

Chapter 2: Modules

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Exercise 2.1. Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$ if m, n are coprime.

It suffices to show that

$$(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$$

where d is the greatest common divisor of m and n .

Outlines.

- (1) Define $\tilde{\varphi}$ by

$$\begin{array}{ccc} \tilde{\varphi}: & (\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z}) & \longrightarrow \mathbb{Z}/d\mathbb{Z} \\ & \Downarrow & \Downarrow \\ & (x + m\mathbb{Z}, y + n\mathbb{Z}) & \longmapsto xy + d\mathbb{Z}. \end{array}$$

$\tilde{\varphi}$ is well-defined and \mathbb{Z} -bilinear.

- (2) By the universal property, $\tilde{\varphi}$ factors through a \mathbb{Z} -bilinear map

$$\varphi: (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \rightarrow \mathbb{Z}/d\mathbb{Z}$$

(such that $\varphi(x \otimes y) = \tilde{\varphi}(x, y)$).

- (3) To show that φ is isomorphic, might find the inverse map $\psi: \mathbb{Z}/d\mathbb{Z} \rightarrow (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$ of φ . Define ψ by

$$\begin{array}{ccc} \psi: & \mathbb{Z}/d\mathbb{Z} & \longrightarrow (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \\ & \Downarrow & \Downarrow \\ & z + d\mathbb{Z} & \longmapsto (z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}). \end{array}$$

ψ is well-defined and \mathbb{Z} -linear.

- (4) $\psi \circ \varphi = \text{id}$.

- (5) $\varphi \circ \psi = \text{id}$.

Proof of (1).

(a) $\tilde{\varphi}$ is well-defined. Say $x' = x + am$ for some $a \in \mathbb{Z}$ and $y' = y + bn$ for some $b \in \mathbb{Z}$. Then $x'y' - xy = yam + xbn + abmn \in \mathbb{Z}/d\mathbb{Z}$. That is, $\tilde{\varphi}$ is independent of coset representative.

(b) $\tilde{\varphi}$ is \mathbb{Z} -bilinear.

(i) For any $\lambda \in \mathbb{Z}$, $\tilde{\varphi}(\lambda x, y) = \tilde{\varphi}(x, \lambda y) = \lambda \tilde{\varphi}(x, y)$. In fact,

$$\begin{aligned}\tilde{\varphi}(\lambda(x + m\mathbb{Z}), y + n\mathbb{Z}) &= \tilde{\varphi}(\lambda x + m\mathbb{Z}, y + n\mathbb{Z}) = \lambda xy + d\mathbb{Z}, \\ \tilde{\varphi}(x + m\mathbb{Z}, \lambda(y + n\mathbb{Z})) &= \tilde{\varphi}(x + m\mathbb{Z}, \lambda y + n\mathbb{Z}) = \lambda xy + d\mathbb{Z}, \\ \tilde{\varphi}(x_1 + m\mathbb{Z}, y + n\mathbb{Z}) &= \lambda(xy + d\mathbb{Z}) = \lambda xy + d\mathbb{Z}.\end{aligned}$$

(ii) $\tilde{\varphi}(x_1 + x_2, y) = \tilde{\varphi}(x_1, y) + \tilde{\varphi}(x_2, y)$. In fact,

$$\begin{aligned}\tilde{\varphi}((x_1 + x_2) + m\mathbb{Z}, y + n\mathbb{Z}) &= (x_1 + x_2)y + d\mathbb{Z}, \\ \tilde{\varphi}(x_1 + m\mathbb{Z}, y + n\mathbb{Z}) + \tilde{\varphi}(x_2 + m\mathbb{Z}, y + n\mathbb{Z}) &= (x_1y + d\mathbb{Z}) + (x_2y + d\mathbb{Z}) \\ &= (x_1 + x_2)y + d\mathbb{Z}.\end{aligned}$$

(iii) $\tilde{\varphi}(x, y_1 + y_2) = \tilde{\varphi}(x, y_1) + \tilde{\varphi}(x, y_2)$. Similar to (ii).

□

Proof of (3).

(a) ψ is well-defined. Say $z' = z + cd$ for some $c \in \mathbb{Z}$. Note that $d = \alpha m + \beta n$ for some $\alpha, \beta \in \mathbb{Z}$. Thus

$$\begin{aligned}\psi(z' + d\mathbb{Z}) &= \psi(z + cd + d\mathbb{Z}) \\ &= \psi(z + c(\alpha m + \beta n) + d\mathbb{Z}) \\ &= (z + c(\alpha m + \beta n) + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\ &= (z + c\beta n + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\ &= (z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) + (c\beta n + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\ &= \psi(z + d\mathbb{Z}) + (1 + m\mathbb{Z}) \otimes (c\beta n + n\mathbb{Z}) \\ &= \psi(z + d\mathbb{Z}).\end{aligned}$$

(b) ψ is \mathbb{Z} -linear.

