

Chapter 9: Functions of Several Variables

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Exercise 9.1. If S is a nonempty subset of a vector space X , prove (as asserted in Section 9.1) that the span of S is a vector space.

Denote the span of S by $\text{span}(S)$.

Proof.

- (1) Since $S \neq \emptyset$, there is $\mathbf{z} \in S$. So $1\mathbf{z} = \mathbf{z} \in \text{span}(S) \neq \emptyset$. (In fact, $\text{span}(S) \supseteq S$.)
- (2) If $\mathbf{x}, \mathbf{y} \in \text{span}(S)$, then there exist elements $\mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{y}_1, \dots, \mathbf{y}_n \in S$ and scalars $a_1, \dots, a_m, b_1, \dots, b_n$ such that

$$\mathbf{x} = a_1\mathbf{x}_1 + \dots + a_m\mathbf{x}_m,$$

$$\mathbf{y} = b_1\mathbf{y}_1 + \dots + b_n\mathbf{y}_n.$$

Then

$$\mathbf{x} + \mathbf{y} = a_1\mathbf{x}_1 + \dots + a_m\mathbf{x}_m + b_1\mathbf{y}_1 + \dots + b_n\mathbf{y}_n$$

is a linear combination of the elements of S . For any scalar c ,

$$c\mathbf{x} = (ca_1)\mathbf{x}_1 + \dots + (ca_m)\mathbf{x}_m$$

is again linear combination of the elements of S .

- (3) By (1)(2), $\text{span}(S)$ is a vector space.

□

Note. Any subspace of X that contains S must also contain $\text{span}(S)$.

Exercise 9.2. Prove (as asserted in Section 9.6) that BA is linear if A and B are linear transformations. Prove also that A^{-1} is linear and invertible if A is invertible.

Proof. Use the notation in Definitions 9.6.

- (1) Show that BA is linear if A and B are linear transformations. Let X, Y, Z be vector spaces, $A \in L(X, Y)$ and $B \in L(Y, Z)$.

(a) Given any $\mathbf{x}_1, \mathbf{x}_2 \in X$.

$$\begin{aligned}
 (BA)(\mathbf{x}_1 + \mathbf{x}_2) &= B(A(\mathbf{x}_1 + \mathbf{x}_2)) \\
 &= B(A\mathbf{x}_1 + A\mathbf{x}_2) && (A \text{ is a linear transformation}) \\
 &= B(A\mathbf{x}_1) + B(A\mathbf{x}_2) && (B \text{ is a linear transformation}) \\
 &= (BA)\mathbf{x}_1 + (BA)\mathbf{x}_2.
 \end{aligned}$$

(b) For any $\mathbf{x} \in X$ and scalar c ,

$$\begin{aligned}
 (BA)(c\mathbf{x}) &= B(A(c\mathbf{x})) \\
 &= B(cA\mathbf{x}) && (A \text{ is a linear transformation}) \\
 &= cB(A\mathbf{x}) && (B \text{ is a linear transformation}) \\
 &= c(BA)\mathbf{x}.
 \end{aligned}$$

By (a)(b), $BA \in L(X, Z)$.

(2) Show that A^{-1} is linear if A is invertible.

(a) Given any $\mathbf{y}_1, \mathbf{y}_2 \in X$. Since A is surjective, there exist $\mathbf{x}_1, \mathbf{x}_2 \in X$ such that

$$\begin{aligned}
 \mathbf{y}_1 &= A\mathbf{x}_1 \\
 \mathbf{y}_2 &= A\mathbf{x}_2.
 \end{aligned}$$

So

$$\begin{aligned}
 A^{-1}\mathbf{y}_1 &= A^{-1}(A\mathbf{x}_1) = \mathbf{x}_1 \\
 A^{-1}\mathbf{y}_2 &= A^{-1}(A\mathbf{x}_2) = \mathbf{x}_2
 \end{aligned}$$

(by Definitions 9.4). Hence

$$\begin{aligned}
 A^{-1}(\mathbf{y}_1 + \mathbf{y}_2) &= A^{-1}(A\mathbf{x}_1 + A\mathbf{x}_2) \\
 &= A^{-1}(A(\mathbf{x}_1 + \mathbf{x}_2)) && (A \text{ is a linear transformation}) \\
 &= \mathbf{x}_1 + \mathbf{x}_2 && (\text{Definitions 9.4}) \\
 &= A^{-1}\mathbf{y}_1 + A^{-1}\mathbf{y}_2.
 \end{aligned}$$

(b) For any $\mathbf{y} \in X$ and scalar c , there is a corresponding $\mathbf{x} \in X$ such that $\mathbf{y} = A\mathbf{x}$ since A is surjective. So $A^{-1}\mathbf{y} = \mathbf{x}$ by Definition 9.4. Hence

$$\begin{aligned}
 A^{-1}(c\mathbf{y}) &= A^{-1}(cA\mathbf{x}) \\
 &= A^{-1}(A(c\mathbf{x})) && (A \text{ is a linear transformation}) \\
 &= c\mathbf{x} && (\text{Definitions 9.4}) \\
 &= cA^{-1}\mathbf{y}.
 \end{aligned}$$

By (a)(b), $A^{-1} \in L(X)$.

