# Chapter 3: Lebesgue Measure

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## Section 3.1: Introduction

**Problem 3.1.** *If $A$ and $B$ are two sets in $\mathfrak{M}$ with $A \subseteq B$, then $mA \leq mB$. This property is called monotonicity.*

*Proof.* Write

$$B = B \cap X = B \cap (A \cup \widetilde{A}) = (B \cap A) \cup (B \cap \widetilde{A}) = A \cup (B - A).$$

Here $B \cap A = A$ comes from $A \subseteq B$ (Problem 1.9). Notice that $A$ and $B - A$ are disjoint. Since $m$ is a countably additive measure ($m$ is nonnegative) on a $\sigma$-algebra $\mathfrak{M}$,

$$mB = mA + m(B - A) \geq mA.$$

□

**Problem 3.2.** *Let $\langle E_n \rangle$ be any sequence of sets in $\mathfrak{M}$. Then $m(\bigcup E_n) \leq \sum mE_n$. (Hint: Use Proposition 1.2) This property of a measure is called countable subadditivity.*

As the argument in Problem 3.1.

*Proof.* Since $\langle E_n \rangle$ is a sequence of sets in $\sigma$-algebra $\mathfrak{M}$, by Proposition 1.2 and its proof, there is a sequence $\langle F_n \rangle$ of sets in $\sigma$-algebra $\mathfrak{M}$ such that all $F_n$ are pairwise disjoint, $F_n \subseteq E_n$, and

$$\bigcup E_n = \bigcup F_n.$$

Since $m$ is a countably additive measure on a $\sigma$-algebra $\mathfrak{M}$,

$$m\left(\bigcup E_n\right) = m\left(\bigcup F_n\right) = \sum mF_n \geq \sum mE_n.$$

The last inequality holds by applying Problem 3.1 on $F_n \subseteq E_n$ for any $n$. □

**Problem 3.3.** *If there is a set $A$ in $\mathfrak{M}$ such that $mA < \infty$, then $m\varnothing = 0$.*

*Proof.* For such $A$, write $A = A \cup \varnothing$. $A$ and $\varnothing$ are disjoint. Since $m$ is a countably additive measure on a $\sigma$-algebra $\mathfrak{M}$,

$$mA = mA + m\varnothing.$$

Since $mA < \infty$, we can cancel out $mA$ on the both sides to get $m\varnothing = 0$. $\square$

**Problem 3.4.** *Let $nE$ be $\infty$ for an infinite set $E$ and be equal to the number of elements of $E$ for a finite set. Show that $n$ is a countably additive set function that is translation invariant and defined for all sets of real numbers. This measure is called the **counting measure**.*

*Proof.*

(1) *Show that $n$ is a countably additive set function.* Note that $n$ is defined on any subset of real numbers since the finiteness is defined on any subset of real numbers. Suppose $\{E_m\}$ is a sequence of disjoint sets of real numbers. We need to show that $n\left(\bigcup E_m\right) = \sum nE_m$.

(2) If $E_m$ is infinite for some $m = k$, then $\bigcup E_m$ is also infinite. Hence, $n\left(\bigcup E_m\right) = \infty$, and $\sum nE_m \geq nE_k = \infty \Longrightarrow \sum nE_m = \infty$.

(3) Suppose all $E_n$ are finite. Note that $\bigcup E_m$ is infinite if and only if all but finitely many $E_m \neq \varnothing$ if and only if $\sum nE_m = \infty$. Besides, if $\bigcup E_m$ is finite, then all but finitely many $E_m = \varnothing$ and thus

$$n\left(\bigcup_m E_m\right) = \sum_{E_m \neq \varnothing} nE_m = \sum_m nE_m < \infty.$$

(4) Since

$$\begin{aligned} n(E+y) &= n(\{x+y : x \in E\}) \\ &= \text{the number of elements } x \in E \\ &= n(E), \end{aligned}$$

$n$ is translation invariant.

