In [1]:
```
from sympy import *
```

Vetores (Vectors)

In [2]:
```
vector = Matrix([1,2,3])
vector
```

Out[2]:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

In [3]:
```
x = symbols('x')
y = symbols('y')
z = symbols('z')
w = symbols('w')
```

Igualdade de Vetores (Vector Equality)

In [4]:
```
vector1 = Matrix([1-x, 2])
vector2 = Matrix([4, y-1])
solve(vector1-vector2)
```

Out[4]:
```
{x: -3, y: 3}
```

Subtração de Vetores (Vector Subtraction)

In [5]:
```
vector1 = Matrix([-1, 2])
vector2 = Matrix([2, 1])
```

In [6]:
```
vector2 - vector1
```

Out[6]:

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

In [7]:
```
vector1 - vector2
```

Out[7]:

$$\begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

Note que as subtrações resultaram em vetores opostos.
Note that the subtractions resulted in oposite vectors.

Soma de Vetores (Sum of Vectors)

In [8]:
```
vector1 = Matrix([10, 5])
vector2 = Matrix([-2, 4])
```

In [9]:
```python
vector1+vector2
```

Out[9]:
$$\begin{bmatrix} 8 \\ 9 \end{bmatrix}$$

In [10]:
```python
vector2+vector1
```

Out[10]:
$$\begin{bmatrix} 8 \\ 9 \end{bmatrix}$$

Multiplicação de Vetores - Produto Escalar (Vector Multiplication - Scalar Product or Dot Product)

(x1,y1,z1,w1) . (x2,y2,z2,w2) = (x1x2+y1y2+w1w2+z1z2) = Scalar (single number)

In [11]:
```python
vector1 = Matrix([1, 2])
vector2 = Matrix([5, 3])
```

In [12]:
```python
vector1.dot(vector2)
```

Out[12]: 11

In [13]:
```python
vector2.dot(vector1)
```

Out[13]: 11

In [14]:
```python
x1 = symbols('x1')
y1 = symbols('y1')
x2 = symbols('x2')
y2 = symbols('y2')
vector1 = Matrix([x1,y1])
vector2 = Matrix([x2, y2])
```

In [15]:
```python
vector2.dot(vector1)
```

Out[15]: $x_1x_2 + y_1y_2$

Multiplicação por Escalar (Scalar Multiplication)

In [16]:
```python
vector1 = Matrix([x, y])
alpha = symbols('alpha')
```

In [17]:
```python
scalar_vector1 = alpha*vector1
```

In [18]:
```python
scalar_vector1
```

Out[18]:
$$\begin{bmatrix} \alpha x \\ \alpha y \end{bmatrix}$$

In [19]:
```
vector2 = Matrix([5,-3])
3*vector2
```

Out[19]:
$$\begin{bmatrix} 15 \\ -9 \end{bmatrix}$$

Módulo de um Vetor (Magnitude or Modulus of a Vector)

In [20]:
```
vector1 = Matrix([5,6])
```

In [21]:
```
vector1.norm()
```

Out[21]: $\sqrt{61}$

In [22]:
```
N(vector1.norm())
```

Out[22]: 7.81024967590665

In [23]:
```
vector2 = Matrix([2,-3])
```

In [24]:
```
vector2.norm()
```

Out[24]: $\sqrt{13}$

In [25]:
```
N(vector2.norm())
```

Out[25]: 3.60555127546399

In [26]:
```
vector3 = Matrix([1,4,5])
```

In [27]:
```
vector3.norm()
```

Out[27]: $\sqrt{42}$

In [28]:
```
N(vector3.norm())
```

Out[28]: 6.48074069840786

In [29]:
```
vector4 = Matrix([x,y,z,w])
vector4.norm()
```

Out[29]:
$$\sqrt{\left|w\right|^{2} + \left|x\right|^{2} + \left|y\right|^{2} + \left|z\right|^{2}}$$

Ângulo (Teta θ) entre 2 Vetores (Angle (Theta θ) between 2

vectors)

Formula: $\frac{u*v}{|u|*|v|}$ = cos θ

In [30]:
```
vector1 = Matrix([2,2])
vector2 = Matrix([0, -2])
```

In [31]:
```
line1 = Line(Point(0,0), Point(vector1[0], vector1[1]))
line2 = Line(Point(0,0), Point(vector2[0], vector2[1]))
```

In [32]:
```
line1.angle_between(line2)
```

Out[32]: $\frac{3\pi}{4}$

In [33]:
```
from math import degrees
degrees(line1.angle_between(line2))
```

Out[33]:
```
135.0
```

In [34]:
```
vector1 = Matrix([1,2,3])
vector2 = Matrix([-5, 1, 1])
line1 = Line(Point3D(0,0,0), Point3D(vector1[0], vector1[1], vector1[2]))
line2 = Line(Point3D(0,0,0), Point3D(vector2[0], vector2[1], vector2[2]))
```

In [35]:
```
line1.angle_between(line2)
```

Out[35]: $\frac{\pi}{2}$

In [36]:
```
degrees(line1.angle_between(line2))
```

Out[36]:
```
90.0
```

Vetores Colineares ou Paralelos (Collinear or Parallel Vectors)

u = (x1, y1)

v = (x2, y2)

Se $\frac{x1}{x2}$ = $\frac{y1}{y2}$ então u e v são vetores colineares

If $\frac{x1}{x2}$ = $\frac{y1}{y2}$ then u and v are collinear vectors

In [37]:
```
vector1 = Matrix([-3, 2])
vector2 = Matrix([6,-4])

scalar1 = vector1[0]/vector1[1]
scalar2 = vector2[0]/vector2[1]

scalar1==scalar2
```

Out[37]:
```
True
```

In [38]:
```
Point.is_collinear(Point(0,0), Point(vector1[0], vector1[1]), Point(vector2[0], vector2[1]
```
Out[38]:
```
True
```
In [39]:
```
vector1 = Matrix([1, 2])
vector2 = Matrix([2, 3])

scalar1 = vector1[0]/vector1[1]
scalar2 = vector2[0]/vector2[1]

scalar1==scalar2
```
Out[39]:
```
False
```
In [40]:
```
Point.is_collinear(Point(0,0), Point(vector1[0], vector1[1]), Point(vector2[0], vector2[1]
```
Out[40]:
```
False
```

Vetores Ortogonais (Orthogonal Vectors)

Se o produto escalar de 2 vetores for 0, assumimos que existe ortogonalidade entre os 2.
If the scalar product between 2 vectors is 0, we assume that there is orthogonality between the 2.

In [41]:
```
vector1 = Matrix([1, 2])
vector2 = Matrix([-2, 1])

vector1.dot(vector2)
```
Out[41]: 0

In [42]:
```
vector3 = Matrix([2, y, -3])
vector4 = Matrix([y-1, 2, 4])

vector3.dot(vector4)
```
Out[42]: $4y - 14$

In [43]:
```
orth_scalar = vector3.dot(vector4)
solve(orth_scalar)
```
Out[43]:
```
[7/2]
```
In [44]:
```
orth_scalar.subs(y, 7/2)
```
Out[44]: 0

Vetores Perpendiculáres (Perpendicular Vectors)

Seguem a mesma lógica dos vetores ortogonais, com a diferença que ortogonalidade é aplicada em 2 dimensões, enquanto perpendicularidade pode ser identificada em múltiplas dimensões

They follow the same logic of the orthogonal vectors, with the difference that orthogonality is applied in 2 dimensions, while perpendicularity can be identified in multiple dimensions.

In [45]:
```
vector1 = Matrix([0, 1, -1])
vector2 = Matrix([2, y, 3*y-2])
```

In [46]:
```
perp_scalar = vector1.dot(vector2)
solve(perp_scalar)
```

Out[46]: [1]

In [47]:
```
perp_scalar = vector1.dot(vector1+vector2)
solve(perp_scalar)
```

Out[47]: [2]

A diferença é que dois segmentos de reta que formam um ângulo reto entre si serão sempre ortogonais mas, só serão perpendiculares se eles se tocarem em algum ponto. Desta forma teremos que todos os segmentos de reta perpendiculares serão sempre ortogonais mas, nem todos os ortogonais serão perpendiculares necessariamente!

The difference is that two line segments that form a right angle to each other will always be orthogonal, but they will only be perpendicular if they touch at some point. In this way, we will have that all perpendicular line segments will always be orthogonal, but not all orthogonal ones will necessarily be perpendicular!

Projeção Ortogonal (Orthogonal Projection)

proj(u)w = $\frac{w*u}{u*u}$ * u

Projeção do vetor w sobre o vetor u Projection of the vector w on vector u

In [48]:
```
k = symbols('k')
```

In [49]:
```
vector1 = Matrix([0, 3, 5])
vector2 = Matrix([0, 1, 1])
proj12 = (vector2.dot(vector1)/vector1.dot(vector1))*vector1
proj12
```

Out[49]:

$$\begin{bmatrix} 0 \\ \frac{12}{17} \\ \frac{20}{17} \end{bmatrix}$$

Exercício: Considere os vetores u=(0, -1, 1) e w=(2, 0, k) de R3. Determine todos os valores de K de modo que a projeção ortogonal do vetor w sobre o vetor u seja igual ao vetor -5u

Exercise: Consider the vectors u=(0, -1, 1) and w=(2, 0, k) in R3. Find all the values of K that satisfy the orthogonal projection of w over u where proj(u)w = -5u

In [50]:
```
vector3 = Matrix([0, -1, 1])
vector4 = Matrix([2, 0, k])
```

```
proj34 = ((vector4.dot(vector3)/vector3.dot(vector3))*vector3) + 5*vector3
proj34
```

Out[50]:

$$\begin{bmatrix} 0 \\ -\frac{k}{2} - 5 \\ \frac{k}{2} + 5 \end{bmatrix}$$

In [51]:

```
solve(proj34)
```

Out[51]:

```
{k: -10}
```

Combinação Linear (Linear Combination)

v = α *v1* + β v2 + γ * v3

In [52]:

```
alpha, beta, gamma, delta = symbols('alpha beta gamma delta')
```

In [53]:

```
v = Matrix([1,2,4])
v1 = Matrix([1,2,1])
v2 = Matrix([1,0,2])
v3 = Matrix([1,1,0])
```

In [54]:

```
equation = alpha*v1 + beta*v2 + gamma*v3 - v
solve(equation)
```

Out[54]:

```
{alpha: 2, beta: 1, gamma: -2}
```

Sistema Linear (Linear System)

Passo (Step) 1:

α + β + γ = 1 } L1

2α + 0 + γ = 2 } L2

α + 2β + 0 = 4 } L3

Possível (Possible) Passo (Step) 2:

α + β + γ = 1 } L1

2α + 0 + γ = 2 } L2

0 + β - γ = 3 } L3 - L1

Não excluiu nenhuma variável, reinicia

Did'nt excluded any variable, restart

Possível (Possible) Passo (Step) 2:

α + β + γ = 1 } L1

α - β + 0 = 1 } L2 - L1

α + 2β + 0 = 4 } L3

Não excluiu nenhuma variável, reinicia

Did'nt excluded any variable, restart

Passo (Step) 2:

$-\alpha$ + β + 0 = -1 } L1 - L2 } L1^

2α + 0 + γ = 2 } L2

α + 2β + 0 = 4 } L3

Passo (Step) 3: $-\alpha$ + β + 0 = -1 } L1^

2α + 0 + γ = 2 } L2

0 + 3β + 0 = 3 } L3 + L1^ } L3^

3β = 3 --> β = 1

$-\alpha$ + 1 + 0 = -1 --> $-\alpha$= -2 --> α = 2

2α + 0 + γ = 2 --> 4 + 0 + γ = 2 --> γ = -2

Exercício: Seja v1 = (1, -3, 2) e v2 = (2, 4, -1), dois vetores em R3. Determine o valor de k para que o vetor u = (-1, k, -7) seja combinação linear de v1 e v2

Exercise: Let v1 = (1, -3, 2) and v2 = (2, 4, -1), two vectors in R3. Determine the value of k so that the vector u = (-1, k, -7) is a linear combination of v1 and v2

In [55]:
```
v1 = Matrix([1,-3,2])
v2 = Matrix([2,4,-1])
u = Matrix([-1,k,-7])
```

In [56]:
```
solve(alpha*v1+beta*v2-u)
```

Out[56]:
```
{alpha: -3, beta: 1, k: 13}
```

1:

α *1* + β 2 = -1 } L1

α *-3* + β 4 = k } L2

α *2* + β -1 = -7 } L3

2:

α *1* + β 2 = -1 } L1

α *-3* + β 4 = k } L2

0 + β *-5 = -5 } L3 - 2*L1

β *-5 = -5 -->* β *= 1*

α 1 + β *2 = -1 -->* α *+ 2 = -1 -->* α *= -3*

α -3 + β * 4 = k --> 9 + 4 = k --> k=13

Exercício: Seja P2 o espaço vetorial dos polinômios de grau 2. Considere polinômios v1 = -x^2 -6x v2 = x^2 - 3x - 1 e v3 = -3x^2 + 2. Mostre que v1 pode ser escrito como uma combinação linear dos vetores {v2, v3}

Exercise: Let P2 be the vectorial space of the degree 2 polynomials. Consider polynomials v1 = -x^2 -6x v2 = x^2 - 3x - 1 e v3 = -3x^2 + 2. Show that v1 can be written as a linear combination of the vectors {v2, v3}

In [57]:

```
v1 = Matrix([-x**2, -6*x, 0])
v2 = Matrix([x**2, -3*x, -1])
v3 = Matrix([-x**2, 0, 2])

solve(alpha*v2+beta*v3-v1)
```

Out[57]:

```
[{alpha: 2*beta, x: 0}, {alpha: 2, beta: 1, x: 0}]
```

1:

$x^2\alpha$ - 3$x^2\beta$ = -x^2 } L1

-3xα + 0 = -6x } L2

-1α + 2β= 0 } L3

2:

α - 3β = -x^2 } L1/x^2

-3α + 0 = -6 } L2/x

-1α + 2β= 0 } L3

-3α + 0 = -6 --> α = 2

-1α + 2β= 0 --> -2 + 2β= 0 --> 2β = 2 --> β = 1

Dependência e Independência Linear (Linear Dependence and Independence)

{v1,v2} --> αv1 + βv2 = 0

Se (if):

α = 0

and

β = 0

então (then):

{v1,v2} : Linearmente Independente (Linearly Independent)

senão (else):

{v1,v2} : Linearmente Dependente (Linearly Dependent)

Exercício: Seja V = R2, temos os vetores v1=(1,2), v2=(0,1), e v3=(-1,1). Verifique se o conjunto desses vetores são LD ou LI.

Exercise: Let V = R2, given the vectors v1=(1,2), v2=(0,1), and v3=(-1,1). Verify if the set of this vectors is LD or LI.

In [58]:

```
v1 = Matrix([1,2])
v2 = Matrix([0,1])
v3 = Matrix([-1,1])
```

In [59]:

```
equation = alpha*v1+beta*v2+gamma*v3
equation
```

Out[59]:

$$\begin{bmatrix} \alpha - \gamma \\ 2\alpha + \beta + \gamma \end{bmatrix}$$

In [60]:

```
solve(equation)
```

Out[60]:

```
{alpha: gamma, beta: -3*gamma}
```

$\alpha - \gamma$ = 0 } L1

$2\alpha + \beta + \gamma$ = 0} L2

--> $\alpha - \gamma$ = 0 --> $\alpha = \gamma$

--> $2\alpha + \beta + \gamma$ = 0 --> $2\gamma + \beta + \gamma$ = 0 --> β = -3γ

--> $\alpha = \gamma$, β = -3γ, $\gamma = \gamma$ --> $\gamma = \alpha \neq \beta$ --> LD

Exercício: Considere agora somente os vetores v1=(1,2) e v2=(0,1). Verifique se os vetores {v1,v2} são LD ou LI.

Exercise: Now consider only the vectors v1=(1,2) and v2=(0,1). Verify if the vectors {v1,v2} are LD or LI

In [61]:

```
v1 = Matrix([1,2])
v2 = Matrix([0,1])
```

In [62]:

```
equation = alpha*v1+beta*v2
solve(equation)
```

Out[62]:

```
{alpha: 0, beta: 0}
```

In [63]:

```
equation
```

Out[63]:

$$\begin{bmatrix} \alpha \\ 2\alpha + \beta \end{bmatrix}$$

α = 0 } L1

$2\alpha + \beta$ = 0 } L2

-->

α = 0 } L1

β = 0 } L2 - 2*L1

--> α = 0 = β = 0 --> {v1, v2} --> LI

Exercício: Considere o conjunto de vetores {(1,0,-1),(1,1,0),(k,1,-1)}. Determine a condição a ser satisfeita por k para que os vetores do conjunto sejam linearmente independentes.

Exercise: Considerar the set of vectors {(1,0,-1),(1,1,0),(k,1,-1)}. Determine the condition that k must satisfy so that the vectors of the set are linearly independent.

