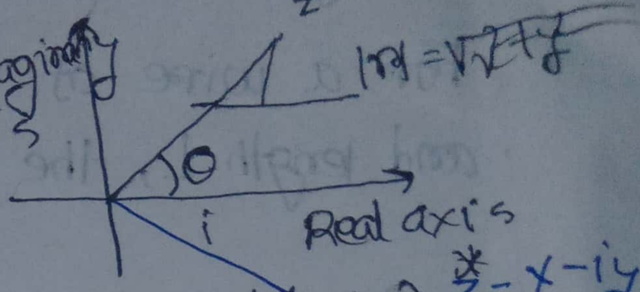
plex variable

$z = x + iy$

ginary part Axis



$\theta = \tan^{-1}(y/x)$

$i r \sin\theta$

$+ i \sin\theta\,]$

→ Euler Formula

$) . (x - iy)$

$- = |z|^2$

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

$$z_1 - z_2 = x_1 + iy_1 - x_2 - iy_2 = (x_1 - x_2) + i(y_1 - y_2)$$

$$z_1 \cdot z_2 = (x_1 + iy_1)\cdot(x_2 + iy_2)$$

$$= x_1 x_2 + iy_1 x_2 + ix_1 y_2 - y_1 \cdot y_2$$

$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

$$\Rightarrow \frac{z_1}{z_2} = ?$$

$$z_1 = r_1 \cos\theta_1 + ir_1 \sin\theta_1$$

$$z_2 = r_2 \cos\theta_2 + ir_2 \sin\theta_2$$

$$z_1 \cdot z_2 = [r_1(\cos\theta_1 + i\sin\theta_1)] \cdot [r_2(\cos\theta_2 + i\sin\theta_2)]$$

$$= (r_1\cos\theta_1 + ir_1\sin\theta_1)(r_2\cos\theta_2 + ir_2\sin\theta_2)$$

$$= r_1 r_2 [\cos\theta_1 \cdot \cos\theta_2 + i\sin\theta_1 \cos_2] + [i\sin\theta_2 \cos\theta_1 \; \cos\theta_1 \sin\theta_2]$$

$$= r_1 \cdot r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

$$= r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

$$= \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

De moivre's theorem:

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$z_1 z_2 = r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

if $z_1 = z_2$ then

$r_1 = r_2 = r$

$\theta_1 = \theta_2 = \theta$

$\therefore z^2 = r^2[\cos 2\theta + i\sin 2\theta]$

$z^n = [r(\cos\theta + i\sin\theta)]^n$

$= r^n[\cos n\theta + i\sin n\theta]$

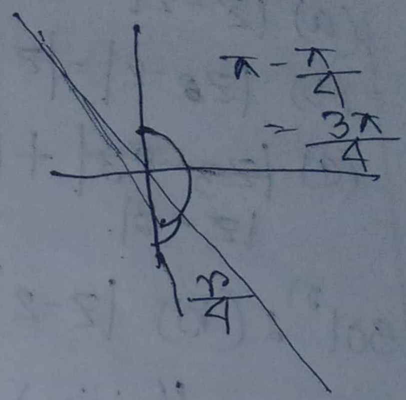
$r^n[\cos(\theta+2\pi k)n + i\sin(\theta+2\pi k)n]$

$k = 0, 1, 2 \ldots (n-1)$

$= r^{\frac{1}{n}}\left[\cos\left(\frac{\theta+2\pi k}{n}\right) + i\sin\left(\frac{\theta+2\pi k}{n}\right)\right]$

(*) Evaluate: $(-1+i)^{\frac{1}{5}}$

let $z = (-1+i)$ —— (1) no



$r = \sqrt{(-1)^2+1^2} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{1}{-1}\right)$

$= \frac{3\pi}{4}$

$z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

Now, $(-1+i)^{\frac{1}{5}} = (\sqrt{2})^{\frac{1}{5}}\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)^{\frac{1}{5}}$

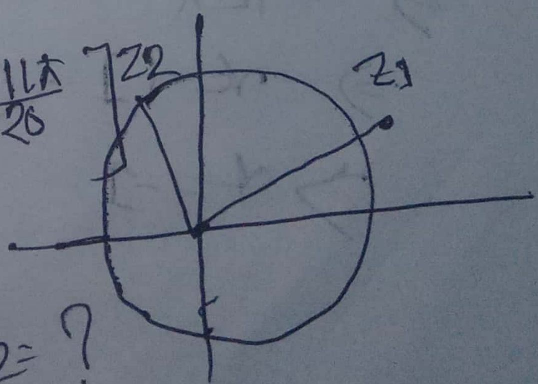
$= (\sqrt{2})^{\frac{1}{5}}\left[\cos\left(\frac{\frac{3\pi}{4}+2\pi k}{5}\right) + i\sin\left(\frac{\frac{3\pi}{4}+2\pi k}{5}\right)\right.$

