Gaussian Elimination Method - (Chapter 1.2)

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

(1) Locate the leftmost coloumn that doesn't consist entirely of zeros

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

(2) Interchange the top row with another row, if necessary, to bring a non-zero entry to the top of the coloumn found in (1)

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

(3) If the entry that is now at the top of the coloumn found in (1) is a, multiply the first row by 1/a order to introduce a leading 1

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \rightarrow r_1' = r_1/2$$

(4) Add suitable multiples of the top row to the row below so that all entries below the leading 1 become zeros.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix} \rightarrow r_3' = r_3 - (2 \times r_1)$$

(5) Now cover the top row in the matrix and begin again with (1) applied to the submatrix that remains. Continue in this way until the entire matrix is now in row enchelon form

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix} \rightarrow r_2' = r_2/-2$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1/2 & 1 \end{bmatrix} \rightarrow r_3' = (r_2 \times -5) + r_3$$

$$\begin{bmatrix} \boxed{1} & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & \boxed{1} & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & \boxed{1} & 2 \end{bmatrix} \rightarrow r_3' = r_3 \times 2$$

$$\begin{cases} x_1 + 2x_2 - 5x_3 + 3x_4 + 6x_5 = 14 \\ x_3 + (-7/2)x_5 = -6 \end{cases}$$

$x_5 = 2$

$$\begin{cases} x_1 = 14 - 2x_2 + 5x_3 - 3x_4 - 6x_5 \\ x_3 = -6 + 7/2 x_5 \\ x_5 = 2 \end{cases}$$

$$\begin{cases} x_3 = (-6\cancel{\times} + 7/2 \times 2) \to 1 \\ x_5 = 2 \\ x_1 = (14 - 2x_2 + (5\times 1) - 3x - (6\times 2)) = 7 - 2x_2 - 3x_4 \end{cases}$$

Let, $x_2 = r$, $x_4 = s$

Sol^n, $x_1 = 7 - 2r - 3s$

$x_2 = r$

$x_3 = 1$

$x_4 = s$

$x_5 = 2$

Gaussian Jordan Elimination

Begining with the last non zero row and working upward, add suitable multiples of each row to the rows above to intro duce zeros above 1's

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow r_2' = r_2 + (7/2 \times r_3)$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \Rightarrow r_1' = r_1 - (6 \times r_3)$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \Rightarrow r_1'' = r_1' + (5 \times r_2)$$

$$\begin{cases} x_1 + 2x_2 + 3x_4 = 7 \\ x_3 = 1 \\ x_5 = 2 \end{cases}$$

$$\begin{cases} x_1 = 7 - 2x_2 - 3x_4 \\ x_3 = 1 \\ x_5 = 2 \end{cases}$$

$$\begin{cases} \text{Let, } x_2 = r, \; x_4 = s \\ x_1 = 7 - 2r - 3s \\ x_2 = r \\ x_3 = 1 \\ x_4 = s \\ x_5 = 2 \end{cases}$$

Real Vector Space and Subspace (chapter (4))

(1) Linear Combination

(2) Linear Span

(3) Linear Independent dependent

lecture-8

(201) page → Problem 8

(a) $u = (2, 1, 4)$

$v = (1, -1, 3)$

$w = (3, 2, 5)$

$(-9, -7, -15)$ is linear combination of u, v, w then there exist some scalar $c_1, c_2, \& c_3$

$$c_1u + c_2v + c_3w = (-9, -7, -15)$$

$$c_1(2, 1, 4) + c_2(1, -1, 3) + c_3(3, 2, 5) = (-9, -7, -15)$$

$$2c_1 + c_2 + 3c_3 = -9$$

$$c_1 - c_2 + 2c_3 = -7$$

$$4c_1 + 3c_2 + 5c_3 = -15$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & -9 \\ 1 & -1 & 2 & -7 \\ 4 & 3 & 5 & -15 \end{array}\right]$$

$$\begin{bmatrix} 1 & \cancel{1/2}\ -1 & 2 & -7 \\ 2 & 1 & 3 & -9 \\ 4 & 3 & 5 & -15 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 & -7 \\ 2 & 1 & 3 & -9 \\ 4 & 3 & 5 & -15 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 2 & -7 \\ 0 & 3 & -1 & 5 \\ 4 & 3 & 5 & -15 \end{bmatrix} \quad r_2' = r_2 - (2 \times r_1)$$

