

The Wavelet Transform



In-Class Assignments

Beginning with the forward Fourier transform equation

$$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-i\omega t} dt$$

use this equation to answer the following:

$$? \iff G(\omega/L)$$

$$? \iff G(\omega - \omega_o)$$

$$? \iff G'(\omega)$$

$$? \iff 1$$

$$? \iff e^{i\omega t_o}$$

$$? \iff \cos(\omega t_o)$$

$$? \iff \int_{-\infty}^{\infty} F(\nu) G(\omega - \nu) d\nu$$



The Reverse Reflections

These reverse reflection equations are:

$$\begin{aligned} Lg(tL) &\iff G(\omega/L) \\ e^{i\omega_o t} g(t) &\iff G(\omega - \omega_o) \\ -itg(t) &\iff G'(\omega) \\ \delta(t) &\iff 1 \\ \delta(t - t_o) &\iff e^{-i\omega t_o} \\ \frac{1}{2}\delta(t - t_o) + \frac{1}{2}\delta(t + t_o) &\iff \cos(\omega t_o) \\ f(t)g(t) &\iff \int_{-\infty}^{\infty} F(\nu)G(\omega - \nu)d\nu \end{aligned}$$

These could also have been inferred from the forward reflection equations by noting the symmetry of the Fourier pair relationship:

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{i\omega t} d\omega, \quad G(\omega) \equiv \int_{-\infty}^{\infty} g(t) e^{-i\omega t} dt.$$



Bandpass Filtering

Previously we looked at the convolution theorem:

$$h(t) = \int_{-\infty}^{\infty} f(\tau)g(t - \tau)d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)G(\omega) e^{i\omega t} d\omega.$$

If $g(t)$ is smooth, the convolution creates a *smoothed version* of $f(t)$.

If $g(t)$ has a Fourier transform $G(\omega)$ is localized in the vicinity of a certain frequency ω_o , then $h(t)$ will also be localized around ω_o .

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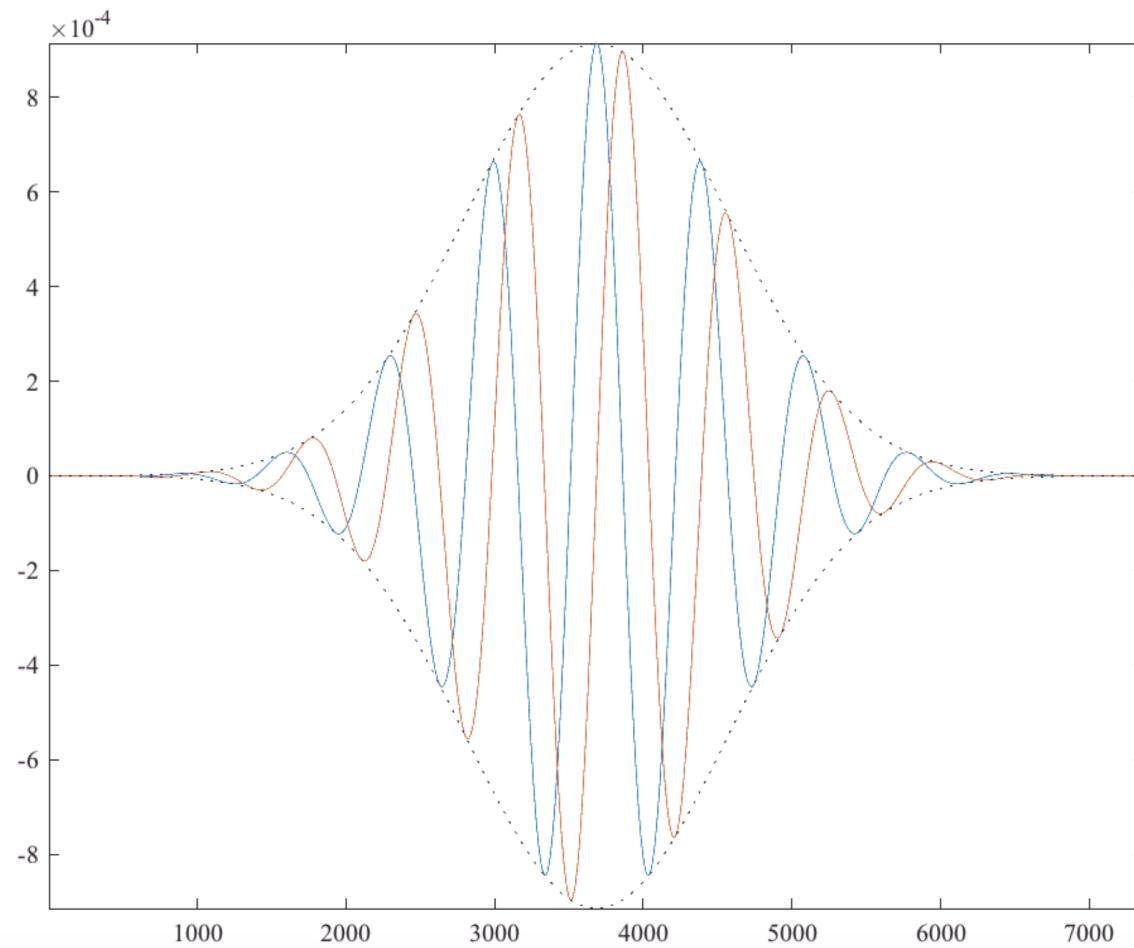
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$$g(t)e^{i\omega_o t} \iff G(\omega - \omega_o)$$

A Modulated Gaussian



- ❖ The function $g(t)e^{i\omega_0 t}$ where $g(t)$ is a Gaussian.

One-Sided Bandpass

If we create a bandpass filter, using the shift theorem, by modulating some function $g(t)$ whose Fourier transform is localized around zero, we obtain the new function

$$\tilde{g}(t) \equiv g(t)e^{i\omega_o t} \iff G(\omega - \omega_o).$$

The bandpass filter $\tilde{g}(t)$ will localize in the vicinity of frequency ω_o .

Thus, if ω_o is chosen to be positive (and sufficiently large), we will isolate a band of positive frequencies, and if ω_o is chosen to be negative, we will isolate a band of negative frequencies.

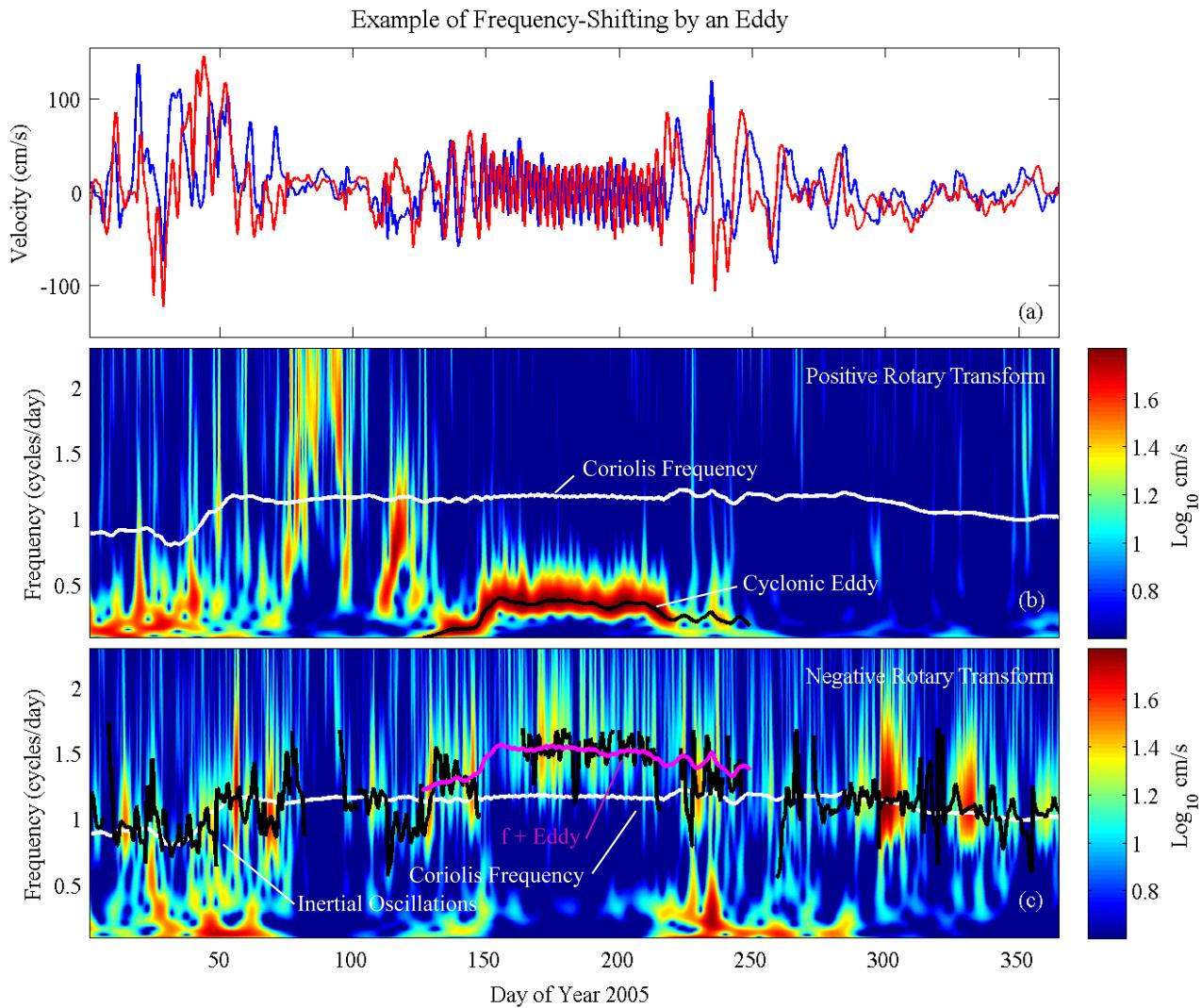
This is called a *one-sided* bandpass filter. An example of a two-sided bandpass filter would be

$$g(t) \cos(\omega_o t) \iff [G(\omega - \omega_o) + G(\omega + \omega_o)]/2.$$

Note that a one-sided bandpass is necessarily complex-valued, because it will be something like a complex exponential with a localizing envelope.



