Chapter 3: Numerical Sequences and Series

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Exercise 3.1. *Prove that the convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?*

Proof.

(1) Since $\{s_n\}$ is convergent, there is $s \in \mathbb{R}^1$ with the following property: given any $\varepsilon > 0$, there is N such that $|s_n - s| < \varepsilon$ whenever $n \geq N$. So

$$||s_n| - |s|| \leq |s_n - s| < \varepsilon$$

(Exercise 1.13). That is, $\{|s_n|\}$ converges to $|s|$.

(2) The converse is not true by considering $s_n = (-1)^{n+1}$.

$\square$

Exercise 3.2. *Calculate* $\lim_{n\to\infty}(\sqrt{n^2+n} - n)$.

Proof.

$$\sqrt{n^2+n} - n = \frac{n}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+\frac{1}{n}}+1} \to \frac{1}{1+1} = \frac{1}{2}$$

as $n \to \infty$. $\square$

Proof ($\varepsilon - N$ argument). Let $s_n = \sqrt{n^2+n} - n$. *Show that the sequence* $\{s_n\}$ *converges to* $s = \frac{1}{2}$. Given any $\varepsilon > 0$, there is $N > \frac{1}{\varepsilon}$ such that

$$\begin{aligned}
|s_n - s| &= \left|(\sqrt{n^2+n} - n) - \frac{1}{2}\right| = \left|\frac{1}{\sqrt{1+\frac{1}{n}}+1} - \frac{1}{2}\right| \\
&= \left|\frac{2 - \left(\sqrt{1+\frac{1}{n}}+1\right)}{2\left(\sqrt{1+\frac{1}{n}}+1\right)}\right| = \left|\frac{1 - \sqrt{1+\frac{1}{n}}}{2\left(\sqrt{1+\frac{1}{n}}+1\right)}\right| \\
&= \left|\frac{1 - \left(1 - \frac{1}{n}\right)}{2\left(\sqrt{1+\frac{1}{n}}+1\right)^2}\right| = \left|\frac{-\frac{1}{n}}{2\left(\sqrt{1+\frac{1}{n}}+1\right)^2}\right| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon
\end{aligned}$$

wheneven $n \geq N$. $\square$

Exercise 3.3. *If* $s_1 = \sqrt{2}$ *and*

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} \ (n = 1, 2, 3, ...),$$

prove that $\{s_n\}$ *converges, and that* $s_n < 2$ *for* $n = 1, 2, 3, ...$

The convergence of $\{s_n\}$ implies there is $s \in \mathbb{R}$ such that $s_n \to s$ where $s = \sqrt{2 + \sqrt{s}}$ and $\sqrt{2} < s \leq 2$. WolframAlpha shows that

$$s = \frac{1}{3}\left(-1 + \sqrt[3]{\frac{1}{2}(79 - 3\sqrt{249})} + \sqrt[3]{\frac{1}{2}(79 + 3\sqrt{249})}\right).$$

Proof (Theorem 3.14).

(1) *Show that* $\{s_n\}$ *is increasing (by mathematical induction).*

   (a) *Show that* $s_2 > s_1$. In fact,

$$s_2 = \sqrt{2 + \sqrt{s_1}} = \sqrt{2 + \sqrt{\sqrt{2}}} < \sqrt{2} = s_1.$$

   (a) *Show that* $s_{n+1} > s_n$ *if* $s_n > s_{n-1}$.

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} > \sqrt{2 + \sqrt{s_{n-1}}} = s_n.$$

   By mathematical induction, $\{s_n\}$ is (strictly) increasing.

(2) *Show that* $\{s_n\}$ *is bounded (by mathematical induction).*

   (a) *Show that* $s_1 \leq 2$. $\sqrt{2} \leq 2$.

   (a) *Show that* $s_{n+1} \leq 2$ *if* $s_n \leq 2$.

