Application of Laplace Transformation

Solve $\begin{cases} x'' + y' + 3x = 15e^{-t} \\ y'' - 4x' + 3y = 15\sin 2t \end{cases}$ subject to $x(0)=35$, $x'(0)=-48$, $y(0)=27$, $y'(0)=-55$

Sol: Taking Laplace Transformation

$L\{x''\} + L\{y'\} + 3L\{x\} = L\{15e^{-t}\}$ — (i)

$L\{y''\} - 4L\{x'\} + 3L\{y\} = L\{15\sin 2t\}$ — (ii)

$\boxed{L\{x\} = x, \quad L\{y\} = y}$

(i) $\Rightarrow \{s^2x - sx(0) - x'(0)\} + \{sy - y(0)\} + 3x = \frac{15}{s+1}$

$\Rightarrow s^2x - 35s + 48 + sy - 27 + 3x = \frac{15}{s+1}$

$\Rightarrow (s^2+3)x + sy = 35s - 21 + \frac{15}{s+1} = \frac{35s^2 + 35s - 21s - 21 + 15}{(s+1)}$

$= \frac{35s^2 + 14s - 6}{(s+1)}$

(ii) $\Rightarrow \{s^2y - sy(0) - y'(0)\} - 4\{sx - x(0)\} + 3y = \frac{15 \cdot 2}{s^2+4}$

$\Rightarrow s^2y - s27 + 55 - 4sx + 35 \cdot 4 + 3y = \frac{30}{s^2+4}$

$\Rightarrow -4sx + (s^2+3)y = 27s - 195 + \frac{30}{s^2+4} = \frac{27s^3 + 108s - 195s^2 - 780 + 30}{s^2+4}$

$= \frac{27s^3 - 195s^2 + 108s - 750}{s^2+4}$

$(s^2+3)x + sy = \frac{35s^2 + 14s - 6}{(s+1)}$

$-4sx + (s^2+3)y = \frac{27s^3 - 195s^2 + 108s - 750}{s^2+4}$

$D_x =$

$D_y =$

$D =$

$x = \frac{D_x}{D}$

$y = \frac{D_y}{D}$

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} \frac{35s^2+14s-6}{(s+1)} & s \\ \frac{27s^3-195s^2+108s-750}{s^2+4} & s^2+3 \end{vmatrix}}{\begin{vmatrix} s^2+3 & s \\ -4s & s^2+3 \end{vmatrix}}$$

$$x = \frac{30s}{s^2+1} - \frac{45}{s^2+9} + \frac{3}{s+1} + \frac{2s}{s^2+4}$$

$\Rightarrow L^{-1}\{x\} = L^{-1}\{ - - - \}$

$\Rightarrow x(t) = 30\cos t - 15\sin 3t + 3e^{-t} + 2\cos 2t$

Similarly:

$y = \frac{D_y}{D} = \ldots$

$y = 30\cos 3t - 60\sin t - 3e^{-t} + \sin 2t$

A particle P of mass 2 grams moves on the x axis and is attracted toward origin 'O' with a force numerically equal to $8x$. If it is initially at rest at $\boxed{x=10}$. Find its position at any subsequent time assuming

a) no other forces act

b) a damping force numerically equal to 8 times the instantaneous velocity acts.

$x(t)$

$v(t) = \frac{dx}{dt}$

$a(t) = \frac{dv(t)}{dt} = \frac{d^2x}{dt^2}$

Solⁿ: (a) (Mass)·(Acceleration) = Net force.

