$$A=\begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

$$A=\begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 0 & -1 & 2 & 12 & 16 & -5 \\ 0 & -1 & 2 & 12 & 16 & -5 \\ 0 & -1 & 2 & 12 & 16 & -5 \end{bmatrix} \quad \begin{bmatrix} r_2' = r_1 \times 3 + r_2 \\ r_3' = r_1 \times 2 + r_3 \\ r_4' = r_1 \times 4 + r_4 \end{bmatrix}$$

$$A=\begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 0 & -1 & 2 & 12 & 16 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} r_3' = r_2 - r_3 \\ r_4' = r_2 - r_4 \end{bmatrix}$$

$$A=\begin{bmatrix} 1 & -2 & 0 & -4 & -5 & 3 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} r_1' = r_1(-1) \\ r_2' = r_2(-1) \end{bmatrix}$$

$$A=\begin{bmatrix} \boxed{1} & 0 & -4 & -28 & -37 & 13 \\ 0 & \boxed{1} & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} r_1' = \\ r_1 + r_2 \times 2 \end{bmatrix}$$

Since this matrix has two leading 1's, so it has rank 2.

In order to find the nullity we have to find the solution space of the system:

$$AX = 0$$

the system is:

$$\boxed{x_1} - 4x_3 - 28x_4 - 37x_5 + 13x_6 = 0$$

$$\boxed{x_2} - 2x_3 - 12x_4 - 16x_5 + 5x_6 = 0$$

leading variable. x_1, x_2.

free variable x_3, x_4, x_5, x_6

let,

$x_3 = t_1$

$x_4 = t_2$

$x_5 = t_3$

$x_6 = t_4$

$x_1 = 4x_3 + 28x_4 + 37x_5 - 13x_6$

$= 4t_1 + 28t_2 + 37t_3 - 13t_4$

$x_2 = 2t_1 + 12t_2 + 16t_3 - 5t_4$

sol^n:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 4t_1 + 28t_2 + 37t_3 - 13t_4 \\ 2t_1 + 12t_2 + 16t_3 - 5t_4 \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} = t_1 \underbrace{\begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{v_1} + t_2 \underbrace{\begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{v_2} + t_3 \underbrace{\begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{v_3} + t_4 \underbrace{\begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{v_4}$$

$$\{v_1, v_2, v_3, v_4\} \downarrow R^6$$

Here, the vectors v_1, v_2, v_3 & v_4 form the basis for the null space. so it has nullity 4.

