Chapter 1: Curves

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Section 1-1: Introduction

Classical differential geometry: the study of local properties of curves and surfaces.

Global differential geometry: the study of the influence of the local properties on the behavior of the entire curve and surface.

No exercises.

Section 1-2: Parametrized Curves

Exercise 1-2.1. *Find a parametrized curve $\alpha(t)$ whose trace is the circle $x^2 + y^2 = 1$ such that $\alpha(t)$ runs clockwise around the circle with $\alpha(0) = (0, 1)$.*

Proof. $\alpha(t) = (\sin t, \cos t)$, $t \in \mathbb{R}$. $\square$

Exercise 1-2.2. *Let $\alpha(t)$ be a parametrized curve which does not pass through the origin. If $\alpha(t_0)$ is the point of the trace of α closest to the origin and $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.*

Proof. Let $f(t) = |\alpha(t)|^2 = \alpha(t) \cdot \alpha(t)$. $f(t)$ is differentiable and $f(t)$ has a local minimum at a point $t = t_0 \in I$. So $f'(t_0) = 0$. [Theorem 5.8 in *W. Rudin, Principles of Mathematical Analysis, 3rd edition.*] Since

$$f'(t) = 2\alpha(t) \cdot \alpha'(t),$$

$f'(t_0) = 2\alpha(t_0) \cdot \alpha'(t_0) = 0$, or $\alpha(t_0) \cdot \alpha'(t_0) = 0$. Since $\alpha(t_0) \neq 0$ and $\alpha'(t_0) \neq 0$, $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$. $\square$

Exercise 1-2.3. *A parametrized curve $\alpha(t)$ has a property that its second derivative $\alpha''(t)$ is identically zero. What can be said about α?*

$\alpha(t)$ is a straight line.

Proof. Since $\alpha''(t)$ is identically zero, $\alpha'(t) = a$ is a constant. [Theorem 5.11 in W. Rudin, Principles of Mathematical Analysis, 3rd edition.] Define

$f(t) = \alpha(t) - at$ (on I). Since $f'(t) = \alpha'(t) - a = 0$, $f(t) = \alpha(t) - at = b$ is a constant again. Therefore, $\alpha(t) = at + b$, which is a straight line (on I). □

Exercise 1-2.4. *Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve and let $v \in \mathbb{R}^3$ be a fixed vector. Assume that $\alpha'(t)$ is orthogonal to v for all $t \in I$ and that $\alpha(0)$ is orthogonal to v. Prove that $\alpha(t)$ is orthogonal to v for all $t \in I$.*

Need to assume that $\alpha(t) \neq 0$ for all $t \in I$.

Proof. Given any $t \neq 0 \in I$. (Nothing to do at $t = 0$.) Define $f : I \to \mathbb{R}$ by $f(t) = \alpha(t) \cdot v$. By the mean value theorem, there exists a point ξ between 0 and t such that

$$f(t) - f(0) = f'(\xi)(t - 0),$$

where $f'(t) = \alpha'(t) \cdot v + \alpha(t) \cdot v' = \alpha'(t) \cdot v$. Note that $f(0) = 0$ since $\alpha(0)$ is orthogonal to v, and $f'(\xi) = 0$ since $\alpha'(t)$ is orthogonal to v. So the identity is reduced to

$$f(t) = 0,$$

or $\alpha(t) \cdot v = 0$, or $\alpha(t)$ is orthogonal to v. □

Exercise 1-2.5. *Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve, with $\alpha'(t) \neq 0$ for all $t \in I$. Show that $|\alpha(t)|$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.*

The same trick in Exercise 1-2.2.

Proof. It is equivalent to *show that $|\alpha(t)|^2$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.* Let

$$f(t) = |\alpha(t)|^2 = \alpha(t) \cdot \alpha(t).$$

Notice that $\alpha'(t) \neq 0$, and thus

$$\begin{aligned}
&|\alpha(t)| \text{ is a nonzero constant} \\
\Longleftrightarrow\ & f(t) = |\alpha(t)|^2 \text{ is a nonzero constant} \\
\Longleftrightarrow\ & f'(t) = 0 \text{ and } f(t) \text{ is a nonzero constant} \\
\Longleftrightarrow\ & \alpha(t) \cdot \alpha'(t) = 0 \text{ and } \alpha(t) \text{ is a nonzero constant} \\
\Longleftrightarrow\ & \alpha(t) \text{ is orthogonal to } \alpha'(t) \text{ for all } t \in I.
\end{aligned}$$

