Chapter-4

General Vector Spaces

Vector space:

$V = \{v_1, v_2, v_3 \dots v_n\}$

প্রত্যেকটা set এর element হচ্ছে vector.

$v_1, v_2 \in V \qquad v_1 + v_2 \in V$

দুইটা ভেক্টর যদি V এর মধ্যে থাকে তাহলে $v_1 + v_2$; V এর মধ্যে থাকবে।

$cv_1 \in V$

(i) $v_1, v_2 \in V$ the $v_1 + v_2 \in V$

(ii) $v_1 + v_2 = v_2 + v_1$

(iii) $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$

(iv) $0 + v_1 = v_1 + 0 = v_1$

(v) $v_1 + (-v_1) = (-v_1) + v_1 = 0$

(vi) $k(v_1 + v_2) = kv_1 + kv_2$

যদি vector space হয় তাহলে এই condition গুলো fulfil করতে হবে।

(vii) $(k+m)v_1 = kv_1 + mv_1$

(viii) $k(mv_1) = (km)v_1$

(ix) $1v_1 = v_1$

Chapter 5

Eigenvalues and Eigenvectors

Definition 1: If A is an $n \times n$ matrix, then a nonzero vector x in R^n is called an eigenvector of A if Ax is a scalar multiple of x; that is,

n dimensional real vector space

$$Ax = \lambda x$$

The scalar λ is called an eigenvalue of A and x is said to be an eigenvector corresponding to λ.

Theorem 5.1.1:

If A is an $n \times n$ matrix, then λ is an eigenvalue of A if and only if it satisfies the equation

$$\det(\lambda I - A) = 0$$

This is called the characteristic equation of A.

Echolon form:

Ways to convert or reduce any matrix to echolon form.

1. The lower triangle will be 0. (All elements)

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & b_1 & b_2 \\ 0 & 0 & c_3 \end{bmatrix}$$

Key Properties of REF(row echolon form) Leading Entries Pivots):
1. The first non-zero number in each row is called a pivot (or leading entry).
2. Each pivot is to the right of the pivot in the row above it.
3. Pivots are always 1 (for simplicity, but sometimes other non-zero numbers are allowed).
Zero Rows at the Bottom:
1. Any rows with all zeros must be at the bottom of the matrix.
2. Entries Below Pivots are Zero:
3. All numbers directly below a pivot must be zero.

Example:

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix} \quad \begin{array}{l} R_1\times(-2) + R_2 \to R_2' \\ R_1\times(-3)+R_3 \to R_3' \\ R_1\times(-6)+R_4 \to R_4' \end{array}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \end{bmatrix} \quad R_2\times(-1)+R_4 = R_4'$$

$$= \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$ $\cancel{R_1 \times (-) + R_4 = R_4'}$

$R_3 - R_4 = R_4'$

$$= \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ → Echolon form of this matrix

* Reduced Row Echelon form এ leading variable এর উপরে এবং নিচে zero থাকতে হবে।

Example 3:

Let the eigenvalue is λ

The characteristic equation is: $\det(\lambda I - A) = 0$

$$\det\left\{\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}\right\} = 0$$

$$\Rightarrow \det\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda-8 \end{bmatrix} = 0$$

$$\Rightarrow \lambda(\lambda^2 - 8\lambda + 17) - 1(4) = 0$$

$$\Rightarrow \lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

$$\Rightarrow \lambda^3 - 4\lambda^2 - 4\lambda^2 + 16\lambda + \lambda - 4 = 0$$

$$\Rightarrow \lambda^2(\lambda - 4) - 4\lambda(\lambda - 4) + 1(\lambda - 4) = 0$$

$$\Rightarrow (\lambda - 4)(\lambda^2 - 4\lambda + 1) = 0$$

$\lambda = 4$; $\lambda^2 - 4\lambda + 1 = 0$

$$\Rightarrow \lambda = \frac{4 \pm \sqrt{16 - 4.1.1}}{2.1}$$

$$= \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$$

Thus, the eigenvalues of A are $\lambda = 4$, $\lambda = 2 \pm \sqrt{3}$.

* Find the eigenvectors :

$Ax = \lambda x$

$\Rightarrow \cancel{(A - \lambda I)} x = 0$ ...... (i)

$(\lambda I - A)$

$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$ $\lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$

$(A - \lambda I) = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 4 & -17 & 8-\lambda \end{bmatrix}$

for $\lambda = 4$ (i) $\Rightarrow$

$\begin{bmatrix} -4 & 1 & 0 \\ 0 & -4 & 1 \\ 4 & -17 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -4 & 1 & 0 \\ 0 & -4 & 1 \\ 4 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $[r_3' = (-4) \times r_2 + r_3]$

$\Rightarrow \begin{bmatrix} -4 & 1 & 0 \\ 0 & -4 & 1 \\ -16 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $[r_3' = (-4) \times r_3 + r_2]$