A Wavelet Transform



The Basic Idea

The wavelet transform is essentially a stack (or set) of one-sided bandpasses at different frequencies.

We bandpass the time series at many different choices of the center of the band, then we look at these all together.

We will work with a particular type of wavelet that is *strictly* one-sided, that is, having no energy at negative frequencies.

Such a wavelet is said to be *analytic*. It is built up entirely from contributions due to positively-rotating Fourier components.

To isolate the negatively-rotating variability we use the complex conjugate of the analytic wavelet. This filter has no support at positive frequencies as is said to be *anti-analytic*.

The Wavelet Transform

The wavelet transform is written as

$$w(t, s) = \int_{-\infty}^{\infty} \frac{1}{s} \psi^* \left(\frac{\tau - t}{s} \right) z(\tau) d\tau.$$

This describes the act of filtering the data with a rescaled versions of a filter $\psi(t)$, where s is the rescaling parameter or *scale*.

Compare the wavelet transform definition with the convolution

$$(f * g)(t) \equiv \int_{-\infty}^{\infty} f(\tau) g(t - \tau) d\tau.$$

What are the differences?



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What are the differences?

- (i) In the wavelet transform, the output depends upon our choice of scale s ,
- (ii) the wavlet transform has a conjugation, and
- (iii) the wavelet transform has $\tau - t$ rather than $t - \tau$.



The Cross-Correlation

In fact the wavelet transform is based on a close relative of the convolution called the *cross-correlation*, defined as

$$(f \star g)(t) = \int_{-\infty}^{\infty} f^*(\tau - t)g(\tau)d\tau = \int_{-\infty}^{\infty} f^*(\tau)g(\tau + t)d\tau$$

which is just a different way to filter one time series with another. Unlike the convolution, the cross-correlation is not symmetric.

The frequency-domain expression of this action is given by the *cross-correlation theorem*,

$$\begin{aligned} (f \star g)(t) &= \int_{-\infty}^{\infty} f^*(\tau - t)g(\tau)d\tau = \int_{-\infty}^{\infty} f^*(\tau)g(\tau + t)d\tau \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega)G(\omega) e^{i\omega t}d\omega \end{aligned}$$

where a conjugation now appears acting on $F(\omega)$.



Convolution vs. Cross-Correlation

The convolution theorem:

$$\begin{aligned}(f * g)(t) &= \int_{-\infty}^{\infty} f(t - \tau)g(\tau)d\tau = \int_{-\infty}^{\infty} f(\tau)g(t - \tau)d\tau \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)G(\omega) e^{i\omega t}d\omega\end{aligned}$$

“A convolution in time is a multiplication in frequency.”

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“A cross-correlation in time is a multiplication in frequency *with the first function conjugated*.”



Cross-Correlation Proof

$$\begin{aligned} h(t) &= \int_{-\infty}^{\infty} f^*(\tau - t)g(\tau)d\tau \\ &= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) e^{-i\omega(\tau-t)} d\omega \right] \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} G(\nu) e^{i\nu\tau} d\nu \right] d\tau \\ &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^*(\omega)G(\nu) e^{i(\nu-\omega)\tau+i\omega t} d\tau d\omega d\nu \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^*(\omega)G(\nu) e^{i\omega t} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\nu-\omega)\tau} d\tau \right] d\omega d\nu \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) e^{i\omega t} \left[\int_{-\infty}^{\infty} G(\nu) \delta(\nu - \omega) d\nu \right] d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega)G(\omega) e^{i\omega t} d\omega \end{aligned}$$



The Wavelet Transform

We see that the wavelet transform is the filtration of $z(t)$ with rescaled versions of $\psi(t)$ using the cross-correlation operator:

$$w(t, s) = \int_{-\infty}^{\infty} \frac{1}{s} \psi^* \left(\frac{t - \tau}{s} \right) z(\tau) d\tau$$

From the cross-correlation theorem and scaling theorems, we have

$$w(t, s) = \int_{-\infty}^{\infty} \frac{1}{s} \psi^* \left(\frac{t - \tau}{s} \right) z(\tau) d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi^*(s\omega) Z(\omega) e^{i\omega t} d\omega$$

for the frequency-domain form of the wavelet transform.

Thus, if $\Psi(\omega)$ is localized around a particular frequency, then $\Psi(s\omega)$ is localized around a *different frequency*.

For this reason we can think of the wavelet transform as a set of bandpasses. For each s value, the wavelet transform $w(t, s)$ will be localized around a different frequency.



Two Key Transforms

Two essential transforms are when the signal of interest consists of delta function in time or a delta function in frequency.

If $z(t) = \delta(t - t_o)$ then we have

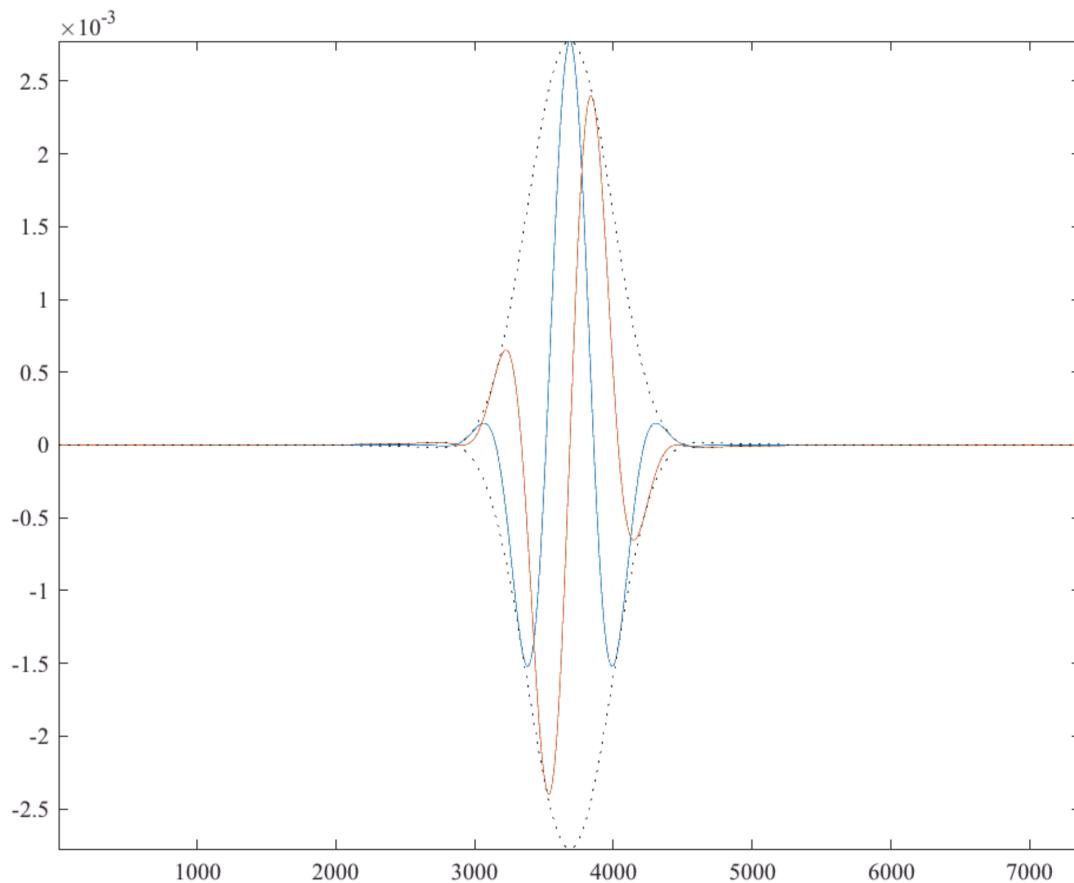
$$\begin{aligned} w(t, s) &= \int_{-\infty}^{\infty} \frac{1}{s} \psi^* \left(\frac{t - \tau}{s} \right) z(\tau) d\tau \\ &= \int_{-\infty}^{\infty} \frac{1}{s} \psi^* \left(\frac{t - \tau}{s} \right) \delta(\tau - t_o) d\tau = \frac{1}{s} \psi^* \left(\frac{t - t_o}{s} \right) \end{aligned}$$

while if $Z(\omega) = 2\pi\delta(\omega - \omega)$ then $z(t) = e^{i\omega_o t}$ and

$$\begin{aligned} w(t, s) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi^*(s\omega) Z(\omega) e^{i\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi^*(s\omega) [2\pi\delta(\omega - \omega_o)] e^{i\omega t} d\omega = \Psi^*(s\omega_o) e^{i\omega_o t}. \end{aligned}$$

- ◆ We see how wavelet itself is imprinted onto the wavelet transform.

