

# Solutions to the book: *Marcus, Number Fields*

Meng-Gen Tsai  
plover@gmail.com

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## Chapter 1: A Special Case of Fermat's Conjecture

*Exercise 1.1-1.9: Define  $N : \mathbb{Z}[i] \rightarrow \mathbb{Z}$  by  $N(a + bi) = a^2 + b^2$ .*

### Exercise 1.1.

*Verify that for all  $\alpha, \beta \in \mathbb{Z}[i]$ ,  $N(\alpha\beta) = N(\alpha)N(\beta)$ , either by direct computation or using the fact that  $N(a + bi) = (a + bi)(a - bi)$ . Conclude that if  $\alpha \mid \gamma$  in  $\mathbb{Z}[i]$ , then  $N(\alpha) \mid N(\gamma)$  in  $\mathbb{Z}$ .*

*Proof.*

(1) *Direct computation.* Write  $\alpha = a + bi, \beta = c + di$  where  $a, b, c, d \in \mathbb{Z}$ . Thus,

$$\begin{aligned} N(\alpha\beta) &= N((a + bi)(c + di)) \\ &= N((ac - bd) + (ad + bc)i) \\ &= (ac - bd)^2 + (ad + bc)^2 \\ &= (a^2c^2 - 2abcd + b^2d^2) + (a^2d^2 + 2abcd + b^2c^2) \\ &= a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2, \\ N(\alpha)N(\beta) &= N(a + bi)N(c + di) \\ &= (a^2 + b^2)(c^2 + d^2) \\ &= a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2. \end{aligned}$$

Therefore,  $N(\alpha\beta) = N(\alpha)N(\beta)$ . (Note that we also get the identity  $(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$ .)

(2) *Using the fact that  $N(a + bi) = (a + bi)(a - bi)$ , or  $N(\alpha) = \alpha\bar{\alpha}$  for any  $\alpha \in \mathbb{Z}[i]$ .* Thus,

$$\begin{aligned} N(\alpha\beta) &= \alpha\beta\overline{\alpha\beta} \\ &= \alpha\beta\bar{\alpha}\bar{\beta} \\ &= \alpha\bar{\alpha}\beta\bar{\beta} \\ &= N(\alpha)N(\beta). \end{aligned}$$

(3) *Show that if  $\alpha \mid \gamma$  in  $\mathbb{Z}[i]$ , then  $N(\alpha) \mid N(\gamma)$  in  $\mathbb{Z}$ .* Write  $\gamma = \alpha\beta$  for some  $\beta \in \mathbb{Z}[i]$ . So  $N(\gamma) = N(\alpha)N(\beta) \in \mathbb{Z}$ , or  $N(\alpha) \mid N(\gamma)$  in  $\mathbb{Z}$ .

□

**Exercise 1.2.**

Let  $\alpha \in \mathbb{Z}[i]$ . Show that  $\alpha$  is a unit iff  $N(\alpha) = 1$ . Conclude that the only units are  $\pm 1$  and  $\pm i$ .

*Proof.*

- (1) ( $\implies$ ) Since  $\alpha$  is a unit, there is  $\beta \in \mathbb{Z}[i]$  such that  $\alpha\beta = 1$ . By Exercise 1.1,  $N(\alpha\beta) = N(1)$ , or  $N(\alpha)N(\beta) = 1$ . Since the image of  $N$  is nonnegative integers,  $N(\alpha) = 1$ .
- (2) ( $\impliedby$ ) By Exercise 1.1,  $N(\alpha) = \alpha\bar{\alpha}$ , or  $1 = \alpha\bar{\alpha}$  since  $N(\alpha) = 1$ . That is,  $\bar{\alpha} \in \mathbb{Z}[i]$  is the inverse of  $\alpha \in \mathbb{Z}[i]$ . (Or by (1), we solve the equation  $N(\alpha) = a^2 + b^2 = 1$ , and show that all four solutions ( $\pm 1$  and  $\pm i$ ) are unit.)

Conclusion: a unit  $\alpha = a+bi$  of  $\mathbb{Z}[i]$  is satisfying the equation  $N(\alpha) = a^2+b^2 = 1$  by (1)(2). That is, the only units of  $\mathbb{Z}[i]$  are  $\pm 1$  and  $\pm i$ .  $\square$

**Exercise 1.3.**

Let  $\alpha \in \mathbb{Z}[i]$ . Show that if  $N(\alpha)$  is a prime in  $\mathbb{Z}$  then  $\alpha$  is irreducible in  $\mathbb{Z}[i]$ . Show that the same conclusion holds if  $N(\alpha) = p^2$ , where  $p$  is a prime in  $\mathbb{Z}$ ,  $p \equiv 3 \pmod{4}$ .

*Proof.*

- (1) Show that if  $N(\alpha)$  is a prime in  $\mathbb{Z}$  then  $\alpha$  is irreducible in  $\mathbb{Z}[i]$ . Write  $\alpha = \beta\gamma$ . Then  $N(\alpha) = N(\beta)N(\gamma)$  is a prime in  $\mathbb{Z}$ . Since each integer prime is irreducible,  $N(\beta) = 1$  or  $N(\gamma) = 1$ . So that  $\beta$  is unit or  $\gamma$  is unit by Exercise 1.2. Hence,  $\alpha$  is irreducible.
- (2) Show that  $\alpha$  is irreducible in  $\mathbb{Z}[i]$  if  $N(\alpha) = p^2$ , where  $p$  is a prime in  $\mathbb{Z}$ ,  $p \equiv 3 \pmod{4}$ . Assume  $\alpha = \beta\gamma$  were not irreducible. Similar to (1),  $N(\alpha) = N(\beta)N(\gamma) = p^2$ . Since  $\beta$  and  $\gamma$  are proper factors of  $\alpha$ ,

$$N(\beta) = N(\gamma) = p.$$

Since any square  $a^2 \equiv 0, 1 \pmod{4}$ , any  $N(a+bi) = a^2 + b^2 \equiv 0, 1, 2 \pmod{4}$ . Especially,  $N(\beta) \equiv 0, 1, 2 \pmod{4}$ , contrary to  $N(\beta) = p \equiv 3 \pmod{4}$  by the assumption. Therefore,  $\alpha$  is irreducible in  $\mathbb{Z}[i]$ .

$\square$

**Supplement.**

- (1) The prime 2 is reducible in  $\mathbb{Z}[i]$  (Exercise 1.4).
- (2) Every prime  $p \equiv 1 \pmod{4}$  is reducible in  $\mathbb{Z}[i]$  (Exercise 1.8).

**Exercise 1.4.**

Show that  $1 - i$  is irreducible in  $\mathbb{Z}$  and that  $2 = u(1 - i)^2$  for some unit  $u$ .

*Proof.*

- (1)  $1 - i$  is irreducible. Since  $N(1 - i) = 2$  is a prime in  $\mathbb{Z}$ ,  $1 - i$  is irreducible by Problem 1.3.
- (2)  $2 = i(1 - i)^2$  where  $i$  is unit in  $\mathbb{Z}$ .

□

**Exercise 1.5.**

Notice that  $(2 + i)(2 - i) = 5 = (1 + 2i)(1 - 2i)$ . How is this consistent with unique factorization?

*Proof.* Since  $2 + i = i(1 - 2i)$  and  $2 - i = (-i)(1 + 2i)$ , the factorization is unique up to order and multiplication of primes by units. □

**Exercise 1.6.**

Show that every nonzero, non-unit Gaussian integer  $\alpha$  is a product of irreducible elements, by induction on  $N(\alpha)$ .

*Proof.* Induction on  $N(\alpha)$ .

- (1)  $n = 2$ . Given  $\alpha \in \mathbb{Z}[i]$  with  $N(\alpha) = 2$ . Since  $N(\alpha) = 2$  is a prime in  $\mathbb{Z}$ ,  $\alpha$  is irreducible (Exercise 1.3).
- (2) Suppose the result holds for  $n \leq k$ . Given  $\alpha \in \mathbb{Z}[i]$  with  $N(\alpha) = k + 1$ . There are only two possible cases.
  - (a)  $\alpha$  is irreducible. Nothing to do.

- (b)  $\alpha$  is reducible. Write  $\alpha = \beta\gamma$  where neither factor is unit. Since  $N(\alpha) = N(\beta)N(\gamma)$  and neither factor is unit,

$$2 \leq N(\beta), N(\gamma) \leq k.$$

By the induction hypothesis, each factor of  $\alpha$  ( $\beta$  and  $\gamma$ ) is a product of irreducible elements. So that  $\alpha$  again is a product of irreducible elements.

In any cases,  $\alpha$  is a product of irreducible elements.

By induction, the result is established.  $\square$

### Exercise 1.7.

Show that  $\mathbb{Z}[i]$  is a principal ideal domain (PID); i.e., every ideal  $I$  is principal. (As shown in Appendix 1, this implies that  $\mathbb{Z}[i]$  is a UFD.)

*Suggestion:* Take  $\alpha \in I \setminus \{0\}$  such that  $N(\alpha)$  is minimized, and consider the multiplies  $\gamma\alpha$ ,  $\gamma \in \mathbb{Z}[i]$ ; show that these are the vertices of an infinite family of squares which fill up the complex plane. (For example, one of the squares has vertices  $0$ ,  $\alpha$ ,  $i\alpha$ , and  $(1+i)\alpha$ ; all others are translates of this one.) Obviously  $I$  contains all  $\gamma\alpha$ ; show by a geometric argument that if  $I$  contains anything else then minimality of  $N(\alpha)$  would be contradicted.

*Proof (without geometric intuition).* Define  $N$  on  $\mathbb{Q}[i]$  by  $N(a + bi) = a^2 + b^2$  where  $a + bi \in \mathbb{Q}[i]$  as usual.

- (1) Show that  $\mathbb{Z}[i]$  is a Euclidean domain. Given  $\alpha = a + bi \in \mathbb{Z}[i]$  and  $\gamma = c + di \in \mathbb{Z}[i]$  with  $\gamma \neq 0$ . It suffices to show there exist  $\delta$  and  $\rho$  such that the identity  $\alpha = \gamma\delta + \rho$  holds and either  $\rho = 0$  or  $N(\rho) < N(\gamma)$ .

- (a) Pick  $\delta \in \mathbb{Z}[i]$ . (Intuition: Pick the ‘integer part’ of  $\frac{\alpha}{\gamma}$  as we did in integer numbers.) Write  $\frac{\alpha}{\gamma} = r + si \in \mathbb{Q}[i]$ . Then we pick  $\delta = m + ni \in \mathbb{Z}[i]$  such that  $|r - m| \leq \frac{1}{2}$  and  $|s - n| \leq \frac{1}{2}$ . Therefore,

$$\begin{aligned} N\left(\frac{\alpha}{\gamma} - \delta\right) &= (r - m)^2 + (s - n)^2 \\ &\leq \frac{1}{4} + \frac{1}{4} \\ &= \frac{1}{2}. \end{aligned}$$

- (b) Pick  $\rho \in \mathbb{Z}[i]$ . Clearly we can pick  $\rho = \alpha - \gamma\delta \in \mathbb{Z}[i]$ . Therefore,

$\rho = 0$  or

$$\begin{aligned}
N(\rho) &= N(\alpha - \gamma\delta) \\
&= N\left(\gamma\left(\frac{\alpha}{\gamma} - \delta\right)\right) \\
&= N(\gamma)N\left(\frac{\alpha}{\gamma} - \delta\right) \\
&\leq \frac{1}{2}N(\gamma) \\
&< N(\gamma).
\end{aligned}$$

(2) *Show that every Euclidean domain  $R$  is a PID.* Given any ideal  $I$  of  $R$ . Take  $\alpha \in I \setminus \{0\}$  such that  $N(\alpha)$  is minimized.

(a)  $R\alpha \subseteq I$  clearly.

(b) Conversely, for any  $\beta \in I$ , there are  $\delta, \rho \in R$  such that  $\beta = \alpha\delta + \rho$ , where either  $\rho = 0$  or  $N(\rho) < N(\alpha)$ . Since  $\rho = \beta - \alpha\delta \in I$ , we cannot have  $N(\rho) < N(\alpha)$  by the minimality of  $N(\alpha)$ . Therefore,  $\rho = 0$  and  $\beta = \alpha\delta \in R\alpha$ , or  $R\alpha \supseteq I$ .

By (1)(2),  $\mathbb{Z}[i]$  is a PID.  $\square$

### Exercise 1.8.

We will use the unique factorization in  $\mathbb{Z}[i]$  to prove that every prime  $p \equiv 1 \pmod{4}$  is a sum of two squares.

- (a) Use the fact that the multiplicative group  $(\mathbb{Z}/p\mathbb{Z})^\times$  of integers mod  $p$  is cyclic to show that if  $p \equiv 1 \pmod{4}$  then  $n^2 \equiv -1 \pmod{p}$  for some  $n \in \mathbb{Z}$ .
- (b) Prove that  $p$  cannot be irreducible in  $\mathbb{Z}[i]$ . (Hint:  $p \mid n^2 + 1 = (n+i)(n-i)$ .)
- (c) Prove that  $p$  is a sum of two squares. (Hint: (b) shows that  $p = (a + bi)(c + di)$  with neither factor a unit. Take norms.)

*Proof of (a).* Since the multiplicative group  $(\mathbb{Z}/p\mathbb{Z})^\times$  of integers mod  $p$  is cyclic,  $(\mathbb{Z}/p\mathbb{Z})^\times$  is generated by (a primitive root)  $g \in \mathbb{Z}/p\mathbb{Z}$ .  $g^{p-1} = 1$ , or

$$(g^{\frac{p-1}{2}} - 1)(g^{\frac{p-1}{2}} + 1) = 0$$

since  $p$  is odd. Since  $\mathbb{Z}/p\mathbb{Z}$  is an integral domain,  $g^{\frac{p-1}{2}} - 1 = 0$  or  $g^{\frac{p-1}{2}} + 1 = 0$ .  $g$  cannot satisfy  $g^{\frac{p-1}{2}} - 1 = 0$  since  $g$  is a generator of  $(\mathbb{Z}/p\mathbb{Z})^\times$ . So,

$$g^{\frac{p-1}{2}} + 1 = 0.$$

Let  $n = g^{\frac{p-1}{4}} \in \mathbb{Z}$  since  $p \equiv 1 \pmod{4}$ . So  $n^2 + 1 = 0 \pmod{p}$ .  $\square$

*Proof of (b).* Since  $n^2 + 1 \equiv 0 \pmod{p}$  by (a),  $p \mid n^2 + 1 = (n+i)(n-i)$ . If  $p$  were irreducible in  $\mathbb{Z}[i]$ ,  $p \mid (n+i)$  or  $p \mid (n-i)$  by using the unique factorization in  $\mathbb{Z}[i]$ . Hence

$$\frac{n+i}{p} = \frac{n}{p} + \frac{1}{p}i \notin \mathbb{Z}[i], \frac{n-i}{p} = \frac{n}{p} - \frac{1}{p}i \notin \mathbb{Z}[i],$$

contrary to the assumption. Therefore,  $p$  is reducible in  $\mathbb{Z}[i]$ .  $\square$

*Proof of (c).* Since  $p$  is reducible in  $\mathbb{Z}[i]$  by (b), write  $p = (a+bi)(c+di)$  with neither factor a unit. Take norms,

$$p^2 = N(p) = N(a+bi)N(c+di).$$

Since neither factor of  $p$  is unit,  $N(a+bi) = p$ , or  $a^2 + b^2 = p$ , or  $p$  is a sum of two squares.  $\square$

### Exercise 1.9.

*Describe all irreducible elements in  $\mathbb{Z}[i]$ .*

*Notice that  $\alpha$  is irreducible if and only if  $\bar{\alpha}$  is irreducible.* (Write  $\alpha = \beta\gamma$ , then  $\bar{\alpha} = \bar{\beta}\bar{\gamma}$ . Besides,  $\bar{\bar{\alpha}} = \alpha$ .)

*Proof. Show that all irreducible elements in  $\mathbb{Z}[i]$  (up to units) are*

- (1)  $1+i$ .
- (2)  $\pi = a+bi$  for each integer prime  $p \equiv 1 \pmod{4}$  with  $p = a^2 + b^2$ .
- (3)  $p$  for each integer prime  $p \equiv 3 \pmod{4}$ .

Let  $\alpha$  be any irreducible element in  $\mathbb{Z}[i]$ . Consider  $N(\alpha) = \alpha\bar{\alpha}$ .  $N(\alpha) \neq 1$  since  $\alpha$  is not unit. By the unique factorization theorem in  $\mathbb{Z}$ ,  $N(\alpha) \in \mathbb{Z}$  is a product of primes in  $\mathbb{Z}$ .

There are three possible cases.

- (a)  $2 \mid N(\alpha)$ . Write  $(1+i)(1-i) \mid \alpha\bar{\alpha}$  in  $\mathbb{Z}[i]$ . Notice that  $1+i$ ,  $1-i$ ,  $\alpha$  and  $\bar{\alpha}$  are all irreducible (Exercise 1.4). By the unique factorization theorem in  $\mathbb{Z}[i]$ ,  $\alpha = 1+i$  (up to units).
- (b)  $p \mid N(\alpha)$  for some prime  $p \equiv 3 \pmod{4}$ . Write  $p \mid \alpha\bar{\alpha}$  in  $\mathbb{Z}[i]$ . Notice that  $p$ ,  $\alpha$  and  $\bar{\alpha}$  are all irreducible (Exercise 1.3). By the unique factorization theorem in  $\mathbb{Z}[i]$ ,  $\alpha = p$  (up to units) or  $\bar{\alpha} = p$  (up to units). So in any cases  $\alpha = p$  (up to units). (Note that  $\bar{p} = p$ .)



- (c)  $p \mid N(\alpha)$  for some prime  $p \equiv 1 \pmod{4}$ . For such  $p$ , there is an irreducible  $\pi \in \mathbb{Z}[i]$  satisfying  $p = \pi\bar{\pi}$  (Exercise 1.8). Now we write  $\pi\bar{\pi} \mid \alpha\bar{\alpha}$  in  $\mathbb{Z}[i]$ . Notice that  $\pi, \bar{\pi}, \alpha$  and  $\bar{\alpha}$  are all irreducible. By the unique factorization theorem in  $\mathbb{Z}[i]$ ,  $\alpha = \pi$  or  $\alpha = \bar{\pi}$ . In any cases,  $\alpha = a + bi$  for integer prime  $p \equiv 1 \pmod{4}$  with  $p = a^2 + b^2$ .

□

*Exercise 1.10 - 1.14:* Let  $\omega = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ . Define  $N : \mathbb{Z}[\omega] \rightarrow \mathbb{Z}$  by  $N(a + b\omega) = a^2 - ab + b^2$ .

**Exercise 1.10.**

Show that if  $a + b\omega$  is written in the form  $u + vi$  where  $u$  and  $v$  are real, then  $N(a + b\omega) = u^2 + v^2$ .

*Proof.* By  $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ , write

$$a + b\omega = \left(a - \frac{1}{2}b\right) + \left(\frac{\sqrt{3}}{2}b\right)i.$$

Here  $u = a - \frac{1}{2}b \in \mathbb{R}$  and  $v = \frac{\sqrt{3}}{2}b \in \mathbb{R}$ . Hence  $u^2 + v^2 = (a - \frac{1}{2}b)^2 + (\frac{\sqrt{3}}{2}b)^2 = a^2 - ab + b^2 = N(a + b\omega)$ . □

**Exercise 1.11.**

Show that for all  $\alpha, \beta \in \mathbb{Z}[\omega]$ ,  $N(\alpha\beta) = N(\alpha)N(\beta)$ , either by direct computation or by using Exercise 1.10. Conclude that if  $\alpha \mid \gamma$  in  $\mathbb{Z}[\omega]$ , then  $N(\alpha) \mid N(\gamma)$  in  $\mathbb{Z}$ .

