

## Chapter 8: Some Special Functions

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**Supplement.** Fourier coefficients in Definition 8.9.

(1) Write

$$f(x) = a_0 + \sum_{n=1}^N (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}$$

(as the textbook Rudin, Principles of Mathematical Analysis, Third Edition).

Then

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx.$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx, n \in \mathbb{Z}^+.$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx, n \in \mathbb{Z}^+.$$

(2) One might write in one different form,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^N (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

The only difference between the new one and the old one is  $a_0$ , so  $a_0$  should be

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx.$$

(3) Again, one might write in one different form,

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^N (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

Similarly,  $a_0$  should be

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x)}{\sqrt{2}} dx.$$

(4) Recall  $f(x) = \sum_{n=-N}^N c_n e^{inx}$  ( $x \in \mathbb{R}$ ) where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

The relations among  $a_n$ ,  $b_n$  of this textbook and  $c_n$  are

$$\begin{aligned} c_0 &= a_0 \\ c_n &= \frac{1}{2}(a_n + ib_n), n \in \mathbb{Z}^+. \end{aligned}$$

- (5) In some textbooks (Henryk Iwaniec, Topics in Classical Automorphic Forms), it is convenient to consider periodic functions  $f$  of period 1. Define

$$e(n) = e^{2\pi i x} = \cos(2\pi x) + i \sin(2\pi x).$$

Any periodic and piecewise continuous function  $f$  has the Fourier series representation

$$f(x) = \sum_{-\infty}^{\infty} a_n e(nx)$$

with coefficients given by

$$a_n = \int_0^1 f(x) e(-nx) dx.$$

Here is one exercise for this representation. *Show that the fractional part of  $x$ ,  $\{x\} = x - [x]$ , is given by*

$$\{x\} = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{\pi n}.$$

**Supplement.** Parseval's theorem 8.16.

- (1) Given

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

Then

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = 2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

- (2) Given

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

Then

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

(3) Given

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

Then

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

**Exercise 8.1.** Define

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

Prove that  $f$  has derivatives of all orders at  $x = 0$ , and that  $f^{(n)}(0) = 0$  for  $n = 1, 2, 3, \dots$

$f(x)$  is an example of non-analytic smooth function, that is, infinitely differentiable functions are not necessarily analytic. In this exercise, we will show that Taylor series of  $f$  at the origin converges everywhere to the zero function. So the Taylor series does not equal  $f(x)$  for  $x \neq 0$ . Consequently,  $f$  is not analytic at  $x = 0$ .

*Proof.*

(1) Show that

$$\lim_{x \rightarrow 0} g(x) e^{-\frac{1}{x^2}} = 0$$

for any rational function  $g(x) \in \mathbb{R}(x)$ .

- (a) Write  $g(x) = \frac{p(x)}{q(x)}$  for some  $p(x), q(x) \in \mathbb{R}[x]$ ,  $g(x) \neq 0$ .
- (b) Write  $q(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_0$ .  $q(x)$  is not identically zero, that is, there exists the unique coefficient of the least power of  $x$  in  $q(x)$  which is non-zero, say  $b_M \neq 0$ .
- (c) Thus,

$$g(x) = \frac{p(x)/x^M}{q(x)/x^M}.$$

The denominator of  $g(x)$  tends to  $b_M \neq 0$  as  $x \rightarrow 0$ . By the similar argument in Theorem 8.6(f), we have

$$\frac{p(x)}{x^M} e^{-\frac{1}{x^2}} \rightarrow 0 \text{ as } x \rightarrow 0.$$

Hence,  $\lim_{x \rightarrow 0} g(x) e^{-\frac{1}{x^2}} = 0$  for any  $g(x) \in \mathbb{R}(x)$ .

(2) Given any real  $x \neq 0$ , show that

$$f^{(n)}(x) = g_n(x) e^{-\frac{1}{x^2}}$$

for some rational function  $g(x) \in \mathbb{R}(x)$ .

- (a) Say  $g_0(x) = 1 \in \mathbb{R}(x)$ .
- (b)  $\mathbb{R}(x)$  is a field. Show that  $g'(x) \in \mathbb{R}(x)$  for any  $g(x) \in \mathbb{R}(x)$ . Write  $g(x) = \frac{p(x)}{q(x)}$  for some  $p(x), q(x) \in \mathbb{R}[x]$ ,  $q(x) \neq 0$ . Thus

$$g'(x) = \frac{p'(x)q(x) - p(x)q'(x)}{q(x)^2}.$$

The numerator of  $g'(x)$  is in  $\mathbb{R}[x]$  since the differentiation operator on  $\mathbb{R}[x]$  is closed in  $\mathbb{R}[x]$ . Also, the denominator of  $g'(x) = \frac{p'(x)q(x) - p(x)q'(x)}{q(x)^2} \neq 0$  since  $\mathbb{R}[x]$  is an integral domain. Therefore,  $g'(x) \in \mathbb{R}(x)$ .

