Solutions to the book:
Fulton, Algebraic Curves

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Contents

Chapter 1: Affine Algebraic Sets

1.1. Algebraic Preliminaries

Problem 1.1.*

Let R be a domain.

(a) *If f, g are forms of degree r, s respectively in $R[x_1, \ldots, x_n]$, show that fg is a form of degree $r + s$.*

(b) *Show that any factor of a form in $R[x_1, \ldots, x_n]$ is also a form.*

Proof of (a).

(1) Write

$$f = \sum_{(i)} a_{(i)} x^{(i)},$$
$$g = \sum_{(j)} b_{(j)} x^{(j)},$$

where $\sum_{(i)}$ is the summation over $(i) = (i_1, \ldots, i_n)$ with $i_1 + \cdots + i_n = r$ and $\sum_{(j)}$ is the summation over $(j) = (j_1, \ldots, j_n)$ with $j_1 + \cdots + j_n = s$.

(2) Hence,

$$\begin{aligned} fg &= \sum_{(i)} \sum_{(j)} a_{(i)} b_{(j)} x^{(i)} x^{(j)} \\ &= \sum_{(i),(j)} a_{(i)} b_{(j)} x^{(k)} \end{aligned}$$

where $(k) = (i_1 + j_1, \ldots, i_n + j_n)$ with $(i_1 + j_1) + \cdots + (i_n + j_n) = r + s$. Each $x^{(k)}$ is the form of degree $r + s$ and $a_{(i)} b_{(j)} \in R$. Hence fg is a form of degree $r + s$.

$\square$

Proof of (b).

(1) Given any form $f \in R[x_1, \ldots, x_n]$, and write $f = gh$. *It suffices to show that g is a form as well.* (So does h.)

(2) Write

$$g = g_0 + \cdots + g_r, \qquad h = h_0 + \cdots + h_s$$

where $g_r \neq 0$ and $h_s \neq 0$. So

$$f = gh = g_0h_0 + \cdots + g_rh_s.$$

Since R is a domain, $R[x_1, \ldots, x_n]$ is a domain and thus $g_rh_s \neq 0$. The maximality of r and s implies that $\deg f = r + s$. Therefore, by the maximality of $r + s$, $f = g_rh_s$, or $g = g_r$, or g is a form.

□

Problem 1.5.*

Let k be any field. Show that there are an infinitely number of irreducible monic polynomials in $k[x]$. (Hint: Suppose $f_1, \ldots, f_n$ were all of them, and factor $f_1 \cdots f_n + 1$ into irreducible factors.)

Proof (Due to Euclid).

(1) If $f_1, \ldots, f_n$ were all irreducible monic polynomials, then we consider

$$g = f_1 \cdots f_n + 1 \in k[x].$$

So there is an irreducible monic polynomial $f = f_i$ dividing g for some i since

$$\deg g = \deg f_1 + \cdots + \deg f_n \geq 1.$$

(2) However, f would divide the difference

$$g - f_1 \cdots f_{i-1}f_if_{i+1} \cdots f_n = 1,$$

contrary to $\deg f_i \geq 1$.

□

Problem 1.6.*

Show that any algebraically closed field is infinite. (Hint: The irreducible monic polynomials are $x - a$, $a \in k$.)

Proof (Due to Euclid).

(1) Let k be an algebraically closed field. If $a_1, \ldots, a_n$ were all elements in k, then we consider a monic polynomials

$$f(x) = (x - a_1) \cdots (x - a_n) + 1 \in k[x].$$

(2) Since k is algebraically closed, there is an element $a \in k$ such that $f(a) = 0$. By assumption, $a = a_i$ for some $1 \leq i \leq n$, and thus $f(a) = f(a_i) = 1$, contrary to the fact that a field is a commutative ring where $0 \neq 1$ and all nonzero elements are invertible.

□

1.2. Affine Space and Algebraic Sets

Problem 1.8.*

Show that the algebraic subsets of $\mathbf{A}^1(k)$ are just the finite subsets, together with $\mathbf{A}^1(k)$ itself.

Proof.

(1) *Show that $k[x]$ is a PID if k is a field.*

(a) Let I be an ideal of $k[x]$.

(b) If $I = \{0\}$ then $I = (0)$ and I is principal.

(c) If $I \neq \{0\}$, then take f to be a polynomial of minimal degree in I. It suffices to show that $I = (f)$. Clearly, $(f) \subseteq I$ since I is an ideal. Conversely, for any $g \in I$,

$$g(x) = f(x)h(x) + r(x)$$

for some $h, r \in k[x]$ with $r = 0$ or $\deg r < \deg f$. Now as

$$r = g - fh \in I,$$

$r = 0$ (otherwise contrary to the minimality of f), we have $g = fh \in (f)$ for all $g \in I$.

