The Two Domains

One Object, Two Domains

In the last lecture we saw that any time series can be represented in terms of its Fourier transform.

The time series and its Fourier transform are linked, like the way an object is linked to its mirror image.

We can say that we are seeing the same object, represented in either the time domain, or the frequency domain.

Any modification we make to a time series will have will also modify in some way its Fourier transform.

In other words, changes in the time domain are reflected by corresponding changes in the frequency domain, and vice-versa.

The Nature of the Link

However, the nature of the link between the time domain and the frequency domain is rather subtle.

The way a modification in one domain is represented in the other domain is not initially obvious.

For example, if we differentiate a time series with respect to time, that is not the same as differentiating its Fourier transform with respect to frequency.

Similarly, if you hold an object in front of a mirror, the mirror accurately represents the object, but in a modified way. As you rotate the object, the mirror image also rotates, but in the opposite direction.

Understanding the way these two domains reflect each other is at the very heart of spectral analysis and is the subject of this lecture.

Orientation

In the last few lectures we have established three fundamental building blocks for understanding Fourier analysis.

These are the sampling model, Euler's Formula, and the discrete Fourier transform equations:

$$z_n = z(n\Delta)$$

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$

$$z_n = \frac{1}{N}\sum_{m=0}^{N-1} Z_m e^{i2\pi mn/N}, \qquad Z_m \equiv \sum_{n=0}^{N-1} z_n e^{-i2\pi mn/N}.$$

Today we will add a fourth building block,

$$z(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z(\omega)\, e^{i\omega t} d\omega, \qquad Z(\omega) \equiv \int_{-\infty}^{\infty} z(t)\, e^{-i\omega t} dt$$

which are the continuous Fourier transform equations. But note, we will have to take care of what $z(t)$ is allowed to be in this case.

Limiting Cases

Before continuing, we consider some limiting cases of our conceptual model. Beginning with the sampling model

$$z_n \equiv z(\Delta n), \qquad n = 0, 1, \ldots, N-1$$

with $T \equiv N\Delta$ being the signal duration, and recalling the Rayleigh and Nyquist frequencies

$$f^{\mathcal{R}} \equiv \frac{1}{N\Delta}, \qquad f^{\mathcal{N}} \equiv \frac{1}{2\Delta}$$

consider the following:

What happens as N tends to infinity with Δ held fixed?

Limiting Cases

Before continuing, we consider some limiting cases of our conceptual model. Beginning with the sampling model

$$z_n \equiv z(\Delta n), \qquad n = 0, 1, \ldots, N-1$$

with $T \equiv N\Delta$ being the signal duration, and recalling the Rayleigh and Nyquist frequencies

$$f^{\mathcal{R}} \equiv \frac{1}{N\Delta}, \qquad f^{\mathcal{N}} \equiv \frac{1}{2\Delta}$$

consider the following:

What happens as N tends to infinity with Δ held fixed?

What happens as N tends to infinity with T held fixed?

A Deterministic Signal

Let g_n denote a discrete time series that is a sampled version of the continous time series $g(t)$:

$$g_n = g(n\Delta)$$

Like $z(t)$, $g(t)$ may be complex-valued, but unlike $z(t)$, $g(t)$ is a deterministic (not stochastic) time series.

This means that $g(t)$ has a definite value at each time t, whereas $z(t)$ is a random variable at each time t.

Also unlike $z(t)$, $g(t)$ does not present variability that goes on and on forever into the past and the future.

One example of such a time series $g(t)$ is a Gaussian curve.

The reason for working with the deterministic time series $g(t)$ will become clear shortly.

Continuous Fourier

Today we will discuss the Fourier transform equations for a deterministic time series $g(t)$ that is a function of *continuous* time:

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, e^{i\omega t} d\omega, \qquad G(\omega) \equiv \int_{-\infty}^{\infty} g(t)\, e^{-i\omega t} dt.$$

These may be compared with the Fourier transform equations for a *discrete* time series g_n that we have been working with.

$$g_n = \frac{1}{N}\sum_{m=0}^{N-1} G_m e^{i2\pi mn/N}, \qquad G_m \equiv \sum_{n=0}^{N-1} g_n e^{-i2\pi mn/N}$$

Please take a few minutes to look closely at these two equations. Assess their commonality and differences.