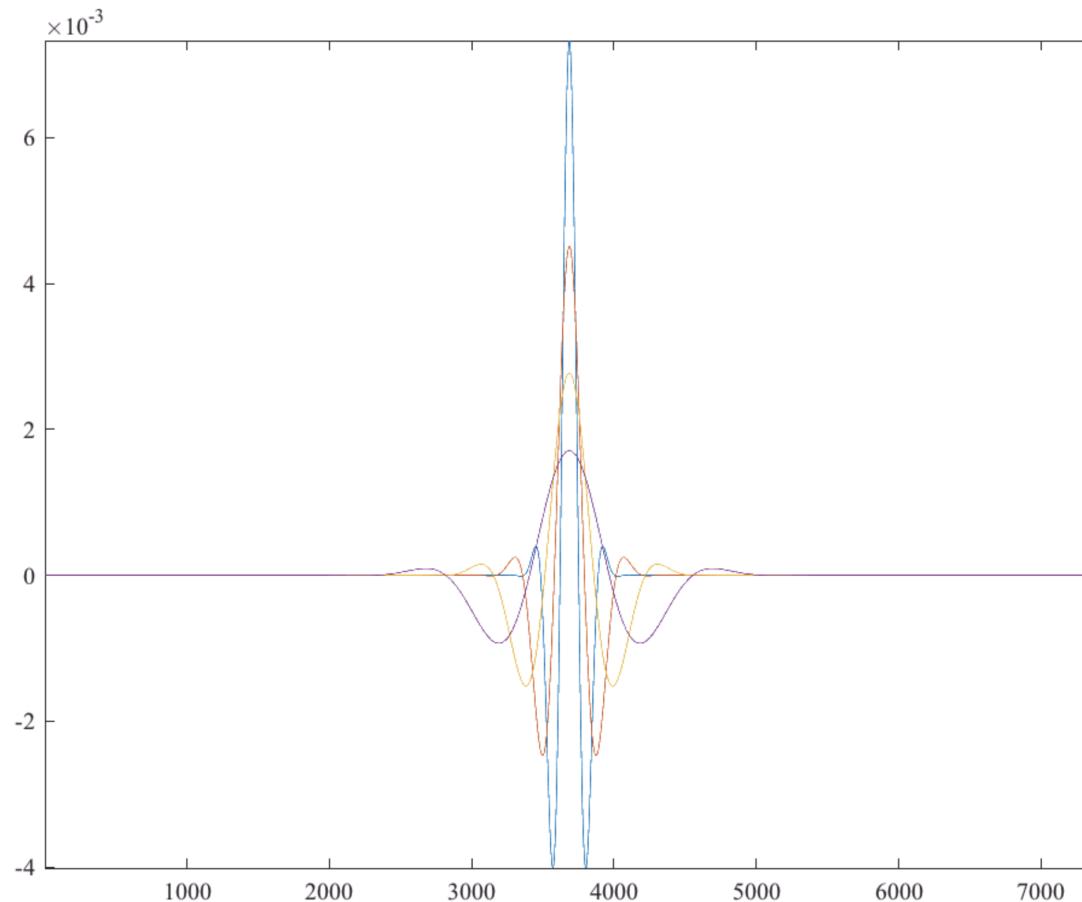
A Wavelet in Time



A typical example of a complex-valued wavelet in the time domain.



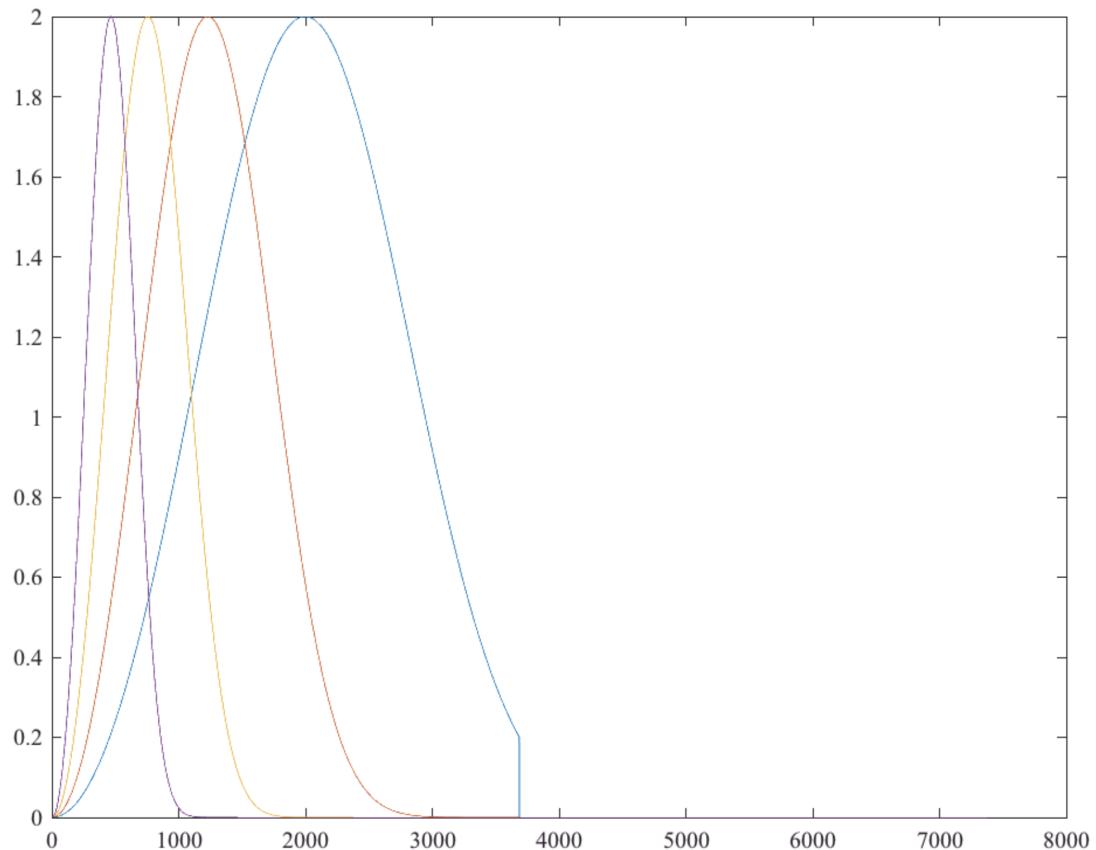
The Role of Scale s



Real part of the time-domain wavelet $\frac{1}{s}\psi(t/s)$ for different s .



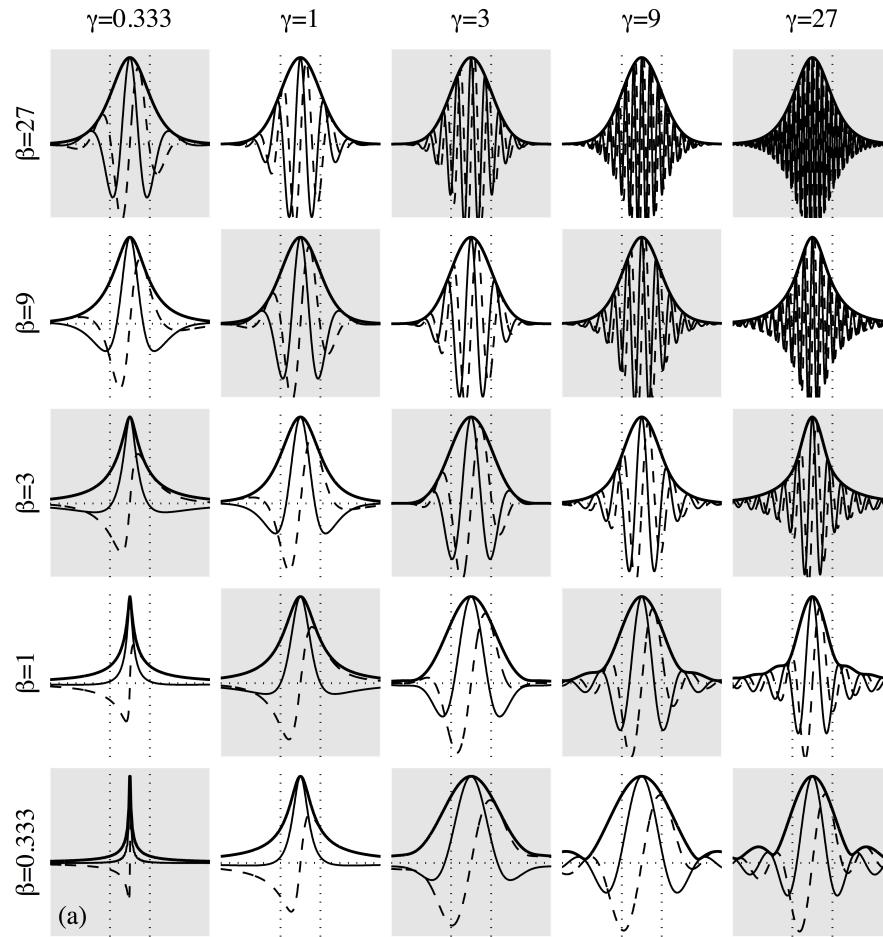
The Role of Scale s



The frequency-domain wavelet $\Psi(s\omega)$ for different choices of s .



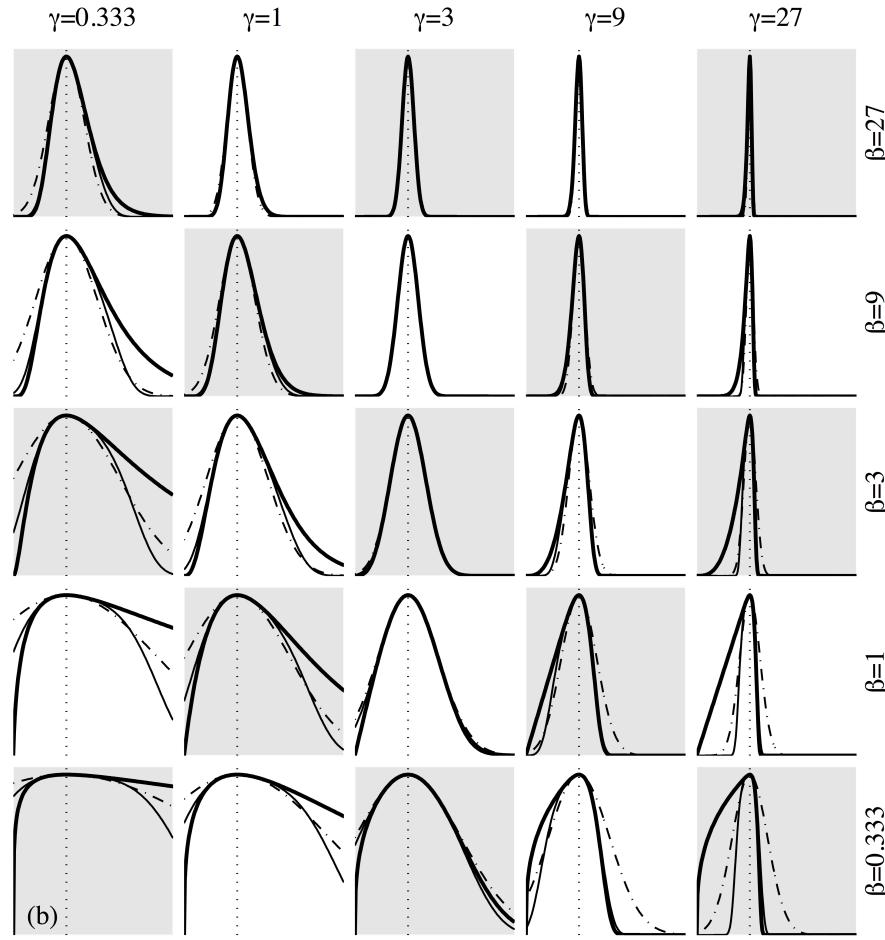
Examples of Wavelets



A broad family of wavelets in the time domain.



