

Chapter 2: Basic Topology

Author: Meng-Gen Tsai

Email: plover@gmail.com

Exercise 2.1. *Prove that the empty set is a subset of every set.*

Proof. By Definitions 1.3,

- (1) The set which contains no element will be called the **empty set**,
- (2) If A and B are sets, and if every element of A is an element of B , we say that A is a **subset** of B ,

every element of the empty set (there are none) belongs to every set. That is, the empty set is a subset of every set. \square

Exercise 2.2. *A complex number z is said to be algebraic if there are integers a_0, \dots, a_n , not all zero, such that*

$$a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic numbers is countable. (Hint: For every positive integer N there are only finitely many equations with

$$n + |a_0| + |a_1| + \dots + |a_n| = N.$$

Might assume $a_0 \neq 0$.

For example, all rational numbers are algebraic since $p = \frac{\alpha}{\beta}$ (where $\alpha, \beta \in \mathbb{Z}$) is a root of $\beta z - \alpha = 0$.

Besides, $z = \sqrt{2} + \sqrt{3}$ is algebraic since $z^4 - 10z^2 + 1 = 0$. In fact, $z = \pm\sqrt{2} \pm \sqrt{3}$ are also algebraic since $z^4 - 10z^2 + 1 = (z - \sqrt{2} - \sqrt{3})(z + \sqrt{2} - \sqrt{3})(z - \sqrt{2} + \sqrt{3})(z + \sqrt{2} + \sqrt{3})$.

Lemma. *The set of all polynomials over \mathbb{Z} is countable implies that the set of algebraic numbers is countable.*

Proof of Lemma. By definition, we write the set of algebraic numbers as

$$S = \bigcup_{f(x) \in \mathbb{Z}[x]} \{z \in \mathbb{C} : f(z) = 0\}.$$

Since each polynomial of degree n has at most n roots, $\{z \in \mathbb{C} : f(z) = 0\}$ is finite for each given $f(x) \in \mathbb{Z}[x]$. So S is a countable union (by assumption) of finite sets, and hence at most countable. S is infinite since every integer α is a root of $f(z) = z - \alpha$. So S is countable. \square

Thus, it suffices to show that *the set of all polynomials over \mathbb{Z} is countable*.

Proof (Hint). For every positive integer N there are only finitely many equations with $n + |a_0| + |a_1| + \cdots + |a_n| = N$. Write

$$P_N = \{f(x) \in \mathbb{Z}[x] : n + |a_0| + |a_1| + \cdots + |a_n| = N\}$$

where $f(x) = a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n$ with $a_0 \neq 0$, and

$$P = \bigcup_{N=1}^{\infty} P_N.$$

P is the set of all polynomials over \mathbb{Z} .

Each P_N is finite for given N (since the equation $n + |a_0| + |a_1| + \cdots + |a_n| = N$ has finitely many solutions $(n, a_0, a_1, \dots, a_n) \in \mathbb{Z}^{n+2}$). So P is a countable union of finite sets, and hence at most countable. P is infinite since \mathbb{Z} is a subring of $\mathbb{Z}[x]$. So P is countable. \square

Proof (Theorem 2.13).

- (1) \mathbb{Z}^N is countable for any integer $N > 0$. Theorem 2.13.
- (2) The set of all polynomials over \mathbb{Z} is countable. Let

$$P_n = \{f \in \mathbb{Z}[x] : \deg f = n\},$$

and

$$P = \bigcup_{n=1}^{\infty} P_n = \mathbb{Z}[x].$$

Claim: P_n is countable. Define a 1-1 map $\varphi_n : P_n \rightarrow \mathbb{Z}^{n+1}$ by

$$\varphi_n(a_0z^n + a_1z^{n-1} + \cdots + a_n) = (a_0, a_1, \dots, a_{n-1}, a_n).$$

By (1) and Theorem 2.8, P_n is countable. (P_n is infinite since $a_n \in \mathbb{Z}$.) Now P is a countable union of countable sets, and hence countable by Theorem 2.12.

\square

Proof (Unique factorization theorem).

- (1) *The set of prime numbers is countable.* Write all primes in the ascending order as $p_1, p_2, \dots, p_n, \dots$ where $p_1 = 2, p_2 = 3, \dots, p_{10001} = 104743, \dots$ (See ProjectEuler 7: 10001st prime. Use sieve of Eratosthenes to get p_{10001} .)
- (2) *The set of all polynomials over \mathbb{Z} is countable.* Let

$$P_n = \{f \in \mathbb{Z}[x] : \deg f = n\},$$

and

$$P = \bigcup_{n=1}^{\infty} P_n = \mathbb{Z}[x].$$

Claim: P_n is countable. Define a map $\varphi_n : P_n \rightarrow \mathbb{Z}^+$ by

$$\varphi_n(a_0 z^n + a_1 z^{n-1} + \dots + a_n) = p_1^{\psi(a_0)} p_2^{\psi(a_1)} \dots p_{n+1}^{\psi(a_n)},$$

where ψ is a 1-1 correspondence from \mathbb{Z} to \mathbb{Z}^+ (Example 2.5). By the unique factorization theorem, φ_n is 1-1. So P_n is countable by Theorem 2.8. (P_n is infinite since $a_n \in \mathbb{Z}$.) Now P is a countable union of countable sets, and hence countable by Theorem 2.12.

□

Exercise 2.3. *Prove that there exist real numbers which are not algebraic.*

Proof (Exercise 2.2). If all real numbers were algebraic, then \mathbb{R} is countable by Exercise 2.2, contrary to the fact that \mathbb{R} is uncountable (Corollary to Theorem 2.43). □

Proof (Liouville, 1844).

- (1) **Lemma.** *If ξ is a real algebraic number of degree $n > 1$, then there is a constant $A > 0$ (depending on ξ) such that*

$$\left| \xi - \frac{h}{k} \right| \geq \frac{A}{k^n}$$

for all rational numbers $\frac{h}{k}$.

- (a) If $\left| \xi - \frac{h}{k} \right| \geq 1$, pick $A = 1 > 0$.
- (b) If $\left| \xi - \frac{h}{k} \right| < 1$, let $f(x) = a_0 + a_1 x + \dots + a_n x^n$ be an irreducible polynomial of degree $n > 1$ over \mathbb{Z} such that $f(\xi) = 0$. By the mean value theorem,

$$f(\xi) - f\left(\frac{h}{k}\right) = \left(\xi - \frac{h}{k}\right) f'(c)$$

for some $c \in \left(\xi - \frac{h}{k}, \xi + \frac{h}{k}\right) \subseteq (\xi - 1, \xi + 1)$. Notice that

- (i) $f(\xi) = 0$ by definition.
- (ii) $f\left(\frac{h}{k}\right) \neq 0$ since $\frac{h}{k}$ cannot be a root of $f(x)$. Otherwise f is of degree 1, contrary to the assumption of f .
- (iii) $\left|f\left(\frac{h}{k}\right)\right| \geq \frac{1}{k^n}$ since

$$\begin{aligned} f\left(\frac{h}{k}\right) &= a_0 + a_1\left(\frac{h}{k}\right) + \cdots + a_n\left(\frac{h}{k}\right)^n \neq 0, \\ k^n f\left(\frac{h}{k}\right) &= a_0 k^n + h k^{n-1} a_1 + \cdots + h^n a_n \neq 0, \\ k^n \left|f\left(\frac{h}{k}\right)\right| &\geq 1. \end{aligned}$$

- (iv) $|f'(c)| \leq \sup_{x \in [\xi-1, \xi+1]} |f'(x)|$ since $c \in [\xi-1, \xi+1]$ and $f'(x)$ is continuous or bounded on a compact set $[\xi-1, \xi+1]$.

By (i)-(iv),

$$\begin{aligned} \left|f(\xi) - f\left(\frac{h}{k}\right)\right| &= \left|\left(\xi - \frac{h}{k}\right) f'(c)\right|, \\ \frac{1}{k^n} &\leq \left|f\left(\frac{h}{k}\right)\right| = \left|\xi - \frac{h}{k}\right| |f'(c)| \leq \left|\xi - \frac{h}{k}\right| \cdot \sup_{x \in [\xi-1, \xi+1]} |f'(x)|. \end{aligned}$$

Pick $A = (1 + \sup_{x \in [\xi-1, \xi+1]} |f'(x)|)^{-1} > 0$.

By (a)(b), we arrange $A = \min(1, (1 + \sup_{x \in [\xi-1, \xi+1]} |f'(x)|)^{-1}) > 0$ to fit the inequality.

- (2) $\xi = \sum_{n=0}^{\infty} 10^{-n!}$ is transcendental.

- (a) Let $k_j = 10^{j!}$, $h_j = 10^{j!} \sum_{n=0}^j 10^{-n!}$. Then

$$\left|\xi - \frac{h_j}{k_j}\right| = \sum_{n=j+1}^{\infty} 10^{-n!} < \sum_{n=(j+1)!}^{\infty} 10^{-n} = \frac{A_j}{k_j^j}$$

where $A_j = \frac{10}{9} \cdot 10^{-j!}$.

