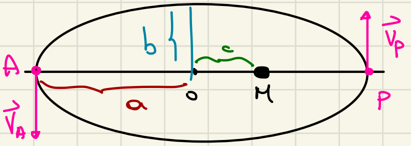
Proposizione: In un orbita ellittica generica di un corpo di massa m intorno a un corpo di massa M valgono i seguenti fatti:



1) $E_{TOT} = -G\frac{mM}{2a}$

2) $L = \frac{2\pi m a b}{T}$

3) $V_A^2 = \frac{GM}{a}\left(\frac{a-c}{b}\right)^2$

4) $V_P^2 = \frac{GM}{a}\left(\frac{a+c}{b}\right)^2$

Dim: L'energia e il momento angolare basta calcolarli in un pto e poi li ho ovunque perché si conservano. Usiamo le due conservaz. in due punti furbi.

$$\begin{cases} E_A = E_P \\ L_A = L_P \end{cases} \qquad \begin{cases} -\frac{G\cancel{m}M}{a+c} + \frac{1}{2}\cancel{m}V_A^2 = -\frac{G\cancel{m}M}{a-c} + \frac{1}{2}\cancel{m}V_P^2 \\ (a+c)\cdot\cancel{m}\cdot V_A = (a-c)\cancel{m}\cdot V_P \end{cases}$$

$$\text{II)}\begin{cases} V_A \overset{(*)}{\boxed{=}} \frac{a-c}{a+c}V_P \\ -GM\left(\frac{1}{a+c} - \frac{1}{a-c}\right) = \frac{1}{2}V_P^2 - \frac{1}{2}\left(\frac{a-c}{a+c}\right)^2 V_P^2 \end{cases}$$

$$\overset{2c\ \cdot\ 2a}{[a+c-(a-c)][a+c+(a-c)]}$$

$$\text{II)}\quad -GM\left(\frac{\cancel{a}-c-\cancel{a}-c}{\underbrace{a^2-c^2}_{b^2}}\right) = \frac{1}{2}V_P^2\left(\frac{(a+c)^2-(a-c)^2}{(a+c)^2}\right)$$

$$\frac{\cancel{2}GM\cancel{c}}{b^2} = \frac{1}{\cancel{2}}V_P^2\,\frac{4a\cancel{c}}{(a+c)^2} \rightsquigarrow \boxed{V_P^2 = \frac{GM}{a}\left(\frac{a+c}{b}\right)^2}$$

$$V_A^2 \overset{(*)}{\boxed{=}} \frac{(a-c)^2}{\cancel{(a+c)^2}}\,\frac{GM}{a}\,\frac{\cancel{(a+c)^2}}{b^2} \rightsquigarrow \boxed{V_A^2 = \frac{GM}{a}\left(\frac{a-c}{b}\right)^2}$$

$$E_{TOT} = U_A + K_A = -\frac{GmM}{a+c} + \frac{1}{2} m \frac{GM}{a} \left(\frac{a-c}{b}\right)^2$$

$$= -GmM \left(\frac{1}{a+c} - \frac{(a-c)^2}{2ab^2} \right)$$

$$= -GmM \left(\frac{2ab^2 - (a+c)(a-c)^2}{2a\, b^2 (a+c)} \right)$$

$$= -\frac{GmM}{2a} \left(\frac{2ab^2 - \overbrace{(a^2-c^2)}^{b^2}(a-c)}{b^2(a+c)} \right)$$

$$= -\frac{GmM}{2a} \left(\frac{\overbrace{2ab^2 - ab^2}^{ab^2} + cb^2}{b^2 (a+c)} \right)$$

$$= -\frac{GmM}{2a} \left(\frac{\cancel{b^2(a+c)}}{\cancel{b^2(a+c)}} \right) = -\frac{GmM}{2a}$$

$$\rightsquigarrow \boxed{E_{TOT} = -\frac{GmM}{2a}}$$

$$L = (a+c)\, m \cdot v_A \quad \rightsquigarrow \quad L^2 = (a+c)^2 m^2 \cdot \frac{GM}{a} \frac{(a-c)^2}{b^2} = \frac{Gm^2M}{a} \frac{b^4}{b^2}$$

$$= Gm^2M \frac{b^2}{a} \boxed{=} \frac{4\pi^2 a^2 b^2}{T^2} m^2$$

$$\downarrow$$

$$\frac{GM}{4\pi^2} = \frac{a^3}{T^2}$$

$$\rightsquigarrow \boxed{L = \frac{2\pi a b m}{T}}$$