# Solutions to the book: *do Carmo, Differential Geometry of Curves and Surfaces*

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## Contents

# Chapter 1: Curves

## 1-1. Introduction

Classical differential geometry: the study of local properties of curves and surfaces.

Global differential geometry: the study of the influence of the local properties on the behavior of the entire curve and surface.

*No exercises.*

## 1-2. Parametrized Curves

**Exercise 1-2.1.**

*Find a parametrized curve $\alpha(t)$ whose trace is the circle $x^2 + y^2 = 1$ such that $\alpha(t)$ runs clockwise around the circle with $\alpha(0) = (0, 1)$.*

*Proof.* $\alpha(t) = (\sin t, \cos t)$, $t \in \mathbb{R}$. $\square$

**Exercise 1-2.2.**

*Let $\alpha(t)$ be a parametrized curve which does not pass through the origin. If $\alpha(t_0)$ is the point of the trace of $\alpha$ closest to the origin and $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.*

*Proof.* Let $f(t) = |\alpha(t)|^2 = \alpha(t) \cdot \alpha(t)$. $f(t)$ is differentiable and $f(t)$ has a local minimum at a point $t = t_0 \in I$. So $f'(t_0) = 0$. [Theorem 5.8 in *W. Rudin, Principles of Mathematical Analysis, 3rd edition.*] Since

$$f'(t) = 2\alpha(t) \cdot \alpha'(t),$$

$f'(t_0) = 2\alpha(t_0) \cdot \alpha'(t_0) = 0$, or $\alpha(t_0) \cdot \alpha'(t_0) = 0$. Since $\alpha(t_0) \neq 0$ and $\alpha'(t_0) \neq 0$, $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$. $\square$

**Exercise 1-2.3.**

*A parametrized curve $\alpha(t)$ has a property that its second derivative $\alpha''(t)$ is identically zero. What can be said about $\alpha$?*

*Proof.*

(1) $\alpha(t)$ is a straight line.

(2) Since $\alpha''(t)$ is identically zero, $\alpha'(t) = a$ is a constant. [Theorem 5.11 in *W. Rudin, Principles of Mathematical Analysis, 3rd edition.*] Define $f(t) = \alpha(t) - at$ (on $I$). Since $f'(t) = \alpha'(t) - a = 0$, $f(t) = \alpha(t) - at = b$ is a constant again. Therefore, $\alpha(t) = at + b$, which is a straight line (on $I$).

□

**Exercise 1-2.4.**

*Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve and let $v \in \mathbb{R}^3$ be a fixed vector. Assume that $\alpha'(t)$ is orthogonal to $v$ for all $t \in I$ and that $\alpha(0)$ is orthogonal to $v$. Prove that $\alpha(t)$ is orthogonal to $v$ for all $t \in I$.*

Need to assume that $\alpha(t) \neq 0$ for all $t \in I$.

*Proof.* Given any $t \neq 0 \in I$. (Nothing to do at $t = 0$.) Define $f : I \to \mathbb{R}$ by $f(t) = \alpha(t) \cdot v$. By the mean value theorem, there exists a point $\xi$ between 0 and $t$ such that

$$f(t) - f(0) = f'(\xi)(t - 0),$$

where $f'(t) = \alpha'(t) \cdot v + \alpha(t) \cdot v' = \alpha'(t) \cdot v$. Note that $f(0) = 0$ since $\alpha(0)$ is orthogonal to $v$, and $f'(\xi) = 0$ since $\alpha'(t)$ is orthogonal to $v$. So the identity is reduced to

$$f(t) = 0,$$

or $\alpha(t) \cdot v = 0$, or $\alpha(t)$ is orthogonal to $v$. □

**Exercise 1-2.5.**

*Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve, with $\alpha'(t) \neq 0$ for all $t \in I$. Show that $|\alpha(t)|$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.*

The same trick in Exercise 1-2.2.

*Proof.* It is equivalent to *show that* $|\alpha(t)|^2$ *is a nonzero constant if and only if* $\alpha(t)$ *is orthogonal to* $\alpha'(t)$ *for all* $t \in I$. Let

$$f(t) = |\alpha(t)|^2 = \alpha(t) \cdot \alpha(t).$$

Notice that $\alpha'(t) \neq 0$, and thus

$$\begin{aligned}
&|\alpha(t)| \text{ is a nonzero constant} \\
\Longleftrightarrow\ & f(t) = |\alpha(t)|^2 \text{ is a nonzero constant} \\
\Longleftrightarrow\ & f'(t) = 0 \text{ and } f(t) \text{ is a nonzero constant} \\
\Longleftrightarrow\ & \alpha(t) \cdot \alpha'(t) = 0 \text{ and } \alpha(t) \text{ is a nonzero constant} \\
\Longleftrightarrow\ & \alpha(t) \text{ is orthogonal to } \alpha'(t) \text{ for all } t \in I.
\end{aligned}$$

□

## 1-3. Regular Curves; Arc Length

### Exercise 1-3.1.

*Show that the tangent lines to the regular parametrized curve* $\alpha(t) = (3t, 3t^2, 2t^3)$ *make a constant angle with the line* $y = 0$, $z = x$.

*Proof.* $\alpha'(t) = (3, 6t, 6t^2)$. The line $y = 0$, $z = x$ is $\beta(t) = (1, 0, 1)$. The cosine of the angle $\theta$ between these to curves is

$$\begin{aligned}
\cos\theta &= \frac{(3, 6t, 6t^2) \cdot (1, 0, 1)}{|(3, 6t, 6t^2)||(1, 0, 1)|} \\
&= \frac{3 + 6t^2}{\sqrt{3^2 + (6t)^2 + (6t^2)^2}\sqrt{2}} \\
&= \frac{3 + 6t^2}{\sqrt{9 + 36t^2 + 36t^4}\sqrt{2}} \\
&= \frac{3 + 6t^2}{\sqrt{(3 + 6t^2)^2}\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}.
\end{aligned}$$

(Notice $3 + 6t^2 > 0$ for all $t \in \mathbb{R}$.) That is, the angle between $\alpha'$ and $\beta$ is a constant $(= \pi/4)$. □

**Exercise 1-3.2. (Cycloid)**

*A circular disk of radius 1 in the plane $xy$ rolls without slipping along the $x$ axis. The figure described by a point of the circumference of of the disk is called a **cycloid** (Figure 1-7 in Mantredo P. do Carmo, Differential Geometry of Curves and Surfaces).*

(a) *Obtain a parametrized curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ the trace of which is the cycloid and determine its singular points.*

(b) *Compute the arc length of the cycloid corresponding to a complete rotation of the disk.*

*Proof of (a).*

(1) Since
$$\begin{cases} x = t - \sin t \\ y = 1 - \cos t, \end{cases}$$
we define $\alpha(t) = (t - \sin t, 1 - \cos t)$.

(2) $\alpha'(t) = (1 - \cos t, \sin t)$. $\alpha'(t) = 0$ if and only if $t = 2n\pi$ where $n \in \mathbb{Z}$. That is, all singular points are $\alpha(2n\pi) = (2n\pi, 0)$ where $n \in \mathbb{Z}$.

