Chapter 3: Numerical Sequences and Series

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Exercise 3.1. *Prove that the convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?*

Proof.

(1) Since $\{s_n\}$ is convergent, there is $s \in \mathbb{R}^1$ with the following property: given any $\varepsilon > 0$, there is N such that $|s_n - s| < \varepsilon$ whenever $n \geq N$. So

$$||s_n| - |s|| \leq |s_n - s| < \varepsilon$$

(Exercise 1.13). That is, $\{|s_n|\}$ converges to $|s|$.

(2) The converse is not true by considering $s_n = (-1)^{n+1}$.

□

Exercise 3.2. *Calculate* $\lim_{n\to\infty}(\sqrt{n^2+n} - n)$.

Proof.

$$\sqrt{n^2+n} - n = \frac{n}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+\frac{1}{n}}+1} \to \frac{1}{1+1} = \frac{1}{2}$$

as $n \to \infty$. □

Proof ($\varepsilon - N$ argument). Let $s_n = \sqrt{n^2+n} - n$. *Show that the sequence $\{s_n\}$ converges to $s = \frac{1}{2}$.* Given any $\varepsilon > 0$, there is $N > \frac{1}{\varepsilon}$ such that

$$\begin{aligned}
|s_n - s| &= \left|(\sqrt{n^2+n} - n) - \frac{1}{2}\right| = \left|\frac{1}{\sqrt{1+\frac{1}{n}}+1} - \frac{1}{2}\right| \\
&= \left|\frac{2 - \left(\sqrt{1+\frac{1}{n}}+1\right)}{2\left(\sqrt{1+\frac{1}{n}}+1\right)}\right| = \left|\frac{1 - \sqrt{1+\frac{1}{n}}}{2\left(\sqrt{1+\frac{1}{n}}+1\right)}\right| \\
&= \left|\frac{1 - \left(1 - \frac{1}{n}\right)}{2\left(\sqrt{1+\frac{1}{n}}+1\right)^2}\right| = \left|\frac{-\frac{1}{n}}{2\left(\sqrt{1+\frac{1}{n}}+1\right)^2}\right| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon
\end{aligned}$$

wheneven $n \geq N$. $\square$

Exercise 3.3. *If* $s_1 = \sqrt{2}$ *and*

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} \ (n = 1, 2, 3, ...),$$

prove that $\{s_n\}$ *converges, and that* $s_n < 2$ *for* $n = 1, 2, 3, ...$

The convergence of $\{s_n\}$ implies there is $s \in \mathbb{R}$ such that $s_n \to s$ where $s = \sqrt{2 + \sqrt{s}}$ and $\sqrt{2} < s \leq 2$. WolframAlpha shows that

$$s = \frac{1}{3}\left(-1 + \sqrt[3]{\frac{1}{2}(79 - 3\sqrt{249})} + \sqrt[3]{\frac{1}{2}(79 + 3\sqrt{249})}\right).$$

Proof (Theorem 3.14).

(1) *Show that* $\{s_n\}$ *is increasing (by mathematical induction).*

   (a) *Show that* $s_2 > s_1$. In fact,

$$s_2 = \sqrt{2 + \sqrt{s_1}} = \sqrt{2 + \sqrt{\sqrt{2}}} < \sqrt{2} = s_1.$$

   (a) *Show that* $s_{n+1} > s_n$ *if* $s_n > s_{n-1}$.

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} > \sqrt{2 + \sqrt{s_{n-1}}} = s_n.$$

   By mathematical induction, $\{s_n\}$ is (strictly) increasing.

(2) *Show that* $\{s_n\}$ *is bounded (by mathematical induction).*

   (a) *Show that* $s_1 \leq 2$. $\sqrt{2} \leq 2$.

   (a) *Show that* $s_{n+1} \leq 2$ *if* $s_n \leq 2$.

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} \leq \sqrt{2 + \sqrt{2}} < 2.$$

   By mathematical induction, $\{s_n\}$ is bounded by 2.

Hence, $\{s_n\}$ converges since $\{s_n\}$ is increasing and bounded (Theorem 3.14). $\square$

Exercise 3.4. *Find the upper and lower limits of the sequences* $\{s_n\}$ *defined by*

$$s_1 = 0; s_{2m} = \frac{s_{2m-1}}{2}; s_{2m+1} = \frac{1}{2} + s_{2m}.$$

Write out the first few terms of $\{s_n\}$:

$$0, 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{3}{8}, \frac{7}{8}, \frac{7}{16}, \frac{15}{16}, ...$$

It suggests us

$$\begin{aligned} s_{2m+1} &= 1 - \frac{1}{2^m} \ \ (m = 0, 1, 2, ...), \\ s_{2m} &= \frac{1}{2} - \frac{1}{2^m} \ \ (m = 1, 2, 3, ...). \end{aligned}$$

Proof.

