

Chapter 8: Some Special Functions

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Supplement. Fourier coefficients in Definition 8.9.

(1) Write

$$f(x) = a_0 + \sum_{n=1}^N (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}$$

(as the textbook Rudin, Principles of Mathematical Analysis, Third Edition).

Then

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx.$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx, n \in \mathbb{Z}^+.$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx, n \in \mathbb{Z}^+.$$

(2) One might write in one different form,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^N (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

The only difference between the new one and the old one is a_0 , so a_0 should be

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx.$$

(3) Again, one might write in one different form,

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^N (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

Similarly, a_0 should be

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x)}{\sqrt{2}} dx.$$

(4) Recall $f(x) = \sum_{n=-N}^N c_n e^{inx}$ ($x \in \mathbb{R}$) where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

The relations among a_n , b_n of this textbook and c_n are

$$\begin{aligned} c_0 &= a_0 \\ c_n &= \frac{1}{2}(a_n + ib_n), n \in \mathbb{Z}^+. \end{aligned}$$

- (5) In some textbooks (Henryk Iwaniec, Topics in Classical Automorphic Forms), it is convenient to consider periodic functions f of period 1. Define

$$e(n) = e^{2\pi i x} = \cos(2\pi x) + i \sin(2\pi x).$$

Any periodic and piecewise continuous function f has the Fourier series representation

$$f(x) = \sum_{-\infty}^{\infty} a_n e(nx)$$

with coefficients given by

$$a_n = \int_0^1 f(x) e(-nx) dx.$$

Here is one exercise for this representation. *Show that the fractional part of x , $\{x\} = x - [x]$, is given by*

$$\{x\} = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{\pi n}.$$

Supplement. Parseval's theorem 8.16.

- (1) Given

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

Then

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = 2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

- (2) Given

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

Then

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

(3) Given

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

Then

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

Exercise 8.1. Define

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

Prove that f has derivatives of all orders at $x = 0$, and that $f^{(n)}(0) = 0$ for $n = 1, 2, 3, \dots$

$f(x)$ is an example of non-analytic smooth function, that is, infinitely differentiable functions are not necessarily analytic. In this exercise, we will show that Taylor series of f at the origin converges everywhere to the zero function. So the Taylor series does not equal $f(x)$ for $x \neq 0$. Consequently, f is not analytic at $x = 0$.

Proof.

(1) Show that

$$\lim_{x \rightarrow 0} g(x) e^{-\frac{1}{x^2}} = 0$$

for any rational function $g(x) \in \mathbb{R}(x)$.

- (a) Write $g(x) = \frac{p(x)}{q(x)}$ for some $p(x), q(x) \in \mathbb{R}[x]$, $g(x) \neq 0$.
- (b) Write $q(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_0$. $q(x)$ is not identically zero, that is, there exists the unique coefficient of the least power of x in $q(x)$ which is non-zero, say $b_M \neq 0$.
- (c) Thus,

$$g(x) = \frac{p(x)/x^M}{q(x)/x^M}.$$

The denominator of $g(x)$ tends to $b_M \neq 0$ as $x \rightarrow 0$. By the similar argument in Theorem 8.6(f), we have

$$\frac{p(x)}{x^M} e^{-\frac{1}{x^2}} \rightarrow 0 \text{ as } x \rightarrow 0.$$

Hence, $\lim_{x \rightarrow 0} g(x) e^{-\frac{1}{x^2}} = 0$ for any $g(x) \in \mathbb{R}(x)$.

(2) Given any real $x \neq 0$, show that

$$f^{(n)}(x) = g_n(x) e^{-\frac{1}{x^2}}$$

for some rational function $g(x) \in \mathbb{R}(x)$.

- (a) Say $g_0(x) = 1 \in \mathbb{R}(x)$.
- (b) $\mathbb{R}(x)$ is a field. Show that $g'(x) \in \mathbb{R}(x)$ for any $g(x) \in \mathbb{R}(x)$. Write $g(x) = \frac{p(x)}{q(x)}$ for some $p(x), q(x) \in \mathbb{R}[x]$, $q(x) \neq 0$. Thus

$$g'(x) = \frac{p'(x)q(x) - p(x)q'(x)}{q(x)^2}.$$

The numerator of $g'(x)$ is in $\mathbb{R}[x]$ since the differentiation operator on $\mathbb{R}[x]$ is closed in $\mathbb{R}[x]$. Also, the denominator of $g'(x) = \frac{p'(x)q(x) - p(x)q'(x)}{q(x)^2} \neq 0$ since $\mathbb{R}[x]$ is an integral domain. Therefore, $g'(x) \in \mathbb{R}(x)$.

- (c) Induction on n . For $n = 1$, we have

$$\begin{aligned} f'(x) &= g'_0(x)e^{-\frac{1}{x^2}} + g_0(x) \cdot \left(-\frac{1}{x^2}\right)' e^{-\frac{1}{x^2}} \\ &= \left(g'_0(x) + g_0(x) \cdot \left(-\frac{1}{x^2}\right)'\right) e^{-\frac{1}{x^2}} \\ &= g_1(x)e^{-\frac{1}{x^2}} \end{aligned}$$

where

$$g_1(x) = g'_0(x) + g_0(x) \cdot \left(-\frac{1}{x^2}\right)' \in \mathbb{R}(x).$$

Now assume that the conclusion holds for $n = k$. As $n = k + 1$, similar to the case $n = 1$,

$$f^{(k+1)}(x) = g_{k+1}(x)e^{-\frac{1}{x^2}}$$

where

$$g_{k+1}(x) = g'_k(x) + g_k(x) \cdot \left(-\frac{1}{x^2}\right)' \in \mathbb{R}(x).$$

By induction, the conclusion is true.

- (3) Induction on n . For $n = 1$, by (1) we have

$$f'(0) = \lim_{t \rightarrow 0} \frac{e^{-\frac{1}{t^2}} - 0}{t} = 0.$$

Now assume that the statement holds for $n = k$. As $n = k + 1$, by (1)(2) we have

$$f^{(k+1)}(0) = \lim_{t \rightarrow 0} \frac{f^{(k)}(t) - f^{(k)}(0)}{t} = \lim_{t \rightarrow 0} \frac{g_k(t)e^{-\frac{1}{t^2}} - 0}{t} = 0.$$

Thus, $f^{(n)}(0) = 0$ for $n \in \mathbb{Z}^+$.

□

Exercise 8.2. Let a_{ij} be the number in the i th row and j th column of the array

$$\begin{array}{ccccc} -1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & -1 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{2} & -1 & 0 & \cdots \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

so that

$$a_{ij} = \begin{cases} 0 & (i < j), \\ -1 & (i = j), \\ 2^{j-i} & (i > j). \end{cases}$$

Prove that

$$\sum_i \sum_j a_{ij} = -2, \quad \sum_j \sum_i a_{ij} = 0.$$

Also see Theorem 8.3.

Proof (Brute-force).

$$\begin{aligned} \sum_i \sum_j a_{ij} &= \sum_{i=1}^{\infty} \left(\sum_{j=i}^{\infty} a_{ij} + \sum_{j<i} a_{ij} \right) \\ &= \sum_{i=1}^{\infty} \left(-1 + \sum_{j=1}^{i-1} 2^{j-i} \right) \\ &= \sum_{i=1}^{\infty} (-1 + (1 - 2^{1-i})) \\ &= \sum_{i=1}^{\infty} -2^{1-i} \\ &= -2. \end{aligned}$$

$$\begin{aligned}
\sum_j \sum_i a_{ij} &= \sum_{j=1}^{\infty} \left(\sum_{i=j}^{\infty} a_{ij} + \sum_{i>j} a_{ij} \right) \\
&= \sum_{j=1}^{\infty} \left(-1 + \sum_{i=j+1}^{\infty} 2^{j-i} \right) \\
&= \sum_{j=1}^{\infty} (-1 + 1) \\
&= \sum_{j=1}^{\infty} 0 \\
&= 0.
\end{aligned}$$

□

Exercise 8.3. *Prove that*

$$\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$$

if $a_{ij} \geq 0$ for all i and j (the case $+\infty = +\infty$ may occur).

Note. It can be proved by Theorem 8.3 if both summations are finite.

Proof.

(1) Let $\mathcal{F}(I)$ be the collection of all finite subsets of I .

(2) Let

$$s = \sup \left\{ \sum_{(i,j) \in E} a_{ij} : E \in \mathcal{F}(\mathbb{N}^2) \right\}$$

(the case $s = +\infty$ may occur). *It suffices to show that $\sum_i \sum_j a_{ij} = s$.*

The case $\sum_j \sum_i a_{ij} = s$ is similar, and thus $\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$.

(3) *Show that $\sum_i \sum_j a_{ij} \geq s$.* Given any $E \in \mathcal{F}(\mathbb{N}^2)$. It is clear that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \geq \sum_{(i,j) \in E} a_{ij}$$

(since $a_{ij} \geq 0$). Thus,

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \geq \sup \left\{ \sum_{(i,j) \in E} a_{ij} : E \in \mathcal{F}(\mathbb{N}^2) \right\} = s.$$

- (4) *Show that $\sum_i \sum_j a_{ij} \leq s$. (Reductio ad absurdum) If $\sum_i \sum_j a_{ij} > s$, especially $s < \infty$, then there exists $\varepsilon > 0$ such that*

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} > s + \varepsilon,$$

or

$$\sum_{i=1}^n \sum_{j=1}^{\infty} a_{ij} > s + \varepsilon$$

for some integer n . Consider two possible cases.

- (a) If there is some $1 \leq i_0 \leq n$ such that

$$\sum_{j=1}^{\infty} a_{i_0 j} = \infty,$$

then there is some m such that

$$\sum_{j=1}^m a_{i_0 j} > s.$$

For $E = \{(i_0, 1), \dots, (i_0, m)\} \in \mathcal{F}(\mathbb{N}^2)$,

$$\sum_{(i,j) \in E} a_{ij} = \sum_{j=1}^m a_{i_0 j} > s,$$

contrary to the supremum of s .