- (c) Induction on  $n$ . For  $n = 1$ , we have

$$\begin{aligned} f'(x) &= g'_0(x)e^{-\frac{1}{x^2}} + g_0(x) \cdot \left(-\frac{1}{x^2}\right)' e^{-\frac{1}{x^2}} \\ &= \left(g'_0(x) + g_0(x) \cdot \left(-\frac{1}{x^2}\right)'\right) e^{-\frac{1}{x^2}} \\ &= g_1(x)e^{-\frac{1}{x^2}} \end{aligned}$$

where

$$g_1(x) = g'_0(x) + g_0(x) \cdot \left(-\frac{1}{x^2}\right)' \in \mathbb{R}(x).$$

Now assume that the conclusion holds for  $n = k$ . As  $n = k + 1$ , similar to the case  $n = 1$ ,

$$f^{(k+1)}(x) = g_{k+1}(x)e^{-\frac{1}{x^2}}$$

where

$$g_{k+1}(x) = g'_k(x) + g_k(x) \cdot \left(-\frac{1}{x^2}\right)' \in \mathbb{R}(x).$$

By induction, the conclusion is true.

- (3) Induction on  $n$ . For  $n = 1$ , by (1) we have

$$f'(0) = \lim_{t \rightarrow 0} \frac{e^{-\frac{1}{t^2}} - 0}{t} = 0.$$

Now assume that the statement holds for  $n = k$ . As  $n = k + 1$ , by (1)(2) we have

$$f^{(k+1)}(0) = \lim_{t \rightarrow 0} \frac{f^{(k)}(t) - f^{(k)}(0)}{t} = \lim_{t \rightarrow 0} \frac{g_k(t)e^{-\frac{1}{t^2}} - 0}{t} = 0.$$

Thus,  $f^{(n)}(0) = 0$  for  $n \in \mathbb{Z}^+$ .

□

**Exercise 8.2.** Let  $a_{ij}$  be the number in the  $i$ th row and  $j$ th column of the array

$$\begin{array}{ccccc} -1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & -1 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{2} & -1 & 0 & \cdots \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

so that

$$a_{ij} = \begin{cases} 0 & (i < j), \\ -1 & (i = j), \\ 2^{j-i} & (i > j). \end{cases}$$

Prove that

$$\sum_i \sum_j a_{ij} = -2, \quad \sum_j \sum_i a_{ij} = 0.$$

Also see Theorem 8.3.

*Proof (Brute-force).*

$$\begin{aligned} \sum_i \sum_j a_{ij} &= \sum_{i=1}^{\infty} \left( \sum_{j=i}^{\infty} a_{ij} + \sum_{j<i} a_{ij} \right) \\ &= \sum_{i=1}^{\infty} \left( -1 + \sum_{j=1}^{i-1} 2^{j-i} \right) \\ &= \sum_{i=1}^{\infty} (-1 + (1 - 2^{1-i})) \\ &= \sum_{i=1}^{\infty} -2^{1-i} \\ &= -2. \end{aligned}$$

$$\begin{aligned}
\sum_j \sum_i a_{ij} &= \sum_{j=1}^{\infty} \left( \sum_{i=j}^{\infty} a_{ij} + \sum_{i>j} a_{ij} \right) \\
&= \sum_{j=1}^{\infty} \left( -1 + \sum_{i=j+1}^{\infty} 2^{j-i} \right) \\
&= \sum_{j=1}^{\infty} (-1 + 1) \\
&= \sum_{j=1}^{\infty} 0 \\
&= 0.
\end{aligned}$$

□

**Exercise 8.3.** PLACEHOLDER.

**Exercise 8.4.** PLACEHOLDER.

**Exercise 8.5.** PLACEHOLDER.

**Exercise 8.6.** Suppose  $f(x)f(y) = f(x+y)$  for all real  $x$  and  $y$ .

(a) Assuming that  $f$  is differentiable and not zero, prove that

$$f(x) = e^{cx}$$

where  $c$  is a constant.

(b) Prove the same thing, assuming only that  $f$  is continuous.

Part (b) implies part (a). We prove part (b) directly.

