# Notes on the book: *Patrick Morandi, Field and Galois Theory*

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## Contents

# I. Galois Theory

## §1. Field Extensions

### Problem 1.1.

*Let $K$ be a field extension of $F$. By defining scalar multiplication for $\alpha \in F$ and $a \in K$ by $\alpha \cdot a = \alpha a$, the multiplication in $K$, show that $K$ is an $F$-vector space.*

*Proof.*

(1) $K$ is an additive group.

(2) *Show that* $(\alpha\beta) \cdot a = \alpha \cdot (\beta \cdot a)$ *for* $\alpha, \beta \in F$ *and* $a \in K$. In fact,

$$\begin{aligned} (\alpha\beta) \cdot a &= \alpha\beta a \in K, \\ \alpha \cdot (\beta \cdot a) &= \alpha\beta a \in K. \end{aligned}$$

(3) *Show that* $(\alpha + \beta) \cdot a = \alpha \cdot a + \beta \cdot a$ *for* $\alpha, \beta \in F$ *and* $a \in K$.

$$\begin{aligned} (\alpha + \beta) \cdot a &= (\alpha + \beta)a \\ &= \alpha a + \beta a \in K, \\ \alpha \cdot a + \beta \cdot a &= \alpha a + \beta a \in K. \end{aligned}$$

(4) *Show that* $\alpha \cdot (a + b) = \alpha \cdot a + \alpha \cdot b$ *for* $\alpha \in F$ *and* $a, b \in K$.

$$\begin{aligned} \alpha \cdot (a + b) &= \alpha(a + b) \\ &= \alpha a + \alpha b \in K, \\ \alpha \cdot a + \alpha \cdot b &= \alpha a + \alpha b \in K. \end{aligned}$$

(5) *Show that* $1 \cdot a = a$ *for* $a \in K$. $1 \cdot a = 1a = a \in K$.

By (1) to (5), $K$ is an $F$-vector space. $\square$

### Problem 1.2.

*If $K$ is a field extension of $F$, prove that $[K : F] = 1$ if and only if $K = F$.*

*Proof.*

(1) $[K : F] = 1 \Longleftarrow K = F$. Take a basis $\{1\}$ for $K$ as an $F$-vector space.

(2) $[K : F] = 1 \implies K = F$. Take a basis $\{a\}$ for $K$ as an $F$-vector space where $a \in K$. Since $1 \in K$ as an $F$-vector space, there exists $\alpha \in F$ such that $1 = \alpha a$. $a = \alpha^{-1} \in F$, or $K \subseteq F$, or $K = F$.

$\square$

**Problem 1.3.**

*Let $K$ be a field extension of $F$, and let $a \in K$. Show that the evaluation map $ev_a : F[x] \to K$ given by $ev_a(f(x)) = f(a)$ is a ring and and $F$-vector space homomorphism. (Such a map is called an $F$-algebra homomorphism.)*

*Proof.*

(1) *$ev_a$ is a ring homomorphism.*

   (a) $\mathrm{ev}_a(f(x) + g(x)) = f(a) + g(a) = \mathrm{ev}_a(f(x)) + \mathrm{ev}_a(g(x))$.
   
   (b) $\mathrm{ev}_a(f(x)g(x)) = g(a)g(b) = \mathrm{ev}_a(f(x))\mathrm{ev}_a(g(x))$.
   
   (c) $\mathrm{ev}_a(1) = 1$.

(2) *$ev_a$ is an $F$-vector space homomorphism.*

   (a) $\mathrm{ev}_a(f(x) + g(x)) = f(a) + g(a) = \mathrm{ev}_a(f(x)) + \mathrm{ev}_a(g(x))$.
   
   (b) Given $c \in F$, $\mathrm{ev}_a(cf(x)) = cf(a) = c\mathrm{ev}_a(f(x))$.

$\square$

**Problem 1.4.**

*Prove Proposition 1.9: Let $K$ be a field extension of $F$ and let $a_1, \ldots, a_n \in K$. Then*

$$F[a_1, \ldots, a_n] = \{f(a_1, \ldots, a_n) : f \in F[x_1, \ldots, x_n]\}$$

*and*

$$F(a_1, \ldots, a_n) = \left\{ \frac{f(a_1, \ldots, a_n)}{g(a_1, \ldots, a_n)} : f, g \in F[x_1, \ldots, x_n], g(a_1, \ldots, a_n) \neq 0 \right\},$$

*so $F(a_1, \ldots, a_n)$ is the quotient field of $F[x_1, \ldots, x_n]$.*

*Proof (Proposition 1.8).*

(1) The evaluation map $\mathrm{ev}_{(a_1, \ldots, a_n)} : F[x_1, \ldots, x_n] \to K$ has image

$$\{f(a_1, \ldots, a_n) : f \in F[x_1, \ldots, x_n]\},$$

so this set is a subring of $K$.

(2) If $R$ is a subring of $K$ that contains $F$ and $a_1, \ldots, a_n$, then

$$f(a_1, \ldots, a_n) \in R$$

for any $f(x_1, \ldots, x_n) \in F[x_1, \ldots, x_n]$ by closure of addition and multiplication.

(3) So $\{f(a_1, \ldots, a_n) : f \in F[x_1, \ldots, x_n]\}$ is contained in all subrings of $K$ that contains $F$ and $a_1, \ldots, a_n$. Hence

$$F[a_1, \ldots, a_n] = \{f(a_1, \ldots, a_n) : f \in F[x_1, \ldots, x_n]\}.$$

(4) The quotient field of $F[a_1, \ldots, a_n]$ is then the set

$$\left\{ \frac{f(a_1, \ldots, a_n)}{g(a_1, \ldots, a_n)} : f, g \in F[x_1, \ldots, x_n], g(a_1, \ldots, a_n) \neq 0 \right\}.$$

It is clearly is contained in any subfield of $K$ that contains $F[a_1, \ldots, a_n]$; hence, it is equal to $F(a_1, \ldots, a_n)$.

$\square$

**Problem 1.5.**

*Show that* $\mathbb{Q}(\sqrt{5}, \sqrt{7}) = \mathbb{Q}(\sqrt{5} + \sqrt{7})$.

*Proof.*

(1) $\mathbb{Q}(\sqrt{5}, \sqrt{7}) \supseteq \mathbb{Q}(\sqrt{5} + \sqrt{7})$ since $\sqrt{5} + \sqrt{7} \in \mathbb{Q}(\sqrt{5}, \sqrt{7})$.

(2)

$$\begin{aligned}
(\sqrt{7} + \sqrt{5})^{-1} &= \frac{1}{\sqrt{7} + \sqrt{5}} \\
&= \frac{\sqrt{7} - \sqrt{5}}{(\sqrt{7} + \sqrt{5})(\sqrt{7} - \sqrt{5})} \\
&= \frac{\sqrt{7} - \sqrt{5}}{2} \in \mathbb{Q}(\sqrt{5} + \sqrt{7}),
\end{aligned}$$

Or $\sqrt{7} - \sqrt{5} \in \mathbb{Q}(\sqrt{5} + \sqrt{7})$. Thus

$$\begin{aligned}
\sqrt{7} &= \frac{1}{2} \cdot ((\sqrt{7} + \sqrt{5}) + (\sqrt{7} - \sqrt{5})) \in \mathbb{Q}(\sqrt{5} + \sqrt{7}), \\
\sqrt{5} &= \frac{1}{2} \cdot ((\sqrt{7} + \sqrt{5}) - (\sqrt{7} - \sqrt{5})) \in \mathbb{Q}(\sqrt{5} + \sqrt{7}).
\end{aligned}$$

Thus, $\mathbb{Q}(\sqrt{5}, \sqrt{7}) \subseteq \mathbb{Q}(\sqrt{5} + \sqrt{7})$.