$\Rightarrow \begin{bmatrix} -4 & 1 & 0 \\ 0 & -4 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $[r_3' = (-4) \times r_1 + r_3]$

the system of eq^n becomes: $~~r_2' = 4 \times r_3~~$

$-4x_1 + x_2 = 0$

$-4x_2 + x_3 = 0$

x_3 is the free variable:

$x_3 = t$

$\therefore -4x_2 + t = 0$

$\Rightarrow -4x_2 = -t$

$\Rightarrow x_2 = \frac{1}{4}t$

$-4x_1 + \frac{1}{4}t = 0$

$\Rightarrow -4x_1 = -\frac{1}{4}t$

$\Rightarrow x_1 = \frac{1}{16}t$

Sol^n is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{16}t \\ \frac{1}{4}t \\ t \end{bmatrix} = \begin{bmatrix} 1/16 \\ 1/4 \\ 1 \end{bmatrix} t$$

eigen vector $\begin{bmatrix} 1/16 \\ 1/4 \\ 1 \end{bmatrix}$

Algebric Multiplicity:

$$(\lambda-1)(\lambda-2)^2=0$$

$\lambda=1$ এর Algebric Multiplicity 1

$\lambda=2$ এর Algebric Multiplicity 2

Example 7:

Let the eigenvalue is λ

The characteristic equation is: $\det(\lambda I - A)=0$

$$\det\left\{\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix}-\begin{bmatrix}0 & 0 & -2\\ 1 & 2 & 1\\ 1 & 0 & 3\end{bmatrix}\right\}=0$$

$$\Rightarrow \det\begin{bmatrix}\lambda & 0 & 2\\ -1 & \lambda-2 & -1\\ -1 & 0 & \lambda-3\end{bmatrix}=0$$

$$\Rightarrow \lambda(\lambda^2-3\lambda-2\lambda+6)+2(\lambda-2)=0$$

$$\Rightarrow \lambda^3-5\lambda^2+6\lambda+2\lambda-4=0$$

$$\Rightarrow \lambda^3-5\lambda^2+8\lambda-4=0$$

$$\lambda=1,2$$

$Ax = \lambda x$

$\Rightarrow (\lambda I - A)x = 0$

$$\begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ ---- (i)}$$

for $\lambda = 2$

$$\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad [r_1' = r_1/2]$$

$[r_2' = r_1 + r_2]$
$[r_3' = r_1 + r_3]$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The system of eq^n becomes:

$$x_1 + x_3 = 0$$

free variable:

$$x_2 = t$$
$$x_3 = s$$
$$x_1 = -s$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s \\ t \\ s \end{bmatrix} = s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are linearly independent.

These vectors form a basis for the eigenspace corresponding to $\lambda = 2$

for, $\lambda = 1$, (i) $\Rightarrow$

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad [r_2' = r_2 + r_1] \quad [r_3' = r_3 + r_1]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad [r_1' = (-2) \times r_2 + r_1]$$

the system of eq^n becomes:

$$x_1 = 0 + 2x_3 = 0$$
$$-x_2 + x_3 = 0$$

x_3 is the free variable:

$$x_3 = s$$

$$x_1 = -2x_3 \qquad \cancel{x_3 = x_2}$$
$$= -2s \qquad x_2 = x_3$$
$$= s$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2s \\ s \\ s \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} s$$

So, $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda = 1$.

→ Geomatric Multiplicity হচ্ছে কতগুলোর eigenvector পেয়েছি। AM $\geq$ GM.

Slide-016

T2. Cayley-Hamilton এ সাহায্যে matrix এর inverse matrix বের করা যায়।

* Show that the following matrices form a basis for M_{22}

$$\underbrace{\begin{bmatrix} 3 & 6 \\ 3 & 6 \end{bmatrix}}_{M_1}, \underbrace{\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}}_{M_2}, \underbrace{\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}}_{M_3}, \underbrace{\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}}_{M_4}, \quad M_{22} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$k_1 M_1 + k_2 M_2 + k_3 M_3 + k_4 M_4 = M_{22}$$

$$k_1 \begin{bmatrix} 3 & 6 \\ 3 & 6 \end{bmatrix} + k_2 \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix} + k_4 \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow k_1 \times 0 + k_2 \times (-1) + k_3 \times (-96) + k_4 \times 2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} 3k_1 + 0 + 0 + k_4 & 6k_1 - k_2 - 8k_3 \\ 3k_1 - k_2 - 12k_3 - k_4 & 6k_1 - 4k_3 + k_4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\boxed{\begin{aligned} 3k_1 + k_4 &= a \\ 6k_1 - k_2 - 8k_3 &= b \\ 3k_1 - k_2 - 12k_3 - k_4 &= c \\ 6k_1 - 4k_3 + k_4 &= d \end{aligned}}$$

$$\begin{bmatrix} \quad \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$$

Sol^n: Characteristic polynomial:

$P(\lambda) = \det(\cancel{A-\lambda}\ \lambda I - A)$.

$$\lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$$

$$(\lambda I - A) = \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -2 & 5 & \lambda-4 \end{bmatrix}$$

$$\therefore P(\lambda) = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -2 & 5 & \lambda-4 \end{vmatrix}$$

$$= \lambda(\lambda^2 - 4\lambda + 5) + 1(-2)$$

$$P(\lambda) = \lambda^3 - 4\lambda^2 + 5\lambda - 2$$

the characteristic eq^n is:

$$P(\lambda) = 0$$

$$\Rightarrow \lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$$

replace λ with A

$$A^3 - 4A^2 + 5A - 2 = 0 \quad \text{-----} \quad (1)$$

Now,

$$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 2 & -5 & 4 \\ 8 & -18 & 11 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 0 & 1 \\ 2 & -5 & 4 \\ 8 & -18 & 11 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -5 & 4 \\ 8 & -18 & 11 \\ 22 & -47 & 26 \end{bmatrix}$$

$(1) \Rightarrow$

$$\begin{bmatrix} 2 & -5 & 4 \\ 8 & -18 & 11 \\ 22 & -47 & 26 \end{bmatrix} - 4\begin{bmatrix} 0 & 0 & 1 \\ 2 & -5 & 4 \\ 8 & -18 & 11 \end{bmatrix} + 5\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix} - 2I = 0$$

$$\Rightarrow \begin{bmatrix} 2 & -5 & 4 \\ 8 & -18 & 11 \\ 22 & -47 & 26 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -4 \\ -8 & 20 & -16 \\ -32 & 72 & -44 \end{bmatrix} + \begin{bmatrix} 0 & 5 & 0 \\ 0 & 0 & 5 \\ 10 & -25 & 20 \end{bmatrix} = 0$$

$-2I$

$$\Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - 2I = 0$$

exam

Example:

Let V be the set of 2×2 matrices with real entries and take the vector space operations on V.

$\Rightarrow$ $u, v \in V$

$$u = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \quad v = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$$

$$u + v = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$$

$$= \cancel{u_{11}v_{11} + u_{12}v_{21}} \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix} \rightarrow M_{22}$$

$$v + u = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$$

$$= \begin{bmatrix} v_{11} + u_{11} & v_{12} \cancel{u_{1}} + u_{12} \\ v_{21} + u_{21} & v_{22} + u_{22} \end{bmatrix}$$

$\therefore u + v = v + u$

$k \in \mathbb{R}$

$$ku = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}_{2\times 2}$$

$$u + (-u) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} - u_{11} & u_{12} - u_{12} \\ u_{21} - u_{21} & u_{22} - u_{22} \end{bmatrix} = 0$$

$$Iu = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}_{(2\times 2)} = u$$

9টি condition check করব এবং দেখব যে 2×2 matrix আছে কিনা। এবং যদি আছে তাহলে vector space হবে

Vector subspace:

Def^n: A subset W of a vector space V is called a subspace of V if W is itself a vector space under addition and scalar multiplication defined on V.

theorem: 4.2.1

If W is a set of one or more vectors in a vector space V, then W is a subspace of V if and only if the following conditions are satisfied.

ⓐ If u and v are vectors in W, then let v is in W. /closed under addition.

ⓑ If k is a scaler and u is a vector in W. then ku is in W. /closed under multiplication

এই দুইটার condition fulfill করাতে হবে যদি vector subspace হয়।

(i) $u+v \in W$ if $u, v \in W$

(ii) $ku \in W$ if $k \in \mathbb{R}, c$ and $u \in W$

$k = 0$; $0 \cdot u = 0 \in W$

$(-1) \cdot u = -u \in W$

R^2-space:

x-axis $\rightarrow (x, 0)$
$\downarrow$
$(0, 0)$

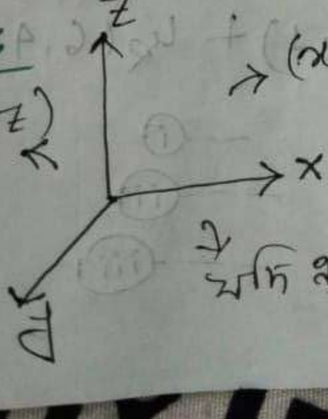
$\boxed{y = x+1}$

$(1,2) \rightarrow (3,5)$

এখানে (3,4) হবে কিন্তু হওয়া (3,5) সুতরাং subspace হবে না।

R^3-space:

$(0, y, z)$



$(x, 0, z) \rightarrow$ এখানের subspace

যদি এটার নেই তাহলে হবে $(x, y, 0)$

Linear Combination:

If w is a vector in a vector space V, then w is said to be a linear combination of the vectors $v_1, v_2, \ldots v_r$ in V if w can be expressed in the form:

$$k_1 v_1 + k_2 v_2 + \ldots + k_r v_r = w$$

Where $k_1, k_2, \ldots k_r$ are scalars. This scalars are called the coefficients of the linear combination.

