

Solutions to the book: *Fulton, Algebraic Curves*

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Chapter 1: Affine Algebraic Sets

1.1. Algebraic Preliminaries

Problem 1.1.*

Let R be a domain.

- (a) If f, g are forms of degree r, s respectively in $R[x_1, \dots, x_n]$, show that fg is a form of degree $r + s$.
- (b) Show that any factor of a form in $R[x_1, \dots, x_n]$ is also a form.

Proof of (a).

- (1) Write

$$f = \sum_{(i)} a_{(i)} x^{(i)},$$
$$g = \sum_{(j)} b_{(j)} x^{(j)},$$

where $\sum_{(i)}$ is the summation over $(i) = (i_1, \dots, i_n)$ with $i_1 + \dots + i_n = r$ and $\sum_{(j)}$ is the summation over $(j) = (j_1, \dots, j_n)$ with $j_1 + \dots + j_n = s$.

- (2) Hence,

$$fg = \sum_{(i)} \sum_{(j)} a_{(i)} b_{(j)} x^{(i)} x^{(j)}$$
$$= \sum_{(i),(j)} a_{(i)} b_{(j)} x^{(k)}$$

where $(k) = (i_1 + j_1, \dots, i_n + j_n)$ with $(i_1 + j_1) + \dots + (i_n + j_n) = r + s$. Each $x^{(k)}$ is the form of degree $r + s$ and $a_{(i)} b_{(j)} \in R$. Hence fg is a form of degree $r + s$.

□

Proof of (b).

- (1) Given any form $f \in R[x_1, \dots, x_n]$, and write $f = gh$. It suffices to show that g is a form as well. (So does h .)
- (2) Write

$$g = g_0 + \dots + g_r, \quad h = h_0 + \dots + h_s$$

where $g_r \neq 0$ and $h_s \neq 0$. So

$$f = gh = g_0h_0 + \cdots + g_rh_s.$$

Since R is a domain, $R[x_1, \dots, x_n]$ is a domain and thus $g_rh_s \neq 0$. The maximality of r and s implies that $\deg f = r + s$. Therefore, by the maximality of $r + s$, $f = g_rh_s$, or $g = g_r$, or g is a form.

□

Problem 1.2.*

Let R be a UFD, K the quotient field of R . Show that every element z of K may be written $z = a/b$, where $a, b \in R$ have no common factors; this representative is unique up to units of R .

Proof.

- (1) Show that every element z of K may be written $z = a/b$, where $a, b \in R$ have no common factors. Given any $z = a/b \in K$ where $a, b \in R$. Write

$$\begin{aligned} a &= p_1 \cdots p_n, \\ b &= q_1 \cdots q_m \end{aligned}$$

where all $p_1, \dots, p_n, q_1, \dots, q_m$ are irreducible in R . (It is possible since R is a UFD.) For each i , suppose $p_i \mid q_j$ for some i, j . Write $q_j = p_i u$ for some $u \in R$. By the irreducibility of p_i and q_j , u is a unit. So

$$z = \frac{a}{b} = \frac{p_1 \cdots \widehat{p_i} \cdots p_n}{q_1 \cdots \widehat{q_j} \cdots q_m} = \frac{p_1 \cdots \widehat{p_i} \cdots p_n}{u q_1 \cdots \widehat{q_j} \cdots q_m}.$$

Continue this method we can write $z = \frac{a'}{b'}$ where a' and b' have no common factors.

- (2) Write $z = a/b = a'/b'$ where

- (a) $a, b, a', b' \in R$,
- (b) a and b have no common factors,
- (c) a' and b' have no common factors.

Write

$$\begin{aligned} a &= p_1 \cdots p_n, \\ b &= q_1 \cdots q_m, \\ a' &= p'_1 \cdots p'_{n'}, \\ b' &= q'_1 \cdots q'_{m'} \end{aligned}$$

where all $p_i, q_j, p'_{i'}, q'_{j'}$ are irreducible in R . As $z = a/b = a'/b'$, $ab' = a'b$ or

$$p_1 \cdots p_n q'_1 \cdots q'_{m'} = p'_1 \cdots p'_{n'} q_1 \cdots q_m.$$

- (3) For $i = 1$, $p_1 = u_1 p'_{i'}$ for some unit $u_1 \in R$ since a and b have no common factors and all $p_1, q_j, p'_{i'}$ are irreducible. Hence

$$u_1 \widehat{p_1} p_2 \cdots p_n q'_1 \cdots q'_{m'} = p'_1 \cdots \widehat{p'_{i'}} \cdots p'_{n'} q_1 \cdots q_m.$$

Continue this method, we have $n \leq n'$ and all p_1, \dots, p_n are canceled.

- (4) Conversely, we can apply the argument in (3) to $i' = 1, \dots, n'$ to conclude that $n' \leq n$. Therefore, $n = n'$ and

$$\underbrace{u_1 \cdots u_n}_{\text{a unit in } R} q'_1 \cdots q'_{m'} = q_1 \cdots q_m.$$

Hence, $b = ub'$ where $u = u_1 \cdots u_n$ is a unit in R . Similarly, $a = va'$ where v is a unit in R . So the representative of $z \in K$ is unique up to units of R .

□

Problem 1.3.*

Let R be a PID. Let \mathfrak{p} be a nonzero, proper, prime ideal in R .

- (a) Show that \mathfrak{p} is generated by an irreducible element.
- (b) Show that \mathfrak{p} is maximal.

Proof of (a).

- (1) Let $\mathfrak{p} = (a)$ be a nonzero, proper, prime ideal in R . It suffices to show that a is irreducible.
- (2) Suppose $a = bc$. By the primality of \mathfrak{p} , $b \in \mathfrak{p}$ or $c \in \mathfrak{p}$. Suppose $b \in \mathfrak{p} = (a)$. (The case $c \in \mathfrak{p}$ is similar.) Then there is a $d \in R$ such that $b = ad$. Hence, $a = bc = adc$ or $(1 - dc)a = 0$.
- (3) Since R is a domain, $1 = dc$ or $a = 0$. $a = 0$ implies that $\mathfrak{p} = (0)$ is a zero ideal, contrary to the assumption. Therefore, $1 = dc$, or c is a unit, or a is irreducible.

□

Proof of (b).

- (1) Given any ideal $I = (b)$ of R containing $\mathfrak{p} = (a)$. As the generator a of \mathfrak{p} is in $\mathfrak{p} \subseteq I$, there is some $c \in R$ such that $a = bc$. By the irreducibility of a (in (a)), b is a unit or c is a unit.
- (2) b is a unit implies that $I = R$. c is a unit implies that $I = \mathfrak{p}$. In any case, we conclude that \mathfrak{p} is maximal.

□

Problem 1.4.*

Let k be an infinite field, $f \in k[x_1, \dots, x_n]$. Suppose $f(a_1, \dots, a_n) = 0$ for all $a_1, \dots, a_n \in k$. Show that $f = 0$. (Hint: Write

$$f = \sum f_i x_n^i, \quad f_i \in k[x_1, \dots, x_{n-1}].$$

Use induction on n , and the fact that $f(a_1, \dots, a_{n-1}, x_n)$ has only a finite number of roots if any $f_i(a_1, \dots, a_{n-1}) \neq 0$.)

Proof.

- (1) Induction on n . The case $n = 1$. (Reductio ad absurdum) If there were a nonzero $f \in k[x_1]$ such that $f(a) = 0$ for all $a \in k$. Note that f has at most $\deg f < \infty$ roots, contrary to the infinity of k .
- (2) Assume that the conclusion holds for $n - 1$, then for any $f \in k[x_1, \dots, x_n]$ we can write

$$f = \sum f_i x_n^i, \quad f_i \in k[x_1, \dots, x_{n-1}]$$

as $f \in (k[x_1, \dots, x_{n-1}])[x_n]$. Suppose $f(a_1, \dots, a_n) = 0$ for all $a_1, \dots, a_n \in k$. For fixed a_1, \dots, a_{n-1} , the polynomial $f(a_1, \dots, a_{n-1}, x_n) \in k[x_n]$ has all distinct roots in an infinite field k . By (1), $f(a_1, \dots, a_{n-1}, x_n) = 0 \in k[x_n]$, or each $f_i(a_1, \dots, a_{n-1}) = 0$. As all a_1, \dots, a_{n-1} run over k , we can apply the induction hypothesis each $f_i(x_1, \dots, x_{n-1}) = 0 \in k[x_1, \dots, x_{n-1}]$. Hence, $f = 0 \in k[x_1, \dots, x_n]$.

□

Note. If k is a finite field of order $q = p^k$, then the polynomial $f(x) = x^q - x$ has q distinct roots in k .

Problem 1.5.*

Let k be any field. Show that there are an infinitely number of irreducible monic polynomials in $k[x]$. (Hint: Suppose f_1, \dots, f_n were all of them, and factor $f_1 \cdots f_n + 1$ into irreducible factors.)

Proof (Due to Euclid).

- (1) If f_1, \dots, f_n were all irreducible monic polynomials, then we consider

$$g = f_1 \cdots f_n + 1 \in k[x].$$

So there is an irreducible monic polynomial $f = f_i$ dividing g for some i since

$$\deg g = \deg f_1 + \cdots + \deg f_n \geq 1$$

and $k[x]$ is a UFD.

- (2) However, f would divide the difference

$$g - f_1 \cdots f_{i-1} f_{i+1} \cdots f_n = 1,$$

contrary to $\deg f_i \geq 1$.

□

Problem 1.6.*

Show that any algebraically closed field is infinite. (Hint: The irreducible monic polynomials are $x - a$, $a \in k$.)

Proof (Due to Euclid).

- (1) Let k be an algebraically closed field. If a_1, \dots, a_n were all elements in k , then we consider a monic polynomials

$$f(x) = (x - a_1) \cdots (x - a_n) + 1 \in k[x].$$

- (2) Since k is algebraically closed, there is an element $a \in k$ such that $f(a) = 0$. By assumption, $a = a_i$ for some $1 \leq i \leq n$, and thus $f(a) = f(a_i) = 1$, contrary to the fact that a field is a commutative ring where $0 \neq 1$ and all nonzero elements are invertible.

□

Problem 1.7.*

Let k be a field, $f \in k[x_1, \dots, x_n]$, $a_1, \dots, a_n \in k$.

(a) Show that

$$f = \sum \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}, \quad \lambda_{(i)} \in k.$$

(b) If $f(a_1, \dots, a_n) = 0$, show that $f = \sum_{i=1}^n (x_i - a_i)g_i$ for some (not unique) g_i in $k[x_1, \dots, x_n]$.

Proof of (a).

(1) Regard $k[x_1, \dots, x_n]$ as $(k[x_1, \dots, x_{n-1}])[x_n]$. Since $(k[x_1, \dots, x_{n-1}])[x_n]$ is a Euclidean domain with a function

$$f \in (k[x_1, \dots, x_{n-1}])[x_n] \mapsto \deg_{x_n} f \in \mathbb{Z}_{\geq 0}$$

satisfying the division-with-remainder property.

(2) Apply the division algorithm for f and nonzero $x_n - a_n$ to produce a quotient q and remainder r with $f = (x_n - a_n)q + r$ and either $r = 0$ or $\deg_{x_n}(r) < \deg_{x_n}(x_n - a_n) = 1$. That is, $r \in k[x_1, \dots, x_{n-1}]$ is a constant in $(k[x_1, \dots, x_{n-1}])[x_n]$. Continue this process to get that f is of the form

$$f = \sum_{i_n} f_{i_n} (x_n - a_n)^{i_n}$$

where $f_{i_n} \in k[x_1, \dots, x_{n-1}]$.

