Introduction to Bayesian Inference

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Outline I

Statistical Inference

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Statistical inference is concerned with drawing conclusions, from *random* numerical data, about quantities that are not observed.

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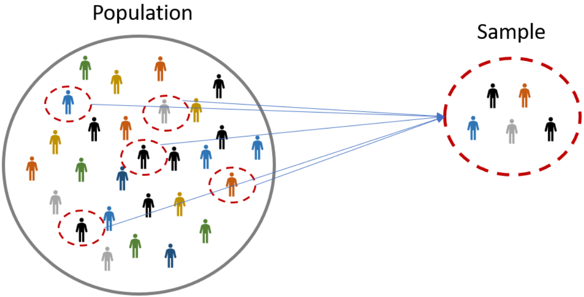
For example, we may collect data to observe the average height of adults in France.

- However, it is not practical to observe the whole *population* of France.
- Instead, we collect a finite set of observations or *samples* from the population.



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Inference Methods

Parametric Inference

Parametric methods are based on the assumption that the sample comes from a population that can be modelled by a probability distribution with fixed set of *parameters*.

For example, *likelihood-based approaches, Bayesian approaches.*

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Non-parametric Inference

Non-parametric methods are used when we may not have any distributional assumption.

For example, *order statistics, quantiles.*

Likelihood

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Let X denotes a random variable, so that we have an associated probability density function (probability mass function for discrete) $f_X(\cdot \mid \theta)$.

Now, let, $x_1, x_2, \cdots, x_n$ be n observations of X. Then the joint probability of the observed data is called *likelihood function* and is denoted by $\mathcal{L}(\theta \mid \tilde{x})$ so that

$$
\begin{aligned}
&\mathcal{L}(\theta \mid \tilde{x}) && (1)\\
&= f_X(\tilde{x} \mid \theta) && (2)\\
&= f_X(x_1 \mid x_2, \cdots, x_n, \theta) \cdot f_X(x_2 \mid x_3, \cdots, x_n, \theta) \cdots f_X(x_n \mid \theta) && (3)\\
&= \prod_{i=1}^{n} f_X(x_i \mid \theta) \quad \text{when } x_i\text{'s are independent.} && (4)
\end{aligned}
$$

Example

Let $x_1, x_2, \cdots, x_n$ are i.i.d. normally distributed variables with mean μ and variance σ^2.

Then the likelihood function is given by:

$$\mathcal{L}(\mu, \sigma^2 \mid \tilde{x}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x_i - \mu)^2}{2\sigma^2}\right) \tag{5}$$

$$= \frac{1}{(\sqrt{2\pi}\sigma)^n} \exp\left(-\sum_{i=1}^{n} \frac{(x_i - \mu)^2}{2\sigma^2}\right) \tag{6}$$

How can we estimate this μ and σ^2?

Maximum Likelihood Estimation (MLE)

We can estimate the parameter θ from the likelihood function by maximising it.

$$\hat{\theta} = \arg\max_{\theta} \mathcal{L}(\theta \mid \tilde{x}). \tag{7}$$

In many cases, we work with natural logarithm of the likelihood function and we denote it by $\ell(\theta \mid \tilde{x})$ so that

$$\ell(\theta \mid \tilde{x}) = \log\left(\mathcal{L}(\theta \mid \tilde{x})\right). \tag{8}$$

Necessary and Sufficient Conditions

Necessary condition: For p different parameters

$$\frac{\partial \ell}{\partial \theta_1} = \frac{\partial \ell}{\partial \theta_2} = \cdots = \frac{\partial \ell}{\partial \theta_p} = 0 \tag{9}$$

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Sufficient condition: Let

$$H(\theta) = \begin{bmatrix} \frac{\partial^2 \ell}{\partial \theta_1^2} & \frac{\partial^2 \ell}{\partial \theta_1 \partial \theta_2} & \cdot & \frac{\partial^2 \ell}{\partial \theta_1 \partial \theta_p} \\ \frac{\partial^2 \ell}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 \ell}{\partial \theta_2^2} & \cdots & \frac{\partial^2 \ell}{\partial \theta_2 \partial \theta_p} \\ \vdots & & & \\ \frac{\partial^2 \ell}{\partial \theta_p \partial \theta_1} & \frac{\partial^2 \ell}{\partial \theta_p \partial \theta_2} & \cdots & \frac{\partial^2 \ell}{\partial \theta_p^2} \end{bmatrix}. \tag{10}$$

Then $H(\hat{\theta})$ has to be negative (semi)definite.

