Chapter 10: Integration of Differential Forms

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Exercise 10.1. ...

Proof.

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Exercise 10.2. *For $i = 1, 2, 3, \ldots$, let $\varphi_i \in \mathscr{C}(\mathbb{R}^1)$ have support in $(2^{-i}, 2^{1-i})$, such that $\int \varphi_i = 1$. Put*

$$f(x, y) = \sum_{i=1}^{\infty} [\varphi_i(x) - \varphi_{i+1}(x)] \varphi_i(y)$$

Then f has compact support in $\mathbb{R}^2$, f is continuous except at $(0, 0)$, and

$$\int dy \int f(x, y) dx = 0 \qquad but \qquad \int dx \int f(x, y) dy = 1.$$

Observe that f is unbounded in every neighborhood of $(0, 0)$.

Proof.

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Exercise 10.3. ...

Proof.

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Exercise 10.4. *For* $(x, y) \in \mathbb{R}^2$, *define*

$$\mathbf{F}(x, y) = (e^x \cos y - 1, e^x \sin y)$$

Prove that $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$, *where*

$$\mathbf{G}_1(x, y) = (e^x \cos y - 1, y)$$
$$\mathbf{G}_2(u, v) = (u, (1 + u) \tan v)$$

are primitive in some neighborhood of $(0, 0)$. *Compute the Jacobians of* $\mathbf{G}_1$, $\mathbf{G}_2$, $\mathbf{F}$ *at* $(0, 0)$. *Define*

$$\mathbf{H}_2(x, y) = (x, e^x \sin y)$$

and find

$$\mathbf{H}_1(u, v) = (h(u, v), v)$$

so that $\mathbf{F} = \mathbf{H}_1 \circ \mathbf{H}_2$ *is in some neighborhood of* $(0, 0)$.

Proof.

(1) By Definition 10.5,

$$\mathbf{G}_1(x, y) = (e^x \cos y - 1)\mathbf{e}_1 + y\mathbf{e}_2,$$
$$\mathbf{G}_2(u, v) = u\mathbf{e}_1 + ((1 + u) \tan v)\mathbf{e}_2$$

are primitive in some neighborhood of $(0, 0)$.

(2) *Show that* $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$. Given any $(x, y) \in \mathbb{R}^2$, we have

$$
\begin{aligned}
(\mathbf{G}_2 \circ \mathbf{G}_1)(x, y) &= \mathbf{G}_2(\mathbf{G}_1(x, y)) \\
&= \mathbf{G}_2(e^x \cos y - 1, y) \\
&= (e^x \cos y - 1, (1 + (e^x \cos y - 1)) \tan y) \\
&= (e^x \cos y - 1, e^x \sin y) \\
&= \mathbf{F}(x, y).
\end{aligned}
$$

(3) Since

$$J_{\mathbf{G}_1}(x, y) = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ 0 & 1 \end{bmatrix}$$

$$J_{\mathbf{G}_2}(x, y) = \begin{bmatrix} 1 & 0 \\ \tan y & (1 + x) \sec^2 y \end{bmatrix}$$

$$J_{\mathbf{F}}(x, y) = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix},$$

$$J_{\mathbf{G}_1}(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$J_{\mathbf{G}_2}(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$J_{\mathbf{F}}(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(4) Define $h(u,v) = \sqrt{e^{2u} - v^2} - 1$ on

$$B\left((0,0); \frac{1}{64}\right) \subseteq \mathbb{R}^2.$$

$h(u,v)$ is well-defined since $e^{2u} - v^2 > 0$ for all $(u,v) \in B\left((0,0); \frac{1}{64}\right)$.

(5) Given any $(x,y) \in \mathbb{R}^2$, we have

$$\begin{aligned}
(\mathbf{H}_1 \circ \mathbf{H}_2)(x,y) &= \mathbf{H}_1(\mathbf{H}_2(x,y)) \\
&= \mathbf{H}_1(x, e^x \sin y) \\
&= (\sqrt{e^{2x} - (e^x \sin y)^2} - 1, e^x \sin y) \\
&= (e^x \cos y - 1, e^x \sin y) \\
&= \mathbf{F}(x,y).
\end{aligned}$$

□

Exercise 10.5. ...

Proof.

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Exercise 10.6. ...

Proof.

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Exercise 10.7. ...

Proof.

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Exercise 10.8. *Let H be the parallelogram in $\mathbb{R}^2$ whose vertices are $(1,1)$, $(3,2)$, $(4,5)$, $(2,4)$. Find the affine map T which sends $(0,0)$ to $(1,1)$, $(1,0)$ to $(3,2)$, $(1,1)$ to $(4,5)$, $(0,1)$ to $(2,4)$. Show that $J_T = 5$. Use T to convert the integral*

$$\alpha = \int_H e^{x-y}\,dxdy$$

to an integral over I^2 and thus compute α.

