IRE-303: Signal and System

Signal Classification

Book Reference: Chapter-02 (Digital Signal Processing- Proakis

Discrete Time Signals

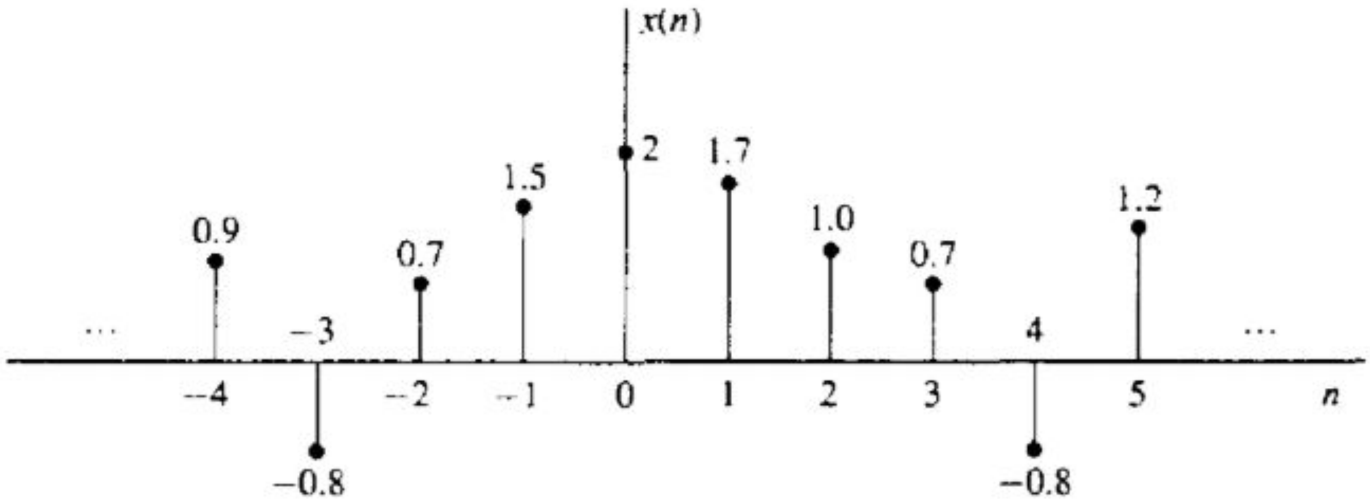


Figure 2.1 Graphical representation of a discrete-time signal.

1. Functional representation, such as

$$x(n) = \begin{cases} 1, & \text{for } n = 1, 3 \\ 4, & \text{for } n = 2 \\ 0, & \text{elsewhere} \end{cases}$$

2. Tabular representation, such as

n	$\cdots$	-2	-1	0	1	2	3	4	5	$\ldots$
$x(n)$	$\cdots$	0	0	0	1	4	1	0	0	$\ldots$

Discrete Time Signals

3. Sequence representation

An infinite-duration signal or sequence with the time origin ($n = 0$) indicated by the symbol $\uparrow$ is represented as

$$x(n) = \{\ldots 0, \underset{\uparrow}{0}, 1, 4, 1, 0, 0, \ldots\} \tag{2.1.2}$$

A sequence $x(n)$, which is zero for $n < 0$, can be represented as

$$x(n) = \{\underset{\uparrow}{0}, 1, 4, 1, 0, 0, \ldots\} \tag{2.1.3}$$

A finite-duration sequence can be represented as

$$x(n) = \{3, -1, \underset{\uparrow}{-2}, 5, 0, 4, -1\} \tag{2.1.4}$$

whereas a finite-duration sequence that satisfies the condition $x(n) = 0$ for $n < 0$ can be represented as

$$x(n) = \{\underset{\uparrow}{0}, 1, 4, 1\} \tag{2.1.5}$$

Some Elementary Discrete Time Signals

1. The *unit sample sequence* is denoted as $\delta(n)$ and is defined as

$$\delta(n) \equiv \begin{cases} 1, & \text{for } n = 0 \\ 0, & \text{for } n \neq 0 \end{cases} \tag{2.1.6}$$

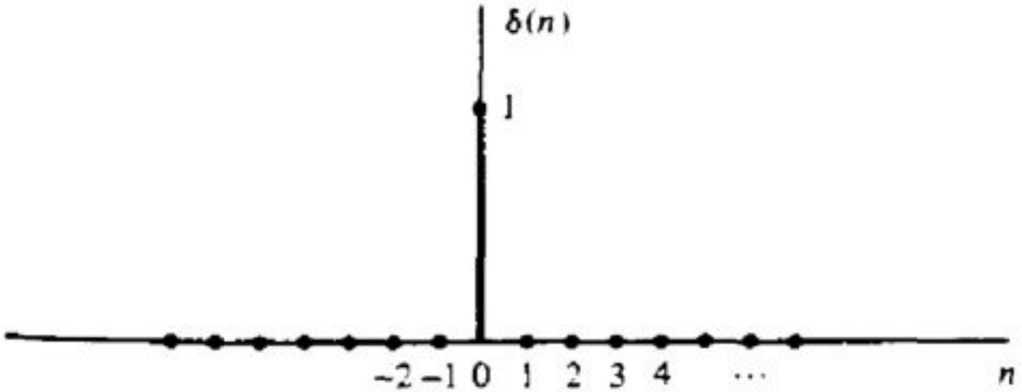


Figure 2.2 Graphical representation of the unit sample signal.

Some Elementary Discrete Time Signals

2. The *unit step signal* is denoted as $u(n)$ and is defined as

$$u(n) \equiv \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases} \tag{2.1.7}$$

Figure 2.3 illustrates the unit step signal.

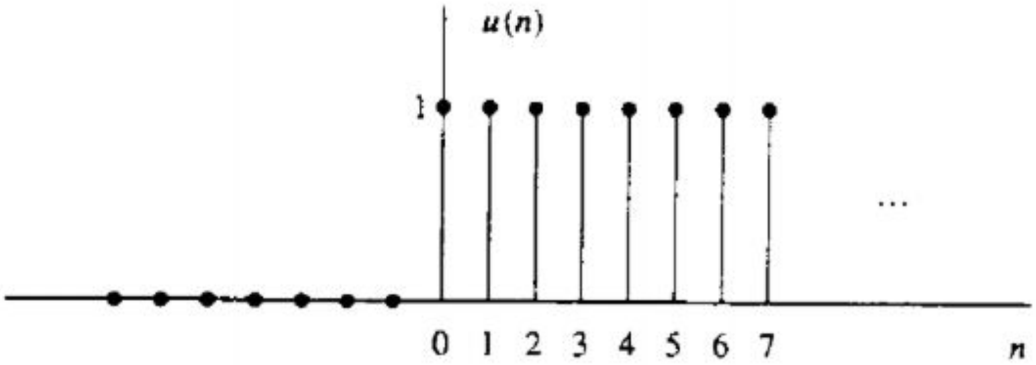


Figure 2.3 Graphical representation of the unit step signal.

Some Elementary Discrete Time Signals

3. The *unit ramp signal* is denoted as $u_r(n)$ and is defined as

$$u_r(n) \equiv \begin{cases} n, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases} \tag{2.1.8}$$

This signal is illustrated in Fig. 2.4.

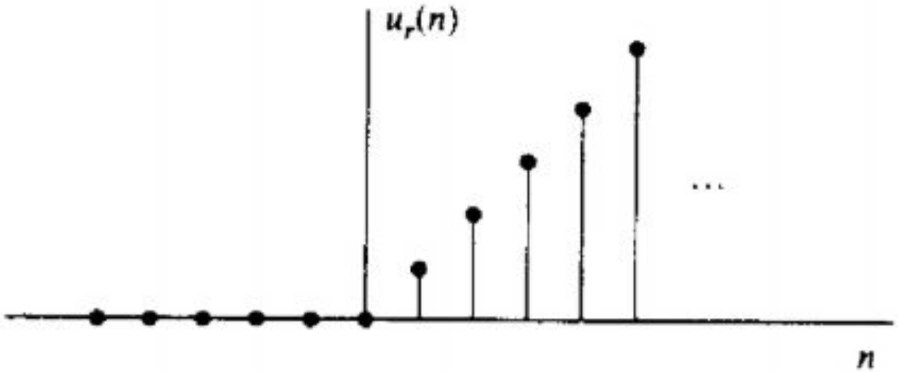


Figure 2.4 Graphical representation of the unit ramp signal.

Some Elementary Discrete Time Signals

4. The *exponential signal* is a sequence of the form

$$x(n) = a^n \qquad \text{for all } n \tag{2.1.9}$$

If the parameter a is real, then $x(n)$ is a real signal. Figure 2.5 illustrates $x(n)$ for various values of the parameter a.