By (1)(2), $\mathbb{Q}(\sqrt{5}, \sqrt{7}) = \mathbb{Q}(\sqrt{5} + \sqrt{7})$. $\square$

**Problem 1.9.**

*If $K$ is an extension of $F$ such that $[K : F]$ is prime, show that there are no intermediate fields between $K$ and $F$.*

*Proof.* Let $L$ be any field such that $F \subseteq L \subseteq K$. By Proposition 1.20,

$$[K : F] = [K : L][L : F].$$

Since $[K : F]$ is prime, $[K : L] = 1$ or $[L : F] = 1$. By Problem 1.2, $L = K$ or $L = F$, or there are no intermediate fields between $K$ and $F$. $\square$

**Problem 1.11.**

*If $K$ is an algebraic extension of $F$ and if $R$ is a subring of $K$ with $F \subseteq R \subseteq K$, show that $R$ is a field.*

*Proof.*

(1) $R$ is a domain since $R$ is contained in a field $K$. To show $R$ is a field, it suffices to show that every nonzero element $\alpha \in R$ has an inverse in $R$.

(2) Since $\alpha \in R \subseteq K$ is algebraic over $F$, there is a minimal polynomial

$$f(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0$$

such that $f(\alpha) = 0$, where each $b_i \in F$ and $b_0 \neq 0$ by the minimality of $f$.

(3) Note that

$$\begin{aligned} & f(\alpha) = 0 \\ \Longleftrightarrow\, & b_n\alpha^n + b_{n-1}\alpha^{n-1} + \cdots + b_0 = 0 \\ \Longleftrightarrow\, & b_n\alpha^n + b_{n-1}\alpha^{n-1} + \cdots + b_1\alpha = -b_0 \\ \Longleftrightarrow\, & \alpha(b_n\alpha^{n-1} + b_{n-1}\alpha^{n-2} + \cdots + b_1) = -b_0 \\ \Longleftrightarrow\, & \alpha(\underbrace{(-b_0)^{-1}b_n\alpha^{n-1} + (-b_0)^{-1}b_{n-1}\alpha^{n-2} + \cdots + (-b_0)^{-1}b_1}_{:=\alpha'}) = 1. \end{aligned}$$

Hence $\alpha' \in F[\alpha] \subseteq R$. Therefore $\alpha'$ is the inverse of $\alpha$ in $R$.

$\square$

**Problem 1.12.**

*Show that $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ are not isomorphic as fields but are isomorphic as vector spaces over $\mathbb{Q}$.*

*Proof.*

(1) *Show that $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ are not isomorphic as fields.* (Reductio ad absurdum) If $\varphi : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{3})$ were an isomorphism as fields, then $\varphi$ is an identity map on $\mathbb{Q}$, and

$$\begin{aligned}
&\varphi(\sqrt{2}) = a + b\sqrt{3} \text{ for some } a, b \in \mathbb{Q} \\
\Longrightarrow &\varphi(\sqrt{2})\varphi(\sqrt{2}) = (a + b\sqrt{3})^2 \\
\Longrightarrow &\varphi(\sqrt{2}\sqrt{2}) = (a + b\sqrt{3})^2 \\
\Longrightarrow &\varphi(2) = a^2 + 3b^2 + 2ab\sqrt{3} \\
\Longrightarrow &2 = a^2 + 3b^2 + 2ab\sqrt{3}.
\end{aligned}$$

If $2ab \neq 0$, then $\sqrt{3} = \frac{2-a^2-3b^2}{2ab} \in \mathbb{Q}$, which is absurd. Hence $2ab = 0$.

(a) $a = 0$. Write $b = \frac{m}{n} \in \mathbb{Q}$ where $m, n \in \mathbb{Z}$ and $(m, n) = 1$. Hence

$$2n^2 = 3m^2.$$

So $2 \mid 3m^2$, $2 \mid m^2$, $2 \mid m$. So $4 \mid 2n^2$, $2 \mid n^2$, $2 \mid n$. Hence $2 \mid (m, n)$, contrary to the assumption that $(m, n) = 1$.

(b) $b = 0$. $2 = a^2$. Write $a = \frac{m}{n} \in \mathbb{Q}$ where $m, n \in \mathbb{Z}$ and $(m, n) = 1$. Similar to the argument in (a), we will reach a contradiction.

By (a)(b), no such isomorphism $\varphi$, that is, $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ are not isomorphic as fields.

(2) *Show that $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ are isomorphic as $\mathbb{Q}$-vector spaces.* $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] = 2$. There is a natural map $\varphi : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{3})$ defined by $\varphi(a+b\sqrt{2}) = a+b\sqrt{3}$. Clearly $\varphi$ is well-defined, linear, injective and surjective.

$\square$

**Problem 1.16.**

*Let $\mathbb{A}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Prove that $[\mathbb{A} : \mathbb{Q}] = \infty$.*

*Proof (Example 1.16).* By Example 1.16, $[\mathbb{Q}(\sqrt[n]{2}) : \mathbb{Q}] = n$. Therefore,

$$[\mathbb{A} : \mathbb{Q}] = [\mathbb{A} : \mathbb{Q}(\sqrt[n]{2})][\mathbb{Q}(\sqrt[n]{2}) : \mathbb{Q}] = [\mathbb{A} : \mathbb{Q}(\sqrt[n]{2})]n$$

for arbitrary $n \in \mathbb{Z}^+$. Hence $[\mathbb{A} : \mathbb{Q}] = \infty$. $\square$

*Proof (Example 1.16).* Given a prime number $p$. By Example 1.16, $[\mathbb{Q}(\rho) : \mathbb{Q}] = p - 1$ where $\rho = \exp(2\pi i/p)$. Therefore,

$$[\mathbb{A} : \mathbb{Q}] = [\mathbb{A} : \mathbb{Q}(\rho)][\mathbb{Q}(\rho) : \mathbb{Q}] = [\mathbb{A} : \mathbb{Q}(\rho)](p-1)$$

for arbitrary prime $p$. Hence $[\mathbb{A} : \mathbb{Q}] = \infty$. $\square$

**Problem 1.23.**

*Recall that the characteristic of a ring $R$ with identity is the smallest positive integer $n$ for which $n \cdot 1 = 0$, if such an $n$ exists, or else the characteristic is $0$. Let $R$ be a ring with identity. Define $\varphi : \mathbb{Z} \to R$ by $\varphi(n) = n \cdot 1$, where $1$ is the identity of $R$. Show that $\varphi$ is a ring homomorphism and that $\ker(\varphi) = m\mathbb{Z}$ for a unique nonnegative integer $m$, and show that $m$ is the characteristic of $R$.*

*Proof.*

(1) *$\varphi$ is a ring homomorphism.*

  (a) $\varphi(a+b) = \varphi(a)+\varphi(b)$. $\varphi(a+b) = (a+b)\cdot 1 = a\cdot 1+b\cdot 1 = \varphi(a)+\varphi(b)$.

  (b) $\varphi(ab) = \varphi(a)\varphi(b)$. $\varphi(ab) = (ab) \cdot 1 = (a \cdot 1)(b \cdot 1) = \varphi(a)\varphi(b)$ since $1 \times 1 = 1$. (Here $\times$ is the multiplication operator of $R$.)

(2) $\ker(\varphi) = m\mathbb{Z}$ *for a unique nonnegative integer $m$.* Since $\ker(\varphi)$ is an ideal of a PID $\mathbb{Z}$, there is a unique nonnegative integer $m$ such that $\ker(\varphi) = m\mathbb{Z}$.

(3) *$m$ is the characteristic of $R$.* There are only two possible cases, $\mathrm{char}(R) = 0$ or else $\mathrm{char}(R) > 0$.

  (a) *char*$(R) = 0$. $\ker(\varphi) = 0$. Thus $m = 0 = \mathrm{char}(R)$.

