Chapter 9: Functions of Several Variables

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Exercise 9.1. *If S is a nonempty subset of a vector space X, prove (as asserted in Section 9.1) that the span of S is a vector space.*

Denote the span of S by $\mathrm{span}(S)$.

Proof.

(1) Since $S \neq \varnothing$, there is $\mathbf{z} \in S$. So $1\mathbf{z} = \mathbf{z} \in \mathrm{span}(S) \neq \varnothing$. (In fact, $\mathrm{span}(S) \supseteq S$.)

(2) If $\mathbf{x}, \mathbf{y} \in \mathrm{span}(S)$, then there exist elements $\mathbf{x}_1, \ldots, \mathbf{x}_m$, $\mathbf{y}_1, \ldots, \mathbf{y}_n \in S$ and scalars $a_1, \ldots, a_m$, $b_1, \ldots, b_n$ such that

$$\begin{aligned}\mathbf{x} &= a_1\mathbf{x}_1 + \cdots + a_m\mathbf{x}_m, \\ \mathbf{y} &= b_1\mathbf{y}_1 + \cdots + b_n\mathbf{y}_n.\end{aligned}$$

Then

$$\mathbf{x} + \mathbf{y} = a_1\mathbf{x}_1 + \cdots + a_m\mathbf{x}_m + b_1\mathbf{y}_1 + \cdots + b_n\mathbf{y}_n$$

is a linear combination of the elements of S. For any scalar c,

$$c\mathbf{x} = (ca_1)\mathbf{x}_1 + \cdots + (ca_m)\mathbf{x}_m$$

is again linear combination of the elements of S.

(3) By (1)(2), $\mathrm{span}(S)$ is a vector space.

$\square$

Note. Any subspace of X that contains S must also contain $\mathrm{span}(S)$.

Exercise 9.2. *Prove (as asserted in Section 9.6) that BA is linear if A and B are linear transformations. Prove also that A^{-1} is linear and invertible if A is invertible.*

Proof. Use the notation in Definitions 9.6.

(1) *Show that BA is linear if A and B are linear transformations.* Let X, Y, Z be vector spaces, $A \in L(X, Y)$ and $B \in L(Y, Z)$.

(a) Given any $\mathbf{x}_1, \mathbf{x}_2 \in X$.

$$\begin{aligned}(BA)(\mathbf{x}_1 + \mathbf{x}_2) &= B(A(\mathbf{x}_1 + \mathbf{x}_2)) \\ &= B(A\mathbf{x}_1 + A\mathbf{x}_2) && (A \text{ is a linear transformation}) \\ &= B(A\mathbf{x}_1) + B(A\mathbf{x}_2) && (B \text{ is a linear transformation}) \\ &= (BA)\mathbf{x}_1 + (BA)\mathbf{x}_2.\end{aligned}$$

(b) For any $\mathbf{x} \in X$ and scalar c,

$$\begin{aligned}(BA)(c\mathbf{x}) &= B(A(c\mathbf{x})) \\ &= B(cA\mathbf{x}) && (A \text{ is a linear transformation}) \\ &= cB(A\mathbf{x}) && (B \text{ is a linear transformation}) \\ &= c(BA)\mathbf{x}.\end{aligned}$$

By (a)(b), $BA \in L(X, Z)$.

(2) *Show that A^{-1} is linear if A is invertible.*

(a) Given any $\mathbf{y}_1, \mathbf{y}_2 \in X$. Since A is surjective, there exist $\mathbf{x}_1, \mathbf{x}_2 \in X$ such that

$$\begin{aligned}\mathbf{y}_1 &= A\mathbf{x}_1 \\ \mathbf{y}_2 &= A\mathbf{x}_2.\end{aligned}$$

So

$$\begin{aligned}A^{-1}\mathbf{y}_1 &= A^{-1}(A\mathbf{x}_1) = \mathbf{x}_1 \\ A^{-1}\mathbf{y}_2 &= A^{-1}(A\mathbf{x}_2) = \mathbf{x}_2\end{aligned}$$

