QOSF mentorship program screening task 2

Author

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Tools

Jupyter notebook and Qiskit

Task 2

The bit-flip code and the sign-flip code are two very simple circuits able to detect and fix the bit-flip and the sign-flip errors, respectively.

1. Build the circuit to prepare the Bell state.
2. Now add, right before the CNOT gate and for each of the two qubits, an arbitrary “error gate”. By error gate we mean that with a certain probability (that you can decide but must be non-zero for all the choices) you have a 1 qubit unitary which can be either the identity, or the X gate (bit-flip error) or the Z gate (sign-flip error).
3. Encode each of the two qubits with a sign-flip or a bit-flip code, in such a way that all the possible choices for the error gates described in 2), occurring on the logical qubits, can be detected and fixed. Motivate your choice. This is the most non-trivial part of the problem, so do it with a lot of care!
4. Test your solution by making many measurements over the final state and testing that the results are in line with the expectations.

Content:

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1. **Basic tasks:**

    - Buld the circuit to prepare Bell state
    - Add the error gate
    - Design the error correction circuit
    - Simulate mutiple times and visulize the results

2. **A more general case**

3. **Conclusion**

Basic tasks

Let's first define all the fuctions needed.

In [712…

```
from qiskit import QuantumCircuit, QuantumRegister, ClassicalRegister
from qiskit.quantum_info import Statevector, state_fidelity
from qiskit.visualization import plot_state_qsphere, plot_histogram
from qiskit import Aer, execute

import collections
from collections import Counter
import random
import matplotlib.pyplot as plt
```

In [749…

```
# This is the function that will add an error gate of ...
# ... X (with probability p=p_error_X/100), or Z (p=p_error_Z/100) or I (p=(100-p_error_X-p_error_Z)/100)...
# ... to the qubit named qubit_applied in a circuit named given_circuit.

def error_gate(given_circuit, qubit_applied,p_error_X,p_error_Z):
    p_I=100-p_error_X-p_error_Z
    for i in qubit_applied:
        n=random.choice([1]*p_error_X + [2]*p_error_Z + [3]*p_I)
        if (n==1):
            given_circuit.x(i)
        elif (n==2):
            given_circuit.z(i)
        else:
            given_circuit.i(i)

    return given_circuit


# This is the function that will generate the error-correction circuit for a single logical qubit.
# The error gate is already included in this circuit and is applied only on the logical qubit.
# qubits_repeat are the ancillary qubits needed for the logical qubit called qubit_error_applied

def error_correction_simple(given_circuit, qubit_error_applied,qubits_repeat,p_error_x,p_error_z):
```

```
    given_circuit.cx(qubit_error_applied,qubits_repeat[1])
    given_circuit.h(qubit_error_applied)
    given_circuit.h(qubits_repeat[1])
    given_circuit.barrier(qubit_error_applied,qubits_repeat)
    given_circuit.cx(qubit_error_applied,qubits_repeat[0])

    # call the function to generate the error gate
    error_gate(given_circuit, qubit_error_applied,p_error_x,p_error_z)

    given_circuit.cx(qubit_error_applied,qubits_repeat[0])
    given_circuit.cx(qubits_repeat[0],qubit_error_applied)
    given_circuit.barrier(qubit_error_applied,qubits_repeat)
    given_circuit.h(qubit_error_applied)
    given_circuit.h(qubits_repeat[1])
    given_circuit.cx(qubit_error_applied,qubits_repeat[1])
    given_circuit.cx(qubits_repeat[1],qubit_error_applied)

    return given_circuit


# This is the function that will generate the error-correction circuit for a single logical qubit.
# The error gate is included in this circuit already and is applied on the logical qubit as well as the ancilla

def error_correction_shor(given_circuit, qubit_error_applied,qubits_repeat,p_error_x,p_error_z):
    given_circuit.cx(qubit_error_applied,qubits_repeat[2])
    given_circuit.cx(qubit_error_applied,qubits_repeat[5])
    given_circuit.h(qubit_error_applied)
    given_circuit.h(qubits_repeat[2])
    given_circuit.h(qubits_repeat[5])
    given_circuit.cx(qubit_error_applied,qubits_repeat[0])
    given_circuit.cx(qubit_error_applied,qubits_repeat[1])
    given_circuit.cx(qubits_repeat[2],qubits_repeat[3])
    given_circuit.cx(qubits_repeat[2],qubits_repeat[4])
    given_circuit.cx(qubits_repeat[5],qubits_repeat[6])
    given_circuit.cx(qubits_repeat[5],qubits_repeat[7])

    # call the function to generate the error gate
    error_gate(given_circuit, qubit_error_applied,p_error_x,p_error_z)
    error_gate(given_circuit, qubits_repeat,p_error_x,p_error_z)

    given_circuit.cx(qubit_error_applied,qubits_repeat[0])
    given_circuit.cx(qubit_error_applied,qubits_repeat[1])
    given_circuit.cx(qubits_repeat[2],qubits_repeat[3])
    given_circuit.cx(qubits_repeat[2],qubits_repeat[4])
    given_circuit.cx(qubits_repeat[5],qubits_repeat[6])
    given_circuit.cx(qubits_repeat[5],qubits_repeat[7])
```