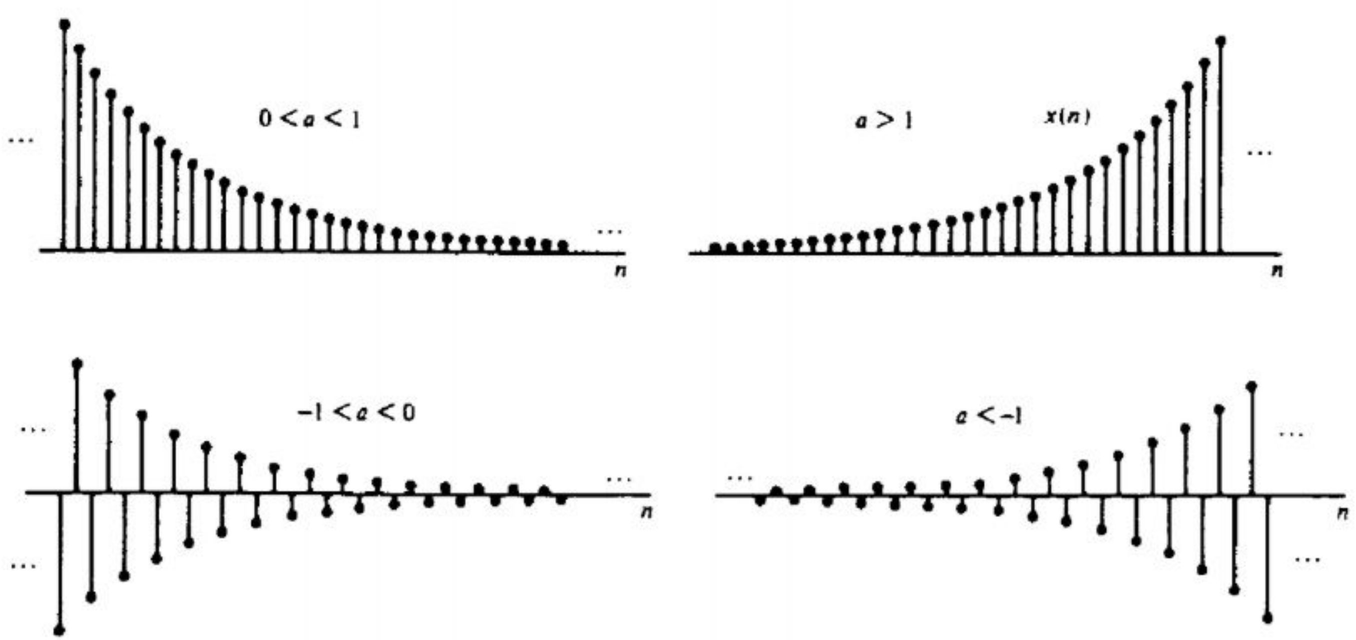


Figure 2.5 Graphical representation of exponential signals.

Classification of Discrete Time Signals

Energy signals and power signals. The energy E of a signal $x(n)$ is defined as

$$E \equiv \sum_{n=-\infty}^{\infty} |x(n)|^2 \qquad (2.1.15)$$

Many signals that possess infinite energy, have a finite average power. The average power of a discrete-time signal $x(n)$ is defined as

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 \qquad (2.1.16)$$

If we define the signal energy of $x(n)$ over the finite interval $-N \leq n \leq N$ as

$$E_N \equiv \sum_{n=-N}^{N} |x(n)|^2 \qquad (2.1.17)$$

then we can express the signal energy E as

$$E \equiv \lim_{N \to \infty} E_N \qquad (2.1.18)$$

and the average power of the signal $x(n)$ as

$$P \equiv \lim_{N \to \infty} \frac{1}{2N+1} E_N \qquad (2.1.19)$$

Classification of Discrete Time Signals

Clearly, if E is finite, $P = 0$. On the other hand, if E is infinite, the average power P may be either finite or infinite. If P is finite (and nonzero), the signal is called a *power signal*. The following example illustrates such a signal.

Example 2.1.1

Determine the power and energy of the unit step sequence. The average power of the unit step signal is

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} u^2(n)$$

$$= \lim_{N\to\infty} \frac{N+1}{2N+1} = \lim_{N\to\infty} \frac{1+1/N}{2+1/N} = \frac{1}{2}$$

Consequently, the unit step sequence is a power signal. Its energy is infinite.

Similarly, it can be shown that the complex exponential sequence $x(n) = Ae^{j\omega_0 n}$ has average power A^2, so it is a power signal. On the other hand, the unit ramp sequence is neither a power signal nor an energy signal.

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} |n|^2$$

$$= \lim_{N\to\infty} \left(\frac{1}{(2N+1)} \times \frac{N(N+1)(2N+1)}{6} \right)$$

$$= \lim_{N\to\infty} \frac{N(N+1)}{6} \approx \infty$$

$\therefore$ $u_r(N)$ neither power nor energy signal.

Classification of Discrete Time Signals

Periodic signals and aperiodic signals. As defined on Section 1.3, a signal $x(n)$ is periodic with period $N (N > 0)$ if and only if

$$x(n+N) = x(n) \text{ for all } n \tag{2.1.20}$$

Symmetric (even) and antisymmetric (odd) signals. A real-valued signal $x(n)$ is called symmetric (even) if

$$x(-n) = x(n) \tag{2.1.24}$$

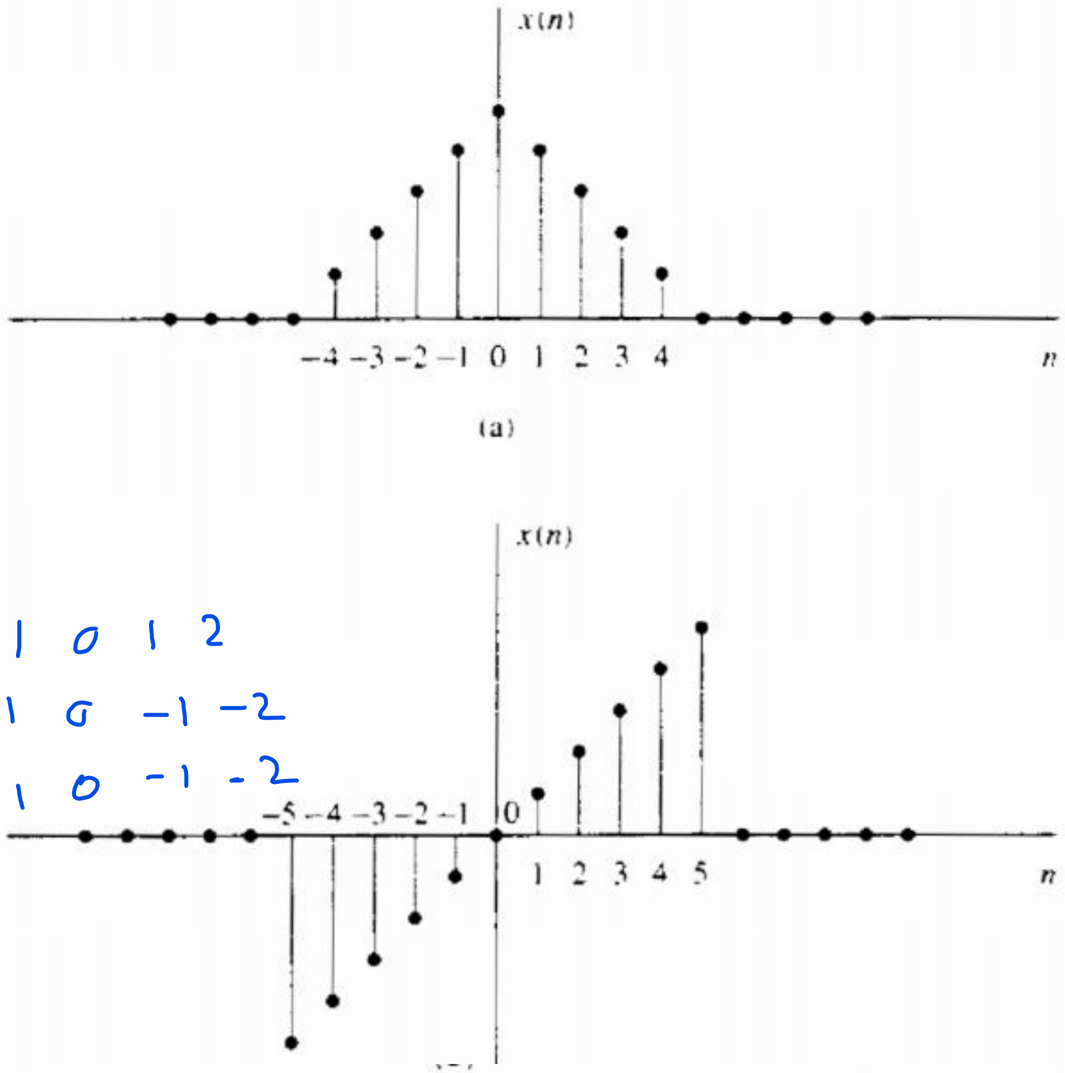
On the other hand, a signal $x(n)$ is called antisymmetric (odd) if

$$x(-n) = -x(n) \tag{2.1.25}$$

We note that if $x(n)$ is odd, then $x(0) = 0$. Examples of signals with even and odd symmetry are illustrated in Fig. 2.8.

asymmetric is neither symmetric nor antisymmetric.

Classification of Discrete Time Signals



n = −2 −1 0 1 2

x(−n) = 2 1 0 −1 −2

−x(n) = 2 1 0 −1 −2

Figure 2.8 Example of even (a) and odd (b) signals.

