

Chapter 11: The Lebesgue Theory

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Exercise 11.1. If $f \geq 0$ and $\int_E f d\mu = 0$, prove that $f(x) = 0$ almost everywhere on E . (Hint: Let E_n be the subset of E on which $f(x) > \frac{1}{n}$. Write $A = \bigcup E_n$. Then $\mu(A) = 0$ if and only if $\mu(E_n) = 0$ for every n .)

Might assume that f is measurable on E .

Proof (Hint).

- (1) Define $A = \{x \in E : f(x) > 0\}$. So $f(x) = 0$ almost everywhere on E if and only if $\mu(A) = 0$.
- (2) Define

$$E_n = \left\{x \in E : f(x) > \frac{1}{n}\right\}$$

for $n = 1, 2, 3, \dots$. Note that $E_1 \subseteq E_2 \subseteq E_3 \subseteq \dots$ and

$$A = \bigcup_{n=1}^{\infty} E_n.$$

Since μ is a measure,

$$\lim_{n \rightarrow \infty} \mu(E_n) = \mu(A)$$

(Theorem 11.3).

- (3) (Reductio ad absurdum) If $\mu(A) > 0$, there is an integer N such that $\mu(E_n) \geq \frac{\mu(A)}{2}$ whenever $n \geq N$ (by (2)). In particular, take $n = N$ to get

$$\begin{aligned} \int_E f d\mu &\geq \int_{E_N} f d\mu && (\mu \text{ is a measure and } E_N \subseteq E) \\ &\geq \frac{1}{N} \cdot \mu(E_N) && (\text{Remarks 11.23(b)}) \\ &\geq \frac{1}{N} \cdot \frac{\mu(A)}{2} \\ &> 0, \end{aligned}$$

contrary to the assumption that $\int_E f d\mu = 0$.

□

Note. Compare to Exercise 6.2.

Exercise 11.2. *If $\int_A f d\mu = 0$ for every measurable subset A of a measurable set E , then $f(x) = 0$ almost everywhere on E .*

Might assume that f is measurable on E .

Proof.

- (1) Define

$$A = \{x \in E : f(x) \geq 0\} \quad \text{and} \quad B = \{x \in E : f(x) \leq 0\}.$$

A and B are measurable subsets of a measurable set E since f is measurable.

- (2) Apply Exercise 11.1 to the fact that $f \geq 0$ on A (by construction) and $\int_A f d\mu = 0$ (by assumption), we have $f(x) = 0$ almost everywhere on A .
- (3) Similarly, apply Exercise 11.1 to the fact that $-f \geq 0$ on B and $\int_B (-f) d\mu = -\int_B f d\mu = 0$, we have $f(x) = 0$ almost everywhere on B .
- (4) As $E = A \cup B$, $f(x) = 0$ almost everywhere on E by (2)(3).

□

Exercise 11.3. *If $\{f_n\}$ is a sequence of measurable functions, prove that the set of points x at which $\{f_n(x)\}$ converges is measurable.*

Proof.

- (1) It suffices to show that

$$E = \{x : \{f_n(x)\} \text{ is convergent}\} = \{x : \{f_n(x)\} \text{ is Cauchy}\}$$

is measurable (since \mathbb{R}^1 is complete).

- (2) Write

$$E = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n,m \geq N} \left\{ x : |f_n(x) - f_m(x)| \leq \frac{1}{k} \right\}$$

Since $\{f_n\}$ is a sequence of measurable functions, $x \mapsto |f_n(x) - f_m(x)|$ is measurable (Theorem 11.16 and Theorem 11.18). Hence

$$\left\{ x : |f_n(x) - f_m(x)| \leq \frac{1}{k} \right\}$$

is measurable (Theorem 11.15). Therefore E is measurable.

□

Exercise 11.4. If $f \in \mathcal{L}(\mu)$ on E and g is bounded and measurable on E , then $fg \in \mathcal{L}(\mu)$ on E .

Proof (Theorem 11.27).

- (1) fg is measurable since both f and g are measurable (Theorem 11.18).
- (2) $|g| \leq M$ for some real $M \in \mathbb{R}^1$ by the boundedness of g . Hence

$$|fg| \leq M|f|$$

on E .

- (3) To apply Theorem 11.27, it suffices to show that $M|f| \in \mathcal{L}(\mu)$ on E . Theorem 11.26 implies that $|f| \in \mathcal{L}(\mu)$ if $f \in \mathcal{L}(\mu)$. And Remarks 11.23(d) implies that $M|f| \in \mathcal{L}(\mu)$ if $|f| \in \mathcal{L}(\mu)$.

□

Note (Riemann integral). If $f \in \mathcal{R}$ on $[a, b]$ and g is bounded and measurable on $[a, b]$, then fg might be not Riemann integrable.