$= (\sqrt{2})^{\frac{1}{5}}\left[\cos\left(\frac{8\pi k+3\pi}{20}\right) + i\sin\left(\frac{8\pi k+3\pi}{20}\right)\right.$

$k = 0, 1, 2, 3, 4$

$k=0$, $\underbrace{(-1+i)^{\frac{1}{5}}}_{z_1} = \underbrace{(\sqrt{2})^{\frac{1}{5}}}_{\text{modulas}}\left[\cos\frac{3\pi}{20} + i\sin\frac{3\pi}{20}\right]$

$k=1$, $z_2 = (\sqrt{2})^{\frac{1}{5}}\left[\cos\frac{11\pi}{20} + i\sin\frac{11\pi}{20}\right]$



$k=2$

$k=3$

$k=4$

$z_2 = ?$

* Graph the following eq^n:

$z = x + iy$

(a) $|z-2| = 3$

(b) $|z-2| = |z+4|$

(c) $\cancel{|z-3|} + |z+3| = 10$

$|z-3|$

sol^n: (b) $|z-2| = |z+4|$

$\Rightarrow |(x+iy)-2| = |(x+iy)+4|$

$\Rightarrow |(x-2)+iy| = |(x+4)+iy|$

$\Rightarrow \sqrt{(x-2)^2+y^2} = \sqrt{(x+4)^2+y^2}$

$\Rightarrow (x-2)^2+y^2 = (x+4)^2+y^2$

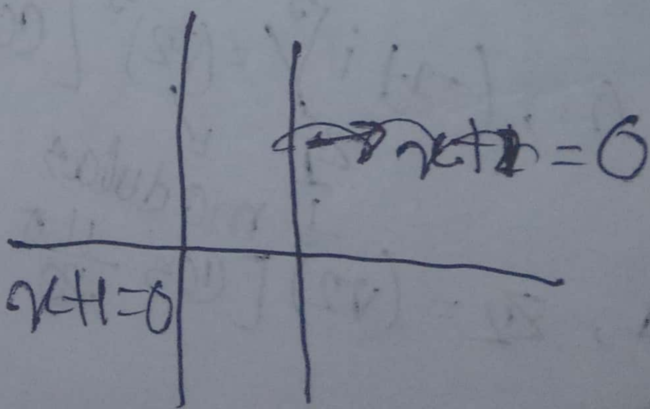
$\Rightarrow x^2-4x+4 = x^2+8x+16$

$\Rightarrow -4x-8x = 16-4$

$\Rightarrow -12x = 12$

$\Rightarrow x+1 = 0$

$\Rightarrow x = -1$



Function, limit & continuity:

$z = x+iy$

$f(z) = f(x+iy) = u(x,y) + iv(x,y)$

* $f(z) = \sin z$

$f(x+iy) = \sin(x+iy)$

$= \sin x \cdot \cos iy + \cos x \sin iy$

$= \sin x \frac{1}{2}[e^{y}+e^{-y}] + \cos x \frac{1}{2}[e^{y}-e^{-y}]$

$= \sin x \cdot \cosh y + \cos x \cdot \cancel{\sinh y}$

$= u(x,y) + iv(x,y)$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

$\cosh x = \frac{1}{2}(e^{x}+e^{-x})$

$\sinh x = \frac{1}{2}(e^{x}-e^{-x})$

$\cancel{= -\frac{1}{2}(}$

$= -\frac{1}{2}(e^{-x}-e^{x})$

$(e^{-x}-e^{x}) = -\sinh x$

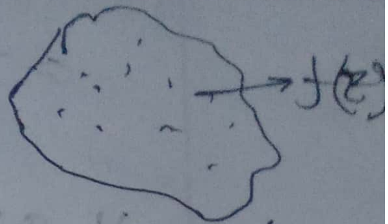
$\frac{1}{i} = \frac{i}{i^2} = -i$

$= \frac{1}{ii}$

$(e^{-x}-e^{x})$

Derivations:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$



let, $z = x + iy$

$\Delta z = \Delta x + i\Delta y$.

$$f'(z) = \lim_{(\Delta x, \Delta y) \to 0} \frac{f(x+iy+\Delta x+i\Delta y) - f(x+iy)}{(\Delta x + i\Delta y)}$$

Cauchy-Rieman-eqn

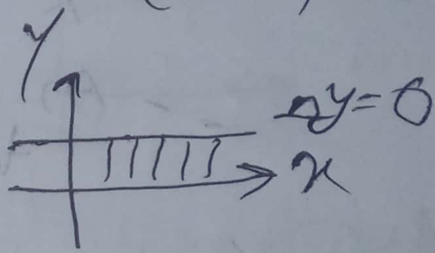
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Along x axis.

