Chapter 4: Limits and Continuity

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Continuity of real-valued functions

Exercise 4.19. *Let f be continuous on $[a, b]$ and define g as follows: $g(a) = f(a)$ and, for $a < x \leq b$, let $g(x)$ be the maximum value of f in the subinterval $[a, x]$. Show that g is continuous on $[a, b]$.*

Indeed, $g(x) = \max_{a \leq t \leq x} f(t)$ for $x \in [a, b]$.

Proof.

(1) f is continuous on $[a, b]$ at a point $p \iff$ Given any $\epsilon' > 0$, there exists $\delta' > 0$ such that $|f(x) - f(p)| < \epsilon'$ whenever $|x - p| < \delta'$ (and $x \in [a, b]$). We left ϵ' and δ' undecided temporarily.

(2) To estimate g on
$$[p - \delta', p + \delta'] \cap [a, b],$$
we need to study the behavior of f on $[a, p + \delta'] \cap [a, b]$ (by the definition of $g(x)$), and then use the continuity of f to establish the desired result.

(3) Look at where f takes the maximum value over on $[a, p + \delta'] \cap [a, b]$ at. There are two possible cases (might overlapped):

   (a) *At a point in* $[a, p - \delta'] \cap [a, b]$. In this case g is constant on $[p - \delta', p + \delta'] \cap [a, b]$, or $|g(x) - g(p)| = 0$.

   (b) *At a point* $q \in (p - \delta', p + \delta'] \cap [a, b]$. For any $x \in [p - \delta', p + \delta'] \cap [a, b]$,

      (i) $f(p) - \epsilon' < g(x)$ by the maximality of g on $[a, x]$.

      (ii) $g(x) \leq f(q) < f(p) + \epsilon'$ since g is an increasing function and f takes the maximum value over on $[a, p + \delta'] \cap [a, b]$ at $q \in (p - \delta', p + \delta'] \cap [a, b]$.

      By (i)(i),
      $$f(p) - \epsilon' < g(x) < f(p) + \epsilon'$$
      for any $x \in [p - \delta', p + \delta'] \cap [a, b]$ (especially $x = p$). Therefore,
      $$|g(x) - g(p)| < 2\epsilon' \text{ whenever } |x - p| < \delta' (\text{and } x \in [a, b]).$$

   By (a)(b), we have $|g(x) - g(p)| < 2\epsilon'$ whenever $|x - p| < \delta'$(and $x \in [a, b]$) in any cases.

(4) Retake $\epsilon' = \frac{\epsilon}{2} > 0$ and $\delta = \delta' > 0$.

$\square$

Continuity in metric spaces

In Exercise 4.29 through 4.33, we assume that $f : S \to T$ is a function from one metric space (S, d_S) to another (T, d_T).

Exercise 4.29. *Prove that f is continuous on S if and only if*

$$f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ \quad \textit{for every subset } B \textit{ of } T.$$

Denote the interior of any set S by S° for convenience sake.

Proof. Only assume S and T are topological spaces.

(1) ($\Longrightarrow$) Given any $x \in f^{-1}(B^\circ)$. $f(x) \in B^\circ$. So there is an open neighborhood $V \subseteq B^\circ \subseteq B$ containing $f(x)$. So $x \in f^{-1}(V) \subseteq f^{-1}(B)$. Since f is continuous, the inverse image $f^{-1}(V)$ is open in S. Hence, $f^{-1}(V)$ is an open neighborhood containing x in $f^{-1}(B)$. $x \in (f^{-1}(B))^\circ$.

(2) ($\Longleftarrow$) Given any open subset V of T. $V = V^\circ$ clearly. So

$$f^{-1}(V) = f^{-1}(V^\circ) \subseteq (f^{-1}(V))^\circ \subseteq f^{-1}(V),$$

that is, $f^{-1}(V) = (f^{-1}(V))^\circ$, or the inverse image $f^{-1}(V)$ is open in S for every open subset V of T, or f is continuous.

$\square$