Chapter 1: Set Theory

Author: Meng-Gen Tsai
Email: plover@gmail.com

Problem 1.1. *Show that $\{x : x \neq x\} = \varnothing$.*

Proof. Every element x of $\{x : x \neq x\}$ satisfying $x \neq x$, contrary to $x = x$. That is, there are no elements in $\{x : x \neq x\}$, or $\{x : x \neq x\} = \varnothing$. □

Problem 1.2. *Show that if $x \in \varnothing$, then x is a green-eyed lion.*

It is a wrong statement because there is a green-eyed lion in the world. In fact, we can show that *if $x \in \varnothing$, then x is the Chinese Communist Party which is kind to people.*

Proof. Liu Xiaobo knew it. □

Problem 1.4. *Show that the well-ordering principle implies the principle of mathematical induction. (Hint: Consider the set $\{n \in \mathbb{N} : P(n)$ is false$\}$.)*

Proof (Hint). Suppose that

(1) $P(n)$ be a proposition defined for each $n \in \mathbb{N}$,

(2) $P(1)$ is true,

(3) $[P(n) \Rightarrow P(n+1)]$ is true.

Consider the set
$$S = \{n \in \mathbb{N} : P(n) \text{ false}\} \subseteq \mathbb{N}.$$

Want to show *S is empty, or the principle of mathematical induction holds.* If S were nonempty, by the well-ordering principle S has a smallest element m. m cannot be 1 by (2). Say $m > 1$. Therefore, $m - 1 \in \mathbb{N}$ and $P(m-1)$ is true by the minimality of m. By (3), $P((m-1)+1) = P(m)$ is true, which is absurd. □

Problem 1.5. *Use mathematical induction to establish that the well-ordering principle. (Hint: Given a set S of positive integers, let $P(n)$ be the proposition 'If $n \in S$, then S has a least element'.)*

Proof (Modified hint).

(1) Given a set S of positive integers, let $P(n)$ be the proposition 'If $m \in S$ for some $m \leq n$, then S has a least element'. Want to show $P(n)$ is true for all $n \in \mathbb{N}$.

   (a) $P(1)$ is true. For $m \in S$ with $m \leq n = 1$, or $m = 1$ by the minimality of $1 \in \mathbb{N}$, S has a least element 1 (m itself) in $\mathbb{N}$.

   (b) Suppose $P(n)$ is true. If $n + 1 \in S$, then there are only two possible cases.

      (i) There is a positive integer $m \in S$ less than $n+1$. So $n \geq m \in S$. Since $P(n)$ is true, S has a least element.

      (ii) There is no positive integer $m \in S$ less than $n + 1$. In this case $n + 1$ is the least element in S.

      In any cases (i)(ii), S has a least element, or $P(n + 1)$ is true.

   By mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

(2) *Show that the well-ordering principle holds.* Let T be a nonempty subset of $\mathbb{N}$, so there exists a positive integer $k \in T$. Notice that $P(k)$ is true by (1), thus T has a least element since $k \leq k$.

$\square$

Problem 1.9. *Show that $A \subseteq B \Leftrightarrow A \cap B = A \Leftrightarrow A \cup B = B$.*

Proof.

(1) $A \subseteq B \Longleftrightarrow A \cap B = A$.

   (a) ($\Longrightarrow$) It suffices to show $A \cap B \supseteq A$. For any $x \in A$, $x \in B$ by $A \subseteq B$, so $x \in A \cap B$, so $A \cap B \supseteq A$.

   (b) ($\Longleftarrow$) $A = A \cap B \subseteq B$.

(2) $A \subseteq B \Leftrightarrow A \cup B = B$.

   (a) ($\Longrightarrow$) It suffices to show $A \cup B \subseteq B$. For any $x \in A \cup B$, $x \in A$ or $x \in B$. By $A \subseteq B$, $x \in B$ or $x \in B$. $x \in B$, so $A \cup B \subseteq B$.

   (b) ($\Longleftarrow$) $A \subseteq A \cup B = B$.

$\square$

Problem 1.9. *Show that $A \subseteq B \Leftrightarrow \widetilde{B} \subseteq \widetilde{A}$.*

Proof.

$$A \subseteq B \iff x \in A \Rightarrow x \in B$$
$$\iff x \notin B \Rightarrow x \notin A$$
$$\iff \widetilde{B} \subseteq \widetilde{A}.$$

□

Problem 1.14. *Show that*

$$B \cap \left[\bigcup_{A \in \mathscr{C}} A \right] = \bigcup_{A \in \mathscr{C}} (B \cap A).$$

Proof.

$$x \in B \cap \left[\bigcup_{A \in \mathscr{C}} A \right] \iff x \in B \text{ and } x \in \bigcup_{A \in \mathscr{C}} A$$
$$\iff x \in B \text{ and } x \in A \text{ for some } A \in \mathscr{C}$$
$$\iff x \in B \cap A \text{ for some } A \in \mathscr{C}$$
$$\iff x \in \bigcup_{A \in \mathscr{C}} (B \cap A).$$

□