□

Section 1-3: Regular Curves; Arc Length

Exercise 1-3.1. *Show that the tangent lines to the regular parametrized curve $\alpha(t) = (3t, 3t^2, 2t^3)$ make a constant angle with the line $y = 0$, $z = x$.*

Proof. $\alpha'(t) = (3, 6t, 6t^2)$. The line $y = 0$, $z = x$ is $\beta(t) = (1, 0, 1)$. The cosine of the angle θ between these to curves is

$$\begin{aligned}
\cos\theta &= \frac{(3, 6t, 6t^2) \cdot (1, 0, 1)}{|(3, 6t, 6t^2)||(1, 0, 1)|} \\
&= \frac{3 + 6t^2}{\sqrt{3^2 + (6t)^2 + (6t^2)^2}\sqrt{2}} \\
&= \frac{3 + 6t^2}{\sqrt{9 + 36t^2 + 36t^4}\sqrt{2}} \\
&= \frac{3 + 6t^2}{\sqrt{(3 + 6t^2)^2}\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}.
\end{aligned}$$

(Notice $3 + 6t^2 > 0$ for all $t \in \mathbb{R}$.) That is, the angle between α' and β is a constant $(= \pi/4)$. $\square$

Exercise 1-3.2. *A circular disk of radius 1 in the plane xy rolls without slipping along the x axis. The figure described by a point of the circumference of of the disk is called a **cycloid** (Figure 1-7 in Mantredo P. do Carmo, Differential Geometry of Curves and Surfaces).*

(a) *Obtain a parametrized curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ the trace of which is the cycloid and determine its singular points.*

(b) *Compute the arc length of the cycloid corresponding to a complete rotation of the disk.*

Proof of (a).

(1) Since

$$\begin{cases} x = t - \sin t \\ y = 1 - \cos t, \end{cases}$$

we define $\alpha(t) = (t - \sin t, 1 - \cos t)$.

(2) $\alpha'(t) = (1 - \cos t, \sin t)$. $\alpha'(t) = 0$ if and only if $t = 2n\pi$ where $n \in \mathbb{Z}$. That is, all singular points are $\alpha(2n\pi) = (2n\pi, 0)$ where $n \in \mathbb{Z}$.

$\square$

Proof of (b). The arc length of the cycloid corresponding to a complete rotation of the disk is

$$\begin{aligned}
\int_0^{2\pi} |\alpha'(t)|dt &= \int_0^{2\pi} \sqrt{(1-\cos t)^2 + (\sin t)^2}dt \\
&= \int_0^{2\pi} \sqrt{2}\sqrt{1-\cos t}dt \\
&= \int_0^{2\pi} 2\sin\frac{t}{2}dt \\
&= \left[-4\cos\frac{t}{2}\right]_{t=0}^{t=2\pi} \\
&= 8.
\end{aligned}$$

□

Supplement. The cycloid is not an algebraic curve.

Exercise 1-3.4. *Let $\alpha : (0, \pi) \to \mathbb{R}^2$ be given by*

$$\alpha(t) = \left(\sin t, \cos t + \log\tan\frac{t}{2}\right),$$

*where t is the angle that the y axis makes with the vector $\alpha(t)$. The trace of α is called the **tractrix**. (Figure 1-9 in Mantredo P. do Carmo, Differential Geometry of Curves and Surfaces). Show that*

(a) *α is a differentiable parametrized curve, regular except at $t = \frac{\pi}{2}$.*

(b) *The length of the segment of the tangent of the tractrix between the point of tangency and the y axis is constantly equal to 1.*

Proof of (a).

$$\begin{aligned}
\alpha'(t) &= \left(\cos t, -\sin t + \frac{1}{\tan\frac{t}{2}}\frac{1}{\cos^2\frac{t}{2}}\frac{1}{2}\right) \\
&= \left(\cos t, -\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right) \\
&= \left(\cos t, \frac{\cos^2 t}{\sin t}\right)
\end{aligned}$$

exists. And $\alpha'(t) = 0$ if and only if $t = \frac{\pi}{2}$. That is, there is an unique singular point at $t = \frac{\pi}{2}$. □

Proof of (b). The the tangent line of the tractrix through the regular point t is parametrized by $\beta : \mathbb{R} \to \mathbb{R}^2$ which is defined by

$$\begin{aligned}\beta(u) &= u\alpha'(t) + \alpha(t) \\ &= \left(u\cos t + \sin t, u\frac{\cos^2 t}{\sin t} + \cos t + \log\tan\frac{t}{2} \right).\end{aligned}$$