$$k_1 v_1 + k_2 v_2 + k_3 v_3 + \cdots + k_n v_n = 0$$

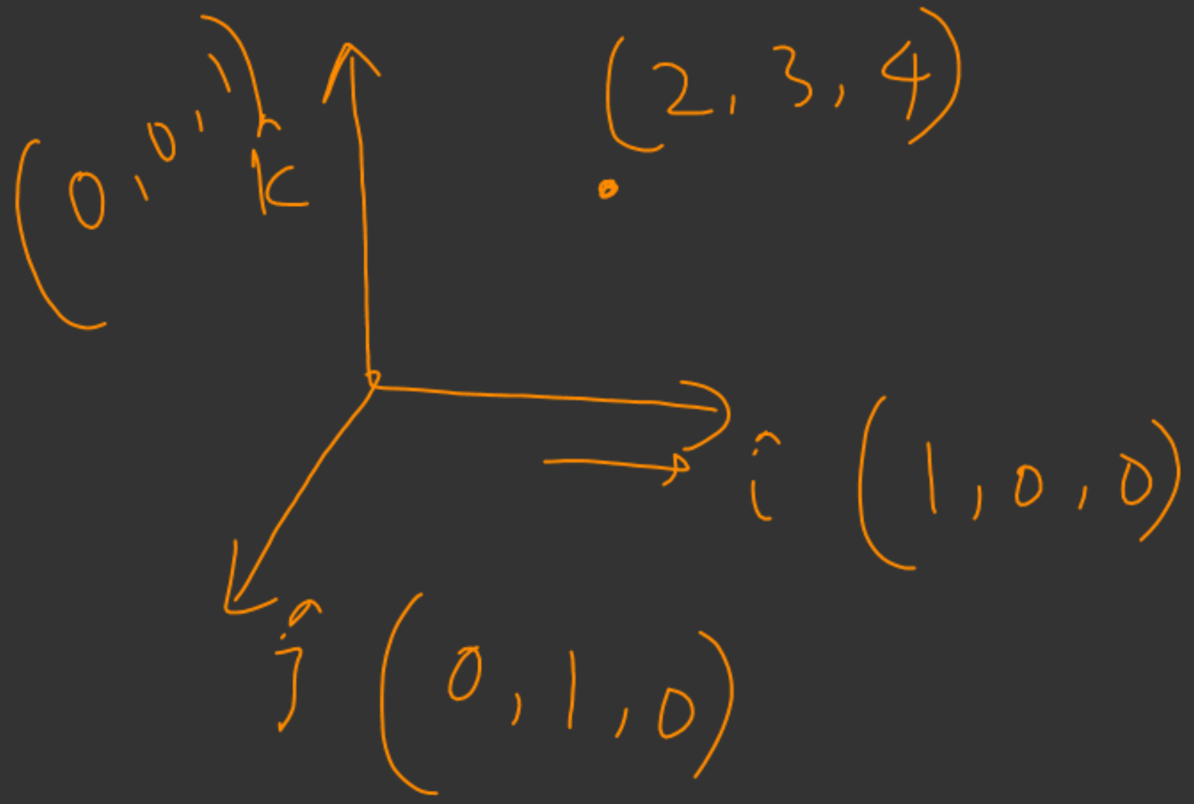
$$2\hat{i} + 3\hat{j} + 4\hat{k}$$

$(2, 3, 4)$

R^4

R^2

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Eigenvalues and eigenvectors.

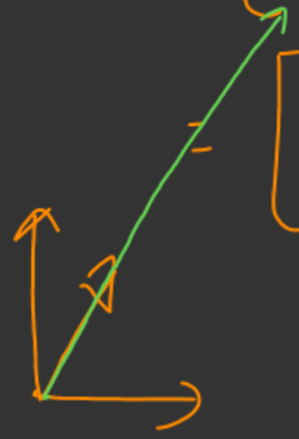
$$\underbrace{A}_{\text{Matrix}}\underbrace{x}_{\text{vector}} = \underbrace{\lambda}_{\text{scalar}}\underbrace{x}_{\text{vector}}$$

vector

vector

$$V = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$



$$AV = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3+0 \\ 8-2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$= 3\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$AV = \underbrace{3}_{\text{eigen-value}}V$$

$$Ax = \lambda x$$

$$\Rightarrow Ax = \lambda I x$$

$$\Rightarrow (A - \lambda I)x = 0$$

solvable iff $\boxed{\det(A - \lambda I) = 0}$

theorem: If A is an $n \times n$ matrix, then λ is an eigen-value of A if and only if it satisfies the eqn:

$$\det(\lambda I - A) = \det(A - \lambda I) = 0$$

this eqn is also called the characteristic eqn of A and $P(\lambda) = \det(A - \lambda I)$ is the characteristic polynomial.

Find the eigenvalues of $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$

Soln: let, the eigen value is λ.

characteristic eqn is: $\det(A - \lambda I) = 0$ —— (1)

Here, $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$

$\lambda I = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$

$\Rightarrow (1) \Rightarrow \det(A - \lambda I) = 0$

$\Rightarrow \det \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 4 & -17 & 8-\lambda \end{pmatrix} = 0$

$\Rightarrow -\lambda(-8\lambda + \lambda^2 + 17) - 1(-4) = 0$

$\Rightarrow 8\lambda^2 - \lambda^3 - 17\lambda + 4 = 0$

$\Rightarrow \lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$

Solving this we get,

$\lambda_1 = 4$

$\lambda_2 = 2 + \sqrt{3}$, $\lambda_3 = 2 - \sqrt{3}$

$$A=\begin{bmatrix}-1 & 2 & 0 & 4 & 5 & -3\\ 3 & -7 & 2 & 0 & 1 & 4\\ 2 & -5 & 2 & 4 & 6 & 1\\ 4 & -9 & 2 & -4 & -4 & 7\end{bmatrix}$$