- (b) Otherwise, for each $1 \leq i \leq n$ we have

$$\sum_{j=1}^{\infty} a_{ij} < \infty,$$

or there exists some m_i such that

$$\sum_{j=1}^{m_i} a_{ij} > \sum_{j=1}^{\infty} a_{ij} - \frac{\varepsilon}{n}.$$

For $E = \bigcup_{1 \leq i \leq n} \{(i, 1), \dots, (i, m_i)\} \in \mathcal{F}(\mathbb{N}^2)$,

$$\begin{aligned}
\sum_{(i,j) \in E} a_{ij} &= \sum_{i=1}^n \sum_{j=1}^{m_i} a_{ij} \\
&> \sum_{i=1}^n \left(\sum_{j=1}^{\infty} a_{ij} - \frac{\varepsilon}{n} \right) \\
&= \sum_{i=1}^n \sum_{j=1}^{\infty} a_{ij} - \sum_{i=1}^n \frac{\varepsilon}{n} \\
&> s + \varepsilon - \varepsilon \\
&= s,
\end{aligned}$$

contrary to the supremum of s .

Therefore, $\sum_i \sum_j a_{ij} \leq s$.

- (5) By (3)(4), $\sum_i \sum_j a_{ij} = s$. Similarly, $\sum_j \sum_i a_{ij} = s$. Hence, $\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$ (including the case $+\infty = +\infty$).

□

Exercise 8.4. *Prove the following limit relations:*

(a) $\lim_{x \rightarrow 0} \frac{b^x - 1}{x} = \log b \quad (b > 0).$

(b) $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1.$

(c) $\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e.$

(d) $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x.$

Proof of (a).

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{b^x - 1}{x} &= \lim_{x \rightarrow 0} \frac{\exp(x \log b) - 1}{x} \\
&= \left. \frac{d}{dx} \exp(x \log b) \right|_{x=0} \\
&= \exp(x \log b) \cdot \log b \Big|_{x=0} \\
&= \log b.
\end{aligned}$$

□

Proof of (b).

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} &= \frac{d}{dx} \log(1+x) \Big|_{x=0} \\ &= \frac{1}{x+1} \Big|_{x=0} \\ &= 1.\end{aligned}$$

□

Proof of (c).

$$\begin{aligned}\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} &= \lim_{x \rightarrow 0} \exp\left(\frac{\log(1+x)}{x}\right) \\ &= \exp\left(\lim_{x \rightarrow 0} \frac{\log(1+x)}{x}\right) \\ &= \exp(1) \\ &= e.\end{aligned}$$

□

Proof of (d).

$$\begin{aligned}\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n &= \lim_{n \rightarrow \infty} \left(\left(1 + \frac{x}{n}\right)^{\frac{n}{x}}\right)^x \\ &= \left(\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^{\frac{n}{x}}\right)^x \\ &= \left(\lim_{y \rightarrow 0} (1+y)^{\frac{1}{y}}\right)^x \\ &= \exp(x).\end{aligned}$$

□

Exercise 8.5. Find the following limits

(a) $\lim_{x \rightarrow 0} \frac{e - (1+x)^{\frac{1}{x}}}{x}.$

(b) $\lim_{n \rightarrow \infty} \frac{n}{\log n} \left[n^{\frac{1}{n}} - 1 \right].$

(c) $\lim_{x \rightarrow 0} \frac{\tan x - x}{x(1 - \cos x)}.$

(d) $\lim_{x \rightarrow 0} \frac{x - \sin x}{\tan x - x}.$

Proof of (a). By L'Hospital's rule (Theorem 5.13),

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{e - (1+x)^{\frac{1}{x}}}{x} &= \lim_{x \rightarrow 0} \frac{-(1+x)^{\frac{1}{x}} \cdot \frac{\frac{x}{x+1} - \log(x+1)}{x^2}}{1} \\
&= \lim_{x \rightarrow 0} \left(-(1+x)^{\frac{1}{x}} \cdot \frac{\frac{x}{x+1} - \log(x+1)}{x^2} \right) \\
&= - \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} \cdot \lim_{x \rightarrow 0} \frac{\frac{x}{x+1} - \log(x+1)}{x^2} \\
&= -e \cdot \lim_{x \rightarrow 0} \frac{\frac{x}{x+1} - \log(x+1)}{x^2} \quad (\text{Exercise 8.4(c)}) \\
&= -e \cdot \lim_{x \rightarrow 0} \frac{-\frac{x}{(x+1)^2}}{2x} \\
&= e \cdot \lim_{x \rightarrow 0} \frac{1}{2(x+1)^2} \\
&= e \cdot \frac{1}{2} \\
&= \frac{e}{2}.
\end{aligned}$$

Here

$$\begin{aligned}
\frac{d}{dx} \left(e - (1+x)^{\frac{1}{x}} \right) &= \frac{d}{dx} \left(e - \exp \left(\frac{\log(x+1)}{x} \right) \right) \\
&= - \exp \left(\frac{1}{x} \log(x+1) \right) \cdot \frac{\frac{1}{x+1} \cdot x - \log(x+1) \cdot 1}{x^2} \\
&= -(1+x)^{\frac{1}{x}} \cdot \frac{\frac{x}{x+1} - \log(x+1)}{x^2},
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dx} \left(\frac{x}{x+1} - \log(x+1) \right) &= \frac{(x+1) - x}{(x+1)^2} - \frac{1}{x+1} \\
&= -\frac{x}{(x+1)^2}.
\end{aligned}$$

□

Proof of (b).

(1) Let $x = \frac{\log n}{n}$. Note that $\lim_{n \rightarrow \infty} \frac{\log n}{n} = 0$.

(2)

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{n}{\log n} \left[n^{\frac{1}{n}} - 1 \right] &= \lim_{n \rightarrow \infty} \frac{n}{\log n} \left[\exp \left(\frac{\log n}{n} \right) - 1 \right] \\ &= \lim_{x \rightarrow 0} \frac{\exp(x) - 1}{x} \\ &= \frac{d}{dx} \exp(x) \Big|_{x=0} \\ &= \exp(x) \Big|_{x=0} \\ &= 1.\end{aligned} \tag{1)}$$

□

Proof of (c) (L'Hospital's rule). By L'Hospital's rule (Theorem 5.13) three times,

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\tan x - x}{x(1 - \cos x)} &= \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{1 - \cos x + x \sin x} \\ &= \lim_{x \rightarrow 0} \frac{2 \sec x (\tan x \sec x)}{\sin x + \sin x + x \cos x} \\ &= \lim_{x \rightarrow 0} \frac{2 \tan x \sec^2 x}{2 \sin x + x \cos x} \\ &= \lim_{x \rightarrow 0} \frac{2[\sec^2 x \sec^2 x + \tan x \cdot 2 \sec x (\tan x \sec x)]}{2 \cos x + \cos x - x \sin x} \\ &= \lim_{x \rightarrow 0} \frac{2 \sec^4 x + 2 \sec^2 x \tan^2 x}{3 \cos x - x \sin x} \\ &= \frac{2}{3}.\end{aligned}$$

□

Proof of (c) (Taylor series). Since

$$\begin{aligned}\cos x &= 1 - \frac{x^2}{2} + O(x^4) \\ \tan x &= x + \frac{x^3}{3} + O(x^5),\end{aligned}$$

we have

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x(1 - \cos x)} = \lim_{x \rightarrow 0} \frac{\frac{x^3}{3} + O(x^5)}{\frac{x^3}{2} + O(x^5)} = \frac{2}{3}.$$

□

Proof of (d) (L'Hospital's rule). By L'Hospital's rule (Theorem 5.13) three times,

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{x - \sin x}{\tan x - x} &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{\sec^2 x - 1} \\
&= \lim_{x \rightarrow 0} \frac{\sin x}{2 \sec x (\tan x \sec x)} \\
&= \lim_{x \rightarrow 0} \frac{\sin x}{2 \tan x \sec^2 x} \\
&= \lim_{x \rightarrow 0} \frac{\cos x}{2 \tan x \sec^2 x} \\
&= \lim_{x \rightarrow 0} \frac{\cos x}{2 [\sec^2 x \sec^2 x + \tan x \cdot 2 \sec x (\tan x \sec x)]} \\
&= \lim_{x \rightarrow 0} \frac{\cos x}{2 \sec^4 x + 2 \sec^2 x \tan^2 x} \\
&= \frac{1}{2}.
\end{aligned}$$

□

Proof of (d) (Taylor series). Since

$$\begin{aligned}
\sin x &= x - \frac{x^3}{6} + O(x^5) \\
\tan x &= x + \frac{x^3}{3} + O(x^5),
\end{aligned}$$

we have

$$\lim_{x \rightarrow 0} \frac{x - \sin x}{\tan x - x} = \lim_{x \rightarrow 0} \frac{\frac{x^3}{6} + O(x^5)}{\frac{x^3}{3} + O(x^5)} = \frac{1}{2}.$$

□

Exercise 8.6. Suppose $f(x)f(y) = f(x+y)$ for all real x and y .

(a) Assuming that f is differentiable and not zero, prove that

$$f(x) = e^{cx}$$

where c is a constant.

(b) Prove the same thing, assuming only that f is continuous.

Part (b) implies part (a). We prove part (b) directly.

Proof of (b).

- (1) Since $f(x)$ is not zero, there exists $x_0 \in \mathbb{R}$ such that $f(x_0) \neq 0$. So $f(0)f(x_0) = f(x_0)$, or $f(0) = 1$ by cancelling $f(x_0) \neq 0$.
- (2) Next, $f(\frac{n}{m}) = f(\frac{1}{m})^n$ for $m \in \mathbb{Z}$, $n \in \mathbb{Z}^+$. Since f is continuous at $x = 0$, f is positive in the neighborhood of $x = 0$. That is, there exists $N \in \mathbb{Z}^+$ such that $f(\frac{1}{m}) > 0$ whenever $|m| \geq N$. So, $f(\frac{n}{m}) = f(\frac{1}{m})^n > 0$. (Since $f(\frac{n}{m}) = f(\frac{kn}{km})$ for any $k \in \mathbb{Z}^+$, we can rescale m to km such that $|km| \geq N$.) That is, f is positive on \mathbb{Q} . Since \mathbb{Q} is dense in \mathbb{R} and f is continuous on \mathbb{R} , f is positive on \mathbb{R} .
- (3) Now let $c = \log f(1)$ (which is well-defined since $f > 0$). We write $f(1)$ in the two ways. Firstly, $f(1) = f(\frac{n}{n}) = f(\frac{1}{n})^n$ where $n \in \mathbb{Z}^+$. Secondly, $f(1) = e^c = (e^{\frac{c}{n}})^n$. Since the positive n -th root is unique (Theorem 1.21), $f(\frac{1}{n}) = e^{\frac{c}{n}}$ for $n \in \mathbb{Z}^+$. By $f(x)f(-x) = f(0) = 1$ or $f(-x) = \frac{1}{f(x)}$, $f(-\frac{1}{n}) = \frac{1}{f(\frac{1}{n})} = e^{-\frac{c}{n}}$ for $n \in \mathbb{Z}^+$. Therefore,

$$f\left(\frac{1}{m}\right) = e^{\frac{c}{m}} \text{ where } m \in \mathbb{Z}.$$

- (4) By using $f(\frac{n}{m}) = f(\frac{1}{m})^n$ for $m \in \mathbb{Z}$, $n \in \mathbb{Z}^+$ again, $f(\frac{n}{m}) = e^{c\frac{n}{m}}$ where $m \in \mathbb{Z}, n \in \mathbb{Z}^+$, or

$$f(x) = e^{cx} \text{ where } x \in \mathbb{Q}.$$

Since $g(x) = f(x) - e^{cx}$ vanishes on a dense set of \mathbb{Q} and g is continuous on \mathbb{R} , g vanishes on \mathbb{R} . Therefore, $f(x) = e^{cx}$ for $x \in \mathbb{R}$.