(i) For any $\lambda \in \mathbb{Z}$, $\psi(\lambda z) = \lambda \psi(z)$. In fact,

$$\begin{aligned}\psi(\lambda(z + d\mathbb{Z})) &= \psi(\lambda z + d\mathbb{Z}) = (\lambda z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}), \\ \lambda \psi(z + d\mathbb{Z}) &= \lambda((z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z})) = (\lambda z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}).\end{aligned}$$

(ii) $\psi(z_1 + z_2) = \psi(z_1) + \psi(z_2)$.

$$\begin{aligned}\psi((z_1 + z_2) + d\mathbb{Z}) &= (z_1 + z_2 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}), \\ \psi(z_1 + d\mathbb{Z}) + \psi(z_2 + d\mathbb{Z}) &= (z_1 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) + (z_2 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\ &= (z_1 + z_2 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}).\end{aligned}$$

□

Proof of (4). For any $(x + m\mathbb{Z}) \otimes (y + n\mathbb{Z}) \in (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$,

$$\begin{aligned}\psi(\varphi((x + m\mathbb{Z}) \otimes (y + n\mathbb{Z}))) &= \psi(xy + d\mathbb{Z}) \\ &= (xy + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\ &= (x + m\mathbb{Z}) \otimes (y + n\mathbb{Z}).\end{aligned}$$

□

Proof of (5). For any $z + d\mathbb{Z} \in \mathbb{Z}/d\mathbb{Z}$,

$$\begin{aligned}\varphi(\psi(z + d\mathbb{Z})) &= \varphi((z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z})) \\ &= z + d\mathbb{Z}.\end{aligned}$$

□

Exercise 2.2. Let A be a ring, \mathfrak{a} an ideal, M an A -module. Show that $(A/\mathfrak{a}) \otimes_A M$ is isomorphic to $M/\mathfrak{a}M$. (Hint: Tensor the exact sequence $0 \rightarrow \mathfrak{a} \rightarrow A \rightarrow A/\mathfrak{a} \rightarrow 0$ with M .)

Proof (Hint). There is a natural exact sequence E :

$$E : 0 \rightarrow \mathfrak{a} \xrightarrow{i} A \xrightarrow{\pi} A/\mathfrak{a} \rightarrow 0$$

where i is the inclusion map (and π is the projection map). Tensor E with M :

$$E' : \mathfrak{a} \otimes_A M \xrightarrow{i \otimes 1} A \otimes_A M \xrightarrow{\pi \otimes 1} (A/\mathfrak{a}) \otimes_A M \rightarrow 0$$

is exact, or

$$(A/\mathfrak{a}) \otimes_A M \cong A \otimes_A M / \text{im}(i \otimes 1).$$

By Proposition 2.14, There is an unique isomorphism $A \otimes_A M \rightarrow M$ defined by $a \otimes x \mapsto ax$. This isomorphism sends $\text{im}(i \otimes 1)$ to $\mathfrak{a}M$. Therefore,

$$(A/\mathfrak{a}) \otimes_A M \cong M/\mathfrak{a}M.$$

□

Proof (Brute-force).

(1) Define $\tilde{\varphi}$ by

$$\begin{array}{ccc}\tilde{\varphi} : & A/\mathfrak{a} \times M & \longrightarrow M/\mathfrak{a}M \\ & \Downarrow & \Downarrow \\ & (a + \mathfrak{a}, x) & \longmapsto ax + \mathfrak{a}M.\end{array}$$

$\tilde{\varphi}$ is well-defined and A -bilinear.

- (2) By the universal property, $\tilde{\varphi}$ factors through a A -bilinear map

$$\varphi : A/\mathfrak{a} \otimes_A M \rightarrow M/\mathfrak{a}M$$

(such that $\varphi(a \otimes x) = \tilde{\varphi}(a, x)$).

- (3) To show that φ is isomorphic, might find the inverse map $\psi : M/\mathfrak{a}M \rightarrow A/\mathfrak{a} \otimes_A M$ of φ . Define ψ by

$$\begin{array}{ccc} \psi : & M/\mathfrak{a}M & \longrightarrow & A/\mathfrak{a} \otimes_A M \\ & \downarrow & & \downarrow \\ & x + \mathfrak{a}M & \longmapsto & (1 + \mathfrak{a}) \otimes x. \end{array}$$

ψ is well-defined and A -linear.

- (4) $\psi \circ \varphi = \text{id}$.

- (5) $\varphi \circ \psi = \text{id}$.

□

Exercise 2.3. Let A be a local ring, M and N finitely generated A -modules. Prove that if $M \otimes_A N = 0$, then $M = 0$ or $N = 0$. (Hint: Let \mathfrak{m} be the maximal ideal, $k = A/\mathfrak{m}$ the residue field. Let $M_k = k \otimes_A M \cong M/\mathfrak{m}M$ by Exercise 2.2. By Nakayama's lemma, $M_k = 0 \implies M = 0$. But $M \otimes_A N = 0 \implies (M \otimes_A N)_k = 0 \implies M_k \otimes_k N_k = 0 \implies M_k = 0$ or $N_k = 0$ since M_k, N_k are vector spaces over a field.)

The conclusion might be false if A is not local. For example, Exercise 2.1.

Proof (Hint). Let \mathfrak{m} be the maximal ideal, $k = A/\mathfrak{m}$ the residue field. Let $M_k = k \otimes_A M$.

- (1) (*Base extension*) Show that $(M \otimes_A N)_k = M_k \otimes_k N_k$. In fact, by Proposition 2.14

$$\begin{aligned} (M \otimes_A N)_k &= k \otimes_A (M \otimes_A N) \\ &= (k \otimes_A M) \otimes_A N \\ &= M_k \otimes_A N \\ &= (M_k \otimes_k k) \otimes_A N \\ &= M_k \otimes_k (k \otimes_A N) \\ &= M_k \otimes_k N_k. \end{aligned}$$

(2)

$$\begin{aligned} M \otimes_A N = 0 &\implies (M \otimes_A N)_k = 0 \\ &\implies M_k \otimes_k N_k = 0 && ((1)) \\ &\implies M_k = 0 \text{ or } N_k = 0 && (M_k, N_k: \text{ vector spaces}) \\ &\implies M/\mathfrak{m}M = 0 \text{ or } N/\mathfrak{m}N = 0 && (\text{Exercise 2.2}) \\ &\implies M = 0 \text{ or } N = 0. && (\text{Nakayama's lemma}) \end{aligned}$$

□