By construction, this tangent line $\beta(u)$ meets the tractrix at $u = 0$, and meets the y-axis when $u\cos t + \sin t = 0$ or $u = -\tan t$. So the length of the segment is

$$\begin{aligned}|\beta(0) - \beta(-\tan t)| &= \sqrt{(-\tan t\cos t)^2 + \left(-\tan t\frac{\cos^2 t}{\sin t}\right)^2} \\ &= \sqrt{(\sin t)^2 + (\cos t)^2} \\ &= 1.\end{aligned}$$

□

Exercise 1-3.8. *Let $\alpha : I \to \mathbb{R}^3$ be a differentiable curve and let $[a, b] \subseteq I$ be a closed interval. For every partition*

$$a = t_0 < t_1 < \cdots < t_n = b$$

of $[a, b]$, consider the sum

$$\sum_{i=1}^{n} |\alpha(t_i) - \alpha(t_{i-1})| = l(\alpha, P),$$

where P stands for the given partition. The norm $|P|$ of a partition P is defined as

$$|P| = \max(t_i - t_{i-1}), i = 1, \cdots, n.$$

Geometrically, $l(\alpha, P)$ is the length of a polygon inscribed in $\alpha([a, b])$ with vertices in $\alpha(t_i)$ (see Figure 1-3 in Mantredo P. do Carmo, Differential Geometry of Curves and Surfaces). The point of the exercise is to show that the arc length of $\alpha([a, b])$ is, in some sense, a limit of lengths of inscribed polygons. Prove that given $\varepsilon > 0$ there exists $\delta > 0$ such that if $|P| < \delta$ then

$$\left| \int_a^b |\alpha'(t)|dt - l(\alpha, P) \right| < \varepsilon.$$

Assume that $\alpha'(t)$ is continuous.

Proof. Given $\varepsilon > 0$.

(1) Since $\alpha'(t)$ is continuous on a compact set $[a, b]$, $\alpha'(t)$ is uniformly continuous, that is, there there exists $\delta > 0$ such that

$$|\alpha'(s) - \alpha'(t)| < \frac{\varepsilon}{2(b-a)} \text{ whenever } |s - t| < \delta.$$

(2) Let $P = \{a = t_0, t_1, \ldots, t_n = b\}$ be a partition of $[a, b]$, with $\Delta t_i = t_i - t_{i-1} < \delta$ for all $i = 1, \ldots, n$. If $t_{i-1} \le t \le t_i$, it follows that

$$|\alpha'(t_i)| - \frac{\varepsilon}{2(b-a)} \le |\alpha'(t)| \le |\alpha'(t_i)| + \frac{\varepsilon}{2(b-a)}.$$

Hence,

$$\begin{aligned}
&\int_{t_{i-1}}^{t_i} |\alpha'(t)|dt \\
\ge&|\alpha'(t_i)|\Delta t_i - \frac{\varepsilon}{2(b-a)}\Delta t_i \\
=&\left|\int_{t_{i-1}}^{t_i} [\alpha'(t) + \alpha'(t_i) - \alpha'(t)]dt\right| - \frac{\varepsilon}{2(b-a)}\Delta t_i \\
\ge&\left|\int_{t_{i-1}}^{t_i} \alpha'(t)dt\right| - \left|\int_{t_{i-1}}^{t_i} [\alpha'(t_i) - \alpha'(t)]dt\right| - \frac{\varepsilon}{2(b-a)}\Delta t_i \\
\ge&|\alpha(t_i) - \alpha(t_{i-1})| - \frac{\varepsilon}{b-a}\Delta t_i
\end{aligned}$$

and

$$\begin{aligned}
&\int_{t_{i-1}}^{t_i} |\alpha'(t)|dt \\
\le&|\alpha'(t_i)|\Delta t_i + \frac{\varepsilon}{2(b-a)}\Delta t_i \\
=&\left|\int_{t_{i-1}}^{t_i} [\alpha'(t) + \alpha'(t_i) - \alpha'(t)]dt\right| + \frac{\varepsilon}{2(b-a)}\Delta t_i \\
\le&\left|\int_{t_{i-1}}^{t_i} \alpha'(t)dt\right| + \left|\int_{t_{i-1}}^{t_i} [\alpha'(t_i) - \alpha'(t)]dt\right| + \frac{\varepsilon}{2(b-a)}\Delta t_i \\
\le&|\alpha(t_i) - \alpha(t_{i-1})| + \frac{\varepsilon}{b-a}\Delta t_i.
\end{aligned}$$

(3) If we add these inequalities, we obtain

$$l(\alpha, P) - \varepsilon \le \int_a^b |\alpha'(t)|dt \le l(\alpha, P) + \varepsilon.$$

□

Exercise 1-3.10. *(Straight Lines as Shortest.) Let* $\alpha : I \to \mathbb{R}^3$ *be a parametrized curve. Let* $[a, b] \subseteq I$ *and set* $\alpha(a) = p$, $\alpha(b) = q$.