  (b) *char*$(R) = n > 0$. $n \in \ker(\varphi)$, so $m > 0$ and $m \mid n$. By the minimality of $n$, $m = n = \mathrm{char}(R)$.

$\square$

**Problem 1.24.**

*For any positive integer $n$, give an example of a ring of characteristic $n$.*

*Proof.* The ring $\mathbb{Z}/n\mathbb{Z}$. $\square$

**Problem 1.25.**

*If $R$ is an integral domain, show that either char$(R) = 0$ or char$(R)$ is prime.*

*Proof.*

(1) 1 has infinite order. $\mathrm{char}(R) = 0$. (Nothing to do.)

(2) 1 has finite order $n$. Want to show $n$ is prime. If $n = ab$ where $a, b \in \mathbb{Z}^+$, then

$$0 = n \cdot 1 = (a \cdot 1)(b \cdot 1).$$

Since $R$ is an integral domain, $a \cdot 1 =$ or $b \cdot 1 = 0$. By the minimality of $n$, $a \geq n$ or $b \geq n$. $a = n$ or $b = n$. That is, $n$ is prime.

□

## §2. Automorphisms

**Problem 2.1.**

*Show that the only automorphism of $\mathbb{Q}$ is the identity.*

*Proof.* Given any $\sigma \in \mathrm{Aut}(\mathbb{Q})$.

(1) *Show that* $\sigma(1) = 1$. Since $1^2 = 1$, $\sigma(1)\sigma(1) = \sigma(1)$. $\sigma(1) = 0$ or 1. There are only two possible cases.

   (a) Assume that $\sigma(1) = 0$. So

   $$\sigma(a) = \sigma(a \cdot 1) = \sigma(a) \cdot \sigma(1) = \sigma(a) \cdot 0 = 0$$

   for any $a \in \mathbb{Q}$. That is, $\sigma = 0 \in \mathrm{Aut}(\mathbb{Q})$, which is absurd.

   (b) Therefore, $\sigma(1) = 1$.

(2) *Show that* $\sigma(n) = n$ *for all* $n \in \mathbb{Z}^+$. Write $n = 1 + 1 + \cdots + 1$ ($n$ times 1). Applying the additivity of $\sigma$, we have

$$\sigma(n) = \sigma(1) + \sigma(1) + \cdots + \sigma(1) = 1 + 1 + \cdots + 1 = n.$$

(Might use induction on $n$ to eliminate $\cdots$ symbols.)

(3) *Show that* $\sigma(n) = n$ *for all* $n \in \mathbb{Z}$. By the additivity of $\sigma$, $\sigma(-n) = -\sigma(n) = -n$ for $n \geq 0$. The result is established.

For any $a = \frac{n}{m} \in \mathbb{Q}$ ($m, n \in \mathbb{Z}$, $n \neq 0$), applying the multiplication of $\sigma$ on $am = n$, that is, $\sigma(a)\sigma(m) = \sigma(n)$. By (3), we have $\sigma(a)m = n$, or

$$\sigma(a) = \frac{m}{n} = a$$

provided $n \neq 0$, or $\sigma$ is the identity. □

**Problem 2.2.**

*Show that the only automorphism of $\mathbb{R}$ is the identity. (Hint: If $\sigma$ is an automorphism, show that $\sigma|_{\mathbb{Q}} = id$, and if $a > 0$, then $\sigma(a) > 0$. It is an interesting fact that there are infinitely many automorphisms of $\mathbb{C}$, even thought $[\mathbb{C} : \mathbb{R}] = 2$. Why is this fact not a contradiction to this problem?)*

*Proof (Hint).* Given any $\sigma \in \mathrm{Aut}(\mathbb{R})$.

(1) Apply the same argument in Problem 2.1, we have $\sigma|_{\mathbb{Q}} = \mathrm{id}$. Notice that $\sigma(a) \neq 0$ for any $a \neq 0$.

(2) *Show that $\sigma(a) > 0$ if $a > 0$.* Given any $a > 0$. Write $a = \sqrt{a}\sqrt{a}$ (well-defined) and then apply $\sigma$ on the both sides,

$$\sigma(a) = \sigma(\sqrt{a})\sigma(\sqrt{a}) = \sigma(\sqrt{a})^2 > 0$$

(since $\sqrt{a} \neq 0$ and thus $\sigma(\sqrt{a})$ cannot be zero).

(3) *Show that $\sigma(a) > \sigma(b)$ if $a > b$.* It is a corollary to (2) by applying $\sigma$ on $a - b > 0$. ($\sigma(a - b) > 0$, or $\sigma(a) - \sigma(b) > 0$, or $\sigma(a) > \sigma(b)$.)

(4) For any real number $x \in \mathbb{R}$, choose two sequences $\{p_n\}, \{q_n\}$ of rational numbers such that $p_n < x < q_n$ and $p_n, q_n \to x$ as $n \to \infty$. Take $\sigma$ on the inequality, $\sigma(p_n) < \sigma(x) < \sigma(q_n)$. So $p_n < \sigma(x) < q_n$ since $\sigma|_{\mathbb{Q}} = \mathrm{id}$. Let $n \to \infty$, we get $x \leq \sigma(x) \leq x$, or $\sigma(x) = x$.

□

**Supplement.** Automorphisms of the Complex Numbers. by Paul B. Yale (Pomona College) [Link].

**Problem 2.4.**

*Let $B$ be an integral domain with quotient field $F$. If $\sigma : B \to B$ is a ring automorphism, show that $\sigma$ induces a ring automorphism $\sigma' : F \to F$ defined by $\sigma'(a/b) = \sigma(a)/\sigma(b)$ if $a, b \in B$ with $b \neq 0$.*

*Proof.*

(1) *Show that $\sigma'$ is well-defined.*

(a) *$\sigma' : F \to F$ is defined.* $\sigma(a), \sigma(b) \in B$ since $\sigma$ is a homomorphism. $\sigma(b) \neq 0$ since $b \neq 0$ and $\sigma$ is a one-on-one homomorphism.

(b) *$\sigma'$ is independent of the representation of $a/b \in F$.* Suppose $a/b = c/d$ where $a, b, c, d \in B$ and $b, d \neq 0$. Hence,

$$\begin{aligned}
a/b = c/d &\Longleftrightarrow ad = bc \\
&\Longleftrightarrow \sigma(ad) = \sigma(bc) \\
&\Longleftrightarrow \sigma(a)\sigma(d) = \sigma(b)\sigma(c) && (\sigma\text{: homomorphism}) \\
&\Longleftrightarrow \sigma(a)/\sigma(d) = \sigma(c)/\sigma(d) && (\sigma(b), \sigma(d) \neq 0) \\
&\Longleftrightarrow \sigma'(a/b) = \sigma'(c/d).
\end{aligned}$$

(2) *Show that $\sigma'$ is a ring homomorphism.*

(a) *Show that $\sigma'(a/b + c/d) = \sigma'(a/b) + \sigma'(c/d)$.*

$$\begin{aligned}
\sigma'(a/b + c/d) &= \sigma'((ad + bc)/(bd)) \\
&= \sigma(ad + bc)/\sigma(bd) \\
&= (\sigma(a)\sigma(d) + \sigma(b)\sigma(c))/(\sigma(b)\sigma(d)) \quad (\sigma\text{: homomorphism}) \\
&= \sigma(a)/\sigma(b) + \sigma(c)/\sigma(d) \\
&= \sigma'(a/b) + \sigma'(c/d).
\end{aligned}$$

(b) *Show that $\sigma'(a/b \cdot c/d) = \sigma'(a/b) \cdot \sigma'(c/d)$.*

$$\begin{aligned}
\sigma'(a/b \cdot c/d) &= \sigma'((ac)/(bd)) \\
&= \sigma(ac)/\sigma(bd) \\
&= (\sigma(a)\sigma(c))/(\sigma(b)\sigma(d)) && (\sigma\text{: homomorphism}) \\
&= \sigma(a)/\sigma(b) \cdot \sigma(c)/\sigma(d) \\
&= \sigma'(a/b) \cdot \sigma'(c/d).
\end{aligned}$$