Continuous Fourier

Today we will discuss the Fourier transform equations for a deterministic time series $g(t)$ that is a function of *continuous* time:

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega)\, e^{i\omega t} d\omega, \qquad G(\omega) \equiv \int_{-\infty}^{\infty} g(t)\, e^{-i\omega t} dt.$$

These may be compared with the Fourier transform equations for a *discrete* time series g_n that we have been working with.

$$g_n = \frac{1}{N} \sum_{m=0}^{N-1} G_m e^{i2\pi mn/N}, \qquad G_m \equiv \sum_{n=0}^{N-1} g_n e^{-i2\pi mn/N}$$

Please take a few minutes to look closely at these two equations. Assess their commonality and differences.

These are discrete and continuous expressions of the same basic idea of representing a time series in terms of sinuosoids.

✧ Differences: sum vs. integral, cyclic vs. radian, $1/N$ vs. $1/(2\pi)$.

Why Both?

A fair question is why we need to work with both the discrete Fourier transform and continuous Fourier transform.

Essential conceptual results are much easier to see when we work with the continuous version, even if the basic idea also applies to the discrete version.

These continuous Fourer transform equations are the key to understanding the link between the time domain and the frequency domain, and to understanding spectral analysis.

However, when we implement things on a computer, we do so using the discrete version.

Therefore it is important to be familiar with both.

Discrete vs. Continuous

Note we can express the mth Fourier frequency equivalently as

$$f_m \equiv m/N, \qquad \omega_m \equiv 2\pi m/N$$

in terms of either cyclic or radian frequencies.

The difference between two successive radian frequencies is

$$\delta\omega \equiv \omega_{m+1} - \omega_m = 2\pi/N$$

which is just the Rayleigh frequency expressed in radian units.

Thus, we can write the discrete Fourier transform as

$$g_n = \frac{1}{N}\sum_{m=0}^{N-1} G_m e^{i2\pi n f_m} = \frac{1}{N\delta\omega}\sum_{m=0}^{N-1} G_m e^{i\omega_m n}\delta\omega = \frac{1}{2\pi}\sum_{m=0}^{N-1} G_m e^{i\omega_m n}\delta\omega$$

in which form we can better see its connection to the integral.

Discrete vs. Continuous

Rewritten in cyclic frequency, the DFT equations become

$$g_n = \frac{1}{2\pi} \sum_{m=0}^{N-1} G_m e^{i\omega_m n} \delta\omega, \qquad G_m \equiv \sum_{n=0}^{N-1} g_n e^{-i\omega_m n}.$$

The left-hand expression is clearly a summation approximating an integral over frequency. The right hand expression is a summation approximating an integral over time, with sample interval $\Delta = 1$.

Comparing these with the continuous Fourier transform equations

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega)\, e^{i\omega t} d\omega, \qquad G(\omega) \equiv \int_{-\infty}^{\infty} g(t)\, e^{-i\omega t} dt$$

we can now see the very close connection between the two sets.

Note that the placement of the 2π is subject to convention. This argument shows why it is natural to have the 2π where it is.

A Symmetric DFT Form

The inverse discrete Fourier transform, with $f_m \equiv m/N$,

$$g_n = \frac{1}{N} \sum_{m=0}^{N-1} G_m e^{i2\pi n f_m}$$

can be written in the symmetric form

$$g_n = \frac{1}{N} \sum_{m=-(N/2-1)}^{N/2-1} G_m e^{i2\pi n f_m} + G_{N/2}(-1)^n \qquad N \text{ even}$$

$$g_n = \frac{1}{N} \sum_{m=-(N-1)/2}^{(N-1)/2} G_m e^{i2\pi n f_m} \qquad N \text{ odd}$$

where we *define* G_m with $m < 0$ as $G_m \equiv G_{N-m}$ in the latter two summations. Note the summations are now symmetric in m.