- (b) If ξ were a real algebraic number of degree $d > 1$, then by Lemma and (a),

$$\frac{A}{k_j^d} < \left|\xi - \frac{h_j}{k_j}\right| < \frac{A_j}{k_j^j} < \frac{A_j}{k_j^d}$$

for some $A > 0$ and $j \geq d$, or $0 < A < A_j$. Since j is arbitrary, $A_j \rightarrow 0$ as $j \rightarrow \infty$, contrary to $A > 0$.

- (c) If ξ were a real algebraic number of degree $d = 1$, $\xi = \frac{h}{k}$ is a rational number. So

$$\left|\xi - \frac{h_j}{k_j}\right| = \left|\frac{h}{k} - \frac{h_j}{k_j}\right| = \left|\frac{hk_j - kh_j}{kk_j}\right| \geq \left|\frac{1}{kk_j}\right| = \frac{|k|^{-1}}{k_j}$$

for all j . (It is impossible that $hk_j - kh_j = 0$ or $\frac{h}{k} = \frac{h_j}{k_j}$ since $|\frac{h}{k} - \frac{h_j}{k_j}| = \sum_{n=j+1}^{\infty} 10^{-n!} > 0$ for all j .) Again by (a),

$$\frac{|k|^{-1}}{k_j} \leq \left| \xi - \frac{h_j}{k_j} \right| < \frac{A_j}{k_j^j} < \frac{A_j}{k_j},$$

or $0 < |k|^{-1} < A_j$. (Similar to (b).) Since j is arbitrary, $A_j \rightarrow 0$ as $j \rightarrow \infty$, contrary to $|k|^{-1} > 0$.

□

Exercise 2.4. *Is the set of all irrational real numbers countable?*

Proof (Reductio ad absurdum). If $\mathbb{R} - \mathbb{Q}$ were countable, then $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} - \mathbb{Q})$ is countable (Theorem 2.12), contrary to the fact that \mathbb{R} is uncountable (Corollary to Theorem 2.43). □

Exercise 2.5. *Construct a bounded set of real numbers with exactly three limit points.*

Proof (Exercise 2.12). Let

$$K_p = \{p\} \cup \left\{ p + \frac{1}{n} : n \in \mathbb{Z}^+ \right\} \subseteq \mathbb{R}^1$$

be a compact set of \mathbb{R}^1 with exactly one limit point $p \in \mathbb{R}^1$ (Exercise 2.12). Then

$$K_{1989} \cup K_6 \cup K_4$$

is a compact set of \mathbb{R}^1 with exactly three limit points $1989, 6, 4 \in \mathbb{R}^1$. □

Exercise 2.6. *Let E' be the set of all limit points of a set E . Prove that E' is closed. Prove that E and \overline{E} have the same limit points. (Recall that $\overline{E} = E \cup E'$.) Do E and E' always have the same limit points?*

Proof.

(1) *Show that E' is closed.*

(a) *Use Definition 2.18 (d).*

(i) It suffices to show every limit point of E' is a limit point of E . Given a limit point p of E' , so that every open neighborhood U of p contains a point $q_0 \neq p$ such that $q_0 \in E'$.

- (ii) Since q_0 is a limit point of E , there is an open neighborhood V of q_0 contains a point $q \neq q_0$ such that $q \in E$, where

$$V = U \cap B\left(q_0; \frac{1}{2}d_E(p, q_0)\right) \subseteq U$$

($B(x; r)$ is the open ball with center at x and radius r).

- (iii) By the construction of V , for such open neighborhood U of p , there is $q \neq p$ and $q \in V \subseteq U$ and $q \in E$. That is, p is a limit point of E .

(b) Use Definition 2.18 (e).

- (i) To show E' is closed or $X - E'$ is open, it suffices to show every point of $X - E'$ is an interior point of $X - E'$.
- (ii) Given a point $p \in X - E'$, or p is not a limit point of E . There is an open neighborhood U of p contains no point $q \neq p$ such that $q \in E$.
- (iii) To show U is an open neighborhood of p such that $U \subseteq X - E'$, it suffices to no point $q \neq p$ such that $q \in E'$. If there were a limit point q of E such that $q \neq p$ and $q \in U$, then

$$V = U \cap B\left(q; \frac{1}{2}d_E(p, q)\right) \subseteq U$$

is an open neighborhood of q contains no point of E , contrary to the assumption $q \in E'$. So $U \subseteq X - E'$ is an open neighborhood of $p \in X - E'$.

- (2) Show that $E' = \overline{E}'$. It suffices to show $E' \supseteq \overline{E}'$. ($E' \subseteq \overline{E}'$ holds trivially since $E \subseteq \overline{E}$). Given a limit point p of $\overline{E} = E \cup E'$.

- (a) p is a limit point of E . Nothing to do.
- (b) p is a limit point of E' . Since p is a limit point of E' and E' is a closed set, $p \in E'$, or p is a limit point of E .

In any case, $E' \supseteq \overline{E}'$.

- (3) E and E' might not have the same limit points. Let

$$E = \left\{ \frac{1}{n} : n \in \mathbb{Z}^+ \right\} \subseteq \mathbb{R}^1.$$

Then $E' = \{0\}$ and thus $(E')' = \emptyset$.

□

Exercise 2.7. Let A_1, A_2, A_3, \dots be subsets of a metric space.

- (a) If $B_n = \bigcup_{i=1}^n A_i$, prove that $\overline{B_n} = \bigcup_{i=1}^n \overline{A_i}$, for $n = 1, 2, 3, \dots$
(b) If $B = \bigcup_{i=1}^{\infty} A_i$, prove that $\overline{B} \supseteq \bigcup_{i=1}^{\infty} \overline{A_i}$.

Show, by an example, that this inclusion can be proper.

Proof of (a).

- (1) Show that $\overline{B_n} \subseteq \bigcup_{i=1}^n \overline{A_i}$. Since $A_i \subseteq \overline{A_i}$ for any i , we have

$$B_n = \bigcup_{i=1}^n A_i \subseteq \bigcup_{i=1}^n \overline{A_i}.$$

Since $\bigcup_{i=1}^n \overline{A_i}$ is a union of finitely many closed set $\overline{A_i}$, $\bigcup_{i=1}^n \overline{A_i}$ is closed (Theorem 2.24(d)). By Theorem 2.27(c), $\overline{B_n} \subseteq \bigcup_{i=1}^n \overline{A_i}$.

- (2) Show that $\overline{B_n} \supseteq \bigcup_{i=1}^n \overline{A_i}$. Same argument in the proof of (b).

□

Proof of (b). Since $\bigcup_{j=1}^{\infty} A_j \supseteq A_i$ for any i , by the monotonicity of closure, we have $\overline{\bigcup_{j=1}^{\infty} A_j} \supseteq \overline{A_i}$ for any i , or $\overline{B} \supseteq \bigcup_{i=1}^{\infty} \overline{A_i}$. □

Proof of proper inclusion in (b). Let

$$A_n = \left(\frac{1}{n}, \infty \right) \subseteq \mathbb{R}^1$$

for any $n \in \mathbb{Z}^+$. Then

$$\begin{aligned} \bigcup_{n=1}^{\infty} A_n = (0, \infty) &\implies \overline{\bigcup_{n=1}^{\infty} A_n} = \overline{(0, \infty)} = [0, \infty), \\ \overline{A_n} = \left[\frac{1}{n}, \infty \right) &\implies \bigcup_{n=1}^{\infty} \overline{A_n} = \bigcup_{n=1}^{\infty} \left[\frac{1}{n}, \infty \right) = (0, \infty). \end{aligned}$$

□

Exercise 2.8. Is every point of every open set $E \subseteq \mathbb{R}^2$ a limit point of E ? Answer the same question for closed sets in \mathbb{R}^2 .

It is not true for all metric spaces X . The (discrete) metric in Exercise 2.10 implies no limit point exists in X .

Proof.

- (1) Show that for every open set $E \subseteq \mathbb{R}^k$, $E \subseteq E'$. Given any point $\mathbf{p} \in E$, we shall show \mathbf{p} is a limit point of E .

- (a) Since E is open, there is an open neighborhood $B(\mathbf{p}; r_0) \subseteq E$ for some $r_0 > 0$.
- (b) In particular, given any $s \in \mathbb{R}$ such that $0 < s < r_0$, we can find

$$\mathbf{q} \in B(\mathbf{p}; s) \subseteq B(\mathbf{p}; r_0) \subseteq E$$

such that $\mathbf{q} \neq \mathbf{p}$. Explicitly, write

$$\mathbf{p} = (p_1, \dots, p_k)$$

and choose

$$\mathbf{q} = \left(p_1 + \frac{s}{89}, p_2, \dots, p_k \right) \neq \mathbf{p}$$

(since $s > 0$). Clearly, \mathbf{q} is well-defined in \mathbb{R}^k and $|\mathbf{q} - \mathbf{p}| = \frac{s}{89} < s$ or $\mathbf{q} \in B(\mathbf{p}; s)$.