*Proof of (b).*

- (1) Since  $f(x)$  is not zero, there exists  $x_0 \in \mathbb{R}$  such that  $f(x_0) \neq 0$ . So  $f(0)f(x_0) = f(x_0)$ , or  $f(0) = 1$  by cancelling  $f(x_0) \neq 0$ .
- (2) Next,  $f(\frac{n}{m}) = f(\frac{1}{m})^n$  for  $m \in \mathbb{Z}$ ,  $n \in \mathbb{Z}^+$ . Since  $f$  is continuous at  $x = 0$ ,  $f$  is positive in the neighborhood of  $x = 0$ . That is, there exists  $N \in \mathbb{Z}^+$  such that  $f(\frac{1}{m}) > 0$  whenever  $|m| \geq N$ . So,  $f(\frac{n}{m}) = f(\frac{1}{m})^n > 0$ . (Since  $f(\frac{n}{m}) = f(\frac{kn}{km})$  for any  $k \in \mathbb{Z}^+$ , we can rescale  $m$  to  $km$  such that

$|km| \geq N$ .) That is,  $f$  is positive on  $\mathbb{Q}$ . Since  $\mathbb{Q}$  is dense in  $\mathbb{R}$  and  $f$  is continuous on  $\mathbb{R}$ ,  $f$  is positive on  $\mathbb{R}$ .

- (3) Now let  $c = \log f(1)$  (which is well-defined since  $f > 0$ ). We write  $f(1)$  in the two ways. Firstly,  $f(1) = f(\frac{n}{n}) = f(\frac{1}{n})^n$  where  $n \in \mathbb{Z}^+$ . Secondly,  $f(1) = e^c = (e^{\frac{c}{n}})^n$ . Since the positive  $n$ -th root is unique (Theorem 1.21),  $f(\frac{1}{n}) = e^{\frac{c}{n}}$  for  $n \in \mathbb{Z}^+$ . By  $f(x)f(-x) = f(0) = 1$  or  $f(-x) = \frac{1}{f(x)}$ ,  $f(-\frac{1}{n}) = \frac{1}{e^{\frac{c}{n}}} = e^{-\frac{c}{n}}$  for  $n \in \mathbb{Z}^+$ . Therefore,

$$f\left(\frac{1}{m}\right) = e^{\frac{c}{m}} \text{ where } m \in \mathbb{Z}.$$

- (4) By using  $f(\frac{n}{m}) = f(\frac{1}{m})^n$  for  $m \in \mathbb{Z}$ ,  $n \in \mathbb{Z}^+$  again,  $f(\frac{n}{m}) = e^{c\frac{n}{m}}$  where  $m \in \mathbb{Z}$ ,  $n \in \mathbb{Z}^+$ , or

$$f(x) = e^{cx} \text{ where } x \in \mathbb{Q}.$$

Since  $g(x) = f(x) - e^{cx}$  vanishes on a dense set of  $\mathbb{Q}$  and  $g$  is continuous on  $\mathbb{R}$ ,  $g$  vanishes on  $\mathbb{R}$ . Therefore,  $f(x) = e^{cx}$  for  $x \in \mathbb{R}$ .

□

**Supplement.** *Proof of (a).*

- (1) Since  $f(x)$  is not zero, there exists  $x_0 \in \mathbb{R}$  such that  $f(x_0) \neq 0$ . So  $f(0)f(x_0) = f(x_0)$ , or  $f(0) = 1$  by cancelling  $f(x_0) \neq 0$ .
- (2) Since  $f$  is differentiable, for any  $x \in \mathbb{R}$ ,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x)f(h) - f(x)}{h} \\ &= f(x) \lim_{h \rightarrow 0} \frac{f(h) - 1}{h} \\ &= f(x) \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} \\ &= f(x)f'(0). \end{aligned}$$

Let  $c = f'(0)$  be a constant. Then  $f'(x) = cf(x)$ . So  $f(x) = e^{cx}$  for  $x \in \mathbb{R}$ . (To see this, let  $g(x) = \frac{f(x)}{e^{cx}}$  be well-defined on  $\mathbb{R}$ .  $g(0) = 1$ .  $g'(x) = 0$  since  $f'(x) = cf(x)$ . So  $g(x)$  is a constant, or  $g(x) = 1$  since  $g(0) = 1$ . Therefore,  $f(x) = e^{cx}$  on  $\mathbb{R}$ .)

□

**Supplement.** Cauchy's functional equation.

- (1) (*Cauchy's functional equation.*) Suppose  $f(x) + f(y) = f(x + y)$  for all real  $x$  and  $y$ . Assuming that  $f$  is continuous, prove that  $f(x) = cx$  where  $c$  is a constant.

Notice that we cannot let  $g(x) = \log f(x)$  and apply Cauchy's functional equation on  $g(x)$  to prove Exercise 8.6 since  $f(x)$  is not necessarily positive and thus  $g(x) = \log f(x)$  might be meaningless. However, this wrong approach gives you some useful ideas such as you need to prove that  $f(x)$  is positive first, and  $f(x)$  should be equal to  $e^{cx}$  where  $c = g(1) = \log f(1)$ .