$$= \begin{bmatrix} 1 & -1 & 2 & -7 \\ 0 & 3 & -1 & 5 \\ 0 & 7 & -3 & -13 \end{bmatrix} \quad r_3' = r_3 - (4 \times r_1)$$

$$= \begin{bmatrix} 1 & -1 & 2 & -7 \\ 0 & 1 & -1/3 & 5/3 \\ 0 & 7 & -3 & -13 \end{bmatrix} \quad r_2'' = r_2/3$$

$$= \begin{bmatrix} 1 & -1 & 2 & -7 \\ 0 & 1 & -1/3 & 5/3 \\ 0 & 0 & -2/3 & 4/2 \end{bmatrix} \quad r_3' = r_3 - 7 \times r_2$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 2 & -7 \\ 0 & 1 & -1/3 & 5/3 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad r_3' = r_3 \times -3/2$$

$$\begin{cases} c_1 - c_2 + 2c_3 = -7 \\ c_2 - \frac{1}{3}c_3 = \frac{5}{3} \\ c_3 = -2 \end{cases}$$

$$c_2 = +\frac{1}{3} \times -2 + \frac{5}{3}$$

$$= \frac{-2+5}{3}$$

$$= \frac{3}{3} = 1$$

$$c_1 = c_2 - 2c_3 - 7$$

$$= 1 + 2 \times -2 - 7$$

$$= 1 + 4 - 7$$

$$= 5 - 7$$

$$= -2$$

$$\begin{cases} c_1 = -2 \\ c_2 = 1 \\ c_3 = -2 \end{cases}$$

Determine whether the vectors $v_1 = (1,1,2)$ $v_2 = (1,0,1)$ and $v_3 = (2,1,3)$ span the vector space R_3

Soln: Let the vector be $v = (a,b,c)$
If the vectors v_1, v_2, v_3 span the vector space R^3 then following eqn must be satisfied

$$c_1v_1 + c_2v_2 + c_3v_3 = v$$

$$c_1(1,1,2) + c_2(1,0,1) + c_3(2,1,3) = (a,b,c)$$

System of eqn is:

$$\left.\begin{aligned} c_1 + c_2 + 2c_3 &= a \\ c_1 + c_3 &= b \\ 2c_1 + c_2 + 3c_3 &= c \end{aligned}\right\} \text{Solve it using gaussian elimination}$$

The co-efficient matrix of the system is

$$M = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

det $\neq 0$; then it has solution Invert Matrix

$$\det(M) = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 0$$

As, determinant is zero, so the system has no ~~unique~~ solution.
So, it doesn't span R^3

* Determinte whether the vectors

$v_1 = (1, -2, 3)$, $v_2 = (5, 6, -1)$, $v_3 = (3, 2, 1)$

are linearly independent or linearly dependent in R^3

Soln= Let there are some scalar k_1, k_2, k_3. for linear independence it should satisfy the following eqn=

$$k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$$

$$k_1(1, -2, 3) + k_2(5, 6, -1) + k_3(3, 2, 1) = 0$$

The system of eqn is:

$$\cancel{k_1 - 2k_2 + 3k_3 = 0}$$

$$\cancel{5k_2 + 6k_2 - k_3 = 0}$$

$$\cancel{3k_1 + 2k_2 + 1k_3 = 0}$$

$$\left.\begin{aligned} k_1 + 5k_2 + 3k_3 &= 0 \\ -2k_1 + 6k_2 + 2k_3 &= 0 \\ 3k_1 - k_2 + k_3 &= 0 \end{aligned}\right\} \quad \begin{aligned} k_1 &= 0 \\ k_2 &= 0 \\ k_3 &= 0 \end{aligned}$$

So, it is linearly independent.

vector-4 basis → Spans, linearly independent.

4.4 Co-ordinate vector

Example $\{(v_1=(1,2,1)), (v_2=(2,9,0)), (v_3=(3,3,4))\}$

(a) we showed in Ex-8 that the vectors from a basis for R^3. Find the co-ordinate vector of $v=(5,-1,9)$ relative to the basis $S=\{(v_1,v_2,v_3)\}$

(b) Find the vectors v in R^3 whose coordinates vector relative to S is $v(s)=(-1,3,2)$

Soln

(a) let, the vectors are $v_1(1,2,1)$, $v_2(2,9,0)$
$v_3=(3,3,4)$

The vectors form a basis in R^3 if

(1) They are linearly independent

(2) They spans the vector space R^3

for, linearly independent let the coefficient vectors k_1, k_2, k_3 then

$K_1V_1 + K_2V_2 + K_3V_3 = 0$

$K_1(1,2,1) + K_2(2,9,0) + K_3(3,3,4) = 0$

$\Rightarrow K_1 + 2K_2 + 3K_3 = 0$

$2K_1 + 9K_2 + 3K_3 = 0$

$K_1 + 4K_3 = 0$

$K_1 = K_2 = K_3 = 0$ (linearly Independent)

Coefficient Matrix

$$M = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{vmatrix} =$$

$\det(M) = -1 \neq 0$

As the value of determinant is non-zero

So, it has a unique solution so it spans

the vector space R^3

Hence, they form basis in R^3.