(1) *Show that*

$$\begin{aligned} s_{2m+1} &= 1 - \frac{1}{2^m} \ \ (m = 0, 1, 2, ...), \\ s_{2m} &= \frac{1}{2} - \frac{1}{2^m}. \ \ (m = 1, 2, 3, ...) \end{aligned}$$

Apply mathematical induction.

(2) The upper limit is 1.

(3) The lower limit is $\frac{1}{2}$.

□

Exercise 3.5. *For any two real sequences $\{a_n\}$, $\{b_n\}$, prove that*

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

provided the sum of the right is not of the form $\infty - \infty$.

Proof. Write $\alpha = \limsup_{n\to\infty} a_n$ and $\beta = \limsup_{n\to\infty} b_n$.

(1) $\alpha = \infty$ *and* $\beta = \infty$. Nothing to do.

(2) $\alpha = -\infty$ *and* $\beta = -\infty$. Since $\alpha = -\infty < \infty$, there exists M' such that $a_n < M'$ for all n. For any real M, $a_n > M - M'$ for at most a finite number of values of n (Theorem 3.17(a)). Hence $a_n + b_n > M$ for at most a finite number of values of n. Hence $\limsup_{n\to\infty}(a_n + b_n) = -\infty$, or

$$\limsup_{n\to\infty}(a_n + b_n) = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

in this case.

(3) *α and β are finite.* (Similar to the argument in Theorem 3.37.) Choose $\alpha' > \alpha$ and $\beta' > \beta$. There is an integer N such that

$$\alpha' \geq a_n \text{ and } \beta' \geq b_n$$

whenever $n \geq N$. Hence

$$a_n + b_n \leq \alpha' + \beta'$$

whenever $n \geq N$. Take lim sup to get Hence

$$\limsup_{n\to\infty}(a_n + b_n) \leq \alpha' + \beta'.$$

Since the inequality is true for every $\alpha' > \alpha$ and $\beta' > \beta$, we have

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

$\square$

Exercise 3.7. *Prove that the convergence of $\sum a_n$ implies the convergence of*

$$\sum \frac{\sqrt{a_n}}{n},$$

if $a_n \geq 0$.

Proof (Cauchy's inequatity).

(1) *Show that $\sum \frac{\sqrt{a_n}}{n}$ is bounded.* For any $k \in \mathbb{Z}^+$,

$$\begin{aligned}\left(\sum_{n=1}^{k} \frac{\sqrt{a_n}}{n}\right)^2 &\leq \left(\sum_{n=1}^{k} a_n\right)\left(\sum_{n=1}^{k} \frac{1}{n^2}\right) && \text{(Cauchy's inequatity)} \\ &\leq \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right). && \left(\sum a_n, \sum \frac{1}{n^2}\text{: convergent}\right)\end{aligned}$$

Thus, $\left(\sum_{n=1}^{k} \frac{\sqrt{a_n}}{n}\right)^2$ is bounded, or $\sum_{n=1}^{k} \frac{\sqrt{a_n}}{n}$ is bounded.

(2) *Show that $\sum_{n=1}^{k} \frac{\sqrt{a_n}}{n}$ is increasing.* It is clear due to $\frac{\sqrt{a_n}}{n} \geq 0$.

By Theorem 3.14, $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges. $\square$

Proof (AM-GM inequality). Show that $\sum \frac{\sqrt{a_n}}{n}$ is bounded.

$$\begin{aligned}
\frac{\sqrt{a_n}}{n} &\leq \frac{1}{2}\left(a_n + \frac{1}{n^2}\right) && \text{(AM-GM inequality)} \\
\sum_{n=1}^{k} \frac{\sqrt{a_n}}{n} &\leq \frac{1}{2}\left(\sum_{n=1}^{k} a_n + \sum_{n=1}^{k} \frac{1}{n^2}\right) \\
&\leq \frac{1}{2}\left(\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} \frac{1}{n^2}\right). && \left(\sum a_n, \sum \frac{1}{n^2}\text{: convergent}\right)
\end{aligned}$$

Thus, $\sum_{n=1}^{k} \frac{\sqrt{a_n}}{n}$ is bounded. The rest proof is the same as previous. □

Exercise 3.12. *Suppose $a_n > 0$ and $\sum a_n$ converges. Put*

$$r_n = \sum_{m=n}^{\infty} a_m.$$

(a) *Prove that*
$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$
if $m < n$, and deduce that $\sum \frac{a_n}{r_n}$ diverges.

(b) *Prove that*
$$\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$$
and deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

Note.

(1) Each r_n is positive and finite (since $a_n > 0$ and $\sum a_n$ converges).

(2) $\{r_n\}$ is monotonic decreasing (since $a_n > 0$).

(3) $\{r_n\}$ converges to 0 (since $\sum a_n$ converges).

Proof of (a).