Examples of Wavelets



The same family of wavelets in the frequency domain.



That's Pretty Much It

So the wavelet transform is simply a stack of one-sided bandpasses.

This means the magnitude-squared of the wavelets transform, $|w(t, s)|^2$, describes the variability in the vicinity of a certain time t and also a certain period of oscillation, set by the scale s .

It is convenient to convert the scale s into an equivalent frequency, the *scale frequency*, which ends up being proportional to $1/s$.

The wavelet can be thought of as a *definition* of an oscillatory signal, and the magnitude of the wavelet transform as the magnitude of the local *projection* onto time and frequency shifted versions of that test signal.



Which Wavelet to Use?

One stumbling block used to be determining which wavelet to use.

“A practical question is which wavelet to use for a particular application. In addition to the popular but only approximately analytic Morlet wavelet, a variety of analytic wavelets have been proposed, including the Cauchy-Klauder-Morse-Paul, Derivative of Gaussian, lognormal or log Gabor, Shannon, and Bessel wavelets. However, this profusion of wavelet types serves as a barrier to acquiring the knowledge necessary to carry out practical applications. In the absence of a suitable unifying theory for wavelet behaviors, the choice of a particular wavelet for a particular problem may even appear arbitrary. As attempts to address this, one may find in the literature very many examples of papers—frequently with overlapping content—in which different types of wavelets are applied to the same signal, in order to decide which wavelet is ‘better’ in some sense. The large effort spent on such case-by-case analyses underscores the need for a systematic and unified framework for understanding analytic wavelets and their properties.” *Lilly and Olhede (2012b)*



Which Wavelet to Use?

In *Lilly and Olhede (2009a, 2012b)*, we showed the following:

1. All known analytic wavelet types can be subsumed into one much broader family.
2. Within that family, one can objectively say which choice is the ‘best’ for general use.

The name of this broad ‘superfamily’ of wavelets is the *generalized Morse wavelets*.

The recommended choice within this wavelet family is that which maximizes frequency-domain symmetry.

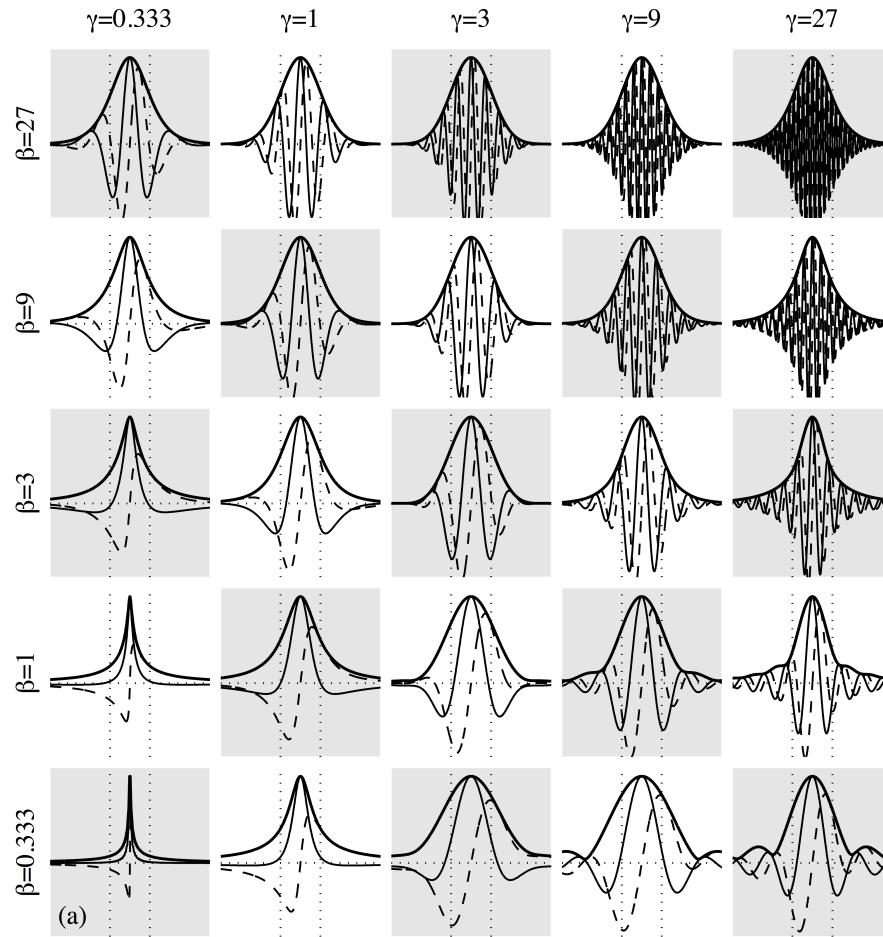
The generalized Morse wavelets have a simple Fourier-domain form

$$\Psi_{\beta,\gamma}(\omega) = a_{\beta,\gamma} \omega^\beta e^{-\omega^\gamma}$$

for $\omega > 0$, with $\Psi_{\beta,\gamma}(\omega) = 0$ elsewhere. Here $a_{\beta,\gamma}$ is a normalization.



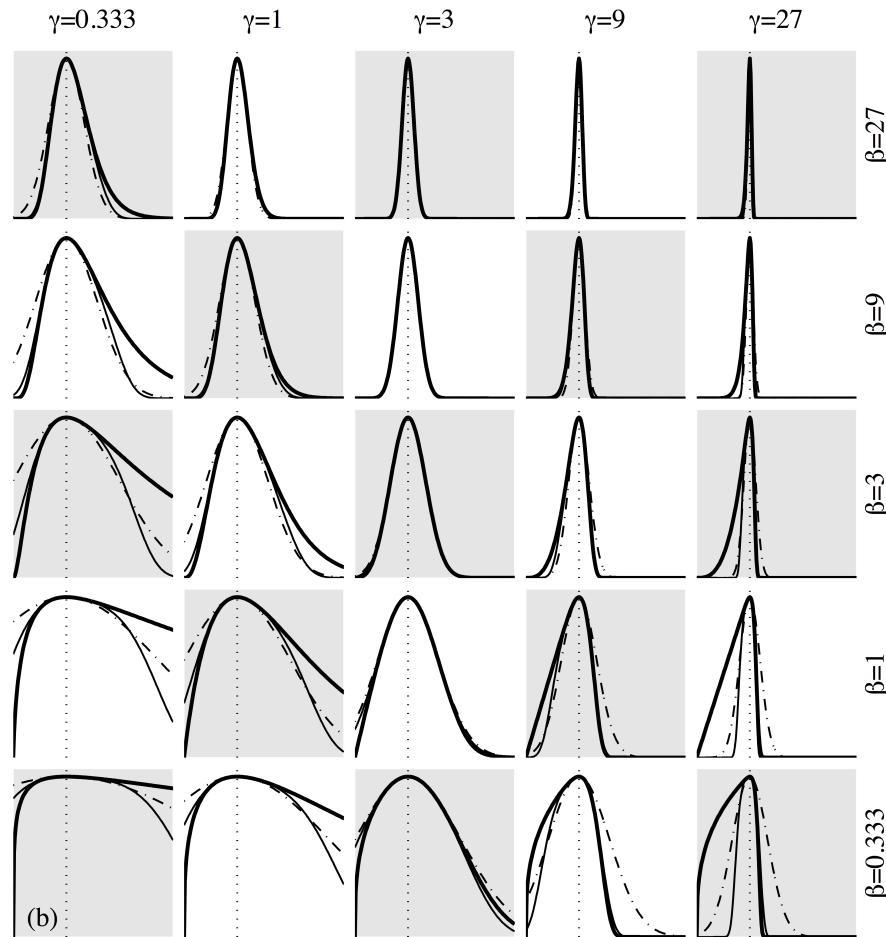
Examples of Wavelets



The generalized Morse wavelets in the time domain.



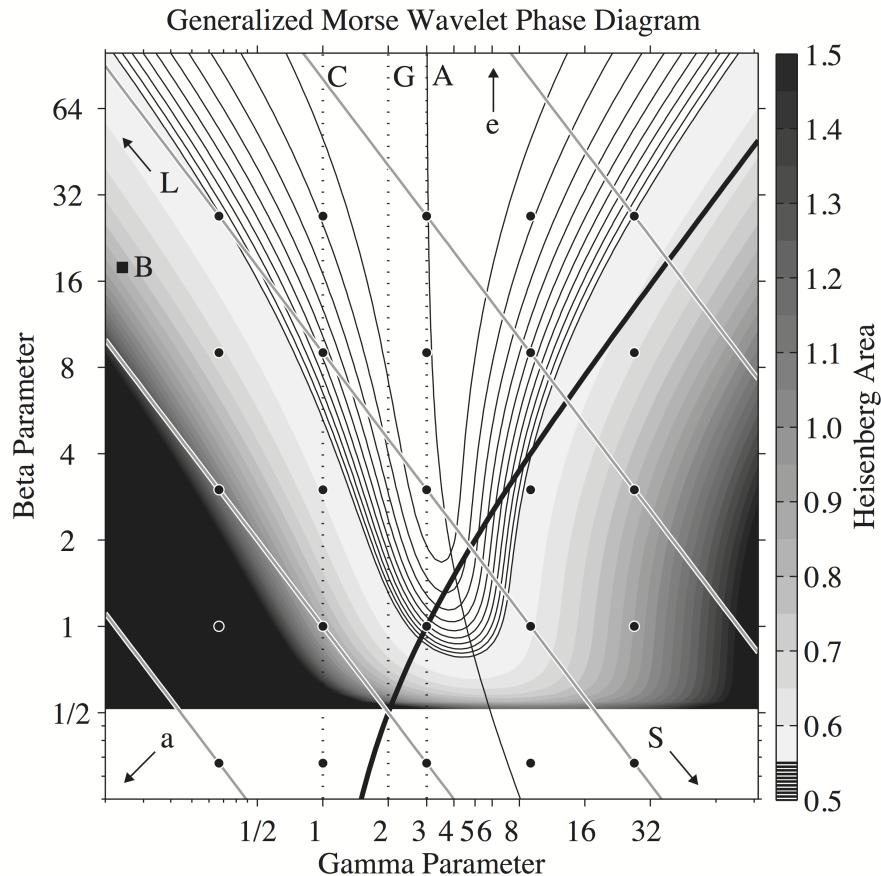
Examples of Wavelets



The generalized Morse wavelets in the frequency domain.