*Proof.*

- (1) *Direct computation.* Note that  $1 + \omega + \omega^2 = 0$  or  $\omega^2 = -1 - \omega$ . Write  $\alpha = a + b\omega, \beta = c + d\omega$  where  $a, b, c, d \in \mathbb{Z}$ . Thus,

$$\begin{aligned} N(\alpha\beta) &= N((a + b\omega)(c + d\omega)) \\ &= N(ac + (ad + bc)\omega + bd\omega^2) \\ &= N(ac + (ad + bc)\omega + bd(-1 - \omega)) \\ &= N((ac - bd) + (ad + bc - bd)\omega) \\ &= (ac - bd)^2 - (ac - bd)(ad + bc - bd) + (ad + bc - bd)^2 \\ &= (a^2 - ab + b^2)(c^2 - cd + d^2), \\ N(\alpha)N(\beta) &= N(a + b\omega)N(c + d\omega) \\ &= (a^2 - ab + b^2)(c^2 - cd + d^2). \end{aligned}$$

- (2) *Exercise 1.10.* The result is established by Exercise 1.10 and Exercise 1.1.
- (3) *Using the fact that  $N(a+b\omega) = (a+b\omega)\overline{(a+b\omega)}$ .* Similar to the argument of Exercise 1.1.
- (4) *Show that if  $\alpha \mid \gamma$  in  $\mathbb{Z}[\omega]$ , then  $N(\alpha) \mid N(\gamma)$  in  $\mathbb{Z}$ .* Similar to the argument of Exercise 1.1.

□

**Exercise 1.12.**

Let  $\alpha \in \mathbb{Z}[\omega]$ . Show that  $\alpha$  is a unit iff  $N(\alpha) = 1$ , and find all units in  $\mathbb{Z}[\omega]$ . (There are six of them.)

*Proof.*

- (1) ( $\implies$ ) Since  $\alpha$  is a unit, there is  $\beta \in \mathbb{Z}[\omega]$  such that  $\alpha\beta = 1$ . By Exercise 1.11,  $N(\alpha\beta) = N(1)$ , or  $N(\alpha)N(\beta) = 1$ . Since the image of  $N$  is nonnegative integers,  $N(\alpha) = 1$ .
- (2) ( $\impliedby$ ) By Exercise 1.10,  $N(\alpha) = \alpha\bar{\alpha}$ , or  $1 = \alpha\bar{\alpha}$  since  $N(\alpha) = 1$ . That is,  $\bar{\alpha} \in \mathbb{Z}[\omega]$  is the inverse of  $\alpha \in \mathbb{Z}[\omega]$ .
- (3) By (1), we solve the equation  $N(\alpha) = a^2 - ab + b^2 = 1$ , or  $4 = (2a-b)^2 + 3b^2$ . There are 2 possible cases.
  - (a)  $2a - b = \pm 1, b = \pm 1$ .
  - (b)  $2a - b = \pm 2, b = \pm 0$ .

Solve these 6 pairs of equations yields the result  $\pm 1, \pm\omega, \pm\omega^2$ .

□

**Exercise 1.13.**

Show that  $1 - \omega$  is irreducible in  $\mathbb{Z}[\omega]$ , and that  $3 = u(1 - \omega)^2$  for some unit  $u$ .

3 is not irreducible in  $\mathbb{Z}[\omega]$ .

*Proof.*

- (1)  $N(1 - \omega) = 3$  is an integer prime. Similar to the argument in Exercise 1.3,  $1 - \omega$  is irreducible in  $\mathbb{Z}[\omega]$ .

- (2) Note that  $1 + \omega + \omega^2 = 0$ . So  $(1 - \omega)^2 = 1 - 2\omega + \omega^2 = 3(-\omega)$ , or  $(-\omega^2)(1 - \omega)^2 = 3$ . By Exercise 1.12,  $-\omega^2$  is unit. Hence  $3 = u(1 - \omega)^2$  for some unit  $u = -\omega^2$ .

□

#### Exercise 1.14.

Modify Exercise 1.7 to show that  $\mathbb{Z}[\omega]$  is a PID, hence a UFD. Here the squares are replaced by parallelograms; one of them has vertices  $0, \alpha, \omega\alpha, (\omega+1)\alpha$ , and all others are translates of this one. Use Exercise 1.10 for the geometric argument at the end.

Similar to Exercise 1.7.

*Proof (without geometric intuition).* Define  $N$  on  $\mathbb{Q}[\omega]$  by  $N(a+b\omega) = a^2 - ab + b^2$  where  $a + b\omega \in \mathbb{Q}[\omega]$  as usual.

- (1) Show that  $\mathbb{Z}[\omega]$  is a Euclidean domain. Given  $\alpha = a + b\omega \in \mathbb{Z}[\omega]$  and  $\gamma = c + d\omega \in \mathbb{Z}[\omega]$  with  $\gamma \neq 0$ . It suffices to show there exist  $\delta$  and  $\rho$  such that the identity  $\alpha = \gamma\delta + \rho$  holds and either  $\rho = 0$  or  $N(\rho) < N(\gamma)$ .
- (a) Pick  $\delta \in \mathbb{Z}[\omega]$ . (Intuition: Pick the ‘integer part’ of  $\frac{\alpha}{\gamma}$  as we did in integer numbers.) Write  $\frac{\alpha}{\gamma} = r + s\omega \in \mathbb{Q}[\omega]$ . Then we pick  $\delta = m + n\omega \in \mathbb{Z}[\omega]$  such that  $|r - m| \leq \frac{1}{2}$  and  $|s - n| \leq \frac{1}{2}$ . Therefore,

$$\begin{aligned} N\left(\frac{\alpha}{\gamma} - \delta\right) &\leq |r - m|^2 + |r - m||s - n| + |s - n|^2 \\ &\leq \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \\ &= \frac{3}{4}. \end{aligned}$$

- (b) Pick  $\rho \in \mathbb{Z}[\omega]$ . Clearly we can pick  $\rho = \alpha - \gamma\delta \in \mathbb{Z}[\omega]$ . Therefore,  $\rho = 0$  or

$$\begin{aligned} N(\rho) &= N(\alpha - \gamma\delta) \\ &= N\left(\gamma\left(\frac{\alpha}{\gamma} - \delta\right)\right) \\ &= N(\gamma)N\left(\frac{\alpha}{\gamma} - \delta\right) \\ &\leq \frac{3}{4}N(\gamma) \\ &< N(\gamma). \end{aligned}$$

(2) Show that every Euclidean domain  $R$  is a PID. Given any ideal  $I$  of  $R$ . Take  $\alpha \in I \setminus \{0\}$  such that  $N(\alpha)$  is minimized.

(a)  $R\alpha \subseteq I$  clearly.

(b) Conversely, for any  $\beta \in I$ , there are  $\delta, \rho \in R$  such that  $\beta = \alpha\delta + \rho$ , where either  $\rho = 0$  or  $N(\rho) < N(\alpha)$ . Since  $\rho = \beta - \alpha\delta \in I$ , we cannot have  $N(\rho) < N(\alpha)$  by the minimality of  $N(\alpha)$ . Therefore,  $\rho = 0$  and  $\beta = \alpha\delta \in R\alpha$ , or  $R\alpha \supseteq I$ .

By (1)(2),  $\mathbb{Z}[\omega]$  is a PID.  $\square$

### Exercise 1.15.

Here is a proof of Fermat's conjecture for  $n = 4$ : If  $x^4 + y^4 = z^4$  has a solution in positive integers, then so does  $x^4 + y^4 = w^2$ . Let  $x, y, w$  be a solution with smallest possible  $w$ . Then  $x^2, y^2, w$  is a primitive Pythagorean triple. Assuming (without loss of generality) that  $x$  is odd, we can write

$$x^2 = m^2 - n^2, y^2 = 2mn, w = m^2 + n^2$$

with  $m$  and  $n$  are relatively prime positive integers, not both odd.

(a) Show that

$$x = r^2 - s^2, n = 2rs, m = r^2 + s^2$$

with  $r$  and  $s$  are relatively prime positive integers, not both odd.

(b) Show that  $r, s$  and  $m$  are pairwise relatively prime. Using  $y^2 = 4rsm$ , conclude that  $r, s$  and  $m$  are all squares, say  $a^2, b^2$  and  $c^2$ .

(c) Show that  $a^4 + b^4 = c^2$ , and that this contradicts minimality of  $w$ .

*Proof of (a).* Write  $x^2 + n^2 = m^2$  by moving  $n^2$  of  $x^2 = m^2 - n^2$  to the left side. Notice that  $x$  is odd, and thus  $x = r^2 - s^2, n = 2rs, m = r^2 + s^2$  with  $r$  and  $s$  are relatively prime positive integers, not both odd.  $\square$

*Proof of (b).*

(1) It suffices to show that  $(r, m) = 1$ . By assumption,  $(r, s) = 1$ . So  $(r, s) = 1 \Rightarrow (r, s^2) = 1 \Rightarrow (r, r^2 + s^2) = 1$  and note that  $m = r^2 + s^2$  to get the result.

(2)  $y^2 = 2mn = 2m(2rs) = 4rsm$  by (a). Since  $r, s$  and  $m$  are pairwise relatively prime,  $r, s$  and  $m$  are all squares.

□

*Proof of (c).* By (b),  $r = a^2$ ,  $s = b^2$ ,  $m = c^2$ . By (a),  $m = r^2 + s^2$ , or  $c^2 = (a^2)^2 + (b^2)^2 = a^4 + b^4$ . However,  $w = m^2 + n^2 > m^2 > m = c^2 > c$ , contrary to the minimality of  $w$ . □

*Exercise 1.16-1.28:* Let  $p$  be an odd prime,  $\omega = e^{\frac{2\pi i}{p}}$ .

**Exercise 1.16.**

Show that

$$(1 - \omega)(1 - \omega^2) \cdots (1 - \omega^{p-1}) = p$$

by considering equation  $t^p - 1 = (t - 1)(t - \omega)(t - \omega^2) \cdots (t - \omega^{p-1})$ .

*Proof.* Note that  $t^p - 1 = (t - 1)(t^{p-1} + t^{p-2} + \cdots + t + 1)$ . Cancel out  $t - 1$  of Equation (2),

$$t^{p-1} + t^{p-2} + \cdots + t + 1 = (t - \omega)(t - \omega^2) \cdots (t - \omega^{p-1}).$$

Put  $t = 1$  to get  $p = (1 - \omega)(1 - \omega^2) \cdots (1 - \omega^{p-1})$ . □

**Exercise 1.17.**

Let  $x^p + y^p = z^p$ . Suppose that  $\mathbb{Z}[\omega]$  is a UFD and  $\pi \mid x + y\omega$ , and  $\pi$  is a prime in  $\mathbb{Z}[\omega]$ . Show that  $\pi$  does not divide any of the other factors on the left side of

$$(x + y)(x + y\omega)(x + y\omega^2) \cdots (x + y\omega^{p-1}) = z^p$$

by showing that if it did, then  $\pi$  would divide both  $z$  and  $yp$  (Hint: Use Exercise 1.16); but  $z$  and  $yp$  are relatively prime (assuming  $p$  divides none of  $x, y, z$ ), hence  $zm + ypn = 1$  for some  $m, n \in \mathbb{Z}$ . How is this a contradiction?

*Proof.* Write

$$z = u\pi_1^{e_1} \cdots \pi_m^{e_m}$$

where  $u$  is unit and  $\pi_k$  ( $1 \leq k \leq m$ ) are distinct primes in  $\mathbb{Z}[\omega]$  and  $e_k \in \mathbb{Z}^+$  ( $1 \leq k \leq m$ ). Since  $\mathbb{Z}[\omega]$  is a UFD by assumption, the factorization of  $z$  is unique up to order and units.

(1) Show that  $\pi \mid z$ . Since  $\pi \mid x + y\omega$ ,  $\pi \mid z^p$ . The factorization of  $z^p$  is

$$z^p = u^p \pi_1^{pe_1} \cdots \pi_m^{pe_m}.$$

$u^p$  is unit, and  $\pi \mid z^p$  implies that  $\pi = \pi_k$  for some  $k$ , that is,  $\pi \mid z$ .

(2) Show that  $\pi \mid yp$  if  $\pi$  were divide any of the other factors on the left side of  $(x+y)(x+y\omega)(x+y\omega^2)\cdots(x+y\omega^{p-1}) = z^p$ . Say  $\pi \mid x+y\omega^k$  for some  $k \neq 1$ . So that  $\pi \mid ((x+y\omega) - (x+y\omega^k))$ , or  $\pi \mid y(\omega - \omega^k)$ . Since  $k \neq 1$ , there are two possible cases.

(a)  $k > 1$ .  $\pi \mid y\omega(1 - \omega^{k-1})$ . By Exercise 1.16,  $\pi \mid y\omega p$ , or  $\pi \mid yp$  since  $\omega$  is unit. ( $\omega^{p-1}$  is the inverse of  $\omega$  since  $\omega \cdot \omega^{p-1} = 1$ .)

(b)  $k = 0$ .  $\pi \mid y(\omega - 1)$ , or  $\pi \mid y(1 - \omega)$ . By Exercise 1.16,  $\pi \mid yp$ .

In any case,  $\pi \mid yp$ .

(3) Note that  $z$  and  $yp$  are integers, and they are relatively prime by the assumption that  $p$  divides none of  $x, y, z$ . Therefore, on  $\mathbb{Z}$  we have  $zm + ypn = 1$  for some  $m, n \in \mathbb{Z}$ .

(4)  $zm + ypn = 1$  is also true in  $\mathbb{Z}[\omega]$ . Therefore, by (1)(2) we have  $\pi \mid (zm + ypn)$  or  $\pi \mid 1$ , or  $\pi$  is unit, contrary to the primality of  $\pi$ .

□

### Exercise 1.18.

Use Exercise 1.17 to show that if  $\mathbb{Z}[\omega]$  is a UFD then  $x + y\omega = u\alpha^p$ ,  $\alpha \in \mathbb{Z}[\omega]$ ,  $u$  a unit in  $\mathbb{Z}[\omega]$ .

*Proof.*

(1) Write  $z = u\pi_1^{e_1} \cdots \pi_m^{e_m}$  as Exercise 1.17. So

$$z^p = u^p \pi_1^{pe_1} \cdots \pi_m^{pe_m}.$$

(2) Factorize  $x + y\omega = vq_1^{f_1} \cdots q_n^{f_n}$ , where  $v$  is unit and all  $q_h$  ( $1 \leq h \leq n$ ) are distinct primes in  $\mathbb{Z}[\omega]$  and  $f_h \in \mathbb{Z}^+$ . Since  $\mathbb{Z}[\omega]$  is a UFD, for every  $q_h \mid x + y\omega$ , there is some  $k(h)$  such that  $q_h = \pi_{k(h)}$  and also  $q_h^{f_h} = \pi_{k(h)}^{pe_{k(h)}}$  or  $f_h = pe_{k(h)}$ .

(3) Hence,

$$x + y\omega = v \left( \pi_{k(1)}^{e_{k(1)}} \cdots \pi_{k(n)}^{e_{k(n)}} \right)^p,$$

where  $\alpha = \pi_{k(1)}^{e_{k(1)}} \cdots \pi_{k(n)}^{e_{k(n)}} \in \mathbb{Z}[\omega]$  and  $v$  is unit.

□

**Exercise 1.19.**

Dropping the assumption that  $\mathbb{Z}[\omega]$  is a UFD but using the fact that ideals factor uniquely (up to order) into prime ideals, show that the principal ideal  $(x + y\omega)$  has no prime ideal factor in common with any of the other principal ideals on the left side of the equation

$$(x + y)(x + y\omega)(x + y\omega^2) \cdots (x + y\omega^{p-1}) = (z)^p$$

in which all factors are interpreted as principal ideals. (Hint: Modify the proof of Exercise 1.17 appropriately, using the fact that if  $A$  is an ideal dividing another ideal  $B$ , then  $A \supseteq B$ .)

*Proof.* Write

$$(z) = \pi_1^{e_1} \cdots \pi_m^{e_m}$$

where  $\pi_k$  ( $1 \leq k \leq m$ ) are distinct prime ideals of  $\mathbb{Z}[\omega]$  and  $e_k \in \mathbb{Z}^+$  ( $1 \leq k \leq m$ ). By assumption that  $\mathbb{Z}[\omega]$  is a Dedekind domain, the factorization of  $z$  is unique up to order.

- (1) Show that  $\pi \mid (z)$ . Since  $\pi \mid (x + y\omega)$ ,  $\pi \mid (z)^p$ . The factorization of  $(z)^p$  is

$$(z)^p = \pi_1^{pe_1} \cdots \pi_m^{pe_m}.$$

$\pi \mid (z)^p$  implies that  $\pi = \pi_k$  for some  $k$ , that is,  $\pi \mid (z)$ .

- (2) Show that  $\pi \mid (yp)$  if  $\pi$  were divide any of the other factors on the left side of  $(x + y)(x + y\omega)(x + y\omega^2) \cdots (x + y\omega^{p-1}) = (z)^p$ . Say  $\pi \mid (x + y\omega^k)$  for some  $k \neq 1$ . So that  $x + y\omega \in \pi$  and  $x + y\omega^k \in \pi$ , or  $y(\omega - \omega^k) \in \pi$ . Since  $k \neq 1$ , there are two possible cases.

- (a)  $k > 1$ .  $y\omega(1 - \omega^{k-1}) \in \pi$ . By Exercise 1.16,  $y\omega p \in \pi$ , or  $yp \in \pi$  since  $\omega$  is unit. ( $\omega^{p-1}$  is the inverse of  $\omega$  since  $\omega \cdot \omega^{p-1} = 1$ .)  
(b)  $k = 0$ .  $y(\omega - 1) \in \pi$ , or  $y(1 - \omega) \in \pi$ . By Exercise 1.16,  $yp \in \pi$ .

In any case,  $yp \in \pi$ , or  $\pi \mid (yp)$ .

- (3) Note that  $z$  and  $yp$  are integers, and they are relatively prime by the assumption that  $p$  divides none of  $x, y, z$ . Therefore, on  $\mathbb{Z}$  we have  $zm + ypn = 1$  for some  $m, n \in \mathbb{Z}$ .  
(4)  $zm + ypn = 1$  is also true in  $\mathbb{Z}[\omega]$ . Therefore, by (1)(2) we have  $z \in \pi$  and  $yp \in \pi$ . So  $zm + ypn \in \pi$  since  $\pi$  is an ideal. So  $1 \in \pi$  or  $\pi = (1)$ , contrary to the primality of  $\pi$ .

□

**Exercise 1.20.**

Use Exercise 1.19 to show that  $(x + y\omega) = I^p$  for some ideal  $I$ .

*Proof.*

- (1) Write  $(z) = \pi_1^{e_1} \cdots \pi_m^{e_m}$  as Exercise 1.17. So

$$(z)^p = \pi_1^{pe_1} \cdots \pi_m^{pe_m}.$$

- (2) Factorize  $(x + y\omega) = q_1^{f_1} \cdots q_n^{f_n}$ , where every  $q_h$  ( $1 \leq h \leq n$ ) are distinct prime ideals of  $\mathbb{Z}[\omega]$  and  $f_h \in \mathbb{Z}^+$ . By assumption that  $\mathbb{Z}[\omega]$  is a Dedekind domain, for every  $q_h \mid (x + y\omega)$ , there is some  $k(h)$  such that  $q_h = \pi_{k(h)}$  and also  $q_h^{f_h} = \pi_{k(h)}^{pe_{k(h)}}$  or  $f_h = pe_{k(h)}$ .