$\square$

## Section 3.2: Outer Measure

**Problem 3.5.** *Let $A$ be the set of rational numbers between $0$ and $1$, and let $\{I_n\}$ be a finite collection of open intervals covering $A$. Then $\sum \ell(I_n) \geq 1$.*

*Idea.* If $\{I_n\}$ is a covering of $[0,1]$ then we are done since the length of $[0,1]$ is 1. However, $\{I_n\}$ only covers $A$ and not necessarily covers $[0,1]$. (For example, $\{I_n\} = \left\{\left(-89, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, 64\right)\right\}$ covers $A$ but not $\frac{1}{\sqrt{2}}$.) Hence, it is natural to consider the closure of $A$ and the closure of $I_n$. Now $\{\overline{I_n}\}$ is a (closed) covering of $\overline{A} = [0,1]$.

*Proof.*

$$\begin{aligned}
1 &= m^*[0,1] && \text{(Proposition 3.1)} \\
&= m^*\overline{A} && \text{($A$ is dense in $[0,1]$)} \\
&\leq m^*\left(\overline{\bigcup I_n}\right) && \text{(Proposition 2.10)} \\
&= m^*\left(\bigcup \overline{I_n}\right) && \text{(Proposition 2.10)} \\
&\leq \sum m^*(\overline{I_n}) && \text{(Proposition 3.2)} \\
&= \sum \ell(\overline{I_n}) && \text{(Proposition 3.1)} \\
&= \sum \ell(I_n). && \text{(Definition of length)}
\end{aligned}$$

□

**Supplement.** Exercise about considering the closure. (Exercise 4.52 in the textbook: *T. M. Apostol, Mathematical Analysis, 2nd edition.*) *Assume that $f$ is uniformly continuous on a bounded set $S$ in $\mathbb{R}^n$. Prove that $f$ must be bounded on $S$.*

*Proof.*

(1) Since $f : S \to T$ is uniformly continuous, given any $\varepsilon > 0$, there is $\delta > 0$ such that $d_T(f(x), f(y)) < \varepsilon$ whenever $d_S(x,y) < \delta$. Choose $\varepsilon = 1 > 0$.

(2) For such $\delta > 0$, construct an open covering of $\overline{S} \subseteq \mathbb{R}^n$. Pick a collection $\mathscr{F}$ of open balls $B(a;\delta) \subseteq \mathbb{R}^n$ where $a$ runs over all elements of $S$. $\mathscr{F}$ covers $\overline{S}$ (by the definition of accumulation points). Since $\overline{S}$ is closed and bounded (since $S$ is bounded), $\overline{S}$ is compact So there is a finite subcollection $\mathscr{F}'$ of $\mathscr{F}$ also covers $\overline{S}$, say

$$\mathscr{F}' = \{B(a_1;\delta)), B(a_2;\delta), \ldots, B(a_m;\delta)\}.$$

(3) Given any $x \in S \subseteq \overline{S}$, there is some $a_i \in S$ $(1 \leq i \leq m)$ such that $x \in B(a_i;\delta)$. In such ball, $d_S(x,a_i) < \delta$. By (1), $\|f(x) - f(a_i)\| < 1$, or $\|f(x)\| < 1 + \|f(a_i)\|$. Therefore, for any $x \in S$,

$$\|f(x)\| < 1 + \max_{1 \leq i \leq m} \|f(a_i)\|.$$

$\square$

**Problem 3.6.** *Prove Proposition 5: Given any set $A$ and any $\varepsilon > 0$, there is an open set $O$ such that $A \subseteq O$ and $m^*O \leq m^*A + \varepsilon$. There is a $G \in G_\delta$ such that $m^*G = m^*A$.*

*Proof.*

(1) *Use the definition of the outer measure.* By the definition of $m^*$, for such $\varepsilon > 0$ there exists a countable collection $\{I_n\}$ of open intervals that covers $A$ and

$$m^*A + \varepsilon \geq \sum \ell(I_n).$$

(2) *Construct an open set $O$.* Let $O = \bigcup I_n \supseteq A$ which is the union of any collection of open sets $I_n$. By Proposition 2.7, $O$ is open.

(3) *Show that $m^*O \leq m^*A + \varepsilon$.* By Proposition 3.2 and 3.1,

$$m^*O = m^*\left(\bigcup I_n\right) \leq \sum m^*I_n = \sum \ell(I_n) \leq m^*A + \varepsilon.$$

Therefore, given any set $A$ and any $\varepsilon > 0$, there is an open set $O$ such that $A \subseteq O$ and $m^*O \leq m^*A + \varepsilon$.

