Electromagnetic Induction

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13.1 Magnetic Flux

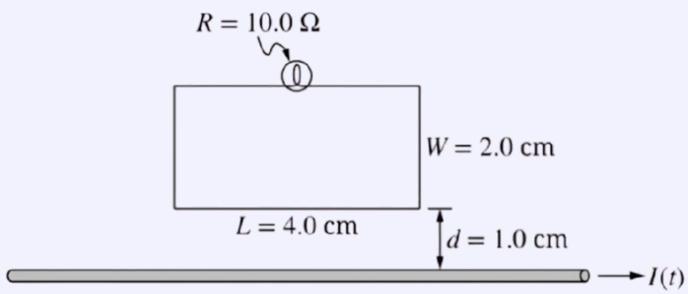
Flux was introduced in Unit 8 as the amount of field passing perpendicularly through an area. We can apply the same concept to magnetic fields with the following equation:

$$\phi_B = \iint \vec{B} \cdot d\vec{A}$$

Don't confuse this with Gauss's Law for magnetism, which says that the flux through a closed area (think of a 3D solid with no openings) is zero. Here, the area does not have to be closed. For simplicity, College Board (and these notes) will write this equation with only one integral.

Problem

What is the magnetic flux through the loop of wire if 4 Amps of current is flowing through the straight wire at the bottom?



Solution. The magnetic field from a infinitely straight length of wire is $\frac{\mu_0 I}{2\pi r}$ (from the last unit), and $dA = Ldw$. We can also make the substitution $dw = dr$.

$$\begin{aligned}\phi_B &= \int \vec{B} \cdot d\vec{A} \\ &= \int \frac{\mu_0 I}{2\pi r} L dr \\ &= \frac{\mu_0 I}{2\pi} \int_d^{d+w} \frac{1}{r} dr \\ &= \frac{\mu_0 I}{2\pi} \ln r|_d^{d+w} = 5.52 \times 10^-8 \; Tm^2\end{aligned}$$

13.2 Electromagnetic Induction

Magnitude of Induced Current

In 13.2, we are introduced to Faraday's Law, which states that a changing magnetic flux induces an electromotive force in a system.

$$emf = -\frac{d\phi_B}{dt}$$

Problem

The wire from the previous example now has a current $I(t) = C - Dt$ flowing through it. Calculate the magnitude of the induced emf generated around the loop at time $t = 3s$ if $C = 10A$ and $D = 2A/s$.

Solution. We can replace I with $C - Dt$, and take the negative of the derivative of ϕ_B with respect to time, then solve at $t = 3$

$$\begin{aligned}\phi_B &= \frac{\mu_0 I}{2\pi}\ln|\frac{d+w}{d}| \\ &= \frac{\mu_0}{2\pi}\ln|\frac{d+w}{d}|(C - Dt)\end{aligned}$$

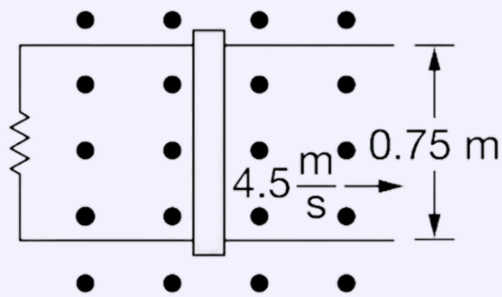
$$\begin{aligned}-\frac{d\phi_B}{dt} &= \frac{d}{dt}\left[\frac{\mu_0}{2\pi}\ln|\frac{d+w}{d}|(C - Dt)\right] \\ &= \frac{\mu_0 D}{2\pi}\ln|\frac{d+w}{d}| \\ &= \boxed{1.758 \times 10^{-8}V}\end{aligned}$$

Direction of Induced Current

Lenz's Law states that the direction of the induced EMF (and thus the induced current) is such that the magnetic field it creates opposes the change in magnetic flux that produced it. This opposition is represented by the negative sign in Faraday's Law. For example, if the magnetic flux through a loop is decreasing, the induced current will produce a magnetic field in the same direction as the original field to oppose the decrease. Conversely, if the magnetic flux through a loop is increasing, the induced current will produce a magnetic field in the opposite direction as the original field to oppose the increase.