(3) *Show that $\sigma'$ is injective.*

$$\begin{aligned}
\sigma'(a/b) = 0 &\Longleftrightarrow \sigma(a)/\sigma(b) = 0 \\
&\Longleftrightarrow \sigma(a) = 0 \\
&\Longleftrightarrow a = 0 && (\sigma\text{: injective}) \\
&\Longleftrightarrow a/b = 0/b = 0 \in F
\end{aligned}$$

(4) *Show that $\sigma'$ is a surjective.* Given any $c/d \in F$, want to show there is $a/b \in F$ such that $\sigma'(a/b) = c/d$.

$$\begin{aligned}
c/d \in F &\Longrightarrow c, d \in B \\
&\Longrightarrow \exists\, a, b \in B \text{ such that } \sigma(a) = c, \sigma(b) = d \quad (\sigma\text{: surjective}) \\
&\Longrightarrow \exists\, a, b \in B \text{ such that } \sigma(a)/\sigma(b) = c/d \\
&\Longrightarrow \exists\, a, b \in B \text{ such that } \sigma'(a/b) = c/d.
\end{aligned}$$

# II. Some Galois Extensions

## §10. Hilbert Theorem 90 and Group Cohomology

**Supplement.**

(1) Corollary 10.4 (Cohomological Hilbert Theorem 90). Let $K$ be a cyclic Galois extension of $F$. Then $H^1(\mathrm{Gal}(K/F), K^\times) = 0$.

(2) *(Exercise 10.24 in the textbook: Rudin, Principles of Mathematical Analysis, 3rd edition.) Let $\omega = \sum a_i(\mathbf{x})dx_i$ be a 1-form of class $\mathcal{C}''$ in a convex open set $E \subseteq \mathbb{R}^n$. Assume $d\omega = 0$ and prove that $\omega$ is exact in $E$.* Hence the first de Rham cohomology $H^1_{\mathrm{dR}}(E) = 0$.

(3) $H^1_{\mathrm{dR}}(E) = 0$ if $E$ is simply connected. (The converse is not true.)

(4) *(Exercise 10.21 in the textbook: Rudin, Principles of Mathematical Analysis, 3rd edition.) Consider the 1-form*

$$\eta = \frac{xdy - ydx}{x^2 + y^2}$$

*in* $\mathbb{R}^2 - \{\mathbf{0}\}$.

(a) *Carry out the computation that leads to*

$$\int_\gamma \eta = 2\pi \neq 0,$$

*and prove that $d\eta = 0$.*

(b) *Let $\gamma(t) = (r\cos t, r\sin t)$, for some $r > 0$, and let $\Gamma$ be a $\mathcal{C}''$-curve in $\mathbb{R}^2 - \{\mathbf{0}\}$, with parameter interval $[0, 2\pi]$, with $\Gamma(0) = \Gamma(2\pi)$, such that the intervals $[\gamma(t), \Gamma(t)]$ do not contain $\mathbf{0}$ for any $t \in [0, 2\pi]$. Prove that*

$$\int_\Gamma \eta = 2\pi.$$

(c) *Take $\Gamma(t) = (a\cos t, b\sin t)$ where $a > 0$, $b > 0$ are fixed. Show that*

$$\int_0^{2\pi} \frac{ab}{a^2\cos^2 t + b^2\sin^2 t} dt = 2\pi.$$

(d) *Show that*

$$\eta = d\left(\arctan\frac{y}{x}\right)$$

*in any convex open set in which $x \neq 0$, and that*

$$\eta = d\left(-\arctan \frac{x}{y}\right)$$

*in any convex open set in which $y \neq 0$. Explain why this justifies the notation $\eta = d\theta$, in spite of the fact that $\eta$ is not exact in $\mathbb{R}^2 - \{0\}$.*

(5) *(Exercise 10.22 in the textbook: Rudin, Principles of Mathematical Analysis, 3rd edition.) Define $\zeta$ in $\mathbb{R}^3 - \{\mathbf{0}\}$ by*

$$\zeta = \frac{x dy \wedge dz + y dz \wedge dx + z dx \wedge dy}{r^3}$$

*where $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$, let $D$ be the rectangle given by $0 \leq u \leq \pi$, $0 \leq v \leq 2\pi$, and let $\Sigma$ be the 2-surface in $\mathbb{R}^3$, with parameter domain $D$, given by*

$$x = \sin u \cos v, \qquad y = \sin u \sin v, \qquad z = \cos u.$$

(a) *Prove that $d\zeta = 0$ in $\mathbb{R}^3 - \{\mathbf{0}\}$.*

(b) *Let $S$ denote the restriction of $\Sigma$ to a parameter domain $E \subseteq D$. Prove that*

$$\int_S \zeta = \int_E \sin u \, du \, dv = A(S),$$

*where $A$ denotes area, as in Section 10.46. Note that this contains*

$$\int_\Sigma \zeta = \int_D \sin u \, du \, dv = 4\pi \neq 0$$

*as a special case.*

(c) *Suppose $g, h_1, h_2, h_3$, are $\mathcal{C}''$-functions on $[0, 1]$, $g > 0$. Let $(x, y, z) = \Phi(s, t)$ define a 2-surface $\Phi$, with parameter domain $I^2$, by*

$$x = g(t)h_1(s), \qquad y = g(t)h_2(s), \qquad z = g(t)h_3(s).$$

*Prove that*

$$\int_\Phi \zeta = 0.$$

*Note the shape of the range of $\Phi$: For fixed $s$, $\Phi(s, t)$ runs over an interval on a line through $\mathbf{0}$. The range of $\Phi$ thus lies in a "cone" with vertex at the origin.*

(d) *Let $E$ be a closed rectangle in $D$, with edges parallel to those of $D$. Suppose $f \in \mathcal{C}''(D)$, $f > 0$. Let $\Omega$ be the 2-surface with parameter domain $E$, defined by*

$$\Omega(u, v) = f(u, v)\Sigma(u, v).$$

*Define $S$ as in (b) and prove that*

$$\int_\Omega \zeta = \int_S \zeta = A(S).$$

(e) *Put* $\lambda = -\frac{z}{r}\eta$, *where*

$$\eta = \frac{xdy - ydx}{x^2 + y^2}.$$

*Then* $\lambda$ *is a 1-form in the open set* $V \subseteq \mathbb{R}^3$ *in which* $x^2 + y^2 > 0$. *Show that* $\zeta$ *is exact in* $V$ *by showing that*

$$\zeta = d\lambda.$$

(f) *Is* $\zeta$ *exact in the complement of every line through the origin?*

(6) *(Exercise 10.23 in the textbook: Rudin, Principles of Mathematical Analysis, 3rd edition.) Fix* $n$. *Define* $r_k = (x_1^2 + \cdots + x_k^2)^{\frac{1}{2}}$ *for* $1 \leq k \leq n$, *let* $E_k$ *be the set of all* $\mathbf{x} \in \mathbb{R}^n$ *at which* $r_k > 0$, *and let* $\omega_k$ *be the* $(k-1)$*-form defined in* $E_k$ *by*

$$\omega_k = (r_k)^{-k} \sum_{i=1}^{k} (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k$$

*Note that* $\omega_2 = \eta$, $\omega_3 = \zeta$ *in the terminology of Exercise 10.21 and Exercise 10.22. Note also that*

$$E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n = \mathbb{R}^n.$$

(a) *Prove that* $d\omega_k = 0$ *in* $E_k$.

