# Chapter 5: Differentiation

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**Exercise 5.1.** *Let $f$ be defined for all real $x$, and suppose that*

$$|f(x) - f(y)| \leq (x - y)^2$$

*for all real $x$ and $y$. Prove that $f$ is a constant.*

*Proof.*

(1) Write
$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq |x - y|$$
if $x \neq y$.

(2) Given any $y \in \mathbb{R}$,
$$\left|\frac{f(x) - f(y)}{x - y}\right| \to 0 \quad \text{as} \quad x \to y,$$
or $|f'(y)| = 0$.

(3) Or using $\varepsilon$-$\delta$ argument. Fix $y \in \mathbb{R}$. Given any $\varepsilon > 0$, there exists $\delta = \varepsilon > 0$ such that
$$\left|\frac{f(x) - f(y)}{x - y} - 0\right| \leq |x - y| < \delta = \varepsilon$$
whenever $|x - y| < \delta$. That is, $|f'(y)| = 0$.

(4) So $f'(y) = 0$ for any $y \in \mathbb{R}$. By Theorem 5.11 (b), $f$ is a constant.

□

**Exercise 5.2.** *Suppose $f'(x) > 0$ in $(a, b)$. Prove that $f$ is strictly increasing in $(a, b)$, and let $g$ be its inverse function. Prove that $g$ is differentiable, and that*

$$g'(f(x)) = \frac{1}{f'(x)} \qquad (a < x < b).$$

*Proof.* Let $E = (a, b)$.

(1) Theorem 5.10 implies that for any $a < p < q < b$ there exists $\xi \in (p, q)$ such that
$$f(p) - f(q) = (p - q)f'(\xi).$$
Since $\xi \in (p, q) \subseteq E$, by assumption $f'(\xi) > 0$. Hence $f(p) - f(q) = (p - q)f'(\xi) < 0$ (here $p - q < 0$), or
$$f(p) < f(q)$$
if $p < q$. Therefore, $f$ is strictly increasing in $(a, b)$.

(2) *Show that $f$ is one-to-one in $E$ if $f$ is strictly increasing in $E$.* If $f(p) = f(q)$, then it cannot be $p > q$ or $p < q$ ((1)). So that $p = q$, or $f$ is injective.

(3) *Show that $g$ is well-defined.* Theorem 5.2 and Theorem 4.17.

(4) *Show that $g'(f(x)) = \frac{1}{f'(x)}$.* Given $y \in f(E)$, say $y = f(x)$ for some $x \in E$. Given any $s \in f(E)$ with $s \neq y$. Here $s = f(t)$ for some $t \in E$ and $t \neq x$.
$$\begin{aligned}
\lim_{s \to y} \frac{g(s) - g(y)}{s - y} &= \lim_{f(t) \to f(x)} \frac{g(f(t)) - g(f(x))}{f(t) - f(x)} \\
&= \lim_{t \to x} \frac{t - x}{f(t) - f(x)} \\
&= \lim_{t \to x} \frac{1}{\frac{f(t) - f(x)}{t - x}} \\
&= \frac{1}{f'(x)}. && (f' > 0)
\end{aligned}$$
Here $s \to y$ if and only if $t \to x$ since both $f$ and $g$ are continuous and one-to-one. Hence $g$ is differentiable and $g'(f(x)) = \frac{1}{f'(x)}$.

$\square$

**Exercise 5.3.** *Suppose $g$ is a real function on $\mathbb{R}^1$, with bounded derivative (say $|g'| \leq M$). Fix $\varepsilon > 0$, and define $f(x) = x + \varepsilon g(x)$. Prove that $f$ is one-to-one if $\varepsilon$ is small enough. (A set of admissible values of $\varepsilon$ can be determined which depends only on $M$.)*

*Proof.*

(1) Note that $f'(x) = 1 + \varepsilon g'(x)$ (Theorem 5.3). Since $|g'| \leq M$,
$$1 - \varepsilon M \leq f'(x) \leq 1 + \varepsilon M.$$

(2) Pick
$$\varepsilon = \frac{1}{M + 1} > 0.$$

Thus,

$$f'(x) \geq \frac{1}{M+1} > 0.$$

By Exercise 5.2, $f(x)$ is strictly increasing in $\mathbb{R}$ or one-to-one in $\mathbb{R}$.

$\square$

**Exercise 5.4.** *If*

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$

*where* $C_0, ..., C_n$ *are real constants, prove that the equation*

$$C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n = 0$$

*has at least one real root between* 0 *and* 1.

*Proof.* Let

$$g(x) = C_0 x + \frac{C_1}{2} x^2 + \cdots + \frac{C_{n-1}}{n} x^n + \frac{C_n}{n+1} x^{n+1} \in \mathbb{R}[x].$$

Then $g(0) = g(1) = 0$, and $g'(x) = C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n$. By the mean value theorem (Theorem 5.10), there exists a point $\xi \in (0, 1)$ at which

$$g(1) - g(0) = g'(\xi)(1 - 0),$$

or $g'(\xi) = 0$. That is, there exists a real root $x = \xi$ between 0 and 1 at which $C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n = 0$. $\square$

**Exercise 5.5.** *Suppose* $f$ *is defined and differentiable for every* $x > 0$, *and* $f'(x) \to 0$ *as* $x \to +\infty$. *Put* $g(x) = f(x+1) - f(x)$. *Prove that* $g(x) \to 0$ *as* $x \to +\infty$.