Problem

A conducting rod is sliding at a speed of $4m/s$ along conducting rails that are $0.75m$ apart. The rails are attached to a 5Ω resistor. The loop is inside a $250mT$ magnetic field directed out of the page. What is the current generated in the resistor?



Solution. First, we must determine the flux enclosed by the circuit:

$$\phi_B = \int BdA = B\int dA = BA = BLW$$

Then, we can find the emf. Everything is constant with respect to time except for W, and $\frac{dW}{dt} = v$:

$$\begin{aligned} emf &= -\frac{d\phi_B}{dt} \\ &= -\frac{d}{dt}[BLW] \\ &= -BLv \end{aligned}$$

$$I = -\frac{BLv}{R} = 0.17A$$

Since the area of the loop is increasing, the flux through the loop is also increasing. Therefore, the current generated should create a magnetic field in the opposite direction as the original field. Using the "thumbs up" right hand rule, we find that the current travels clockwise.

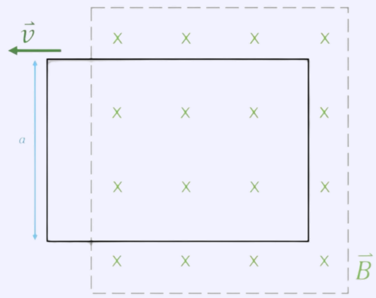
13.3 Induced Currents and Magnetic Forces

The induced current in a wire may also cause a force, described by the Lorentz force equation covered in the last unit:

$$\vec{F_B} = \int I(d\vec{l} \times \vec{B})$$

Problem

Describe the force on the loop of wire in the following scenario.



Solution. We can find the electromotive force (ϵ) induced by the changing magnetic flux, then apply the Lorentz force equation to the left side of the wire to find the force. We apply a modified Faraday's Law in the second equation to represent having multiple loops (N is the number of loops).

$$\phi_B = \int B\,dA = BA = Baw$$

$$\begin{aligned} |\epsilon| &= N|\frac{d\phi_B}{dt}| \\ &= N\frac{d(Baw)}{dt} = NBa\frac{dw}{dt} \\ &= NBav \end{aligned}$$

$$I = \frac{\epsilon}{R} = \frac{NBav}{R}$$

$$\begin{aligned} \vec{F_B} &= \int I(d\vec{l} \times \vec{B}) \\ &= \frac{NB^2av}{R}\int_0^a dl = \boxed{\frac{NB^2a^2v}{R}} \end{aligned}$$

Notice how the strength of the magnetic force induced is like a resistive force, where the magnitude of the force is proportional to the velocity ($F_r = -kv$). We can apply this to model how a cart carrying a rectangular loop of wire moves through a magnetic field region that has the same width as the loop.

$$ma = F$$
$$ma = \boxed{-\frac{NB^2a^2v}{R}}v$$

For now, we can assign the constant $-k$ to the boxed expression above. This is negative because it is in the opposite direction as the velocity. We solve a differential equation by setting integration bounds so that the velocity at $t = 0$ is v_0 and at time t is $v(t)$.

$$m\frac{dv}{dt} = -kv$$
$$\frac{dv}{v} = \frac{-k}{m}dt$$
$$\int_{v_0}^{v(t)} \frac{dv}{v} = \int_0^t \frac{-k}{m}dt$$
$$\ln(\frac{v(t)}{v_0}) = -\frac{kt}{m}$$
$$\frac{v(t)}{v_0} = e^{-\frac{kt}{m}}$$
$$v(t) = v_0e^{-\frac{kt}{m}}$$

13.4 Inductance

Inductance is the tendency of a conductor to resist a change in current.