In [64]:

```
vector1 = Matrix([1,0,-1])
vector2 = Matrix([1,1,0])
vector3 = Matrix([k,1,-1])
```

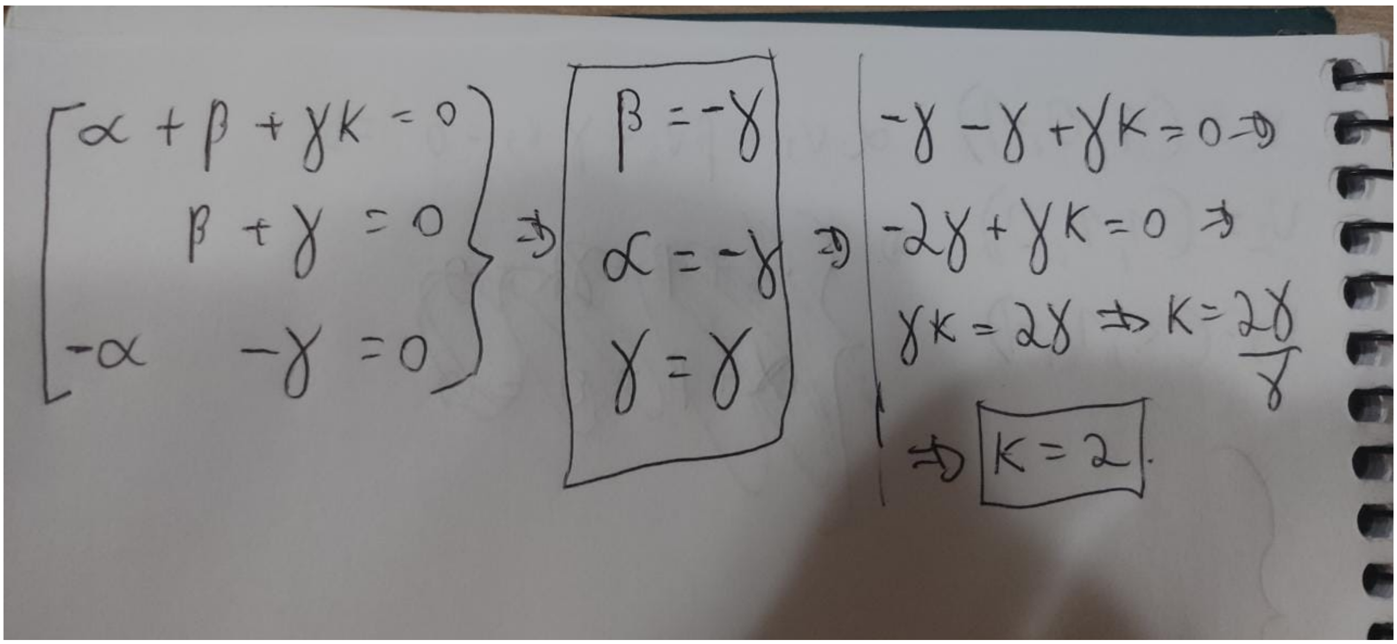
In [65]:

```
equation = alpha*vector1 + beta*vector2 + gamma*vector3
equation
```

Out[65]:

$$\begin{bmatrix} \alpha + \beta + \gamma k \\ \beta + \gamma \\ -\alpha - \gamma \end{bmatrix}$$

In [66]:
```
solve(equation,k)
```
Out[66]:
```
{k: (-alpha - beta)/gamma}
```
In [67]:
```
solve(equation.subs(k,(-alpha - beta)/gamma))
```
Out[67]:
```
{alpha: -gamma, beta: -gamma}
```
Logo, a condição para que o conjunto de vetores seja linearmente independente é que k seja diferente de (-alpha - beta)/gamma
Therefore, the condition for the set of vector to be linearly independent is that k be different then (-alpha - beta)/gamma



Exercício: Responda, justificando, se cada um dos conjuntos abaixo é LI ou LD.
a)
{| 1 2 | | 3 6|}
{|-4 -3 | , |-12 -9|}
b)
{(-1,-2,0,3), (2,-1,0,0), (1,0,0,0)}
c)
{1 + 2x - x^2, 2 - x + 3x^2, 3 - 4x + 7x^2}

Exercise: Answer, justifying, if each one of the sets below is LI or LD.
a)
{| 1 2 | | 3 6|}
{|-4 -3 | , |-12 -9|}
b)
{(-1,-2,0,3), (2,-1,0,0), (1,0,0,0)}
c)
{1 + 2x - x^2, 2 - x + 3x^2, 3 - 4x + 7x^2}

In [68]:
```
# a)
matrix_1 = Matrix([[1,2], [-4,-3]])
matrix_2 = Matrix([[3,6], [-12,-9]])
```

```python
3*matrix_1 == matrix_2
```

Out[68]:

```
True
```

--> matrix_1 == 3 * matrix_2 --> LD

In [69]:

```python
# b)
vector_1 = Matrix([-1,-2,0,3])
vector_2 = Matrix([2,-1,0,0])
vector_3 = Matrix([1,0,0,0])

equation = alpha*vector_1+beta*vector_2+gamma*vector_3
equation
```

Out[69]:

$$\begin{bmatrix} -\alpha + 2\beta + \gamma \\ -2\alpha - \beta \\ 0 \\ 3\alpha \end{bmatrix}$$

In [70]:

```python
solve(equation)
```

Out[70]:

```
{alpha: 0, beta: 0, gamma: 0}
```

$$(-1,-2,0,3) \quad (2,-1,0,0) \quad (1,0,0,0) \Rightarrow \begin{cases} -\alpha+2\beta+1\gamma=0 \\ -2\alpha-\beta=0 \\ //////=0 \\ 3\alpha=0 \end{cases} \Rightarrow$$

$$3\alpha=0 \Rightarrow -2(0)-\beta=0 \Rightarrow 0-\beta=0 \Rightarrow \beta=0 \Rightarrow -(0)+2(0)+\gamma=0 \Rightarrow \gamma=0 \Rightarrow LI$$

$$\alpha=\beta=\gamma=0$$

In [71]:

```python
# c)
vector1 = Matrix([-x**2, 2*x, 1])
vector2 = Matrix([3*x**2, -x, 2])
vector3 = Matrix([7*x**2, -4*x, 3])
```

In [72]:

```python
equation = alpha*vector1+beta*vector2+gamma*vector3
equation
```

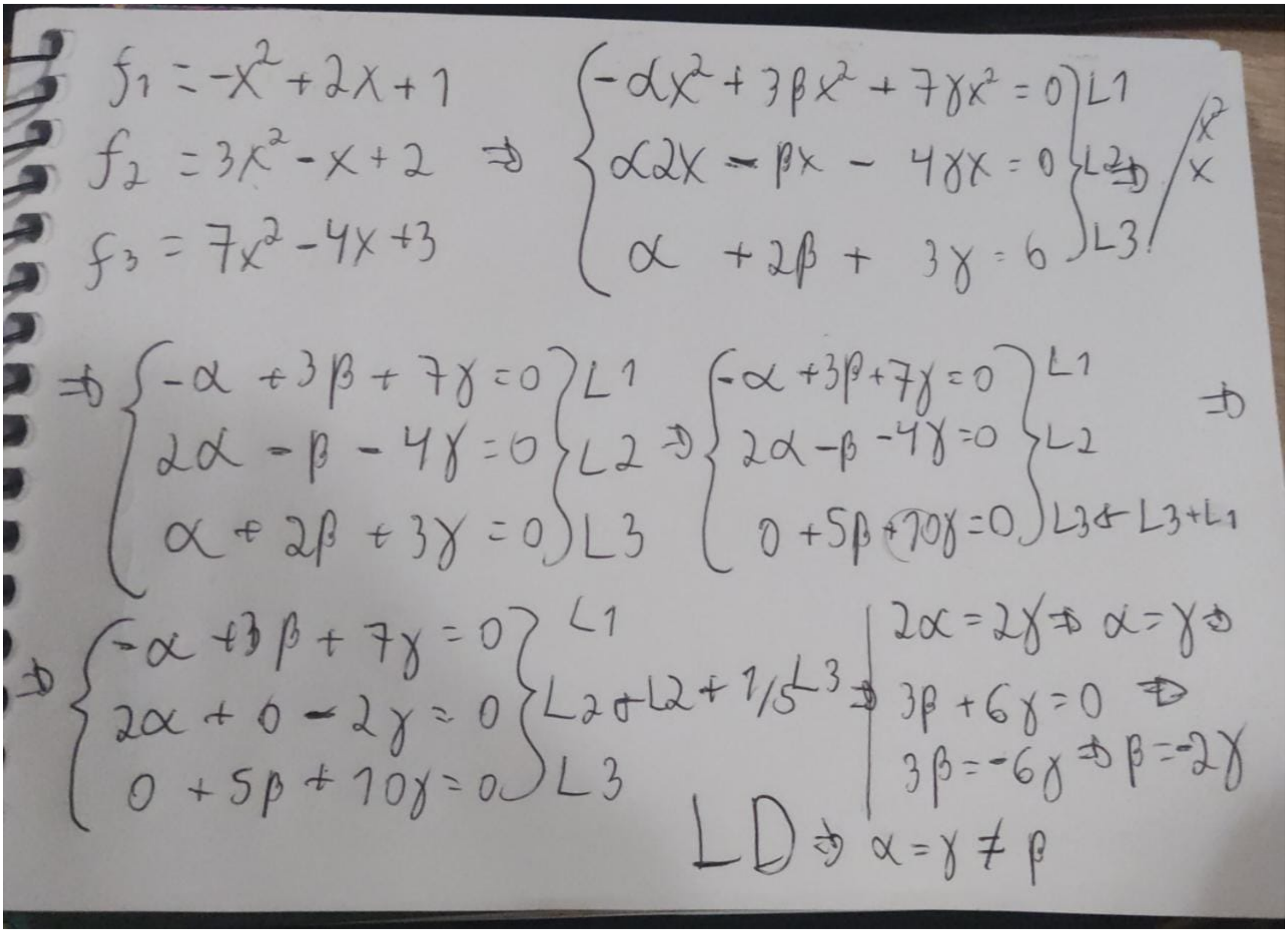
Out[72]:

$$\begin{bmatrix} -\alpha x^2 + 3\beta x^2 + 7\gamma x^2 \\ 2\alpha x - \beta x - 4\gamma x \\ \alpha + 2\beta + 3\gamma \end{bmatrix}$$

In [73]:

```python
try:
    solve(equation)
except:
    print("LD")
```

LD



$$f_1 = -x^2 + 2x + 1$$
$$f_2 = 3x^2 - x + 2 \Rightarrow \begin{cases} -\alpha x^2 + 3\beta x^2 + 7\gamma x^2 = 0 & L1 \\ \alpha 2x - \beta x - 4\gamma x = 0 & L2 \Rightarrow /x^2, /x \\ \alpha + 2\beta + 3\gamma = 0 & L3 \end{cases}$$
$$f_3 = 7x^2 - 4x + 3$$

$$\Rightarrow \begin{cases} -\alpha + 3\beta + 7\gamma = 0 & L1 \\ 2\alpha - \beta - 4\gamma = 0 & L2 \\ \alpha + 2\beta + 3\gamma = 0 & L3 \end{cases} \Rightarrow \begin{cases} -\alpha + 3\beta + 7\gamma = 0 & L1 \\ 2\alpha - \beta - 4\gamma = 0 & L2 \\ 0 + 5\beta + 10\gamma = 0 & L3 \leftarrow L3 + L1 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} -\alpha + 3\beta + 7\gamma = 0 & L1 \\ 2\alpha + 0 - 2\gamma = 0 & L2 \leftarrow L2 + 1/5\, L3 \\ 0 + 5\beta + 10\gamma = 0 & L3 \end{cases} \Rightarrow$$

$$2\alpha = 2\gamma \Rightarrow \alpha = \gamma \Rightarrow$$
$$3\beta + 6\gamma = 0 \Rightarrow$$
$$3\beta = -6\gamma \Rightarrow \beta = -2\gamma$$

$$LD \Rightarrow \alpha = \gamma \neq \beta$$

Espaço e Subespaço Vetorial (Vector Space and Subspace)

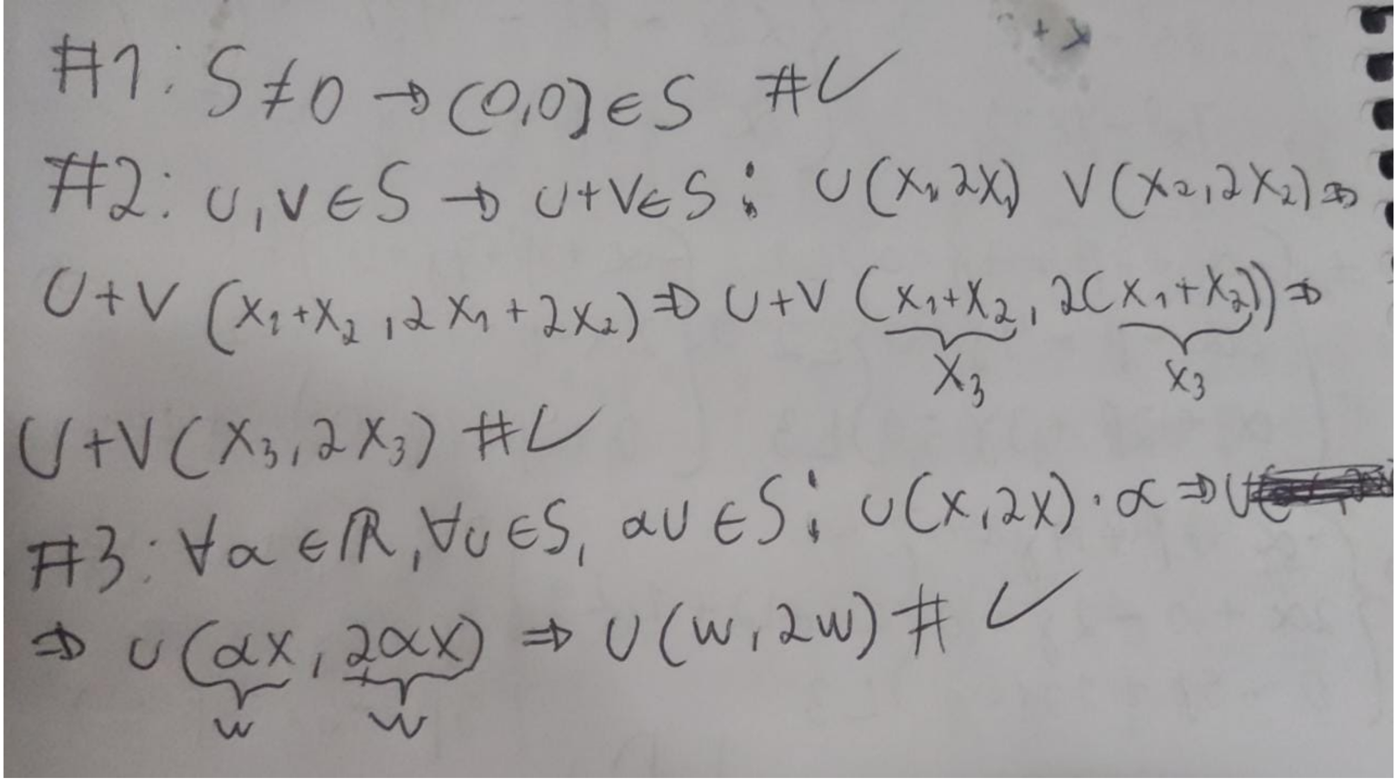
#1: S $\neq$ 0 --> (0,0) $\in$ S

#2: u, v $\in$ S --> u+v $\in$ S

#3: $\forall \alpha \in$ R, $\forall$u $\in$ S, αu $\in$ S

Exercício: V = R2 e S = {(x,y) $\in$ R2; y = 2x}. S é um subespaço vetorial de V?

Exercise: V = R2 and S = {(x,y) $\in$ R2; y = 2x}. Is S a vectorial subspace of V?



S é um subespaço vetorial de V pois respeita as 3 regras.
S is a vectorial subspace of V because it respects the 3 rules.

Exercício: Considere o conjunto B = {v1, v2}, onde v1 = (1,2,3) e v2 = (-5, 1, 1). Determine o espaço gerado pelos vetores v1 e v2 de B.
Exercise: Consider the set B = {v1, v2}, where v1 = (1,2,3) and v2 = (-5, 1, 1). Determine the space generated by the vectors v1 and v2 in B.

In [74]:
```
v1 = Matrix([1, 2, 3])
v2 = Matrix([-5, 1, 1])
vB = Matrix([x, y, z])
```

In [75]:
```
equation = (alpha*v1+beta*v2)-vB
equation
```

Out[75]:
$$\begin{bmatrix} \alpha - 5\beta - x \\ 2\alpha + \beta - y \\ 3\alpha + \beta - z \end{bmatrix}$$

In [76]:
```
solve(equation)
```

Out[76]:
```
{alpha: -y + z, beta: 3*y - 2*z, x: -16*y + 11*z}
```

$$V_1 = (1,2,3) \quad V_2 = (-5,1,1) \Rightarrow \alpha V_1 + \beta V_2 = (x,y,z) \Rightarrow \begin{cases} \alpha + (-5)\beta = x & L1 \\ 2\alpha + \beta = y & L2 \\ 3\alpha + \beta = z & L3 \end{cases}$$

$$\Rightarrow \begin{cases} \alpha - 5\beta = x & L1 \\ 0 + 11\beta = y - 2x & L2 \leftarrow L2 - 2L1 \\ 0 + 16\beta = z - 3x & L3 \leftarrow L3 - 3L1 \end{cases} \Rightarrow \beta = \frac{y-2x}{11} = \frac{z-3x}{16}$$

$$\Rightarrow 16(y-2x) = 11(z-3x) \Rightarrow 16y - 32x = 11z - 33x \Rightarrow x = 11z - 16y$$

$$[V_1, V_2] = \{(x,y,z) \in \mathbb{R}^3 \mid x = 11z - 16y\}$$

[v1, v2] = {(x, y, z) ∈ R3 | x = 11z - 16y}

(3.0)3. Responda, justificando, se cada um dos subconjuntos abaixo é um subspaço vetorial do $\mathbb{R}^2$.

(a) $S = \{(x, y)/y = -x\}$.

(b) $S = \{(x, x^2)/x \in \mathbb{R}\}$.

(c) $S = \{(x, y)/x \geq 0\}$.

Exercício: Responda se cada um dos subconjuntos abaixo é um subespapço vetorial de R2.

Exercise: Anwser if each of the subsets below is a vectorial subspace of R2.