- (c) Now given every open neighborhood $B(\mathbf{p}, r)$ of \mathbf{p} . We can choose $s \in \mathbb{R}$ such that $0 < s < \min\{r_0, r\} \leq r_0$. (might pick $s = \frac{1}{64} \min\{r_0, r\}$.) By (b), there exists $\mathbf{q} \neq \mathbf{p}$ such that

$$\mathbf{q} \in B(\mathbf{p}; s) \subseteq B(\mathbf{p}; r) \subseteq E.$$

- (2) Give an example of a closed set $E \subseteq \mathbb{R}^k$ such that $E \not\subseteq E'$. Pick $E = \{\mathbf{0}\}$. So $E' = \emptyset$ and thus $E \not\subseteq E'$.

□

Exercise 2.9. Let E° denote the set of all interior points of a set E . [See Definition 2.18(e); E° is called the interior of E .]

- (a) Prove that E° is always open.
- (b) Prove that E is open if and only if $E^\circ = E$.
- (c) If G is contained in E and G is open, prove that G is contained in E° .
- (d) Prove that the complement of E° is the closure of the complement of E .
- (e) Do E and \overline{E} always have the same interiors?
- (f) Do E and E° always have the same closures?

Similar to Theorem 2.27.

Proof of (a). It is equivalent to show that $E^\circ \subseteq (E^\circ)^\circ$.

- (1) Given any point $x \in E^\circ$, there is $r > 0$ such that $B(x; r) \subseteq E$.

- (2) It suffices to show that $B(x; \frac{2}{r}) \subseteq E^\circ$. Given any point $y \in B(x; \frac{2}{r})$, we will show that there is an open neighborhood $B(y; \frac{2}{r})$ of y such that $B(y; \frac{2}{r}) \subseteq E$.
- (3) Given any point $z \in B(y; \frac{2}{r})$, we have

$$d(z, x) \leq d(z, y) + d(y, x) < \frac{2}{r} + \frac{2}{r} = r,$$

or $z \in B(x; r) \subseteq E$. Therefore, $B(y; \frac{2}{r}) \subseteq E$, or $y \in E^\circ$, or $B(x; \frac{2}{r}) \subseteq E^\circ$, or $x \in (E^\circ)^\circ$, or $E^\circ \subseteq (E^\circ)^\circ$.

□

Proof of (b).

- (1) (\implies)(Definition 2.18) Since E is open, every point of E is an interior point of E . Hence $E \subseteq E^\circ$. Note that $E^\circ \subseteq E$ is trivial, and thus $E^\circ = E$.
- (2) (\Leftarrow)(a) By (a), $E = E^\circ$ is always open.
- (3) (\Leftarrow)(Definition 2.18) Every point of E is an interior point of E since $E = E^\circ$. Hence E is open by Definition 2.18(f).

□

Proof of (c). $G \subseteq E$ implies $G^\circ \subseteq E^\circ$. $G = G^\circ$ since G is open ((b)). Hence $G = G^\circ \subseteq E^\circ$, that is, E° is the largest open set contained in E . (Similarly, \overline{E} is the smallest closed set containing E .) □

Proof of (d). Show that $X - E^\circ = \overline{X - E}$ and $(X - E)^\circ = X - \overline{E}$.

- (1) (Theorem 2.27 and (c))

$$\begin{aligned}
X - E^\circ &= X - \bigcup_{\text{Open } V \subseteq E} V \\
&= \bigcap_{\text{Open } V \subseteq E} (X - V) \\
&= \bigcap_{\text{Closed } W \supseteq X - E} W \\
&= \overline{X - E}. \\
X - \overline{E} &= X - \bigcap_{\text{Closed } W \supseteq E} W \\
&= \bigcup_{\text{Closed } W \supseteq E} (X - W) \\
&= \bigcup_{\text{Open } V \subseteq X - E} V \\
&= (X - E)^\circ.
\end{aligned}$$

(2) (Brute-force)

$$\begin{aligned}
x \in E^\circ &\iff \exists r > 0 \text{ such that } B(x; r) \subseteq E \\
&\iff \exists r > 0 \text{ such that } B(x; r) \cap (X - E) = \emptyset \\
&\iff x \notin \overline{X - E} \\
&\iff x \in X - \overline{X - E}. \\
x \in (X - E)^\circ &\iff \exists r > 0 \text{ such that } B(x; r) \subseteq (X - E) \\
&\iff \exists r > 0 \text{ such that } B(x; r) \cap E = \emptyset \\
&\iff x \notin \overline{E} \\
&\iff x \in X - \overline{E}.
\end{aligned}$$

Note that $X - E^\circ = \overline{X - E}$ is equivalent to $(X - E)^\circ = X - \overline{E}$ by mapping $E \mapsto X - E$. \square

Proof of (e). No.

- (1) Let $X = \mathbb{R}^1$ equipped with the Euclidean metric, and $E = \mathbb{Q} \subseteq X$.
- (2) $E^\circ = \emptyset$ since $\tilde{\mathbb{Q}}$ is dense in \mathbb{R} .
- (3) $(\overline{E})^\circ = (\mathbb{R}^1)^\circ = \mathbb{R}^1$ since \mathbb{Q} is dense in \mathbb{R} and \mathbb{R}^1 is open.

\square

Proof of (f). No.

- (1) Let $X = \mathbb{R}^1$ equipped with the Euclidean metric, and $E = \mathbb{Q} \subseteq X$.
- (2) $\overline{E} = \mathbb{R}^1$ since \mathbb{Q} is dense in \mathbb{R} .
- (3) $\overline{E^\circ} = \overline{\emptyset} = \emptyset$ since $\tilde{\mathbb{Q}}$ is dense in \mathbb{R} .

\square

Exercise 2.10. Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p, q) = \begin{cases} 1 & (\text{if } p \neq q) \\ 0 & (\text{if } p = q). \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

(The statement holds even if X is finite.) We called d the discrete metric, and the corresponding topology on X induces the discrete topology. Conversely, if X has the discrete topology, X is always metrizable by the discrete metric.

Proof.

- (1) $d(p, q)$ is a metric.
- (a) $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$. Trivial.
 - (b) $d(p, q) = d(q, p)$. Trivial.
 - (c) $d(p, q) \leq d(p, r) + d(r, q)$ for any $r \in X$. If $p = q$, nothing to do. If $p \neq q$, $r \neq p$ or $r \neq q$ for any $r \in X$. (Assume not true, $r = p$ and $r = q$ implies that $p = q$ which is a contradiction.) In any case $d(p, r) + d(r, q) \geq 1 = d(p, q)$.
- (2) *Every subset is open.* Let E be any subset of X . Then every point $p \in E$ is an interior point of E . In fact, we can pick one open neighborhood $U = B(p; \frac{1}{2})$ of p containing only one point $p \in E$ or $U = \{p\}$, and such open neighborhood U is a subset of E . So every subset of X is open.
- (3) *Every subset is closed.* Since every subset is open, every subset is closed by Theorem 2.23.

Supplement. Might use Definition 2.18 (d) to prove directly since there are no limit points in X if we consider one open neighborhood $U = B(p; \frac{1}{2})$ of p . Therefore, every subset is closed. Again we apply Theorem 2.23 to get that every subset is open without using Definition 2.18 (f).

- (4) *A subset is compact iff it is finite.*
- (a) *Any finite subset is compact.* Say $E = \{p_1, p_2, \dots, p_k\}$, and $\{G_\alpha\}$ be an open covering of E . From $\{G_\alpha\}$ we pick G_{α_1} containing p_1 , G_{α_2} containing p_2 , ..., and G_{α_k} containing p_k . This process can be done in the finitely many steps. Therefore,

$$\{G_{\alpha_1}, G_{\alpha_2}, \dots, G_{\alpha_k}\}$$

is a finite subcovering of $\{G_\alpha\}$ covering E .

- (b) *Any infinite subset is not compact.* Take a collection

$$\mathcal{G} = \{G_p = \{p\}\}$$

of open subsets where p runs all points in E . Clearly, $\{G_p\}$ is an open covering. Assume

$$\mathcal{G}' = \{G_{p_1}, G_{p_2}, \dots, G_{p_k}\}$$

is any finite subcovering of \mathcal{G} . Since E is infinite, there exist a point $p \in E$ such that $p \neq p_1, p \neq p_2, \dots, p \neq p_k$. Therefore, \mathcal{G}' does not cover p , or \mathcal{G} does not contain any finite subcovering \mathcal{G}' .

□

Notice that every subset is bounded. Therefore, every subset is closed and bounded, but only finite subset is compact, i.e., Heine-Borel theorem is not true in the infinite discrete topology.