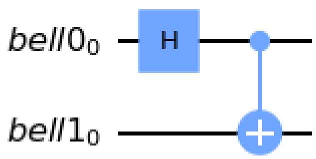
```
    given_circuit.ccx(qubits_repeat[1],qubits_repeat[0],qubit_error_applied)
    given_circuit.ccx(qubits_repeat[4],qubits_repeat[3],qubits_repeat[2])
    given_circuit.ccx(qubits_repeat[7],qubits_repeat[6],qubits_repeat[5])
    given_circuit.h(qubit_error_applied)
    given_circuit.h(qubits_repeat[2])
    given_circuit.h(qubits_repeat[5])
    given_circuit.cx(qubit_error_applied,qubits_repeat[2])
    given_circuit.cx(qubit_error_applied,qubits_repeat[5])
    given_circuit.ccx(qubits_repeat[5],qubits_repeat[2],qubit_error_applied)

    return given_circuit


# This function is for data visualization

def plot_data(data):
    p = collections.Counter(data)
    probability=[]
    # convert counts to probabilities
    for value in p.values():
        probability.extend([value/sum(p.values())])
    xlocs, xlabs = plt.xticks()
    xlocs=[i+1 for i in range(0,len(p.keys()))]
    xlabs=[i/2 for i in range(0,len(p.keys()))]
    plt.bar(p.keys(), probability)
    for i, v in enumerate(probability):
        plt.text(xlocs[i] - 1.15, v+0.01 , str(v))
    plt.title('Distribution of  Measurement Results')
    plt.xlabel('Measurement results')
    plt.ylabel('Probabilities')
    plt.ylim((0,0.6))
```

1. Buld the circuit to prepare Bell state

In [750…

```
bell_qubit0 = QuantumRegister(1,'bell0')
bell_qubit1 = QuantumRegister(1,'bell1')
BellCircuit = QuantumCircuit(bell_qubit0,bell_qubit1)
BellCircuit.h(0)
BellCircuit.cx(0,1)
BellCircuit.draw('mpl')
```

Out[750…

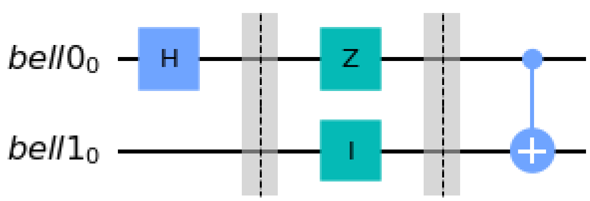


2. Add the error gates

In [751…

```
bell_qubit0 = QuantumRegister(1,'bell0')
bell_qubit1 = QuantumRegister(1,'bell1')
BellCircuit_error = QuantumCircuit(bell_qubit0,bell_qubit1)

# Set the probability in percentage
[p_error_x,p_error_z]=[30,30]  # only integer is allowed. 30 means 30% error probability

BellCircuit_error.h(0)
BellCircuit_error.barrier()
# Randomly (follow the preset probability) add an error gate to each logical qubits
error_gate(BellCircuit_error,[bell_qubit0,bell_qubit1],p_error_x,p_error_z)
BellCircuit_error.barrier()
BellCircuit_error.cx(0,1)

BellCircuit_error.draw('mpl')
```