□

Supplement. *Proof of (a).*

- (1) Since $f(x)$ is not zero, there exists $x_0 \in \mathbb{R}$ such that $f(x_0) \neq 0$. So $f(0)f(x_0) = f(x_0)$, or $f(0) = 1$ by cancelling $f(x_0) \neq 0$.
- (2) Since f is differentiable, for any $x \in \mathbb{R}$,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x)f(h) - f(x)}{h} \\ &= f(x) \lim_{h \rightarrow 0} \frac{f(h) - 1}{h} \\ &= f(x) \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} \\ &= f(x)f'(0). \end{aligned}$$

Let $c = f'(0)$ be a constant. Then $f'(x) = cf(x)$. So $f(x) = e^{cx}$ for $x \in \mathbb{R}$. (To see this, let $g(x) = \frac{f(x)}{e^{cx}}$ be well-defined on \mathbb{R} . $g(0) = 1$. $g'(x) = 0$ since $f'(x) = cf(x)$. So $g(x)$ is a constant, or $g(x) = 1$ since $g(0) = 1$. Therefore, $f(x) = e^{cx}$ on \mathbb{R} .)

□

Supplement. Cauchy's functional equation.

- (1) (*Cauchy's functional equation.*) Suppose $f(x) + f(y) = f(x + y)$ for all real x and y . Assuming that f is continuous, prove that $f(x) = cx$ where c is a constant.

Notice that we cannot let $g(x) = \log f(x)$ and apply Cauchy's functional equation on $g(x)$ to prove Exercise 8.6 since $f(x)$ is not necessarily positive and thus $g(x) = \log f(x)$ might be meaningless. However, this wrong approach gives you some useful ideas such as you need to prove that $f(x)$ is positive first, and $f(x)$ should be equal to e^{cx} where $c = g(1) = \log f(1)$.

- (2) Suppose $f(xy) = f(x) + f(y)$ for all positive real x and y . Assuming that f is continuous, prove that $f(x) = c \log x$ where c is a constant.
- (3) Suppose $f(xy) = f(x)f(y)$ for all positive real x and y . Assuming that f is continuous and positive, prove that $f(x) = x^c$ where c is a constant.
- (4) Suppose $f(x + y) = f(x) + f(y) + xy$ for all real x and y . Assuming that f is continuous, prove that $f(x) = \frac{1}{2}x^2 + cx$ where c is a constant.
- (5) (*USA 2002.*) Suppose $f(x^2 - y^2) = xf(x) - yf(y)$ for all real x and y . Assuming that f is continuous, prove that $f(x) = cx$ where c is a constant.

Supplement. Show that the only automorphism of \mathbb{Q} is the identity.

Proof. Given any $\sigma \in \text{Aut}(\mathbb{Q})$.

- (1) Show that $\sigma(1) = 1$. Since $1^2 = 1$, $\sigma(1)\sigma(1) = \sigma(1)$. $\sigma(1) = 0$ or 1 . There are only two possible cases.

- (a) Assume that $\sigma(1) = 0$. So

$$\sigma(a) = \sigma(a \cdot 1) = \sigma(a) \cdot \sigma(1) = \sigma(a) \cdot 0 = 0$$

for any $a \in \mathbb{Q}$. That is, $\sigma = 0 \in \text{Aut}(\mathbb{Q})$, which is absurd.

- (b) Therefore, $\sigma(1) = 1$.

- (2) Show that $\sigma(n) = n$ for all $n \in \mathbb{Z}^+$. Write $n = 1 + 1 + \cdots + 1$ (n times 1). Applying the additivity of σ , we have

$$\sigma(n) = \sigma(1) + \sigma(1) + \cdots + \sigma(1) = 1 + 1 + \cdots + 1 = n.$$

(Might use induction on n to eliminate \cdots symbols.)

- (3) Show that $\sigma(n) = n$ for all $n \in \mathbb{Z}$. By the additivity of σ , $\sigma(-n) = -\sigma(n) = -n$ for $n \geq 0$. The result is established.

For any $a = \frac{n}{m} \in \mathbb{Q}$ ($m, n \in \mathbb{Z}$, $n \neq 0$), applying the multiplication of σ on $am = n$, that is, $\sigma(a)\sigma(m) = \sigma(n)$. By (3), we have $\sigma(a)m = n$, or

$$\sigma(a) = \frac{m}{n} = a$$

provided $n \neq 0$, or σ is the identity. \square

Exercise 8.7. If $0 < x < \frac{\pi}{2}$, prove that

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1.$$

Proof.

(1) Let

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

be a continuous function on $[0, \frac{\pi}{2}]$ (since $\lim_{x \rightarrow 0+} f(x) = 1$). So

$$f'(x) = \frac{x \cos x - \sin x}{x^2} < 0$$

on $(0, \frac{\pi}{2})$ since $\tan x > x$ on $(0, \frac{\pi}{2})$.

(2) Show that $\frac{\sin x}{x} < 1$ on $(0, \frac{\pi}{2})$. Given any $x \in (0, \frac{\pi}{2})$, there exists $\xi_1 \in (0, x)$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(\xi_1) < 0$$

by the mean value theorem (Theorem 5.10). So $f(x) < f(0) = 1$, or $\frac{\sin x}{x} < 1$.

(3) Show that $\frac{\sin x}{x} > \frac{2}{\pi}$ on $(0, \frac{\pi}{2})$. Given any $x \in (0, \frac{\pi}{2})$, there exists $\xi_2 \in (0, x)$ such that

$$\frac{f(\frac{\pi}{2}) - f(x)}{\frac{\pi}{2} - x} = f'(\xi_2) < 0$$

by the mean value theorem (Theorem 5.10). So $f(x) > f(\frac{\pi}{2}) = \frac{2}{\pi}$, or $\frac{\sin x}{x} > \frac{2}{\pi}$.

\square

Exercise 8.8. For $n = 0, 1, 2, \dots$, and x real, prove that

$$|\sin(nx)| \leq n|\sin x|.$$

Note that this inequality may be false for other values of n . For instance,

$$\left| \sin\left(\frac{1}{2}\pi\right) \right| > \frac{1}{2} |\sin \pi|.$$

Proof. Induction on n .

(1) Note that

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

for any $a, b \in \mathbb{R}$.

(2) $n = 0, 1$ are clearly true.

(3) Assume the induction hypothesis that for the single case $n = k$ holds, meaning

$$|\sin(kx)| \leq k |\sin x|$$

is true. It follows that

$$\begin{aligned} |\sin((k+1)x)| &= |\sin(kx) \cos x + \cos(kx) \sin x| && ((1)) \\ &\leq |\sin(kx)| |\cos x| + |\cos(kx)| |\sin x| && (\text{Triangle inequality}) \\ &\leq |\sin(kx)| + |\sin x| && (|\cos(\cdot)| \leq 1) \\ &\leq k |\sin x| + |\sin x| && (\text{Induction hypothesis}) \\ &\leq (k+1) |\sin x|. \end{aligned}$$

□

Exercise 8.9 (The Euler-Mascheroni constant).

(a) Put $s_N = 1 + \frac{1}{2} + \cdots + \frac{1}{N}$. Prove that

$$\lim_{N \rightarrow \infty} (s_N - \log N)$$

exists. (The limit, often denoted by γ , is called Euler's constant. Its numerical value is $0.5772\dots$. It is not known whether γ is rational or not.)

(b) Roughly how large must m be so that $N = 10^m$ satisfies $s_N > 100$?

Proof of (a) (Theorem 3.14).

(1) Note that

$$\begin{aligned}
& \frac{1}{1 + \frac{1}{n}} \leq \frac{1}{x} \leq 1 \text{ for } x \in \left[1, 1 + \frac{1}{n}\right] \\
& \Rightarrow \int_1^{1 + \frac{1}{n}} \frac{dx}{1 + \frac{1}{n}} \leq \int_1^{1 + \frac{1}{n}} \frac{dx}{x} \leq \int_1^{1 + \frac{1}{n}} dx \quad (\text{Theorem 6.12(b)}) \\
& \Rightarrow \frac{1}{n+1} \leq \int_1^{1 + \frac{1}{n}} \frac{dx}{x} \leq \frac{1}{n} \\
& \Rightarrow \frac{1}{n+1} \leq \log \left(1 + \frac{1}{n}\right) \leq \frac{1}{n}. \quad (\text{Equation (39) on page 180})
\end{aligned}$$

(2) Define

$$\gamma_n = s_n - \log n.$$

It suffices to show that $\{\gamma_n\}$ is monotonic and bounded (Theorem 3.14).