Exercise 11.5. Put

$$g(x) = \begin{cases} 0 & (0 \leq x \leq \frac{1}{2}), \\ 1 & (\frac{1}{2} < x \leq 1), \end{cases}$$

and

$$\begin{aligned} f_{2k}(x) &= g(x) & (0 \leq x \leq 1), \\ f_{2k+1}(x) &= g(1-x) & (0 \leq x \leq 1). \end{aligned}$$

Show that

$$\liminf_{n \rightarrow \infty} f_n(x) = 0 \quad (0 \leq x \leq 1),$$

but

$$\int_0^1 f_n(x) dx = \frac{1}{2}.$$

(Compare with the Fatou's theorem.)

Proof.

- (1) Show that $\liminf_{n \rightarrow \infty} f_n(x) = 0$. Note that

$$g(1-x) = \begin{cases} 1 & (0 \leq x < \frac{1}{2}), \\ 0 & (\frac{1}{2} < x \leq 1). \end{cases}$$

Since $f_n(x) \geq 0$ by definition, $\liminf_{n \rightarrow \infty} f_n(x) \geq 0$. Since $f_{2k}(0) = f_{2k+1}(1) = 0$ for all positive integers k , $\liminf_{n \rightarrow \infty} f_n(x) \leq 0$. Therefore the result is established.

(2) Show that $\int_0^1 f_n(x) dx = \frac{1}{2}$. Since

$$\begin{aligned}\int_0^1 f_{2k}(x) dx &= \int_0^1 g(x) dx = \frac{1}{2}, \\ \int_0^1 f_{2k+1}(x) dx &= \int_0^1 g(1-x) dx = \frac{1}{2},\end{aligned}$$

in any case $\int_0^1 f_n(x) dx = \frac{1}{2}$ for all positive integers n .

(3) This example shows that we may have the strict inequality in the Fatou's theorem.

□

Supplement (Similar exercise). Consider the sequence $\{f_n\}$ defined by $f_n(x) = 1$ if $n \leq x < n+1$, with $f_n(x) = 0$ otherwise. Show that we may have the strict inequality in the Fatou's theorem.

Exercise 11.6. Let

$$f_n(x) = \begin{cases} \frac{1}{n} & (|x| \leq n), \\ 0 & (|x| > n). \end{cases}$$

Then $f_n(x) \rightarrow 0$ uniformly on \mathbb{R}^1 , but

$$\int_{-\infty}^{\infty} f_n(x) dx = 2 \quad (n = 1, 2, 3, \dots).$$

(We write $\int_{-\infty}^{\infty}$ in place of $\int_{\mathbb{R}^1}$.) Thus uniform convergence does not imply dominated convergence in the sense of Theorem 11.32. However, on sets of finite measure, uniformly convergent sequences of bounded functions do satisfy Theorem 11.32.

Proof.

(1) Show that $f_n(x) \rightarrow 0$ uniformly on \mathbb{R}^1 . Given any $\varepsilon > 0$, there is an integer $N > \frac{1}{\varepsilon}$ such that

$$|f_n(x) - 0| \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

whenever $n \geq N$ and $x \in \mathbb{R}^1$. Hence $f_n(x) \rightarrow 0$ uniformly.

(2) Show that $\int_{-\infty}^{\infty} f_n(x)dx = 2$.

$$\int_{-\infty}^{\infty} f_n(x)dx = \int_{-n}^n \frac{1}{n}dx = 2.$$

(3) By (1)(2),

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n(x)dx \neq \int_{-\infty}^{\infty} \lim_{n \rightarrow \infty} f_n(x)dx$$

suggests that the Lebesgue's dominated convergence theorem (Theorem 11.32) does not hold in this case. In fact, if there were $g \in \mathcal{L}$ such that $|f_n(x)| \leq g(x)$, then

$$\begin{aligned} \int_{-\infty}^{\infty} g(x)dx &\geq \int_0^{\infty} g(x)dx && \text{(Theorem 11.24)} \\ &= \sum_{n=1}^{\infty} \int_{n-1}^n g(x)dx && \text{(Theorem 11.24)} \\ &\geq \sum_{n=1}^{\infty} \int_{n-1}^n |f_n(x)|dx \\ &= \sum_{n=1}^{\infty} \int_{n-1}^n \frac{1}{n}dx \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \\ &= \infty, \end{aligned}$$

which is absurd.

(4) Show that on sets of finite measure, uniformly convergent sequences of bounded functions $\{f_n\}$ do satisfy Theorem 11.32.

(a) Since $\{f_n\}$ is uniformly convergent, $\{f_n\}$ is uniformly bounded (Exercise 7.1), or there exists a real number M such that

$$|f_n(x)| \leq M$$

for all positive integer n and $x \in E$.

(b) Define $g(x) = M$ on E . It is clear that

$$\int_E g(x)dx = M\mu(E) < +\infty.$$

Now we can apply the Lebesgue's dominated convergence theorem (Theorem 11.32) to get

$$\lim_{n \rightarrow \infty} \int_E f_n d\mu = \int_E \lim_{n \rightarrow \infty} f_n d\mu.$$

□

Exercise 11.7. ...

Proof.

(1)

(2)

□

Exercise 11.8. If $f \in \mathcal{R}$ on $[a, b]$ and if $F(x) = \int_a^x f(t)dt$, prove that $F'(x) = f(x)$ almost everywhere on $[a, b]$.