(b) Let, the co-ordinate vector is (c_1, c_2, c_3) (k_1, k_2, k_3)

$v = (5, -1, 9)$ relative to the

basis vector $S = (v_1, v_2, v_3)$ is (c_1, c_2, c_3) (k_1, k_2, k_3)

$k_1 v_1 + k_2 v_2 + k_3 v_3 = (5, -1, 9)$

$k_1(1,2,1) + k_2(2,9,0) + k_3(3,3,4) = (5,-1,9)$

$k_1 + 2k_2 + 3k_3 = 5$

$2k_1 + 9k_2 + 3k_3 = -1$

$k_1 + 0k_2 + 4k_3 = 9$

Solve using gaussian elimination

Solving the system we get

$k_1 = 1, k_2 = -1, k_3 = 2$

$(k_1, k_2, k_3) = (1, -1, 2)$

So, the co-ordinate vector is $(1, -1, 2)$

(c) Here the co-ordinate vector $(-1, 3, 2)$

and the basis vector $S = (v_1, v_2, v_3)$

let the new vector is v

So, $v = -1v_1 + 3v_2 + 2v_3 = -1(1,2,1) + 3(2,9,0) + 2(3,3,4)$

$= (-1, -2, -1) + (6, 18, 0) + (6, 6, 8)$

$= (11, 31, 7)$

4.5 Dimension

4.7, 4.8

Find a basis for and the dimension of solⁿ space of the homogenous system

$$x_1 + 3x_2 - 2x_3 + 2x_5 = 0$$
$$2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = 0$$
$$5x_3 + 10x_4 + 15x_6 = 0$$
$$2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 0$$

Number of non zero rows are Dimension

Solⁿ

Augmented Matrix

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 15 & 0 & \\ 2 & 6 & 0 & 8 & 4 & 18 & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{bmatrix}$$

$$r_2'' = (-2 \times r_4) + r_2$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 0 & 0 & 0 & 8 & 0 & 18 & 0 \end{bmatrix}$$

$$r_4'' = (-2 \times r_4) + r_4$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 \end{bmatrix}$$

$$\left\{ \begin{array}{l} r_3' = (r_2 \times 5) + r_3 \\ r_4' = (r_2 \times 4) + r_4 \end{array} \right\}$$

4.7 Row Space, Column Space, Rank and Nullity

$$= \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$r_3' \; \cancel{r_3 \times r_3} \; \cancel{r_3} = r_3/6$

$$= \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{cases} x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \\ x_3 + 2x_4 + 3x_6 = 0 \\ x_6 = 0 \end{cases}$$

$$\begin{cases} x_2 = s \\ x_3 = t \\ x_5 = p \end{cases}$$

$$\begin{cases} x_3 = -2t \\ x_6 = 0 \\ x_1 = -3s + 4t - 2p \end{cases}$$

Dimension, Nullity, Rank, Basis

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3s-4t-2p \\ s \\ -2t \\ t \\ p \\ 0 \end{bmatrix} = s\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + p\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

So, dimension is 3

v_1, v_2, v_3 are the basis of null space

Nullity is the number basis vectors in the null space

Nullity = 3

Number of coloumns = rank + nullity

Rank = (6 − 3) = 3

4.7 ⇒ Coloumn Space, Raw Space, Nulity, Rank

Let, $Ax = b$ be the linear system

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Soln, $b = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$

$$c_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$$

linear Combination,

$$c_1x + c_2y + c_3z = b$$

$$x\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + z\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

