The Discrete Fourier Transform

Homework solutions

Deriving the trigonometic sum formulas

$$\begin{aligned}
\cos(A+B) &= \cos A \cos B - \sin A \sin B \\
\sin(A+B) &= \cos A \sin B + \sin A \cos B
\end{aligned}$$

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This implies from Euler's formula, $e^{ix} = \cos x + i \sin x$, that

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Multiplying out the right-hand-side, we find

$$\cos(A+B) + i \sin(A+B) =$$
$$\cos A \cos B - \sin A \sin B + i [\cos A \sin B + i \cos B \sin A]$$

Taking the real and imaginary parts, we have the first two equations.

Homework solutions

Deriving the product formula

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Substituting $-B$ in place of B leads to the difference formula

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because $\sin B$ changes sign but $\cos B$ does not.

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Add these two formulas to obtain

$$\cos(A+B) + \cos(A-B) = 2\cos A \cos B$$

which rearranges to give the sum formula.

Homework solutions

Similarly, the sin version of the product theorem is

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These exercises have shown how Euler's formula

$$e^{ix} = \cos x + i\sin x$$

can be used to readily derive common trigonometic identities.

Homework solutions

Next we'll look at time derivatives of the two sides Euler's formula

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Second deriviative

$$-\omega^2 e^{i\omega t} \quad ? \quad -\omega^2 \cos(\omega t) - i\omega^2 \sin \omega t$$

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nth deriviative

$$(i\omega)^n e^{i\omega t} \quad ? \quad (i\omega)^n \cos(\omega t) + i(i\omega)^n \sin \omega t$$

Homework solutions

Taylor series of cosine about zero:

$$\cos(\omega t) = 1 - \frac{1}{2}\omega^2 t^2 + \ldots = \sum_{n=0}^{\infty} \frac{1}{(2n)!}(-1)^n (\omega t)^{2n}$$

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$$\sin(\omega t) = \omega t - \frac{1}{6}\omega^3 t^3 + \ldots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(-1)^n (\omega t)^{2n+1}$$

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Taylor series of the complex exponential about zero:

$$e^{i\omega t} = 1 + i\omega t - \frac{1}{2}\omega^2 t^2 - i\frac{1}{6}\omega^3 t^3 \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}(i\omega t)^n$$

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We note that the even terms in this last summation correspond to those of cos, whereas the odd terms correspond to those of $i \sin$.

Homework solutions

The point of this exercise is to show you one way that you can verify Euler's formula for yourself

$$e^{i\omega t} = \cos(\omega t) + i \sin \omega t$$

by Taylor-expanding the left- and right-hand sides separately, and showing that they agree.

Homework solutions

Finally I asked you to look at the equation

$$z(t) = e^{i\theta}\left[a\cos\omega t + ib\sin\omega t\right]$$

What is this an equation for?

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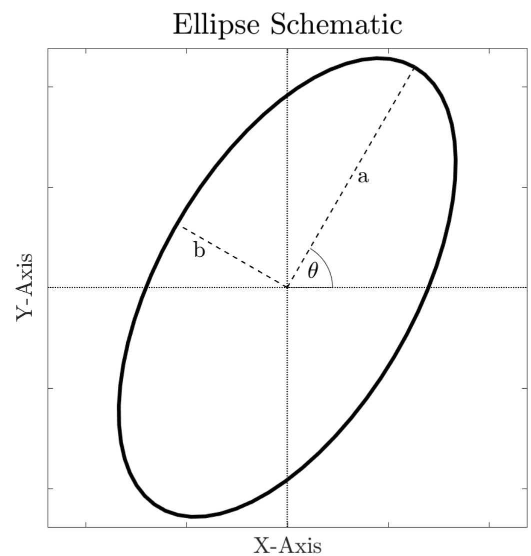

This is the parametric equation for an ellipse with semi-axes lengths a and b, and orientation θ.

Homework solutions

Let's also consider the sum of a positively-rotating and negatively-rotating circle

$$z(t) = Pe^{i\phi}e^{i\omega t} + Ne^{i\phi}e^{-i\omega t}$$

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If we compare to the equation on the previous page

$$z(t) = e^{i\theta}\left[a\cos\omega t + ib\sin\omega t\right]$$

We see that the sum of two opposite-rotating complex exponentials of the same frequency traces out an ellipse.

This fundamental fact will inform how we see Fourier analysis.

Orientation

In the last lecture we looked at two fundamental building blocks of Fourier analysis. The first is the complex exponential itself:

$$e^{i\omega t}.$$

The second is the Euler's formula

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$

which we use so often, we forget how deep and non-obvious it is.

This lecture looks at a third building block, the Discrete Fourier Transform or DFT:

$$z_n = \frac{1}{N}\sum_{m=0}^{N-1} Z_m e^{i2\pi mn/N}, \qquad Z_m \equiv \sum_{n=0}^{N-1} z_n e^{-i2\pi mn/N}.$$

In this lecture, we will take this equation apart, and understand it hopefully on a deeper level.

In-Class Exercises

Before continuing with theory, we will play around a bit in Matlab or Python, exploring some aspects of the discrete Fourier transform in practice.

We will work in small groups on these exercises. Please make sure that everyone in your group agrees on each answer.

1. Constuct the even-length time series $t = [0:1:99]$. You are going to plot $x(t) = \cos(2\pi f t)$ and also the discrete Fourier transform $\mathrm{abs}(\mathrm{fft}(x(t)))$ for (a) $f = 0$ (b) $f = 1/10$ and (c) $f = 1/2$. What do you see? At which locations in the frequency domain do you see something occurring in each of these three cases? Do you have an explanation?
2. Same as #1 but for $z(t) = e^{i2\pi f t}$. What has changed?
3. Same as #1 but for the odd-length time series $t = [0:1:100]$. What has changed?
4. As in #1 but for $x(t) = \sin(2\pi f t)$ for (a) $f = 1/100$ (b) $f = 1/200$ and (c) $f = 1/400$.

The Fourier Transform

Any discrete time series z_n can be built up of a sum of complex exponentials:

$$z_n = \frac{1}{N}\sum_{m=0}^{N-1} Z_m e^{i2\pi n f_m}, \qquad f_m \equiv \frac{m}{N} \qquad n = 0, 1, \ldots N-1$$

where the quantity f_m is called the mth *Fourier frequency*. Note that here the sample interval Δ is implicitly set to one!

The *period* associated with f_m is $1/f_m = N/m$. Thus m tells us *how many oscillations* at this frequency fit in the length N time series.