$$f'(z) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x+iy) - f(x+iy)}{\Delta x}$$

$(x+iy) \to (x,y)$



*) $f(x+iy) = u(x,y) + iv(x,y)$

$f(x+\Delta x+iy) = u(x+\Delta x, y) + iv(x+\Delta x, y)$

) $\Rightarrow f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + iv(x+\Delta x, y) - u(x,y) - iv(x,y)}{\Delta x}$

$$= \left[\lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}\right] + i$$

$$\left[\lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}\right]$$

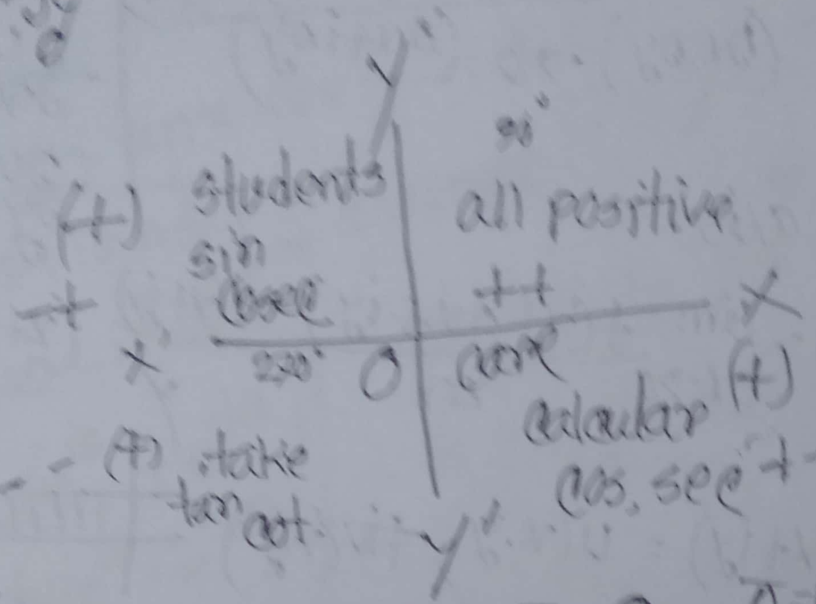
$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$

home ~~task~~

y axis →

$\frac{\partial u}{\partial y} - i \frac{\partial v}{\partial y}$

sin, cos, sec, cosec, tan, cot



২য় চতুর্ভাগে যাবার রাস্তা $= \pi - \theta$ $\pi = 180$

৩য় চতুর্ভাগে " $= \pi + \theta$

৪র্থ " " $= 2\pi - \theta$

$\sin(-\theta) = -\sin\theta$

$\sin(90 - \theta) = \sin\theta / \cos\theta$

$\sin(90 + \theta)$

$\sin(180 - \theta)$

$\sin(180 + \theta)$

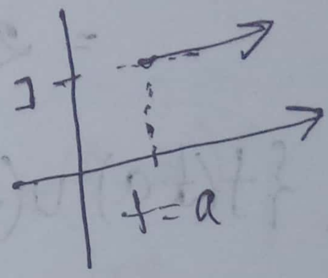
cos θ
−sin θ
−cos θ

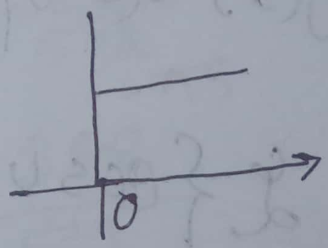
Limit step function:

$$u(t-a)=\begin{cases}0 & 0\le t<a\\ 1 & t\ge a\end{cases}$$

$u(t-a)$

$$u(t)=\begin{cases}1 & t\ge 0\\ 0 & t<0\end{cases}$$





$$f(t)=\begin{cases}g(t) & 0\le t\le a\\ h(t) & t\ge a\end{cases}$$

$$f(t)=g(t)\left[u(t-0)-u(t-a)\right]+h(t)\left[u(t-a)\right]$$

$$=g(t)u(t)-g(t)u(t-a)+h(t)u(t-a)$$

$$\boxed{f(t)=g(t)-g(t)u(t-a)+h(t)u(t-a)}$$

$$f(t) = \begin{cases} 20t & 0 < t < 5 \\ 0 & t > 5 \end{cases}$$ piecewise function