Exercise 2.11. For $x \in \mathbb{R}^1$ and $y \in \mathbb{R}^1$, define

$$\begin{aligned} d_1(x, y) &= (x - y)^2, \\ d_2(x, y) &= \sqrt{|x - y|}, \\ d_3(x, y) &= |x^2 - y^2|, \\ d_4(x, y) &= |x - 2y|, \\ d_5(x, y) &= \frac{|x - y|}{1 + |x - y|}. \end{aligned}$$

Determine, for each of these, whether it is a metric or not.

Proof.

- (1) $d = d_1$ is not a metric. (Reductio ad absurdum) If d were a metric, then

$$d(0, 2) > d(0, 1) + d(1, 2),$$

contrary to Definition 2.15(c) that $d(p, q) \leq d(p, r) + d(r, q)$.

- (2) $d = d_2$ is a metric. It suffices to show that $d'(x, y) = \sqrt{d(x, y)}$ is a metric if $d(x, y)$ is a metric. For any $p, q, r \in \mathbb{R}^1$,

(a) $d'(p, q) = \sqrt{d(p, q)} > 0$ if $p \neq q$; $d'(p, p) = \sqrt{d(p, p)} = 0$.

(b) $d'(p, q) = \sqrt{d(p, q)} = \sqrt{d(q, p)} = d'(q, p)$.

(c)

$$\begin{aligned} \sqrt{d(p, r) + d(r, q)} &\leq \sqrt{d(p, r)} + \sqrt{d(r, q)} \\ \iff (\sqrt{d(p, r)} + \sqrt{d(r, q)})^2 &\leq (\sqrt{d(p, r)} + \sqrt{d(r, q)})^2 \\ \iff d(p, r) + d(r, q) &\leq d(p, r) + d(r, q) + 2\sqrt{d(p, r)}\sqrt{d(r, q)} \\ \iff 0 &\leq 2\sqrt{d(p, r)}\sqrt{d(r, q)}. \end{aligned}$$

(d)

$$\begin{aligned} d'(p, q) &= \sqrt{d(p, q)} \\ &\leq \sqrt{d(p, r) + d(r, q)} && \text{(Triangle inequality)} \\ &\leq \sqrt{d(p, r)} + \sqrt{d(r, q)} && ((c)) \\ &= d'(p, r) + d'(r, q). \end{aligned}$$

By Definition 2.15, d' is a metric.

(3) $d = d_3$ is not a metric. (Reductio ad absurdum) If d were a metric, then

$$d(1, -1) = 0,$$

contrary to Definition 2.15(a): $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$.

(4) $d = d_4$ is not a metric. (Reductio ad absurdum) If d were a metric, then

$$d(1, 1) = 1,$$

contrary to Definition 2.15(a): $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$.

(5) $d = d_5$ is a metric. It suffices to show that $d'(x, y) = \frac{d(x, y)}{1+d(x, y)}$ is a metric if $d(x, y)$ is a metric. For any $p, q, r \in \mathbb{R}^1$,

$$(a) \quad d'(p, q) = \frac{d(p, q)}{1+d(p, q)} > 0 \text{ if } p \neq q; \quad d'(p, p) = \frac{d(p, p)}{1+d(p, p)} = 0.$$

$$(b) \quad d'(p, q) = \frac{d(p, q)}{1+d(p, q)} = \frac{d(q, p)}{1+d(q, p)} = d'(q, p).$$

(c) Write $x = d(p, q)$, $y = d(p, r)$ and $z = d(r, q)$. So $x, y, z \geq 0$ and

$$\begin{aligned} x &\leq y + z \\ \iff x + x(y + z) &\leq y + z + x(y + z) \\ \iff x(1 + y + z) &\leq (1 + x)(y + z) \\ \iff \frac{x}{1 + x} &\leq \frac{y + z}{1 + y + z}. \end{aligned}$$

(d)

$$\begin{aligned} d'(p, q) &= \frac{d(p, q)}{1 + d(p, q)} \\ &\leq \frac{d(p, r) + d(r, q)}{1 + d(p, r) + d(r, q)} && ((c)) \\ &= \frac{d(p, r)}{1 + d(p, r) + d(r, q)} + \frac{d(r, q)}{1 + d(p, r) + d(r, q)} \\ &= \frac{d(p, r)}{1 + d(p, r)} + \frac{d(r, q)}{1 + d(r, q)} \\ &= d'(p, r) + d'(r, q). \end{aligned}$$

(e) Or we can show $d'(p, q) \leq d'(p, r) + d'(r, q)$ by

$$\begin{aligned} \frac{x}{1 + x} &\leq \frac{y}{1 + y} + \frac{z}{1 + z} \\ \iff x(1 + y)(1 + z) &\leq y(1 + z)(1 + x) + z(1 + x)(1 + y) \\ \iff x + xy + xz + xyz &\leq (y + xy + yz + xyz) + (z + xz + yz + xyz) \\ \iff x &\leq y + z + 2yz + xyz \\ \iff x &\leq y + z && (d \text{ is nonnegative}) \end{aligned}$$

By Definition 2.15, d' is a metric.

□

Exercise 2.12. Let $K \subseteq \mathbb{R}^1$ consist of 0 and the numbers $\frac{1}{n}$, for $n = 1, 2, 3, \dots$. Prove that K is compact directly from the definition (without using the Heine-Borel theorem).

Proof. Let $\{G_\alpha\}$ be an open covering of K . There is an open set $G_0 \in \{G_\alpha\}$ containing 0. So there exists an open neighborhood $U = B(0; r)$ of 0 such that $U \subseteq G_0$. So U contains all points $q = \frac{1}{n}$ of K whenever $n > \frac{1}{r}$. To construct a finite subcovering of $\{G_\alpha\}$, we need to pick finitely many open sets from $\{G_\alpha\}$ to cover the remaining points $q = \frac{1}{n}$ where $n = 1, 2, \dots, [\frac{1}{r}]$, say G_1 contains $q = \frac{1}{1}$, G_2 contains $q = \frac{1}{2}$, ..., $G_{[\frac{1}{r}]}$ contains $q = \frac{1}{[\frac{1}{r}]}$. (Might be duplicated.)

Hence,

$$\left\{G_0, G_1, G_2, \dots, G_{[\frac{1}{r}]}\right\}$$

is a finite subcovering of $\{G_\alpha\}$ covering K . □

Proof (Heine-Borel theorem).

(1) K is closed. In fact, the only limit point of K is 0, which is in K .

- (a) $p = 0$ is a limit point. Given $r > 0$. There always exists $n \in \mathbb{Z}^+$ such that $r > \frac{1}{n}$. So any open neighborhood $B(0; r)$ of $p = 0$ contains at least one point $q = \frac{1}{n} \neq 0$ in K .
- (b) $p < 0$ is not a limit point. Pick an open neighborhood $B(p; r)$ of p where $r = |p| > 0$. Then $B(p; r) \cap K = \emptyset$.
- (c) $p > 0$ is not a limit point. There always exists $m \in \mathbb{Z}^+$ such that $p > \frac{1}{m}$ whenever $n \geq m$. Pick an open neighborhood $B(p; r)$ of p where $r = p - \frac{1}{m} > 0$. Then $B(p; r)$ does not have all points $q = \frac{1}{n} \in K$ whenever $n \geq m$. By Theorem 2.20, p cannot be a limit point of K .

(2) K is bounded. There is a real number $M = 2$ and a point $q = 0 \in \mathbb{R}^1$ such that $|p - q| = |p| < 2$ for all $p \in K$.

By Heine-Borel theorem, K is compact in \mathbb{R}^1 . □

Exercise 2.13. Construct a compact set of real numbers whose limit points form a countable set.

Proof (Exercise 2.12). Let $K(p; r) \subseteq \mathbb{R}^1$ be

$$K(p; r) = \left\{p + \frac{r}{n} : n = 2, 3, \dots\right\} \cup \{p\}$$

and

$$K = \left(\bigcup_{i=0}^{\infty} K(2^{-i}; 2^{-i}) \right) \cup \{0\}.$$

- (1) *The set of limit points of K is $K' = \{2^{-i} : i = 0, 1, 2, \dots\} \cup \{0\}$, which is (infinitely) countable.*
 - (a) The unique limit point of $K(2^{-i}; 2^{-i})$ is 2^{-i} for each $i = 0, 1, 2, \dots$ (Exercise 2.12).
 - (b) 0 is a limit point of K .
 - (c) No other limit points of K . Similar to the argument of the proof of Exercise 2.12.
- (2) K is closed. All limit points are in K .
- (3) K is bounded. There is a real number $M = 2$ and a point $q = 0 \in \mathbb{R}^1$ such that $|p - q| = |p| < 2$ for all $p \in K$.