(b) *For* $k = 2, \ldots, n$, *prove that* $\omega_k$ *is exact in* $E_{k-1}$, *by showing that*

$$\omega_k = d(f_k \omega_{k-1}) = df_k \wedge \omega_{k-1}$$

*where* $f_k(\mathbf{x}) = (-1)^k g_k\left(\frac{x_k}{r_k}\right)$ *where*

$$g_k(t) = \int_{-1}^{t} (1 - s^2)^{\frac{k-3}{2}} ds \qquad (-1 < t < 1).$$

(c) *Is* $\omega_n$ *exact in* $E_n$*?*

(7) $H_{\mathrm{dR}}^{n-1}(\mathbb{R}^n - \{\mathbf{0}\}) = \mathbb{R}^1$. (Compare to (5)(6)(7).)

**Problem 10.1.**

*Let* $M$ *be a* $G$*-module. Show that the boundary map* $\delta_n : C^n(G, M) \to C^{n+1}(G, M)$ *defined in this section is a homomorphism.*

*Proof.*

(1) $\delta_n$ is defined by

$$\begin{aligned}\delta_n(f)(\sigma_1,\dots,\sigma_{n+1}) = \sigma_1 f(\sigma_2,\dots,\sigma_{n+1})\\ + \sum_{i=1}^{n}(-1)^i f(\sigma_1,\dots,\sigma_i\sigma_{i+1},\dots,\sigma_{n+1})\\ + (-1)^{n+1} f(\sigma_1,\dots,\sigma_n)\end{aligned}$$

if $n > 0$. If $n = 0$, then the map $\delta_0 : M = C^0(G,M) \to C^1(G,M)$ is defined by $\delta_0(m)(\sigma) = \sigma m - m$.

(2) It suffices to show that $\delta_n(f+g) = \delta_n(f)+\delta_n(g)$ for all $n$ and all $n$-cochains $f$ and $g$.

(3) If $n = 0$, then

$$\begin{aligned}\delta_0(f+g)(\sigma) &= \sigma(f+g) - (f+g)\\ &= \sigma f + \sigma g - f - g && (M\text{: } G\text{-module})\\ &= (\sigma f - f) + (\sigma g - g) && (M\text{: abelian group})\\ &= \delta_0(f) + \delta_0(g).\end{aligned}$$

(4) If $n \geq 1$, then

$$\begin{aligned}&\delta_n(f+g)(\sigma)\\ &= \sigma_1(f+g)(\sigma_2,\dots,\sigma_{n+1}) + \sum_{i=1}^{n}(-1)^i(f+g)(\sigma_1,\dots,\sigma_i\sigma_{i+1},\dots,\sigma_{n+1})\\ &\quad + (-1)^{n+1}(f+g)(\sigma_1,\dots,\sigma_n)\\ &= \sigma_1 f(\sigma_2,\dots,\sigma_{n+1}) + \sigma_1 g(\sigma_2,\dots,\sigma_{n+1})\\ &\quad + \sum_{i=1}^{n}(-1)^i f(\sigma_1,\dots,\sigma_i\sigma_{i+1},\dots,\sigma_{n+1})\\ &\quad + \sum_{i=1}^{n}(-1)^i g(\sigma_1,\dots,\sigma_i\sigma_{i+1},\dots,\sigma_{n+1})\\ &\quad + (-1)^{n+1} f(\sigma_1,\dots,\sigma_n) + (-1)^{n+1} g(\sigma_1,\dots,\sigma_n)\\ &= \Bigg\{\sigma_1 f(\sigma_2,\dots,\sigma_{n+1}) + \sum_{i=1}^{n}(-1)^i f(\sigma_1,\dots,\sigma_i\sigma_{i+1},\dots,\sigma_{n+1})\\ &\quad + (-1)^{n+1} f(\sigma_1,\dots,\sigma_n)\Bigg\} + \Bigg\{\sigma_1 g(\sigma_2,\dots,\sigma_{n+1})\\ &\quad + \sum_{i=1}^{n}(-1)^i g(\sigma_1,\dots,\sigma_i\sigma_{i+1},\dots,\sigma_{n+1}) + (-1)^{n+1} g(\sigma_1,\dots,\sigma_n)\Bigg\}\\ &= \delta_n(f)(\sigma) + \delta_n(g)(\sigma).\end{aligned}$$

(Here note that $C^n(G,M)$ is an abelian group).

□

**Problem 10.2.**

*With notation as in the previous problem, show that $\delta_{n+1} \circ \delta_n$ is the zero map.*

*Proof.*

(1) If $n = 0$, then

$$\begin{aligned}(\delta_1 \circ \delta_0)(f)(\sigma_1, \sigma_2) &= \delta_1(\delta_0(f))(\sigma_1, \sigma_2) \\ &= \sigma_1 \delta_0(f)(\sigma_2) - \delta_0(f)(\sigma_1\sigma_2) + \delta_0(f)(\sigma_1) \\ &= \sigma_1(\sigma_2 f - f) - (\sigma_1\sigma_2 f - f) + (\sigma_1 f - f) \\ &= 0.\end{aligned}$$

(2) If $n \geq 1$, then we write

$$\begin{aligned}&(\delta_{n+1} \circ \delta_n)(f)(\sigma_1, \dots, \sigma_{n+2}) \\ &= \delta_{n+1}(\delta_n(f))(\sigma_1, \dots, \sigma_{n+2}) \\ &= \underbrace{\sigma_1 \delta_n(f)(\sigma_2, \dots, \sigma_{n+2})}_{\text{Part (3)}} \\ &\quad + \sum_{j=1}^{n+1} \underbrace{(-1)^j \delta_n(f)(\sigma_1, \dots, \sigma_j\sigma_{j+1}, \dots, \sigma_{n+2})}_{\text{Parts (4)(5)(6)}} \\ &\quad + \underbrace{(-1)^{n+2} \delta_n(f)(\sigma_1, \dots, \sigma_{n+1})}_{\text{Part (7)}}.\end{aligned}$$

(3) The first term is

$$\begin{aligned}&\sigma_1 \delta_n(f)(\sigma_2, \dots, \sigma_{n+2}) \\ &= \sigma_1\sigma_2 f(\sigma_3, \dots, \sigma_{n+2}) \\ &\quad + \sum_{i=1}^{n} (-1)^i \sigma_1 f(\sigma_2, \dots, \sigma_{i+1}\sigma_{i+2}, \dots, \sigma_{n+2}) \\ &\quad + (-1)^{n+1} \sigma_1 f(\sigma_2, \dots, \sigma_{n+1}).\end{aligned}$$

(4) The first term ($j = 1$) in the summation is

$$\begin{aligned}&(-1)^1 \delta_n(f)(\sigma_1\sigma_2, \dots, \sigma_{n+2}) \\ &= -\sigma_1\sigma_2 f(\sigma_3, \dots, \sigma_{n+2}) \\ &\quad + f(\sigma_1\sigma_2\sigma_3, \dots, \sigma_{n+2}) - \sum_{i=2}^{n} (-1)^i f(\sigma_1\sigma_2, \dots, \sigma_{i+1}\sigma_{i+2}, \dots, \sigma_{n+2}) \\ &\quad - (-1)^{n+1} f(\sigma_1\sigma_2, \dots, \sigma_{n+1})\end{aligned}$$

(5) The $j$th term for $2 \leq j \leq n$ in the summation is

$$\begin{aligned}
&(-1)^j \delta_n(f)(\sigma_1, \dots, \sigma_j\sigma_{j+1}, \dots, \sigma_{n+2}) \\
= {} &(-1)^j \sigma_1 f(\sigma_2, \dots, \sigma_j\sigma_{j+1}, \dots, \sigma_{n+2}) \\
&+ (-1)^j \sum_{i=1}^{j-2} (-1)^i f(\sigma_1, \dots, \sigma_i\sigma_{i+1}, \dots, \sigma_j\sigma_{j+1}, \dots, \sigma_{n+2}) \\
&+ (-1)^j (-1)^{j-1} f(\sigma_1, \dots, \sigma_{j-1}\sigma_j\sigma_{j+1}, \dots, \sigma_{n+2}) \\
&+ (-1)^j (-1)^j f(\sigma_1, \dots, \sigma_j\sigma_{j+1}\sigma_{j+2}, \dots, \sigma_{n+2}) \\
&+ (-1)^j \sum_{i=j+1}^{n} (-1)^i f(\sigma_1, \dots, \sigma_j\sigma_{j+1}, \dots, \sigma_{i+1}\sigma_{i+2}, \dots, \sigma_{n+2}) \\
&+ (-1)^j (-1)^{n+1} f(\sigma_1, \dots, \sigma_j\sigma_{j+1}, \dots, \sigma_{n+1}).
\end{aligned}$$