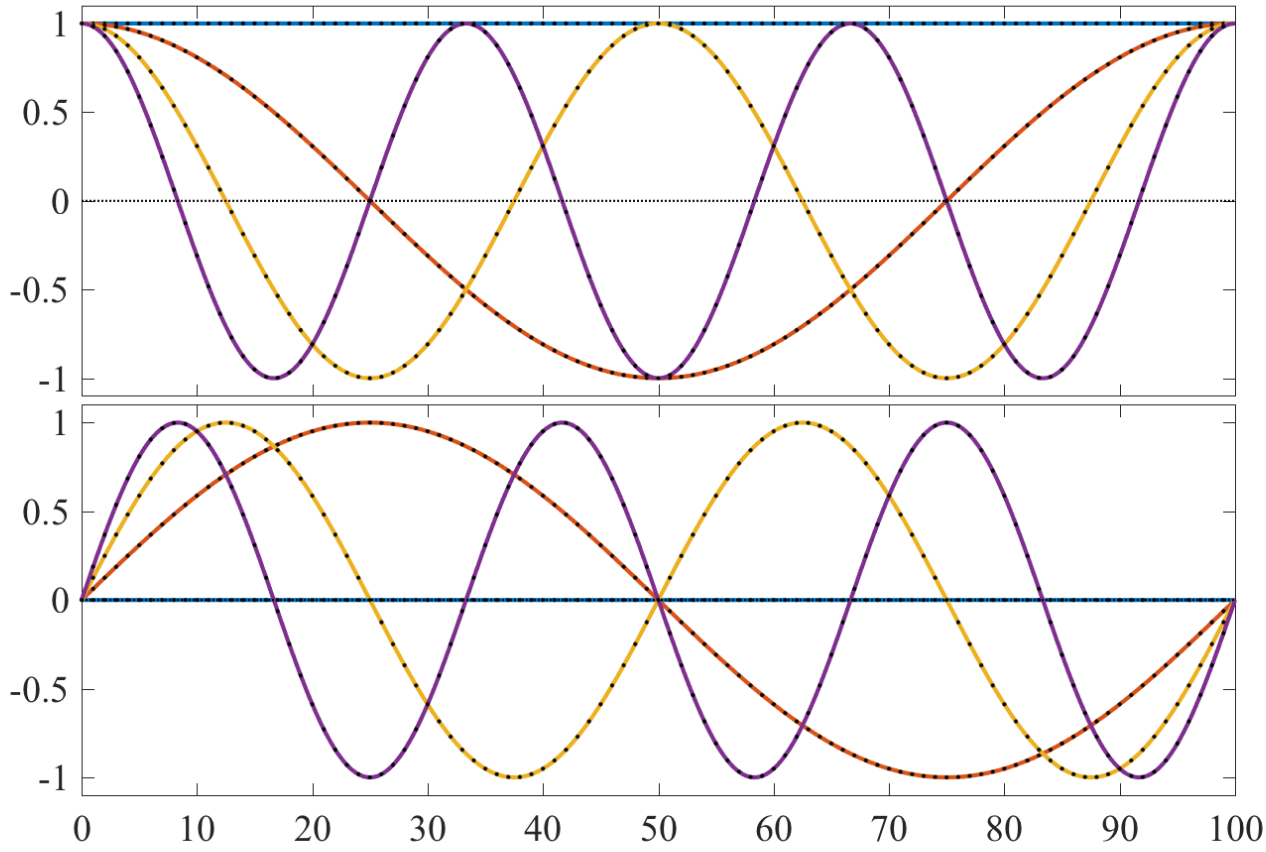
The first few terms in this expansion are

$$z_n = \frac{1}{N}\left[Z_0 + Z_1 e^{i2\pi n\,(1/N)} + Z_2 e^{i2\pi n\,(2/N)} + Z_3 e^{i2\pi n\,(3/N)} + \ldots\right].$$

So the mth term behaves as $e^{i2\pi f_m n} = \cos(2\pi f_m n) + i\sin(2\pi f_m n)$.

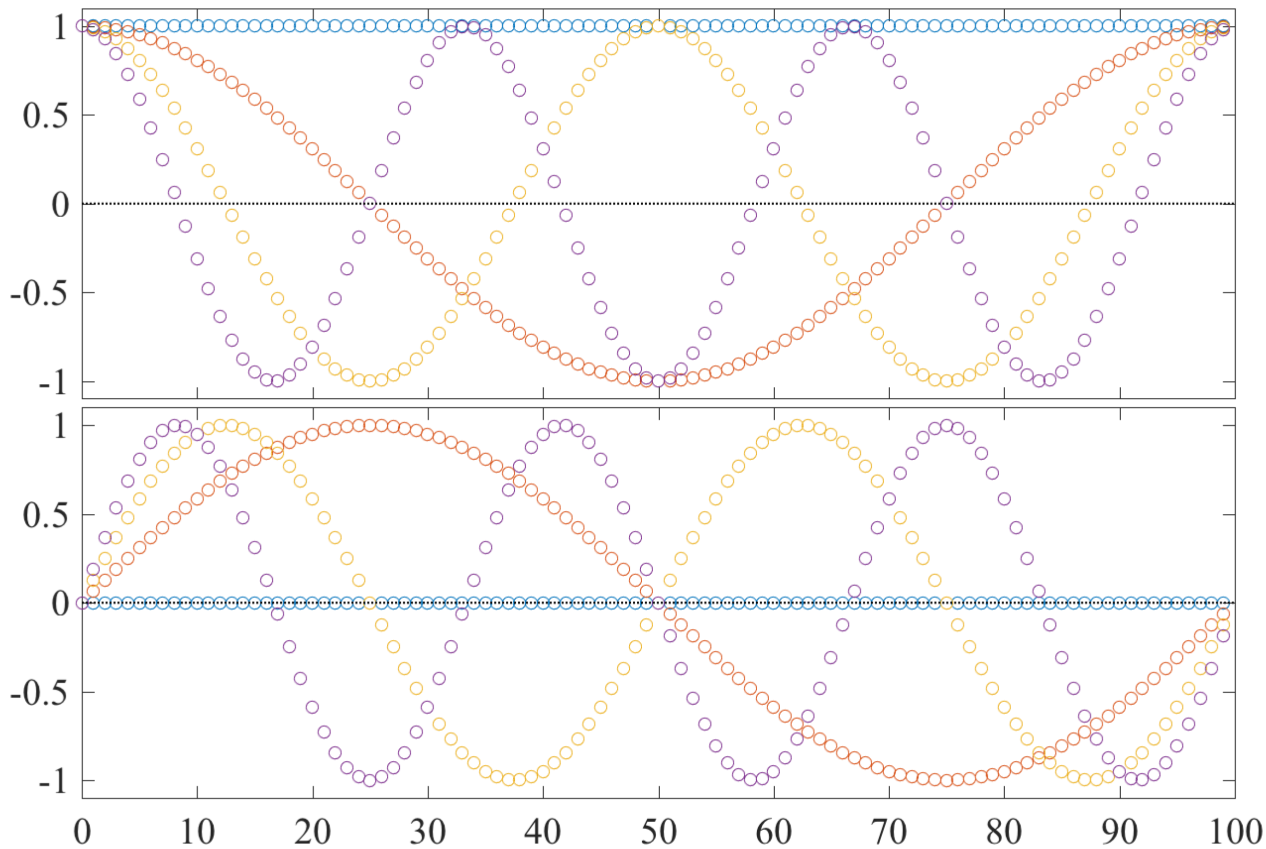
Continuous Time

$$\cos(2\pi f_m t) \text{ and } \sin(2\pi f_m t)\ f_m = 0,\ 1/100,\ 2/100,\ 3/100$$
$$t = [0 \ldots 100]$$