Limiting Forms of the DFT

Consider the case of odd N for simplicity, we can now see the close connections between the discrete and continuous cases:

$$g_n = \frac{1}{2\pi} \sum_{m=-(N-1)/2}^{(N-1)/2} G_m e^{i\omega_m n} \delta\omega, \qquad g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega)\, e^{i\omega t} d\omega$$

It can be shown that the continuous Fourier transform emerges from the DFT as (i) $T \equiv N\Delta \longrightarrow \infty$ together with (ii) $\Delta \longrightarrow 0$.

Note on Transform Validity

Let's take a look at this equation:

$$G(\omega) \equiv \int_{-\infty}^{\infty} g(t)\, e^{-i\omega t} dt.$$

Some conditions must be placed on $g(t)$ in order for this to be valid.

If we assume that $g(t)$ and $G(\omega)$ are both *absolutely integrable*

$$\int_{-\infty}^{\infty} |g(t)|\, dt < \infty, \qquad \int_{-\infty}^{\infty} |G(\omega)|\, d\omega < \infty$$

then it can be shown that the Fourier transform equations hold, and $g(t)$ can be reconstructed* from its Fourier transform.

The Fourier transform does not exist in this usual sense for, e.g., a stochastic process $z(t)$ that extends to infinity in both directions.

(* In the sense of being equal almost everywhere.)

A Simplifying Notation

We introduce the notation

$$A(t) \Longleftrightarrow B(\omega)$$

to mean "A is a Fourier transform pair with B".

In other words, $A(t) \Longleftrightarrow B(\omega)$ means *both* of the two equations

$$A(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} B(\omega)\, e^{i\omega t} d\omega, \qquad B(\omega) = \int_{-\infty}^{\infty} A(t)\, e^{-i\omega t} dt.$$

We will always put time-domain quantity on the left, and the frequency-domain quantity on the right.

Transform of a Gaussian

It can readily be shown that a Gaussian transforms to a Gaussian:

$$g(t) = e^{-\frac{1}{2}\frac{t^2}{L^2}} \quad \Longleftrightarrow \quad G(\omega) = \sqrt{2\pi}\,L\,e^{-\frac{1}{2}L^2\omega^2}.$$

Notice that the Gaussian width in the time domain, L, becomes $1/L$ in the frequency domain.

This means that as you make the Gaussian *more narrow* in time, it becomes *more broad* in the frequency domain, and vice-versa.

This is actually a general result.

The Scaling Theorem

Using the inverse Fourier transform equation, if you scale time in a function $g(t)$, what happens to its Fourier transform?

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega)\, e^{i\omega t}\, d\omega$$

The Scaling Theorem

Using the inverse Fourier transform equation, if you scale time in a function $g(t)$, what happens to its Fourier transform?

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, e^{i\omega t}\, d\omega$$

$$g(t/L) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, e^{i\omega t/L}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} L\, G(\omega)\, e^{i(\omega/L)t}\, d(\omega/L)$$