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} \leq \sqrt{2 + \sqrt{2}} < 2.$$

   By mathematical induction, $\{s_n\}$ is bounded by 2.

Hence, $\{s_n\}$ converges since $\{s_n\}$ is increasing and bounded (Theorem 3.14). $\square$

Exercise 3.4. *Find the upper and lower limits of the sequences* $\{s_n\}$ *defined by*

$$s_1 = 0; s_{2m} = \frac{s_{2m-1}}{2}; s_{2m+1} = \frac{1}{2} + s_{2m}.$$

Write out the first few terms of $\{s_n\}$:

$$0, 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{3}{8}, \frac{7}{8}, \frac{7}{16}, \frac{15}{16}, ...$$

It suggests us

$$\begin{aligned} s_{2m+1} &= 1 - \frac{1}{2^m} \ (m = 0, 1, 2, ...), \\ s_{2m} &= \frac{1}{2} - \frac{1}{2^m} \ (m = 1, 2, 3, ...). \end{aligned}$$

Proof.

(1) *Show that*

$$\begin{aligned} s_{2m+1} &= 1 - \frac{1}{2^m} \ (m = 0, 1, 2, ...), \\ s_{2m} &= \frac{1}{2} - \frac{1}{2^m}. \ (m = 1, 2, 3, ...) \end{aligned}$$

Apply mathematical induction.

(2) The upper limit is 1.

(3) The lower limit is $\frac{1}{2}$.

□

Exercise 3.5. *For any two real sequences $\{a_n\}$, $\{b_n\}$, prove that*

$$\limsup_{n \to \infty} (a_n + b_n) \leq \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n$$

provided the sum of the right is not of the form $\infty - \infty$.

Proof. Write $\alpha = \limsup_{n \to \infty} a_n$ and $\beta = \limsup_{n \to \infty} b_n$.

(1) $\alpha = \infty$ *and* $\beta = \infty$. Nothing to do.

(2) $\alpha = -\infty$ *and* $\beta = -\infty$. Since $\alpha = -\infty < \infty$, there exists M' such that $a_n < M'$ for all n. For any real M, $a_n > M - M'$ for at most a finite number of values of n (Theorem 3.17(a)). Hence $a_n + b_n > M$ for at most a finite number of values of n. Hence $\limsup_{n \to \infty}(a_n + b_n) = -\infty$, or

$$\limsup_{n \to \infty} (a_n + b_n) = \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n$$

in this case.

(3) *α and β are finite.* (Similar to the argument in Theorem 3.37.) Choose $\alpha' > \alpha$ and $\beta' > \beta$. There is an integer N such that

$$\alpha' \geq a_n \text{ and } \beta' \geq b_n$$

whenever $n \geq N$. Hence

$$a_n + b_n \leq \alpha' + \beta'$$

whenever $n \geq N$. Take lim sup to get Hence

$$\limsup_{n \to \infty} (a_n + b_n) \leq \alpha' + \beta'.$$

Since the inequality is true for every $\alpha' > \alpha$ and $\beta' > \beta$, we have

$$\limsup_{n \to \infty} (a_n + b_n) \leq \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n.$$

□

Exercise 3.7. *Prove that the convergence of $\sum a_n$ implies the convergence of*

$$\sum \frac{\sqrt{a_n}}{n},$$

if $a_n \geq 0$.

Proof (Cauchy's inequatity).

(1) *Show that $\sum \frac{\sqrt{a_n}}{n}$ is bounded.* For any $k \in \mathbb{Z}^+$,

$$\begin{aligned} \left(\sum_{n=1}^{k} \frac{\sqrt{a_n}}{n} \right)^2 &\leq \left(\sum_{n=1}^{k} a_n \right) \left(\sum_{n=1}^{k} \frac{1}{n^2} \right) && \text{(Cauchy's inequatity)} \\ &\leq \left(\sum_{n=1}^{\infty} a_n \right) \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right). && \left(\sum a_n, \sum \frac{1}{n^2} \text{: convergent} \right) \end{aligned}$$

Thus, $\left(\sum_{n=1}^{k} \frac{\sqrt{a_n}}{n} \right)^2$ is bounded, or $\sum_{n=1}^{k} \frac{\sqrt{a_n}}{n}$ is bounded.