(3) Show that $\{\gamma_n\}$ is decreasing.

$$\begin{aligned}
\gamma_{n+1} - \gamma_n &= (s_{n+1} - \log(n+1)) - (s_n - \log n) \\
&= (s_{n+1} - s_n) - (\log(n+1) - \log n) \\
&= \frac{1}{n+1} - \log \left(\frac{n+1}{n}\right) \\
&= \frac{1}{n+1} - \log \left(1 + \frac{1}{n}\right) \\
&\leq 0. \quad ((1))
\end{aligned}$$

Note. $\gamma_n \leq \dots \leq \gamma_1 = 1$ for all $n = 1, 2, 3, \dots$

(4) Show that $\gamma_n \geq 0$ for all $n = 1, 2, 3, \dots$ Since

$$\begin{aligned}
\log n &= \sum_{k=1}^{n-1} (\log(k+1) - \log k) \\
&= \sum_{k=1}^{n-1} \log \frac{k+1}{k} \\
&= \sum_{k=1}^{n-1} \log \left(1 + \frac{1}{k}\right) \\
&\leq \sum_{k=1}^{n-1} \frac{1}{k} \quad ((1)) \\
&= s_{n-1},
\end{aligned}$$

we have

$$\gamma_n = s_n - \log n \geq s_n - s_{n-1} = \frac{1}{n} > 0.$$

By (3)(4), $\{\gamma_n\}$ converges to $\lim_{N \rightarrow \infty} (s_N - \log N) = \gamma$. \square

Supplement. Show that if $f \geq 0$ on $[0, \infty)$ and f is monotonically decreasing, and if

$$c_n = \sum_{k=1}^n f(k) - \int_1^n f(x) dx,$$

then $\lim_{n \rightarrow \infty} c_n$ exists. (Exercise 10 of Section 5.2 in the textbook: *R Creighton Buck, Advanced Calculus, 3rd edition*. See page 235.) If this exercise is true, we can get the existence of γ by taking $f(x) = \frac{1}{x}$.

(1) Note that

$$f(n+1) \leq \int_n^{n+1} f(x) dx \leq f(n).$$

(2) Show that $\{c_n\}$ is decreasing.

$$c_{n+1} - c_n = f(n+1) - \int_n^{n+1} f(x) dx \leq 0.$$

(3) Show that $c_n \geq 0$. Since $f(k) \geq \int_k^{k+1} f(x) dx$,

$$\begin{aligned} \sum_{k=1}^n f(k) &\geq \sum_{k=1}^n \int_k^{k+1} f(x) dx \\ &= \int_1^{n+1} f(x) dx \\ &\geq \int_1^n f(x) dx. \end{aligned} \quad (f \geq 0)$$

So that $c_n = \sum_{k=1}^n f(k) - \int_1^n f(x) dx \geq 0$.

(4) By (2)(3), $\{c_n\}$ converges (Theorem 3.14).

\square

Proof of (a) (Limit comparison test). Inspired by this paper: *Philippe Flajolet and Ilan Vardi, Zeta Function Expansions of Classical Constants*.

(1) Rewrite

$$\gamma_n + \log n - \log(n+1) = \sum_{k=1}^n \left(\frac{1}{k} - \log \left(1 + \frac{1}{k} \right) \right)$$

(similar to the argument in (a)(4)(Theorem 3.14)). Let

$$c_k = \frac{1}{k} - \log \left(1 + \frac{1}{k} \right).$$

(2) Show that

$$\lim_{k \rightarrow \infty} \frac{c_k}{\frac{1}{k^2}} = \frac{1}{2}.$$

In fact,

$$\begin{aligned} & \lim_{k \rightarrow \infty} \frac{c_k}{\frac{1}{k^2}} \\ &= \lim_{x \rightarrow 0} \frac{x - \log(1+x)}{x^2} \quad \left(\text{Put } x = \frac{1}{k}\right) \\ &= \lim_{x \rightarrow 0} \frac{1 - \frac{1}{1+x}}{2x} \quad (\text{L'Hospital's rule}) \\ &= \lim_{x \rightarrow 0} \frac{1}{2(x+1)} \\ &= \frac{1}{2}. \end{aligned}$$

(3) By limit comparison test or comparison test, $\sum c_k$ converges since $\sum \frac{1}{k^2}$ converges. Also,

$$\lim_{n \rightarrow \infty} \log n - \log(n+1) = 0.$$

Therefore, $\lim_{n \rightarrow \infty} \gamma_n$ exists.

□

Note. This proof is based on **limit comparison test** (Theorem 8.21) in this textbook: *Tom. M. Apostol, Mathematical Analysis, 2nd edition*. It is easy to prove by the original comparison test.

Proof of (a) (Comparison test).

(1) Note that

$$0 \leq x - \log(x+1) \leq \frac{x^2}{2}$$

for all $x \geq 0$.

(2) Write

$$c_n = \frac{1}{n} - \log\left(1 + \frac{1}{n}\right).$$

as in the the proof of (a) (Limit comparison test). By (1),

$$|c_n| \leq \frac{1}{2n^2}$$

for all $n = 1, 2, \dots$. Hence, by the comparison test (Theorem 3.25(a)), $\sum c_n$ converges since $\sum \frac{1}{n^2}$ converges (to $\frac{\pi^2}{6}$). Use the same argument in the proof of (a) (Limit comparison test), since

$$\gamma_n + \log n - \log(n+1) = \sum c_n \text{ and } \lim_{n \rightarrow \infty} \log n - \log(n+1) = 0,$$

we have the existence of $\lim \gamma_n = \gamma$.

□

Proof of (a) (Uniformly convergence of $\sum \frac{x}{n(x+n)}$). (One example to Exercise 7 of Section 6.2 in the textbook: *R Creighton Buck, Advanced Calculus, 3rd edition*. See pages 270 to 271.)

(1) Let

$$f_n(x) = \frac{x}{n(x+n)} = \frac{1}{n} - \frac{1}{x+n}$$

defined on $E = [0, 1]$.

(2) Note that

$$|f_n(x)| \leq \frac{1}{n^2}$$

for all $x \in [0, 1]$. Since $\sum \frac{1}{n^2}$ converges, $\sum f_n$ converges uniformly on $[0, 1]$ (Theorem 7.10).

(3) Corollary to Theorem 7.16 implies that

$$\begin{aligned} \int_0^1 \sum_{n=1}^{\infty} \frac{x}{n(x+n)} dx &= \sum_{n=1}^{\infty} \int_0^1 \frac{x}{n(x+n)} dx \\ &= \sum_{n=1}^{\infty} \int_0^1 \left(\frac{1}{n} - \frac{1}{x+n} \right) dx \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \log \frac{n+1}{n} \right) \\ &= \lim_{N \rightarrow \infty} \left(\sum_{n=1}^N \frac{1}{n} - \log(N+1) \right) \\ &= \lim_{N \rightarrow \infty} (s_N - \log(N+1)) \end{aligned}$$

exists. Since $\lim_{N \rightarrow \infty} (\log(N+1) - \log N) = 0$,

$$\begin{aligned} \gamma &= \lim_{N \rightarrow \infty} (s_N - \log N) \\ &= \lim_{N \rightarrow \infty} (s_N - \log(N+1)) + \lim_{N \rightarrow \infty} (\log(N+1) - \log N) \end{aligned}$$

exists.

□

Proof of (a) (Existence of $\int_1^{\infty} \frac{\{x\}}{x^2} dx$).

- (1) Define $\{x\} = x - [x]$ where $[x]$ is the greatest integer $\leq x$ (Exercise 6.16).
Show that

$$\int_1^\infty \frac{\{x\}}{x^2} dx < \infty.$$

Use the similar argument in Exercise 6.16(b). Since $\frac{\{x\}}{x^2} \leq \frac{1}{x^2}$ on $[1, \infty)$ and $\int_1^\infty \frac{1}{x^2} dx = 1$ exists, the result is established (Theorem 6.12(b)).

- (2) Show that

$$\int_1^N \frac{[x]}{x^2} dx = s_N - 1.$$

Use the similar argument in Exercise 6.16(a),

$$\begin{aligned} \int_1^N \frac{[x]}{x^2} dx &= \sum_{k=1}^{N-1} \int_k^{k+1} \frac{[x]}{x^2} dx \\ &= \sum_{k=1}^{N-1} \int_k^{k+1} \frac{k}{x^2} dx \\ &= \sum_{k=1}^{N-1} \int_k^{k+1} \frac{k}{x^2} dx \\ &= \sum_{k=1}^{N-1} \frac{1}{k+1} \\ &= \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} \\ &= s_N - 1. \end{aligned}$$

Supplement (Euler's summation formula). (Theorem 7.13 in the textbook: Tom. M. Apostol, *Mathematical Analysis*, 2nd edition.) If f has a continuous derivative f' on $[a, b]$, then we have

$$\sum_{a < n \leq b} f(n) = \int_a^b f(x) dx + \int_a^b f'(x) \{x\} dx + f(a)\{a\} - f(b)\{b\},$$

where $\sum_{a < n \leq b}$ means the sum from $n = [a] + 1$ to $n = [b]$. When a and b are integers, this becomes

$$\sum_{n=a}^b f(n) = \int_a^b f(x) dx + \int_a^b f'(x) \left(\{x\} - \frac{1}{2} \right) dx + \frac{f(a) + f(b)}{2}.$$

By taking $f(x) = \frac{1}{x}$ we can get the same result.

(3) Show that

$$\int_1^N \frac{\{x\}}{x^2} dx = \log N - s_N + 1 = 1 - \gamma_N.$$

In fact,

$$\begin{aligned} \int_1^N \frac{\{x\}}{x^2} dx &= \int_1^N \frac{x - [x]}{x^2} dx \\ &= \int_1^N \frac{1}{x} dx - \int_1^N \frac{[x]}{x^2} dx \\ &= \log N - (s_N - 1) \\ &= \log N - s_N + 1 \\ &= 1 - \gamma_N. \end{aligned}$$

(4) Since

$$\lim_{N \rightarrow \infty} \int_1^N \frac{\{x\}}{x^2} dx = \int_1^\infty \frac{\{x\}}{x^2} dx$$

exists (by (1)), $\gamma = \lim \gamma_N$ exists.

□

Proof of (b). By $s_n - \log n > 0$ in (a)(4)(Theorem 3.14), it suffices to choose $N = 10^m$ such that $s_N \geq \log(N+1) > 100$, or

$$m > \frac{\log(\exp(100) - 1)}{\log 10},$$

or choose m satisfying

$$m > \frac{100}{\log 10} > \frac{\log(\exp(100) - 1)}{\log 10},$$

or $m = 44$. □

Note. The exact value of N is

$$15092688622113788323693563264538101449859497 \approx 1.509 \times 10^{43}.$$

Exercise 8.10. Prove that $\sum \frac{1}{p}$ diverges; the sum extends over all primes.

There are many proofs of this result. We provide some of them.

Proof (Due to hint). Given N .

(1) Show that

$$\sum_{n \leq N} \frac{1}{n} \leq \prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1}.$$

By the unique factorization theorem on $n \leq N$,

$$\sum_{n \leq N} \frac{1}{n} \leq \prod_{p \leq N} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1}.$$

(2) By (1) and the fact that $\sum \frac{1}{n}$ diverges, there are infinitely many primes.

