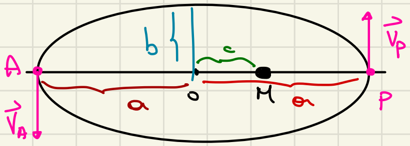
# Formule generali per orbite ellittiche

**Proposizione**: In un orbita ellittica generica di un corpo di massa $m$ intorno a un corpo di massa $M$ valgono i seguenti fatti:




1) $E_{TOT} = -G\dfrac{mM}{2a}$

2) $L = \dfrac{2\pi m a b}{T}$

3) $V_A^2 = \dfrac{GM}{a}\left(\dfrac{a-c}{b}\right)^2$

4) $V_P^2 = \dfrac{GM}{a}\left(\dfrac{a+c}{b}\right)^2$

**Dim**: Si conservano il momento angolare e l'energia. Imponiamo la conservazione in afelio e in perielio

$$\begin{cases} E_A = E_P \\ L_A = L_P \end{cases} \qquad \begin{cases} \frac{1}{2}\cancel{m}V_A^2 + \left(-G\frac{\cancel{m}M}{a+c}\right) = \frac{1}{2}\cancel{m}V_P^2 + \left(-G\frac{\cancel{m}M}{a-c}\right) \\ (a+c)\cancel{m}\cdot V_A = (a-c)\cdot\cancel{m}\cdot V_P \end{cases}$$

$$\begin{cases} V_A \overset{(*)}{\boxed{=}} \frac{a-c}{a+c}V_P \\ \frac{1}{2}\left(\frac{a-c}{a+c}V_P\right)^2 - \frac{1}{2}V_P^2 = -GM\left(\frac{1}{a-c} - \frac{1}{a+c}\right) \end{cases}$$

$$\frac{1}{2}V_P^2\left(\frac{(a-c)^2}{(a+c)^2} - 1\right) = -GM\left(\frac{\overbrace{\cancel{a}+c-\cancel{a}+c}^{2c}}{(a-c)(a+c)}\right)$$

$$V_P^2\left(\frac{\overbrace{(a-c)^2-(a+c)^2}^{(\cancel{a}-c-\cancel{a}-c)(a-\cancel{c}+a+\cancel{c})}}{(a+c)^{\cancel{2}}}\right) = -\frac{4GMc}{(a-c)\cancel{(a+c)}}$$

$$V_P^2\left(\frac{+\cancel{4}a\cancel{c}}{a+c}\right) = +\cancel{4}\frac{GM\cancel{c}}{a-c}$$

$$V_P^2 = GM\,\frac{(a+c)\,(a+c)}{a(a-c)\,(a+c)} = \frac{GM}{a}\,\frac{(a+c)^2}{\underbrace{a^2-c^2}_{b^2}}$$

$$\leadsto \boxed{V_P^2 = \frac{GM}{a}\cdot\left(\frac{a+c}{b}\right)^2}$$

$\leadsto$ Metto questo in $L_A = L_P$ tutto al $\square$

$$\leadsto V_A^2 = \left(\frac{a-c}{a+c}\right)^2 V_P^2 = \frac{(a-c)^2}{\cancel{(a+c)^2}}\cdot\frac{GM}{a}\,\frac{\cancel{(a+c)^2}}{b^2}$$

$$\leadsto \boxed{V_A^2 = \frac{GM}{a}\left(\frac{a-c}{b}\right)^2}$$

Calcolo adesso $E_{TOT}$

$$E_{TOT} = \frac{1}{2}mV_A^2 + \left(-\frac{GmM}{a+c}\right) = \frac{1}{2}m\,\frac{GM}{a}\left(\frac{a-c}{b}\right)^2 - \frac{GmM}{a+c}$$

$$= GMm\left(\frac{(a-c)^2}{2a\underbrace{b^2}_{a^2-c^2=(a-c)(a+c)}} - \frac{1}{a+c}\right)$$

$$= GMm\left(\frac{a^2+c^2-2\cancel{a}c - 2a^2 + 2\cancel{a}c}{2a(a-c)(a+c)}\right)$$

$$= \frac{GMm}{2a}\left(\frac{\overbrace{c^2-a^2}^{-\cancel{b^2}}}{\cancel{b^2}}\right)$$

$$\boxed{E_{TOT} = -\frac{GMm}{2a}}$$

$$L_A^2 = (a+c)^2\cdot m^2\cdot\frac{GM}{a}\left(\frac{a-c}{b}\right)^2 = \frac{\overbrace{[(a+c)(a-c)]^2}^{b^2}}{b^2}\cdot m^2\,\frac{GM}{a}$$

$$a^3 = \frac{GM}{4\pi^2}T^2$$

$$= GM\,m^2\,\frac{b^2}{a}\cdot\frac{a^2}{a^2}$$

$$= \frac{\cancel{GM}\,m^2\,a^2b^2}{\cancel{GM}\,T^2}\cdot 4\pi^2$$

$$\leadsto \boxed{L_A = \frac{2\pi m a b}{T}}$$ facendo la radice

$\square$