Example

Let $x_1, x_2, \cdots, x_n$ are i.i.d. normally distributed variables with mean μ and variance σ^2. Then the likelihood function is given by:

$$\mathcal{L}(\mu, \sigma^2 \mid \tilde{x}) = \frac{1}{(\sqrt{2\pi}\sigma)^n} \exp\left(-\sum_{i=1}^{n} \frac{(x_i - \mu)^2}{2\sigma^2}\right) \tag{11}$$

Then, necessary condition for MLE gives us

$$\frac{\partial \ell(\mu, \sigma^2 \mid \tilde{x})}{\partial \mu} = \frac{2n(\overline{x} - \mu)}{2\sigma^2} \tag{12}$$

$$\frac{\partial \ell(\mu, \sigma^2 \mid \tilde{x})}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(x_i - \mu)^2 \tag{13}$$

where, $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$.
We can show that $\hat{\mu} = \overline{x}$ and $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2$.

Bayes' Rule

For any two event A and B, we have

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}. \tag{14}$$

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Similarly for two continuous random variable X and Y

$$f_{X|Y}(x \mid y) = \frac{f_{Y|X}(y \mid x)f_X(x)}{f_Y(y)}. \tag{15}$$

Then using law of total probability we have

$$f_{X|Y}(x \mid y) = \frac{f_{Y|X}(y \mid x)f_X(x)}{\int_x f_{Y|X}(y \mid x)f_X(x)dx}. \tag{16}$$

Bayesian Inference

Let, $x_1, x_2, \cdots, x_n$ be observations of a random variable with p.d.f $f_X(x \mid \theta)$. Let θ be our parameter of interest. Then

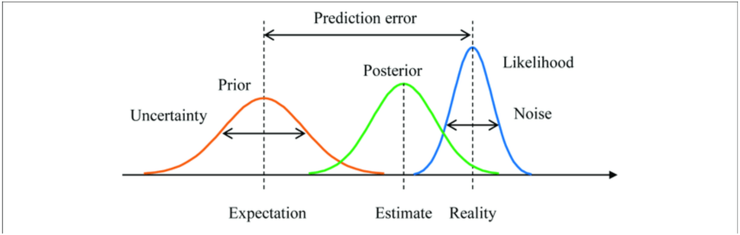
$$f_{\theta|X}(\theta \mid \tilde{x}) = \frac{f_{X|\theta}(\tilde{x} \mid \theta)\pi(\theta)}{\int_\theta f_{X|\theta}(\tilde{x} \mid \theta)\pi(\theta)d\theta} = \frac{\mathcal{L}(\theta \mid \tilde{x})\pi(\theta)}{\int_\theta \mathcal{L}(\theta \mid \tilde{x})\pi(\theta)d\theta}. \tag{17}$$

- $f_{\theta|X}(\theta \mid \tilde{x}) \equiv$ Posterior distribution
- $\pi(\theta) \equiv$ Prior distribution
- $\int_\theta \mathcal{L}(\theta \mid \tilde{x})\pi(\theta)d\theta \equiv$ Marginal likelihood or model evidence

We can simply write

$$\text{Posterior} \propto \text{Likelihood} \times \text{Prior}. \tag{18}$$

Visualisation



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Choice of priors I

- Subjective Priors Subjective priors are usually used to incorporate one's subjective belief about the modelling parameter. Subjective priors are often elicitation-based and allow us to gather information from previous analysis.
- Prior Predictive Before the data x is observed, we can look into the distribution of this unknown but observable data x, which is given by:

$$f_X(x) = \int_\theta f_{X|\theta}(x \mid \theta)\pi(\theta)d\theta \tag{19}$$

where $f_{X|\theta}(x \mid \theta)$ refers to our sampling distribution of some observable quantity x and $\pi(\theta)$ refers to our prior on the parameter θ. We call this distribution $f_X(x)$ the prior predictive distribution.