- (2) Suppose  $f(xy) = f(x) + f(y)$  for all positive real  $x$  and  $y$ . Assuming that  $f$  is continuous, prove that  $f(x) = c \log x$  where  $c$  is a constant.
- (3) Suppose  $f(xy) = f(x)f(y)$  for all positive real  $x$  and  $y$ . Assuming that  $f$  is continuous and positive, prove that  $f(x) = x^c$  where  $c$  is a constant.
- (4) Suppose  $f(x + y) = f(x) + f(y) + xy$  for all real  $x$  and  $y$ . Assuming that  $f$  is continuous, prove that  $f(x) = \frac{1}{2}x^2 + cx$  where  $c$  is a constant.
- (5) (*USA 2002.*) Suppose  $f(x^2 - y^2) = xf(x) - yf(y)$  for all real  $x$  and  $y$ . Assuming that  $f$  is continuous, prove that  $f(x) = cx$  where  $c$  is a constant.

**Supplement.** Show that the only automorphism of  $\mathbb{Q}$  is the identity.

*Proof.* Given any  $\sigma \in \text{Aut}(\mathbb{Q})$ .

- (1) Show that  $\sigma(1) = 1$ . Since  $1^2 = 1$ ,  $\sigma(1)\sigma(1) = \sigma(1)$ .  $\sigma(1) = 0$  or  $1$ . There are only two possible cases.

- (a) Assume that  $\sigma(1) = 0$ . So

$$\sigma(a) = \sigma(a \cdot 1) = \sigma(a) \cdot \sigma(1) = \sigma(a) \cdot 0 = 0$$

for any  $a \in \mathbb{Q}$ . That is,  $\sigma = 0 \in \text{Aut}(\mathbb{Q})$ , which is absurd.

- (b) Therefore,  $\sigma(1) = 1$ .

- (2) Show that  $\sigma(n) = n$  for all  $n \in \mathbb{Z}^+$ . Write  $n = 1 + 1 + \cdots + 1$  ( $n$  times 1). Applying the additivity of  $\sigma$ , we have

$$\sigma(n) = \sigma(1) + \sigma(1) + \cdots + \sigma(1) = 1 + 1 + \cdots + 1 = n.$$

(Might use induction on  $n$  to eliminate  $\cdots$  symbols.)

- (3) Show that  $\sigma(n) = n$  for all  $n \in \mathbb{Z}$ . By the additivity of  $\sigma$ ,  $\sigma(-n) = -\sigma(n) = -n$  for  $n \geq 0$ . The result is established.



For any  $a = \frac{n}{m} \in \mathbb{Q}$  ( $m, n \in \mathbb{Z}$ ,  $n \neq 0$ ), applying the multiplication of  $\sigma$  on  $am = n$ , that is,  $\sigma(a)\sigma(m) = \sigma(n)$ . By (3), we have  $\sigma(a)m = n$ , or

$$\sigma(a) = \frac{m}{n} = a$$

provided  $n \neq 0$ , or  $\sigma$  is the identity.  $\square$

**Exercise 8.7.** PLACEHOLDER.

**Exercise 8.8.** For  $n = 0, 1, 2, \dots$ , and  $x$  real, prove that

$$|\sin(nx)| \leq n|\sin x|.$$

Note that this inequality may be false for other values of  $n$ . For instance,

$$\left| \sin\left(\frac{1}{2}\pi\right) \right| > \frac{1}{2}|\sin \pi|.$$

*Proof.* Induction on  $n$ .

(1) Note that

$$\sin(a + b) = \sin a \cos b + \cos a \sin b$$

for any  $a, b \in \mathbb{R}$ .

(2)  $n = 0, 1$  are clearly true.

(3) Assume the induction hypothesis that for the single case  $n = k$  holds, meaning

$$|\sin(kx)| \leq k|\sin x|$$

is true. It follows that

$$\begin{aligned} |\sin((k+1)x)| &= |\sin(kx) \cos x + \cos(kx) \sin x| && ((1)) \\ &\leq |\sin(kx)| |\cos x| + |\cos(kx)| |\sin x| && (\text{Triangle inequality}) \\ &\leq |\sin(kx)| + |\sin x| && (|\cos(\cdot)| \leq 1) \\ &\leq k|\sin x| + |\sin x| && (\text{Induction hypothesis}) \\ &\leq (k+1)|\sin x|. \end{aligned}$$

$\square$

**Exercise 8.9 (The Euler-Mascheroni constant).**

(a) Put  $s_N = 1 + \frac{1}{2} + \cdots + \frac{1}{N}$ . Prove that

$$\lim_{N \rightarrow \infty} (s_N - \log N)$$

exists. (The limit, often denoted by  $\gamma$ , is called Euler's constant. Its numerical value is  $0.5772\dots$ . It is not known whether  $\gamma$  is rational or not.)