By Heine-Borel theorem, K is compact in \mathbb{R}^1 , and has infinitely countable limit points. \square

Exercise 2.14. *Give an example of an open cover of the segment $(0, 1)$ which has no finite subcover.*

Proof. In \mathbb{R}^1 , take a collection

$$\mathcal{G} = \left\{ G_n = \left(\frac{1}{n}, 1 \right) \right\}$$

of open subsets where $n \in \mathbb{Z}^+$.

- (1) \mathcal{G} is an open covering of $(0, 1) \subseteq \mathbb{R}^1$. Actually, given $x \in (0, 1)$, there exists a positive integer n such that $x > \frac{1}{n}$. That is, $x \in (\frac{1}{n}, 1) = G_n$.
- (2) *There is no finite subcovering of \mathcal{G} .* Assume

$$\mathcal{G}' = \{G_{n_1}, G_{n_2}, \dots, G_{n_k}\}$$

is any finite subcovering of \mathcal{G} where $n_1 < n_2 < \dots < n_k$. Take $x \in \left(0, \frac{1}{n_k}\right) \neq \emptyset$, $x = \frac{1}{2n_k}$ for example. Then $x \notin G_{n_1}$, $x \notin G_{n_2}$, ..., $x \notin G_{n_k}$, which contradicts that \mathcal{G}' is a finite subcovering of \mathcal{G} covering $(0, 1)$.

\square

Exercise 2.15. *Show that Theorem 2.36 and its Corollary become false (in \mathbb{R}^1 , for example) if the word “compact” is replaced by “closed” or by “bounded.”*

Recall:

- (1) Theorem 2.36: If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.
- (2) Corollary: If $\{K_n\}$ is a sequence of nonempty compact sets such that K_n contains K_{n+1} ($n = 1, 2, 3, \dots$), then $\bigcap K_n$ is not empty.

Proof. Let $X = \mathbb{R}^1$ with the usual Euclidean metric.

- (1) For the closeness, let $K_n = [n, \infty) \subseteq X$.
- (2) For the boundedness, let $K_n = (0, \frac{1}{n}) \subseteq X$.

In any case, $K_1 \supseteq K_2 \supseteq \dots$ and $\bigcap K_n = \emptyset$. \square

Exercise 2.16. Regard \mathbb{Q} , the set of all rational numbers, as a metric space, with $d(p, q) = |p - q|$. Let E be the set of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Show that E is closed and bounded in \mathbb{Q} , but that E is not compact. Is E open in \mathbb{Q} ?

Lemma. Assume $S \subseteq T \subseteq M$. Then S is compact in (M, d) if, and only if, S is compact in the metric subspace (T, d) .

Proof of Lemma.

- (1) (\implies) Let \mathcal{F} be an open covering of S in (T, d) , say $S \subseteq \bigcup_{A \in \mathcal{F}} A$ where A is open in T . Then $A = B \cap T$ for some open set B in M (Theorem 3.33). Let \mathcal{G} be the collection of B . Then

$$S \subseteq \bigcup_{A \in \mathcal{F}} A = \bigcup_{B \in \mathcal{G}} (B \cap T) \subseteq \bigcup_{B \in \mathcal{G}} B,$$

or \mathcal{G} be an open covering of S in (M, d) . Since S is compact in (M, d) , \mathcal{G} contains a finite subcovering, say

$$S \subseteq B_1 \cap \dots \cap B_p.$$

So

$$S \cap T \subseteq (B_1 \cap T) \cap \dots \cap (B_p \cap T),$$

or

$$S \subseteq A_1 \cap \dots \cap A_p$$

(since $S \subseteq T$ or $S \cap T = S$). So there is a finite subcovering of \mathcal{F} covering S , or S is compact in (T, d) .

- (2) (\impliedby) Let \mathcal{G} be an open covering of S in (M, d) , say $S \subseteq \bigcup_{B \in \mathcal{G}} B$ where B is open in M . Then $A = B \cap T$ is open in T . Let \mathcal{F} be the collection of A . Then

$$S \cap T \subseteq \bigcup_{B \in \mathcal{G}} (B \cap T) = \bigcup_{A \in \mathcal{F}} A,$$

or \mathcal{F} be an open covering of $S \cap T = S$ in (T, d) . Since S is compact in (T, d) , \mathcal{F} contains a finite subcovering, say

$$S \subseteq A_1 \cap \cdots \cap A_p.$$

Clearly, $S \subseteq B_1 \cap \cdots \cap B_p$ since $A = B \cap T \subseteq B$. So there is a finite subcovering of \mathcal{G} covering S , or S is compact in (M, d) .

□

Proof. Write $E_0 = (\sqrt{2}, \sqrt{3}) \cup (-\sqrt{3}, -\sqrt{2})$, and $E = E_0 \cap \mathbb{Q}$.

- (1) E is a subset of \mathbb{Q} .
- (2) *Show that E is bounded in \mathbb{Q} .* Since \mathbb{Q} is dense in \mathbb{R} , there is $p \in \mathbb{Q}$ such that $\sqrt{2} < p < \sqrt{3}$, or $p \in E$. Let $r = p + \sqrt{3} > 0$. Therefore, $E \subseteq B(p; r)$ for some $r > 0$ and $p \in E$, or E is bounded.
- (3) *Show that E is closed in \mathbb{Q} .* It suffices to show its complement is open in \mathbb{Q} . Given any

$$p \in \tilde{E} = ((-\infty, -\sqrt{3}] \cup [-\sqrt{2}, \sqrt{2}] \cup [\sqrt{3}, \infty)) \cap \mathbb{Q}.$$

$$p \leq -\sqrt{3} \text{ or } -\sqrt{2} \leq p \leq \sqrt{2} \text{ or } p \geq \sqrt{3}.$$

- (a) $p \leq -\sqrt{3}$. $p \neq -\sqrt{3}$ since $p \in \mathbb{Q}$ and $-\sqrt{3}$ is irrational. So $p < -\sqrt{3}$ and thus there exists $q \in \mathbb{Q}$ such that $p < q < -\sqrt{3}$ since \mathbb{Q} is dense in \mathbb{R} . Let $r = \max\{-\sqrt{3} - q, q - p\} > 0$. The ball $B(q; r)$ is contained in \tilde{E} .
- (b) $-\sqrt{2} \leq p \leq \sqrt{2}$. Similar to (a).
- (c) $p \geq \sqrt{3}$. Similar to (a).

By (a)(b), \tilde{E} is open in \mathbb{Q} , or E is closed in \mathbb{Q} .

- (4) *Show that E is not compact in \mathbb{Q} .* (Reductio ad absurdum) If E_0 were compact in the metric space \mathbb{Q} , E_0 is compact in the metric space \mathbb{R} (Lemma), which is absurd.
- (5) *Show that E is open.* Similar to (3).

□

Exercise 2.17. Let E be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in $[0, 1]$? Is E compact? Is E perfect?

Proof.

- (1) *Show that E is uncountable.* Same as Theorem 2.14. Or show that E is perfect and then apply Theorem 2.43.
- (2) *Show that E is not dense in $[0, 1]$.* Note that $E \subseteq [\frac{4}{9}, \frac{7}{9}]$. So

$$B\left(0; \frac{1}{64}\right) \cap E \subseteq B\left(0; \frac{1}{64}\right) \cap \left[\frac{4}{9}, \frac{7}{9}\right] = \emptyset$$

or 0 is not a limit point of E . Hence E is not dense in $[0, 1]$.

- (3) *Show that E is compact.* It is equivalent to show that E is closed and bounded (Theorem 2.41). Let a decimal expansion of $x \in (0, 1)$ be $0.x_1x_2\cdots$.

- (a) *Show that \tilde{E} is open.* Since $E \subseteq [\frac{4}{9}, \frac{7}{9}]$, it suffices to show that every point $x \in (0, 1) \cap \tilde{E}$ is an interior point of \tilde{E} . Say a decimal expansion of x containing at least one digit $x_n \neq 4, 7$. Note that

$$|x - y| \geq 10^{-n} > 0$$

for any $y = 0.y_1y_2\cdots \in E$. Hence there is an open neighborhood $B(x; 10^{-n})$ of x such that $B(x; 10^{-n}) \cap E = \emptyset$, or $B(x; 10^{-n}) \subseteq \tilde{E}$, or x is an interior point of \tilde{E} .

- (b) *Show that E is closed.* Given any limit point $x \in \mathbb{R}^1$ of E , we want to show that $x \in E$. (Reductio ad absurdum) Similar to (a).
- (c) *Show that E is bounded.* $E \subseteq B(0; 1)$.

- (4) *Show that E is perfect.*

- (a) E is closed (by (3)).
- (b) *Show that every point of E is a limit point of E .* Given any $x \in E$. Given any open neighborhood $B(x; r)$ of x , there is a positive integer n such that

$$\frac{3}{10^n} < r.$$

For such n , pick $y = 0.x_1x_2\cdots x_{n-1}y_n\cdots x_{n+1}\cdots \in E$ where

$$y_n = \begin{cases} 4 & (x_n = 7), \\ 7 & (x_n = 4). \end{cases}$$

$y \neq x$, and $|y - x| = \frac{3}{10^n} < r$. So that there is $y \neq x$ such that $y \in B(x; r)$, or x is a limit point of E .

