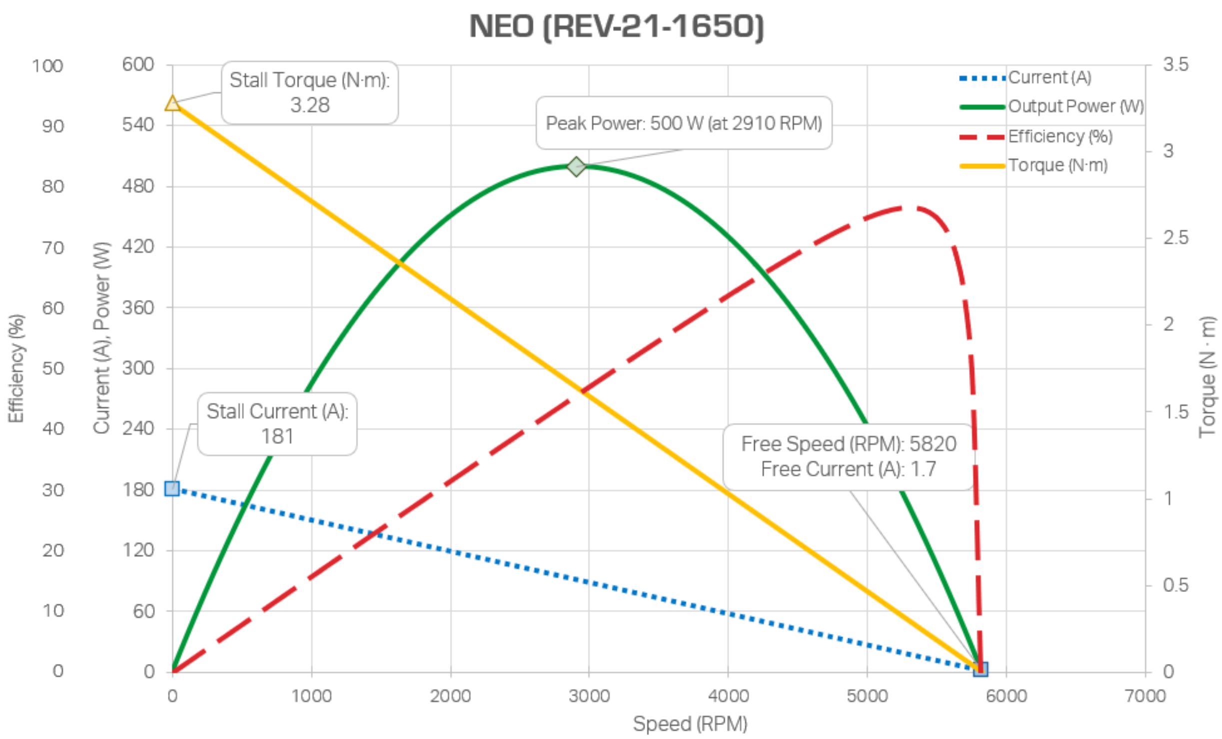

Torque vs Speed curve (remember to convert speed to rad/s first):

Intercept form for a line:

$$\frac{x}{\omega_{max}} + \frac{y}{\tau_{max}} = 1$$

$$x = \omega_{motor}$$

$$y = \tau_{motor}$$

$$\frac{\omega_{motor}}{\omega_{max}} + \frac{\tau_{motor}}{\tau_{max}} = 1$$

$$\tau_{motor} = \tau_{max}\left(1 - \frac{\omega_{motor}}{\omega_{max}}\right)$$

Motor net torque equation:

$$\Sigma\tau = \tau_{motor} - \tau_{load} = \tau_{system}$$

Same axis/rigid body:

$$\alpha_{system} = \alpha_{motor} = \alpha_{load} = \alpha$$

$$\omega_{system} = \omega_{motor} = \omega_{load} = \omega$$

Substitution:

$$\tau_{system} = \tau_{max}\left(1 - \frac{\omega}{\omega_{max}}\right) - \tau_{load}$$

Definition (Newton's second law for rotation):

$$\tau_{system} = I_{system}\alpha$$

Substitution:

$$I_{system}\alpha = \tau_{max}\left(1 - \frac{\omega}{\omega_{max}}\right) - \tau_{load}$$

$$\alpha = \frac{\tau_{max}\left(1 - \frac{\omega}{\omega_{max}}\right) - \tau_{load}}{I_{system}}$$

Perpendicular axis theorem:

$$I_{system} = I_{motor} + I_{load}$$

Definition of angular acceleration:

$$\alpha = \frac{d\omega}{dt}$$

Separation of variables (recall α is a function of ω):

$$dt = \frac{d\omega}{\alpha}$$

$$\int_0^t dt = \int_0^\omega \frac{1}{\alpha} d\omega$$

$$t = \int_0^\omega \frac{I_{system}}{\tau_{max}\left(1 - \frac{\omega}{\omega_{max}}\right) - \tau_{load}} d\omega$$

U substitution:

$$u = 1 - \frac{\omega}{\omega_{max}}$$

$$du = -\frac{1}{\omega_{max}} d\omega$$

$$t = -I_{system}\omega_{max} \int_1^{1-\frac{\omega}{\omega_{max}}} \frac{1}{\tau_{max}u - \tau_{load}} du$$