(6) The last term ($j = n+1$) in the summation is

$$\begin{aligned}
&(-1)^{n+1} \delta_n(f)(\sigma_1, \dots, \sigma_n, \sigma_{n+1}\sigma_{n+2}) \\
= {} &(-1)^{n+1} \sigma_1 f(\sigma_2, \dots, \sigma_{n+1}\sigma_{n+2}) \\
&+ (-1)^{n+1} \sum_{i=1}^{n-1} (-1)^i f(\sigma_1, \dots, \sigma_i\sigma_{i+1}, \dots, \sigma_{n+1}\sigma_{n+2}) \\
&+ (-1)^{n+1} (-1)^n f(\sigma_1, \dots, \sigma_n\sigma_{n+1}\sigma_{n+2}) \\
&+ (-1)^{n+1} (-1)^{n+1} f(\sigma_1, \dots, \sigma_n).
\end{aligned}$$

(7) The last term is

$$\begin{aligned}
&(-1)^{n+2} \delta_n(f)(\sigma_1, \dots, \sigma_{n+1}) \\
= {} &(-1)^{n+2} \sigma_1 f(\sigma_2, \dots, \sigma_{n+1}) \\
&+ (-1)^{n+2} \sum_{i=1}^{n} (-1)^i f(\sigma_1, \dots, \sigma_i\sigma_{i+1}, \dots, \sigma_{n+1}) \\
&+ (-1)^{n+2} (-1)^{n+1} f(\sigma_1, \dots, \sigma_n).
\end{aligned}$$

(8) Hence we have $(\delta_{n+1} \circ \delta_n)(f)(\sigma_1, \dots, \sigma_{n+2}) = 0$.

□

**Supplement.**

(1) *(Theorem 10.20 in the textbook: Rudin, Principles of Mathematical Analysis, 3rd edition.) If $\omega$ is a $k$-form of class $\mathscr{C}''$ in some open set $E \subseteq \mathbb{R}^n$, then $d^2\omega = 0$.*

(2) *(Exercise 10.16 in the textbook: Rudin, Principles of Mathematical Analysis, 3rd edition.) If $k \geq 2$ and $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_k]$ is an oriented affine $k$-simplex, prove that $\partial^2\sigma = 0$, directly from the definition of the boundary operator $\partial$. Deduce from this that $\partial^2\Psi = 0$ for every chain $\Psi$.*

(3) It is not easy to remember the definition of $\delta_n$. See page 112 of the book: *Jean-Pierre Serre, Local Fields.* (Define a covariant cochain to systems of the form $(1, \sigma_1, \sigma_1\sigma_2, \ldots, \sigma_1\sigma_2\cdots\sigma_{n+1})$.)

**Problem 10.3.**

*Let $M$ be a $G$-module, and let $f \in Z^2(G, M)$. Show that $f(1,1) = f(1,\sigma) = \sigma^{-1}f(\sigma, 1)$ for all $\sigma \in G$.*

*Proof.*

(1) $f \in Z^2(G, M)$ if and only if $\delta_2(f) = 0$. So

$$\begin{aligned}\delta_2(f)(\sigma_1, \sigma_2, \sigma_3) &= \sigma_1 f(\sigma_2, \sigma_3) - f(\sigma_1\sigma_2, \sigma_3) + f(\sigma_1, \sigma_2\sigma_3) - f(\sigma_1, \sigma_2)\\ &= 0.\end{aligned}$$

for any $\sigma_1\sigma_2, \sigma_3 \in G$.

(2) Take $\sigma_1 = \sigma_2 = 1$ and $\sigma_3 = \sigma$ to get

$$f(1,\sigma) - f(1,\sigma) + f(1,\sigma) - f(1,1) = 0.$$

So $f(1,1) = f(1,\sigma)$.

(3) Take $\sigma_1 = \sigma$ and $\sigma_2 = \sigma_3 = 1$ to get

$$\sigma f(1,1) - f(\sigma,1) + f(\sigma,1) - f(\sigma,1) = 0.$$

So $\sigma f(1,1) = f(\sigma,1)$ or $f(1,1) = \sigma^{-1}f(\sigma,1)$.

□

**Problem 10.4.**

*If $E$ is a group with an abelian normal subgroup $M$, and if $G = E/M$, show that the action of $G$ on $M$ given by $\sigma m = eme^{-1}$ if $eM = \sigma$ is well-defined and makes $M$ into a $G$-module.*

*Proof.*

(1) *Show that $G \times M \to M$ defined by $\sigma m = eme^{-1}$ is independent of the choice of the coset representation of $\sigma = eM$.* Suppose $\sigma = e_1 M = e_2 M$. $e_2 = e_1 m_1$ for some $m_1 \in M$.

(2) Therefore

$$e_2 m e_2^{-1} = (e_1 m_1) m (e_1 m_1)^{-1} = e_1 m_1 m m_1^{-1} e_1^{-1} = e_1 m e_1^{-1}.$$

Here $(e_1 m_1)^{-1} = m_1^{-1} e_1^{-1}$ holds in a group $E$ and $m_1 m m_1^{-1} = m$ since $M$ is an abelian group.

(3) *Show that $M$ is a $G$-module where $G \times M \to M$ is defined by $\sigma m = eme^{-1}$.*

(a) *Show that $1m = m$.* $1m = 1m1^{-1} = m$ where $1 = 1M \in G = E/M$.

(b) *Show that $\sigma(\tau m) = (\sigma\tau)m$.* Write $\sigma = e_\sigma M$ and $\tau = e_\tau M$. Hence $\sigma\tau = e_\sigma e_\tau M$ and

$$\begin{aligned}\sigma(\tau m) &= \sigma(e_\tau m e_\tau^{-1}) \\ &= e_\sigma(e_\tau m e_\tau^{-1})e_\sigma^{-1} \\ &= (e_\sigma e_\tau) m (e_\sigma e_\tau)^{-1} \\ &= (\sigma\tau)m.\end{aligned}$$

(c) *Show that $\sigma(m_1 + m_2) = \sigma m_1 + \sigma m_2$.*

$$\begin{aligned}\sigma(m_1 + m_2) &= e(m_1 + m_2)e^{-1} \\ &= em_1e^{-1} + em_2e^{-1} \\ &= \sigma m_1 + \sigma m_2\end{aligned}$$

where $\sigma = eM$ for some $e \in E$.

