Notes on the book:
Ash, Probability and Measure Theory, 2nd edition

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August 8, 2022

Contents

Chapter 1: Fundamentals of Measure and Integration Theory

1.1. Introduction

Problem 1.1.1.

Establish formulas (1)-(5).

Formulas.

(1) If $A_n \uparrow A$, then $A_n^c \downarrow A^c$; If $A_n \downarrow A$, then $A_n^c \uparrow A^c$.

(2)

$$\begin{aligned}\bigcup_{i=1}^{n} A_i = A_1 \cup (A_1^c \cap A_2) \cup (A_1^c \cap A_2^c \cap A_3) \\ \cup \cdots \cup (A_1^c \cap \cdots \cap A_{n-1}^c \cap A_n).\end{aligned}$$

(3) Furthermore,

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \Big(A_1^c \cap \cdots \cap A_{n-1}^c \cap A_n \Big).$$

(4) If the A_n form an increasing sequence, then

$$\bigcup_{i=1}^{n} A_i = A_1 \cup (A_2 - A_1) \cup \cdots \cup (A_n - A_{n-1}).$$

(5) If the A_n form an increasing sequence, then

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} (A_n - A_{n-1})$$

(take A_0 as the empty set).

Proof of Formula (1).

(1) Suppose that $A_n \uparrow A$ is an increasing sequence of sets with limit A. Then $A_1 \subset A_2 \subset \cdots$ and $\bigcup_{n=1}^{\infty} A_n = A$. So $A_1^c \supset A_2^c \supset \cdots$ and

$$\bigcap_n A_n^c = \Big(\bigcup_n A_n \Big)^c = A^c$$

by the De Morgan laws. Hence $A_n \uparrow A$ implies that $A_n^c \downarrow A^c$.

(2) Conversely, suppose that $A_n \downarrow A$ is an decreasing sequence of sets with limit A. Then $A_1 \supset A_2 \supset \cdots$ and $\bigcap_{n=1}^{\infty} A_n = A$. So $A_1^c \subset A_2^c \subset \cdots$ and

$$\bigcup_n A_n^c = \left(\bigcap_n A_n \right)^c = A^c$$

by the De Morgan laws. Hence $A_n \downarrow A$ implies that $A_n^c \uparrow A^c$.

□

Proof of Formula (2).

(1) Set

$$B_i = A_1^c \cap \cdots \cap A_{i-1}^c \cap A_i$$

for $i = 1, \cdots, n$. Observe that $B_1 = A_1$. So it is equivalent to show that

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i.$$

(2) Since each B_i is a subset of A_i, $\bigcup_{i=1}^n A_i \supset \bigcup_{i=1}^n B_i$.

(3) Conversely, given any $x \in \bigcup_{i=1}^n A_i$. $x \in A_j$ for some j. Now take the minimal value of j such that $x \in A_j$. The minimality of j implies that $x \notin A_1, A_2, \cdots, A_{j-1}$. Hence

$$x \in A_1^c \cap \cdots \cap A_{j-1}^c \cap A_j = B_j \subset \bigcup_{i=1}^{n} B_i.$$

Therefore, $\bigcup_{i=1}^n A_i \subset \bigcup_{i=1}^n B_i$.

(4) By (2)(3), $\bigcup_{i=1}^n A_i$ and $\bigcup_{i=1}^n B_i$ are equal.

□

Proof of Formula (3). Same as the proof of formula (2) since the minimality of j described in part (3) exists. □

Proof of Formula (4).

(1) As A_n form an increasing sequence, $A_1 \subset A_2 \subset \cdots$ or $A_1^c \supset A_2^c \supset \cdots$. Hence

$$A_1^c \cap \cdots \cap A_{i-1}^c = A_{i-1}^c.$$

Therefore, B_i is reduced to

$$B_i = A_1^c \cap \cdots \cap A_{i-1}^c \cap A_i = A_{i-1}^c \cap A_i = A_i - A_{i-1}.$$

(2) Now formula (2) becomes

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n}(A_i - A_{i-1}).$$

$\square$

Proof of Formula (5). Note that $B_n = A_n - A_{n-1}$ in the proof of formula (4). Formula (3) becomes $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty}(A_n - A_{n-1})$. $\square$

Problem 1.1.2.

Define sets of real numbers as follows. Let $A_n = (-\frac{1}{n}, 1]$ *if* n *is odd, and* $A_n = (-1, \frac{1}{n}]$ *if* n *is even. Find* $\limsup_n A_n$ *and* $\liminf_n A_n$.

Proof.