Sol^n: $f(t) = 20t - 20t\,u(t-5) + 0$

$= 20t - 20t\,u(t-5)$

** $\mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-as}F(s)$

$\mathcal{L}\{g(t)\,u(t-a)\} = e^{-as}\,\mathcal{L}\{g(t+a)\}$

Find: $\mathcal{L}\{\cos u(t-\pi)\}$

$g(t) = \cos t$

$g(t+\pi) = \cos(t+\pi) = -\cos t$

$\mathcal{L}\{\cos t\ u(t-\pi)$

$= e^{-\pi s}\,\mathcal{L}\{-\cos t\}$

$= e^{-\pi s}\,\frac{-s}{s^2+1}$

$= \frac{-s e^{-\pi s}}{s^2+1}$ Ans

Solve $\cancel{y''} \cdot y' + y = f(t)$ $y(0)=5$, where $f(t) = \begin{cases} 0 & 0 \le t < \pi \\ 3\cos t & t \ge \pi \end{cases}$

$f(t) = \cancel{f(t)}\ h(t) \cdot u(t-a) = 3\cos t\ u(t-\pi)$

$y' + y = 3\cos t\ u(t-\pi)$

$\Rightarrow \mathcal{L}\{y'\} + \mathcal{L}\{y\} = 3\mathcal{L}\{\cos t\ u(t-\pi)\}$

$\Rightarrow sy(s) - y(0) + y(s) = 3 \dfrac{-se^{-\pi s}}{s^2+1}$

$\Rightarrow sy(s) - 5 + y(s) = \dfrac{-3se^{-\pi s}}{s^2+1}$

$\Rightarrow y(s+1) = 5 - \dfrac{3se^{-\pi s}}{s^2+1}$

$\Rightarrow y(s+1) = \dfrac{5s^2+5-3se^{-\pi s}}{s^2+1}$

$\Rightarrow y = \dfrac{5s^2+5-3se^{-\pi s}}{(s+1)(s^2+1)}$

$\therefore \dfrac{5s^2+5-3se^{-\pi s}}{(s+1)(s^2+1)} = \dfrac{A}{s+1} + \dfrac{Bs+C}{s^2+1}$

$\Rightarrow 5s^2+5-3se^{-\pi s} = A(s^2+1) + (Bs+C)(s+1)$

$5s^2 + 5 - 3s\,e^{-\pi s} = As^2 + A + Bs^2 + Bs + Cs + C$

$A + B = 5$, $\quad A + C = 5$

$\Rightarrow A = 5 - B$ $\quad \Rightarrow C = 5 - A$

$\Rightarrow C = 5 - 5 + B$

$\Rightarrow C = B$

$B + C = -3e^{-\pi s}$

$\Rightarrow B + B = -3e^{-\pi s}$ $\quad \therefore C = -\frac{3}{2}e^{-\pi s}$

$\Rightarrow 2B = -3e^{-\pi s}$

$\Rightarrow B = -\frac{3}{2}e^{-\pi s}$

$\therefore A = 5 - B$

$= 5 + \frac{3}{2}e^{-\pi s}$

$= \frac{10 + 3e^{-\pi s}}{2}$,

$\therefore \mathcal{L}\ \frac{10 + 3e^{-\pi s}}{2(s+1)} + \frac{\frac{3}{2}e^{-\pi s}}{s^2+1} - \frac{3}{2}\frac{e^{-\pi s}}{(s^2+1)}$

Complex form of fourier series:

$$F(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}} \quad \text{where } C_n = \frac{1}{2l}\int_{-l}^{l} F(x) e^{-\frac{in\pi x}{l}} dx$$

* Find the Fourier series of $F(x) = e^{x}, \ -\pi \le x \le \pi$

sol^n: We know, $F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{\pi n x}{\pi} + b_n \sin\frac{n\pi x}{\pi}\right)$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x) \cos\frac{\pi n x}{\pi} dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{x} \cos nx \, dx \quad \text{--- (ii) no}$$

$$= \cancel{\frac{1}{\pi}\{e^{x}.e}$$

let, $I = \int e^{x} \cos nx \, dx$

$$= e^{x}.\frac{\sin nx}{n} - \int e^{x}.\frac{\sin nx}{n} dx$$

$$= e^{x}.\frac{\sin nx}{n} - \frac{1}{n}\left[-\frac{e^{x}\cos nx}{n} + \int e^{x}\frac{\cos nx}{n}\right.$$

$$= e^{x}.\frac{\sin nx}{n} + \frac{1}{n^2}e^{x}\cos nx - \frac{1}{n^2}I$$

$$= e^{x}.\frac{\sin nx}{n} + \frac{1}{n^2}e^{x}\cos nx - \frac{1}{n^2}I$$

$$\Rightarrow I + \frac{1}{n^2} I = e^x \frac{\sin nx}{n} + \frac{1}{n^2} e^x \cos nx$$

