

Notes on the book:

Atiyah and Macdonald, Introduction to Commutative Algebra

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Contents

Chapter 1: Rings and Ideals	3
Exercise 1.1.	3
Exercise 1.2.	3
Exercise 1.3.	6
Exercise 1.4.	7
Exercise 1.5.	7
Supplement 1.5.1.	9
Exercise 1.6.	10
Exercise 1.7.	11
Exercise 1.8.	11
Exercise 1.9.	12
Exercise 1.10.	12
Exercise 1.11. (Boolean ring)	13
Exercise 1.12.	13
Construction of an algebraic closure of a field (E. Artin)	14
Exercise 1.13.	14
Exercise 1.14.	15
The prime spectrum of a ring	15
Lemma 1.15.1.	15
Exercise 1.15.	16
Exercise 1.16.	17
Exercise 1.17.	19
Exercise 1.18.	23
Exercise 1.19.	24
Exercise 1.20.	25
Exercise 1.21.	26
Exercise 1.22.	29
Exercise 1.23.	30

Chapter 2: Modules	32
Exercise 2.1.	32
Exercise 2.2.	34
Exercise 2.3.	35
Exercise 2.4.	36
Exercise 2.5.	37
Exercise 2.8.	38
Exercise 2.9.	38

Chapter 1: Rings and Ideals

Exercise 1.1.

Let x be a nilpotent element of A . Show that $1 + x$ is a unit of A . Deduce that the sum of a nilpotent element and a unit is a unit.

Proof.

- (1) Suppose $x^m = 0$ for some odd integer $m \geq 0$. Then

$$1 = 1 + x^m = (1 + x)(1 - x + x^2 - \cdots + (-1)^{m-1}x^{m-1}),$$

or $1 + x$ is a unit.

- (2) If u is any unit and x is any nilpotent, $u + x = u \cdot (1 + u^{-1}x)$ is a product of two units (using that $u^{-1}x$ is nilpotent and applying (1)) and hence a unit again.

□

Proof (Proposition 1.9).

- (1) *The nilradical is a subset of the Jacobson radical.*
- (a) The nilradical \mathfrak{N} of A is the intersection of all the prime ideals of A by Proposition 1.8.
 - (b) The Jacobson radical \mathfrak{J} of A is the intersection of all the maximal ideals of A by definition.
- (2) By Proposition 1.9, $x \in \mathfrak{J}$ if and only if $1 - xy$ is a unit in A for all $y \in A$. So $1 + x = 1 - (-x) \cdot 1$ is a unit in A since x is a nilpotent and \mathfrak{J} is an ideal.

□

Exercise 1.2.

Let A be a ring and let $A[x]$ be the ring of polynomials in an indeterminate x , with coefficients in A . Let $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$. Prove that

- (i) f is a unit in $A[x]$ if and only if a_0 is a unit in A and a_1, \dots, a_n are nilpotent. (Hint: If $b_0 + b_1x + \cdots + b_mx^m$ is the inverse of f , prove by induction on r that $a_n^{r+1}b_{m-r} = 0$. Hence show that a_n is nilpotent, and then use Exercise 1.1.)
- (ii) f is nilpotent if and only if a_0, a_1, \dots, a_n are nilpotent.

- (iii) f is a zero-divisor if and only if there exists $a \neq 0$ such that $af = 0$. (Hint: Choose a polynomial $g = b_0 + b_1x + \cdots + b_mx^m$ of least degree m such that $fg = 0$. Then $a_nb_m = 0$, hence $a_ng = 0$ (because a_ng annihilates f and has degree $< m$). Now show by induction that $a_{n-r}g = 0$ ($0 \leq r \leq n$).)
- (iv) f is said to be **primitive** if $(a_0, a_1, \dots, a_n) = (1)$. Prove that if $f, g \in A[x]$, then fg is primitive if and only if f and g are primitive.

Proof of (i).

- (1) (\Leftarrow) holds by Exercise 1.1.
- (2) (\Rightarrow) There exists the inverse g of f , say $g = b_0 + b_1x + \cdots + b_mx^m$ satisfying $1 = fg$. Clearly, $1 = a_0b_0$, or a_0 is a unit in A . Also,

$$\begin{aligned} 0 &= a_nb_m, \\ 0 &= a_nb_{m-1} + a_{n-1}b_m, \\ 0 &= a_nb_{m-2} + a_{n-1}b_{m-1} + a_{n-2}b_m, \\ &\dots \end{aligned}$$

A direct computing shows that

$$\begin{aligned} 0 &= a_n^1b_m, \\ 0 &= a_n(a_nb_{m-1} + a_{n-1}b_m) \\ &= a_n^2b_{m-1} + a_{n-1}a_nb_m \\ &= a_n^2b_{m-1}, \\ 0 &= a_n^2(a_nb_{m-2} + a_{n-1}b_{m-1} + a_{n-2}b_m) \\ &= a_n^3b_{m-2} + a_{n-1}a_n^2b_{m-1} + a_{n-2}a_n^2b_m \\ &= a_n^3b_{m-2}, \\ &\dots \end{aligned}$$

So we might have $a_n^{r+1}b_{m-r} = 0$ for $r = 0, 1, 2, \dots, m$.

- (3) Show that $a_n^{r+1}b_{m-r} = 0$ for $r = 0, 1, 2, \dots, m$ by induction on r .
- (a) As $r = 0$, $a_nb_m = 0$ by comparing the coefficient of $fg = 1$ at x^{n+m} .
- (b) For any $r > 0$, comparing the coefficient of $fg = 1$ at x^{n+m-r} ,

$$0 = a_nb_{m-r} + a_{n-1}b_{m-r+1} + \cdots + a_{n-r}b_m.$$

Multiplying by a_n^r on the both sides,

$$\begin{aligned} 0 &= a_n^{r+1}b_{m-r} + a_{n-1}a_n^rb_{m-r+1} + \cdots + a_{n-r}a_n^rb_m \\ &= a_n^{r+1}b_{m-r}. \end{aligned}$$

by the induction hypothesis.

- (4) a_n is a nilpotent. Putting $r = m$ in $a_n^{r+1}b_{m-r} = 0$ and get $a_n^{m+1}b_0 = 0$. Notice that b_0 is a unit, $a_n^{m+1} = 0$, or a_n is a nilpotent.
- (5) Consider $f - a_n x^n = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$, a polynomial $\in A[x]$ of degree $n-1$. Note that f is a unit and $a_n x^n$ is a nilpotent. By Exercise 1.1, $f - a_n x^n$ is a unit too. Applying the (2)(3)(4) again, a_{n-1} is a nilpotent as $n-1 > 0$, that is, applying descending induction on n then yields the desired property.

□

Proof of (ii).

- (1) (\Leftarrow) holds since the nilradical of any ring is an ideal.
- (2) (\Rightarrow) $f^N = 0$ for some $N > 0$. So $0 = f^N = a_0^N + \cdots + a_n^N x^{nN}$. Compare the coefficient in the lowest term to get $a_0^N = 0$, or a_0 is a nilpotent.
- (3) Note that $f - a_0 = a_1 x + \cdots + a_n x^n \in A[x]$ is nilpotent since f and a_0 are nilpotent. $f - a_0$ is a nilpotent too. Continue the same argument in (2), the result is established.

□

Proof of (iii).

- (1) (\Leftarrow) holds trivially.
- (2) (\Rightarrow) Pick a polynomial $g = b_0 + b_1 x + \cdots + b_m x^m$ of least degree m such that $fg = 0$. Especially, $a_n b_m = 0$.
- (3) Consider

$$\begin{aligned} a_n g &= a_n b_0 + \cdots + a_n b_{m-1} x^{m-1} + a_n b_m x^m \\ &= a_n b_0 + \cdots + a_n b_{m-1} x^{m-1} \end{aligned}$$

(since $a_n b_m = 0$). $a_n g$ is a polynomial over A of having degree strictly less than m . Notice that $f \cdot (a_n g) = a_n \cdot (fg) = 0$. By minimality of m , $a_n g = 0$.

- (4) Induction on the degree n of f .
- (a) As $n = 0$, $f = a_0$. There exists $b_m \neq 0$ such that $b_m f = b_m a_0 = 0$ by (2).
- (b) For any zero-divisor f of degree n , there is a polynomial $g = b_0 + b_1 x + \cdots + b_m x^m$ of least degree m such that $fg = 0$. By (2)(3),

$$\begin{aligned} (f - a_n x^n) \cdot g &= fg - a_n x^n g \\ &= 0 - 0 \\ &= 0. \end{aligned}$$

That is, $f - a_n x^n$ is a zero-divisor of degree $n - 1$. By the induction hypothesis, there exists $b_m \neq 0$ such that $b_m(f - a_n x^n) = 0$. So $b_m f = b_m(f - a_n x^n) + b_m a_n x^n = 0 + 0 = 0$.

(c) By (a)(b), (\implies) holds by mathematical induction.

□

Proof of (iv). Note that

- (1) $f \notin \mathfrak{m}[x]$ for any maximal ideal \mathfrak{m} of A if and only if f is primitive.
- (2) For any maximal ideal \mathfrak{m} of A , A/\mathfrak{m} is a field (or an integral domain).
- (3) $A[x]$ is an integral domain if A is an integral domain.
- (4) $A[x]/\mathfrak{m}[x] \cong (A/\mathfrak{m})[x]$ as a ring isomorphism.

Hence,

$$\begin{aligned}
 f, g : \text{primitive} &\iff f, g \notin \mathfrak{m}[x] \text{ for any maximal ideal } \mathfrak{m} \\
 &\iff f, g \neq 0 \text{ in } (A/\mathfrak{m})[x] \text{ for any maximal ideal } \mathfrak{m} \\
 &\iff fg \neq 0 \text{ in } (A/\mathfrak{m})[x] \text{ for any maximal ideal } \mathfrak{m} \\
 &\iff fg \notin \mathfrak{m}[x] \text{ for any maximal ideal } \mathfrak{m} \\
 &\iff fg : \text{primitive}.
 \end{aligned}$$

□

Exercise 1.3.

Generalize the results of Exercise 1.2 to a polynomial ring $A[x_1, \dots, x_r]$ in several indeterminates.

Generalization. Let

$$f = \sum_{(i)} a_{(i)} x^{(i)} \in A[x_1, \dots, x_r]$$

where $\sum_{(i)}$ is the summation over $(i) = (i_1, \dots, i_r)$ with $i_1 + \dots + i_r = n$. Then

- (i) f is a unit in $A[x_1, \dots, x_r]$ if and only if $a_{(0)}$ is a unit in A and all other $a_{(i)}$ are nilpotent.
- (ii) f is nilpotent if and only if all $a_{(i)}$ are nilpotent.
- (iii) f is a zero-divisor if and only if there exists $a \neq 0$ such that $af = 0$.
- (iv) If $f, g \in A[x_1, \dots, x_r]$, then fg is primitive if and only if f and g are primitive.

Proof. Use the mathematical induction to prove (i)(ii)(iii) and apply the same argument in Exercise 1.2 (iv) to prove (iv). \square

Exercise 1.4.

In the ring $A[x]$, the Jacobson radical is equal to the nilradical.

Proof.

