$$\begin{bmatrix} 1 & 0 & 3 \end{bmatrix}$$

Solⁿ: eigen value: $\lambda = 1, 2, 2$

eigen vector: for $\lambda = 2$: $p_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$; $p_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

for $\lambda = 1$: $p_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

there are three basis vectors in total. So the matrix

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

diagonalizes the matrix A. as,

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D$$

A and D are the similar matrix:

(*) Determinant

(*) if A is invertible then D is also invertible

(*) Same Rank

(*) Same nullity

(*) Trace

(*) characteristic polynomial

(*) Eigenvalues

(*) Power of a Matrix:

$$D = P^{-1}AP$$

$$D^K = (P^{-1}AP)^K$$

$$\Rightarrow D^K = P^{-1}A^K P$$

$$\Rightarrow A^K = PD^K P^{-1}$$

$$B = P^{-1}AP$$

B is similar to A

(*) Same eigen space

Find $A^{13}$:

$$A^{13} = P D^{13} P^{-1}$$

$$= \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{13} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8192 & 0 & 0 \\ 0 & 8192 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -8190 & 0 & -16382 \\ 8191 & 8192 & 8191 \\ 8191 & 0 & 16383 \end{bmatrix}$$

Show that the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ is not diagonalisable.

$$\det(\lambda I - A) = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda-1 & 0 & 0 \\ -1 & \lambda-2 & 0 \\ 3 & -5 & \lambda-2 \end{bmatrix} = (\lambda-1)(\lambda-2)^2$$

$$(\lambda-1)(\lambda-2)^2 = 0$$

$\lambda = 2$

$$\begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ -3 & 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} r_2' = r_1 + r_2; \\ r_3' = r_2 \times 3 + r_3 \end{bmatrix}$$

System,
$-x_1 = 0$
$5x_2 = 0$

free, $x_3$

Let, $x_3 = t$
$x_1 = 0$
$x_2 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(Q) If a Matrix is $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

then find the eigen values and eigen vectors of Matrices.

(a) $A^{-1}$

(b) $A^5$

Sol^n: $A^{-1} \rightarrow 1, \frac{1}{2}, \frac{1}{2}$

$A \rightarrow \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$1, 2, 2$

$A^5 \rightarrow 1^5, 2^5, 2^5 \rightarrow 1, 32, 32$

Thm: If a matrix $A$ has the eigen values and eigen-vectors are $\lambda_1, \lambda_2, \ldots \lambda_k$ and $V_1, V_2, \ldots V_k$ respectively then the eigen values and eigen-vectors of the matrix $A^m$ are $\lambda_1^m, \lambda_2^m, \ldots \lambda_k^m$ and $V_1, V_2, V_3 \ldots V_k$ respectively.