- When current is moving clockwise down a solenoid, a magnetic field pointing into the page is generated.
- Suppose the current is being increased. This causes an increase in magnetic flux.
- The change in magnetic flux causes a current to be induced that creates a magnetic field upwards to oppose that change in flux.

- The induced magnetic field creates a current that is counterclockwise, moving upwards partially countering the effects of the increasing current.

In a solenoid, the formula for inductance is

$$L = \frac{N\phi_B}{I}$$

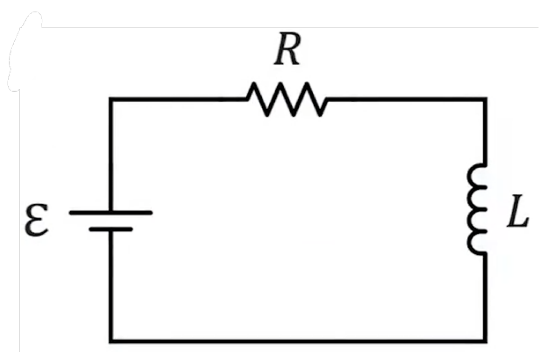
measured in Henry ($T \cdot m^2/A$). We can also substitute ϕ_B with BA and then B with $\mu_0 \frac{N}{l} I$, resulting in another equation for the inductance in a solenoid:

$$L = \frac{\mu_0 N^2 A}{l}$$

In capacitors, we could add a dielectric material between the two plates to increase its capacitance. In inductors, we can place a different material inside the loop to increase its inductance. To calculate the inductance, simply replace μ_0 with the permeability of the core.

Potential energy stored in an inductor

By connecting a battery, resistor, and inductor in series, we can determine the amount of potential energy an inductor has.



We can then apply Kirchhoff's Loop Law and manipulate the equation to show potential energy.

$$\epsilon - \Delta V_R - \epsilon_L = 0$$
$$\epsilon - IR - L\frac{dI}{dt} = 0$$
$$\epsilon \cdot I - IR \cdot I - L\frac{dI}{dt} \cdot I = 0$$

If ϵI represents the power supplied to the circuit, and I^2R represents the power dissipated by the resistor, then $LI\frac{dI}{dt}$ must be the power stored or dissipated by the inductor. We can integrate this with respect to current to find the potential energy stored by the inductor.

$$P = \frac{dU_L}{dt} = LI\frac{dI}{dt}$$
$$\int_0^{U_L} dU_L = \int_0^I LI\,dI$$
$$U_L = \boxed{\frac{1}{2}LI^2}$$

13.5 LR Circuits

Behaviors of Inductors

A qualitative characterization of inductor behavior in a circuit (like one shown above):

- At $t = 0$, the inductor acts as an open switch; no current flows through the resistor.
- At $t > 0$, the current through the inductor slowly approaches the current set by the battery and the resistor.
- At very large values of t, the inductor acts like a closed switch.

We can calculate a more quantitative relationship by applying Kirchhoff's

Loop Rule again but solving for I, the current, as a function of time.

$$\begin{aligned}
\epsilon - IR - L\frac{dI}{dt} &= 0 \\
L\frac{dI}{dt} &= \epsilon - IR \\
\frac{dI}{\epsilon - IR} &= \frac{dt}{L} \\
\int_0^{I(t)} \frac{dI}{\epsilon - IR} &= \int_0^t \frac{dt}{L} \\
\frac{\ln \epsilon}{R} - \frac{\ln \epsilon - IR}{R} &= \frac{t}{L} \\
\ln \frac{\epsilon}{\epsilon - IR} &= \frac{Rt}{L} \\
\frac{\epsilon}{\epsilon - IR} &= e^{\frac{Rt}{L}} \\
&\cdots \\
I(t) &= \boxed{\frac{\epsilon}{R}(1 - e^{-\frac{Rt}{L}})}
\end{aligned}$$

What if the battery is removed, leaving the inductor to discharge? We apply Kirchhoff's Loop Rule again, this time without the battery. We also assume the current is $\frac{\epsilon}{R}$ at time $t = 0$.

