## Complex Variable:

$z = x + iy$ → $x$: Real part, $iy$: imaginary part

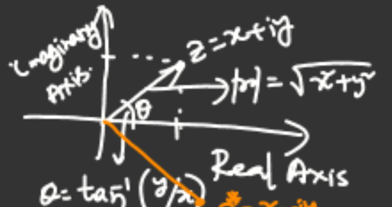
$x = r\cos\theta$

$y = r\sin\theta$.

$\therefore z = r\cos\theta + ir\sin\theta$

$= r[\cos\theta + i\sin\theta]$

$= re^{i\theta}$ → Euler eqn.




$\theta = \tan^{-1}(y/x)$ → Argument.

$zz^* = (x+iy)(x-iy)$

$= x^2 + y^2 = |z|^2$

$|z|^2 = zz^*$

$z^2 = z \cdot z^*$.

$z = x + iy$

$|z| = \sqrt{x^2+y^2}$

$z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

$z_1 - z_2 = x_1 + iy_1 - x_2 - iy_2 = (x_1 - x_2) + i(y_1 - y_2)$.

$z_1 \cdot z_2 = (x_1 + iy_1)\cdot(x_2 + iy_2)$

$= x_1x_2 + iy_1x_2 + ix_1y_2 - y_1y_2$

$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

$\Rightarrow \dfrac{z_1}{z_2} = ??$ $\dfrac{z_1}{z_2} = \dfrac{x_1y_1 + x_2y_2 + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$

$z_1 = r_1\cos\theta_1 + ir_1\sin\theta_1$

$z_2 = r_2\cos\theta_2 + ir_2\sin\theta_2$

$z_1z_2 = [r_1(\cos\theta_1 + i\sin\theta_1)]$
$[r_2(\cos\theta_2 + i\sin\theta_2)]$.

$= r_1r_2[\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2$
$+ i\sin\theta_2\cos\theta_1 - \sin\theta_1\sin\theta_2]$

$= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)$
$+ i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$

$= r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

$= r_1r_2e^{i(\theta_1+\theta_2)}$

$\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$.

$= \dfrac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$.

De moivre's theorem:

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

$z_1 z_2 = r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$.

if $z_1 = z_2$ then..

$r_1 = r_2 = r$

$\theta_1 = \theta_2 = \theta$

$\therefore\ z^2 = r^2[\cos 2\theta + i\sin 2\theta]$.

$z^n = [r(\cos\theta + i\sin\theta)]^n$

$= r^n[\cos n\theta + i\sin n\theta]$.

$= r^n[\cos(\theta + 2\pi k)n + i\sin(\theta+2\pi k)n]$.

$k = 0, 1, \ldots (n-1)$

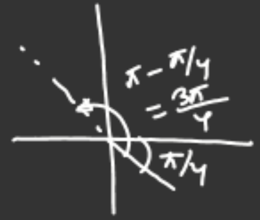
$\therefore z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta+2\pi k}{n}\right) + i\sin\left(\frac{\theta+2\pi k}{n}\right)\right]$

(*) Evaluate : $(-1+i)^{\frac{1}{5}}$

Let, $z = (-1+i)$ —— (1)

$r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{1}{-1}\right) = \frac{3\pi}{4}$




$z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

Now: $(-1+i)^{\frac{1}{5}} = (\sqrt{2})^{\frac{1}{5}}\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]^{1/5}$

$= (\sqrt{2})^{\frac{1}{5}}\left[\cos\left(\frac{\frac{3\pi}{4}+2\pi k}{5}\right) + i\sin\left(\frac{\frac{3\pi}{4}+2\pi k}{5}\right)\right]$

$$z_1 = (\sqrt{2})^{\frac{1}{5}}\left[\cos\left(\frac{8\pi k + 3\pi}{20}\right) + i\sin\left(\frac{8\pi k+3\pi}{20}\right)\right]$$

$k = 0, 1, 2, 3, 4$.

for $k = 0$

$k = 1$

$k = 2$

$k = 3$

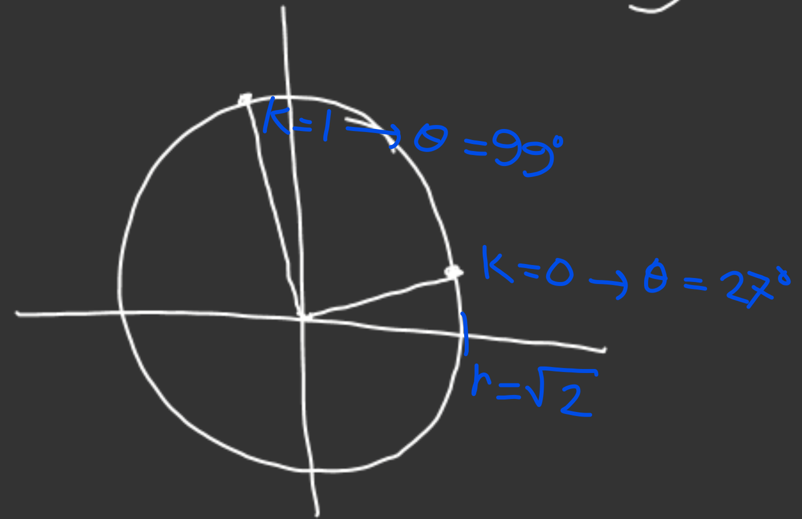
$k = 4$

for $k = 0$

$z_1 = (\sqrt{2})^{\frac{1}{5}}\left[\cos\frac{3\pi}{20} + i\sin\frac{3\pi}{20}\right]$

for $k = 1$

$z_1 = (\sqrt{2})^{\frac{1}{5}}\left[\cos\frac{11\pi}{20} + i\sin\frac{11\pi}{20}\right]$.