□

Exercise 2.18. *Is there a nonempty perfect set in \mathbb{R}^1 which contains no rational number?*

Exercise 2.19.

- (a) If A and B are disjoint closed sets in some metric space X , prove that they are separated.
- (b) Prove the same for disjoint open sets.
- (c) Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p, q) < \delta$, define B similarly, with $>$ in place of $<$. Prove that A and B are separated.
- (d) Prove that every connected metric space with at least two points is uncountable. Hint: Use (c).

Proof of (a). Since

$$\begin{aligned}
 A \cap \overline{B} &= A \cap B && (B \text{ is closed}) \\
 &= \emptyset, && (A \text{ and } B \text{ are disjoint}) \\
 \overline{A} \cap B &= A \cap B && (A \text{ is closed}) \\
 &= \emptyset. && (A \text{ and } B \text{ are disjoint})
 \end{aligned}$$

A and B are separated. \square

Proof of (b)(Theorem 2.27(c)). Note that \tilde{A} is a closed set containing B . Since \overline{B} is the smallest closed set containing B , $\tilde{A} \supseteq \overline{B}$ (Theorem 2.27(c)). Hence

$$A \cap \overline{B} \subseteq A \cap \tilde{A} = \emptyset.$$

Similarly, $\overline{A} \cap B = \emptyset$. Hence A and B are separated. \square

Proof of (c). Since both

$$A = \{q \in X : d(p, q) < \delta\} \text{ and } B = \{q \in X : d(p, q) > \delta\}$$

are open in X , they are separated by (b). \square

Proof of (d). Let X be a connected metric space.

- (1) Let $p, q \in X$ with $p \neq q$. Hence $d_X(p, q) = r > 0$ (Definition 2.15(a)).
- (2) Given any $\delta \in (0, r)$. Define

$$A = \{x \in X : d(p, x) < \delta\} \text{ and } B = \{x \in X : d(p, x) > \delta\}.$$

$$p \in A \neq \emptyset \text{ and } q \in B \neq \emptyset.$$

- (3) If there were no $y_\delta \in X$ such that $d(p, y_\delta) = \delta$, we can write $X = A \cup B$ as a union of two nonempty separated sets ((c)), contrary to the connectedness of X .
- (4) Collect these y as E . Since d is a function, there is a one-to-one map from $(0, r)$ to E defined by $\delta \mapsto y_\delta$ in (3). Since $(0, r)$ is uncountable, $X \supseteq E$ is uncountable.

□

Exercise 2.20. *Are closures and interiors of connected sets always connected? (Look at subsets of \mathbb{R}^2 .)*

Proof.

- (1) *Interiors of connected sets are not always connected.* Let $X = \mathbb{R}^2$ with the usual Euclidean metric be a metric space. Take

$$E = B(89; 1) \bigcup B(64; 1) \bigcup \{(x, 0) \in \mathbb{R}^2 : 64 \leq x \leq 89\}.$$

E is connected and

$$E^\circ = B(89; 1) \bigcup B(64; 1)$$

is disconnected.

- (2) *Closures of connected sets are always connected. It suffices to show that E is disconnected if \overline{E} is disconnected.*

- (a) Write $\overline{E} = A \cup B$ as a union of two nonempty separated sets. Here $A \neq \emptyset$, $B \neq \emptyset$, $A \cap \overline{B} = \emptyset$ and $\overline{A} \cap B = \emptyset$.

- (b) Write

$$E = (A \cap E) \bigcup (B \cap E)$$

and we will show that E is disconnected.

- (c) *Show that $A \cap E$ and $B \cap E$ are separated.* In fact,

$$(A \cap E) \cap \overline{B \cap E} \subseteq A \cap \overline{B} = \emptyset,$$

$$\overline{A \cap E} \cap (B \cap E) \subseteq \overline{A} \cap B = \emptyset.$$

- (d) *Show that $A \cap E$ and $B \cap E$ are nonempty.* (Reductio ad absurdum)
If $A \cap E = \emptyset$, then

$$E = (A \cap E) \bigcup (B \cap E) = B \cap E \implies E \subseteq B.$$

So

$$\begin{aligned} A &= (A \cup B) \bigcap A && (A \subseteq A \cup B) \\ &= \overline{E} \bigcap A \\ &\subseteq \overline{B} \bigcap A && (E \subseteq B) \\ &= \emptyset \end{aligned}$$

which contradicts $A \neq \emptyset$ in (a). Therefore, $A \cap E \neq \emptyset$. Similarly, $B \cap E \neq \emptyset$.

Hence, E is disconnected if \overline{E} is disconnected, or closures of connected sets are always connected.

□

Exercise 2.21. Let A and B be separated subsets of some \mathbb{R}^k , suppose $\mathbf{a} \in A$, $\mathbf{b} \in B$, and define

$$\mathbf{p}(t) = (1-t)\mathbf{a} + t\mathbf{b}$$

for $t \in \mathbb{R}^1$. Put $A_0 = \mathbf{p}^{-1}(A)$, $B_0 = \mathbf{p}^{-1}(B)$. [Thus $t \in A_0$ if and only if $\mathbf{p}(t) \in A$.]

- (a) Prove that A_0 and B_0 are separated subsets of \mathbb{R}^1 .
- (b) Prove that there exists $t_0 \in (0, 1)$ such that $\mathbf{p}(t_0) \notin A \cup B$.
- (c) Prove that every convex subset of \mathbb{R}^k is connected.

Exercise 2.22. A metric space is called separable if it contains a countable dense subset. Show that \mathbb{R}^k is separable. (Hint: Consider the set of points which have only rational coordinates.)

Proof. Let E be the set of points which have only rational coordinates.

- (1) Show that E is countable. \mathbb{Q} is countable and thus $E = \mathbb{Q}^k$ is countable (Theorem 2.13).
- (2) Show that E is dense. Given any $\mathbf{p} = (p_1, \dots, p_k) \in \mathbb{R}^k$. We want to show that \mathbf{p} is a limit point of E .
 - (a) Given any open neighborhood $B(\mathbf{p}; r)$ of \mathbf{p} , $r > 0$.
 - (b) Since \mathbb{Q} is dense in \mathbb{R} (Theorem 1.20), every coordinate of \mathbf{p} is a limit point of \mathbb{Q} . In particular, for every $i = 1, 2, \dots, k$, the open neighborhood $B\left(p_i, \frac{r}{\sqrt{k}}\right)$ of p_i contains a point $q_i \neq p_i$ and $q_i \in \mathbb{Q}$.
 - (c) Collect all q_i in (b) and define $\mathbf{q} = (q_1, \dots, q_k) \in \mathbb{Q}^k = E$. By construction $\mathbf{q} \neq \mathbf{p}$ and

$$\begin{aligned} |\mathbf{p} - \mathbf{q}| &= \sqrt{(p_1 - q_1)^2 + \dots + (p_k - q_k)^2} \\ &< \sqrt{\left(\frac{r}{\sqrt{k}}\right)^2 + \dots + \left(\frac{r}{\sqrt{k}}\right)^2} \\ &= \sqrt{k \cdot \frac{r^2}{k}} \\ &= r \end{aligned}$$

or $\mathbf{q} \in B(\mathbf{p}; r)$.

By (a)(b)(c), E is dense in \mathbb{R}^k .

By (1)(2), \mathbb{R}^k is separable. \square

Exercise 2.23. A collection $\{V_\alpha\}$ of open subsets of X is said to be a base for X if the following is true: For every $x \in X$ and every open set $G \subseteq X$ such that $x \in G$, we have $x \in V_\alpha \subseteq G$ for some α . In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$.

Prove that every separable metric space has a countable base. (Hint: Take all neighborhoods with rational radius and center in some countable dense subset of X .)

Note. \mathbb{R}^k has a countable base (Exercise 2.22).

Proof (Hint). Let X be a separable metric space, and E be a countable dense subset of X . Let \mathcal{B} be a collection of all neighborhoods with rational radius and center in E .

- (1) \mathcal{B} is countable (Theorem 2.12).
- (2) \mathcal{B} is a base for X . Similar to Exercise 2.9(a). Given any $p \in X$ and every open set $G \subseteq X$ such that $p \in G$. Since p is in an open set G , there exists an open neighborhood $B(p; r)$ of p such that $B(p; r) \subseteq G$.
- (3) Let r_0 be rational such that $0 < r_0 < \frac{r}{2}$ (Theorem 1.20(b)). Since E is dense in X , there is $q \in E$ such that $d_X(p, q) < r_0$. For such $r_0 \in \mathbb{Q}$ we pick an open neighborhood $B(q; r_0)$ of q . Clearly, $B(q; r_0) \in \mathcal{B}$.
- (4) $p \in B(q; r_0)$ since $d_X(p, q) < r_0$.
- (5) Show that $B(q; r_0) \subseteq B(p; r) \subseteq G$. For any $z \in B(q; r_0)$, $d_X(z, p) \leq d_X(z, q) + d_X(q, p) < r_0 + r_0 < \frac{r}{2} + \frac{r}{2} = r$. That is, $z \in B(p; r)$.