- (3) Hence,

$$(x + y\omega) = \left( \pi_{k(1)}^{e_{k(1)}} \cdots \pi_{k(n)}^{e_{k(n)}} \right)^p,$$

where  $I = \pi_{k(1)}^{e_{k(1)}} \cdots \pi_{k(n)}^{e_{k(n)}}$  is an ideal of  $\mathbb{Z}[\omega]$ .

□

**Exercise 1.21.**

Show that every number of  $\mathbb{Q}[\omega]$  is uniquely representable in the form

$$a_0 + a_1\omega + a_2\omega^2 + \cdots + a_{p-2}\omega^{p-2}, a_i \in \mathbb{Q} \ \forall i$$

by show that  $\omega$  is a root of the polynomial

$$f(t) = t^{p-1} + t^{p-2} + \cdots + t + 1$$

and that  $f(t)$  is irreducible over  $\mathbb{Q}$ . (Hint: It is enough to show that  $f(t+1)$  is irreducible, which can be established by Eisenstein's criterion. It helps to notice that  $f(t+1) = \frac{(t+1)^p - 1}{t}$ .)

*Proof.*

- (1) Given any number  $\alpha \in \mathbb{Q}[\omega]$ . Show that

$$\alpha = a_0 + a_1\omega + a_2\omega^2 + \cdots + a_{p-2}\omega^{p-2}, a_i \in \mathbb{Q} \ \forall i.$$

Since  $\omega^p = 1$ , we can write

$$\alpha = a'_0 + a'_1\omega + a'_2\omega^2 + \cdots + a'_{p-2}\omega^{p-2} + a'_{p-1}\omega^{p-1}, a_i \in \mathbb{Q} \ \forall i.$$

Note that  $\omega^{p-1} + \omega^{p-2} + \cdots + \omega + 1 = 0$ , and thus we can replace  $\omega^{p-1}$  by  $-\omega^{p-2} - \cdots - \omega - 1$ .



- (2) Show that  $\omega$  is a root of the polynomial  $f(t) = t^{p-1} + t^{p-2} + \cdots + t + 1$ .  
 $f(\omega) = \omega^{p-1} + \omega^{p-2} + \cdots + \omega + 1 = 0$ .
- (3) Show that  $f(t)$  is irreducible over  $\mathbb{Q}$ . It suffices to show that  $f(t+1)$  is irreducible over  $\mathbb{Q}$ . Write  $(t-1)f(t) = t^p - 1$ . So

$$\begin{aligned}
tf(t+1) &= (t+1)^p - 1 && \text{(Put } t \mapsto t+1\text{)} \\
&= \left( \sum_{k=0}^p \binom{p}{k} t^k \right) - 1 && \text{(Binomial theorem)} \\
&= \sum_{k=1}^p \binom{p}{k} t^k, \\
f(t+1) &= \sum_{k=1}^p \binom{p}{k} t^{k-1} \\
&= t^{p-1} + pt^{p-2} + \cdots + \frac{p(p-1)}{2}t + p.
\end{aligned}$$

By Eisenstein's criterion,  $f(t+1)$  is irreducible over  $\mathbb{Q}$ .

- (4) To show the uniqueness, it suffices to show that the relation

$$0 = a_0 + a_1\omega + a_2\omega^2 + \cdots + a_{p-2}\omega^{p-2}$$

implies all  $a_i = 0$ . Say  $g(t) = a_0 + a_1t + a_2t^2 + \cdots + a_{p-2}t^{p-2} \in \mathbb{Q}[t]$ . Clearly  $g(\omega) = 0$ . By the minimality of  $f(t)$ ,  $g(t)$  is identical zero, or all  $a_i = 0$ .

□

### Exercise 1.22.

Use Exercise 1.21 to show that if  $\alpha \in \mathbb{Z}[\omega]$  and  $p \mid \alpha$ , then (writing  $\alpha = a_0 + a_1\omega + \cdots + a_{p-2}\omega^{p-2}$ ,  $a_i \in \mathbb{Z}$ ) all  $a_i$  are divisible by  $p$ .

*Proof.* Since  $p \mid \alpha$ , there is  $\beta \in \mathbb{Z}[\omega]$  such that  $\alpha = p\beta$ . Write

$$\begin{aligned}
\alpha &= a_0 + a_1\omega + \cdots + a_{p-2}\omega^{p-2}, \\
\beta &= b_0 + b_1\omega + \cdots + b_{p-2}\omega^{p-2},
\end{aligned}$$

where  $a_i, b_j \in \mathbb{Z}$ . By  $\alpha = p\beta$  and Exercise 1.21,  $a_i = pb_i$  for every  $1 \leq i \leq p-2$ . So all  $a_i$  are divisible by  $p$ . □

Define congruence mod  $p$  for  $\beta, \gamma \in \mathbb{Z}[\omega]$  as follows:

$$\beta \equiv \gamma \pmod{p} \text{ iff } \beta - \gamma = \delta p \text{ for some } \delta \in \mathbb{Z}[\omega].$$

(Equivalently, this is congruence mod the principal ideal  $p\mathbb{Z}[\omega]$ ).

**Exercise 1.23.**

Show that if  $\beta \equiv \gamma \pmod{p}$ , then  $\bar{\beta} \equiv \bar{\gamma} \pmod{p}$  where the bar denotes complex conjugation.

*Proof.*

(1) Show that  $\bar{\delta} \in \mathbb{Z}[\omega]$  for any  $\delta \in \mathbb{Z}[\omega]$ . Write

$$\delta = a_0 + a_1\omega + \cdots + a_{p-1}\omega^{p-1}$$

where  $a_0, \dots, a_{p-1} \in \mathbb{Z}$ . Take the complex conjugation to get

$$\begin{aligned} \bar{\delta} &= \overline{a_0} + \overline{a_1} \cdot \bar{\omega} + \cdots + \overline{a_{p-1}} \cdot \bar{\omega}^{p-1} \\ &= a_0 + a_1\bar{\omega} + \cdots + a_{p-1}\bar{\omega}^{p-1} && \text{(Every } a_k \in \mathbb{Z}) \\ &= a_0 + a_1\omega^{p-1} + \cdots + a_{p-1}\omega \in \mathbb{Z}[\omega]. && (\omega^p = 1) \end{aligned}$$

(2)

$$\begin{aligned} \beta &\equiv \gamma \pmod{p} \\ \iff \beta - \gamma &= \delta p \text{ for some } \delta \in \mathbb{Z}[\omega] \\ \iff \bar{\beta} - \bar{\gamma} &= \bar{\delta} p \text{ for some } \delta \in \mathbb{Z}[\omega] && \text{(Complex conjugation)} \\ \iff \bar{\beta} - \bar{\gamma} &= \delta' p \text{ for some } \delta' \in \mathbb{Z}[\omega] && ((1)) \\ \iff \bar{\beta} &\equiv \bar{\gamma} \pmod{p} \end{aligned}$$

□

**Exercise 1.24.**

Show that  $(\beta + \gamma)^p \equiv \beta^p + \gamma^p \pmod{p}$  and generalize this to sums of arbitrarily many terms by induction.

*Proof.*

(1) Binomial theorem gives us

$$(\beta + \gamma)^p = \sum_{k=0}^p \binom{p}{k} \beta^k \gamma^{p-k} = \beta^p + \gamma^p + \sum_{k=1}^{p-1} \binom{p}{k} \beta^k \gamma^{p-k}.$$

(2) Note that every binomial coefficient  $\binom{p}{k}$  is divided by  $p$  in  $\mathbb{Z}$  for  $1 \leq k \leq p-1$ . Also, every term  $\beta^k \gamma^{p-k}$  is in  $\mathbb{Z}[\omega]$ . So  $(\beta + \gamma)^p - \beta^p - \gamma^p = \delta p$  for some  $\delta \in \mathbb{Z}[\omega]$ . Hence the result holds.

(3) In general,

$$\left( \sum_{k=1}^n \alpha_k \right)^p \equiv \sum_{k=1}^n \alpha_k^p \pmod{p}.$$

Induction by  $(\alpha_1 + \alpha_2)^p \equiv \alpha_1^p + \alpha_2^p \pmod{p}$  and  $\left( \sum_{k=1}^{n+1} \alpha_k \right)^p \equiv (\sum_{k=1}^n \alpha_k)^p + \alpha_{n+1}^p \equiv (\sum_{k=1}^n \alpha_k^p) + \alpha_{n+1}^p \equiv \sum_{k=1}^{n+1} \alpha_k^p \pmod{p}$ .

□

### Exercise 1.25.

Show that for all  $\alpha \in \mathbb{Z}[\omega]$ ,  $\alpha^p$  is congruent  $\pmod{p}$  to some  $a \in \mathbb{Z}$ . (Hint: Write  $\alpha$  in terms of  $\omega$  and use Exercise 1.24.)

*Proof (Hint).* Write

$$\alpha = a_0 + a_1\omega + \cdots + a_{p-1}\omega^{p-1}$$

where  $a_0, \dots, a_{p-1} \in \mathbb{Z}$ . By Exercise 1.24,

$$\begin{aligned} \alpha^p &\equiv a_0^p + (a_1\omega)^p + \cdots + (a_{p-1}\omega^{p-1})^p \\ &\equiv a_0^p + a_1^p\omega^p + \cdots + a_{p-1}^p(\omega^{p-1})^p \\ &\equiv a_0^p + a_1^p\omega^p + \cdots + a_{p-1}^p(\omega^p)^{p-1} \\ &\equiv a_0^p + a_1^p + \cdots + a_{p-1}^p. \end{aligned} \quad (\omega^p = 1)$$

Here  $a_0^p + a_1^p + \cdots + a_{p-1}^p \in \mathbb{Z}$ , and thus  $\alpha^p$  is congruent  $\pmod{p}$  to some integer. □

*Exercise 1.26-1.28:* Now assume  $p \geq 5$ . We will show that if  $x + y\omega = u\alpha^p \pmod{p}$ ,  $\alpha \in \mathbb{Z}[\omega]$ ,  $u$  a unit in  $\mathbb{Z}[\omega]$ ,  $x$  and  $y$  integers not divisible by  $p$ , then  $x \equiv y \pmod{p}$ . For this we will need the following result, proved by Kummer, on the units of  $\mathbb{Z}[\omega]$ :

*Lemma:* If  $u$  is a unit in  $\mathbb{Z}[\omega]$  and  $\bar{u}$  is its complex conjugate, then  $u/\bar{u}$  is a power of  $\omega$ . (For the proof, see Exercise 2.12.)

**Exercise 1.26.**

Show that  $x + y\omega \equiv u\alpha^p \pmod{p}$  implies

$$x + y\omega \equiv (x + y\omega^{-1})\omega^k \pmod{p}$$

for some  $k \in \mathbb{Z}$ . (Use the Lemma on units and Exercise 1.23 and 1.25. Note that  $\bar{\omega} = \omega^{-1}$ .)

*Proof (Hint).*

$$\begin{aligned} x + y\omega &\equiv u\alpha^p \pmod{p} \\ \implies x + y\omega &\equiv ua \pmod{p} \text{ for some } a \in \mathbb{Z} && \text{(Exercise 1.25)} \\ \implies \overline{x + y\omega} &\equiv \bar{u}\bar{a} \pmod{p} && \text{(Exercise 1.23)} \\ \implies x + y\bar{\omega} &\equiv \bar{u}a \pmod{p} \\ \implies x + y\omega^{-1} &\equiv \bar{u}a \pmod{p} && (\bar{\omega} = \omega^{-1}) \\ \implies x + y\omega^{-1} &\equiv u\omega^{-k}a \pmod{p} \text{ for some } k \in \mathbb{Z} && \text{(Lemma)} \\ \implies ua &\equiv (x + y\omega^{-1})\omega^k \pmod{p} \\ \implies x + y\omega &\equiv (x + y\omega^{-1})\omega^k \pmod{p}. \end{aligned}$$

□

**Exercise 1.27.**

Use Exercise 1.22 to show that a contradiction results unless  $k \equiv 1 \pmod{p}$ . (Recall that  $p \nmid xy$ ,  $p \geq 5$ , and  $\omega^{p-1} + \omega^{p-2} + \cdots + \omega + 1 = 0$ .)

*Proof.* Exercise 1.26 shows

$$x + y\omega \equiv (x + y\omega^{-1})\omega^k \pmod{p}.$$

Multiply  $\omega$  on the both sides to get  $x\omega + y\omega^2 \equiv y\omega^k + x\omega^{k+1} \pmod{p}$ , or

$$p \mid (x\omega + y\omega^2 - y\omega^k - x\omega^{k+1}).$$

If  $k$  were satisfying  $k \not\equiv 1 \pmod{p}$ , then by Exercise 1.22 and  $p \geq 5$  we have  $p \mid x$  or  $p \mid y$ , contrary to the assumption that  $x$  and  $y$  are integers not divisible by  $p$ . □

**Exercise 1.28.**

Finally, show  $x \equiv y \pmod{p}$ .

*Proof.* In the argument of Exercise 1.27 we have

$$p \mid ((x - y)\omega + (y - x)\omega^2)$$

by replacing  $k = 1$ . By Exercise 1.22 and  $p \geq 5$ ,  $x - y$  is divisible by  $p$ , or  $x \equiv y \pmod{p}$  as integers.  $\square$

**Exercise 1.29.**

Let  $\omega = \exp(\frac{2\pi i}{23})$ . Verify that the product

$$(1 + \omega^2 + \omega^4 + \omega^5 + \omega^6 + \omega^{10} + \omega^{11})(1 + \omega + \omega^5 + \omega^6 + \omega^7 + \omega^9 + \omega^{11})$$

is divisible by 2 in  $\mathbb{Z}[\omega]$ , although neither factor is. It can be shown (Exercise 3.17) that 2 is an irreducible element in  $\mathbb{Z}[\omega]$ ; it follows that  $\mathbb{Z}[\omega]$  cannot be a UFD.

*Proof.* Note that  $\sum_{k=0}^{22} \omega^k = 0$ . So

$$\begin{aligned} & (1 + \omega^2 + \omega^4 + \omega^5 + \omega^6 + \omega^{10} + \omega^{11})(1 + \omega + \omega^5 + \omega^6 + \omega^7 + \omega^9 + \omega^{11}) \\ &= 2(\omega^5 + \omega^6 + \omega^7 + \omega^9 + \omega^{10} + 3\omega^{11} + \omega^{12} + \omega^{13} + \omega^{15} + \omega^{16} + \omega^{17}) \end{aligned}$$

is divisible by 2 in  $\mathbb{Z}[\omega]$ , although neither factor is.  $\square$

*Exercise 1.30-1.32:*  $R$  is an integral domain (commutative ring with 1 and no zero divisors).

**Exercise 1.30.**

Show that two ideals in  $R$  are isomorphic as  $R$ -modules iff they are in the same ideal class.

*Proof.* Given any two ideals  $A, B$  in an commutative integral domain  $R$ .

- (1) ( $\implies$ ) Let  $\varphi : A \rightarrow B$  be an  $R$ -module isomorphism. Given any nonzero  $\alpha \in A$ , we have

$$\begin{aligned} \varphi(\alpha)A &= \{\varphi(\alpha)a : a \in A\} \\ &= \{\varphi(\alpha a) : a \in A\} && (\varphi \text{ is a homomorphism}) \\ &= \{\alpha\varphi(a) : a \in A\} && (\varphi \text{ is a homomorphism}) \\ &= \{\alpha b : b \in B\} && (\varphi \text{ is an isomorphism}) \\ &= \alpha B. \end{aligned}$$

Notice that  $\varphi(\alpha) \neq 0$  since  $\alpha \neq 0$  and  $\varphi$  is injective. Therefore,  $A \sim B$ .

(2) ( $\Leftarrow$ ) Given  $A \sim B$ , there are nonzero  $\alpha, \beta \in R$  such that  $\alpha A = \beta B$ . Define a map  $\varphi : A \rightarrow B$  by  $\varphi(a) = b$  if  $\alpha a = \beta b$ .

(a)  $\varphi$  is well-defined.

(i) *Existence of  $b$ .* Since  $\alpha a \in \alpha A = \beta B$ , there is  $b \in B$  such that  $\alpha a = \beta b$ .

(ii) *Uniqueness of  $b$ .* If  $\alpha a = \beta b_1 = \beta b_2$ ,  $\beta(b_1 - b_2) = 0$ . Since  $R$  is an integral domain and  $\beta \neq 0$ ,  $b_1 - b_2 = 0$  or  $b_1 = b_2$ .

(b)  $\varphi$  is an  $R$ -module homomorphism.

(i) Show that  $\varphi(a_1 + a_2) = \varphi(a_1) + \varphi(a_2)$ . Write  $\varphi(a_1) = b_1$  and  $\varphi(a_2) = b_2$ .

$$\begin{aligned} \varphi(a_1) = b_1 \text{ and } \varphi(a_2) = b_2 & \\ \implies \alpha a_1 = \beta b_1 \text{ and } \alpha a_2 = \beta b_2 & \quad (\text{Definition of } \varphi) \\ \implies \alpha a_1 + \alpha a_2 = \beta b_1 + \beta b_2 & \quad (\text{Add together}) \\ \implies \alpha(a_1 + a_2) = \beta(b_1 + b_2) & \\ \implies \varphi(a_1 + a_2) = b_1 + b_2 = \varphi(a_1) + \varphi(a_2). & \quad (\text{Definition of } \varphi) \end{aligned}$$

(ii) Show that  $\varphi(ra) = r\varphi(a)$ . Write  $\varphi(a) = b$ .

$$\begin{aligned} \varphi(a) = b & \implies \alpha a = \beta b & (\text{Definition of } \varphi) \\ \implies r\alpha a = r\beta b & & (\text{Multiply } r) \\ \implies \alpha(ra) = \beta(rb) & & (R \text{ is commutative}) \\ \implies \varphi(ra) = rb = r\varphi(a). & & (\text{Definition of } \varphi) \end{aligned}$$

(c)  $\varphi$  is injective. Given  $\varphi(a) = 0$ . Then  $\alpha a = \beta b = \beta 0 = 0$ . Since  $R$  is an integral domain and  $\alpha \neq 0$ ,  $a = 0$ .

(d)  $\varphi$  is surjective. Given any  $b \in B$ .  $\beta b \in \beta B = \alpha A$ . There is  $a \in A$  such that  $\beta b = \alpha a$ . Such  $a$  satisfies  $\varphi(a) = b$ .

Therefore,  $\varphi : A \rightarrow B$  is an  $R$ -module isomorphism.

□

### Exercise 1.31.

Show that if  $A$  is an ideal in  $R$  and if  $\alpha A$  is principal for some nonzero  $\alpha \in R$ , then  $A$  is principal. Conclude that the principal ideals form an ideal class.