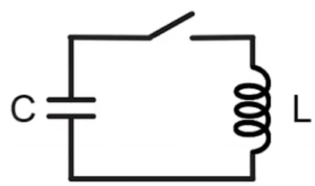
$$\begin{aligned}
-IR - L\frac{dI}{dt} &= 0 \\
L\frac{dI}{dt} &= -IR \\
\frac{dI}{I} &= \frac{-Rdt}{L} \\
\int_{\frac{\epsilon}{R}}^{I(t)} \frac{dI}{I} &= \int_0^t \frac{-Rdt}{L} \\
\ln I - \ln \frac{\epsilon}{R} &= e^{\frac{-Rt}{L}} \\
&\cdots \\
I(t) &= \boxed{\frac{\epsilon}{R}e^{-\frac{Rt}{L}}}
\end{aligned}$$

13.6 LC Circuits

A review of the basic properties of capacitors and inductors:

- A capacitor stores is energy in a electric field, whereas an inductor stores its energy in a magnetic field.
- A capactior acts like a closed switch when current initially flows in the circuit and a open switch when current has been flowing for a long amount of time.
- An inductor acts like a open switch when current initially flows inside the circuit (resisting a change in current) and a closed switch when current has been flowing for a long amount of time.

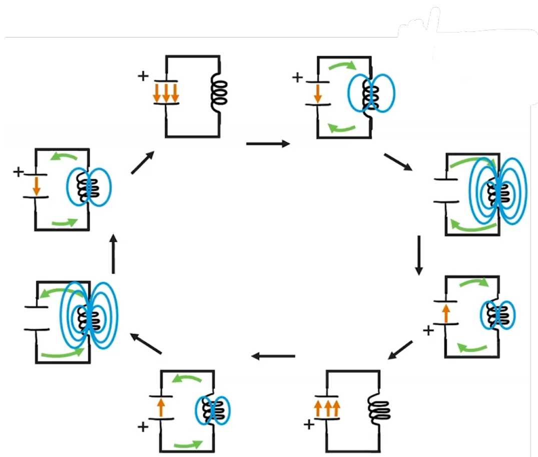
Let's take a look at the following circuit. In this circuit, the capacitor is fully charged, and no current is flowing through the wire.



Initially, energy is stored inside the electric field in the capacitor. When the switch is closed, current begins to flow between the capacitor plates, causing a magnetic field to be produced inside the inductor. As the voltage between the plates of the capacitor decrease, the energy stored in the electric field decreases, while the increase in current causes the energy stored in the magnetic field of the inductor to increase. Since we assume the wire to be a perfect conductor, no energy is lost, and we can apply the conservation of potential energy with the components.

The following equation shows how when the energy stored in the electric field is at its maximum, there is no energy stored in the magnetic field, and vice versa.

$$U_{Cmax} = U_{Lmax}$$
$$\frac{1}{2}CV_{max}^2 = \frac{1}{2}LI_{max}^2$$



SHM of LC Circuits

The oscillations between the magnetic field and electric field can be described as simple harmonic motion. We can use Kirchhoff's Loop rule to derive this.

$$\Delta V_C - \Delta V_L = 0$$
$$L\frac{dI}{dt} = -\frac{q}{C}$$
$$L\frac{d^2q}{dt^2} = -\frac{q}{C}$$

This looks very similar to the other simple harmonic motion equations from mechanics:

Simple Harmonic Motion	
Mass and Spring	$\frac{d^2x}{dt^2} = -\frac{k}{m}x$
Simple Pendulums	$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$
LC Circuits	$\frac{d^2q}{dt^2} = -\frac{1}{LC}q$

Applying this to the general equation for simple harmonic motion ($\frac{d^2x}{dt^2} = -\omega^2 x$), we get that ω, the angular frequency, is $\frac{1}{\sqrt{LC}}$. Therefore, the period is $2\pi\sqrt{LC}$.