Chapter 2: Number Fields and Number Rings

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Exercise 2.28. *Let $f(x) = x^3 + ax + b$, a and $b \in \mathbb{Z}$, and assume f is irreducible over $\mathbb{Q}$. Let α be a root of f.*

(a) *Show that $f'(\alpha) = -\frac{2a\alpha+3b}{\alpha}$.*

(b) *Show that $2a\alpha + 3b$ is a root of*

$$\left(\frac{x-3b}{2a}\right)^3 + a\left(\frac{x-3b}{2a}\right) + b.$$

Use this to find $N^{\mathbb{Q}[\alpha]}_{\mathbb{Q}}(2a\alpha + 3b)$.

(c) *Show that $disc(\alpha) = -(4a^3 + 27b^2)$.*

(d) *Suppose $\alpha^3 = \alpha + 1$. Prove that $\{1, \alpha, \alpha^2\}$ is an integral basis for $\mathbb{A} \cap \mathbb{Q}[\alpha]$. (See Exercise 2.27(e).) Do the same if $\alpha^3 + \alpha = 1$.*

Proof of (a).

(1) *Show that $\alpha \neq 0$.* If α were 0, then $f(\alpha) = f(0) = b$. So $f(x) = x^3 + ax = x(x^2 + a)$ is reducible, contrary to the irreducibility of f.

(2) Since α be a root of f, $f(\alpha) = 0$, or $\alpha^3 + a\alpha + b = 0$, or $\alpha^3 = -a\alpha - b$.

(3)

$$\begin{aligned}
f'(x) = 3x^2 + a \Longrightarrow f'(\alpha) &= 3\alpha^2 + a \\
\Longleftrightarrow \alpha f'(\alpha) &= 3\alpha^3 + a\alpha && (\alpha \neq 0) \\
\Longleftrightarrow \alpha f'(\alpha) &= 3(-a\alpha - b) + a\alpha && (\alpha^3 = -a\alpha - b) \\
\Longleftrightarrow \alpha f'(\alpha) &= -2a\alpha - 3b.
\end{aligned}$$

So $f'(\alpha) = -\frac{2a\alpha+3b}{\alpha}$.

□

Proof of (b).

(1) Since $\alpha^3 + a\alpha + b = 0$,

$$\left(\frac{(2a\alpha+3b)-3b}{2a}\right)^3 + a\left(\frac{(2a\alpha+3b)-3b}{2a}\right) + b = 0.$$

That is, $2a\alpha + 3b$ is a root of $\left(\frac{x-3b}{2a}\right)^3 + a\left(\frac{x-3b}{2a}\right) + b$.

(2) $N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(2a\alpha+3b)$ is the product of three roots of $\left(\frac{x-3b}{2a}\right)^3 + a\left(\frac{x-3b}{2a}\right) + b$. Hence,

$$\begin{aligned} N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(2a\alpha+3b) &= (2a)^3\left[\left(\frac{-3b}{2a}\right)^3 + a\cdot\frac{-3b}{2a} + b\right] \\ &= 8a^3\left[\frac{-27b^3}{8a^3} - \frac{b}{2}\right] \\ &= -27b^3 - 4a^3 b. \end{aligned}$$

□

Proof of (c).

$$\begin{aligned} \operatorname{disc}(\alpha) &= (-1)^{\frac{n(n-1)}{2}} N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(f'(\alpha)) && \text{(Theorem 2.8)} \\ &= -N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}\left(-\frac{2a\alpha+3b}{\alpha}\right) && (n=3 \text{ and (a)}) \\ &= \frac{N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(2a\alpha+3b)}{N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(\alpha)} \\ &= \frac{-27b^3-4a^3b}{b} && \text{((b))} \\ &= -27b^2 - 4a^3. \end{aligned}$$

□

Proof of (d).

(1) (a) $\alpha^3 = \alpha + 1$, or $\alpha^3 - \alpha - 1 = 0$.
  (b) $f(x) = x^3 - x - 1$ is irreducible over $\mathbb{Q}$ since $f(x)$ is irreducible over $\mathbb{Z}/3\mathbb{Z}$.
  (c) $\operatorname{disc}(\alpha) = -23$ (by (c)).
  (d) Since $\operatorname{disc}(\alpha)$ is squarefree, the result is established (Exercise 2.27(e)).

(2) (a) $\alpha^3 + \alpha = 1$, or $\alpha^3 + \alpha - 1 = 0$.
  (b) $f(x) = x^3 + x - 1$ is irreducible over $\mathbb{Q}$ since $f(x)$ is irreducible over $\mathbb{Z}/2\mathbb{Z}$.
  (c) $\operatorname{disc}(\alpha) = -31$ (by (c)).
  (d) Since $\operatorname{disc}(\alpha)$ is squarefree, the result is established (Exercise 2.27(e)).

□

Exercise 2.43. *Let $f(x) = x^5 + ax + b$, a and $b \in \mathbb{Z}$, and assume f is irreducible over $\mathbb{Q}$. Let α be a root of f.*

(a) *Show that* $\mathrm{disc}(\alpha) = 4^4a^5 + 5^4b^4$. *(Suggestion: See Exercise 2.28.)*

(b) *Suppose* $\alpha^5 = \alpha + 1$. *Prove that* $\mathbb{A} \cap \mathbb{Q}[\alpha] = \mathbb{Z}[\alpha]$. *(*$x^5 - x - 1$ *is irreducible over* $\mathbb{Q}$*; this can be shown by reducing* $\pmod 3$*.)*

(c) ...

(d) ...

Proof of (a)(Exercise 2.28). ... □

Exercise 2.44. *Let* $f(x) = x^5 + ax^4 + b$*,* a *and* $b \in \mathbb{Z}$*, and assume* f *is irreducible over* $\mathbb{Q}$*. Let* α *be a root of* f *and let* d_1, d_2, d_3 *and* d_4 *be as in Theorem 2.13.*

(a) *Show that* $\mathrm{disc}(\alpha) = b^3(4^4a^5 + 5^5b)$.

(b) ...

(c) ...

(d) ...

Proof of (a)(Exercise 2.28). ... □

Exercise 2.45. *Obtain a formula for* $\mathrm{disc}(\alpha)$ *if* α *is a root of an irreducible polynomial* $x^n + ax + b$ *over* $\mathbb{Q}$*. Do the same for* $x^n + ax^{n-1} + b$*.*

Proof (Exercise 2.28). ... □