- (1) The nilradical \mathfrak{N} is a subset of the Jacobson radical \mathfrak{J} . It suffices to show that $\mathfrak{J} \subseteq \mathfrak{N}$.

(2)

$$\begin{aligned}
 & f \in \mathfrak{J} \\
 \iff & 1 - fy \text{ is a unit in } A[x] \text{ for all } y \in A[x] && \text{(Proposition 1.9)} \\
 \implies & 1 - xf \text{ is a unit in } A[x] && (y = x) \\
 \implies & \text{All coefficients of } f \text{ are nilpotent} && \text{(Exercise 1.2 (i))} \\
 \implies & f \text{ is nilpotent} && \text{(Exercise 1.2 (ii))} \\
 \implies & f \in \mathfrak{N}.
 \end{aligned}$$

\square

Exercise 1.5.

Let A be a ring and let $A[[x]]$ be the ring of formal power series $f = \sum_{n=0}^{\infty} a_n x^n$ with coefficients in A . Show that

- (i) *f is a unit in $A[[x]]$ if and only if a_0 is a unit in A .*
- (ii) *If f is nilpotent, then a_n is nilpotent for all $n \geq 0$. Is converse true? (See Exercise 7.2.)*
- (iii) *f belongs to the Jacobson radical of $A[[x]]$ if and only if a_0 belongs to the Jacobson radical of A .*
- (iv) *The contraction of a maximal ideal \mathfrak{m} of $A[[x]]$ is a maximal ideal of A , and \mathfrak{m} is generated by \mathfrak{m}^c and x .*
- (v) *Every prime ideal of A is the contraction of a prime ideal of $A[[x]]$.*

Proof of (i).

- (1) (\implies) If $g = \sum_{n=0}^{\infty} b_n x^n$ is an inverse of f , then $fg = 1$ implies that $a_0 b_0 = 1$ so that a_0 is a unit in A .
- (2) (\impliedby) Our goal is to find $g = \sum_{n=0}^{\infty} b_n x^n$ such that the Cauchy product $fg = \sum_{n=0}^{\infty} c_n x^n$ is equal to $1 \in A[x]$. Here $c_n = \sum_{r=0}^n a_r b_{n-r}$. By the assumption we have that $c_0 = 1$ and $c_1 = c_2 = \dots = 0$. Hence

$$\begin{aligned} b_0 &= a_0^{-1} \\ b_1 &= -a_0^{-1} a_1 b_0 \\ &\dots \\ b_n &= a_0^{-1} \sum_{r=1}^n a_r b_{n-r} \\ &\dots \end{aligned}$$

by induction.

□

Proof of (ii).

- (1) The proof is the same as Exercise 1.2 (ii).
- (2) The converse is true if A is Noetherian (by Exercise 7.2).
- (3) The converse is not always true. Take

$$A = \mathbb{F}_2[t, t^{-2}, t^{-2^2}, \dots]/(t)$$

and

$$f(x) = \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} t^{-2^n} x^n \in A[x].$$

Note that A is not Noetherian and all a_n are nilpotent in A . To show f is not nilpotent in $A[x]$, it suffices to show that f^{2^r} is not equal to zero for all positive integers r .

- (4) Note that \mathbb{F}_2 is a field of characteristic 2. So

$$f^{2^r} = \sum_{n=1}^{\infty} a_n^{2^r} x^n = \sum_{n=1}^{\infty} t^{2^{r-n}} x^n = \sum_{n=r+1}^{\infty} t^{2^{r-n}} x^n \neq 0$$

for all r .

□

Proof of (iii).

$$\begin{aligned}
& f \text{ in the Jacobson radical of } A[[x]] \\
& \iff 1 - fg \in A[[x]] \text{ is unit for all } g = \sum_{n=0}^{\infty} b_n x^n \in A[[x]] \quad (\text{Proposition 1.9}) \\
& \iff 1 - a_0 b_0 \in A \text{ is unit for all } b_0 \in A \quad ((i)) \\
& \iff a_0 \text{ belongs to the Jacobson radical of } A. \quad (\text{Proposition 1.9})
\end{aligned}$$

□

Proof of (iv).

- (1) Note that $x = 0 + x$ belongs to the Jacobson radical of $A[[x]]$ since 0 obviously belongs to the Jacobson radical of A (by (iii)).
- (2) So $x \in \mathfrak{m}$ or $(x) \subseteq \mathfrak{m}$ for any maximal ideal in $A[[x]]$. So it is clear that $\mathfrak{m} = \mathfrak{m}^c + (x)$.
- (3) Moreover, \mathfrak{m}^c is a maximal ideal since $A/\mathfrak{m}^c \cong A[[x]]/\mathfrak{m}$ is a field.

□

Proof of (v).

- (1) Similar to (iv). Suppose \mathfrak{p} is a prime ideal of A . Let $\mathfrak{q} = \mathfrak{p} + (x)$ be an ideal of $A[[x]]$.
- (2) $\mathfrak{q}^c = \mathfrak{p}$ clearly. Besides, \mathfrak{q}^c is a prime ideal since

$$A[[x]]/\mathfrak{q}^c \cong A/\mathfrak{p}$$

is an integral domain.

□

Supplement 1.5.1.

(Exercise II.1.2 in the textbook: Jrgen Neukirch, *Algebraic Number Theory*.) A p -adic integer $a = a_0 + a_1 p + a_2 p^2 + \cdots$ is a unit in the ring \mathbb{Z}_p if and only if $a_0 \neq 0$.

Proof.

- (1) (\implies) If $b = b_0 + b_1 p + b_2 p^2 + \cdots$ is an inverse of a , then $ab = 1$ implies that $a_0 b_0 = 1$ so that a_0 is a unit in $\mathbb{Z}/p\mathbb{Z}$ or $a_0 \neq 0$.

(2) (\Leftarrow) Our goal is to find

$$b = b_0 + b_1p + b_2p^2 + \cdots \in \mathbb{Z}_p$$

such that the Cauchy product

$$ab = c_0 + c_1p + c_2p^2 + \cdots$$

is equal to $1 \in \mathbb{Z}_p$. Here $c_n = \sum_{\nu=0}^n a_\nu b_{n-\nu}$. By the assumption we have that $c_0 = 1$ and $c_1 = c_2 = \cdots = 0$. Hence

$$b_0 = a_0^{-1}$$

$$b_1 = -a_0^{-1}a_1b_0$$

$$\dots$$

$$b_n = a_0^{-1} \sum_{\nu=1}^n a_\nu b_{n-\nu}$$

$$\dots$$

by induction.

□

Exercise 1.6.

A ring A is such that every ideal not contained in the nilradical contains a nonzero idempotent (that is, an element e such that $e^2 = e \neq 0$). Prove that the nilradical and Jacobson radical of A are equal.

Proof.

(1) $\mathfrak{N} \subseteq \mathfrak{J}$ clearly.

(2) Since

$$\begin{aligned} a \notin \mathfrak{N} &\implies (a) \not\subseteq \mathfrak{N} \\ &\implies \text{there exists a nonzero idempotent } e \in (a) \\ &\implies e = ar \text{ for some } r \in A \\ &\implies 0 = e - e^2 = e(1 - e) = ar(1 - ar) \\ &\implies 1 - ar \text{ is a zero-divisor, not a unit} \\ &\implies a \notin \mathfrak{J}, \end{aligned} \tag{Proposition 1.9}$$

we have $\mathfrak{J} \subseteq \mathfrak{N}$.

□

Exercise 1.7.

Let A be a ring in which every element satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal in A is maximal.

Proof. It suffices to show that for any prime ideal \mathfrak{p} in A , A/\mathfrak{p} is a field.

- (1) Take any $0 \neq \bar{x} \in A/\mathfrak{p}$, which is represented by $x \in A - \mathfrak{p}$. By assumption there exists $n \geq 2$ such that $x^n = x$. So $\bar{x}^n = \bar{x}$ or $\bar{x}(\bar{x}^{n-1} - 1) = 0$.
- (2) Since \mathfrak{p} is prime, A/\mathfrak{p} is an integral domain. That is, $\bar{x} = 0$ (impossible) or $\bar{x}^{n-1} - 1 = 0$. Write $\bar{x} \cdot \bar{x}^{n-2} = 1$ in A/\mathfrak{p} . So \bar{x}^{n-2} is an inverse of $\bar{x} \neq 0$ in A/\mathfrak{p} , which implies that A/\mathfrak{p} is a field (since \bar{x} is arbitrary).
- (3) A/\mathfrak{p} is a field if and only if \mathfrak{p} is maximal.

□

Exercise 1.8.

Let A be a ring $\neq 0$. Show that the set of prime ideals of A has minimal elements with respect to inclusion.

Similar to Theorem 1.3.

Proof (Zorn's Lemma).

- (1) Let Σ be the set of all prime ideals of A .
- (2) Order Σ by \supseteq , that is, $\mathfrak{p} \leq \mathfrak{q}$ if $\mathfrak{p} \supseteq \mathfrak{q}$.
- (3) Σ is not empty, since every ring $A \neq 0$ has at least one maximal ideal (or prime ideal) (Theorem 1.3).
- (4) To apply Zorn's lemma we must show that every chain in Σ has a lower bound in Σ ; let then (\mathfrak{p}_α) be a chain of prime ideals in Σ , so that for each pair of indices α, β we have either $\mathfrak{p}_\alpha \subseteq \mathfrak{p}_\beta$ or $\mathfrak{p}_\beta \subseteq \mathfrak{p}_\alpha$. Let $\mathfrak{p} = \bigcap_\alpha \mathfrak{p}_\alpha$.
- (5) Show that \mathfrak{p} is a prime ideal. Clearly \mathfrak{p} is an ideal. Given any $xy \in \mathfrak{p}$ and $x \notin \mathfrak{p}$. So xy is in all prime ideals \mathfrak{p}_α . By assumption $x \notin \mathfrak{p}$, there is some β such that $x \notin \mathfrak{p}_\beta$, or $x \notin \mathfrak{p}_\alpha$ whenever $\alpha \geq \beta$. So $y \in \mathfrak{p}_\alpha$ whenever $\alpha \geq \beta$. Since $y \in \mathfrak{p}_\beta$, $y \in \mathfrak{p}_\gamma$ whenever $\beta \geq \gamma$. Therefore, $y \in \mathfrak{p}_\alpha$ for all α , or $y \in \mathfrak{p}$, or \mathfrak{p} is prime.

□

Exercise 1.9.

Let \mathfrak{a} be an ideal $\neq (1)$ in a ring A . Show that $\mathfrak{a} = r(\mathfrak{a}) \iff \mathfrak{a}$ is an intersection of prime ideals.

Proof.

- (1) (\implies) . By Proposition 1.14, $\mathfrak{a} = r(\mathfrak{a})$ is the intersection of the prime ideals which contain \mathfrak{a} .
- (2) (\impliedby) .

$$\begin{aligned}
 \mathfrak{a} &= \bigcap \{\mathfrak{p} \in \text{some subset of } \text{Spec}(A)\} \\
 &= \bigcap \{\mathfrak{p} \in \text{some subset of } \text{Spec}(A) : \mathfrak{p} \supseteq \mathfrak{a}\} \\
 &\supseteq \bigcap \{\mathfrak{p} \in \text{Spec}(A) : \mathfrak{p} \supseteq \mathfrak{a}\} \\
 &= r(\mathfrak{a}) \\
 &\supseteq \mathfrak{a}.
 \end{aligned}$$

□

Exercise 1.10.

