Chapter 11: The Lebesuge Theory

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Exercise 11.1. *If $f \geq 0$ and $\int_E f d\mu = 0$, prove that $f(x) = 0$ almost everywhere on E. (Hint: Let E_n be the subset of E on which $f(x) > \frac{1}{n}$. Write $A = \bigcup E_n$. Then $\mu(A) = 0$ if and only if $\mu(E_n) = 0$ for every n.)*

Might assume that f is measurable on E.

Proof (Hint).

(1) Define $A = \{x \in E : f(x) > 0\}$. So $f(x) = 0$ almost everywhere on E if and only if $\mu(A) = 0$.

(2) Define

$$E_n = \left\{ x \in E : f(x) > \frac{1}{n} \right\}$$

for $n = 1, 2, 3, \ldots$. Note that $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$ and

$$A = \bigcup_{n=1}^{\infty} E_n.$$

Since μ is a measure,

$$\lim_{n \to \infty} \mu(E_n) = \mu(A)$$

(Theorem 11.3).

(3) (Reductio ad absurdum) If $\mu(A) > 0$, there is an integer N such that $\mu(E_n) \geq \frac{\mu(A)}{2}$ whenever $n \geq N$ (by (2)). In particular, take $n = N$ to get

$$\begin{aligned}
\int_E f d\mu &\geq \int_{E_N} f d\mu & &(\mu \text{ is a measure and } E_N \subseteq E) \\
&\geq \frac{1}{N} \cdot \mu(E_N) & &(\text{Remarks 11.23(b)}) \\
&\geq \frac{1}{N} \cdot \frac{\mu(A)}{2} \\
&> 0,
\end{aligned}$$

contrary to the assumption that $\int_E f d\mu = 0$.

$\square$

Note. Compare to Exercise 6.2.

Exercise 11.2. *If $\int_A f d\mu = 0$ for every measurable subset A of a measurable set E, then $f(x) = 0$ almost everywhere on E.*

Might assume that f is measurable on E.

Proof.

(1) Define

$$A = \{x \in E : f(x) \geq 0\} \qquad \text{and} \qquad B = \{x \in E : f(x) \leq 0\}.$$

A and B are measurable subsets of a measurable set E since f is measurable.

(2) Apply Exercise 11.1 to the fact that $f \geq 0$ on A (by construction) and $\int_A f d\mu = 0$ (by assumption), we have $f(x) = 0$ almost everywhere on A.

(3) Similarly, apply Exercise 11.1 to the fact that $-f \geq 0$ on B and $\int_B (-f) d\mu = -\int_B f d\mu = 0$, we have $f(x) = 0$ almost everywhere on B.

(4) As $E = A \cup B$, $f(x) = 0$ almost everywhere on E by (2)(3).

□

Exercise 11.3. *If $\{f_n\}$ is a sequence of measurable functions, prove that the set of points x at which $\{f_n(x)\}$ converges is measurable.*

Proof.

(1) It suffices to show that

$$E = \{x : \{f_n(x)\} \text{ is convergent } \} = \{x : \{f_n(x)\} \text{ is Cauchy } \}$$

is measurable (since $\mathbb{R}^1$ is complete).

(2) Write

$$E = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n,m \geq N} \left\{ x : |f_n(x) - f_m(x)| \leq \frac{1}{k} \right\}$$

Since $\{f_n\}$ is a sequence of measurable functions, $x \mapsto |f_n(x) - f_m(x)|$ is measurable (Theorem 11.16 and Theorem 11.18). Hence

$$\left\{ x : |f_n(x) - f_m(x)| \leq \frac{1}{k} \right\}$$

is measurable (Theorem 11.15). Therefore E is measurable.

□

Exercise 11.4. *If $f \in \mathscr{L}(\mu)$ on E and g is bounded and measurable on E, then $fg \in \mathscr{L}(\mu)$ on E.*

Proof (Theorem 11.27).