System,

$$-x + 3y + 2z = 1$$

$$x + 2y + -3z = -9$$

$$2x + y - 2z = -3$$

$$\begin{bmatrix} -1 & 3 & 2 & 1 \\ 1 & 2 & -3 & -9 \\ 2 & 1 & -2 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & -2 & -1 \\ 0 & 5 & -1 & -8 \\ 0 & 7 & 2 & -1 \end{bmatrix} \quad \begin{aligned} R_1' &= -1 \times R_1 \\ R_2' &= R_1 + R_2 \\ R_3' &= (R_2 \times 2) + R_3 \end{aligned}$$

$$\begin{bmatrix} 1 & -3 & -2 & -1 \\ 0 & 1 & -1/5 & -8/5 \\ 0 & 0 & 17/5 & 51/5 \end{bmatrix} \quad R_3' = -7 \times R_2 + R_3$$

$$\begin{bmatrix} 1 & -3 & -2 & -1 \\ 0 & 1 & -1/5 & -8/5 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad R_3' = R_3 \times 5/17$$

$$x - 3y - 2z = -1$$

$$y - \tfrac{1}{5} z = -8/5$$

$$z = 3$$

$$\Rightarrow y = -8/5 + 3/5$$

$$= \frac{-8+3}{5} \Rightarrow -1$$

$$x = (3 \times -1) + (2 \times 3) - 1 \Rightarrow -3 + 6 - 1 \Rightarrow 2$$

So, the vector b is the linear combination of the coloumn vector In the form of

$$2\begin{bmatrix} -7 \\ 7 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Find the rank and Nullity of the Matrix:

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

Since, this matrix has two leading 1's, So it has rank 2

In order to find the nulling we have to find the soloton spacee of the system.

$$AX = 0$$

the system is

$$x_1 - 4x_3 - 28x_4 - 37x_5 + 13x_6 = 0$$

$$x_2 - 2x_3 - 12x_4 - 16x_5 + 5x_6 = 0$$

leading variable x_1, x_2

free variable, x_3, x_4, x_5, x_6

let, $x_3 = t_1$

$x_4 = t_2$

$x_5 = t_3$

$x_6 = t_4$

$x_1 = 4x_3 + 28x_4 + 37x_5$

$= 4t_1 + 28t_2 + 37t_3 - 13t_4$

$x_2 = 2t_1 + 12t_2 + 16t_3 - 5t_4$

Sol^n =

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 4t_1 + 28t_2 + 37t_3 - 13t_4 \\ 2t_1 + 12t_2 + 16t_3 - 5t_4 \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix}$$

$$\Rightarrow t_1 \underbrace{\begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{v_1} + t_2 \underbrace{\begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{v_2} + t_3 \underbrace{\begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{v_3} + t_4 \underbrace{\begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{v_4}$$

Here, the vectors v_1, v_2, v_3, v_4 from the basis for the null space. so, it has nullity 4

Eigenvalues & Eigenvectors

$A x = \lambda x$

vector ← A (Matrix), x (vectors); λ → scaler, x → vector

Cayley Hamilton Theorem

$V = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

$AV = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 3+0 \\ 8-2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3$ ✓

eigen-vale

Find the eigenvalues of

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

Solⁿ = let, the eigen value is λ

characteristics eqⁿ is; $\det(A - \lambda I) = 0$

Here,

$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$

$\lambda I = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$

$\det(A - \lambda I) = 0$

$\det \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 4 & -17 & 8-\lambda \end{pmatrix} = 0$

$= -\lambda(-8\lambda + \lambda^2 + 17) - 1(-4) = 0$

$= 8\lambda^2 - \lambda^3 - \cancel{17\lambda} \cancel{+ 4} = 0$

$= \lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$

Solving this we get,

$\lambda_1 = 4$

$\lambda_2 = 2 + \sqrt{3}$, $\lambda_3 = 2 - \sqrt{3}$

lecture (10 Sep. 2024)

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

① Find Eigen values of Matrix A or det $(A - \lambda I)$

② Find bases for the eigenspaces of Matrix A.

① $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

$$\lambda I = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix}$$

$$\Rightarrow \{-\lambda(2-\lambda)(3-\lambda)\} + \{(-2) \times -(2-\lambda)\}$$

$$\Rightarrow -\lambda(6 - 2\lambda - 3\lambda + \lambda^2) + (+4 - 2\lambda)$$

$$\Rightarrow -6\lambda + 2\lambda^2 + 3\lambda^2 - \lambda^3 + 4 - 2\lambda$$

$$\Rightarrow -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$$

$$\Rightarrow \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

So, the distinct eigenvalues of A are

$\lambda = 1$ & $\lambda = 2$

(2) By defination,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} ; \quad \therefore (A - \lambda I)x = 0$$

$$\begin{bmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

for, $\lambda = 1$

$$\begin{bmatrix} -1 & 0 & -2 \\ 1 & 2-1 & 1 \\ 1 & 0 & 3-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} r_1' = -1 \times r_1 \\ r_2' = r_2 - r_1 \\ r_3' = r_3 + r_1 \end{matrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore x_1 + 2x_3 = 0$