Let A be a ring, \mathfrak{N} its nilradical. Show the following are equivalent:

- (i) A has exactly one prime ideal;
- (ii) every element of A is either a unit or nilpotent;
- (iii) A/\mathfrak{N} is a field.

Proof.

$$\begin{aligned}
 &A/\mathfrak{N} \text{ is a field} \\
 \implies &\mathfrak{N} \text{ is a maximal ideal} \\
 \implies &\mathfrak{p} = \mathfrak{N} \text{ for every prime ideal } \mathfrak{p} && (\text{Proposition 1.8}) \\
 \implies &A \text{ has exactly one prime ideal } \mathfrak{p} \\
 \implies &\mathfrak{p} = \mathfrak{N} \\
 \implies &A \text{ has exactly one maximal ideal } \mathfrak{p} \\
 \implies &\text{Given any } a \in A, a \text{ is a unit or } a \in \mathfrak{p} = \mathfrak{N}. && (\text{Corollary 1.5}) \\
 \implies &A/\mathfrak{N} \text{ is a field.}
 \end{aligned}$$

□

Exercise 1.11. (Boolean ring)

A ring A is **Boolean** if $x^2 = x$ for all $x \in A$. In a Boolean ring A , show that

- (i) $2x = 0$ for all $x \in A$;
- (ii) every prime ideal \mathfrak{p} is maximal, and A/\mathfrak{p} is a field with two elements;
- (iii) every finitely generated ideal in A is principal.

Proof of (i). Note that $2x = x + x = (x + x)^2 = (2x)^2 = 4x^2 = 4x$. So $2x = 0$. \square

Proof of (ii). Same as Exercise 1.7 with $n = 2$. \square

Proof of (iii).

- (1) By induction, it suffices to show that if $\mathfrak{a} = (x, y)$ is an ideal in A , then $\mathfrak{a} = (z)$ for some $z \in A$.
- (2) Take $z = x + y + xy$. $(z) \subseteq \mathfrak{a}$ obviously.
- (3) Conversely, note that

$$x = x^2 = x(z - y - xy) = xz - \overbrace{xy}^{=2xy=0} - \underbrace{x^2y}_{=xy} = xz \in (z).$$

Also $y \in (z)$ similarly. So $\mathfrak{a} \subseteq (z)$ and thus $\mathfrak{a} = (z)$ is principal.

\square

Exercise 1.12.

A local ring contains no idempotent $\neq 0, 1$.

Proof.

- (1) If e is an idempotent $\neq 0, 1$ in a local ring A with the maximal ideal \mathfrak{m} , then by definition $0 = e(1 - e)$ shows that both $e \neq 0$ and $1 - e \neq 0$ are not unit.
- (2) Thus $e \in \mathfrak{m}$ and $1 - e \in \mathfrak{m}$. So $1 = (1 - e) + e$ is a unit in \mathfrak{m} , which is absurd.

\square

Construction of an algebraic closure of a field (E. Artin)

Exercise 1.13.

Let K be a field and let Σ be the set of all irreducible monic polynomials f in one indeterminate with coefficients in K . Let A be the polynomial ring over K generated by indeterminates x_f , one for each $f \in \Sigma$. Let \mathfrak{a} be the ideal of A generated by the polynomials $f(x_f)$ for all $f \in \Sigma$. Show that $\mathfrak{a} \neq (1)$.

Let \mathfrak{m} be a maximal ideal of A containing \mathfrak{a} and let $K_1 = A/\mathfrak{m}$. Then K_1 is an extension field of K in which each $f \in \Sigma$ has a root. Repeat the construction with K_1 in place of K , obtaining a field K_2 , and so on. Let $L = \bigcup_{n=1}^{\infty} K_n$. Then L is a field in which each $f \in \Sigma$ splits completely into linear factors. Let \overline{K} be the set of all elements of L which are algebraic over K . Then \overline{K} is an algebraic closure of K .

Proof.

- (1) Show that $\mathfrak{a} \neq (1)$. (Reductio ad absurdum) If $\mathfrak{a} = (1)$, then we can write

$$1 = \sum_{i=1}^n g_i(x) f_i(x_{f_i}) \in A$$

where $x = (x_{f_1}, \dots, x_{f_n}, x_{g_1}, \dots, x_{g_r})$ is a tuple with finitely many indeterminates. It is possible since it is a finite sum.

- (2) Let L be an algebraic extension of K such that each f_i has a root $a_i \in L$ ($i = 1, \dots, n$).
- (3) Take $x = (a_1, \dots, a_n, 0, \dots, 0)$ in the equation $1 = \sum_{i=1}^n g_i(x) f_i(x_{f_i})$ to get

$$\begin{aligned} 1 &= \sum_{i=1}^n g_i(a_1, \dots, a_n, 0, \dots, 0) f_i(a_i) \\ &= \sum_{i=1}^n g_i(a_1, \dots, a_n, 0, \dots, 0) \cdot 0 \\ &= 0, \end{aligned}$$

which is absurd.

□

Exercise 1.14.

In a ring A , let Σ be the set of all ideals in which every element is a zero-divisor. Show that the set Σ has maximal elements and that every maximal element of Σ is a prime ideal. Hence the set of zero-divisors in A is a union of prime ideals.

Proof.

- (1) Suppose $1 \neq 0$.
- (2) Show that the set Σ has maximal elements. Order Σ by inclusion. Σ is not empty, since $0 \in \Sigma$. To apply Zorn's lemma we must show that every chain in Σ has an upper bound in Σ ; let then (\mathfrak{a}_α) be a chain of ideals in Σ , so that for each pair of indices α, β we have either $\mathfrak{a}_\alpha \subseteq \mathfrak{a}_\beta$ or $\mathfrak{a}_\beta \subseteq \mathfrak{a}_\alpha$.
- (3) Let $\mathfrak{a} = \bigcup_\alpha \mathfrak{a}_\alpha$. Then \mathfrak{a} is an ideal and every element of \mathfrak{a} is a zero-divisor. Hence $\mathfrak{a} \in \Sigma$, and \mathfrak{a} is an upper bound of the chain. Hence by Zorn's lemma, Σ has maximal elements.
- (4) Show that every maximal element of Σ is a prime ideal. Let \mathfrak{p} be a maximal element in Σ . Suppose $x, y \notin \mathfrak{p}$. Then there are non-zero-divisors in $\mathfrak{p} + (x)$ and $\mathfrak{p} + (y)$, and their product is an element of $\mathfrak{p} + (xy)$ that is again a non-zero-divisor. So $xy \notin \mathfrak{p}$.
- (5) Hence the set of zero-divisors in A is a union of prime ideals (by the construction in (2) and the result of (4)).

□

The prime spectrum of a ring**Lemma 1.15.1.**

For any $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$, $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$.

Proof.

- (1) If $\mathfrak{p} \supseteq \mathfrak{a}$. We are done.
- (2) If $\mathfrak{p} \not\supseteq \mathfrak{a}$, there exists $a \in \mathfrak{a} - \mathfrak{p}$. So for any $b \in \mathfrak{b}$, $b \in \mathfrak{p}$ since $ab \in \mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$ and \mathfrak{p} is a prime ideal, that is, $\mathfrak{p} \supseteq \mathfrak{b}$.

By (1)(2), $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$. □

Exercise 1.15.

Let A be a ring and let X be the set of all prime ideals of A . For each subset E of A , let $V(E)$ denote the set of all prime ideals of A which contain E . Prove that

(i) if \mathfrak{a} is the ideal generated by E , then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$.

(ii) $V(0) = X$, $V(1) = \emptyset$.

(iii) if $(E_i)_{i \in I}$ is any family of subsets of A , then

$$V\left(\bigcup_{i \in I} E_i\right) = \bigcap_{i \in I} V(E_i).$$

(iv) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ for any ideals \mathfrak{a} , \mathfrak{b} of A .

The results show that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. The resulting topology is called the **Zariski topology**. The topological space X is called the **prime spectrum** of A , and is written $\text{Spec}(A)$.

Note that if $E_1 \subseteq E_2$, then $V(E_1) \supseteq V(E_2)$.

Proof of (i).

(1) Show that $V(E) = V(\mathfrak{a})$.

(a) Show that $V(E) \subseteq V(\mathfrak{a})$. Given any $\mathfrak{p} \in V(E)$, $\mathfrak{p} \supseteq E$. For any $a \in \mathfrak{a}$, since \mathfrak{a} is generated by E , we can write a as a finite sum $a = \sum \alpha\beta$ where $\alpha \in A$ and $\beta \in E$. Since $E \subseteq \mathfrak{p}$, all $\beta \in \mathfrak{p}$. Since \mathfrak{p} is an ideal, $a = \sum \alpha\beta \in \mathfrak{p}$. That is, $\mathfrak{p} \supseteq \mathfrak{a}$, or $\mathfrak{p} \in V(\mathfrak{a})$.

(b) $V(E) \supseteq V(\mathfrak{a})$ since $\mathfrak{a} \supseteq E$.

(2) Show that $V(\mathfrak{a}) = V(r(\mathfrak{a}))$.

(a) Show that $V(\mathfrak{a}) \subseteq V(r(\mathfrak{a}))$. Given any $\mathfrak{p} \in V(\mathfrak{a})$,

$$\begin{aligned} \mathfrak{p} \in V(\mathfrak{a}) &\implies \mathfrak{p} \supseteq \mathfrak{a} \\ &\implies \mathfrak{p} \supseteq \text{the intersection of the prime ideals } \mathfrak{p} \supseteq \mathfrak{a} \\ &\implies \mathfrak{p} \supseteq r(\mathfrak{a}) \text{ (by Proposition 1.14)} \\ &\implies \mathfrak{p} \in V(r(\mathfrak{a})). \end{aligned}$$

(b) $V(\mathfrak{a}) \supseteq V(r(\mathfrak{a}))$ since $r(\mathfrak{a}) \supseteq \mathfrak{a}$.

□

Proof of (ii).

- (1) $V(1) = \emptyset$ since no prime ideal contains 1 by definition.
- (2) $V(0) = X$ since 0 is in every ideal (especially in every prime ideal).

□

Proof of (iii).

$$\begin{aligned}
\mathfrak{p} \in V\left(\bigcup_{i \in I} E_i\right) &\iff \mathfrak{p} \supseteq \bigcup_{i \in I} E_i \\
&\iff \mathfrak{p} \supseteq E_i \text{ for all } i \in I \\
&\iff \mathfrak{p} \in V(E_i) \text{ for all } i \in I \\
&\iff \mathfrak{p} \in \bigcap_{i \in I} V(E_i).
\end{aligned}$$

□

Proof of (iv).