(4) *Construct $G \in G_\delta$ in a natural way.* Given any $n \in \mathbb{N}$, there exists an open set $O_n$ such that $O_n \supseteq A$ and $m^*O_n \leq m^*A + \frac{1}{n}$. Let

$$G = \bigcap_{n=1}^{\infty} O_n \in G_\delta.$$

(5) *Show that $m^*G = m^*A$.*

(a) Since $A \subseteq O_n$ for any $n \in \mathbb{N}$, $A \subseteq \bigcap_{n=1}^{\infty} O_n = G$. Thus $m^*A \leq m^*G$.

(b) Since $O_n \supseteq \bigcap_{n=1}^{\infty} O_n = G$ for any $n \in \mathbb{N}$,

$$m^*A + \frac{1}{n} \geq m^*O_n \geq m^*G$$

for any $n \in \mathbb{N}$. Since $n \in \mathbb{N}$ is arbitrary, $m^*A \geq m^*G$.

By (a)(b), $m^*A = m^*G$.

$\square$

**Problem 3.7.** *Prove that $m^*$ is translation invariant.*

*Proof.* Given $E \in \mathfrak{M}$ and $y \in \mathbb{R}$.

(1) $m^*(E+y) \le m^*E$. Let $\{I_n\}$ of open intervals that cover $E$. Then $\{I_n+y\}$ of open intervals that cover $E+y$. Notice that the definition of $m^*$ and $\ell(I_n+y) = \ell(I_n)$, then

$$m^*(E+y) \le \sum \ell(I_n+y) = \sum \ell(I_n).$$

Take the infimum of all such sum $\sum \ell(I_n)$, $m^*(E+y) \le m^*E$.

(2) $m^*(E) \le m^*(E+y)$. Similar to (1).

By (1)(2), $m^*(E+y) = m^*E$, that is, $m^*$ is translation invariant. □

**Problem 3.8.** *Prove that if $m^*A = 0$, then $m^*(A \cup B) = m^*B$.*

*Proof.*

(1) $m^*(A \cup B) \ge m^*B$ since $A \cup B \supseteq B$ and the definition of $m^*$. (Any covering of $A \cup B$ by open intervals is also a covering of $B$ so that the latter infimum is taken over a larger collection than the former.)

(2) $m^*(A \cup B) \le m^*B$. By Proposition 3.2,

$$m^*(A \cup B) \le m^*A + m^*B = 0 + m^*B = m^*B.$$

By (1)(2), $m^*(A \cup B) = m^*B$. □

## Section 3.3: Measurable Sets and Lebesgue Measure

**Problem 3.9.** *Show that if $E$ is a measurable set, then each translate $E+y$ of $E$ is also measurable.*

*Proof.*

(1) *$E$ is measurable if and only if for each set $A$, each $y \in \mathbb{R}$,*

$$m^*(A+y) = m^*((A+y) \cap E) + m^*((A+y) \cap \widetilde{E}).$$

   (a) ($\Longrightarrow$) $E$ is measurable and $A+y$ is a set (for any set $A$ and $y \in \mathbb{R}$).

   (b) ($\Longleftarrow$) $A = (A-y)+y$ for any set $A$ and $y \in \mathbb{R}$.

(2) For any set $E$ and $y \in \mathbb{R}$, $\widetilde{E+y} = \widetilde{E}+y$ by the definition of translation.

(3) For any sets $E_1$, $E_2$ and $y \in \mathbb{R}$, $(E_1 \cap E_2)+y = (E_1+y)+(E_2+y)$ by the definition of translation.

(4) For each set $A$ and $y \in \mathbb{R}$,

$$
\begin{aligned}
& m^*((A+y)\cap(E+y)) + m^*((A+y)\cap(\widetilde{E+y})) \\
&= m^*((A+y)\cap(E+y)) + m^*((A+y)\cap(\widetilde{E}+y)) && ((2)) \\
&= m^*((A\cap E)+y) + m^*((A\cap\widetilde{E})+y) && ((3)) \\
&= m^*(A\cap E) + m^*(A\cap\widetilde{E}) && \text{(Problem 3.7)} \\
&= m^*A && \text{(Measurability of } E\text{)} \\
&= m^*(A+y). && \text{(Problem 3.7)}
\end{aligned}
$$

By (1), $E+y$ is measurable.