$$A=\begin{bmatrix}-1 & 2 & 0 & 4 & 5 & -3\\ 0 & -1 & 2 & 12 & 16 & -5\\ 0 & -1 & 2 & 12 & 16 & -5\\ 0 & -1 & 2 & 12 & 16 & -5\end{bmatrix}\quad \begin{bmatrix} r_2' = r_1\times 3 + r_2\\ r_3' = r_1 \times 2 + r_3\\ r_4' = r_1\times 4 + r_4\end{bmatrix}$$

$$A=\begin{bmatrix}-1 & 2 & 0 & 4 & 5 & -3\\ 0 & -1 & 2 & 12 & 16 & -5\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}\quad \begin{bmatrix} r_3' = r_2 - r_3\\ r_4' = r_2 - r_4\end{bmatrix}$$

$$A=\begin{bmatrix}1 & -2 & 0 & -4 & -5 & 3\\ 0 & 1 & -2 & -12 & -16 & 5\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}\quad \begin{bmatrix} r_1' = r_1(-1)\\ r_2' = r_2(-1)\end{bmatrix}$$

$$A=\begin{bmatrix}\boxed{1} & 0 & -4 & -28 & -37 & 13\\ 0 & \boxed{1} & -2 & -12 & -16 & 5\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}\quad \begin{bmatrix} r_1' = \\ r_1 + r_2 \times 2\end{bmatrix}$$

Since this matrix has two leading 1's, so it has rank 2.

In order to find the nullity we have to find the solution space of the system:

$$AX = 0$$

the system is:

$$\boxed{x_1} - 4x_3 - 28x_4 - 37x_5 + 13x_6 = 0$$

$$\boxed{x_2} - 2x_3 - 12x_4 - 16x_5 + 5x_6 = 0$$

leading variable. x_1, x_2.

free variable x_3, x_4, x_5, x_6

let,

$x_3 = t_1$

$x_4 = t_2$

$x_5 = t_3$

$x_6 = t_4$

$$x_1 = 4x_3 + 28x_4 + 37x_5 - 13x_6$$

$$= 4t_1 + 28t_2 + 37t_3 - 13t_4$$

$$x_2 = 2t_1 + 12t_2 + 16t_3 - 5t_4$$

sol^n:

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{bmatrix} = \begin{bmatrix}4t_1+28t_2+37t_3-13t_4\\ 2t_1+12t_2+16t_3-5t_4\\ t_1\\ t_2\\ t_3\\ t_4\end{bmatrix} = t_1\underbrace{\begin{bmatrix}4\\2\\1\\0\\0\\0\end{bmatrix}}_{v_1} + t_2\underbrace{\begin{bmatrix}28\\12\\0\\1\\0\\0\end{bmatrix}}_{v_2} + t_3\underbrace{\begin{bmatrix}37\\16\\0\\0\\1\\0\end{bmatrix}}_{v_3} + t_4\underbrace{\begin{bmatrix}-13\\-5\\0\\0\\0\\1\end{bmatrix}}_{v_4}$$

