# Chapter 6: The Riemann-Stieltjes Integral

**Exercise 6.1.** *Suppose $\alpha$ increases on $[a,b]$, $a \leq x_0 \leq b$, $\alpha$ is continuous at $x_0$, $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathscr{R}(\alpha)$ and that $\int f d\alpha = 0$.*

Given any partition $P = \{a = p_0, p_1, ..., p_{n-1}, p_n = b\}$, where $a = p_0 \leq p_1 \leq \cdots \leq p_{n-1} \leq p_n = b$. We might compute $L(P, f, \alpha)$ and $U(P, f, \alpha)$ by using $\epsilon$-$\delta$ argument since we are hinted by the condition that $\alpha$ is continuous. A function which is continuous at $x_0$ has a nice property near $x_0$ and this property would help us estimate $U(P, f, \alpha)$ near $x_0$. On the contrary, if both $f$ and $\alpha$ are discontinuous at $x_0$, it might be $f \notin \mathscr{R}(\alpha)$. Besides, if $f$ has too many points of discontinuity ($f(x) = 0$ if $x \in \mathbb{Q}$ and $f(x) = 1$ otherwise, for example), then $f$ might not be Riemann-integrable on $[0, 1]$.

**Claim 1.** $L(P, f, \alpha) = 0$.
*Proof of Claim 1.* $m_i = 0$ since $\inf f(x) = 0$ on any subinterval of $[a, b]$. So $L(P, f, \alpha) = \sum m_i \Delta\alpha_i = 0$. Here we don't need the condition that $\alpha$ is continuous at $x_0$. $\square$

**Claim 2.** *For any $\epsilon > 0$, there exists a partition $P$ such that $U(P, f, \alpha) < \epsilon$.*
*Proof of Claim 2.* Let $x_0 \in [p_{i_0-1}, p_{i_0}]$ for some $i_0$. Then $M_i = \sup_{p_{i-1} \leq x \leq p_i} f(x) = 0$ if $i \neq i_0$, and $M_{i_0} = 1$. So

$$U(P, f, \alpha) = \sum M_i \Delta\alpha_i = \Delta\alpha_{i_0}.$$

It is not true for any arbitrary $\alpha$. (For example, $\alpha$ has a jump on $x = x_0$.) In fact, Exercise 6.3 shows this. Luckily, $\alpha$ is continuous at $x_0$. So for $\epsilon > 0$, there exists $\delta > 0$ such that $|\alpha(x) - \alpha(x_0)| < \frac{\epsilon}{2}$ whenever $|x - x_0| < \delta$ (and $x \in [a, b]$). Now we pick a nice partition

$$P = \{a, x_0 - \delta_1, x_0 + \delta_2, b\},$$

where $\delta_1 = \min(\delta, x_0 - a) \geq 0$ and $\delta_2 = \min(\delta, b - x_0) \geq 0$. (It is a trick about resizing "$\delta$" to avoid considering the edge cases $x_0 = a$ or $x_0 = b$ or $a = b$.) Then $x_0 \in [x_0 - \delta_1, x_0 + \delta_2]$ and $\Delta\alpha$ on $[x_0 - \delta_1, x_0 + \delta_2]$ is

$$\begin{aligned}\alpha(x_0 + \delta_2) - \alpha(x_0 - \delta_1) &= (\alpha(x_0 + \delta_2) - \alpha(x_0)) + (\alpha(x_0) - \alpha(x_0 - \delta_1)) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Therefore, $U(P, f, \alpha) < \epsilon$. $\square$

*Proof (Definition 6.2).* By Claim 1 and 2 and notice that $U(P, f, \alpha) \geq 0$ for any partition $P$,

$$\overline{\int_a^b} f d\alpha = \inf U(P, f, \alpha) = 0,$$
$$\underline{\int_a^b} f d\alpha = \sup L(P, f, \alpha) = 0,$$

the inf and sup again being taken over all partitions. Hence $f \in \mathscr{R}(\alpha)$ and that $\int f d\alpha = 0$ by Definition 6.2. $\square$

*Proof (Theorem 6.5).* By Claim 1 and 2,

$$0 \leq U(P, f, \alpha) - L(P, f, \alpha) < \epsilon.$$

Hence $f \in \mathscr{R}(\alpha)$ by Theorem 6.5. Furthermore,

$$\int f d\alpha = \underline{\int_a^b} f d\alpha = \sup L(P, f, \alpha) = 0.$$

$\square$

*Proof (Theorem 6.10).* $f \in \mathscr{R}(\alpha)$ by Theorem 6.10. Thus, by Claim 1

$$\int f d\alpha = \underline{\int_a^b} f d\alpha = \sup L(P, f, \alpha) = 0.$$

$\square$