Chapter 2: Linear Transformations and Matrices

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Section 2.4: Invertibility and Isomorphisms

Exercise 2.4.8. *Let A and B be $n \times n$ matrices such that $AB = I_n$. Prove*

(a) A and B are invertible.

(b) $A = B^{-1}$ (and hence $B = A^{-1}$). (We are in effect saying that for square matrices, a "one-sided" inverse is a "two-sided" inverse.)

(c) State and prove analogous results for linear transformations defined on finite-dimensional vector spaces.

Proof of (a). Regard $\mathsf{V} = \mathsf{M}_{n\times n}(F)$ as a finite-dimensional vector space over F. Given $X \in \mathsf{M}_{n\times n}(F)$, consider the subset V_X of V defined by

$$\mathsf{V}_X = \{XY : Y \in \mathsf{M}_{n\times n}(F)\}.$$

(1) $\mathsf{V}_0 = 0$.

(2) $\mathsf{V}_{I_n} = \mathsf{V}$. In general, $\mathsf{V}_X = \mathsf{V}$ for any invertible matrix $X \in \mathsf{M}_{n\times n}(F)$.

(3) V_X is a subspace of V for any $X \in \mathsf{M}_{n\times n}(F)$.

(4) There is a descending sequence of subspaces

$$\mathsf{V} \supseteq \mathsf{V}_X \supseteq \cdots \supseteq \mathsf{V}_{X^k} \supseteq \cdots$$

This sequence must be stationary since V is finite-dimensional, that is,

$$\mathsf{V}_{X^k} = \mathsf{V}_{X^{k+1}} = \cdots$$

for some k. (Descending chain condition.) In particular, $B^k = B^{k+1}C$ for some $C \in \mathsf{V}$. Multiply with A^k on the left to get $I_n = BC$. ($A^k B^k = A^{k-1}(AB)B^{k-1} = A^{k-1}B^{k-1} = \cdots = I_n$.)

(4) Since $AB = I_n$ and $BC = I_n$, $A = AI_n = A(BC) = (AB)C = I_nC = C$, or $AB = BA = I_n$. By definition of invertibility, A and B are invertible.

□

Proof of (b). By (a), $A = B^{-1}$ and $B = A^{-1}$. □

Proof of (c). Let V *be a finite-dimensional vector space, and let* $\mathsf{S}, \mathsf{T} : \mathsf{V} \to \mathsf{V}$ *be linear such that* ST *is invertible. Show that* S *and* T *are invertible.* Let

$$\beta = \{\beta_1, ..., \beta_n\}$$

be an ordered basis for V where $n = \dim(\mathsf{V})$. Let $A = [\mathsf{S}]_\beta$ and $B = [\mathsf{T}]_\beta$. So

$$AB = [\mathsf{S}]_\beta [\mathsf{T}]_\beta = [\mathsf{ST}]_\beta = [\mathsf{I}_\mathsf{V}]_\beta = I_n$$

(Theorem 2.11). By (a), $A = [\mathsf{S}]_\beta$ and $B = [\mathsf{T}]_\beta$ are invertible, or S and T are invertible (Theorem 2.18). $\square$