Out[751…



3. Design the error correction circuit

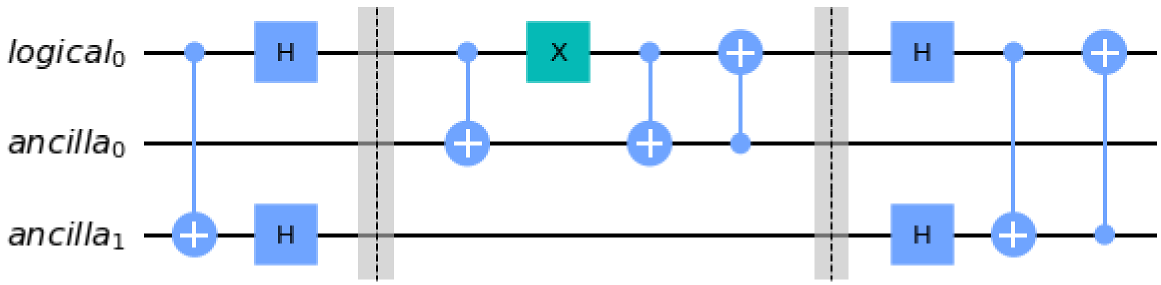
For now, let's assume the error occurs only on the logical qubit (bell0 and bell1), not on the ancillary qubits that are used to detect and correct errors. For each logical qubit, I encode it with an error-correction code, which is a combination of the bit-flip code and sign-flip code. Since we assume the error occurs only on the logical qubits, this code is a simplified version of Shor code. Now let's build the error-correction circuit for a logical qubit (in an arbitrary initial state) first and I will explain why it works.

As is shown in the circuit below, logical0 is the qubit that matters and that errors will act on. The ancilla0 and 1 are two ancilary qubits that will help detect and correct error.

In [755…

```
logical_qubit = QuantumRegister(1,'logical')
qubits_repeat = QuantumRegister(2,'ancilla')
# set the error probability
[p_error_x,p_error_z]=[30,30]

Circuit_ErrorCorrection = QuantumCircuit(logical_qubit,qubits_repeat)
# apply the error correction code together with the error gate
error_correction_simple(Circuit_ErrorCorrection, logical_qubit,qubits_repeat,p_error_x,p_error_z)

Circuit_ErrorCorrection.draw('mpl')
```

Out[755…



This circuit is a combination of bit-flip code and sign-flip code. Now let's discuss why it works. Assume we have an arbitrary single logical qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and two ancillary qubits both in state $|0\rangle$. The initial state of the whole system is therefore $|\phi_0\rangle = |\psi\rangle|00\rangle$. Right before the error occurs, the systems state has evolved into:

$$|\phi_1\rangle = 1/\sqrt{2}\big(\alpha|00+\rangle + \alpha|11+\rangle + \beta|00-\rangle - \beta|11-\rangle\big)$$

Let's consider two senarios:

1. A bit-flip error occurs on the logical qubit:

This means right after the error gate, the state becomes $|\phi_2\rangle = 1/\sqrt{2}\big(\alpha|10+\rangle + \alpha|01+\rangle + \beta|10-\rangle - \beta|01-\rangle\big)$. Then at the end of the circuit, we can get the final state: $|\phi_3\rangle = |\psi\rangle|10\rangle$, which means we have corrected the error that occurs on $|\psi\rangle$. And the ancilla0 qubit being 1 is an indication of a bit-flip error occurred.

1. A sign-flip error occurs on the logical qubit:

This means right after the error gate, the state becomes $|\phi_2'\rangle = 1/\sqrt{2}(\alpha|00+\rangle - \alpha|11+\rangle + \beta|00-\rangle + \beta|11-\rangle)$. Then at the end of the circuit, we can get the final state: $|\phi_3'\rangle = |\psi\rangle|01\rangle$, which means we have corrected the error that occurs on $|\psi\rangle$. And the ancilla1 qubit being 1 will indicate a sign-flip error occurred.

Let's integrate the error-correction circuit with the circuit that generates the Bell state and run the simulations.