□

*Proof of (b).* The arc length of the cycloid corresponding to a complete rotation of the disk is
$$\begin{aligned}
\int_0^{2\pi} |\alpha'(t)|dt &= \int_0^{2\pi} \sqrt{(1-\cos t)^2 + (\sin t)^2}dt \\
&= \int_0^{2\pi} \sqrt{2}\sqrt{1-\cos t}dt \\
&= \int_0^{2\pi} 2\sin\frac{t}{2}dt \\
&= \left[-4\cos\frac{t}{2}\right]_{t=0}^{t=2\pi} \\
&= 8.
\end{aligned}$$

□

**Supplement.** The cycloid is not an algebraic curve.

**Exercise 1-3.3. (Cissoid of Diocles)**

*Let $0A = 2a$ be the diameter of a circle $\mathbb{S}^1$ and $0Y$ and $AV$ be the tangents to $\mathbb{S}^1$ at $0$ and $A$, respectively. A half-line $r$ is drawn from $0$ which meets the circle $\mathbb{S}^1$ at $C$ and the line $AV$ at $B$. On $0B$ mark off the segment $0p = CB$. If we rotate $r$ about $0$, the point $p$ will describe a curve called the* ***cissoid of Diocles****. By taking $0A$ as the $x$ axis and $0Y$ as the $y$ axis, prove that*

(a) *The tract of*
$$\alpha(t) = \left(\frac{2at^2}{1+t^2}, \frac{2at^3}{1+t^2}\right), \qquad t \in \mathbb{R},$$
*is the cissoid of Diocles ($t = \tan\theta$; see Figure 1-8 in Mantredo P. do Carmo, Differential Geometry of Curves and Surfaces).*

(b) *The origin $(0,0)$ is a singular point of the cissoid.*

(c) *As $t \to \infty$, $\alpha(t)$ approaches the line $x = 2a$, and $\alpha'(t) \to (0, 2a)$. Thus, as $t \to \infty$, the curve and its tangent approach the line $x = 2a$; we say that $x = 2a$ is an* ***asymptote*** *to the cissoid.*

*Proof of (a).*

(1) The polar equations of the circle $\mathbb{S}^1$ and the half-line $r$ is
$$\begin{aligned} r &= 2a\cos\theta, \\ r &= 2a\sec\theta, \end{aligned}$$
respectively.

(2) By construction, the polar equation of the cissoid is
$$r = 2a\sec\theta - 2a\cos\theta = 2a\frac{\sin^2\theta}{\cos\theta} = 2a\sin\theta\tan\theta.$$

(3) Put $t = \tan\theta$, we have
$$\begin{aligned} x &= r\cos\theta = 2a\sin^2\theta = \frac{2at^2}{1+t^2}, \\ y &= r\sin\theta = tx = \frac{2at^3}{1+t^2}. \end{aligned}$$
So
$$\alpha(t) = (x, y) = \left(\frac{2at^2}{1+t^2}, \frac{2at^3}{1+t^2}\right).$$

□

**Supplement.** The cissoid is an algebraic curve $= V((x^2+y^2)x = 2ay^2)$.

*Proof of (b).* Note that $\alpha(0) = (0,0)$ and

$$\alpha'(t) = \left( \frac{4at}{(t^2+1)^2}, \frac{2at^2(t^2+3)}{(t^2+1)^2} \right).$$

Hence $\alpha'(0) = (0,0)$. That is, $(0,0)$ is a singular point of the cissoid. (In fact, the origin is the unique singular point of the cissoid.) □

*Proof of (c).*

(1) Note that

$$\lim_{t\to\pm\infty} x(t) = \lim_{t\to\pm\infty} \frac{2at^2}{1+t^2} = 2a,$$
$$\lim_{t\to\pm\infty} y(t) = \lim_{t\to\pm\infty} \frac{2at^3}{1+t^2} = \pm\infty.$$

Hence, $\alpha(t)$ approaches the line $x = 2a$ as $t \to \pm\infty$.

(2) Similarly,

$$\lim_{t\to\pm\infty} x'(t) = \lim_{t\to\pm\infty} \frac{4at}{(t^2+1)^2} = 0,$$
$$\lim_{t\to\pm\infty} y'(t) = \lim_{t\to\pm\infty} \frac{2at^2(t^2+3)}{(t^2+1)^2} = 2a.$$

Therefore, $\alpha'(t) \to (0, 2a)$ as $t \to \pm\infty$.

(3) By (1)(2), the curve and its tangent approach the line $x = 2a$ as $t \to \pm\infty$, or $x = 2a$ is an asymptote to the cissoid.

□

**Exercise 1-3.4. (Tractrix)**

*Let $\alpha : (0, \pi) \to \mathbb{R}^2$ be given by*

$$\alpha(t) = \left( \sin t, \cos t + \log\tan\frac{t}{2} \right),$$

*where $t$ is the angle that the $y$ axis makes with the vector $\alpha(t)$. The trace of $\alpha$ is called the **tractrix**. (Figure 1-9 in Mantredo P. do Carmo, Differential Geometry of Curves and Surfaces). Show that*

(a) *$\alpha$ is a differentiable parametrized curve, regular except at $t = \frac{\pi}{2}$.*

(b) *The length of the segment of the tangent of the tractrix between the point of tangency and the $y$ axis is constantly equal to 1.*

*Proof of (a).*

$$\begin{aligned}\alpha'(t) &= \left(\cos t, -\sin t + \frac{1}{\tan\frac{t}{2}}\frac{1}{\cos^2\frac{t}{2}}\frac{1}{2}\right)\\ &= \left(\cos t, -\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right)\\ &= \left(\cos t, \frac{\cos^2 t}{\sin t}\right)\end{aligned}$$

exists. And $\alpha'(t) = 0$ if and only if $t = \frac{\pi}{2}$. That is, there is an unique singular point at $t = \frac{\pi}{2}$. □

*Proof of (b).* The the tangent line of the tractrix through the regular point $t$ is parametrized by $\beta : \mathbb{R} \to \mathbb{R}^2$ which is defined by

$$\begin{aligned}\beta(u) &= u\alpha'(t) + \alpha(t)\\ &= \left(u\cos t + \sin t, u\frac{\cos^2 t}{\sin t} + \cos t + \log\tan\frac{t}{2}\right).\end{aligned}$$

By construction, this tangent line $\beta(u)$ meets the tractrix at $u = 0$, and meets the $y$-axis when $u\cos t + \sin t = 0$ or $u = -\tan t$. So the length of the segment is

$$\begin{aligned}|\beta(0) - \beta(-\tan t)| &= \sqrt{(-\tan t\cos t)^2 + \left(-\tan t\frac{\cos^2 t}{\sin t}\right)^2}\\ &= \sqrt{(\sin t)^2 + (\cos t)^2}\\ &= 1.\end{aligned}$$