Concentration



The time-frequency concentration (Heisenberg area) is maximized near $\gamma = 3$, where the wavelets are also symmetric in frequency.



Important Parameters

It can be readily shown that the generalized Morse wavelet

$$\Psi_{\beta,\gamma}(\omega) = a_{\beta,\gamma} \omega^{\beta} e^{-\omega^{\gamma}}$$

has a peak in the frequency domain at frequency $\omega_{\beta,\gamma} \equiv (\beta/\gamma)^{1/\gamma}$.

Scale can therefore be converted to frequency by $\omega = s/\omega_{\beta,\gamma}$.

The *half-width* of this peak, as a fraction of its location $\omega_{\beta,\gamma}$, is roughly $1/P_{\beta,\gamma}$ where $P_{\beta,\gamma} \equiv \sqrt{\beta\gamma}$.

Frequency-domain symmetry, time-frequency concentration, and similarity to a Gaussian are all maximized near $\gamma = 3$. This means that the only parameter that needs to be adjusted is β .

Increasing β has the effect of increasing the number of oscillations in the time domain, or *decreasing* the bandwidth, thus making the wavelet narrower in the frequency domain.



Narrowband Filtering

The wavelet transform can be used to isolate localized bands of variability, such as those due to tides or internal waves.

To do this, we note that the wavelet transform at a particular scale s_o , or equivalently a particular frequency $\omega_o = s_o/\omega_{\beta,\gamma}$, is simply a *one-sided* (or analytic) bandpass in the vicinity of frequency ω_o .

Thus the real part of the wavelet transform at a particular frequency

$$\hat{x}(t) = \Re \{ w(t, s_o) \}$$

is a bandpassed version of the signal at frequency ω_o . Increasing β makes this bandpass more narrow in the frequency domain.

To remove an isolated peak, (i) take the wavelet transform for only *one* choice of frequency, then (ii) take the spectrum of the residual $x(t) - \hat{x}(t)$. Adjust the choice of β such that the spectrum of the residual shows that the peak has been effectively removed.



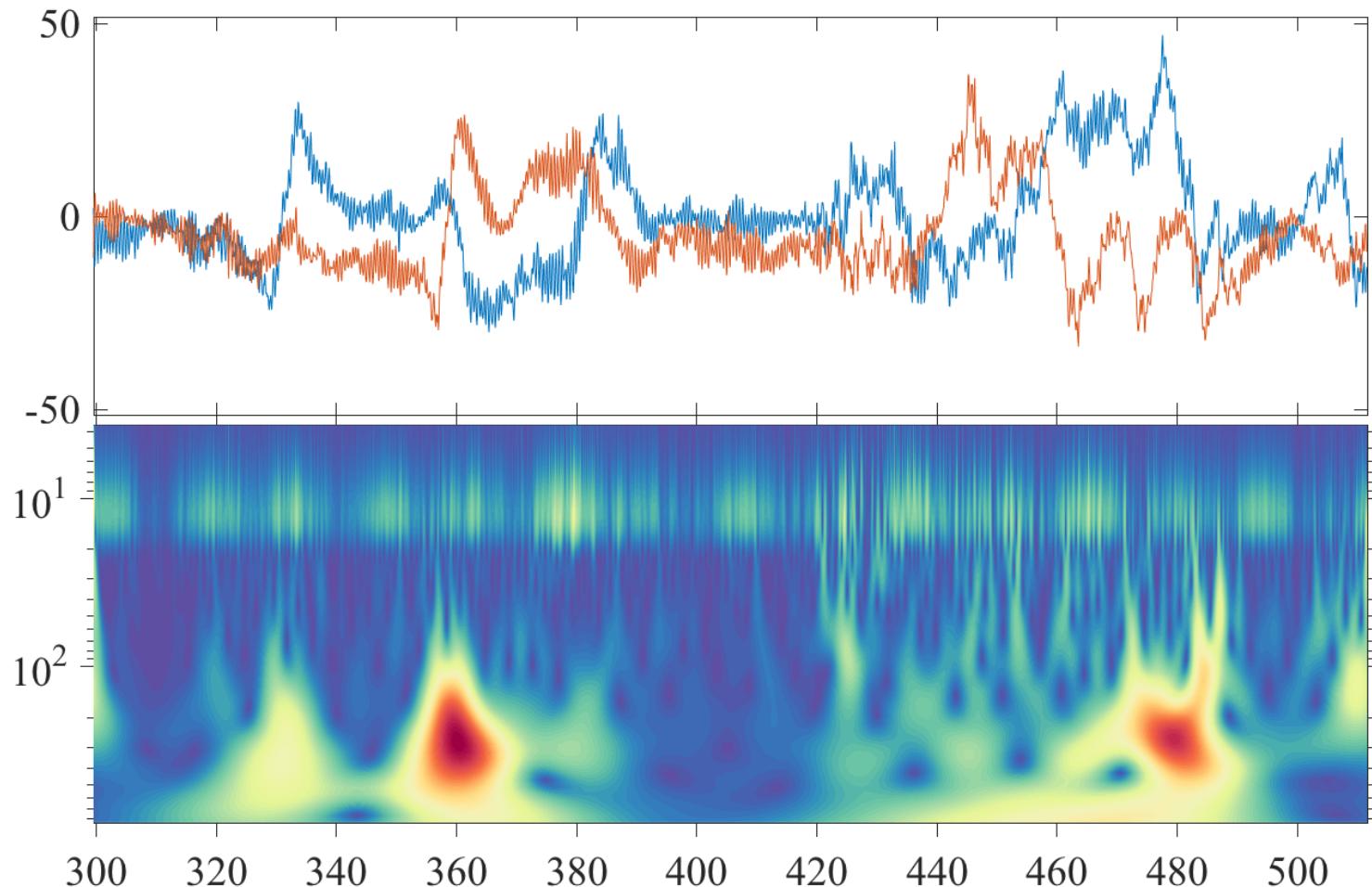
Implementation

Wavelet analysis using generalized Morse wavelets is implemented in my toolbox, {jLab}, available at {www.jmlilly.net}.

They are also now the default wavelets used in Matlab's continuous wavelet transform routine in their wavelet toolbox.



Another Example



Always Say Which Wavelet

The wavelet transform is a great way to visualize your data, “unwrapping” the variability at different scales.

However, it is essential to keep in mind that all time series methods represent an *interaction* between your signal and the analyzing method.

In wavelet analysis, the wavelet transform is the result of *smoothing* the signal with the wavelet. It is not a property of the signal! It is an *interaction* between the signal and the wavelet.

Therefore, whenever you show a wavelet transform, you *must* also (i) show the time series that you started with, and (ii) say exactly which wavelet you are using.

In a paper, it is a good idea to actually show the wavelet you are using in both time and frequency.



Understanding Artifacts

You should also always take the wavelet transform of a delta function and a sinusoid with your wavelet of choice for time series of the same length as your data.

This helps you to understand what structure is due to the signal and what is due to the wavelet.

To understand what the wavelet transform of noise looks like, take the wavelet transform of your data, but with the time order randomly reshuffled using `randperm`. Any structure you see here is due entirely to the wavelet! Compare this with the transform of the original data to check what is signal and what is artifact.

A wavelet is like a *lens* through which we view the variability in both time *and* frequency. All lenses will distort in some way, but you can't *not* choose a lens!

Therefore, you have to understand the nature of the artifacts introduced by the lens.



Summary, Part I

(i) The sampling model

$$z_n \equiv z(\Delta n), \quad n = 0, 1, \dots, N - 1$$

(ii) Euler's formula

$$e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$$

(iii) The discrete Fourier transform

$$z_n = \frac{1}{N} \sum_{m=0}^{N-1} Z_m e^{i2\pi nm/N}, \quad Z_m \equiv \sum_{n=0}^{N-1} z_n e^{-i2\pi nm/N}$$

(iv) The continuous Fourier transform

$$z(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z(\omega) e^{i\omega t} d\omega, \quad Z(\omega) \equiv \int_{-\infty}^{\infty} z(t) e^{-i\omega t} dt$$



Note (iii) becomes (iv) as a limit; (iv) gives the reflection formula

Summary, Part II

(v) The wavelet transform

$$w(t, s) = \int_{-\infty}^{\infty} \frac{1}{s} \psi^* \left(\frac{t - \tau}{s} \right) z(\tau) d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi^*(s\omega) Z(\omega) e^{i\omega t} d\omega$$

The right hand side of (v) follows from reflection formula; then (v) gives wavelet transform of delta and sinusoid



Important Extensions

Wavelet analysis is great for visualizing the variability contained in your time series.

By itself, however, it is more a visualization tool than a quantitative analysis.

