

Solutions to the book:
Robin Hartshorne, Algebraic Geometry

Meng-Gen Tsai
plover@gmail.com

July 18, 2021

Contents

Chapter I: Varieties	2
Problem I.1.6.	2

Chapter I: Varieties

Problem I.1.6.

Any nonempty open subset of an irreducible topological space is dense and irreducible. If Y is a subset of a topological space X , which is irreducible in its induced topology, then the closure \overline{Y} is also irreducible.

Proof.

- (1) Show that any nonempty open subset of an irreducible topological space is dense. It suffices to show that $U_1 \cap U_2 \neq \emptyset$ for any nonempty open subsets of an irreducible topological space.

$$\begin{aligned}
 & \forall \text{ nonempty open sets } U_1 \text{ and } U_2, U_1 \cap U_2 \neq \emptyset \\
 \iff & \forall \text{ nonempty open sets } U_1 \text{ and } U_2, X - (U_1 \cap U_2) \neq X \\
 \iff & \forall \text{ nonempty open sets } U_1 \text{ and } U_2, (X - U_1) \cup (X - U_2) \neq X \\
 \iff & \forall \text{ proper closed sets } Y_1 \text{ and } Y_2, Y_1 \cup Y_2 \neq X \\
 \iff & \nexists \text{ proper closed sets } Y_1 \text{ and } Y_2, Y_1 \cup Y_2 = X.
 \end{aligned}$$

- (2) Show that any nonempty open subset of an irreducible topological space is irreducible. Given any open subset U of an irreducible topological space X . Write $U \subseteq Y_1 \cup Y_2$ where Y_1 and Y_2 are closed in X .

$$\begin{aligned}
 U \subseteq Y_1 \cup Y_2 & \implies \overline{U} \subseteq \overline{Y_1 \cup Y_2} \\
 & \implies X \subseteq Y_1 \cup Y_2 && (U \text{ is dense, } Y_1 \cup Y_2 \text{ is closed}) \\
 & \implies Y_1 = X \supseteq U \text{ or } Y_2 = X \supseteq U && (X \text{ is irreducible}) \\
 & \implies U \text{ is irreducible.}
 \end{aligned}$$

- (3) Show that if Y is a subset of a topological space X , which is irreducible (in its induced topology), then the closure \overline{Y} is also irreducible. (Reductio ad absurdum) If \overline{Y} were reducible, there are two closed sets Y_1 and Y_2 such that

$$\overline{Y} \subseteq Y_1 \cup Y_2, \overline{Y} \not\subseteq Y_i (i = 1, 2).$$

- (a) $Y \subseteq \overline{Y} \subseteq Y_1 \cup Y_2$.
 (b) $Y \not\subseteq Y_i (i = 1, 2)$. If not, $Y \subseteq Y_i$ for some i . Take closure to get $\overline{Y} \subseteq \overline{Y_i} = Y_i$ (since Y_i is closed), contrary to the assumption.

By (a)(b), Y is reducible, which is absurd.

□