(3) Use the same argument in (2) for each $f_{i_n} \in k[x_1, \dots, x_{n-1}]$, we have

$$\begin{aligned} f_{i_n} &= \sum_{\substack{i_{n-1} \\ \in k[x_1, \dots, x_{n-2}]}} \underbrace{f_{i_n, i_{n-1}}}_{\in k[x_1, \dots, x_{n-2}]} (x_{n-1} - a_{n-1})^{i_{n-1}} \\ f_{i_n, i_{n-1}} &= \sum_{\substack{i_{n-2} \\ \in k[x_1, \dots, x_{n-3}]}} \underbrace{f_{i_n, i_{n-1}, i_{n-2}}}_{\in k[x_1, \dots, x_{n-3}]} (x_{n-2} - a_{n-2})^{i_{n-2}}, \\ &\dots \\ f_{i_n, \dots, i_2} &= \sum_{\substack{i_1 \\ \in k}} \underbrace{f_{i_n, \dots, i_1}}_{\in k} (x_1 - a_1)^{i_1}. \end{aligned}$$

Note that $f_{i_n, \dots, i_1} \in k$, we can write

$$f = \sum \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}, \quad \lambda_{(i)} \in k.$$

by replacing all f_{i_n, \dots, i_k} by $f_{i_n, \dots, i_{k-1}}$ for $k = n, n-1, \dots, 2$.

(4) Or use the induction on n .

□

Proof of (b).

(1) Write

$$f = \sum \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}, \quad \lambda_{(i)} \in k$$

by (a).

(2) As $f(a_1, \dots, a_n) = 0$, $\lambda_{(i)} = 0$ if all i_1, \dots, i_n are zero, that is, there is no nonzero constant term in the representation of f . Hence, for each term

$$f_{(i)} := \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}$$

with $\lambda_{(i)} \neq 0$, there exists one $i_k > 0$ for some $1 \leq k \leq n$. So we can write

$$f_{(i)} = (x_k - a_k) \underbrace{(\lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_k - a_k)^{i_k-1} \cdots (x_n - a_n)^{i_n})}_{:= g_{(i)} \in k[x_1, \dots, x_n]}.$$

Note that the expression of $f_{(i)}$ is not unique since there may exist more than one $i_k > 0$ as $1 \leq k \leq n$.

(3) Now we iterate each nonzero term in f , apply the factorization in (2), and then group by each $x_k - a_k$. Therefore, we can write

$$f = \sum_{i=1}^n (x_i - a_i) g_i$$

for some $g_i \in k[x_1, \dots, x_n]$.

(4) The expression of f is not unique. For example, take $f(x, y) = x^2 + 2xy + y^2 \in k[x, y]$. As $f(0, 0) = 0$, we can write

$$\begin{aligned} f(x, y) &= x \cdot \underbrace{(x + 2y)}_{g_1} + y \cdot \underbrace{y}_{g_2}, \text{ or} \\ &= x \cdot \underbrace{(x + y)}_{g_1} + y \cdot \underbrace{(x + y)}_{g_2}, \text{ or} \\ &= x \cdot \underbrace{x}_{g_1} + y \cdot \underbrace{(2x + y)}_{g_2}. \end{aligned}$$

□

1.2. Affine Space and Algebraic Sets

Problem 1.8.*

Show that the algebraic subsets of $\mathbf{A}^1(k)$ are just the finite subsets, together with $\mathbf{A}^1(k)$ itself.

Proof.

(1) Show that $k[x]$ is a PID if k is a field.

- (a) Let I be an ideal of $k[x]$.
- (b) If $I = \{0\}$ then $I = (0)$ and I is principal.
- (c) If $I \neq \{0\}$, then take f to be a polynomial of minimal degree in I . It suffices to show that $I = (f)$. Clearly, $(f) \subseteq I$ since I is an ideal. Conversely, for any $g \in I$,

$$g(x) = f(x)h(x) + r(x)$$

for some $h, r \in k[x]$ with $r = 0$ or $\deg r < \deg f$ (as $k[x]$ is a Euclidean domain). Now as

$$r = g - fh \in I,$$

$r = 0$ (otherwise contrary to the minimality of f), we have $g = fh \in (f)$ for all $g \in I$.

(2) Let Y be an algebraic subset of $\mathbf{A}^1(k)$, say $Y = V(I)$ for some ideal I of $k[x]$. Since $k[x]$ is a PID, $I = (f)$ for some $f \in k[x]$.

- (a) If $f = 0$, then $I = (0)$ and $Y = V(0) = \mathbf{A}^1(k)$.
- (b) If $f \neq 0$, then $f(x) = 0$ has finitely many roots in k , say $a_1, \dots, a_m \in k$. Hence,

$$Y = V(I) = V(f) = \{f(a) = 0 : a \in k\} = \{a_1, \dots, a_m\}$$

is a finite subsets of $\mathbf{A}^1(k)$.

By (a)(b), the result is established.

□

Notes.

- (1) By the Hilbert basis theorem, $k[x]$ is Noetherian as k is Noetherian. Hence, for any algebraic subset $Y = V(I)$ of $\mathbf{A}^1(k)$, we can write $I = (f_1, \dots, f_m)$. Note that

$$Y = V(I) = V(f_1) \cap \dots \cap V(f_m).$$

Now apply the same argument to get the same conclusion.

- (2) Suppose $k = \bar{k}$. $\mathbf{A}^1(k)$ is irreducible, because its only proper closed subsets are finite, yet it is infinite (because k is algebraically closed, hence infinite).

Problem 1.9.

If k is a finite field, show that every subset of $\mathbf{A}^n(k)$ is algebraic.

Proof.

- (1) Every subset of $\mathbf{A}^n(k)$ is finite since $|\mathbf{A}^n(k)| = |k|^n$ is finite.
- (2) Note that $V(x_1 - a_1, \dots, x_n - a_n) = \{(a_1, \dots, a_n)\} \subseteq \mathbf{A}^n(k)$ (property (5) in this section) and any finite union of algebraic sets is algebraic (property (4) in this section). Thus, every subset of $\mathbf{A}^n(k)$ is algebraic (by (1)).

□

Problem 1.10.

Give an example of a countable collection of algebraic sets whose union is not algebraic.

Proof.

- (1) Let $k = \mathbb{Q}$ be an infinite field. $V(x - a) = \{a\}$ is an algebraic sets for all $a \in \mathbb{Q}$. In particular, $V(x - a) = \{a\}$ is algebraic for all $a \in \mathbb{Z}$.
- (2) Note that

$$Y := \bigcup_{a \in \mathbb{Z}} V(x - a) = \mathbb{Z}$$

is a countable union of algebraic sets. Since Y is a proper subset of $k = \mathbb{Q}$, it cannot be algebraic by Problem 1.8.

□

Problem 1.11.

Show that the following are algebraic sets:

- (a) $\{(t, t^2, t^3) \in \mathbf{A}^3(k) : t \in k\}$;
- (b) $\{(\cos(t), \sin(t)) \in \mathbf{A}^2(\mathbb{R}) : t \in \mathbb{R}\}$;
- (c) the set of points in $\mathbf{A}^2(\mathbb{R})$ whose polar coordinates (r, θ) satisfy the equation $r = \sin(\theta)$.

Proof of (a).

- (1) The twisted cubic curve

$$Y = \{(t, t^2, t^3) \in \mathbf{A}^3(k) : t \in k\} = V(x^2 - y) \cap V(x^3 - z)$$

is algebraic. We say that Y is given by the parametric representation $x = t, y = t^2, z = t^3$.

- (2) The generators for the ideal $I(Y)$ are $x^2 - y$ and $x^3 - z$.
 (3) Y is an affine variety of dimension 1.
 (4) The affine coordinate ring $A(Y)$ is isomorphic to a polynomial ring in one variable over k .

□

Proof of (b). The circle

$$\{(\cos(t), \sin(t)) \in \mathbf{A}^2(\mathbb{R}) : t \in \mathbb{R}\} = V(x^2 + y^2 - 1)$$

is algebraic. □

Proof of (c). The circle

$$\{(r, \theta) : r = \sin(\theta)\} = V(x^2 + y^2 - y)$$

is algebraic again. □

Problem 1.12.

Suppose C is an affine plane curve, and L is a line in $\mathbf{A}^2(k)$, $L \not\subseteq C$. Suppose $C = V(f)$, $f \in k[x, y]$ a polynomial of degree n . Show that $L \cap C$ is a finite set of no more than n points. (Hint: Suppose $L = V(y - (ax + b))$, and consider $f(x, ax + b) \in k[x]$.)

Proof.

- (1) Say $L = V(y - (ax + b))$ be a line in $\mathbf{A}^2(k)$. (The case $L = V(x - (ay + b))$ is similar.)
 (2) Note that $L \not\subseteq C$ implies that $(y - (ax + b)) \nmid f$. Hence, the polynomial

$$g : x \mapsto f(x, ax + b) \in k[x]$$

is nonzero and $\deg g \leq n$. Therefore, the number of roots of g in k is no more than n .

(3) Hence,

$$\begin{aligned}
L \cap C &= V(y - (ax + b)) \cap V(f) \\
&= \{(x, y) \in \mathbf{A}^2(k) : y = ax + b \text{ and } f(x, y) = 0\} \\
&= \{(x, y) \in \mathbf{A}^2(k) : f(x, ax + b) = 0\}
\end{aligned}$$

is finite of no more than n points.

□

Problem 1.13.

Show that each of the following sets is not algebraic:

- (a) $\{(x, y) \in \mathbf{A}^2(\mathbb{R}) : y = \sin(x)\}$.
- (b) $\{(z, w) \in \mathbf{A}^2(\mathbb{C}) : |z|^2 + |w|^2 = 1\}$, where $|x + iy|^2 = x^2 + y^2$ for $x, y \in \mathbb{R}$.
- (c) $\{(\cos(t), \sin(t), t) \in \mathbf{A}^3(\mathbb{R}) : t \in \mathbb{R}\}$.

Proof of (a).

- (1) (Reductio ad absurdum) If

$$Y := \{(x, y) \in \mathbf{A}^2(\mathbb{R}) : y = \sin(x)\}$$

were algebraic, then there is a subset S of $\mathbb{R}[x, y]$ such that

$$Y = V(S) = \bigcap_{f \in S} V(f).$$

- (2) $S \neq \emptyset$ since $Y \neq \mathbf{A}^2(\mathbb{R})$. ($(89, 64) \in \mathbf{A}^2(\mathbb{R}) - Y$.)
- (3) Take a fixed line $L = V(y)$ in $\mathbf{A}^2(\mathbb{R})$. For each affine curve $f \in S$, we have

$$V(f) \cap L \supseteq \bigcap_{f \in S} V(f) \cap L = Y \cap L = \{(n\pi, 0) \in \mathbf{A}^2(\mathbb{R}) : n \in \mathbb{Z}\},$$

which is infinite. By problem 1.12, $y \mid f$. As f runs over S , $Y \subseteq V(y) = L$, contradicts that $(0, \frac{\pi}{2}) \in L - Y$.

□

Proof of (b).