□

**Exercise 1-3.5. (Folium of Descartes)**

*Let $\alpha : (-1, +\infty) \to \mathbb{R}^2$ be given by*

$$\alpha(t) = \left(\frac{3at}{1+t^3}, \frac{3at^2}{1+t^3}\right).$$

*Prove that:*

(a) *For $t = 0$, $\alpha$ is tangent to the $x$ axis.*

(b) *As $t \to +\infty$, $\alpha(t) \to (0,0)$ and $\alpha'(t) = (0,0)$.*

(c) *Take the curve the the opposite orientation. Now, as $t \to -1$, the curve and its tangent approach the line $x + y + a = 0$.*

*The figure obtained by completing the trace of $\alpha$ in such a way that it becomes symmetric relative the the line $y = x$ is called the* ***folium of Descartes*** *(See Figure 1-10 in Mantredo P. do Carmo, Differential Geometry of Curves and Surfaces).*

*Proof of (a).* Note that

$$\alpha'(t) = \left( \frac{3a(1-2t^3)}{(1+t^3)^2}, \frac{3at(2-t^3)}{(1+t^3)^2} \right).$$

Hence, $\alpha'(0) = (3a, 0)$, or $\alpha$ is tangent to the $x$ axis when $t = 0$. $\square$

*Proof of (b).*

(1)

$$\begin{aligned}
\lim_{t\to+\infty} \alpha(t) &= \lim_{t\to+\infty} \left( \frac{3at}{1+t^3}, \frac{3at^2}{1+t^3} \right) \\
&= \left( \lim_{t\to+\infty} \frac{3at}{1+t^3}, \lim_{t\to+\infty} \frac{3at^2}{1+t^3} \right) \\
&= (0, 0).
\end{aligned}$$

(2)

$$\begin{aligned}
\lim_{t\to+\infty} \alpha'(t) &= \lim_{t\to+\infty} \left( \frac{3a(1-2t^3)}{(1+t^3)^2}, \frac{3at(2-t^3)}{(1+t^3)^2} \right) \\
&= \left( \lim_{t\to+\infty} \frac{3a(1-2t^3)}{(1+t^3)^2}, \lim_{t\to+\infty} \frac{3at(2-t^3)}{(1+t^3)^2} \right) \\
&= (0, 0).
\end{aligned}$$

$\square$

*Proof of (c).*

(1) Note that

$$\begin{aligned}
\lim_{t\to-1^+} \alpha(t) &= \lim_{t\to-1^+} \left( \frac{3at}{1+t^3}, \frac{3at^2}{1+t^3} \right) \\
&= \left( \lim_{t\to-1^+} \frac{3at}{1+t^3}, \lim_{t\to-1^+} \frac{3at^2}{1+t^3} \right) \\
&= (-\infty, +\infty)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{t\to-1^+} (x(t) + y(t)) &= \lim_{t\to-1^+} \left( \frac{3at}{1+t^3} + \frac{3at^2}{1+t^3} \right) \\
&= \lim_{t\to-1^+} \frac{3at}{1-t+t^2} \\
&= -a.
\end{aligned}$$

Therefore, as $t \to -1$, the curve approaches the line $x + y + a = 0$.

(2) Note that

$$\begin{aligned}
\lim_{t \to -1^+} \frac{y'(t)}{x'(t)} &= \lim_{t \to -1^+} \frac{\frac{3a(1-2t^3)}{(1+t^3)^2}}{\frac{3at(2-t^3)}{(1+t^3)^2}} \\
&= \lim_{t \to -1^+} \frac{1 - 2t^3}{t(2 - t^3)} \\
&= -1.
\end{aligned}$$

Hence, as $t \to -1$, its tangent also approaches the line $x + y + a = 0$.

□

**Exercise 1-3.6. (Logarithmic spiral)**

*Let $\alpha(t) = (ae^{bt} \cos t, ae^{bt} \sin t)$, $t \in \mathbb{R}$, $a$ and $b$ constants, $a > 0$, $b < 0$, be a parametrized curve.*

(a) *Show that as $t \to +\infty$, $\alpha(t)$ approaches the origin 0, spiraling around it (because of this, the trace of $\alpha$ is called the **logarithmic spiral**; See Figure 1-11 in Mantredo P. do Carmo, Differential Geometry of Curves and Surfaces).*

(b) *Show that $\alpha'(t) \to (0, 0)$ as $t \to +\infty$ and that*

$$\lim_{t \to +\infty} \int_{t_0}^{t} |\alpha'(t)| dt$$

*is finite; that is, $\alpha$ has finite arc length in $[t_0, \infty)$.*

*Proof of (a).*

(1) Note that

$$\lim_{t \to +\infty} x(t) = \lim_{t \to +\infty} \frac{\overbrace{a \cos t}^{\text{bounded}}}{\underbrace{e^{-bt}}_{\to +\infty}} = 0$$

and $\lim_{t \to +\infty} y(t) = 0$ (by the similar argument). Hence $\alpha(t)$ approaches the origin 0 as $t \to +\infty$.

(2) $\alpha(t) = (ae^{bt} \cos t, ae^{bt} \sin t)$ is moving in counter-clockwise on a circle path and sweeping out a length $ae^{bt}$ as $t$ is moving from $t_0$ to $+\infty$. Note that $t \mapsto ae^{bt}$ is decreasing strictly (as $t$ is moving from $t_0$ to $+\infty$). Hence $\alpha$ spiraling around the origin.

□

*Proof of (b).*

(1) Note that

$$\alpha'(t) = (ae^{bt}(\underbrace{b\cos t - \sin t}_{\text{bounded}}), ae^{bt}(\underbrace{b\sin t + \cos t}_{\text{bounded}})).$$

As $t \to +\infty$, $\alpha'(t) \to (0,0)$.

(2) As

$$\begin{aligned}\int_{t_0}^{+\infty} |\alpha'(t)|dt &= \int_{t_0}^{+\infty} ae^{bt}\sqrt{b^2+1}dt \\ &= \left[\frac{a}{b}e^{bt}\sqrt{b^2+1}\right]_{t=t_0}^{t=+\infty} \\ &= -\frac{a}{b}e^{bt_0}\sqrt{b^2+1} \\ &< +\infty,\end{aligned}$$

$\alpha$ has finite arc length in $[t_0, \infty)$.

□

**Exercise 1-3.7.**

*A map $\alpha : I \to \mathbb{R}^3$ is called **a curve of class $\mathcal{C}^k$** if each of the coordinate functions in the expression $\alpha(t) = (x(t), y(t), z(t))$ has continuous derivatives up to order $k$. If $\alpha$ is merely continuous, we say that $\alpha$ is of class $\mathcal{C}^0$. A curve $\alpha$ is called **simple** is the map $\alpha$ is one-to-one. Thus, the curve $\alpha(t) = (t^3 - 4t, t^2 - 4)$ ($t \in \mathbb{R}$) is not simple.*

*Let $\alpha : I \to \mathbb{R}^3$ be a simple curve of class $\mathcal{C}^0$. We say that $\alpha$ has a **weak tangent** at $t = t_0 \in I$ if the line determined by $\alpha(t_0 + h)$ and $\alpha(t_0)$ has a limit position when $h \to 0$. We say that $\alpha$ has a **strong tangent** at $t = t_0 \in I$ if the line determined by $\alpha(t_0 + h)$ and $\alpha(t_0 + k)$ has a limit position when $h, k \to 0$. Show that*