(1)

$$\begin{aligned}
\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} &> \frac{a_m}{r_m} + \cdots + \frac{a_n}{r_m} && (r_m > r_k \text{ for } k = m+1, \ldots, n) \\
&= \frac{a_m + \cdots + a_n}{r_m} \\
&= \frac{r_m - r_{n+1}}{r_m} && (\text{Definition of } r_k) \\
&> \frac{r_m - r_n}{r_m} && (r_n > r_{n+1}) \\
&= 1 - \frac{r_n}{r_m}.
\end{aligned}$$

(2) (Reductio ad absurdum) If $\sum \frac{a_n}{r_n}$ were converged, then given $\varepsilon = \frac{1}{64} > 0$ there is an integer N such that

$$\left| \frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} \right| < \frac{1}{64} \text{ whenever } n \geq m \geq N$$

(Theorem 3.22). By (1), let $m = N$ to get

$$1 - \frac{r_n}{r_N} < \frac{1}{64} \text{ whenever } n \geq N,$$

or

$$r_n > \frac{63}{64} r_N,$$

contrary to the assumption that $\{r_n\}$ converges to 0 (since $\sum a_n$ converges).

$\square$

Proof of (b).

(1) Note that each r_n is positive and finite, and thus

$$\begin{aligned}
\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}}) &\Longleftrightarrow \frac{r_n - r_{n+1}}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}}) \\
&\Longleftrightarrow \frac{\sqrt{r_n} + \sqrt{r_{n+1}}}{\sqrt{r_n}} < 2 \\
&\Longleftrightarrow \sqrt{r_n} + \sqrt{r_{n+1}} < 2\sqrt{r_n} \\
&\Longleftrightarrow \sqrt{r_{n+1}} < \sqrt{r_n} \\
&\Longleftrightarrow r_{n+1} < r_n.
\end{aligned}$$

The last statement holds since $\{r_n\}$ is monotonic decreasing.

(2) (a) Each term $\frac{a_n}{\sqrt{r_n}}$ of $\sum \frac{a_n}{\sqrt{r_n}}$ is nonnegative.

(b) The partial sum

$$\sum_{k=1}^{n} \frac{a_k}{\sqrt{r_k}} < \sum_{k=1}^{n} 2(\sqrt{r_k} - \sqrt{r_{k+1}}) = 2(\sqrt{r_1} - \sqrt{r_{n+1}}) < 2\sqrt{r_1}$$

is bounded by $2\sqrt{r_1}$.

By (a)(b), $\sum \frac{a_n}{\sqrt{r_n}}$ converges (Theorem 3.24).

$\square$

Exercise 3.13. *Prove that the Cauchy product of two absolutely convergent series converges absolutely.*

Proof.

(1) Given two absolutely convergent series $\sum a_n$ and $\sum b_n$. The Cauchy product is $\sum c_n$ where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \ \ (n = 0, 1, 2, \ldots).$$

Let $\sum |a_n| = A < \infty$ and $\sum |b_n| = B < \infty$.

(2) Each term $|c_k|$ of $\sum_{k=0}^{n} |c_k|$ is nonnegative.

(3) Thus,

$$\begin{aligned}
\sum_{k=0}^{n} |c_k| &= \sum_{k=0}^{n} \left| \sum_{m=0}^{k} a_m b_{k-m} \right| \\
&\leq \sum_{k=0}^{n} \sum_{m=0}^{k} |a_m||b_{k-m}| \\
&= \sum_{k=0}^{n} |a_k| \sum_{m=0}^{n-k} |b_m| \\
&\leq \sum_{k=0}^{n} |a_k| B \\
&\leq AB \\
&< \infty.
\end{aligned}$$

(4) By (2)(3), $\sum_{k=0}^{n} |c_k|$ converges (Theorem 3.24), or $\sum_{k=0}^{n} c_k$ converges absolutely.

$\square$

Exercise 3.14 (Cesàro convergence). *If $\{s_n\}$ is a complex sequence, define its arithmetic means σ_n by*

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1} \quad (n = 0, 1, 2, \ldots).$$

(a) *If* $\lim s_n = s$, *prove that* $\lim \sigma_n = s$.

(b) *Construct a sequence $\{s_n\}$ which does not converge, although* $\lim \sigma_n = 0$.

(c) *Can it happen that $s_n > 0$ for all n and that* $\limsup s_n = \infty$, *although* $\lim \sigma_n = 0$*?*

(d) *Put $a_n = s_n - s_{n-1}$, for $n \geq 1$. Show that*

$$s_n - \sigma_n = \frac{1}{n+1} \sum_{k=1}^{n} k a_k.$$

Assume that $\lim(na_n) = 0$ *and that $\{\sigma_n\}$ converges. Prove that $\{s_n\}$ converges. [This gives a converse of (a), but under the additional assumption that $na_n \to 0$.]*

(e) *Since ε was arbitrary,* $\lim s_n = \sigma$.

Proof of (a). Given any $\varepsilon > 0$.