Example 14:

$u = (1, 2, -1)$ in R^3 $w = (9, 2, 7)$

$v = (6, 4, 2)$ $w' = (4, -1, 8)$

Show that w is a linear combination of u and v

$\Rightarrow$ By the def^n: $k_1 u + k_2 v = w$

$$\Rightarrow k_1(1, 2, -1) + k_2(6, 4, 2) = (9, 2, 7)$$

$k_1 + 6k_2 = 9$ ——— (i)

$2k_1 + 4k_2 = 2$ ——— (ii)

$-k_1 + 2k_2 = 7$ ——— (iii)

(i) + (iii) =>

$$u_1 + 6u_2 = 9$$
$$-u_1 + 2u_2 = 7$$
$$8u_2 = 16$$
$$\Rightarrow u_2 = 2$$

In eq. (i) =>

$$u_1 + 6 \times 2 = 9$$
$$\Rightarrow u_1 = -3$$

$\therefore (u_1, u_2) = (-3, 2)$

So, w is a linear combination of u and v.

$$u_1 u + u_2 v = w'$$
$$\Rightarrow u_1(1, 2, -1) + u_2(6, 4, 2) = (4, -1, 8)$$

$$u_1 + 6u_2 = 4 \quad \text{---(iv)}$$
$$2u_1 + 4u_2 = -1 \quad \text{---(v)}$$
$$-u_1 + 2u_2 = 8 \quad \text{---(vi)}$$

(iv) + (vi) =>

$$u_1 + 6u_2 = 4$$
$$-u_1 + 2u_2 = 8$$
$$8u_2 = 12$$
$$\Rightarrow u_2 = \frac{3}{2}$$

In eq. (iv) => $u_1 + 6 \times \frac{3}{2} = 4$

$$\Rightarrow u_1 = -5$$

$\therefore (u_1, u_2) = (-5, \frac{3}{2})$

~~So, W' is not a~~

In (v) eq.

$$2\times(-5) + 4 \times \frac{3}{2}$$
$$= -4 \neq -1$$

So, W' is not a linear combination of u and v.

Linear Span:

If $S = \{w_1, w_2, \ldots w_r\}$ is a non-empty set of vectors in a vector space V, then the subspace W of V that consists of all possible linear combinations of the vectors in S is called the subspace of V generated by S and say that the vectors $w_1, w_2 \ldots w_r$ span W. We denote this subspace as

$$W = \text{span}\{w_1, w_2, \ldots w_r\} \text{ or } W = \text{span}(S).$$

* The first four Lagrange polynomials are:

$\{1, 1-t, 2-4t+t^2, 6-18t+9t^2-t^3\}$

(a) Show that the first four laguerre polynomials form a basis for P_3.

(b) Let B be the basis in part (a). find the Co-ordinate vector of the polynomial $p(t) = -10t+9t^2-t^3$ relative to B.

* $\boxed{a_0+a_1t+a_2t^2+a_3t^3 \rightarrow \text{3rd order polynomial}}$

Hints:

$$k_1P_1' + k_2P_2' + k_3P_3' + k_4P_4' = a_0+a_1t+a_2t^2+a_3t^3$$

Span:

Let, $P_1' = 1$

$P_2' = \cancel{2-4t+t^2}\ 1-t$

$P_3' = \cancel{6-18t+9t^2-t^3}\ 2-4t+t^2$

$P_4' = 6-18t+9t^2-t^3$

$\therefore \cancel{k_1\times 1 + k_2}$

$k_1\times 1 + k_2\times(1-t) + k_3(2-4t+t^2) + k_4(6-18t+9t^2-t^3) = a_0+a_1t+a_2t^2+a_3t^3$

$u_1 + u_2 + 2u_3 + 6u_4 = a_0$ —— (i)

$-u_2 - 4u_3 - 18u_4 = a_1$ —— (ii)

$u_3 + 9u_4 = a_2$ —— (iii)

$-u_4 = a_3$ —— (iv)

$\Rightarrow u_4 = -a_3$

In eq (iii)

$u_3 - 9a_3 = a_2$

$\Rightarrow u_3 = a_2 + 9a_3$

In eq. (ii) $-u_2 - 4(a_2 + 9a_3) + 18a_3 = a_1$

$\Rightarrow -u_2 - 4a_2 - 36a_3 + 18a_3 = a_1$

$\Rightarrow -u_2 = a_1 + 4a_2 + 36a_3 - 18a_3$

$\Rightarrow u_2 = -a_1 - 4a_2 - 18a_3$

In eq (i)

$u_1 - a_1 - 4a_2 - 18a_3 + 2a_2 + 18a_3 - 6a_3 = a_0$

$\Rightarrow u_1 = a_0 + a_1 + 4a_2 + \cancel{18a_3} - 2a_2 - \cancel{18a_3} + 6a_3$

$\Rightarrow u_1 = a_0 + a_1 + 2a_2 + 6a_3$

So, the first four Laguerre polynomials are linearly independent.

