Chapter 1: Vector Spaces

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Section 1.6: Bases and Dimension

Exercise 1.6.19. *Let V be a vector space having dimension n, and let S be a subset of V that generates V.*

(a) Prove that there is a subset of S that is a basis for V. (Be careful not to assume that S is finite.)

(b) Prove that S contains at least n elements.

Proof of (a). Similar to the argument in Theorem 1.9.

(1) If $S = \varnothing$ or $S = \{0\}$, then $\mathsf{V} = \{0\}$ and $\varnothing$ is a subset of S that is a basis for V.

(2) Otherwise S contains a nonzero element u_1. $\{u_1\}$ is a linearly independent set. Continue, if possible, choosing elements $u_2, ..., u_k$ in S such that $\{u_1, u_2, ..., u_k\}$ is linearly independent. By the Replacement Theorem (Theorem 1.10), we must eventually reach a stage at which $\beta = \{u_1, u_2, ..., u_k\}$ is a linearly independent subset of S with $k \leq n$.

(3) β generates S by the construction of β, and S generates V. Therefore, β generates V (and thus $k = n$ by the definition of dimension).

Therefore, there is a subset of S that is a basis for V. $\square$

Proof of (b). By (a), there is a subset $\beta \subseteq S$ of size n that is a basis for V. So S contains at least n elements of β. $\square$