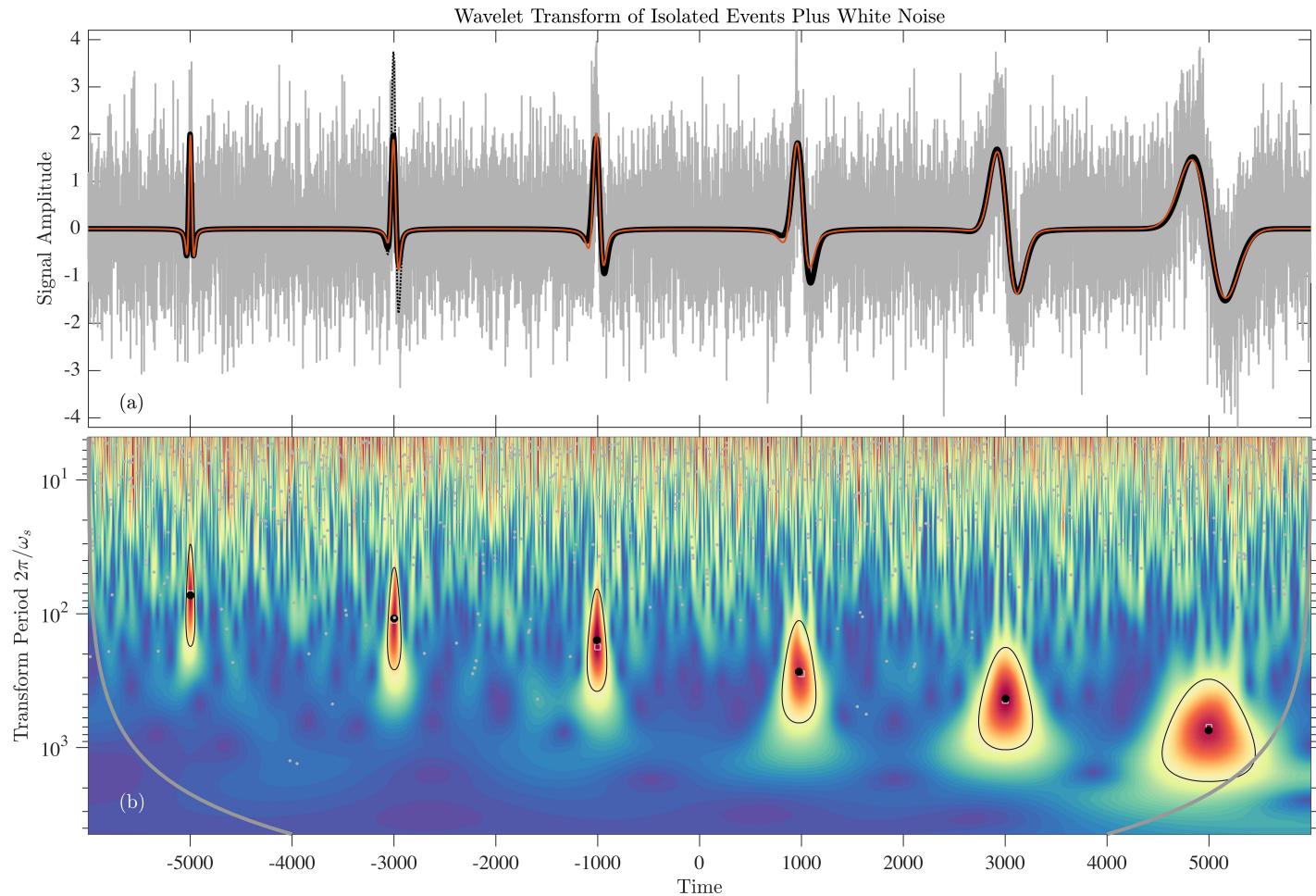
It is possible to go further with very powerful analysis methods that use wavelet analysis as their foundation:

- *Wavelet ridge analysis* is a technique for extracting the properties of amplitude and-frequency modulated oscillations from noisy signals. *Lilly and Olhede (2009b, 2010, 2012a)*
- *Element analysis* is a method for analyzing *impulsive* or *time-localized* features which themselves resemble wavelets, including significance testing. *Lilly (2017)*

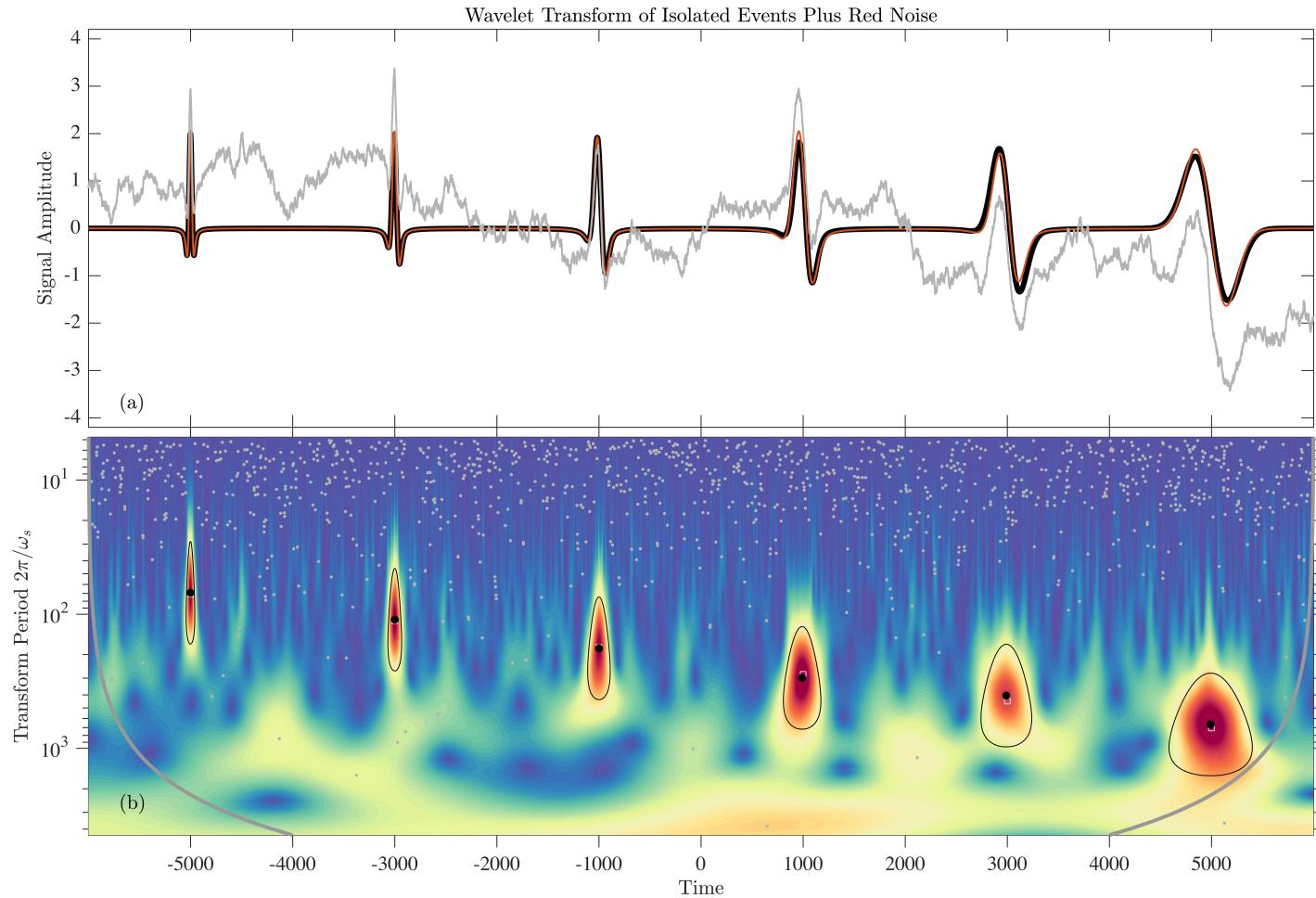
These have in common that they add to the wavelet analysis a *conceptual model* for what that signal is like.