(b) Roughly how large must  $m$  be so that  $N = 10^m$  satisfies  $s_N > 100$ ?

*Proof of (a) (Theorem 3.14).*

(1) Note that

$$\begin{aligned} \frac{1}{1 + \frac{1}{n}} &\leq \frac{1}{x} \leq 1 \text{ for } x \in \left[1, 1 + \frac{1}{n}\right] \\ \Rightarrow \int_1^{1 + \frac{1}{n}} \frac{dx}{1 + \frac{1}{n}} &\leq \int_1^{1 + \frac{1}{n}} \frac{dx}{x} \leq \int_1^{1 + \frac{1}{n}} dx && \text{(Theorem 6.12(b))} \\ \Rightarrow \frac{1}{n+1} &\leq \int_1^{1 + \frac{1}{n}} \frac{dx}{x} \leq \frac{1}{n} \\ \Rightarrow \frac{1}{n+1} &\leq \log \left(1 + \frac{1}{n}\right) \leq \frac{1}{n}. && \text{(Equation (39) on page 180)} \end{aligned}$$

(2) Define

$$\gamma_n = s_n - \log n.$$

It suffices to show that  $\{\gamma_n\}$  is monotonic and bounded (Theorem 3.14).

(3) Show that  $\{\gamma_n\}$  is decreasing.

$$\begin{aligned} \gamma_{n+1} - \gamma_n &= (s_{n+1} - \log(n+1)) - (s_n - \log n) \\ &= (s_{n+1} - s_n) - (\log(n+1) - \log n) \\ &= \frac{1}{n+1} - \log \left( \frac{n+1}{n} \right) \\ &= \frac{1}{n+1} - \log \left( 1 + \frac{1}{n} \right) \\ &\leq 0. && ((1)) \end{aligned}$$

*Note.*  $\gamma_n \leq \cdots \leq \gamma_1 = 1$  for all  $n = 1, 2, 3, \dots$

(4) Show that  $\gamma_n \geq 0$  for all  $n = 1, 2, 3, \dots$ . Since

$$\begin{aligned}
 \log n &= \sum_{k=2}^n (\log k - \log(k-1)) \\
 &= \sum_{k=2}^n \log \frac{k}{k-1} \\
 &= \sum_{k=2}^n \log \left( 1 + \frac{1}{k-1} \right) \\
 &\leq \sum_{k=2}^n \frac{1}{k-1} \\
 &= s_{n-1},
 \end{aligned} \tag{1)$$

we have

$$\gamma_n = s_n - \log n \geq s_n - s_{n-1} = \frac{1}{n} > 0.$$

By (3)(4),  $\{\gamma_n\}$  converges to  $\lim_{N \rightarrow \infty} (s_N - \log N) = \gamma$ .  $\square$

*Note.* This proof is based on **integral test** (Theorem 8.23) in this textbook: Apostol, *Mathematical Analysis, 2nd edition*.

*Proof of (a) (Limit comparison test).*

(1) Rewrite

$$\gamma_n = \sum_{k=1}^n \left( \frac{1}{k} - \log \left( 1 + \frac{1}{k} \right) \right)$$

(similar to the argument in (a)(4)(Theorem 3.14)).

(2) Show that

$$\lim_{k \rightarrow \infty} \frac{\frac{1}{k} - \log \left( 1 + \frac{1}{k} \right)}{\frac{1}{k^2}} = \frac{1}{2}.$$

In fact,

$$\begin{aligned}
 &\lim_{k \rightarrow \infty} \frac{\frac{1}{k} - \log \left( 1 + \frac{1}{k} \right)}{\frac{1}{k^2}} \\
 &= \lim_{x \rightarrow 0} \frac{x - \log(1+x)}{x^2} \quad \text{(Put } x = \frac{1}{k} \text{)} \\
 &= \lim_{x \rightarrow 0} \frac{1 - \frac{1}{1+x}}{2x} \quad \text{(L'Hospital's rule)} \\
 &= \lim_{x \rightarrow 0} \frac{1}{2(x+1)} \\
 &= \frac{1}{2}.
 \end{aligned}$$

- (3) By limit comparison test or comparison test,  $\sum \left(\frac{1}{k} - \log\left(1 + \frac{1}{k}\right)\right)$  converges since  $\sum \frac{1}{k^2}$  converges.

□

*Note.* This proof is based on **limit comparison test** (Theorem 8.21) in this textbook: *Apostol, Mathematical Analysis, 2nd edition. It is easy to prove by the original comparison test.*

*Proof of (b).* By  $s_n - \log n > 0$  in (a)(4)(Theorem 3.14), it suffices to choose  $N = 10^m$  such that  $s_N \geq \log(N + 1) > 100$ , or

$$m > \frac{\log(\exp(100) - 1)}{\log 10},$$

or choose  $m$  satisfying

$$m > \frac{100}{\log 10} > \frac{\log(\exp(100) - 1)}{\log 10},$$

or  $m = 44$ . □

**Exercise 8.10.** Prove that  $\sum \frac{1}{p}$  diverges; the sum extends over all primes.

There are many proofs of this result. We provide some of them.