- (1) Similar to (a). (Reductio ad absurdum) If

$$Y := \{(x, y) \in \mathbf{A}^2(\mathbb{C}) : |x|^2 + |y|^2 = 1\}$$

were algebraic, then there is a subset S of $\mathbb{C}[x, y]$ such that

$$Y = V(S) = \bigcap_{f \in S} V(f).$$

- (2) $S \neq \emptyset$ since $Y \neq \mathbf{A}^2(\mathbb{C})$. $((89, 64) \in \mathbf{A}^2(\mathbb{C}) - Y)$
(3) Take a fixed line $L = V(x)$ in $\mathbf{A}^2(\mathbb{C})$. For each affine curve $f \in S$, we have

$$V(f) \cap L \supseteq \bigcap_{f \in S} V(f) \cap L = Y \cap L = \{(0, y) \in \mathbf{A}^2(\mathbb{C}) : |y| = 1\},$$

which is infinite (since Y contains a unit circle in the complex plane). By problem 1.12, $x \mid f$. As f runs over S , $Y \subseteq V(x) = L$, contradicts that the origin $(0, 0) \in L - Y$.

□

Proof of (c).

- (1) Similar to (a) and (b).
(2) Suppose C is an affine plane curve, and L is a line in $\mathbf{A}^3(k)$, $L \not\subseteq C$. Suppose $C = V(f)$, $f \in k[x, y, z]$ a polynomial of degree n . Show that $L \cap C$ is a finite set of no more than n points. The proof is similar to Problem 1.12.
(a) Say $L = V(y - (ax + b), z - (cx + d))$ be a line in $\mathbf{A}^3(k)$.
(b) Note that $L \not\subseteq C$ implies that $(y - (ax + b)) \nmid f$ and $(z - (cx + d)) \nmid f$. Hence, the polynomial

$$g : x \mapsto f(x, ax + b, cx + d) \in k[x]$$

is nonzero and $\deg g \leq n$. Therefore, the number of roots of g in k is no more than n .

- (c) Hence,

$$\begin{aligned} L \cap C &= V(y - (ax + b), z - (cx + d)) \cap V(f) \\ &= \{(x, y) \in \mathbf{A}^2(k) : y = ax + b, z = cx + d \text{ and } f(x, y) = 0\} \\ &= \{(x, y) \in \mathbf{A}^2(k) : f(x, ax + b, cx + d) = 0\} \end{aligned}$$

is finite of no more than n points.

(3) (Reductio ad absurdum) If

$$Y := \{(\cos(t), \sin(t), t) \in \mathbf{A}^3(\mathbb{R}) : t \in \mathbb{R}\}$$

were algebraic, then there is a subset S of $\mathbb{R}[x, y, z]$ such that

$$Y = V(S) = \bigcap_{f \in S} V(f).$$

(4) $S \neq \emptyset$ since $Y \neq \mathbf{A}^3(\mathbb{R})$. ((1989, 6, 4) $\in \mathbf{A}^3(\mathbb{R}) - Y$.)

(5) Take a fixed line $L = V(x - 1, y)$ in $\mathbf{A}^3(\mathbb{R})$. For each affine curve $f \in S$, we have

$$V(f) \cap L \supseteq \bigcap_{f \in S} V(f) \cap L = Y \cap L = \{(1, 0, 2n\pi) \in \mathbf{A}^3(\mathbb{R}) : n \in \mathbb{Z}\},$$

which is infinite. By (2), $(x - 1) \mid f$ and $y \mid f$. As f runs over S , $Y \subseteq V(x - 1, y) = L$, contradicts that $(1, 0, \pi) \in L - Y$.

□

Supplement. A circular disk of radius 1 in the plane xy rolls without slipping along the x axis. The figure described by a point of the circumference of the disk is called a **cycloid**. The parametrized curve $\alpha : \mathbb{R} \rightarrow \mathbb{R}^2$ is

$$\begin{cases} x = t - \sin t \\ y = 1 - \cos t. \end{cases}$$

The cycloid is not algebraic (as (a)).

Problem 1.14.*

Let f be a nonconstant polynomial in $k[x_1, \dots, x_n]$, k algebraically closed. Show that $\mathbf{A}^n(k) - V(f)$ is infinite if $n \geq 1$, and $V(f)$ is infinite if $n \geq 2$. Conclude that the complement of any proper algebraic set is infinite. (Hint: See Problem 1.4.)

Proof.

(1) Show that $\mathbf{A}^n(k) - V(f)$ is infinite if $n \geq 1$. Since f is a nonconstant polynomial in $k[x_1, \dots, x_n]$, we may assume that $\deg_{x_n}(f) > 0$. Hence

$$x_n \mapsto f(1, \dots, 1, x_n)$$

is a nonconstant polynomial of degree $\deg_{x_n}(f) > 0$ in $k[x_n]$. So f has finitely many roots in k , say ξ_1, \dots, ξ_m ($m \geq 0$). Hence,

$$(1, \dots, 1, x_n) \neq 0$$

whenever $x_n \neq \xi_m$. Such subset in $\mathbf{A}^1(k)$ is infinite since $k = \bar{k}$ (Problem 1.6). Therefore,

$$\begin{aligned}\mathbf{A}^n(k) - V(f) &= \{(a_1, \dots, a_n) \in \mathbf{A}^n(k) : f(a_1, \dots, a_n) \neq 0\} \\ &\supseteq \{a_n \in \mathbf{A}^1(k) : f(1, \dots, 1, x_n) \neq 0\}\end{aligned}$$

is infinite.

(2) Show that $V(f)$ is infinite if $n \geq 2$.

(a) Similar to (1). Since f is a nonconstant polynomial in $k[x_1, \dots, x_n]$, we may assume that $m := \deg_{x_n}(f) > 0$. Write

$$f = \sum_{i=0}^m f_i(x_1, \dots, x_{n-1})x_n^i.$$

Note that each f_i is well-defined since $n \geq 2$.

(b) If f_n is constant in $k[x_1, \dots, x_{n-1}]$, then f_n is nonzero (since $m > 0$) or $V(f_n) = \emptyset$. If f_n is nonconstant in $k[x_1, \dots, x_{n-1}]$, then the set $\mathbf{A}^{n-1}(k) - V(f_n)$ is infinite by (1). In any case,

$$\mathbf{A}^{n-1}(k) - V(f_n)$$

is infinite.

(c) For each $P = (a_1, \dots, a_{n-1}) \in \mathbf{A}^{n-1}(k) - V(f_n)$,

$$g_P : x_n \mapsto f(P, x_n) = f(a_1, \dots, a_{n-1}, x_n)$$

defines a polynomial in $k[x_n]$ of degree $m > 0$. Since $k = \bar{k}$, g_P has at least one root $Q \in k$. Hence

$$V(f) \supseteq \{(P, Q) \in \mathbf{A}^n(k) : P \in \mathbf{A}^{n-1}(k) - V(f_n), g_P(Q) = 0\}$$

is infinite since the set $\mathbf{A}^{n-1}(k) - V(f_n)$ is infinite.

Note. It is not true if $k \neq \bar{k}$. For example, $V(x^2 + y^2 + 1) = \emptyset$ in $\mathbf{A}^2(\mathbb{R})$.

(3) Note that

$$\mathbf{A}^n(k) - V(S) = \mathbf{A}^n(k) - \bigcap_{f \in S} V(f) = \bigcup_{f \in S} (\mathbf{A}^n(k) - V(f)).$$

Thus the complement of any proper algebraic set is infinite by (1).

□

Problem 1.15.*

Let $V \subseteq \mathbf{A}^n(k)$, $W \subseteq \mathbf{A}^m(k)$ be algebraic sets. Show that

$$V \times W = \{(a_1, \dots, a_n, b_1, \dots, b_m) : (a_1, \dots, a_n) \in V, (b_1, \dots, b_m) \in W\}$$

is an algebraic set in $\mathbf{A}^{n+m}(k)$. It is called the **product** of V and W .

Proof.

(1) Write

$$\begin{aligned} V &= V(S_V) = \{P \in \mathbf{A}^n(k) : f(P) = 0 \forall f \in S_V\} \\ W &= V(S_W) = \{Q \in \mathbf{A}^m(k) : g(Q) = 0 \forall g \in S_W\}, \end{aligned}$$

where $S_V \subseteq k[x_1, \dots, x_n]$ and $S_W \subseteq k[y_1, \dots, y_m]$. It suffices to show that

$$V \times W = V(S),$$

where $S \subseteq k[x_1, \dots, x_n, y_1, \dots, y_m]$ is the union of S_V and S_W .

(2) Here we can identify S_V with the subset of $k[x_1, \dots, x_n, y_1, \dots, y_m]$ by noting that

$$k[x_1, \dots, x_n] \hookrightarrow (k[y_1, \dots, y_m])[x_1, \dots, x_n] = k[x_1, \dots, x_n, y_1, \dots, y_m].$$

Here we regard k as a subring of $k[y_1, \dots, y_m]$. Similar treatment to S_W .

(3) By construction, $V \times W \subseteq V(S)$. Conversely, given any $(P, Q) \in V(S) \subseteq \mathbf{A}^{n+m}(k)$, we have $h(P, Q) = 0$ for all $h \in S = S_V \cup S_W$ (by (2)). By construction, $f(P) = 0$ for all $f \in S_V$ since f only involve x_1, \dots, x_n . Hence, $P \in V$. Similarly, $Q \in W$. Therefore, $(P, Q) \in V \times W$.

□

1.3. The Ideal of a Set of Points

Problem 1.16.*

Let V, W be algebraic sets in $\mathbf{A}^n(k)$. Show that $V = W$ if and only if $I(V) = I(W)$.

Proof.

(1) (Proof of Equation (6) in this section.) Show that if $X \subseteq Y$, then $I(X) \supseteq I(Y)$. If $f \in I(Y)$ then $f(P) = 0$ for all $P \in Y$. So $f(P) = 0$ for all $P \in X \subseteq Y$ or $f \in I(X)$.

- (2) (Proof of Equation (8) in this section.) $I(V(S)) \supseteq S$ for any set S of polynomials; $V(I(X)) \supseteq X$ for any set X of points.
- (a) If $f \in S$ then f vanishes on $V(S)$, hence $f \in I(V(S))$.
 - (b) If $P \in X$ then every polynomial in $I(X)$ vanishes at P , so P belongs to the zero set of $I(X)$.
- (3) (Proof of Equation (9) in this section.) $V(I(V(S))) = V(S)$ for any set S of polynomials, and $I(V(I(X))) = I(X)$ for any set X of points. So if V is an algebraic set, $V = V(I(V))$, and if I is the ideal of an algebraic set, $I = I(V(I))$.
- (a) In each case, it suffices to show that the left side is a subset of the right side. (by Equations (6)(8) in this section).
 - (b) If $P \in V(S)$ then $f(P) = 0$ for all $f \in I(V(S))$, so $P \in V(I(V(S)))$.
 - (c) If $f \in I(X)$ then $f(P) = 0$ for all $P \in V(I(X))$. Thus f vanishes on $V(I(X))$, so $f \in I(V(I(X)))$.
- (4) Show that $V = W$ if and only if $I(V) = I(W)$.
- (a) By Equation (6) in this section, $I(V) \supseteq I(W)$ if $V \subseteq W$ and $I(V) \subseteq I(W)$ if $V \supseteq W$. Thus, $I(V) = I(W)$ if $V = W$.
 - (b) Conversely, $I(V) = I(W)$ implies that $V(I(V)) = V(I(W))$ by Equation (3) in the previous section and similar argument in (a). By Equation (9) in this section, $V(I(V)) = V$ and $V(I(W)) = W$. Thus, $V = W$.