By (3)(4)(5), (2) is established. By (1)(2), \mathcal{B} is a countable base for X . \square

Supplement.

- (1) In topology, a second-countable space, also called a completely separable space, is a topological space whose topology has a countable base.
- (2) Every second-countable space is separable.
- (3) The reverse implication of (2) does not hold in general. However, for metric spaces the properties of being second-countable and separable are equivalent.
- (4) Show that every second-countable metric space X is separable.

- (a) Let $\mathcal{B} = \{B_n : n \in \mathbb{Z}^+\}$ be a countable base of X .
- (b) For every $B_n \in \mathcal{B}$, pick any point p_n of B_n and collect them as

$$E = \{p_n : p_n \in B_n \text{ for } n \in \mathbb{Z}^+\}.$$

- (c) E is countable.
- (d) *Show that E is dense.* Given any $x \in X$. For any open neighborhood $B(x)$ of x , $B(x)$ is a union of subcollection of \mathcal{B} . That is, there is always a point in E by the construction of E .

□

Exercise 2.24. Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable.

(Hint: Fix $\delta > 0$, and pick $x_1 \in X$. Having chosen $x_1, \dots, x_j \in X$, choose x_{j+1} , if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \dots, j$. Show that this process must stop after finite number of steps, and that X can therefore be covered by finite many neighborhoods of radius δ . Take $\delta = \frac{1}{n}$ ($n = 1, 2, 3, \dots$) and consider the centers of the corresponding neighborhoods.)

Note. The reverse implication does not hold (Exercise 2.10).

Proof (Hint).

- (1) Fix $\delta > 0$, and pick $x_1 \in X$. Show that every limit point compact metric space X is totally bounded.
 - (a) Having chosen $x_1, \dots, x_j \in X$, choose x_{j+1} , if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \dots, j$. Let E_δ be the set of these x_i .
 - (b) *Show that this process must stop after finite number of steps, and that X can therefore be covered by finite many neighborhoods of radius δ .* (Reductio ad absurdum)
 - (i) If not, E_δ is an infinite subset of X . By assumption there is a limit point of E_δ , say $p \in X$.
 - (ii) In particular, an open neighborhood $B(p; \frac{\delta}{64})$ of p contains a point $x_n \in E_\delta$ with $p \neq x_n$.
 - (iii) The neighborhood $B(p; \frac{\delta}{64})$ contains no other point $x_m \in E_\delta$ with $m \neq n$. If so,

$$d_X(x_n, x_m) \leq d_X(x_n, p) + d_X(p, x_m) < \frac{\delta}{64} + \frac{\delta}{64} < \delta,$$

contrary to the construction of E_δ .

- (iv) Note that $p \notin E_\delta$ as a corollary to (iii).

- (v) So another open neighborhood $B(p; r)$ of p with $r = d_X(p, x_n) > 0$ contains no points $x_m \in E_\delta$ with $p \neq x_m$, contrary to the assumption that p is a limit point of E_δ .
- (2) Show that every totally bounded metric space X is separable. Take $\delta = \frac{1}{n}$ ($n = 1, 2, 3, \dots$) in (1), and union all $E_{\frac{1}{n}}$ as

$$E = \bigcup_{n=1}^{\infty} E_{\frac{1}{n}} \subseteq X.$$

Show that E is a countable dense subset of X .

- (a) Show that E is countable. Since E is the countable union of finite set $E_{\frac{1}{n}}$, E is countable (Theorem 2.12).
- (b) Show that E is dense in X . Given any $p \in X$. It suffices to show that given any open neighborhood $B(p; r)$ of $p \in X - E$, there exists $q \in E$ such that $q \in B(p; r)$. Pick any $n \in \mathbb{Z}^+$ such that $\frac{1}{n} < r$ (Theorem 1.20(a)). By the construction of $E_{\frac{1}{n}}$, there is $q \in E_{\frac{1}{n}}$ such that $p \in B(q; \frac{1}{n})$, or $d_X(p, q) < \frac{1}{n} < r$, or $q \in B(p; r)$.

□

Supplement.

- (1) A topological space X is said to be limit point compact or weakly countably compact if every infinite subset of X has a limit point in X .
- (2) In a metric space, limit point compactness, compactness, and sequential compactness are all equivalent. For general topological spaces, however, these three notions of compactness are not equivalent.
- (3) A metric space X is totally bounded if and only if for every real number $\delta > 0$, there exists a finite collection of open balls in X of radius δ whose union contains X .

Exercise 2.25. Prove that every compact metric space K has a countable base, and that K is therefore separable. (Hint: For every positive integer n , there are finitely many neighborhood of radius $\frac{1}{n}$ whose union covers K .)

Proof (Exercise 2.24(a)).

- (1) Show that every compact metric space K is limit point compact. Given any subset $E \subseteq K$. It suffices to show that if E has no limit point, then E must be finite.
 - (a) Since E has no limit point, E is closed.

- (b) For any point $p \in E$. Since p is not a limit point, there is an open neighborhood $B(p)$ such that $B(p)$ contains no point other than p .
- (c) Similar to the proof of Theorem 2.35, let

$$\mathcal{F} = \{B(p) : p \in E \text{ with } B(p) \cap E = \{p\}\} \bigcup \tilde{E}.$$

Hence \mathcal{F} is an open covering of K .

- (d) Since K is compact by assumption, there is a finitely subcovering \mathcal{F}' of K . Since \tilde{E} does not intersect E , each $B(p) \in \mathcal{F}'$ contains only one point of E and so E is finite.

- (2) Since K is limit point compact, K is separable (Theorem 2.24).

□

Proof (Exercise 2.24(b)).

- (1) Show that every compact metric space K is totally bounded. Given any real number $\delta > 0$, define an open covering \mathcal{F} of K by

$$\mathcal{F} = \{B(p; \delta) : p \in K\}.$$

Since K is compact, there exists a finite subcovering \mathcal{F}' of K . \mathcal{F}' is our desired finite collection of open balls in X of radius δ whose union contains X .

- (2) Since K is totally bounded, K is separable (Theorem 2.24).

□

Proof (Hint).

- (1) Given any positive integer $n > 0$, define an open covering \mathcal{F}_n of K by

$$\mathcal{F}_n = \left\{ B\left(p; \frac{1}{n}\right) : p \in K \right\}.$$

Since K is compact, there exists a finite subcovering \mathcal{G}_n of K .

- (2) Show that every compact metric space K is second-countable.

- (a) Define

$$\mathcal{B} = \bigcup_{n \geq 1} \mathcal{G}_n$$

be a collection. Since \mathcal{B} is a countable union of finite set \mathcal{G}_n , \mathcal{B} is countable. Hence it suffices to show that for every $p \in K$ and every open set $G \subseteq K$ such that $p \in G$, there is $B \in \mathcal{B}$ such that $x \in B \subseteq G$.

- (b) Since G is open, there is an open neighborhood $B(p; r)$ of p such that $B(p; r) \subseteq G$.
- (c) For such $r > 0$, there is $n \in \mathbb{Z}^+$ with $0 < \frac{1}{n} < \frac{r}{2}$ (Theorem 1.20(a)). So p is in some $B(q; \frac{1}{n}) \in \mathcal{G}_n \subseteq \mathcal{B}$ since \mathcal{G}_n is a subcovering of K .
- (d) Show that $B(q; \frac{1}{n}) \subseteq B(p; r) \subseteq G$. For any $z \in B(q; \frac{1}{n})$,

$$d_K(z, p) \leq d_K(z, q) + d_K(q, p) < \frac{1}{n} + \frac{1}{n} < \frac{r}{2} + \frac{r}{2} = r.$$

That is, $z \in B(p; r)$, or $B(q; \frac{1}{n}) \subseteq B(p; r) \subseteq G$.

By (a)(b)(c)(d), K is second-countable.

- (3) Show that every second-countable metric space is separable. Supplement (4) to Exercise 2.23.

□

Exercise 2.26. Let X be a metric space in which every infinite subsets has a limit point. Prove that X is compact.

By Exercises 2.23 and 2.24, X has a countable base. It follows that every open cover of X has a countable subcovering $\{G_n\}$, $n = 1, 2, 3, \dots$. If no finite subcollection of $\{G_n\}$ covers X , then the complement F_n of $G_1 \cup \dots \cup G_n$ is nonempty for each n , but $\bigcap F_n$ is empty. If E is a set contains a point from each F_n , consider a limit point of E , and obtain a contradiction.