4. Simulate mutiple times and visulize the results

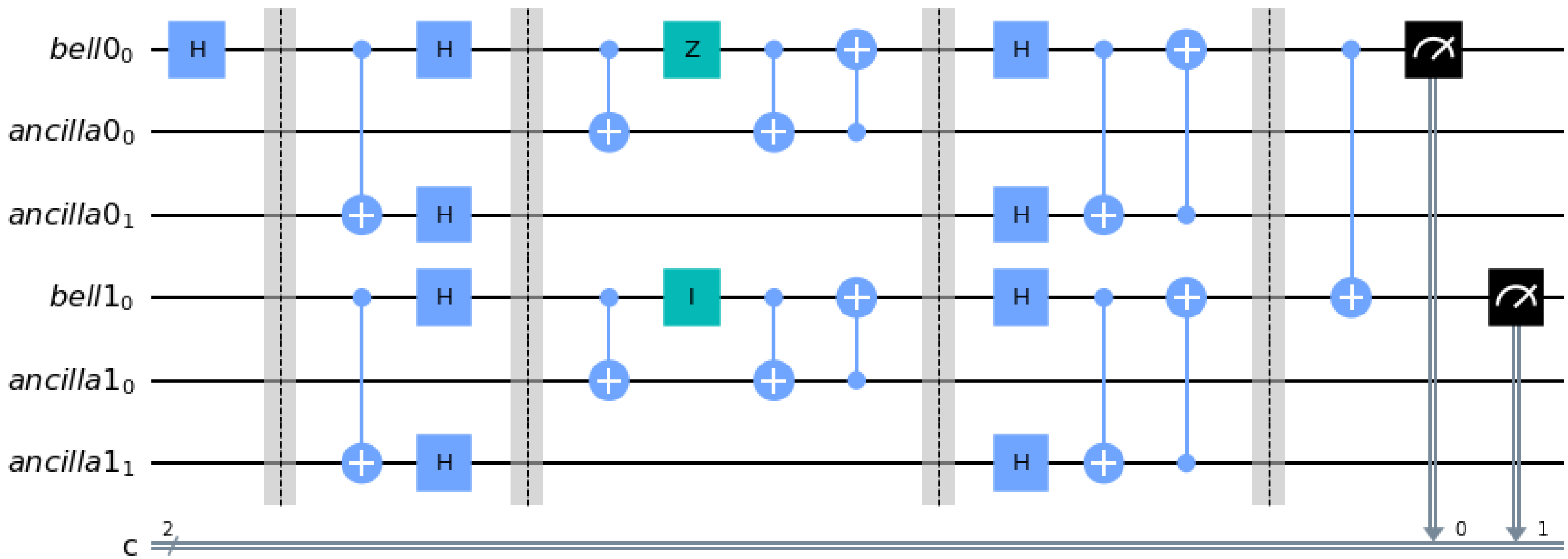
In [756…

```
# Set the simulator and number of simulations
simulator = Aer.get_backend('qasm_simulator')
# number of simulations
N=1000
# to store the results
data = []

bell_qubit0 = QuantumRegister(1,'bell0')
bell_qubit0_repeat = QuantumRegister(2,'ancilla0')
bell_qubit1 = QuantumRegister(1,'bell1')
bell_qubit1_repeat = QuantumRegister(2,'ancilla1')
cl = ClassicalRegister(2,'c')

BellCircuit_ShorErrorCorrection.draw('mpl')

for x in range(N):
    BellCircuit_ErrorCorrection = QuantumCircuit(bell_qubit0,bell_qubit0_repeat,bell_qubit1,bell_qubit1_repeat,
    BellCircuit_ErrorCorrection.h(0)
    BellCircuit_ErrorCorrection.barrier()
    error_correction_simple(BellCircuit_ErrorCorrection, bell_qubit0,bell_qubit0_repeat,p_error_x,p_error_z)
    error_correction_simple(BellCircuit_ErrorCorrection, bell_qubit1,bell_qubit1_repeat,p_error_x,p_error_z)
    BellCircuit_ErrorCorrection.barrier()
    BellCircuit_ErrorCorrection.cx(0,3)
    BellCircuit_ErrorCorrection.measure([0,3], [0,1])
    result = execute(BellCircuit_ErrorCorrection, simulator, shots=1, memory=True).result()
    data.extend(result.get_memory(BellCircuit_ErrorCorrection))
```