(3) Show that

$$\prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1} \leq \exp \left(\sum_{p \leq N} \frac{2}{p} \right).$$

By applying the inequality $(1 - x)^{-1} < e^{2x}$ where $x \in (0, \frac{1}{2}]$ on any prime p ,

$$\left(1 - \frac{1}{p}\right)^{-1} < \exp \left(\frac{2}{p} \right).$$

Now multiplying the inequality over all primes $p \leq N$ and noticing that $\exp(x) \cdot \exp(y) = \exp(x + y)$, we have

$$\prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1} \leq \exp \left(\sum_{p \leq N} \frac{2}{p} \right).$$

(4) By (1)(3),

$$\sum_{n \leq N} \frac{1}{n} \leq \exp \left(\sum_{p \leq N} \frac{2}{p} \right).$$

Since $\sum_{n \leq N} \frac{1}{n}$ diverges, the result holds.

□

Proof (Due to Kenneth Ireland and Michael Rosen). The proof in Kenneth Ireland and Michael Rosen, A Classical Introduction to Modern Number Theory, Second Edition (Theorem 3 in Chapter 2) does not use the inequality $(1 - x)^{-1} < e^{2x}$ ($x \in (0, \frac{1}{2}]$) directly. Instead, the authors take the logarithm on $(1 - p^{-1})^{-1}$ and estimate it. (So the length of proof is longer than the proof due to hint.)

That is,

$$\begin{aligned}
-\log(1 - p^{-1}) &= \sum_{n=1}^{\infty} \frac{p^{-n}}{n} \\
&= \frac{1}{p} + \sum_{n=2}^{\infty} \frac{p^{-n}}{n} \\
&< \frac{1}{p} + \sum_{n=2}^{\infty} p^{-n} \\
&= \frac{1}{p} + \frac{p^{-2}}{1 - p^{-1}} \\
&< \frac{1}{p} + 2 \cdot \frac{1}{p^2}.
\end{aligned}$$

Now we sum over all primes $p \leq N$,

$$\log \left(\prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1} \right) < \sum_{p \leq N} \frac{1}{p} + 2 \sum_{p \leq N} \frac{1}{p^2}.$$

So

$$\log \sum_{n \leq N} \frac{1}{n} < \sum_{p \leq N} \frac{1}{p} + 2 \sum_{p \leq N} \frac{1}{p^2}.$$

Notice that $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{p^2}$ converges (since $\sum \frac{1}{n^2}$ converges). Therefore, $\sum \frac{1}{p}$ diverges. \square

Proof (Due to I. Niven). It is an exercise in Kenneth Ireland and Michael Rosen, A Classical Introduction to Modern Number Theory, Second Edition. See Exercise 27 in Chapter 2.

- (1) Show that $\sum' \frac{1}{n}$, the sum being over square free integers, diverges. For any positive integers n , we can write $n = a^2 b$ where $a \in \mathbb{Z}^+$ and b is a square free integer. Given N ,

$$\sum_{n \leq N} \frac{1}{n} \leq \left(\sum_{a=1}^{\infty} \frac{1}{a^2} \right) \left(\sum'_{b \leq N} \frac{1}{b} \right).$$

Notice that $\sum_{a=1}^{\infty} \frac{1}{a^2}$ converges. Since $\sum_{n \leq N} \frac{1}{n} \rightarrow \infty$ as $N \rightarrow \infty$, $\sum'_{b \leq N} \frac{1}{b} \rightarrow \infty$ as $N \rightarrow \infty$.

- (2) Show that

$$\prod_{p \leq N} \left(1 + \frac{1}{p}\right) \rightarrow \infty \text{ as } N \rightarrow \infty.$$

By the unique factorization theorem on $n \leq N$,

$$\prod_{p \leq N} \left(1 + \frac{1}{p}\right) \geq \sum'_{n \leq N} \frac{1}{n}.$$

Since $\sum'_{n \leq N} \frac{1}{n} \rightarrow \infty$ as $N \rightarrow \infty$ by (1), the conclusion is established.

(3) By applying the inequality $e^x > 1 + x$ on any prime p ,

$$\exp\left(\frac{1}{p}\right) > 1 + \frac{1}{p}.$$

Now multiplying the inequality over all primes $p \leq N$ and noticing that $\exp(x) \cdot \exp(y) = \exp(x + y)$, we have

$$\exp\left(\sum_{p \leq N} \frac{1}{p}\right) > \prod_{p \leq N} \left(1 + \frac{1}{p}\right).$$

By (2), $\exp\left(\sum_{p \leq N} \frac{1}{p}\right) \rightarrow \infty$ as $N \rightarrow \infty$, or $\sum_{p \leq N} \frac{1}{p} \rightarrow \infty$ as $N \rightarrow \infty$.

□

Exercise 8.11. Suppose $f \in \mathcal{R}$ on $[0, A]$ for all $A < \infty$, and $f(x) \rightarrow 1$ as $x \rightarrow +\infty$. Prove that

$$\lim_{t \rightarrow 0} t \int_0^\infty e^{-tx} f(x) dx = 1 \quad (t > 0),$$

It is similar to Exercise 3.14(a).

Proof. Given any $\varepsilon > 0$.

(1) The integral $\int_0^\infty e^{-tx} f(x) dx$ is well-defined. (It suffices to show that $\int_0^\infty e^{-tx} f(x) dx$ converges absolutely in the sense of Exercise 6.8. It is quite easy since $f(x) \rightarrow 1$ as $x \rightarrow +\infty$ and well-behavior of $\int_{A_0}^\infty e^{-tx} f(x) dx$ for any $A_0 > 0$.)

(2) Note that

$$t \int_0^\infty e^{-tx} dx = 1$$

for any $t > 0$.

(3) Since $f(x) \rightarrow 1$ as $x \rightarrow +\infty$, there is $A_0 > 0$ such that

$$|f(x) - 1| < \frac{\varepsilon}{64} \text{ whenever } x \geq A_0.$$

(4) Since $f \in \mathcal{R}$ on $[0, A_0]$, f is bounded on $[0, A_0]$, or $|f| \leq M$ on $[0, A_0]$ for some M (Theorem 6.7(c)).

(5) As $t > 0$,

$$\begin{aligned} & \left| \left(t \int_0^\infty e^{-tx} f(x) dx \right) - 1 \right| \\ &= \left| t \int_0^\infty e^{-tx} (f(x) - 1) dx \right| \end{aligned} \quad ((2))$$

$$\leq t \int_0^\infty e^{-tx} |f(x) - 1| dx \quad ((1) \text{ with Theorem 6.13})$$

$$\begin{aligned} &= t \int_0^{A_0} e^{-tx} |f(x) - 1| dx + t \int_{A_0}^\infty e^{-tx} |f(x) - 1| dx \\ &\leq t \int_0^{A_0} (M + 1) dx + t \int_{A_0}^\infty e^{-tx} |f(x) - 1| dx \end{aligned} \quad ((3) \text{ and } e^{-tx} \leq 1)$$

$$\leq t \int_0^{A_0} (M + 1) dx + t \int_{A_0}^\infty e^{-tx} \frac{\varepsilon}{64} dx \quad ((4))$$

$$\begin{aligned} &= t A_0 (M + 1) + \exp(-A_0 t) \frac{\varepsilon}{64} \\ &\leq t A_0 (M + 1) + \frac{\varepsilon}{64}. \end{aligned} \quad (e^{-tx} \leq 1)$$

Since t is arbitrary, take $t = \frac{\varepsilon}{89 A_0 (M + 1)} > 0$ to get

$$\left| \left(t \int_0^\infty e^{-tx} f(x) dx \right) - 1 \right| < \frac{\varepsilon}{89} + \frac{\varepsilon}{64} < \varepsilon,$$

or

$$\lim_{t \rightarrow 0^+} t \int_0^\infty e^{-tx} f(x) dx = 1.$$

□

Exercise 8.12. Suppose $0 < \delta < \pi$,

$$f(x) = \begin{cases} 1 & \text{if } |x| \leq \delta, \\ 0 & \text{if } \delta < |x| \leq \pi, \end{cases}$$

and $f(x + 2\pi) = f(x)$ for all x .

(a) Compute the Fourier coefficients of f .

(b) Compute that

$$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2} \quad (0 < \delta < \pi).$$

(c) Deduce from Parseval's theorem that

$$\sum_{n=1}^{\infty} \frac{(\sin(n\delta))^2}{n^2\delta} = \frac{\pi - \delta}{2}.$$

(d) Let $\delta \rightarrow 0$ and prove that

$$\int_0^{\infty} \left(\frac{\sin x}{x} \right)^2 dx = \frac{\pi}{2}.$$

(e) Put $\delta = \frac{\pi}{2}$ in (c). What do you get?

It is a centered square pulse around $x = 0$ with shift δ . Besides, $f(x)$ is an even function.

Proof of (a).

$$\begin{aligned} c_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \\ &= \frac{1}{2\pi} \int_{-\delta}^{\delta} dx \\ &= \frac{\delta}{\pi}. \end{aligned}$$

For $0 \neq n \in \mathbb{Z}$,

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \\ &= \frac{1}{2\pi} \int_{-\delta}^{\delta} e^{-inx} dx \\ &= \frac{1}{2\pi} \cdot \frac{2 \sin(n\delta)}{n} \\ &= \frac{\sin(n\delta)}{n\pi}. \end{aligned}$$

□

Supplement. Find a_n and b_n of this textbook.

By (a), $a_0 = \frac{\delta}{\pi}$, $a_n = \frac{2 \sin(n\delta)}{n\pi}$, $b_n = 0$ for $n \in \mathbb{Z}^+$. Surely, we can compute a_n

and b_n ($n > 0$) directly. Since $f(x)$ is an even function, $b_n = 0$. And

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \\ &= \frac{2}{\pi} \int_0^{\delta} \cos(nx) dx \\ &= \frac{2 \sin(n\delta)}{n\pi}. \end{aligned}$$

Proof of (b). Given $x = 0$, there are constants $\delta' = \delta > 0$ and $M = 1 < \infty$ such that

$$|f(0+t) - f(0)| \leq M|t|$$

for all $t \in (-\delta', \delta')$. By Theorem 8.14,

$$\sum_{-\infty}^{\infty} c_n = f(0).$$

Notice that $c_{-n} = c_n$ for $n \in \mathbb{Z}^+$, so

$$\begin{aligned} \frac{\delta}{\pi} + 2 \sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n\pi} &= 1 \\ \sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} &= \frac{\pi - \delta}{2}. \end{aligned}$$

□

We can also use the expression a_n and b_n to prove the same thing. Besides, taking $\delta = 1$ yields

$$\sum_{n=1}^{\infty} \frac{\sin n}{n} = \frac{\pi - 1}{2}.$$

Proof of (c). Since $f(x)$ is a Riemann-integrable function with period 2π , by Parseval's theorem

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{-\infty}^{\infty} |c_n|^2.$$

So

$$\frac{\delta}{\pi} = \frac{\delta^2}{\pi^2} + 2 \sum_{n=1}^{\infty} \frac{(\sin(n\delta))^2}{n^2 \pi^2},$$

or

$$\sum_{n=1}^{\infty} \frac{(\sin(n\delta))^2}{n^2 \delta} = \frac{\pi - \delta}{2}.$$

□

Notices that

$$\sum_{n=1}^{\infty} \frac{(\sin n)^2}{n^2} = \frac{\pi - 1}{2}$$

as $\delta = 1$.