- (1) *Show that $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$.*
 - (a) $V(\mathfrak{a} \cap \mathfrak{b}) \subseteq V(\mathfrak{a}\mathfrak{b})$ since $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b}$.
 - (b) *Show that $V(\mathfrak{a} \cap \mathfrak{b}) \supseteq V(\mathfrak{a}\mathfrak{b})$.* Given any $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$, $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$. By Lemma 15.1.1, $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$. Notice that $\mathfrak{a} \supseteq \mathfrak{a} \cap \mathfrak{b}$ and $\mathfrak{b} \supseteq \mathfrak{a} \cap \mathfrak{b}$. In any case, $\mathfrak{p} \supseteq \mathfrak{a} \cap \mathfrak{b}$, $\mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b})$.
- (2) *Show that $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.*
 - (a) *Show that $V(\mathfrak{a}\mathfrak{b}) \subseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$.* Given any $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$, $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$. By Lemma 15.1.1, $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$, $\mathfrak{p} \in V(\mathfrak{a})$ or $\mathfrak{p} \in V(\mathfrak{b})$, $\mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$.
 - (b) *Show that $V(\mathfrak{a}\mathfrak{b}) \supseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$.* Given any $\mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$, $\mathfrak{p} \in V(\mathfrak{a})$ or $\mathfrak{p} \in V(\mathfrak{b})$, $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$. Notice that $\mathfrak{a} \supseteq \mathfrak{a}\mathfrak{b}$ and $\mathfrak{b} \supseteq \mathfrak{a}\mathfrak{b}$. In any cases, $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$, or $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$.

□

Exercise 1.16.

Draw pictures of $\text{Spec}(\mathbb{Z})$, $\text{Spec}(\mathbb{R})$, $\text{Spec}(\mathbb{C}[x])$, $\text{Spec}(\mathbb{R}[x])$, $\text{Spec}(\mathbb{Z}[x])$.

Proof.

- (1) *Show that $\text{Spec}(\mathbb{Z}) = \{(0)\} \cup \{(p) : p \text{ is a rational prime}\}$.* Note that \mathbb{Z} is a PID. So all non-trivial prime ideals are of the form (π) where π are irreducible.

- (2) Show that $\text{Spec}(\mathbb{R}) = \{(0)\}$. Note that \mathbb{R} is a field.
- (3) Show that $\text{Spec}(\mathbb{C}[x]) = \{(0)\} \cup \{(x - z) : z \in \mathbb{C}\}$. Note that $\mathbb{C}[x]$ is a PID and \mathbb{C} is algebraically closed. Hence all non-trivial prime ideals are of the form $(x - z)$ where $z \in \mathbb{C}$.
- (4) Show that $\text{Spec}(\mathbb{R}[x])$ are
- (i) (0) .
 - (ii) $\{(x - r) : r \in \mathbb{R}\}$.
 - (iii) $\{(x - z)(x - \bar{z}) : z \in \mathbb{C}, \text{Im}(z) > 0\}$.

Here is the proof.

- (a) Note that $\mathbb{R}[x]$ is a PID and all non-trivial prime ideals are of the form (f) where f are irreducible. Might assume f is monic. By the fundamental theorem of algebra, f has a root $z \in \mathbb{C}$.
 - (b) The case $r := z \in \mathbb{R}$. $x - r$ is a factor of f . Hence $f = x - r$.
 - (c) The case $z \in \mathbb{C} \setminus \mathbb{R}$. Since the conjugate of f is also in $\mathbb{R}[x]$, \bar{z} is also a root of f . So $(x - z)(x - \bar{z}) \in \mathbb{R}[x]$ is an irreducible factor of f . Hence $f = (x - z)(x - \bar{z})$ by the irreducibility of f .
- (5) Show that $\text{Spec}(\mathbb{Z}[x])$ are
- (i) (0) .
 - (ii) (p) where p are rational primes.
 - (iii) (f) where $f \in \mathbb{Z}[x]$ are irreducible.
 - (iv) (p, f) where p are rational primes and $f \in \mathbb{Z}[x]$ are irreducible when viewed in $\mathbb{F}_p[x]$.

Before giving a proof, it is worth taking a look at the book: *David Mumford, The red book of varieties and schemes*.

- (a) Let $\phi : \mathbb{Z} \rightarrow \mathbb{Z}[x]$ be the natural inclusion map. Hence $\phi^* : \text{Spec}(\mathbb{Z}[x]) \rightarrow \text{Spec}(\mathbb{Z})$ is continuous (Exercise 1.21). Suppose $\mathfrak{P} \in \text{Spec}(\mathbb{Z}[x])$, then $\phi^*(\mathfrak{P}) = (0)$ or (p) where p is a rational prime.
- (b) The case $\phi^*(\mathfrak{P}) = (0)$. A non-trivial prime ideal \mathfrak{P} must be generated by a set of nonconstant polynomials which, since \mathfrak{P} is prime, may be assumed to be irreducible in $\mathbb{Z}[x]$. Note that $\mathbb{Z}[x]$ is not a PID.
- (c) By Gauss' lemma, these polynomials are also irreducible in $\mathbb{Q}[x]$. Since $\mathbb{Q}[x]$ is a Euclidean domain, if there are at least two distinct irreducible polynomials f, g generating \mathfrak{P} , then $1 = af + bg$ for some $a, b \in \mathbb{Q}[x]$. Clearing all denominators to get that $n = \tilde{a}f + \tilde{b}g$ for some $\tilde{a}, \tilde{b} \in \mathbb{Z}[x]$ and some $n \in \mathbb{Z} \setminus \{0\}$, contrary to $\phi^*(\mathfrak{P}) = (0)$. Therefore, $\mathfrak{P} = (f)$ for one irreducible polynomial $f \in \mathbb{Z}[x]$.

(d) The case $\phi^*(\mathfrak{P}) = (p)$ where p is a rational prime. Note that

$$\begin{aligned}\mathbb{Z}[x]/\mathfrak{P} &\cong (\mathbb{Z}[x]/p\mathbb{Z}[x])/(\mathfrak{P}/p\mathbb{Z}[x]) \\ &\cong \underbrace{(\mathbb{Z}/p\mathbb{Z})[x]}_{:=\mathbb{F}_p}/(\mathfrak{P}/p\mathbb{Z}[x])\end{aligned}$$

is an integral domain (since \mathfrak{P} is prime). So $\mathfrak{P}/p\mathbb{Z}[x]$ is a prime ideal in $\mathbb{F}_p[x]$. Note that $\mathbb{F}_p[x]$ is a PID and all non-trivial prime ideals are of the form (f) where f are irreducible.

(e) As $\mathfrak{P}/p\mathbb{Z}[x] = (0)$, $\mathfrak{P} = p\mathbb{Z}[x] = (p) \in \mathbb{Z}[x]$.

(f) As $\mathfrak{P}/p\mathbb{Z}[x] = (f)$ where $f \in \mathbb{Z}[x]$ is irreducible when viewed in $\mathbb{F}_p[x]$, $\mathfrak{P} = (p, f)$.

□

Exercise 1.17.

For each $f \in A$, let X_f denote the complement of $V(f)$ in $X = \text{Spec}(A)$. The sets X_f are open. Show that they form a basis of open sets for the Zariski topology, and that

- (i) $X_f \cap X_g = X_{fg}$.
- (ii) $X_f = \emptyset \iff f$ is nilpotent.
- (iii) $X_f = X \iff f$ is a unit.
- (iv) $X_f = X_g \iff r((f)) = r((g))$.
- (v) X is quasi-compact (compact), that is, every open covering of X has a finite subcovering.
- (vi) More generally, each X_f is quasi-compact.
- (vii) An open subset of X is quasi-compact if and only if it is a finite union of sets X_f .

The sets X_f are called basic open sets of $X = \text{Spec}(A)$.

(Hint: To prove (v), remark that it is enough to consider a covering of X by basic open sets X_{f_i} ($i \in I$). Show that the f_i generate the unit ideal and hence that there is an equation of the form

$$1 = \sum_{i \in J} g_i f_i \quad (g_i \in A)$$

where J is some finite subset of I . Then the X_{f_i} ($i \in J$) cover X .)

Proof of basis. It is equivalent to Exercise 1.15 (iii). Given any open set O in X . Write $O = X - V(\mathfrak{a})$ for some ideal \mathfrak{a} of A . Since

$$V(\mathfrak{a}) = V\left(\bigcup_{f \in \mathfrak{a}} (f)\right) = \bigcap_{f \in \mathfrak{a}} V(f),$$

we have

$$O = X - V(\mathfrak{a}) = X - \bigcap_{f \in \mathfrak{a}} V(f) = \bigcup_{f \in \mathfrak{a}} (X - V(f)) = \bigcup_{f \in \mathfrak{a}} X_f,$$

or any open set is a union of basic open sets. \square

Proof of (i). $X_f \cap X_g = X_{fg} \iff V(f) \cup V(g) = V(fg)$ holds by Exercise 1.15 (iv). \square

Proof of (ii).

$$\begin{aligned} X_f = \emptyset &\iff V(f) = X \\ &\iff f \in \mathfrak{p} \text{ for all prime ideal } \mathfrak{p} \text{ of } A \\ &\iff f \in \mathfrak{N}, \text{ the nilradical of } A \text{ (Proposition 1.8)} \\ &\iff f \text{ is nilpotent (Proposition 1.7)} \end{aligned}$$

\square

Proof of (ii) (Using (iv)).

$$\begin{aligned} X_f = \emptyset &\iff X_f = X_0 && \text{(Exercise 15(ii))} \\ &\iff r(f) = r(0) && \text{((iv))} \\ &\iff f \in r(f) = r(0) \\ &\iff f^m = 0 \text{ for some } m > 0 \\ &\iff f \text{ is nilpotent} \end{aligned}$$

\square

Proof of (iii).

$$\begin{aligned} X_f = X &\iff V(f) = \emptyset \\ &\iff f \notin \mathfrak{p} \text{ for all prime ideal } \mathfrak{p} \text{ of } A \\ &\iff f \text{ is unit (Corollary 1.5)} \end{aligned}$$

\square

Proof of (iii)(Using (iv)).

$$\begin{aligned}
X_f = X &\iff X_f = X_1 && \text{(Exercise 15(ii))} \\
&\iff r(f) = r(1) && \text{((iv))} \\
&\iff f \in r(f) = r(1) \\
&\iff f^m = 1 \text{ for some } m > 0 \\
&\iff f \text{ is unit}
\end{aligned}$$

□

Proof of (iv).

(1) *Show that $X_f \subseteq X_g \iff r((f)) \subseteq r((g))$. Actually,*

$$\begin{aligned}
X_f \subseteq X_g &\implies V(f) \supseteq V(g) \\
&\implies \{\mathfrak{p} \in \text{Spec}(A) : \mathfrak{p} \supseteq (f)\} \supseteq \{\mathfrak{p} \in \text{Spec}(A) : \mathfrak{p} \supseteq (g)\} \\
&\implies \bigcap_{(f) \subseteq \mathfrak{p} \in \text{Spec}(A)} \mathfrak{p} \subseteq \bigcap_{(g) \subseteq \mathfrak{p} \in \text{Spec}(A)} \mathfrak{p} \\
&\stackrel{1.14}{\implies} r(f) \subseteq r(g) \\
&\implies V(r(f)) \supseteq V(r(g)) \\
&\implies V(f) \supseteq V(g) \\
&\implies X_f \subseteq X_g.
\end{aligned}$$

(2) By (1),

$$\begin{aligned}
X_f \subseteq X_g &\iff r((f)) \subseteq r((g)), \\
X_f \supseteq X_g &\iff r((f)) \supseteq r((g)).
\end{aligned}$$

Hence,

$$X_f = X_g \iff r((f)) = r((g)).$$

□

Proof of (v). Notice that it is enough to consider a covering of X by basic open sets X_{f_i} ($i \in I$).