*Proof (Due to hint).* Given  $N$ .

- (1) Show that

$$\sum_{n \leq N} \frac{1}{n} \leq \prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1}.$$

By the unique factorization theorem on  $n \leq N$ ,

$$\sum_{n \leq N} \frac{1}{n} \leq \prod_{p \leq N} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1}.$$

- (2) By (1) and the fact that  $\sum \frac{1}{n}$  diverges, there are infinitely many primes.

- (3) Show that

$$\prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1} \leq \exp\left(\sum_{p \leq N} \frac{2}{p}\right).$$

By applying the inequality  $(1-x)^{-1} < e^{2x}$  where  $x \in (0, \frac{1}{2}]$  on any prime  $p$ ,

$$\left(1 - \frac{1}{p}\right)^{-1} < \exp\left(\frac{2}{p}\right).$$

Now multiplying the inequality over all primes  $p \leq N$  and noticing that  $\exp(x) \cdot \exp(y) = \exp(x+y)$ , we have

$$\prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1} \leq \exp\left(\sum_{p \leq N} \frac{2}{p}\right).$$

(4) By (1)(3),

$$\sum_{n \leq N} \frac{1}{n} \leq \exp\left(\sum_{p \leq N} \frac{2}{p}\right).$$

Since  $\sum_{n \leq N} \frac{1}{n}$  diverges, the result holds.

□

*Proof (Due to Kenneth Ireland and Michael Rosen).* The proof in Kenneth Ireland and Michael Rosen, A Classical Introduction to Modern Number Theory, Second Edition (Theorem 3 in Chapter 2) does not use the inequality  $(1-x)^{-1} < e^{2x}$  ( $x \in (0, \frac{1}{2}]$ ) directly. Instead, the authors take the logarithm on  $(1-p^{-1})^{-1}$  and estimate it. (So the length of proof is longer than the proof due to hint.) That is,

$$\begin{aligned} -\log(1-p^{-1}) &= \sum_{n=1}^{\infty} \frac{p^{-n}}{n} \\ &= \frac{1}{p} + \sum_{n=2}^{\infty} \frac{p^{-n}}{n} \\ &< \frac{1}{p} + \sum_{n=2}^{\infty} p^{-n} \\ &= \frac{1}{p} + \frac{p^{-2}}{1-p^{-1}} \\ &< \frac{1}{p} + 2 \cdot \frac{1}{p^2}. \end{aligned}$$

Now we sum over all primes  $p \leq N$ ,

$$\log\left(\prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1}\right) < \sum_{p \leq N} \frac{1}{p} + 2 \sum_{p \leq N} \frac{1}{p^2}.$$

So

$$\log \sum_{n \leq N} \frac{1}{n} < \sum_{p \leq N} \frac{1}{p} + 2 \sum_{p \leq N} \frac{1}{p^2}.$$

Notice that  $\sum \frac{1}{n}$  diverges and  $\sum \frac{1}{p^2}$  converges (since  $\sum \frac{1}{n^2}$  converges). Therefore,  $\sum \frac{1}{p}$  diverges.  $\square$

*Proof (Due to I. Niven).* It is an exercise in Kenneth Ireland and Michael Rosen, A Classical Introduction to Modern Number Theory, Second Edition. See Exercise 27 in Chapter 2.

- (1) Show that  $\sum' \frac{1}{n}$ , the sum being over square free integers, diverges. For any positive integers  $n$ , we can write  $n = a^2 b$  where  $a \in \mathbb{Z}^+$  and  $b$  is a square free integer. Given  $N$ ,

$$\sum_{n \leq N} \frac{1}{n} \leq \left( \sum_{a=1}^{\infty} \frac{1}{a^2} \right) \left( \sum'_{b \leq N} \frac{1}{b} \right).$$

Notice that  $\sum_{a=1}^{\infty} \frac{1}{a^2}$  converges. Since  $\sum_{n \leq N} \frac{1}{n} \rightarrow \infty$  as  $N \rightarrow \infty$ ,  $\sum'_{b \leq N} \frac{1}{b} \rightarrow \infty$  as  $N \rightarrow \infty$ .