□

Problem 1.17.*

- (a) Let V be an algebraic set in $\mathbf{A}^n(k)$, $P \in \mathbf{A}^n(k)$ a point not in V . Show that there is a polynomial $f \in k[x_1, \dots, x_n]$ such that $f(Q) = 0$ for all $Q \in V$, but $f(P) = 1$. (Hint: $I(V) \neq I(V \cup \{P\})$.)
- (b) Let P_1, \dots, P_r be distinct points in $\mathbf{A}^n(k)$, not in an algebraic set V . Show that there are polynomials $f_1, \dots, f_r \in I(V)$ such that $f_i(P_j) = 0$ if $i \neq j$, and $f_i(P_i) = 1$. (Hint: Apply (a) to the union of V and all but one point.)
- (c) With P_1, \dots, P_r and V as in (b), and $a_{ij} \in k$ for $1 \leq i, j \leq r$, show that there are $g_i \in I(V)$ with $g_i(P_j) = a_{ij}$ for all i and j . (Hint: Consider $\sum_j a_{ij} f_j$.)

Proof of (a).

- (1) Since $I(V) \supsetneq I(V \cup \{P\})$ (by Problem 1.16), there is a polynomial $f \in k[x_1, \dots, x_n]$ such that $f(Q) = 0$ for all $Q \in V$, but $f(P) \neq 0$.
- (2) Since k is a field, $(f(P))^{-1} \in k$. Consider the polynomial $(f(P))^{-1}f \in k[x_1, \dots, x_n]$. It is well-defined. Also, $((f(P))^{-1}f)(Q) = (f(P))^{-1}f(Q) = 0$ for all $Q \in V$, but $(f(P))^{-1}f(P) = (f(P))^{-1}f(P) = 1$.

□

Proof of (b).

- (1) For $1 \leq i \leq$, define

$$\begin{aligned} W &= V \cup \{P_1, \dots, P_r\} \\ W_i &= V \cup \{P_1, \dots, \widehat{P_i}, \dots, P_r\}. \end{aligned}$$

Here $W = W_i \cup \{P_i\} \neq W_i$.

- (2) By (a), there is a polynomial $f_i \in k[x_1, \dots, x_n]$ such that $f_i(Q) = 0$ for all $Q \in W_i$, but $f_i(P_i) = 1$. Here $f_i \in I(V)$ and $f_i(P_j) = \delta_{ij}$ where δ_{ij} is the Kronecker delta.

□

Proof of (c).

- (1) For each $1 \leq i \leq r$, define

$$g_i = \sum_j a_{ij} f_j \in k[x_1, \dots, x_n].$$

- (2) $g_i \in I(V)$ since g_i is a linear combination of f_j and $I(V)$ is an ideal.

- (3) Also,

$$g_i(P_j) = \sum_{j'} a_{ij'} f_{j'}(P_j) = \sum_{j'} a_{ij'} \delta_{j'j} = a_{ij}.$$

□

Problem 1.18.*

Let I be an ideal in a ring R . If $a^n \in I$, $b^m \in I$, show that $(a+b)^{n+m} \in I$. Show that $\text{rad}(I)$ is an ideal, in fact a radical ideal. Show that any prime ideal is radical.

Proof.

- (1) Show that $(a + b)^{n+m} \in I$ if $a^n \in I$, $b^m \in I$. By the binomial theorem,

$$(a + b)^{n+m} = \sum_{i=0}^{n+m} a^i b^{n+m-i}.$$

For each term $a^i b^{n+m-i}$, either $i \geq n$ holds or $n + m - i \geq m$ holds, and thus $a^i b^{n+m-i} \in I$ (since $a^n \in I$, $b^m \in I$ and I is an ideal). Hence, the result is established.

- (2) Show that $\text{rad}(I)$ is an ideal.

- (a) $0 \in \text{rad}(I)$ since $0 = 0^1 \in I$ for any ideal in R .
- (b) $(a + b)^{n+m} \in I$ if $a^n \in I$, $b^m \in I$ by (1).
- (c) $(-a)^{2n} = (a^n)^2 \in I$ if $a^n \in I$ (since I is an ideal).
- (d) $(ra)^n = r^n a^n \in I$ if $a^n \in I$ and $r \in R$ (since I is an ideal and R is commutative).

- (3) Show that $\text{rad}(\text{rad}(I)) = \text{rad}(I)$. It suffices to show $\text{rad}(\text{rad}(I)) \subseteq \text{rad}(I)$. Given any $a \in \text{rad}(\text{rad}(I))$. By definition $a^n \in \text{rad}(I)$ for some positive integer n . Again by definition $(a^n)^m = a^{nm} \in I$ for some positive integer m . As nm is a positive integer, $a \in \text{rad}(I)$.

- (4) Show that every prime ideal \mathfrak{p} is radical. Given any $a \in \text{rad}(\mathfrak{p})$, that is, $a^n \in \mathfrak{p}$ for some positive integer. Write $a^n = aa^{n-1}$ if $n > 1$. By the primality of \mathfrak{p} , $a \in \mathfrak{p}$ or $a^{n-1} \in \mathfrak{p}$. If $a \in \mathfrak{p}$, we are done. If $a^{n-1} \in \mathfrak{p}$, we continue this descending argument (or the mathematical induction) until the power of a is equal to 1. Hence \mathfrak{p} is radical.

□

Problem 1.19.

Show that $I = (x^2 + 1) \subseteq \mathbb{R}[x]$ is a radical (even a prime) ideal, but I is not the ideal of any set in $\mathbf{A}^1(\mathbb{R})$.

Proof.

- (1) Show that $I = (x^2 + 1)$ is a prime ideal in $\mathbb{R}[x]$. Given any $fg \in I$. It suffices to show that $f \in I$ or $g \in I$. By definition of I , there is a polynomial $h \in \mathbb{R}[x]$ such that $fg = (x^2 + 1)h$. So $(x^2 + 1) \mid f$ or $(x^2 + 1) \mid g$ since $x^2 + 1$ is irreducible in a unique factorization domain $\mathbb{R}[x]$. Therefore, $f \in I$ or $g \in I$.
- (2) Show that I is not the ideal of any set in $\mathbf{A}^1(\mathbb{R})$. Since $x^2 + 1$ has no roots in \mathbb{R} , I cannot be the ideal of any nonempty set in $\mathbf{A}^1(\mathbb{R})$. Besides, $I(\emptyset) = (1) \neq (x^2 + 1)$.

□

Problem 1.20.*

Show that for any ideal I in $k[x_1, \dots, x_n]$, $V(I) = V(\text{rad}(I))$, and $\text{rad}(I) \subseteq I(V(I))$.

Proof.

- (1) Show that $V(I) = V(\text{rad}(I))$. Since $I \subseteq \text{rad}(I)$, it suffices to show that $V(I) \subseteq V(\text{rad}(I))$. Given any $P \in V(I)$. For any $f \in \text{rad}(I)$, $f^n \in I$ for some positive integer $n > 0$. Note that

$$0 = (f^n)(P) = f(P)^n$$

since $f^n \in I$ and $P \in V(I)$. As k is a domain, $f(P)^n = 0$ implies $f(P) = 0$. So $P \in V(\text{rad}(I))$.

- (2) By Equations (6) and (8) in this section,

$$I(V(I)) = I(V(\text{rad}(I))) \supseteq \text{rad}(I).$$

□

Note.

- (1) By the Hilbert's Nullstellensatz, $I(V(I)) = \text{rad}(I)$ if $k = \bar{k}$.
 (2) Take $I = (x^2 + 1)$ as an ideal in $\mathbb{R}[x]$. Note that $I(V(I)) = I(\emptyset) = (1)$ and $\text{rad}(I) = I = (x^2 + 1)$. So the equality in $\text{rad}(I) \subsetneq I(V(I))$ might not hold if $k \neq \bar{k}$. (See Problem 1.19.)

Problem 1.21.*

Show that $I = (x_1 - a_1, \dots, x_n - a_n) \subseteq k[x_1, \dots, x_n]$ is a maximal ideal, and that the natural homomorphism from k to $k[x_1, \dots, x_n]/I$ is an isomorphism.

Proof.

- (1) Show that I is a maximal ideal. Suppose that J is an ideal such that $J \supsetneq I$. Take any $f \in J - I$. By Problem 1.7(a),

$$f = \sum \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}.$$

As $f \notin I$, there is a nonzero constant term in f , say $\lambda \in k - \{0\}$. Note that $f - \lambda \in I \subsetneq J$. Hence,

$$\lambda = f - (f - \lambda) \in J$$

since J is an ideal. As $\lambda \neq 0$, $J = k[x_1, \dots, x_n]$ is not a proper ideal containing I .

- (2) Let $\varphi : k \rightarrow k[x_1, \dots, x_n]/I$ be the natural homomorphism. (That is, $\varphi : \lambda \rightarrow \lambda + I \in k[x_1, \dots, x_n]/I$.)
- (3) Show that φ is surjective. Given any $f + I \in k[x_1, \dots, x_n]/I$. By Problem 1.7(a),

$$f = \sum \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}.$$

So

$$\begin{aligned} f + I &= \sum \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n} + I \\ &= \left(f(a_1, \dots, a_n) + \sum_{\text{nonconstant}} \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n} \right) + I \\ &= f(a_1, \dots, a_n) + I. \end{aligned}$$

(Here the summation over all nonconstant terms is in I .) Hence

$$\varphi : f(a_1, \dots, a_n) \in k \mapsto f + I.$$

- (4) Show that φ is injective. $\ker(\varphi) = \{\lambda \in k : \lambda \in I\} = k \cap I = \{0\}$ since I is a proper ideal.
- (5) By (2)(3)(4), $\varphi : k \rightarrow k[x_1, \dots, x_n]/(x_1 - a_1, \dots, x_n - a_n)$ is an isomorphism.

□

1.4. The Hilbert Basis Theorem

Problem 1.22.*

Let I be an ideal in a ring R , $\pi : R \rightarrow R/I$ the natural homomorphism.

- (a) Show that for every ideal J' of R/I , $\pi^{-1}(J') = J$ is an ideal of R containing I , and for every ideal J of R containing I , $\pi(J) = J'$ is an ideal of R/I . This sets up a natural one-to-one correspondence between $\{\text{ideals of } R/I\}$ and $\{\text{ideals of } R \text{ that contain } I\}$.
- (b) Show that J' is a radical ideal if and only if J is radical. Similarly for prime and maximal ideals.

- (c) Show that J' is finitely generated if J is. Conclude that R/I is Noetherian if R is Noetherian. Any ring of the form $k[x_1, \dots, x_n]/I$ is Noetherian.

Proof of (a).

- (1) Show that for every ideal J' of R/I , $\pi^{-1}(J') = J$ is an ideal of R containing I .

- (a) Show that J contains I . Note that $\pi^{-1}(0) = I \subseteq \pi^{-1}(J') = J$. So J contains I . In particular, $J \neq \emptyset$ since $I \neq \emptyset$.
(b) Show that J is a additive subgroup of R . It suffices to show that $a - b \in J$ for any $a \in J$ and $b \in J$. Actually,

$$\pi(a - b) = \pi(a) - \pi(b) \in J'$$

implies $a - b \in \pi^{-1}(J') = J$.