Proof of (d). Given $\varepsilon > 0$. By Exercise 6.8,

$$\int_0^{\infty} \left(\frac{\sin x}{x} \right)^2 dx$$

exists. So there exists $b > 0$ such that

$$\left| \int_0^b \left(\frac{\sin x}{x} \right)^2 dx - \int_0^{\infty} \left(\frac{\sin x}{x} \right)^2 dx \right| < \frac{\varepsilon}{4}$$

By Supplement in Chapter 6, there exists $\delta > 0$ such that for any partition $P_m = \{0, \frac{b}{m}, \frac{2b}{m}, \dots, \frac{(m-1)b}{m}, b\}$ of $[0, b]$ with $\|P\| = \frac{b}{m} < \delta$, or $m > \frac{b}{\delta}$, we have

$$\begin{aligned} \left| \sum_{n=1}^m \frac{(\sin(n \frac{b}{m}))^2}{(n \frac{b}{m})^2} \cdot \frac{b}{m} - \int_0^b \left(\frac{\sin x}{x} \right)^2 dx \right| &< \frac{\varepsilon}{4}, \\ \left| \sum_{n=1}^m \frac{(\sin(n \frac{b}{m}))^2}{n^2 \frac{b}{m}} - \int_0^b \left(\frac{\sin x}{x} \right)^2 dx \right| &< \frac{\varepsilon}{4}. \end{aligned}$$

For simplicity we resize δ to $\delta < \pi$ to make $0 < \frac{b}{m} < \delta < \pi$. Besides, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, there exists $N > 0$ such that

$$\left| \sum_{n=1}^{\infty} \frac{(\sin(n \frac{b}{m}))^2}{n^2 \frac{b}{m}} - \sum_{n=1}^m \frac{(\sin(n \frac{b}{m}))^2}{n^2 \frac{b}{m}} \right| < \frac{\varepsilon}{4}$$

whenever $m \geq N$. By (c),

$$\left| \frac{\pi - \frac{b}{m}}{2} - \sum_{n=1}^m \frac{(\sin(n \frac{b}{m}))^2}{n^2 \frac{b}{m}} \right| < \frac{\varepsilon}{4}$$

whenever $m \geq N$. Last, it is easy to get

$$\left| \frac{\pi}{2} - \frac{\pi - \frac{b}{m}}{2} \right| < \frac{\varepsilon}{4}$$

whenever $m > \frac{2b}{\varepsilon}$. Now we have

$$\left| \frac{\pi}{2} - \int_0^{\infty} \left(\frac{\sin x}{x} \right)^2 dx \right| < \varepsilon$$

whenever $m > \max(\frac{b}{\delta}, N, \frac{2b}{\varepsilon})$. Since ε is arbitrary, $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$. \square

Proof of (e).

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

Write

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^2} &= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \\ &= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}, \end{aligned}$$

so

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{6}.$$

\square

Exercise 8.13. Put $f(x) = x$ if $0 \leq x < 2\pi$, and apply Parseval's theorem to conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi}{6}.$$

Proof.

$$\begin{aligned} c_0 &= \frac{1}{2\pi} \int_0^{2\pi} x dx \\ &= \pi, \end{aligned}$$

For $n \neq 0$,

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_0^{2\pi} x e^{-inx} dx \\ &= \frac{1}{2\pi} \left(\left[-\frac{1}{in} x e^{-inx} \right]_{x=0}^{x=2\pi} - \int_0^{2\pi} -\frac{1}{in} e^{-inx} dx \right) \\ &= \frac{i}{n}. \end{aligned}$$

Since $f(x)$ is a Riemann-integrable function with period 2π , by Parseval's theorem

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

So

$$\frac{1}{2\pi} \cdot \frac{(2\pi)^3}{3} = \pi^2 + 2 \sum_{n=1}^{\infty} \frac{1}{n^2},$$

or

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

□

Supplement. Put $f(x) = x^n$ if $n \in \mathbb{Z}^+$ and $0 \leq x < 2\pi$. Might get

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!}.$$

Exercise 8.14. PLACEHOLDER.

Exercise 8.15. With the Dirichlet kernel D_n as defined by

$$D_n(x) = \sum_{k=-n}^n \exp(ikx) = \frac{\sin(n + \frac{1}{2})x}{\sin(\frac{x}{2})},$$

put the **Fejér kernel**

$$K_N(x) = \frac{1}{N+1} \sum_{n=0}^N D_n(x).$$

Prove that

$$K_N(x) = \frac{1}{N+1} \cdot \frac{1 - \cos(N+1)x}{1 - \cos x}$$

and that

- (a) $K_N \geq 0$,
- (b) $\frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(x) dx = 1$,
- (c) $K_N(x) \leq \frac{1}{N+1} \cdot \frac{2}{1 - \cos \delta}$ if $0 < \delta \leq |x| \leq \pi$.

If $s_N = s_N(f; x)$ is the N th partial sum of the Fourier series of f , consider the arithmetic means

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_N}{N+1}.$$

Prove that

$$s_N(f; x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) K_N(t) dt,$$

and hence prove **Fejér's theorem**:

If f is continuous, with period 2π , then $\sigma_N(f; x) \rightarrow f(x)$ uniformly on $[-\pi, \pi]$.

(Hint: Use properties (a), (b), (c) to proceed as in Theorem 7.26.)

Proof of $K_N(x) = \frac{1}{N+1} \cdot \frac{1 - \cos(N+1)x}{1 - \cos x}$. Since

$$\begin{aligned} (1 - \cos x)K_N(x) &= 2 \left(\sin \frac{x}{2} \right)^2 \frac{1}{N+1} \sum_{n=0}^N \frac{\sin\left(n + \frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)} \\ &= \frac{1}{N+1} \sum_{n=0}^N 2 \sin \frac{x}{2} \sin\left(n + \frac{1}{2}\right)x \\ &= \frac{1}{N+1} \sum_{n=0}^N (\cos(nx) - \cos(n+1)x) \\ &= \frac{1 - \cos(N+1)x}{N+1}, \\ K_N(x) &= \frac{1}{N+1} \cdot \frac{1 - \cos(N+1)x}{1 - \cos x} \end{aligned}$$

if $x \neq 2k\pi$ for $k \in \mathbb{Z}$. \square

Proof of (a). It is clear since $\cos x \leq 1$ for all $x \in \mathbb{R}$. Or we may write

$$K_N(x) = \frac{1}{N+1} \left(\frac{\sin \frac{(N+1)x}{2}}{\sin \frac{x}{2}} \right)^2 \geq 0.$$

\square

Proof of (b). By the definition of $D_n(x)$,

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(x) dx &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{N+1} \sum_{n=0}^N D_n(x) dx \\ &= \frac{1}{N+1} \sum_{n=0}^N \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(x) dx \\ &= \frac{1}{N+1} \sum_{n=0}^N 1 \\ &= 1. \end{aligned}$$

\square

Proof of (c). Since $\cos x$ is bounded by 1 and monotonically decreasing on $(0, \pi]$,

$$\begin{aligned} K_N(x) &= \frac{1}{N+1} \cdot \frac{1 - \cos(N+1)x}{1 - \cos x} \\ &\leq \frac{1}{N+1} \cdot \frac{2}{1 - \cos \delta}. \end{aligned}$$

□

Proof of $s_N(f; x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)K_N(t)dt$.

$$\begin{aligned} \sigma_N(f; x) &= \frac{1}{N+1} \sum_{n=0}^N s_N(f; x) \\ &= \frac{1}{N+1} \sum_{n=0}^N \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)D_N(t)dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) \left(\frac{1}{N+1} \sum_{n=0}^N D_N(t) \right) dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)K_N(t)dt. \end{aligned}$$

□

Proof of Fejér's theorem. Given any $\varepsilon > 0$.

(1)

$$\begin{aligned} |\sigma_N(f; x) - f(x)| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)K_N(t)dt - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)K_N(t)dt \right| \\ &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x-t) - f(x))K_N(t)dt \right| \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x-t) - f(x)|K_N(t)dt. \end{aligned}$$

(2) Since f is continuous on a compact set $[-\pi, \pi]$, f is continuous uniformly. For such $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(y) - f(x)| < \frac{\varepsilon}{2}$$

whenever $x, y \in [-\pi, \pi]$ and $|y - x| < \delta$.

(3) Since f is continuous on a compact set $[-\pi, \pi]$, f is bounded on $[-\pi, \pi]$, say $M = \sup |f(x)|$.

(4) Therefore,

$$\begin{aligned}
& |\sigma_N(f; x) - f(x)| \\
& \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x-t) - f(x)| K_N(t) dt \\
& = \frac{1}{2\pi} \int_{-\pi}^{-\delta} |f(x-t) - f(x)| K_N(t) dt \\
& \quad + \frac{1}{2\pi} \int_{-\delta}^{\delta} |f(x-t) - f(x)| K_N(t) dt \\
& \quad + \frac{1}{2\pi} \int_{\delta}^{\pi} |f(x-t) - f(x)| K_N(t) dt \\
& \leq \frac{1}{2\pi} \int_{-\pi}^{-\delta} 2M \cdot \frac{1}{N+1} \cdot \frac{2}{1-\cos \delta} dt \\
& \quad + \frac{1}{2\pi} \int_{-\delta}^{\delta} \frac{\varepsilon}{2} K_N(t) dt \\
& \quad + \frac{1}{2\pi} \int_{\delta}^{\pi} 2M \cdot \frac{1}{N+1} \cdot \frac{2}{1-\cos \delta} dt \\
& = \frac{4M(\pi-\delta)}{(N+1)(1-\cos \delta)\pi} + \frac{\varepsilon}{2} \cdot \frac{1}{2\pi} \int_{-\delta}^{\delta} K_N(t) dt \\
& \leq \frac{4M(\pi-\delta)}{(N+1)(1-\cos \delta)\pi} + \frac{\varepsilon}{2}.
\end{aligned}$$

(5) Since N is arbitrary, we can take an integer $N > \frac{4M(\pi-\delta)}{(1-\cos \delta)\pi\varepsilon} - 1$ so that

$$\begin{aligned}
|\sigma_N(f; x) - f(x)| & \leq \frac{4M(\pi-\delta)}{(N+1)(1-\cos \delta)\pi} + \frac{\varepsilon}{2} \\
& < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
& = \varepsilon.
\end{aligned}$$

Therefore, the conclusion holds.