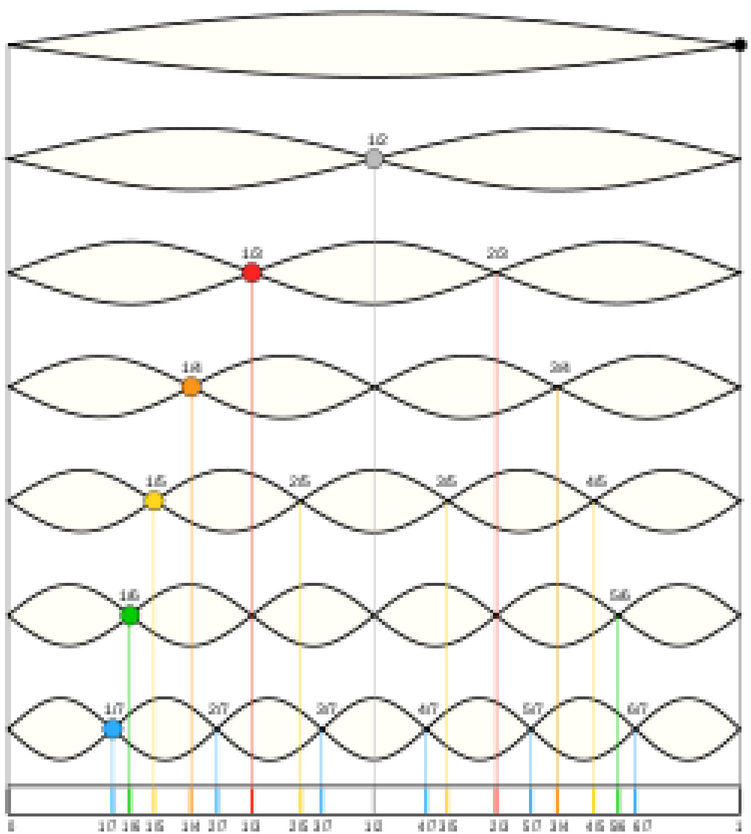
Discrete Time

$$\cos(2\pi f_m n) \text{ and } \sin(2\pi f_m n)\ f_m = 0,\ 1/100,\ 2/100,\ 3/100$$
$$n = 0, 1, 2, \ldots 99$$



Musical Analogy

We can think of the Fourier transform in musical terminology.



Fundamental (First Harmonic)

First Overtone (2nd Harmonic)

2nd Overtone (3rd Harmonic)

...

We can imagine that the sine terms in the Fourier transform describe the vibration of a string. Note that in Fourier analysis there is no fundamental, and we must also include the cosine terms having *anti-nodes* at the endpoints.

A Sum of Phasors

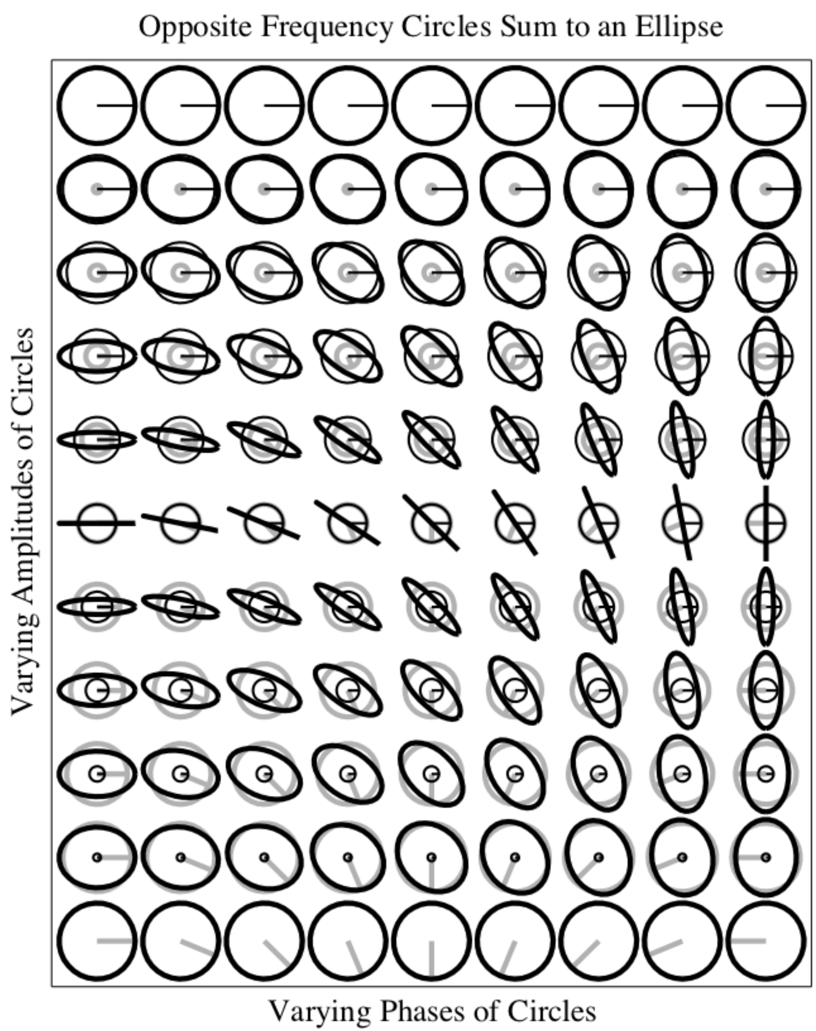
If we think of our time series z_n as a curve traced out on the u/v plane, the discrete Fourier transform is literally seen as being the instruction to add up a bunch of circles with different amplitudes and different frequencies,

$$z_n = \frac{1}{N}\sum_{m=0}^{N-1} Z_m e^{i2\pi n f_m}, \qquad f_m \equiv \frac{m}{N} \qquad n = 0, 1, \ldots N-1.$$