(a) *Show that, for any constant vector* v, $|v| = 1$,

$$(q - p) \cdot v = \int_a^b \alpha'(t) \cdot v dt \leq \int_a^b |\alpha'(t)| dt.$$

(b) *Set*

$$v = \frac{q - p}{|q - p|}$$

and show that

$$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)| dt;$$

that is, the curve of shortest length from $\alpha(a)$ *to* $\alpha(b)$ *is the straight line joining these points.*

Assume $p \neq q$ (otherwise $v = \frac{q-p}{|q-p|}$ is meaningless).

Proof of (a). Let $f(t) = \alpha(t) \cdot v$ defined on I. By the fundamental theorem of calculus,

$$\int_a^b f'(t) dt = f(b) - f(a).$$

Since $f'(t) = \alpha'(t) \cdot v$,

$$(\alpha(b) - \alpha(a)) \cdot v = \int_a^b \alpha'(t) \cdot v dt.$$

Therefore,

$$\begin{aligned} (q - p) \cdot v &= \int_a^b \alpha'(t) \cdot v dt \\ &\leq \int_a^b |\alpha'(t) \cdot v| dt \\ &\leq \int_a^b |\alpha'(t)||v| dt \\ &= \int_a^b |\alpha'(t)| dt. \end{aligned}$$

□

Proof of (b). $|v| = \frac{|q-p|}{|q-p|} = 1$. So,

$$\begin{aligned}(q-p)\cdot\frac{q-p}{|q-p|} &\leq \int_a^b |\alpha'(t)|dt,\\ |q-p| &\leq \int_a^b |\alpha'(t)|dt.\end{aligned}$$

□

Section 1-4: The Vector Product in $\mathbb{R}^3$

Exercise 1-4.1. *Check whether the following bases are positive:*

(a) *The basis* $\{(1,3),(4,2)\}$ *in* $\mathbb{R}^2$.

(b) *The basis* $\{(1,3,5),(2,3,7),(4,8,3)\}$ *in* $\mathbb{R}^3$.

Proof of (a). Write $u = (1,3)$ and $v = (4,2)$. Then

$$\det(u,v) = \begin{vmatrix}1 & 3\\ 4 & 2\end{vmatrix} = -10 < 0.$$

Thus $\{u,v\}$ is negative w.r.t. the natural order basis $\{e_1 = (1,0), e_2 = (0,1)\}$. □

Proof of (b). Write $u = (1,3,5)$, $v = (2,3,7)$, $w = (4,8,3)$. Then

$$\det(u,v,w) = \begin{vmatrix}1 & 3 & 5\\ 2 & 3 & 7\\ 4 & 8 & 3\end{vmatrix} = 39 > 0.$$

Thus $\{u,v,w\}$ is positive w.r.t. the natural order basis $\{e_1,e_2,e_3\}$. □

Exercise 1-4.2. *A plane P contained in $\mathbb{R}^3$ is given by the equation $ax+by+cz+d=0$. Show that the vector $v=(a,b,c)$ is perpendicular to the plane and that $|d|/\sqrt{a^2+b^2+c^2}$ measures the distance from the plane to the origin* $(0,0,0)$.

Say v is a normal vector of E.

In general, the distance from the plane E to any point $(x_0,y_0,z_0)\in\mathbb{R}^3$ is

$$\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}.$$

Proof.