In [758…

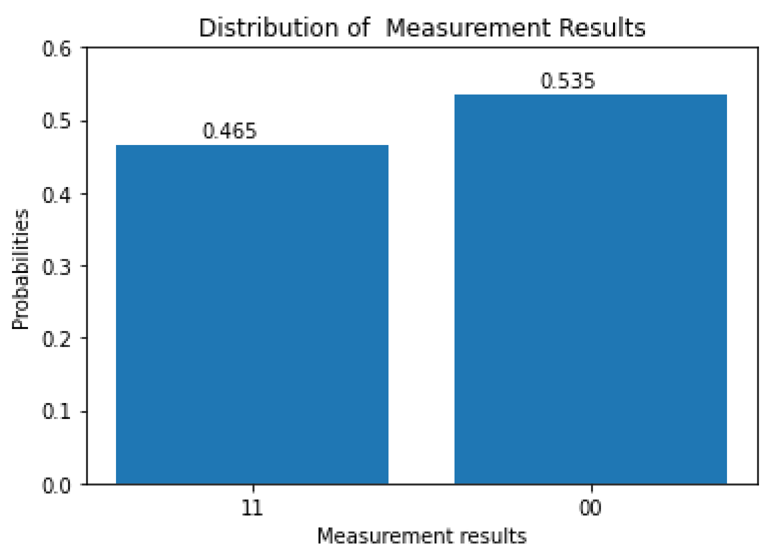
```
# Show the circuit of the last run as an example
BellCircuit_ErrorCorrection.draw('mpl')
```

Out[758…



In [759…

```
# Visualize the results
plot_data(data)
```



As we can see, the results (00 and 11) and their distribution (close to 1:1) are in line with the expectations of measuring a Bell state, even when the probability of error X and Z are as high as 30%, respectively. Actually, no matter what the error probability is, the circuit defined in function error_correction_simple can always successfully correct the error. This is becasue the error only occurs on logical qubits.

A more general and realistic case is that both the logical qubit and the ancillary qubits will experience error, becasue usually they pass through the noisy channel together. This brings us to the next section.

A more general case

Now we will assume all qubits, including logical and ancillary qubits, will experience error (For simplicity, we assume they have the same probability distribution in the types of errors.). This is what Shor code is about and we are going to implement Shor code with circuit.

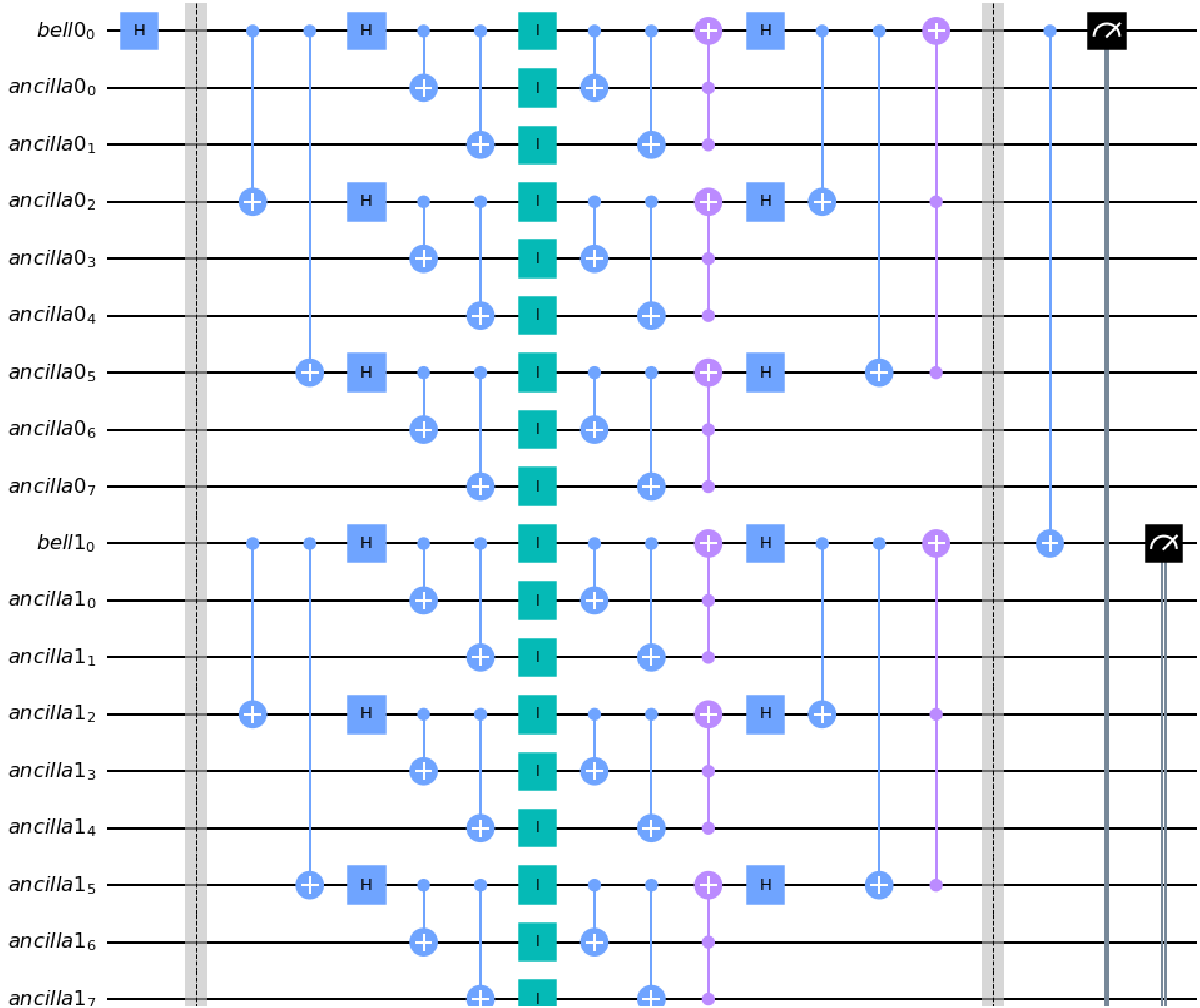
In [765…

```
simulator = Aer.get_backend('qasm_simulator')
# number of simulations
N=1000
data = []

bell_qubit0 = QuantumRegister(1,'bell0')
bell_qubit0_repeat = QuantumRegister(8,'ancilla0')
bell_qubit1 = QuantumRegister(1,'bell1')
bell_qubit1_repeat = QuantumRegister(8,'ancilla1')
cl = ClassicalRegister(2,'c')

[p_error_x,p_error_z]=[3,3]

for x in range(N):

    BellCircuit_ShorErrorCorrection = QuantumCircuit(bell_qubit0,bell_qubit0_repeat,bell_qubit1,bell_qubit1_rep
    BellCircuit_ShorErrorCorrection.h(0)
    BellCircuit_ShorErrorCorrection.barrier()
    error_correction_shor(BellCircuit_ShorErrorCorrection, bell_qubit0,bell_qubit0_repeat,p_error_x,p_error_z)
    error_correction_shor(BellCircuit_ShorErrorCorrection, bell_qubit1,bell_qubit1_repeat,p_error_x,p_error_z)
    BellCircuit_ShorErrorCorrection.barrier()
    BellCircuit_ShorErrorCorrection.cx(0,9)
    BellCircuit_ShorErrorCorrection.measure([0,9], [0,1])
    result = execute(BellCircuit_ShorErrorCorrection, simulator, shots=1, memory=True).result()
    data.extend(result.get_memory(BellCircuit_ShorErrorCorrection))
```