- (c) Show that for every $r \in R$ and every $a \in J$, the product $ra \in J$. In fact,

$$\pi(ra) = \pi(r)\pi(a) \in J'$$

implies $ra \in \pi^{-1}(J') = J$.

- (2) Show that for every ideal J of R containing I , $\pi(J) = J'$ is an ideal of R/I .

- (a) Show that J' is nonempty. Note that $\pi(a) = 0 \in \pi(I) \subseteq \pi(J) = J'$ for any $a \in I$. So J' is nonempty since J is nonempty.
(b) Show that J' is a additive subgroup of R/I . It suffices to show that $\pi(a) - \pi(b) \in J'$ for any $\pi(a) \in J'$, $\pi(b) \in J'$, $a \in J$ and $b \in J$. It is trivial since

$$\pi(a) - \pi(b) = \pi(a - b) \in \pi(J) = J',$$

π is a ring homomorphism and J is an ideal.

- (c) Show that for every $\pi(r) \in R/I$ ($r \in R$) and every $\pi(a) \in J'$ ($a \in J$), the product $\pi(r)\pi(a) \in J'$. It is trivial since

$$\pi(r)\pi(a) = \pi(ra) \in \pi(J) = J',$$

π is a ring homomorphism and J is an ideal.

- (3) By (1)(2), we setup the correspondence between

$$\{\text{ideals of } R/I\} \longleftrightarrow \{\text{ideals of } R \text{ that contain } I\}.$$

Note that this correspondence preserves the subset relation, and thus this correspondence is one-to-one.

□

Proof of (b).

- (1) *Show that J' is radical if J is radical.* It suffices to show that $(a + I)^n = a^n + I \in J'$ implies that $a + I \in J'$. Note that

$$(a + I)^n = a^n + I \in J'$$

implies that $a^n \in J$ or $a \in J$ since J is radical. Hence $a + I \in J/I = J'$.

- (2) *Show that J is radical if J' is radical.* It suffices to show that $a^n \in J$ implies that $a \in J$. Note that

$$\pi(a^n) = \pi(a)^n \in J'$$

implies that $\pi(a) \in J'$ since J' is radical. $a \in \pi^{-1}(J') = J$.

- (3) *Show that J' is prime if J is prime.* It suffices to show that $(a + I)(b + I) = ab + I \in J'$ implies that $a + I \in J'$ or $b + I \in J'$. Note that

$$(a + I)(b + I) = ab + I \in J'$$

implies that $ab \in J$. So $a \in J$ or $b \in J$ by the primality of J . Hence $a + I \in J'$ or $b + I \in J'$.

- (4) *Show that J is prime if J' is prime.* It suffices to show that $ab \in J$ implies that $a \in J$ or $b \in J$. Note that

$$\pi(ab) = \pi(a)\pi(b) \in J'$$

implies that $\pi(a) \in J'$ or $\pi(b) \in J'$ by the primality of J' . So $a \in \pi^{-1}(J') = J$ or $b \in \pi^{-1}(J') = J$.

- (5) *Show that J' is maximal if J is maximal.* Suppose \mathfrak{m} is an ideal containing J' . By (a), $\pi^{-1}(\mathfrak{m})$ is an ideal containing J . So $\pi^{-1}(\mathfrak{m}) = J$ or $\pi^{-1}(\mathfrak{m}) = R$ by the maximality of J . Hence, $\mathfrak{m} = \pi(J) = J'$ or $\mathfrak{m} = \pi(R) = R/I$.

- (6) *Show that J is maximal if J' is maximal.* Suppose \mathfrak{m} is an ideal containing J . By (a), $\pi(\mathfrak{m})$ is an ideal containing J' . So $\pi(\mathfrak{m}) = J'$ or $\pi(\mathfrak{m}) = R/I$ by the maximality of J' . Hence, $\mathfrak{m} = \pi^{-1}(J') = J$ or $\mathfrak{m} = \pi^{-1}(R/I) = R$.

□

Note.

- (1) Note that

$$R/J \cong (R/I)/(J/I)$$

if J is an ideal of R such that $I \subseteq J$.

- (2) Hence, J is prime iff $R/J \cong (R/I)/(J/I)$ is a domain iff J/I is prime.
(3) Also, J is maximal iff $R/J \cong (R/I)/(J/I)$ is a field iff J/I is maximal.

Proof of (c).

- (1) *Show that J' is finitely generated if J is.* Suppose J is generated by a_1, \dots, a_m . It suffices to show that J' is generated by

$$a_1 + I, \dots, a_m + I \in J/I.$$

Given any $a + I \in J'$ where $a \in J$. Write $a = \sum_{1 \leq i \leq m} r_i a_i$ for some $r_i \in R$. Then

$$a + I = \sum r_i a_i + I = \sum (r_i + I)(a_i + I)$$

is generated by $a_1 + I, \dots, a_m + I$.

- (2) *Show that R/I is Noetherian if R is Noetherian.* Note that R is an ideal of itself.
(3) *Show that any ring of the form $k[x_1, \dots, x_n]/I$ is Noetherian.* By the corollary to the Hilbert basis theorem, $k[x_1, \dots, x_n]$ is Noetherian. By (2), the ring $k[x_1, \dots, x_n]/I$ is Noetherian.

□

1.5. Irreducible Components of an Algebraic Set

Problem 1.23.

Give an example of a collection of ideals \mathcal{S} ideals in a Noetherian ring such that no maximal member of \mathcal{S} is a maximal ideal.

Proof.

- (1) Let R be any Noetherian ring. Let \mathcal{S} be any collection of ideals containing R itself. Then the only maximal member of \mathcal{S} is R , which is not a maximal ideal.
(2) Or let R be any Noetherian ring and R is not a field. ($R = k[x_1, \dots, x_n]$ where k is a field for example.) Let $\mathcal{S} = \{(0)\}$. Then the only maximal member of \mathcal{S} is (0) , which is not maximal since R is not a field.

□

Problem 1.24.

Show that every proper ideal in a Noetherian ring is contained in a maximal ideal. (Hint: If I is the ideal, apply the lemma to $\{\text{proper ideals that contain } I\}$.)

Proof.

- (1) Say I be any proper ideal in a Noetherian ring. Let

$$\mathcal{S} = \{\text{proper ideals that contain } I\}.$$

Apply the lemma to \mathcal{S} to get that \mathcal{S} has a maximal member $\mathfrak{m} \in \mathcal{S}$.

- (2) Show that \mathfrak{m} is maximal. Since $\mathfrak{m} \in \mathcal{S}$, \mathfrak{m} is a proper ideal in R . Suppose $\mathfrak{m}' \supsetneq \mathfrak{m}$ is a proper ideal containing \mathfrak{m} . As \mathfrak{m} contains I , \mathfrak{m}' also contains I or $\mathfrak{m}' \in \mathcal{S}$. By the maximality of \mathfrak{m} , $\mathfrak{m}' \subseteq \mathfrak{m}$. So $\mathfrak{m}' = \mathfrak{m}$.

□

Problem 1.25.

- (a) Show that $V(y - x^2) \subseteq \mathbf{A}^2(\mathbb{C})$ is irreducible, in fact, $I(V(y - x^2)) = (y - x^2)$.
- (b) Decompose $V(y^4 - x^2, y^4 - x^2y^2 + xy^2 - x^3) \subseteq \mathbf{A}^2(\mathbb{C})$ into irreducible components.

Proof of (a).

- (1) Let $I = (y - x^2)$ be an ideal of $\mathbb{C}[x, y]$. Since \mathbb{C} is algebraically closed,

$$I(V(I)) = \text{rad}(I)$$

by the Hilbert's Nullstellensatz. It suffices to show that I is prime, or to show that $y - x^2$ is prime. Since $\mathbb{C}[x, y]$ is a UFD, it suffices to show that $y - x^2$ is irreducible.

- (2) Show that $y - x^2$ is irreducible in $\mathbb{C}[x, y]$. Write

$$y - x^2 \in (\mathbb{C}[y])[x].$$

Note that $\mathbb{C}[y]$ is a UFD and y is the constant term. If we can show that y is prime in $\mathbb{C}[y]$, then by the Eisenstein's criterion we can say $y - x^2$ is irreducible in $(\mathbb{C}[y])[x]$.

- (3) As $\mathbb{C}[y]/(y) \cong \mathbb{C}$ is a field or a domain, (y) is maximal or prime. Hence, $y - x^2$ is irreducible.

(4) Or apply Corollary 1 to Proposition 2 in the next section to (2)(3).

□

Proof of (b).

(1) Write

$$\begin{aligned}
 Y &:= V(y^4 - x^2, y^4 - x^2y^2 + xy^2 - x^3) \\
 &= V((y^2 - x)(y^2 + x), (y^2 - x^2)(y^2 + x)) \\
 &= V(y^2 + x) \cup V(y^2 - x, y^2 - x^2) \\
 &= V(y^2 + x) \cup V(y^2 - x, x(x - 1)) \\
 &= V(y^2 + x) \cup V(x, y) \cup V(y + 1, x - 1) \cup V(y - 1, x - 1).
 \end{aligned}$$

(2) Here $V(y^2 + x)$ is irreducible as (a). Besides, $V(x, y)$, $V(y + 1, x - 1)$ and $V(y - 1, x - 1)$ are irreducible since all corresponding ideals are maximal (by the Hilbert's Nullstellensatz and Problem 1.21).

□

Problem 1.26.

Show that $f = y^2 + x^2(x - 1)^2 \in \mathbb{R}[x, y]$ is an irreducible polynomial, but $V(f)$ is reducible.

Proof.

(1) Show that f is an irreducible polynomial.

(a) Suppose

$$f = (f_2(x)y^2 + f_1(x)y + f_0(x)) \cdot g(x)$$

for some $f_i(x), g(x) \in \mathbb{R}[x]$. So

$$f_2(x)g(x) = 1, \quad f_1(x)g(x) = 0, \quad f_0(x)g(x) = x^2(x - 1)^2.$$

Hence,

$$f_2(x)y^2 + f_1(x)y + f_0(x) = uf, \quad g(x) = u^{-1},$$

where u is a unit in \mathbb{R} .

(b) Suppose

$$f = (f_1(x)y + f_0(x)) \cdot (g_1(x)y + g_0(x))$$

for some $f_i(x), g_j(x) \in \mathbb{R}[x]$. So

$$\begin{aligned} f_1(x)g_1(x) &= 1, \\ f_1(x)g_0(x) + f_0(x)g_1(x) &= 0, \\ f_0(x)g_0(x) &= x^2(x-1)^2. \end{aligned}$$

So $f_1(x) = u$, $g_1(x) = u^{-1}$ for some unit $u \in \mathbb{R}$. Hence,

$$u^2 g_0(x)^2 = -x^2(x-1)^2,$$

which is absurd since \mathbb{R} is not algebraically closed.

(c) By (a)(b), f is irreducible in $\mathbb{R}[x, y]$.

- (2) Show that $V(f)$ is reducible. $V(f) = \{(0, 0), (1, 0)\} = V(x, y) \cup V(x-1, y)$.
Here $V(x, y)$ and $V(x-1, y)$ are all proper algebraic sets in $V(f)$.

□

Problem 1.27.