(1) To show $v = (a, b, c)$ is perpendicular to the plane, it suffices to show that $v \cdot u = 0$ for any vector u lying on the plane E. Write $u = \overrightarrow{PQ}$ where $P = (x_1, y_1, z_1) \in E$ and $Q = (x_2, y_2, z_2) \in E$. Hence $u = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

$$\begin{aligned} v \cdot u &= (a, b, c) \cdot (x_2 - x_1, y_2 - y_1, z_2 - z_1) \\ &= a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) \\ &= (ax_2 + by_2 + cz_2) - (ax_1 + by_1 + cz_1) \\ &= (-d) - (-d) \\ &= 0. \end{aligned}$$

(2) Pick any point $(x_1, y_1, z_1) \in E$. The distance from the plane E to the point (x_0, y_0, z_0) is

$$\begin{aligned} &\left| (x_1 - x_0, y_1 - y_0, z_1 - z_0) \cdot \frac{v}{|v|} \right| \\ =&\left| (x_1 - x_0, y_1 - y_0, z_1 - z_0) \cdot \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}} \right| \\ =&\frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ =&\frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ =&\frac{|-d - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ =&\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}. \end{aligned}$$

□

Exercise 1-4.3. *Determine the angle of intersection of the two planes* $5x + 3y + 2z - 4 = 0$ *and* $3x + 4y - 7z = 0$.

Proof.

(1) The angle of intersection of the two planes is equal to a angle between two normal vectors of planes.

(2) Let

   (a) the angle of intersection of the two planes be θ.
   (b) the normal vector of $5x + 3y + 2z - 4 = 0$ be $n_1 = (5, 3, 2)$.
   (c) the normal vector of $3x + 4y - 7z = 0$ be $n_2 = (3, 4, -7)$.

(3) Hence,

$$\cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{13}{2\sqrt{703}}.$$

$\theta = \cos^{-1}\left(\frac{13}{2\sqrt{703}}\right)$.

□

Exercise 1-4.13. *Let $u(t) = (u_1(t), u_2(t), u_3(t))$ and $v(t) = (v_1(t), v_2(t), v_3(t))$ be differentiable maps from the interval (a, b) into $\mathbb{R}^3$. If the derivatives $u'(t)$ and $v'(t)$ satisfy the conditions*

$$u'(t) = au(t) + bv(t), v'(t) = cu(t) - av(t),$$

where a, b, and c are constants, show that $u(t) \wedge v(t)$ is a constant vector.

Proof. Since

$$\begin{aligned}
\frac{d}{dt}(u(t) \wedge v(t)) &= u'(t) \wedge v(t) + u(t) \wedge v'(t) \\
&= (au(t) + bv(t)) \wedge v(t) + u(t) \wedge (cu(t) - av(t)) \\
&= au(t) \wedge v(t) + u(t) \wedge (-av(t)) \\
&= a(u(t) \wedge v(t)) + (-a)(u(t) \wedge v(t)) \\
&= (0, 0, 0),
\end{aligned}$$

$u(t) \wedge v(t)$ is a constant vector. □

Section 1-5: The Local Theory of Curves Parametrized by Arc Length

Exercise 1-5.2. *Show that the torsion τ of α is given by*

$$\tau(s) = -\frac{\alpha'(s) \wedge \alpha''(s) \cdot \alpha'''(s)}{|\kappa(s)|^2}.$$

Proof.

(1) Take inner product $n(s)$ to the definition of torsion $\tau(s)n(s) = b'(s)$, we have

$$\tau(s) = b'(s) \cdot n(s).$$

Since $b'(s) = t(s) \wedge n'(s)$, we have to compute $n'(s)$ first.

(2) Compute $n'(s)$.

$$n'(s) = \frac{d}{ds}\left(\frac{\alpha''(s)}{\kappa(s)}\right) = \frac{\alpha'''(s)}{\kappa(s)} - \frac{\alpha''(s)\kappa'(s)}{\kappa(s)^2}.$$

(3) By (1)(2),

$$\begin{aligned}
\tau(s) &= b'(s) \cdot n(s) \\
&= (t(s) \wedge n'(s)) \cdot n(s) \\
&= \left(\alpha'(s) \wedge \left(\frac{\alpha'''(s)}{\kappa(s)} - \frac{\alpha''(s)\kappa'(s)}{\kappa(s)^2}\right)\right) \cdot \frac{\alpha''(s)}{\kappa(s)} \\
&= \left(\alpha'(s) \wedge \frac{\alpha'''(s)}{\kappa(s)}\right) \cdot \frac{\alpha''(s)}{\kappa(s)} \\
&= \frac{\alpha'(s) \wedge \alpha'''(s) \cdot \alpha''(s)}{|\kappa(s)|^2},
\end{aligned}$$

or

$$\tau(s) = \frac{\alpha'(s) \wedge \alpha'''(s) \cdot \alpha''(s)}{\alpha''(s)^2}.$$

□

Section 1-6: The Local Canonical Form

Section 1-7: Global Properties of Plane Curves