

Chapter 5: Differentiation

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Exercise 5.1. Let f be defined for all real x , and suppose that

$$|f(x) - f(y)| \leq (x - y)^2$$

for all real x and y . Prove that f is a constant.

Proof.

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq |x - y|$$

for $x \neq y$. Given any $y \in \mathbb{R}$, $\left| \frac{f(x) - f(y)}{x - y} \right| \rightarrow 0$ as $x \rightarrow y$, or $|f'(y)| = 0$. (Or using ϵ - δ argument. Fix $y \in \mathbb{R}$. Given any $\epsilon > 0$, there exists $\delta = \epsilon > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} - 0 \right| \leq |x - y| < \delta = \epsilon$$

whenever $|x - y| < \delta$. That is, $|f'(y)| = 0$.) So $f'(y) = 0$ for any $y \in \mathbb{R}$. By Theorem 5.11 (b), f is a constant. \square

Exercise 5.4. If

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$

where C_0, \dots, C_n are real constants, prove that the equation

$$C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n = 0$$

has at least one real root between 0 and 1.

Proof. Let

$$g(x) = C_0x + \frac{C_1}{2}x^2 + \cdots + \frac{C_{n-1}}{n}x^n + \frac{C_n}{n+1}x^{n+1} \in \mathbb{R}[x].$$

Then $g(0) = g(1) = 0$, and $g'(x) = C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n$. By the mean value theorem (Theorem 5.10), there exists a point $\xi \in (0, 1)$ at which

$$g(1) - g(0) = g'(\xi)(1 - 0),$$

or $g'(\xi) = 0$. That is, there exists a real root $x = \xi$ between 0 and 1 at which $C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n = 0$. \square