$$\Rightarrow I\left(\frac{1+n^2}{n^2}\right) = \frac{1}{n^2}\left[n e^x \sin nx + e^x \cos nx\right]$$

$$\Rightarrow I = \frac{e^x(\cos nx + n \sin nx)}{1+n^2}$$

$$a_n = \frac{1}{\pi}\left[\frac{e^x(\cos nx + n \sin nx)}{1+n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{\pi}(\cos n\pi + 0) - e^{-\pi}(\cos n\pi + 0)\right]$$

$$= \frac{1}{\pi(1+n^2)}\left[(e^{\pi} - e^{-\pi})(-1)^n\right]$$

$$= \frac{(-1)^n(e^{\pi} - e^{-\pi})}{\pi(1+n^2)}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cdot dx = \frac{1}{\pi}\left[e^x\right]_{-\pi}^{\pi}$$

$$= \left(\frac{e^{\pi} - e^{-\pi}}{\pi}\right)$$

$$= \frac{?}{\pi}$$

Now $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x) \frac{\sin n\pi x}{\pi}\, dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{x} \sin nx \, dx$$

$$b_n = \frac{(-1)^{n+1}}{\pi(1+n^2)}\, n\left(e^{\pi} - e^{-\pi}\right)$$

$$F(x) = \frac{e^{\pi} - e^{-\pi}}{2\pi} + \sum_{n=1}^{\infty} \frac{(-1)^n}{\pi(1+n^2)}\left(e^{\pi} - e^{-\pi}\right)$$

$$\cos nx + \frac{(-1)^{n+1}}{\pi(1+n^2)}\, n\left(e^{\pi} - e^{-\pi}\right)\sin nx$$

Ans

$F(x) \rightarrow$ এর Graph:

ODD and even Function:

$F(-x) = F(x) \longrightarrow$ even Function $\longrightarrow \cos x, x^2, x^4$

$F(-x) = -F(x) \longrightarrow$ Odd $\longrightarrow x^5, x^3, \sin x$

$\cos(181)$

$\cos(-181)$

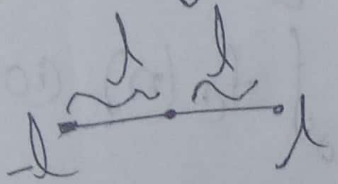
$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)\right]$$

even: $b_n = 0$, $F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$

odd $a_n = 0$ $F(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

Half Range sine and cosine series:

Half range sine: $a_n = a_0 = 0$; $b_n = \frac{1}{l}\int_{-l}^{l} F(x) \sin\frac{n\pi x}{l}\,dx$



$$= \frac{2\cdot 1}{l}\int_0^l F(x) \sin\frac{n\pi x}{l}\,dx$$

Half range cosine:

$b_n = 0$

$$a_n = \frac{2}{l}\int_0^l F(x) \cos\frac{n\pi x}{l}\,dx$$

~~Infinite Fourier~~

Fourier transformation:

Finite Fourier sine transformation: $\int_0^l F(x) \sin \frac{n\pi x}{l}$

finit Inverse sine transform $F(x) = \frac{2}{l} \sum_{n=1}^{\infty} f_s(n)$

Finite Fourier Cosine $= f_c(n) = \int_0^l F(x) \cos \frac{n\pi x}{l} dx$

Finit fourier Inverse ... $F(x) = \frac{1}{l} f_c(0) + \frac{2}{l} \sum_{n=1}^{\infty} f_c(n)$

Infinite F. sine: $f_s(n) = \int_0^{\infty} F(x) \sin nx \, dx$ $0 < x < \infty$

" Inverse: $F(x) = \frac{2}{\pi} \int_0^{\infty} f_s(n) \sin nx \, dn$

Infinite F. Cosine: $f_c(n) = \int_0^{\infty} F(x) \cos nx \, dx$

Inverse F. Cosine $F(x) = \frac{2}{\pi} \int_0^{\infty} f_c(n) \cos nx \, dn$

tutorial - 13 $\quad UV = U \int \cos nx$

Find the Fourier cosine transform e^{-x}, $x \geq 0$ $\quad (F(x))$

$$f_c(n) = \int_0^\infty F(x) \cos nx \, dx$$

$$= \int_0^\infty e^{-x} \cos nx \, dx$$

$$= \left[e^{-x} \frac{\sin nx}{n} - \int \frac{d}{dx}(e^{-x}) \int \cos nx \, dx \, dx \right]_0^\infty$$