*Proof.* Given any $x > 0$. Since f is differentiable for every $x > 0$, f is differentiable on $[x, x+1]$. By Theorem 5.2 and Theorem 5.10 (the mean value theorem), there is a point $\xi \in (x, x+1)$ at which

$$f(x+1) - f(x) = [(x+1) - x] f'(\xi)$$

or

$$g(x) = f'(\xi).$$

As $x \to +\infty$, $\xi \to +\infty$. Hence

$$\lim_{x \to +\infty} g(x) = \lim_{\xi \to +\infty} f'(\xi) = 0.$$

□

**Exercise 5.6.** *Suppose*

(a) $f$ *is continuous for* $x \geq 0$,

(b) $f'(x)$ *exists for* $x > 0$,

(c) $f(0) = 0$,

(d) $f'$ *is monotonically increasing.*

*Put*

$$g(x) = \frac{f(x)}{x} \qquad (x > 0)$$

*and prove that* $g$ *is monotonically increasing.*

*Proof.*

(1) It suffices to show that $g'(x) \geq 0$ for $x > 0$ (Theorem 5.11(a)), that is, to show that
$$g'(x) = \frac{xf'(x) - f(x)}{x^2} \geq 0 \qquad (x > 0),$$
or
$$xf'(x) - f(x) \geq 0 \qquad (x > 0)$$
since $x^2 > 0$ for all nonzero $x$.

(2) Given $x > 0$. By (a)(b), we apply the mean value theorem (Theorem 5.10) on $f$ to get
$$f(x) - f(0) = (x - 0)f'(\xi)$$
for some $\xi \in (0, x)$. By (c),
$$f(x) = xf'(\xi).$$
By (d),
$$f(x) = xf'(\xi) \leq xf'(x).$$
Hence $xf'(x) - f(x) \geq 0$, or $g$ is monotonically increasing.

□

*Note.* $g$ is increasing strictly if $f$ is increasing strictly.

**Exercise 5.7.** *Suppose* $f'(x)$, $g'(x)$ *exist,* $g'(x) \neq 0$*, and* $f(x) = g(x) = 0$. *Prove that*
$$\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}.$$

*(This holds also for complex functions.)*

*Proof.*

$$\begin{aligned}\frac{f'(t)}{g'(t)} &= \frac{\lim_{t\to x}\frac{f(t)-f(x)}{t-x}}{\lim_{t\to x}\frac{g(t)-g(x)}{t-x}} \\ &= \lim_{t\to x}\frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}} && \text{(Both limits exist and } g' \neq 0\text{)} \\ &= \lim_{t\to x}\frac{f(t)}{g(t)}. && (f(x) = g(x) = 0)\end{aligned}$$

This proof is also true for complex functions. $\square$

**Exercise 5.8.** *Suppose $f'(x)$ is continuous on $[a,b]$ and $\varepsilon > 0$. Prove that there exists $\delta > 0$ such that*

$$\left|\frac{f(t)-f(x)}{t-x} - f'(x)\right| < \varepsilon$$

*whenever $0 < |t-x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$. (This could be expressed by saying $f$ is **uniformly differentiable** on $[a,b]$ if $f'$ is continuous on $[a,b]$.) Does this hold for vector-valued functions too?*

*Proof.*

(1) Since $f'(x)$ is continuous on a compact set $[a,b]$, $f'(x)$ is uniformly continuous on $[a,b]$. So given any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f'(t) - f'(x)| < \varepsilon$$

whenever $0 < |t-x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$.

(2) For such $t < x$ in (1), by the mean value theorem (Theorem 5.10), there exists a point $\xi \in (t,x)$ at which

$$f'(\xi) = \frac{f(t)-f(x)}{t-x}.$$

Note that $\xi$ is also satisfying $0 < |t-\xi| < |t-x| < \delta$ and $a \leq \xi \leq b$. Hence by (1) we also have

$$|f'(\xi) - f'(x)| < \varepsilon,$$

or

$$\left|\frac{f(t)-f(x)}{t-x} - f'(x)\right| < \varepsilon.$$

(3) *Suppose* $\mathbf{f}'(x)$ *is continuous on* $[a,b]$ *and* $\varepsilon > 0$. *Prove that there exists* $\delta > 0$ *such that*

$$\left| \frac{\mathbf{f}(t) - \mathbf{f}(x)}{t - x} - \mathbf{f}'(x) \right| < \varepsilon$$

*whenever* $0 < |t - x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$.

