

Solutions to the book:  
*Jürgen Neukirch, Algebraic Number  
Theory*

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# Chapter I: Algebraic Integers

## I.1. The Gaussian Integers

### Exercise I.1.1.

$\alpha \in \mathbb{Z}[i]$  is a unit if and only if  $N(\alpha) = 1$ .

*Proof.*

- (1) ( $\implies$ ) Since  $\alpha$  is a unit, there is  $\beta \in \mathbb{Z}[i]$  such that  $\alpha\beta = 1$ . So  $N(\alpha\beta) = N(1)$ , or  $N(\alpha)N(\beta) = 1$ . Since the image of  $N$  is nonnegative integers,  $N(\alpha) = 1$ .
- (2) ( $\impliedby$ )  $N(\alpha) = \alpha\bar{\alpha}$ , or  $1 = \alpha\bar{\alpha}$  since  $N(\alpha) = 1$ . That is,  $\bar{\alpha} \in \mathbb{Z}[i]$  is the inverse of  $\alpha \in \mathbb{Z}[i]$ . (Or we solve the equation  $N(\alpha) = a^2 + b^2 = 1$ , and show that all four solutions ( $\pm 1$  and  $\pm i$ ) are unit.)
- (3) Conclusion: a unit  $\alpha = a + bi$  of  $\mathbb{Z}[i]$  is satisfying the equation  $N(\alpha) = a^2 + b^2 = 1$  by (1)(2). That is, the only unit of  $\mathbb{Z}[i]$  are  $\pm 1$  and  $\pm i$ .

□