Classification of Discrete Time Signals

every signal is composed of an even and an odd part!!!

x(n) = xe(n) + xo(n)
x(-n) = xe(n) - xo(n)
----
x(n)+x(-n)=2xe(n)
xe(n) = ...

x(n) = xe(n) + xo(n)
x(-n) = xe(n) - xo(n)
----
x(n)-x(-n)=2xo(n)
xo(n) = ...

$$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$$

$$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$

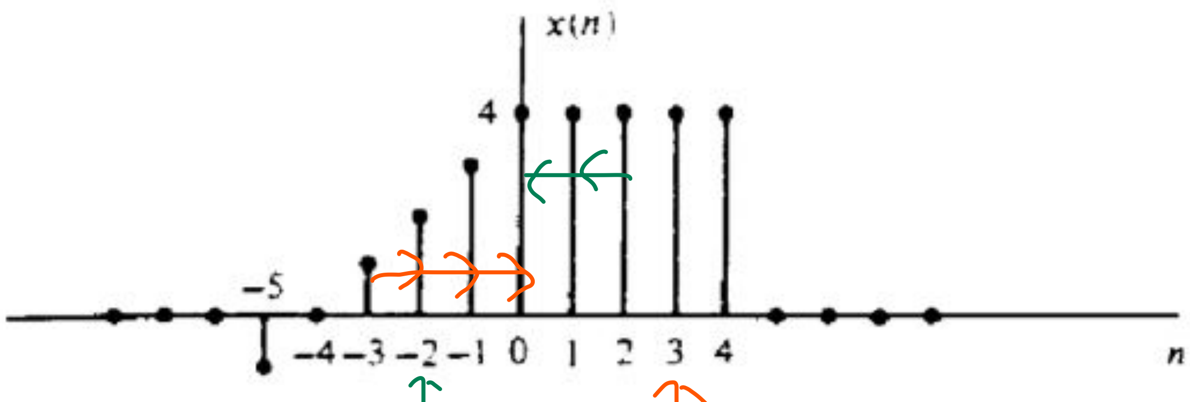
$$x(n) = x_e(n) + x_o(n)$$

Simple Manipulation of Discrete Time Signals

Transformation of the independent variable (time). A signal $x(n)$ may be shifted in time by replacing the independent variable n by $n-k$, where k is an integer. If k is a positive integer, the time shift results in a delay of the signal by k units of time. If k is a negative integer, the time shift results in an advance of the signal by $|k|$ units in time.

Example 2.1.2

A signal $x(n)$ is graphically illustrated in Fig. 2.9a. Show a graphical representation of the signals $x(n-3)$ and $x(n+2)$.



(a)

Simple Manipulation of Discrete Time Signals

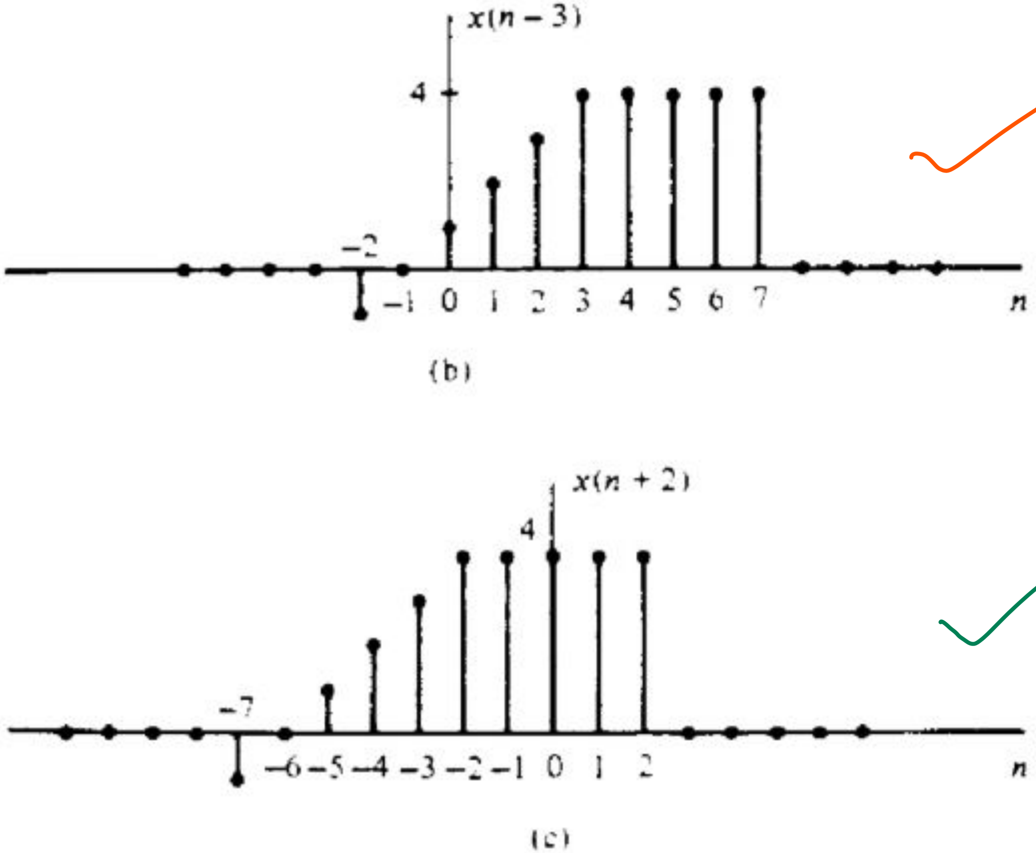


Figure 2.9 Graphical representation of a signal, and its delayed and advanced versions.

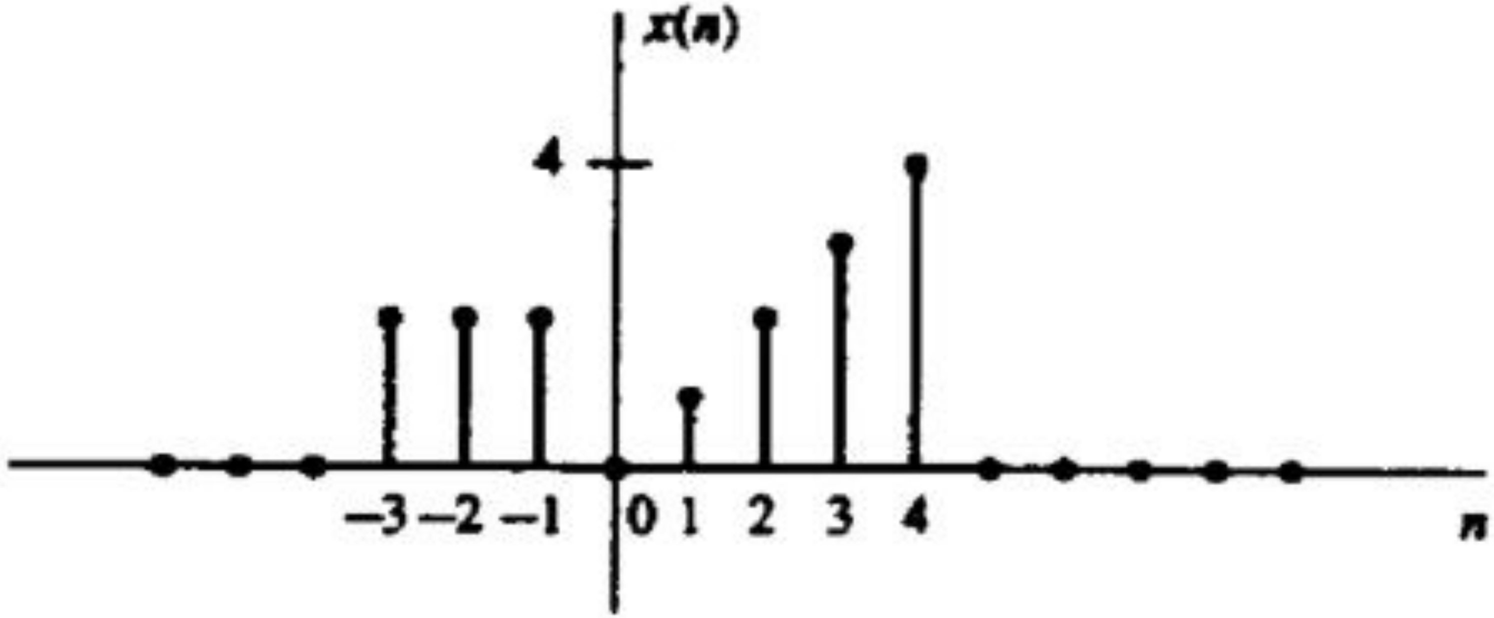
$n - 3 = 0$
$n = 3$

$n + 2 = 0$
$n = -2$

Simple Manipulation of Discrete Time Signals

Example 2.1.3

Show the graphical representation of the signal $x(-n)$ and $x(-n+2)$, where $x(n)$ is the signal illustrated in Fig. 2.10a.