(a) Write

$$\mathbf{f}(x) = (f_1(x), \dots, f_k(x)) \in \mathbb{R}^k.$$

By Remarks 5.16, $\mathbf{f}(x)$ is differentiable at a point $x$ if and only if each $f_1, \dots, f_k$ is differentiable at $x$. So that

$$\mathbf{f}'(x) = (f_1'(x), \dots, f_k'(x)) \in \mathbb{R}^k.$$

By Theorem 4.10, $\mathbf{f}'(x)$ is continuous if and only if each $f_1, \dots, f_k$ is continuous.

(b) Similar to (1)(2), Since $f_i'(x)$ is continuous on a compact set $[a,b]$ where $1 \leq i \leq k$, $f_i'(x)$ is uniformly continuous on $[a,b]$. So given any $\varepsilon > 0$ there exists $\delta_i > 0$ such that

$$|f_i'(t) - f_i'(x)| < \frac{\varepsilon}{\sqrt{k}}$$

whenever $0 < |t - x| < \delta_i$, $a \leq x \leq b$, $a \leq t \leq b$. Take $\delta = \min_{1 \leq i \leq k} \delta_i > 0$.

(c) For such $t < x$ in (1), by the mean value theorem (Theorem 5.10), there exists a point $\xi_i \in (t, x)$ at which

$$f_i'(\xi_i) = \frac{f_i(t) - f_i(x)}{t - x}.$$

Note that $\xi_i$ is also satisfying $0 < |t - \xi_i| < |t - x| < \delta$ and $a \leq \xi_i \leq b$. Hence by (1) we also have

$$|f_i'(\xi_i) - f_i'(x)| < \frac{\varepsilon}{\sqrt{k}},$$

or

$$\left| \frac{f_i(t) - f_i(x)}{t - x} - f_i'(x) \right| < \frac{\varepsilon}{\sqrt{k}}.$$

(d) Hence

$$\left| \frac{\mathbf{f}(t) - \mathbf{f}(x)}{t - x} - \mathbf{f}'(x) \right| = \left( \sum_{i=1}^{k} \left| \frac{f_i(t) - f_i(x)}{t - x} - f_i'(x) \right|^2 \right)^{\frac{1}{2}} < \varepsilon.$$

$\square$

**Exercise 5.9.** *Let $f$ be a continuous real function on $\mathbb{R}^1$, of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$. Dose it follow that $f'(0)$ exists?*

*Proof.*

(1) *Show that $f'(0) = 3$.* It is equivalent to show that

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 3.$$

Write $F(x) = f(x) - f(0)$ and $G(x) = x - 0$ on $\mathbb{R}^1$. So that

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{F(x)}{G(x)} = 0.$$

(2) Note that

$$\lim_{x \to 0} \frac{F'(x)}{G'(x)} = \lim_{x \to 0} \frac{f'(x)}{1} = 3.$$

(3) Since $f$ is continuous on $\mathbb{R}^1$, $F$ is continuous on $\mathbb{R}^1$. Hence

$$\lim_{x \to 0} F(x) = F(\lim_{x \to 0} x) = F(0) = 0.$$

Also, $G$ is continuous on $\mathbb{R}^1$ implies that

$$\lim_{x \to 0} G(x) = G(\lim_{x \to 0} x) = G(0) = 0.$$

(4) Apply L'Hospital's rule (Theorem 5.13) to (2)(3), we have

$$\lim_{x \to 0} \frac{F(x)}{G(x)} = 3,$$

or $f'(0) = 3$.

□

**Exercise 5.10.**

**Exercise 5.11.** *Suppose $f$ is defined in a neighborhood of $x$, and suppose $f''(x)$ exists. Show that*

$$\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

*Show by an example that the limit may exist even if $f''(x)$ dose not. (Hint: Use Theorem 5.13.)*

*Proof (Theorem 5.13).*

(1) Write $F(h) = f(x+h) + f(x-h) - 2f(x)$ and $G(h) = h^2$. It is equivalent to show that

$$\lim_{h\to 0} \frac{F(h)}{G(h)} = f''(x).$$

We might apply Theorem 5.13 (L'Hospital rule) to get it.

(2) *Show that* $\lim_{h\to 0} F(h) = 0$ *and* $\lim_{h\to 0} G(h) = 0$. It is clear that $\lim_{h\to 0} G(h) = \lim_{h\to 0} h^2 = 0$ since $x \mapsto x^2$ is continuous on $\mathbb{R}^1$. Besides, since $f$ is continuous at $x$ (by applying Theorem 5.2 twice),

$$\lim_{h\to 0} F(h) = f(x) + f(x) - 2f(x) = 0.$$

(3) *Show that*

$$\lim_{h\to 0} \frac{F'(h)}{G'(h)} = \lim_{h\to 0} \frac{f'(x+h) - f'(x-h)}{2h}$$

*is well-defined.* Since $f''(x)$ exists in a neighborhood $B(x;r)$ of $x$ (where $r > 0$), $f'(x)$ exists and is continuous in $B(x;r)$ (Theorem 5.2). As $0 < |h| < \frac{r}{2}$,

$$x + h \in B\left(x+h; \frac{r}{2}\right) \subseteq B(x;r)$$

and

$$x - h \in B\left(x-h; \frac{r}{2}\right) \subseteq B(x;r).$$

So $f'(x+h)$ and $f'(x-h)$ exist in $B(x;r)$ as $0 < |h| < \frac{r}{2}$. Hence

$$\lim_{h\to 0} \frac{F'(h)}{G'(h)} = \lim_{h\to 0} \frac{f'(x+h) - f'(x-h)}{2h}$$

is well-defined (Theorem 5.3 and Theorem 5.5 (the chain rule)).