*Proof.*

(1) Write  $\alpha A = (b)$  for some  $b \in \alpha A$ . That is, there is  $a \in A$  such that

$$b = \alpha a.$$

- (2) *Show that  $A = (a)$  is principal.*  $(a) \subseteq A$  holds trivially since  $a \in A$  and  $A$  is an ideal. Given any  $x \in A$ ,  $\alpha x \in \alpha A = (b)$ , and thus there is  $y \in R$  such that  $\alpha x = by$ . Replace  $b$  by  $b = \alpha a$  to get  $\alpha x = \alpha ay$  or

$$\alpha(x - ay) = 0.$$

Since  $\alpha \neq 0$  and  $R$  is an integral domain,  $x - ay = 0$  or  $x = ay \in (a)$  or  $A \subseteq (a)$ . Hence  $A = (a)$  is principal.

- (3) *Show that the principal ideals form an ideal class.* Given any  $A = (a) \neq 0$  and  $B = (b) \neq 0$ , we have  $bA = aB = (ab)$  for  $a, b \in R$  or  $A \sim B$ .

□

### Exercise 1.32.

*Show that the ideal classes in  $R$  form a group iff for every ideal  $A$  there is an ideal  $B$  such that  $AB$  is principal.*

*Note.* The Picard group of the spectrum of a Dedekind domain is its ideal class group.

*Proof.* Let  $[A]$  be the ideal class representing by a nonzero ideal  $A$  of  $R$ . Let

$$\text{Pic}(R) = \{[A] : A \text{ is an ideal of } R\}$$

be the set of all ideal classes. Define the operation  $\cdot : \text{Pic}(R) \times \text{Pic}(R) \rightarrow \text{Pic}(R)$  by  $[A] \cdot [B] \mapsto [AB]$ .

- (1) *(Closure) Show that the operation  $[A] \cdot [B] \mapsto [AB]$  is well-defined.* Trivial due to the definition of the ideal class. Note that  $[A] \cdot [B] = [B] \cdot [A]$  by the commutativity of  $R$ .
- (2) *(Associativity) Show that  $([A] \cdot [B]) \cdot [C] = [A] \cdot ([B] \cdot [C])$ .* Trivial due to the definition of the ideal class.
- (3) *(Identity element) Show that the non-zero principal ideals form the ideal class  $[1]$ .* Exercise 1.30 and note that  $(1)$  is principal too.
- (4) *Show that the set  $\text{Pic}(R)$  forms an (abelian) group with  $[1]$  as the identity element if and only if every  $[A]$  has an inverse in  $\text{Pic}(R)$ .* By (1)(2)(3), the set  $\text{Pic}(R)$  forms an (abelian) group iff every element has an inverse element. The conclusion is established.

□

## Chapter 2: Number Fields and Number Rings

### Exercise 2.1.

- (a) Show that every number field of degree 2 over  $\mathbb{Q}$  is one of the quadratic fields  $\mathbb{Q}[\sqrt{m}]$ ,  $m \in \mathbb{Z}$ .
- (b) Show that the fields  $\mathbb{Q}[\sqrt{m}]$ ,  $m$  squarefree, are pairwise distinct. (Hint: Consider the equation  $\sqrt{m} = a + b\sqrt{n}$ ; use this to show that they are in fact pairwise non-isomorphic).

*Proof of (a).* Let  $f(x) = ax^2 + bx + c$ ,  $a, b, c \in \mathbb{Z}$  ( $a \neq 0$ ) and assume  $f$  is irreducible over  $\mathbb{Q}$ . Let  $\alpha$  be a root of  $f(x)$ . So

$$\alpha = \frac{-b \pm \sqrt{m}}{2a}$$

where  $m = b^2 - 4ac \in \mathbb{Z}$ . Therefore,

$$\mathbb{Q}[\alpha] = \mathbb{Q}\left[\frac{-b \pm \sqrt{m}}{2a}\right] = \mathbb{Q}[\sqrt{m}].$$

□

*Proof of (b).* Show that  $\mathbb{Q}[\sqrt{m}]$  and  $\mathbb{Q}[\sqrt{n}]$  are not isomorphic as fields if  $m$  and  $n$  are squarefree and  $m \neq n$ . Reductio ad absurdum.

- (1) If  $\varphi : \mathbb{Q}[\sqrt{m}] \rightarrow \mathbb{Q}[\sqrt{n}]$  were an isomorphism as fields, then  $\varphi$  is an identity map on  $\mathbb{Q}$ , and

$$\begin{aligned} \varphi(\sqrt{m}) &= a + b\sqrt{n} \text{ for some } a, b \in \mathbb{Q} \\ \implies \varphi(\sqrt{m})\varphi(\sqrt{m}) &= (a + b\sqrt{n})^2 \\ \implies \varphi(\sqrt{m}\sqrt{m}) &= (a + b\sqrt{n})^2 \\ \implies \varphi(m) &= a^2 + nb^2 + 2ab\sqrt{n} \\ \implies m &= a^2 + nb^2 + 2ab\sqrt{n}. \end{aligned}$$

If  $2ab \neq 0$ , then  $\sqrt{n} = \frac{m - a^2 - nb^2}{2ab} \in \mathbb{Q}$ , contrary to the assumption that  $n$  is squarefree. Hence  $2ab = 0$ .

- (2)  $a = 0$ . Write  $b = \frac{r}{s} \in \mathbb{Q}$  where  $r, s \in \mathbb{Z}$  and  $(r, s) = 1$ . So

$$ms^2 = nr^2.$$

Hence

$$\begin{aligned} b \neq 0 &\implies s^2 > 0 \text{ and } r^2 > 0 \\ &\implies m \text{ and } n \text{ have the same sign} \\ &\implies (\exists \text{ prime } p \mid m, p \nmid n) \text{ or } (\exists \text{ prime } q \mid n, q \nmid m) \text{ since } m \neq n. \end{aligned}$$



(a) *There is a prime  $p \mid m$  but  $p \nmid n$ .*

$$\begin{aligned}
p \mid m &\implies \text{Write } m = pm_1 \text{ for some } m_1 \in \mathbb{Z} \\
&\implies (pm_1)s^2 = nr^2 && (ms^2 = nr^2) \\
&\implies p \mid nr^2 \\
&\implies p \mid r^2 && (p \nmid n \text{ by assumption}) \\
&\implies p \mid r && (p \text{ is a prime}) \\
&\implies \text{Write } r = pr_1 \text{ for some } r_1 \in \mathbb{Z} \\
&\implies (pm_1)s^2 = n(pr_1)^2 && (ms^2 = nr^2) \\
&\implies m_1s^2 = npr_1^2 \\
&\implies p \mid m_1s^2 \\
&\implies p \mid m_1 && ((r, s) = 1 \text{ and } p \mid r) \\
&\implies \text{Write } m_1 = pm_2 \text{ for some } m_2 \in \mathbb{Z} \\
&\implies m = p^2m_2,
\end{aligned}$$

contrary to the assumption that  $m$  is squarefree.

(b) *There is a prime  $q \mid n$  but  $q \nmid m$ .* Similar to (a).

(3)  $b = 0$ .  $m = a^2$ . Write  $a = \frac{r}{s} \in \mathbb{Q}$  where  $r, s \in \mathbb{Z}$  and  $(r, s) = 1$ . Hence  $ms^2 = r^2$ . Similar to the argument in (2).

(4) By (2)(3), no such isomorphism  $\varphi$ , that is,  $\mathbb{Q}[\sqrt{m}]$  and  $\mathbb{Q}[\sqrt{n}]$  are not isomorphic as fields.

□

### Supplement. (Isomorphic as vector spaces)

Show that  $\mathbb{Q}[\sqrt{m}]$  and  $\mathbb{Q}[\sqrt{n}]$  are isomorphic as  $\mathbb{Q}$ -vector spaces.

*Proof.*  $[\mathbb{Q}[\sqrt{m}] : \mathbb{Q}] = [\mathbb{Q}[\sqrt{n}] : \mathbb{Q}] = 2$ . There is a natural map  $\varphi : \mathbb{Q}[\sqrt{m}] \rightarrow \mathbb{Q}[\sqrt{n}]$  defined by  $\varphi(a + b\sqrt{m}) = a + b\sqrt{n}$ . Clearly  $\varphi$  is well-defined, linear, injective and surjective. □

### Exercise 2.2.

Let  $I$  be the ideal generated by 2 and  $1 + \sqrt{-3}$  in the ring  $\mathbb{Z}[\sqrt{-3}] = \{a + b\sqrt{-3} : a, b \in \mathbb{Z}\}$ . Show that  $I \neq (2)$  but  $I^2 = 2I$ . Conclude that ideals in  $\mathbb{Z}[\sqrt{-3}]$  do not factor uniquely into prime ideals. Show moreover that  $I$  is the unique prime ideal containing  $(2)$  and conclude that  $(2)$  is not a product of prime ideals.

*Proof.*

(1) Show that  $I \neq (2)$ .

(a) Show that  $I \supseteq (2)$ .  $2 \in (2, 1 + \sqrt{-3}) = I$ .

(b) Show that  $I \not\subseteq (2)$ . Consider  $1 + \sqrt{-3} \in I$ . (Reductio ad absurdum)  
If  $1 + \sqrt{-3}$  were in  $(2)$ , then there exists  $a + b\sqrt{-3}$  such that

$$1 + \sqrt{-3} = 2(a + b\sqrt{-3}) = 2a + 2b\sqrt{-3}.$$

Thus,  $a = \frac{1}{2}$  and  $b = \frac{1}{2}$ , which is absurd.

(2) Show that  $I^2 = 2I$ .

(a) Show that  $I^2 \supseteq 2I$ . Since  $2 \in (2, 1 + \sqrt{-3}) = I$ ,  $2I \subseteq I^2$ .

(b) Show that  $I^2 \subseteq 2I$ . All elements of  $I^2$  are generated by

$$2 \cdot 2, 2(1 + \sqrt{-3}) \text{ and } (1 + \sqrt{-3})^2.$$

Clearly,  $2 \cdot 2, 2(1 + \sqrt{-3}) \in 2I$ . Besides,

$$(1 + \sqrt{-3})^2 = -2 + 2\sqrt{-3} = 2(-(2) + (1 + \sqrt{-3})) \in 2I.$$

Hence  $I^2 \subseteq 2I$ .

(3) Show that ideals in  $\mathbb{Z}[\sqrt{-3}]$  do not factor uniquely into prime ideals. It is followed by  $I^2 = 2I$  and  $I \neq (2)$ .

(4) Show that  $I$  is the unique prime ideal containing  $(2)$ .

(a) Show that  $I = (2, 1 + \sqrt{-3})$  is a prime ideal containing  $(2)$ . Note that

$$\mathbb{Z}[\sqrt{-3}]/(2) = (\mathbb{Z}/2\mathbb{Z})[\sqrt{-3}] = \{0, 1, \sqrt{-3}, 1 + \sqrt{-3}\}$$

and

$$I/(2) = (1 + \sqrt{-3})$$

is an ideal of  $\mathbb{Z}[\sqrt{-3}]/(2)$ . So

$$\mathbb{Z}[\sqrt{-3}]/I = (\mathbb{Z}[\sqrt{-3}]/(2))/(I/(2)) = \{0, 1\} = \mathbb{Z}/2\mathbb{Z}$$

is an integral domain. Hence  $I$  is a prime ideal containing  $(2)$ .

(b) Suppose  $I'$  is a prime ideal containing  $(2)$ . Similar to part (a),

$$\begin{aligned} \mathbb{Z}[\sqrt{-3}]/I' &= (\mathbb{Z}[\sqrt{-3}]/(2))/(I'/(2)) \\ &= \{0, 1, \sqrt{-3}, 1 + \sqrt{-3}\}/(I'/(2)) \end{aligned}$$

must be an integral domain.

(c) Since  $\{0, 1, \sqrt{-3}, 1 + \sqrt{-3}\}$  is not an integral domain,  $I'/(2) \neq (0)$  or  $I' \neq (2)$ . Also,  $I'/(2) \neq \{0, 1, \sqrt{-3}, 1 + \sqrt{-3}\}$  implies that  $I'/(2) \neq (1) = (\sqrt{-3})$ . Therefore we must have  $I'/(2) = (1 + \sqrt{-3})$ . Here the existence is guaranteed by part (a).

- (5) Show that  $(2)$  is not a product of prime ideals. (Reductio ad absurdum)  
 Suppose  $(2)$  were a product of prime ideals. By part (4), we might write  $(2) = I^n$  for some positive integer  $n$ . Since  $I \neq (2)$  and  $I^2 = 2I$ ,

$$(2) = (2)I^{n-1} \subseteq (2)I.$$

for some  $n \geq 2$ .

- (6) Take  $2 \in (2) \subseteq (2)I$ . Write

$$2 = 2a_1 + \cdots + 2a_k = 2 \underbrace{(a_1 + \cdots + a_k)}_{:=a \in I}$$

where  $a_1, \dots, a_k \in I$ . We take the norm of the both sides to get  $N(a) = 1$ .  $a$  is a unit in  $\mathbb{Z}[\sqrt{-3}]$ .  $I = \mathbb{Z}[\sqrt{-3}]$ , which is absurd. Therefore  $(2)$  is not a product of prime ideals.

□

### Exercise 2.3.

Complete the proof of Corollary 2, Theorem 2.1.

Corollary 2: Let  $m$  be a squarefree integer. The set of algebraic integers in the quadratic field  $\mathbb{Q}[\sqrt{m}]$  is

$$\begin{aligned} & \{a + b\sqrt{m} : a, b \in \mathbb{Z}\} \text{ if } m \equiv 2, 3 \pmod{4}, \\ & \left\{ \frac{a + b\sqrt{m}}{2} : a, b \in \mathbb{Z}, a \equiv b \pmod{2} \right\} \text{ if } m \equiv 1 \pmod{4}. \end{aligned}$$

*Proof.*

- (1) Let  $\alpha = r + s\sqrt{m}$ ,  $r, s \in \mathbb{Q}$ . If  $s \neq 0$ , then the monic irreducible polynomial over  $\mathbb{Q}$  having  $\alpha$  as a root is

$$x^2 - 2rx + r^2 - ms^2.$$

Thus  $\alpha$  is an algebraic integer iff  $2r$  and  $r^2 - ms^2$  are both integers.

- (2) Hence  $4(r^2 - ms^2) = (2r)^2 - m(2s)^2 \in \mathbb{Z}$ .  $m(2s)^2 \in \mathbb{Z}$  since  $2r \in \mathbb{Z}$ . Hence  $2s \in \mathbb{Z}$  since  $m$  is squarefree. Let  $a = 2r, b = 2s \in \mathbb{Z}$ . Then  $a^2 - mb^2 = 4(r^2 - ms^2) \equiv 0 \pmod{4}$ . Note that a square  $\equiv 0, 1 \pmod{4}$  and thus we consider the following two cases.

(3) If  $m \equiv 1 \pmod{4}$ , then

$$\begin{aligned} a^2 - mb^2 &\equiv a^2 - b^2 \pmod{4} \\ \implies a \text{ and } b \text{ has the same parity} \\ \implies \alpha = r + s\sqrt{m} &= \frac{a + b\sqrt{m}}{2}, a, b \in \mathbb{Z}, a \equiv b \pmod{2}. \end{aligned}$$

(4) If  $m \equiv 2, 3 \pmod{4}$ , then

$$\begin{aligned} a^2 - mb^2 &\equiv a^2 + 2b^2 \text{ or } a^2 + b^2 \pmod{4} \\ \implies \text{both } a \text{ and } b \text{ are even} \\ \implies \text{both } r \text{ and } s \text{ are rational integers} \\ \implies \alpha = r + s\sqrt{m}, r, s \in \mathbb{Z}. \end{aligned}$$

□

### Supplement.

(Exercise I.2.4 in [Jürgen Neukirch, *Algebraic Number Theory*].) Let  $D$  be a squarefree rational integer  $\neq 0, 1$  and  $d$  the discriminant of the quadratic number field  $K = \mathbb{Q}(\sqrt{D})$ . Show that

$$d = \begin{cases} D & \text{if } D \equiv 1 \pmod{4}, \\ 4D & \text{if } D \equiv 2, 3 \pmod{4}. \end{cases}$$

and that an integral basis of  $K$  is given by  $\{1, \sqrt{D}\}$  in the second case, by  $\left\{1, \frac{1+\sqrt{D}}{2}\right\}$  in the first case, and by  $\left\{1, \frac{d+\sqrt{d}}{2}\right\}$  in both case.

*Proof.*

- (1) The Galois group of  $K|\mathbb{Q}$  has two elements, the identity and an automorphism sending  $\sqrt{D}$  to  $-\sqrt{D}$ .
- (2) Note that  $\alpha \in \mathcal{O}_K$  iff  $\text{Tr}_{K|\mathbb{Q}}(\alpha), N_{K|\mathbb{Q}}(\alpha) \in \mathbb{Z}$  (by noting that the equation  $x^2 - \text{Tr}_{K|\mathbb{Q}}(\alpha)x + N_{K|\mathbb{Q}}(\alpha) = 0$  has a root  $x = \alpha$ ). So given  $\alpha = x + y\sqrt{D} \in \mathcal{O}_K$ , we have

$$\begin{aligned} \text{Tr}_{K|\mathbb{Q}}(\alpha) &= 2x \in \mathbb{Z}, \\ N_{K|\mathbb{Q}}(\alpha) &= x^2 - Dy^2 \in \mathbb{Z}. \end{aligned}$$

- (3) So  $4(x^2 - Dy^2) = (2x)^2 - D(2y)^2 \in \mathbb{Z}$ . So  $D(2y)^2 \in \mathbb{Z}$  since  $2x \in \mathbb{Z}$ . So  $2y \in \mathbb{Z}$  since  $D$  is squarefree  $\neq 0, 1$ . Let  $r = 2x, s = 2y$ . Then  $r^2 - Ds^2 = 4(x^2 - Dy^2) \equiv 0 \pmod{4}$ . Note that a square  $\equiv 0, 1 \pmod{4}$  and thus we consider the following two cases.

(4) If  $D \equiv 1 \pmod{4}$ , then

$$\begin{aligned}
& r^2 - Ds^2 \equiv r^2 - s^2 \pmod{4} \\
& \implies r \text{ and } s \text{ has the same parity} \\
& \implies \mathcal{O}_K = \left\{ \frac{r + s\sqrt{D}}{2} : r \equiv s \pmod{2} \right\} \\
& \implies \mathcal{O}_K = \left\{ \frac{r-s}{2} + s \cdot \frac{1+\sqrt{D}}{2} : r \equiv s \pmod{2} \right\} \\
& \implies \mathcal{O}_K = \mathbb{Z} + \mathbb{Z} \frac{1+\sqrt{D}}{2}.
\end{aligned}$$

So  $\left\{1, \frac{1+\sqrt{D}}{2}\right\}$  is an integral basis of  $K$ . Hence

$$d = \begin{vmatrix} 1 & \frac{1+\sqrt{D}}{2} \\ 1 & \frac{1-\sqrt{D}}{2} \end{vmatrix}^2 = D.$$

(5) If  $D \equiv 2, 3 \pmod{4}$ , then

$$\begin{aligned}
& r^2 - Ds^2 \equiv r^2 + 2s^2 \text{ or } r^2 + s^2 \pmod{4} \\
& \implies \text{both } r \text{ and } s \text{ are even} \\
& \implies \text{both } x \text{ and } y \text{ are rational integers} \\
& \implies \mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\sqrt{D}.
\end{aligned}$$

So  $\{1, \sqrt{D}\}$  is an integral basis of  $K$ . Hence

$$d = \begin{vmatrix} 1 & \sqrt{D} \\ 1 & -\sqrt{D} \end{vmatrix}^2 = 4D.$$

(6) By (4)(5),  $\left\{1, \frac{d+\sqrt{d}}{2}\right\}$  is an integral basis of  $K$  for any case.