(1) For such $\varepsilon > 0$, there is an integer $N' \geq 1$ such that

$$|s_n - s| < \frac{\varepsilon}{64} \text{ whenever } n \geq N'.$$

(2) For such N', $\sum_{n=0}^{N'} |s_n - s|$ is finite. Let N'' be an integer such that

$$\sum_{n=0}^{N'} |s_n - s| < \frac{N''\varepsilon}{89}$$

(by taking $N'' = \left\lfloor \frac{89}{\varepsilon} \sum_{n=0}^{N'} |s_n - s| \right\rfloor + 1$).

(3) Note that

$$\begin{aligned}
|\sigma_n - s| &= \left| \left(\frac{1}{n+1} \sum_{k=0}^{n} s_k \right) - s \right| \\
&= \left| \frac{1}{n+1} \sum_{k=0}^{n} (s_k - s) \right| \\
&\leq \frac{1}{n+1} \sum_{k=0}^{n} |s_k - s|
\end{aligned}$$

holds for each $n = 0, 1, 2, \ldots$. In particular, for $n \geq N = \max\{N', N''\} \geq 1$, we have

$$\begin{aligned}
|\sigma_n - s| &\leq \frac{1}{n+1} \sum_{k=0}^{n} |s_k - s| \\
&\leq \left(\frac{1}{n+1} \sum_{k=0}^{N'} |s_k - s| \right) + \left(\frac{1}{n+1} \sum_{k=N'+1}^{n} |s_k - s| \right) \\
&< \frac{1}{n+1} \cdot \frac{N''\varepsilon}{89} + \frac{1}{n+1} \cdot \frac{(n-N')\varepsilon}{64} \\
&< \frac{\varepsilon}{89} + \frac{\varepsilon}{64} \\
&< \varepsilon.
\end{aligned}$$

Therefore, $\lim \sigma_n = s$.

□

Proof of (b). Define $\{s_n\}$ by $s_n = (-1)^{n+1}$. □

Proof of (c). □

Proof of (d). □

Proof of (e). □

Exercise 3.20. *Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X, and some subsequence $\{p_{n_i}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p.*

Proof. Given any $\varepsilon > 0$.

(1) Since $\{p_n\}$ is a Cauchy sequence, there exists a positive integer N_1 such that
$$d(p_n, p_m) < \frac{\varepsilon}{2} \text{ whenever } n, m \geq N_1.$$

(2) Since the subsequence $\{p_{n_i}\}$ converges to a point $p \in X$, there exists a positive integer N_2 such that
$$d(p_{n_i}, p) < \frac{\varepsilon}{2} \text{ whenever } n_i \geq N_2.$$

(3) Let $N = \max\{N_1, N_2\}$ be a positive integer. So
$$\begin{aligned}
d(p_n, p) &\leq d(p_n, p_{n_i}) + d(p_{n_i}, p) && \text{(Definition 2.15(c))} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \text{ whenever } n, n_i \geq N && \text{((1)(2))} \\
&= \varepsilon \text{ whenever } n \geq N.
\end{aligned}$$

Hence the full sequence $\{p_n\}$ converges to p.

$\square$

Exercise 3.21. *Prove the following analogue of Theorem 3.10(b): If $\{E_n\}$ is a sequence of closed and bounded sets in a complete metric space X, if $E_n \supseteq E_{n+1}$, and if*

$$\lim_{n \to \infty} \operatorname{diam}(E_n) = 0,$$

then $\bigcap_{n=1}^{\infty} E_n$ consists of exactly one point.

Assume $E_n \neq \varnothing$. It is unnecessary to assume that E_n is bounded since we have the condition that $\lim_{n \to \infty} \operatorname{diam}(E_n) = 0$.

Note. Every compact metric space is complete, but complete spaces need not be compact. In fact, a metric space is compact if and only if it is complete and totally bounded.

Proof.

(1) Pick $p_n \in E_n$ for $n = 1, 2, \ldots$.

(2) *Show that $\{p_n\}$ is a Cauchy sequence.* Given any $\varepsilon > 0$. There is a positive integer N such that $\operatorname{diam}(E_n) < \varepsilon$ whenever $n \geq N$. Especially,

$$\operatorname{diam}(E_N) < \varepsilon.$$

As $m, n \geq N$, $p_m \in E_m \subseteq E_N$ and $p_n \in E_n \subseteq E_N$. By the definition of the diameter of E_N,

$$d(p_m, p_n) \leq \operatorname{diam}(E_N) < \varepsilon \text{ whenever } m, n \geq N.$$

(3) Since X is complete, $\{p_n\}$ converges to a point $p \in X$.

(4) *Show that $p \in \bigcap_{n=1}^{\infty} E_n$.* (Reductio ad absurdum) If there were some n such that $p \notin E_n$. Consider the subsequence

$$p_n, p_{n+1}, p_{n+2}, \ldots.$$

Note that all $p_n, p_{n+1}, \ldots$ are in E_n. By (3), it converges to p. Thus p is a limit point of E_n. Since E_n is closed, $p \in E_n$, which is absurd.