I heard you like U substitution:

$$v = \tau_{max}u - \tau_{load}$$

$$dv = \tau_{max}du$$

$$t = -\frac{I_{system}\omega_{max}}{\tau_{max}} \int_{\tau_{max}-\tau_{load}}^{\tau_{max}\left(1-\frac{\omega}{\omega_{max}}\right)-\tau_{load}} \frac{1}{v} dv$$

$$t = -\frac{I_{system}\omega_{max}}{\tau_{max}} [\ln|v|]_{\tau_{max}-\tau_{load}}^{\tau_{max}\left(1-\frac{\omega}{\omega_{max}}\right)-\tau_{load}}$$

$$t = -\frac{I_{system}\omega_{max}}{\tau_{max}} \left[\ln\left|\tau_{max}\left(1-\frac{\omega}{\omega_{max}}\right)-\tau_{load}\right| - \ln|\tau_{max}-\tau_{load}|\right]$$

$$t = -\frac{I_{system}\omega_{max}}{\tau_{max}} \ln\left|\frac{\tau_{max}\left(1-\frac{\omega}{\omega_{max}}\right)-\tau_{load}}{\tau_{max}-\tau_{load}}\right|$$

The above formula gives time as a function of angular velocity. Plotting the curve shows that the function is asymptotic and will never reach maximum speed. However, spin up time can be calculated as the time it takes for the system to get within a certain percentage of that maximum speed.

From earlier:

$$\tau_{motor} - \tau_{load} = \tau_{system}$$

Since τ_{motor} is a line, in terms of transformations, applying a load torque shifts the τ_{motor} vs ω graph downward. Thus, the ω intercept, the maximum speed, decreases. Setting the torque of our system to 0 will allow us to find our new ω intercept. Note: the follow expression was found earlier.

$$\tau_{system} = 0 = \tau_{max}\left(1-\frac{\omega}{\omega_{max}}\right) - \tau_{load}$$

$$\tau_{load} = \tau_{max}\left(1-\frac{\omega}{\omega_{max}}\right)$$

$$\frac{\tau_{load}}{\tau_{max}} = 1 - \frac{\omega}{\omega_{max}}$$

$$\frac{\omega}{\omega_{max}} = 1 - \frac{\tau_{load}}{\tau_{max}}$$

$$\omega_{int} = \omega_{max}\left(1-\frac{\tau_{load}}{\tau_{max}}\right)$$

Let ω = 99% of its new maximum speed

$$\omega = 0.99\omega_{max}\left(1-\frac{\tau_{load}}{\tau_{max}}\right)$$

Substitution:

$$t = -\frac{I_{system}\omega_{max}}{\tau_{max}} \ln \left| \frac{\tau_{max}\left(1 - \frac{0.99\omega_{max}\left(1 - \frac{\tau_{load}}{\tau_{max}}\right)}{\omega_{max}}\right) - \tau_{load}}{\tau_{max} - \tau_{load}} \right|$$

$$t = -\frac{I_{system}\omega_{max}}{\tau_{max}} \ln \left| \frac{\tau_{max}\left(1 - 0.99\left(1 - \frac{\tau_{load}}{\tau_{max}}\right)\right) - \tau_{load}}{\tau_{max} - \tau_{load}} \right|$$

$$t = -\frac{I_{system}\omega_{max}}{\tau_{max}} \ln \left| \frac{\tau_{max}\left(1 - 0.99 + 0.99\frac{\tau_{load}}{\tau_{max}}\right) - \tau_{load}}{\tau_{max} - \tau_{load}} \right|$$

$$t = -\frac{I_{system}\omega_{max}}{\tau_{max}} \ln \left| \frac{\tau_{max} - 0.99\tau_{max} + 0.99\tau_{load} - \tau_{load}}{\tau_{max} - \tau_{load}} \right|$$

$$t = -\frac{I_{system}\omega_{max}}{\tau_{max}} \ln \left| \frac{\tau_{max}(1 - 0.99) + \tau_{load}(0.99 - 1)}{\tau_{max} - \tau_{load}} \right|$$

$$t = -\frac{I_{system}\omega_{max}}{\tau_{max}} \ln \left| \frac{0.01\tau_{max} - 0.01\tau_{load}}{\tau_{max} - \tau_{load}} \right|$$

$$\boxed{t = -\frac{I_{system}\omega_{max}}{\tau_{max}} \ln 0.01}$$

In words, the spin up time of a motor is equal to the moment of inertia of the system multiplied by the maximum angular velocity capable by the motor multiplied by the natural log of the compliment of the percent of maximum speed divided by the maximum torque capable by the motor.

From this, we can see that the spin up time increases with the load moment of inertia but not with load torque. Aka. Attaching a heavy wheel to the motor will take the motor longer to reach max speed but applying friction to the rotor shaft will not affect the time to reach max speed. What the load torque does affect is the magnitude of that max speed.

$$\omega_{int} = \omega_{max}\left(1 - \frac{\tau_{load}}{\tau_{max}}\right)$$

The equation for time in terms of angular velocity can be solved for angular velocity in terms of time, not shown. This produces a graph like the one below. The red line is the angular velocity vs time graph. The blue is 99% of max speed. The purple is the spin up time.

The above graph follows a similar shape to that found experimentally for nerf motors that I found on reddit.