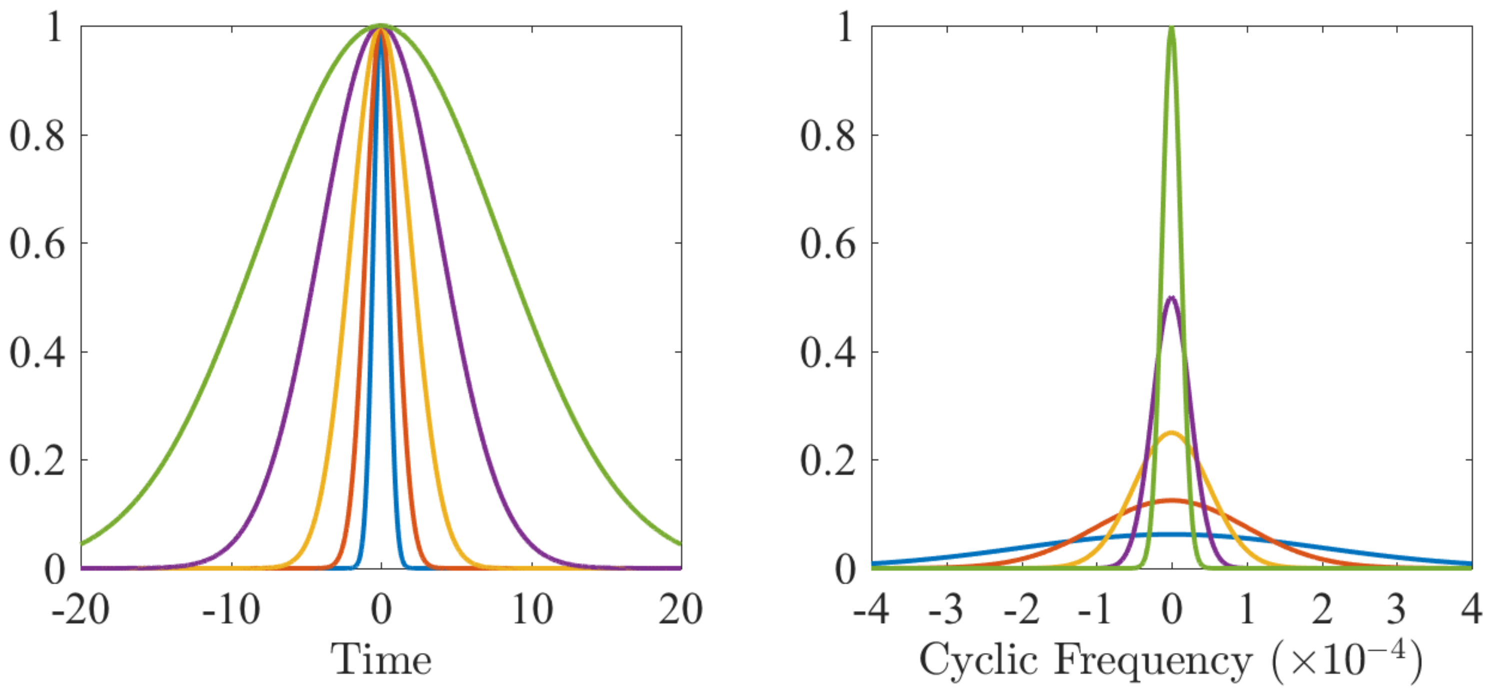
$$(\text{with } \nu \equiv \omega/L \text{ followed by } \omega \equiv \nu) \qquad = \frac{1}{2\pi}\int_{-\infty}^{\infty} L\, G(\omega L)\, e^{i\omega t}\, d\omega$$

This is a general result that we will call the *scaling theorem*:

$$g(t/L) \quad \Longleftrightarrow \quad L\, G(L\omega).$$

This states that making any function *more narrow* in the time domain makes its Fourier transform *more broad*, and vice-versa.

The Scaling Theorem



Stretching a function in time compresses it in frequency, and also increases the frequency-domain amplitude.

$$g(t/L) \quad \Longleftrightarrow \quad LG(L\omega).$$

The Shift Theorem

Using the inverse Fourier transform equation, if you shift a function $z(t)$ in time, what happens to its Fourier transform?

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega)\, e^{i\omega t}\, d\omega$$

The Shift Theorem

Using the inverse Fourier transform equation, if you shift a function $z(t)$ in time, what happens to its Fourier transform?

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, e^{i\omega t}\, d\omega$$

$$g(t - t_o) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, e^{i\omega(t-t_o)}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[e^{-i\omega t_o} G(\omega)\right] e^{i\omega t}\, d\omega$$

Thus, the effect in Fourier domain of a time shift is to *modify the phases* of the Fourier transform without changing its magnitude.

Such an action is called a *phase modulation*. The word *modulate* means to adjust or vary.

This result, the *shift theorem*, can be compactly expressed as

$$g(t - t_o) \quad \Longleftrightarrow \quad e^{-i\omega t_o} G(\omega).$$

Time Derivatives

Using the inverse Fourier transform equation, if you take a time derivative of $g(t)$, what happens to its Fourier transform?

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega)\, e^{i\omega t}\, d\omega$$

Time Derivatives

Using the inverse Fourier transform equation, if you take a time derivative of $g(t)$, what happens to its Fourier transform?