(5) *Show that $\bigcap_{n=1}^{\infty} E_n = \{p\}$.* (Reductio ad absurdum) If there were $q \in \bigcap_{n=1}^{\infty} E_n$ with $q \neq p$, then $d(p, q) > 0$ (Definition 2.15(a)). It implies that

$$\operatorname{diam}(E_n) \geq d(p, q) > 0 \text{ for all } n,$$

contrary to $\lim_{n \to \infty} \operatorname{diam}(E_n) = 0$.

$\square$

Exercise 3.22 (Baire category theorem). *Suppose X is a complete metric space, and $\{G_n\}$ is a sequence of dense open subsets of X. Prove Baire's theorem, namely, that $\bigcap_1^\infty G_n$ is not empty. (In fact, it is dense in X.) (Hint: Find a shrinking sequence of neighborhoods E_n such that $\overline{E_n} \subseteq G_n$, and apply Exercise 3.21.)*

Proof. Given any open set G_0 in X, will show that

$$\bigcap_{n=0}^{\infty} G_n \neq \varnothing.$$

(1) Since G_1 is dense, $G_0 \cap G_1$ is nonempty. Take any one point p_1 in the open set $G_0 \cap G_1$, then there exists a closed neighborhood

$$V_1 = \{q \in X : d(q, p_1) < r_1\}$$

of p_1 with $r_1 < 1$ such that

$$V_1 \subseteq G_0 \cap G_1.$$

Take $U_1 \subseteq E_1 \subseteq V_1$ such that

$$E_1 = \left\{q \in X : d(q, p_1) \leq \frac{r_1}{64}\right\} \subseteq V_1,$$
$$U_1 = \left\{q \in X : d(q, p_1) < \frac{r_1}{89}\right\} \subseteq E_1.$$

(2) Suppose V_n, E_n, U_n have been constructed, take any one point p_{n+1} in the open set $U_n \cap G_{n+1}$, there exists an open neighborhood

$$V_{n+1} = \{q \in X : d(q, p_{n+1}) < r_{n+1}\}$$

of p_{n+1} with r_{n+1} with $r_{n+1} < \frac{1}{n+1}$ such that

$$V_{n+1} \subseteq U_n \cap G_{n+1}.$$

Take $U_1 \subseteq E_1 \subseteq V_1$ such that

$$E_{n+1} = \left\{q \in X : d(q, p_{n+1}) \leq \frac{r_{n+1}}{64}\right\} \subseteq V_{n+1},$$
$$U_{n+1} = \left\{q \in X : d(q, p_{n+1}) < \frac{r_{n+1}}{89}\right\} \subseteq E_{n+1}.$$

(3) Note that

(a) E_n is closed and nonempty (since $p_n \in E_n$).

(b) $\lim_{n\to\infty} \mathrm{diam}(E_n) = 0$ (since $\mathrm{diam}(E_n) \leq 2 \cdot \frac{r_n}{64} < r_n < \frac{1}{n}$.)
(c) $E_1 \supseteq E_2 \supseteq \cdots$ (since $E_{n+1} \subseteq V_{n+1} \subseteq U_n \cap G_{n+1} \subseteq U_n \subseteq E_n$).

Since X is complete, by Exercise 3.21,

$$\bigcap_{n=1}^{\infty} E_n = \{p\}$$

for some $p \in X$.

(4) Hence

$$\begin{aligned} p \in \bigcap_{n=1}^{\infty} E_n &\Longleftrightarrow p \in E_n \text{ for all } n = 1, 2, 3, \dots \\ &\Longrightarrow p \in E_1 \subseteq G_0 \cap G_1 \text{ and } p \in E_{n+1} \subseteq U_n \cap G_{n+1} \subseteq G_{n+1} \\ &\Longrightarrow p \in G_0 \cap G_1 \cap \cdots = \bigcap_{n=0}^{\infty} G_n \\ &\Longrightarrow \bigcap_{n=0}^{\infty} G_n \neq \varnothing. \end{aligned}$$

□

Exercise 3.23. *Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $\{d(p_n, q_n)\}$ converges. (Hint: For any m, n,*

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n);$$

it follows that

$$|d(p_n, q_n) - d(p_m, q_m)|$$

is small if m and n are large.)

Proof. Given any $\varepsilon > 0$.

(1) Since $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences, there exists N such that

$$d(p_n, p_m) < \frac{\varepsilon}{2} \text{ and } d(q_m, q_n) < \frac{\varepsilon}{2}$$

whenever $m, n \geq N$.

(2) Note that

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n).$$

It follows that

$$|d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_m, q_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $\{d(p_n, q_n)\}$ is a Cauchy sequence in $\mathbb{R}^1$ (not in X).

(3) Since $\mathbb{R}^1$ is a complete metric space, $\{d(p_n, q_n)\}$ converges.