The determinant of the system is:

$$\Delta = \begin{vmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -1 & -4 & -18 \\ 0 & 1 & 9 \\ 0 & 0 & -1 \end{vmatrix} - 1 \begin{vmatrix} 0 & -4 & -18 \\ 0 & 1 & 9 \\ 0 & 0 & -1 \end{vmatrix} + 2 \begin{vmatrix} 0 & -1 & -18 \\ 0 & 0 & 9 \\ 0 & 0 & -1 \end{vmatrix} - 6 \begin{vmatrix} 0 & -1 & -4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix}$$

$$= 1 + 0 + 0 + 0$$

$$= 1 \neq 0$$

So, the system has a unique. So, the first four laguerre polynomials span P_3

As, the set of vectors $\{P_1', P_2', P_3'\}$ are linearly independent and span P_3, so it forms a basis for P_3.

$Ax = 0$ হলে Solution শূন্য Nullity value

Inverse formula:

$$[A|I] \longrightarrow [I|A^{-1}]$$

Augment
অর্থ সংযুক্ত

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

Inversion form: $[A|I]$

$$= \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] \quad [r_2' = (-2) \times r_1 + r_2] \quad [r_3' = (-1) \times r_1 + r_3]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \quad [r_1' = (-2) \times r_2 + r_1] \quad [r_3' = 2 \times r_2 + r_3]$$

$$r_3' = (-1) \times r_3$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \quad \begin{matrix} [r_1' = (-9) \times r_3 + r_1] \\ [r_2'' = 3 \times r_3 + r_2] \end{matrix}$$

$$= [I \mid A^{-1}]$$

LU factorization

$A = LU$

L is lower triangular and U is upper triangular.

Example:

Solve the following system of equations using LU factorization.

$$\underbrace{\begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix}}_{b}$$

$Ax = b$

$\Rightarrow x = A^{-1}b$ ---- (i) $A^{-1} =$ calculator

$$A^{-1} = \frac{adj(A)}{|A|}$$

~~Det A = -14~~ 2

$Adj(A) =$

$$A^{-1} = \begin{bmatrix} 0.7 & 0.3 & 0.4 \\ 0.1 & -0.1 & 0.2 \\ 0.1 & 0.4 & 0.2 \end{bmatrix}$$

$$x = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Lecture-1

Lineare Algebra

* $3x+2y=11$

$x-2y=1$

$Ax=b$

$$\begin{bmatrix}3 & 2\\ 1 & -2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}11\\ 1\end{bmatrix}$$

Row Echelon and Reduced Row Echelon form:

$$\begin{bmatrix}1 & 3 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \rightarrow \text{Reduced Row Echelon form}$$

$$\begin{bmatrix}1 & 3 & 1\\ 0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \rightarrow \text{Row Echelon}$$

also
L=[1 0 0]
[l 1 0]
[l l 1]
&
U=[u u u]
[0 u u]
[0 0 u]

Youtube:

LU Factorization

$$8x - 3y + 2z = 20$$
$$4x + 11y - z = 33$$
$$6x + 3y + 12z = 36$$

$$Ax = B$$

$$A = \begin{bmatrix} 8 & -3 & 2 \\ 4 & 11 & -1 \\ 6 & 3 & 12 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad B = \begin{bmatrix} 20 \\ 33 \\ 36 \end{bmatrix}$$

$$A = L \times U$$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \times \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & -3 & 2 \\ 4 & 11 & -1 \\ 6 & 3 & 12 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix} = \begin{bmatrix} 8 & -3 & 2 \\ 4 & 11 & -1 \\ 6 & 3 & 12 \end{bmatrix}$$

$l_{11} = 8$

$l_{11}u_{12} = -3$

$\Rightarrow u_{12} = -\frac{3}{8}$

$l_{11}u_{13} = 2$

$\Rightarrow u_{13} = \frac{1}{4}$

$l_{21} = 4$ $\quad l_{21} u_{12} + l_{22} = 11$ $\quad l_{21} u_{13} + l_{22} u_{23} = -1$

$\Rightarrow 4 \times (-\frac{3}{8}) + l_{22} = 11$ $\quad \Rightarrow 4 \times \frac{1}{4} + \frac{25}{2} \times u_{23} = -1$

$\Rightarrow l_{22} = \frac{25}{2}$ $\quad \Rightarrow u_{23} = -\frac{4}{25}$

$l_{31} = 6$ $\quad l_{31} u_{12} + l_{32} = 3$ $\quad l_{31} u_{13} + l_{32} u_{23} + l_{33} = 12$

$\Rightarrow 6 \times (-\frac{3}{8}) + l_{32} = 3$ $\quad \Rightarrow 6 \times \frac{1}{4} + \frac{21}{4} \times (-\frac{4}{25}) + l_{33} = 12$