(4) *Show that*

$$\lim_{h\to 0} \frac{f'(x+h) - f'(x-h)}{2h} = f''(x).$$

Since $f''(x)$ exists, by definition

$$\lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} = f''(x)$$

and

$$\lim_{h\to 0} \frac{f'(x-h) - f'(x)}{-h} = f''(x).$$

Sum up two expressions to get

$$2f''(x) = \lim_{h\to 0} \frac{f'(x-h) - f'(x-h)}{h}.$$

(5) By (2)(3)(4) and Theorem 5.13 (L'Hospital rule), the result is established.

(6) *Given $f(x) = x|x|$ on $\mathbb{R}^1$. Show that*

$$\lim_{h \to 0} \frac{f(h) + f(-h) - 2f(0)}{h^2} = 0$$

*but $f''(x)$ does not exist at $x = 0$.* Clearly,

$$\begin{aligned}
\lim_{h \to 0} \frac{f(h) + f(-h) - 2f(0)}{h^2} &= \lim_{h \to 0} \frac{h|h| + (-h)| - h| - 2 \cdot 0}{h^2} \\
&= \lim_{h \to 0} \frac{h|h| - h|h| - 0}{h^2} \\
&= \lim_{h \to 0} 0 \\
&= 0.
\end{aligned}$$

But $f''(x)$ does not exist by Exercise 5.12.

$\square$

**Exercise 5.12.** *If $f(x) = |x|^3$, compute $f'(x)$, $f''(x)$ for all real $x$, and show that $f^{(3)}(0)$ does not exist.*

*Proof.*

(1) Write

$$f(x) = \begin{cases} x^3 & (x \geq 0), \\ -x^3 & (x \leq 0). \end{cases}$$

(2) *Show that $f'(x) = 3x|x|$.* It is trivial that

$$f'(x) = \begin{cases} 3x^2 & (x > 0), \\ -3x^2 & (x < 0). \end{cases}$$

Note that

$$\lim_{x \to 0} f'(x) = 0.$$

Apply the same argument in Exercise 5.9, we have

$$f'(0) = 0.$$

Hence $f'$ exists and $f'(x) = 3x|x|$ for any $x \in \mathbb{R}$.

(3) *Show that $f''(x) = 6|x|$.* Similar to (2).

$$f''(x) = \begin{cases} 6x & (x > 0), \\ -6x & (x < 0). \end{cases}$$

Note that
$$\lim_{x\to 0} f''(x) = 0.$$
Apply the same argument in Exercise 5.9, we have
$$f''(0) = 0.$$
Hence $f''$ exists and $f''(x) = 6|x|$ for any $x \in \mathbb{R}$.

(4) *Show that $f^{(3)}(0)$ does not exist.*
$$f'''(x) = \begin{cases} 6 & (x > 0), \\ -6 & (x < 0). \end{cases}$$
There are some proofs for showing that $f^{(3)}(0)$ does not exist.

(a) Since
$$\lim_{t\to 0+} \frac{f''(t) - f''(0)}{t - 0} = \lim_{t\to 0+} \frac{6t}{t} = 6$$
and
$$\lim_{t\to 0-} \frac{f''(t) - f''(0)}{t - 0} = \lim_{t\to 0-} \frac{-6t}{t} = -6,$$
$f^{(3)}(0)$ does not exist.

(b) (Reductio ad absurdum) If $f$ were differentiable on $\mathbb{R}^1$, then
$$\lim_{t\to 0+} f'''(t) = 6$$
and
$$\lim_{t\to 0-} f'''(t) = -6,$$
or $f'''$ has a simple discontinuity at $x = 0$, contrary to Corollary to Theorem 5.12.

□

*Note.* Given $k > 0$. We can construct one real function $f$ on $\mathbb{R}^1$, say
$$f(x) = \begin{cases} |x|^k & (k \text{ is odd}), \\ x|x|^{k-1} & (k > 0 \text{ is even}), \end{cases}$$
such that all $f^{(0)}(0) = \cdots = f^{(k-1)}(0) = 0$ exist but $f^{(k)}(0)$ does not exist.