$\Rightarrow 2 \cdot \frac{d^2x}{dt^2} = -8x$

$\Rightarrow \frac{d^2x}{dt^2} + 4x = 0$

$x(0) = 10$

$x'(0) = 0$

$\mathcal{L}\{\frac{d^2x}{dt^2}\} + 4\mathcal{L}\{x\} = 0$

$\Rightarrow s^2x - sx(0) + x'(0) - 4x = 0$

$\Rightarrow s^2x - 10s - 4x = 0$

$\Rightarrow x = \frac{10s}{s^2+4}$

$\Rightarrow \mathcal{L}^{-1}\{x\} = 10\mathcal{L}^{-1}\{\frac{s}{s^2+4}\} = 10\cos 2t$

$\therefore \boxed{X(t) = 10\cos 2t}$

(b) considering damping force $= -8\frac{dX}{dt}$

diff: $2\frac{d^2X}{dt^2} = -8X - 8\frac{dX}{dt}$

$X(0) = 10$; $X'(0) =$

$\Rightarrow 2X'' + 8X' + 8X = 0$

$\Rightarrow \boxed{\mathcal{L}\{X'' + 4X' + 4X\} = \mathcal{L}\{0\}} \Longrightarrow X(t) = 10e^{-2t} + 20te^{-2t}$

$\Rightarrow s^2x(t) - sx(0) - x'(0) + 4sx(t) - 4x(0) + 4x(t) = 0$

$\Rightarrow s^2x(t) - 10s + 4sx(t) - 40 + 4x(t) = 0$

$\Rightarrow x(s^2 + 4s + 4) - 10s - 40 = 0$

$\Rightarrow x = \frac{10s+40}{(s+2)^2} = 10\frac{s+4}{(s+2)^2} = 10\left(\frac{s+2}{(s+2)^2} + \frac{2}{(s+2)^2}\right)$

$= 10\left(\frac{1}{s+2} + \frac{2}{(s+2)^2}\right)$

$= 10\left(\mathcal{L}(e^{-2t}) + 2\mathcal{L}(te^{-2t})\right)$

$\therefore X = 10e^{-2t} + 20te^{-2t}$

Electrical Circuits

$Q(0) = 0$

$I(0) = 0$

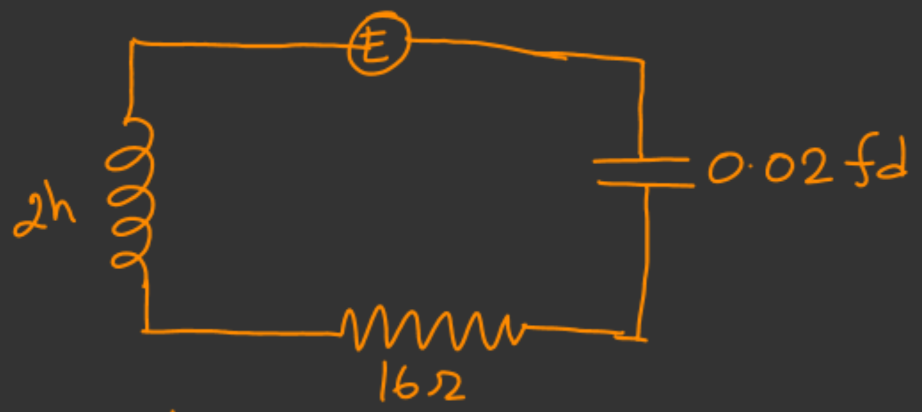
$I = \frac{Q(t)}{t} = Q'(t)$

$Q'(0) = 0$

Find the charge and current at any time $t > 0$

if a) $E = 300$ volts

(b) $E = 100 \sin 3t$ volts.



Q : instantaneous charge

I = - - - current.

$$H \frac{dI}{dt} + RI + \frac{Q}{C} = E$$

$$\Rightarrow H \cdot \frac{d^2Q}{dt^2} + R \cdot \frac{dQ}{dt} + \frac{Q}{C} = E$$

(a)

$$\Rightarrow 2 \frac{d^2Q}{dt^2} + 16 \frac{dQ}{dt} + \frac{Q}{0.02} = 300$$

$$\Rightarrow 2Q'' + 16Q' + 50Q = 300$$

$Q(0) = Q'(0) = 0$

$$\vdots$$

$$Q(t) = 6 - 6e^{-4t} \cos 3t - 8e^{-4t} \sin 3t.$$

$$I(t) = \frac{dQ(t)}{dt}$$