$\Rightarrow l_{32} = \frac{21}{4}$ $\quad \Rightarrow l_{33} = \frac{567}{50}$

$A = L \times U$ $\quad AX = B$

$L\underbrace{UX}_{Y} = B$

$$\begin{bmatrix} 8 & 0 & 0 \\ 4 & 25/2 & 0 \\ 6 & 21/4 & 567/50 \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 20 \\ 33 \\ 36 \end{bmatrix}$$

$8y_1 = 20$ $\quad \Rightarrow y_1 = \frac{5}{2}$

$4y_1 + 25/2\, y_2 = 33$ $\quad \Rightarrow y_2 = \frac{46}{25}$

$6y_1 + 21/4\, y_2 + 567/50\, y_3 = 36$ $\quad \Rightarrow y_3 = 1$

Now, $UX = Y$

$$\begin{bmatrix} 1 & -3/8 & 1/4 \\ 0 & 1 & -4/25 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 46/25 \\ 1 \end{bmatrix}$$

$x_1 - 3/8\, x_2 + 1/4\, x_3 = 5/2 \Rightarrow x_1 = 3$

$x_2 - 4/25\, x_3 = 46/25 \Rightarrow x_2 = 2$

$x_3 = 1$

$x_1 = x = 3$

$x_2 = y = 2$

$x_3 = z = 1$

Example:

$$\begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix}$$

$Ax = B$

$$A = \begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad B = \begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix}$$

$A = L \times U$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \times \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix}$$

$l_{11} = 2$ $\quad l_{11}u_{12} = -2$ $\quad l_{11}u_{13} = -2$

$\Rightarrow u_{12} = -1$ $\quad \Rightarrow u_{13} = -1$

$l_{21} = 0$ $\quad l_{21}u_{12} + l_{22} = -2$ $\quad l_{21}u_{13} + l_{22}u_{23} = 2$

$\Rightarrow l_{22} = -2$ $\quad \Rightarrow u_{23} = -1.$

$l_{31} = -1$ $\quad l_{31}u_{12} + l_{32} = 5$ $\quad l_{31}u_{13} + l_{32}u_{23} + l_{33} = 2$

$\Rightarrow l_{32} = 4$ $\quad \Rightarrow l_{33} = 5$

$A = L \times U$ $\quad AX = B$

$L\underbrace{UX}_{Y} = B$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix}$$

$\Rightarrow 2y_1 = -4 \Rightarrow y_1 = -2$

$\cancel{y_1} =$

$-2y_2 = -2 \Rightarrow y_2 = 1$

$-y_1 + 4y_2 + 5y_3 = 6 \Rightarrow y_3 = \cancel{8/5}\ 0$

Now, $UX = Y$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ \cancel{8}50 \end{bmatrix}$$

$x_1 - x_2 - x_3 = -2 \Rightarrow x_1 = -1$

$x_2 - x_3 = 1 \Rightarrow x_2 = 1$

$x_3 = 80$

Example-2

$$C = \begin{bmatrix} 0.5 & 0.1 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.1 & 0.3 & 0.4 \end{bmatrix} \quad d = \begin{bmatrix} 7900 \\ 3950 \\ 1975 \end{bmatrix}$$

$$\therefore I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\because (I - C)x = d$

$\Rightarrow x = (I - C)^{-1} d$

$$\therefore (I - C) = \begin{bmatrix} 0.5 & -0.1 & -0.1 \\ -0.2 & 0.5 & -0.3 \\ -0.1 & -0.3 & 0.6 \end{bmatrix}$$

$$x = (I - C)^{-1} d = \begin{bmatrix} 2.658 & 1.1392 & 1.0125 \\ 1.8987 & 3.6708 & 2.1518 \\ 1.3924 & 2.0253 & 2.9113 \end{bmatrix} \begin{bmatrix} 7900 \\ 3950 \\ 1975 \end{bmatrix}$$

$x_1 = 2.658 \times 7900 + 1.1392 \times 3950 + 1.0125 \times 1975$

$= 27497.72$

$x_2 = 1.8987 \times 7900 + 3.6708 \times 3950 + 2.1518 \times 1975$

$= 33749.19$

$x_3 = 1.3924 \times 7900 + 2.0253 \times 3950 + 2.9113 \times 1975$

$= 24749.71$

(a)

Yes, the economy can meet this demand.