(3) *Show that A^{-1} is invertible if A is invertible.* It suffices to show that A^{-1} is injective and surjective.

(a) *Show that A^{-1} is injective.* Given any $\mathbf{y}_1, \mathbf{y}_2 \in X$. Since A is surjective, there exist $\mathbf{x}_1, \mathbf{x}_2 \in X$ such that

$$\begin{aligned}\mathbf{y}_1 &= A\mathbf{x}_1 \\ \mathbf{y}_2 &= A\mathbf{x}_2.\end{aligned}$$

Suppose $A^{-1}\mathbf{y}_1 = A^{-1}\mathbf{y}_2$. So $A^{-1}(A\mathbf{x}_1) = A^{-1}(A\mathbf{x}_2)$, or $\mathbf{x}_1 = \mathbf{x}_2$, or $\mathbf{y}_1 = A\mathbf{x}_1 = A\mathbf{x}_2 = \mathbf{y}_2$.

(b) *Show that A^{-1} is surjective.* For any $\mathbf{x} \in X$, there exists $A\mathbf{x} \in X$ such that $A^{-1}(A\mathbf{x}) = \mathbf{x}$ by Definitions 9.4.

□

Exercise 9.3. Assume $A \in L(X, Y)$ and $A\mathbf{x} = \mathbf{0}$ only when $\mathbf{x} = \mathbf{0}$. Prove that A is then 1-1.

Proof. Suppose $A\mathbf{x} = A\mathbf{y}$. Since A is a linear transformation, $A(\mathbf{x} - \mathbf{y}) = A\mathbf{x} - A\mathbf{y} = \mathbf{0}$. By assumption, $\mathbf{x} - \mathbf{y} = \mathbf{0}$ or $\mathbf{x} = \mathbf{y}$. □

Exercise 9.4. Prove (as asserted in Section 9.30) that null spaces and ranges of linear transformations are vector spaces.

Proof. Use the notation in Definitions 9.30. Suppose X, Y are vector spaces, and $A \in L(X, Y)$, as in Definition 9.6.

(1) *Show that $\mathcal{N}(A)$ is a vector space in X .*

(a) Note that $\mathbf{0} \in X$. Since $A\mathbf{0} = \mathbf{0}$, $\mathbf{0} \in \mathcal{N}(A) \neq \emptyset$.

(b) Suppose $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{N}(A)$. Then

$$\begin{aligned}A(\mathbf{x}_1 + \mathbf{x}_2) &= A\mathbf{x}_1 + A\mathbf{x}_2 && (A \text{ is a linear transformation}) \\ &= \mathbf{0} + \mathbf{0} && (\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{N}(A)) \\ &= \mathbf{0}.\end{aligned}$$

So $\mathbf{x}_1 + \mathbf{x}_2 \in \mathcal{N}(A)$.

(c) Suppose $\mathbf{x} \in \mathcal{N}(A)$ and c is a scalar. Then

$$\begin{aligned}A(c\mathbf{x}) &= cA\mathbf{x} && (A \text{ is a linear transformation}) \\ &= c\mathbf{0} && (\mathbf{x} \in \mathcal{N}(A)) \\ &= \mathbf{0}.\end{aligned}$$

So $c\mathbf{x} \in \mathcal{N}(A)$.

By (a)(b)(c), $\mathcal{N}(A)$ is a vector space.

(2) Show that $\mathcal{R}(A)$ is a vector space in Y .

(a) Note that $\mathbf{0} \in X$. So $A\mathbf{0} = \mathbf{0} \in \mathcal{R}(A) \neq \emptyset$.

(b) Suppose $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{R}(A)$. Then there exist $\mathbf{x}_1, \mathbf{x}_2 \in X$ such that $A\mathbf{x}_1 = \mathbf{y}_1$ and $A\mathbf{x}_2 = \mathbf{y}_2$. Hence

$$\begin{aligned}\mathbf{y}_1 + \mathbf{y}_2 &= A\mathbf{x}_1 + A\mathbf{x}_2 \\ &= A(\mathbf{x}_1 + \mathbf{x}_2) \quad (A \text{ is a linear transformation}).\end{aligned}$$

So $\mathbf{y}_1 + \mathbf{y}_2 \in \mathcal{R}(A)$.

(c) Suppose $\mathbf{y} \in \mathcal{R}(A)$ and c is a scalar. Then there exists $\mathbf{x} \in X$ such that $A\mathbf{x} = \mathbf{y}$. Hence

$$\begin{aligned}c\mathbf{y} &= cA\mathbf{x} \\ &= A(c\mathbf{x}) \quad (A \text{ is a linear transformation}).\end{aligned}$$

So $c\mathbf{y} \in \mathcal{R}(A)$.

By (a)(b)(c), $\mathcal{R}(A)$ is a vector space.

□