Notes on the book: *Apostol, Modular Functions and Dirichlet Series in Number Theory, 2nd edition*

Meng-Gen Tsai
plover@gmail.com

July 23, 2021

Contents

Chapter 1: Elliptic functions

Exercise 1.11.

If $k \geq 2$ and $\tau \in H$ prove that the Eisenstein series

$$G_{2k}(\tau) = \sum_{(m,n)\neq(0,0)} (m+n\tau)^{-2k}$$

has the Fourier expansion

$$G_{2k}(\tau) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)e^{2\pi i n\tau}.$$

Proof.

(1) Similar to Lemma 1.3 on page 19, we have

$$(2k-1)!\sum_{m=-\infty}^{+\infty}\frac{1}{(\tau+m)^{2k}} = (2\pi i)^{2k}\sum_{r=1}^{\infty} r^{2k-1}e^{2\pi i r\tau}.$$

(2) Similar to Theorem 1.18, we have

$$\begin{aligned}
G_{2k}(\tau) &= \sum_{(m,n)\neq(0,0)} (m+n\tau)^{-2k} \\
&= \sum_{\substack{m=-\infty \\ m\neq 0 (n=0)}}^{+\infty} m^{-2k} + \sum_{n=1}^{\infty}\sum_{m=-\infty}^{+\infty}\left((m+n\tau)^{-2k} + (m-n\tau)^{-2k}\right) \\
&= 2\zeta(2k) + 2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{+\infty}(m+n\tau)^{-2k} \\
&= 2\zeta(2k) + 2\sum_{n=1}^{\infty}\frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{r=1}^{\infty} r^{2k-1}e^{2\pi i n r\tau} \\
&= 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\underbrace{\sum_{d|n} d^{2k-1}}_{=\sigma_{2k-1}(n)} e^{2\pi i n\tau}.
\end{aligned}$$

In the last double sum we collect together those terms for which nr is constant.

$\square$

Exercise 1.12.

Refer to Exercise 1.11. If $\tau \in H$ *prove that*

$$G_{2k}\left(-\frac{1}{\tau}\right) = \tau^{2k} G_{2k}(\tau)$$

and deduce that

$$\begin{aligned} G_{2k}\left(\frac{i}{2}\right) &= (-4)^k G_{2k}(2i) && \textit{for all } k \geq 2, \\ G_{2k}(i) &= 0 && \textit{if } k \textit{ is odd}, \\ G_{2k}(e^{\frac{2\pi i}{3}}) &= 0 && \textit{if } k \not\equiv 0 \pmod 3. \end{aligned}$$

Proof.

(1)

$$\begin{aligned} G_{2k}\left(-\frac{1}{\tau}\right) &= \sum_{(m,n)\neq(0,0)} \left(m - \frac{n}{\tau}\right)^{-2k} \\ &= \tau^{2k} \sum_{(m,n)\neq(0,0)} (\tau m - n)^{-2k} \\ &= \tau^{2k} G_{2k}(\tau). \end{aligned}$$

(2) Let $\tau = 2i$. We have $G_{2k}\left(\frac{i}{2}\right) = (-4)^k G_{2k}(2i)$.

(3) Let $\tau = i$. We have $G_{2k}(i) = (-1)^k G_{2k}(i)$. Hence $G_{2k}(i) = 0$ if k is odd.

(4) Let $\tau = e^{\frac{\pi i}{3}}$. We have $G_{2k}(e^{\frac{2\pi i}{3}}) = e^{\frac{2k\pi i}{3}} G_{2k}(e^{\frac{\pi i}{3}})$. Since

$$e^{\frac{2\pi i}{3}} = -1 + e^{\frac{\pi i}{3}}$$

and each Eisenstein series is a periodic function of τ of period 1, we have $G_{2k}(e^{\frac{2\pi i}{3}}) = G_{2k}(e^{\frac{\pi i}{3}})$. So $G_{2k}(e^{\frac{2\pi i}{3}}) = e^{\frac{2k\pi i}{3}} G_{2k}(e^{\frac{2\pi i}{3}})$. Therefore $G_{2k}(e^{\frac{2\pi i}{3}}) = 0$ if $k \not\equiv 0 \pmod 3$.

□