(1) Since X is covered by $X_{f_i} (i \in I)$,

$$\begin{aligned}
X = \bigcup_{i \in I} X_{f_i} &\implies X - V(1) = \bigcup_{i \in I} (X - V(f_i)) \\
&\implies V(1) = \bigcap_{i \in I} V(f_i) \\
&\implies V(1) = V\left(\sum_{i \in I} f_i\right) \\
&\implies r(1) = r\left(\sum_{i \in I} f_i\right).
\end{aligned}$$

Hence, $1 \in r(1) = r\left(\sum_{i \in I} f_i\right)$ can be expressed as

$$1 = 1^m = \sum_{j \in J} g_j f_j$$

where J is a finite subset of I and $g_j \in A$. That is, $(1) = \sum_{j \in J} f_j$.

(2) Hence, $V(1) = V\left(\sum_{j \in J} f_j\right)$. Therefore, X is covered by finite subcovering $\{X_{f_j}\} (j \in J)$.

□

Proof of (v)(Using (vi)). Since $X = X_1$, X is quasi-compact by (vi). □

Proof of (vi). Notice that it is enough to consider a covering of X_f by basic open sets $X_{f_i} (i \in I)$.

(1) Since X_f is covered by $X_{f_i} (i \in I)$,

$$\begin{aligned}
X_f = \bigcup_{i \in I} X_{f_i} &\implies X - V(f) = \bigcup_{i \in I} (X - V(f_i)) \\
&\implies V(f) = \bigcap_{i \in I} V(f_i) \\
&\implies V(f) = V\left(\sum_{i \in I} f_i\right) \\
&\implies r(f) = r\left(\sum_{i \in I} f_i\right).
\end{aligned}$$

Hence, $f \in r(f) = r\left(\sum_{i \in I} f_i\right)$ can be expressed as

$$f^m = \sum_{j \in J} g_j f_j$$

where J is a finite subset of I and $g_j \in A$. That is, $f^m \in \sum_{j \in J} f_j$.

(2) Show that $V\left(\sum_{j \in J} f_j\right) = V(f)$.

(a) (\subseteq) For any prime ideal $\mathfrak{p} \supseteq \sum_{j \in J} f_j$, $f^m \in \mathfrak{p}$ or $f \in \mathfrak{p}$ (since \mathfrak{p} is prime). So $\mathfrak{p} \supseteq (f)$, or $V\left(\sum_{j \in J} f_j\right) \subseteq V(f)$.

(b) (\supseteq)

$$\sum_{j \in J} f_j \subseteq \sum_{i \in I} f_i \implies V\left(\sum_{j \in J} f_j\right) \supseteq V\left(\sum_{i \in I} f_i\right) = V(f).$$

(3) Therefore, X_f is covered by finite subcovering $\{X_{f_j}\}(j \in J)$.

□

Proof of (vi) (Using (v)). Exercise 3.21 (i) shows that X_f is the spectrum of A_f . By (v), X_f is quasi-compact. □

Proof of (vii).

(1) (\implies) Given an open subset O . Since X_f form a basis of open sets,

$$O = \bigcup_{f \in \mathfrak{a}} X_f \text{ for some ideal } \mathfrak{a} \text{ of } A$$

Especially, $\{X_f\}_{f \in \mathfrak{a}}$ is an open covering of O . Since O is quasi-compact, there exists a finite subcovering $\{X_f\}_{f \in J}$ of O , where J is a finite subset of \mathfrak{a} (as a set). That is, $O = \bigcup_{f \in J} X_f$ is a finite union of sets X_f .

(2) (\impliedby) Since X_f is quasi-compact, any finite union of quasi-compact sets is quasi-compact again.

□

Exercise 1.18.

For psychological reasons it is sometimes convenient to denote a prime ideal of A by a letter such as x or y when thinking of it as a point of $X = \text{Spec}(A)$. When thinking of x as a prime ideal of A , we denote it by \mathfrak{p}_x (logically, of course, it is the same thing). Show that

- (i) The set $\{x\}$ is closed (we say that x is a “closed point”) in $\text{Spec}(A)$ if and only if \mathfrak{p}_x is maximal;
- (ii) $\overline{\{x\}} = V(\mathfrak{p}_x)$;
- (iii) $y \in \overline{\{x\}}$ if and only if $\mathfrak{p}_x \subseteq \mathfrak{p}_y$;

- (iv) X is a T_0 -space (this means that if x, y are distinct points of X , then either there is a neighborhood of x which does not contain y , or else there is a neighborhood of y which does not contain x).

Proof of (i).

$$\{x\} = \overline{\{x\}} \stackrel{(ii)}{\iff} \{x\} = V(\mathfrak{p}_x) \iff \mathfrak{p}_x \text{ is maximal.}$$

□

Proof of (ii). Since $\overline{\{x\}}$ is the intersection of all closed sets containing x and Exercise 1.15 (iii), we have

$$\overline{\{x\}} = \bigcap_{\mathfrak{a} \subseteq \mathfrak{p}_x} V(\mathfrak{a}) = V\left(\sum_{\mathfrak{a} \subseteq \mathfrak{p}_x} \mathfrak{a}\right) = V(\mathfrak{p}_x).$$

□

Proof of (iii).

$$y \in \overline{\{x\}} \stackrel{(ii)}{\iff} y \in V(\mathfrak{p}_x) \iff \mathfrak{p}_y \supseteq \mathfrak{p}_x.$$

□

Proof of (iv).

- (1) Suppose x and y are two points in X such that $y \in \overline{\{x\}}$ and $x \in \overline{\{y\}}$. Note that $x = y$ implies that X is a T_0 -space. So it suffices to show that $x = y$.
- (2) By (iii), $\mathfrak{p}_y \supseteq \mathfrak{p}_x$ and $\mathfrak{p}_x \supseteq \mathfrak{p}_y$. So $\mathfrak{p}_x = \mathfrak{p}_y$ or $x = y$.

□

Exercise 1.19.

A topological space X is said to be irreducible if $X \neq \emptyset$ and if every pair of non-empty open sets in X intersect, or equivalently if every non-empty open set is dense in X . Show that $\text{Spec}(A)$ is irreducible if and only if the nilradical of A is a prime ideal.

Proof. Use the notations in Proposition 1.7 and Exercise 1.17.

$\text{Spec}(A)$ is irreducible

$$\begin{aligned} &\iff X_f \cap X_g \neq \emptyset \text{ for nonempty } X_f, X_g \in \text{Spec}(A) \\ &\iff X_{fg} \neq \emptyset \text{ for nonempty } X_f, X_g \in \text{Spec}(A) && \text{(Exercise 1.17 (i))} \\ &\iff fg \notin \mathfrak{N} \text{ for } f, g \notin \mathfrak{N} && \text{(Exercise 1.17 (ii))} \\ &\iff \mathfrak{N} \text{ is prime.} \end{aligned}$$

□

Exercise 1.20.

Let X be a topological space.

- (i) If Y is an irreducible subspace of X , then the closure \overline{Y} of Y in X is irreducible.
- (ii) Every irreducible subspace of X is contained in a maximal irreducible subspace.
- (iii) The maximal irreducible subspaces of X are closed and cover X . They are called the irreducible components of X . What are the irreducible components of a Hausdorff space?
- (iv) If A is a ring and $X = \text{Spec}(A)$, then the irreducible components of X are the closed sets $V(\mathfrak{p})$, where \mathfrak{p} is a minimal prime ideal of A (Exercise 1.8).

Proof of (i).

- (1) Y is irreducible if and only if Y cannot be represented as the union of two proper closed subspaces.

$$\begin{aligned}
 & \forall \text{ nonempty open sets } U_1 \text{ and } U_2, U_1 \cap U_2 \neq \emptyset \\
 \iff & \forall \text{ nonempty open sets } U_1 \text{ and } U_2, X - (U_1 \cap U_2) \neq X \\
 \iff & \forall \text{ nonempty open sets } U_1 \text{ and } U_2, (X - U_1) \cup (X - U_2) \neq X \\
 \iff & \forall \text{ proper closed sets } Y_1 \text{ and } Y_2, Y_1 \cup Y_2 \neq X \\
 \iff & \nexists \text{ proper closed sets } Y_1 \text{ and } Y_2, Y_1 \cup Y_2 = X.
 \end{aligned}$$

- (2) If \overline{Y} were reducible, there are two closed set Y_1 and Y_2 such that

$$\overline{Y} \subseteq Y_1 \cup Y_2, \quad \overline{Y} \not\subseteq Y_i (i = 1, 2).$$

- (a) $Y \subseteq \overline{Y} \subseteq Y_1 \cup Y_2$.
- (b) $Y \not\subseteq Y_i (i = 1, 2)$. If not, $Y \subseteq Y_i$ for some i . Take closure to get $\overline{Y} \subseteq \overline{Y_i} = Y_i$ (since Y_i is closed), contrary to the assumption.

By (a)(b), Y is reducible, which is absurd.

□

Proof of (ii).

- (1) This is a standard application of Zorn's lemma.

- (2) Suppose Y is an irreducible subspace of X . Let Σ be the set of all irreducible subspaces of X containing Y . Order Σ by inclusion. Σ is not empty, since $Y \in \Sigma$. To apply Zorn's lemma we must show that every chain in Σ has an upper bound in Σ ; let then (Y_α) be a chain in Σ . Let $Z = \bigcup_\alpha Y_\alpha$. $Z \supseteq Y$ clearly.
- (3) *Show that Z is irreducible.* Given two non-empty open sets U and V contained in $Z = \bigcup_\alpha Y_\alpha$. Then $U \cap Y_\alpha \neq \emptyset$ and $V \cap Y_\beta \neq \emptyset$ for some α, β . Since (Y_α) is a chain, we might have $V \cap Y_\alpha \supseteq V \cap Y_\beta \neq \emptyset$ if $\beta \leq \alpha$. (The case $\alpha \leq \beta$ is similar.) So $U \cap V \cap Z \supseteq U \cap V \cap Y_\alpha \neq \emptyset$ since Z contains an irreducible subspace Y_α in X .
- (4) Hence $Z \in \Sigma$, and Z is an upper bound of the chain (Y_α) . Hence by Zorn's lemma Σ has a maximal element.

□

Proof of (iii).

- (1) *Show that the maximal irreducible subspaces of X are closed.* Suppose Y is a maximal irreducible subspaces of X . So \overline{Y} of Y in X is irreducible (by part (i)). The maximality of Y implies that $Y = \overline{Y}$.
- (2) *Show that the maximal irreducible subspaces of X cover X .* Note that each element $P \in X$ forms an irreducible subset $\{P\}$ and thus $\{P\}$ is contained in one irreducible component (by (ii)).
- (3) One point subsets are the irreducible components of a Hausdorff space.

