Lab 11

Derivative and Gradient

Chain rule

Calculate dz/dt for each of the following functions:

1. $z = f(x, y) = 4x^2 + 3y^2,\ x = x(t) = \sin t,\ y = y(t) = \cos t$
2. $z = f(x, y) = \sqrt{x^2 - y^2},\ x = x(t) = e^{2t},\ y = y(t) = e^{-t}$

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Answer:

1. $2 \sin t \ \cos t$
2. $$\frac{2e^{6t} + 1}{e^t \sqrt{e^{6t} - 1}}$$

Chain rule for two variables

Calculate partial derivatives $\partial z / \partial u,\ \partial z / \partial v$ using the following functions:

$$z = f(x, y) = 3x^2 - 2xy + y^2, x = x(u, v) = 3u + 2v, y = y(u, v) = 4u - v$$

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Answer:

$$\frac{\partial z}{\partial u} = 38u + 18v$$

$$\frac{\partial z}{\partial v} = 18u + 34v$$

Gradient (partial derivative)

Find the gradient of the following function

$$f(x, y) = x^2 - xy + 3y^2$$

Answer: $\nabla f(x, y) = [2x - y, -x + 6y]$

More about Gradient

Find the gradient of the following

$$f(x, y, z) = e^{-2z} \sin 2x \cos 2y$$

Answer:

$$\nabla f(x, y, z) = 2e^{-2z} \cdot [\cos 2x \cos 2y, -\sin 2x \sin 2y, -\sin 2x \cos 2y]$$

Directional derivative

Let $\theta = \arccos(3/5)$, find the directional derivative $\nabla_v f(x,y)$ of

$f(x,y) = x^2 - xy + 3y^2$ In the direction of $v = (\cos\theta,\ \sin\theta)$

Connect with the previous problem $\nabla_v f(x) = \nabla f(x) \cdot v$

Directional derivative

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Connect with the previous problem $\nabla_v f(x) = \nabla f(x) \cdot v$

Answer:

Partial derivative of f is $[2x - y, -x + 6y]$

$$\nabla_v f(x, y) = (2x - y)\frac{3}{5} + (-x + 6y)\frac{4}{5} = \frac{2x + 21y}{5}$$

Gradient of a least-squares loss in a linear model

Consider the linear model $$y = X \cdot \theta$$

where theta is a parameter vector of length D, X is an n by D input feature matrix and y are the corresponding observations of length n.

Optimizing such a model can be considered as solving

$$\min_{\theta \in \mathbb{R}^{\mathbb{D}}} \left(||y - X \cdot \theta||^2 \right)$$

Gradient of a least-squares loss in a linear model

$$\min_{\theta \in \mathbb{R}^{\mathbb{D}}} \left(||y - X \cdot \theta||^2 \right)$$

This can be solved by computing the gradient of $L = ||e||^2, \ e = y - X \cdot \theta$

$$\frac{\partial L}{\partial e} = 2e^T \qquad \frac{\partial e}{\partial \theta} = -X$$

$$\frac{\partial L}{\partial \theta} = \frac{\partial L}{\partial e} \frac{\partial e}{\partial \theta} = -2\left(y^T - \theta^T X^T\right) X$$

* Solving derivative equals 0 is sufficient to minimize the loss, since the Hassian of L equals $X^T X$ is PSD.