**Exercise 5.13.**

**Exercise 5.14.** *Let $f$ be a differentiable real function defined in $(a, b)$. Prove that $f$ is convex if and only if $f'$ is monotonically increasing. Assume next $f''(x)$ exists for every $x \in (a, b)$, and prove that $f$ is convex if and only if $f''(x) \geq 0$ for all $x \in (a, b)$.*

*Proof.*

(1) *Show that $f'$ is monotonically increasing if $f$ is convex.*

(a) Since $f$ is convex, by definition (Exercise 4.23)

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

whenever $a < x < b$, $a < y < b$, $0 < \lambda < 1$.

(b) As $x \neq y$, we have

$$\begin{aligned} f(y) - f(x) &\geq \frac{f(x + \lambda(y-x)) - f(x)}{\lambda} \\ &= \frac{f(x + \lambda(y-x)) - f(x)}{\lambda(y-x)} \cdot (y-x) \end{aligned}$$

and let $\lambda \to 0$ to get

$$f(y) - f(x) \geq f'(x)(y-x)$$

(since $f'(x)$ exists). Similarly, we have

$$f(x) - f(y) \geq f'(y)(x-y).$$

(c) Given any $y > x$, we have

$$f'(y)(y-x) \geq f(y) - f(x) \geq f'(x)(y-x).$$

Hence $f'(y) \geq f'(x)$ whenever $y > x$, or $f'$ is monotonically increasing.

(2) *Show that $f$ is convex if $f'$ is monotonically increasing.* Given any $y > x$ and any $0 < \lambda < 1$.

(a) By Theorem 5.10 (the mean value theorem), there is a point $x < \xi < y$ such that

$$f(y) - f(x) = f'(\xi)(y-x).$$

Since $f'$ is monotonically increasing,

$$f'(y)(y-x) \geq f(y) - f(x) \geq f'(x)(y-x).$$

(b) Write $z = \lambda x + (1-\lambda)y$. Hence

$$\begin{aligned} f(y) - f(z) &\geq f'(z)(y-z), \\ f(z) - f(x) &\leq f'(z)(z-x), \end{aligned}$$

or

$$\begin{aligned} f(y) &\geq f(z) + f'(z)(y-z), \\ f(x) &\geq f(z) + f'(z)(x-z), \end{aligned}$$

or

$$\begin{aligned}\lambda f(x)+(1-\lambda)f(y) \geq& \lambda[f(z)+f'(z)(x-z)] \\ &+(1-\lambda)[f(z)+f'(z)(y-z)] \\ =& f(z) \\ =& f(\lambda x+(1-\lambda)y).\end{aligned}$$

Hence $f$ is convex.

(3) *Show that $f''(x) \geq 0$ if $f$ is convex and $f''$ exists.* By (1), $f'$ is monotonically increasing since $f$ is convex. Given any $x \neq y$, we have

$$\frac{f'(y)-f'(x)}{y-x} \geq 0.$$

Let $y \to x$, we have $f''(x) \geq 0$ if $f''$ exists.

(4) *Show that $f$ is convex if $f''$ exists and $f''(x) \geq 0$.* By Theorem 5.11(a), $f'$ is monotonically increasing. By (2), $f$ is convex.

□

**Exercise 5.15 (Landau-Kolmogorov inequality on the half-line).** *Suppose $a \in \mathbb{R}^1$, $f$ is a twice-differentiable real function on $(a,\infty)$, and $M_0$, $M_1$, $M_2$ are the least upper bounds of $|f(x)|$, $|f'(x)|$, $|f''(x)|$, respectively, on $(a,\infty)$. Prove that*

$$M_1^2 \leq 4M_0M_2.$$

*(Hint: If $h>0$, Taylor's theorem shows that*

$$f'(x)=\frac{1}{2h}[f(x+2h)-f(x)]-hf''(\xi)$$

*for some $\xi \in (x,x+2h)$. Hence*

$$|f'(x)| \leq hM_2+\frac{M_0}{h}.)$$

*To show that $M_1^2 = 4M_0M_2$ can actually happen, take $a=-1$, define*

$$f(x)=\begin{cases} 2x^2-1 & (-1<x<0), \\ \frac{x^2-1}{x^2+1} & (0 \leq x < \infty), \end{cases}$$

*and show that $M_0=1$, $M_1=4$, $M_2=4$. Does $M_1^2 \leq 4M_0M_2$ hold for vector-valued functions too?*

*Note.*

(1) Write

$$M_1 \leq 2M_0^{\frac{1}{2}} M_2^{\frac{1}{2}}.$$

2 is called the Landau-Kolmogorov constant, which is the best possible by the above example.

(2) In general, suppose $a \in \mathbb{R}^1$, $f$ is a $n$th differentiable real function on $(a, \infty)$, and $M_0$, $M_k$, $M_n$ are the least upper bounds of $|f(x)|$, $\left|f^{(k)}(x)\right|$, $\left|f^{(n)}(x)\right|$, respectively, on $(a, \infty)$ where $1 \leq k < n$. Then

$$M_k \leq C(n, k) M_0^{1-\frac{k}{n}} M_n^{\frac{k}{n}}.$$

*Proof.*

(1) Consider some trivial cases.

