1. (Simon & Blume - Exercício 8.1) Sejam

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \end{bmatrix},\ B = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -1 & 2 \end{bmatrix},\ C = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix},\ D = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},\ E = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(a) Compute cada uma das seguintes matrizes se ela for definida:

$A+B$, $B+C$, $B-A$, $3B$, AB, CA, CE, EC, B^T, $(CA)^T$, A^TC^T.

1a $A+B$ $\begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -1 \\ 4 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 0 \\ 4 & -2 & 4 \end{bmatrix}$

$B+C$ = Indefinido

$B-A$ $\begin{bmatrix} 0 & 1 & -1 \\ 4 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -2 & -2 \\ 4 & 0 & 0 \end{bmatrix}$

$3 \cdot B$ $\begin{bmatrix} 0 & 1 & -1 \\ 4 & -1 & 2 \end{bmatrix} \cdot 3 = \begin{bmatrix} 0 & 3 & -3 \\ 12 & -3 & 6 \end{bmatrix}$

$\underset{2\times 3}{A} \cdot \underset{2\times 3}{B}$ = Indefinido

$$C \cdot A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \end{bmatrix} \Rightarrow$$

$2\times 2 \quad 2\times 3$

$1\cdot 2 + 2\cdot 0 = 2$ $\quad CA_{11}$

$3\cdot 1 + 2\cdot -1 = 1$ $\quad 12$

$1\cdot 1 + 2\cdot 2 = 5$ $\quad 13$

$3\cdot 2 - 1\cdot 0 = 6$ $\quad 21$

$3\cdot 3 - 1\cdot -1 = 10$ $\quad 22$

$3\cdot 1 - 1\cdot 2 = 1$ $\quad 23$

$$\begin{bmatrix} 2 & 1 & 5 \\ 6 & 10 & 1 \end{bmatrix}$$

$$C \cdot E = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow$$

$2\times 2 \quad 2\times 1$

$1\cdot 1 + 2\cdot -1 = -1$ $\quad 11$

$3\cdot 1 - 1\cdot -1 = 4$ $\quad 21$

$$CE = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$E \cdot C$ = Indefinido

$2\times 1 \quad 2\times 2$

$$B^T = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 4 \\ 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$(CA)^T = \begin{bmatrix} 2 & 1 & 5 \\ 6 & 10 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 6 \\ 1 & 10 \\ 5 & 1 \end{bmatrix}$$

$$A^T C^T = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \Rightarrow$$

$$\underset{3\times 2}{\begin{bmatrix} 2 & 0 \\ 3 & -1 \\ 1 & 2 \end{bmatrix}} \cdot \underset{2\times 2}{\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}} \Rightarrow$$