S = {(x,y)|y=-x}

S = {(x,x^2)|x ∈ R}

S = {(x,y)|x>=0}

$U=(x_1,-x_1)$ $V=(x_2,-x_2)$ $\Rightarrow U+V=(x_1+x_2,-x_1-x_2)$

$\Rightarrow U+V=(\underbrace{x_1+x_2}_{x_3},-(\underbrace{x_1+x_2}_{x_3})) \Rightarrow U+V=(x_3,-x_3)$ #2 ✓

$U=(x,-x) \Rightarrow U\cdot\alpha \Rightarrow (\underbrace{\alpha x}_{x_2},-\underbrace{\alpha x}_{x_2}) \Rightarrow U\cdot\alpha=(x_2,-x_2)$ #3 ✓

$U=(x_1,x_1^2)$ $V=(x_2,x_2^2)$ $\Rightarrow U+V=(x_1+x_2,x_1^2+x_2^2) \Rightarrow$

#2 X

$V=(x_1,y_1) \mid x \geq 0$ #3 X

Base de um Espaço e Subespaço Vetorial (Basis of a Vector Space and Subspace)

#1 B = LI

#2 B gera (generates) V

Exercício: Verifique se B = {(1,0) and (1,1)} é uma base de R2

Exercise: Verify if B = {(1,0) and (1,1)} is a base of R2

In [77]:
```
vector1 = Matrix([1,0])
vector2 = Matrix([1,1])
```

In [78]:
```
equation1= alpha*vector1+beta*vector2
equation1
```

Out[78]:
$$\begin{bmatrix}\alpha+\beta\\ \beta\end{bmatrix}$$

In [79]:
```
solve(equation1)
#1 Ok
```

Out[79]:
```
{alpha: 0, beta: 0}
```

In [80]:
```
subspace = Matrix([x,y])
equation2 = equation1 - subspace
equation2
```

Out[80]:
$$\begin{bmatrix}\alpha+\beta-x\\ \beta-y\end{bmatrix}$$

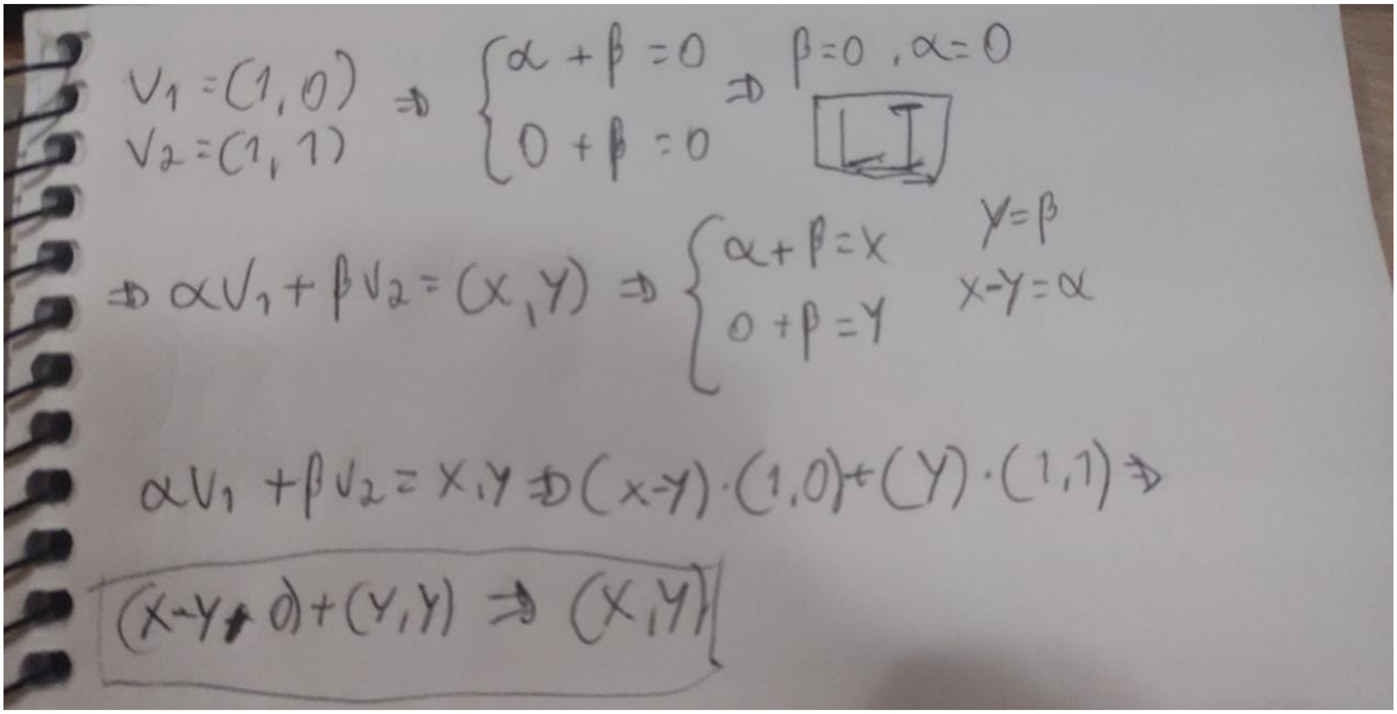
In [81]:

```
solve(equation2)
```

Out[81]: {alpha: x - y, beta: y}

In [82]:
```
equation1.subs(solve(equation2))
#2 Ok
```

Out[82]: $\begin{bmatrix} x \\ y \end{bmatrix}$



Resposta: É uma base

Answer: It's a base

Base Canônica de um Espaço Vetorial (Canonical Base of a Vectorial Space)

Exercício: Verifique se B = {(1,0) e (0,1)} é uma base de R2

Exercise: Verify if B = {(1,0) and (0,1)} is a base of R2

In [83]:
```
#1 B = LI
vector1 = Matrix([1,0])
vector2 = Matrix([0,1])
equation = alpha*vector1+beta*vector2
solve(equation)
```

Out[83]: {alpha: 0, beta: 0}

In [84]:
```
#2 B -> V
vs = Matrix([x,y])
equation2 = equation-vs
equation.subs(solve(equation2))
```

Out[84]: $\begin{bmatrix} x \\ y \end{bmatrix}$

Dimensão de um Espaço Vetorial (Dimension of a Vectorial Space)

1) dim R = 1

2) dim Rn = n

3) dim M2x2 = 4

4) dim Mmxn = m*n

5) dim Pn = n+1

P = polinômio (polynomial)

Exercício: Considere o seguinte subspaço vetorial do R3:

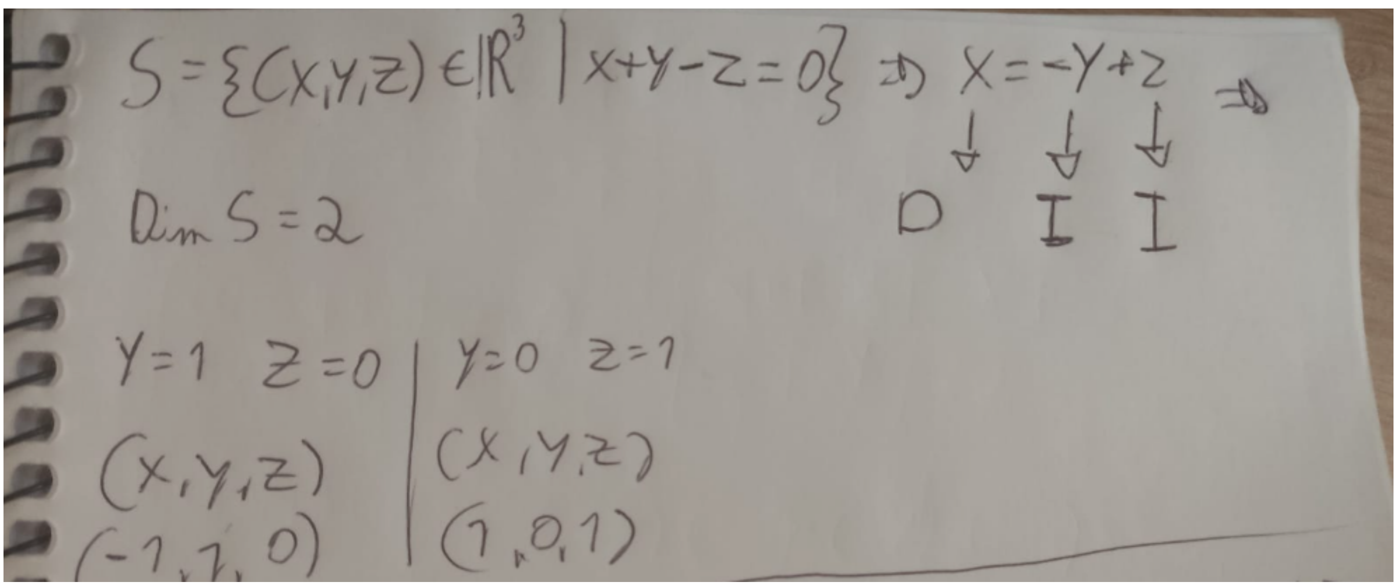
S = {(x,y,z) ∈ R3| x+y-z = 0}

Determinar a dimensão e a uma base B de S.

Exercise: Considerar the following vectorial subspace of R3:

S = {(x,y,z) ∈ R3| x+y-z = 0}

Determine the dimension and the base B of S.



Exercício: Considere o seguinte subspaço vetorial do R4:

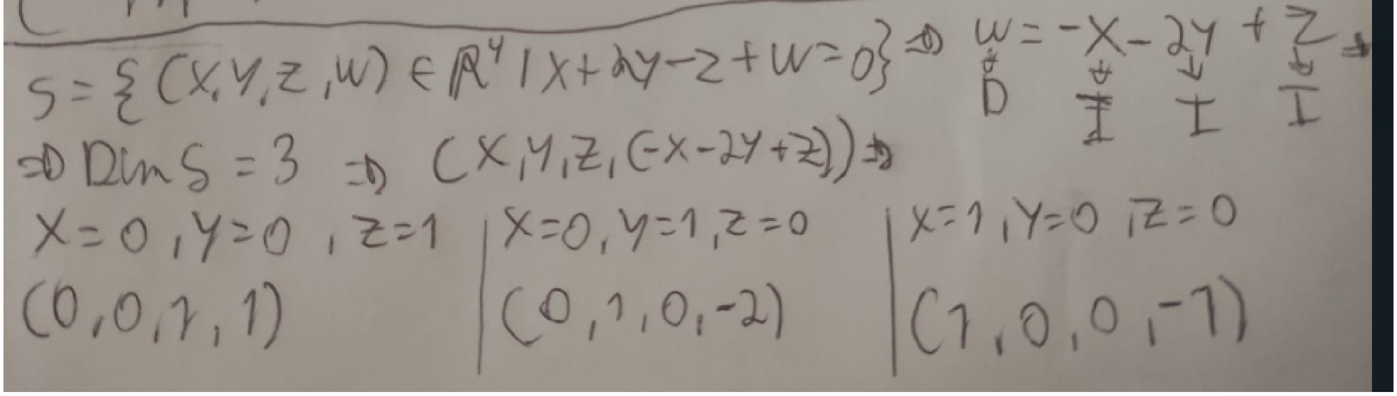
S = {(x,y,z,w) ∈ R4| x+2y-z+w = 0}

Determinar a dimensão e a uma base B de S.

Exercise: Considerar the following vectorial subspace of R4:

S = {(x,y,z,w) ∈ R4| x+2y-z+w = 0}

Determine the dimension and the base B of S.


$S = \{(x,y,z,w) \in \mathbb{R}^4 \mid x+2y-z+w=0\} \Rightarrow w = -x-2y+z \Rightarrow$

$\Rightarrow \operatorname{Dim} S = 3 \Rightarrow (x,y,z,(-x-2y+z)) \Rightarrow$

$x=0, y=0, z=1$ | $x=0, y=1, z=0$ | $x=1, y=0, z=0$

$(0,0,1,1)$ | $(0,1,0,-2)$ | $(1,0,0,-1)$

Bases Ortogonais (Orthogonal Bases)

#1 LI = u * v = 0

Exercício: B = {(1,2,-3);(3,0,1);(1,-5,-3) ⊂ R3 é um conjunto ortogonal?

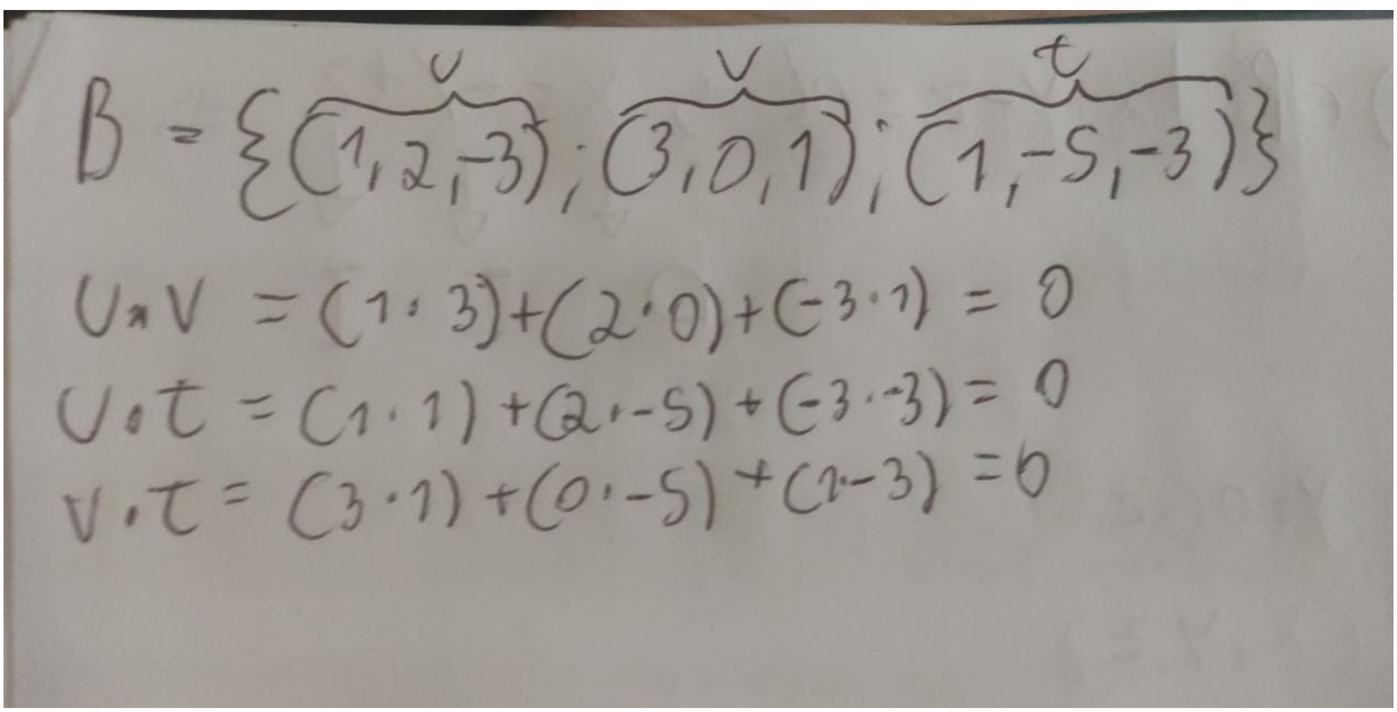
Exercise: B = {(1,2,-3);(3,0,1);(1,-5,-3) ⊂ R3 is a orthogonal set?

In [85]:
```
u = Matrix([1,2,-3])
v = Matrix([3,0,1])
t = Matrix([1,-5,-3])
```

In [86]:
```
print(u.dot(v))
print(u.dot(t))
print(v.dot(t))

print(u.dot(v) == u.dot(t) == v.dot(t) == 0)
```

```
0
0
0
True
```


$B = \{\underbrace{(1,2,-3)}_{u}; \underbrace{(3,0,1)}_{v}; \underbrace{(1,-5,-3)}_{t}\}$

$u * v = (1 \cdot 3) + (2 \cdot 0) + (-3 \cdot 1) = 0$

$u \cdot t = (1 \cdot 1) + (2 \cdot -5) + (-3 \cdot -3) = 0$

$v \cdot t = (3 \cdot 1) + (0 \cdot -5) + (1 \cdot -3) = 0$

Bases Ortonormais (Orthonormal Bases)

#1 Tem que ser Ortogonal (Must be Orthogonal)

#2 Norma = 1 (Norm = 1) ---> u*u=1

Exercício: B = {(1,0);(0,1)} é uma base ortonormal do R2?

Exercise: B = {(1,0);(0,1)} is a orthogonal base of R2?

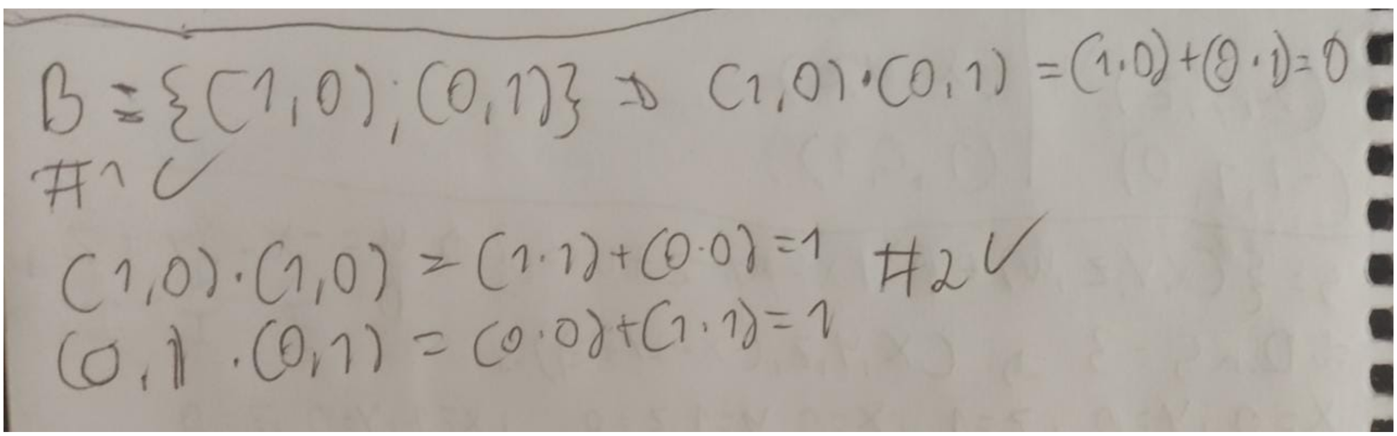
In [87]:
```
vector1 = Matrix([1,0])
vector2 = Matrix([0,1])
```

In [88]:
```
#1
vector1.dot(vector2)
```

Out[88]: 0

In [89]:
```
#2
print(vector1.dot(vector1))
print(vector2.dot(vector2))
```

```
1
1
```



Tranformação de base Ortogonal para Ortonormal (Orthogonal to Orthonormal base transformation)

$\hat{B} = \{\frac{v1}{|v1|}, \frac{v2}{|v2|}, \ldots, \frac{vn}{|vn|}\}$

Exercício: B = {v1, v2, v3}, sendo v1 = (1,1,1), v2 = (-2,1,1) e v3 = (0,-1,1).

Exercise: B = {v1, v2, v3}, where v1 = (1,1,1), v2 = (-2,1,1) and v3 = (0,-1,1).