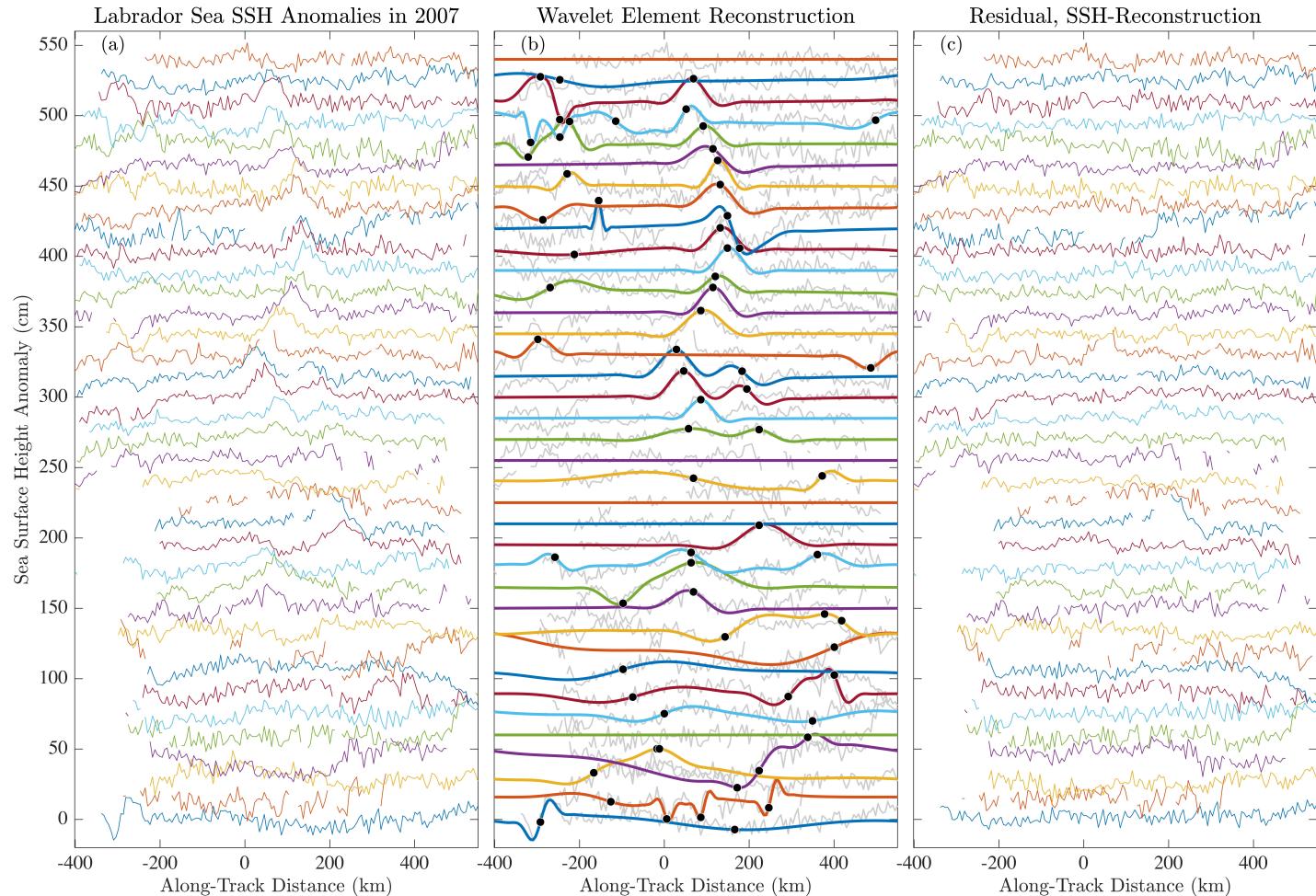
Element Analysis



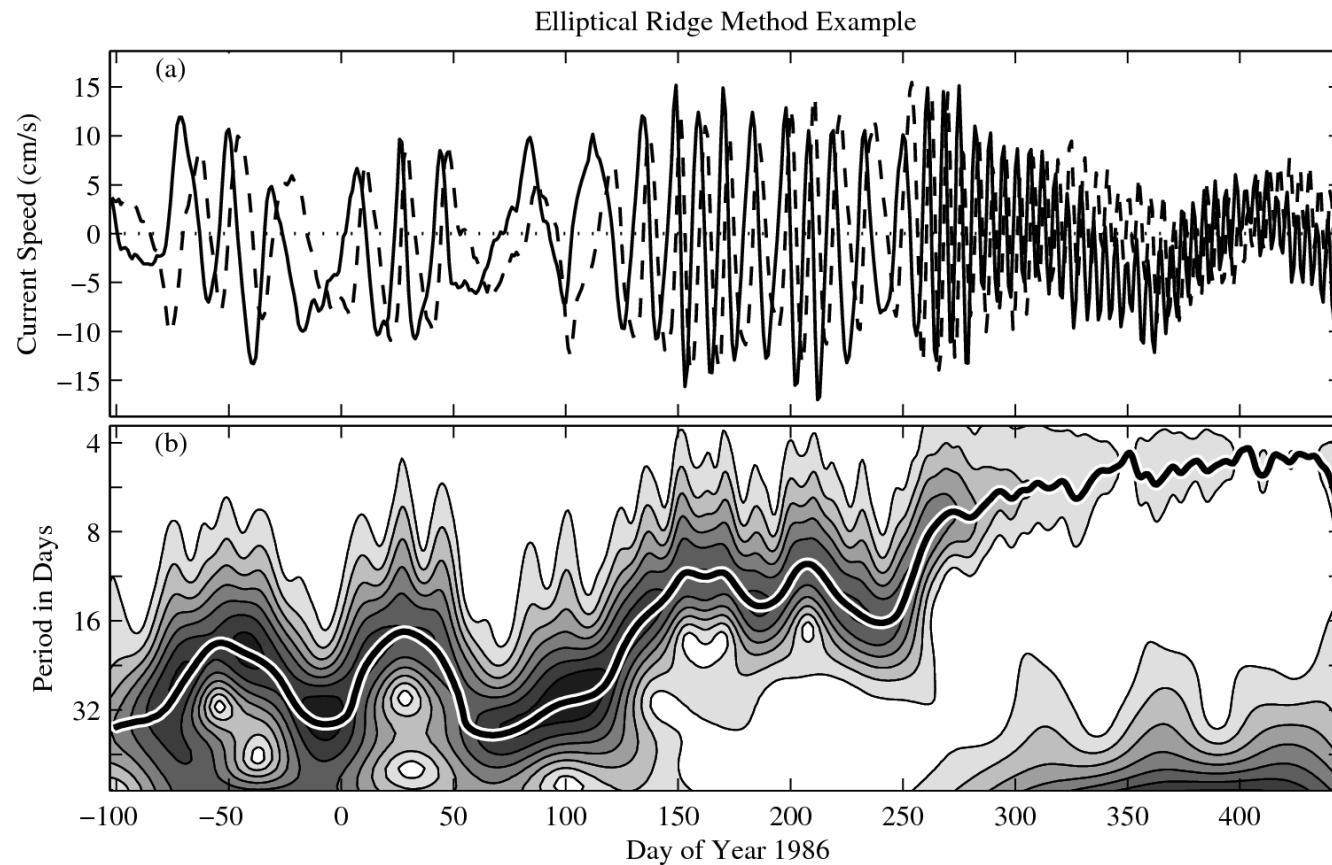
Element Analysis



Element Analysis



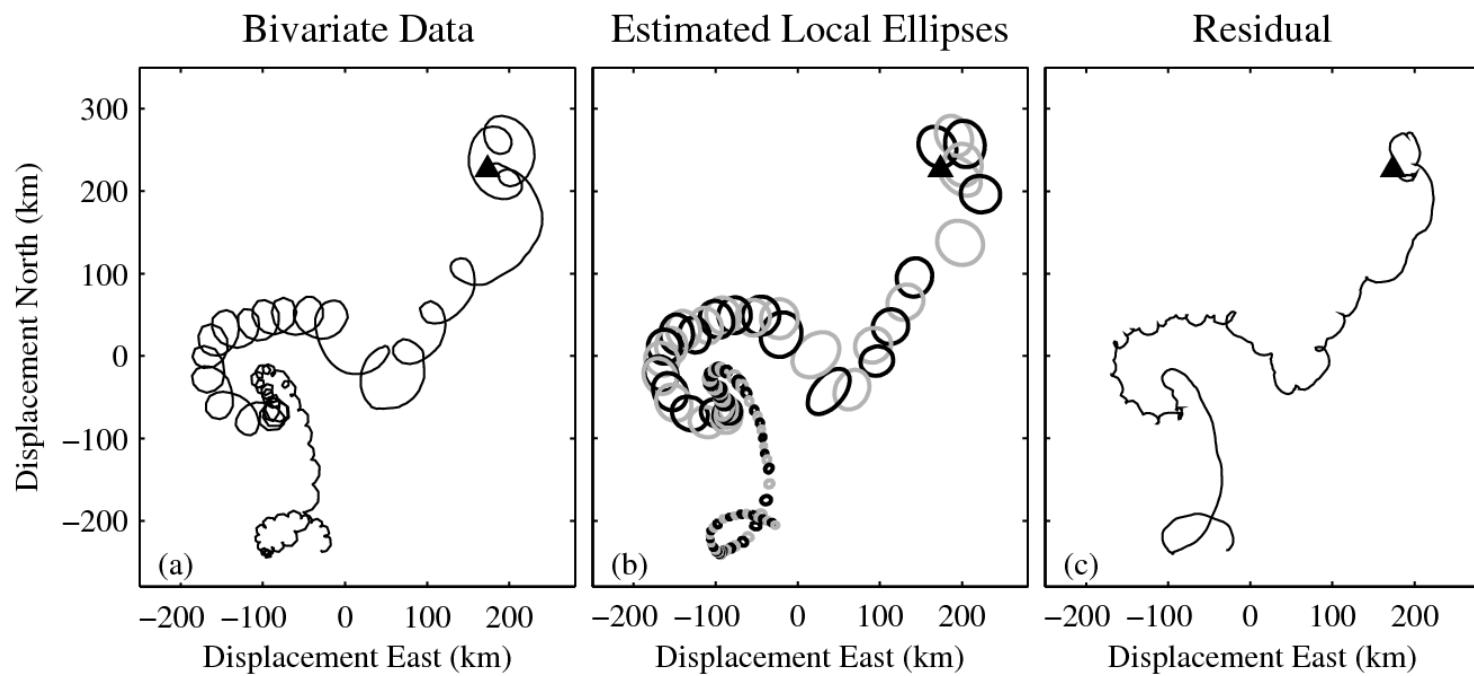
Example of Ridge Analysis



The *ridge* is the curve made by tracing out the maximum modulus of the wavelet transform.



Example of Ridge Analysis

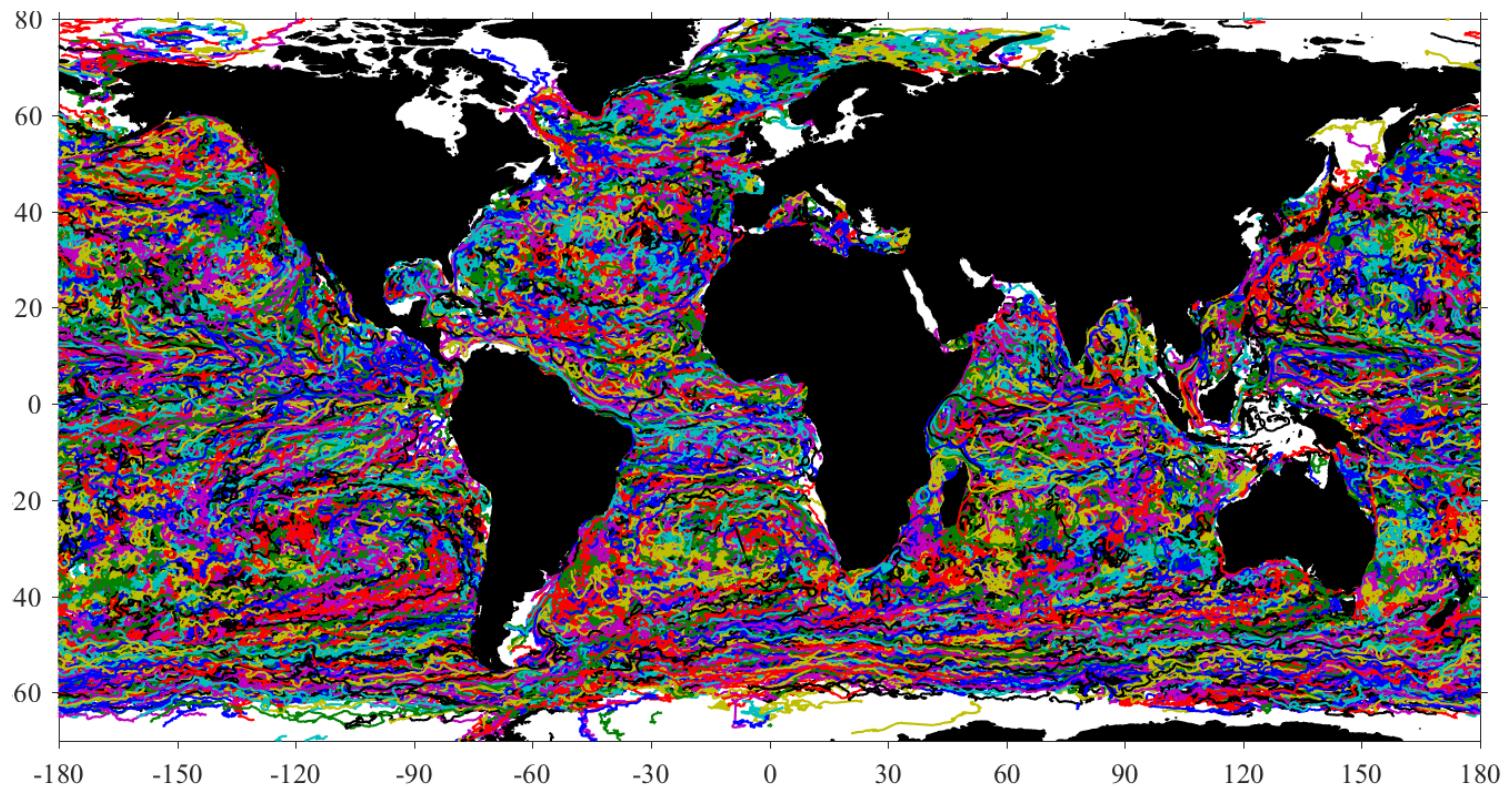


The transform value along this curve is an estimate of the oscillatory signal component, visualized as ellipses.