$2\cdot 1 + 0\cdot 2 = 2 \quad {}_{11}$

$2\cdot 3 + 0\cdot -1 = 6 \quad {}_{12}$

$3\cdot 1 + -1\cdot 2 = 1 \quad {}_{21}$

$3\cdot 3 + -1\cdot -1 = 10 \quad {}_{22}$

$1\cdot 1 + 2\cdot 2 = 5 \quad {}_{31}$

$1\cdot 3 + 2\cdot -1 = 1 \quad {}_{32}$

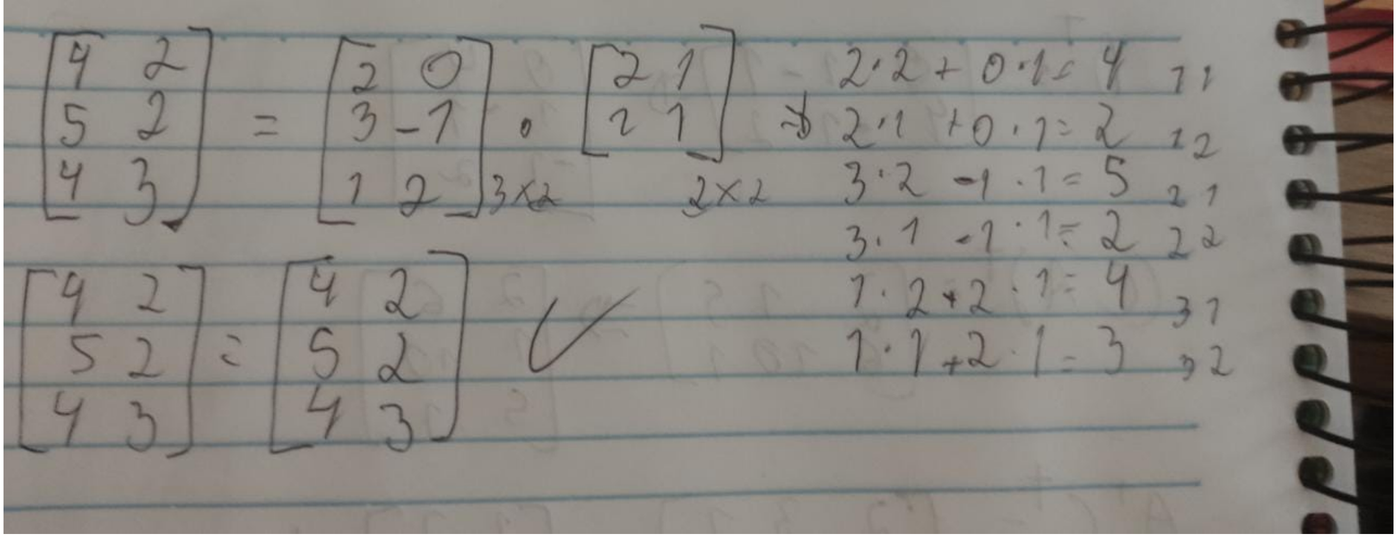
$$\begin{bmatrix} 2 & 6 \\ 1 & 10 \\ 5 & 1 \end{bmatrix} = A^T C^T$$

(b) Verifique que $(DA)^T = A^T D^T$.

$$1b \quad (DA)^T = A^T D^T \Rightarrow$$

$$\underset{2\times 2 \;\; 2\times 3}{DA} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 4 \\ 2 & 2 & 3 \end{bmatrix}$$

$$(DA)^T = \begin{bmatrix} 4 & 2 \\ 5 & 2 \\ 4 & 3 \end{bmatrix} = \underset{3\times 2}{\begin{bmatrix} 2 & 0 \\ 3 & -1 \\ 1 & 2 \end{bmatrix}} \cdot \underset{2\times 2}{\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}} \Rightarrow$$