(b)

The exact production vector x is

$$x = \begin{bmatrix} 27497.72 \\ 33749.19 \\ 24749.71 \end{bmatrix}$$

This means the open sector needs to produce 27497.72 ~~wot~~ worth of manufacturing products, 33749.19 worth of agricultural products and ~~27~~ 24749.71 worth of utilities products.

$$x_1 + 3x_2 - 2x_3 + 2x_5 = 0$$

$$2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1$$

$$5x_3 + 10x_4 + 15x_6 = 5$$

$$2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6$$

$$Ax = b$$

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ -1 \\ 5 \\ 6 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

Augmented form $= [A \mid B]$

$$A = \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right]$$

$$= \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right]$$

$$= \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] \quad \begin{array}{l} [r_2' = (-2)\times r_1 + r_2] \\ [r_4' = (-2)\times r_1 + r_2] \end{array}$$

$$= \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] \quad [r_2' = (-1)\times r_2]$$

$$= \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right] \quad \begin{array}{l} [r_3' = (-5)\times r_2 + r_3] \\ [r_4' = (-4)\times r_2 + r_4] \end{array}$$

$$= \left[\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 6 & 2 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right] \quad [r_1' = 2\times r_2 + r_1]$$

$$= \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \text{Row echelon form}$$

$$= \left[\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 6 & 2 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad [r_1' = 2\times r_2 + r_1]$$

$$= \left[\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 6 & 2 \\ 0 & 0 & -2 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad [r_2' = (-2)\times r_2 + r_1]$$

$[r_1' = (-6)\times r_3 + r_1$

$r_2' =$

$$= \begin{array}{cccccc} 1 & 3 & \cancel{0} & 4 & \cancel{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}$$

$$=\left[\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 6 & 2 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad [R_2' = (-3)\times R_3 + R_2]$$

$$=\left[\begin{array}{cccccc|c} \boxed{1} & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & \boxed{1} & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad [R_1' = (-6)\times R_3 + R_1]$$

leading variable

The system converted to :

$$x_1 + 3x_2 + 4x_4 + 2x_5 = 0$$

$$x_3 + 2x_4 = 0$$

$$x_6 = \frac{1}{3}$$

leading variable (x_1, x_3, x_6)

free variable $\to x_2, \cancel{x_4, x_5} = r$

$$x_4 = s$$

$$x_5 = t$$

Now:

$x_6 = \frac{1}{3}$

$x_5 = t$

$x_4 = s$

$x_3 = -2x_4 = -2s$

$x_2 = r$

$x_1 = -3x_2 - 4x_4 - 2x_5$

$\quad = -3r - 4s - 2t$

So, the solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3r-4s-2t \\ r \\ -2s \\ s \\ t \\ \frac{1}{3} \end{bmatrix}$$

$$= r\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{3} \end{bmatrix}$$

Example-15:

Determine whether the vectors $v_1 = (1,1,2)$, $v_2 = (1,0,1)$ and $v_3 = (2,1,3)$ span the vector space R^3.

$\Rightarrow R^3 \rightarrow v = (a,b,c)$

$$u_1 v_1 + u_2 v_2 + u_3 v_3 = V$$

$$\Rightarrow u_1(1,1,2) + u_2(1,0,1) + u_3(2,1,3) = (a,b,c)$$

$$u_1 + u_2 + 2u_3 = a \quad \text{---- (i)}$$

$$u_1 + u_3 = b \quad \text{---- (ii)}$$

$$2u_1 + u_2 + 3u_3 = c \quad \text{---- (iii)}$$

Suppose, $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ $X = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ $B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$\therefore AX = B$

$\Rightarrow X = A^{-1}B$

$\Rightarrow X =$

(i) $\times 2$ − (iii) $\Rightarrow$

$$2u_1 + 2u_2 + 4u_3 = 2a$$
$$2u_1 + 2u_2 + 3u_3 = c$$
$$\overline{\qquad u_3 = 2a - c}$$

In eq. (ii) $\Rightarrow$

$$u_1 + 2a - c = b$$
$$\Rightarrow u_1 = \cancel{a}\, b + c - 2a$$

In eq (i)

$b+c-2a+u_2+2(2a-c)=a$

$\Rightarrow u_2+4a-2c=a-b-c+2a$

$\Rightarrow u_2=a-b-c+2a-4a+2c$

$\Rightarrow u_2=-a-b+c$

Linear Independence or dependence:

Defn: If $S=\{v_1, v_2 \ldots v_r\}$ is a set of two or more vectors in a vector space V, then S is said to be a linearly dependent set if no vector in S can be expressed as linear combination to others. A set of vectors that is not linearly independent is said to be linearly.

Theorem:

A nonempty set $S=\{v_1, v_2 \ldots v_r\}$ in a vector space V is linearly independent if and only if the coefficients satisfying the vector equation

$$k_1 v_1 + k_2 v_2 + \dots k_n v_n = 0$$

are $k_1 = k_2 \dots = k_n = 0$ (trivial solⁿ)

এর solution হবে 0 আসবে।

* independent শুধু যদি determine non-zero হয়।

Basis Vector:

If $S = (v_1, v_2 \dots v_n)$ is a set of vectors in a finite dimensional vector space V, then S is called a basis for V if

ⓐ S span V.

ⓑ S is linearly independent.