$\frac{1}{1+x^2}$

$$= [0] - \int_0^\infty -e^{-x} \frac{\sin nx}{n} dx$$

$$= \int_0^\infty e^{-x} \frac{\sin nx}{n} dx$$

let, $I' = \frac{1}{n} \int_0^\infty e^{-x} \sin nx \, dx$

$$= \frac{1}{n} \left[e^{-x} \cdot \frac{-\cos nx}{n} - \int \frac{d}{dx}(e^{-x}) \int \sin nx \, dx \, dx \right]$$

$$= \frac{1}{n} \left[e^{-x} \cdot \frac{-\cos nx}{n} - \int -e^{-x} \cdot \frac{-\cos nx}{n} dx \right]$$

$$= \frac{1}{n^2} e^{-x} \cdot -\cos nx + \frac{1}{n^2} \int e^{-x} \cos nx \, dx$$

$$= \frac{1}{n^2} e^{-x} - \cos nx - \frac{1}{n} \cdot I$$

$\int v\,du = uv - \int v\,du$

* Find the Fourier cosine transform of

$f(x) = e^{-x^2}, \; x \geq 0$, $f_c(n) = \int_0^\infty e^{-x^2} \cos nx \, dx$

$I = \int_0^\infty e^{-x^2} \cos nx \, dx$

$\int u\,dv = uv - \int u\,dv$

$\frac{d}{dx}\left(e^{-x^2}\right) = e^{-x^2}.(-2x)$

$\frac{dI}{dn} = \int_0^\infty e^{-x^2} (-x \sin nx) \, dx$

$= \frac{1}{2} \int_0^\infty (-2x) e^{-x^2} . \sin nx \, dx = \frac{1}{2} \int_0^\infty \sin nx \, d\left(e^{-x^2}\right) dx$

$= \frac{1}{2} \left\{ \left[\sin nx . e^{-x^2} \right]_0^\infty - \int_0^\infty n e^{-x^2} \cos nx \, dx \right\}$

$= \frac{1}{2} \left[0 \quad - nI \right]$

$= -\frac{1}{2} nI$

$\frac{dI}{dn} = -\frac{1}{2} nI$

$\Rightarrow \frac{dI}{I} = \frac{-1}{2} n \, dn$

$$\Rightarrow \ln I = -\frac{1}{2}\frac{n^2}{2} + \ln c$$

$$\Rightarrow \ln\left(\frac{I}{c}\right) = -\frac{n^2}{4}$$

$$\Rightarrow I = ce^{-\frac{n^2}{4}}$$

$$\Rightarrow \int_0^\infty e^{-x^2}\cos nx\,dx = ce^{-\frac{n^2}{4}}$$

putting, $n=0$

$$\int_0^\infty e^{-x^2}dx = c$$

$\therefore$ let

$$\Rightarrow x = \sqrt{p} = p^{\frac{1}{2}}$$

$$\Rightarrow 2x\,dx = dp$$

$$\Rightarrow dx = \frac{1}{2}\frac{1}{x}dp = \frac{1}{2}p^{-\frac{1}{2}}dp$$

x	0	∞
p	0	∞

$$\Rightarrow c = \int_0^\infty e^{-p} \cdot \frac{1}{2} p^{-\frac{1}{2}} \, dp$$

$$\Rightarrow c = \frac{1}{2} \int_0^\infty e^{-p} p^{\left(\frac{1}{2}-1\right)} \, dp$$

$$\Rightarrow c = \frac{1}{2} \int_0^\infty e^{-p} p^{-\frac{1}{2}} \, dp$$

$$= \frac{1}{2} \Gamma\frac{1}{2} = \frac{\sqrt{\pi}}{2}$$

Gamma Functi

$$\Gamma n = \int_0^\infty e^{-x} x^{(n-1)} \, dx$$

$$\Rightarrow \cancel{\Gamma\frac{n}{2} = \sqrt{\pi}}$$

$$\Rightarrow \Gamma\frac{n}{2} = \sqrt{\pi}$$

$$\therefore \int_0^\infty e^{-x^2} \cos nx \, dx$$

$$= \frac{\sqrt{\pi}}{2} e^{-\frac{n^2}{4}}$$

Ans

Find the Fourier sine transform of $F(x) = \frac{e^{-ax}}{x}$
where $a>0$ and $x \geq 0$

$$f_s(n) = \int_0^\infty \frac{e^{-ax}}{x} \sin nx \, dx$$

** Complex Fourier transform:**

$$f(n) = \int_{-\infty}^{\infty} F(x) e^{-inx} \, dx$$

Find the complex Fourier transform
$F(x) = e^{-a|x|}$ where $a>0$

$$|x| = \begin{cases} -x & \text{if } x<0 \\ x & \text{if } x>0 \end{cases}$$

Sol^n: $f(n) = \int_{-\infty}^{\infty} e^{-a|x|} e^{-inx} \, dx$

$$= \int_{-\infty}^{0} e^{-(-x)} e^{-inx} \, dx + \int_0^\infty e^{-ax} e^{inx} \, dx$$

$$= \int_{-\infty}^{0} e^{(a-in)x} \, dx + \int_0^\infty e^{-(a+in)x} \, dx$$

$$= \left[\frac{e^{(a-in)x}}{(a-in)}\right]_{-\infty}^{0} + \left[\frac{e^{-(a+in)x}}{-(a+in)}\right]_{0}^{\infty}$$

$$\cancel{= \left[e^{0} - e^{-\infty}\right] + \left[\right.}$$

$$= \frac{1}{a-in}\left[e^{0} - e^{-\infty}\right] - \frac{1}{(a+in)}\left[e^{-\infty} - e^{0}\right]$$