(a)

Simple Manipulation of Discrete Time Signals

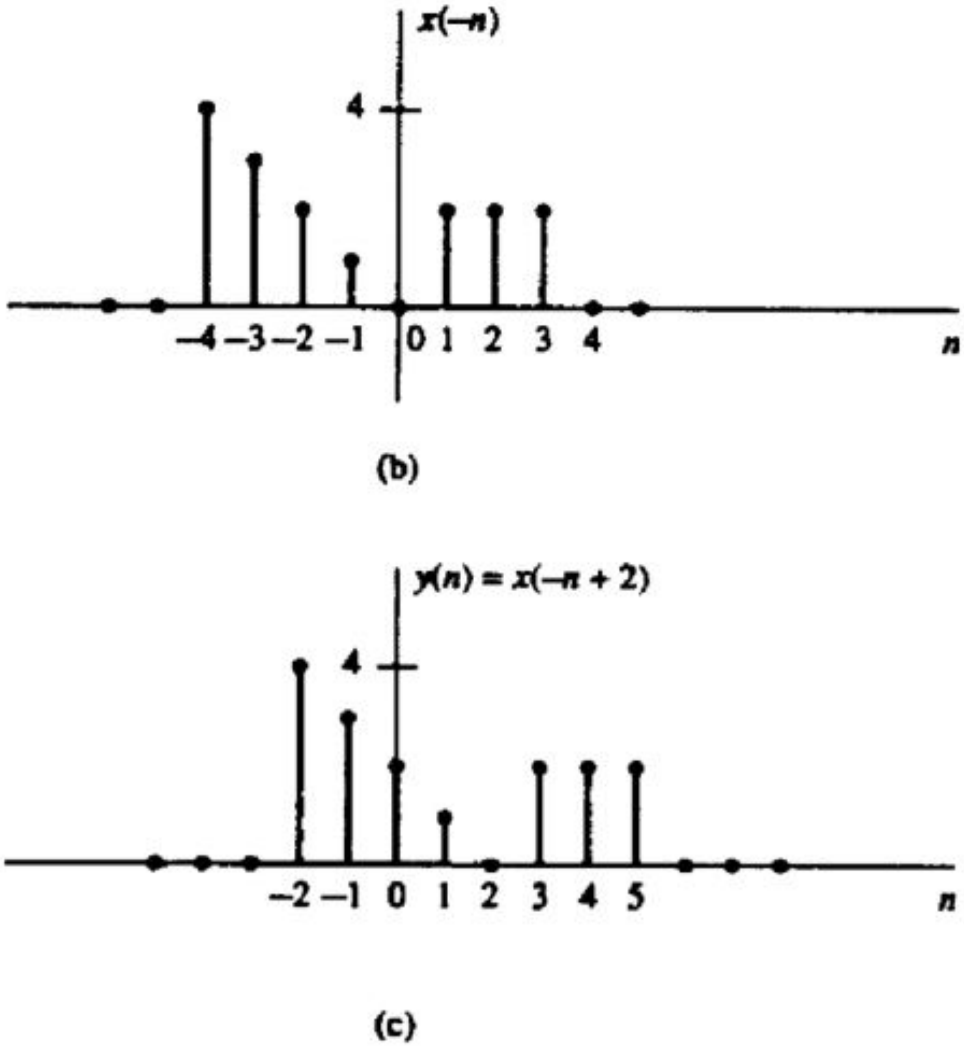
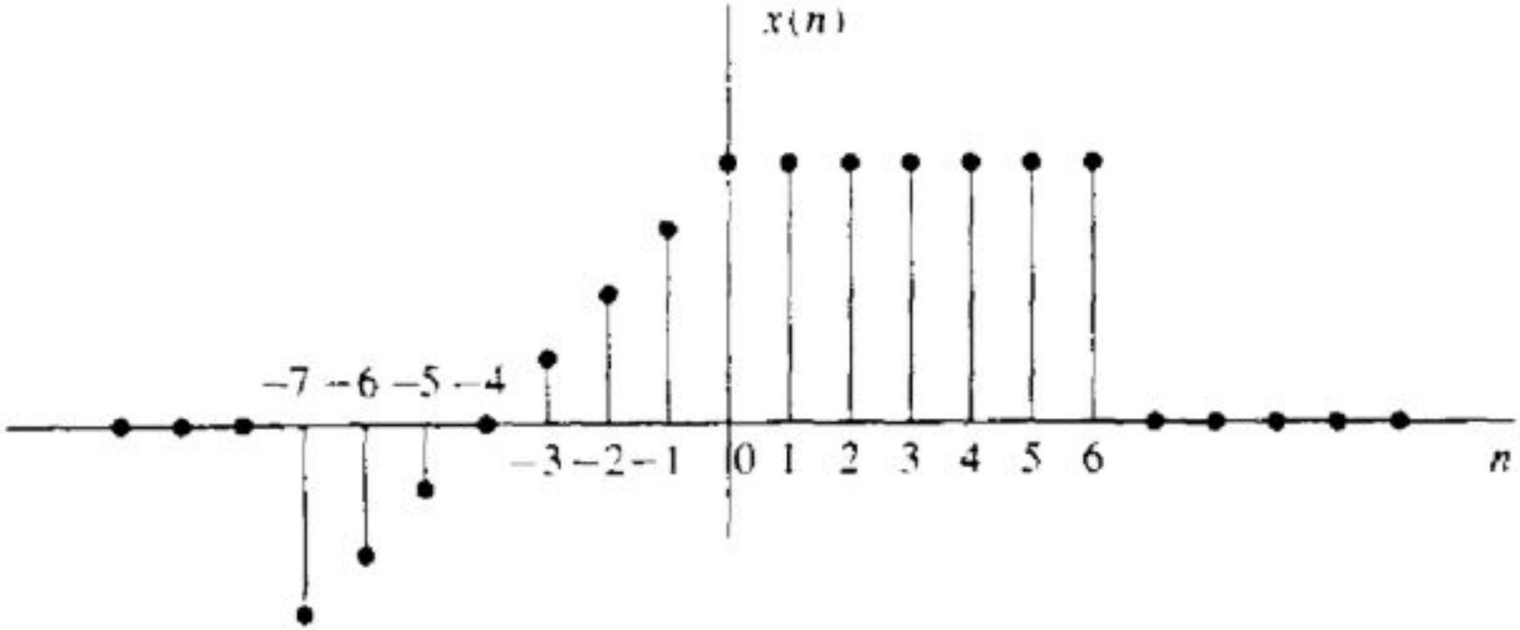


Figure 2.10 Graphical illustration of the folding and shifting operations.

Simple Manipulation of Discrete Time Signals

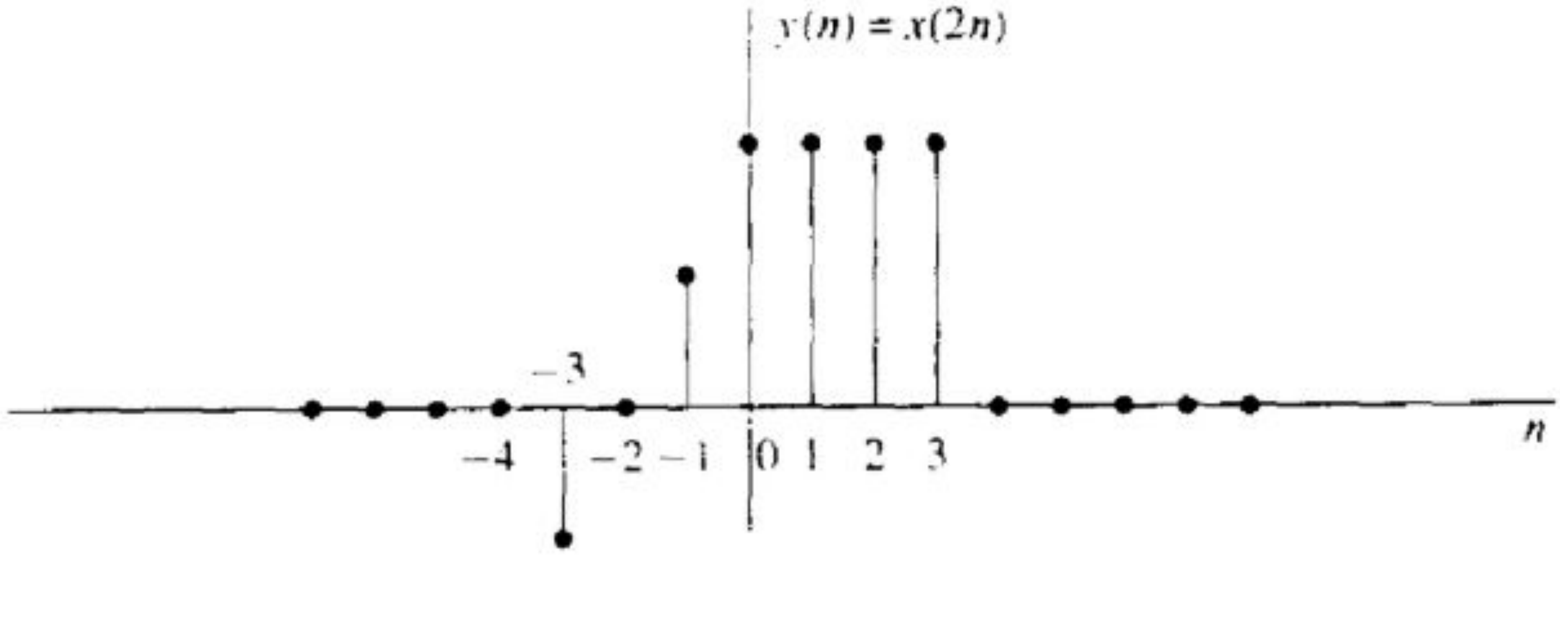
Example 2.1.4

Show the graphical representation of the signal $y(n) = x(2n)$, where $x(n)$ is the signal illustrated in Fig. 2.11a.