$$\{v_1, v_2, v_3, v_4\} \to R^6$$

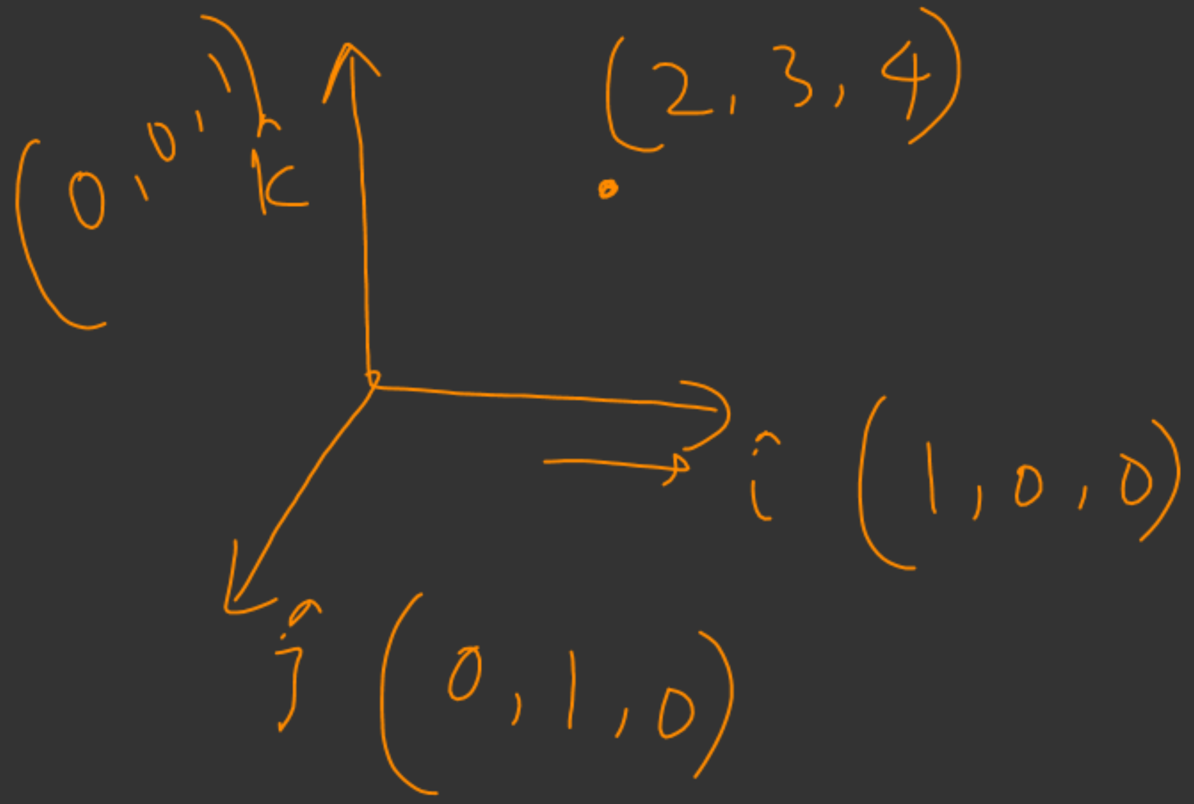
Here, the vectors v_1, v_2, v_3 & v_4 form the basis for the null space. So it has nullity 4.

$$k_1 v_1 + k_2 v_2 + k_3 v_3 + \cdots + k_n \underline{v_n} = 0$$

$$2\hat{i} + 3\hat{j} + 4\hat{k}$$

R^4

R^2



$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Eigenvalues and eigenvectors.

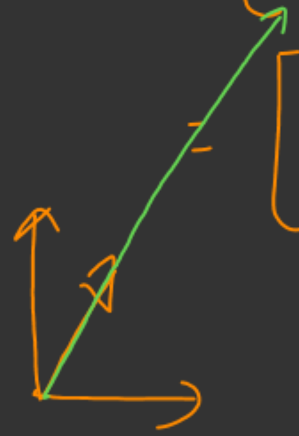
$$Ax = \lambda x$$

vector, vector, vector

Matrix, vector, scalar

$$V = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$



$$AV = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3+0 \\ 8-2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$AV = 3V$$

eigen-value.

$$Ax = \lambda x$$
$$\Rightarrow Ax = \lambda I x$$
$$\Rightarrow (A - \lambda I)x = 0$$

$\Rightarrow$ solvable iff $\boxed{\det(A - \lambda I) = 0}$

theorem: If A is an $n \times n$ matrix, then λ is an eigen-value of A if and only if it satisfies the eqn:

$$\det(\lambda I - A) = \det(A - \lambda I) = 0$$

this eqn is also called the characteristic eqn of A.

and $P(\lambda) = \det(A - \lambda I)$ is the characteristic polynomial.