□

Proof of (iv).

- (1) Suppose Y is an irreducible components of X . *Show that $Y = V(\mathfrak{p})$ where \mathfrak{p} is a prime ideal.* Similar to the proof of Exercise 1.19.
- (2) *Show that \mathfrak{p} is a minimal prime ideal of A .* Suppose $\mathfrak{q} \subseteq \mathfrak{p}$. Then $V(\mathfrak{q}) \supseteq V(\mathfrak{p})$. By the maximality of $Y = V(\mathfrak{p})$, $V(\mathfrak{q}) = V(\mathfrak{p})$ or $r(\mathfrak{q}) = r(\mathfrak{p})$ or $\mathfrak{q} = \mathfrak{p}$. Hence \mathfrak{p} is a minimal prime ideal of A .

□

Exercise 1.21.

Let $\phi : A \rightarrow B$ be a ring homomorphism. Let $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$. If $\mathfrak{q} \in Y$, then $\phi^{-1}(\mathfrak{q})$ is a prime ideal of A , i.e., a point of X . Hence ϕ induces a mapping $\phi^* : Y \rightarrow X$. Show that

- (i) If $f \in A$ then $\phi^{*-1}(X_f) = Y_{\phi(f)}$, and hence that ϕ^* is continuous.
- (ii) If \mathfrak{a} is an ideal of A , then $\phi^{*-1}(V(\mathfrak{a})) = V(\mathfrak{a}^e)$.
- (iii) If \mathfrak{b} is an ideal of B , then $\overline{\phi^*(V(\mathfrak{b}))} = V(\mathfrak{b}^c)$.
- (iv) If ϕ is surjective, then ϕ^* is a homeomorphism of Y onto the closed subset $V(\ker(\phi))$ of X . (In particular, $\text{Spec}(A)$ and $\text{Spec}(A/\mathfrak{N})$ (where \mathfrak{N} is the nilradical of A) are naturally homeomorphic.)
- (v) If ϕ is injective, then $\phi^*(Y)$ is dense in X . More precisely, $\phi^*(Y)$ is dense in X if and only if $\ker(\phi) \subseteq \mathfrak{N}$.
- (vi) Let $\psi : B \rightarrow C$ be another ring homomorphism. Then $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.
- (vii) Let A be an integral domain with just one nonzero prime ideal \mathfrak{p} , and let K be the field of fractions of A . Let $B = (A/\mathfrak{p}) \times K$. Define $\phi : A \rightarrow B$ by $\phi(x) = (\bar{x}, x)$, where \bar{x} is the image of x in A/\mathfrak{p} . Show that ϕ^* is bijective but not a homeomorphism.

Proof of (i). Since

$$\begin{aligned}
\mathfrak{q} &\in Y_{\phi(f)} = Y - V(\phi(f)) \\
&\iff \mathfrak{q} \notin V(\phi(f)) = \{\text{all prime ideals in } B \text{ containing } \phi(f)\} \\
&\iff \phi(f) \notin \mathfrak{q} \\
&\iff f \notin \phi^{-1}(\mathfrak{q}) \\
&\iff \phi^{-1}(\mathfrak{q}) \notin V(f) = \{\text{all prime ideals in } A \text{ containing } f\} \\
&\iff \phi^*(\mathfrak{q}) = \phi^{-1}(\mathfrak{q}) \in X_f,
\end{aligned}$$

ϕ^* is continuous. \square

Proof of (ii).

- (1) Use the same notation of Proposition 1.17. Show that

$$\mathfrak{b}^c \supseteq \mathfrak{a} \iff \mathfrak{b} \supseteq \mathfrak{a}^e.$$

Suppose $\mathfrak{b}^c \supseteq \mathfrak{a}$, then $\mathfrak{b}^{ce} \supseteq \mathfrak{a}^e$. Proposition 1.17 (i) suggests that $\mathfrak{b} \supseteq \mathfrak{b}^{ce} \supseteq \mathfrak{a}^e$. The converse is similar.

- (2) So

$$\begin{aligned}
&\mathfrak{q} \in \phi^{*-1}(V(\mathfrak{a})) \\
&\iff \phi^*(\mathfrak{q}) \in V(\mathfrak{a}) = \{\text{all prime ideals containing } \mathfrak{a}\} \\
&\iff \phi^*(\mathfrak{q}) \supseteq \mathfrak{a} \\
&\iff \mathfrak{q}^c \supseteq \mathfrak{a} \\
&\iff \mathfrak{q} \supseteq \mathfrak{a}^e \\
&\iff \mathfrak{q} \in V(\mathfrak{a}^e) = \{\text{all prime ideals containing } \mathfrak{a}^e\}.
\end{aligned} \tag{1}$$

□

Proof of (iii).

- (1) Might assume that $\mathfrak{b} = r(\mathfrak{b})$ is radical by Exercise 1.15 (i).
- (2) Show that $\overline{\phi^*(V(\mathfrak{b}))} \supseteq V(\mathfrak{b}^c)$. Write $\overline{\phi^*(V(\mathfrak{b}))} = V(\mathfrak{a})$ for some radical ideal \mathfrak{a} in A since $\phi^*(V(\mathfrak{b}))$ is closed. So

$$\begin{aligned}
 V(\mathfrak{a}^e) &= \phi^{*-1}(V(\mathfrak{a})) = \phi^{*-1}(\overline{\phi^*(V(\mathfrak{b}))}) \supseteq V(\mathfrak{b}) & ((ii)) \\
 \implies r(\mathfrak{a}^e) &\subseteq r(\mathfrak{b}) \\
 \implies r(\mathfrak{a})^e &\subseteq r(\mathfrak{a}^e) \subseteq r(\mathfrak{b}) \\
 \implies \mathfrak{a}^e &\subseteq \mathfrak{b} \\
 \implies \mathfrak{a} &\subseteq \mathfrak{b}^c \\
 \implies V(\mathfrak{a}) &\supseteq V(\mathfrak{b}^c).
 \end{aligned}$$

- (3) Show that $\overline{\phi^*(V(\mathfrak{b}))} \subseteq V(\mathfrak{b}^c)$. It suffices to show that $\phi^*(V(\mathfrak{b})) \subseteq V(\mathfrak{b}^c)$ since $V(\mathfrak{b}^c)$ is closed. Suppose $\mathfrak{p} \in \phi^*(V(\mathfrak{b}))$. Then there is $\mathfrak{q} \in V(\mathfrak{b})$ such that

$$\mathfrak{p} = \phi^*(\mathfrak{q}) = \mathfrak{q}^c \supseteq \mathfrak{b}^c.$$

So $\mathfrak{p} \in V(\mathfrak{b}^c)$.

□

Proof of (iv). Note that $A/\ker \phi \cong B$ since ϕ is surjective. The correspondence theorem shows that $\phi^* : Y \rightarrow V(\ker \phi)$ is bijective. As the continuity of ϕ^* is given by (i), ϕ^* is a homeomorphism of Y onto $V(\ker(\phi)) \subseteq X$. □

Proof of (v).

- (1) It suffices to show that $\phi^*(Y)$ is dense in X if and only if $\ker(\phi) \subseteq \mathfrak{N}$.
- (2)

$$\begin{aligned}
 &\phi^*(Y) \text{ is dense in } X \\
 \iff &X = \overline{\phi^*(Y)} = \overline{\phi^*(V(0))} = V(0^c) = V(\ker \phi) \\
 \iff &\ker \phi \text{ is contained in every prime ideal of } A \\
 \iff &\ker \phi \subseteq \mathfrak{N}.
 \end{aligned}$$

□

Proof of (vi).

$$(\psi \circ \phi)^*(\mathfrak{p}) = (\psi \circ \phi)^{-1}(\mathfrak{p}) = \phi^{-1}(\psi^{-1}(\mathfrak{p})) = \phi^*(\psi^*(\mathfrak{p})) = (\phi^* \circ \psi^*)(\mathfrak{p})$$

for every prime ideal \mathfrak{p} in $\text{Spec}(C)$. □

Proof of (vii).

- (1) Show that ϕ^* is bijective. Note that

$$\begin{aligned} X &= \text{Spec}(A) = \{(0), \mathfrak{p}\} \\ Y &= \text{Spec}(B) = \{A/\mathfrak{p} \times (0), (0) \times K\} \end{aligned}$$

and thus

$$\begin{aligned} \phi^*(A/\mathfrak{p} \times (0)) &= (0) \\ \phi^*((0) \times K) &= (\mathfrak{p}). \end{aligned}$$

Hence ϕ^* is a bijection.

- (2) Show that ϕ^* is not a homeomorphism. Note that $\overline{\{(0)\}} = X$ (Exercise 1.18 (iii)) and Y is equipped with the discrete topology since each prime ideal of B is maximal (Exercise 1.18 (i)). So ϕ^* cannot be a homeomorphism.

□

Exercise 1.22.

Let $A = \prod_{i=1}^n A_i$ be a direct product of rings A_i . Show that $\text{Spec}(A)$ is the disjoint union of open (and closed) subspaces X_i , where X_i is canonically homeomorphic with $\text{Spec}(A_i)$.

Conversely, let A be any ring. Show that the following statements are equivalent:

- (i) $X = \text{Spec}(A)$ is disconnected.
- (ii) $A \cong A_1 \times A_2$ where neither of the rings A_1, A_2 is the zero ring.
- (iii) A contains an idempotent $\neq 0, 1$. In particular, the spectrum of a local ring is always connected (Exercise 1.12).

Proof.

- (1) Show that $\text{Spec}(A)$ is the union of closed subspaces X_i , where $X_i \cong \text{Spec}(A_i)$. Let $\phi_i : A \rightarrow A_i$ be the projection map. So

$$\ker \phi_i = A_1 \times \cdots \times A_{i-1} \times 0 \times A_{i+1} \times \cdots \times A_n.$$

So

$$\text{Spec}(A) = V(0) = V\left(\bigcap_{i=1}^n \ker \phi_i\right) = \bigcup_{i=1}^n V(\ker \phi_i)$$

where $X_i := V(\ker \phi_i) \cong \text{Spec}(A_i)$ (Exercise 1.21).

(2) Show that $V(\ker \phi_i)$ and $V(\ker \phi_j)$ are disjoint if $i \neq j$.

$$V(\ker \phi_i) \cap V(\ker \phi_j) = V(\ker \phi_i + \ker \phi_j) = V(A) = V(1) = \emptyset.$$

(3) Show that $V(\ker \phi_i)$ is open. $\text{Spec}(A) = \bigcup_{j=1}^n V(\ker \phi_j)$ and $V(\ker \phi_i) \cap V(\ker \phi_j) = \emptyset$ (if $i \neq j$) implies that $\text{Spec}(A) \setminus V(\ker \phi_i) = \bigcup_{j \neq i} V(\ker \phi_j)$ is closed. Thus $V(\ker \phi_i)$ is open.

(4) ((ii) \implies (i)) See (1)(2)(3).