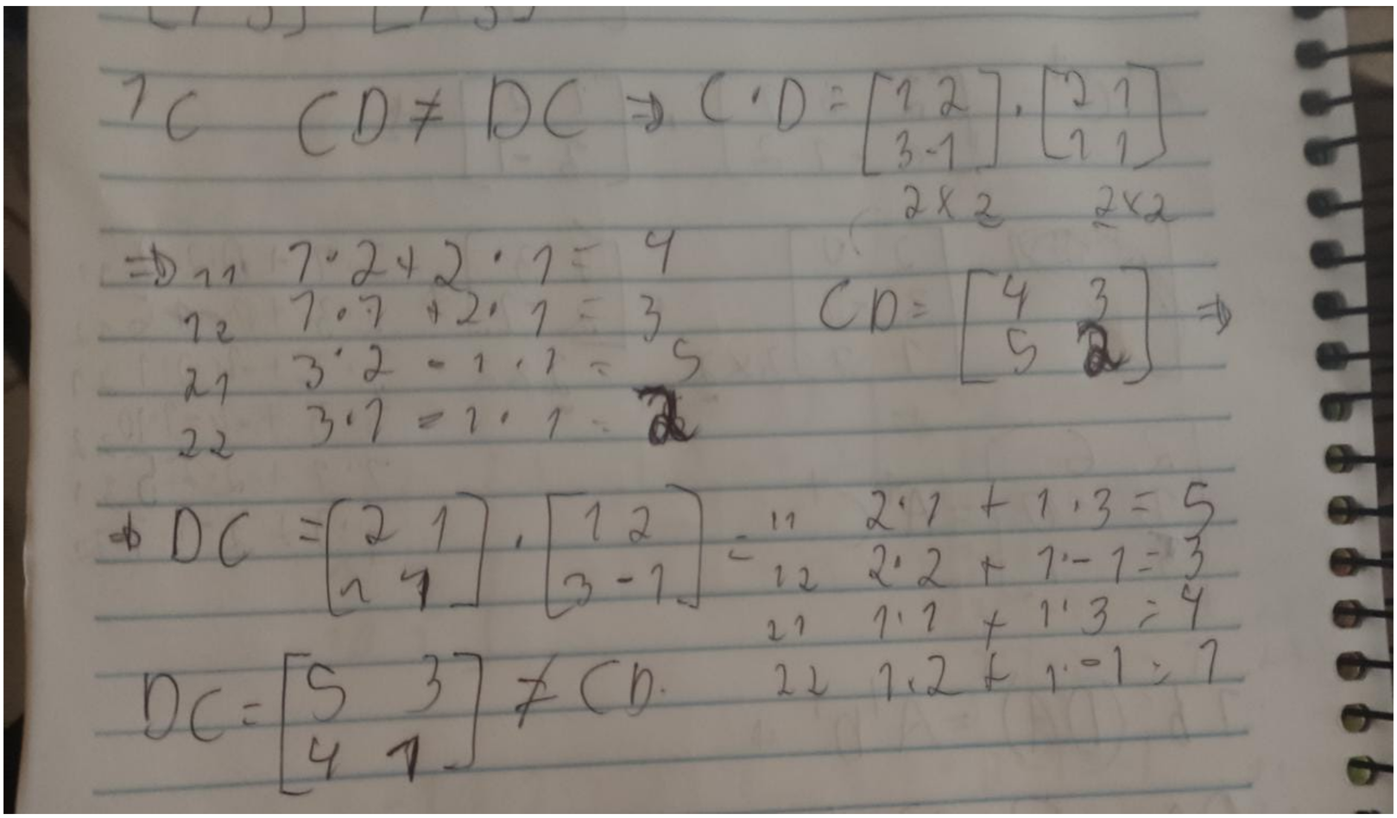


$$\begin{bmatrix} 4 & 2 \\ 5 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 3 & -1 \\ 1 & 2 \end{bmatrix}_{3\times 2} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}_{2\times 2} \Rightarrow$$

$2\cdot 2 + 0\cdot 1 = 4$ 11
$2\cdot 1 + 0\cdot 1 = 2$ 12
$3\cdot 2 - 1\cdot 1 = 5$ 21
$3\cdot 1 - 1\cdot 1 = 2$ 22
$1\cdot 2 + 2\cdot 1 = 4$ 31
$1\cdot 1 + 2\cdot 1 = 3$ 32

$$\begin{bmatrix} 4 & 2 \\ 5 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 5 & 2 \\ 4 & 3 \end{bmatrix} \checkmark$$

(c) Verifique que $CD \neq DC$.



1c $CD \neq DC \Rightarrow C\cdot D = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}_{2\times 2} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}_{2\times 2}$

$\Rightarrow$ 11 $1\cdot 2 + 2\cdot 1 = 4$
12 $1\cdot 1 + 2\cdot 1 = 3$
21 $3\cdot 2 - 1\cdot 1 = 5$
22 $3\cdot 1 - 1\cdot 1 = 2$

$CD = \begin{bmatrix} 4 & 3 \\ 5 & 2 \end{bmatrix} \Rightarrow$

$\Rightarrow DC = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} =$
11 $2\cdot 1 + 1\cdot 3 = 5$
12 $2\cdot 2 + 1\cdot -1 = 3$
21 $1\cdot 1 + 1\cdot 3 = 4$
22 $1\cdot 2 + 1\cdot -1 = 1$

$DC = \begin{bmatrix} 5 & 3 \\ 4 & 1 \end{bmatrix} \neq CD.$

2. (Simon & Blume - Exercício 8.5) Às vezes acontece de $AB = BA$. Cheque isso para $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ e $B = \begin{bmatrix} 3 & -4 \\ -4 & 3 \end{bmatrix}$.

