ISC3, Fall 2022 (A22)
Computer works report TP 2, 26/09/2022

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1 Stem

From a dataset point cloud, we want to achieve a regression using the following regression function defined on [0, 1]:

$$\tilde{f}_d(x) = \sum_{j=1}^{d} u_j \Lambda_j(x) \qquad \textit{where} \qquad \Lambda_j(x) = \max\left(0, 1-(d-1)\left|x - \frac{j-1}{d-1}\right|\right) \tag{1}$$

This function defines a piecewise linear function. Try and analyze the following Scilab script:

```
clear
function y = piecewiselinear(x,d,u)
y = zeros(x);
for j = 1:d
y = y + u(j)*max( 0, 1-(d-1)*abs(x-(j-1)/(d-1)) );
end
endfunction
//
//Example
d=10;
xi = linspace(0,1,d);
u = sin(2*%pi*xi);
x = linspace(0,1,200);
y = piecewiselinear(x, d, u);
plot(x, y, '-'); plot(xi, u, 'o'); xgrid();
```

Consider the following dataset generated by the following Scilab script

```
N=100;
xi = rand(N,1);
yi = sin(2*%pi*xi)+0.2*rand(N,1, "normal");
```

Then we want to find the coefficients uj , j = 1, ..., d, that minimize the least square error

$$\min_{(u_1,\dots,u_d)} \frac{1}{2}\sum_{j=1}^{N}\left(\tilde{f}_d(x_i) - y_i\right)^2 \tag{2}$$

(the coefficients u_j are in the definition of f).

2 Questions

a) On the paper, determine what is A_{ij} . Then in a Scilab script, assemble the matrix $A \in M_{Nd}(R)$.
Solution :

$$A_{ij} = \Lambda_j(x_i) = \max\left(0, 1-(d-1)\left|x_i - \frac{j-1}{d-1}\right|\right)$$

b) Solve the normal equations $A^T Au = A^T y$.
Code pour cette question :

```
function y = piecewiselinear(x,d,u)
    y = zeros(x);
    for j = 1:d
        y = y + u(j)*max( 0, 1-(d-1)*abs(x-(j-1)/(d-1)) );
    end
endfunction

N=100;d=20
xi = rand(N,1);xi = gsort(xi);
yi = sin(2*%pi*xi)+0.2*rand(N,1, "normal");

A = zeros(N,d)
for j=1:d
    A(:,j) = max( 0, 1-(d-1)*abs(xi-(j-1)/(d-1)) );
end

coefs = (A'*A)\(A'*yi)
```

c) By using the function piecewiselinear(), plot the resulting regression function in solid line. On the same graphics, plot also the point cloud $(x_i, y_i)_{i=1,\ldots,N}$ with circles for each point. Check if the resulting function $f(x)$is a good regression function.
Solution :

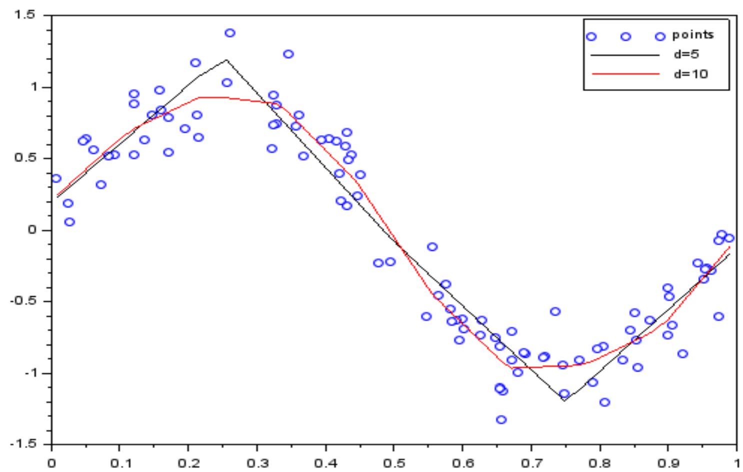

Figure 1: L'image de la function piecewiselinear

Next, we would like to add a Tykhonov regularization term to the least square minimization problem, and study the effect of the regularization coefficient $\mu < 0$.