Let V, W be algebraic sets in $\mathbf{A}^n(k)$ with $V \subseteq W$. Show that each irreducible component of V is contained in some irreducible component of W .

Proof.

- (1) Write two decompositions of V, W into irreducible components as

$$\begin{aligned} V &= V_1 \cup \cdots \cup V_r, \\ W &= W_1 \cup \cdots \cup W_s, \end{aligned}$$

- (2) For each irreducible component V_i of V , consider $V_i \cap W$:

$$V_i \cap W = (V_i \cap W_1) \cup \cdots \cup (V_i \cap W_s).$$

By the irreducibility of V_i , there is only one j such that $V_i \cap W_j = V_i$ and other intersections are empty. Therefore, each irreducible component V_i is contained in some irreducible component W_j of W .

□

Problem 1.28.

If $V = V_1 \cup \cdots \cup V_r$ is the decomposition of an algebraic set into irreducible components, show that $V_i \not\subseteq \bigcup_{j \neq i} V_j$.

Proof.

- (1) (Reductio ad absurdum) If

$$V_i \subseteq \bigcup_{j \neq i} V_j$$

for some i , then

$$V = V_1 \cup \cdots \cup \widehat{V_i} \cup \cdots \cup V_r$$

is another decomposition of an algebraic set into irreducible components.

- (2) By Theorem 2 in this section, the number of irreducible components is uniquely determined, contrary to the assumption and (1).

□

Problem 1.29.*

Show that $\mathbf{A}^n(k)$ is irreducible if k is infinite.

Proof.

- (1) (Reductio ad absurdum) If $\mathbf{A}^n(k)$ were reducible, then $\mathbf{A}^n(k) = V_1 \cup V_2$ where V_1, V_2 are algebraic sets in $\mathbf{A}^n(k)$, V_1 and V_2 are nonempty and proper in $\mathbf{A}^n(k)$.
- (2) Take $P_i \in V_i$ for $i = 1, 2$. By Problem 1.17, there are two polynomials $f_1, f_2 \in k[x_1, \dots, x_n]$ such that $f_i(Q) = 0$ for all $Q \in V_i$ and $f_1(P_2) = f_2(P_1) = 1$.
- (3) By construction, $(f_1 f_2)(a_1, \dots, a_n) = 0$ for any $a_1, \dots, a_n \in k$. As k is infinite, $f_1 f_2 = 0$ by Problem 1.4. Since $k[x_1, \dots, x_n]$ is a domain, $f_1 = 0$ or $f_2 = 0$, contrary to $f_1(P_2) = f_2(P_1) \neq 0$.

□

Note. $\mathbf{A}^n(k)$ is reducible if k is finite.

1.6. Algebraic Subsets of the Plane

Problem 1.30.

Let $k = \mathbb{R}$.

- (a) Show that $I(V(x^2 + y^2 + 1)) = (1)$.
- (b) Show that every algebraic subset of $\mathbf{A}^2(\mathbb{R})$ is equal to $V(f)$ for some $f \in \mathbb{R}[x, y]$.

This indicates why we usually require that k be algebraically closed.

Proof of (a). $I(V(x^2 + y^2 + 1)) = I(\emptyset) = (1)$ since $x^2 + y^2 + 1 \geq 1$ is never zero for any $x, y \in \mathbb{R}$. \square

Proof of (b).

- (1) Given any algebraic subset V of $\mathbf{A}^2(\mathbb{R})$. $V = V(1)$ if $V = \emptyset$. $V = V(0)$ if $V = \mathbf{A}^2(\mathbb{R})$. Now suppose V is a nonempty proper algebraic subset V of $\mathbf{A}^2(\mathbb{R})$. Write $V = V_1 \cup \cdots \cup V_m$, where each V_i is irreducible. Here $V_i \neq \emptyset$ and $V_i \neq \mathbf{A}^2(\mathbb{R})$ for all i .
- (2) As $k = \mathbb{R}$ is infinite, Corollary 2 to Proposition 2 implies that each V_i is either a point or an irreducible plane curves $V(f_i)$, where f_i is an irreducible polynomial and $V(f_i)$ is infinite.
- (3) If $V_i = \{(a_i, b_i)\}$ is a point, then define

$$f_i(x, y) = (x - a_i)^2 + (y - b_i)^2.$$

By the property of \mathbb{R} , $V_i = V(f_i)$.

- (4) Define $f = f_1 \cdots f_m \in \mathbb{R}[x, y]$. Hence,

$$\begin{aligned} V &= V_1 \cup \cdots \cup V_m \\ &= V(f_1) \cup \cdots \cup V(f_m) \\ &= V(f_1 \cdots f_m) \\ &= V(f). \end{aligned}$$

\square

Problem 1.31.

- (a) Find the irreducible components of $V(y^2 - xy - x^2y + x^3)$ in $\mathbf{A}^2(\mathbb{R})$, and also in $\mathbf{A}^2(\mathbb{C})$.

(b) Do the same for $V(y^2 - x(x^2 - 1))$, and for $V(x^3 + x - x^2y - y)$.

Proof of (a).

(1) Note that

$$\begin{aligned} V(y^2 - xy - x^2y + x^3) &= V((y - x^2)(y - x)) \\ &= V(y - x^2) \cup V(y - x). \end{aligned}$$

(2) Note that $y - x^2$ and $y - x$ are irreducible in $\mathbb{C}[x, y]$ and thus also in $\mathbb{R}[x, y]$ by the similar argument in Problem 1.25(a). Also, $V(y - x^2)$ and $V(y - x)$ are infinite in $\mathbf{A}^2(\mathbb{R})$ and thus also in $\mathbf{A}^2(\mathbb{C})$.

(3) Therefore, $V(y - x^2)$ and $V(y - x)$ are the irreducible components of $V(y^2 - xy - x^2y + x^3)$ in $\mathbf{A}^2(\mathbb{R})$ and also in $\mathbf{A}^2(\mathbb{C})$.

□

Outline of (b).

- (1) The elliptic curve $V(y^2 - x(x + 1)(x - 1))$ is irreducible over $\mathbf{A}^2(\mathbb{R})$.
- (2) The elliptic curve $V(y^2 - x(x + 1)(x - 1))$ is irreducible over $\mathbf{A}^2(\mathbb{C})$.
- (3) The irreducible component of $V(x^3 + x - x^2y - y)$ over $\mathbf{A}^2(\mathbb{R})$ is $V(x - y)$.
- (4) The irreducible components of $V(x^3 + x - x^2y - y)$ over $\mathbf{A}^2(\mathbb{C})$ are $V(x + i)$, $V(x - i)$ and $V(x - y)$.

Proof of (b).

(1) Similar to Problem 1.25. To show $y^2 - x(x + 1)(x - 1)$ is irreducible in $\mathbb{C}[x, y]$, we write

$$y^2 - x(x + 1)(x - 1) \in (\mathbb{C}[x])[y].$$

Note that $\mathbb{C}[x]$ is a UFD and $-x(x + 1)(x - 1)$ is the constant term. As $\mathbb{C}[x]/(x) \cong \mathbb{C}$ is a domain, (x) is prime. Clearly, $x \mid x(x + 1)(x - 1)$ but $x^2 \nmid x(x + 1)(x - 1)$. By the Eisenstein's criterion, we can say $y^2 - x(x + 1)(x - 1)$ is irreducible over $(\mathbb{C}[x])[y]$.

- (2) Moreover, $V(y^2 - x(x + 1)(x - 1))$ is infinite over $\mathbf{A}^2(\mathbb{R})$ and thus also over $\mathbf{A}^2(\mathbb{C})$. ($y = f(x) = \sqrt{x(x + 1)(x - 1)}$ is continuous and strictly increasing on $[1, \infty)$ in the sense of calculus. As the measure of $[1, \infty)$ is ∞ , the set $V(y^2 - x(x + 1)(x - 1))$ is infinite over $\mathbf{A}^2(\mathbb{R})$.)
- (3) By Corollary 1 to Proposition 2, $V(y^2 - x(x^2 - 1))$ itself is irreducible over $\mathbf{A}^2(\mathbb{R})$ or $\mathbf{A}^2(\mathbb{C})$.

- (4) Consider $V(x^3 + x - x^2y - y) \subseteq \mathbf{A}^2(\mathbb{R})$.

$$\begin{aligned} V(x^3 + x - x^2y - y) &= V((x^2 + 1)(x - y)) \\ &= V(x^2 + 1) \cup V(x - y) \\ &= \emptyset \cup V(x - y) \\ &= V(x - y). \end{aligned}$$

Here we use that fact that $x^2 + 1 = 0$ has no real solution $x \in \mathbb{R}$. Similar to (a), $V(x - y)$ is the only irreducible component of $V(x^3 + x - x^2y - y)$ in $\mathbf{A}^2(\mathbb{R})$.

- (5) Consider $V(x^3 + x - x^2y - y) \subseteq \mathbf{A}^2(\mathbb{C})$.

$$\begin{aligned} V(x^3 + x - x^2y - y) &= V((x + i)(x - i)(x - y)) \\ &= V(x + i) \cup V(x - i) \cup V(x - y). \end{aligned}$$

Similar to (a), $V(x \pm i)$ and $V(x - y)$ are the irreducible components of $V(x^3 + x - x^2y - y)$ in $\mathbf{A}^2(\mathbb{C})$.

□

1.7. Hilbert's Nullstellensatz

Problem 1.32.

Show that both theorems and all of the corollaries are false if k is not algebraically closed.

Proof.

- (1) Weak Nullstellensatz: $I = (x^2 + 1)$ is a proper ideal in $\mathbb{R}[x]$ but $V(I) = \emptyset$.
- (2) Hilbert's Nullstellensatz: Let $I = (y^2 + x^2(x - 1)^2)$ be an ideal in $\mathbb{R}[x, y]$. Hence,

$$\begin{aligned} I(V(I)) &= I(\{(0, 0), (1, 0)\}) && \text{(Problem 1.26.)} \\ &= (x(x - 1), y) \\ &\neq I \\ &= \text{rad}(I). \end{aligned}$$

The last equality holds since f is irreducible in a UFD $\mathbb{R}[x, y]$ and thus I is a prime ideal.

- (3) Corollary 1: Same example in the case Hilbert's Nullstellensatz. If $I = (y^2 + x^2(x - 1)^2)$ is a radical ideal in $\mathbb{R}[x, y]$. Then $I(V(I)) \neq I$.

- (4) Corollary 2: Same example in the case Hilbert's Nullstellensatz. If $I = (y^2 + x^2(x-1)^2)$ is a prime ideal in $\mathbb{R}[x, y]$, then

$$V(I) = \{(0, 0), (1, 0)\} = V(x, y) \cup V(x-1, y)$$

is reducible. Next, consider a prime ideal $J = (x^2 + y^2)$ in $\mathbb{R}[x, y]$. (Use the same argument in Problem 1.26 to get the irreducibility of $x^2 + y^2$.) $V(J) = \{(0, 0)\}$ is a point but J is not a maximal ideal (since $J \subsetneq (x^2 + y^2, x) \subsetneq (1)$).

- (5) Corollary 3: Same example in Corollary 2.
- (6) Corollary 4: Let $I = (x^2 + y^2)$ be an ideal in $\mathbb{R}[x, y]$. Then $V(I) = \{(0, 0)\}$ is a finite set. But $\mathbb{R}[x, y]/(x^2 + y^2)$ is an infinite dimensional vector space over \mathbb{R} . In fact, the monomials

$$\{\overline{x^m}, \overline{x^m y} : m = 0, 1, 2, \dots\}$$

is a basis for $\mathbb{R}[x, y]/(x^2 + y^2)$.