□

**Problem 3.10.** *Show that if $E_1$ and $E_2$ are measurable, then*

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = mE_1 + mE_2.$$

*Proof.* Since the collection $\mathfrak{M}$ of measurable sets is a $\sigma$-algebra (Theorem 3.10) and $m$ is countable additive (Proposition 3.13),

$$
\begin{aligned}
m(E_1 \cup E_2) + m(E_1 \cap E_2) &= \Big(m(E_1) + m(E_2 \cap \widetilde{E_1})\Big) + m(E_2 \cap E_1) \\
&= m(E_1) + \Big(m(E_2 \cap \widetilde{E_1}) + m(E_2 \cap E_1)\Big) \\
&= m(E_1) + m(E_2).
\end{aligned}
$$

($E_1$ and $E_2 \cap \widetilde{E_1}$ are disjoint. $E_2 \cap \widetilde{E_1}$ and $E_2 \cap E_1$ are disjoint too.) □

**Problem 3.11.** *Show that the condition $mE_1 < \infty$ is necessary in Proposition 3.14 by giving a decreasing sequence $\langle E_n \rangle$ of measurable sets with $\varnothing = \bigcap E_n$ and $mE_n = \infty$ for each $n$.*

*Proof.* Set

$$E_n = (n, \infty)$$

for each $n \in \mathbb{N}$.

(1) *$\langle E_n \rangle$ is a decreasing sequence of measurable sets.* $E_n \supseteq E_{n+1}$ by definition. Besides, each $E_n$ is measurable by Lemma 3.11.

(2) *$\bigcap E_n = \varnothing$.* For each $x \in \mathbb{R}$, $x \notin E_1$ if $x \leq 1$; $x \notin E_{[x]}$ if $x \geq 1$ where $x \mapsto [x]$ is the floor function.

(3) *$mE_n = \infty$ for each $n$.* The length of each $E_n$ is $\infty$ (Proposition 3.1).

□

**Problem 3.12.** *Let $\langle E_n \rangle$ be a sequence of disjoint measurable sets and $A$ any set. Then $m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(A \cap E_i)$.*

*Proof.*

(1) $A \cap \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty}(A \cap E_i)$ (Problem 1.14).

(2) $m^*\left(\bigcup_{i=1}^{\infty}(A \cap E_i)\right) \leq \sum_{i=1}^{\infty} m^*(A \cap E_i)$ by the subadditivity of $m^*$ (Proposition 3.2).

(3) By Lemma 3.9,

$$m^*\left(\bigcup_{i=1}^{n}(A \cap E_i)\right) = \sum_{i=1}^{n} m^*(A \cap E_i)$$

for any $n \in \mathbb{N}$. Since $\bigcup_{i=1}^{\infty}(A \cap E_i) \supseteq \bigcup_{i=1}^{n}(A \cap E_i)$, $m^*\left(\bigcup_{i=1}^{\infty}(A \cap E_i)\right) \geq m^*\left(\bigcup_{i=1}^{n}(A \cap E_i)\right)$ by the monotonicity of $m^*$. Thus,

$$m^*\left(\bigcup_{i=1}^{\infty}(A \cap E_i)\right) \geq \sum_{i=1}^{n} m^*(A \cap E_i)$$

for any $n \in \mathbb{N}$. Since $\sum_{i=1}^{n} m^*(A \cap E_i)$ is bounded and increasing (by the non-negativity of $m^*$),

$$m^*\left(\bigcup_{i=1}^{\infty}(A \cap E_i)\right) \geq \sum_{i=1}^{\infty} m^*(A \cap E_i).$$

By (2)(3), $m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(A \cap E_i)$. □

## Section 3.4: A Nonmeasurable Set

## Section 3.5: Measurable Functions

## Section 3.6: Littlewood's Three Principles