□

#### Exercise 2.4.

Suppose  $a_0, \dots, a_{n-1}$  are algebraic integers and  $\alpha$  is a complex number satisfying

$$\alpha^n + a_{n-1}\alpha^{n-1} + \dots + a_1\alpha + a_0 = 0.$$

Show that the ring  $\mathbb{Z}[a_0, \dots, a_{n-1}, \alpha]$  has a finitely generated additive group. (Hint: Consider the products  $a_0^{m_0} a_1^{m_1} \dots a_{n-1}^{m_{n-1}} \alpha^m$  and show that only finitely many values of the exponents are needed.) Conclude that  $\alpha$  is an algebraic

integer.

*Proof.* Let  $V = \mathbb{Z}[a_0, \dots, a_{n-1}, \alpha]$ . Let  $n_k$  be the degree of the algebraic integer  $a_k$  where  $0 \leq k \leq n-1$ .

- (1) *Show that  $V$  is finitely generated as an additive subgroup of  $\mathbb{C}$ . It suffices to show that  $V$  is generated by*

$$a_0^{m_0} a_1^{m_1} \dots a_{n-1}^{m_{n-1}} \alpha^m$$

where  $0 \leq m_k < n_k$  and  $0 \leq m < n$ . Given any  $x \in V$ ,  $x$  is a finite sum of the product  $a_0^{m_0} a_1^{m_1} \dots a_{n-1}^{m_{n-1}} \alpha^m$  with  $m_k \geq 0$  and  $m \geq 0$ .

If  $m \geq n$ , replace  $\alpha^m$  by

$$\begin{aligned} \alpha^m &= \alpha^{m-n} \alpha^n \\ &= \alpha^{m-n} (-a_{n-1} \alpha^{n-1} - \dots - a_1 \alpha - a_0) \\ &= -a_{n-1} \alpha^{m-1} - \dots - a_1 \alpha^{m-n+1} - a_0 \alpha^{m-n}. \end{aligned}$$

Repeat this process to reduce the degree of  $\alpha^m$  less than  $n$ . Therefore, we can write  $x$  as a finite sum of the product  $a_0^{m'_0} a_1^{m'_1} \dots a_{n-1}^{m'_{n-1}} \alpha^{m'}$  with  $m'_k \geq 0$  and  $0 \leq m' < n$ .

Once the degree of  $\alpha^m$  is reduced, continue to reduce the degree of each  $a_k^{m'_k}$  without affecting other  $a_h$  ( $h \neq k$ ) and  $\alpha$ . Now replace  $a_k^{m'_k}$  by

$$a_k^{m'_k} = \sum_{i=0}^{n_k-1} b_{k,i} a_k^i$$

where  $b_{k,i} \in \mathbb{Z}$ . Therefore, we can write  $x$  as a finite sum of the product  $a_0^{m''_0} a_1^{m''_1} \dots a_{n-1}^{m''_{n-1}} \alpha^{m'}$  with  $0 \leq m''_k < n_k$  and  $0 \leq m' < n$ .

- (4) *Show that  $\alpha$  is an algebraic integer.* Since  $\alpha \in V$ ,  $\alpha V \subseteq V$ . Thus  $\alpha$  is an algebraic integer (Theorem 2.2).

□

### Exercise 2.5.

*Show that if  $f$  is any polynomials over  $\mathbb{Z}/p\mathbb{Z}$  ( $p$  a prime) then  $f(x^p) = (f(x))^p$ . (Suggestion: Use induction on the number of terms.)*

*Proof.*

(1) Let

$$\binom{p}{k} = \frac{p!}{k!(p-k)!}$$

be a binomial coefficient. If  $1 \leq k \leq p-1$ , show that  $p$  divides  $\binom{p}{k}$ .

(a) If  $1 \leq k \leq p-1$ , then  $p \nmid k!$  and  $p \nmid (p-k)!$  since  $p$  is a prime.

(b) Write  $a = \frac{p!}{k!(p-k)!} \in \mathbb{Z}$ . Hence,

$$\begin{aligned} a = \frac{p!}{k!(p-k)!} &\iff p! = ak!(p-k)! \\ &\implies p \mid p! \text{ or } p \mid ak!(p-k)! \\ &\implies p \mid a \text{ by (a).} \end{aligned}$$

Hence  $p$  divides  $\binom{p}{k}$  if  $1 \leq k \leq p-1$ .

(2) Note that  $a^p = a \in \mathbb{Z}/p\mathbb{Z}$  for all  $a \in \mathbb{Z}/p\mathbb{Z}$ .

(3) Write

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}/p\mathbb{Z}[x].$$

Induction on  $n$ .

(a)  $n = 0$ . So  $f(x) = a_0$ , and thus  $f(x)^p = a_0^p = a_0$  by (2).

(b)  $n = 1$ . By  $f(x) = a_1 x + a_0$ ,

$$\begin{aligned} f(x)^p &= (a_1 x + a_0)^p \\ &= a_1^p x^p + \sum_{k=1}^{p-1} \binom{p}{k} (a_1 x)^k a_0^{p-k} + a_0^p \quad (\text{Binomial theorem}) \\ &= a_1^p x^p + a_0^p \quad ((1)) \\ &= a_1 x^p + a_0 \quad ((2)) \\ &= f(x^p). \end{aligned}$$

(c) If the statement holds for  $n-1$ , then

$$\begin{aligned} f(x)^p &= (a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0)^p \\ &= [a_n x^n + (a_{n-1} x^{n-1} + \cdots + a_1 x + a_0)]^p \\ &= (a_n x^n)^p + (a_{n-1} x^{n-1} + \cdots + a_1 x + a_0)^p \quad (\text{Same as (b)}) \\ &= a_n (x^p)^n + (a_{n-1} x^{n-1} + \cdots + a_1 x + a_0)^p \quad ((2)) \\ &= a_n (x^p)^n + a_{n-1} (x^p)^{n-1} + \cdots + a_1 x^p + a_0 \quad (\text{Induction hypothesis}) \\ &= f(x^p). \end{aligned}$$

The inductive step is established.

By induction,  $f(x)^p = f(x^p)$  holds for any  $n \geq 0$ .

□

**Exercise 2.6.**

Show that if  $f$  and  $g$  are polynomials over a field  $K$  and  $f^2 \mid g$  in  $K[x]$ , then  $f \mid g'$ . (Hint: Write  $g = f^2h$  and differentiate.)

*Proof (Hint).* Since  $f^2 \mid g$  in  $K[x]$ , there exists  $h \in K[x]$  such  $g = f^2h$ . Differentiate to get  $g' = 2ff'h + f^2h' = f(2f'h + fh')$ , or  $f \mid g'$  in  $K[x]$ . □

**Exercise 2.7.**

Complete the proof of Corollary 2, Theorem 2.3.

Corollary 2: The galois group of  $\mathbb{Q}[\omega]$  over  $\mathbb{Q}$  is isomorphic to the multiplicative group of integer  $(\text{mod } m)$

$$(\mathbb{Z}/m\mathbb{Z})^* = \{k : 1 \leq k \leq m, (k, m) = 1\}.$$

For each  $k \in (\mathbb{Z}/m\mathbb{Z})^*$ , the corresponding automorphism in the galois group sends  $\omega$  to  $\omega^k$  (and hence  $g(\omega) \rightarrow g(\omega^k)$  for each  $g \in \mathbb{Z}[x]$ ).

*Proof.*

- (1) An automorphism of  $\mathbb{Q}[\omega]$  is uniquely determined by the image of  $\omega$ , and Theorem 2.3 shows that  $\omega$  can be sent to any of the  $\omega^k$ ,  $(k, m) = 1$ . (Clearly it can't be sent anywhere else.) This established the one-to-one correspondence between the galois group and the multiplicative group of integer  $(\text{mod } m)$ , say

$$\alpha : \text{Gal}(\mathbb{Q}[\omega]/\mathbb{Q}) \rightarrow (\mathbb{Z}/m\mathbb{Z})^*.$$

- (2) The composition of automorphisms corresponds to multiplication  $(\text{mod } m)$  in the natural way. That is, if  $\sigma, \tau \in \text{Gal}(\mathbb{Q}[\omega]/\mathbb{Q})$  with  $\sigma(\omega) = \omega^k$  and  $\tau(\omega) = \omega^h$ , then

$$(\sigma\tau)(\omega) = \sigma(\omega^h) = \omega^{kh} \xrightarrow{\alpha} kh.$$

Hence  $\alpha$  is a group homomorphism.

□



**Exercise 2.8.**

- (a) Let  $\omega = e^{\frac{2\pi i}{p}}$ ,  $p$  an odd prime. Show that  $\mathbb{Q}[\omega]$  contains  $\sqrt{p}$  if  $p \equiv 1 \pmod{4}$ , and  $\sqrt{-p}$  if  $p \equiv 3 \pmod{4}$ . (Hint: Recall that we have shown that  $\text{disc}(\omega) = \pm p^{p-2}$  with  $+$  holding iff  $p \equiv 1 \pmod{4}$ .) Express  $\sqrt{-3}$  and  $\sqrt{5}$  as polynomials in the appropriate  $\omega$ .
- (b) Show that the eighth cyclotomic field contains  $\sqrt{2}$ .
- (c) Show that every quadratic field is contained in a cyclotomic field: In fact,  $\mathbb{Q}[\sqrt{m}]$  is contained in the  $d$ -th cyclotomic field, where  $d = \text{disc}(\mathcal{O}_{\mathbb{Q}[\sqrt{m}]})$ . (More generally, Kronecker and Weber proved that every abelian extension of  $\mathbb{Q}$  (normal with abelian Galois group) is contained in a cyclotomic field. See the Chapter 4 exercises. Hilbert and others investigated the abelian extensions of an arbitrary number field; their results are known as **class field theory**, which will be discussed in Chapter 8.)

*Proof of (a).*

- (1) Recall that we have shown that

$$\text{disc}(\omega) = \prod_{1 \leq r < s \leq p} (\omega_r - \omega_s)^2 = (-1)^{\frac{p-1}{2}} p^{p-2} = (-1)^{\frac{p-1}{2}} p \cdot p^{p-3}$$

where  $\omega_1 = \omega, \dots, \omega_p$  are the conjugates of  $\omega$  over  $\mathbb{Q}$ . Hence

$$\prod_{1 \leq r < s \leq p} (\omega_r - \omega_s) = \pm \sqrt{(-1)^{\frac{p-1}{2}} p \cdot p^{\frac{p-3}{2}}} \in \mathbb{Q}[\omega].$$

Note that  $p^{\frac{p-3}{2}} \in \mathbb{Q}$  as  $p \geq 3$  is odd and  $\pm$  is unrelated as  $\mathbb{Q}[\omega]$  is a field. Therefore

$$\sqrt{(-1)^{\frac{p-1}{2}} p} \in \mathbb{Q}[\omega].$$

- (2) Express  $\sqrt{-3}$  as polynomials in the appropriate  $\omega$ . Take  $\omega = e^{\frac{2\pi i}{3}}$ . A direct computing shows that

$$\begin{aligned} \prod_{1 \leq r < s \leq 3} (\omega_r - \omega_s) &= \prod_{1 \leq r < s \leq 3} (\omega^r - \omega^s) \\ &= (1 - \omega)(1 - \omega^2)(\omega - \omega^2) \\ &= 3(-\omega^2 + \omega) \\ &= 3\sqrt{-3}. \end{aligned}$$

Hence  $\sqrt{-3} = -\omega^2 + \omega$ .

- (3) Express  $\sqrt{5}$  as polynomials in the appropriate  $\omega$ . Take  $\omega = e^{\frac{2\pi i}{5}}$ . A direct computing shows that

$$\begin{aligned}\prod_{1 \leq r < s \leq 5} (\omega_r - \omega_s) &= \prod_{1 \leq r < s \leq 5} (\omega^r - \omega^s) \\ &= 3(\omega - \omega^2) \\ &= -25(\omega^4 - \omega^3 - \omega^2 + \omega) \\ &= -25\sqrt{5}.\end{aligned}$$

Hence  $\sqrt{5} = \omega^4 - \omega^3 - \omega^2 + \omega$ .

- (4) (Another proof) The quadratic Gauss sum shows that

$$\sum_{n=0}^{p-1} e^{\frac{2\pi i n^2}{p}} = \sqrt{(-1)^{\frac{p-1}{2}} p}.$$

So  $\sqrt{-3} = 2\omega_3 + 1$  and  $\sqrt{5} = 2\omega_5^4 + 2\omega_5 + 1$ .

□

*Proof of (b).*

- (1) A root of eighth unity is  $\omega = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ .  
(2) Hence

$$\omega + \omega^{-1} = \left( \frac{\sqrt{2}}{2} + \frac{\sqrt{-2}}{2} \right) + \left( \frac{\sqrt{2}}{2} - \frac{\sqrt{-2}}{2} \right) = \sqrt{2} \in \mathbb{Q}[\omega].$$

□

*Proof of (c).*

- (1) Note that  $\mathbb{Q}[\omega_a, \omega_b] = \mathbb{Q}[\omega_{ab}]$  if  $a, b \in \mathbb{Z}$  are relatively prime. Might assume that  $m$  is squarefree since  $\mathbb{Q}[\sqrt{ab^2}] = \mathbb{Q}[\sqrt{a}]$ . Consider the following four cases.  
(2) Suppose  $m > 0$  and  $2 \nmid m$ . Write

$$m = p_1 \cdots p_r \cdot q_1 \cdots q_s$$

as a product of distinct primes where  $p_i \equiv 1 \pmod{4}$  and  $q_j \equiv 3 \pmod{4}$ . Part (a) shows that

$$\sqrt{p_1}, \dots, \sqrt{p_r}, \sqrt{-q_1}, \dots, \sqrt{-q_s} \in \mathbb{Q}[\omega_{p_1}, \dots, \omega_{p_r}, \omega_{q_1}, \dots, \omega_{q_s}].$$

So  $\sqrt{(-1)^s m} \in \mathbb{Q}[\omega_m]$ . If  $s$  is even, then  $\sqrt{m} \in \mathbb{Q}[\omega_m]$  or  $\mathbb{Q}[\sqrt{m}] \subseteq \mathbb{Q}[\omega_m]$ . If  $s$  is odd, then  $\sqrt{m} \in \mathbb{Q}[\omega_m, \omega_4] = \mathbb{Q}[\omega_{4m}]$  (since  $\sqrt{-1} \in \mathbb{Q}[\omega_4]$ ). In any case,  $\mathbb{Q}[\sqrt{m}]$  is contained in the  $d$ -th cyclotomic field, where  $d = \text{disc}(\mathcal{O}_{\mathbb{Q}[\sqrt{m}]})$ . (See Supplement to Exercise 2.3.)

(3) Suppose  $m < 0$  and  $2 \nmid m$ . Similar to (2).

(4) Suppose  $m > 0$  and  $2 \mid m$ . Write

$$m = 2 \cdot p_1 \cdots p_r \cdot q_1 \cdots q_s$$

as a product of distinct primes where  $p_i \equiv 1 \pmod{4}$  and  $q_j \equiv 3 \pmod{4}$ . Parts (a)(b) show that

$$\sqrt{2}, \sqrt{p_1}, \dots, \sqrt{p_r}, \sqrt{-q_1}, \dots, \sqrt{-q_s} \in \mathbb{Q}[\omega_8, \omega_{p_1}, \dots, \omega_{p_r}, \omega_{q_1}, \dots, \omega_{q_s}].$$

So  $\sqrt{(-1)^s m} \in \mathbb{Q}[\omega_{4m}]$ . Note that  $\sqrt{(-1)^s} \in \mathbb{Q}[\omega_4] \subseteq \mathbb{Q}[\omega_{4m}]$ . Hence  $\sqrt{m} \in \mathbb{Q}[\omega_{4m}]$  is contained in the  $d$ -th cyclotomic field, where  $d = 4m = \text{disc}(\mathcal{O}_{\mathbb{Q}[\sqrt{m}]})$ .

(5) Suppose  $m < 0$  and  $2 \mid m$ . Same as (4).

□

### Exercise 2.9.

With notation as in the proof of Corollary 3, Theorem 2.3, show that there exist integers  $u$  and  $v$  such that  $e^{\frac{2\pi i}{r}} = \omega^u \theta^v$ .

*Proof.*

(1) Recall  $\omega = e^{\frac{2\pi i}{m}}$ ,  $\theta = e^{\frac{2\pi i}{k}}$  and  $r$  is the least common multiple of  $k$  and  $m$ .

(2) As  $r$  is the least common multiple of  $k$  and  $m$ , there exist coprime integers  $a$  and  $b$  such that  $r = am = bk$ . As  $(a, b) = 1$ , there exist integers  $u$  and  $v$  such that  $au + bv = 1$ .

(3) Hence,

$$\begin{aligned} \omega^u \theta^v &= e^{\frac{2\pi i u}{m}} \cdot e^{\frac{2\pi i v}{k}} \\ &= e^{\frac{2\pi i a u}{r}} \cdot e^{\frac{2\pi i b v}{r}} \\ &= e^{\frac{2\pi i (a u + b v)}{r}} \\ &= e^{\frac{2\pi i}{r}}. \end{aligned}$$

□

**Exercise 2.10.**

Complete the proof of Corollary 3 to Theorem 2.3, by showing if  $m$  is even,  $m \mid r$ , and  $\varphi(r) \leq \varphi(m)$ , then  $r = m$ .

*Proof.*

- (1) Since  $m$  is even, write the unique factorization of  $m$  as

$$m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

where  $p_1 = 2$ , all  $\alpha_i \geq 1$  ( $1 \leq i \leq k$ ), and all  $p_i$  ( $1 \leq i \leq k$ ) are distinct prime numbers.

- (2) Since  $m \mid r$ , write  $r = mm_1$  for some  $m_1 \in \mathbb{Z}$ . Thus we can write the unique factorization of  $r$  as

$$r = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k} q_1^{\gamma_1} \cdots q_h^{\gamma_h}$$

where all  $\beta_i \geq \alpha_i \geq 1$  ( $1 \leq i \leq k$ ) and all  $p_i$  ( $1 \leq i \leq k$ ) and  $q_j$  ( $1 \leq j \leq h$ ) are distinct prime numbers. Here  $h$  might be zero if  $m_1 = 1$ , and all  $q_j \mid m_1$  but  $q_j \nmid m$ .

- (3) Thus,

$$\begin{aligned} \varphi(m) &= m \left(1 - \frac{1}{2}\right) \cdots \left(1 - \frac{1}{p_k}\right) \\ \varphi(r) &= mm_1 \left(1 - \frac{1}{2}\right) \cdots \left(1 - \frac{1}{p_k}\right) \left(1 - \frac{1}{q_1}\right) \cdots \left(1 - \frac{1}{q_h}\right) \\ &= \varphi(m)m_1 \left(1 - \frac{1}{q_1}\right) \cdots \left(1 - \frac{1}{q_h}\right) \\ &\geq \varphi(m)(q_1 \cdots q_h) \left(1 - \frac{1}{q_1}\right) \cdots \left(1 - \frac{1}{q_h}\right) \\ &\geq \varphi(m)(q_1 - 1) \cdots (q_h - 1). \end{aligned}$$

- (4) Since all  $q_j \neq 2$  ( $1 \leq j \leq h$ ),  $q_j - 1 > 1$ . Hence by (3) and assumption that  $\varphi(r) \leq \varphi(m)$ ,  $h = 0$  or  $m_1 = 1$  or  $r = m$ .