(1) Write

$$\begin{aligned}
\bigcup_{k=n}^{\infty} A_k &= \left(\bigcup_{k=\lfloor \frac{n}{2} \rfloor}^{\infty} A_{2k+1} \right) \cup \left(\bigcup_{k=\lfloor \frac{n+1}{2} \rfloor}^{\infty} A_{2k} \right) \\
&= \left(\bigcup_{k=\lfloor \frac{n}{2} \rfloor}^{\infty} \left(-\frac{1}{2k+1}, 1 \right] \right) \cup \left(\bigcup_{k=\lfloor \frac{n+1}{2} \rfloor}^{\infty} \left(-1, \frac{1}{2k} \right] \right) \\
&= \left(-\frac{1}{2\lfloor \frac{n}{2} \rfloor + 1}, 1 \right] \cup \left(-1, \frac{1}{2\lfloor \frac{n+1}{2} \rfloor} \right] \\
&= (-1, 1]
\end{aligned}$$

for each k. Hence

$$\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \bigcap_{n=1}^{\infty} (-1, 1] = (-1, 1].$$

(2) Similarly, for each k we have

$$\begin{aligned}
\bigcap_{k=n}^{\infty} A_k &= \left(\bigcap_{k=\lfloor \frac{n}{2} \rfloor}^{\infty} A_{2k+1} \right) \cap \left(\bigcap_{k=\lfloor \frac{n+1}{2} \rfloor}^{\infty} A_{2k} \right) \\
&= \left(\bigcap_{k=\lfloor \frac{n}{2} \rfloor}^{\infty} \left(-\frac{1}{2k+1}, 1 \right] \right) \cap \left(\bigcap_{k=\lfloor \frac{n+1}{2} \rfloor}^{\infty} \left(-1, \frac{1}{2k} \right] \right) \\
&= [0, 1] \cup (-1, 0] \\
&= \{0\}.
\end{aligned}$$

Hence

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} \{0\} = \{0\}.$$

□

Problem 1.1.5.

Establish formulas (10)-(13).

Formulas.

(10)

$$\left(\limsup_n A_n\right)^c = \liminf_n A_n^c.$$

(11)

$$\left(\liminf_n A_n\right)^c = \limsup_n A_n^c.$$

(12)

$$\liminf_n A_n \subset \limsup_n A_n.$$

(13) If $A_n \uparrow A$ or $A_n \downarrow A$, then $\liminf_n A_n = \limsup_n A_n = A$.

Proof of Formula (10). The De Morgan laws shows that

$$\begin{aligned}\left(\limsup_n A_n\right)^c &= \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right)^c \\ &= \bigcup_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k\right)^c \\ &= \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k^c \\ &= \limsup_n A_n^c.\end{aligned}$$

□

Proof of Formula (11). Similar to the proof of formula (10).

$$\begin{aligned}
\left(\liminf_n A_n\right)^c &= \left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k\right)^c \\
&= \bigcap_{n=1}^{\infty}\left(\bigcap_{k=n}^{\infty} A_k\right)^c \\
&= \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k^c \\
&= \limsup_n A_n^c.
\end{aligned}$$

□

Proof of Formula (12). Formulas (7) and (9) give all. □

Proof of Formula (13).

(1) If $A_n \uparrow A$, then

$$\limsup_n A_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k = \bigcap_{n=1}^{\infty} A = A$$

and

$$\liminf_n A_n = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} A_n = A.$$

(2) If $A_n \downarrow A$, then

$$\limsup_n A_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k = \bigcap_{n=1}^{\infty} A_n = A$$

and

$$\liminf_n A_n = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} A = A.$$

□

Problem 1.1.6.

Let $A = (a, b)$ and $B = (c, d)$ be disjoint open intervals of $\mathbb{R}$, and let $C_n = A$ if n is odd, $C_n = B$ if n is even. Find $\limsup_n C_n$ and $\liminf_n C_n$.

Proof.

(1)

$$\limsup_n C_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} C_k = \bigcap_{n=1}^{\infty} (A \cup B) = A \cup B.$$

(2)

$$\liminf_n C_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} C_k = \bigcup_{n=1}^{\infty} \varnothing = \varnothing.$$

□

1.2. Fields, σ-Fields, and Measures

Problem 1.2.5.

Let μ be a nonnegative, finitely additive set function on the field $\mathscr{F}$. If $A_1, A_2, \dots$ are disjoint sets in $\mathscr{F}$ and $\bigcup_{n=1}^{\infty} A_n \in \mathscr{F}$, show that

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \geq \sum_{n=1}^{\infty} \mu(A_n).$$

Proof.

(1) Note that μ is a nonnegative, finitely additive set function on $\mathscr{F}$. Hence,

$$\begin{aligned} \mu\left(\bigcup_{n=1}^{\infty} A_n\right) &\geq \mu\left(\bigcup_{n=1}^{m} A_n\right) && \text{(Theorem 1.2.5)} \\ &= \sum_{n=1}^{m} \mu(A_n) \end{aligned}$$

for every m.

(2) Since $\sum_{n=1}^{m} \mu(A_n)$ is bounded by $\mu\big(\bigcup_{n=1}^{\infty} A_n\big)$ and μ is nonnegative, the result is established as letting $m \to \infty$.

□