(a) *$\alpha(t) = (t^3, t^2)$, $t \in \mathbb{R}$, has a weak tangent but not a strong tangent at $t = 0$.*

(b) *If $\alpha : I \to \mathbb{R}^3$ is of class $\mathcal{C}^1$ and regular at $t = t_0$, then it has a strong tangent at $t = t_0$.*

(c) *The curve given by*

$$\alpha(t) = \begin{cases} (t^2, t^2), & t \geq 0, \\ (t^2, -t^2), & t \leq 0, \end{cases}$$

*is of class $\mathcal{C}^1$ but not of class $\mathcal{C}^2$. Draw a sketch of the curve and its tangent vectors.*

*Proof of (a).*

(1) Note that $\alpha(0) = (0,0)$ and $\alpha(h) = (h^3, h^2)$. The line passing $\alpha(0)$ and $\alpha(h)$ is

$$\begin{aligned}&(x-0)(h^2-0)-(y-0)(h^3-0)=0\\ \Longleftrightarrow\ & x-hy=0.\end{aligned}$$

As $h \to 0$, the line has a limit position $x = 0$. Therefore, $\alpha(t)$ has a weak tangent.

(2) The line passing $\alpha(h)$ and $\alpha(k)$ is

$$\begin{aligned}&(x-k^2)(h^2-k^2)-(y-k^3)(h^3-k^3)=0\\ \Longleftrightarrow\ &(x-k^2)(h+k)-(y-k^3)(h^2+hk+k^2)=0.\end{aligned}$$

As $h \to 0$, the line has a limit position

$$\begin{aligned}&(x-k^2)-(y-k^3)k=0\\ \Longleftrightarrow\ & x-ky+k^4-k^2=0.\end{aligned}$$

As $k \to 0$, the line has a limit position $x = 0$.

(3) On the other hand, as $h = -k$ we have $y - k^3 = 0$. As $k \to 0$, the line has a limit position $y = 0$, contrary to (2). Therefore, $\alpha(t)$ has a strong tangent.

$\square$

*Proof of (b).*

(1) The line $L$ passing $\alpha(t_0 + h)$ and $\alpha(t_0 + k)$ is

$$\begin{aligned}x(s)&=x(t_0)+\frac{x(t_0+h)-x(t_0+k)}{h-k}s,\\ y(s)&=y(t_0)+\frac{y(t_0+h)-y(t_0+k)}{h-k}s,\\ z(s)&=z(t_0)+\frac{z(t_0+h)-z(t_0+k)}{h-k}s.\end{aligned}$$

(2) By the mean value theorem,

$$\frac{x(t_0+h)-x(t_0+k)}{h-k}=x'(t_0+\xi)$$

for some $\xi$ between $h$ and $k$. Since $\alpha \in \mathcal{C}^1$, $x(t) \in \mathcal{C}^1$. Hence

$$\begin{aligned}
\lim_{h,k\to 0} \frac{x(t_0+h)-x(t_0+k)}{h-k} &= \lim_{h,k\to 0} x'(t_0+\xi) \\
&= \lim_{\xi\to 0} x'(t_0+\xi) \\
&= x'(t_0).
\end{aligned}$$

Similarly, we have $\lim_{h,k\to 0} \frac{y(t_0+h)-y(t_0+k)}{h-k} = y'(t_0)$ and $\lim_{h,k\to 0} \frac{z(t_0+h)-z(t_0+k)}{h-k} = z'(t_0)$. Since $\alpha$ is regular, $\lim_{h,k\to 0} L$ is a non degenerate line

$$\begin{aligned}
x(s) &= x(t_0) + x'(t_0)s, \\
y(s) &= y(t_0) + y'(t_0)s, \\
z(s) &= z(t_0) + z'(t_0)s
\end{aligned}$$

and thus $\lim_{h,k\to 0} L$ is a strong tangent at $t = t_0$.

□

*Proof of (c).*

(1) Since

$$\alpha'(t) = \begin{cases} (2t, 2t), & t \geq 0, \\ (2t, -2t), & t \leq 0, \end{cases}$$

$\alpha$ is of class $\mathcal{C}^1$.

(2) Since

$$\alpha''(t) = \begin{cases} (2, 2), & t > 0, \\ \text{undefined}, & t = 0 \\ (2, -2), & t < 0, \end{cases}$$

$\alpha$ is not of class $\mathcal{C}^2$.

(Skip drawing a sketch of the curve and its tangent vectors.) □

**Exercise 1-3.8.**

*Let $\alpha : I \to \mathbb{R}^3$ be a differentiable curve and let $[a, b] \subseteq I$ be a closed interval. For every partition*

$$a = t_0 < t_1 < \cdots < t_n = b$$

*of $[a, b]$, consider the sum*

$$\sum_{i=1}^{n} |\alpha(t_i) - \alpha(t_{i-1})| = l(\alpha, P),$$

*where $P$ stands for the given partition. The norm $|P|$ of a partition $P$ is defined as*

$$|P| = \max(t_i - t_{i-1}), i = 1, \cdots, n.$$

*Geometrically, $l(\alpha, P)$ is the length of a polygon inscribed in $\alpha([a, b])$ with vertices in $\alpha(t_i)$ (see Figure 1-3 in Mantredo P. do Carmo, Differential Geometry of Curves and Surfaces). The point of the exercise is to show that the arc length of $\alpha([a, b])$ is, in some sense, a limit of lengths of inscribed polygons. Prove that given $\varepsilon > 0$ there exists $\delta > 0$ such that if $|P| < \delta$ then*

$$\left| \int_a^b |\alpha'(t)| dt - l(\alpha, P) \right| < \varepsilon.$$

Assume that $\alpha'(t)$ is continuous.

*Proof.* Given $\varepsilon > 0$.

(1) Since $\alpha'(t)$ is continuous on a compact set $[a, b]$, $\alpha'(t)$ is uniformly continuous, that is, there there exists $\delta > 0$ such that

$$|\alpha'(s) - \alpha'(t)| < \frac{\varepsilon}{2(b-a)} \text{ whenever } |s - t| < \delta.$$

(2) Let $P = \{a = t_0, t_1, \ldots, t_n = b\}$ be a partition of $[a, b]$, with $\Delta t_i = t_i - t_{i-1} < \delta$ for all $i = 1, \ldots, n$. If $t_{i-1} \leq t \leq t_i$, it follows that

$$|\alpha'(t_i)| - \frac{\varepsilon}{2(b-a)} \leq |\alpha'(t)| \leq |\alpha'(t_i)| + \frac{\varepsilon}{2(b-a)}.$$