□

**Exercise 2.11.**

- (a) Suppose all roots of a monic polynomial  $f \in \mathbb{Q}[x]$  has absolute value 1. Show that the coefficient of  $x^r$  has absolute value  $\leq \binom{n}{r}$ , where  $n$  is the degree of  $f$  and  $\binom{n}{r}$  is the binomial coefficient.

- (b) Show that there are only finitely many algebraic integers  $\alpha$  of fixed degree  $n$ , all of whose conjugates (including  $\alpha$ ) have absolute value 1. (Note: If you don't use Theorem 2.1, your proof is probably wrong.)
- (c) Show that  $\alpha$  must be a root of 1. (Show that its powers are restricted to a finite set.)

*Proof of (a).*

(1) Write  $f(x) = (x - \alpha_1) \cdots (x - \alpha_n)$  where  $\alpha_i \in \mathbb{C}$ ,  $|\alpha_i| = 1$  for  $i = 1, 2, \dots, n$ .

(2) So

$$f(x) = x^n - s_1 x^{n-1} + s_2 x^{n-2} + \cdots + (-1)s_n$$

where

$$s_r = \sum_{1 \leq j_1 < \cdots < j_r \leq n} \alpha_{j_1} \cdots \alpha_{j_r} \in \mathbb{C}.$$

Let  $c_r = (-1)^r s_{n-r}$  be the coefficient of  $x^r$ .

(3)

$$\begin{aligned} |c_r| &= |(-1)^r s_{n-r}| \\ &= \left| \sum_{1 \leq j_1 < \cdots < j_{n-r} \leq n} \alpha_{j_1} \cdots \alpha_{j_{n-r}} \right| \\ &\leq \sum_{1 \leq j_1 < \cdots < j_{n-r} \leq n} |\alpha_{j_1} \cdots \alpha_{j_{n-r}}| \\ &= \sum_{1 \leq j_1 < \cdots < j_{n-r} \leq n} |\alpha_{j_1}| \cdots |\alpha_{j_{n-r}}| \\ &= \sum_{1 \leq j_1 < \cdots < j_{n-r} \leq n} 1 \\ &= \binom{n}{n-r} \\ &= \binom{n}{r}. \end{aligned}$$

□

*Proof of (b).*

- (1) Let  $f$  be an irreducible monic polynomial over  $\mathbb{Z}$  of degree  $n$  such that  $f(\alpha) = 0$ . So  $f$  is irreducible over  $\mathbb{Q}$  (Theorem 2.1), and thus all the conjugates of  $\alpha$  (including  $\alpha$ ) are roots of  $f$ .

- (2) By (a), all the coefficient of  $x^r$  has absolute value  $\leq \binom{n}{r}$ . Since all the coefficient of  $x^r$  are integers, there are finitely many irreducible monic polynomials  $f \in \mathbb{Z}[x]$  such that  $f(\alpha) = 0$  with  $|\alpha| = 1$ .
- (3) For each such  $f$ , there are only finitely many roots. Therefore, there are only finitely many such algebraic integers  $\alpha$ .

□

*Proof of (c).*

- (1) If  $\alpha_1, \dots, \alpha_n$  are the roots of  $f$  of degree  $n$  over  $\mathbb{Q}$ , then for every  $r \in \mathbb{Z}^+$ ,  $\alpha_1^r, \dots, \alpha_n^r$  are all the roots of some monic polynomial  $f_r$  of degree  $n$  over  $\mathbb{Q}$  (Fundamental theorem of symmetric polynomials).
- (2) Now we consider the powers of  $\alpha$ . All the powers of  $\alpha$  ( $\alpha^r$ ) are algebraic integers (Theorem 2.2), and of degree at most  $n$ . (Let  $g \in \mathbb{Z}[x]$  be the minimal polynomial of  $\alpha^r$  over  $\mathbb{Q}$ . By (1),  $f_r(\alpha^r) = 0$ , and thus  $g \mid f_r$ . Hence  $\deg(g) \leq \deg(f_r) = n$ .)
- (3) By (b), the powers of  $\alpha$  are restricted to a finite set, say  $\alpha^r = \alpha^s$  for some  $s > r \geq 1$ . So  $\alpha^{s-r} = 1$  with  $s - r \geq 1$ . That is,  $\alpha$  is a root of unity.

□

### Exercise 2.12. (Kummer's Lemma)

Now we can prove Kummer's lemma on units in the  $p$ -th cyclotomic field, as stated before Exercise 1.26: Let  $\omega = e^{\frac{2\pi i}{p}}$ ,  $p$  an odd prime, and suppose  $u$  is a unit in  $\mathbb{Z}[\omega]$ .

- (a) Show that  $u/\bar{u}$  is a root of 1. (Use Exercise 2.11(c) above and observe that complex conjugation is a member of the Galois group of  $\mathbb{Z}[\omega]$  over  $\mathbb{Q}$ .) Conclude that  $u/\bar{u} = \pm \omega^k$  for some  $k$ .
- (b) Show that the + sign holds: Assuming  $u/\bar{u} = -\omega^k$ , we have  $u^p = -\bar{u}^p$ ; show that this implies that  $u^p$  is divisible by  $p$  in  $\mathbb{Z}[\omega]$ . (Use Exercise 1.23 and 1.25) But this is impossible since  $u^p$  is a unit.

*Proof of (a).* Write  $\alpha = u/\bar{u}$ . Then

$$\begin{aligned} |\alpha| = 1 &\implies \alpha \text{ is a root of unity} && \text{(Exercise 2.11)} \\ &\implies \alpha \text{ is a } 2p\text{-th root of unity} && \text{(Corollary 3 to Theorem 2.3)} \\ &\implies \alpha = \pm \omega^k \text{ for some } k \in \mathbb{Z} \end{aligned}$$

□

*Proof of (b).* (Reductio ad absurdum) Assume that  $u/\bar{u} = -\omega^k$ , then

$$\begin{aligned} u/\bar{u} = -\omega^k &\implies (u/\bar{u})^p = (-\omega^k)^p \\ &\implies u^p/\bar{u}^p = (-1)^p \omega^{pk} = -1 \quad (p \text{ is odd}) \\ &\implies u^p = -\bar{u}^p = -\overline{u^p} \end{aligned}$$

By Exercise 1.25,  $u^p \equiv a \pmod{p}$  for some  $a \in \mathbb{Z}$ . By Exercise 1.23,  $\bar{u}^p \equiv \bar{a} \equiv a \pmod{p}$ . Thus

$$\begin{aligned} u^p = -\bar{u}^p &\implies a \equiv -a \pmod{p} \\ &\implies 2a \equiv 0 \pmod{p} \\ &\implies a \equiv 0 \pmod{p} \quad (p \text{ is odd}) \end{aligned}$$

or  $u^p \equiv 0 \pmod{p}$ , contradicts the assumption that  $u$  is a unit. Hence  $u/\bar{u} = \omega^k$  for some  $k$ .  $\square$

### Exercise 2.13.

Show that 1 and  $-1$  are the only units in the ring  $\mathcal{O}_{\mathbb{Q}[\sqrt{m}]}$ ,  $m$  squarefree,  $m < 0$ ,  $m \neq -1, -3$ . What if  $m = -1$  or  $-3$ ?

*Proof.*

- (1) Let  $K = \mathbb{Q}[\sqrt{m}]$ . Define a norm  $N$  on  $K$  by

$$N(a + b\sqrt{m}) = (a + b\sqrt{m})(a - b\sqrt{m}) = a^2 + |m|b^2.$$

- (2) Corollary 2 to Theorem 2.1 shows that

$$\mathcal{O}_K = \begin{cases} \{a + b\sqrt{m} : a, b \in \mathbb{Z}\} & (m \equiv 2, 3 \pmod{4}), \\ \left\{ \frac{a+b\sqrt{m}}{2} : a, b \in \mathbb{Z}, a \equiv b \pmod{2} \right\} & (m \equiv 1 \pmod{4}). \end{cases}$$

Clearly,  $N$  maps  $\mathcal{O}_K$  to nonnegative integers. That is,  $u$  is a unit in  $\mathcal{O}_K$  if and only if  $N(u) = 1$  (by the fact that  $N(u) = u\bar{u}$ ).

- (3) If  $m \equiv 2, 3 \pmod{4}$  and  $u = a + b\sqrt{m} \in \mathcal{O}_K$  is a unit ( $a, b \in \mathbb{Z}$ ), then

$$N(u) = 1 = a^2 + |m|b^2.$$

- (a)  $m = -1$  or  $|m| = 1$ .  $1 = a^2 + b^2$  or  $(a, b) = (\pm 1, 0), (0, \pm 1)$ . Hence all units in  $\mathcal{O}_K$  are

$$\pm 1, \pm \sqrt{-1}.$$

- (b)  $m < -1$  or  $|m| > 1$ .  $1 = a^2 + |m|b^2$  implies that  $b^2 = 0$ . Hence all units in  $\mathcal{O}_K$  are  $\pm 1$ .

- (4) If  $m \equiv 1 \pmod{4}$  and  $u = \frac{a+b\sqrt{m}}{2} \in \mathcal{O}_K$  is a unit ( $a, b \in \mathbb{Z}, a \equiv b \pmod{2}$ ), then  $N(u) = 1 = (\frac{a}{2})^2 + |m|(\frac{b}{2})^2$  or

$$4 = a^2 + |m|b^2.$$

- (a)  $m = -3$  or  $|m| = 3$ .  $4 = a^2 + 3b^2$  or  $(a, b) = (\pm 2, 0), (\pm 1, \pm 1)$ . Hence all units in  $\mathcal{O}_K$  are

$$\pm 1, \frac{\pm 1 \pm \sqrt{-3}}{2}.$$

- (b)  $m < -3$  or  $|m| > 3$ .  $4 = a^2 + |m|b^2$  implies that  $b^2 = 0$ . Hence all units in  $\mathcal{O}_K$  are  $\pm 1$ .

- (5) By (3)(4), all units in  $\mathcal{O}_K$  are

$$\begin{cases} \pm 1 & (m \neq -1, -3), \\ \pm 1, \pm \sqrt{-1} & (m = -1), \\ \pm 1, \frac{\pm 1 \pm \sqrt{-3}}{2} & (m = -3). \end{cases}$$

□

#### Exercise 2.14.

Show that  $1 + \sqrt{2}$  is a unit in  $\mathbb{Z}[\sqrt{2}]$ . Use the powers of  $1 + \sqrt{2}$  to generate infinitely many solutions to the diophantine equation  $a^2 - 2b^2 = \pm 1$ . (It will be shown in Chapter 5 that all units in  $\mathbb{Z}[\sqrt{2}]$  are of the form  $\pm(1 + \sqrt{2})^k$ ,  $k \in \mathbb{Z}$ .)

Might assume to find nonnegative solutions to the Pell's equation  $a^2 - 2b^2 = \pm 1$ .

*Proof.*

- (1) Show that  $1 + \sqrt{2}$  is a unit in  $\mathbb{Z}[\sqrt{2}]$ . There is  $-1 + \sqrt{2} \in \mathbb{Z}[\sqrt{2}]$  such that

$$(1 + \sqrt{2})(-1 + \sqrt{2}) = 1 \in \mathbb{Z}[\sqrt{2}].$$

Hence  $1 + \sqrt{2}$  is a unit.

- (2)  $N(a + b\sqrt{2}) = |a^2 - 2b^2|$  is a norm on  $\mathbb{Z}[\sqrt{2}]$ . To prove this, use the same argument as Exercise 1.1 and note that

$$N(a + b\sqrt{2}) = |(a + b\sqrt{2})(a - b\sqrt{2})|.$$



- (3) By (1)(2), all  $(1+\sqrt{2})^k$  with  $k \geq 0$  are distinct solutions to the diophantine equation  $a^2 - 2b^2 = \pm 1$ . Explicitly, let

$$\begin{aligned}(a_0, b_0) &= (1, 0), \\(a_1, b_1) &= (1, 1), \\(a_2, b_2) &= (3, 2), \\(a_3, b_3) &= (7, 5), \\&\dots \\(a_k, b_k) &= (a_{k-1} + 2b_{k-1}, a_{k-1} + b_{k-1}), \\&\dots\end{aligned}$$

Note that all  $(a_k, b_k)$  are distinct and satisfying  $a_k^2 - 2b_k^2 = \pm 1$ . Hence we get infinitely many solutions to the Pell's equation  $a^2 - 2b^2 = \pm 1$ .

*Note.* Suppose that all units in  $\mathbb{Z}[\sqrt{2}]$  are of the form  $\pm(1+\sqrt{2})^k$ ,  $k \in \mathbb{Z}$ . Note that  $(1+\sqrt{2})^k = (-1+\sqrt{2})^{-k}$ . Thus we can find all nonnegative solutions to the Pell's equation  $a^2 - 2b^2 = \pm 1$  are exactly the same as (3).  $\square$

**Supplement. (Exercise I.1.6 in Jürgen Neukirch, *Algebraic Number Theory*)**

Show that the ring  $\mathbb{Z}[\sqrt{d}] = \mathbb{Z} + \mathbb{Z}\sqrt{d}$ , for any squarefree rational integer  $d > 1$ , has infinitely many units.

*Proof.* The proof is quoted from Proposition 17.5.2 in the book: Ireland and Rosen, *A Classical Introduction to Modern Number Theory*, 2nd Ed.

- (1) Define the norm of  $z = x + y\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$  by  $N(z) = z\bar{z}$  or

$$N(x + y\sqrt{d}) = \underbrace{(x + y\sqrt{d})}_{=: z} \underbrace{(x - y\sqrt{d})}_{:= \bar{z}} = x^2 - dy^2.$$

Note that a norm is multiplicative. Similar to Exercise I.1.1,  $\alpha \in \mathbb{Z}[\sqrt{d}]$  is a unit if and only if  $N(\alpha) = \pm 1$ .

- (2) To show  $\mathbb{Z}[\sqrt{d}]$  has infinitely many units, it suffices to show the equation  $x^2 - dy^2 = 1$  has infinitely many  $(x, y)$  solutions.
- (3) If  $\xi$  is irrational then there are infinitely many rational numbers  $\frac{x}{y}$ ,  $(x, y) = 1$  such that  $\left| \frac{x}{y} - \xi \right| < \frac{1}{y^2}$ . It is followed by the pigeonhole principle.
- (4) If  $d$  is a positive squarefree integer then there is a constant  $M := 2\sqrt{d} + 1$  such that  $|x^2 - dy^2| < M$  has infinitely many solutions over  $\mathbb{Z}$ . Write  $x^2 - dy^2 = (x + y\sqrt{d})(x - y\sqrt{d})$ . By part (3), there exist infinitely many

pairs of relatively prime integers  $(x, y)$ ,  $y > 0$  satisfying  $|x - y\sqrt{d}| < \frac{1}{y}$ .  
Hence

$$\begin{aligned} |x^2 - dy^2| &= |x + y\sqrt{d}| |x - y\sqrt{d}| \\ &\leq (|x - y\sqrt{d}| + 2y\sqrt{d}) |x - y\sqrt{d}| \\ &\leq 2\sqrt{d} + 1. \end{aligned}$$

- (5) By part (4), there is an integer  $m$  such that  $x^2 - dy^2 = m$  for infinitely many solutions over  $\mathbb{Z}$ . Here  $m \neq 0$ . We might assume  $x, y > 0$  and  $x$  components of solutions are distinct.
- (6) The pigeonhole principle shows that there are two distinct solutions  $(x_1, y_1)$ ,  $(x_2, y_2)$  with  $x_1 \neq x_2$  such that

$$x_1 \equiv x_2 \pmod{|m|}, \quad y_1 \equiv y_2 \pmod{|m|}.$$

Let  $\alpha = x_1 - y_1\sqrt{d}$ ,  $\beta = x_2 + y_2\sqrt{d}$  and  $\gamma = \alpha\beta$ . Hence

$$\begin{aligned} \gamma &= (x_1 - y_1\sqrt{d})(x_2 + y_2\sqrt{d}) \\ &= \underbrace{(x_1x_2 - dy_1y_2)}_{\equiv 0 \pmod{|m|}} + \underbrace{(x_1y_2 - x_2y_1)}_{\equiv 0 \pmod{|m|}} \sqrt{d} \\ &:= m(u + v\sqrt{d}) \end{aligned}$$

for some  $u + v\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$ . Taking norms of  $\gamma = \alpha\beta$  gives  $N(\gamma) = N(\alpha)N(\beta)$  or

$$m^2(u + v\sqrt{d}) = m^2.$$

Hence  $u + v\sqrt{d} = 1$ . By construction of  $x_1, x_2$ ,  $v \neq 0$ . Therefore the equation  $x^2 - dy^2 = 1$  has one solution with  $x, y > 0$ .

- (7) By part (6), we might take a unit  $\varepsilon = x + y\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$  with  $x, y > 0$ . Note that  $\varepsilon \geq 1 + \sqrt{d} > 1$  (over the ordered field  $\mathbb{R}$ ). Hence there are infinitely many units

$$\varepsilon, \varepsilon^2, \varepsilon^3, \dots$$

in  $\mathbb{Z}[\sqrt{d}]$ .

□

*Note. Furthermore, show that there is a unit  $\varepsilon$  such that every unit has the form  $\pm\varepsilon^n$ ,  $n \in \mathbb{Z}$ .*

*Proof.*

- (1) By the well-ordering principle, there is a unit  $\varepsilon = x_1 + y_1\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$  such that  $x_1, y_1 > 0$  and  $(x_1, y_1)$  is the smallest solution of  $x^2 - dy^2 = \pm 1$  with  $x, y > 0$ .

- (2) Now given any unit  $\varepsilon' = x + y\sqrt{d}$ ,  $x, y > 0$ , it suffices to show that there is a positive integer  $n$  such that  $\varepsilon' = \varepsilon^n$ .
- (3) (Reductio ad absurdum) If not, there were a positive integer  $n$  such that  $\varepsilon^n < \varepsilon' < \varepsilon^{n+1}$ . Hence  $1 < \varepsilon^{-n}\varepsilon' < \varepsilon$ . Say  $\varepsilon^{-n}\varepsilon' := x' + y'\sqrt{d}$ . As  $\varepsilon^{-n}\varepsilon' > 1 > 0$ , the inverse is satisfying  $x' - y'\sqrt{d} > 0$ . Hence  $x' > 0$ .
- (4) As the inverse is satisfying  $x' - y'\sqrt{d} < 1$ ,  $y' \geq 0$ . Note that  $y' \neq 0$  (since  $\varepsilon > 1$ ). Hence the existence of  $\varepsilon^{-n}\varepsilon'$  contradicts the minimality of  $\varepsilon$ .
- (5) Now suppose a unit  $\varepsilon' = x + y\sqrt{d}$  is of the form  $x > 0$ ,  $y < 0$ . Then  $\varepsilon'^{-1} = x - y\sqrt{d} = \varepsilon^n$  for some positive integer  $n$  by (2)(3)(4). Hence  $\varepsilon' = \varepsilon^{-n}$  for some positive integer  $n$ . Other two cases of  $\varepsilon' = x + y\sqrt{d}$  are similar. Therefore, every unit has the form  $\pm\varepsilon^n$ ,  $n \in \mathbb{Z}$ .

□

**Supplement. (Exercise I.1.7 in Jürgen Neukirch, *Algebraic Number Theory*)**

Show that the ring  $\mathbb{Z}[\sqrt{2}] = \mathbb{Z} + \mathbb{Z}\sqrt{2}$  is euclidean. Show furthermore that its units are given by  $\pm(1 + \sqrt{2})^n$ ,  $n \in \mathbb{Z}$ , and determine its prime elements.