□

Problem 1.33.

- (a) Decompose $V(x^2 + y^2 - 1, x^2 - z^2 - 1) \subseteq \mathbf{A}^3(\mathbb{C})$ into irreducible components.
- (b) Let $V = \{(t, t^2, t^3) \in \mathbf{A}^3(\mathbb{C}) : t \in \mathbb{C}\}$. Find $I(V)$, and show that V is irreducible.

Proof of (a).

- (1) Write

$$\begin{aligned} & V(x^2 + y^2 - 1, x^2 - z^2 - 1) \\ &= V(x^2 + y^2 - 1, y^2 + z^2) \\ &= V(x^2 + y^2 - 1, (y + iz)(y - iz)) \\ &= V(x^2 + y^2 - 1, y + iz) \cup V(x^2 + y^2 - 1, y - iz). \end{aligned}$$

By the Hilbert's Nullstellensatz, it suffices to show that $(x^2 + y^2 - 1, y + iz)$ and $(x^2 + y^2 - 1, y - iz)$ are prime.

- (2) Show that $I = (x^2 + y^2 - 1, y + iz)$ is prime in $\mathbb{C}[x, y, z]$. Note that

$$\mathbb{C}[x, y, z]/I \cong \mathbb{C}[x, y]/(x^2 + y^2 - 1)$$

is a ring isomorphism defined by

$$f(x, y, z) + I \mapsto f(x, y, -iy) + (x^2 + y^2 - 1).$$

(Use the similar argument in (b) to prove it is indeed an isomorphism.)
So it suffices to show that

$$x^2 + y^2 - 1 \in \mathbb{C}[x, y]$$

is irreducible. (Thus, $\mathbb{C}[x, y]/(x^2 + y^2 - 1) \cong \mathbb{C}[x, y, z]/I$ is a domain, or I is prime.) We can use the similar argument in Problem 1.31 (b) to show $x^2 + y^2 - 1 = y^2 + (x+1)(x-1)$ is irreducible as showing the irreducibility of $y^2 - x(x+1)(x-1)$.

- (3) Similarly, $I = (x^2 + y^2 - 1, y - iz)$ is prime. Therefore, the irreducible components of $V(x^2 + y^2 - 1, x^2 - z^2 - 1)$ are $V(x^2 + y^2 - 1, y + iz)$ and $V(x^2 + y^2 - 1, y - iz)$.

□

Proof of (b).

- (1) Write

$$V = \{(t, t^2, t^3) \in \mathbf{A}^3(\mathbb{C}) : t \in \mathbb{C}\} = V(x^2 - y, x^3 - z).$$

Let $I = (x^2 - y, x^3 - z)$ in $\mathbb{C}[x, y, z]$. By the Hilbert's Nullstellensatz, $I(V) = \text{rad}(I)$. So it suffices to show that $I = (x^2 - y, x^3 - z)$ is prime (and thus V is irreducible).

- (2) *Show that*

$$\mathbb{C}[x, y, z]/I \cong \mathbb{C}[t]$$

is a domain, and thus $I = (x^2 - y, x^3 - z)$ is a prime ideal.

- (a) Define a ring homomorphism $\alpha : \mathbb{C}[x, y, z]/I \rightarrow \mathbb{C}[t]$ by

$$\alpha : f(x, y, z) + I \mapsto f(t, t^2, t^3).$$

α is well-defined since $\alpha((x^2 - y) + I) = 0$ and $\alpha((x^3 - z) + I) = 0$.

- (b) *Show that α is surjective.*

$$\alpha : g(x) + I \in \mathbb{C}[x, y, z]/I \mapsto g(t) \in \mathbb{C}[t]$$

for any $g(t)$.

- (c) *Show that α is injective.* Suppose $\alpha(f(x, y, z) + I) = 0$. Write

$$\begin{aligned} f(x, y, z) + I &= \sum_{(i)} \lambda_{(i)} x^{i_1} (y - x^2)^{i_2} (z - x^3)^{i_3} + I \\ &= \sum_i \lambda_i x^i + I. \end{aligned}$$

So

$$0 = \alpha(f(x, y, z) + I) = \alpha\left(\sum_i \lambda_i x^i + I\right) = \sum_i \lambda_i t^i.$$

Hence, $\ker(\alpha) = I$.

□

Problem PLACEHOLDER

PLACEHOLDER

Proof.

(1) PLACEHOLDER

□

Problem PLACEHOLDER

PLACEHOLDER

Proof.

(1) PLACEHOLDER

□

Problem PLACEHOLDER

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Proof.

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Proof.

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Proof.

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Problem PLACEHOLDER

PLACEHOLDER

Proof.

- (1) PLACEHOLDER

□

Problem PLACEHOLDER

PLACEHOLDER

Proof.

- (1) PLACEHOLDER

□

1.8. Modules; Finiteness Conditions

Problem 1.41.*

If S is module-finite over R , then S is ring-finite over R .

Proof.

- (1) $S = \sum R s_i$ for some $s_1, \dots, s_n \in S$ since S is module-finite over R .
(2) Let I be the minimal subset of $\{s_1, \dots, s_n\}$ which also spans S , say $\{t_1, \dots, t_m\}$ with $m \leq n$. Clearly we can write

$$S = R[t_1, \dots, t_m],$$

that is, S is ring-finite over R .

(3) The converse is not true (Problem 1.42).

□

Problem 1.42.

Show that $S = R[x]$ (the ring of polynomials in one variable) is ring-finite over R , but not module-finite.

Proof.

(1) $S = R[x]$ is ring-finite over R by definition (as $x \in S$).

(2) (Reductio ad absurdum) If $S = \sum Rs_i$ for some $s_1, \dots, s_n \in S$ were module-finite over R . Any element $s \in \sum Rs_i$ is of degree

$$\deg s \leq \max_{1 \leq i \leq n} \deg s_i := m.$$

So that $x^{m+1} \in S = R[x]$ but not in $\sum Rs_i$, which is absurd.

□

Problem 1.43.* (WIP)

If L is ring-finite over K (K, L fields) then L is a finitely generated field extension of K .

Proof.

(1) $L = K[v_1, \dots, v_n]$ for some $v_i \in L$. To show $L = K[v_1, \dots, v_n] = K(v_1, \dots, v_n)$, it suffices to show that all v_i are algebraic over L .

(2)

□

Problem PLACEHOLDER

PLACEHOLDER

Proof.

(1) PLACEHOLDER

□

Problem PLACEHOLDER

PLACEHOLDER

Proof.

(1) PLACEHOLDER

□

1.9. Integral Elements

Problem PLACEHOLDER

PLACEHOLDER

Proof.

(1) PLACEHOLDER

□

Problem PLACEHOLDER

PLACEHOLDER

Proof.

(1) PLACEHOLDER

□

Problem PLACEHOLDER

PLACEHOLDER

Proof.

(1) PLACEHOLDER

□

Problem PLACEHOLDER

PLACEHOLDER

Proof.

(1) PLACEHOLDER

□

Problem PLACEHOLDER

PLACEHOLDER

Proof.

(1) PLACEHOLDER

□

1.10. Field Extensions

Problem PLACEHOLDER

PLACEHOLDER

Proof.

(1) PLACEHOLDER

□

Problem PLACEHOLDER

PLACEHOLDER

Proof.

(1) PLACEHOLDER

□

Problem PLACEHOLDER

PLACEHOLDER

Proof.

(1) PLACEHOLDER

□

Problem PLACEHOLDER

PLACEHOLDER

Proof.

(1) PLACEHOLDER

□

Chapter 2: Affine Varieties

2.1. Coordinate Rings

Problem 2.1.*

Show that the map which associates to each $f \in k[x_1, \dots, x_n]$ a polynomial function in $\mathcal{F}(V, k)$ is a ring homomorphism whose kernel is $I(V)$.

Proof.

- (1) Define a map $\alpha : k[x_1, \dots, x_n] \rightarrow \mathcal{F}(V, k)$. Every polynomial $f \in k[x_1, \dots, x_n]$ defines a function from V to k by

$$\alpha(f)(a_1, \dots, a_n) = f(a_1, \dots, a_n)$$

for all $(a_1, \dots, a_n) \in V$.

- (2) α is a ring homomorphism by construction in (1).
(3) Show that $\ker(\alpha) = I(V)$. In fact, given any $f \in k[x_1, \dots, x_n]$, we have $\alpha(f) = 0$ (sending all $a \in V$ to $0 \in k$) if and only if $f(a) = 0$ for all $a \in V$ if and only if $f \in I(V)$.
(4) Hence $k[x_1, \dots, x_n]/I(V) = \Gamma(V) \hookrightarrow \mathcal{F}(V, k)$ is an injective homomorphism.

□

Problem PLACEHOLDER

PLACEHOLDER

Proof.

- (1) PLACEHOLDER

2.2. Polynomial Maps

2.3. Coordinate Changes

2.4. Rational Functions and Local Rings

2.5. Discrete Valuation Rings

2.6. Forms

2.7. Direct Products of Rings

2.8. Operations with Ideals

Problem 2.39.*

Prove the following relations among ideals I_i , J in a ring R :

(a) $(I_1 + I_2)J = I_1J + I_2J$.

(b) $(I_1 \cdots I_N)^n = I_1^n \cdots I_N^n$.

Proof of (a).

(1) Note that $(I_1 + I_2)J$ and $I_1J + I_2J$ are ideals.

(2) *Show that $(I_1 + I_2)J \subseteq I_1J + I_2J$.* Given any

$$(x_1 + x_2)y \in (I_1 + I_2)J$$

where $x_i \in I_i$ and $y \in J$. It suffices to show that $(x_1 + x_2)y \in I_1J + I_2J$ (by (1)). In fact,

$$(x_1 + x_2)y = x_1y + x_2y \in I_1J + I_2J.$$

(3) *Show that $(I_1 + I_2)J \supseteq I_1J + I_2J$.* Given any

$$x_1y_1 + x_2y_2 \in I_1J + I_2J$$

where $x_i \in I_i$ and $y_i \in J$. It suffices to show that $x_1y_1 + x_2y_2 \in (I_1 + I_2)J$ (by (1)). In fact,

$$x_1y_1 + x_2y_2 = (x_1 + \underbrace{0}_{\in I_2})y_1 + (\underbrace{0}_{\in I_1} + x_2)y_2 \in (I_1 + I_2)J$$

since $(I_1 + I_2)J$ is an ideal.

□

Proof of (b).