(a) If $M_0 = 0$, then $f(x) = 0$ on $(a, +\infty)$. So that $f'(x) = f''(x) = 0$ on $(a, +\infty)$, or $M_1 = M_2 = 0$. The inequality holds.

(b) If $M_2 = 0$, then $f''(x) = 0$ on $(a, +\infty)$. So that $f'(x) = \alpha$ for some constant $\alpha \in \mathbb{R}^1$ (Theorem 5.11(b)), and $f(x) = \alpha x + \beta$ for some constant $\beta \in \mathbb{R}^1$ (by applying Theorem 5.11(b) to $x \mapsto f(x) - \alpha x$). Hence $M_1 = |\alpha|$ and

$$M_0 = \begin{cases} +\infty & (\alpha \neq 0), \\ |\beta| & (\alpha = 0). \end{cases}$$

In any case, the inequality holds.

(c) If $M_0 = +\infty$ and $M_2 \neq 0$, there is nothing to do.

(d) If $M_2 = +\infty$ and $M_0 \neq 0$, there is nothing to do.

(2) By (1), we suppose that $0 < M_0 < +\infty$ and $0 < M_2 < +\infty$. Given $x \in (a, +\infty)$ and $h > 0$. By Taylor's theorem (Theorem 5.15):

$$f(x + 2h) = f(x) + 2hf'(x) + 2h^2 f''(\xi)$$

for some $\xi \in (x, x + 2h) \subseteq (a, +\infty)$. Thus

$$\begin{aligned} 2h|f'(x)| &\leq |f(x + 2h)| + |f(x)| + 2h^2|f''(\xi)| \\ &\leq 2M_0 + 2h^2 M_2, \\ |f'(x)| &\leq \frac{M_0}{h} + hM_2 \end{aligned}$$

holds for all $h > 0$. In particular, take

$$h = \sqrt{\frac{M_0}{M_2}}$$

to get

$$|f'(x)| \leq 2\sqrt{M_0M_2}.$$

Thus $2\sqrt{M_0M_2}$ is an upper bound of $|f'(x)|$ for all $x \in (a, +\infty)$. Hence

$$M_1 \leq 2\sqrt{M_0M_2}$$

or

$$M_1^2 \leq 4M_0M_2.$$

(3) *Define*

$$f(x) = \begin{cases} 2x^2 - 1 & (-1 < x < 0), \\ \frac{x^2-1}{x^2+1} & (0 \leq x < \infty). \end{cases}$$

*Show that $M_0 = 1$, $M_1 = 4$, $M_2 = 4$.* Similar to Exercise 5.12,

$$f'(x) = \begin{cases} 4x & (-1 < x \leq 0), \\ \frac{4x}{(x^2+1)^2} & (0 \leq x < \infty). \end{cases}$$

(Here $\lim_{x\to 0+} f'(x) = 0$ and $\lim_{x\to 0-} f'(x) = 0$. So $f'(0) = 0$ by Exercise 5.9.) Also,

$$f''(x) = \begin{cases} 4 & (-1 < x \leq 0), \\ \frac{-12x^2+4}{(x^2+1)^3} & (0 \leq x < \infty). \end{cases}$$

(Here $\lim_{x\to 0+} f''(x) = 4$ and $\lim_{x\to 0-} f''(x) = 4$. So $f''(0) = 4$ by Exercise 5.9.) Hence, $M_0 = 1$, $M_1 = 4$, $M_2 = 4$.

(4) *Given*

$$\mathbf{f}(x) = (f_1(x), \ldots, f_k(x))$$

*be a twice-differentiable vector-valued function from $(a, \infty)$ to $\mathbb{R}^k$. and $M_0$, $M_1$, $M_2$ are the least upper bounds of $|\mathbf{f}(x)|$, $|\mathbf{f}'(x)|$, $|\mathbf{f}''(x)|$, respectively, on $(a, \infty)$. Show that*

$$M_1^2 \leq 4M_0M_2.$$

Similar to (1), we suppose that $0 < M_0 < +\infty$ and $0 < M_2 < +\infty$. Given any $\mathbf{v} = (v_1, \ldots, v_k) \in \mathbb{R}^k$, $\mathbf{v} \cdot \mathbf{f}$ is a twice-differentiable real function on $(a, \infty)$. Similar to (2), Given $x \in (a, +\infty)$ and $h > 0$. By Taylor's theorem (Theorem 5.15):

$$(\mathbf{v} \cdot \mathbf{f})(x + 2h) = (\mathbf{v} \cdot \mathbf{f})(x) + 2h(\mathbf{v} \cdot \mathbf{f})'(x) + 2h^2(\mathbf{v} \cdot \mathbf{f})''(\xi)$$

for some $\xi \in (x, x + 2h) \subseteq (a, +\infty)$. Thus by the Schwarz inequality (Theorem 1.35)