(2) *Show that $\sum_{n=1}^{k} \frac{\sqrt{a_n}}{n}$ is increasing.* It is clear due to $\frac{\sqrt{a_n}}{n} \geq 0$.

By Theorem 3.14, $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges. □

Proof (AM-GM inequality). Show that $\sum \frac{\sqrt{a_n}}{n}$ *is bounded.*

$$\begin{aligned}
\frac{\sqrt{a_n}}{n} &\leq \frac{1}{2}\left(a_n + \frac{1}{n^2}\right) && \text{(AM-GM inequality)} \\
\sum_{n=1}^{k} \frac{\sqrt{a_n}}{n} &\leq \frac{1}{2}\left(\sum_{n=1}^{k} a_n + \sum_{n=1}^{k} \frac{1}{n^2}\right) \\
&\leq \frac{1}{2}\left(\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} \frac{1}{n^2}\right). && \left(\sum a_n, \sum \frac{1}{n^2}\text{: convergent}\right)
\end{aligned}$$

Thus, $\sum_{n=1}^{k} \frac{\sqrt{a_n}}{n}$ is bounded. The rest proof is the same as previous. □

Exercise 3.20. *Suppose* $\{p_n\}$ *is a Cauchy sequence in a metric space* X*, and some subsequence* $\{p_{n_i}\}$ *converges to a point* $p \in X$*. Prove that the full sequence* $\{p_n\}$ *converges to* p*.*

Proof. Given any $\varepsilon > 0$.

(1) Since $\{p_n\}$ is a Cauchy sequence, there exists a positive integer N_1 such that

$$d(p_n, p_m) < \frac{\varepsilon}{2} \text{ whenever } n, m \geq N_1.$$

(2) Since the subsequence $\{p_{n_i}\}$ converges to a point $p \in X$, there exists a positive integer N_2 such that

$$d(p_{n_i}, p) < \frac{\varepsilon}{2} \text{ whenever } n_i \geq N_2.$$

(3) Let $N = \max\{N_1, N_2\}$ be a positive integer. So

$$\begin{aligned}
d(p_n, p) &\leq d(p_n, p_{n_i}) + d(p_{n_i}, p) && \text{(Definition 2.15(c))} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \text{ whenever } n, n_i \geq N && \text{((1)(2))} \\
&= \varepsilon \text{ whenever } n \geq N.
\end{aligned}$$

Hence the full sequence $\{p_n\}$ converges to p.

□

Exercise 3.21. *Prove the following analogue of Theorem 3.10(b): If* $\{E_n\}$ *is a sequence of closed and bounded sets in a complete metric space* X*, if* $E_n \supseteq E_{n+1}$*, and if*

$$\lim_{n\to\infty} \operatorname{diam}(E_n) = 0,$$

then $\bigcap_{n=1}^{\infty} E_n$ *consists of exactly one point.*

Assume $E_n \neq \varnothing$. It is unnecessary to assume that E_n is bounded since we have the condition that $\lim_{n\to\infty} \operatorname{diam}(E_n) = 0$.

Note. Every compact metric space is complete, but complete spaces need not be compact. In fact, a metric space is compact if and only if it is complete and totally bounded.

Proof.

(1) Pick $p_n \in E_n$ for $n = 1, 2, \ldots$.

(2) *Show that $\{p_n\}$ is a Cauchy sequence.* Given any $\varepsilon > 0$. There is a positive integer N such that $\operatorname{diam}(E_n) < \varepsilon$ whenever $n \geq N$. Especially,

$$\operatorname{diam}(E_N) < \varepsilon.$$

As $m, n \geq N$, $p_m \in E_m \subseteq E_N$ and $p_n \in E_n \subseteq E_N$. By the definition of the diameter of E_N,

$$d(p_m, p_n) \leq \operatorname{diam}(E_N) < \varepsilon \text{ whenever } m, n \geq N.$$

(3) Since X is complete, $\{p_n\}$ converges to a point $p \in X$.