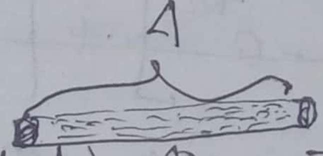
$$= \frac{1}{a-in} + \frac{1}{a+in}$$

$$= \frac{2a}{a^2-(in)^2} = \frac{2a}{a^2+n^2} \text{ Ans.}$$

Application: Use Finite Fourier transform to solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ u(0,t) = 0;\ u(4,t) = 0;\ u(x,0) = 2x$$

where $0 < x < 4,\ t > 0$



x

$u(0,t) = 0$ $\quad$ $u(4,0) = 0$

$u(x,0) = 2x$

$u(x,t) = ??$

Sol^n: taking Fourier sine transformation on both side

$$\int_0^4 \frac{\partial u}{\partial t} \sin\frac{n\pi x}{4}\,dx = \int_0^4 \frac{\partial^2 u}{\partial x^2} \sin\frac{n\pi x}{4}\,dx$$

let, $\displaystyle\int_0^4 u(x,t) \sin\frac{n\pi x}{4}\,dx = U(n,t) = U$

then: $\displaystyle\frac{dU}{dt} = \int_0^4 \frac{\partial u}{\partial t} \sin\frac{n\pi x}{4}\,dx$

$$= \int_0^4 \frac{\partial^2 u}{\partial x^2} \sin\frac{n\pi x}{4}\,dx$$

$$= \left[\sin \frac{n\pi x}{4} \frac{\partial u}{\partial x} \right]_0^4 - \int \cos \frac{n\pi x}{4} \left(\frac{n\pi}{4}\right) \frac{\partial u}{\partial x} dx$$

$$= 0 - \frac{n\pi}{4} \left\{ \left[\cos \frac{n\pi x}{4} \, u(x,t) \right]_0^4 + \int_0^4 \sin \frac{n\pi x}{4} \frac{n\pi}{4} \, u(x,t) \, dx \right\}$$

$$= 0 - \frac{n^2\pi^2}{16} \int_0^4 u(x,t) \sin \frac{n\pi x}{4} dx$$

$$\frac{dU}{dt} = -\frac{n^2\pi^2}{16} U$$

$$\Rightarrow \frac{dU}{U} = -\frac{n^2\pi^2}{16} dt$$

$$\Rightarrow U = A e^{-\frac{n^2\pi^2}{16} t}$$

$$\Rightarrow U(n,t) = A e^{-\frac{n^2\pi^2}{16} t}$$

$$\text{at } t = 0; \quad U(n,0) = A \quad \text{—(1)}$$

$$U = U(n,t) = \int_0^4 u(x,t) \sin \frac{n\pi x}{4} dx$$

$$U = U(n,0) = \int_0^4 u(x,0) \sin \frac{n\pi^2 x}{4} dx$$

dx

$$\pi x \Big) \cdot \frac{4}{n\pi}\Big]_0^4 + \int_0^4 2\cos\frac{n\pi x}{4}\cdot\frac{4}{n\pi}\,dx$$

$$+\frac{8}{n\pi}\int_0^4 \cos\frac{n\pi x}{4}\,dx$$

$$+\frac{8}{n\pi}\left[\sin\frac{n\pi x}{4}\cdot\frac{4}{n\pi}\right]_0^4$$

$n\pi$

$$s\, n\pi \; e^{-\frac{n^2\pi^2 t}{16}}$$

nverse Fourier transformation

$$2 \sum_{n=1} u(n,t)\sin\frac{n\pi x}{4}$$

$$2\sum_{n=1} \frac{-32}{n\pi}\cos n\pi \; e^{-\frac{\pi^2 n^2 t}{16}} \sin\frac{n\pi x}{4}$$

Ans.