*Proof.*

- (1) Show that  $\mathbb{Z}[\sqrt{2}]$  is euclidean with respect to the function  $N : \mathbb{Z}[\sqrt{2}] \rightarrow \mathbb{N} \cup \{0\}$ ,  $\alpha \mapsto \alpha\bar{\alpha}$ . For  $\alpha, \beta \neq 0 \in \mathbb{Z}[\sqrt{2}]$ , one has to find  $\gamma, \rho \in \mathbb{Z}[\sqrt{2}]$  such that

$$\alpha = \gamma\beta + \rho, \quad N(\rho) < N(\beta).$$

- (2) Extend the norm function  $N$  to  $\mathbb{Q}[\sqrt{2}]$ . Write

$$\frac{\alpha}{\beta} = x + y\sqrt{2} \in \mathbb{Q}[\sqrt{2}].$$

Take  $\gamma = u + v\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$  such that  $u, v$  are satisfying  $|u - x| \leq \frac{1}{2}$ ,  $|v - y| \leq \frac{1}{2}$ . Now take  $\rho = \alpha - \gamma\beta$ .

- (3) Hence,

$$N\left(\frac{\alpha}{\beta} - \gamma\right) = (u - x)^2 + 2(v - y)^2 \leq \left(\frac{1}{2}\right)^2 + 2 \cdot \left(\frac{1}{2}\right)^2 < 1$$

and thus

$$N(\rho) = N(\alpha - \gamma\beta) = N(\beta)N\left(\frac{\alpha}{\beta} - \gamma\right) < N(\beta).$$

- (4) Show that its units are given by  $\pm(1 + \sqrt{2})^n$ ,  $n \in \mathbb{Z}$ .  $\varepsilon = 1 + \sqrt{2} \in \mathbb{Z}[\sqrt{2}]$  is a unit such that  $(1, 1)$  is the smallest solution of  $x^2 - 2y^2 = \pm 1$  with  $x, y > 0$ . By the note in Exercise I.1.6, all units are given by  $\pm(1 + \sqrt{2})^n$ ,  $n \in \mathbb{Z}$ .
- (5) For all prime numbers  $p \neq 2$ , one has  $p = a^2 - 2b^2$  ( $a, b \in \mathbb{Z}$ ) if and only if  $p \equiv 1, 7 \pmod{8}$ . Similar to the proof of Proposition I.1.1, it suffices to show that a prime number  $p \equiv 1, 7 \pmod{8}$  of  $\mathbb{Z}$  does not remain a prime element in the ring  $\mathbb{Z}[\sqrt{2}]$ . (Reductio ad absurdum) Note that the congruence

$$2 \equiv x^2 \pmod{p}$$

admits a solution (by the law of quadratic reciprocity). Thus we have  $p \mid x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2})$ . Hence  $\frac{x}{p} \pm \frac{\sqrt{2}}{p} \in \mathbb{Z}[\sqrt{2}]$ , which is absurd.

- (6) The prime element  $\pi$  of  $\mathbb{Z}[\sqrt{2}]$ , up to associated elements, are given as follows.
- (i)  $\pi = \sqrt{2}$ ,
  - (ii)  $\pi = a + \sqrt{2}b$  with  $a^2 - 2b^2 = p$ ,  $p \equiv 1, 7 \pmod{8}$ ,
  - (iii)  $\pi = p$ ,  $p \equiv 3, 5 \pmod{8}$ .

Here,  $p$  denotes a prime number of  $\mathbb{Z}$ . The proof is exactly the same as Theorem I.1.4.

□

### Exercise 2.15.

- (a) Show that  $\mathbb{Z}[\sqrt{-5}]$  contains no element whose norm is 2 or 3.
- (b) Verify that  $2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$  is an example of non-unique factorization in the number ring  $\mathbb{Z}[\sqrt{-5}]$ .

*Proof of (a).* Since  $N(a + b\sqrt{-5}) = a^2 + 5b^2 \equiv a^2 \equiv 0, 1, 4 \pmod{5}$ , there is no element whose norm is 2 or 3. □

*Proof of (b).*

- (1) Show that  $2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$ .

$$2 \cdot 3 = 6 \text{ and } (1 + \sqrt{-5})(1 - \sqrt{-5}) = 6.$$

- (2) *Show that 2 is irreducible.* Suppose  $2 = \alpha\beta$  where  $\alpha, \beta \in \mathbb{Z}[\sqrt{-5}]$ . Take norm to get

$$\begin{aligned} N(2) = N(\alpha)N(\beta) &\implies 4 = N(\alpha)N(\beta) \\ &\implies N(\alpha) = 1 \text{ or } N(\beta) = 1 \\ &\implies \alpha \text{ or } \beta \text{ is unit.} \end{aligned} \quad ((1))$$

- (3) *Show that 3 is irreducible.* Similar to (2).

- (4) *Show that  $1 \pm \sqrt{-5}$  is irreducible.* Since  $N(1 \pm \sqrt{-5}) = 2$  is prime,  $1 \pm \sqrt{-5}$  is irreducible.

Hence 6 has a non-unique factorization in the number ring  $\mathbb{Z}[\sqrt{-5}]$ .  $\square$

### Exercise 2.16.

Set  $\alpha = \sqrt[4]{2}$ . Use the trace  $T = T^{\mathbb{Q}[\alpha]}$  to show that  $\sqrt{3} \notin \mathbb{Q}[\alpha]$ . (Hint: Write  $\sqrt{3} = a + b\alpha + c\alpha^2 + d\alpha^3$  and successively show that  $a = 0$ ;  $b = 0$  (what is  $T\left(\frac{\sqrt{3}}{\alpha}\right)$ );  $c = 0$ ; and finally obtain a contradiction.)

*Proof.*

- (1) Let  $K = \mathbb{Q}[\alpha]$ . (Reductio ad absurdum) If  $\sqrt{3} \in K$ , then we can write  $\sqrt{3} = a + b\alpha + c\alpha^2 + d\alpha^3$  for some integers  $a, b, c$  and  $d$ .
- (2) Note that  $K = \mathbb{Q}[\alpha] = \mathbb{Q}[\sqrt{3}]$  by assumption. Hence

$$\begin{aligned} T^{\mathbb{Q}[\sqrt{3}]}(\sqrt{3}) &= T^{\mathbb{Q}[\alpha]}(a + b\alpha + c\alpha^2 + d\alpha^3) \\ &\implies 0 = 4a \\ &\implies a = 0. \end{aligned}$$

$$\text{So } \sqrt{3} = b\alpha + c\alpha^2 + d\alpha^3.$$

- (3)  $\sqrt{3} = b\alpha + c\alpha^2 + d\alpha^3$  implies that

$$\underbrace{\sqrt{3}\alpha^3}_{=\sqrt[4]{72}} = 2b + 2c\alpha + 2d\alpha^2.$$

Since  $\mathbb{Q}[\sqrt[4]{72}] \subseteq K$  and  $[\mathbb{Q}[\sqrt[4]{72}] : \mathbb{Q}] = [\mathbb{Q}[\sqrt[4]{2}] : \mathbb{Q}] = 4$ ,  $K = \mathbb{Q}[\sqrt[4]{72}]$ . Hence

$$\begin{aligned} T^{\mathbb{Q}[\sqrt[4]{72}]}(\sqrt[4]{72}) &= T^{\mathbb{Q}[\alpha]}(2b + 2c\alpha + 2d\alpha^2) \\ &\implies 0 = 8b \\ &\implies b = 0. \end{aligned}$$

$$\text{So } \sqrt{3} = c\alpha^2 + d\alpha^3.$$

(4) Similar to (3).  $\sqrt{3} = c\alpha^2 + d\alpha^3$  implies that

$$\underbrace{\sqrt{3}\alpha^2}_{=\sqrt{6}} = 2c + 2d\alpha.$$

Since  $\mathbb{Q}[\sqrt{6}] \subseteq K$  and  $[\mathbb{Q}[\sqrt{6}] : \mathbb{Q}] = [\mathbb{Q}[\sqrt{3}] : \mathbb{Q}] = 2$ ,  $K = \mathbb{Q}[\sqrt{6}]$ . Hence

$$T^{\mathbb{Q}[\sqrt{6}]}(\sqrt{6}) = T^{\mathbb{Q}[\alpha]}(2c + 2d\alpha) \implies 0 = 8c \implies c = 0.$$

So  $\sqrt{3} = d\alpha^3$ .

(5) Similar to (3)(4),  $d = 0$  and thus  $\sqrt{3} = 0$ , which is absurd.

□

*Proof (Field theory).*

(1) (Reductio ad absurdum) If  $\sqrt{3} \in \mathbb{Q}[\sqrt[4]{2}]$ , then  $\mathbb{Q}[\sqrt{3}, \sqrt{2}] \subseteq \mathbb{Q}[\sqrt[4]{2}]$ . As  $[\mathbb{Q}[\sqrt{3}, \sqrt{2}] : \mathbb{Q}] = [\mathbb{Q}[\sqrt[4]{2}] : \mathbb{Q}] = 4$ ,  $\mathbb{Q}[\sqrt{3}, \sqrt{2}] = \mathbb{Q}[\sqrt[4]{2}]$ .

(2) Note that  $\mathbb{Q}[\sqrt{3}, \sqrt{2}]$  is normal over  $\mathbb{Q}$  but  $\mathbb{Q}[\sqrt[4]{2}]$  is not normal over  $\mathbb{Q}$ .

□

### Supplement.

(1) Give an example of fields  $F \subseteq K \subseteq L$  where  $L/K$  and  $K/F$  are normal but  $L/F$  is not normal.

(2) Show that  $\sqrt[3]{3} \notin \mathbb{Q}[\sqrt[3]{2}]$ .

(3) Show that  $1 + 5\sqrt[3]{2} - \sqrt[3]{4}$  is not a perfect square in  $\mathbb{Q}[\sqrt[3]{2}]$ .

### Exercise 2.19. (Vandermonde determinant)

Let  $R$  be a commutative ring and fix elements  $a_1, a_2, \dots \in R$ . We will prove by induction that the Vandermonde determinant

$$\begin{vmatrix} 1 & a_1 & \cdots & a_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & \cdots & a_n^{n-1} \end{vmatrix}$$

is equal to the product  $\prod_{1 \leq r < s \leq n} (a_s - a_r)$ . Assuming that the result holds for some  $n$ , consider the determinant

$$\begin{vmatrix} 1 & a_1 & \cdots & a_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & \cdots & a_n^n \\ 1 & a_{n+1} & \cdots & a_{n+1}^n \end{vmatrix}.$$

Show that this is equal to

$$\begin{vmatrix} 1 & a_1 & \cdots & f(a_1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & \cdots & f(a_n) \\ 1 & a_{n+1} & \cdots & f(a_{n+1}) \end{vmatrix}$$

for any monic polynomial  $f$  over  $R$  of degree  $n$ . Then choose  $f$  cleverly so that the determinant is easily calculated.

*Proof.*

(1) Let

$$V_n = \begin{pmatrix} 1 & a_1 & \cdots & a_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & \cdots & a_n^{n-1} \end{pmatrix}$$

be the Vandermonde matrix. We will apply the induction to show that  $\det(V_n) = \prod_{1 \leq r < s \leq n} (a_s - a_r)$ .

(2) Nothing to do for  $n = 1, 2$ . Now Assuming that the result holds for some  $n$ , consider the determinant

$$\det(V_{n+1}) = \begin{vmatrix} 1 & a_1 & \cdots & a_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & \cdots & a_n^n \\ 1 & a_{n+1} & \cdots & a_{n+1}^n \end{vmatrix}.$$

(3) Show that

$$\det(V_{n+1}) = \begin{vmatrix} 1 & a_1 & \cdots & f(a_1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & \cdots & f(a_n) \\ 1 & a_{n+1} & \cdots & f(a_{n+1}) \end{vmatrix}$$

for any monic polynomial  $f$  over  $R$  of degree  $n$ . Note that  $\det(V_{n+1})$  is unchanged by adding a multiple of one column of  $V_{n+1}$  to another column of  $V_{n+1}$ . In particular, we add a multiple of the  $i$ -th column of  $V_{n+1}$  to the last column of  $V_{n+1}$  for  $i = 1, 2, \dots, n$ . Then we obtain the equation

$$\det(V_{n+1}) = \begin{vmatrix} 1 & a_1 & \cdots & f(a_1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & \cdots & f(a_n) \\ 1 & a_{n+1} & \cdots & f(a_{n+1}) \end{vmatrix}.$$

(4) In particular, we take

$$f(x) = (x - a_1)(x - a_2) \cdots (x - a_n).$$

Therefore

$$\begin{aligned}
\det(V_{n+1}) &= \begin{vmatrix} 1 & a_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & \cdots & 0 \\ 1 & a_{n+1} & \cdots & \prod_{1 \leq r \leq n} (a_{n+1} - a_r) \end{vmatrix} \\
&= (-1)^{(n+1)+(n+1)} \prod_{1 \leq r \leq n} (a_{n+1} - a_r) \begin{vmatrix} 1 & a_1 & \cdots & a_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & \cdots & a_n^{n-1} \end{vmatrix} \\
&= \prod_{1 \leq r \leq n} (a_{n+1} - a_r) \prod_{1 \leq r < s \leq n} (a_s - a_r) \\
&= \prod_{1 \leq r < s \leq n+1} (a_s - a_r).
\end{aligned}$$

By induction, the result is established.

□

**Exercise 2.20.**

Let  $f$  be a monic irreducible polynomial over a number field  $K$  and let  $\alpha$  be one of its roots in  $\mathbb{C}$ . Show that  $f'(\alpha) = \prod_{\beta \neq \alpha} (\alpha - \beta)$  with the product taken over all roots  $\beta \neq \alpha$ . (Hint: Write  $f(x) = (x - \alpha)g(x)$ .)

*Proof.*

- (1) Note that  $f$  has no repeated roots in  $\mathbb{C}$  by the irreducibility of  $f$ . So we can write

$$f(x) = (x - \alpha)g(x) = (x - \alpha) \prod_{\beta \neq \alpha} (x - \beta).$$

- (2) So

$$f'(x) = g(x) + (x - \alpha)g'(x)$$

by the Leibniz rule. Take  $x = \alpha$  to get

$$f'(\alpha) = g(\alpha) = \prod_{\beta \neq \alpha} (\alpha - \beta).$$

□



**Exercise 2.22. (Stickelberger's criterion)**

Let  $K$  be a number field of degree  $n$  over  $\mathbb{Q}$  and fix algebraic integers  $\alpha_1, \dots, \alpha_n \in K$ . We know that  $d = \text{disc}(\alpha_1, \dots, \alpha_n)$  is in  $\mathbb{Z}$ ; we will show that  $d \equiv 0$  or  $1 \pmod{4}$ . Letting  $\sigma_1, \dots, \sigma_n$  denote the embeddings of  $K$  in  $\mathbb{C}$ , we know that  $d$  is the square of the determinant  $|\sigma_i(\alpha_j)|$ . This determinant is a sum of  $n!$  terms, one for each permutation of  $\{1, \dots, n\}$ . Let  $P$  denote the sum of the terms corresponding to even permutations, and let  $N$  denote the sum of the terms (without negative signs) corresponding to odd permutations. Thus  $d = (P - N)^2 = (P + N)^2 - 4PN$ . Complete the proof by showing that  $P + N$  and  $PN$  are in  $\mathbb{Z}$ . (Suggestion: Show that they are algebraic integers and that they are in  $\mathbb{Q}$ ; for the latter, extend all  $\sigma_i$  to some normal extension  $L$  of  $\mathbb{Q}$  so that they become automorphisms of  $L$ .)

In particular we have  $\text{disc}(\mathcal{O}_K) \equiv 0$  or  $1 \pmod{4}$ . This is known as **Stickelberger's criterion**.

*Proof.*

- (1) Let  $\sigma_1, \dots, \sigma_n$  be the embeddings of  $K$  in  $\mathbb{C}$ .
- (2) Note that

$$\begin{aligned} |\sigma_i \alpha_j| &= \sum_{\pi \in S_n} \left( \text{sgn}(\pi) \prod_{i=1}^n \sigma_i \alpha_{\pi(i)} \right) \\ &= \underbrace{\sum_{\pi \in A_n} \prod_{i=1}^n \sigma_i \alpha_{\pi(i)}}_{:=P} - \underbrace{\sum_{\pi \in S_n - A_n} \prod_{i=1}^n \sigma_i \alpha_{\pi(i)}}_{:=N} \end{aligned}$$

where  $S_n$  is the symmetric group of degree  $n$  and  $A_n$  is the alternating group of degree  $n$ .

- (3) Note that  $\sigma_i(P + N) = P + N$  and  $\sigma_i(PN) = PN$  for all  $\sigma_i$ . Hence  $P + N, PN \in \mathbb{Q}$  by extending all  $\sigma_i$  to some normal extension  $L$  of  $\mathbb{Q}$  so that they become automorphisms of  $L$ . Therefore  $P + N, PN \in \mathbb{Q} \cap \mathcal{O}_K = \mathbb{Z}$ .
- (4) By (2)(3),

$$\begin{aligned} d &= |\sigma_i \omega_j|^2 \\ &= (P - N)^2 \\ &= (P + N)^2 - 4PN \\ &\equiv 0, 1 \pmod{4}. \end{aligned}$$

In particular,  $\text{disc}(\mathcal{O}_K) \equiv 0, 1 \pmod{4}$ .

□

**Supplement.**

(Exercise I.2.7 (Stickelberger's discriminant relation) in [Jürgen Neukirch, *Algebraic Number Theory*].) The discriminant  $d_K$  of an algebraic number field  $K$  is always  $\equiv 0 \pmod{4}$  or  $\equiv 1 \pmod{4}$ . (Hint: The discriminant  $\det(\sigma_i \omega_j)$  of an integral basis  $\omega_j$  is a sum of terms, each prefixed by a positive or a negative sign. Writing  $P$  (resp.  $N$ ) for the sum of the positive (resp. negative) terms, one find  $d_K = (P - N)^2 = (P + N)^2 - 4PN$ .)

*Proof (Hint).*

- (1) Let  $S_n$  be the symmetric group of degree  $n$ , and  $A_n$  be the alternating group of degree  $n$ . So

$$\begin{aligned} \det(\sigma_i \omega_j) &= \sum_{\pi \in S_n} \left( \operatorname{sgn}(\pi) \prod_{i=1}^n \sigma_i \omega_{\pi(i)} \right) \\ &= \underbrace{\sum_{\pi \in A_n} \prod_{i=1}^n \sigma_i \omega_{\pi(i)}}_{:=P} - \underbrace{\sum_{\pi \in S_n - A_n} \prod_{i=1}^n \sigma_i \omega_{\pi(i)}}_{:=N}. \end{aligned}$$

- (2) Note that  $\sigma_i(P + N) = P + N$  and  $\sigma_i(PN) = PN$  for all  $\sigma_i$ . Hence  $P + N, PN \in \mathbb{Q}$  by extending all  $\sigma_i$  to some normal extension  $L$  of  $\mathbb{Q}$  so that they become automorphisms of  $L$ . Therefore  $P + N, PN \in \mathbb{Q} \cap \mathcal{O}_K = \mathbb{Z}$ .
- (3) By (1)(2),

$$\begin{aligned} d_K &= \det(\sigma_i \omega_j)^2 \\ &= (P - N)^2 \\ &= (P + N)^2 - 4PN \\ &\equiv 0, 1 \pmod{4}. \end{aligned}$$

□

**Exercise 2.24.**

Let  $G$  be a free abelian group of rank  $n$  and let  $H$  be a subgroup. Without loss of generality we take  $G = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$  ( $n$  times). We will show by induction that  $H$  is a free abelian group of rank  $\leq n$ . First prove it for  $n = 1$ . Then, assuming the result holds for  $n - 1$ , let  $\pi : G \rightarrow \mathbb{Z}$  denote the obvious projection of  $G$  on the first factor (so that an  $n$ -tuple of integers gets sent to its first component). Let  $K$  denote the kernel of  $\pi$ .