- (2) Show that

$$\prod_{p \leq N} \left( 1 + \frac{1}{p} \right) \rightarrow \infty \text{ as } N \rightarrow \infty.$$

By the unique factorization theorem on  $n \leq N$ ,

$$\prod_{p \leq N} \left( 1 + \frac{1}{p} \right) \geq \sum'_{n \leq N} \frac{1}{n}.$$

Since  $\sum'_{n \leq N} \frac{1}{n} \rightarrow \infty$  as  $N \rightarrow \infty$  by (1), the conclusion is established.

- (3) By applying the inequality  $e^x > 1 + x$  on any prime  $p$ ,

$$\exp \left( \frac{1}{p} \right) > 1 + \frac{1}{p}.$$

Now multiplying the inequality over all primes  $p \leq N$  and noticing that  $\exp(x) \cdot \exp(y) = \exp(x + y)$ , we have

$$\exp \left( \sum_{p \leq N} \frac{1}{p} \right) > \prod_{p \leq N} \left( 1 + \frac{1}{p} \right).$$

By (2),  $\exp \left( \sum_{p \leq N} \frac{1}{p} \right) \rightarrow \infty$  as  $N \rightarrow \infty$ , or  $\sum_{p \leq N} \frac{1}{p} \rightarrow \infty$  as  $N \rightarrow \infty$ .

□

**Exercise 8.11.** PLACEHOLDER.

**Exercise 8.12.** Suppose  $0 < \delta < \pi$ ,

$$f(x) = \begin{cases} 1 & \text{if } |x| \leq \delta, \\ 0 & \text{if } \delta < |x| \leq \pi, \end{cases}$$

and  $f(x + 2\pi) = f(x)$  for all  $x$ .

(a) Compute the Fourier coefficients of  $f$ .

(b) Compute that

$$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2} \quad (0 < \delta < \pi).$$

(c) Deduce from Parseval's theorem that

$$\sum_{n=1}^{\infty} \frac{(\sin(n\delta))^2}{n^2\delta} = \frac{\pi - \delta}{2}.$$

(d) Let  $\delta \rightarrow 0$  and prove that

$$\int_0^{\infty} \left( \frac{\sin x}{x} \right)^2 dx = \frac{\pi}{2}.$$

(e) Put  $\delta = \frac{\pi}{2}$  in (c). What do you get?

It is a centered square pulse around  $x = 0$  with shift  $\delta$ . Besides,  $f(x)$  is an even function.

*Proof of (a).*

$$\begin{aligned} c_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \\ &= \frac{1}{2\pi} \int_{-\delta}^{\delta} dx \\ &= \frac{\delta}{\pi}. \end{aligned}$$

For  $0 \neq n \in \mathbb{Z}$ ,

$$\begin{aligned}
c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \\
&= \frac{1}{2\pi} \int_{-\delta}^{\delta} e^{-inx} dx \\
&= \frac{1}{2\pi} \cdot \frac{2 \sin(n\delta)}{n} \\
&= \frac{\sin(n\delta)}{n\pi}.
\end{aligned}$$

□

**Supplement.** Find  $a_n$  and  $b_n$  of this textbook.

By (a),  $a_0 = \frac{\delta}{\pi}$ ,  $a_n = \frac{2 \sin(n\delta)}{n\pi}$ ,  $b_n = 0$  for  $n \in \mathbb{Z}^+$ . Surely, we can compute  $a_n$  and  $b_n$  ( $n > 0$ ) directly. Since  $f(x)$  is an even function,  $b_n = 0$ . And

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \\
&= \frac{2}{\pi} \int_0^{\delta} \cos(nx) dx \\
&= \frac{2 \sin(n\delta)}{n\pi}.
\end{aligned}$$

*Proof of (b).* Given  $x = 0$ , there are constants  $\delta' = \delta > 0$  and  $M = 1 < \infty$  such that

$$|f(0+t) - f(0)| \leq M|t|$$

for all  $t \in (-\delta', \delta')$ . By Theorem 8.14,

$$\sum_{-\infty}^{\infty} c_n = f(0).$$

Notice that  $c_{-n} = c_n$  for  $n \in \mathbb{Z}^+$ , so

$$\begin{aligned}
\frac{\delta}{\pi} + 2 \sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n\pi} &= 1 \\
\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} &= \frac{\pi - \delta}{2}.
\end{aligned}$$

□

We can also use the expression  $a_n$  and  $b_n$  to prove the same thing. Besides, taking  $\delta = 1$  yields

$$\sum_{n=1}^{\infty} \frac{\sin n}{n} = \frac{\pi - 1}{2}.$$



*Proof of (c).* Since  $f(x)$  is a Riemann-integrable function with period  $2\pi$ , by Parseval's theorem

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

So

$$\frac{\delta}{\pi} = \frac{\delta^2}{\pi^2} + 2 \sum_{n=1}^{\infty} \frac{(\sin(n\delta))^2}{n^2 \pi^2},$$

or

$$\sum_{n=1}^{\infty} \frac{(\sin(n\delta))^2}{n^2 \delta} = \frac{\pi - \delta}{2}.$$

□

Notices that

$$\sum_{n=1}^{\infty} \frac{(\sin n)^2}{n^2} = \frac{\pi - 1}{2}$$

as  $\delta = 1$ .