Hence,

$$\begin{aligned}
&\int_{t_{i-1}}^{t_i} |\alpha'(t)| dt \\
&\geq |\alpha'(t_i)| \Delta t_i - \frac{\varepsilon}{2(b-a)} \Delta t_i \\
&= \left| \int_{t_{i-1}}^{t_i} [\alpha'(t) + \alpha'(t_i) - \alpha'(t)] dt \right| - \frac{\varepsilon}{2(b-a)} \Delta t_i \\
&\geq \left| \int_{t_{i-1}}^{t_i} \alpha'(t) dt \right| - \left| \int_{t_{i-1}}^{t_i} [\alpha'(t_i) - \alpha'(t)] dt \right| - \frac{\varepsilon}{2(b-a)} \Delta t_i \\
&\geq |\alpha(t_i) - \alpha(t_{i-1})| - \frac{\varepsilon}{b-a} \Delta t_i
\end{aligned}$$

and

$$
\begin{aligned}
&\int_{t_{i-1}}^{t_i} |\alpha'(t)|dt \\
&\leq |\alpha'(t_i)|\Delta t_i + \frac{\varepsilon}{2(b-a)}\Delta t_i \\
&= \left| \int_{t_{i-1}}^{t_i} [\alpha'(t) + \alpha'(t_i) - \alpha'(t)]dt \right| + \frac{\varepsilon}{2(b-a)}\Delta t_i \\
&\leq \left| \int_{t_{i-1}}^{t_i} \alpha'(t)dt \right| + \left| \int_{t_{i-1}}^{t_i} [\alpha'(t_i) - \alpha'(t)]dt \right| + \frac{\varepsilon}{2(b-a)}\Delta t_i \\
&\leq |\alpha(t_i) - \alpha(t_{i-1})| + \frac{\varepsilon}{b-a}\Delta t_i.
\end{aligned}
$$

(3) If we add these inequalities, we obtain

$$
l(\alpha, P) - \varepsilon \leq \int_a^b |\alpha'(t)|dt \leq l(\alpha, P) + \varepsilon.
$$

□

**Exercise 1-3.9.**

(a) *Let $\alpha : I \to \mathbb{R}^3$ be a curve of class $\mathcal{C}^0$ (compare Exercise 1-3.7). Use the approximation by polygons described in Exercise 1-3.8 to give a reasonable defintion of arc length of $\alpha$.*

(b) *(A Nonrectifiable Curve.) The following example shows that, with any reasonable definition, the arc length of a $\mathcal{C}^0$ curve in a closed interval may be unbounded. Let $\alpha : [0,1] \to \mathbb{R}^2$ be given as $\alpha(t) = (t, t\sin(\frac{\pi}{t}))$ if $t \neq 0$, and $\alpha(0) = (0,0)$. Show, geometrically, that the arc length of the portion of the curve corresponding to $\frac{1}{n+1} \leq t \leq \frac{1}{n}$ is at least $\frac{2}{n+\frac{1}{2}}$. Use this to show that the length of curve in the interval $\frac{1}{N} \leq t \leq 1$ is greater than $2\sum_{n=1}^{N-1} \frac{1}{n+1}$, and thus it tends to infinity as $N \to \infty$.*

*Proof of (a).* Define

$$
l(\alpha) = \sup\{l(\alpha, P) : P \text{ is a partition of } [a,b]\}.
$$

□

*Note.* (Theorem 6.17 in *Tom. M. Apostol, Mathematical Analysis, 2nd edition.*). $\alpha$ is rectifiable if and only $\alpha$ is of bounded variation on $[a,b]$.

*Proof of (b).*

(1) Consider a partition $P = \left\{\frac{1}{n+1}, \frac{1}{n+\frac{1}{2}}, \frac{1}{n}\right\}$ of $\left[\frac{1}{n+1}, \frac{1}{n}\right]$. So that $\alpha(\frac{1}{n+1}) = \alpha(\frac{1}{n}) = 0$ and $\alpha(\frac{1}{n+\frac{1}{2}}) = \pm 1$.

(2) Thus,

$$\begin{aligned}
&\text{The arc length of the portion of } \alpha \text{ over } \left[\frac{1}{n+1}, \frac{1}{n}\right] \\
\geq\ &\text{The sum of each length of the individual chords} \\
=\ &\sqrt{\left(\frac{1}{n+\frac{1}{2}} - \frac{1}{n+1}\right)^2 + \left(\frac{1}{n+\frac{1}{2}}\right)^2} \\
&+ \sqrt{\left(\frac{1}{n} - \frac{1}{n+\frac{1}{2}}\right)^2 + \left(\frac{1}{n+\frac{1}{2}}\right)^2} \\
\geq\ &\frac{2}{n+\frac{1}{2}}.
\end{aligned}$$

(3) So

$$\begin{aligned}
&\text{The arc length of } \alpha \text{ over } \left[\frac{1}{N}, 1\right] \\
=\ &\sum_{n=1}^{N-1} \left\{\text{The arc length of } \alpha \text{ over } \left[\frac{1}{n+1}, \frac{1}{n}\right]\right\} \\
\geq\ &\sum_{n=1}^{N-1} \frac{2}{n+\frac{1}{2}} \\
>\ &2\sum_{n=1}^{N-1} \frac{1}{n+1}.
\end{aligned}$$

It tends to infinity as $N \to \infty$, or $\alpha$ is nonrectifiable.

□

**Exercise 1-3.10. (Straight Lines as Shortest)**

*Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve. Let $[a, b] \subseteq I$ and set $\alpha(a) = p$, $\alpha(b) = q$.*

(a) *Show that, for any constant vector $v$, $|v| = 1$,*

$$(q - p) \cdot v = \int_a^b \alpha'(t) \cdot v dt \leq \int_a^b |\alpha'(t)| dt.$$

(b) *Set*

$$v = \frac{q-p}{|q-p|}$$

*and show that*

$$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)|dt;$$

*that is, the curve of shortest length from $\alpha(a)$ to $\alpha(b)$ is the straight line joining these points.*

Assume $p \neq q$ (otherwise $v = \frac{q-p}{|q-p|}$ is meaningless).

*Proof of (a).* Let $f(t) = \alpha(t) \cdot v$ defined on $I$. By the fundamental theorem of calculus,

$$\int_a^b f'(t)dt = f(b) - f(a).$$

Since $f'(t) = \alpha'(t) \cdot v$,

$$(\alpha(b) - \alpha(a)) \cdot v = \int_a^b \alpha'(t) \cdot v dt.$$

Therefore,

$$\begin{aligned}
(q-p) \cdot v &= \int_a^b \alpha'(t) \cdot v dt \\
&\leq \int_a^b |\alpha'(t) \cdot v| dt \\
&\leq \int_a^b |\alpha'(t)||v| dt \\
&= \int_a^b |\alpha'(t)| dt.
\end{aligned}$$

□

*Proof of (b).* $|v| = \frac{|q-p|}{|q-p|} = 1$. So,

$$\begin{aligned}
(q-p) \cdot \frac{q-p}{|q-p|} &\leq \int_a^b |\alpha'(t)| dt, \\
|q-p| &\leq \int_a^b |\alpha'(t)| dt.
\end{aligned}$$

□

## 1-4. The Vector Product in $\mathbb{R}^3$

**Exercise 1-4.1.**

*Check whether the following bases are positive:*

(a) *The basis* $\{(1,3),(4,2)\}$ *in* $\mathbb{R}^2$.

(b) *The basis* $\{(1,3,5),(2,3,7),(4,8,3)\}$ *in* $\mathbb{R}^3$.