□

**Problem 10.5.**

*With $E$, $M$, $G$ as in the previous problem, if $e_\sigma$ is a coset representative of $\sigma$, show that the function defined by $f(\sigma, \tau) = e_\sigma e_\tau e_{\sigma\tau}^{-1}$ is a 2-cocycle.*

*Proof.* It suffices to show that $\delta_2(f)(\sigma, \tau, \upsilon) = 0$ for any $\sigma, \tau, \upsilon \in G$. That is,

$$\begin{aligned}&\delta_2(f)(\sigma, \tau, \upsilon) \\ &= \sigma f(\tau, \upsilon) f(\sigma\tau, \upsilon)^{-1} f(\sigma, \tau\upsilon) f(\sigma, \tau)^{-1} \\ &= \sigma f(\tau, \upsilon) f(\sigma, \tau\upsilon) f(\sigma\tau, \upsilon)^{-1} f(\sigma, \tau)^{-1} && (M\text{: abelian}) \\ &= \sigma(e_\tau e_\upsilon e_{\tau\upsilon}^{-1})(e_\sigma e_{\tau\upsilon} e_{\sigma\tau\upsilon}^{-1})(e_{\sigma\tau} e_\upsilon e_{\sigma\tau\upsilon}^{-1})^{-1}(e_\sigma e_\tau e_{\sigma\tau}^{-1})^{-1} \\ &= (e_\sigma e_\tau e_\upsilon e_{\tau\upsilon}^{-1} e_\sigma^{-1})(e_\sigma e_{\tau\upsilon} e_{\sigma\tau\upsilon}^{-1})(e_{\sigma\tau\upsilon} e_\upsilon^{-1} e_{\sigma\tau}^{-1})(e_{\sigma\tau} e_\tau^{-1} e_\sigma^{-1}) \\ &= 1.\end{aligned}$$

□

**Problem 10.6.**

*Suppose that $M$ is a $G$-module. For each $\sigma \in G$, let $m_\sigma \in M$. Show that the cochain $f$ defined by $f(\sigma, \tau) = m_\sigma + \sigma m_\tau - m_{\sigma\tau}$ is a coboundary.*

*Proof.*

(1) To show $f$ is a 2-coboundary, it suffices to show that there is a $g \in C^1(G, M)$ such that $f = \delta_1(g)$.

(2) Actually, we can define $g : G \to M$ by $\sigma \mapsto m_\sigma$. So

$$\delta_1(g)(\sigma, \tau) = \sigma g(\tau) - g(\sigma\tau) + g(\sigma) = \sigma m_\tau - m_{\sigma\tau} + m_\sigma = f(\sigma, \tau)$$

for all $\sigma, \tau \in G$. Hence $f \in B^2(G, M)$.

□

**Problem 10.7.**

*If $M$ is a $G$-module and $f \in Z^2(G, M)$, show that $E_f = M \times G$ with multiplication defined by*

$$(m, \sigma)(n, \tau) = (m \cdot \sigma n \cdot f(\sigma, \tau), \sigma\tau)$$

*makes $E_f$ into a group.*

*Proof.*

(1) The multiplication is a binary operation on $E_f$.

(2) (Associativity) *Show that*

$$((m, \sigma)(n, \tau))(k, \upsilon) = (m, \sigma)((n, \tau)(k, \upsilon)).$$

*for all* $(m, \sigma), (n, \tau), (k, \upsilon)$. Note that

$$\begin{aligned}
((m, \sigma)(n, \tau))(k, \upsilon) &= (m \cdot \sigma n \cdot f(\sigma, \tau), \sigma\tau)(k, \upsilon) \\
&= (m \cdot \sigma n \cdot f(\sigma, \tau) \cdot \sigma\tau k \cdot f(\sigma\tau, \upsilon), \sigma\tau\upsilon) \\
&= (m \cdot \sigma n \cdot \sigma\tau k \cdot f(\sigma, \tau) \cdot f(\sigma\tau, \upsilon), \sigma\tau\upsilon)
\end{aligned}$$

and

$$\begin{aligned}
(m, \sigma)((n, \tau)(k, \upsilon)) &= (m, \sigma)(n \cdot \tau k \cdot f(\tau, \upsilon), \tau\upsilon) \\
&= (m \cdot \sigma(n \cdot \tau k \cdot f(\tau, \upsilon)) \cdot f(\sigma, \tau\upsilon), \sigma\tau\upsilon) \\
&= (m \cdot \sigma n \cdot \sigma\tau k \cdot \underbrace{\sigma f(\tau, \upsilon) \cdot f(\sigma, \tau\upsilon)}_{= f(\sigma, \tau) \cdot f(\sigma\tau, \upsilon)}, \sigma\tau\upsilon)
\end{aligned}$$

(since $f \in Z^2(G, M)$).

(3) (Identity element) *Show that there exists an element*

$$1 := (f(1,1)^{-1}, 1) \in E_f$$

*such that* $1(m, \sigma) = (m, \sigma)1 = (m, \sigma)$ *for every* $(m, \sigma) \in E_f$. Same as Problem 10.3. Note that

$$\begin{aligned}
(m, \sigma)(f(1,1)^{-1}, 1) &= (m \cdot \sigma \underbrace{f(1,1)^{-1}}_{=\sigma^{-1} f(\sigma,1)^{-1}} \cdot f(\sigma, 1), \sigma) \\
&= (m \cdot \sigma(\sigma^{-1} f(\sigma, 1)^{-1}) \cdot f(\sigma, 1), \sigma) \\
&= (m \cdot (\sigma\sigma^{-1}) f(\sigma, 1)^{-1} \cdot f(\sigma, 1), \sigma) \\
&= (m, \sigma)
\end{aligned}$$

and

$$\begin{aligned}
(f(1,1)^{-1}, 1)(m, \sigma) &= (f(1,1)^{-1} \cdot m \cdot f(1, \sigma), \sigma) \\
&= (f(1, \sigma)^{-1} \cdot m \cdot f(1, \sigma), \sigma) \\
&= (m, \sigma).
\end{aligned}$$

(4) (Inverse element) *Show that for each* $(m, \sigma) \in E_f$*, there exists an element*

$$(n, \tau) := \left(\sigma^{-1} \left\{ f(\sigma, \sigma^{-1})^{-1} \cdot m^{-1} \cdot f(1,1)^{-1} \right\}, \sigma^{-1}\right) \in E_f$$

*such that* $(m, \sigma)(n, \tau) = (n, \tau)(m, \sigma) = 1$*, where* $1$ *is the identity element in* $E_f$. (To find the inverse element, we might apply the same argument in the following note.) A direct calculation with the fact that $f \in Z^2(G, M)$ gives

$$\begin{aligned}
&(m, \sigma)\left(\sigma^{-1} \left\{ f(\sigma, \sigma^{-1})^{-1} \cdot m^{-1} \cdot f(1,1)^{-1} \right\}, \sigma^{-1}\right) \\
&= \left(m \cdot \sigma \left(\sigma^{-1} \left\{ f(\sigma, \sigma^{-1})^{-1} \cdot m^{-1} \cdot f(1,1)^{-1} \right\}\right) \cdot f(\sigma, \sigma^{-1}), 1\right) \\
&= (m \cdot f(\sigma, \sigma^{-1})^{-1} \cdot m^{-1} \cdot f(1,1)^{-1} \cdot f(\sigma, \sigma^{-1}), 1) \\
&= (f(1,1)^{-1}, 1)
\end{aligned}$$

and

$$\begin{aligned}
&\left(\sigma^{-1} \left\{ f(\sigma, \sigma^{-1})^{-1} \cdot m^{-1} \cdot f(1,1)^{-1} \right\}, \sigma^{-1}\right)(m, \sigma) \\
&= \left(\sigma^{-1} \left\{ f(\sigma, \sigma^{-1})^{-1} \cdot m^{-1} \cdot f(1,1)^{-1} \right\} \cdot \sigma^{-1} m \cdot f(\sigma^{-1}, \sigma), 1\right) \\
&= (\sigma^{-1} f(\sigma, \sigma^{-1})^{-1} \cdot f(\sigma^{-1}, \sigma) \cdot \sigma^{-1} f(1,1)^{-1}, 1) \\
&= (f(1,1)^{-1} \cdot \underbrace{\sigma^{-1} f(1,1) \cdot \sigma^{-1} f(1,1)^{-1}}_{=1}, 1) \\
&= (f(1,1)^{-1}, 1).
\end{aligned}$$

Here we take $(\sigma_1, \sigma_2, \sigma_3) \mapsto (\sigma^{-1}, \sigma, \sigma^{-1})$ in part (1) of the proof of Problem 10.3 to get

$$\begin{aligned}\sigma^{-1} f(\sigma, \sigma^{-1})^{-1} \cdot f(\sigma^{-1}, \sigma) &= f(1, \sigma^{-1})^{-1} \cdot f(\sigma^{-1}, 1) \\ &= f(1,1)^{-1} \cdot \sigma^{-1} f(1,1).\end{aligned}$$

□

*Note.* To find the identity element, we need to find $(n, \tau)$ such that $(m, \sigma)(n, \tau) = (m, \sigma)$. So

$$(m, \sigma)(n, \tau) = (m \cdot \sigma n \cdot f(\sigma, \tau), \sigma\tau) = (m, \sigma)$$

implies that $\tau = 1 \in G$ and thus $m \cdot \sigma n \cdot f(\sigma, 1) = m$. Hence

$$n = \sigma^{-1} f(\sigma, 1)^{-1} = (\sigma^{-1} f(\sigma, 1))^{-1} = f(1,1)^{-1}$$

(in the multiplicative notation).