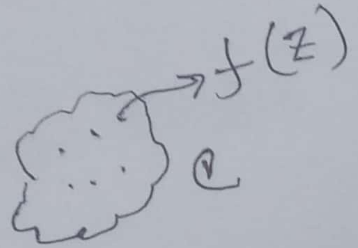
finite fourier sine

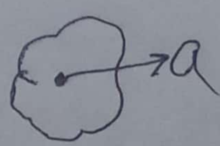
$$f_s(n) = \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$$

Inverse:

$$f(x) = \frac{2}{l} \sum_{n=1}^{\infty} f_s(n) \sin \frac{n\pi x}{l}$$

Cauchys theorem	Cauchys Integral Formula
let 'C' be a simple closed curve. If $f(z)$ is analytic within the region bounded by 'C' as well as on 'C' then the cauchy's theorem $$\oint_C f(z)\,dz = 0$$	If $f(z)$ is analytic with on on a simple closed curve 'C' and 'a' is any point interior to 'C', then $$f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-a)}\,dz$$ where 'C' is traversed in the positive (counterclockwise). Also, the nth derivative of $f(z)$ at $z = a$ is given by $$f^{(n)}(a) = \frac{n!}{2\pi i}\oint \frac{f(z)}{(z-a)^{n+1}}\,dz$$





Cauchy's theorem:

$f(z) \rightarrow$ Analytic

$\downarrow$

Cauchy Reimann equation

$$z = x + iy$$

$$f(z) = u + iv \rightarrow u(x,y) + iv(x,y)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\therefore \oint f(z)\, dz = 0$$

Proof: $\oint_c f(z)\, dz$

$$= \oint_c (u + iv)(dx + idy)$$

$$= \oint_c (u\,dx + iv\,dx + u\,i\,dy - v\,dy)$$

$$= \oint_c (u\,dx - v\,dy) + i\oint_c (v\,dx + u\,dy)$$

$$= \oint_c [u(x,y)\,dx - v(x,y)\,dy] + i\oint_c [v(x,y)\,dx + u(x,y)\,dy]$$

line integral

Green theorem:

$$\oint_C P(x,y)\,dx + Q(x,y)\,dy$$

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\Rightarrow \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) = 0$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0$$

again:

$$= \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$$= 0$$

$$\oint_C f(z)\,dz = 0$$

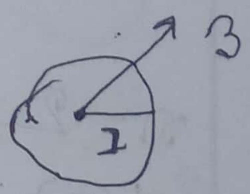
analytic হলে প্রতিটি point এ derivative হবে।

problem: Evaluate $\oint_C \frac{dz}{(z-3)}$ where C is

(a) the circle $|z| = 1$

(b) the circle $|z+i| = 4$

(a) $\oint_C f(z)\,dz = \oint_C \frac{1}{(z-3)}\,dz$



$\oint_C \frac{dz}{z-3} = 0$

$(x-0)^2 + (y-1)^2 = r^2$

$|x + iy + i| = 4$

$\Rightarrow |x + i(y+1)| = 4$

$\Rightarrow x^2 + (1+y)^2 = 4^2$

$(0, -1)$

* $\oint_C \frac{f(z)}{(z-a)}\,dz = 2\pi i f(a)$

$\therefore \oint_C \frac{f(z)}{(z-a)}\,dz = 2\pi i$

constant $f(z) = 1$

$f(3) = 1$

* Evaluate $\oint_C \frac{5z^2 - 3z + 2}{(z-1)^3}$

Solⁿ: According to cauchy's integral formula.

Get, $f^n(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz$

$n = 2$

$f(z) = 5z^2 - 3z + 2$

$f'(z) = 10z - 3$

$f''(z) = 10$

$f^2(a=1) = 10$

$\oint_C \frac{5z^2 - 3z + 2}{(z-1)^3} dz$

$= \frac{10 \cdot 2\pi i}{2!}$

$= 10\pi$ Ans.

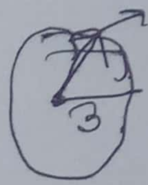
(24) Evaluate (a) $\oint_C \frac{\cos z}{(z-\pi)} dz$ (b) $\oint_C \frac{e^z}{z(z+1)} dz$, where C is the circle $|z-1| = 3$.

(a) $\oint_C \frac{\cos z}{(z-\pi)} dz = 2\pi i f(z).$ $\rightarrow f(z) = \cos z$

$f(\pi) = -1$

$= -2\pi i$

$z = \pi \rightarrow 3.14$



(b) $\oint_C \frac{e^z}{z(z+1)} dz$

$= \oint_C e^z \left(\frac{1}{z} - \frac{1}{z+1}\right) dz$ $\quad f(z) = e^z$

$f(0) = e^0$

$= \oint_C \left(\frac{e^z}{z} - \frac{e^z}{z+1}\right) dz = 2\pi i\left[e^0 - e^{-1}\right]$ $\quad f(-1) = e^{-1}$

$= 2\pi i\left(1 - e^{-1}\right)$ Ans