□

Exercise 8.16. Prove a pointwise version of Fejér's theorem: If $f \in \mathcal{R}$ and $f(x+)$, $f(x-)$ exist for some x , then

$$\lim_{N \rightarrow \infty} \sigma_N(f; x) = \frac{1}{2}[f(x+) + f(x-)].$$

Proof. Given any $\varepsilon > 0$.

(1) Since $K_N(-t) = K_N(t)$, we have

$$\sigma_N(f; x) = \frac{1}{2\pi} \int_0^\pi f(x-t)K_N(t)dt + \frac{1}{2\pi} \int_0^\pi f(x+t)K_N(t)dt$$

and

$$\frac{1}{2\pi} \int_0^\pi K_N(t)dt = \frac{1}{2}.$$

(2) Since $f \in \mathcal{B}$, f is bounded on $[-\pi, \pi]$, say $M = \sup |f(x)|$.

(3) Therefore,

$$\begin{aligned} & \left| \frac{1}{2\pi} \int_0^\pi f(x-t)K_N(t)dt - \frac{1}{2}f(x-) \right| \\ &= \left| \frac{1}{2\pi} \int_0^\pi (f(x-t) - f(x-))K_N(t)dt \right| \\ &\leq \frac{1}{2\pi} \int_0^\pi |f(x-t) - f(x-)|K_N(t)dt. \end{aligned}$$

Since $f(x-)$ exists, for fixed $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(y) - f(x-)| < \frac{\varepsilon}{2}$$

whenever $y \in (x - \delta, x) \cap [-\pi, \pi]$. Hence,

$$\begin{aligned} & \left| \frac{1}{2\pi} \int_0^\pi f(x-t)K_N(t)dt - \frac{1}{2}f(x-) \right| \\ &\leq \frac{1}{2\pi} \int_0^\pi |f(x-t) - f(x-)|K_N(t)dt \\ &= \frac{1}{2\pi} \int_0^\delta |f(x-t) - f(x-)|K_N(t)dt \\ &\quad + \frac{1}{2\pi} \int_\delta^\pi |f(x-t) - f(x-)|K_N(t)dt \\ &\leq \frac{1}{2\pi} \int_0^\delta \frac{\varepsilon}{2} K_N(t)dt + \frac{1}{2\pi} \int_\delta^\pi 2M \cdot \frac{1}{N+1} \cdot \frac{2}{1-\cos\delta} dt \\ &= \frac{\varepsilon}{2} \cdot \frac{1}{2\pi} \int_0^\delta K_N(t)dt + \frac{2M(\pi-\delta)}{(N+1)(1-\cos\delta)\pi} \\ &\leq \frac{\varepsilon}{4} + \frac{2M(\pi-\delta)}{(N+1)(1-\cos\delta)\pi}. \end{aligned}$$

(4) Since N is arbitrary, we can take an integer $N_1 > \frac{8M(\pi-\delta)}{(1-\cos \delta)\pi\varepsilon} - 1$ such that

$$\begin{aligned} \left| \frac{1}{2\pi} \int_0^\pi f(x-t)K_n(t)dt - \frac{1}{2}f(x-) \right| &\leq \frac{\varepsilon}{4} + \frac{2M(\pi-\delta)}{(n+1)(1-\cos \delta)\pi} \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &= \frac{\varepsilon}{2} \end{aligned}$$

whenever $n \geq N_1$. Similarly, we can take an integer N_2 such that

$$\begin{aligned} \left| \frac{1}{2\pi} \int_0^\pi f(x+t)K_n(t)dt - \frac{1}{2}f(x+) \right| &\leq \frac{\varepsilon}{4} + \frac{2M(\pi-\delta)}{(n+1)(1-\cos \delta)\pi} \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &= \frac{\varepsilon}{2}. \end{aligned}$$

whenever $n \geq N_2$.

(5) Hence,

$$\begin{aligned} &\left| \sigma_n(f; x) - \frac{1}{2}[f(x+) + f(x-)] \right| \\ &\leq \left| \frac{1}{2\pi} \int_0^\pi f(x-t)K_n(t)dt - \frac{1}{2}f(x-) \right| \\ &\quad + \left| \frac{1}{2\pi} \int_0^\pi f(x+t)K_n(t)dt - \frac{1}{2}f(x+) \right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

whenever $n \geq \max\{N_1, N_2\}$. Hence, $\lim \sigma_n(f; x) = \frac{1}{2}[f(x+) + f(x-)]$.

□

Supplement. Poisson's equation. (Theorem 1 of Section 2.2 in the textbook: *Lawrence C. Evans, Partial Differential Equations.*) Let the fundamental solution of Laplace's equation be

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log |x| & (n=2) \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & (n \geq 3), \end{cases}$$

where $x \in \mathbb{R}^n$, $x \neq 0$. Let

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)dy.$$

Then $-\Delta u = f$ in \mathbb{R}^n . Note that $\Phi(x)$ blows up at 0. To calculate $\Delta u(x)$, we need to isolate this singularity inside a small ball, say $B(0; \varepsilon)$. Therefore,

$$\Delta u(x) = \int_{B(0; \varepsilon)} \Phi(y) \Delta_x f(x - y) dy + \int_{\mathbb{R}^n - B(0; \varepsilon)} \Phi(y) \Delta_x f(x - y) dy,$$

and we can continue estimating two integrals individually as the textbook did.

Exercise 8.17. PLACEHOLDER.

Exercise 8.18. PLACEHOLDER.

Exercise 8.19. PLACEHOLDER.

Exercise 8.20. *The following simple computation yields a good approximation to Stirling's formula. For $m = 1, 2, 3, \dots$, define*

$$f(x) = (m + 1 - x) \log m + (x - m) \log(m + 1)$$

if $m \leq x \leq m + 1$, and define

$$g(x) = \frac{x}{m} - 1 + \log m$$

if $m - \frac{1}{2} \leq x < m + \frac{1}{2}$. Draw the graphs of f and g . Note that $f(x) \leq \log x \leq g(x)$ if $x \geq 1$ and that

$$\int_1^n f(x) dx = \log(n!) - \frac{1}{2} \log n > -\frac{1}{8} + \int_1^n g(x) dx.$$

Integrate $\log x$ over $[1, n]$. Conclude that

$$\frac{7}{8} < \log(n!) - \left(n + \frac{1}{2}\right) \log n + n < 1$$

for $n = 2, 3, 4, \dots$ (Note: $\log \sqrt{2\pi} \approx 0.918 \dots$) Thus

$$e^{\frac{7}{8}} < \frac{n!}{\left(\frac{n}{e}\right)^n \sqrt{n}} < e.$$

Proof.

- (1) Omit the graphs of f and g . Note that the concavity of $\log(x)$ implies that $f(x) \leq \log(x)$. Here the equality holds if and only if $x \in \mathbb{Z}^+$. Besides, since $g(x)$ is the tangent line at $(x, \log x)$ whenever $x \in \mathbb{Z}^+$, $g(x) \geq \log(x)$ and the equality holds if and only if $x \in \mathbb{Z}^+$.

(2)

$$\begin{aligned}
\int_1^n f(x)dx &= \sum_{m=1}^{n-1} \int_m^{m+1} f(x)dx \\
&= \sum_{m=1}^{n-1} \int_m^{m+1} (m+1-x) \log m + (x-m) \log(m+1) dx \\
&= \sum_{m=1}^{n-1} \int_m^{m+1} (\log(m+1) - \log m)x + (m+1) \log m - m \log(m+1) dx \\
&= \sum_{m=1}^{n-1} (\log(m+1) - \log m) \left(\frac{(m+1)^2 - m^2}{2} \right) + (m+1) \log m - m \log(m+1) \\
&= \sum_{m=1}^{n-1} \log m + \frac{1}{2} \sum_{m=1}^{n-1} (\log(m+1) - \log m) \\
&= \log((n-1)!) + \frac{1}{2} \log n \\
&= \log(n!) - \frac{1}{2} \log n.
\end{aligned}$$

(3) Write

$$\int_1^n g(x)dx = \left(\sum_{m=1}^n \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} g(x)dx \right) - \int_{\frac{1}{2}}^1 g(x)dx - \int_n^{n+\frac{1}{2}} g(x)dx.$$

(a)

$$\begin{aligned}
\sum_{m=1}^n \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} g(x)dx &= \sum_{m=1}^n \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} \left(\frac{x}{m} - 1 + \log m \right) dx \\
&= \sum_{m=1}^n \log m \\
&= \log(n!).
\end{aligned}$$

(b)

$$\int_{\frac{1}{2}}^1 g(x)dx = \int_{\frac{1}{2}}^1 (x - 1 + \log 1)dx = -\frac{1}{8}.$$

(c)

$$\int_n^{n+\frac{1}{2}} g(x)dx = \int_{\frac{1}{2}}^1 \left(\frac{x}{n} - 1 + \log n \right) dx = \frac{1}{2} \log n - \frac{1}{8n}.$$

By (a)(b)(c),

$$\int_1^n g(x)dx = \log(n!) - \frac{1}{2} \log n + \frac{1}{8} \left(1 - \frac{1}{n} \right) < \log(n!) - \frac{1}{2} \log n + \frac{1}{8}.$$

- (4) Since $f(x) \leq \log x \leq g(x)$ and the equality holds if and only if $x \in \mathbb{Z}^+$ (by (1)),

$$\int_1^n f(x)dx \leq \int_1^n \log x dx \leq \int_1^n g(x)dx$$

for all $n = 1, 2, 3, \dots$. The equality holds if and only if $n = 1$. Hence by (2)(3)

$$\log(n!) - \frac{1}{2} \log n \leq n \log n - n + 1 \leq \log(n!) - \frac{1}{2} \log n + \frac{1}{8}.$$

Arrange the inequality to get

$$\frac{7}{8} < \log(n!) - \left(n + \frac{1}{2}\right) \log n + n \leq 1$$

for $n = 1, 2, 3, \dots$. Note that the equality holds if and only if $n = 1$. Therefore

$$\frac{7}{8} < \log(n!) - \left(n + \frac{1}{2}\right) \log n + n < 1$$

for $n = 2, 3, \dots$

- (5) Exponentiate to get

$$\exp\left(\frac{7}{8}\right) < \exp\left[\log(n!) - \left(n + \frac{1}{2}\right) \log n + n\right] < \exp(1),$$

or

$$e^{\frac{7}{8}} < \frac{\exp(\log(n!)) \exp(n)}{\exp\left[\left(n + \frac{1}{2}\right) \log n\right]} < e,$$

or $e^{\frac{7}{8}} < \frac{n!}{\left(\frac{n}{e}\right)^n \sqrt{n}} < e$ (since $\exp(x)$ is a strictly increasing function of x).