**Problem 10.8.**

*If $M$ is a $G$-module, show that the group extensions constructed from 2-cocycles $f, g \in Z^2(G, M)$ are isomorphic if $f$ and $g$ are cohomologous.*

*Proof.*

(1) Say $f \cdot g^{-1} = \delta_1(h) \in B^2(G, M)$ for some $h \in C^1(G, M)$, i.e.,

$$f(\sigma, \tau) \cdot g^{-1}(\sigma, \tau) = \delta_1(h)(\sigma, \tau) = \sigma h(\tau) \cdot h(\sigma\tau)^{-1} \cdot h(\sigma).$$

(2) By the help of $h$, define a map $\alpha : E_f \to E_g$ by

$$\alpha((m, \sigma)) = (m \cdot h(\sigma), \sigma).$$

Now it suffices to show that $\alpha$ is a group isomorphism.

(3) *Show that $\alpha$ is a group homomorphism.* Note that

$$\begin{aligned}\alpha((m, \sigma)(n, \tau)) &= \alpha((m \cdot \sigma n \cdot f(\sigma, \tau), \sigma\tau)) \\ &= (m \cdot \sigma n \cdot f(\sigma, \tau) \cdot h(\sigma\tau), \sigma\tau)\end{aligned}$$

and

$$\begin{aligned}\alpha((m, \sigma))\alpha((n, \tau)) &= (m \cdot h(\sigma), \sigma)(n \cdot h(\tau), \tau) \\ &= (m \cdot h(\sigma) \cdot \sigma(n \cdot h(\tau)) \cdot f(\sigma, \tau), \sigma\tau) \\ &= (m \cdot \sigma n \cdot f(\sigma, \tau) \cdot \underbrace{\sigma h(\tau) \cdot h(\sigma)}_{=h(\sigma\tau)}, \sigma\tau). \qquad ((1))\end{aligned}$$

Hence $\alpha((m, \sigma)(n, \tau)) = \alpha((m, \sigma))\alpha((n, \tau))$.

(3) *Show that $\alpha$ is injective.* $\alpha((m,\sigma)) = \alpha((n,\tau))$ implies that $(m \cdot h(\sigma), \sigma) = (n \cdot h(\tau), \tau)$. So $\sigma = \tau$, $h(\sigma) = h(\tau)$, and thus $m = n$.

(4) *Show that $\alpha$ is surjective.* Given any $(m,\sigma) \in E_g$, we have

$$\alpha((m \cdot h(\sigma)^{-1}, \sigma)) = (m, \sigma).$$

□

*Note.*

(1) A 2-cocycle is also called a **factor set**.

(2) $H^2(G, M)$ is the set of isomorphism classes of extension of $G$ by $M$.

**Problem 10.9.**

*In the crossed product construction given in this section, show that the multiplicative identity is $f(1,1)^{-1}x_{\mathrm{id}}$.*

*Proof.*

(1)

$$\begin{aligned}(f(1,1)^{-1}x_{\mathrm{id}}) \sum_{\sigma \in G} a_\sigma x_\sigma &= \sum_{\sigma \in G} f(1,1)^{-1}\mathrm{id}(a_\sigma) f(\mathrm{id}, \sigma) x_{\mathrm{id}\cdot\sigma} \\ &= \sum_{\sigma \in G} f(1,1)^{-1} a_\sigma f(1,\sigma) x_\sigma \\ &= \sum_{\sigma \in G} a_\sigma x_\sigma\end{aligned}$$

for all $\sum_{\sigma \in G} a_\sigma x_\sigma \in A = (K/F, G, f)$.

(2)

$$\begin{aligned}\left(\sum_{\sigma \in G} a_\sigma x_\sigma\right)(f(1,1)^{-1}x_{\mathrm{id}}) &= \sum_{\sigma \in G} a_\sigma \sigma(f(1,1)^{-1}) f(\sigma, \mathrm{id}) x_{\sigma\cdot\mathrm{id}} \\ &= \sum_{\sigma \in G} a_\sigma \sigma f(1,1)^{-1} f(\sigma, 1) x_\sigma \\ &= \sum_{\sigma \in G} a_\sigma x_\sigma\end{aligned}$$

for all $\sum_{\sigma \in G} a_\sigma x_\sigma \in A = (K/F, G, f)$.

□

**Problem 10.10.**

*A **normalized cocycle** is a cocycle $f$ that satisfies $f(1,\sigma)=f(\sigma,1)=1$ for all $\sigma\in G$. Let $A=(K/F,G,f)$ be a crossed product algebra. Show that $x_{\mathrm{id}}=1$ if and only if $f$ is a normalized cocycle.*

*Proof.*

$$\begin{aligned}
& f \text{ is a normalized cocycle} \\
\iff & f(1,\sigma)=\sigma^{-1}f(\sigma,1)=1 \text{ for all } \sigma\in G \\
\iff & f(1,\sigma)=f(\sigma,1)=1 \text{ for all } \sigma\in G \\
\iff & f(1,1)=1 && \text{(Problem 10.3)} \\
\iff & \text{the multiplicative identity is } f(1,1)^{-1}x_{\mathrm{id}}=x_{\mathrm{id}}. && \text{(Problem 10.9)}
\end{aligned}$$

□

**Problem 10.11.**

*In the construction of group extensions, show that if $e_{\mathrm{id}}$ is chosen to be $1$, then the resulting cocycle is a normalized cocycle.*

*Proof.* Suppose $f\in Z^2(G,M)$. In Problem 10.5, we take $\sigma=\tau=\mathrm{id}$ in $f(\sigma,\tau)=e_\sigma e_\tau e_{\sigma\tau}^{-1}$ to reach

$$f(1,1)=e_{\mathrm{id}}e_{\mathrm{id}}e_{\mathrm{id}}^{-1}=e_{\mathrm{id}}=1.$$

Problem 10.10 implies that $f$ is a normalized cocycle. □

**Problem 10.12.**

*Show that any 2-cocycle is cohomologous to a normalized cocycle.*

*Proof.*

(1) Similar to Problem 10.8, we need to find a normalized cocycle $g$ and $h\in C^1(G,M)$ such that $f\cdot g^{-1}=\delta_1(h)$. So

$$f(\sigma,\tau)\cdot g^{-1}(\sigma,\tau)=\delta_1(h)(\sigma,\tau)=\sigma h(\tau)\cdot h(\sigma\tau)^{-1}\cdot h(\sigma).$$

(2) Let $\tau=1$ to get

$$f(\sigma,1)=\underbrace{g(\sigma,1)}_{=1}\cdot\sigma h(1)\cdot\underbrace{h(\sigma)^{-1}\cdot h(\sigma)}_{=1}=\sigma h(1)=h(\sigma).$$

Hence we might define $h(\sigma):=f(\sigma,1)$ and $g:=f\cdot\delta_1(h)^{-1}$.

□