d) Consider a set of regularization coefficients $\mu_k = 10^k, k = -8, \ldots, 2$.
For each k solve the regularized normal equations

$$\left(A^T A + \mu_k I\right) u_k = A^T y \tag{3}$$

(u now depends on k, so we add an index k in u_k to notice the dependency). Then plot the regression function $f_d(x)$ and the point cloud on the same graphics as before.
Observe the (possible) influence of the regularization coefficient μ_k on the solution.
Solution :

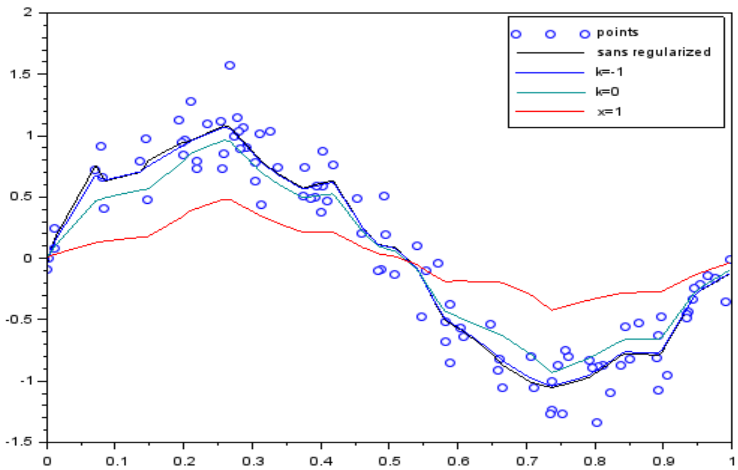

Figure 2: la function piecewiselinear ave la regularization

Code pour cette question :

```
coefs2=zeros(d,10)
for k=1:10
    uk=10^(k-9)
    coefs2(:,k) = (A'*A+uk.*eye(d,d))\(A'*yi)
end
plot2d(xi,piecewiselinear(xi,d,coefs2(:,8)),style=2)
plot2d(xi,piecewiselinear(xi,d,coefs2(:,9)),style=16)
plot2d(xi,piecewiselinear(xi,d,coefs2(:,10)),style=5)
legend("points","sans regularized", "k=-1","k=0","x=1");
```

e) On another graphics, plot the parametric curve

$$\mu_k \to (||Au_k - y||, ||u_k||)^T \tag{4}$$

still for $\mu_k = 10^k, k = -8, \ldots, 2$, and plot the parametric curve in log-log scale. Inyour opinion, what could be the best empirical value of μ_k ?

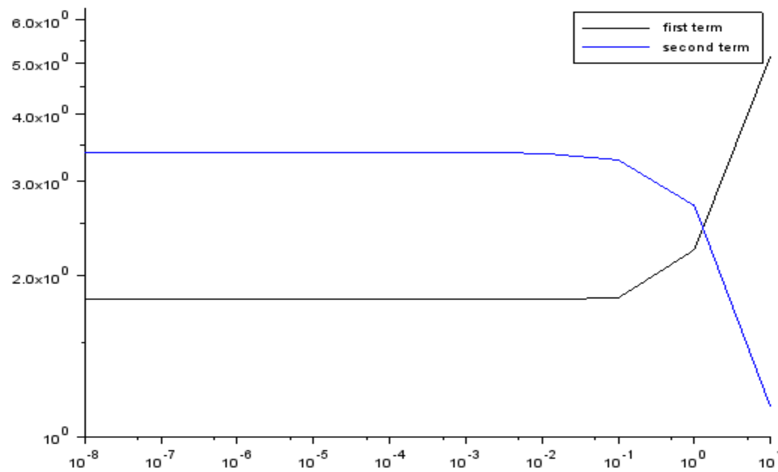

Figure 3: the parametric curve

Nous choisissons le point d'intersection des deux lignes, qui est proche de $x = 0$, lorsque $k = 0, u_k = 0$

Résumé

Une dimensionnalité élevée crée des problèmes d'ajustement excessif, qui peuvent être résolus en réduisant la dimensionnalité ou en ajoutant une régularisation afin d'atténuer l'ajustement excessif.