(by Definitions 9.4). Hence

$$\begin{aligned}A^{-1}(\mathbf{y}_1 + \mathbf{y}_2) &= A^{-1}(A\mathbf{x}_1 + A\mathbf{x}_2) \\ &= A^{-1}(A(\mathbf{x}_1 + \mathbf{x}_2)) && (A \text{ is a linear transformation}) \\ &= \mathbf{x}_1 + \mathbf{x}_2 && (\text{Definitions 9.4}) \\ &= A^{-1}\mathbf{y}_1 + A^{-1}\mathbf{y}_2.\end{aligned}$$

(b) For any $\mathbf{y} \in X$ and scalar c, there is a corresponding $\mathbf{x} \in X$ such that $\mathbf{y} = A\mathbf{x}$ since A is surjective. So $A^{-1}\mathbf{y} = \mathbf{x}$ by Definition 9.4. Hence

$$\begin{aligned}A^{-1}(c\mathbf{y}) &= A^{-1}(cA\mathbf{x}) \\ &= A^{-1}(A(c\mathbf{x})) && (A \text{ is a linear transformation}) \\ &= c\mathbf{x} && (\text{Definitions 9.4}) \\ &= cA^{-1}\mathbf{y}.\end{aligned}$$

By (a)(b), $A^{-1} \in L(X)$.

(3) *Show that A^{-1} is invertible if A is invertible.* It suffices to show that A^{-1} is injective and surjective.

(a) *Show that A^{-1} is injective.* Given any $\mathbf{y}_1, \mathbf{y}_2 \in X$. Since A is surjective, there exist $\mathbf{x}_1, \mathbf{x}_2 \in X$ such that

$$\begin{aligned} \mathbf{y}_1 &= A\mathbf{x}_1 \\ \mathbf{y}_2 &= A\mathbf{x}_2. \end{aligned}$$

Suppose $A^{-1}\mathbf{y}_1 = A^{-1}\mathbf{y}_2$. So $A^{-1}(A\mathbf{x}_1) = A^{-1}(A\mathbf{x}_2)$, or $\mathbf{x}_1 = \mathbf{x}_2$, or $\mathbf{y}_1 = A\mathbf{x}_1 = A\mathbf{x}_2 = \mathbf{y}_2$.

(b) *Show that A^{-1} is surjective.* For any $\mathbf{x} \in X$, there exists $A\mathbf{x} \in X$ such that $A^{-1}(A\mathbf{x}) = \mathbf{x}$ by Definitions 9.4.

$\square$

Exercise 9.3. *Assume $A \in L(X, Y)$ and $A\mathbf{x} = \mathbf{0}$ only when $\mathbf{x} = \mathbf{0}$. Prove that A is then* 1-1.

Proof. Suppose $A\mathbf{x} = A\mathbf{y}$. Since A is a linear transformation, $A(\mathbf{x} - \mathbf{y}) = A\mathbf{x} - A\mathbf{y} = \mathbf{0}$. By assumption, $\mathbf{x} - \mathbf{y} = \mathbf{0}$ or $\mathbf{x} = \mathbf{y}$. $\square$

Exercise 9.4. *Prove (as asserted in Section 9.30) that null spaces and ranges of linear transformations are vector spaces.*

Proof. Use the notation in Definitions 9.30. Suppose X, Y are vector spaces, and $A \in L(X, Y)$, as in Definition 9.6.

(1) *Show that $\mathscr{N}(A)$ is a vector space in X.*

(a) Note that $\mathbf{0} \in X$. Since $A\mathbf{0} = \mathbf{0}$, $\mathbf{0} \in \mathscr{N}(A) \neq \varnothing$.

(b) Suppose $\mathbf{x}_1, \mathbf{x}_2 \in \mathscr{N}(A)$. Then

$$\begin{aligned} A(\mathbf{x}_1 + \mathbf{x}_2) &= A\mathbf{x}_1 + A\mathbf{x}_2 && (A \text{ is a linear transformation}) \\ &= \mathbf{0} + \mathbf{0} && (\mathbf{x}_1, \mathbf{x}_2 \in \mathscr{N}(A)) \\ &= \mathbf{0}. \end{aligned}$$

So $\mathbf{x}_1 + \mathbf{x}_2 \in \mathscr{N}(A)$.

