Chapter 1: Set Theory

Author: Meng-Gen Tsai
Email: plover@gmail.com

Problem 1.1. *Show that* $\{x : x \neq x\} = \varnothing$.

Proof. Every element x of $\{x : x \neq x\}$ satisfying $x \neq x$, contrary to $x = x$. That is, there are no elements in $\{x : x \neq x\}$, or $\{x : x \neq x\} = \varnothing$. $\square$

Problem 1.2. *Show that if* $x \in \varnothing$, *then* x *is a green-eyed lion.*

Proof. $\varnothing \subseteq \{\text{a green-eyed lion}\}$. $\square$

Problem 1.4. *Show that the well-ordering principle implies the principle of mathematical induction. (Hint: Consider the set* $\{n \in \mathbb{N} : P(n)$ *is false*$\}$*.)*

Proof (Hint). Suppose that

(1) $P(n)$ be a proposition defined for each $n \in \mathbb{N}$,

(2) $P(1)$ is true,

(3) $[P(n) \Rightarrow P(n+1)]$ is true.

Consider the set
$$S = \{n \in \mathbb{N} : P(n) \text{ is false}\} \subseteq \mathbb{N}.$$
Want to show S *is empty, or the principle of mathematical induction holds.* If S were nonempty, by the well-ordering principle S has a smallest element m. m cannot be 1 by (2). Say $m > 1$. Therefore, $m - 1 \in \mathbb{N}$ and $P(m-1)$ is true by the minimality of m. By (3), $P((m-1)+1) = P(m)$ is true, which is absurd. $\square$

Problem 1.5. *Use mathematical induction to establish that the well-ordering principle. (Hint: Given a set* S *of positive integers, let* $P(n)$ *be the proposition 'If* $n \in S$*, then* S *has a least element'.)*

Proof (Modified hint).

(1) Given a set S of positive integers, let $P(n)$ be the proposition 'If $m \in S$ for some $m \leq n$, then S has a least element'. Want to show $P(n)$ is true for all $n \in \mathbb{N}$.

(a) $P(1)$ is true. For $m \in S$ with $m \le n = 1$, or $m = 1$ by the minimality of $1 \in \mathbb{N}$, S has a least element 1 (m itself) in $\mathbb{N}$.

(b) Suppose $P(n)$ is true. If $n + 1 \in S$, then there are only two possible cases.

  (i) There is a positive integer $m \in S$ less than $n+1$. So $n \ge m \in S$. Since $P(n)$ is true, S has a least element.

  (ii) There is no positive integer $m \in S$ less than $n + 1$. In this case $n + 1$ is the least element in S.

  In any cases (i)(ii), S has a least element, or $P(n + 1)$ is true.

By mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

(2) *Show that the well-ordering principle holds.* Let T be a nonempty subset of $\mathbb{N}$, so there exists a positive integer $k \in T$. Notice that $P(k)$ is true by (1), thus T has a least element since $k \le k$.

□

Problem 1.9. *Show that $A \subseteq B \Leftrightarrow A \cap B = A \Leftrightarrow A \cup B = B$.*

Proof.

(1) $A \subseteq B \Longleftrightarrow A \cap B = A$.

  (a) ($\Longrightarrow$) It suffices to show $A \cap B \supseteq A$. For any $x \in A$, $x \in B$ by $A \subseteq B$, so $x \in A \cap B$, so $A \cap B \supseteq A$.

  (b) ($\Longleftarrow$) $A = A \cap B \subseteq B$.

(2) $A \subseteq B \Leftrightarrow A \cup B = B$.

  (a) ($\Longrightarrow$) It suffices to show $A \cup B \subseteq B$. For any $x \in A \cup B$, $x \in A$ or $x \in B$. By $A \subseteq B$, $x \in B$ or $x \in B$. $x \in B$, so $A \cup B \subseteq B$.

  (b) ($\Longleftarrow$) $A \subseteq A \cup B = B$.

□

Problem 1.11. *Show that $A \subseteq B \Leftrightarrow \widetilde{B} \subseteq \widetilde{A}$.*

Proof.

$$\begin{aligned} A \subseteq B &\Longleftrightarrow x \in A \Rightarrow x \in B \\ &\Longleftrightarrow x \notin B \Rightarrow x \notin A \\ &\Longleftrightarrow \widetilde{B} \subseteq \widetilde{A}. \end{aligned}$$

□

Problem 1.14. *Show that*

$$B \cap \left[\bigcup_{A \in \mathscr{C}} A \right] = \bigcup_{A \in \mathscr{C}} (B \cap A).$$

Proof.

$$\begin{aligned} x \in B \cap \left[\bigcup_{A \in \mathscr{C}} A \right] &\iff x \in B \text{ and } x \in \bigcup_{A \in \mathscr{C}} A \\ &\iff x \in B \text{ and } x \in A \text{ for some } A \in \mathscr{C} \\ &\iff x \in B \cap A \text{ for some } A \in \mathscr{C} \\ &\iff x \in \bigcup_{A \in \mathscr{C}} (B \cap A). \end{aligned}$$

□