Co-ordinate vector:

If $S = \{v_1, v_2, \dots v_n\}$ is a basis vector for a vector space V and $v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$ is the expression for a vector v in terms of the basis S; then the scalars $c_1, c_2, \dots c_n$

are called the co-ordinates of v relative to the basis S. The vector $(c_1, c_2, \ldots, c_n)$ in R^n constructed from these coordinates is called the co-ordinate vector of v relative to S; it is denoted by

$$(v)_S = (c_1, c_2, \ldots, c_n)$$

Problems:

Show that the vectors $v_1 = (1, 2, 1)$, $v_2 = (2, 9, 0)$ and $v_3 = (3, 3, 4)$ form a basis for R^3.

Soln:

Linear independent:

$$u_1 v_1 + u_2 v_2 + u_3 v_3 = 0$$

$$\Rightarrow u_1(1, 2, 1) + u_2(2, 9, 0) + u_3(3, 3, 4) = (0, 0, 0)$$

System:

$$u_1 + 2u_2 + 3u_3 = 0$$

$$2u_1 + 9u_2 + 3u_3 = 0$$

$$u_1 + \cancel{4u_3}\, 0 \cdot u_2 + 4u_3 = 0$$

the system can be written as:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 9 & 3 & 0 \\ 1 & 0 & 4 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 5 & -3 & 0 \\ 0 & -2 & 1 & 0 \end{array}\right] \quad \begin{array}{l} \cancel{R_2' = (-2)\times R_1 + R_2} \; [r_2' = (-2)\times r_1 + r_2] \\ \cancel{R_3' = R_3 - R_1} \; [r_3' = r_3 - r_1] \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -3/5 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] \quad \begin{array}{l} \cancel{R_2' = R_2/5} \; [r_2' = r_2/5] \\ \cancel{R_3' = R_3 + R_1} \; [r_3' = r_3 + r_1] \end{array}$$

$$[r_1' = (-2)\times r_2 + r_1]$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 21/5 & 0 \\ 0 & 1 & -3/5 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] \quad [r_3' = r_3/4]$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 21/5 & 0 \\ 0 & 1 & -3/5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$[r_1' = (-21/5)\times r_3 + r_1]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad [r_2' = \tfrac{3}{5}\times r_3 + r_2]$$

$u_1 = 0$
$u_2 = 0$
$u_3 = 0$

So, the vectors v_1, v_2, v_3 are linearly independent.

2nd condition:

$S = \{v_1, v_2, v_3\}$ span R^3 if the following system has a unique soln:

$k_1 v_1 + k_2 v_2 + k_3 v_3 = (a, b, c)$

$k_1 + 2k_2 + 3k_3 = a$

$2k_1 + 9k_2 + 3k_3 = b$

$k_1 + 0.k_2 + 4k_3 = c$

→ unique solution হবে যদি determine non-zero হয়।

The determinant of the system is:

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{vmatrix}$$

$$= 1(36-0) - 2(8-3) + 3(-9)$$

$$= -1 \neq 0$$

So, the system has a unique. So v_1, v_2, v_3 span R^3.

As the set of vectors $S = \{v_1, v_2, v_3\}$ are linearly independent and span R^3. So it forms a basis for R^3.

Example-9

(a) We showed in Example 3 that the vectors

$$v_1 = (1,2,1),\ v_2 = (2,9,0),\ v_3 = (3,3,4)$$

form a basis for R^3. Find the coordinate vector find the c's of $v = (5,-1,9)$ relative to the basis $S = \{v_1, v_2, v_3\}$

(b) Find the vector v in R^3 whose coordinate vector relative to S is $(v)_S = (-1,3,2)$.
given value of the c's

a

By the def^n of co-ordinate vector:

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = v$$

$$\Rightarrow c_1(1,2,1) + c_2(2,9,0) + c_3(3,3,4) = (5,-1,9)$$

$$c_1 + 2c_2 + 3c_3 = 5$$

$$2c_1 + 9c_2 + 3c_3 = -1$$

$$c_1 + 0.c_2 + 4c_3 = 9$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix} \quad x = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \quad B = \begin{bmatrix} 5 \\ -1 \\ 9 \end{bmatrix}$$

$$Ax = B$$

$$\Rightarrow x = A^{-1} B$$

$$\Rightarrow x = \begin{bmatrix} -36 & 8 & 21 \\ 5 & -1 & -3 \\ 9 & -2 & -5 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 9 \end{bmatrix}.$$

$$\Rightarrow x = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$\therefore c_1 = 1, \quad c_2 = -1, \quad c_3 = 2$

$\therefore (v)_S = (1, -1, 2)$

b

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = v$$

$$\Rightarrow -1(1,2,1) + 3(2,9,0) + 2(3,3,4) = v$$

$$\Rightarrow (-1,-2,-1) + (6,27,0) + (6,6,8) = v$$

$$\Rightarrow (11, 31, 7) = v$$