

# Chapter 1: Vector Spaces

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## Section 1.2: Vector Spaces

**Exercise 1.2.2.** Write the zero vector of  $M_{3 \times 4}(F)$ .

*Proof.*  $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in M_{3 \times 4}(F)$ .  $\square$

**Exercise 1.2.22.** How many elements are there in the vector space  $M_{m \times n}(\mathbb{Z}/2\mathbb{Z})$ ?

*Proof.*  $2^{mn}$ .  $\square$

## Section 1.6: Bases and Dimension

**Exercise 1.6.19.** Let  $V$  be a vector space having dimension  $n$ , and let  $S$  be a subset of  $V$  that generates  $V$ .

- (a) Prove that there is a subset of  $S$  that is a basis for  $V$ . (Be careful not to assume that  $S$  is finite.)
- (b) Prove that  $S$  contains at least  $n$  elements.

*Proof of (a).* Similar to the argument in Theorem 1.9.

- (1) If  $S = \emptyset$  or  $S = \{0\}$ , then  $V = \{0\}$  and  $\emptyset$  is a subset of  $S$  that is a basis for  $V$ .
- (2) Otherwise  $S$  contains a nonzero element  $u_1$ .  $\{u_1\}$  is a linearly independent set. Continue, if possible, choosing elements  $u_2, \dots, u_k$  in  $S$  such that  $\{u_1, u_2, \dots, u_k\}$  is linearly independent. By the Replacement Theorem (Theorem 1.10), we must eventually reach a stage at which  $\beta = \{u_1, u_2, \dots, u_k\}$  is a linearly independent subset of  $S$  with  $k \leq n$ .
- (3)  $\beta$  generates  $S$  by the construction of  $\beta$ , and  $S$  generates  $V$ . Therefore,  $\beta$  generates  $V$  (and thus  $k = n$  by the definition of dimension).

Therefore, there is a subset of  $S$  that is a basis for  $V$ .  $\square$

*Proof of (b).* By (a), there is a subset  $\beta \subseteq S$  of size  $n$  that is a basis for  $V$ . So  $S$  contains at least  $n$  elements of  $\beta$ .  $\square$