In [90]:
```
v1 = Matrix([1,1,1])
v2 = Matrix([-2,1,1])
v3 = Matrix([0,-1,1])
```

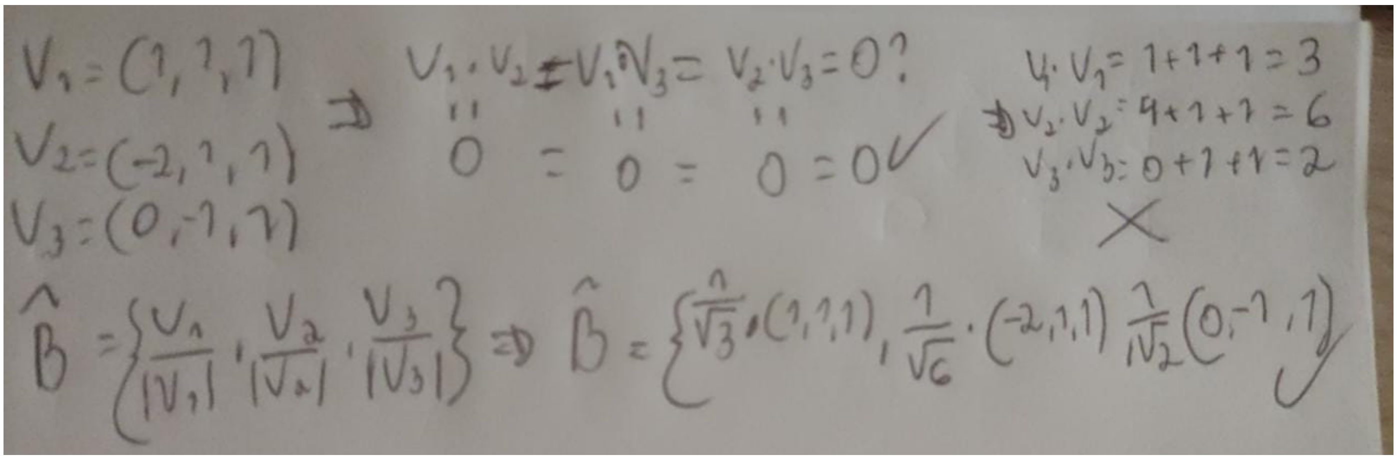
In [91]:
```
# Teste de Ortogonalidade (Orthogonality test)
v1.dot(v2) == v2.dot(v3) == v3.dot(v1) == 0
```

Out[91]: True

In [92]:
```python
# Teste de Ortonormalidade (Orthonormality test)
print(v1.dot(v1))
print(v2.dot(v2))
print(v3.dot(v3))
# Falhou (Failed)
```

```
3
6
2
```

In [93]:
```python
# B_hat
v1_n = v1/v1.norm()
v2_n = v2/v2.norm()
v3_n = v3/v3.norm()
```



Processo de ortogonalização de Gram-Schmidt (Gram-Schmidt ortogonalization process)

B = {v1, v2,..., vn}: base de V, então, existe uma base ortogonal tal que $\hat{B}$ = {w1, w2,..., wn} para V.

w1 = v1

w2 = v2 - ($\frac{v2*w1}{w1*w1}$) *w1*

w3 = v3 - ($\frac{ v3* w2}{ w2 *w2}$) w2 - ($\frac{v3*w1}{w1*w1}$) w1

In [94]:
```python
v1 = Matrix([1,0])
v2 = Matrix([1,1])
```

In [95]:
```python
w1, w2 = GramSchmidt([v1, v2])
print(w1)
print(w2)
```

```
Matrix([[1], [0]])
Matrix([[0], [1]])
```

In [96]:
```python
w1, w2 = GramSchmidt([v1, v2], orthonormal=True)
print(w1)
print(w2)
```

```
Matrix([[1], [0]])
Matrix([[0], [1]])
```

$B = \{v_1, v_2\} \in \mathbb{R}^2 \Rightarrow v_1 = (1,0) \Rightarrow w_1 = v_1 = (1,0)$

$v_2 = (1,1) \quad w_2 = v_2 - \left(\frac{v_2 \cdot w_1}{w_1 \cdot w_1}\right) \cdot w_1 \Rightarrow$

$w_2 = (1,1) - \left(\frac{(1,1)\cdot(1,0)}{(1,0)^2}\right) \cdot (1,0) \Rightarrow (1,1) - \left(\frac{1}{1}\right) \cdot (1,0) \Rightarrow$

$\Rightarrow (1,1) - (1,0) \Rightarrow w_2 = (0,1)$ ✓

ON $\Rightarrow w_1/|w_1| \Rightarrow (1,0)/\sqrt{(1,0)^2} \Rightarrow (1,0)/1 \Rightarrow (1,0)$ ✓

$w_2/|w_2| \Rightarrow (0,1)/\sqrt{(0,1)^2} \Rightarrow (0,1)/1 \Rightarrow (0,1)$

Sistemas Lineares (Linear Systems)

Equação Linear (Linear Equation)

$\alpha 1\ x1 + \alpha 2\ x2 + ... + \alpha n * xn = b$

Método de Eliminação de Gauss (Gauss Elimination Method)

Considere o sistema linear:

$$\begin{cases} \boxed{x_1 + 3x_2 = 5} \\ \boxed{2x_1 - x_2 = 3} \end{cases}$$

LINHA 1 L1

LINHA 2 L2

L2 ← L2 - 2*L1

$$\begin{cases} x_1 + 3x_2 = 5 \\ 0 \quad -7x_2 = -7 \end{cases}$$

$$\begin{cases} [x_1] + 3x_2 = 5 \\ 2x_1 - [x_2] = 3 \end{cases}$$

L2 <- L2 - 2* L1

$$\begin{cases} [x_1] + 3x_2 = 5 \\ 0 - [7x_2] = -7 \end{cases}$$

$x_2 = 1$
$x_1 = 2$

[Pivot]

$$\begin{cases} x_1 + 3x_2 = 5 & L1 \\ 2x_1 - x_2 = 3 & L2 \end{cases} \Rightarrow \begin{cases} x_1 + 3x_2 = 5 & L1 \\ 0 - 7x_2 = -7 & L2 \leftarrow L2 - 2L1 \end{cases}$$

$\bigcirc$ = Pivot

$$x_2 = 1 \Rightarrow x_1 + 3 \cdot 1 = 5 \Rightarrow x_1 = 2$$

Uma, Nenhuma e Infinitas Soluções (One, None and Infinite Solutions)

Uma solução (One Solution)

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ 2x_1 + 2x_2 - x_3 = 3 \\ 4x_1 + 4x_2 + 2x_3 = 2 \end{cases} \Rightarrow \begin{cases} x_1 - x_2 + x_3 = 0 & L1 \\ 0 + 4x_2 - 3x_3 = 3 & L2 \leftarrow L2 - 2L1 \\ 0 + 8x_2 - 2x_3 = 2 & L3 \leftarrow L3 - 4L3 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x_1 - x_2 + x_3 = 0 & L1 \\ 0 + 4x_2 - 3x_3 = 3 & L2 \\ 0 + 0 + 4x_3 = -4 & L3 \leftarrow L3 - 2L2 \end{cases} \Rightarrow \begin{array}{l} x_3 = -1 \Rightarrow 4x_2 + 3 = 3 \\ x_2 = 0 \Rightarrow x_1 - 0 - 1 = 0 \\ x_1 = 1 \end{array}$$

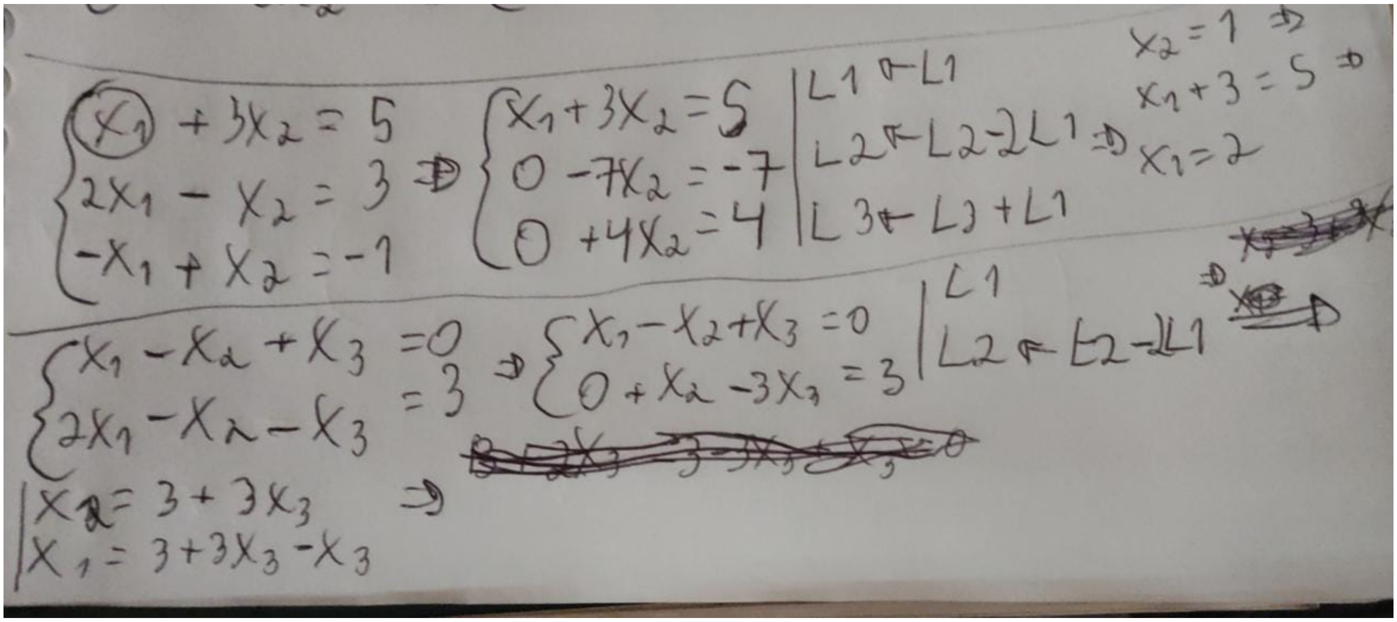
Nenhuma solução (No Solution)

$$\begin{cases} x_1 + 3x_2 = 5 \\ 3x_1 + 9x_2 = -2 \end{cases} \Rightarrow \begin{cases} x_1 + 3x_2 = 5 & L1 \\ 0 + 0 = -17 & L2 \leftarrow L2 - 3L1 \end{cases}$$

Infinitas soluções (Infinite Solutions)

$$\begin{cases} x_1 + 3x_2 = 5 \\ 2x_1 + 6x_2 = 10 \end{cases} \Rightarrow \begin{cases} x_1 + 3x_2 = 5 & L1 \\ 0 + 0 = 0 & L2 \leftarrow L2 - 2L1 \end{cases} \Rightarrow x_1 = 5 - 3x_2$$

Exercícios (Exercises):



Método de Eliminação de Gauss com Pivoteamento (Gauss Method with Pivoting)

Para cada vez que se trocam as linhas (pivoteamento), adiciona-se um traço sobre o nome da matriz. Se a quatidade de traços for par, o determinante é multiplicado por 1, se for impar, é multiplicado por -1. O critério para decidir sobre pivoteamento consiste em sempre colocar o pivot com maior valor ABUSOLUTO encima.
For each time that we change rows (pivoting), add a trace over the matrix name. If the quantity of traces is even, the determinant is multiplied by 1, and if it is odd, is multiplied by -1. The criterion to decide over pivoting consists in always putting the pivot with the highest ABSOLUTE value above.

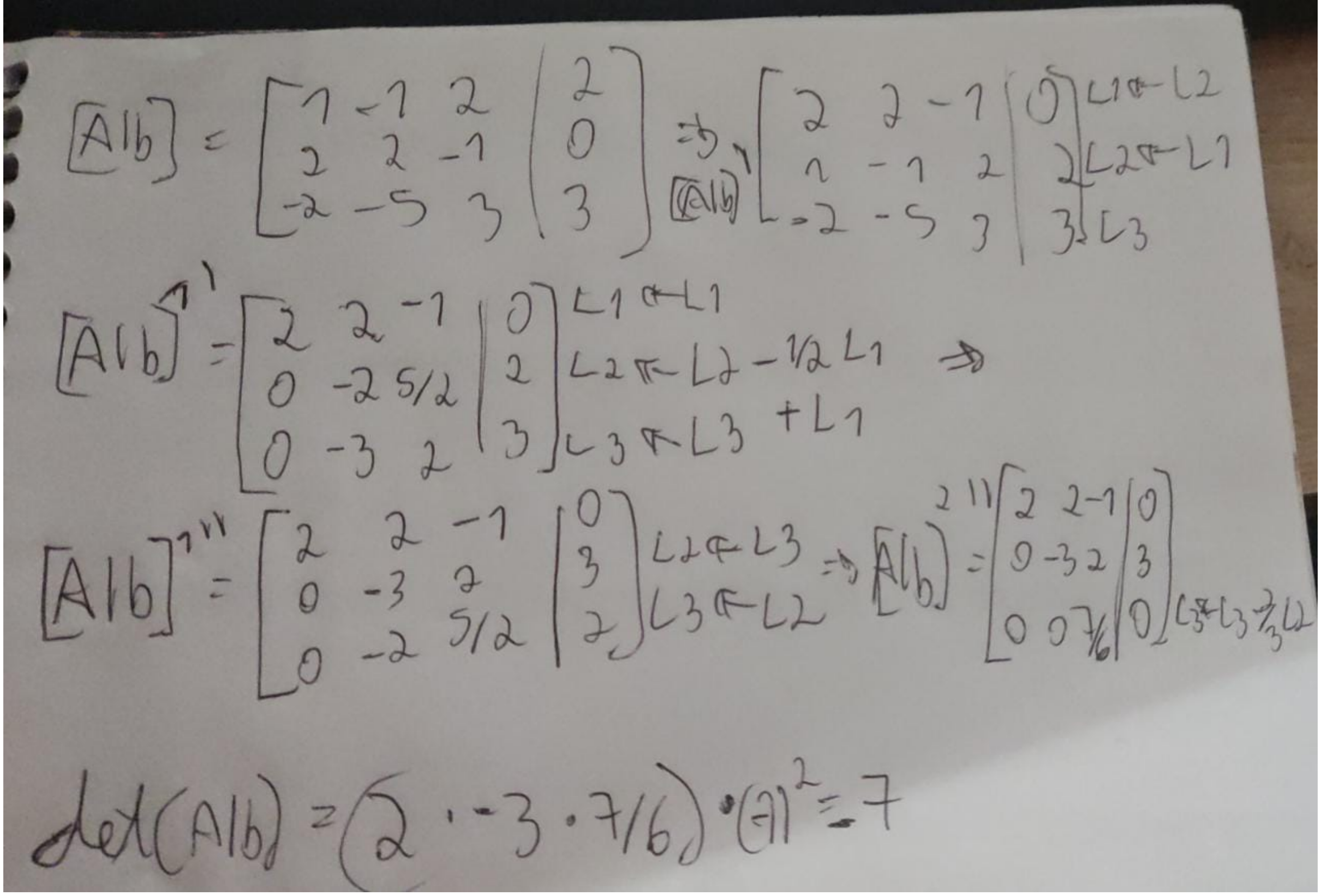
$det(A) = (\prod a_{ij})(-1)^{traces} \quad i = j$

$2x2, 3x3, 4x4, 5x5, \ldots NxN$

In [97]:

```
A = Matrix([[1, -1, 2],
            [2, 2, -1],
            [-2, -5, 3]])
A.det()
```

Out[97]: -7



In [98]:

```
A = Matrix([[1,2,-1,0],
            [0,-1,1,-1],
            [-2,-1,4,2],
            [4,3,0,1]])
```

In [99]:

```
A.det()
```

Out[99]: -29

Matrizes (Matrices)

Igualdade de Matrizes (Matrices Equality)

$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & x & 0 \end{bmatrix}, B = \begin{bmatrix} y & 2 & z \\ t & -1 & 0 \end{bmatrix}$

A = B se (if): y = 1, z = 1, t = 0, x = -1

Soma, Subtração e Multiplicação por Escalar (Sum, Subtraction and Multiplication by a Scalar)

Soma (Sum)

$$A = \begin{bmatrix}1 & 2 & 1\\0 & -1 & 0\end{bmatrix}, B = \begin{bmatrix}2 & 5 & 3\\-1 & -2 & -5\end{bmatrix}$$

$$C = A + B$$
$$C_{i,j} = A_{i,j} + B_{i,j}$$

$$A + B = C = \begin{bmatrix}3 & 7 & 4\\-1 & -3 & -5\end{bmatrix}$$

In [100…

```
A = Matrix([[1,2,1]
          ,[0,-1,0]])
B = Matrix([[2,5,3]
          ,[-1,-2,-5]])

A+B
```

Out[100…

$$\begin{bmatrix}3 & 7 & 4\\-1 & -3 & -5\end{bmatrix}$$

Subtração (Subtraction)

$$A = \begin{bmatrix}1 & 2 & 1\\0 & -1 & 0\end{bmatrix}, B = \begin{bmatrix}2 & 5 & 3\\-1 & -2 & -5\end{bmatrix}$$

$$C = A - B$$
$$C_{i,j} = A_{i,j} - B_{i,j}$$

$$A - B = C = \begin{bmatrix}-1 & -3 & -2\\1 & 1 & 5\end{bmatrix}$$

In [101…

```
A-B
```

Out[101…

$$\begin{bmatrix}-1 & -3 & -2\\1 & 1 & 5\end{bmatrix}$$

Multiplicação com Escalar (Multiplication with Scalar)

$$A = \begin{bmatrix}1 & 2 & 1\\0 & -1 & 0\end{bmatrix}, \alpha = 2$$

$$B = A\alpha$$
$$B_{i,j} = A_{i,j}\alpha$$

$$B = \begin{bmatrix}2 & 4 & 2\\0 & -2 & 0\end{bmatrix}$$

In [102…

```
A*alpha.subs(alpha, 2)
```

Out[102…

$$\begin{bmatrix} 2 & 4 & 2 \\ 0 & -2 & 0 \end{bmatrix}$$

Base das Matrizes (Base of Matrices)

#1 B =LI

#2 B -> M

In [103…

```
matrix1 = Matrix([[1,0]
                 ,[0,0]])
matrix2 = Matrix([[0,1]
                 ,[0,0]])
matrix3 = Matrix([[0,0]
                 ,[1,0]])
matrix4 = Matrix([[0,0]
                 ,[0,1]])
```

In [104…

```
# 1 B = LI
equation = alpha*matrix1+beta*matrix2+gamma*matrix3+delta*matrix4
equation
```

Out[104…

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

In [105…

```
solve(equation)
```

Out[105…

```
{alpha: 0, beta: 0, gamma: 0, delta: 0}
```

In [106…

```
# 2 B --> M22
matrix5 = Matrix([[x,y]
                 ,[z,w]])
equation2 = equation - matrix5
equation.subs(solve(equation2))
```

Out[106…

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

Matriz Transporta e Matrix Simétrica (Symmetric and Transposed Matrix)

In [107…

```
A = Matrix([[1,0]
           ,[2,-3]
           ,[0,5]])
A
```

Out[107…

$$\begin{bmatrix} 1 & 0 \\ 2 & -3 \\ 0 & 5 \end{bmatrix}$$

In [108…

```
# Transposta (Transposed)
A.T
```

Out[108…

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 5 \end{bmatrix}$$

Produto Escalar de Matrizes (Matrices Scalar Product)

In [109…

```
a = Matrix([[1,5,-3]])
b = Matrix([[3]
           ,[0]
           ,[-2]])
```

In [110…

```
a
```

Out[110…

$$\begin{bmatrix} 1 & 5 & -3 \end{bmatrix}$$

In [111…

```
b
```

Out[111…

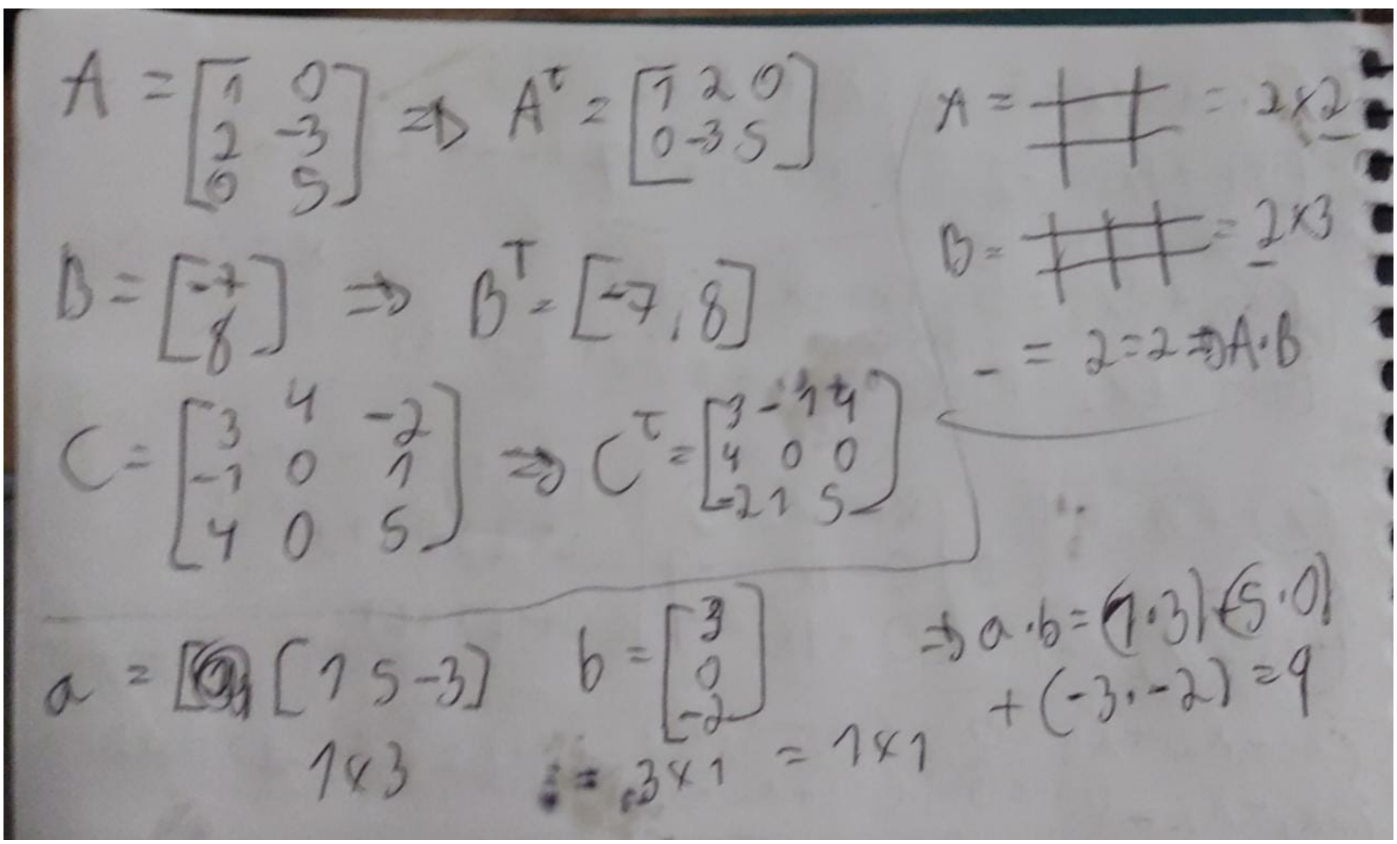
$$\begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