- (a) Show that  $H \cap K$  is a free abelian group of rank  $\leq n - 1$ .

- (b) The image  $\pi(H) \subseteq \mathbb{Z}$  is either  $\{0\}$  or infinite cyclic. If it is  $\{0\}$ , then  $H = H \cap K$ ; otherwise fix  $h \in H$  such that  $\pi(h)$  generates  $\pi(H)$  and show that  $H$  is the direct sum of its subgroups  $\pi(H) = \pi(h)\mathbb{Z}$  and  $H \cap K$ .

*Proof.*

- (1) Induction on  $n$ . If  $n = 1$ , then  $H$  is a subgroup of  $G = \mathbb{Z}$ . Thus  $H = 0$  or  $H = h\mathbb{Z} \cong \mathbb{Z}$  for some integer  $h > 0$ . In any case,  $H$  is a free abelian group of rank  $\leq 1$ .
- (2) Assume the result holds for  $n - 1$ . Suppose  $G = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$  ( $n$  times). Let  $\pi : G \rightarrow \mathbb{Z}$  denote the obvious projection of  $G$  on the first factor, say

$$\pi((g_1, \dots, g_n)) \mapsto g_1.$$

So the kernel of  $\pi$  is

$$\{(g_1, \dots, g_n) \in G : g_1 = 0\} = \{(0, g_2, \dots, g_n) \in G\} \cong \mathbb{Z}^{n-1}$$

is a free abelian group of rank  $n - 1$ .

- (3) (Part (a)) Show that  $H \cap K$  is a free abelian group of rank  $\leq n - 1$ . Note that  $H \cap K$  is a subgroup of a free abelian group  $K = \ker(\pi)$  of rank  $n - 1$ . The induction hypothesis shows that  $H \cap K$  is a free abelian group of rank  $\leq n - 1$ .
- (4) Show that the image  $\pi(H) \subseteq \mathbb{Z}$  is either  $\{0\}$  or infinite cyclic. As  $\pi$  is a group homomorphism,  $\pi(H)$  is a subgroup of  $\mathbb{Z}$ . Thus  $\pi(H)$  is a free abelian group of rank  $\leq 1$ .
- (5) Show that  $H = \pi(H) \oplus (H \cap K)$ . If  $\pi(H) = 0$ , then  $H = H \cap K = \pi(H) \oplus (H \cap K)$ . If  $\pi(H)$  is infinite cyclic, we might assume that  $\pi(H)$  is generated by  $\pi(h_0)$  for some  $h_0 \in H$ .
- (6) Observe that

$$\pi|_H : H \rightarrow \pi(H)$$

is surjective and  $\ker(\pi|_H) = K \cap H$ . Given any  $h \in H$ , we have  $\pi(h) = \pi(h_0) \cdot a = \pi(ah_0)$  for some integer  $a$ . So  $h - ah_0 \in H \cap K$ . Since  $H \cap K$  is a free abelian group  $K$  of rank  $\leq n - 1$ , we might write

$$h - ah_0 = b_1k_1 + \cdots + b_rk_r$$

where  $\{k_1, \dots, k_r\}$  is a basis of  $H \cap K$  ( $r \leq n - 1$ ) and  $b_1, \dots, b_r \in \mathbb{Z}$ . Therefore

$$h = ah_0 + b_1k_1 + \cdots + b_rk_r$$

is generated by a basis  $\{h_0, k_1, \dots, k_r\}$  (since  $\pi(h_0) \neq 0$  by assumption). Hence  $H = \pi(H) \oplus (H \cap K)$ .

□

**Exercise 2.25.**

Show that for any algebraic number  $\alpha$ , there exists  $m \in \mathbb{Z}$ ,  $m \neq 0$ , such that  $m\alpha$  is an algebraic integer. (Hint: Obtain  $f \in \mathbb{Z}[x]$  such that  $f(\alpha) = 0$  and take  $m$  to be a power of the leading coefficient.) Use this to show that for every finite set of algebraic numbers  $\alpha_i$ , there exists  $m \in \mathbb{Z}$ ,  $m \neq 0$ , such that all  $m\alpha_i \in \mathcal{O}_{\mathbb{Q}}$ .

*Proof.*

- (1) As  $\alpha$  is an algebraic number, there is a monic polynomial

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{Q}[x]$$

such that  $f(\alpha) = 0$ . Eliminating all denominators of  $a_{n-1}, \dots, a_0$ , we might assume that

$$f(x) = mx^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$$

such that  $f(\alpha) = 0$  where  $m \neq 0$ .

- (2) Hence

$$\begin{aligned} m^n \alpha^n + m^{n-1} a_{n-1} \alpha^{n-1} + \cdots + m^{n-1} a_0 &= 0 \\ \implies (m\alpha)^n + \underbrace{a_{n-1}}_{\in \mathbb{Z}} (m\alpha)^{n-1} + \underbrace{ma_{n-2}}_{\in \mathbb{Z}} (m\alpha)^{n-2} + \cdots + \underbrace{m^{n-1} a_0}_{\in \mathbb{Z}} &= 0. \end{aligned}$$

Therefore  $m\alpha$  ( $m \neq 0$ ) is an algebraic integer.

- (3) Given finitely many algebraic numbers  $\alpha_1, \dots, \alpha_r$ . There exist  $m_i \in \mathbb{Z}$ ,  $m_i \neq 0$ , such that  $m_i \alpha_i \in \mathcal{O}_{\mathbb{Q}}$  for all  $i = 1, \dots, r$ . Take  $m = m_1 \cdots m_r$ . Hence all  $m\alpha_i$  are algebraic integers again.

□

**Exercise 2.28.**

Let  $f(x) = x^3 + ax + b$ ,  $a$  and  $b \in \mathbb{Z}$ , and assume  $f$  is irreducible over  $\mathbb{Q}$ . Let  $\alpha$  be a root of  $f$ .

- (a) Show that  $f'(\alpha) = -\frac{2a\alpha+3b}{\alpha}$ .  
 (b) Show that  $2a\alpha + 3b$  is a root of

$$\left( \frac{x-3b}{2a} \right)^3 + a \left( \frac{x-3b}{2a} \right) + b.$$

Use this to find  $N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(2a\alpha + 3b)$ .

- (c) Show that  $\text{disc}(\alpha) = -(4a^3 + 27b^2)$ .
- (d) Suppose  $\alpha^3 = \alpha + 1$ . Prove that  $\{1, \alpha, \alpha^2\}$  is an integral basis for  $\mathcal{O}_{\mathbb{Q}[\alpha]}$ .  
(See Exercise 2.27(e).) Do the same if  $\alpha^3 + \alpha = 1$ .

*Proof of (a).*

- (1) Show that  $\alpha \neq 0$ . If  $\alpha$  were 0, then  $f(\alpha) = f(0) = b$ . So  $f(x) = x^3 + ax = x(x^2 + a)$  is reducible, contrary to the irreducibility of  $f$ .
- (2) Since  $\alpha$  be a root of  $f$ ,  $f(\alpha) = 0$ , or  $\alpha^3 + a\alpha + b = 0$ , or  $\alpha^3 = -a\alpha - b$ .
- (3)

$$\begin{aligned}
 f'(x) = 3x^2 + a &\implies f'(\alpha) = 3\alpha^2 + a \\
 &\iff \alpha f'(\alpha) = 3\alpha^3 + a\alpha & (\alpha \neq 0) \\
 &\iff \alpha f'(\alpha) = 3(-a\alpha - b) + a\alpha & (\alpha^3 = -a\alpha - b) \\
 &\iff \alpha f'(\alpha) = -2a\alpha - 3b.
 \end{aligned}$$

$$\text{So } f'(\alpha) = -\frac{2a\alpha + 3b}{\alpha}.$$

□

*Proof of (b).*

- (1) Since  $\alpha^3 + a\alpha + b = 0$ ,

$$\left(\frac{(2a\alpha + 3b) - 3b}{2a}\right)^3 + a\left(\frac{(2a\alpha + 3b) - 3b}{2a}\right) + b = 0.$$

That is,  $2a\alpha + 3b$  is a root of  $\left(\frac{x-3b}{2a}\right)^3 + a\left(\frac{x-3b}{2a}\right) + b$ .

- (2)  $N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(2a\alpha + 3b)$  is the product of three roots of  $\left(\frac{x-3b}{2a}\right)^3 + a\left(\frac{x-3b}{2a}\right) + b$ .  
Hence,

$$\begin{aligned}
 N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(2a\alpha + 3b) &= (2a)^3 \left[ \left(\frac{-3b}{2a}\right)^3 + a \cdot \frac{-3b}{2a} + b \right] \\
 &= 8a^3 \left[ \frac{-27b^3}{8a^3} - \frac{b}{2} \right] \\
 &= -27b^3 - 4a^3b.
 \end{aligned}$$

□

*Proof of (c).*

$$\begin{aligned}
\text{disc}(\alpha) &= (-1)^{\frac{n(n-1)}{2}} N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(f'(\alpha)) && \text{(Theorem 2.8)} \\
&= -N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]} \left( -\frac{2a\alpha + 3b}{\alpha} \right) && (n = 3 \text{ and (a)}) \\
&= \frac{N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(2a\alpha + 3b)}{N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(\alpha)} \\
&= \frac{-27b^3 - 4a^3b}{b} && ((b)) \\
&= -27b^2 - 4a^3.
\end{aligned}$$

□

*Proof of (d).*

- (1) Write  $\alpha^3 = \alpha + 1$  as  $\alpha^3 - \alpha - 1 = 0$ . Note that  $f(x) = x^3 - x - 1$  is irreducible over  $\mathbb{Q}$  since  $f(x)$  is irreducible over  $\mathbb{Z}/3\mathbb{Z}$ . So  $\text{disc}(\alpha) = -23$  (by (c)). Since  $\text{disc}(\alpha)$  is squarefree, the result is established (Exercise 2.27(e)).
- (2) Similar to (1). Write  $\alpha^3 + \alpha = 1$  as  $\alpha^3 + \alpha - 1 = 0$ . Note that  $f(x) = x^3 + x - 1$  is irreducible over  $\mathbb{Q}$  since  $f(x)$  is irreducible over  $\mathbb{Z}/2\mathbb{Z}$ . So  $\text{disc}(\alpha) = -31$  (by (c)). Since  $\text{disc}(\alpha)$  is squarefree, the result is established (Exercise 2.27(e)).

□

### Exercise 2.43.

Let  $f(x) = x^5 + ax + b$ ,  $a$  and  $b \in \mathbb{Z}$ , and assume  $f$  is irreducible over  $\mathbb{Q}$ . Let  $\alpha$  be a root of  $f$ .

- (a) Show that  $\text{disc}(\alpha) = 4^4a^5 + 5^4b^4$ . (Suggestion: See Exercise 2.28.)
- (b) Suppose  $\alpha^5 = \alpha + 1$ . Prove that  $\mathcal{O}_{\mathbb{Q}[\alpha]} = \mathbb{Z}[\alpha]$ . ( $x^5 - x - 1$  is irreducible over  $\mathbb{Q}$ ; this can be shown by reducing (mod 3).)

*Proof of (a) (Exercise 2.28).*

- (1) Show that  $f'(\alpha) = -\frac{4a\alpha + 5b}{\alpha}$ .
  - (a) Show that  $\alpha \neq 0$ . If  $\alpha$  were 0, then  $f(\alpha) = f(0) = b$ . So  $f(x) = x^5 + ax = x(x^4 + a)$  is reducible, contrary to the irreducibility of  $f$ .
  - (b) Since  $\alpha$  be a root of  $f$ ,  $f(\alpha) = 0$ , or  $\alpha^5 + a\alpha + b = 0$ , or  $\alpha^5 = -a\alpha - b$ .

(c)

$$\begin{aligned}
f'(x) = 5x^4 + a &\implies f'(\alpha) = 5\alpha^4 + a \\
&\iff \alpha f'(\alpha) = 5\alpha^5 + a\alpha & (\alpha \neq 0) \\
&\iff \alpha f'(\alpha) = 5(-a\alpha - b) + a\alpha & (\alpha^5 = -a\alpha - b) \\
&\iff \alpha f'(\alpha) = -4a\alpha - 5b.
\end{aligned}$$

$$\text{So } f'(\alpha) = -\frac{4a\alpha + 5b}{\alpha}.$$

(2) Show that  $4a\alpha + 5b$  is a root of

$$\left(\frac{x - 5b}{4a}\right)^5 + a\left(\frac{x - 5b}{4a}\right) + b.$$

Use this to show that  $N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(4a\alpha + 5b) = -4^4a^5b - 5^5b^5$ .

(a) Since  $\alpha^5 + a\alpha + b = 0$ ,

$$\left(\frac{(4a\alpha + 5b) - 5b}{4a}\right)^5 + a\left(\frac{(4a\alpha + 5b) - 5b}{4a}\right) + b = 0.$$

That is,  $4a\alpha + 5b$  is a root of  $\left(\frac{x-5b}{4a}\right)^5 + a\left(\frac{x-5b}{4a}\right) + b$ .

(b)  $N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(4a\alpha + 5b)$  is the product of 5 roots of  $\left(\frac{x-5b}{4a}\right)^5 + a\left(\frac{x-5b}{4a}\right) + b$ .  
Hence,

$$\begin{aligned}
N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(4a\alpha + 5b) &= (4a)^5 \left[ \left(\frac{-5b}{4a}\right)^5 + a \cdot \frac{-5b}{4a} + b \right] \\
&= 4^5a^5 \left[ \frac{-5^5b^5}{4^5a^5} - \frac{b}{4} \right] \\
&= -5^5b^5 - 4^4a^5b.
\end{aligned}$$

(3) Show that  $\text{disc}(\alpha) = 4^4a^5 + 5^4b^4$ .

$$\begin{aligned}
\text{disc}(\alpha) &= (-1)^{\frac{n(n-1)}{2}} N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(f'(\alpha)) && (\text{Theorem 2.8}) \\
&= N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]} \left( -\frac{4a\alpha + 5b}{\alpha} \right) && (n = 5 \text{ and (1)}) \\
&= -\frac{N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(4a\alpha + 5b)}{N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(\alpha)} \\
&= -\frac{-4^4a^5b - 5^5b^5}{b} && ((2)) \\
&= 4^4a^5 + 5^4b^4.
\end{aligned}$$

□

*Proof of (b)(Exercise 2.28).* Write  $\alpha^5 = \alpha + 1$  as  $\alpha^5 - \alpha - 1 = 0$ . Note that  $f(x) = x^5 - x - 1$  is irreducible over  $\mathbb{Q}$  since  $f(x)$  is irreducible over  $\mathbb{Z}/3\mathbb{Z}$ . So  $\text{disc}(\alpha) = 881$  (by (a)). Since  $\text{disc}(\alpha)$  is squarefree (a prime number), the result is established (Exercise 2.27(e)). □

### Exercise 2.45.

Obtain a formula for  $\text{disc}(\alpha)$  if  $\alpha$  is a root of an irreducible polynomial  $x^n + ax + b$  over  $\mathbb{Q}$ . Do the same for  $x^n + ax^{n-1} + b$ .

Assume that  $n \geq 2$ .

*Proof of  $x^n + ax + b$  (Exercise 2.28).*

(1) Show that  $f'(\alpha) = -\frac{(n-1)a\alpha + nb}{\alpha}$ .

- (a) Show that  $\alpha \neq 0$ . If  $\alpha$  were 0, then  $f(\alpha) = f(0) = b$ . So  $f(x) = x^n + ax = x(x^{n-1} + a)$  is reducible, contrary to the irreducibility of  $f$ .
- (b) Since  $\alpha$  be a root of  $f$ ,  $f(\alpha) = 0$ , or  $\alpha^n + a\alpha + b = 0$ , or  $\alpha^n = -a\alpha - b$ .
- (c)

$$\begin{aligned} f'(x) = nx^{n-1} + a &\implies f'(\alpha) = n\alpha^{n-1} + a \\ &\iff \alpha f'(\alpha) = n\alpha^n + a\alpha & (\alpha \neq 0) \\ &\iff \alpha f'(\alpha) = n(-a\alpha - b) + a\alpha & (\alpha^n = -a\alpha - b) \\ &\iff \alpha f'(\alpha) = -(n-1)a\alpha - nb. \end{aligned}$$

$$\text{So } f'(\alpha) = -\frac{(n-1)a\alpha + nb}{\alpha}.$$

(2) Let  $\beta = (n-1)a\alpha + nb$ . Show that  $\beta$  is a root of

$$\left(\frac{x - nb}{(n-1)a}\right)^n + a\left(\frac{x - nb}{(n-1)a}\right) + b.$$

Use this to show that

$$N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(\beta) = -(n-1)^{n-1}a^n b + (-1)^n n^n b^n.$$

(a) Since  $\alpha^n + a\alpha + b = 0$ ,

$$\left(\frac{\beta - nb}{(n-1)a}\right)^n + a\left(\frac{\beta - nb}{(n-1)a}\right) + b = 0.$$

That is,  $\beta$  is a root of  $\left(\frac{x - nb}{(n-1)a}\right)^n + a\left(\frac{x - nb}{(n-1)a}\right) + b$ .



(b)  $N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(\beta)$  is the product of  $n$  roots of  $\left(\frac{x-nb}{(n-1)a}\right)^n + a\left(\frac{x-nb}{(n-1)a}\right) + b$ .

Hence,

$$\begin{aligned} N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(\beta) &= ((n-1)a)^n \left[ \left( \frac{-nb}{(n-1)a} \right)^n + a \cdot \frac{-nb}{(n-1)a} + b \right] \\ &= (n-1)^n a^n \left[ \frac{(-1)^n n^n b^n}{(n-1)^n a^n} - \frac{b}{n-1} \right] \\ &= (-1)^n n^n b^n - (n-1)^{n-1} a^n b. \end{aligned}$$

(3) Show that  $\text{disc}(\alpha) = (-1)^{\frac{(n-1)(n-2)}{2}} (n-1)^{n-1} a^n + (-1)^{\frac{n(n-1)}{2}} n^n b^{n-1}$ .

$$\text{disc}(\alpha) = (-1)^{\frac{n(n-1)}{2}} N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(f'(\alpha)) \quad (\text{Theorem 2.8})$$

$$= (-1)^{\frac{n(n-1)}{2}} N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]} \left( -\frac{(n-1)a\alpha + nb}{\alpha} \right) \quad ((1))$$

$$= (-1)^{\frac{n(n-1)}{2}} (-1)^n \frac{N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}((n-1)a\alpha + nb)}{N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(\alpha)}$$

$$= (-1)^{\frac{n(n-1)}{2}} (-1)^n \frac{-(n-1)^{n-1} a^n b + (-1)^n n^n b^n}{b} \quad ((2))$$

$$= (-1)^{\frac{(n-1)(n-2)}{2}} (n-1)^{n-1} a^n + (-1)^{\frac{n(n-1)}{2}} n^n b^{n-1}.$$

□