(4) *Show that $p \in \bigcap_{n=1}^{\infty} E_n$.* (Reductio ad absurdum) If there were some n such that $p \notin E_n$. Consider the subsequence

$$p_n, p_{n+1}, p_{n+2}, \ldots.$$

Note that all $p_n, p_{n+1}, \ldots$ are in E_n. By (3), it converges to p. Thus p is a limit point of E_n. Since E_n is closed, $p \in E_n$, which is absurd.

(5) *Show that $\bigcap_{n=1}^{\infty} E_n = \{p\}$.* (Reductio ad absurdum) If there were $q \in \bigcap_{n=1}^{\infty} E_n$ with $q \neq p$, then $d(p, q) > 0$ (Definition 2.15(a)). It implies that

$$\operatorname{diam}(E_n) \geq d(p, q) > 0 \text{ for all } n,$$

contrary to $\lim_{n\to\infty} \operatorname{diam}(E_n) = 0$.

$\square$

Exercise 3.22 (Baire category theorem). *Suppose X is a complete metric space, and $\{G_n\}$ is a sequence of dense open subsets of X. Prove Baire's theorem, namely, that $\bigcap_1^{\infty} G_n$ is not empty. (In fact, it is dense in X.) (Hint: Find a shrinking sequence of neighborhoods E_n such that $\overline{E_n} \subseteq G_n$, and apply Exercise 3.21.)*

Proof. Given any open set G_0 in X, will show that

$$\bigcap_{n=0}^{\infty} G_n \neq \varnothing.$$

(1) Since G_1 is dense, $G_0 \cap G_1$ is nonempty. Take any one point p_1 in the open set $G_0 \cap G_1$, then there exists a closed neighborhood

$$V_1 = \{q \in X : d(q, p_1) < r_1\}$$

of p_1 with $r_1 < 1$ such that

$$V_1 \subseteq G_0 \cap G_1.$$

Take $U_1 \subseteq E_1 \subseteq V_1$ such that

$$\begin{aligned} E_1 &= \left\{ q \in X : d(q, p_1) \leq \frac{r_1}{64} \right\} \subseteq V_1, \\ U_1 &= \left\{ q \in X : d(q, p_1) < \frac{r_1}{89} \right\} \subseteq E_1. \end{aligned}$$

(2) Suppose V_n, E_n, U_n have been constructed, take any one point p_{n+1} in the open set $U_n \cap G_{n+1}$, there exists an open neighborhood

$$V_{n+1} = \{q \in X : d(q, p_{n+1}) < r_{n+1}\}$$

of p_{n+1} with r_{n+1} with $r_{n+1} < \frac{1}{n+1}$ such that

$$V_{n+1} \subseteq U_n \cap G_{n+1}.$$

Take $U_1 \subseteq E_1 \subseteq V_1$ such that

$$\begin{aligned} E_{n+1} &= \left\{ q \in X : d(q, p_{n+1}) \leq \frac{r_{n+1}}{64} \right\} \subseteq V_{n+1}, \\ U_{n+1} &= \left\{ q \in X : d(q, p_{n+1}) < \frac{r_{n+1}}{89} \right\} \subseteq E_{n+1}. \end{aligned}$$

(3) Note that

(a) E_n is closed and nonempty (since $p_n \in E_n$).

(b) $\lim_{n \to \infty} \operatorname{diam}(E_n) = 0$ (since $\operatorname{diam}(E_n) \leq 2 \cdot \frac{r_n}{64} < r_n < \frac{1}{n}$.)