Note. In every metric space, we have

$$\begin{aligned} \{\text{compact}\} &\iff \{\text{limit point compact}\} \\ &\implies \{\text{totally bounded}\} \\ &\implies \{\text{separable}\} \\ &\iff \{\text{second-countable}\} \\ &\iff \{\text{Lindelof}\}. \end{aligned}$$

Proof (Hint).

- (1) Since X is limit point compact, X is separable (Exercise 2.24). Since X is separable, X is second-countable (Exercise 2.23).
- (2) Show that X is Lindelof if X is second-countable. Let X be a second-countable metric space. Let $\mathcal{B} = \{B_n\}$ be a countable base of X . Given any open covering \mathcal{F} of X .

- (a) Iterate each $B_n \in \mathcal{B}$, pick one $G_n \in \mathcal{F}$ containing B_n , and collect them as

$$\mathcal{G} = \{G_n : G_n \supseteq B_n \text{ for } n \in \mathbb{Z}^+\}.$$

(G_n might be duplicated.)

- (b) \mathcal{G} is a countable subset of \mathcal{F} .
- (c) \mathcal{G} covers X since \mathcal{B} is a countable base of X .
- (3) Hence, given any open covering \mathcal{F} of X , there is a countable subcovering $\mathcal{G} = \{G_n\}$ of X . (Reductio ad absurdum) If there were no finite subcovering of \mathcal{G} , then the complement F_n of $G_1 \cup \cdots \cup G_n$ is nonempty for each n , but $\cap F_n$ is empty.
- (4) Let E be a set contains a point from each F_n . E is infinite and thus E has a limit point, say p . $p \in G_n$ for some n since $\mathcal{G} = \{G_n\}$ is an open covering of X . Since G_n is open, there is an open neighborhood $B(p)$ of p such that $B(p) \subseteq G_n$. By the construction of F_n ,

$$B(p) \cap F_m = \emptyset$$

whenever $m \geq n$, contrary to the assumption that p is a limit point of E .

Hence, X is compact if X is limit point compact. \square

Supplement.

- (1) Lindelof space is a topological space in which every open covering has a countable subcovering.
- (2) Show that X is second-countable if X is Lindelof. Same as the Proof (Hint) of Exercise 2.25 except changing the word “compact” to “Lindelof” and “finite” to “countable.” \square
- (3) In every metric space, we have

$$\{\text{compact}\} \iff \{\text{limit point compact}\} \iff \{\text{sequentially compact}\}.$$

Exercise 2.27. Define a point p in a metric space X to be a condensation point of a set $E \subseteq X$ if every neighborhood of p contains uncountably many points of E .

Suppose $E \subseteq \mathbb{R}^k$, E is uncountable, and let P be the set of all condensation points of E . Prove that P is perfect and that at most countably many points of E are not in P . In other words, show that $\widetilde{P} \cap E$ is at most countable.

(Hint: Let $\{V_n\}$ be a countable base of \mathbb{R}^k , let W be the union of those V_n for which $E \cap V_n$ is at most countable, and show that $P = \widetilde{W}$.)

Note. The statement is also true for separable metric space.

Proof.

(1) Let $\{V_n\}$ be a countable base of \mathbb{R}^k (Exercise 2.22 and 2.23). Let W be the union of those V_n for which $E \cap V_n$ is at most countable.

(2) Show that $P = \widetilde{W}$.

(a) ($P \subseteq \widetilde{W}$) Given any $x \in P$.

$$\begin{aligned} x \in P &\implies x \text{ is a condensation point of } E \\ &\implies \forall V_n \ni x, \exists B(x) \subseteq V_n \text{ such that } E \cap B(x) \text{ is uncountable} \\ &\implies E \cap V_n \text{ is uncountable} \\ &\implies x \notin W. \end{aligned}$$

(b) ($P \supseteq \widetilde{W}$) Given any $x \in \widetilde{W}$. Let $P(V_n)$ be the proposition that $E \cap V_n$ is at most countable.

$$\begin{aligned} x \in \widetilde{W} &\implies x \notin W = \bigcup_{P(V_n)} V_n \\ &\implies x \notin V_n \text{ for which } E \cap V_n \text{ is at most countable} \\ &\implies \forall B(x) \text{ of } x, x \in V_m \subseteq B(x) \text{ for some } V_m \quad (\{V_n\}: \text{ base of } X) \\ &\implies E \cap V_m \text{ is uncountable} \\ &\implies E \cap B(x) \supseteq E \cap V_m \text{ is uncountable} \\ &\implies x \text{ is a condensation point of } E \\ &\implies x \in P. \end{aligned}$$

(3) Show that P is closed. P is the complement of an open subset W .

(4) Show that $P \subseteq P'$. (Reductio ad absurdum)

(a) If there were an isolated point $x \in P$, then there exists an open neighborhood $B(x)$ of x such that $B(x) \cap P = \{x\}$.

(b) Since x is a condensation point of E , there are uncountably many points of E in $B(x)$, and such points y are not a condensation points of E except $y = x$.

(c) Given any point $y \in E \cap B(x)$ with $y \neq x$. Since y is not a condensation point, there exists a neighborhood $B(y)$ of y such that $B(y) \cap E$ is at most countable. Since $\{V_n\}$ is a base, for each $B(y)$ there exists $V_{n(y)}$ such that $y \in V_{n(y)} \subseteq B(y)$. Hence

$$V_{n(y)} \cap E \subseteq B(y) \cap E$$

is at most countable.

(d) Hence,

$$\begin{aligned} E \cap B(x) - \{x\} &\subseteq \bigcup_{y \in E \cap B(x) - \{x\}} V_{n(y)} \\ &= \bigcup_{n(y)} V_{n(y)} \end{aligned}$$

is a countable union of at most countable sets, which is countable. Hence $E \cap B(x) - \{x\}$ or $E \cap B(x)$ is countable, contrary to the assumption that $E \cap B(x)$ is uncountable.

(5) Show that $E \cap \tilde{P}$ is at most countable.

$$E \cap \tilde{P} = E \cap \left(\bigcup_{P(V_n)} V_n \right) = \bigcup_{P(V_n)} (E \cap V_n)$$

is at most countable.

□

Exercise 2.28. Prove that every closed set in a separable metric space is the union of a (possible empty) perfect set and a set which is at most countable. (Corollary: Every countable closed set in \mathbb{R}^k has isolated points.) (Hint: Use Exercise 2.27.)

Proof (Exercise 2.27). Let E be a closed set in a separable metric space.

- (1) E contains all limit points of E , especially contains all condensation points of E . So we can write

$$E = P \cup (E - P)$$

where P is the set of all condensation points of E .

- (2) By Exercise 2.27, P is perfect and $E - P = E \cap \tilde{P}$ is at most countable.

□

Cantor-Bendixson theorem.

- (1) Closed sets of a Polish space X have the perfect set property in a particularly strong form: any closed subset of X may be written uniquely as the disjoint union of a perfect set and a countable set.
- (2) A Polish space is a separable completely metrizable topological space; that is, a space homeomorphic to a complete metric space that has a countable dense subset.

Exercise 2.29. Prove that every open set in \mathbb{R}^1 is the union of an at most countable collection of disjoint segments. (Hint: Use Exercise 2.22.)

Proof. Let E be an open subset of \mathbb{R}^1 .

- (1) For each $x \in E$, let I_x denote the largest open interval containing x and contained in E . More precisely, since E is open, x is contained in some small (non-trivial) interval, and therefore if

$$a_x = \inf\{a < x : (a, x) \subseteq E\} \text{ and } b_x = \sup\{b > x : (x, b) \subseteq E\}$$

we must have $a_x < x < b_x$ (with possibly infinite values for a_x and b_x).

- (2) Let $I_x = (a_x, b_x)$, then by construction we have $x \in I_x$ as well as $I_x \subseteq E$. Hence

$$E = \bigcup_{I_x \in \mathcal{F}} I_x,$$

where $\mathcal{F} = \{I_x\}_{x \in E}$.

- (3) Suppose that two intervals I_x and I_y intersect. Then their union (which is also an open interval) is contained in E and contains x (and y). Since I_x is maximal, $I_x \cup I_y \subseteq I_x$, and similarly $I_x \cup I_y \subseteq I_y$. This can happen only if $I_x = I_y$.
- (4) Therefore, any two distinct intervals in \mathcal{F} must be disjoint. Hence \mathcal{F} is countable since each open interval $I_x \in \mathcal{F}$ contains a rational number.

□

Exercise 2.30. Imitate the proof of Theorem 2.43 to obtain the following result:

If $\mathbb{R}^k = \bigcup_{n=1}^{\infty} F_n$, where each F_n is a closed subset of \mathbb{R}^k , then at least one F_n has a nonempty interior.

Equivalent statement: If G_n is a dense open subset of \mathbb{R}^k , for $n = 1, 2, 3, \dots$, then $\bigcup_{n=1}^{\infty} G_n$ is not empty (in fact, it is dense in \mathbb{R}^k).

(This is a special case of Baire's theorem; see Exercise 3.22 for the general case.)