2

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}_{2\times 2} \quad B = \begin{bmatrix} 3 & -4 \\ -4 & 3 \end{bmatrix}_{2\times 2} \Rightarrow AB: \begin{array}{l} 2\cdot 3 + 1\cdot -4 = 2 \\ 2\cdot -4 + 1\cdot 3 = -5 \\ 1\cdot 3 + 2\cdot -4 = -5 \\ 1\cdot -4 + 2\cdot 3 = 2 \end{array}$$

$$AB = \begin{bmatrix} 2 & -5 \\ -5 & 2 \end{bmatrix} \quad B\cdot A = \begin{array}{l} 3\cdot 2 - 4\cdot 1 = 2 \\ 3\cdot 1 - 4\cdot 2 = -5 \\ -4\cdot 2 + 3\cdot 1 = -5 \\ -4\cdot 1 + 3\cdot 2 = 2 \end{array} \quad BA = \begin{bmatrix} 2 & -5 \\ -5 & 2 \end{bmatrix}$$

$$AB = BA$$

3. (Simon & Blume - Exercício 8.7) Mostre que $A = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}$ e $B = \begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix}$ são idempotentes.

3

Seg Ter Qua Qui Sex Sáb Dom / /

$$A = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \quad A\cdot A = \begin{array}{ll} 11 & -1\cdot -1 + 2\cdot -1 = -1 \\ 12 & -1\cdot 2 + 2\cdot 2 = 2 \\ 21 & -1\cdot -1 + 2\cdot -1 = -1 \\ 22 & -1\cdot 2 + 2\cdot 2 = 2 \end{array}$$

//

$$A^2 = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix} \quad B\cdot B = \begin{array}{ll} 11 & 3\cdot 3 + 6\cdot -1 = 3 \\ 12 & 3\cdot 6 + 6\cdot -2 = 6 \\ 21 & -1\cdot 3 - 2\cdot -1 = -1 \\ 22 & -1\cdot 6 - 2\cdot -2 = -2 \end{array}$$

//

$$B^2 = \begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix}$$

4. (Simon & Blume - Exercício 8.9) Quantas matrizes de permutação $n \times n$ existem?

n!

5. (Simon & Blume - Exercício 8.14)

(a) Prove a seguinte afirmação. Se P é uma matriz de permutação $m \times m$ e A é $m \times n$, então PA é a matriz A com suas linhas permutadas de acordo com P. Se $p_{ij} = 1$, então a i-ésima linha de PA será a j-ésima linha de A.

$$P = \begin{array}{c} \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \\ \begin{matrix} C_1 & & C_i & & C_j & & C_n \end{matrix} \end{array} \begin{matrix} L_1 \\ \\ L_i \\ \\ L_j \\ \\ L_m \end{matrix}$$

$$A = \begin{array}{c} \begin{bmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots \\ a_{i1} & \cdots & a_{ii} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots \\ a_{j1} & \cdots & a_{ji} & \cdots & a_{jj} & \cdots & a_{in} \\ \vdots \\ a_{m1} & \cdots & a_{mi} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \\ \begin{matrix} C_1 & & C_i & & C_j & & C_n \end{matrix} \end{array} \begin{matrix} L_1 \\ \\ L_i \\ \\ L_j \\ \\ L_m \end{matrix}$$

$$p_{ij} = 1 \quad \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

$P_{ij=1}A \Rightarrow$

$_{11}P_a = a_{11}$ $\quad$ $_{i1}P_a = a_{j1}$ $\quad$ $_{j1}P_a = a_{i1}$ $\quad$ $_{m1}P_a = a_{m1}$

$\vdots$

$_{1i}P_a = a_{1i}$ $\quad$ $_{ii}P_a = a_{ji}$ $\quad$ $_{ji}P_a = a_{ii}$ $\quad$ $_{mi}P_a = a_{mi}$