(c) $E_1 \supseteq E_2 \supseteq \cdots$ (since $E_{n+1} \subseteq V_{n+1} \subseteq U_n \cap G_{n+1} \subseteq U_n \subseteq E_n$).

Since X is complete, by Exercise 3.21,

$$\bigcap_{n=1}^{\infty} E_n = \{p\}$$

for some $p \in X$.

(4) Hence

$$\begin{aligned}
p \in \bigcap_{n=1}^{\infty} E_n &\Longleftrightarrow p \in E_n \text{ for all } n = 1, 2, 3, \dots \\
&\Longrightarrow p \in E_1 \subseteq G_0 \cap G_1 \text{ and } p \in E_{n+1} \subseteq U_n \cap G_{n+1} \subseteq G_{n+1} \\
&\Longrightarrow p \in G_0 \cap G_1 \cap \cdots = \bigcap_{n=0}^{\infty} G_n \\
&\Longrightarrow \bigcap_{n=0}^{\infty} G_n \neq \varnothing.
\end{aligned}$$

□

Exercise 3.23. *Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $\{d(p_n, q_n)\}$ converges. (Hint: For any m, n,*

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n);$$

it follows that

$$|d(p_n, q_n) - d(p_m, q_m)|$$

is small if m and n are large.)

Proof. Given any $\varepsilon > 0$.

(1) Since $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences, there exists N such that

$$d(p_n, p_m) < \frac{\varepsilon}{2} \text{ and } d(q_m, q_n) < \frac{\varepsilon}{2}$$

whenever $m, n \geq N$.

(2) Note that

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n).$$

It follows that

$$|d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_m, q_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $\{d(p_n, q_n)\}$ is a Cauchy sequence in $\mathbb{R}^1$ (not in X).

(3) Since $\mathbb{R}^1$ is a complete metric space, $\{d(p_n, q_n)\}$ converges.

□

Exercise 3.24. *Let X be a metric space.*

(a) *Call two Cauchy sequences* $\{p_n\}$, $\{q_n\}$ *in* X *equivalent if*

$$\lim_{n\to\infty} d(p_n, q_n) = 0.$$

Prove that this is an equivalence relation.

(b) *Let* X^* *be the set of all equivalence classes so obtained. If* $P \in X^*$, $Q \in X^*$, $\{p_n\} \in P$, $\{q_n\} \in Q$, *define*

$$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n);$$

by Exercise 3.23, this limit exists. Show that the number $\Delta(P, Q)$ *is unchanged if* $\{p_n\}$ *and* $\{q_n\}$ *are replaced by equivalent sequences, and hence that* Δ *is a distance function in* X^*.

(c) *Prove that the resulting metric space* X^* *is complete.*

(d) *For each* $p \in X$, *there is a Cauchy sequence all of whose terms are* p; *let* P_p *be the element of* X^* *which contains this sequence. Prove that*

$$\Delta(P_p, P_q) = d(p, q)$$

for all $p, q \in X$. *In other words, the mapping* φ *defined by* $\varphi(p) = P_p$ *is an isometry (i.e., a distance-preserving mapping) of* X *into* X^*.

(e) *Prove that* $\varphi(X)$ *is dense in* X^*, *and that* $\varphi(X) = X^*$ *if* X *is complete. By (d), we may identify* X *and* $\varphi(X)$ *and thus regard* X *as embedded in the complete metric space* X^*. *We call* X^* *the* ***completion*** *of* X.

Proof of (a). Given Cauchy sequences $\{p_n\}$, $\{q_n\}$, $\{r_n\}$ in X.

(1) *(Reflexivity)*

$$\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} 0 = 0$$

by the reflexivity of the metric function d.

(2) *(Symmetry)*

$$\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(q_n, p_n) = 0$$

by the symmetry of the metric function d.