(5) ((i) \implies (iii)) Write X as a disjoint union of two nonempty closed sets $V(\mathfrak{a}), V(\mathfrak{b})$ where $\mathfrak{a}, \mathfrak{b}$ are radical ideals in A (Exercise 1.15). Since

$$\begin{aligned} V(0) = X &= V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) \\ V(1) = \emptyset &= V(\mathfrak{a}) \cap V(\mathfrak{b}) = V(\mathfrak{a} + \mathfrak{b}), \end{aligned}$$

there exist $a \in \mathfrak{a}, b \in \mathfrak{b}$ such that $a + b = 1$ and $(ab)^n = 0$ for one positive integer n . So $ab = 0$ since $\mathfrak{a}\mathfrak{b}$ is radical. (Note that $\mathfrak{a} + \mathfrak{b} = 1$ and Exercise 1.13 on page 9.) So

$$a^2 = a(1 - b) = a - ab = a$$

is an idempotent. Also $a \neq 0, 1$ since $V(\mathfrak{a}), V(\mathfrak{b})$ are proper subsets of X .

(6) ((iii) \implies (ii)) Take an idempotent $e \neq 0, 1$ in A . Two ideals (e) and $(1 - e)$ are proper and coprime. So $(e) \cap (1 - e) = (e)(1 - e) = (0)$ (Proposition 1.10 (i)). Proposition 1.10 (ii) and (iii) imply that the ring homomorphism

$$A \rightarrow A/(e) \times A/(1 - e)$$

is an isomorphism. Also $A/(e), A/(1 - e) \neq 0$ since $e \neq 0, 1$.

□

Exercise 1.23.

Let A be a Boolean ring (Exercise 1.11), and let $X = \text{Spec}(A)$.

- (i) For each $f \in A$, the set X_f (Exercise 1.17) is both open and closed in X .
- (ii) Let $f_1, \dots, f_n \in A$. Show that $X_{f_1} \cup \dots \cup X_{f_n} = X_f$ for some $f \in A$.
- (iii) The sets X_f are the only open subsets of X which are both open and closed.
- (iv) X is a compact Hausdorff space.

Proof of (i).

- (1) Show that X is the disjoint union of subspaces X_f and X_{1-f} . Note that every element in a Boolean ring is an idempotent. Hence

$$\begin{aligned} X_f \cap X_{1-f} &= X_{f(1-f)} = X_0 = \emptyset \\ X_f \cup X_{1-f} &= X \setminus (V(f) \cap V(1-f)) = X \setminus \underbrace{V(f + (1-f))}_{=V(1)=\emptyset} = X. \end{aligned}$$

- (2) Hence $X_f = X \setminus X_{1-f}$ is both open and closed.

□

Proof of (ii). Similar to (i),

$$\begin{aligned} X_{f_1} \cup \cdots \cup X_{f_n} &= X \setminus (V(f_1) \cap \cdots \cap V(f_n)) \\ &= X \setminus V(f_1, \dots, f_n) \\ &= X \setminus V(f) && \text{(Exercise 1.11 (iii))} \\ &= X_f \end{aligned}$$

for some $f \in A$. □

Proof of (iii).

- (1) Suppose Y is both open and closed in X .
- (2) Since Y is closed and X is quasi-compact (Exercise 1.17 (vi)), Y is quasi-compact.
- (3) Since Y is open, Y is a finite union of sets X_{f_i} for $i = 1, \dots, n$ (Exercise 1.17 (vii)). Hence $Y = X_f$ for some $f \in A$ (by (ii)).

□

Proof of (iv).

- (1) The compactness of X is followed by Exercise 1.17 (v).
- (2) Show that X is Hausdorff. Exercise 1.18 shows that X is a T_0 -space. This means that if x, y are distinct points of X , we might assume that there is a neighborhood U of x which does not contain y .
- (3) Write $U = X_f$ for some $f \in A$ (by Exercise 1.17 and (ii)). As $x \in X_f$, $y \in X \setminus X_f = X_{1-f}$ and $X_f \cap X_{1-f} = \emptyset$ by (i). Hence X is Hausdorff.

□

Chapter 2: Modules

Exercise 2.1.

Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$ if m, n are coprime.

It suffices to show that

$$(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$$

where d is the greatest common divisor of m and n .

Outlines.

- (1) Define $\tilde{\varphi}$ by

$$\begin{array}{ccc} \tilde{\varphi}: & (\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z}) & \longrightarrow \mathbb{Z}/d\mathbb{Z} \\ & \Downarrow & \Downarrow \\ & (x + m\mathbb{Z}, y + n\mathbb{Z}) & \longmapsto xy + d\mathbb{Z}. \end{array}$$

$\tilde{\varphi}$ is well-defined and \mathbb{Z} -bilinear.

- (2) By the universal property, $\tilde{\varphi}$ factors through a \mathbb{Z} -bilinear map

$$\varphi: (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \rightarrow \mathbb{Z}/d\mathbb{Z}$$

(such that $\varphi(x \otimes y) = \tilde{\varphi}(x, y)$).

- (3) To show that φ is isomorphic, might find the inverse map $\psi: \mathbb{Z}/d\mathbb{Z} \rightarrow (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$ of φ . Define ψ by

$$\begin{array}{ccc} \psi: & \mathbb{Z}/d\mathbb{Z} & \longrightarrow (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \\ & \Downarrow & \Downarrow \\ & z + d\mathbb{Z} & \longmapsto (z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}). \end{array}$$

ψ is well-defined and \mathbb{Z} -linear.

- (4) $\psi \circ \varphi = \text{id}$.

- (5) $\varphi \circ \psi = \text{id}$.

Proof of (1).

- (a) $\tilde{\varphi}$ is well-defined. Say $x' = x + am$ for some $a \in \mathbb{Z}$ and $y' = y + bn$ for some $b \in \mathbb{Z}$. Then $x'y' - xy = yam + xbn + abmn \in d\mathbb{Z}$. That is, $\tilde{\varphi}$ is independent of coset representative.

(b) $\tilde{\varphi}$ is \mathbb{Z} -bilinear.

(i) For any $\lambda \in \mathbb{Z}$, $\tilde{\varphi}(\lambda x, y) = \tilde{\varphi}(x, \lambda y) = \lambda \tilde{\varphi}(x, y)$. In fact,

$$\begin{aligned}\tilde{\varphi}(\lambda(x + m\mathbb{Z}), y + n\mathbb{Z}) &= \tilde{\varphi}(\lambda x + m\mathbb{Z}, y + n\mathbb{Z}) = \lambda xy + d\mathbb{Z}, \\ \tilde{\varphi}(x + m\mathbb{Z}, \lambda(y + n\mathbb{Z})) &= \tilde{\varphi}(x + m\mathbb{Z}, \lambda y + n\mathbb{Z}) = \lambda xy + d\mathbb{Z}, \\ \tilde{\varphi}(x_1 + m\mathbb{Z}, y + n\mathbb{Z}) &= \lambda(xy + d\mathbb{Z}) = \lambda xy + d\mathbb{Z}.\end{aligned}$$

(ii) $\tilde{\varphi}(x_1 + x_2, y) = \tilde{\varphi}(x_1, y) + \tilde{\varphi}(x_2, y)$. In fact,

$$\begin{aligned}\tilde{\varphi}((x_1 + x_2) + m\mathbb{Z}, y + n\mathbb{Z}) &= (x_1 + x_2)y + d\mathbb{Z}, \\ \tilde{\varphi}(x_1 + m\mathbb{Z}, y + n\mathbb{Z}) + \tilde{\varphi}(x_2 + m\mathbb{Z}, y + n\mathbb{Z}) &= (x_1 y + d\mathbb{Z}) + (x_2 y + d\mathbb{Z}) \\ &= (x_1 + x_2)y + d\mathbb{Z}.\end{aligned}$$

(iii) $\tilde{\varphi}(x, y_1 + y_2) = \tilde{\varphi}(x, y_1) + \tilde{\varphi}(x, y_2)$. Similar to (ii).

□

Proof of (3).

(a) ψ is well-defined. Say $z' = z + cd$ for some $c \in \mathbb{Z}$. Note that $d = \alpha m + \beta n$ for some $\alpha, \beta \in \mathbb{Z}$. Thus

$$\begin{aligned}\psi(z' + d\mathbb{Z}) &= \psi(z + cd + d\mathbb{Z}) \\ &= \psi(z + c(\alpha m + \beta n) + d\mathbb{Z}) \\ &= (z + c(\alpha m + \beta n) + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\ &= (z + c\beta n + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\ &= (z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) + (c\beta n + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\ &= \psi(z + d\mathbb{Z}) + (1 + m\mathbb{Z}) \otimes (c\beta n + n\mathbb{Z}) \\ &= \psi(z + d\mathbb{Z}).\end{aligned}$$

(b) ψ is \mathbb{Z} -linear.

(i) For any $\lambda \in \mathbb{Z}$, $\psi(\lambda z) = \lambda \psi(z)$. In fact,

$$\begin{aligned}\psi(\lambda(z + d\mathbb{Z})) &= \psi(\lambda z + d\mathbb{Z}) = (\lambda z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}), \\ \lambda \psi(z + d\mathbb{Z}) &= \lambda((z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z})) = (\lambda z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}).\end{aligned}$$

(ii) $\psi(z_1 + z_2) = \psi(z_1) + \psi(z_2)$.

$$\begin{aligned}\psi((z_1 + z_2) + d\mathbb{Z}) &= (z_1 + z_2 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}), \\ \psi(z_1 + d\mathbb{Z}) + \psi(z_2 + d\mathbb{Z}) &= (z_1 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) + (z_2 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\ &= (z_1 + z_2 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}).\end{aligned}$$

□

Proof of (4). For any $(x + m\mathbb{Z}) \otimes (y + n\mathbb{Z}) \in (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$,

$$\begin{aligned}\psi(\varphi((x + m\mathbb{Z}) \otimes (y + n\mathbb{Z}))) &= \psi(xy + d\mathbb{Z}) \\ &= (xy + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\ &= (x + m\mathbb{Z}) \otimes (y + n\mathbb{Z}).\end{aligned}$$

□

Proof of (5). For any $z + d\mathbb{Z} \in \mathbb{Z}/d\mathbb{Z}$,

$$\begin{aligned}\varphi(\psi(z + d\mathbb{Z})) &= \varphi((z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z})) \\ &= z + d\mathbb{Z}.\end{aligned}$$

□

Exercise 2.2.

Let A be a ring, \mathfrak{a} an ideal, M an A -module. Show that $(A/\mathfrak{a}) \otimes_A M$ is isomorphic to $M/\mathfrak{a}M$. (Hint: Tensor the exact sequence $0 \rightarrow \mathfrak{a} \rightarrow A \rightarrow A/\mathfrak{a} \rightarrow 0$ with M .)

Proof (Hint). There is a natural exact sequence E :

$$E : 0 \rightarrow \mathfrak{a} \xrightarrow{i} A \xrightarrow{\pi} A/\mathfrak{a} \rightarrow 0$$

where i is the inclusion map (and π is the projection map). Tensor E with M :

$$E' : \mathfrak{a} \otimes_A M \xrightarrow{i \otimes 1} A \otimes_A M \xrightarrow{\pi \otimes 1} (A/\mathfrak{a}) \otimes_A M \rightarrow 0$$

is exact, or

$$(A/\mathfrak{a}) \otimes_A M \cong A \otimes_A M / \text{im}(i \otimes 1).$$

By Proposition 2.14, There is an unique isomorphism $A \otimes_A M \rightarrow M$ defined by $a \otimes x \mapsto ax$. This isomorphism sends $\text{im}(i \otimes 1)$ to $\mathfrak{a}M$. Therefore,

$$(A/\mathfrak{a}) \otimes_A M \cong M/\mathfrak{a}M.$$

□

Proof (Brute-force).