□

PLACEHOLDER.

Exercise 8.21 (Norm of Dirichlet kernel). *Let*

$$L_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)| dt \quad (n = 1, 2, 3, \dots).$$

Prove that there exists a constant $C > 0$ such that

$$L_n > C \log n \quad (n = 1, 2, 3, \dots),$$

or, more precisely, that the sequence

$$\left\{ L_n - \frac{4}{\pi^2} \log n \right\}$$

is bounded.

Proof.

(1) Write

$$\begin{aligned}
L_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)| dt \\
&= \frac{1}{\pi} \int_0^{\pi} |D_n(t)| dt && (D_n(-t) = D_n(t)) \\
&= \frac{1}{\pi} \int_0^{\pi} \frac{|\sin(n + \frac{1}{2})t|}{\sin(\frac{t}{2})} dt. && (\sin(\frac{t}{2}) \geq 0 \text{ on } [0, \pi])
\end{aligned}$$

(2) So,

$$\begin{aligned}
L_n &= \frac{1}{\pi} \int_0^{\pi} \frac{|\sin(n + \frac{1}{2})t|}{\sin(\frac{t}{2})} dt \\
&= \frac{1}{\pi} \int_0^{\pi} \left| \sin\left(n + \frac{1}{2}\right)t \right| \left(\frac{1}{\sin(\frac{t}{2})} - \frac{1}{t} + \frac{1}{t} \right) dt \\
&= \underbrace{\frac{1}{\pi} \int_0^{\pi} \left| \sin\left(n + \frac{1}{2}\right)t \right| \left(\frac{1}{\sin(\frac{t}{2})} - \frac{1}{t} \right) dt}_{:=I_n} + \underbrace{\frac{2}{\pi} \int_0^{\pi} \frac{|\sin(n + \frac{1}{2})t|}{t} dt}_{:=J_n}.
\end{aligned}$$

(3) Show that I_n is uniformly bounded. Note that $f(x) = \frac{1}{\sin(x)} - \frac{1}{x}$ is bounded (since $\lim_{x \rightarrow 0} f(x) = 0$ by using L'Hospital's rule twice). Also, $|\sin(n + \frac{1}{2})t| \leq 1$ for any n . Hence

$$0 \leq I_n < \sup(f(x)) = \frac{2}{\pi}.$$

(4) Show that $J_n - \frac{4}{\pi^2} \log n$ is uniformly bounded. Since

$$\begin{aligned}
J_n &= \frac{2}{\pi} \int_0^{\pi} \frac{|\sin(n + \frac{1}{2})t|}{t} dt \\
&= \frac{2}{\pi} \int_0^{(n+\frac{1}{2})\pi} \frac{|\sin x|}{x} dx, && (\text{Let } x = \left(n + \frac{1}{2}\right)t)
\end{aligned}$$

we have

$$\frac{2}{\pi} \sum_{k=0}^{n-1} \underbrace{\int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x} dx}_{:=J_n^{(1)}} \leq J_n \leq \frac{2}{\pi} \sum_{k=0}^n \underbrace{\int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x} dx}_{:=J_n^{(2)}}.$$

So

$$\begin{aligned}
J_n^{(1)} &\geq \frac{2}{\pi} \sum_{k=0}^{n-1} \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{(k+1)\pi} dx \\
&= \frac{2}{\pi} \sum_{k=0}^{n-1} \frac{2}{(k+1)\pi} \quad \left(\int_0^\pi |\sin x| dx = 0 \right) \\
&\geq \frac{4}{\pi^2} \log n, \quad (\text{Exercise 8.9})
\end{aligned}$$

and

$$\begin{aligned}
J_n^{(2)} &= \frac{2}{\pi} \int_0^\pi \frac{|\sin x|}{x} dx + \frac{2}{\pi} \sum_{k=1}^n \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x} dx \\
&\leq \frac{2}{\pi} \int_0^\pi \frac{|\sin x|}{x} dx + \frac{2}{\pi} \sum_{k=1}^n \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{k\pi} dx \\
&= \frac{2}{\pi} \int_0^\pi \frac{|\sin x|}{x} dx + \frac{2}{\pi} \sum_{k=1}^n \frac{2}{k\pi} \\
&\leq \frac{2}{\pi} \int_0^\pi \frac{|\sin x|}{x} dx + \frac{4}{\pi^2} (\log n + 1) \\
&= \frac{4}{\pi^2} \log n + \frac{4}{\pi^2} + \frac{2}{\pi} \int_0^\pi \frac{|\sin x|}{x} dx.
\end{aligned}$$

Hence,

$$0 \leq J_n - \frac{4}{\pi^2} \log n \leq \frac{4}{\pi^2} + \frac{2}{\pi} \int_0^\pi \frac{|\sin x|}{x} dx.$$

(5) By (3)(4),

$$0 \leq L_n - \frac{4}{\pi^2} \log n \leq \frac{2}{\pi} + \frac{4}{\pi^2} + \frac{2}{\pi} \int_0^\pi \frac{|\sin x|}{x} dx.$$

□

Exercise 8.22. PLACEHOLDER.

Exercise 8.23. PLACEHOLDER.

Exercise 8.24. PLACEHOLDER.

Exercise 8.25. PLACEHOLDER.

Exercise 8.26. PLACEHOLDER.

Exercise 8.27. PLACEHOLDER.

Exercise 8.28. PLACEHOLDER.

Exercise 8.29. PLACEHOLDER.

Exercise 8.30. Use Stirling's formula to prove that

$$\lim_{x \rightarrow \infty} \frac{\Gamma(x+c)}{x^c \Gamma(x)} = 1$$

for every real constant c .

Proof. By Stirling's formula,

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\Gamma(x+c)}{\left(\frac{x+c-1}{e}\right)^{x+c-1} \sqrt{2\pi(x+c-1)}} &= 1 \\ \lim_{x \rightarrow \infty} \frac{\Gamma(x)}{\left(\frac{x-1}{e}\right)^{x-1} \sqrt{2\pi(x-1)}} &= 1, \end{aligned}$$

we have

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\Gamma(x+c)}{x^c \Gamma(x)} &= \lim_{x \rightarrow \infty} \frac{\Gamma(x+c)}{x^c \Gamma(x)} \\ &\quad \times \lim_{x \rightarrow \infty} \frac{\left(\frac{x+c-1}{e}\right)^{x+c-1} \sqrt{2\pi(x+c-1)}}{\Gamma(x+c)} \\ &\quad \times \lim_{x \rightarrow \infty} \frac{\Gamma(x)}{\left(\frac{x-1}{e}\right)^{x-1} \sqrt{2\pi(x-1)}} \\ &= \lim_{x \rightarrow \infty} \frac{\left(\frac{x+c-1}{e}\right)^{x+c-1} \sqrt{2\pi(x+c-1)}}{x^c \left(\frac{x-1}{e}\right)^{x-1} \sqrt{2\pi(x-1)}} \\ &= \lim_{x \rightarrow \infty} \frac{\left(\frac{x+c-1}{e}\right)^c}{x^c} \frac{\left(\frac{x+c-1}{e}\right)^{x-1}}{\left(\frac{x-1}{e}\right)^{x-1}} \sqrt{\frac{x+c-1}{x-1}} \\ &= \frac{1}{e^c} \cdot e^c \cdot 1 \\ &= 1 \end{aligned}$$

since

(1)

$$\lim_{x \rightarrow \infty} \frac{\left(\frac{x+c-1}{e}\right)^c}{x^c} = \frac{1}{e^c} \lim_{x \rightarrow \infty} \left(\frac{x+c-1}{x}\right)^c = \frac{1}{e^c}.$$

(2)

$$\lim_{x \rightarrow \infty} \frac{\left(\frac{x+c-1}{e}\right)^{x-1}}{\left(\frac{x-1}{e}\right)^{x-1}} = \lim_{x \rightarrow \infty} \left(\frac{x+c-1}{x-1}\right)^{x-1} = \lim_{x \rightarrow \infty} \left(1 + \frac{c}{x-1}\right)^{x-1} = e^c.$$

(3) and

$$\lim_{x \rightarrow \infty} \sqrt{\frac{x+c-1}{x-1}} = \lim_{x \rightarrow \infty} \sqrt{1 + \frac{c}{x-1}} = 1.$$

□

Exercise 8.31. In the proof of Theorem 7.26 it was shown that

$$\int_{-1}^1 (1-x^2)^n dx \geq \frac{4}{3\sqrt{n}}$$

for $n = 1, 2, 3, \dots$. Use Theorem 8.20 and Exercise 8.30 to show the more precise result

$$\lim_{n \rightarrow \infty} \sqrt{n} \int_{-1}^1 (1-x^2)^n dx = \sqrt{\pi}.$$

Proof.

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sqrt{n} \int_{-1}^1 (1-x^2)^n dx \\ &= \lim_{n \rightarrow \infty} \sqrt{n} \int_0^1 u^{-\frac{1}{2}} (1-u)^n dx && (u = x^2) \\ &= \lim_{n \rightarrow \infty} \sqrt{n} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma(n+1)}{\Gamma\left(n + \frac{3}{2}\right)} && (\text{Theorem 8.20}) \\ &= \Gamma\left(\frac{1}{2}\right) \lim_{n \rightarrow \infty} \frac{n^{\frac{1}{2}} \Gamma(n+1)}{\Gamma\left(n + \frac{3}{2}\right)} \\ &= \Gamma\left(\frac{1}{2}\right) && (\text{Exercise 8.30}) \\ &= \sqrt{\pi}. && (\text{Some consequences 8.21}) \end{aligned}$$

□