(3) *(Transitivity)* Suppose that $\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(q_n, r_n) = 0$. By the triangle inequality of the metric function d, we have

$$0 \leq d(p_n, r_n) \leq d(p_n, q_n) + d(q_n, r_n).$$

Take limit to get

$$\begin{aligned}0 &\leq \lim_{n\to\infty} d(p_n, r_n)\\ &\leq \lim_{n\to\infty} (d(p_n, q_n) + d(q_n, r_n))\\ &= \lim_{n\to\infty} d(p_n, q_n) + \lim_{n\to\infty} d(q_n, r_n)\\ &= 0\end{aligned}$$

or $\lim_{n\to\infty} d(p_n, r_n) = 0$.

□

Proof of (b).

(1) *Show that* Δ *is well-defined.* Given any $\{p_n\}, \{p'_n\} \in P$ and $\{q_n\}, \{q'_n\} \in Q$.

   (a) $\lim_{n\to\infty} d(p_n, p'_n) = 0$ since $\{p_n\}$ and $\{p'_n\}$ are in the same equivalence class.

   (b) $\lim_{n\to\infty} d(q_n, q'_n) = 0$ (similar to (a)).

   (c) *Show that* $\lim_{n\to\infty} d(p_n, q_n) \leq \lim_{n\to\infty} d(p'_n, q'_n)$. Since $d(p_n, q_n) \leq d(p_n, p'_n) + d(p'_n, q'_n) + d(q'_n, q_n)$, take limit to get

$$\begin{aligned}\lim_{n\to\infty} d(p_n, q_n) &\leq \lim_{n\to\infty} (d(p_n, p'_n) + d(p'_n, q'_n) + d(q'_n, q_n))\\ &= \lim_{n\to\infty} d(p_n, p'_n) + \lim_{n\to\infty} d(p'_n, q'_n) + \lim_{n\to\infty} d(q'_n, q_n)\\ &= 0 + \lim_{n\to\infty} d(p'_n, q'_n) + 0\\ &= \lim_{n\to\infty} d(p'_n, q'_n)\end{aligned}$$

   since (a)(b).

   (d) *Show that* $\lim_{n\to\infty} d(p_n, q_n) \geq \lim_{n\to\infty} d(p'_n, q'_n)$. Similar to (c).

   By (c)(d), $\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(p'_n, q'_n)$, or $\Delta(P, Q)$ is well-defined.

(2) *Show that* Δ *is a metric.*

   (a) *Show that* $\Delta(P, Q) > 0$ *if* $P \neq Q$*;* $\Delta(P, P) = 0$. It is the definition of Δ.

   (b) *Show that* $\Delta(P, Q) = \Delta(Q, P)$. Similar to the argument in (a)(2).

   (c) *Show that* $\Delta(P, Q) \leq \Delta(P, R) + \Delta(R, Q)$. Similar to the argument in (a)(3).

□

Proof of (c). Show that $\{P_k\}_{k=1}^{\infty}$ *converges to* P *in* (X^*, Δ) *for any given Cauchy sequence* $\{P_k\}$.

(1) Take a Cauchy sequence $\{p_n^{(k)}\}_{n=1}^{\infty}$ to represent P_k for each k. *We will construct a Cauchy sequence $\{p_k\}$ in (X, d) such that $\{P_k\}$ converges to P which is the equivalent class of $\{p_k\}$.*

(2) For each k, there exists N_k such that

$$d\left(p_m^{(k)}, p_n^{(k)}\right) < \frac{1}{k} \text{ whenever } m, n \geq N_k.$$

Especially,

$$d\left(p_m^{(k)}, p_{N_k}^{(k)}\right) < \frac{1}{k} \text{ whenever } m \geq N_k.$$

Let $p_k = p_{N_k}^{(k)}$ and collect all p_k as $\{p_k\}_{k=1}^{\infty}$.

