

Notes on the book:  
*Apostol, Introduction to Analytic  
Number Theory*

Meng-Gen Tsai  
plover@gmail.com

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## Chapter 1: The Fundamental Theorem of Arithmetic

### Exercise 1.15.

*Prove that every  $n \geq 12$  is the sum of two composite numbers.*

*Proof.* Write  $n = 2m$  (resp.  $n = 2m + 1$ ) where  $m \in \mathbb{Z}$ ,  $m \geq 6$ . Then  $n = 8 + 2(m - 4)$  (resp.  $n = 9 + 2(m - 4)$ ) is the sum of two composite numbers.  $\square$

### Exercise 1.30.

*If  $n > 1$  prove that the sum*

$$\sum_{k=1}^n \frac{1}{k}$$

*is not an integer.*

*Proof.*

(1) (Reductio ad absurdum) Suppose

$$H := \sum_{k=1}^n \frac{1}{k}$$

were an integer.

(2) Let  $s$  be the largest integer such that  $2^s \leq n$ . So the integer number

$$\begin{aligned} 2^{s-1}H &= \sum_{k=1}^n \frac{2^{s-1}}{k} \\ &= 2^{s-1} + 2^{s-2} + \frac{2^{s-1}}{3} + 2^{s-3} + \frac{2^{s-1}}{5} + \frac{2^{s-2}}{3} + \cdots + \frac{1}{2} + \cdots. \end{aligned}$$

has only one term of even denominators (as  $n > 1$ ) if we write all terms in irreducible fractions. That is,

$$2^{s-1}H = \frac{1}{2} + \frac{c}{d} \in \mathbb{Z}$$

where  $\frac{c}{d}$  is an irreducible fraction with odd  $d$ . Hence it suffices to show that  $2 \nmid d$  to get a contradiction.

(3) By

$$\frac{1}{2} + \frac{c}{d} = \frac{d+2c}{2d} \in \mathbb{Z}$$

we have  $d + 2c = 2dd'$  for some  $d' \in \mathbb{Z}$ . Note that 2 is a prime. So  $2 \mid (d + 2c)$  or  $2 \mid d$ , which is absurd.

□

## Chapter 2: Arithmetical functions and Dirichlet multiplication

### Exercise 2.4.

Prove that  $\varphi(n) > \frac{n}{6}$  for all  $n$  with at most 8 distinct prime factors.

*Proof.*

(1)

$$\begin{aligned}
 \varphi(n) &= n \prod_{p|n} \left(1 - \frac{1}{p}\right) && \text{(Theorem 2.4)} \\
 &\geq n \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{5}\right) \left(1 - \frac{1}{7}\right) \\
 &\quad \left(1 - \frac{1}{11}\right) \left(1 - \frac{1}{13}\right) \left(1 - \frac{1}{17}\right) \left(1 - \frac{1}{19}\right) \\
 &= \frac{55296}{323323} n \\
 &> \frac{n}{6}.
 \end{aligned}$$

(2) The conclusion does not hold if  $n$  has more than 9 distinct prime factors.

□