(c) Suppose $\mathbf{x} \in \mathscr{N}(A)$ and c is a scalar. Then

$$\begin{aligned} A(c\mathbf{x}) &= cA\mathbf{x} && (A \text{ is a linear transformation}) \\ &= c\mathbf{0} && (\mathbf{x} \in \mathscr{N}(A)) \\ &= \mathbf{0}. \end{aligned}$$

So $c\mathbf{x} \in \mathscr{N}(A)$.

By (a)(b)(c), $\mathscr{N}(A)$ is a vector space.

(2) *Show that $\mathscr{R}(A)$ is a vector space in Y.*

(a) Note that $\mathbf{0} \in X$. So $A\mathbf{0} = \mathbf{0} \in \mathscr{R}(A) \neq \varnothing$.

(b) Suppose $\mathbf{y}_1, \mathbf{y}_2 \in \mathscr{R}(A)$. Then there exist $\mathbf{x}_1, \mathbf{x}_2 \in X$ such that $A\mathbf{x}_1 = \mathbf{y}_1$ and $A\mathbf{x}_2 = \mathbf{y}_2$. Hence

$$\begin{aligned}\mathbf{y}_1 + \mathbf{y}_2 &= A\mathbf{x}_1 + A\mathbf{x}_2 \\ &= A(\mathbf{x}_1 + \mathbf{x}_2) \qquad (A \text{ is a linear transformation}).\end{aligned}$$

So $\mathbf{y}_1 + \mathbf{y}_2 \in \mathscr{R}(A)$.

(c) Suppose $\mathbf{y} \in \mathscr{R}(A)$ and c is a scalar. Then there exists $\mathbf{x} \in X$ such that $A\mathbf{x} = \mathbf{y}$. Hence

$$\begin{aligned}c\mathbf{y} &= cA\mathbf{x} \\ &= A(c\mathbf{x}) \qquad (A \text{ is a linear transformation}).\end{aligned}$$

So $c\mathbf{y} \in \mathscr{R}(A)$.

By (a)(b)(c), $\mathscr{R}(A)$ is a vector space.

$\square$

Exercise 9.5. *Prove that to every $A \in L(\mathbb{R}^n, \mathbb{R}^1)$ corresponds a unique $\mathbf{y} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$. Prove also that $\|A\| = |\mathbf{y}|$. (Hint: Under certain conditions, equality holds in the Schwarz inequality.)*

Proof.

(1) Recall that $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$ (Definitions 9.1). Given any $\mathbf{x} \in \mathbb{R}^n$, write $\mathbf{x} = (x_1, \dots, x_n)$ as $\mathbf{x} = \sum x_j \mathbf{e}_j$.

(2) *Show that $\mathbf{y}$ exists.* Since A is a linear transformation,

$$\begin{aligned}A\mathbf{x} &= A\left(\sum x_j \mathbf{e}_j\right) \\ &= \sum x_j A\mathbf{e}_j \\ &= (x_1, \dots, x_n) \cdot (A\mathbf{e}_1, \dots, A\mathbf{e}_n) \\ &= \mathbf{x} \cdot \sum (A\mathbf{e}_j)\mathbf{e}_j.\end{aligned}$$

Define $\mathbf{y} = \sum (A\mathbf{e}_j)\mathbf{e}_j \in \mathbb{R}^n$ so that $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$.

(3) *Show that $\mathbf{y}$ is unique.* Suppose there exists some $\mathbf{z} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{x} \cdot \mathbf{z}$. So

$$\begin{aligned}0 &= A\mathbf{x} - A\mathbf{x} \\ &= \mathbf{x} \cdot \mathbf{y} - \mathbf{x} \cdot \mathbf{z} \\ &= \mathbf{x} \cdot (\mathbf{y} - \mathbf{z})\end{aligned}$$

for any $\mathbf{x} \in \mathbb{R}^n$. In particular, take $\mathbf{x} = \mathbf{y} - \mathbf{z} \in \mathbb{R}^n$ to get

$$0 = (\mathbf{y} - \mathbf{z}) \cdot (\mathbf{y} - \mathbf{z}) = |\mathbf{y} - \mathbf{z}|^2$$

or $\mathbf{y} - \mathbf{z} = \mathbf{0}$ or $\mathbf{y} = \mathbf{z}$.