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, e^{i\omega t}\, d\omega$$

$$\begin{aligned} g'(t) &= \frac{d}{dt} g(t) = \frac{d}{dt}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, e^{i\omega t}\, d\omega\right] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, \left[\frac{d}{dt} e^{i\omega t}\right] d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} i\omega G(\omega)\, e^{i\omega t}\, d\omega \end{aligned}$$

We will call this result the *differentiation theorem*:

$$g'(t) \quad \Longleftrightarrow \quad i\omega G(\omega).$$

Note that the differentiation theorem readily generalizes

$$g^{(n)}(t) \quad \Longleftrightarrow \quad (i\omega)^n G(\omega).$$

Summary So Far

We have learned the scaling theorem, the shift theorem, and the differentiation theorem:

$$\begin{aligned} g(t/L) \quad &\Longleftrightarrow \quad L\,G(L\omega) \\ g(t-t_o) \quad &\Longleftrightarrow \quad e^{-i\omega t_o}G(\omega) \\ g'(t) \quad &\Longleftrightarrow \quad i\omega G(\omega). \end{aligned}$$

These three results are fundamental in describing how changes in the time domain are reflected in the frequency domain.

All of these results are derivable with a few lines of algebra from the Fourier representation, or inverse Fourier transform equation,

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, e^{i\omega t} d\omega.$$

After a while, you can just look at this equation and you can see those results inside of it.

The Delta Function

The Dirac delta function, $\delta(t)$, is a special type of function that comes up frequently in Fourier analysis.

One can visualize $\delta(t)$ as an infinite spike at time $t = 0$, but with an integated value of one, $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$.

The fundamental property of a delta function is that its product with another function $f(t)$ integrates to a particular value of $f(t)$:

$$\int_{-\infty}^{\infty} \delta\left(t - t_o\right)\, f(t)\, dt = f(t_o).$$

Thus $\delta\left(t - t_o\right)$ plays the role of *collapsing* the integral, *choosing* the value of $f(t)$ at time $t = t_o$.

An important detail is that $f(t)$ must be "smooth" in some sense. It can't be a stochastic process like a random walk, or another delta function, or a fractal. These vary at infinitesimally small scales.

Delta Function Check

Let's make sure we understand how the delta function works. Please answer the following:

$$\int_{-\infty}^{\infty} \delta\left(t - t_o\right)\, \cos(\omega t)\, dt \, = ?$$

$$\int_{-\infty}^{\infty} \delta\left(t\right)\, \cos(\omega t)\, dt \, = ?$$

$$\int_{-\infty}^{\infty} \delta\left(\omega - \omega_o\right) e^{i\omega t}\, d\omega \, = ?$$

$$\frac{1}{2}\int_{-\infty}^{\infty} \left[\delta\left(\omega - \omega_o\right) + \delta\left(\omega + \omega_o\right)\right] e^{i\omega t}\, d\omega \, = ?$$

$$\frac{-i}{2}\int_{-\infty}^{\infty} \left[\delta\left(\omega - \omega_o\right) - \delta\left(\omega + \omega_o\right)\right] e^{i\omega t}\, d\omega \, = ?$$

Fourier Transform of Delta

Setting $G(\omega) = 2\pi\delta(\omega)$ in the inverse Fourier transform leads to

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, e^{i\omega t}\, d\omega, \qquad 1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega)\, e^{i\omega t} d\omega$$

using the fundamental property of a delta function. This indicates that the corresponding forward Fourier transform should be

$$2\pi\delta(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} dt$$

and this can in fact be proven, but is beyond the scope of this class.

Thus we have identified the Fourier transform pair of a constant value in the time domain, and a delta function at zero frequency:

$$1 \quad \Longleftrightarrow \quad 2\pi\delta(\omega).$$

You can see this as a special case of the Gaussian pair for $L \to \infty$.

Deltas and Sinuosoids

Delta functions are important because *shifted* delta functions are the Fourier transforms of complex exponentials.