(1) fg is measurable since both f and g are measurable (Theorem 11.18).

(2) $|g| \leq M$ for some real $M \in \mathbb{R}^1$ by the boundedness of g. Hence

$$|fg| \leq M|f|$$

on E.

(3) To apply Theorem 11.27, it suffices to show that $M|f| \in \mathscr{L}(\mu)$ on E. Theorem 11.26 implies that $|f| \in \mathscr{L}(\mu)$ if $f \in \mathscr{L}(\mu)$. And Remarks 11.23(d) implies that $M|f| \in \mathscr{L}(\mu)$ if $|f| \in \mathscr{L}(\mu)$.

□

Note (Riemann integral). *If $f \in \mathscr{R}$ on $[a, b]$ and g is bounded and measurable on $[a, b]$, then fg might be not Riemann integrable.*

Exercise 11.5. *Put*

$$g(x) = \begin{cases} 0 & (0 \leq x \leq \frac{1}{2}), \\ 1 & (\frac{1}{2} < x \leq 1), \end{cases}$$

and

$$\begin{aligned} f_{2k}(x) &= g(x) & (0 \leq x \leq 1), \\ f_{2k+1}(x) &= g(1-x) & (0 \leq x \leq 1). \end{aligned}$$

Show that

$$\liminf_{n \to \infty} f_n(x) = 0 \qquad (0 \leq x \leq 1),$$

but

$$\int_0^1 f_n(x) dx = \frac{1}{2}.$$

(Compare with the Fatou's theorem.)

Proof.

(1) *Show that* $\liminf_{n \to \infty} f_n(x) = 0$. Note that

$$g(1-x) = \begin{cases} 1 & (0 \leq x < \frac{1}{2}), \\ 0 & (\frac{1}{2} < x \leq 1). \end{cases}$$

Since $f_n(x) \geq 0$ by definition, $\liminf_{n\to\infty} f_n(x) \geq 0$. Since $f_{2k}(0) = f_{2k+1}(1) = 0$ for all positive integers k, $\liminf_{n\to\infty} f_n(x) \leq 0$. Therefore the result is established.

(2) *Show that* $\int_0^1 f_n(x)dx = \frac{1}{2}$. Since

$$\int_0^1 f_{2k}(x)dx = \int_0^1 g(x)dx = \frac{1}{2},$$
$$\int_0^1 f_{2k+1}(x)dx = \int_0^1 g(1-x)dx = \frac{1}{2},$$

in any case $\int_0^1 f_n(x)dx = \frac{1}{2}$ for all positive integers n.

(3) This example shows that we may have the strict inequality in the Fatou's theorem.

□

Supplement (Similar exercise). *Consider the sequence* $\{f_n\}$ *defined by* $f_n(x) = 1$ *if* $n \leq x < n+1$, *with* $f_n(x) = 0$ *otherwise. Show that we may have the strict inequality in the Fatou's theorem.*

Exercise 11.6. ...

Proof.

(1)

(2)

□

Exercise 11.7. ...

Proof.

(1)

(2)

□

Exercise 11.8. ...

Proof.

(1)

(2)

□

Exercise 11.9. ...

Proof.

(1)

(2)

□

Exercise 11.10. *If $\mu(X) < +\infty$ and $f \in \mathscr{L}^2(\mu)$ on X, prove that $f \in \mathscr{L}$ on X. If*

$$\mu(X) = +\infty,$$

this is false. For instance, if

$$f(x) = \frac{1}{1+|x|},$$

then $f^2 \in \mathscr{L}$ on $\mathbb{R}^1$, but $f \notin \mathscr{L}$ on $\mathbb{R}^1$.

Proof.