*Proof of (a).* Write $u=(1,3)$ and $v=(4,2)$. Then

$$\det(u,v)=\begin{vmatrix}1 & 3\\ 4 & 2\end{vmatrix}=-10<0.$$

Thus $\{u,v\}$ is negative w.r.t. the natural order basis $\{e_1=(1,0),e_2=(0,1)\}$. □

*Proof of (b).* Write $u=(1,3,5)$, $v=(2,3,7)$, $w=(4,8,3)$. Then

$$\det(u,v,w)=\begin{vmatrix}1 & 3 & 5\\ 2 & 3 & 7\\ 4 & 8 & 3\end{vmatrix}=39>0.$$

Thus $\{u,v,w\}$ is positive w.r.t. the natural order basis $\{e_1,e_2,e_3\}$. □

**Exercise 1-4.2.**

*A plane $P$ contained in $\mathbb{R}^3$ is given by the equation $ax+by+cz+d=0$. Show that the vector $v=(a,b,c)$ is perpendicular to the plane and that $|d|/\sqrt{a^2+b^2+c^2}$ measures the distance from the plane to the origin* $(0,0,0)$.

Say $v$ is a normal vector of $E$.

In general, the distance from the plane $E$ to any point $(x_0,y_0,z_0)\in\mathbb{R}^3$ is

$$\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}.$$

*Proof.*

(1) To show $v=(a,b,c)$ is perpendicular to the plane, it suffices to show that $v\cdot u=0$ for any vector $u$ lying on the plane $E$. Write $u=\overrightarrow{PQ}$ where $P=(x_1,y_1,z_1)\in E$ and $Q=(x_2,y_2,z_2)\in E$. Hence $u=(x_2-x_1,y_2-$

$y_1, z_2 - z_1)$.

$$\begin{aligned} v \cdot u &= (a, b, c) \cdot (x_2 - x_1, y_2 - y_1, z_2 - z_1) \\ &= a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) \\ &= (ax_2 + by_2 + cz_2) - (ax_1 + by_1 + cz_1) \\ &= (-d) - (-d) \\ &= 0. \end{aligned}$$

(2) Pick any point $(x_1, y_1, z_1) \in E$. The distance from the plane $E$ to the point $(x_0, y_0, z_0)$ is

$$\begin{aligned} &\left|(x_1 - x_0, y_1 - y_0, z_1 - z_0) \cdot \frac{v}{|v|}\right| \\ =& \left|(x_1 - x_0, y_1 - y_0, z_1 - z_0) \cdot \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}}\right| \\ =& \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ =& \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ =& \frac{|-d - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ =& \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}. \end{aligned}$$

□

**Exercise 1-4.3.**

*Determine the angle of intersection of the two planes* $5x + 3y + 2z - 4 = 0$ *and* $3x + 4y - 7z = 0$.

*Proof.*

(1) The angle of intersection of the two planes is equal to a angle between two normal vectors of planes.

(2) Let

  (a) the angle of intersection of the two planes be $\theta$.
  (b) the normal vector of $5x + 3y + 2z - 4 = 0$ be $n_1 = (5, 3, 2)$.
  (c) the normal vector of $3x + 4y - 7z = 0$ be $n_2 = (3, 4, -7)$.

(3) Hence,

$$\cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{13}{2\sqrt{703}}.$$

$\theta = \cos^{-1}\left(\frac{13}{2\sqrt{703}}\right)$.

□

**Exercise 1-4.4.**

*Given two planes $a_i x + b_i y + c_i z + d_i = 0$, $i = 1, 2$, prove that a necessary and sufficient condition for them to be parallel is*

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2},$$

*where the convention is made that if a denominator is zero, the corresponding numerator is also zero (we say that two planes are parallel if they either coincide or do not intersect).*

*Proof.*

(1) Write

$$E_i : a_i x + b_i y + c_i z + d_i = 0.$$

By Exercise 1-4.2, the vector $(a_i, b_i, c_i)$ is perpendicular to the plane $E_i$.

(2) Hence,

$$\begin{aligned} E_1 \text{ is parallel to } E_2 &\Longleftrightarrow (a_1, b_1, c_1) \text{ is parallel to } (a_2, b_2, c_2) \\ &\Longleftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}. \end{aligned}$$

(where the convention is made that if a denominator is zero, the corresponding numerator is also zero).

□

**Exercise 1-4.5.**

*Show that the equation of a plane passing through three noncolinear points $p_i = (x_i, y_i, z_i)$, $i = 1, 2, 3$ is given by*

$$(p - p_1) \wedge (p - p_2) \cdot (p - p_3) = 0.$$

*where $p = (x, y, z)$ is an arbitrary point of the plane and $p - p_1$, for instance, means the vector $(x - x_1, y - y_1, z - z_1)$.*

*Proof.*

(1) By Exercise 1-4.11(a), the volume $V$ of a parallelepiped generated by $p - p_1, p - p_2, p - p_3 \in \mathbb{R}^3$ is given by

$$V = (p - p_1) \wedge (p - p_2) \cdot (p - p_3).$$

(2) Since all vectors $p - p_1, p - p_2, p - p_3$ are lying on the same plane, $V = 0$. Therefore, the equation of a plane is $(p - p_1) \wedge (p - p_2) \cdot (p - p_3) = 0$.

□

**Exercise 1-4.6.**

*Given two nonparallel planes $a_i x + b_i y + c_i z + d_i = 0$, $i = 1, 2$, show that their line of intersection may be parametrized as*

$$x - x_0 = u_1 t, \qquad y - y_0 = u_2 t, \qquad z - z_0 = u_3 t,$$

*where $(x_0, y_0, z_0)$ belongs to the intersection and $u = (u_1, u_2, u_3)$ is the vector product $u = v_1 \wedge v_2$, $v_i = (a_i, b_i, c_i)$, $i = 1, 2$.*

*Proof.*

(1) Suppose that the line of intersection is

$$L : x - x_0 = u_1' t, \qquad y - y_0 = u_2' t, \qquad z - z_0 = u_3' t$$

where $(x_0, y_0, z_0)$ belongs to the intersection.

(2) By Exercise 1-4.2, the vector $v_i = (a_i, b_i, c_i)$ is perpendicular to the plane $E_i : a_i x + b_i y + c_i z + d_i = 0$. Hence $v_i$ is perpendicular to $u' = (u_1', u_2', u_3')$. Since $v_i \cdot (v_1 \wedge v_2) = 0$, we may choose $u' = v_1 \wedge v_2 = u$.