From the fundamental property of a delta function, and setting $G(\omega) = 2\pi\delta(\omega - \omega_o)$ in the inverse Fourier transform, we have

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, e^{i\omega t} d\omega, \quad e^{i\omega_o t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta\left(\omega - \omega_o\right) e^{i\omega t} d\omega$$

for the Fourier transform of a shifted delta function, or

$$e^{i\omega_o t} \quad \Longleftrightarrow \quad 2\pi\delta(\omega - \omega_o).$$

Recall that $\cos\theta = \Re\left\{e^{i\theta}\right\} = \frac{1}{2}\left[e^{i\theta} + e^{-i\theta}\right]$. It follows at once that

$$\cos(\omega_o t) \quad \Longleftrightarrow \quad \pi\delta\left(\omega + \omega_o\right) + \pi\delta\left(\omega - \omega_o\right)$$

and the Fourier transform of a cosine is the sum of *two* delta functions, one at ω_o and one at $-\omega_o$.

But Wait ...

I thought you said we have to work with functions that are absolutely integrable? Sines and cosines are not absolutely integrable because

$$\int_{-\infty}^{\infty} |\cos(\omega t)| dt = \infty.$$

But Wait ...

I thought you said we have to work with functions that are absolutely integrable? Sines and cosines are not absolutely integrable because

$$\int_{-\infty}^{\infty} |\cos(\omega t)| dt = \infty.$$

True. These are a special case. They can be treated as, say, a Gaussian times a cosine---which is absolutely integrable---in the limit as the Gaussian becomes infinitely wide.

A special theory, called *the theory of distributions*, is needed to work with delta-functions, for example, to prove the fundamental property of delta functions.

In general, we don't need to worry about this, just to be aware of it.

The Convolution Integral

A few days ago we worked with simple smoothing of our discrete time series z_n, which we represented mathematically as

$$\tilde{z}_n = \sum_{m=-(M-1)/2}^{(M-1)/2} z_{n-m}\, g_m$$

We might choose g_m to be constant, for example, to implement a running mean.

Changing notation slightly, this becomes

$$h_n \equiv \sum_{m=-(M-1)/2}^{(M-1)/2} f_{n-m}\, g_m, \qquad h(t) \equiv \int_{-\infty}^{\infty} f(t-\tau)g(\tau)\, d\tau.$$

where we have written the continuous-time equivalent on the right. This is a very important type of operation called a *convolution*.

The Convolution

"Convolution" is essentially just a fancy name for the filtering action we do when we smooth a time series or take a running mean.

Let $f(t)$ be a time series of interest, and $g(t)$ be smoothing function, such as a boxcar. Then a smoothed version of $f(t)$ is given by

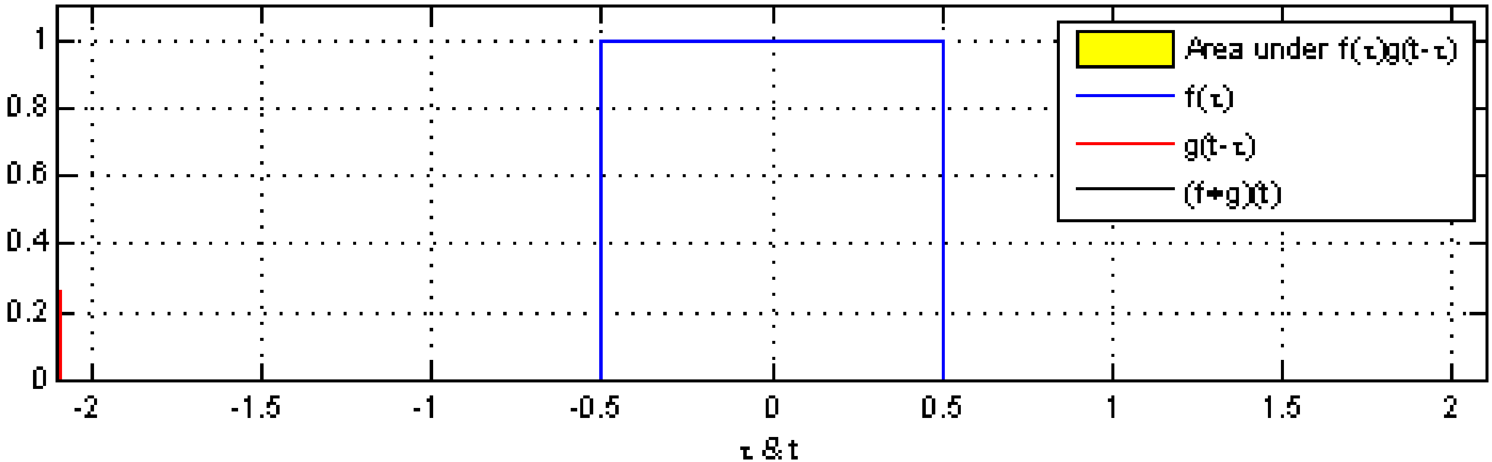
$$h(t) \equiv (f * g)(t) \equiv \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

where the integral in the above is called a *convolution*. The notation $(f * g)(t)$ is confusing but conventional.