In [112…

```
a*b
```

Out[112…

$$\begin{bmatrix} 9 \end{bmatrix}$$



In [113…

```
A = Matrix([[4,3,1]
           ,[2,5,-1]])
B = Matrix([[5,-1]
           ,[-2,4]
           ,[1,3]])
```

In [114…

```
A
```

Out[114…

$$\begin{bmatrix} 4 & 3 & 1 \\ 2 & 5 & -1 \end{bmatrix}$$

In [115…

```
B
```

Out[115…

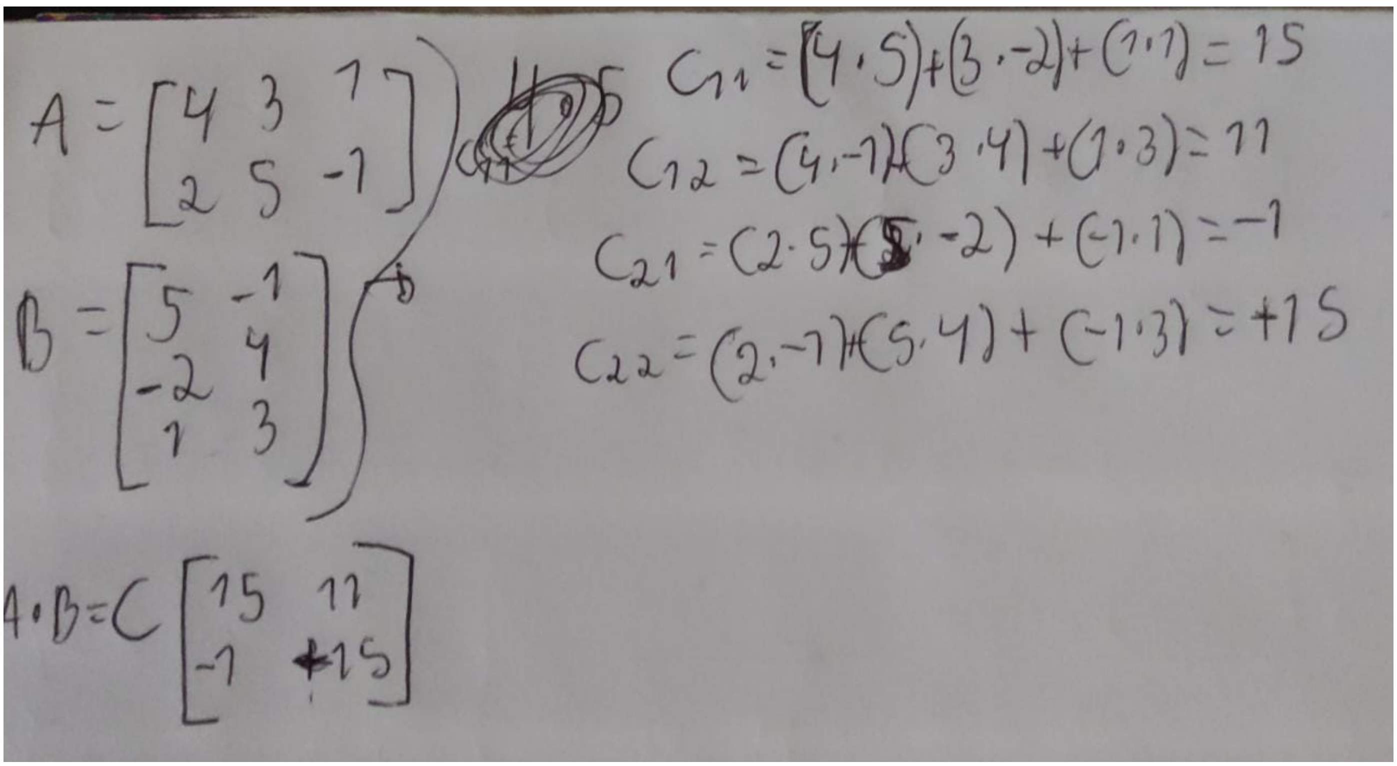
$$\begin{bmatrix} 5 & -1 \\ -2 & 4 \\ 1 & 3 \end{bmatrix}$$

In [116…

```
A*B
```

Out[116…

$$\begin{bmatrix} 15 & 11 \\ -1 & 15 \end{bmatrix}$$



Dadas as Matrizes

Dica: Nunc
nº de colunas d

$$\mathbf{A} = \begin{bmatrix} 7 & 0 & 9 \\ 1 & 5 & 3 \\ \boxed{-1 \quad 2 \quad 8} \end{bmatrix} \quad \text{e} \quad \mathbf{B} = \begin{bmatrix} 3 & -2 \\ -2 & 1 \\ 1 & 5 \end{bmatrix}$$

Encontre o elemento $(3,2)$ *do produto* $\mathbf{AB}$.

In [117…

```
A = Matrix([[7,0,9]
```

```
            ,[1,5,3]
            ,[-1,2,8]])
B = Matrix([[3,-2]
             ,[-2,1]
            ,[1,5]])
```

In [118…

```
C = A*B
C
```

Out[118…

$$\begin{bmatrix} 30 & 31 \\ -4 & 18 \\ 1 & 44 \end{bmatrix}$$

$$A = \begin{bmatrix} 7 & 0 & 9 \\ 1 & 5 & 3 \\ -1 & 2 & 8 \end{bmatrix}_{\underline{3 \times 3}} \cdot B = \begin{bmatrix} 3 & -2 \\ -2 & 1 \\ 1 & 5 \end{bmatrix}_{\underline{3 \times 2}} \Rightarrow C = \begin{bmatrix} 30 & 31 \\ -4 & 18 \\ 1 & 44 \end{bmatrix}_{\underline{3 \times 2}}$$

$\Rightarrow$ $7 \cdot 3 + 0 \cdot -2 + 9 \cdot 1 = 30 = C_{11}$

$7 \cdot -2 \quad 0 \cdot 1 \quad 9 \cdot 5 = 31 = C_{12}$

$1 \cdot 3 \quad 5 \cdot -2 \quad 3 \cdot 1 = -4 = C_{21}$

$1 \cdot -2 \quad 5 \cdot 1 \quad 3 \cdot 5 = 18 = C_{22}$

$-1 \cdot 3 \quad 2 \cdot -2 \quad 8 \cdot 1 = 1 = C_{31}$

$-1 \cdot -2 \quad 2 \cdot 1 \quad 8 \cdot 5 = 44 = C_{32}$

Encontre o produto **AB**.

$$\mathbf{AB} = \begin{bmatrix} 5 & 3 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} (5)(5) + (3)(-3) & (5)(-1) + (3)(4) \\ (1)(5) + (7)(-3) & (1)(-1) + (7)(4) \end{bmatrix} = \begin{bmatrix} 16 & 7 \\ -16 & 27 \end{bmatrix}$$

Encontre o produto **BA**.

$$\mathbf{BA} = \begin{bmatrix} 5 & -1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} (5)(5) + (-1)(1) & (5)(3) + (-1)(7) \\ (-3)(5) + (4)(1) & (-3)(3) + (4)(7) \end{bmatrix}$$

In [119…

```
A = Matrix([[5,3]
           ,[1,7]])
```

```
B = Matrix([[5,-1]
           ,[-3,4]])

# A*B != B*A
```

In [120…

```
AB = A*B
AB
```

Out[120…

$$\begin{bmatrix} 16 & 7 \\ -16 & 27 \end{bmatrix}$$

In [121…

```
BA = B*A
BA
```

Out[121…

$$\begin{bmatrix} 24 & 8 \\ -11 & 19 \end{bmatrix}$$

In [122…

```
AB == BA
```

Out[122…

```
False
```

Matriz Aumentada e Sistemas Lineares (Augmented Matrix and Linear Systems)

In [123…

```
A = Matrix([[3,2,-4],
            [1,-5,1],
            [-1,3,1]])
b = Matrix([[5],
            [2],
            [6]])
X = Matrix([[x],
            [y],
            [z]])

(A*X)-b
```

Out[123…

$$\begin{bmatrix} 3x + 2y - 4z - 5 \\ x - 5y + z - 2 \\ -x + 3y + z - 6 \end{bmatrix}$$

$$\begin{cases} 3x_1 + 2x_2 - 4x_3 = 5 \\ x_1 - 5x_2 + x_3 = 2 \\ -x_1 + 3x_2 + x_3 = 6 \end{cases} \Rightarrow A = \begin{bmatrix} 3 & 2 & -4 \\ 1 & -5 & 1 \\ -1 & 3 & 1 \end{bmatrix} \quad \boxed{A|b}$$

$$\Rightarrow [A|b] = \left[\begin{array}{ccc|c} 3 & 2 & -4 & 5 \\ 1 & -5 & 1 & 2 \\ -1 & 3 & 1 & 6 \end{array}\right] \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ e } b = \begin{bmatrix} 5 \\ 2 \\ 6 \end{bmatrix}$$

$$Ax = b$$

Matriz Identidade (Identidy Matrix)

In [124…

```
A = Matrix([[3,2,-4],
            [1,-5,1],
            [-1,3,1]])
```

In [125…

```
I = eye(3)
I
```

Out[125…

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In [126…

```
I*A
```

Out[126…

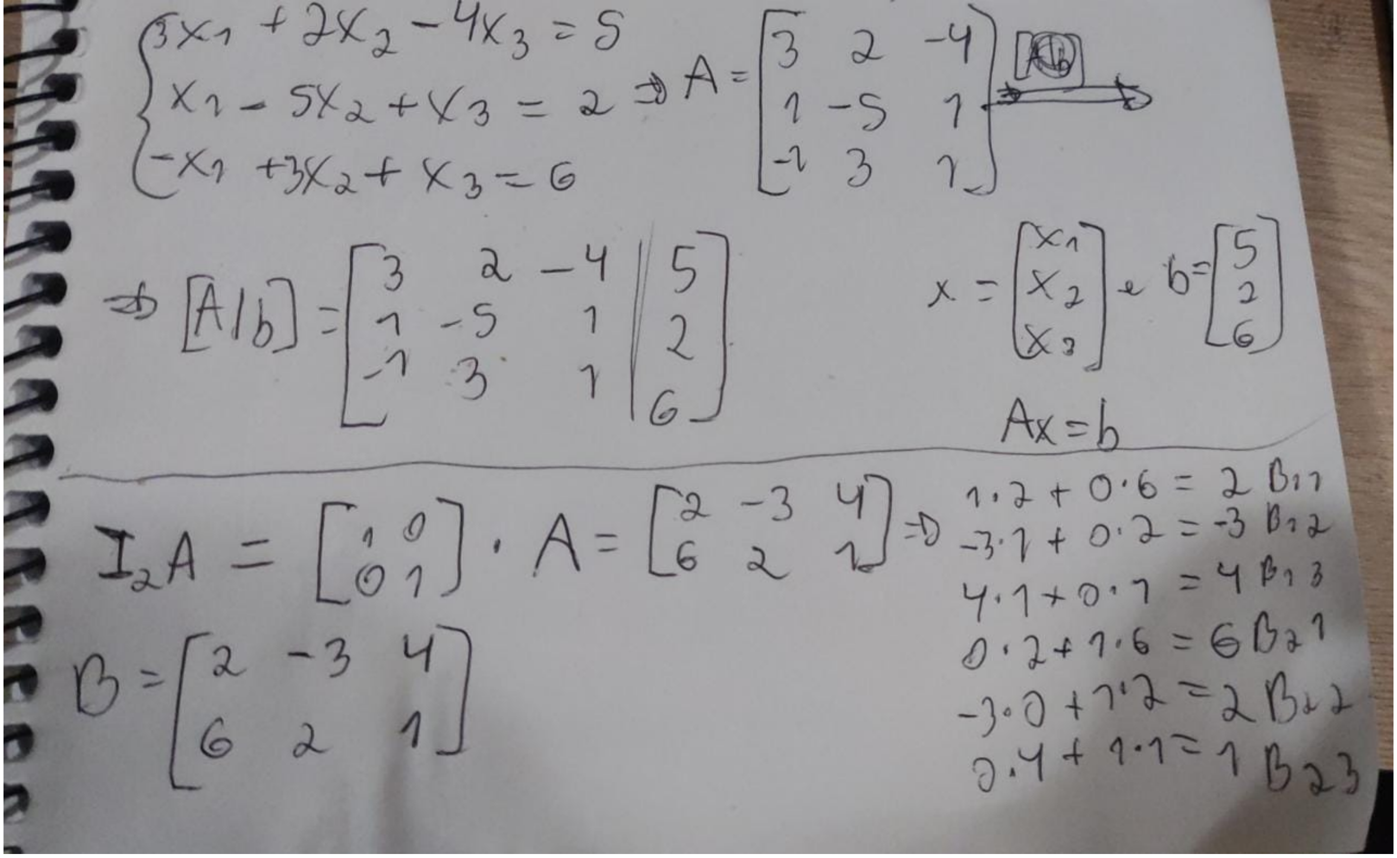
$$\begin{bmatrix} 3 & 2 & -4 \\ 1 & -5 & 1 \\ -1 & 3 & 1 \end{bmatrix}$$

In [127…

```
I*A == A
```

Out[127…

```
True
```



$$\begin{cases}3x_1+2x_2-4x_3=5\\ x_1-5x_2+x_3=2\\ -x_1+3x_2+x_3=6\end{cases} \Rightarrow A=\begin{bmatrix}3&2&-4\\1&-5&1\\-1&3&1\end{bmatrix} \quad [A|b]$$

$$\Rightarrow [A|b]=\left[\begin{array}{ccc|c}3&2&-4&5\\1&-5&1&2\\-1&3&1&6\end{array}\right] \qquad x=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \text{ e } b=\begin{bmatrix}5\\2\\6\end{bmatrix}$$

$$Ax=b$$

$$I_2A=\begin{bmatrix}1&0\\0&1\end{bmatrix}\cdot A=\begin{bmatrix}2&-3&4\\6&2&1\end{bmatrix} \Rightarrow$$

$1\cdot 2+0\cdot 6=2 \; B_{11}$

$-3\cdot 1+0\cdot 2=-3 \; B_{12}$

$4\cdot 1+0\cdot 1=4 \; B_{13}$

$0\cdot 2+1\cdot 6=6 \; B_{21}$

$-3\cdot 0+1\cdot 2=2 \; B_{22}$

$0\cdot 4+1\cdot 1=1 \; B_{23}$

$$B=\begin{bmatrix}2&-3&4\\6&2&1\end{bmatrix}$$

Potencialização de Matrizes (Matrix Empowerment)

In [128…
```python
A = Matrix([[1,-1],
            [1,2]])
```

In [129…
```python
A
```

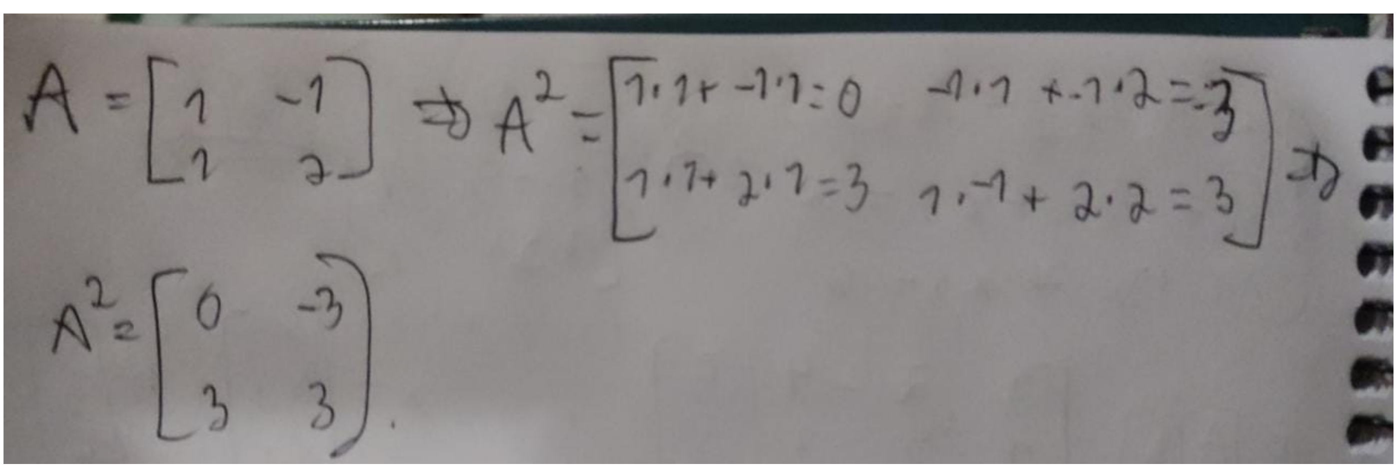
Out[129…

$$\begin{bmatrix}1&-1\\1&2\end{bmatrix}$$

In [130…
```python
A**2
```

Out[130…

$$\begin{bmatrix}0&-3\\3&3\end{bmatrix}$$



$$A=\begin{bmatrix}1&-1\\1&2\end{bmatrix} \Rightarrow A^2=\begin{bmatrix}1\cdot 1+-1\cdot 1=0 & -1\cdot 1+-1\cdot 2=-3\\ 1\cdot 1+2\cdot 1=3 & 1\cdot -1+2\cdot 2=3\end{bmatrix} \Rightarrow$$

$$A^2=\begin{bmatrix}0&-3\\3&3\end{bmatrix}.$$

Introdução a Forma Ecalonada Por Linhas (Método de Gauss

Jordan) (Introduction to the Row Echelon Form (Gauss Jordan Method))