Proof.

(1) By Affine simplexes 10.26,

$$T(\mathbf{x}) = T(\mathbf{0}) + A\mathbf{x},$$

where $A \in L(\mathbb{R}^2, \mathbb{R}^2)$, say $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Note that $T : \begin{bmatrix} 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus

$$T : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 + ax + by \\ 1 + cx + dy \end{bmatrix}.$$

(2) By $T : (1,0) \mapsto (3,2)$ and $T : (0,1) \mapsto (2,4)$, we can solve A as

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

It is easy to verify such

$$T : \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{x}} \mapsto \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{T(\mathbf{0})} + \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{x}} = \begin{bmatrix} 1 + 2x + y \\ 1 + x + 3y \end{bmatrix}$$

satisfying our requirement.

(3)

$$J_T = \det \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = 5.$$

(4) We cannot apply Theorem 10.9 directly as Exercise 10.9. Luckily, we might use the first part of the proof in Theorem 10.9 to show that

$$\int_{\mathbb{R}^2} f(\mathbf{y})d\mathbf{y} = \int_{\mathbb{R}^2} f(T(\mathbf{x}))|J_T(\mathbf{x})|d\mathbf{x}$$

holds for affine maps by Theorem 10.2 again. Hence

$$
\begin{aligned}
\int_H e^{x-y}dxdy &= \int_{[0,1]^2} e^{(1+2u+v)-(1+u+3v)}|J_T|dudv \\
&= 5\int_{[0,1]^2} e^{u-2v}dudv \\
&= 5\left\{\int_0^1 e^u du\right\}\left\{\int_0^1 e^{-2v}dv\right\} \qquad \text{(Theorem 10.2)} \\
&= \frac{5}{2}(e-1)(1-e^{-2}).
\end{aligned}
$$

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Exercise 10.9. ...

Proof.

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Exercise 10.10. ...

Proof.

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Exercise 10.11. ...

Proof.

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Exercise 10.12. ...

Proof.

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Exercise 10.13. ...

Proof.

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Exercise 10.14. ...

Proof.

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Exercise 10.15. *If ω and λ are k- and m-forms, respectively, prove that*

$$\omega \wedge \lambda = (-1)^{km} \lambda \wedge \omega.$$

Proof.

  (1) Write
$$\omega = \sum_I b_I(\mathbf{x})dx_I, \qquad \lambda = \sum_J c_J(\mathbf{x})dx_J$$

    in the stardard presentations, where I and J range over all increasing k-indices and over all increasing m-indices taken from the set $\{1, \ldots, n\}$.

(2) *Show that $dx_I \wedge dx_J = (-1)^{km}dx_J \wedge dx_I$.*

$$\begin{aligned}
dx_I \wedge dx_J &= dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_J \\
&= (-1)^m dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}} \wedge dx_J \wedge dx_{i_k} \\
&= (-1)^{2m} dx_{i_1} \wedge \cdots \wedge dx_{i_{k-2}} \wedge dx_J \wedge dx_{i_{k-1}} \wedge dx_{i_k} \\
&\cdots \\
&= (-1)^{km} dx_J \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&= (-1)^{km} dx_J \wedge dx_I.
\end{aligned}$$

(3)

$$\begin{aligned}
\omega \wedge \lambda &= \sum_{I,J} b_I(\mathbf{x}) c_J(\mathbf{x}) dx_I \wedge dx_J \\
&= (-1)^{km} \sum_{J,I} c_J(\mathbf{x}) b_I(\mathbf{x}) dx_J \wedge dx_I \\
&= (-1)^{km} \lambda \wedge \omega.
\end{aligned}$$

$\square$

Exercise 10.16. ...

Proof.

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$\square$

Exercise 10.17. ...

Proof.

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$\square$

Exercise 10.18. ...

Proof.

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Exercise 10.19. ...

Proof.

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Exercise 10.20. ...

Proof.

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Exercise 10.21. ...

Proof.

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Exercise 10.22. ...

Proof.

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Exercise 10.23. ...

Proof.

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Exercise 10.24. ...

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Exercise 10.25. ...

Proof.

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Exercise 10.26. ...

Proof.

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Exercise 10.27. ...

Proof.

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Exercise 10.28. ...

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Exercise 10.29. ...

Proof.

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Exercise 10.30. ...

Proof.

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Exercise 10.31. ...

Proof.

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Exercise 10.32. ...

Proof.

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