Solve the following initial value problem by Laplace Transformation:

① $y'' - 3y' + 2y = 4e^{2t}$; $y(0) = -3$; $y'(0) = 5$

ii) $y'' + 2y' + 5y = e^{-t}\sin t$, $y(0) = 0$; $y'(0) = 1$

Note:

$\mathcal{L}\{y''\} = s^2 y - s y(0) - y'(0)$

(*) <u>problem:</u>

$$Y'' + 2Y' + 5Y = e^{-t}\sin t \qquad Y(0) = 0 \;;\; Y'(0) = 1$$

$$\mathcal{L}\{F(t)\} = f(s)$$

$$\xrightarrow{\text{LT}} \boxed{\text{ODE}} \longrightarrow \boxed{\text{Algebraic eq}^{n}} \longrightarrow \boxed{\text{solve}} \longrightarrow \text{ILT}$$

$Y'' + 2Y' + 5Y = e^{-t}\sin t \quad ; Y(0)=0 ; Y'(0)=1$

Sol: $\mathcal{L}\{Y''\} + 2\mathcal{L}\{Y'\} + 5\mathcal{L}\{Y\} = \mathcal{L}\{e^{-t}\sin t\}$

$\Rightarrow \{s^2 y - sY(0) - Y'(0)\} + 2\{sy - Y(0)\} + 5y = \frac{1}{(s+1)^2+1}$

$\Rightarrow s^2 y - 0 - 1 + 2sy + 5y = \frac{1}{s^2+2s+1+1} = \frac{1}{s^2+2s+2}$

$\Rightarrow (s^2+2s+5)y = \frac{1+s^2+2s+2}{(s^2+2s+2)} = \frac{s^2+2s+3}{(s^2+2s+2)}$

$\Rightarrow y = \frac{(s^2+2s+3)}{(s^2+2s+2)(s^2+2s+5)}$

$\Rightarrow Y = \mathcal{L}^{-1}\left\{\frac{(s^2+2s+3)}{(s^2+2s+2)(s^2+2s+5)}\right\} \longrightarrow (*)$

Let, $P = \frac{(s^2+2s+3)}{(s^2+2s+2)(s^2+2s+5)}$

$= \frac{As+B}{(s^2+2s+2)} + \frac{Cs+D}{(s^2+2s+5)}$

$\Rightarrow (s^2+2s+3) = (As+B)(s^2+2s+5) + (Cs+D)(s^2+2s+2)$

$\Rightarrow (s^2+2s+3) = s^3(A+C) + s^2(B+2A+D+2C) + s(5A+2B+2D+2C) + (5B+2D)$

equating both sides we get:

$A + C = 0$

$B + 2A + 2C + D = 1$

$5A + 2B + 2C + 2D = 2$

$5B + 2D = 3$

solving the equations we get:

$A = C = 0$

$B = \frac{1}{3}; D = \frac{2}{3}$

$(*) \Rightarrow Y = \mathcal{L}^{-1}\left\{\frac{1/3}{(s^2+2s+2)} + \frac{2/3}{s^2+2s+5}\right\}$

$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} + \frac{2}{3}\cdot\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\}$

$\Rightarrow Y = \frac{1}{3}e^{-t}\sin t$

$= \frac{1}{3}e^{-t}\sin 2t$

$= \frac{1}{3}e^{-t}(\sin t + \sin 2t)$

(Ans)

We know $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)}\right\} = \cos at = F(t)$

$\frac{1}{a}\mathcal{L}^{-1}\left\{\frac{a}{(s^2+a^2)}\right\} = \frac{\sin at}{a} = G(t)$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \int_0^t \cos au \cdot \frac{\sin a(t-u)}{a}\,du$$

$$= \frac{1}{a}\int_0^t \cos au \sin(at-au)\,du$$

$$= \frac{1}{a}\int_0^t \cos au\left[\sin at \cos au - \cos at \sin au\right]du$$

$$= \frac{1}{a}\int_0^t \cos^2 au \sin at\,du - \frac{1}{a}\int_0^t \cos au \sin au \cos at\,du$$

$$= \frac{\sin at}{a\cdot 2}\int_0^t 2\cos^2 au\,du - \frac{\cos at}{a\cdot 2}\int_0^t 2\sin au \cos au\,du$$

$$= \frac{\sin at}{2a}\int_0^t (1+\cos 2au)\,du - \frac{\cos at}{2a}\int_0^t \sin 2au\,du$$

$$= \frac{\sin at}{2a}\left[u + \frac{\sin 2au}{2a}\right]_0^t - \frac{\cos at}{2a}\left[\frac{-\cos 2au}{2a}\right]_0^t$$

$$= \frac{\sin at}{2a}\left[t + \frac{\sin 2at}{2a}\right] - \frac{\cos at}{2a}\left[\frac{-\cos 2at}{2a} + \frac{1}{2a}\right]$$

$$= \frac{\sin at}{2a}\cdot\frac{1}{2a}\left(2at + \sin 2at\right) - \frac{1\cdot\cos at}{2a\cdot 2a}\left[2\sin^2 at\right]$$

$$= \frac{1}{4a^2}\left[2at\sin at + 2\sin^2 at\cos at\right] - \frac{1}{4a^2}\left[2\cos at\sin^2 at\right]$$

$$= \frac{1}{4a^2}\left[2at\sin at\right]$$