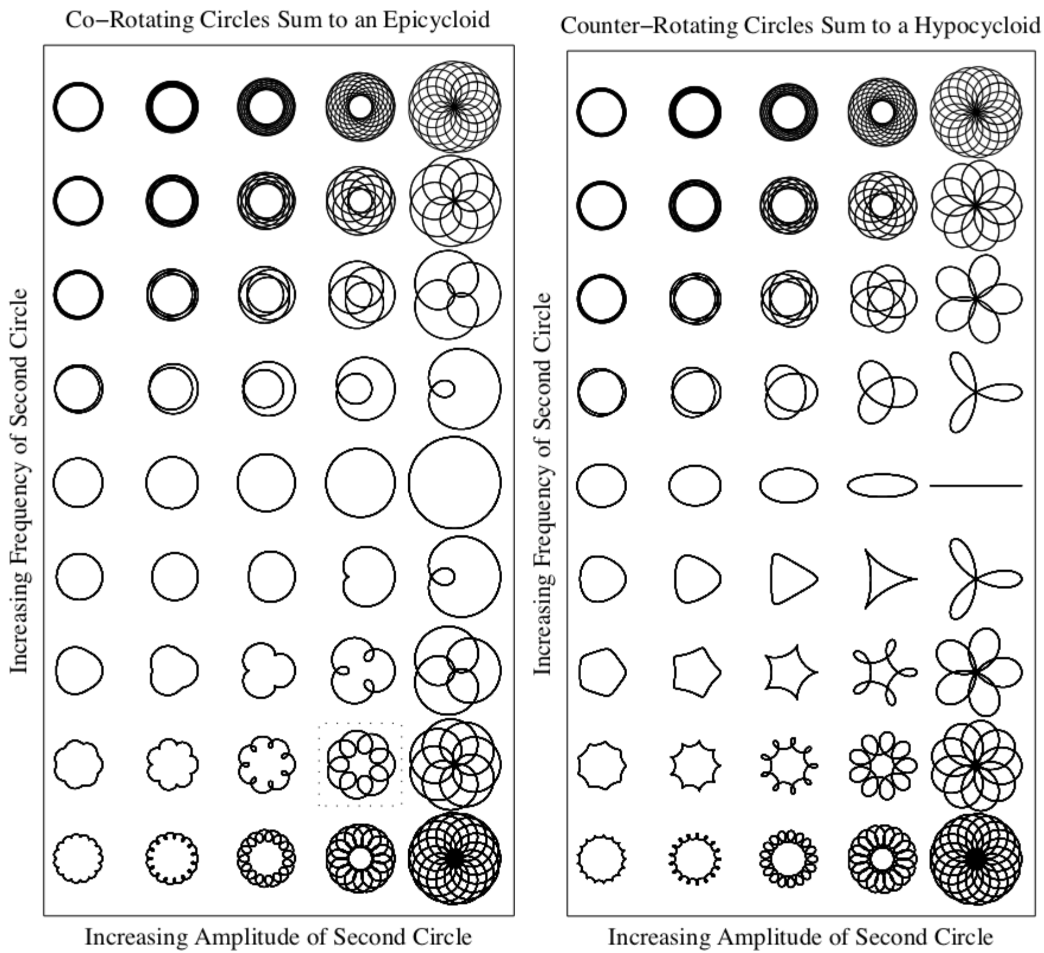
This makes the Fourier transform actually *easier* to understand if we are dealing with complex-valued data such as velocity.

If you have real-valued data, the phases of the Fourier coefficients must arrange themselves such that z_n is real.

Opposing Frequencies



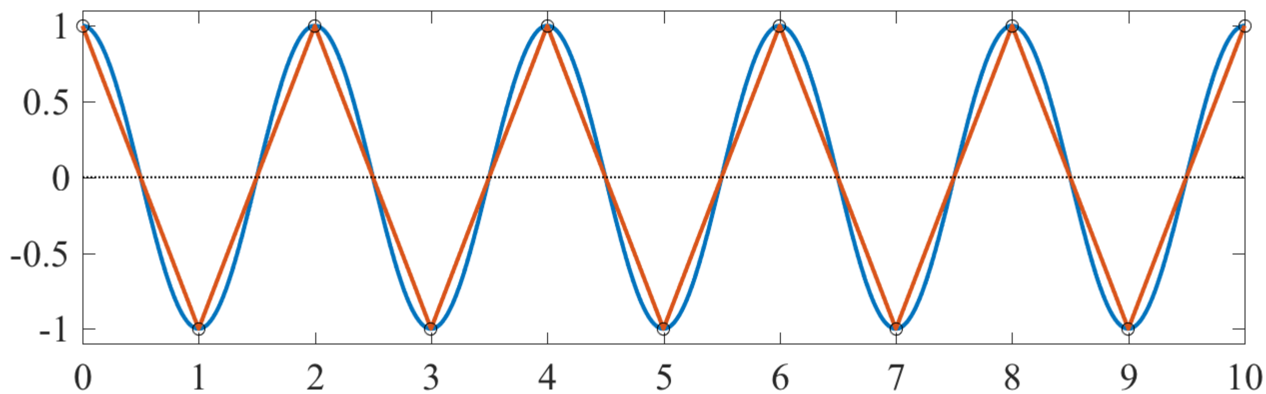
Two Different Frequences



The Nyquist Frequency

The single most important frequency is the *highest resolvable* frequency, the *Nyquist frequency*.

$$f^{\mathcal{N}} \equiv \frac{1}{2\Delta} \qquad \omega^{\mathcal{N}} \equiv \frac{2\pi}{2\Delta} = \frac{\pi}{\Delta}$$



The highest resolvable frequency is *one cycle per two data points*.

$$e^{i2\pi f_m n\Delta} = e^{i2\pi\cdot 1/(2\Delta)\cdot n\Delta} = e^{i\pi n} = (-1)^n = 1, -1, 1, -1, \ldots$$

Note that there is no “sine” component at Nyquist!

Aliasing

Q: What if you try to observe a *higher* frequency than the Nyquist?

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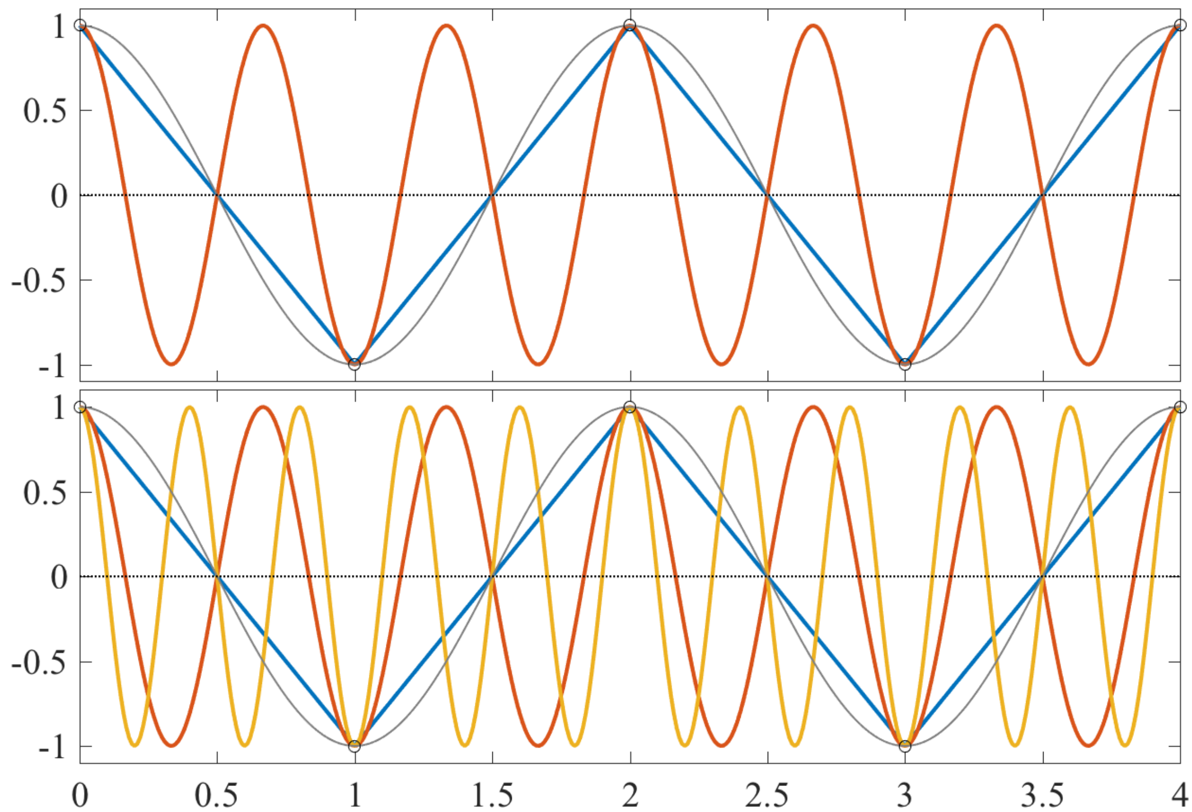