If $g(t)$ is a rectangle function, the convolution produces a running mean. But in general, $g(t)$ need not become smoother! We could apply, for example, bandpass filter.

As with the Fourier transform equation, for the convolution integral to be well-defined, we need to make some conditions on $f(t)$ and $g(t)$. It is well-defined if both are square-integrable.

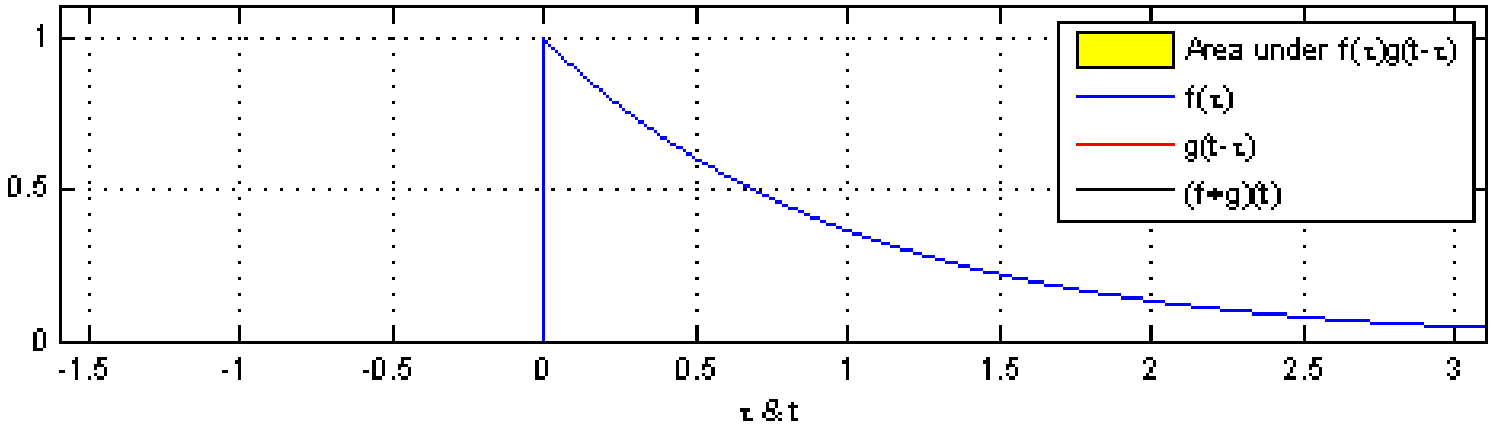
Example of Convolution



$$(f * g)(t) \equiv \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

This {animation} is from Wikipedia, by Brian Amberg and modified by Tinos, redistributed under the {CC BY-SA 3.0} license.

Another Example



$$(f * g)(t) \equiv \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

This {animation} is from Wikipedia, by Brian Amberg and modified by Tinos, redistributed under the {CC BY-SA 3.0} license.

Symmetry of Convolution

In convolution, order does not matter. Convolving $f(t)$ with $g(t)$ is the same as convolving $g(t)$ with $f(t)$. Define $h(t)$ as

$$h(t) \equiv \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau.$$

With $u \equiv t - \tau$, which implies $du = -d\tau$ and $\tau = t - u$, we find

$$h(t) = -\int_{\infty}^{-\infty} f(t-u)g(u)du = \int_{-\infty}^{\infty} f(t-u)g(u)du$$

after changing the limits of integration, i.e. noting $\int_a^b = -\int_b^a$. The variable of integration u can be replaced with with τ. This shows

$$h(t) \equiv \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau = \int_{-\infty}^{\infty} g(\tau)f(t-\tau)d\tau$$

so that in convolution, the *order does not matter*.

