Summary

The sampling model

$$z_n = z(n\Delta)$$

Euler's formula

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$

The discrete Fourier transform

$$z_n = \frac{1}{N}\sum_{m=0}^{N-1} Z_m e^{i2\pi nm/N}, \qquad Z_m \equiv \sum_{n=0}^{N-1} z_n e^{-i2\pi nm/N}$$

Continuous Fourier transform for deterministic signals

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, e^{i\omega t} d\omega, \qquad G(\omega) \equiv \int_{-\infty}^{\infty} g(t)\, e^{-i\omega t} dt$$

The Cramér spectral representation of a stochastic process

$$z(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} dZ(\omega)$$

Spectrum and autocovariance, a Fourier transform pair

$$R(\tau) \equiv \mathrm{E}\{z(t+\tau)\, z^*(t)\}, \qquad S(\omega)\delta(\omega-\nu)\, d\omega d\nu \equiv \frac{1}{2\pi}\mathrm{E}\left\{dZ(\omega)dZ^*(\nu)\right.$$

$$S(\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau} R(\tau) d\tau, \qquad R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\tau} S(\omega)\, d\omega$$

The origin of broadbias

We consider a truncated version of our continuously sampled time series, $z_T(t) \equiv z(t)\Pi_T(t)$. Its expected autovariance is

$$R_T(\tau) \equiv \mathrm{E}\left\{\frac{1}{T}\int_{-T/2}^{T/2} z_T(t+\tau)z_T^*(t)dt\right\}$$

which corresponds to an expected periodogram-like spectral estimate that is a filtered version of the true spectrum

$$R_T(\tau) = R(\tau)\Lambda_T(\tau) \Longleftrightarrow S_T(\omega) \equiv \int_{-\infty}^{\infty} S(\omega-\nu)F_T(\nu)d\nu$$

where $F_T(\omega) \equiv \int_{-\infty}^{\infty} \Lambda_T(t)e^{i\omega\tau}\,d\tau = \frac{1}{T}\frac{\sin^2(\omega T/2)}{(\omega/2)^2}$ is the Fejér kernel.

$$\Pi_T(t) \equiv \begin{cases} 1, & t \leq T/2 \\ 0, & t > T/2 \end{cases}, \qquad \Lambda_T(t) \equiv \begin{cases} 1 - \frac{|\tau|}{T}, & t \leq T \\ 0, & t > T \end{cases}$$

Multitaper spectral estimation

Let $\psi_n^{\{k\}}$ be $K = 2P - 1$ length-N orthogonal functions that are optimally concentrated in a frequency band $2P$ Rayleigh frequencies wide centered at zero.

K spectral estimates, known as the "eigenspectra", are defined as

$$\widehat{S}_m^{\{k\}} \equiv \left| \sum_{n=0}^{N-1} \psi_n^{\{k\}} z_n \, e^{-i2\pi mn/N} \right|^2, \qquad n = 0, 1, 2, \ldots, N-1.$$

are averaged to give the *multitaper spectral estimate*

$$\widehat{S}_m^{\psi} \equiv \frac{1}{K} \sum_{k=1}^{K} \widehat{S}_m^{\{k\}}.$$

This reduces bias by tapering the data with functions that minimize broadband leakage from *outside* the $2P$ band, while reducing variance through approximating an average over independent Fourier coefficients *within* the $2P$ band.

The continuous wavelet transform

The Morse wavelets are defined in terms of their Fourier transform

$$\psi_{\beta,\gamma}(t) \Longleftrightarrow \Psi_{\beta,\gamma}(\omega) = \begin{cases} a_{\beta,\gamma}\,\omega^{\beta} e^{-\omega^{\gamma}}, & \omega > 0 \\ 0, & \omega \leq 0 \end{cases}$$

Since $\Psi_{\beta,\gamma}(\omega)$ is real, $\psi_{\beta,\gamma}(t) = \psi^{*}_{\beta,\gamma}(-t)$. The wavelet transform is

$$w(t,s) = \int_{-\infty}^{\infty} \frac{1}{s}\psi\left(\frac{\tau - t}{s}\right) z(\tau) d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi(s\omega) Z(\omega) e^{i\omega t} d\omega$$

and scale is mapped to frequency by $\omega = s/\omega_{\beta,\gamma}$ with $\omega_{\beta,\gamma} \equiv (\beta/\gamma)^{1/\gamma}$.

Note, this is a *joint function* of the signal and the wavelet!!

$$z(t) = \delta(t - t_o) \qquad \Longrightarrow \qquad w(t,s) = \frac{1}{s}\psi^{*}\left(\frac{t - t_o}{s}\right)$$

$$\left.\begin{aligned} z(t) &= e^{i\omega_o t} \\ Z(\omega) &= 2\pi\delta(\omega - \omega_o) \end{aligned}\right\} \qquad \Longrightarrow \qquad w(t,s) = \Psi(s\omega_o) e^{i\omega_o t}$$