$\square$

Exercise 3.24. *Let X be a metric space.*

(a) *Call two Cauchy sequences $\{p_n\}$, $\{q_n\}$ in X equivalent if*

$$\lim_{n\to\infty} d(p_n, q_n) = 0.$$

Prove that this is an equivalence relation.

(b) *Let X^* be the set of all equivalence classes so obtained. If $P \in X^*$, $Q \in X^*$, $\{p_n\} \in P$, $\{q_n\} \in Q$, define*

$$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n);$$

by Exercise 3.23, this limit exists. Show that the number $\Delta(P, Q)$ is unchanged if $\{p_n\}$ and $\{q_n\}$ are replaced by equivalent sequences, and hence that Δ is a distance function in X^*.

(c) *Prove that the resulting metric space X^* is complete.*

(d) *For each $p \in X$, there is a Cauchy sequence all of whose terms are p; let P_p be the element of X^* which contains this sequence. Prove that*

$$\Delta(P_p, P_q) = d(p, q)$$

for all $p, q \in X$. In other words, the mapping φ defined by $\varphi(p) = P_p$ is an isometry (i.e., a distance-preserving mapping) of X into X^*.

(e) *Prove that $\varphi(X)$ is dense in X^*, and that $\varphi(X) = X^*$ if X is complete. By (d), we may identify X and $\varphi(X)$ and thus regard X as embedded in the complete metric space X^*. We call X^* the **completion** of X.*

Proof of (a). Given Cauchy sequences $\{p_n\}$, $\{q_n\}$, $\{r_n\}$ in X.

(1) *(Reflexivity)*

$$\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} 0 = 0$$

by the reflexivity of the metric function d.

(2) *(Symmetry)*

$$\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(q_n, p_n) = 0$$

by the symmetry of the metric function d.

(3) *(Transitivity)* Suppose that $\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(q_n, r_n) = 0$. By the triangle inequality of the metric function d, we have

$$0 \leq d(p_n, r_n) \leq d(p_n, q_n) + d(q_n, r_n).$$

Take limit to get

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} d(p_n, r_n) \\ &\leq \lim_{n\to\infty} (d(p_n, q_n) + d(q_n, r_n)) \\ &= \lim_{n\to\infty} d(p_n, q_n) + \lim_{n\to\infty} d(q_n, r_n) \\ &= 0 \end{aligned}$$

or $\lim_{n\to\infty} d(p_n, r_n) = 0$.

□

Proof of (b).

(1) *Show that Δ is well-defined.* Given any $\{p_n\}, \{p'_n\} \in P$ and $\{q_n\}, \{q'_n\} \in Q$.

(a) $\lim_{n\to\infty} d(p_n, p'_n) = 0$ since $\{p_n\}$ and $\{p'_n\}$ are in the same equivalence class.

(b) $\lim_{n\to\infty} d(q_n, q'_n) = 0$ (similar to (a)).

(c) *Show that* $\lim_{n\to\infty} d(p_n, q_n) \leq \lim_{n\to\infty} d(p'_n, q'_n)$. Since $d(p_n, q_n) \leq d(p_n, p'_n) + d(p'_n, q'_n) + d(q'_n, q_n)$, take limit to get

$$\begin{aligned} \lim_{n\to\infty} d(p_n, q_n) &\leq \lim_{n\to\infty} (d(p_n, p'_n) + d(p'_n, q'_n) + d(q'_n, q_n)) \\ &= \lim_{n\to\infty} d(p_n, p'_n) + \lim_{n\to\infty} d(p'_n, q'_n) + \lim_{n\to\infty} d(q'_n, q_n) \\ &= 0 + \lim_{n\to\infty} d(p'_n, q'_n) + 0 \\ &= \lim_{n\to\infty} d(p'_n, q'_n) \end{aligned}$$

since (a)(b).

(d) *Show that* $\lim_{n\to\infty} d(p_n, q_n) \geq \lim_{n\to\infty} d(p'_n, q'_n)$. Similar to (c).

By (c)(d), $\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(p'_n, q'_n)$, or $\Delta(P, Q)$ is well-defined.

(2) *Show that Δ is a metric.*

(a) *Show that* $\Delta(P, Q) > 0$ *if* $P \neq Q$*;* $\Delta(P, P) = 0$. It is the definition of Δ.

(b) *Show that* $\Delta(P, Q) = \Delta(Q, P)$. Similar to the argument in (a)(2).

(c) *Show that* $\Delta(P, Q) \leq \Delta(P, R) + \Delta(R, Q)$. Similar to the argument in (a)(3).

$\square$

Proof of (c). Show that $\{P_k\}_{k=1}^\infty$ *converges to* P *in* (X^*, Δ) *for any given Cauchy sequence* $\{P_k\}$.