(4) *Show that* $\|A\| = |\mathbf{y}|$. By the Schwarz inequality (Theorem 1.37(d)),

$$|A\mathbf{x}| = |\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}| \leq |\mathbf{y}|$$

as $|\mathbf{x}| \leq 1$. Take the sup over all $|\mathbf{x}| \leq 1$ to get

$$\|A\| \leq |\mathbf{y}|.$$

If $\mathbf{y} = \mathbf{0}$, then $\|A\| = |\mathbf{y}| = 0$. If $\mathbf{y} \neq \mathbf{0}$, then the equality holds when $\mathbf{x} = \frac{\mathbf{y}}{|\mathbf{y}|} \in \mathbb{R}^n$. (Here $|\mathbf{x}| = 1$.)

□

Exercise 9.6. *If* $f(0,0) = 0$ *and*

$$f(x,y) = \frac{xy}{x^2+y^2} \qquad \textit{if } (x,y) \neq (0,0),$$

prove that $(D_1 f)(x,y)$ *and* $(D_2 f)(x,y)$ *exist at every point of* $\mathbb{R}^2$, *although* f *is not continuous at* $(0,0)$.

Proof.

(1) *Show that*

$$(D_1 f)(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0), \\ \frac{y(y^2-x^2)}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0). \end{cases}$$

Write

$$\begin{aligned}(D_1 f)(x,y) &= \lim_{t \to 0} \frac{f((x,y) + t(1,0)) - f(x,y)}{t} \\ &= \lim_{t \to 0} \frac{f(x+t,y) - f(x,y)}{t}.\end{aligned}$$

If $(x,y) = (0,0)$,

$$(D_1 f)(0,0) = \lim_{t \to 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t \to 0} \frac{0-0}{t} = 0.$$

If $(x, y) \neq (0, 0)$,

$$\begin{aligned}
(D_1 f)(x, y) &= \lim_{t \to 0} \frac{f(x+t, y) - f(x, y)}{t} \\
&= \lim_{t \to 0} \frac{\frac{(x+t)y}{(x+t)^2+y^2} - \frac{xy}{x^2+y^2}}{t} \\
&= \lim_{t \to 0} \frac{y(y^2 - x^2) - txy}{((x+t)^2 + y^2)(x^2 + y^2)} \\
&= \frac{y(y^2 - x^2)}{(x^2 + y^2)^2}.
\end{aligned}$$

(2) *Show that*

$$(D_2 f)(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0), \\ \frac{x(x^2-y^2)}{(x^2+y^2)^2} & \text{if } (x, y) \neq (0, 0). \end{cases}$$

Similar to (1).

(3) *Show that f is not continuous at* $(0, 0)$. Note that

$$\lim_{n \to \infty} f\left(\frac{1}{n}, \frac{1}{n}\right) = \lim_{n \to \infty} \frac{\frac{1}{n} \cdot \frac{1}{n}}{\frac{1}{n^2} + \frac{1}{n^2}} = \lim_{n \to \infty} \frac{1}{2} = \frac{1}{2}$$

and

$$\lim_{n \to \infty} f\left(\frac{1}{n}, 0\right) = \lim_{n \to \infty} \frac{0}{\frac{1}{n^2} + 0} = \lim_{n \to \infty} 0 = 0.$$

Hence the limit $\lim_{(x,y) \to (0,0)} f(x, y)$ does not exist.

□

Exercise 9.7. ...

Proof.

(1)

(2)

□

Exercise 9.8. ...

Proof.

(1)

(2)

□

Exercise 9.9. *...*

Proof.

(1)

(2)

□

Exercise 9.10. *...*

Proof.