$x_2 - x_3 = 0$

$x_3 = t$

$\therefore x_2 = t$

$\therefore x_1 + 2t = 0$

$\Rightarrow x_1 = -2t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

for $\lambda = 2$

$$\begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad r_1' = r_1/-2, \quad r_2' = r_2$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + x_3 = 0$

$x_2 = t$

$x_3 = s$

$\therefore x_1 = -s$

$x_2 = t$

$x_3 = s$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix}$$

$$= s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore \lambda = 2$; $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (basis for 2) $\therefore$

$\lambda = 1$, $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ (basis for 1)

Inverse Matrix

$$A=\begin{bmatrix}0 & 0 & -2\\ 1 & 2 & 1\\ 1 & 0 & 3\end{bmatrix}$$

$$x^3-5x^2+8x-4=0$$

$$\Rightarrow A^3-5A^2+8A-4I=0$$

$$\Rightarrow A^3\cdot A^{-1}-5A^2\cdot A^{-1}+8A\cdot A^{-1}-4IA^{-1}=0$$

$$\Rightarrow A^2-5A+8I-4A^{-1}=0$$

$$\Rightarrow A^{-1}=\frac{1}{4}\left(A^2-5A+8I\right)$$

$$A^2-5A+8I$$

$$\Rightarrow \begin{bmatrix}0 & 0 & -2\\ 1 & 2 & 1\\ 1 & 0 & 3\end{bmatrix}\begin{bmatrix}0 & 0 & -2\\ 1 & 2 & 1\\ 1 & 0 & 3\end{bmatrix}-5\begin{bmatrix}0 & 0 & -2\\ 1 & 2 & 1\\ 1 & 0 & 3\end{bmatrix}+8\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}6 & 0 & 4\\ -2 & 2 & -2\\ -2 & 0 & 0\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{4}\begin{bmatrix}6 & 0 & 4\\ -2 & 2 & -2\\ -2 & 0 & 0\end{bmatrix}$$

diagonalizes

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Soln: $\lambda = 2;\ P_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},\ P_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\lambda = 1;\ P_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Geometric Multiplicity:

Number of eigenvectors corresponding to an eigenvalue

Algebraic Multiplicity:

Number of times an eigenvalue occurs.

AM = GM (Diagnalizable)

$GM \leq AM$

Diagonalization and Power of a Matrix:

find a Matrix P that diagonalizes.

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Sol^n eigen values $\lambda = 1, 2, 2$

for $\lambda = 2$; $P_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $P_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\lambda = 1$; $P_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

There are three basis vector in total, So the matrix

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

diagonalizes the matrix. A as

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A & D are the similar Matrix

(*) Determinant

(*) if A is invertible then D is also invertible

(*) Same Rank

(*) Same Nullity

(*) Trace

(*) Characteristic Polynomial

(*) Eigenvalues

(*) Same eigen space

* Power of a Matrix

$D = P^{-1} A P$

$D^K = (P^{-1} A P)^K$

$\Rightarrow A^K = P D^K P^{-1}$

find A^{13}, $A^{13} = P D^{13} P^{-1}$

$$= \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{13} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -8190 & 0 & -16382 \\ 8191 & 8192 & 8191 \\ 8191 & 0 & 16383 \end{bmatrix}$$

* Show the Matrix.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$$

Soln, let the eigenvalue is λ

characteristics eqn is $\det(A - \lambda I) = 0$

there,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$$

$$\lambda I = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ -3 & 5 & 2-\lambda \end{bmatrix}$$

$$= \{(1-\lambda)(2-\lambda)(2-\lambda)\}$$

$$= \{(1-\lambda)(4 - 2\lambda - 2\lambda + \lambda^2)\}$$

$$= \{4 - 2\lambda - 2\lambda + \lambda^2 - 4\lambda + 2\lambda^2 + 2\lambda^2 - \lambda^3\}$$

$$\Rightarrow 5\lambda^2 - \lambda^3 - 8\lambda + 4$$

$$\Rightarrow \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$A^S = 15, 25, 2^5 \rightarrow 1, 32, 382$