(1) Take a Cauchy sequence $\{p_n^{(k)}\}_{n=1}^\infty$ to represent P_k for each k. *We will construct a Cauchy sequence* $\{p_k\}$ *in* (X, d) *such that* $\{P_k\}$ *converges to* P *which is the equivalent class of* $\{p_k\}$.

(2) For each k, there exists N_k such that

$$d\left(p_m^{(k)}, p_n^{(k)}\right) < \frac{1}{k} \text{ whenever } m, n \geq N_k.$$

Especially,

$$d\left(p_m^{(k)}, p_{N_k}^{(k)}\right) < \frac{1}{k} \text{ whenever } m \geq N_k.$$

Let $p_k = p_{N_k}^{(k)}$ and collect all p_k as $\{p_k\}_{k=1}^\infty$.

(3) *Show that* $\{p_k\}$ *is a Cauchy sequence in* (X, d). Note that for any k, we have

$$\begin{aligned} d(p_m, p_n) &= d\left(p_{N_m}^{(m)}, p_{N_n}^{(n)}\right) \\ &\leq d\left(p_{N_m}^{(m)}, p_k^{(m)}\right) + d\left(p_k^{(m)}, p_k^{(n)}\right) + d\left(p_k^{(n)}, p_{N_n}^{(n)}\right). \end{aligned}$$

Let $k \to \infty$, we have

$$\begin{aligned} d(p_m, p_n) &\leq \limsup_{k \to \infty} \left[d\left(p_{N_m}^{(m)}, p_k^{(m)}\right) + d\left(p_k^{(m)}, p_k^{(n)}\right) + d\left(p_k^{(n)}, p_{N_n}^{(n)}\right) \right] \\ &\leq \frac{1}{m} + \Delta(P_m, P_n) + \frac{1}{n} \end{aligned}$$

for any m, n (by (2)). Let $m, n \to \infty$, we establish the result (since $\{P_k\}$ is Cauchy).

(4) *Show that* $\{P_k\}$ *converges to* $P \ni \{p_k\}$. Given any $\varepsilon > 0$. Since $\{p_k\}$ is Cauchy (3), there is $N > \frac{2}{\varepsilon}$ such that

$$d(p_m, p_n) < \frac{\varepsilon}{2} \text{ whenever } m, n \geq N.$$

Note that

$$\begin{aligned} d\left(p_n^{(k)}, p_n\right) &= d\left(p_n^{(k)}, p_{N_n}^{(n)}\right) \\ &\leq d\left(p_n^{(k)}, p_{N_k}^{(k)}\right) + d\left(p_{N_k}^{(k)}, p_{N_n}^{(n)}\right). \end{aligned}$$

For any $k \geq N$, let $n \to \infty$ to get

$$\begin{aligned}
\Delta(P_k, P) &= \lim_{n\to\infty} d\left(p_n^{(k)}, p_n\right) \\
&\leq \limsup_{n\to\infty} d\left(p_n^{(k)}, p_{N_k}^{(k)}\right) + \limsup_{n\to\infty} d\left(p_{N_k}^{(k)}, p_{N_n}^{(n)}\right) \\
&< \frac{1}{k} + \frac{\varepsilon}{2} \\
&\leq \frac{1}{N} + \frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&< \varepsilon.
\end{aligned}$$

Hence, (X^*, Δ) is complete. □

Proof of (d).

(1) Define $\{p_n\}$ by $p_n = p$ $(n = 1, 2, \ldots)$ for any $p \in X$.

(2) *Show that $\{p_n\}$ is a Cauchy sequence.* $d(p_m, p_n) = d(p, p) = 0$.

(3) Take $\{p\} \in P_p$ and $\{q\} \in P_q$. Then

$$\Delta(P_p, P_q) = \lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(p, q) = d(p, q).$$

□

Proof of (e).

(1) *Show that $\varphi(X)$ is dense in X^*.* Given any $P \in X^*$, any $\{p_n\} \in P$ and any $\varepsilon > 0$. Since $\{p_n\}$ is Cauchy, there is N such that

$$d(p_m, p_n) < \frac{\varepsilon}{64} \text{ whenever } m, n \geq N.$$

Note that $p_N \in X$. Pick $\{p_N\} \in P_{p_N} = \varphi(p_N) \in \varphi(X)$. So

$$\Delta(P, P_{p_N}) = \lim_{n\to\infty} d(p_n, p_N) \leq \frac{\varepsilon}{64} < \varepsilon.$$

Hence $\varphi(X)$ is dense in X^*.

(2) *Show that $\varphi(X) = X^*$ if X is complete.* Given any $P \in X^* \ni \{p_n\}$. Since X is complete, a Cauchy sequence $\{p_n\}$ converges to $p \in X$. Pick $\{p\} \in P_p = \varphi(p) \in \varphi(X)$. So

$$\Delta(P, P_p) = \lim_{n\to\infty} d(p_n, p) = 0,$$

or $P = P_p$, or $\varphi(X) = X^*$.