(2) Let Y be an algebraic subset of $\mathbf{A}^1(k)$, say $Y = V(I)$ for some ideal I of $k[x]$. Since $k[x]$ is a PID, $I = (f)$ for some $f \in k[x]$.

(a) If $f = 0$, then $I = (0)$ and $Y = V(0) = \mathbf{A}^1(k)$.

(b) If $f \neq 0$, then $f(x) = 0$ has finitely many roots in k, say $a_1, \ldots, a_m \in k$. Hence,

$$Y = V(I) = V(f) = \{f(a) = 0 : a \in k\} = \{a_1, \ldots, a_m\}$$

is a finite subsets of $\mathbf{A}^1(k)$.

By (a)(b), the result is established.

□

Notes.

(1) By the Hilbert basis theorem, $k[x]$ is Noetherian as k is Noetherian. Hence, for any algebraic subset $Y = V(I)$ of $\mathbf{A}^1(k)$, we can write $I = (f_1, \cdots, f_m)$. Note that

$$Y = V(I) = V(f_1) \cap \cdots \cap V(f_m).$$

Now apply the same argument to get the same conclusion.

(2) Suppose $k = \overline{k}$. $\mathbf{A}^1(k)$ is irreducible, because its only proper closed subsets are finite, yet it is infinite (because k is algebraically closed, hence infinite).

Problem 1.11.

Show that the following are algebraic sets:

(a) $\{(t, t^2, t^3) \in \mathbf{A}^3(k) : t \in k\}$;

(b) $\{(\cos(t), \sin(t)) \in \mathbf{A}^2(\mathbb{R}) : t \in \mathbb{R}\}$;

(c) *the set of points in* $\mathbf{A}^2(\mathbb{R})$ *whose polar coordinates* (r, θ) *satisfy the equation* $r = \sin(\theta)$.

Proof of (a).

(1) The twisted cubic curve

$$Y = \{(t, t^2, t^3) \in \mathbf{A}^3(k) : t \in k\} = V(x^2 - y) \cap V(x^3 - z)$$

is algebraic. We say that Y is given by the parametric representation $x = t$, $y = t^2$, $z = t^3$.

(2) The generators for the ideal $I(Y)$ are $x^2 - y$ and $x^3 - z$.

(3) Y is an affine variety of dimension 1.

(4) The affine coordinate ring $A(Y)$ is isomorphic to a polynomial ring in one variable over k.

□

Proof of (b). The circle

$$\{(\cos(t), \sin(t)) \in \mathbf{A}^2(\mathbb{R}) : t \in \mathbb{R}\} = V(x^2 - y^2 - 1)$$

is algebraic. □

Proof of (c). The circle

$$\{(r,\theta) : r = \sin(\theta)\} = V(x^2 + y^2 - y)$$

is algebraic again. $\square$

Problem 1.15.*

Let $V \subseteq \mathbf{A}^n(k)$, $W \subseteq \mathbf{A}^m(k)$ be algebraic sets. Show that

$$V \times W = \{(a_1, \dots, a_n, b_1, \dots, b_m) : (a_1, \dots, a_n) \in V, (b_1, \dots, b_m) \in W\}$$

*is an algebraic set in $\mathbf{A}^{n+m}(k)$. It is called the **product** of V and W.*

Proof.

(1) Write

$$\begin{aligned} V = V(S_V) &= \{a \in \mathbf{A}^n(k) : f(a) = 0 \; \forall f \in S_V\} \\ W = V(S_W) &= \{b \in \mathbf{A}^m(k) : g(b) = 0 \; \forall g \in S_W\}, \end{aligned}$$

where $S_V \subseteq k[x_1, \dots, x_n]$ and $S_W \subseteq k[y_1, \dots, y_m]$. It suffices to show that

$$V \times W = V(S),$$

where $S \subseteq k[x_1, \dots, x_n, y_1, \dots, y_m]$ is the union of S_V and S_W.

(2) Here we can regard S_V as a subset of $k[x_1, \dots, x_n, y_1, \dots, y_m]$ since

$$k[x_1, \dots, x_n] \hookrightarrow (k[y_1, \dots, y_m])[x_1, \dots, x_n] = k[x_1, \dots, x_n, y_1, \dots, y_m].$$

(Similar treatment to S_W.)

(3) By construction, $V \times W \subseteq V(S)$. Conversely, given any $(a, b) \in V(S)$, we have $h(a, b) = 0$ for all $h \in S = S_V \cup S_W$ (by (2)). By construction, $f(a) = 0$ for all $f \in S_V$ since f only involve $x_1, \dots, x_n$. Hence, $a \in V$. Similarly, $b \in W$. Therefore, $(a, b) \in V \times W$.

$\square$

Chapter TODO

Section TODO

Problem TODO

TODO

Proof.

(1) TODO