*Proof of (d).* Given  $\varepsilon > 0$ . By Exercise 6.8,

$$\int_0^{\infty} \left( \frac{\sin x}{x} \right)^2 dx$$

exists. So there exists  $b > 0$  such that

$$\left| \int_0^b \left( \frac{\sin x}{x} \right)^2 dx - \int_0^{\infty} \left( \frac{\sin x}{x} \right)^2 dx \right| < \frac{\varepsilon}{4}$$

By Supplement in Chapter 6, there exists  $\delta > 0$  such that for any partition  $P_m = \{0, \frac{b}{m}, \frac{2b}{m}, \dots, \frac{(m-1)b}{m}, b\}$  of  $[0, b]$  with  $\|P\| = \frac{b}{m} < \delta$ , or  $m > \frac{b}{\delta}$ , we have

$$\begin{aligned} \left| \sum_{n=1}^m \frac{(\sin(n \frac{b}{m}))^2}{(n \frac{b}{m})^2} \cdot \frac{b}{m} - \int_0^b \left( \frac{\sin x}{x} \right)^2 dx \right| &< \frac{\varepsilon}{4}, \\ \left| \sum_{n=1}^m \frac{(\sin(n \frac{b}{m}))^2}{n^2 \frac{b}{m}} - \int_0^b \left( \frac{\sin x}{x} \right)^2 dx \right| &< \frac{\varepsilon}{4}. \end{aligned}$$

For simplicity we resize  $\delta$  to  $\delta < \pi$  to make  $0 < \frac{b}{m} < \delta < \pi$ . Besides, since  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges, there exists  $N > 0$  such that

$$\left| \sum_{n=1}^{\infty} \frac{(\sin(n \frac{b}{m}))^2}{n^2 \frac{b}{m}} - \sum_{n=1}^m \frac{(\sin(n \frac{b}{m}))^2}{n^2 \frac{b}{m}} \right| < \frac{\varepsilon}{4}$$

whenever  $m \geq N$ . By (c),

$$\left| \frac{\pi - \frac{b}{m}}{2} - \sum_{n=1}^m \frac{(\sin(n\frac{b}{m}))^2}{n^2 \frac{b}{m}} \right| < \frac{\varepsilon}{4}$$

whenever  $m \geq N$ . Last, it is easy to get

$$\left| \frac{\pi}{2} - \frac{\pi - \frac{b}{m}}{2} \right| < \frac{\varepsilon}{4}$$

whenever  $m > \frac{2b}{\varepsilon}$ . Now we have

$$\left| \frac{\pi}{2} - \int_0^\infty \left( \frac{\sin x}{x} \right)^2 dx \right| < \varepsilon$$

whenever  $m > \max(\frac{b}{\delta}, N, \frac{2b}{\varepsilon})$ . Since  $\varepsilon$  is arbitrary,  $\int_0^\infty \left( \frac{\sin x}{x} \right)^2 dx = \frac{\pi}{2}$ .  $\square$

*Proof of (e).*

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

Write

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^2} &= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \\ &= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}, \end{aligned}$$

so

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{6}.$$

$\square$

**Exercise 8.13.** Put  $f(x) = x$  if  $0 \leq x < 2\pi$ , and apply Parseval's theorem to conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi}{6}.$$

*Proof.*

$$\begin{aligned} c_0 &= \frac{1}{2\pi} \int_0^{2\pi} x dx \\ &= \pi, \end{aligned}$$

For  $n \neq 0$ ,

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_0^{2\pi} x e^{-inx} dx \\ &= \frac{1}{2\pi} \left( \left[ -\frac{1}{in} x e^{-inx} \right]_{x=0}^{x=2\pi} - \int_0^{2\pi} -\frac{1}{in} e^{-inx} dx \right) \\ &= \frac{i}{n}. \end{aligned}$$

Since  $f(x)$  is a Riemann-integrable function with period  $2\pi$ , by Parseval's theorem

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{-\infty}^{\infty} |c_n|^2.$$

So

$$\frac{1}{2\pi} \cdot \frac{(2\pi)^3}{3} = \pi^2 + 2 \sum_{n=1}^{\infty} \frac{1}{n^2},$$

or

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

□

**Supplement.** Put  $f(x) = x^n$  if  $n \in \mathbb{Z}^+$  and  $0 \leq x < 2\pi$ . Might get

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!}.$$

**Exercise 8.14-8.31.** PLACEHOLDER.