(a)

Simple Manipulation of Discrete Time Signals



(b)

Figure 2.11 Graphical illustration of down-sampling operation.

Discrete Time(DT) Systems

DT system operates of DT signals

$$y(n) = \Gamma[x(n)]$$

transformation

Operations:

- Adder
- Constant multiplier
- Signal multiplier
- Unit delay
- Unit advance

Example: $y(n) = 0.25\, y(n-1) + 0.5\, x(n) + x(n-2) + \ldots$

Input-Output Description System

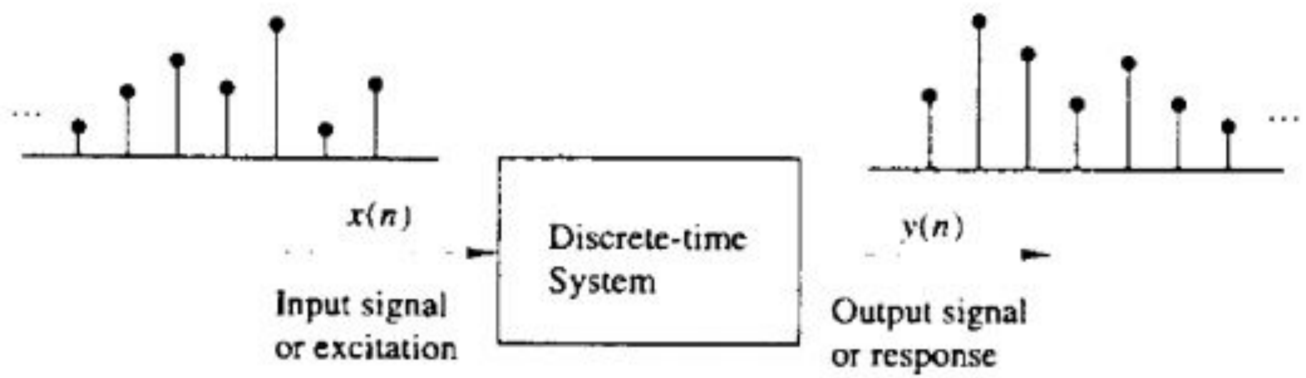


Figure 2.12 Block diagram representation of a discrete-time system.

$$x(n) \xrightarrow{\mathcal{T}} y(n) \quad (2.2.2)$$

which simply means that $y(n)$ is the response of the system $\mathcal{T}$ to the excitation $x(n)$. The following examples illustrate several different systems.

Example 2.2.1

Determine the response of the following sytems to the input signal

$$x(n) = \begin{cases} |n|, & -3 \le n \le 3 \\ 0, & \text{otherwise} \end{cases}$$

(a) $y(n) = x(n)$

(b) $y(n) = x(n-1)$

(c) $y(n) = x(n+1)$

(d) $y(n) = \frac{1}{3}[x(n+1) + x(n) + x(n-1)]$

(e) $y(n) = max\{x(n+1), x(n), x(n-1)\}$

(f) $y(n) = \sum_{k=-\infty}^{n} x(k) = x(n) + x(n-1) + x(n-2) + \cdots$ (2.2.3)

Solution First, we determine explicitly the sample values of the input signal

$$x(n) = \{\ldots, 0, 3, 2, 1, \underset{\uparrow}{0}, 1, 2, 3, 0, \ldots\}$$

Next, we determine the output of each system using its input–output relationship.

(a) In this case the output is exactly the same as the input signal. Such a system is known as the *identity* system.

(b) This system simply delays the input by one sample. Thus its output is given by

$$x(n) = \{\ldots, 0, 3, 2, \underset{\uparrow}{1}, 0, 1, 2, 3, 0, \ldots\}$$

(c) In this case the system "advances" the input one sample into the future. For example, the value of the output at time $n = 0$ is $y(0) = x(1)$. The response of this system to the given input is

$$x(n) = \{\ldots, 0, 3, 2, 1, 0, \underset{\uparrow}{1}, 2, 3, 0, \ldots\}$$

(d) The output of this system at any time is the mean value of the present, the immediate past, and the immediate future samples. For example, the output at time $n = 0$ is

$$y(0) = \tfrac{1}{3}[x(-1) + x(0) + x(1)] = \tfrac{1}{3}[1 + 0 + 1] = \tfrac{2}{3}$$

Repeating this computation for every value of n, we obtain the output signal

$$y(n) = \{\ldots 0, 1, \tfrac{5}{3}, 2, 1, \underset{\uparrow}{\tfrac{2}{3}}, 1, 2, \tfrac{5}{3}, 1, 0, \ldots\}$$

(e) This system selects as its output at time n the maximum value of the three input samples $x(n-1)$, $x(n)$, and $x(n+1)$. Thus the response of this system to the input signal $x(n)$ is

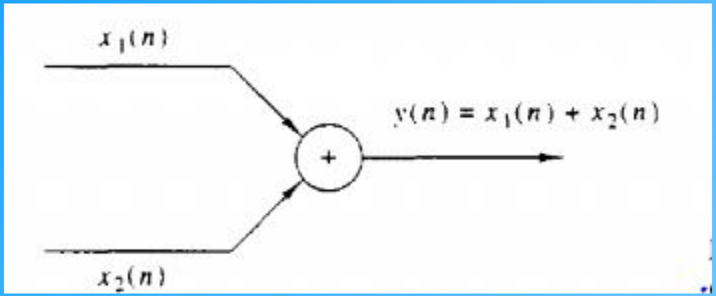
$$y(n) = \{0, 3, 3, 3, 2, \underset{\uparrow}{1}, 2, 3, 3, 3, 0, \ldots\}$$

(f) This system is basically an *accumulator* that computes the running sum of all the past input values up to present time. The response of this system to the given input is

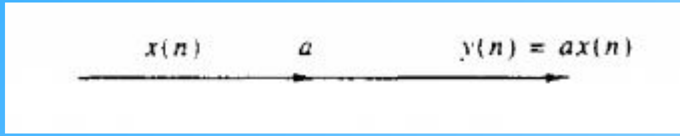
$$y(n) = \{\ldots 0, 3, 5, 6, \underset{\uparrow}{6}, 7, 9, 12, 0, \ldots\}$$