(1) Since $\mu(X) < +\infty$, $1 \in \mathscr{L}^2(\mu)$ on X. By Theorem 11.35, $f \in \mathscr{L}(\mu)$, and

$$\int_X |f| d\mu \leq \|f\| \|1\|.$$

(2) *Show that $f^2 \in \mathscr{L}$ on $\mathbb{R}^1$.* To apply Theorem 11.33, we might restrict the measure space $X = \mathbb{R}^1$ to some interval $[a, b]$. Then apply the Lebesgue's monotone convegence theorem (Theorem 11.28) to get the conclusion.

(a) Write

$$f(x)^2 = \left(\frac{1}{1+|x|} \right)^2 = \frac{1}{1+2|x|+x^2} \leq \frac{1}{1+x^2}.$$

By Theorem 11.27, *it suffices to show that $\frac{1}{1+x^2} \in \mathscr{L}$ on $\mathbb{R}^1$.*

(b) Consider the sequence $\{f_n\}$ defined by

$$f_n(x) = \frac{1}{1+x^2} \chi_{[-n,n]}(x).$$

(Here $\chi_{[-n,n]} = K_{[-n,n]}$ is the characteristic function of $[-n,n]$ defined in Definition 11.19.) By construction,

$$0 \leq f_1(x) \leq f_2(x) \leq \cdots \qquad (x \in \mathbb{R}^1)$$

and

$$f_n(x) \to \frac{1}{1+x^2} \qquad (x \in \mathbb{R}^1).$$

(c) Hence

$$\begin{aligned}
\int_{\mathbb{R}^1} \frac{1}{1+x^2} dx &= \lim_{n\to\infty} \int_{\mathbb{R}^1} f_n(x) dx && \text{(Theorem 11.28)} \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^1} \frac{1}{1+x^2} \chi_{[-n,n]}(x) dx \\
&= \lim_{n\to\infty} \int_{-n}^{n} \frac{1}{1+x^2} dx \\
&= \lim_{n\to\infty} \mathscr{R} \int_{-n}^{n} \frac{1}{1+x^2} dx && \text{(Theorem 11.33)} \\
&= \lim_{n\to\infty} 2 \arctan(n) \\
&= \pi < \infty.
\end{aligned}$$

(4) *Show that* $f \notin \mathscr{L}$ *on* $\mathbb{R}^1$.

(a) Consider the sequence $\{f_n\}$ defined by

$$f_n(x) = f(x)\chi_{[-n,n]}(x) = \frac{1}{1+|x|}\chi_{[-n,n]}(x).$$

By construction,

$$0 \leq f_1(x) \leq f_2(x) \leq \cdots \qquad (x \in \mathbb{R}^1)$$

and

$$f_n(x) \to f(x) \qquad (x \in \mathbb{R}^1).$$

(b) Hence

$$\begin{aligned}
\int_{\mathbb{R}^1} f(x) dx &= \lim_{n\to\infty} \int_{\mathbb{R}^1} f_n(x) dx && \text{(Theorem 11.28)} \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^1} \frac{1}{1+|x|} \chi_{[-n,n]}(x) dx \\
&= \lim_{n\to\infty} \int_{-n}^{n} \frac{1}{1+|x|} dx \\
&= \lim_{n\to\infty} \mathscr{R} \int_{-n}^{n} \frac{1}{1+|x|} dx && \text{(Theorem 11.33)} \\
&= \lim_{n\to\infty} 2 \log(n+1) \\
&= \infty,
\end{aligned}$$

or $f \notin \mathscr{L}$ on $\mathbb{R}^1$.

□

Note. Compare to Exercise 6.5.

Exercise 11.11. ...

Proof.

(1)

(2)

□

Exercise 11.12. ...

Proof.

(1)

(2)

□

Exercise 11.13. ...

Proof.

(1)

(2)

□

Exercise 11.14. ...

Proof.

(1)

(2)

□

Exercise 11.15. ...

Proof.

(1)

(2)

□

Exercise 11.16. ...

Proof.

(1)

(2)

□

Exercise 11.17. ...

Proof.

(1)

(2)

□

Exercise 11.18. ...

Proof.

(1)

(2)

□