$\vdots$

$_{1j}P_a = a_{1j}$ $\quad$ $_{ij}P_a = a_{jj}$ $\quad$ $_{jj}P_a = a_{ij}$ $\quad$ $_{mj}P_a = a_{mj}$

$\vdots$

$_{1n}P_a = a_{1n}$ $\quad$ $_{in}P_a = a_{jn}$ $\quad$ $_{jn}P_a = a_{in}$ $\quad$ $_{mn}P_a = a_{mn}$

$$P_{ij=1}A = \begin{array}{cc} \begin{bmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ a_{j1} & \cdots & a_{ji} & \cdots & e_{jj} & \cdots & a_{jn} \\ \vdots & & & \ddots & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ii} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mi} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} & \begin{array}{c} L_1 \\ \\ L_i \\ \\ L_j \\ \\ L_m \end{array} \\ \begin{array}{ccccccc} C_1 & & C_i & & C_j & & C_n \end{array} & \end{array}$$

(b) Enuncie e prove uma afirmação similar sobre a permutação de colunas pela multiplicação AP.

$$P = \begin{array}{c} \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \\ \begin{array}{cccc} C_1 & C_i & C_j & C_n \end{array} \end{array} \begin{array}{c} L_1 \\ \\ L_i \\ \\ L_j \\ \\ L_m \end{array}$$

$$A = \begin{array}{c} \begin{bmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & & & & & \\ a_{i1} & \cdots & a_{ii} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & & & & & \\ a_{j1} & \cdots & a_{ji} & \cdots & a_{jj} & \cdots & a_{in} \\ a_{m1} & \cdots & a_{mi} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \\ \begin{array}{cccc} C_1 & C_i & C_j & C_n \end{array} \end{array} \begin{array}{c} L_1 \\ \\ L_i \\ \\ L_j \\ L_m \end{array}$$

$\cancel{PA}$ $$P_{ij} = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

$11ap = a11 \;\cdots\; 1iap = a1j \;\cdots\; 1jap = a1i \;\cdots\; 1nap = a1n$

$i1ap = ai1 \;\cdots\; iiap = aij \;\cdots\; ijap = aii \;\cdots\; inap = ain$

$j1ap = aj1 \;\cdots\; jiap = ajj \;\cdots\; jjap = aji \;\cdots\; jnap = ajn$

$m1ap = am1 \;\cdots\; miap = amj \;\cdots\; mjap = ami \;\cdots\; mnap = amn$

$$A\,P_{ij=1} = \begin{bmatrix} a11 & \cdots & a1j & \cdots & a1i & \cdots & a1n \\ ai1 & \cdots & aij & \cdots & aii & \cdots & ain \\ aj1 & \cdots & ajj & \cdots & aji & \cdots & ajn \\ am1 & \cdots & amj & \cdots & ami & \cdots & amn \end{bmatrix}$$

Se $pij = 1$ então a i-ésima coluna de AP será a j-ésima coluna de A.

6. Seja X uma matriz $n \times k$.

   (a) Mostre que a matriz $P \equiv X(X'X)^{-1}X'$ é idempotente. Lembre-se que P é a chamada *matriz de projeção* do estimador de mínimos quadrados ordinários.

$$P = X(X'X)^{-1}X'$$

$$P^2 = X\underbrace{(X'X)^{-1}X'X}(X'X)^{-1}X'$$

$$P^2 = X(X'X)^{-1}X'$$

(b) Mostre que a matriz $M \equiv I - P$, onde I é a matrix identidade $n \times n$, é idempotente. Lembre-se que M é a chamada *matriz aniquiladora* do estimador de mínimos quadrados ordinários.

$$M = I - P \qquad P^2 = P$$

$$M^2 = (I - P)(I - P)$$

$$M^2 = I - P - P + P^2$$

$$M^2 = I - 2P + P$$

$$M^2 = I - P$$

$$M^2 = M \Rightarrow I - P = I - P.$$