$$\begin{aligned} 2h|(\mathbf{v} \cdot \mathbf{f})'(x)| &\leq |(\mathbf{v} \cdot \mathbf{f})(x + 2h)| + |(\mathbf{v} \cdot \mathbf{f})(x)| + 2h^2|(\mathbf{v} \cdot \mathbf{f})''(\xi)| \\ &\leq |\mathbf{v}||\mathbf{f}(x + 2h)| + |\mathbf{v}||\mathbf{f}(x)| + 2h^2|\mathbf{v}||\mathbf{f}''(\xi)| \\ &\leq (2M_0 + 2h^2M_2)|\mathbf{v}|, \\ |(\mathbf{v} \cdot \mathbf{f})'(x)| &\leq \left(\frac{M_0}{h} + hM_2\right)|\mathbf{v}| \end{aligned}$$

holds for any $\mathbf{v}$ and $h > 0$. In particular, we take

$$\mathbf{v} = \mathbf{f}'(y)$$

and

$$h = \sqrt{\frac{M_0}{M_2}}$$

to get

$$|\mathbf{f}'(x) \cdot \mathbf{f}'(y)| \leq 2\sqrt{M_0 M_2}|\mathbf{f}'(y)| \leq 2M_1\sqrt{M_0 M_2}.$$

Note that $x$ and $y$ are arbitrary (in $(a, +\infty)$). In particular, we take $x = y$ to get

$$|\mathbf{f}'(x)|^2 \leq 2M_1\sqrt{M_0 M_2}.$$

Thus $2M_1\sqrt{M_0 M_2}$ is an upper bound of $|\mathbf{f}'(x)|^2$ for all $x \in (a, +\infty)$. Hence

$$M_1^2 \leq 2M_1\sqrt{M_0 M_2}$$

or

$$M_1^2 \leq 4M_0 M_2.$$

□

**Supplement (Landau-Kolmogorov inequality on the real line).** *Suppose $f$ is a twice-differentiable real function on $(-\infty, +\infty)$, and $M_0$, $M_1$, $M_2$ are the least upper bounds of $|f(x)|$, $|f'(x)|$, $|f''(x)|$, respectively, on $(-\infty, +\infty)$. Prove that*

$$M_1^2 \leq 2M_0 M_2.$$

*Proof.*

(1) Similar to (1) in Landau-Kolmogorov inequality on the half-line, we suppose that $0 < M_0 < +\infty$ and $0 < M_2 < +\infty$.

(2) Similar to (2) in Landau-Kolmogorov inequality on the half-line. Given $x \in \mathbb{R}^1$ and $h > 0$. By Taylor's theorem (Theorem 5.15):

$$f(x + 2h) = f(x) + 2hf'(x) + 2h^2 f''(\xi_1) \tag{I}$$

$$f(x - 2h) = f(x) - 2hf'(x) + 2h^2 f''(\xi_2) \tag{II}$$

for some $\xi_1 \in (x, x + 2h)$ and $\xi_2 \in (x, x - 2h)$. So (I) subtracts (II):

$$f(x + 2h) - f(x - 2h) = 4hf'(x) + 2h^2 f''(\xi_1) - 2h^2 f''(\xi_2).$$

Thus

$$\begin{aligned}
4h|f'(x)| &\leq |f(x + 2h)| + |f(x - 2h)| + 2h^2|f''(\xi_1)| + 2h^2|f''(\xi_2)| \\
&\leq 2M_0 + 4h^2 M_2, \\
|f'(x)| &\leq \frac{M_0}{2h} + hM_2
\end{aligned}$$

holds for all $h > 0$. In particular, take

$$h = \sqrt{\frac{M_0}{2M_2}}$$

to get

$$|f'(x)| \leq \sqrt{2M_0M_2}.$$

Thus $\sqrt{2M_0M_2}$ is an upper bound of $|f'(x)|$ for all $x \in \mathbb{R}^1$. Hence

$$M_1 \leq \sqrt{2M_0M_2}$$

or

$$M_1^2 \leq 2M_0M_2.$$

□

*Note.*

(1) Write

$$M_1 \leq \sqrt{2}M_0^{\frac{1}{2}}M_2^{\frac{1}{2}}.$$

$\sqrt{2}$ is called the Landau-Kolmogorov constant, which is the best possible.

(2) In general, suppose $f$ is a $n$th differentiable real function on $\mathbb{R}^1$, and $M_0$, $M_k$, $M_n$ are the least upper bounds of $|f(x)|$, $\left|f^{(k)}(x)\right|$, $\left|f^{(n)}(x)\right|$, respectively, on $\mathbb{R}^1$ where $1 \leq k < n$. Then

$$M_k \leq C(n,k)M_0^{1-\frac{k}{n}}M_n^{\frac{k}{n}}.$$

**Exercise 5.16.** *Suppose $f$ is twice-differentiable on $(0, \infty)$, $f''$ is bounded on $(0, \infty)$, and $f(x) \to 0$ as $x \to \infty$. Prove that $f'(x) \to 0$ as $x \to \infty$. (Hint: Let $a \to \infty$ in Exercise 5.15.)*

*Proof.*

(1) Write $|f''| \leq M$ for some real $M$ since $f''$ is bounded on $(0, \infty)$.