Choice of priors II

- Objective Priors Objective prior is an alternative method for describing a prior where we usually use objective source of information about the modelling parameter such as parameter support or sign of the modelling parameter. We often consider these priors as non-informative priors/uninformative priors as they do not posses any other descriptive information.
- Improper Priors Improper priors can also be classified as objective priors. However, improper priors may not integrate to 1. To give some intuition, we can consider an unbounded parameter, then a uniform distribution will result to an improper prior.
- We also have conjugate priors which is used the most because of convenience.

Conjugate Priors

In Bayesian inference, if the posterior distribution $f_{\theta|X}(\theta \mid \tilde{x})$ is in the same probability distribution family as the prior probability distribution $\pi(\theta)$, then the prior and posterior are then called conjugate distributions, and the prior is called a conjugate prior for the likelihood function $\mathcal{L}(\theta \mid \tilde{x})$.
For example, Inverse-gamma distribution is a conjugate prior for the variance of normal distribution

Exponential Family

Let $\theta := (\theta_1, \cdots, \theta_p)$ be a vector of parameters. Then the exponential family of distributions is defined by:

$$f(x \mid \theta) = h(x) \exp\left(\sum_{i=1}^{p} a_i(\theta) T_i(x) - b(\theta)\right) \tag{20}$$

where h, a, T and b are fixed functions for each probability distribution.

Example

In case of a normal distribution, the probability density function is given by:

$$\begin{aligned} f(x \mid \mu, \sigma^2) &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) && (21) \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2 - 2x\mu + \mu^2}{2\sigma^2} - \ln\sigma\right) && (22) \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(\left(-\frac{1}{2\sigma^2}, \frac{\mu}{\sigma^2}\right) \cdot (x^2, x)^T - \frac{\mu^2}{2\sigma^2} - \ln\sigma\right). && (23) \end{aligned}$$

Therefore, $h(x) := \frac{1}{\sqrt{2\pi}}$, $a(\mu, \sigma^2) := \left(-\frac{1}{2\sigma^2}, \frac{\mu}{\sigma^2}\right)$, $T(x) := (x^2, x)$ and $b(\mu, \sigma^2) := \left(\frac{\mu^2}{2\sigma^2} + \ln\sigma\right)$.

Jeffrey's Prior

In Bayesian inference, Jeffrey's prior is an objective prior distribution for a parameter space. Its probability density function is proportional to the square root of the determinant of the Fisher information matrix ($I(\theta)$).

For log-likelihood $\ell(\theta \mid \tilde{x})$, the Fisher information matrix is given by:

$$I(\theta) = \mathbb{E}\left[\left(\frac{\partial \ell(\theta \mid \tilde{x})}{\partial \theta}\right)^2 \mid \theta\right] \tag{24}$$

under regularity conditions

$$= -\mathbb{E}\left[\frac{\partial^2 \ell(\theta \mid \tilde{x})}{\partial \theta^2} \mid \theta\right]. \tag{25}$$

Estimation

- Posterior Mean The most common and convenient way to learn from the posterior distribution is to check the posterior mean given by:

$$\mathbb{E}(\theta \mid X) = \int_{\theta} \theta f_{\theta|X}(\theta \mid \tilde{x}) d\theta. \tag{26}$$

- Posterior Mode Besides posterior mean, we sometimes look for the maximum a posteriori (MAP) estimates. That is we look for the value that achieves greatest posterior density. We look for MAP in the following way:

$$\theta_{\text{MAP}} = \arg\max_{\theta} f_{\theta|X}(\theta \mid \tilde{x}). \tag{27}$$