- (1) Note that $(I_1 \cdots I_N)^n$ and $I_1^n \cdots I_N^n$ are ideals.
- (2) Show that $(I_1 \cdots I_N)^n \subseteq I_1^n \cdots I_N^n$. Given any

$$x = x_1 \cdots x_n$$

where $x_i \in I_1 \cdots I_N$. It suffices to show that $x \in I_1^n \cdots I_N^n$ (by (1)). For each $x_i \in I_1 \cdots I_N$, write

$$x_i = \sum_{j(i)} x_{j(i),1} \cdots x_{j(i),N}$$

where $x_{j(i),k} \in I_k$ for $1 \leq k \leq N$. Hence

$$\begin{aligned} x &= x_1 \cdots x_n \\ &= \left(\sum_{j(1)} x_{j(1),1} \cdots x_{j(1),N} \right) \cdots \left(\sum_{j(n)} x_{j(n),1} \cdots x_{j(n),N} \right) \\ &= \sum_{j(1), \dots, j(n)} (x_{j(1),1} \cdots x_{j(1),N}) \cdots (x_{j(n),1} \cdots x_{j(n),N}) \\ &= \sum_{j(1), \dots, j(n)} \underbrace{(x_{j(1),1} \cdots x_{j(n),1})}_{\in I_1^n} \cdots \underbrace{(x_{j(1),N} \cdots x_{j(n),N})}_{\in I_N^n} \\ &\in I_1^n \cdots I_N^n. \end{aligned}$$

- (3) Show that $(I_1 \cdots I_N)^n \supseteq I_1^n \cdots I_N^n$. Given any

$$x = x_1 \cdots x_N \in I_1^n \cdots I_N^n$$

where $x_i \in I_i^n$ ($1 \leq i \leq N$). It suffices to show that $x \in (I_1 \cdots I_N)^n$ (by (1)). For each $x_i \in I_i^n$, write

$$x_i = \sum_{j(i)} x_{j(i),1} \cdots x_{j(i),n}$$

where $x_{j(i),k} \in I_i$ for $1 \leq k \leq n$. Hence

$$\begin{aligned}
x &= x_1 \cdots x_N \\
&= \left(\sum_{j(1)} x_{j(1),1} \cdots x_{j(1),n} \right) \cdots \left(\sum_{j(N)} x_{j(N),1} \cdots x_{j(N),n} \right) \\
&= \sum_{j(1), \dots, j(N)} (x_{j(1),1} \cdots x_{j(1),n}) \cdots (x_{j(N),1} \cdots x_{j(N),n}) \\
&= \sum_{j(1), \dots, j(N)} \underbrace{(x_{j(1),1} \cdots x_{j(N),1})}_{\in I_1 \cdots I_N} \cdots \underbrace{(x_{j(1),n} \cdots x_{j(N),n})}_{\in I_1 \cdots I_N} \\
&\in (I_1 \cdots I_N)^n.
\end{aligned}$$

□

Problem 2.41.*

Let I, J be ideals in R . Suppose I is finitely generated and $I \subseteq \text{rad}(J)$. Show that $I^n \subseteq J$ for some n .

Proof.

- (1) Let I be generated by $x_1, \dots, x_m \in I$. As $I \subseteq \text{rad}(J)$, there are integers $n_i > 0$ such that $x_i^{n_i} \in J$.
- (2) Let $N = n_1 + \cdots + n_m$. Given any $x = \sum_{i=1}^m r_i x_i \in I$, so

$$\begin{aligned}
x^N &= \left(\sum_{i=1}^m r_i x_i \right)^N \\
&= \sum_{k_1 + \cdots + k_m = N} \binom{N}{k_1, \dots, k_m} r_1^{k_1} x_1^{k_1} \cdots r_m^{k_m} x_m^{k_m}.
\end{aligned}$$

- (3) Note that for each term there is some j such that $k_j \geq n_j$. Hence,

$$\begin{aligned}
x_j^{k_j} &= x_j^{k_j - n_j} x_j^{n_j} \in J && (J \text{ is an ideal}) \\
\Rightarrow r_1^{k_1} x_1^{k_1} \cdots r_m^{k_m} x_m^{k_m} &\in J \text{ for each term} && (J \text{ is an ideal}) \\
\Rightarrow x^N &\in J. && (J \text{ is an ideal}) \\
\Rightarrow I^N &\subseteq J.
\end{aligned}$$

□

Supplement. (Exercise 1.13 in the textbook: Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*.) Suppose that I is an ideal in a commutative ring. Show that if $\text{rad}(I)$ is finitely generated, then for some integer N we have $(\text{rad}(I))^N \subseteq I$. Conclude that in a Noetherian ring the ideals I and J have the same radical iff there is some integer N such that $I^N \subseteq J$ and $J^N \subseteq I$. Use the Nullstellensatz to deduce that if $I, J \subseteq S = k[x_1, \dots, x_n]$ are ideals and k is algebraically closed, then $Z(I) = Z(J)$ iff $I^N \subseteq J$ and $J^N \subseteq I$ for some N .

Proof.

- (1) Show that if $\text{rad}(I)$ is finitely generated, then for some integer N we have $(\text{rad}(I))^N \subseteq I$. Say $x_1, \dots, x_m \in \text{rad}(I)$ generate $\text{rad}(I)$.

(a) For each i , there exists an integer $n_i > 0$ such that $x_i^{n_i} \in I$ (since $\text{rad}(I)$ is radical).

(b) Let $N = n_1 + \dots + n_m$. Given any $x = \sum_{i=1}^m r_i x_i \in \text{rad}(I)$, so

$$\begin{aligned} x^N &= \left(\sum_{i=1}^m r_i x_i \right)^N \\ &= \sum_{k_1 + \dots + k_m = N} \binom{N}{k_1, \dots, k_m} r_1^{k_1} x_1^{k_1} \dots r_m^{k_m} x_m^{k_m}. \end{aligned}$$

(c) Note that for each term there is some j such that $k_j \geq n_j$. Hence,

$$\begin{aligned} x_j^{k_j} &= x_j^{k_j - n_j} x_j^{n_j} \in I && (I \text{ is an ideal}) \\ \implies r_1^{k_1} x_1^{k_1} \dots r_m^{k_m} x_m^{k_m} &\in I \text{ for each term} && (I \text{ is an ideal}) \\ \implies x^N &\in I. && (I \text{ is an ideal}) \\ \implies (\text{rad}(I))^N &\subseteq I. \end{aligned}$$

- (2) Show that in a Noetherian ring the ideals I and J have the same radical iff there is some integer N such that $I^N \subseteq J$ and $J^N \subseteq I$.

(a) (\implies) Since in a Noetherian ring every ideal is finitely generated, $\text{rad}(I)$ and $\text{rad}(J)$ are finitely generated. By (1), there is a common integer N such that

$$(\text{rad}(I))^N \subseteq I \quad \text{and} \quad (\text{rad}(J))^N \subseteq J.$$

Note that $I^N \subseteq (\text{rad}(I))^N$ and $J^N \subseteq (\text{rad}(J))^N$. Since $\text{rad}(I) = \text{rad}(J)$ by assumption,

$$\begin{aligned} I^N &\subseteq (\text{rad}(I))^N = (\text{rad}(J))^N \subseteq J, \\ J^N &\subseteq (\text{rad}(J))^N = (\text{rad}(I))^N \subseteq I. \end{aligned}$$

- (b) (\Longleftarrow) It suffices to show that $\text{rad}(I) \subseteq \text{rad}(J)$. $\text{rad}(J) \subseteq \text{rad}(I)$ is similar. Given any $x \in \text{rad}(I)$, there is an integer $M > 0$ such that $x^M \in I$. Hence $x^{MN} \in I^N \subseteq J$, or $x \in \text{rad}(J)$.
- (3) Show that if $I, J \subseteq S = k[x_1, \dots, x_n]$ are ideals and k is algebraically closed, then $Z(I) = Z(J)$ iff $I^N \subseteq J$ and $J^N \subseteq I$ for some N . Note that S is Noetherian and we can apply part (2). By the Nullstellensatz, $Z(I) = Z(J)$ iff $\text{rad}(I) = \text{rad}(J)$ iff $I^N \subseteq J$ and $J^N \subseteq I$ for some N .

□

2.9. Ideals with a Finite Number of Zeros

2.10. Quotient Modules and Exact Sequences

Problem 2.51.

Let

$$0 \longrightarrow V_1 \longrightarrow \cdots \longrightarrow V_n \longrightarrow 0$$

be an exact sequence of finite-dimensional vector spaces. Show that $\sum (-1)^i \dim(V_i) = 0$.

Proof (Proposition 7 in this section).

- (1) For $i = 0, \dots, n$, by the rank-nullity theorem for a linear transformation $\varphi_i : V_i \rightarrow V_{i+1}$, we have

$$\dim V_i = \dim \text{im}(\varphi_i) + \dim \ker(\varphi_i).$$

(Here $V_0 = V_{n+1} := 0$ by convention.)

- (2) By the exactness of the sequence, we have

$$(a) \quad \text{im}(\varphi_i) = \ker(\varphi_{i+1}) \text{ for } i = 0, \dots, n-1. \text{ In particular, } \ker(\varphi_1) = \text{im}(\varphi_0) = 0.$$

$$(b) \quad \ker(\varphi_n) = V_n.$$

Hence,

$$\begin{aligned}
\sum_{i=1}^{n-1} (-1)^i \dim(V_i) &= \sum_{i=1}^{n-1} (-1)^i \dim \operatorname{im}(\varphi_i) + \sum_{i=1}^{n-1} (-1)^i \dim \operatorname{ker}(\varphi_i) \\
&= \sum_{i=1}^{n-1} (-1)^i \dim \operatorname{ker}(\varphi_{i+1}) + \sum_{i=1}^{n-1} (-1)^i \dim \operatorname{ker}(\varphi_i) \\
&= (-1)^{n-1} \underbrace{\dim \operatorname{ker}(\varphi_n)}_{=V_n} + (-1)^1 \underbrace{\dim \operatorname{ker}(\varphi_1)}_{=0} \\
&= -(-1)^n \dim V_n,
\end{aligned}$$

or $\sum (-1)^i \dim(V_i) = 0$.

□

2.11. Free Modules

Chapter 3: Local Properties of Plane Curves

3.1. Multiple Points and Tangent Lines

Problem PLACEHOLDER

PLACEHOLDER

Proof.

(1) PLACEHOLDER

□

3.2. Multiplicities and Local Rings

3.3. Intersection Numbers

Chapter 4: Projective Varieties

4.1. Projective Space

Problem PLACEHOLDER

PLACEHOLDER

Proof.

(1) PLACEHOLDER

□

4.2. Projective Algebraic Sets

4.3. Affine and Projective Varieties

4.4. Multiprojective Space

Chapter 5: Projective Plane Curves

5.1. Definitions

Problem PLACEHOLDER

PLACEHOLDER

Proof.

(1) PLACEHOLDER

□

5.2. Linear Systems of Curves

5.3. Bézout's Theorem

5.4. Multiple Points

5.5. Max Noether's Fundamental Theorem

5.6. Applications of Noether's Theorem

Chapter 6: Varieties, Morphisms, and Rational Maps

6.1. The Zariski Topology

6.2. Varieties

6.3. Morphisms of Varieties

6.4. Products and Graphs

6.5. Algebraic Function Fields and Dimension of Varieties

6.6. Rational Maps

Chapter 7: Resolution of Singularities

7.1. Rational Maps of Curves

Problem PLACEHOLDER

PLACEHOLDER

Proof.

(1) PLACEHOLDER

□

7.2. Blowing up a Point in A^2

7.3. Blowing up a Point in P^2

7.4. Quadratic Transformations

7.5. Nonsingular Models of Curves

Chapter 8: Riemann-Roch Theorem

8.1. Divisors

Problem PLACEHOLDER

PLACEHOLDER

Proof.

(1) PLACEHOLDER

□

8.2. The Vector Spaces $L(D)$

8.3. Riemann's Theorem

8.4. Derivations and Differentials

8.5. Canonical Divisors

8.6. Riemann-Roch Theorem