(3) *Show that $\{p_k\}$ is a Cauchy sequence in (X, d).* Note that for any k, we have

$$\begin{aligned} d(p_m, p_n) &= d\left(p_{N_m}^{(m)}, p_{N_n}^{(n)}\right) \\ &\leq d\left(p_{N_m}^{(m)}, p_k^{(m)}\right) + d\left(p_k^{(m)}, p_k^{(n)}\right) + d\left(p_k^{(n)}, p_{N_n}^{(n)}\right). \end{aligned}$$

Let $k \to \infty$, we have

$$\begin{aligned} d(p_m, p_n) &\leq \limsup_{k\to\infty} \left[d\left(p_{N_m}^{(m)}, p_k^{(m)}\right) + d\left(p_k^{(m)}, p_k^{(n)}\right) + d\left(p_k^{(n)}, p_{N_n}^{(n)}\right)\right] \\ &\leq \frac{1}{m} + \Delta(P_m, P_n) + \frac{1}{n} \end{aligned}$$

for any m, n (by (2)). Let $m, n \to \infty$, we establish the result (since $\{P_k\}$ is Cauchy).

(4) *Show that $\{P_k\}$ converges to $P \ni \{p_k\}$.* Given any $\varepsilon > 0$. Since $\{p_k\}$ is Cauchy (3), there is $N > \frac{2}{\varepsilon}$ such that

$$d(p_m, p_n) < \frac{\varepsilon}{2} \text{ whenever } m, n \geq N.$$

Note that

$$\begin{aligned} d\left(p_n^{(k)}, p_n\right) &= d\left(p_n^{(k)}, p_{N_n}^{(n)}\right) \\ &\leq d\left(p_n^{(k)}, p_{N_k}^{(k)}\right) + d\left(p_{N_k}^{(k)}, p_{N_n}^{(n)}\right). \end{aligned}$$

For any $k \geq N$, let $n \to \infty$ to get

$$\begin{aligned}
\Delta(P_k, P) &= \lim_{n\to\infty} d\left(p_n^{(k)}, p_n\right) \\
&\leq \limsup_{n\to\infty} d\left(p_n^{(k)}, p_{N_k}^{(k)}\right) + \limsup_{n\to\infty} d\left(p_{N_k}^{(k)}, p_{N_n}^{(n)}\right) \\
&< \frac{1}{k} + \frac{\varepsilon}{2} \\
&\leq \frac{1}{N} + \frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&< \varepsilon.
\end{aligned}$$

Hence, (X^*, Δ) is complete. □

Proof of (d).

(1) Define $\{p_n\}$ by $p_n = p$ $(n = 1, 2, \ldots)$ for any $p \in X$.

(2) *Show that $\{p_n\}$ is a Cauchy sequence.* $d(p_m, p_n) = d(p, p) = 0$.

(3) Take $\{p\} \in P_p$ and $\{q\} \in P_q$. Then

$$\Delta(P_p, P_q) = \lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(p, q) = d(p, q).$$

□

Proof of (e).

(1) *Show that $\varphi(X)$ is dense in X^*.* Given any $P \in X^* \ni \{p_n\}$ and any $\varepsilon > 0$. Since $\{p_n\}$ is Cauchy, there is N such that

$$d(p_m, p_n) < \frac{\varepsilon}{64} \text{ whenever } m, n \geq N.$$

Note that $p_N \in X$. Pick $\{p_N\} \in P_{p_N} = \varphi(p_N) \in \varphi(X)$. So

$$\Delta(P, P_{p_N}) = \lim_{n\to\infty} d(p_n, p_N) \leq \frac{\varepsilon}{64} < \varepsilon.$$

Hence $\varphi(X)$ is dense in X^*.

(2) *Show that $\varphi(X) = X^*$ if X is complete.* Given any $P \in X^* \ni \{p_n\}$. Since X is complete, a Cauchy sequence $\{p_n\}$ converges to $p \in X$. Pick $\{p\} \in P_p = \varphi(p) \in \varphi(X)$. So

$$\Delta(P, P_p) = \lim_{n\to\infty} d(p_n, p) = 0,$$

or $P = P_p$, or $\varphi(X) = X^*$.

□