Regras:

#1 Se existirem linhas nulas, elas estarão sempre abaixo das não-nulas

#2 O primeiro elemento da linha (além dos 0) deve ser 1

#3 Deve apresentar forma de Escada (Escalonada)

#4 Encontrando o primeiro elemento da linha não-nula, os outros elementos da Coluna devem ser iguais a 0

Rules:

#1 If there are null rows, they must always be below the non-null ones

#2 The firs element of the row (besides 0) must be 1

#3 Must have an echelon form

#4 Once the first non-null element of the row is found, the other elements of the column must be equal to 0

Exemplos (Exemples):

$$A = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \#1, \#2, \#3, \#4$$

$$B = \begin{bmatrix} 1 & -1 & 0 & 0 & 9 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 8 \end{bmatrix} \quad \#1, \#2, \#3, \#4$$

$$C = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \#1, \#2, \#3, \#4$$

$$D = \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 3 & 2 & 5 \end{bmatrix} \quad \#1, \color{red}{\#2, \#3, \#4}$$

$$E = \begin{bmatrix} 1 & -1 & 3 & 4 & 9 \\ 0 & 1 & 4 & 3 & 1 \\ 0 & 0 & 1 & 1 & 8 \end{bmatrix} \quad \#1, \#2, \#3, \color{red}{\#4}$$

$$F = \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 5 \end{bmatrix} \quad \color{red}{\#1, \#2, \#3, \#4}$$

Escalonamento (Echelon)

$$A=\begin{bmatrix}0&1&2&-1&0\\2&6&0&-2&4\\-2&2&1&-1&0\\4&0&0&2&1\end{bmatrix} \Rightarrow \begin{bmatrix}\boxed{0}&1&2&-1&0\\\boxed{2}&6&0&-2&4\\\boxed{-2}&2&1&-1&0\\\boxed{4}&0&0&2&1\end{bmatrix} \Rightarrow$$

$$\Rightarrow A^1=\begin{bmatrix}2&6&0&-2&4\\0&1&2&-1&0\\-2&2&1&-1&0\\4&0&0&2&1\end{bmatrix}\begin{matrix}L_1\leftarrow L_2\\L_2\leftarrow L_1\\L_3\\L_4\end{matrix} \Rightarrow A^2\begin{bmatrix}1&3&0&-1&2\\0&1&2&-1&0\\-2&2&1&-1&0\\4&0&0&2&1\end{bmatrix}\begin{matrix}L_1\leftarrow \frac{1}{2}L_1\\L_2\\L_3\\L_4\end{matrix}$$

$$\Rightarrow A^3\begin{bmatrix}1&3&0&-1&2\\0&1&2&-1&0\\0&8&1&-3&4\\4&0&0&2&1\end{bmatrix}\begin{matrix}L_1\\L_2\\L_3\leftarrow L_3+2L_1\\L_4\end{matrix} \Rightarrow A^4\begin{bmatrix}1&3&0&-1&2\\0&1&2&-1&0\\0&8&1&-3&4\\0&-12&0&6&-7\end{bmatrix}\begin{matrix}L_1\\L_2\\L_3\\L_4\leftarrow L_4-4L_1\end{matrix}$$

$$A^5\begin{bmatrix}1&3&0&-1&2\\0&1&2&-1&0\\0&0&-15&5&4\\0&0&24&-6&-7\end{bmatrix}\begin{matrix}L_1\\L_2\\L_3\leftarrow L_3-8L_2\\L_4\leftarrow L_4+12L_2\end{matrix} \Rightarrow A^6\begin{bmatrix}1&3&0&-1&2\\0&1&2&-1&0\\0&0&1&-1/3&-4/15\\0&0&24&-6&-7\end{bmatrix}\begin{matrix}L_1\\L_2\\L_3\leftarrow -\frac{1}{15}L_3\\L_4\end{matrix}$$

$$\Rightarrow A^7\begin{bmatrix}1&3&0&-1&2\\0&1&2&-1&0\\0&0&1&-1/3&-4/15\\0&0&0&2&-\frac{77}{15}\end{bmatrix}\begin{matrix}L_1\\L_2\\L_3\\L_4\leftarrow L_4-24L_3\end{matrix} \quad A^8\Rightarrow\begin{bmatrix}1&3&0&-1&2\\0&1&2&-1&0\\0&0&1&-1/3&-4/15\\0&0&0&1&-\frac{77}{30}\end{bmatrix}\begin{matrix}L_1\\L_2\\L_3\\L_4\leftarrow \frac{1}{2}L_4\end{matrix}$$

$$A^9\begin{bmatrix}1&3&0&0&-\frac{17}{30}\\0&1&2&0&-\frac{77}{30}\\0&0&1&0&-\frac{101}{90}\\0&0&0&1&-\frac{77}{30}\end{bmatrix}\begin{matrix}L_1\leftarrow L_1+L_4\\L_2\leftarrow L_2+L_4 \Rightarrow\\L_3\leftarrow L_3+\frac{1}{3}L_4\\L_4\end{matrix}$$

$$A^{10}\begin{array}{cl}\left[\begin{array}{ccccc}1 & 3 & 0 & 0 & -17/30\\ 0 & 1 & 0 & 0 & -18/45\\ 0 & 0 & 1 & 0 & -107/90\\ 0 & 0 & 0 & 1 & -77/30\end{array}\right] & \begin{array}{l}L1\\ L2 \leftarrow L2-2L3\\ L3\\ L4\end{array}\end{array}$$

$$\Rightarrow A^{11}\begin{array}{cl}\left[\begin{array}{ccccc}1 & 0 & 0 & 0 & 19/30\\ 0 & 1 & 0 & 0 & -18/45\\ 0 & 0 & 1 & 0 & -107/90\\ 0 & 0 & 0 & 1 & -77/30\end{array}\right] & \begin{array}{l}L1 \leftarrow L1-3L2\\ L2\\ L3\\ L4\end{array}\end{array}$$

In [131…

```
A = Matrix([[1,0,0,5],
            [0,1,0,-1],
            [0,0,1,2]])
b = Matrix([[0],
            [3],
            [2]])
X = Matrix([[x],
            [y],
            [z],
            [w]])
```

In [132…

```
X
```

Out[132…

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

In [133…

```
X.T
```

Out[133…

$$\begin{bmatrix} x & y & z & w \end{bmatrix}$$

In [134…

```
linear_system = A*X-b
linear_system
```

Out[134…

$$\begin{bmatrix} 5w + x \\ -w + y - 3 \\ 2w + z - 2 \end{bmatrix}$$

$$\begin{cases} 2x_1 - 2x_2 + 4x_3 - 2x_4 = 6 \\ x_1 + 0 + x_3 - x_4 = 2 \\ -x_1 - x_2 + 2x_3 - x_4 = 1 \\ 0 - x_2 + 3x_3 - 2x_4 = 3 \end{cases}$$

$$[A|b] = \left[\begin{array}{cccc|c} 2 & -2 & 4 & -2 & 6 \\ 1 & 0 & 1 & -1 & 2 \\ -1 & -1 & 2 & -1 & 1 \\ 0 & -1 & 3 & -2 & 3 \end{array}\right] \Rightarrow [A|b]^1 = \begin{array}{c} \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 3 \\ 1 & 0 & 1 & -1 & 2 \\ -1 & -1 & 2 & -1 & 1 \\ 0 & -1 & 3 & -2 & 3 \end{array}\right] \end{array} \begin{array}{l} L_1 \leftarrow \frac{1}{2}L_1 \\ L_2 \\ L_3 \\ L_4 \end{array} \Rightarrow$$

$$[A|b]^2 = \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 3 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & -2 & 4 & -2 & 4 \\ 0 & -1 & 3 & -2 & 3 \end{array}\right] \begin{array}{l} L_1 \\ L_2 \leftarrow L_2 - L_1 \\ L_3 \leftarrow L_3 + L_1 \\ L_4 \end{array} \Rightarrow$$

$$[A|b]^3 \Rightarrow \begin{array}{c} \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 3 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right] \end{array} \begin{array}{l} L1 \\ L2 \\ L3 \leftarrow L3 + 2L2 \\ L4 \leftarrow L4 + L2 \end{array} \Rightarrow$$

$$[A|b]^4 \Rightarrow \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 3 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} L1 \\ L2 \\ L3 \leftarrow \frac{1}{2} L3 \\ L4 \leftarrow L4 - L3 \end{array} \Rightarrow$$

$$[A|b]^4 \Rightarrow \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} L1 \leftarrow L1 - 2L3 \\ L2 \leftarrow L2 + L3 \\ L3 \\ L4 \end{array} \Rightarrow$$

$$[A|b]^5 \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} L1 \leftarrow L1 + L2 \\ L2 \\ L3 \\ L4 \end{array} \Rightarrow \begin{cases} x_1 = 1 \\ x_2 - x_4 = 0 \\ x_3 - x_4 = 1 \end{cases}$$

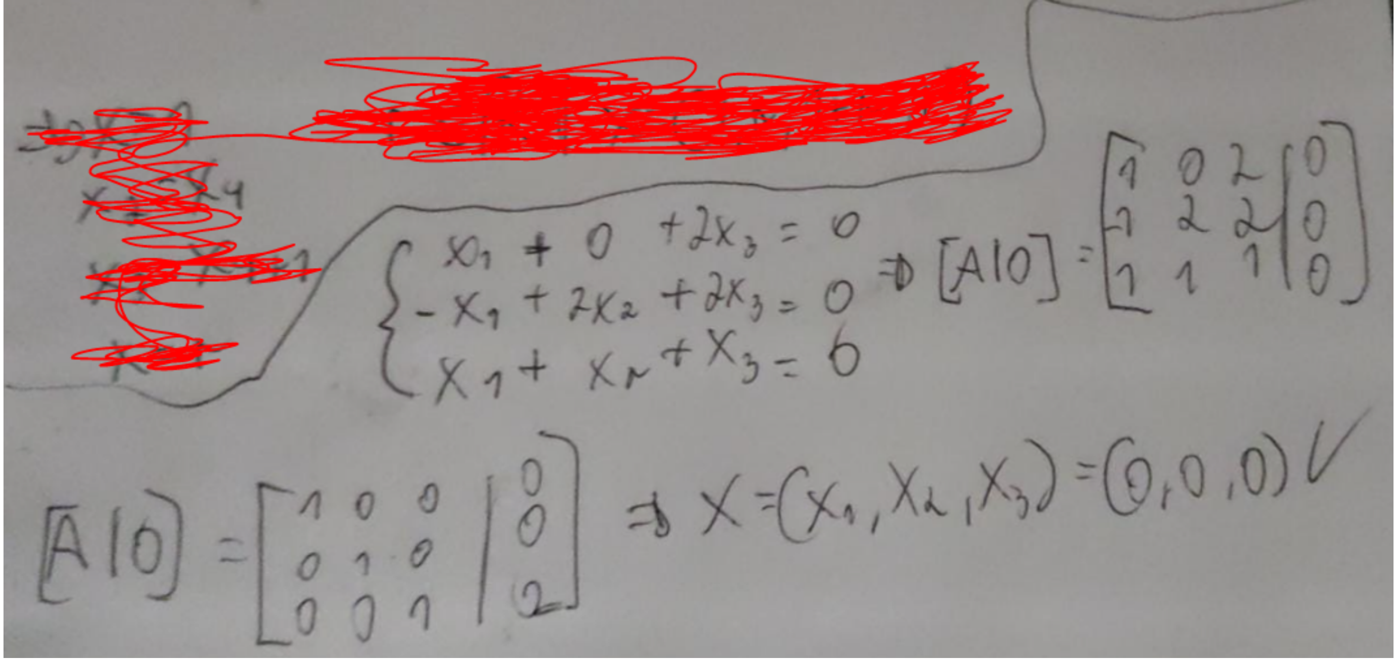
$$\Rightarrow x = 1 \qquad r \in \mathbb{R} \mid x = (1, r, 1+r, r).$$

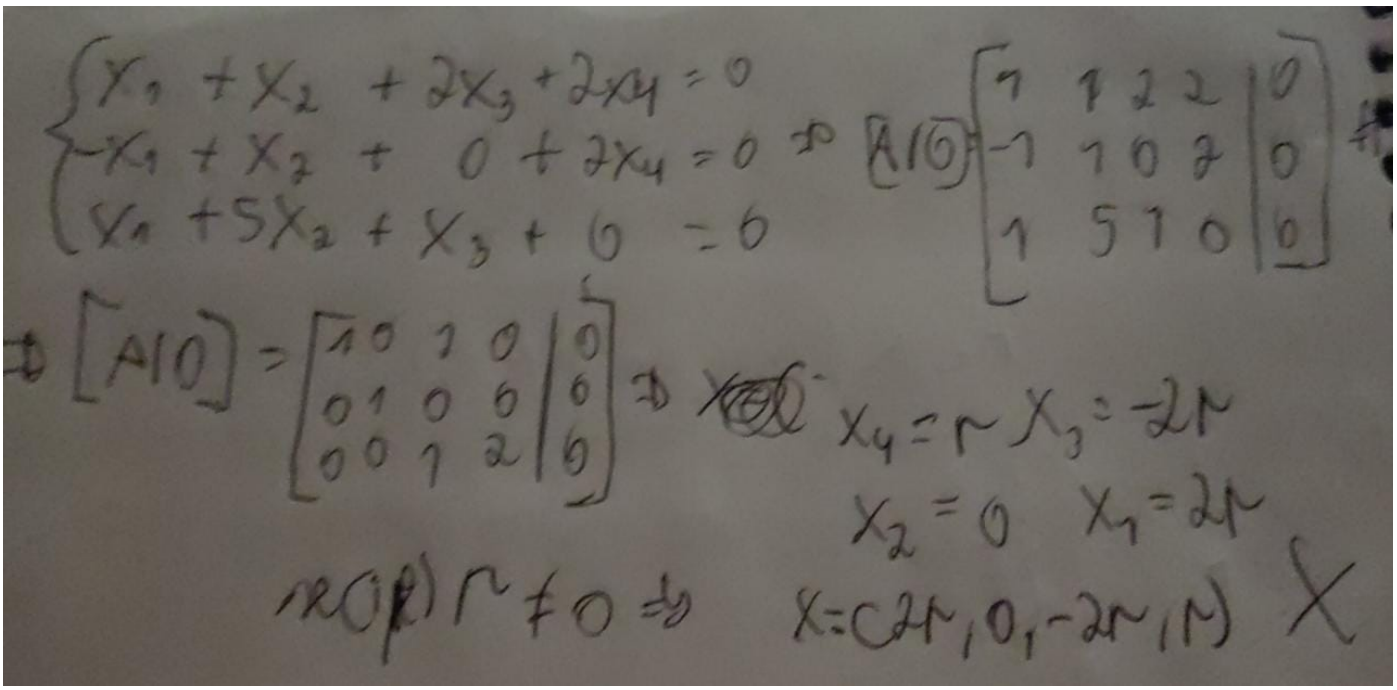
$$x_2 = x_4$$

$$x_3 = x_4 + 1$$

$$x = r$$

Solução Trivial e Sistemas Homogêneos (Homogenous Systems and Trivial Solution)





Matriz Inversa (Inverse Matrix)

In [135…

```
A = Matrix([[1,0,0,5],
            [0,1,0,-1],
            [0,0,1,2],
           [3,1,1,0]])
```

In [136…

```
A.inv()
```

Out[136…

$$\begin{bmatrix} \frac{1}{16} & -\frac{5}{16} & -\frac{5}{16} & \frac{5}{16} \\ \frac{3}{16} & \frac{17}{16} & \frac{1}{16} & -\frac{1}{16} \\ -\frac{3}{8} & -\frac{1}{8} & \frac{7}{8} & \frac{1}{8} \\ \frac{3}{16} & \frac{1}{16} & \frac{1}{16} & -\frac{1}{16} \end{bmatrix}$$

In [137…

```
A
```

Out[137…

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 3 & 1 & 1 & 0 \end{bmatrix}$$

In [138…

```
A*A.inv()
```

Out[138…

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In [139…

```
A = Matrix([[1,2,-1],
            [-1,0,4],
            [2,2,1]])
```

In [140…

```
A.inv()
```

Out[140…

$$\begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 0 & 4 \\ 2 & 2 & 1 \end{bmatrix} \Rightarrow [A|I] \neq \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -1 & 0 & 4 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow$$

$$[A|I]^1 = \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & -2 & 3 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} L1 \\ L2 \leftarrow L2 + L1 \\ L3 \leftarrow L3 - 2L1 \end{array} \Rightarrow$$

$$[A|I]^2 = \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & 1/2 & 1/2 & 0 \\ 0 & 0 & 6 & -1 & 1 & 1 \end{array}\right] \begin{array}{l} L1 \\ L2 \leftarrow \frac{1}{2}L2 \\ L3 \leftarrow L3 + L2 \end{array} \Rightarrow$$

$$[A|I]^3 = \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & -1/6 & 1/6 & 1/6 \end{array}\right] \begin{array}{l} L1 \\ L2 \\ L3 \leftarrow \frac{1}{6}L3 \end{array} \Rightarrow$$

$$[A|I]^4 = \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 5/6 & 1/6 & 1/6 \\ 0 & 1 & 0 & 3/4 & 1/4 & -1/4 \\ 0 & 0 & 1 & -1/6 & 1/6 & 1/6 \end{array}\right] \begin{array}{l} L1 \leftarrow L1 + L3 \\ L2 \leftarrow L2 - \frac{3}{2}L3 \\ L3 \end{array}$$

$$[A|I]^5 = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/3 & -1/3 & 2/3 \\ 0 & 1 & 0 & 3/4 & 1/4 & -1/4 \\ 0 & 0 & 1 & -1/6 & 1/6 & 1/6 \end{array}\right] \begin{array}{l} L1 \leftarrow L1 - 2L2 \\ L2 \\ L3 \end{array} \Rightarrow$$

$$A^{-1} = \begin{bmatrix} -2/3 & -1/3 & 2/3 \\ 3/4 & 1/4 & -1/4 \\ -1/6 & 1/6 & 1/6 \end{bmatrix} \checkmark$$

In [141…

```
A = Matrix([[1,1,2],
            [2,1,3],
            [-1,2,1]])
```

In [142…

```
try:
    A.inv()
except:
    print("Não Invertível (Not Invetible)")
```

Não Invertível (Not Invetible)

Determinantes (Determinants)