□

**Exercise 1-4.7.**

*Prove that a necessary and sufficient condition for the plane*

$$ax + by + cz + d = 0$$

*and the line*

$$x - x_0 = u_1 t, \qquad y - y_0 = u_2 t, \qquad z - z_0 = u_3 t$$

*to be parallel is*

$$au_1 + bu_2 + cu_3 = 0.$$

*Proof.* Write

$$E : ax + by + cz + d = 0$$
$$L : x - x_0 = u_1 t, \qquad y - y_0 = u_2 t, \qquad z - z_0 = u_3 t.$$

By Exercise 1-4.2, the vector $(a, b, c)$ is perpendicular to the plane $E$. Hence,

$$\begin{aligned} E \text{ is parallel to } L &\Longleftrightarrow (a,b,c) \text{ is perpendicular to } L \\ &\Longleftrightarrow (a,b,c) \text{ is perpendicular to } u = (u_1, u_2, u_3) \\ &\Longleftrightarrow 0 = (a,b,c) \cdot (u_1, u_2, u_3) = au_1 + bu_2 + cu_3. \end{aligned}$$

□

**Exercise 1-4.8.**

*Prove that the distance $\rho$ between the nonparallel lines*

$$x - x_0 = u_1 t, \qquad y - y_0 = u_2 t, \qquad z - z_0 = u_3 t,$$
$$x - x_0 = v_1 t, \qquad y - y_0 = v_2 t, \qquad z - z_0 = v_3 t$$

*is given by*

$$\rho = \frac{|(u \wedge v) \cdot r|}{|u \wedge v|}$$

*where $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$, $r = (x_0 - x_1, y_0 - y_1, z_0 - z_1)$.*

*Proof.*

$$\rho = |r||\cos \angle(u \wedge v, r)| = |r| \frac{|(u \wedge v) \cdot r|}{|u \wedge v||r|} = \frac{|(u \wedge v) \cdot r|}{|u \wedge v|}.$$

It is well-defined ($|u \wedge v| > 0$) since two lines are nonparallel. □

**Exercise 1-4.9.**

*Determine the angle of intersection of the plane $3x + 4y + 7z + 8 = 0$ and the line $x - 2 = 3t$, $y - 3 = 5t$, $z - 5 = 9t$.*

*Proof.*

(1) The angle of intersection is equal to $\pi/2$ minus an acute angle $\theta$ between the normal vector of the plane and the direction vector of a line.

(2)
$$\cos\theta = \frac{(3,4,7) \cdot (3,5,9)}{|(3,4,7)||(3,5,9)|} = \frac{92}{\sqrt{74}\sqrt{115}}.$$

Hence, the angle of intersection is

$$\pi/2 - \arccos\left(\frac{92}{\sqrt{74}\sqrt{115}}\right).$$

□

**Exercise 1-4.10. (Oriented area)**

*The natural orientation of $\mathbb{R}^2$ makes it possible to associate a sign to the area $A$ of a parallelogram generated by two linearly independent vectors $u, v \in \mathbb{R}^2$, and write $u = u_1e_1 + u_2e_2$, $v = v_1e_1 + v_2e_2$. Observe the matrix relation*

$$\begin{bmatrix} u \cdot u & u \cdot v \\ v \cdot u & v \cdot v \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$$

*and conclude that*

$$A^2 = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2.$$

*Since the last determinant has the same sign as the basis $\{u, v\}$, we can say that $A$ is positive or negative according to whether the orientation of $\{u, v\}$ is positive or negative. This is called the **oriented area** in $\mathbb{R}^2$.*

*Proof.*

$$\begin{aligned} A^2 &= \begin{vmatrix} u \cdot u & u \cdot v \\ v \cdot u & v \cdot v \end{vmatrix} \\ &= \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \\ &= \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2 \end{aligned}$$

since $\det A^T = \det A$. □

**Exercise 1-4.11. (Oriented volume)**

(a) *Show that the volume $V$ of a parallelepiped generated by three linearly independent vectors $u, v, w \in \mathbb{R}^3$ is given by $V = |(u \wedge v) \cdot w|$, and introduce an **oriented volume** in $\mathbb{R}^3$.*

(b) *Prove that*

$$V^2 = \begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix}.$$

*Proof of (a).*

(1) We can calculate the volume $V$ by multiplying the area of the base $|u \wedge v|$ and the height $h$.

(2) Note that

$$h = |w||\cos \angle(u \wedge v, w)| = |w| \frac{|(u \wedge v) \cdot w|}{|u \wedge v||w|} = \frac{|(u \wedge v) \cdot w|}{|u \wedge v|}.$$

It is well-defined since $u$ and $v$ are linearly independent. Therefore,

$$V = |u \wedge v| h = |(u \wedge v) \cdot w|.$$

(3) The oriented volume is defined by

$$(u \wedge v) \cdot w.$$

□

*Proof of (b).* Recall $(u \wedge v) \cdot w = \det(u, v, w)$. Also note that $\det\big((u, v, w)^T\big) = \det(u, v, w)$. So

$$\begin{aligned}
\begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix} &= \det\big((u, v, w) \cdot (u, v, w)^T\big) \\
&= \det(u, v, w) \det\big((u, v, w)^T\big) \\
&= \det(u, v, w)^2 \\
&= ((u \wedge v) \cdot w)^2 \\
&= V^2.
\end{aligned}$$

□

**Exercise 1-4.12.**

*Given the vectors $v \neq 0$ and $w$, show that there exists a vector $u$ such that $u \wedge v = w$ if and only if $v$ is perpendicular to $w$. Is this vector $u$ uniquely determined? If not, what is the most general solution?*

*Proof.*

(1) Suppose that there exists a vector $u$ such that $u \wedge v = w$. So

$$v \cdot w = w \cdot v = (u \wedge v) \cdot v = 0$$

or $v$ is perpendicular to $w$.

(2) Suppose that $v$ is perpendicular to $w$. Linear algebra says that there exists a vector $u$ such that $\{\widetilde{u}, v, w\}$ is a basis of $\mathbb{R}^3$. Note that $\widetilde{u} \wedge v$ is parallel to $w$, say $\widetilde{u} \wedge v = cw$ for some nonzero constant $c \in \mathbb{R}$. Take $u = c^{-1}\widetilde{u}$ and thus
$$u \wedge v = c^{-1}\widetilde{u} \wedge v = c^{-1}cw = w.$$
(Note that $\{u, v, w\}$ is also a basis of $\mathbb{R}^3$.)

(3) Such vector $u$ is not uniquely determined. Let $L$ be a line passing the point $u = (u_1, u_2, u_3)$ and parallel to the vector $v$. By the definition of vector product, for any point $p = (p_1, p_2, p_3) \in L$ we have $p \wedge v = w$ as a vector.