□

Exercise 3.25. *Let X be the metric space whose points are rational numbers, with the metric $d(x,y) = |x-y|$. What is the completion of this space? (Compare Exercise 3.24.)*

Proof. By Exercise 3.24, we can identify one completion (X^*, Δ) with $(\mathbb{R}, |\cdot|)$ (Theorem 3.11(c) and Theorem 1.20(b)). □

Supplement (Uniqueness of completion). *Show that a completion of a metric space is unique up to isometry.*

Outline. Suppose there are two completions $\{\varphi_i, (X_i^*, d_i^*)\}$ $(i = 1, 2)$ of (X, d). Let

$$\psi = \varphi_2 \circ \varphi_1^{-1} : \varphi_1(X) \to \varphi_2(X)$$

be an isometry from $\varphi_1(X)$ into $\varphi_2(X)$ The sets $\varphi_i(X)$ $(i = 1, 2)$ are dense in X_i^*. So we can extend ψ (continuously) to a map $\psi : X_1^* \to X_2^*$.

Proof.

(1) Given any $P \in X_1^*$, there is a Cauchy sequence $\{P_{p_n}\} = \{\varphi_1(p_n)\}$ in $\varphi_1(X)$ converging to P. Define $\psi(P)$ by

$$\psi(P) = \lim_{n\to\infty} \psi(P_{p_n}).$$

(2) *Show that ψ is well-defined.* Note that

$$\begin{aligned}
\Delta_2(\psi(P_{p_m}), \psi(P_{p_n})) &= \Delta_2(\psi(\varphi_1(p_m)), \psi(\varphi_1(p_n))) \\
&= \Delta_2(\varphi_2(p_m), \varphi_2(p_n)) \\
&= d(p_n, p_m) && (\varphi_2 \text{ is isometric}) \\
&= \Delta_1(\varphi_1(p_m), \varphi_1(p_n)) && (\varphi_1 \text{ is isometric}) \\
&= \Delta_1(P_{p_m}, P_{p_n}).
\end{aligned}$$

So $\{\psi(P_{p_n})\}$ is a Cauchy sequence in $\varphi_2(X)$ if (and only if) $\{P_{p_n}\}$ is a Cauchy sequence in $\varphi_1(X)$. Since X_2^* is complete, $\{\psi(P_{p_n})\}$ converges to $\psi(P)$. The limit $\psi(P)$ is uniquely determined since Δ_2 is a metric function.

(3) Since ψ is an isometry from $\varphi_1(X)$ into $\varphi_2(X)$,

$$\psi^{-1} = \varphi_1 \circ \varphi_2^{-1} : \varphi_2(X) \to \varphi_1(X)$$

is an isometry from $\varphi_2(X)$ into $\varphi_1(X)$. Besides, $\psi^{-1} \circ \psi = 1_{\varphi_1(X)}$ and $\psi \circ \psi^{-1} = 1_{\varphi_2(X)}$.

(4) *Show that ψ is surjective.* Given any $Q \in X_2^*$, there is a Cauchy sequence $\{P_{q_n}\} = \{\varphi_2(q_n)\}$ in $\varphi_2(X)$ converging to Q. Define

$$P_{p_n} = \psi^{-1}(P_{q_n}) \in \varphi_1(X).$$

$\psi(P_{p_n}) = 1_{\varphi_2(X)}(P_{q_n}) = P_{q_n}$. Besides, similar to argument in (2), $\{P_{p_n}\}$ is a Cauchy sequence in $\varphi_1(X)$. Since X_1^* is complete, $\{P_{p_n}\}$ converges to $P \in X_1^*$. It is easy to verify that $\psi(P) = Q$.

(5) *Show that ψ is injective.* Given any $P \in X_1^*$ and $Q \in X_1^*$, there are Cauchy sequences

$$\{P_{p_n}\} = \{\varphi_1(p_n)\} \to P \text{ and } \{P_{q_n}\} = \{\varphi_1(q_n)\} \to Q.$$

So

$$\begin{aligned}
\psi(P) = \psi(Q) &\Longrightarrow \lim_{n\to\infty} \psi(P_{p_n}) = \lim_{n\to\infty} \psi(P_{q_n}) \\
&\Longrightarrow 0 = \lim_{n\to\infty} \Delta_2(\psi(P_{p_n}), \psi(P_{q_n})) \\
&\Longrightarrow 0 = \lim_{n\to\infty} \Delta_2(\psi(\varphi_1(p_n)), \psi(\varphi_1(q_n))) \\
&\Longrightarrow 0 = \lim_{n\to\infty} \Delta_2(\varphi_2(p_n), \varphi_2(q_n)) \\
&\Longrightarrow 0 = \lim_{n\to\infty} d(p_n, q_n). && (\varphi_2 \text{ is isometric})
\end{aligned}$$

Thus $\{p_n\} \in P$ and $\{q_n\} \in Q$ in the same equivalence class. Thus $P = Q$.

□