

Problem Set 12, Dec 9, 2021 (Solutions to SVD Theory Questions)

Problem 1 (How to compute U and S efficiently):

In class, we saw that solving the eigenvector/value problem for the matrix $\mathbf{X}\mathbf{X}^\top$ gives us a way to compute U and S . But in some instances $D \gg N$. In those cases, is there a way to accomplish this computation more efficiently?

Solution 1:

Consider the $N \times N$ matrix $\mathbf{X}^\top \mathbf{X}$. Similarly as before, we have

$$\mathbf{X}^\top \mathbf{X} = \mathbf{V} \mathbf{S}^\top \mathbf{S} \mathbf{V}^\top.$$

Let \mathbf{v}_i , $i = 1, \dots, D$, denote the columns of \mathbf{V} . Then

$$\mathbf{X}^\top \mathbf{X} \mathbf{v}_j = \mathbf{V} \mathbf{S}^\top \mathbf{S} \mathbf{V}^\top \mathbf{v}_j = s_j^2 \mathbf{v}_j. \quad (1)$$

So we see that the j -th column of \mathbf{V} is an eigenvector of $\mathbf{X}^\top \mathbf{X}$ with eigenvalue s_j^2 . Therefore, solving the eigenvector/value problem for the matrix $\mathbf{X}^\top \mathbf{X}$ gives us a way to compute \mathbf{V} and \mathbf{S} .

Now multiply the identity (1) from the left by the matrix \mathbf{X} . We get

$$\mathbf{X} \mathbf{X}^\top (\mathbf{X} \mathbf{v}_j) = s_j^2 (\mathbf{X} \mathbf{v}_j).$$

We see therefore that $\mathbf{u}_j = \mathbf{X} \mathbf{v}_j$ and so we can compute the desired eigenvectors \mathbf{u}_j from the eigenvectors \mathbf{v}_j without having to solve the $D \times D$ eigenvector/value problem.

Problem 2 (Positive semi-definite):

Show that if \mathbf{X} is a $N \times N$ symmetric matrix then the SVD has the form $\mathbf{U} \mathbf{S} \mathbf{U}^\top$, where \mathbf{U} is a $N \times N$ unitary matrix and \mathbf{S} is a $N \times N$ diagonal matrix with non-necessarily positive entries (i.e can be zeros). Show that if \mathbf{X} is positive semi-definite, then the diagonal entries of \mathbf{S} are positive.

Solution 2:

Consider $\mathbf{A} = \mathbf{X} \mathbf{X}^\top$ and $\mathbf{B} = \mathbf{X}^\top \mathbf{X}$. By the SVD we know that $\mathbf{X} = \mathbf{U} \mathbf{S} \mathbf{V}^\top$. As we discussed in the course, the columns of \mathbf{U} are eigenvectors of the first matrix and the columns of \mathbf{V} are eigenvectors of the second matrix. But note that $\mathbf{A} = \mathbf{B}$ since \mathbf{X} is symmetric. Hence the eigenspace associated to each distinct eigenvalue of \mathbf{A} is equal to the eigenspace associated to the same eigenvalue of \mathbf{B} .

Set $\mathbf{U} = \mathbf{V}$ and let the columns of \mathbf{U} be eigenvectors of \mathbf{A} . Compute $\mathbf{U}^\top \mathbf{X} \mathbf{V}$. This gives us a diagonal matrix which we can define to be \mathbf{S} . It's entries are not necessarily positive.

If the matrix is in addition positive semi-definite then the diagonal entries of \mathbf{S} must in fact must be positive – multiplying the matrix from the left by \mathbf{u}_j^\top and from the right by \mathbf{u}_j gives s_j which must be positive if the quadratic form given by the matrix is assumed to be positive-definite.