□

**Exercise 1-4.13.**

*Let $u(t) = (u_1(t), u_2(t), u_3(t))$ and $v(t) = (v_1(t), v_2(t), v_3(t))$ be differentiable maps from the interval $(a, b)$ into $\mathbb{R}^3$. If the derivatives $u'(t)$ and $v'(t)$ satisfy the conditions*
$$u'(t) = au(t) + bv(t), \qquad v'(t) = cu(t) - av(t),$$
*where $a$, $b$, and $c$ are constants, show that $u(t) \wedge v(t)$ is a constant vector.*

*Proof.* Since
$$\begin{aligned}
\frac{d}{dt}(u(t) \wedge v(t)) &= u'(t) \wedge v(t) + u(t) \wedge v'(t) \\
&= (au(t) + bv(t)) \wedge v(t) + u(t) \wedge (cu(t) - av(t)) \\
&= au(t) \wedge v(t) + u(t) \wedge (-av(t)) \\
&= a(u(t) \wedge v(t)) + (-a)(u(t) \wedge v(t)) \\
&= (0, 0, 0),
\end{aligned}$$
$u(t) \wedge v(t)$ is a constant vector. □

**Exercise 1-4.14.**

*Find all unit vectors which are perpendicular to the vector $(2, 2, 1)$ and parallel to the plane determined by the points $(0, 0, 0)$, $(1, -2, 1)$, $(-1, 1, 1)$.*

*Proof.*

(1) Let $E$ be the plane determined by the points $p_0 = (0, 0, 0)$, $p_1 = (1, -2, 1)$, $p_2(-1, 1, 1)$. The normal vector of $E$ is

$$(p_1 - p_0) \wedge (p_2 - p_0) = (1, -2, 1) \wedge (-1, 1, 1) = (-3, -2, -1).$$

(2) All unit vectors which are perpendicular to $(2, 2, 1)$ and parallel to $E$ are all unit vectors which are perpendicular to $(2, 2, 1)$ and $(-3, -2, -1)$. Note that such vector is in the direction of

$$(2, 2, 1) \wedge (-3, -2, -1) = (0, -1, 2).$$

Hence, the desired unit vectors are

$$\pm \left(0, -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right).$$

□

## 1-5. The Local Theory of Curves Parametrized by Arc Length

*Unless explicity stated, $\alpha : I \to \mathbb{R}^3$ is a curve parametrized by arc length $s$, with curvature $\kappa(s) \neq 0$, for all $s \in I$.*

**Exercise 1-5.1.**

*Given the parametrized curve (helix)*

$$\alpha(s) = \left(a \cos \frac{s}{c}, a \sin \frac{s}{c}, b\frac{s}{c}\right), \qquad s \in \mathbb{R},$$

*where $c^2 = a^2 + b^2$.*

(a) *Show that the parameter $s$ is the arc length.*

(b) *Determine the curvature and the torsion of $\alpha$.*

(c) *Determine the osculating plane of $\alpha$.*

(d) *Show that the lines containing $n(s)$ and passing through $\alpha(s)$ meet the $z$ axis under a constant angle equal to $\frac{\pi}{2}$.*

(e) *Show that the tangent lines to $\alpha$ make a constant angle with the $z$ axis.*

*Proof of (a).* Since
$$\alpha'(s) = \left(-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}\right)$$
or $|\alpha'(s)| = 1$,
$$s(t) = \int_0^t |\alpha'(u)|du = \int_0^t du = t$$
is indeed the arc length. $\square$

*Proof of (b).*

(1) Note that
$$\alpha''(s) = \left(-\frac{a}{c^2}\cos\frac{s}{c}, -\frac{a}{c^2}\sin\frac{s}{c}, 0\right).$$
So the curvature of $\alpha$ is
$$\kappa(s) = |\alpha''(s)| = \frac{|a|}{c^2}.$$

(2) Note that
$$n(s) = \frac{1}{\kappa(s)}\alpha''(s) = -\mathrm{sgn}(a)\left(\cos\frac{s}{c}, \sin\frac{s}{c}, 0\right).$$
Hence,
$$\begin{aligned} b(s) &= t(s) \wedge n(s) = \mathrm{sgn}(a)\left(\frac{b}{c}\sin\frac{s}{c}, \frac{b}{c}\cos\frac{s}{c}, \frac{a}{c}\right) \\ \Longrightarrow b'(s) &= \mathrm{sgn}(a)\left(\frac{b}{c^2}\cos\frac{s}{c}, -\frac{b}{c^2}\sin\frac{s}{c}, 0\right) \\ \Longrightarrow \tau(s) &= |b'(s)| = \frac{|b|}{c^2}. \end{aligned}$$

$\square$

*Proof of (c).* Since the binormal vector $b(s)$ is normal to the osculating plane, the osculating plane is
$$\left(\frac{b}{c}\sin\frac{s}{c}\right)x + \left(\frac{b}{c}\cos\frac{s}{c}\right)y + \left(\frac{a}{c}\right)z = \frac{ab}{c}\sin\frac{2s}{c} + \frac{ab}{c^2}s$$
for $s \in I$. $\square$

*Proof of (d).*

(1) The line $L$ containing $n(s)$ and passing through $\alpha(s)$ is

$$\begin{aligned} x &= a\cos\frac{s}{c} + \cos\frac{s}{c}t, \\ y &= a\sin\frac{s}{c} + \sin\frac{s}{c}t, \\ z &= b\frac{s}{c}, \end{aligned}$$

for all $t \in \mathbb{R}$. Hence $L$ meets the $z$ axis at $t = -a$.

(2) The directional vector of $L$ (resp. the $z$ axis) is $\left(\cos\frac{s}{c}, \sin\frac{s}{c}, 0\right)$ (resp. $(0,0,1)$). Since the inner product

$$\left(\cos\frac{s}{c}, \sin\frac{s}{c}, 0\right) \cdot (0,0,1) = 0,$$

$L$ meets the $z$ axis under a constant angle $= \frac{\pi}{2}$.

□

*Proof of (e).* Note that the directional vector of the tangent line (resp. the $z$ axis) is $t(s)$ (resp. $(0,0,1)$). Since the inner product of $t(s)$ and $(0,0,1)$ is a constant $\frac{b}{c}$, the conclusion holds. □

**Exercise 1-5.2.**

*Show that the torsion $\tau$ of $\alpha$ is given by*

$$\tau(s) = -\frac{\alpha'(s) \wedge \alpha''(s) \cdot \alpha'''(s)}{|\kappa(s)|^2}.$$

*Proof.*

(1) Take inner product $n(s)$ to the definition of torsion $\tau(s)n(s) = b'(s)$, we have

$$\tau(s) = b'(s) \cdot n(s).$$

Since $b'(s) = t(s) \wedge n'(s)$, we have to compute $n'(s)$ first.

(2) Compute $n'(s)$.

$$n'(s) = \frac{d}{ds}\left(\frac{\alpha''(s)}{\kappa(s)}\right) = \frac{\alpha'''(s)}{\kappa(s)} - \frac{\alpha''(s)\kappa'(s)}{\kappa(s)^2}.$$

(3) By (1)(2),

$$\begin{aligned}
\tau(s) &= b'(s) \cdot n(s) \\
&= (t(s) \wedge n'(s)) \cdot n(s) \\
&= \left(\alpha'(s) \wedge \left(\frac{\alpha'''(s)}{\kappa(s)} - \frac{\alpha''(s)\kappa'(s)}{\kappa(s)^2}\right)\right) \cdot \frac{\alpha''(s)}{\kappa(s)} \\
&= \left(\alpha'(s) \wedge \frac{\alpha'''(s)}{\kappa(s)}\right) \cdot \frac{\alpha''(s)}{\kappa(s)} \\
&= \frac{\alpha'(s) \wedge \alpha'''(s) \cdot \alpha''(s)}{|\kappa(s)|^2},
\end{aligned}$$

or

$$\tau(s) = \frac{\alpha'(s) \wedge \alpha'''(s) \cdot \alpha''(s)}{\alpha''(s)^2}.$$

□

## 1-6. The Local Canonical Form

## 1-7. Global Properties of Plane Curves