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

$- > det(A) == -det(B)$

2x2

det(A) = $a_{11}a_{22} - a_{12}a_{21}$

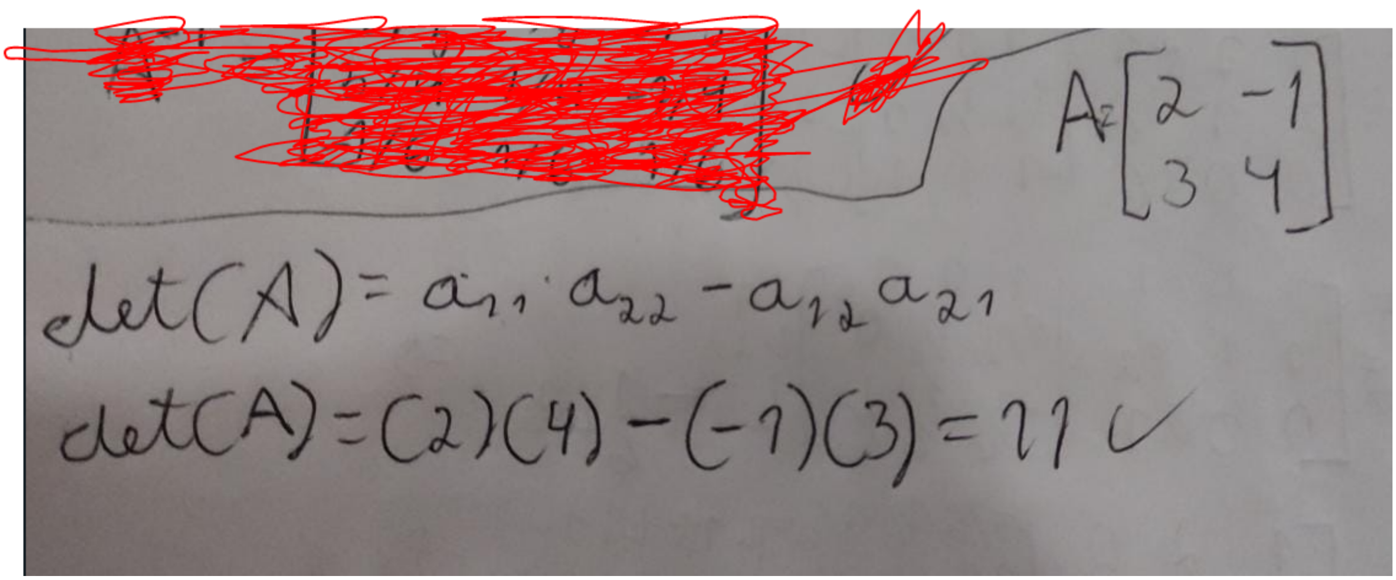
In [143…

```
A = Matrix([[2,-1],
            [3,4]])
```

In [144…

```
A.det()
```

Out[144… 11



3x3

(1)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

(2)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{bmatrix}$$

$det(A) = \{a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}\} - \{a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} + a_{13}a_{22}a_{31}\}$

det(A) == det(A.T)

In [145…

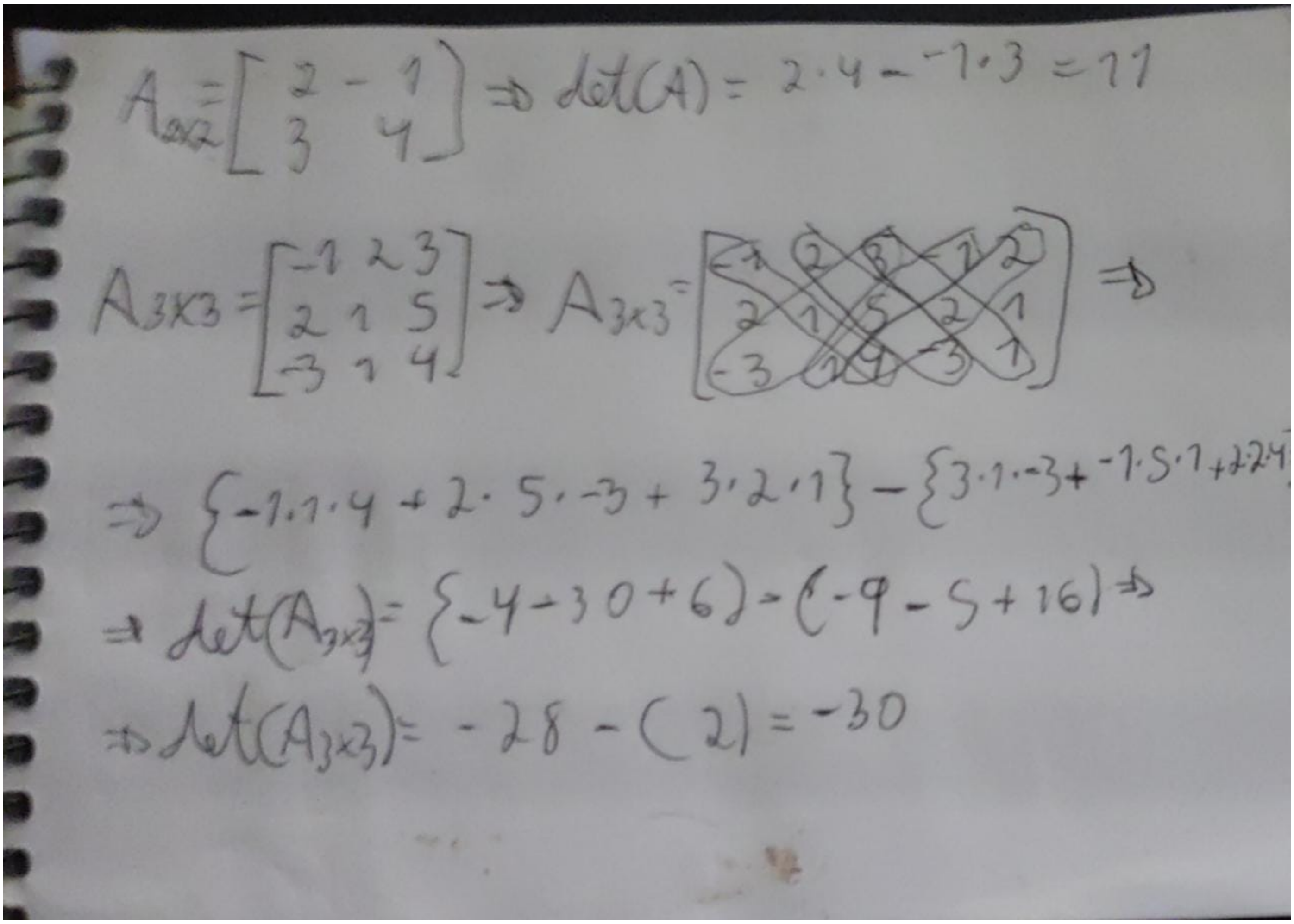
```
A = Matrix([[-1,2,3],
            [2,1,5],
            [-3,1,4]])
```

In [146...
```
A.det()
```

Out[146... -30

In [147...
```
A.T.det()
```

Out[147... -30



det(AB) = det(A)det(B)

In [148...
```
A = Matrix([[1,1,2],
            [2,2,3],
            [-1,2,1]])
B = Matrix([[2,3,2],
            [1,-1,-1],
            [-1,-2,3]])
```

In [149...
```
A.det()
```

Out[149... 3

In [150...
```
B.det()
```

Out[150... -22

In [151...
```
(A*B).det()
```

Out[151... -66

In [152...
```
A.det()*B.det()
```

Out[152... -66

det(A.inv()) = 1/det(A)

In [153...
```
A.inv().det()
```

Out[153... $\frac{1}{3}$

In [154...
```
1/A.det()
```

Out[154... $\frac{1}{3}$

Expansão de Cofatores e Laplace (Cofactor Expansion and Laplace)

In [155...
```
A = Matrix([[1,-2,0,5],
            [2,3,1,-2],
            [0,2,-1,3],
            [3,1,0,2]])
```

In [156...
```
A.det()
```

Out[156... 54

$$A = \begin{bmatrix} 1 & -2 & 0 & 5 \\ 2 & 3 & 1 & -2 \\ 0 & 2 & -1 & 3 \\ 3 & 1 & 0 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -2 & 0 & 5 \\ 2 & 3 & 1 & -2 \\ 0 & 2 & -1 & 3 \\ 3 & 1 & 0 & 2 \end{bmatrix} \Rightarrow M_{13} = \begin{bmatrix} 2 & 3 & -2 \\ 0 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \det(M_{13}) = 41$$

$$A = \begin{bmatrix} 1 & -2 & 0 & 5 \\ 2 & 3 & 1 & -2 \\ 0 & 2 & -1 & 3 \\ 3 & 1 & 0 & 2 \end{bmatrix} \Rightarrow M_{23} = \begin{bmatrix} 1 & -2 & 5 \\ 0 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \det(M_{23}) = -47$$

$$A = \begin{bmatrix} 1 & -2 & 0 & 5 \\ 2 & 3 & 1 & -2 \\ 0 & 2 & -1 & 3 \\ 3 & 1 & 0 & 2 \end{bmatrix} \Rightarrow M_{33} \begin{bmatrix} 1 & -2 & 5 \\ 2 & 3 & -2 \\ 3 & 1 & 2 \end{bmatrix} = \det(M_{33}) = -7$$

$$A = \begin{bmatrix} 1 & -2 & 0 & 5 \\ 2 & 3 & 1 & 2 \\ 0 & 2 & -1 & 3 \\ 3 & 1 & 0 & 2 \end{bmatrix} \Rightarrow M_{43} \begin{bmatrix} 1 & -2 & 5 \\ 2 & 3 & -2 \\ 0 & 2 & 3 \end{bmatrix} = \det(M_{43}) = 45$$

$$A_{13} = (-1)^{1+3} \cdot 41 = 41$$

$$A_{23} = (-1)^{2+3} \cdot -47 = 47$$

$$A_{33} = (-1)^{3+3} \cdot -7 = -7$$

$$A_{34} = (-1)^{3+4} \cdot 45 = -45$$

$$\det(A) = 0 \cdot 41 + 1 \cdot 47 + -1 \cdot -7 + 0 \cdot -45 \Rightarrow 0 + 47 + 7 + 0 = 54$$

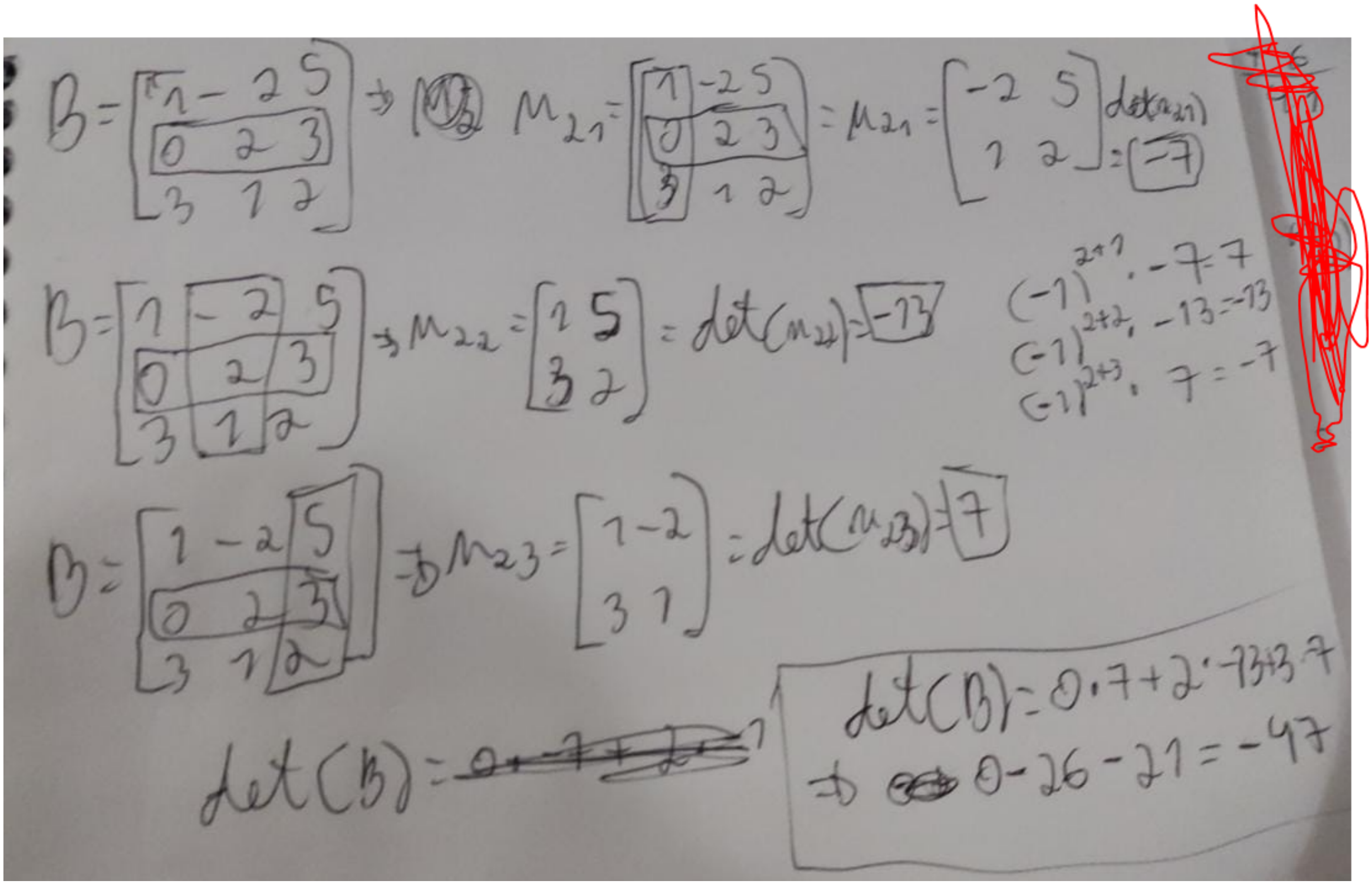
In [157…

```
B = Matrix([[1,-2,5],
            [0,2,3],
            [3,1,2]])
```

In [158…

```
B.det()
```

Out[158… -47



In [159…

```
C = Matrix([[1,-2,5],
           [2,3,-2],
           [3,1,2]])
```

In [160…

```
C.det()
```

Out[160… -7

$$C = \begin{bmatrix} 1 & -2 & 5 \\ 2 & 3 & -2 \\ 3 & 1 & 2 \end{bmatrix} \Rightarrow M_{11} = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix} \quad \det(M_{11}) = 8$$

$$C = \begin{bmatrix} 1 & -2 & 5 \\ 2 & 3 & -2 \\ 3 & 1 & 2 \end{bmatrix} \Rightarrow M_{12} = \begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix} \quad \det(M_{12}) = 10$$

$$C = \begin{bmatrix} 1 & -2 & 5 \\ 2 & 3 & -2 \\ 3 & 1 & 2 \end{bmatrix} \Rightarrow M_{13} = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \quad \det(M_{13}) = -7$$

$(-1)^2 \cdot 8 = 8$

$(-1)^3 \cdot 10 = -10$

$(-1)^4 \cdot -4 = -7$

$\det(C) = 1 \cdot 8 + -2 \cdot -10 + 5 \cdot -7$

$8 \quad 20 \quad -35$

$\det(C) = 8 - 7$

Cálculo de Determinante Introduzindo 0 (Determinant Calculation Introducing 0)

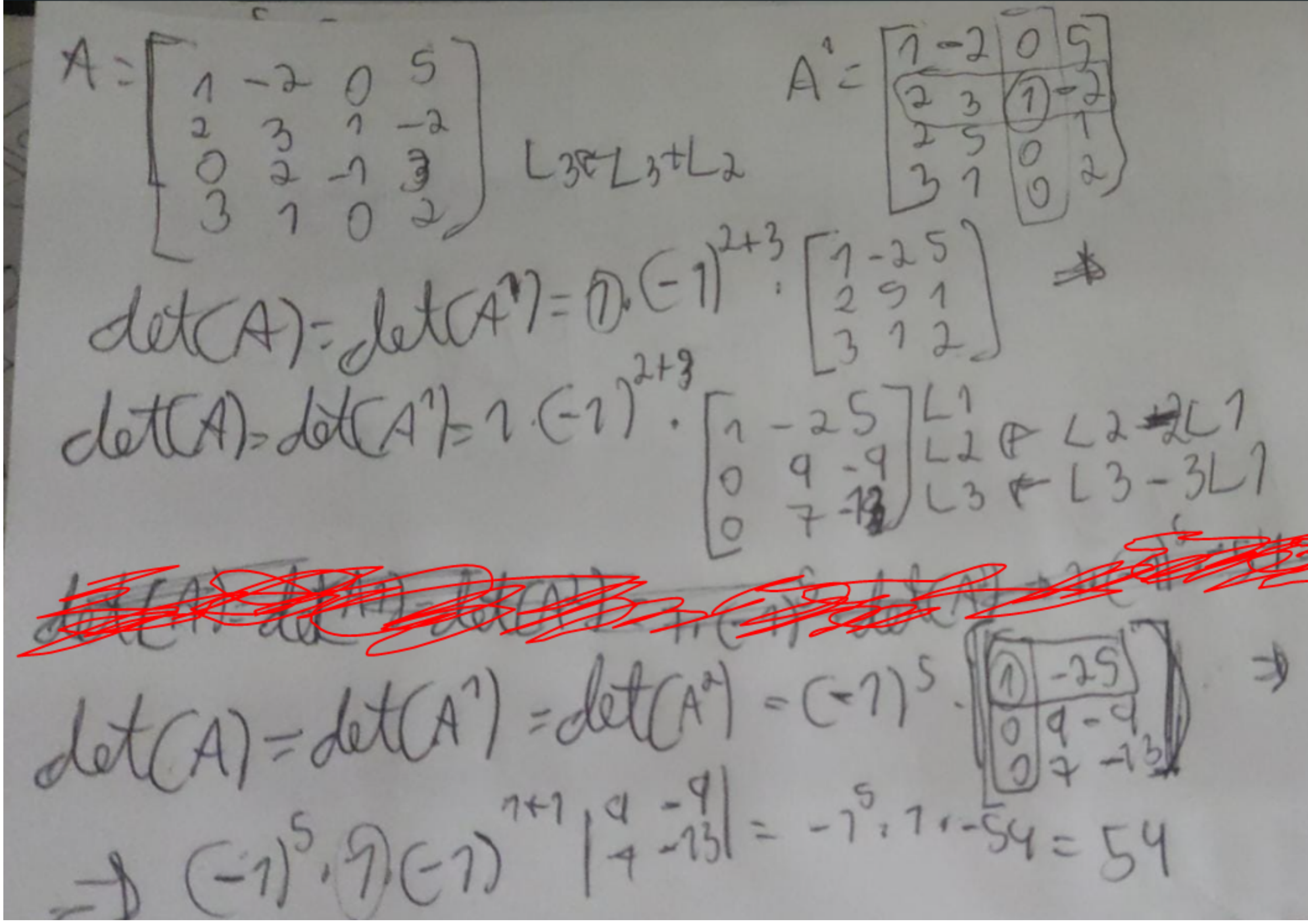
In [161…

```
A = Matrix([[1,-2,0,5],
            [2,3,1,-2],
            [0,2,-1,3],
            [3,1,0,2]])
```