(1) Define $\tilde{\varphi}$ by

$$\begin{array}{ccc} \tilde{\varphi}: & A/\mathfrak{a} \times M & \longrightarrow M/\mathfrak{a}M \\ & \Downarrow & \Downarrow \\ & (a + \mathfrak{a}, x) & \longmapsto ax + \mathfrak{a}M. \end{array}$$

$\tilde{\varphi}$ is well-defined and A -bilinear.

- (2) By the universal property, $\tilde{\varphi}$ factors through a A -bilinear map

$$\varphi: A/\mathfrak{a} \otimes_A M \rightarrow M/\mathfrak{a}M$$

(such that $\varphi(a \otimes x) = \tilde{\varphi}(a, x)$).

- (3) To show that φ is isomorphic, might find the inverse map $\psi: M/\mathfrak{a}M \rightarrow A/\mathfrak{a} \otimes_A M$ of φ . Define ψ by

$$\begin{array}{ccc} \psi: & M/\mathfrak{a}M & \longrightarrow A/\mathfrak{a} \otimes_A M \\ & \Downarrow & \Downarrow \\ & x + \mathfrak{a}M & \longmapsto (1 + \mathfrak{a}) \otimes x. \end{array}$$

ψ is well-defined and A -linear.

- (4) $\psi \circ \varphi = \text{id}$.

- (5) $\varphi \circ \psi = \text{id}$.

□

Exercise 2.3.

Let A be a local ring, M and N finitely generated A -modules. Prove that if $M \otimes_A N = 0$, then $M = 0$ or $N = 0$. (Hint: Let \mathfrak{m} be the maximal ideal, $k = A/\mathfrak{m}$ the residue field. Let $M_k = k \otimes_A M \cong M/\mathfrak{m}M$ by Exercise 2.2. By Nakayama's lemma, $M_k = 0 \implies M = 0$. But $M \otimes_A N = 0 \implies (M \otimes_A N)_k = 0 \implies M_k \otimes_k N_k = 0 \implies M_k = 0$ or $N_k = 0$ since M_k, N_k are vector spaces over a field.)

The conclusion might be false if A is not local. For example, Exercise 2.1.

Proof (Hint). Let \mathfrak{m} be the maximal ideal, $k = A/\mathfrak{m}$ the residue field. Let $M_k = k \otimes_A M$.

- (1) (*Base extension*) Show that $(M \otimes_A N)_k = M_k \otimes_k N_k$. In fact, by Proposition 2.14

$$\begin{aligned}
 (M \otimes_A N)_k &= k \otimes_A (M \otimes_A N) \\
 &= (k \otimes_A M) \otimes_A N \\
 &= M_k \otimes_A N \\
 &= (M_k \otimes_k k) \otimes_A N \\
 &= M_k \otimes_k (k \otimes_A N) \\
 &= M_k \otimes_k N_k.
 \end{aligned}$$

(2)

$$\begin{aligned}
 M \otimes_A N = 0 &\implies (M \otimes_A N)_k = 0 \\
 &\implies M_k \otimes_k N_k = 0 && ((1)) \\
 &\implies M_k = 0 \text{ or } N_k = 0 && (M_k, N_k: \text{ vector spaces}) \\
 &\implies M/\mathfrak{m}M = 0 \text{ or } M/\mathfrak{m}M = 0 && (\text{Exercise 2.2}) \\
 &\implies M = 0 \text{ or } N = 0. && (\text{Nakayama's lemma})
 \end{aligned}$$

□

Exercise 2.4.

Let M_i ($i \in I$) be any family of A -modules, and let M be their direct sum. Prove that M is flat \Leftrightarrow each M_i is flat.

Proof. Given any A -module homomorphism $f : N' \rightarrow N$.

- (1) Similar to Proposition 2.14(iii), we have two isomorphisms

(a)

$$\varphi : \bigoplus_{i \in I} (N' \otimes M_i) \cong N' \otimes_A \bigoplus_{i \in I} M_i$$

defined by

$$\varphi((x \otimes m_i)_{i \in I}) = x \otimes (m_i)_{i \in I}$$

where $x \in N'$, $m_i \in M_i$ ($i \in I$).

(b)

$$\psi : N \otimes_A \bigoplus_{i \in I} M_i \cong \bigoplus_{i \in I} (N \otimes M_i)$$

defined by

$$\psi(y \otimes (m_i)_{i \in I}) = (y \otimes m_i)_{i \in I}$$

where $y \in N$, $m_i \in M_i$ ($i \in I$).

(2) $f : N' \rightarrow N$ induces an A -module homomorphism

$$f \otimes \text{id}_M : N' \otimes_A M \rightarrow N \otimes_A M.$$

(3) $\psi \circ f \otimes \text{id}_M \circ \varphi$ defines an A -module homomorphism

$$\psi \circ f \otimes \text{id}_M \circ \varphi : \bigoplus_{i \in I} (N' \otimes M_i) \rightarrow \bigoplus_{i \in I} (N \otimes M_i)$$

which sends $(x \otimes m_i)_{i \in I}$ to $(f(x) \otimes m_i)_{i \in I}$. That is,

$$\psi \circ f \otimes \text{id}_M \circ \varphi = \bigoplus_{i \in I} f \otimes \text{id}_{M_i}$$

(4) Show that M is flat if and only if each M_i is flat. Suppose f is injective.

$$\begin{aligned} & M_i \text{ is flat } \forall i \in I \\ \iff & f \otimes \text{id}_{M_i} \text{ is injective } \forall i \in I \\ \iff & \bigoplus_{i \in I} f \otimes \text{id}_{M_i} \text{ is injective} && \text{(Injectivity)} \\ \iff & \psi \circ f \otimes \text{id}_M \circ \varphi \text{ is injective} && ((3)) \\ \iff & f \otimes \text{id}_M \text{ is injective} && (\varphi, \psi \text{ are isomorphic}) \\ \iff & M \text{ is flat.} \end{aligned}$$

□

Exercise 2.5.

Let $A[x]$ be the ring of polynomials in one indeterminate over a ring A . Prove that $A[x]$ is a flat A -algebra. (Hint: Use Exercise 2.4.)

Proof (Hint).

(1) A is a flat A -module by Proposition 2.14(iv).

(2) As an A -module,

$$A[x] \cong \bigoplus_{n \in \mathbb{Z}^+} Ax^n \cong \bigoplus_{n \in \mathbb{Z}^+} A$$

(since $Ax^n \cong A$).

(3) By Exercise 2.4, $A[x] \cong \bigoplus_{n \in \mathbb{Z}^+} A$ is flat.

□

Exercise 2.8.

- (i) If M and N are flat A -modules, then so is $M \otimes_A N$.
- (ii) If B is a flat A -algebra and N is a flat B -module, then N is flat as A -module.

Proof of (i). Given any exact sequence of A -modules $0 \rightarrow N_1 \rightarrow N_2 \rightarrow N_3 \rightarrow 0$. Since M is flat,

$$0 \rightarrow N_1 \otimes_A M \rightarrow N_2 \otimes_A M \rightarrow N_3 \otimes_A M \rightarrow 0$$

is exact. Since N is flat,

$$0 \rightarrow (N_1 \otimes_A M) \otimes_A N \rightarrow (N_2 \otimes_A M) \otimes_A N \rightarrow (N_3 \otimes_A M) \otimes_A N \rightarrow 0$$

is exact. By Proposition 2.14 (ii),

$$0 \rightarrow N_1 \otimes_A (M \otimes_A N) \rightarrow N_2 \otimes_A (M \otimes_A N) \rightarrow N_3 \otimes_A (M \otimes_A N) \rightarrow 0$$

is exact, or $M \otimes_A N$ is flat. \square

Proof of (ii). Given any exact sequence of A -modules $0 \rightarrow N_1 \rightarrow N_2 \rightarrow N_3 \rightarrow 0$. Since B is a flat A -algebra (A -module),

$$0 \rightarrow N_1 \otimes_A B \rightarrow N_2 \otimes_A B \rightarrow N_3 \otimes_A B \rightarrow 0$$

is exact. Since N is a flat B -module,

$$0 \rightarrow (N_1 \otimes_A B) \otimes_B N \rightarrow (N_2 \otimes_A B) \otimes_B N \rightarrow (N_3 \otimes_A B) \otimes_B N \rightarrow 0$$

is exact. By “Exercise 2.15” on page 27,

$$0 \rightarrow N_1 \otimes_A (B \otimes_B N) \rightarrow N_2 \otimes_A (B \otimes_B N) \rightarrow N_3 \otimes_A (B \otimes_B N) \rightarrow 0$$

is exact. By Proposition 2.14 (iv),

$$0 \rightarrow N_1 \otimes_A N \rightarrow N_2 \otimes_A N \rightarrow N_3 \otimes_A N \rightarrow 0$$

is exact, or N is flat. \square

Exercise 2.9.

Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be an exact sequence of A -modules. If M' and M'' are finitely generated, then so is M .

Proof.

(1) Write

$$0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0.$$

Also write

x_1, \dots, x_n as generators of M' ,

z_1, \dots, z_m as generators of M''

(since M' and M'' are finitely generated).

(2) Since the map $g : M \rightarrow M''$ is surjective, there exists $y_j \in M$ such that $g(y_j) = z_j$ for $j = 1, \dots, m$.

(3) Show that M is generated by

$$f(x_1), \dots, f(x_n), y_1, \dots, y_m.$$

Given any $y \in M$.

$$\begin{aligned} y \in M &\implies g(y) \in M'' \\ &\implies g(y) = \sum_{j=1}^m s_j z_j \text{ where } s_j \in A \\ &\implies g(y) = \sum_{j=1}^m s_j g(y_j) \\ &\implies g(y) = g\left(\sum_{j=1}^m s_j y_j\right) \\ &\implies y - \sum_{j=1}^m s_j y_j \in \ker(g) = \text{im}(f) \\ &\implies \exists x \in M' \text{ such that } f(x) = y - \sum_{j=1}^m s_j y_j \end{aligned}$$

Write $x = \sum_{i=1}^n r_i x_i$ where $r_i \in A$. So,

$$\begin{aligned} y \in M &\implies f\left(\sum_{i=1}^n r_i x_i\right) = y - \sum_{j=1}^m s_j y_j \\ &\implies \sum_{i=1}^n r_i f(x_i) = y - \sum_{j=1}^m s_j y_j \\ &\implies y = \sum_{i=1}^n r_i f(x_i) + \sum_{j=1}^m s_j y_j. \end{aligned}$$

Hence, every $y \in M$ is a linear combination of $f(x_1), \dots, f(x_n), y_1, \dots, y_m$, or M is finitely generated (by $f(x_1), \dots, f(x_n), y_1, \dots, y_m$).