In [766…

```
# Show the circuit of the last run
BellCircuit_ShorErrorCorrection.draw('mpl')
```

Out[766…


bell0₀
ancilla0₀
ancilla0₁
ancilla0₂
ancilla0₃
ancilla0₄
ancilla0₅
ancilla0₆
ancilla0₇
bell1₀
ancilla1₀
ancilla1₁
ancilla1₂
ancilla1₃
ancilla1₄
ancilla1₅
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H
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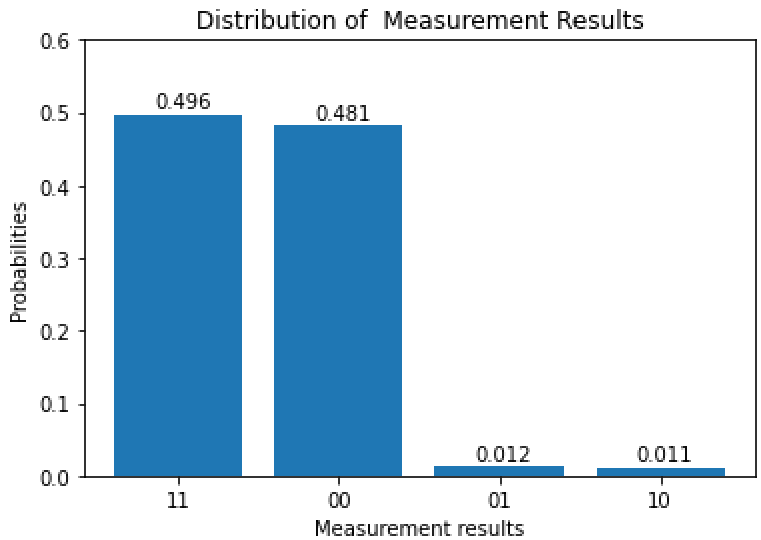
In [767…

```
# Visualize the results
plot_data(data)
```