In [162…

```
A.det()
```

Out[162… 54



Regra de Cramer (Cramer's Rule)

det(A) != 0

In [163…

```python
x1, x2, x3, x4 = symbols("x1 x2 x3 x4")

A = Matrix([[-x1, 2*x2, x3],
            [2*x1, x2, -x3],
            [x1, x2, 2*x3]])
b = Matrix([[2, -3, 1]])
```

$$\begin{cases} -x_1 + 2x_2 + x_3 = 2 \\ 2x_1 + x_2 - x_3 = -3 \\ x_1 + x_2 + 2x_3 = 1 \end{cases}$$

$$A = \begin{bmatrix} -1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

$$\det(A) = -12$$

$$A_1 = \begin{bmatrix} 2 & 2 & 1 \\ -3 & 1 & -1 \\ 1 & 1 & 2 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 & 2 & 1 \\ 2 & -3 & -1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$x_i = \frac{\det(A_i)}{\det(A)}$$

$$A_3 = \begin{bmatrix} -1 & 2 & 2 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\det(A_1) = 12$$
$$\det(A_2) = 0$$
$$\det(A_3) = -12$$

$$x_1 = \frac{12}{-12} = -1$$

$$x_2 = \frac{0}{-12} = 0$$

$$x_3 = \frac{-12}{-12} = \frac{12}{12} = 1$$

Matriz Adjunta (Adjugate Matrix)

In [164…

```
A = Matrix([[2,-4,1],
            [-2,6,3],
            [1,2,-1]])
```

In [165…

```
A.adjugate()
```

Out[165…

$$\begin{bmatrix} -12 & -2 & -18 \\ 1 & -3 & -8 \\ -10 & -8 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -4 & 1 \\ -2 & 6 & 3 \\ 1 & 2 & -1 \end{bmatrix}$$

$$A_{11} \begin{vmatrix} 6 & 3 \\ 2 & -1 \end{vmatrix} = (-6)-(6) = -12 \cdot 1$$

$$A_{12} \begin{vmatrix} -2 & 3 \\ 1 & -1 \end{vmatrix} = (2)-(3) = -1 \cdot -1$$

$$A_{31} \begin{vmatrix} -4 & 1 \\ 6 & 3 \end{vmatrix} = (-12)-(6) = -18 \cdot 1$$

$$A_{13} \begin{vmatrix} -2 & 6 \\ 1 & 2 \end{vmatrix} = (-4)-(6) = -10 \cdot 1$$

$$A_{32} \begin{vmatrix} 2 & 1 \\ -2 & 3 \end{vmatrix} = (6)-(-2) = 8 \cdot -1$$

$$A_{21} \begin{vmatrix} -4 & 1 \\ 2 & -1 \end{vmatrix} = (4)-(2) = 2 \cdot -1$$

$$A_{33} \begin{vmatrix} 2 & -4 \\ -2 & 6 \end{vmatrix} = (12)-(8) = 4 \cdot 1$$

$$A_{22} \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = (-2)-(1) = -3 \cdot 1$$

$$A_{23} \begin{vmatrix} 2 & -4 \\ 1 & 2 \end{vmatrix} = (4)-(-4) = 8 \cdot -1$$

$A_{11} = -12$

$A_{12} = +1$

$A_{13} = -10$

$A_{21} = -2$

$A_{22} = -3$

$A_{23} = -8$

$A_{31} = -18$

$A_{32} = -8$

$A_{33} = +4$

$$\operatorname{adj}A^t \begin{bmatrix} -12 & 1 & -10 \\ -2 & -3 & -8 \\ -18 & -8 & +4 \end{bmatrix} \Rightarrow \operatorname{adj}A \begin{bmatrix} -12 & -2 & -18 \\ 1 & -3 & -8 \\ -10 & -8 & 4 \end{bmatrix}$$

Matriz Inversa usando Matriz Adjunta (Inverse Matrix using Adjugate Matrix)

$A^{-1} = \frac{1}{det(A)} adj(A)$

In [166…
```
A.det()
```

Out[166… -38

In [167…
```
(1/A.det())*A.adjugate()
```

Out[167…

$$\begin{bmatrix} \frac{6}{19} & \frac{1}{19} & \frac{9}{19} \\ -\frac{1}{38} & \frac{3}{38} & \frac{4}{19} \\ \frac{5}{19} & \frac{4}{19} & -\frac{2}{19} \end{bmatrix}$$

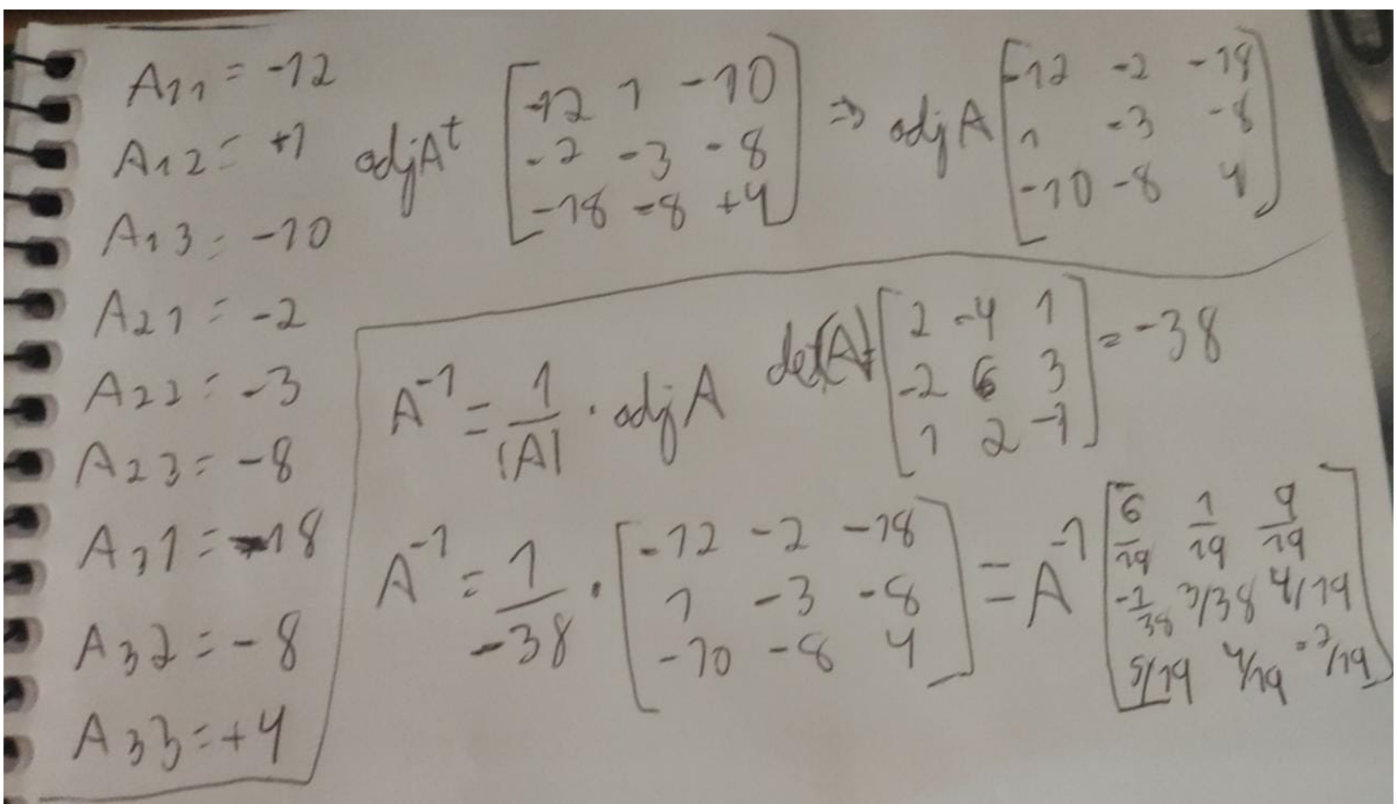
In [168…
```
A.inv()
```

Out[168…

$$\begin{bmatrix} \frac{6}{19} & \frac{1}{19} & \frac{9}{19} \\ -\frac{1}{38} & \frac{3}{38} & \frac{4}{19} \\ \frac{5}{19} & \frac{4}{19} & -\frac{2}{19} \end{bmatrix}$$

In [169…
```
(1/A.det())*A.adjugate() == A.inv()
```

Out[169…
```
True
```



Transformações Lineares (Linear Mapping)

R1 => R2 => R3

$\#1 \quad T(u+v) = T(u) + T(v), \quad \forall u, v \in V$

$\#2 \quad T(\alpha u) = \alpha T(u), \quad \forall \alpha \in R, \quad \forall u \in V$

$T : \mathbb{R}^2 \to \mathbb{R}^3$

$V = (x, y) \in \mathbb{R}^2 \quad w = (x, y, z) \in \mathbb{R}^3$

$\# \; T(x, y) = (3x, -2y, x-y)$

$T(-1, 3) = (3 \cdot -1, -2 \cdot 3, -1-3) \Rightarrow (-3, -6, -4)$

$T(0, 0) = (0, 0, 0)$

$T(2, 1) = (6, -2, 1)$

#1

$T(x, y) = (3x, -2y, x-y) \quad \mathbb{R}^2 \to \mathbb{R}^3$

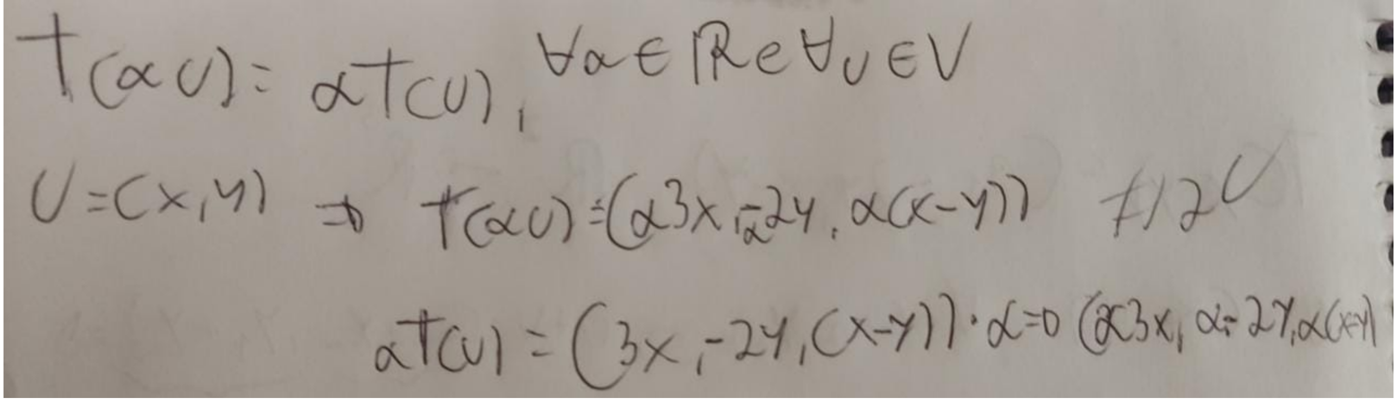
$U = (x_1, y_1), \quad V = (x_2, y_2) \Rightarrow U+V = (\underbrace{x_1 + x_2}_{x}, \underbrace{y_1 + y_2}_{y}) \Rightarrow$

$T(U+V) = (3(x_1 + x_2), -2(y_1 + y_2), (x_1 + x_2) - (y_1 + y_2))$

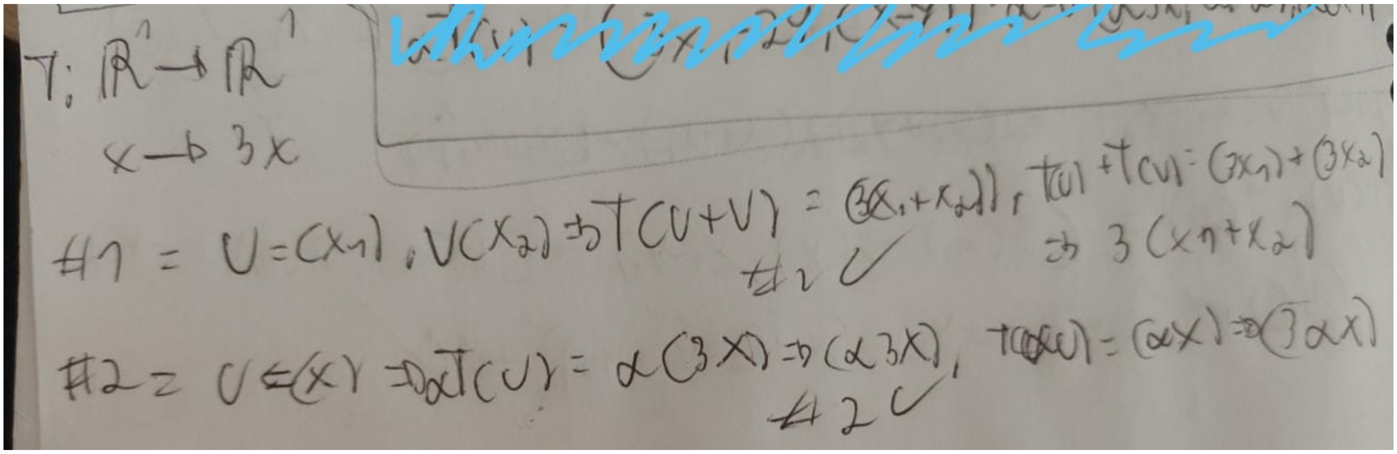
$T(U) + T(V) = (3(x_1), -2(y_1), x_1 - y_1) + (3(x_2), -2(y_2), x_2 - y_2)$

$= (3(x_1 + x_2), -2(y_1 + y_2), (x_1 + x_2) - (y_1 + y_2)) \; \#1$

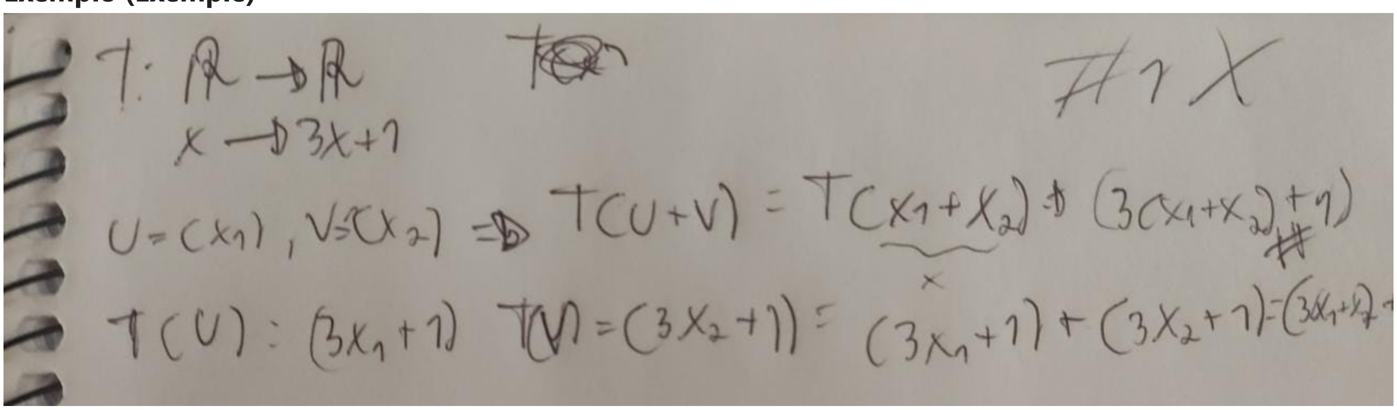
#2



Exemplo (Exemple)



Exemplo (Exemple)



Núcleo (Kernel)

$N(T) = Ker(T) = v \in V; T(v) = 0$

Definição de Núcleo: São os elementos que, através da transformação, resultam em uma imagem de vetor nulo
Definition of Kernel: They are the elements that, through the tranformation, result in an image of a null vector

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$(x,y) \to T(x,y) = (x+y, 2x-y)$

$(x+y, 2x-y) = (0,0)$

$$\begin{cases} x+y=0 \\ 2x-y=0 \end{cases} \Rightarrow \left|\begin{array}{cc|c} 1 & +1 & 0 \\ 2 & -1 & 0 \end{array}\right| \Rightarrow \left|\begin{array}{cc|c} 1 & +1 & 0 \\ 0 & -3 & 0 \end{array}\right| \begin{array}{l} L_1 \\ L_2 = L_2 - 2L_1 \end{array} \Rightarrow$$

$-3y = 0 \Rightarrow y = \frac{0}{-3} = y = 0 \qquad x+y=0 \Rightarrow x+0=0 \Rightarrow x=0$

$Ker(T) = \{(0,0)\}$

Exemplo (Exemple)

$T: \mathbb{R}^3 \to \mathbb{R}^2$

$(x,y,z) \to T(x,y,z) = (x-y+4z, 3x+y+8z)$

$$\begin{cases} x-y+4z=0 \\ 3x+y+8z=0 \end{cases} \Rightarrow \left|\begin{array}{ccc|c} 1 & -1 & 4 & 0 \\ 3 & 1 & 8 & 0 \end{array}\right| \Rightarrow \left|\begin{array}{ccc|c} 1 & -1 & 4 & 0 \\ 0 & 4 & -4 & 0 \end{array}\right| \begin{array}{l} L_1 \\ L_2 = L_2 - 3L_1 \end{array}$$

$y = z \qquad x - z + 4z = 0 \Rightarrow x + 3z = 0 \quad x = -3z$

$x = -3z$

$z = z$

$Ker(T) = \{(-3z, z, z)\} \quad z \in \mathbb{R}$

$a(-3, 1, 1) = Base$

Injetora (Injector)

#1:

Uma transformação é injetora se o núcleo é igual a 0: N(T) = {0}.

A transformation is injector if the kernel is equal to 0: Ker(T) = {0}.

$$N(T) = Ker(T) = \{v \in V;\ T(v) = 0\}$$

$$\mathbb{R}^3 \to \mathbb{R}^3$$

$$(x, y, z) \to (z, x - y, -z)$$

$$\begin{matrix} z = 0 \\ x - y = 0 \\ -z = 0 \end{matrix} \quad \left|\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right| = \begin{matrix} z = 0 \\ x = y \\ y = x \end{matrix} \qquad Ker(T) = \{(x, x, 0)\}$$

$$Base(T) = x(1, 1, 0)\ x \in \mathbb{R}$$

$$\#1 \times \quad (x, x, 0) \neq (0, 0, 0)$$

AutoValores e AutoVetores (EigenValues and EigenVectors)

$T(v) = \lambda V$

$$T(v) = \lambda(v)$$

$$T: \mathbb{R}^2 \to \mathbb{R}^2$$

$$(x, y) \to T(x, y) = (4x + 5y, 2x + y)$$

$$v = (5, 2) \Rightarrow T(v) = (20 + 10, 10 + 2) \Rightarrow T(v) = (30, 12)$$

$$\frac{30}{5} = 6 \quad \frac{12}{2} = 6 \Rightarrow T(v) = (30, 12) = \lambda(v) = 6(5, 2)\ \# \checkmark$$

$$T: \mathbb{R}^2 \to \mathbb{R}^2$$

$$(x,y) \to T(x,y) = (4x+5y,\ 2x+y)$$

$$v = (1,1) \Rightarrow (4+5,\ 2+1) \Rightarrow (9,3).$$

$$\underbrace{9 \div 1}_{9} \neq \underbrace{3 \div 1}_{3} \Rightarrow \nexists\, \lambda \qquad T(v) \neq \lambda(v).$$

In [ ]:

```
```