ⓐ Find the fourier cosine Transform of $e^{-x}, x \geq 0$.

$$f_c(n) = \int_0^\infty f(x) \cos nx \, dx$$

$$= \int_0^\infty e^{-x} \cos nx \, dx \longrightarrow \frac{1}{1+n^2}$$

ⓑ Find the fourier cosine Transform of $f(x) = e^{-x^2}, x \geq 0$

$$f_c(n) = \int_0^\infty e^{-x^2} \cos(nx) \, dx.$$

$\int e^{-x^2} dx \, (-2x)$ $\quad$ $\int e^{-x^2} dx$ $\quad$ $f(x) = e^{x}$, $h(x) = x^2$ $\quad$ $f(h(x)) = (f \circ h)(x)$

let, $I = \int_0^\infty e^{-x^2} \cos nx \, dx$

$$\Rightarrow \frac{dI}{dn} = \int_0^\infty e^{-x^2} (-x \sin nx) \, dx$$

$$= \frac{1}{2} \int_0^\infty (-2x) e^{-x^2} \sin nx \, dx = \frac{1}{2} \int_0^\infty \underbrace{\sin nx}_{u} \, d(\underbrace{e^{-x^2}}_{v}) dx$$

$\int u \, dv = uv - \int v \, du.$

$\frac{d}{dx}(e^{-x^2}) = \boxed{e^{-x^2} \cdot (-2x)}$ $\quad$ v $\quad$ dv

$$= \frac{1}{2} \left\{ \left[\sin nx \cdot e^{-x^2} \right]_0^\infty - \int_0^\infty n e^{-x^2} \cos nx \, dx \right\}$$

$\cos nx \boxed{\int e^{-x^2}} dx + \int \left\{ n \sin nx \boxed{\int e^{-x^2} dx} \right\} dx.$

$$\frac{dI}{dn} = \frac{1}{2} \left[0 - n \cdot I \right]$$

$$\Rightarrow \frac{dI}{dn} = -\frac{1}{2} n I$$

$$\Rightarrow \frac{dI}{I} = -\frac{1}{2} n \, dn$$

$$\Rightarrow \ln I = -\frac{1}{2} \cdot \frac{n^2}{2} + \boxed{\ln c}$$

$$\Rightarrow \ln\left(\frac{I}{c}\right) = \frac{-n^2}{4}.$$

$$\Rightarrow I = c e^{-n^2/4}$$

$$\Rightarrow \int_0^\infty e^{-x^2} \cos nx \, dx = c e^{-n^2/4}. \longrightarrow (*)$$

putting $n = 0$:

$$\int_0^\infty e^{-x^2} dx = c$$

let, $x^2 = p \Rightarrow x = \sqrt{p} = p^{1/2}$

$\Rightarrow 2x \, dx = dp$

$\Rightarrow dx = \frac{1}{2} \frac{1}{x} dp$

$= \frac{1}{2} p^{-1/2} dp.$

x	0	∞
p	0	∞

$$\Rightarrow c = \int_0^\infty e^{-p} \cdot \frac{1}{2} p^{-1/2} dp$$

$$= \frac{1}{2} \int_0^\infty e^{-p} p^{\left(\frac{1}{2}-1\right)} dp.$$

$$= \frac{1}{2} \Gamma\tfrac{1}{2}$$

$$= \frac{\sqrt{\pi}}{2}$$

Gamma Function:

$$\Gamma n = \int_0^\infty e^{-x} x^{(n-1)} dx$$

$$\Gamma\tfrac{1}{2} = \sqrt{\pi}$$

$$\therefore \int_0^\infty e^{-x^2} \cos nx \, dx = \frac{\sqrt{\pi}}{2} e^{-n^2/4}$$

(Ans).

Find the Fourier Sine transform of $F(x) = \frac{e^{-ax}}{x}$ where $a>0$ and $x>0$.

$$f_s(n) = \int_0^\infty \frac{e^{-ax}}{x} \sin nx \, dx.$$

⊛ <u>Complex Fourier Transform:</u> $f(n) = \int_{-\infty}^{\infty} F(x)\, e^{-inx} dx.$

Find the complex Fourier transform of $F(x) = e^{-a|x|}$ where $a>0$

$$|x| = \begin{cases} -x & \text{if } x<0 \\ x & \text{if } x>0. \end{cases}$$

<u>Soln</u>: $f(n) = \int_{-\infty}^{\infty} e^{-a|x|} e^{-inx} dx$

$$= \int_{-\infty}^{0} e^{-a(-x)} e^{-inx} dx + \int_0^\infty e^{-ax} e^{-inx} dx.$$

$$= \int_{-\infty}^{0} e^{(a-in)x} dx + \int_0^\infty e^{-(a+in)x} dx.$$

$$= \left[\frac{e^{(a-in)x}}{(a-in)}\right]_{-\infty}^{0} + \left[\frac{e^{-(a+in)x}}{-(a+in)}\right]_0^\infty$$

$$= \frac{1}{(a-in)}\left[e^0 - e^{-\infty}\right] - \frac{1}{(a+in)}\left[e^{-\infty} - e^0\right]$$

$$= \frac{1}{a-in} + \frac{1}{a+in} = \frac{2a}{a^2-(in)^2} = \boxed{\frac{2a}{a^2+n^2}} \text{ (Ans).}$$

