^2+2s+5}$$

$$\Rightarrow (s^2+2s+3) = (As+B)(s^2+2s+5) + (Cs+D)(s^2+2s+D)$$

$$\Rightarrow (s^2+2s+3) = s^3(A+C) + s^2(B+2A+D+2S) + s\,(5A+2B+2D+2C) + (5B+2D)$$

equating both sides

$A+C=0$, $B+2A+D+2S=1$; $5A+2B+2D+2C=2$

$A=-C$ $\Rightarrow$ $A=C=0$

$B=\frac{1}{3}$, $D=\frac{2}{3}$

$5B+2D=3$

(1) no

$$* \; y = \mathcal{L}^{-1}\left\{\frac{\frac{1}{3}}{s^2+2s+2} + \frac{\frac{2}{3}}{s^2+2s+5}\right\}$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} + \frac{1}{2}\cdot\frac{2}{3}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\}$$

$$y = \frac{1}{3}e^{-t}\sin t$$

$$= \frac{1}{3}e^{-t}\sin 2t$$

$$y = \frac{1}{3}e^{-t}(\sin t + \sin 2t)$$

Convolution property:

If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ and $\mathcal{L}^{-1}\{g(s)\} = G(t)$, then

$$\mathcal{L}^{-1}\{f(s)g(s)\} = \int_0^t F(u)\,G(t-u)\,du = F*G$$

Solve: $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)} \cdot \frac{1}{s^2+a^2}\right\}$$

we know, $\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at = F(t)$

$$\frac{1}{a}\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \frac{\sin at}{a} = G(t)$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \int_0^t \cos au \cdot \frac{\sin a(t-u)}{a}\,du$$

$$= \frac{1}{a}\int_0^t \cos au \cdot \sin(at-au)\,du$$

$$= \frac{1}{a}\int_0^t \cos au\left[\sin at\cos au - \cos at\sin au\right]du$$

$$= \frac{1}{a}\int_0^t \cos^2 au \sin at\,du - \frac{1}{a}\int_0^t \cos au \cos at \sin au\,du$$

$$= \frac{1}{a \cdot 2}\sin at\int_0^t 2\cos^2 au\,du - \frac{1}{a \cdot 2}\cos at\int_0^t 2\cos au \sin au\,du$$

$$= \frac{\sin at}{2a}\int_0^t (1+\cos 2au)\,du - \frac{\cos at}{2a}\int_0^t \sin 2au\,du$$

$$\frac{\sin at}{2a}\left[u + \frac{\sin 2au}{2a}\right]_0^t - \frac{\cos at}{2a}\left[\frac{-\sin 2au}{2a}\right]_0^t$$

$$= \frac{\sin at}{2a}\left[t + \frac{\sin 2at}{2a}\right] - \frac{\cos at}{2a}\left[\frac{-\sin 2at}{2a} - 0\right]$$

$$= \frac{1}{2a \cdot 2a}\left[2at \sin at + \sin^2 at + \cos^2 at \sin at\right]$$

$$\cancel{= \frac{1}{4a^2}\left[2at \sin at + \sin at\right]}$$

$$= \frac{1}{4a^2}\left[2at \sin at + 2\sin^2 at \cancel{\cos at} + 2\sin at \cos at\right]$$

$$\cancel{= \frac{1}{4a^2}\left[2at \sin at + 2\sin at\right]}$$

$$= \frac{\sin 2at}{2a}\left[u + \frac{\sin 2au}{2a}\right]_0^t - \frac{\cos at}{2a}\left[\frac{-\cos 2au}{2a}\right]_0^t$$

$$= \frac{\sin 2at}{2a}\left[t + \frac{\sin 2at}{2a}\right] - \frac{\cos at}{2a}\left[\frac{-\cos 2at}{2a} + \frac{1}{2a}\right]$$

$$= \frac{\sin 2at}{2a} \cdot \frac{1}{2a}\left(2at + \sin 2at\right) - \frac{1 \cdot \cos at}{2a \cdot 2a}\left[2\sin^2 at\right]$$

$$\frac{1}{4a^2}\left(2ad\sin ad + 2\sin^2 ad\cos ad\right] - \frac{1}{4a^2}\left(2\cos ad\,\frac{1}{\sin ad}\right\}$$

$$= \frac{1}{4a^2}\left[2ad\sin ad\right]$$

$$= \frac{d\sin ad}{2a} \text{ (Ans)}$$