The Convolution Theorem

A key to understanding spectral analysis is to know what happens in the Fourier domain when you perform a time-domain smoothing.

The *convolution theorem* states convolving $f(t)$ and $g(t)$ in the time domain is the same as a *multiplication* in the frequency domain.

$$h(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G(\omega)\, e^{i\omega t}d\omega$$

which can be written more compactly as

$$\int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau \quad \Longleftrightarrow \quad F(\omega)G(\omega).$$

We see clearly here that order does not matter to the convolution.

The convolution theorem is *the single more important* result in all of Fourier analysis. It takes a little while to really sink in.

Convolution Proof

$$
\begin{aligned}
h(t) &= \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega\tau} d\omega\right] \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\nu)\, e^{i\nu(t-\tau)} d\nu\right] d\tau \\
&= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\omega)G(\nu)\, e^{i(\omega-\nu)\tau+i\nu t} d\omega\, d\nu\, d\tau \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\omega)G(\nu)\, e^{i\nu t} \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega-\nu)\tau} d\tau\right] d\omega\, d\nu \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) \left[\int_{-\infty}^{\infty} G(\nu)\, e^{i\nu t} \delta(\nu-\omega) d\nu\right] d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G(\omega)\, e^{i\omega t} d\omega
\end{aligned}
$$

The Reflection Equations

We have seen how key elementary functions and operations in the time domain are reflected in the frequency domain.

$$
\begin{aligned}
g(t/L) &\iff L\,G(L\omega) \\
g(t-t_o) &\iff e^{-i\omega t_o} G(\omega) \\
g'(t) &\iff i\omega G(\omega) \\
1 &\iff 2\pi\delta(\omega) \\
e^{i\omega_o t} &\iff 2\pi\delta(\omega-\omega_o) \\
\cos(\omega_o t) &\iff \pi\delta\left(\omega+\omega_o\right)+\pi\delta\left(\omega-\omega_o\right) \\
\int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau &\iff F(\omega)G(\omega)
\end{aligned}
$$

We will call these the *reflection equations*. All of them are readily derivable from the Fourier decomposition of $g(t)$,

$$g(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\,e^{i\omega t}d\omega.$$

The Reflection Equations

To skillfully employ Fourier analysis, it is essential to understand the reflection equations on a conceptual level.

$$
\begin{aligned}
\text{stretching} &\iff \text{compression and rescaling} \\
\text{time shift} &\iff \text{phase modulation} \\
\text{time derivative} &\iff \text{multiply by } i\omega \\
\text{constant} &\iff \text{delta function} \\
\text{complex exponential} &\iff \text{shifted delta function} \\
\text{cosinusoid} &\iff \text{two shifted delta functions} \\
\text{convolution} &\iff \text{multiplication}
\end{aligned}
$$

Note that the terminology “reflection equations” is not standard. It's a term I came up with to try to give an intuitive feeling of the meaning of these equations.

The Time/Frequency Link

All of this is about unpacking the information contained within our fourth foundation, the Fourier transform pair equations,

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, e^{i\omega t} d\omega, \qquad G(\omega) \equiv \int_{-\infty}^{\infty} g(t)\, e^{-i\omega t} dt.$$

Once you conceptually understand the reflection equations, it's not necessary to remember them, because they can be easily derived from the Fourier transform pair equations.

Of all the reflection equations, the most important is this:

$$\begin{aligned} \text{convolution} \quad &\Longleftrightarrow \quad \text{multiplication} \\ \text{multiplication} \quad &\Longleftrightarrow \quad \text{convolution} \end{aligned}$$

Convolutions in one domain are a multiplication in the other.

Homework

The reflection theorems very important results for anyone working with spectral analysis.

1. Please commit them to memory—by which I mean the basic idea that you can say in words as on page 34, *not* the symbols.
2. You should be able to derive all of the reflection equations, see page 33, from the inverse Fourier transform equation by yourself. Please practice the algebra involved so that it becomes familiar, and clarify any steps that don't make sense to you.
3. Make sure to include the algebra behind the convolution theorem on page 32.