Use finite Fourier transform to solve:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2};\ u(0,t)=0;\ u(4,t)=0;\ \boxed{u(x,0)=2x}\ \text{where } 0<x<4,\ t>0.$$

$u(0,t)=0$ $u(4,t)=0$

$u(x,0) = 2x$

$\boxed{u(x,t) = ??}$

$\boxed{\text{Sol}^n:}$ Taking Fourier 'sine' Transformation on both sides

$$\int_0^4 \frac{\partial u}{\partial t} \sin\frac{n\pi x}{4}dx = \int_0^4 \frac{\partial^2 u}{\partial x^2} \sin\frac{n\pi x}{4}dx.$$

Let. $\int_0^4 \underline{u(x,t)} \sin\frac{n\pi x}{4}dx = u(n,t) = u$

Then: $\frac{du}{dt} = \int_0^4 \frac{\partial u}{\partial t} \sin\frac{n\pi x}{4}dx = \int_0^4 \frac{\partial^2 u}{\partial x^2} \sin\frac{n\pi x}{4}dx.$

$$= \left[\sin\frac{n\pi x}{4}\frac{\partial u}{\partial x}\right]_0^4 - \int_0^4 \cos\frac{n\pi x}{4}\cdot\left(\frac{n\pi}{4}\right)\cdot\frac{\partial u}{\partial x}dx.$$

$$= 0 - \frac{n\pi}{4}\left\{\underbrace{\left[\cos\frac{n\pi x}{4}\cdot u(x,t)\right]_0^4}\ + \int_0^4 \sin\frac{n\pi x}{4}\cdot\frac{n\pi}{4}\cdot u(x,t)dx\right\}$$

$$= 0 - \frac{n^2\pi^2}{16}\int_0^4 u(x,t)\sin\frac{n\pi x}{4}dx.$$

$$\frac{du}{dt} = -\frac{n^2\pi^2}{16}u$$

$$\Rightarrow \frac{du}{u} = \frac{-n^2\pi^2}{16}dt$$

$$\Rightarrow u = Ae^{-\frac{n^2\pi^2}{16}t}.$$

$$\therefore u(n,t) = Ae^{-\frac{n^2\pi^2 t}{16}}$$

$t=0$; $u(n,0) = A.$ — (1)

$$u = u(n,t) = \int_0^4 u(x,t)\sin\frac{n\pi x}{4}dx$$

$$u \to u(n,0) = \int_0^4 u(x,0)\sin\frac{n\pi x}{4}dx$$

$$= \int_0^4 \underline{\underline{2x}}\sin\frac{n\pi x}{4}dx.$$

$$= \left[2x\cdot\left(-\cos\frac{n\pi x}{4}\right)\cdot\frac{4}{n\pi}\right]_0^4 + \int_0^4 2\cdot\cos\frac{n\pi x}{4}\cdot\frac{4}{n\pi}dx.$$

$$= \frac{-32}{n\pi}\cos n\pi + \frac{8}{n\pi}\int_0^4 \cos\frac{n\pi x}{4}dx.$$

$$= \frac{-32}{n\pi}\cos n\pi + \frac{8}{n\pi}\left[\sin\frac{n\pi x}{4}\cdot\frac{4}{n\pi}\right]_0^4.$$

$$= \frac{-32}{n\pi}\cos n\pi.$$

$$\therefore A = \frac{-32}{n\pi}\cos n\pi.$$

$$\therefore \boxed{u(n,t)} = \frac{-32}{n\pi}\cos n\pi\ e^{-\frac{n^2\pi^2 t}{16}}$$

Now taking the inverse Fourier Transformation:

$$u(x,t) = \frac{2}{4}\sum_{n=1}^{\infty} u(n,t)\sin\frac{n\pi x}{4} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{-32}{n\pi}\cos n\pi\, e^{-\frac{n^2\pi^2 t}{16}}\sin\frac{n\pi x}{4}$$

(Ans).

* Finite Fourier Sine:

$$f_s(n) = \int_0^l F(x) \sin\frac{n\pi x}{l}\, dx$$

Inverse:

$$F(x) = \frac{2}{l} \sum_{n=1}^{\infty} f_s(n) \sin\frac{n\pi x}{l}.$$