A. You will think you see things that *aren't really there*.

The Wagon-Wheel Effect

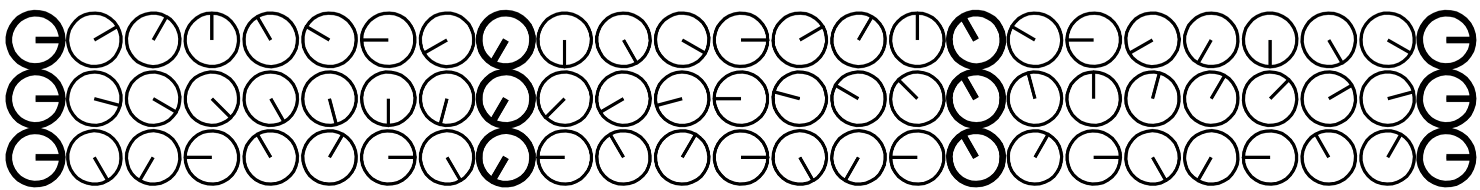
Aliasing is a kind of *optical illusion*. In film, it's known as the *wagon-wheel effect*.

Thanks to {Dora} for posting.

Aliasing = Wagon Wheel

Aliasing and the wagon-wheel effect are essentially the *same thing*. Unresolved frequencies are said to be *aliased into* resolved ones.

Examples of postive and negative rotary aliasing



More than one *true* rotation rate could correspond to the *apparent* rotation rate—even in the opposite direction!

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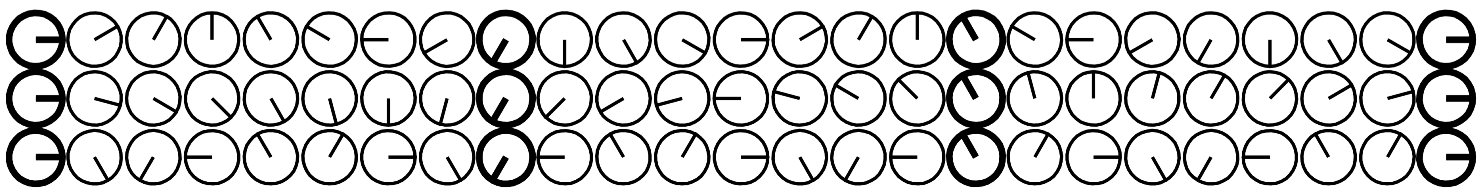

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What is the Nyquist frequency of your time series in cycles per unit time? In radians per unit time?

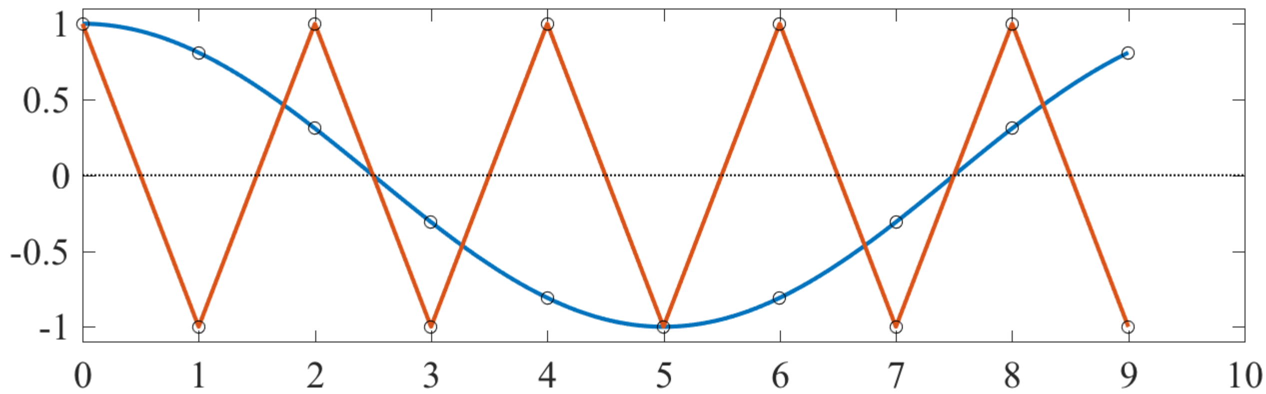
$$f^{\mathcal{N}} \equiv \frac{1}{2\Delta} \qquad \omega^{\mathcal{N}} \equiv \frac{2\pi}{2\Delta} = \frac{\pi}{\Delta}$$

Anything happening at higher frequencies will be aliased—it will appear to occur at a *different* frequency!

The Rayleigh Frequency

The second most important frequency is the *lowest resolvable* frequency, the *Rayleigh frequency*.

$$f^{\mathcal{R}} \equiv \frac{1}{N\Delta} \qquad \omega^{\mathcal{R}} \equiv \frac{2\pi}{N\Delta}$$



The lowest resolvable frequency is one cycle over the *entire record*. Here the sample interval is $\Delta = 1$ and the # of points is $N = 10$.

If we think of the data as a vibrating string, the Rayleigh frequency is the *first overtone*. The fundamental does not appear in the DFT.

Importance of Rayleigh

The Rayleigh frequency $f^{\mathcal{R}}$ is important because it controls the *spacing between* the Fourier frequencies, in other words, the *frequency-domain resolution*.

With general sample interval Δ, the Fourier frequencies are

$$f_0 = 0, \quad f_1 = \frac{1}{N\Delta}, \quad f_2 = \frac{2}{N\Delta}, \ldots$$
$$f_n = n\, f^{\mathcal{R}}, \qquad f^{\mathcal{R}} = \frac{1}{N\Delta}.$$

Thus if you want to distiguish two closely spaced peaks (e.g. tidal components), you need the dataset *duration* to be sufficiently *large* so that the Rayleigh frequency is sufficiently *small*.

The ratio of the Rayleigh to Nyquist frequencies tells you how many *different frequencies* you can resolve.

$$\frac{f^{\mathcal{N}}}{f^{\mathcal{R}}} = \frac{N\Delta}{2\Delta} = \frac{N}{2}$$

Summary So Far

Let's review what we've learned yesterday and today.

Cyclic vs. radian (or angular) frequency

$$\cos(2\pi f t) \qquad vs. \qquad \cos(\omega t)$$

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Beating; or, the sum of two different frequency sinusoids

$$\cos(\omega_1 t) + \cos(\omega_2 t) = 2\cos\left(\frac{1}{2}(\omega_1 + \omega_2)t\right)\cos\left(\frac{1}{2}(\omega_1 - \omega_2)t\right)$$

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The sum of two oppositely-rotating complex exponentials is an ellipse. We saw this both geometrically and algebraically.

Summary So Far

The discrete Fourier transform

$$z_n = \frac{1}{N}\sum_{m=0}^{N-1} Z_m e^{i2\pi nm/N}, \qquad\qquad Z_m \equiv \sum_{n=0}^{N-1} z_n e^{-i2\pi nm/N}$$

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Frequencies present in the data above the Nyquist frequency experience the phenomenon of *aliasing*.