Find the eigenvalues of $A=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$

Soln: Let, the eigen value is λ.

characteristic eqn is: $\det(A-\lambda I)=0 \quad \text{—} ①$

Here, $A=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$

$\lambda I = \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$

$\Rightarrow ① \Rightarrow \det(A-\lambda I)=0$

$\Rightarrow \det\begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 4 & -17 & 8-\lambda \end{pmatrix}=0$

$\Rightarrow -\lambda(-8\lambda+\lambda^2+17)-1(-4)=0$

$\Rightarrow 8\lambda^2-\lambda^3-17\lambda+4=0$

$\Rightarrow \lambda^3-8\lambda^2+17\lambda-4=0$

Solving this we get,

$\lambda_1 = 4$

$\lambda_2 = 2+\sqrt{3}, \ \lambda_3 = 2-\sqrt{3}$

Find the bases for the eigen-spaces of the Matrix

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \qquad (A-\lambda I) = \begin{bmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{bmatrix}$$

Soln hints:

(i) eigen value: $\det(A-\lambda I)=0$

(ii) eigen vector: $(A-\lambda I)\underline{x}=0$

solve $\underline{x}$ for each eigen value.

$$\det(A-\lambda I) = \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\Rightarrow (\lambda-1)(\lambda-2)^2 = 0$$

$$\lambda = 1, 2, 2$$

For eigen-vector:

$$\begin{bmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

for $\lambda = 1$

$$\begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} r_2' = r_1 + r_2 \\ r_3' = r_1 + r_3 \\ r_1' = (-1) r_1 \end{bmatrix}$$

System: $x_1 + 2x_3 = 0$

$x_2 - x_3 = 0$

let free variable: $x_3 = t$

$\therefore x_2 = t$

$x_1 = -2x_3 = -2t$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

So $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda = 1$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$A^3 - 5A^2 + 8A - 4I = 0$$

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} -2 & 0 & -6 \\ 3 & 4 & 3 \\ 3 & 0 & 7 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -6 & 0 & -14 \\ 7 & 8 & 7 \\ 7 & 0 & 15 \end{bmatrix}$$

Characteristic polynomial:

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$A^3 - 5A^2 + 8A - 4I = 0$$

$$A^3 \cdot A^{-1} - 5A^2 \cdot A^{-1} + 8AA^{-1} - 4IA^{-1} = 0$$

$$\Rightarrow A^2 - 5A + 8I - 4A^{-1} = 0$$

$$\Rightarrow 4A^{-1} = A^2 - 5A + 8I$$

$$\Rightarrow A^{-1} = \frac{1}{4}\left(A^2 - 5A + 8I\right) \longrightarrow *$$

$$AA^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^2 - 5A + 8I = \begin{bmatrix} -2 & 0 & -6 \\ 3 & 4 & 3 \\ 3 & 0 & 7 \end{bmatrix} - 5\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -5 & -16 \\ -2 & 2 & -2 \\ -2 & 0 & 0 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/4 & -5/4 & -4 \\ -1/2 & 1/2 & -1/2 \\ -1/2 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{aligned} r_2' &= r_2 - r_3 \\ r_3' &= r_3 - r_2 \end{aligned}$$

System, $-2x_1 - 2x_3 = 0$

$\Rightarrow \boxed{2x_1 + 2x_3 = 0}$

free, x_2, x_3

Let,

$x_2 = t$

$x_3 = p$

$x_1 = -\frac{2p}{p}$

$= -p$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -p \\ t \\ p \end{bmatrix} = p \underset{(*)}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}} + t \underset{(*)}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$$

Cayley Hamilton theorem: every square matrix satisfies its own characteristic equation: that is if A is an $n \times n$ matrix whose characteristic eqn is

$$\lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \cdots + c_n = 0$$

then $A^n + c_1 A^{n-1} + \cdots + c_n = 0$

$\boxed{\text{Diagonalization:}}$

(*) If A and B are square matrices, then we say that B is similar to A if there is an invertible matrix P such that $B = P^{-1} A P$

Def: A square matrix A is said to be diagonalizable if it is similar to some diagonal matrix, that is, if there exists an invertible matrix P such that $P^{-1} A P$ is diagonal. In this case the matrix P is said to diagonalize A.

Find a matrix P thas diagonalizes

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Soln: $\lambda = 2$; $P_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$; $P_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\lambda = 1$; $P_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$(\lambda-1)(\lambda-2)^2 = 0$$

$$\lambda = 1, 2, 2$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(*) Algebraic Multiplicity :

Geometric Multiplicity :

$\lambda = 1$: $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

$\lambda = 2$; $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$