Block Diagram Representation

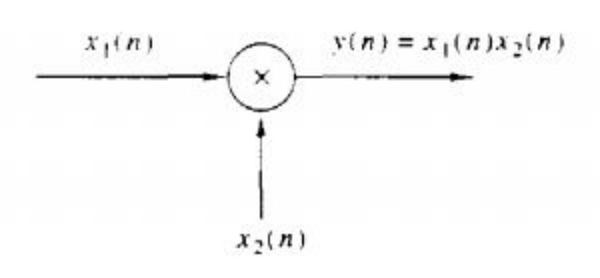
- An adder



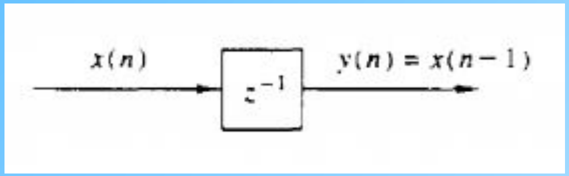
- A constant Multiplier



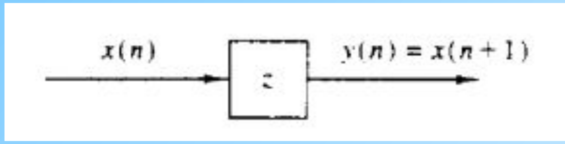
- A signal multiplier



- A unit delay system



- A unit advance system

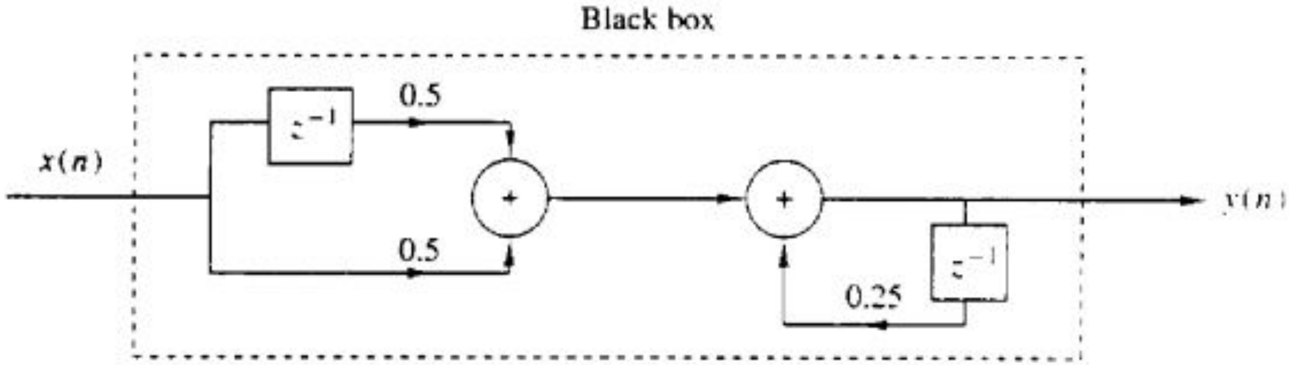


Example 2.2.3

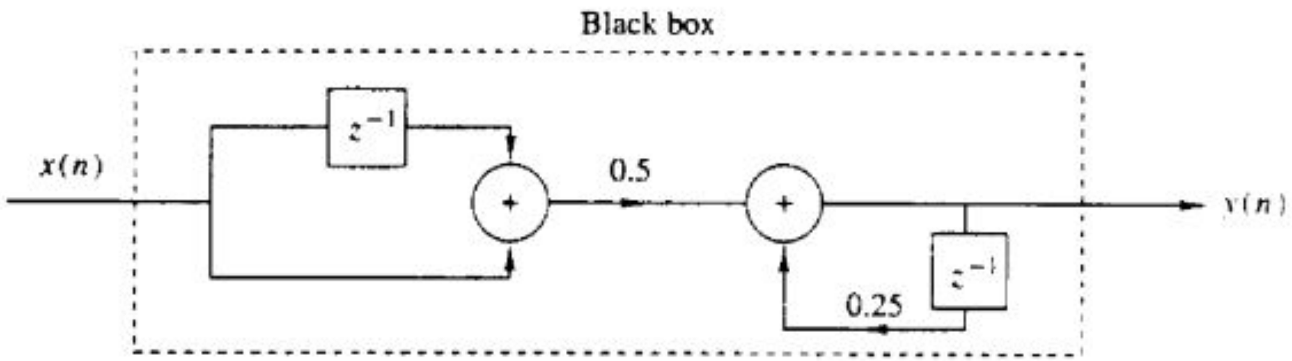
Using basic building blocks introduced above, sketch the block diagram representation of the discrete-time system described by the input–output relation.

$$y(n) = \frac{1}{4}y(n-1) + \frac{1}{2}x(n) + \frac{1}{2}x(n-1) \qquad (2.2.5)$$

where $x(n)$ is the input and $y(n)$ is the output of the system.



(a)



(b)

Classification of Systems

Static versus dynamic systems.

The systems described by the following input–output equations

$$y(n) = ax(n) \tag{2.2.7}$$

$$y(n) = nx(n) + bx^3(n) \tag{2.2.8}$$

are both static or memoryless. Note that there is no need to store any of the past or future inputs or outputs in order to compute the present output. On the other hand, the systems described by the following input–output relations

$$y(n) = x(n) + 3x(n-1) \tag{2.2.9}$$

$$y(n) = \sum_{k=0}^{n} x(n-k) \tag{2.2.10}$$

$$y(n) = \sum_{k=0}^{\infty} x(n-k) \tag{2.2.11}$$

are dynamic systems or systems with memory. The systems described by (2.2.9)

We observe that static or memoryless systems are described in general by input–output equations of the form

$$y(n) = \mathcal{T}[x(n), n] \tag{2.2.12}$$

and they do not include delay elements (memory).
past

Causal versus noncausal systems.

Definition. A system is said to be *causal* if the output of the system at any time n [i.e., $y(n)$] depends only on present and past inputs [i.e., $x(n)$, $x(n-1)$, $x(n-2), \ldots$], but does not depend on future inputs [i.e., $x(n+1)$, $x(n+2), \ldots$]. In mathematical terms, the output of a causal system satisfies an equation of the form

$$y(n) = F[x(n), x(n-1), x(n-2), \ldots] \tag{2.2.44}$$

where $F[\cdot]$ is some arbitrary function.

If a system does not satisfy this definition, it is called *noncausal*. Such a system has an output that depends not only on present and past inputs but also on future inputs.

Example 2.2.6

Determine if the systems described by the following input–output equations are causal or noncausal.

(a) $y(n) = x(n) - x(n-1)$ **(b)** $y(n) = \sum_{k=-\infty}^{n} x(k)$ **(c)** $y(n) = ax(n)$

(d) $y(n) = x(n) + 3x(n+4)$ **(e)** $y(n) = x(n^2)$ **(f)** $y(n) = x(2n)$

(g) $y(n) = x(-n)$

Solution The systems described in parts (a), (b), and (c) are clearly causal, since the output depends only on the present and past inputs. On the other hand, the systems in parts (d), (e), and (f) are clearly noncausal, since the output depends on future values of the input. The system in (g) is also noncausal, as we note by selecting, for example, $n = -1$, which yields $y(-1) = x(1)$. Thus the output at $n = -1$ depends on the input at $n = 1$, which is two units of time into the future.

Block Diagram Representation

Time-invariant versus time-variant systems.

Definition. A relaxed system $\mathcal{T}$ is *time invariant* or *shift invariant* if and only if

$$x(n) \xrightarrow{\mathcal{T}} y(n)$$

implies that

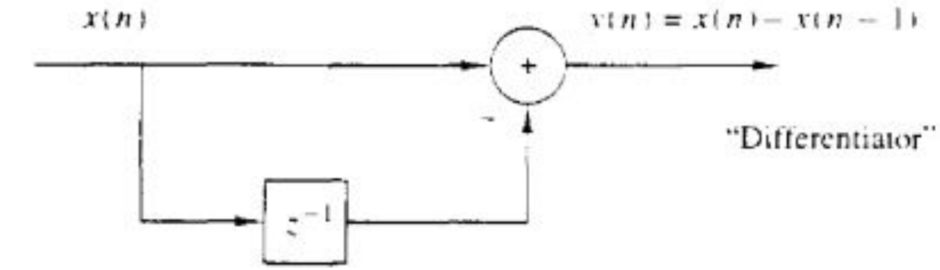
$$x(n-k) \xrightarrow{\mathcal{T}} y(n-k) \tag{2.2.14}$$

for every input signal $x(n)$ and every time shift k.

To determine if any given system is time invariant, we need to perform the test specified by the preceding definition. Basically, we excite the system with an arbitrary input sequence $x(n)$, which produces an output denoted as $y(n)$. Next we delay the input sequence by same amount k and recompute the output. In general, we can write the output as

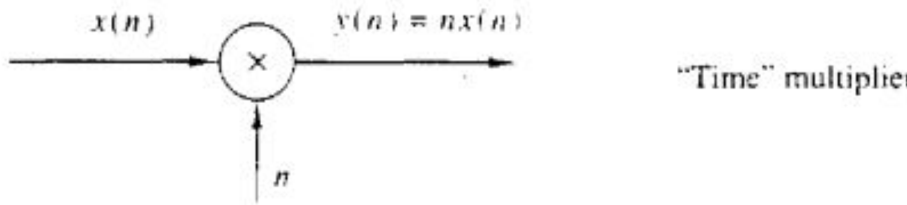
$$y(n,k) = \mathcal{T}[x(n-k)]$$

Now if this output $y(n,k) = y(n-k)$, for all possible values of k, the system is time invariant. On the other hand, if the output $y(n,k) \neq y(n-k)$, even for one value of k, the system is time variant.



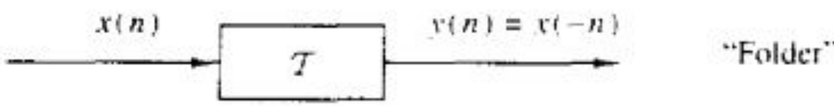
x(n)
y(n) = x(n) − x(n − 1)
+
−
z^{-1}
"Differentiator"

(a)



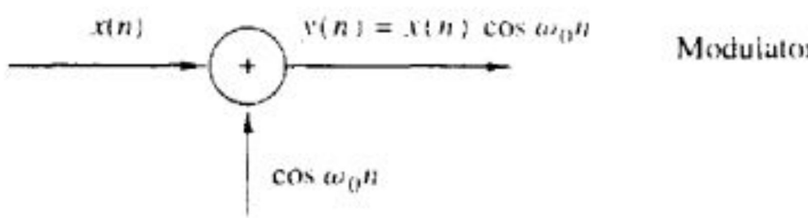
x(n)
y(n) = nx(n)
×
n
"Time" multiplie

(b)



x(n)
T
y(n) = x(−n)
"Folder"

(c)



x(n)
y(n) = x(n) cos $\omega_0 n$
+
cos $\omega_0 n$
Modulator

(d)

Example 2.2.4

Determine if the systems shown in Fig. 2.19 are time invariant or time variant.