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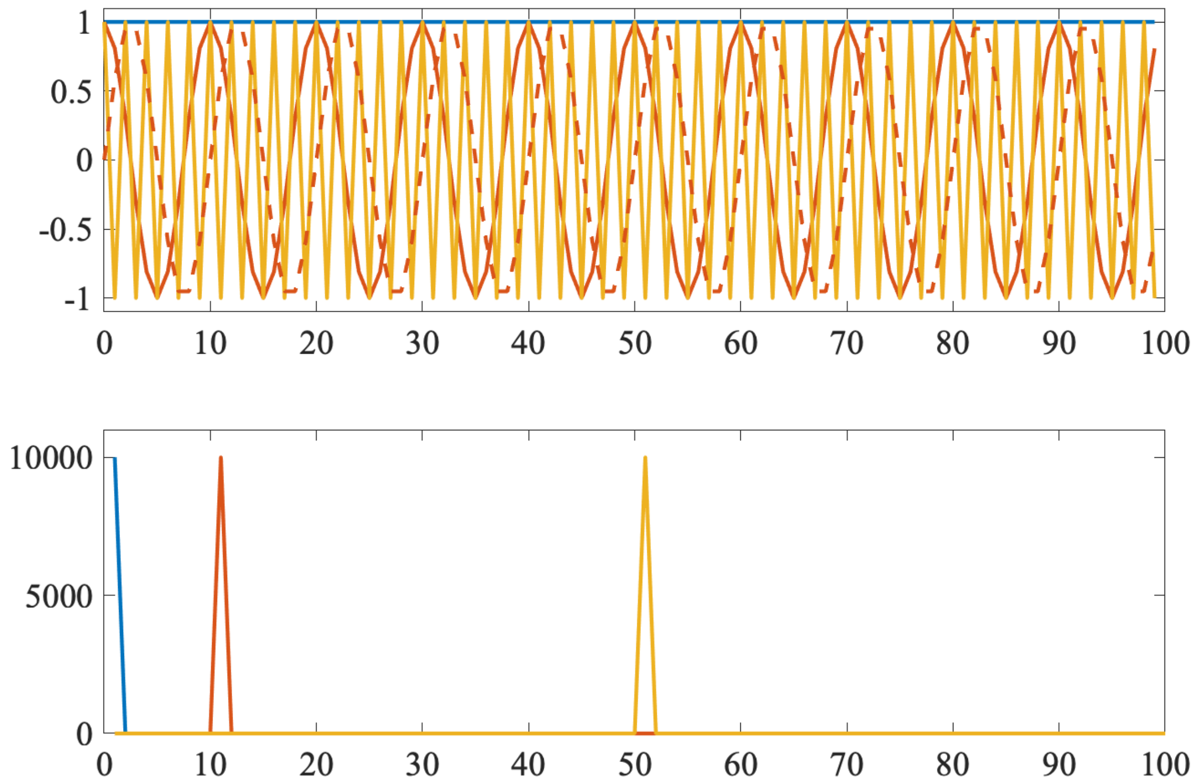
The Rayleigh frequency $f^{\mathcal{R}} = \frac{1}{N\Delta}$

This sets both the *lowest* Fourier frequency, and the interval *between* adjacent frequencies—a.k.a. the frequency resolution.

Some Fourier Transforms

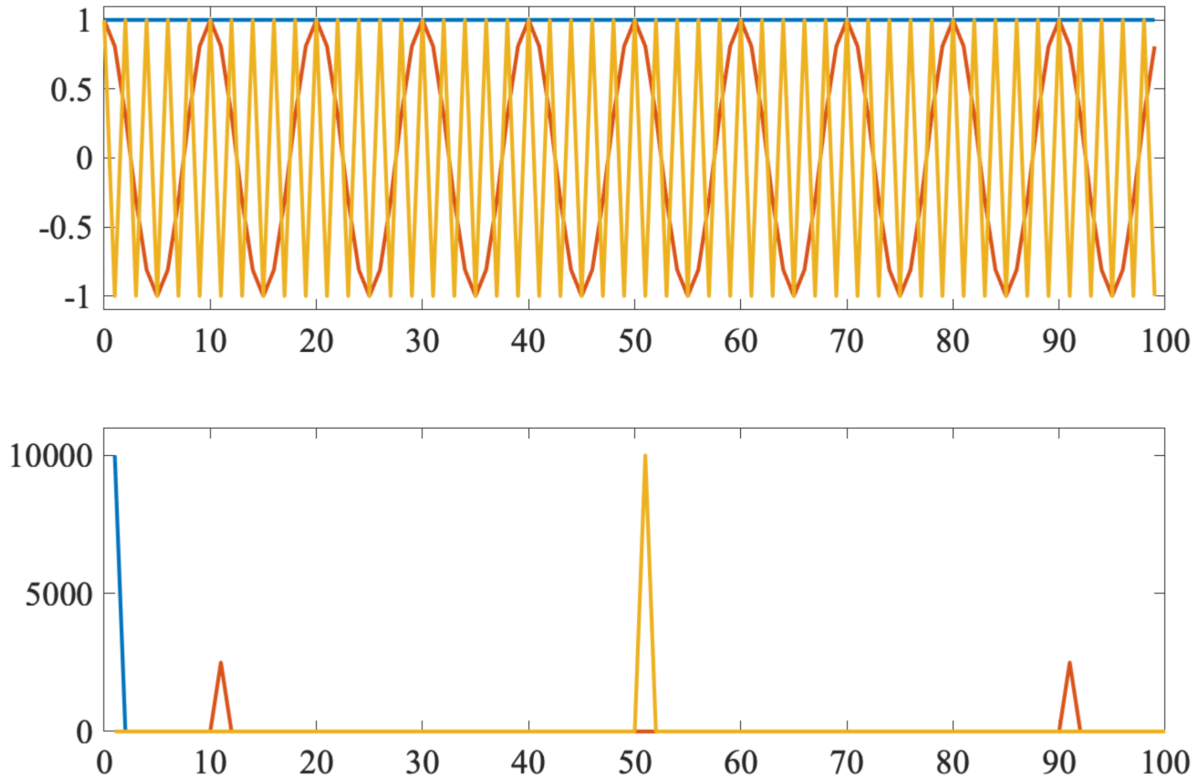
$e^{i2\pi nf}$ for $n = 0, 1, \ldots, 99$ and $f = 0, 1/10$, and $1/2$



Peaks at 1, 11, and 51

Some Fourier Transforms

$\cos(2\pi n f)$ for $n = 0, 1, \ldots, 99$ and $f = 0, 1/10$, and 1/2



Peaks at 1, 11 and 91, and 51

The Fourier Series

Any discrete time series z_n can be built up of a sum of complex exponentials. Assuming a unit sample interval, $\Delta = 1$, we have

$$z_n = \frac{1}{N}\sum_{m=0}^{N-1} Z_m e^{i2\pi n f_m}, \qquad f_m \equiv \frac{m}{N} \qquad n = 0, 1, \ldots N-1$$

If there are N points in z_n, then you also need N different complex exponentials to completely describe z_n.

Subtlety (i): z_n may contain frequencies not present in the sum

Subtlety (ii): z_n may be real-valued, but Z_m is complex.