(1)

(2)

□

Exercise 9.11. *If f and g are differentiable real functions in $\mathbb{R}^n$, prove that*

$$\nabla(fg) = f\nabla g + g\nabla f$$

and that

$$\nabla\left(\frac{1}{f}\right) = -\frac{1}{f^2}\nabla f$$

whenever $f \neq 0$.

Proof. Recall Example 9.18:

$$(\nabla(f))(\mathbf{x}) = \sum_{i=1}^{n}(D_i f)(\mathbf{x})\mathbf{e}_i.$$

(1) *Show that* $\nabla(fg) = f\nabla g + g\nabla f$. For any $\mathbf{x} \in \mathbb{R}^n$,

$$\begin{aligned}
(\nabla(fg))(\mathbf{x}) &= \sum_{i=1}^{n} (D_i(fg))(\mathbf{x})\mathbf{e}_i \\
&= \sum_{i=1}^{n} (g(D_i f) + f(D_i g))(\mathbf{x})\mathbf{e}_i && \text{(Theorem 5.3(b))} \\
&= \sum_{i=1}^{n} \left[g(\mathbf{x})(D_i f)(\mathbf{x}) + f(\mathbf{x})(D_i g)(\mathbf{x}) \right] \mathbf{e}_i \\
&= g(\mathbf{x}) \sum_{i=1}^{n} (D_i f)(\mathbf{x})\mathbf{e}_i + f(\mathbf{x}) \sum_{i=1}^{n} (D_i g)(\mathbf{x})\mathbf{e}_i \\
&= g(\mathbf{x})(\nabla f)(\mathbf{x}) + f(\mathbf{x})(\nabla g)(\mathbf{x}) \\
&= (f\nabla g + g\nabla f)(\mathbf{x}).
\end{aligned}$$

(2) *Show that*

$$\nabla\left(\frac{1}{f}\right) = -\frac{1}{f^2}\nabla f$$

whenever $f \neq 0$. Note that $\nabla(1) = 0$ since

$$\nabla(1)(\mathbf{x}) = \sum (D_i 1)(\mathbf{x})\mathbf{e}_i = \sum (0)(\mathbf{x})\mathbf{e}_i = \sum 0\mathbf{e}_i = 0.$$

Hence as $f \neq 0$, we have

$$\begin{aligned}
0 &= \nabla(1) \\
&= \nabla\left(f\frac{1}{f}\right) && (f \neq 0) \\
&= f\nabla\left(\frac{1}{f}\right) + \frac{1}{f}\nabla f && ((1)),
\end{aligned}$$

or $\nabla\left(\frac{1}{f}\right) = -\frac{1}{f^2}\nabla f$.

□

Exercise 9.12. ...

Proof.

(1)

(2)

□

Exercise 9.13. ...

Proof.

(1)

(2)

□

Exercise 9.14. ...

Proof.

(1)

(2)

□

Exercise 9.15. ...

Proof.

(1)

(2)

□

Exercise 9.16. ...

Proof.

(1)

(2)

□

Exercise 9.17. ...

Proof.

(1)

(2)

□

Exercise 9.18. ...

Proof.

(1)

(2)

□

Exercise 9.19. ...

Proof.

(1)

(2)

□

Exercise 9.20. ...

Proof.

(1)

(2)

□

Exercise 9.21. ...

Proof.

(1)

(2)

□

Exercise 9.22. ...

Proof.

(1)

(2)

□

Exercise 9.23. ...

Proof.

(1)

(2)

□

Exercise 9.24. ...

Proof.

(1)

(2)

□

Exercise 9.25. ...

Proof.

(1)

(2)

□

Exercise 9.26. ...

Proof.

(1)

(2)

□

Exercise 9.27. ...

Proof.

(1)

(2)

□

Exercise 9.28. ...

Proof.

(1)

(2)

□

Exercise 9.29. ...

Proof.

(1)

(2)

□

Exercise 9.30. ...

Proof.

(1)

(2)

$\square$

Exercise 9.31. ...

Proof.

(1)

(2)

$\square$