Solution

(a) This system is described by the input–output equations

$$y(n) = \mathcal{T}[x(n)] = x(n) - x(n-1) \tag{2.2.15}$$

Now if the input is delayed by k units in time and applied to the system, it is clear from the block diagram that the output will be

$$y(n, k) = x(n-k) - x(n-k-1) \tag{2.2.16}$$

On the other hand, from (2.2.14) we note that if we delay $y(n)$ by k units in time, we obtain

$$y(n-k) = x(n-k) - x(n-k-1) \tag{2.2.17}$$

Since the right-hand sides of (2.2.16) and (2.2.17) are identical, it follows that $y(n, k) = y(n-k)$. Therefore, the system is time invariant.

(b) The input–output equation for this system is

$$y(n) = \mathcal{T}[x(n)] = nx(n) \tag{2.2.18}$$

The response of this system to $x(n-k)$ is

$$y(n,k) = nx(n-k) \tag{2.2.19}$$

Now if we delay $y(n)$ in (2.2.18) by k units in time, we obtain

$$\begin{aligned} y(n-k) &= (n-k)x(n-k) \\ &= nx(n-k) - kx(n-k) \end{aligned} \tag{2.2.20}$$

This system is time variant, since $y(n,k) \neq y(n-k)$.

(c) This system is described by the input–output relation

$$y(n) = \mathcal{T}[x(n)] = x(-n) \tag{2.2.21}$$

The response of this system to $x(n-k)$ is

$$y(n,k) = \mathcal{T}[x(n-k)] = x(-n-k) \tag{2.2.22}$$

Now, if we delay the output $y(n)$, as given by (2.2.21), by k units in time, the result will be

$$y(n-k) = x(-n+k) \tag{2.2.23}$$

Since $y(n,k) \neq y(n-k)$, the system is time variant.

(d) The input–output equation for this system is

$$y(n) = x(n)\cos\omega_0 n \tag{2.2.24}$$

The response of this system to $x(n-k)$ is

$$y(n,k) = x(n-k)\cos\omega_0 n \tag{2.2.25}$$

If the expression in (2.2.24) is delayed by k units and the result is compared to (2.2.25), it is evident that the system is time variant.

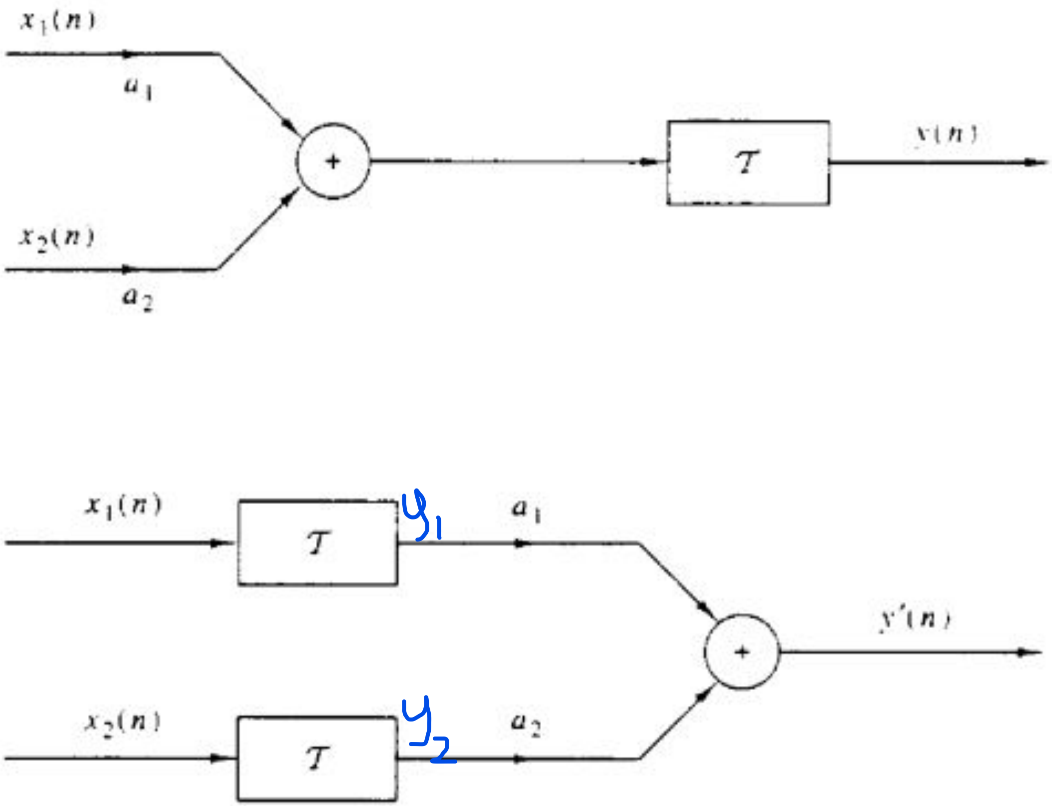
Linear versus nonlinear systems.

Definition. A relaxed $\mathcal{T}$ system is linear if and only if

$$\mathcal{T}[a_1x_1(n) + a_2x_2(n)] = a_1\mathcal{T}[x_1(n)] + a_2\mathcal{T}[x_2(n)] \tag{2.2.26}$$

for any arbitrary input sequences $x_1(n)$ and $x_2(n)$, and any arbitrary constants a_1 and a_2.

Figure 2.20 gives a pictorial illustration of the superposition principle.



Example 2.2.5

Determine if the systems described by the following input–output equations are linear or nonlinear.

(a) $y(n) = nx(n)$ **(b)** $y(n) = x(n^2)$ **(c)** $y(n) = x^2(n)$

(d) $y(n) = Ax(n) + B$ **(e)** $y(n) = e^{x(n)}$

Solution

(a) For two input sequences $x_1(n)$ and $x_2(n)$, the corresponding outputs are

$$\begin{aligned} y_1(n) &= nx_1(n) \\ y_2(n) &= nx_2(n) \end{aligned} \tag{2.2.31}$$

A linear combination of the two input sequences results in the output

$$\begin{aligned} y_3(n) &= \mathcal{T}[a_1x_1(n) + a_2x_2(n)] = n[a_1x_1(n) + a_2x_2(n)] \\ &= a_1nx_1(n) + a_2nx_2(n) \end{aligned} \tag{2.2.32}$$

On the other hand, a linear combination of the two outputs in (2.2.31) results in the output

$$a_1y_1(n) + a_2y_2(n) = a_1nx_1(n) + a_2nx_2(n) \tag{2.2.33}$$

Since the right-hand sides of (2.2.32) and (2.2.33) are identical, the system is linear.

(b) As in part (a), we find the response of the system to two separate input signals $x_1(n)$ and $x_2(n)$. The result is

$$\begin{aligned} y_1(n) &= x_1(n^2) \\ y_2(n) &= x_2(n^2) \end{aligned} \tag{2.2.34}$$

The output of the system to a linear combination of $x_1(n)$ and $x_2(n)$ is

$$y_3(n) = \mathcal{T}[a_1x_1(n) + a_2x_2(n)] = a_1x_1(n^2) + a_2x_2(n^2) \tag{2.2.35}$$

Finally, a linear combination of the two outputs in (2.2.36) yields

$$a_1y_1(n) + a_2y_2(n) = a_1x_1(n^2) + a_2x_2(n^2) \tag{2.2.36}$$

By comparing (2.2.35) with (2.2.36), we conclude that the system is linear.

(c) The output of the system is the square of the input. (Electronic devices that have such an input–output characteristic and are called square-law devices.) From our previous discussion it is clear that such a system is memoryless. We now illustrate that this system is nonlinear.

$$y_1(n) = x_1^2(n)$$
$$y_2(n) = x_2^2(n) \tag{2.2.37}$$

The response of the system to a linear combination of these two input signals is

$$\begin{aligned} y_3(n) &= \mathcal{T}[a_1x_1(n) + a_2x_2(n)] \\ &= [a_1x_1(n) + a_2x_2(n)]^2 \\ &= a_1^2x_1^2(n) + 2a_1a_2x_1(n)x_2(n) + a_2^2x_2^2(n) \end{aligned} \tag{2.2.38}$$

On the other hand, if the system is linear, it would produce a linear combination of the two outputs in (2.2.37), namely,

$$a_1y_1(n) + a_2y_2(n) = a_1x_1^2(n) + a_2x_2^2(n) \tag{2.2.39}$$

Since the actual output of the system, as given by (2.2.38), is not equal to (2.2.39), the system is nonlinear.

(d) Assuming that the system is excited by $x_1(n)$ and $x_2(n)$ separately, we obtain the corresponding outputs

$$\begin{aligned} y_1(n) &= Ax_1(n) + B \\ y_2(n) &= Ax_2(n) + B \end{aligned} \tag{2.2.40}$$

A linear combination of $x_1(n)$ and $x_2(n)$ produces the output

$$\begin{aligned} y_3(n) &= \mathcal{T}[a_1x_1(n) + a_2x_2(n)] \\ &= A[a_1x_1(n) + a_2x_2(n)] + B \\ &= Aa_1x_1(n) + a_2Ax_2(n) + B \end{aligned} \tag{2.2.41}$$

On the other hand, if the system were linear, its output to the linear combination of $x_1(n)$ and $x_2(n)$ would be a linear combination of $y_1(n)$ and $y_2(n)$, that is,

$$a_1y_1(n) + a_2y_2(n) = a_1Ax_1(n) + a_1B + a_2Ax_2(n) + a_2B \tag{2.2.42}$$

Clearly, (2.2.41) and (2.2.42) are different and hence the system fails to satisfy the linearity test.