As we can see, except for 00 and 11 (which still dominant in the results), we also measured 01 and 10. This is because Shor code is based on the assumption that the probablity of error is very low so that at most one qubit (out of the group of a logical qubit and its ancillas) will experience the error. However, it is still possible that more than one error will occur. When that happens, not all errors can be fixed, so we will get the result of 01 and 10 which indicates that we fail to generate Bell state. The lower the probability of the error, the higher chance that we will successfully correct the error and generate Bell state. For example, let's lower p_error_x and p_error_z to 1% and simulate again.

In [768…

```
simulator = Aer.get_backend('qasm_simulator')
# number of simulations
N=1000
data = []

bell_qubit0 = QuantumRegister(1,'bell0')
bell_qubit0_repeat = QuantumRegister(8,'ancilla0')
bell_qubit1 = QuantumRegister(1,'bell1')
bell_qubit1_repeat = QuantumRegister(8,'ancilla1')
```

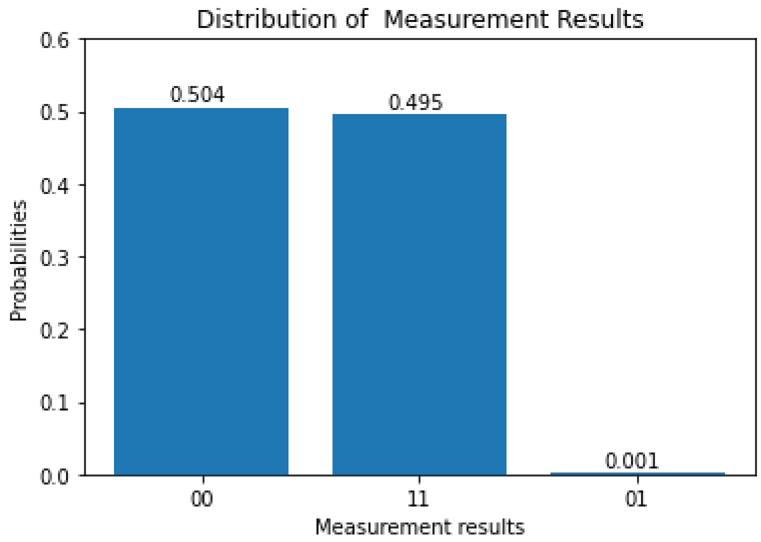
```
cl = ClassicalRegister(2,'c')

[p_error_x,p_error_z]=[1,1]

for x in range(N):

    BellCircuit_ShorErrorCorrection = QuantumCircuit(bell_qubit0,bell_qubit0_repeat,bell_qubit1,bell_qubit1_rep
    BellCircuit_ShorErrorCorrection.h(0)
    BellCircuit_ShorErrorCorrection.barrier()
    error_correction_shor(BellCircuit_ShorErrorCorrection, bell_qubit0,bell_qubit0_repeat,p_error_x,p_error_z)
    error_correction_shor(BellCircuit_ShorErrorCorrection, bell_qubit1,bell_qubit1_repeat,p_error_x,p_error_z)
    BellCircuit_ShorErrorCorrection.barrier()
    BellCircuit_ShorErrorCorrection.cx(0,9)
    BellCircuit_ShorErrorCorrection.measure([0,9], [0,1])
    result = execute(BellCircuit_ShorErrorCorrection, simulator, shots=1, memory=True).result()
    data.extend(result.get_memory(BellCircuit_ShorErrorCorrection))

plot_data(data)
```



The probability that we get the wrong output (indicated by 10 and 01) becomes much lower.

Conclusion

1. If the error gate only acts on the logical qubit, then only two ancillary qubits are needed to detect and correct the error of each logical bit. Also, the error correction circuit will 100% work, no matter how high the error probability is.

2. If the error gates apply on all the qubits involved (logical and ancillary qubits), then we need Shor code to correct the error, which means there will be 8 ancillary qubits for each logical qubit. Also, the correctness is not guaranteed. The higher the probability of the error gate, the lower chance we will get the correct logical qubit. If the error probability is low enough so that only one error will occur, then it's guaranteed to obtain the right result.

Reference

https://en.wikipedia.org/wiki/Quantum_error_correction