Chapter 6: The Riemann-Stieltjes Integral

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Supplement. Another definition of Riemann-Stieltjes integral.
(Exercise 7.3, 7.4 of the book T. M. Apostol, Mathematical Analysis, Second Edition.) Let P be a partition of $[a, b]$. The norm of a partition P is the length of the largest subinterval $[x_{i-1}, x_i]$ of P and is denoted by $\|P\|$.

We say $f \in \mathscr{R}(\alpha)$ if there exists $A \in \mathbb{R}$ having the property that for any $\epsilon > 0$, there exists $\delta > 0$ such that for any partition P of $[a, b]$ with norm $\|P\| < \delta$ and for any choice of $t_i \in [x_{i-1}, x_i]$, we have $|\sum_{i=1}^n f(t_i)\Delta\alpha_i - A| < \epsilon$.

Cliam. *$f \in \mathscr{R}$ in the sense of Definition 6.2 implies that $f \in \mathscr{R}$ in the sense of this another definition.*
Proof of Claim. Let $A = \int f dx$, $M > 0$ be one upper bound of $|f|$ on $[a, b]$. Given $\epsilon > 0$, there exists a partition $P_0 = \{a = x_0, x_1, ..., x_{N-1}, x_N = b\}$ such that $U(P_0, f) \le A + \frac{\epsilon}{2}$. Let $\delta = \frac{\epsilon}{2MN} > 0$. Then for any partition P with norm $\|P\| < \delta$, write

$$U(P, f) = \sum_{i=1}^n M_i \Delta x_i = S_1 + S_2,$$

where S_1 is the sum of terms arising from those subintervals of P containing no point of P_0, S_2 is the sum of the remaining terms. Then

$$S_1 \le U(P_0, f) < A + \frac{\epsilon}{2},$$
$$S_2 \le NM\|P\| < NM\delta < \frac{\epsilon}{2}.$$

Therefore, $U(P, f) < A + \epsilon$. Similarly, $L(P, f) > A - \epsilon$ whenever $\|P\| < \delta'$. Hence, $|\sum_{i=1}^n f(t_i)\Delta x_i - A| < \epsilon$ whenever $\|P\| < \min(\delta, \delta')$. (Copy Apostol's hint and ensure $M > 0$. M in Apostol's hint might be zero if $f = 0$.) $\square$

This supplement will be used in computing $\int_0^\infty (\frac{\sin x}{x})^2 dx = \frac{\pi}{2}$ in Exercise 8.12.

Exercise 6.1. *Suppose α increases on $[a, b]$, $a \le x_0 \le b$, α is continuous at x_0, $f(x_0) = 1$, and $f(x) = 0$ if $x \ne x_0$. Prove that $f \in \mathscr{R}(\alpha)$ and that $\int f d\alpha = 0$.*

Given any partition $P = \{a = p_0, p_1, ..., p_{n-1}, p_n = b\}$, where $a = p_0 \le p_1 \le \cdots \le p_{n-1} \le p_n = b$. We might compute $L(P, f, \alpha)$ and $U(P, f, \alpha)$ by using ϵ-δ argument since we are hinted by the condition that α is continuous. A function which is continuous at x_0 has a nice property near x_0 and this property would

help us estimate $U(P, f, \alpha)$ near x_0. On the contrary, if both f and α are discontinuous at x_0, it might be $f \notin \mathscr{R}(\alpha)$. Besides, if f has too many points of discontinuity ($f(x) = 0$ if $x \in \mathbb{Q}$ and $f(x) = 1$ otherwise, for example), then f might not be Riemann-integrable on $[0, 1]$.

Claim 1. $L(P, f, \alpha) = 0$.
Proof of Claim 1. $m_i = 0$ since $\inf f(x) = 0$ on any subinterval of $[a, b]$. So $L(P, f, \alpha) = \sum m_i \Delta \alpha_i = 0$. Here we don't need the condition that α is continuous at x_0. $\square$

Claim 2. *For any $\epsilon > 0$, there exists a partition P such that $U(P, f, \alpha) < \epsilon$.*
Proof of Claim 2. Let $x_0 \in [p_{i_0 - 1}, p_{i_0}]$ for some i_0. Then $M_i = \sup_{p_{i-1} \leq x \leq p_i} f(x) = 0$ if $i \neq i_0$, and $M_{i_0} = 1$. So

$$U(P, f, \alpha) = \sum M_i \Delta \alpha_i = \Delta \alpha_{i_0}.$$

It is not true for any arbitrary α. (For example, α has a jump on $x = x_0$.) In fact, Exercise 6.3 shows this. Luckily, α is continuous at x_0. So for $\epsilon > 0$, there exists $\delta > 0$ such that $|\alpha(x) - \alpha(x_0)| < \frac{\epsilon}{2}$ whenever $|x - x_0| < \delta$ (and $x \in [a, b]$). Now we pick a nice partition

$$P = \{a, x_0 - \delta_1, x_0 + \delta_2, b\},$$

where $\delta_1 = \min(\delta, x_0 - a) \geq 0$ and $\delta_2 = \min(\delta, b - x_0) \geq 0$. (It is a trick about resizing "δ" to avoid considering the edge cases $x_0 = a$ or $x_0 = b$ or $a = b$.) Then $x_0 \in [x_0 - \delta_1, x_0 + \delta_2]$ and $\Delta \alpha$ on $[x_0 - \delta_1, x_0 + \delta_2]$ is

$$\alpha(x_0 + \delta_2) - \alpha(x_0 - \delta_1) = (\alpha(x_0 + \delta_2) - \alpha(x_0)) + (\alpha(x_0) - \alpha(x_0 - \delta_1))$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore, $U(P, f, \alpha) < \epsilon$. $\square$

Proof (Definition 6.2). By Claim 1 and 2 and notice that $U(P, f, \alpha) \geq 0$ for any partition P,

$$\overline{\int_a^b} f \, d\alpha = \inf U(P, f, \alpha) = 0,$$
$$\underline{\int_a^b} f \, d\alpha = \sup L(P, f, \alpha) = 0,$$

the inf and sup again being taken over all partitions. Hence $f \in \mathscr{R}(\alpha)$ and that $\int f \, d\alpha = 0$ by Definition 6.2. $\square$

Proof (Theorem 6.5). By Claim 1 and 2,

$$0 \leq U(P, f, \alpha) - L(P, f, \alpha) < \epsilon.$$

Hence $f \in \mathscr{R}(\alpha)$ by Theorem 6.5. Furthermore,

$$\int f d\alpha = \underline{\int_a^b} f d\alpha = \sup L(P, f, \alpha) = 0.$$

$\square$

Proof (Theorem 6.10). $f \in \mathscr{R}(\alpha)$ by Theorem 6.10. Thus, by Claim 1

$$\int f d\alpha = \underline{\int_a^b} f d\alpha = \sup L(P, f, \alpha) = 0.$$

$\square$