(2) Given any $a > 0$. As in Exercise 5.15, define $M_0, M_1, M_2$ are the least upper bounds of $|f(x)|$, $|f'(x)|$, $|f''(x)|$ on $(a, \infty)$. Note that $M_2 \leq M$ for any $a > 0$ (by (1)). So that

$$M_1^2 \leq 4M_0M_2 \leq 4MM_0$$

for any $a > 0$.

(3) By assumption, $M_0 \to 0$ as $a \to \infty$. (So given any $\varepsilon > 0$, there exists a real $A$ such that
$$0 \leq M_0 < \frac{\varepsilon}{4M+1}$$
whenever $a \geq A$. Hence
$$M_1^2 \leq 4MM_0 \leq 4M \cdot \frac{\varepsilon}{4M+1} < \varepsilon.$$
whenever $a \geq A$.) Therefore $M_1^2 \to 0$ as $a \to \infty$, or $f'(x) \to 0$ as $x \to \infty$.

□

**Exercise 5.17.**
**Exercise 5.18.**
**Exercise 5.19.**
**Exercise 5.20.**
**Exercise 5.21.** *Let $E$ be a closed subset of $\mathbb{R}^1$. We saw in Exercise 4.22, that there is a real continuous function $f$ on $\mathbb{R}^1$ whose zero set is $E$. Is it possible, for each closed set $E$, to find such an $f$ which is differentiable on $\mathbb{R}^1$, or one which is $n$ times differentiable, or even one which has derivatives of all orders on $\mathbb{R}^1$?*

**Exercise 5.22.**
**Exercise 5.23.**
**Exercise 5.24.**
**Exercise 5.25.**
**Exercise 5.26.** *Suppose $f$ is differentiable on $[a,b]$, $f(a) = 0$, and there is a real number $A$ such that $|f'(x)| \leq A|f(x)|$ on $[a,b]$. Prove that $f(x) = 0$ for all $x \in [a,b]$. (Hint: Fix $x_0 \in [a,b]$, let*
$$M_0 = \sup |f(x)|, \qquad M_1 = \sup |f'(x)|$$
*for $a \leq x \leq x_0$. For any such $x$,*
$$|f(x)| \leq M_1(x_0 - a) \leq A(x_0 - a)M_0.$$
*Hence $M_0 = 0$ if $A(x_0 - a) < 1$. That is, $f = 0$ on $[a, x_0]$. Proceed.)*

*Proof (Hint).*

(1) If $A = 0$, then $f'(x) = 0$ or $f(x)$ is constant on $[a,b]$ (Theorem 5.11(b)). Since $f(a) = 0$, $f(x) = 0$ on $[a,b]$.

(2) Suppose that $A > 0$. Fix $x_0 \in [a,b]$, let
$$M_0 = \sup |f(x)|, \qquad M_1 = \sup |f'(x)|$$

for $a \leq x \leq x_0$. Since $|f'(x)| \leq A|f(x)|$ on $[a,b]$,

$$|f'(x)| \leq A|f(x)| \leq AM_0.$$

Since $AM_0$ is an upper bound for $|f'(x)|$,

$$M_1 \leq AM_0.$$

(3) Given any $x \in [a, x_0]$. Since $f$ is differentiable on $[a, x_0] \subseteq [a,b]$, by the mean value theorem (Theorem 5.10), there is $\xi \in (a, x)$ such that

$$f(x) - f(a) = f'(\xi)(x-a).$$

Note that $f(a) = 0$ by assumption. So that

$$\begin{aligned}
|f(x)| &= |f'(\xi)|(x-a) & \\
&\leq M_1(x-a) & (\text{Definition of } M_1) \\
&\leq AM_0(x-a) & ((2)) \\
&\leq AM_0(x_0-a). & (x \in [a, x_0])
\end{aligned}$$

Since $AM_0(x_0 - a)$ is an upper bound for $|f(x)|$,

$$M_0 \leq AM_0(x_0 - a).$$

Take

$$x_0 = \min\left\{\frac{1}{2A} + a, b\right\}$$

so that $M_0 \leq AM_0(x_0 - a) \leq \frac{M_0}{2}$. $M_0 = 0$ or $f(x) = 0$ on $[a, x_0]$.

(4) Take a partition

$$P = \{a = x_{-1}, x_0, \ldots, x_n = b\}$$

of $[a,b]$ such that each subinterval $[x_{i-1}, x_i]$ satisfying $\Delta x_i = x_i - x_{i-1} < \frac{1}{2A}$. By (3), $f(x) = 0$ on $[x_{-1}, x_0]$. Apply the same argument in (3), $f(x) = 0$ on $[x_0, x_1]$. Continue this process, $f(x) = 0$ on each subinterval and thus on the whole interval $[a,b]$.

□

**Exercise 5.27.**
**Exercise 5.28.**
**Exercise 5.29.**