Subtlety (iii): There are N different numbers in a real-valued z_n, but $2N$ different numbers in $Z_m = \Re\{Z_m\} + i\Im\{Z_m\}$.

Points (ii) and (iii) are resolved by noting that of half the information in Z_m is redundant if z_n is real-valued.

The Fourier Frequencies

The first few Fourier frequencies $f_m = m/N$, and the last, are

$$f_0 = \frac{0}{N}, \quad f_1 = \frac{1}{N}, \quad f_2 = \frac{2}{N}, \ldots \qquad f_{N-1} = \frac{N-1}{N} = 1 - \frac{1}{N}.$$

While the corresponding Fourier terms $e^{i2\pi n f_m}$ are

$$e^{i2\pi f_0} = e^0 = 1, \quad e^{i2\pi n/N}, \quad e^{i4\pi n/N} \ldots \qquad e^{i2\pi(N-1)n/N}.$$

The first, $m = 0$, term is just a constant. The second, $m = 1$, term is a complex exponential whose period is the whole signal duration.

But notice that the last Fourier term becomes

$$e^{i2\pi(N-1)n/N} = e^{i2\pi n(1-1/N)} = e^{i2\pi n}e^{-i2\pi n/N} = e^{-i2\pi n/N}$$

as $e^{i2\pi n} = 1$ for integer n. This is a low-frequency oscillation at the Rayleigh frequency. It is just the conjugate of the $m = 1$ term!

The Fourier Frequencies

Similarly in the vicinity of $m = N/2$ for even N we have

$$f_{N/2-1} = \frac{1}{2} - \frac{1}{N}, \quad f_{N/2} = \frac{1}{2}, \quad f_{N/2+1} = \frac{1}{2} + \frac{1}{N}, \ldots$$

but actually the first frequency higher than the Nyquist is the *highest negative frequency*

$$f_{N/2-1} = \frac{1}{2} - \frac{1}{N}, \quad f_{N/2} = \frac{1}{2}, \quad f_{N/2+1} = -\left(\frac{1}{2} - \frac{1}{N}\right), \ldots.$$

Thus the positive frequencies and negative frequencies occur in twins that both increase *toward the middle* of the frequency array.

For this reason Matlab provides `fftshift`, to shifts the zero frequency, *not* the Nyquist, to be in the middle of the array.

The Fourier Frequencies

Thus the Fourier components $e^{i2\pi nm/N}$ are (for even N)

$$\underset{m=0}{1} \quad \underset{m=1}{e^{i2\pi n(1/N)}} \quad \underset{m=2}{e^{i2\pi(2/N)}} \quad \ldots \quad \underset{m=N/2}{e^{i2\pi(1/2)}} \quad \ldots \quad \underset{m=N-2}{e^{-i2\pi n(2/N)}} \quad \underset{m=N-1}{e^{-i2\pi(1/N)}}$$

Mean Rayleigh Nyquist Negative Rayleigh

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Mean Rayleigh Nyquist Negative Rayleigh

Recalling $e^{i\omega t} + e^{-i\omega t} = 2\cos(\omega t)$, we can now see why our Fourier transform of a real-valued oscillation led to two peaks. One is for $e^{i\omega t}$ and the other for $e^{-i\omega t}$, which add together to give $\cos(\omega t)$.

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Note that this can be understood as aliasing. Frequencies higher than the Nyquist don't explicitly appear in our representation. Instead, those terms are wrapped around into *negatively-rotating* terms at frequencies lower than the Nyquist.

One-Sided vs. Two-Sided

Because of this wrapping—or aliasing—of high positive frequencies into negative frequencies, we can write the discrete Fourier transform in the two equivalent forms (for an even value of N)

$$z_n = \frac{1}{N}\sum_{m=0}^{N-1} Z_m e^{i2\pi n f_m}$$

$$z_n = \frac{1}{N}Z_0 + \frac{1}{N}\sum_{m=1}^{N/2-1} Z_m e^{i2\pi n f_m} + \frac{1}{N}\sum_{m=1}^{N/2-1} Z_{N-m} e^{-i2\pi n f_m} + Z_{N/2}(-1)^n.$$

where the former is said to be a *two-sided* representation, and the latter to be *one-sided*.

In the one-sided representation, the summation only includes terms up to $N/2-1$ and frequency $f^{\mathcal{N}} - f^{\mathcal{R}}$. However, we need two such sums, one for positive frequencies and one for negative frequencies.

Twin Frequencies

We will refer to frequencies with the same magnitude but opposite sign as *twin frequencies*.

It is useful to write out the discrete Fourier transform, grouping twin frequencies together, and writing out $f_m = m/N$, as

$$z_n = \frac{1}{N}\Big\{Z_0 + \Big[Z_1 e^{i2\pi n/N} + Z_{N-1} e^{-i2\pi n/N}\Big] + \Big[Z_2 e^{i2\pi n(2/N)} + Z_{N-2} e^{-i2\pi n}$$
$$+ \Big[Z_3 e^{i2\pi n(3/N)} + Z_{N-3} e^{-i2\pi n(3/N)}\Big] \ldots + Z_{N/2}$$

If z_n is complex-valued, like a velocity $z_n = u_n + iv_n$, these pairs are telling us about *positively* and *negatively* rotating circles at the same frequency. As we will see later, each pair adds up to an *ellipse*.

If our time series z_n is real-valued, the Fourier coefficients at twin frequencies need to have a particular relationship to each other. We must have $Z_1 = Z_{N-1}^*$, $Z_2 = Z_{N-2}^*$, etc. Thus, for real z_n, the Fourier components occur in *conjugate pairs*.

The Case of Real-Valued z_n

For real-valued z_n, the usual discrete inverse Fourier transform

$$z_n = \frac{1}{N}\sum_{m=0}^{N-1} Z_m e^{i2\pi n f_m}$$

represents the time series as involving a sum of *positively-rotating* and *negatively-rotating* contributions, occuring in conjugate pairs such that the sum of each pair is real-valued.

This seems somewhat roundabout! Instead, the one-sided representation can be written as phase-shifted real-valued sinusoids

$$z_n = \frac{1}{N} Z_0 + \frac{2}{N}\sum_{m=1}^{N/2-1} A_m \cos(2\pi n f_m + \Phi_m) + Z_{N/2}(-1)^n$$

where A_n and Φ_n are an *amplitude* and *phase*, with $Z_n = A_n e^{i\Phi_n}$. Note the zero frequency Z_0 and Nyquist $Z_{N/2}$ must be real-valued.

The Case of Complex z_n

If, however, we have complex-valued data $z_n = x_n + iy_n$, for example representing velocity, then the Fourier coefficents Z_n defined by

$$z_n = \frac{1}{N}\sum_{m=0}^{N-1} Z_m e^{i2\pi mn/N}, \qquad Z_m \equiv \sum_{n=0}^{N-1} z_n e^{-i2\pi mn/N}.$$

no longer possess a conjugate symmetry.

In this case, the representation of z_n in terms of positively-rotating and negatively rotating circles is exactly what we want!

For example, in the northern hemisphere, inertial oscillations or anticyclonic rotations occur in the mathematically *negative sense*, and thus correspond to $e^{-i\omega t}$ with $\omega > 0$.

Thus all the complications in dealing with real-valued time series actually become *easier* with complex-valued time series!