Ridge analysis separates modulated elliptical signal from low-frequency meandering and higher-frequency variability.

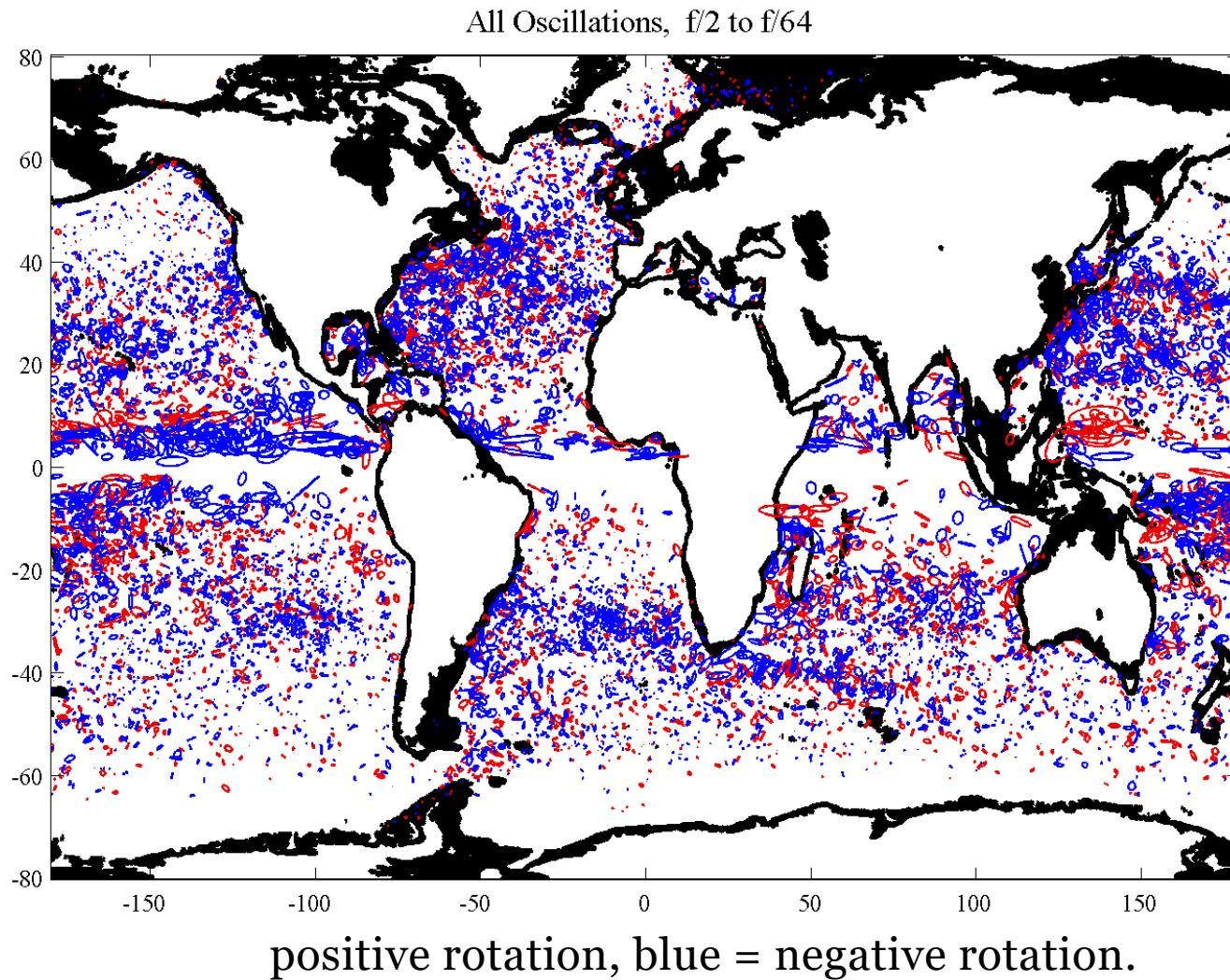


Application to the Global Drifter Dataset

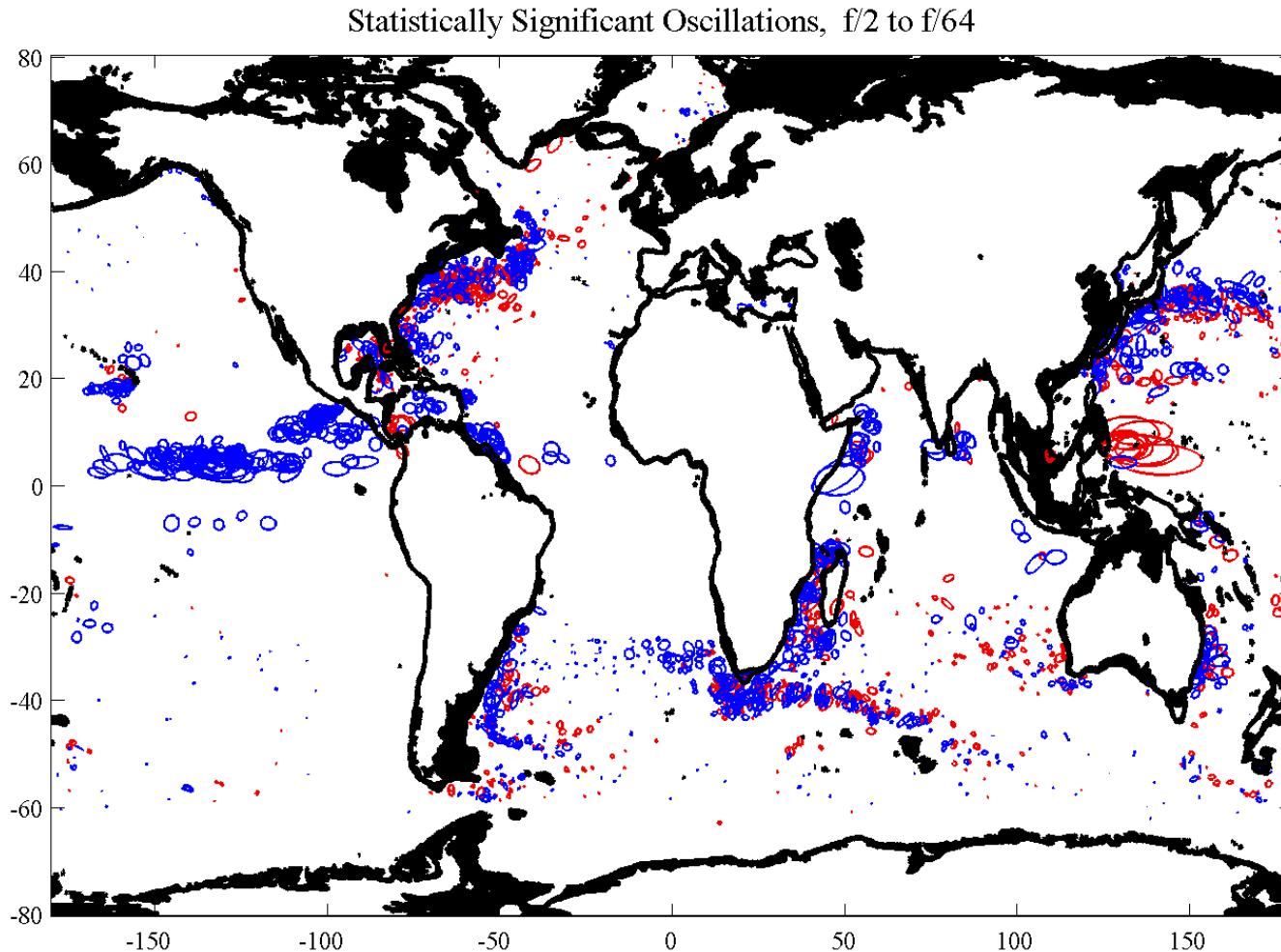


Apply the wavelet ridge analysis method to 20K time series.

Vortex Detection



Statistical Significance



After

comparison with a null hypothesis of red noise.



Ridge Analysis Details

- The notion of the *analytic signal* lets an *instantaneous* amplitude and frequency be associated with a given time series.
Gabor (1946), Vakman and Vainshtein (1977), Picinbono (1997)
- Similarly, a bivariate (x,y) signal defines the geometrical properties of a time-varying ellipse. *Lilly and Olhede (2010a)*
- This may be extended to 3D (e.g. seismographs) or N-D signals.
Lilly (2011), Lilly and Olhede (2012a)
- Modulated oscillations can be extracted using *wavelet ridge analysis*, a local best fit onto an oscillatory test function. *Delprat et al. (1992), Mallat (1999), Lilly and Olhede (2010b)*
- Modulated multivariate oscillations can be treated similarly.
Lilly and Olhede (2012a)
- Errors are proportional to modulation strength, and are minimized by a suitable choice of wavelet. *Lilly and Olhede (2010b), Lilly and Olhede (2012a)*