(*) Graph the following eqⁿ:

$z = x + iy$

(a) $|z-2| = 3$

(b) $|z-2| = |z+4|$

(c) $|z-3| + |z+3| = 10$.

Solⁿ: (b) $|z-2| = |z+4|$

$\Rightarrow |(x+iy)-2| = |(x+iy)+4|$

$\Rightarrow |(x-2)+iy| = |(x+4)+iy|$

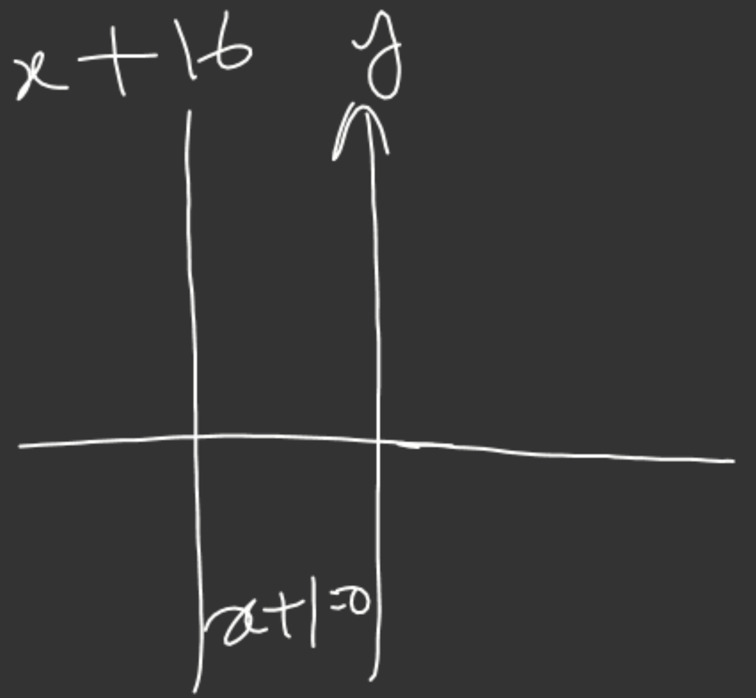
$\Rightarrow \sqrt{(x-2)^2 + y^2} = \sqrt{(x+4)^2 + y^2}$

$\Rightarrow (x-2)^2 + \cancel{y^2} = (x+4)^2 + \cancel{y^2}$

$\Rightarrow \cancel{x^2} - 4x + 4 = \cancel{x^2} + 8x + 16$

$\Rightarrow 12x + 12 = 0$

$\Rightarrow \boxed{x + 1 = 0}$



# Function, Limit & continuity:

$z = x + iy$

$f(z) = f(x+iy) = u(x,y) + iv(x,y)$

$* \ f(z) = \sin z$

$f(x+iy) = \sin(x+iy)$

$= \sin x \cos iy + \cos x \sin iy$

$= \sin x \cdot \underbrace{\frac{1}{2}\left[e^{-y} + e^{y}\right]} + \cos x \, \frac{1}{2i}\left[e^{-y} - e^{y}\right]$

$= \sin x \cdot \cosh y + i \cos x \sinh y$

$= u(x,y) + iv(x,y).$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$

$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right).$

$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right).$

$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$

$\frac{1}{2}\left(e^{-x} - e^{x}\right) = -\frac{1}{2}\left(e^{-x} - e^{x}\right).$

$= -\sinh x.$

$\frac{1}{i}$

$= \frac{i}{i \cdot i}$

$= \frac{i}{i^2}$

$= -i$

$\frac{1}{2i}\left(e^{-x} - e^{x}\right)$

$= (-i)(-1)$

$\sinh x$

$= i \sinh x.$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

let, $z = x + iy$

$\Delta z = \Delta x + i\Delta y$.

$$f'(z) = \lim_{(\Delta x, \Delta y) \to 0} \frac{f(x+iy+\Delta x+i\Delta y) - f(x+iy)}{(\Delta x + i\Delta y)}$$


Cauchy-Rieman eqn:

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}}$$

Along x axis.

$$f'(z) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x+iy) - f(x+iy)}{\Delta x} \quad \text{— (1)}$$


$x + iy \to (x, y)$

$\Delta y = 0$

(*) $f(x+iy) = u(x,y) + iv(x,y)$.

$f(x+\Delta x+iy) = u(x+\Delta x, y) + iv(x+\Delta x, y)$.

$$(1) \Rightarrow f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + iv(x+\Delta x, y) - u(x,y) - iv(x,y)}{\Delta x}$$

$$= \left[\lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}\right] + i\left[\lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}\right]$$

$$= \boxed{\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}}$$

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} - i\frac{\partial v}{\partial y}.$$

$\boxed{\text{y axis}} \longrightarrow \frac{\partial u}{\partial y} - i\frac{\partial v}{\partial y}$.