Proof.

(1) Theorem 6.20 implies that $F'(x_0) = f(x_0)$ if f is continuous at $x_0 \in [a, b]$.

(2) Since $f \in \mathcal{R}$ on $[a, b]$, f is bounded on $[a, b]$. Theorem 11.33 implies that f is continuous almost everywhere on $[a, b]$.

By (1)(2), $F'(x) = f(x)$ almost everywhere on $[a, b]$. □

Exercise 11.9. ...

Proof.

(1)

(2)

□

Exercise 11.10. If $\mu(X) < +\infty$ and $f \in \mathcal{L}^2(\mu)$ on X , prove that $f \in \mathcal{L}$ on X . If

$$\mu(X) = +\infty,$$

this is false. For instance, if

$$f(x) = \frac{1}{1 + |x|},$$

then $f^2 \in \mathcal{L}$ on \mathbb{R}^1 , but $f \notin \mathcal{L}$ on \mathbb{R}^1 .

Proof.

- (1) Since $\mu(X) < +\infty$, $1 \in \mathcal{L}^2(\mu)$ on X . By Theorem 11.35, $f \in \mathcal{L}(\mu)$, and

$$\int_X |f| d\mu \leq \|f\| \|1\|.$$

- (2) Show that $f^2 \in \mathcal{L}$ on \mathbb{R}^1 . To apply Theorem 11.33, we might restrict the measure space $X = \mathbb{R}^1$ to some interval $[a, b]$. Then apply the Lebesgue's monotone convergence theorem (Theorem 11.28) to get the conclusion.

- (a) Write

$$f(x)^2 = \left(\frac{1}{1+|x|} \right)^2 = \frac{1}{1+2|x|+x^2} \leq \frac{1}{1+x^2}.$$

By Theorem 11.27, it suffices to show that $\frac{1}{1+x^2} \in \mathcal{L}$ on \mathbb{R}^1 .

- (b) Consider the sequence $\{f_n\}$ defined by

$$f_n(x) = \frac{1}{1+x^2} \chi_{[-n,n]}(x).$$

(Here $\chi_{[-n,n]} = K_{[-n,n]}$ is the characteristic function of $[-n, n]$ defined in Definition 11.19.) By construction,

$$0 \leq f_1(x) \leq f_2(x) \leq \cdots \quad (x \in \mathbb{R}^1)$$

and

$$f_n(x) \rightarrow \frac{1}{1+x^2} \quad (x \in \mathbb{R}^1).$$

- (c) Hence

$$\begin{aligned} \int_{\mathbb{R}^1} \frac{1}{1+x^2} dx &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^1} f_n(x) dx && \text{(Theorem 11.28)} \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^1} \frac{1}{1+x^2} \chi_{[-n,n]}(x) dx \\ &= \lim_{n \rightarrow \infty} \int_{-n}^n \frac{1}{1+x^2} dx \\ &= \lim_{n \rightarrow \infty} \mathcal{R} \int_{-n}^n \frac{1}{1+x^2} dx && \text{(Theorem 11.33)} \\ &= \lim_{n \rightarrow \infty} 2 \arctan(n) \\ &= \pi < \infty. \end{aligned}$$

- (4) Show that $f \notin \mathcal{L}$ on \mathbb{R}^1 .

- (a) Consider the sequence $\{f_n\}$ defined by

$$f_n(x) = f(x) \chi_{[-n,n]}(x) = \frac{1}{1+|x|} \chi_{[-n,n]}(x).$$

By construction,

$$0 \leq f_1(x) \leq f_2(x) \leq \cdots \quad (x \in \mathbb{R}^1)$$

and

$$f_n(x) \rightarrow f(x) \quad (x \in \mathbb{R}^1).$$

(b) Hence

$$\begin{aligned} \int_{\mathbb{R}^1} f(x) dx &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^1} f_n(x) dx && \text{(Theorem 11.28)} \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^1} \frac{1}{1+|x|} \chi_{[-n,n]}(x) dx \\ &= \lim_{n \rightarrow \infty} \int_{-n}^n \frac{1}{1+|x|} dx \\ &= \lim_{n \rightarrow \infty} \mathcal{R} \int_{-n}^n \frac{1}{1+|x|} dx && \text{(Theorem 11.33)} \\ &= \lim_{n \rightarrow \infty} 2 \log(n+1) \\ &= \infty, \end{aligned}$$

or $f \notin \mathcal{L}$ on \mathbb{R}^1 .

□

Note. Compare to Exercise 6.5.

Exercise 11.11. ...

Proof.

(1)

(2)

□

Exercise 11.12. ...

Proof.

(1)

(2)

□

Exercise 11.13. ...

Proof.

(1)

(2)

□

Exercise 11.14. ...

Proof.

(1)

(2)

□

Exercise 11.15. ...

Proof.

(1)

(2)

□

Exercise 11.16. ...

Proof.

(1)

(2)

□

Exercise 11.17. ...

Proof.

(1)

(2)

□

Exercise 11.18. ...

Proof.

(1)

(2)

□