(e) Note that the system described by the input–output equation

$$y(n) = e^{x(n)} \tag{2.2.43}$$

is relaxed. If $x(n) = 0$, we find that $y(n) = 1$. This is an indication that the system is nonlinear. This, in fact, is the conclusion reached when the linearity test, is applied.

Stable versus unstable systems.

Definition. An arbitrary relaxed system is said to be bounded input–bounded output (BIBO) stable if and only if every bounded input produces a bounded output.

The conditions that the input sequence $x(n)$ and the output sequence $y(n)$ are bounded is translated mathematically to mean that there exist some finite numbers, say M_x and M_y, such that

$$|x(n)| \leq M_x < \infty \qquad |y(n)| \leq M_y < \infty \tag{2.2.45}$$

for all n. If, for some bounded input sequence $x(n)$, the output is unbounded (infinite), the system is classified as unstable.

Example 2.2.7

Consider the nonlinear system described by the input–output equation

$$y(n) = y^2(n-1) + x(n)$$

As an input sequence we select the bounded signal

$$x(n) = C\delta(n)$$

where C is a constant. We also assume that $y(-1) = 0$. Then the output sequence is

$$y(0) = C, \quad y(1) = C^2, \quad y(2) = C^4, \quad \ldots, \quad y(n) = C^{2^n}$$

Clearly, the output is unbounded when $1 < |C| < \infty$. Therefore, the system is BIBO unstable, since a bounded input sequence has resulted in an unbounded output.

Convolution

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

Operations

Folding: Find *h(-k)* from *h(k)*

Shifting: Find *h(n-k)* from *h(-k)*; right shift for +ve *n* and left shift for –ve *n* opposite for when h(k)

Multiplication: Multiply *x(k)* by *h(n-k)* to obtain the product *v(n)*

Summation: Sum all the values of product sequence to obtain the output value

Properties

Commutative $\square$ $x(n) * h(n) = h(n) * x(n)$

Associative $\square$ $[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$

Distributive $\square$ $x(n) * [h_1(n) + h_2(n)]$

$= x(n) * h_1(n) + x(n) * h_2(n)$

Response of LTI Systems to Arbitrary Inputs: The Convolution Sum

Ex-2.3.2

Given, $h(n)=\{1,\underset{\uparrow}{2},1,-1\}$ & $x(n)=\{\underset{\uparrow}{1},2,3,1\}$

The convoluted signal, $y(n)=\sum_{K=-\infty}^{\infty} x(K)h(n-K)$

Here, $x(K)=\{\underset{\uparrow}{1},2,3,1\}$ & $h(-K)=\{-1,1,\underset{\uparrow}{2},1\}$

$$\therefore y(0)=\sum_{K=-2}^{3} x(K)h(-K) = x(-2)h(2)+x(-1)h(1)+x(0)h(0)+x(1)h(1)+x(2)h(2)+x(3)h(3)$$

$$= 0+0+(1\times 2)+(2\times 1)+0+0+0 = 4$$

Now, $h(1-K)=\{-1,\underset{\uparrow}{1},2,1\}$

$$\therefore y(1)=\sum x(K)\,h(1-K)=(1\times 1)+(2\times 2)+(3\times 1)=8$$

$h(2-K)=\{\underset{\uparrow}{-1},1,2,1\}$

$$\therefore y(2)=\sum x(K)h(2-K)=(-1\times 1)+(1\times 2)+(2\times 3)+(1\times 1)=8$$

Similarly, $y(3)=3$; $y(4)=-2$; $y(5)=-1$; $y(6)=0$.

$y(-1)=1$; $y(-2)=0$.

$$\therefore y(n)=\{\dots 0,1,\underset{\uparrow}{4},8,8,3,-2,-1,0,\dots\}$$

Tabular method:-

$x(n)$ \ $h(n)$	1	2	1	-1
1	1	2	1	-1
2	2	4	2	-2
3	3	6	3	-3
1	1	2	1	-1

$$y(n) = \{1, \underset{\uparrow}{4}, 8, 8, 3, -2, -1\}$$

Correlation

Measurement of the degree of similarity between two signals, cross-correlation and auto-correlation

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-l) = \sum_{n=-\infty}^{\infty} x(n+l)y(n) \qquad l = 0, \pm 1, \pm 2, \ldots$$

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} x(n-l)y(n) = \sum_{n=-\infty}^{\infty} x(n)y(n+l)$$

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)x(n-l) = \sum_{n=-\infty}^{\infty} x(n+l)x(n) \qquad r_{xx}(0) = \sum_{n=-\infty}^{\infty} x(n)x(n) = \sum_{n=-\infty}^{\infty} x(n)^2 = E_x$$

$$r_{xy}(l) = \sum_{n=i}^{N-k-1} x(n)y(n-l) \qquad r_{xx}(l) = \sum_{n=i}^{N-k-1} x(n)x(n-l)$$

i = l, k = 0 for $l \geq 0$, and i = 0, k = l for $l < 0$

Example 2.6.1

Determine the crosscorrelation sequence $r_{xy}(l)$ of the sequences

$$x(n) = \{\ldots, 0, 0, 2, -1, 3, 7, \underset{\uparrow}{1}, 2, -3, 0, 0, \ldots\}$$

$$y(n) = \{\ldots, 0, 0, 1, -1, 2, -2, \underset{\uparrow}{4}, 1, -2, 5, 0, 0, \ldots\}$$

Solution Let us use the definition in (2.6.3) to compute $r_{xy}(l)$. For $l = 0$ we have

$$r_{xy}(0) = \sum_{n=-\infty}^{\infty} x(n)y(n)$$

The product sequence $v_0(n) = x(n)y(n)$ is

$$v_0(n) = \{\ldots, 0, 0, 2, 1, 6, -14, \underset{\uparrow}{4}, 2, 6, 0, 0, \ldots\}$$

and hence the sum over all values of n is

$$r_{xy}(0) = 7$$

For $l > 0$, we simply shift $y(n)$ to the right relative to $x(n)$ by l units, compute the product sequence $v_l(n) = x(n)y(n-l)$, and finally, sum over all values of the product sequence. Thus we obtain

$$r_{xy}(1) = 13, \qquad r_{xy}(2) = -18, \qquad r_{xy}(3) = 16, \qquad r_{xy}(4) = -7$$

$$r_{xy}(5) = 5, \qquad r_{xy}(6) = -3, \qquad r_{xy}(l) = 0, \qquad l \geq 7$$

For $l < 0$, we shift $y(n)$ to the left relative to $x(n)$ by l units, compute the product sequence $v_l(n) = x(n)y(n-l)$, and sum over all values of the product sequence. Thus we obtain the values of the crosscorrelation sequence

$$r_{xy}(-1) = 0, \qquad r_{xy}(-2) = 33, \qquad r_{xy}(-3) = -14, \qquad r_{xy}(-4) = 36$$

$$r_{xy}(-5) = 19, \qquad r_{xy}(-6) = -9, \qquad r_{xy}(-7) = 10, \qquad r_{xy}(l) = 0, \ l \leq -8$$

Therefore, the crosscorrelation sequence of $x(n)$ and $y(n)$ is

$$r_{xy}(l) = \{10, -9, 19, 36, -14, 33, \underset{\uparrow}{0}, 7, 13, -18, 16, -7, 5, -3\}$$

Example 2.6.2

Compute the autocorrelation of the signal

$$x(n) = a^n u(n),\ 0 < a < 1$$

Solution Since $x(n)$ is an infinite-duration signal, its autocorrelation also has infinite duration. We distinguish two cases.

If $l \geq 0$, from Fig. 2.39 we observe that

$$r_{xx}(l) = \sum_{n=l}^{\infty} x(n)x(n-l) = \sum_{n=l}^{\infty} a^n a^{n-l} = a^{-l} \sum_{n=l}^{\infty} (a^2)^n$$

Since $a < 1$, the infinite series con erges and we obtain

$$r_{xx}(l) = \frac{1}{1-a^2} a^l \qquad l \geq 0$$

For $l < 0$ we have

$$r_{xx}(l) = \sum_{n=0}^{\infty} x(n)x(n-l) = a^{-l} \sum_{n=0}^{\infty} (a^2)^n = \frac{1}{1-a^2} a^{-l} \qquad l < 0$$

But when l is negative, $a^{-l} = a^{|l|}$. Thus the two relations for $r_{xx}(l)$ can be combined into the following expression:

$$r_{xx}(l) = \frac{1}{1-a^2} a^{|l|} \qquad -\infty < l < \infty \tag{2.6.20}$$

The sequence $r_{xx}(l)$ is shown in Fig. 2.42(d). We observe that

$$r_{xx}(-l) = r_{xx}(l)$$

and

$$r_{xx}(0) = \frac{1}{1-a^2}$$

Therefore, the normalized autocorrelation sequence is

$$\rho_{xx}(l) = \frac{r_{xx}(l)}{r_{xx}(0)} = a^{|l|} \qquad -\infty < l < \infty \tag{2.6.21}$$