A Twist for Odd N

There is a subtle difference for even and odd N.

If N is odd, the Nyquist frequency $f_\mathcal{N} = 1/(2\Delta)$, or $f_\mathcal{N} = 1/2$ with $\Delta = 1$, *does not appear* in the Fourier transform.

Instead the highest positive frequency occurs at position $m = (N-1)/2$ in zero-based counting, and the highest negative frequency occurs at the next position $m = (N+1)/2$.

Thus the highest frequencies resolved with an *odd* choice of N are

$$\frac{1}{N\Delta}\frac{N-1}{2} = \frac{1}{2\Delta} - \frac{1}{2N\Delta} = f_\mathcal{N} - \frac{1}{2}f_\mathcal{R}.$$

Thus there are *two* highest resolved frequencies, which are each one-half of the Rayleigh frequency away from the Nyquist.

Again, the m's given above are in zero-based numbering, so one must add one to find these positions in a Matlab array.

Summary

The inverse Fourier transform

$$z_n = \frac{1}{N}\sum_{m=0}^{N-1} Z_m e^{i2\pi nm/N}$$

represents a time series as being composed of contributions from complex exponentials at plus or minus all the Fourier frequencies from the Rayleigh to the Nyquist frequency, together with zero.

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For real-valued time series, this decomposition can be expressed as a sum over sines and cosines. These emerge because of a cancellation arising from conjugate pairs of Fourier coefficients.

For complex-valued signals, positively and negatively rotating Fourier components at the same frequency can usefully be thought of as generating an ellipse.

Forward & Inverse DFT

The Fourier transform equations occur in a pair

$$z_n = \frac{1}{N}\sum_{m=0}^{N-1} Z_m e^{i2\pi mn/N}, \qquad Z_m \equiv \sum_{n=0}^{N-1} z_n e^{-i2\pi mn/N}.$$

The second of these is called the *discrete Fourier transform* of z_n. It *transforms* z_n from the time domain to the frequency domain.

The DFT *defines* a sequence of N complex-valued numbers, Z_n, for $n = 0, 1, 2, \ldots N-1$, which are termed the *Fourier coefficients*.

The first expression is the *inverse discrete Fourier transform,* or perhaps more intuitively, the *Fourier representation* of z_n.

It expresses how z_n may be *constructed* by a sum of complex exponentials at different frequencies, with the right coefficients Z_m.

Note the symmetry of these equations.

Forward & Inverse DFT

However, considering the discrete Fourier transform equations

$$z_n = \frac{1}{N} \sum_{m=0}^{N-1} Z_m e^{i2\pi mn/N}, \qquad Z_m \equiv \sum_{n=0}^{N-1} z_n e^{-i2\pi mn/N}.$$

it remains to be shown that the coefficients from forward transform (which we regard as a definition) allow us to reconstruct our original time series using the inverse equation.

To understand this we need to review the notion of orthogonality.

Review: Orthogonality

If you have complex exponentials at two different frequencies, $f_p \equiv p/N$ and $f_q \equiv q/N$, and you multiply one by *the conjugate* of the other and sum over all times n, you have

$$\sum_{n=0}^{N-1} e^{i2\pi n f_p} e^{-i2\pi n f_q} = \sum_{n=0}^{N-1} e^{i2\pi n(p-q)/N} = N\delta_{pq}$$

where δ_{pq} is called the *Kronecker delta-function:*

$$\delta_{pq} = \begin{cases} 1 & p = q \\ 0 & p \neq q \end{cases}$$

Thus, the sum over the product these two cosines is N if their frequencies are the same, and *zero* otherwise.

This occurs because $e^{i2\pi n(p-q)/N}$ executes an integral number of *complete periods* unless $p = q$, thus summing to zero.

Proof of Orthogonality

Using $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$, we find

$$\sum_{n=0}^{N-1} \cos(2\pi pn/N) \cos(2\pi qn/N)$$

$$= \sum_{n=0}^{N-1} \frac{1}{2}[\cos(2\pi(p+q)n/N) + \cos(2\pi(p-q)n/N)] \,.$$

Both terms are of the form $\cos(2\pi kn/N)$ where is an integer. If $k \neq 0$, both terms execute an integer number of complete periods as n varies from 0 to $N-1$, so that $\sum_{n=0}^{N-1} \cos(2\pi kn/N) = 0$. If $p = q$ however, then the second term is always $\cos(0) = 1$.

If N is even and $k = 1$, $\cos(2\pi(n + N/2)/N) = \cos(2\pi n/N + \pi)$, the first $N/2$ terms cancel the last $N/2$ terms.

Thus we have $N/2$ if $p = q$ and 0 otherwise, or $\frac{N}{2}\delta_{pq}$.

The Fourier Coefficients

How do we know the values of the Fourier coefficients Z_m?

Multiply z_n by *another* complex exponential, $e^{-i2\pi n f_p}$ with frequency $f_p \equiv p/N$, and sum over all N.

$$z_n e^{-i2\pi n f_p} = \left[\frac{1}{N} \sum_{m=0}^{N-1} Z_m e^{i2\pi n f_m} \right] e^{-i2\pi n f_p}$$

$$\sum_{n=0}^{N-1} z_n e^{-i2\pi n f_p} = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} Z_m e^{i2\pi n (m-p)/N}$$

$$\sum_{n=0}^{N-1} z_n e^{-i2\pi n f_p} = \frac{1}{N} \sum_{m=0}^{N-1} Z_m N \delta_{pm} = Z_p$$

Because of orthogonality, all Fourier components vanishes except Z_p, which will be the coefficient of $e^{i2\pi n f_p}$. This gives a *value* for Z_p.

Note the “p” is just a label, so we can use the letter “m” instead.

Comment on Notation

It is admittedly a possible point of confusion that we choose a numbering system for z_n and Z_m such that $m = 1$, the Rayleigh frequency, occurs as the *second* element in a vector in Matlab.

If we had made the alternate choice that z_n and Z_m run from 1 to N instead of 0 to $N - 1$, we would have obtained

$$Z_m \equiv \sum_{n=1}^{N} z_n e^{-i2\pi(m-1)(n-1)/N}, \qquad z_n = \frac{1}{N}\sum_{m=1}^{N} Z_m e^{i2\pi(m-1)(n-1)/N}.$$

Thus the convenience of one-based numbering for keeping track of our arrays leads to a notation inconvenience in more complicated expressions for the complex exponentials.

The choice of notation is of a matter of taste. However, in my experience the simplification of $e^{i2\pi pn/N}$ in the zero-based numbering system is worth the tradeoff.

Homework

1. Determine the sample rate, Nyquist frequency, and Rayleigh frequency for your time series.
2. Are there important processes that are not resolved by your sampling? Are the major tidal components resolved? The annual cycle? The diurnal cycle? Think of important processes and explain why these are or not these are resolved. Don't forget to consider processes that are both too fast and too slow for you to resolve.
3. Work through the algebra (for even N) showing that frequencies above the Nyquist can be thought of as negatively-rotating circles.
4. Work through the algebra (for even N) showing that the discrete Fourier transform of a real-valued z_n can be rewritten as a sum over sines and cosines.
5. Work through the algebra showing that oppositely-rotating circles sum to an ellipse; see slide 9 of this lecture.