Chapter 5: Differentiation

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Exercise 5.1. *Let f be defined for all real x, and suppose that*

$$|f(x) - f(y)| \leq (x - y)^2$$

for all real x and y. Prove that f is a constant.

Proof.

(1) Write
$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq |x - y|$$
if $x \neq y$.

(2) Given any $y \in \mathbb{R}$,
$$\left| \frac{f(x) - f(y)}{x - y} \right| \to 0 \quad \text{as} \quad x \to y,$$
or $|f'(y)| = 0$.

(3) Or using ε-δ argument. Fix $y \in \mathbb{R}$. Given any $\varepsilon > 0$, there exists $\delta = \varepsilon > 0$ such that
$$\left| \frac{f(x) - f(y)}{x - y} - 0 \right| \leq |x - y| < \delta = \varepsilon$$
whenever $|x - y| < \delta$. That is, $|f'(y)| = 0$.

(4) So $f'(y) = 0$ for any $y \in \mathbb{R}$. By Theorem 5.11 (b), f is a constant.

□

Exercise 5.2. *Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b), and let g be its inverse function. Prove that g is differentiable, and that*

$$g'(f(x)) = \frac{1}{f'(x)} \qquad (a < x < b).$$

Proof. Let $E = (a, b)$.

(1) Theorem 5.10 implies that for any $a < p < q < b$ there exists $\xi \in (p, q)$ such that

$$f(p) - f(q) = (p - q)f'(\xi).$$

Since $\xi \in (p, q) \subseteq E$, by assumption $f'(\xi) > 0$. Hence $f(p) - f(q) = (p - q)f'(\xi) < 0$ (here $p - q < 0$), or

$$f(p) < f(q)$$

if $p < q$. Therefore, f is strictly increasing in (a, b).

(2) *Show that f is one-to-one in E if f is strictly increasing in E.* If $f(p) = f(q)$, then it cannot be $p > q$ or $p < q$ ((1)). So that $p = q$, or f is injective.

(3) *Show that g is well-defined.* Theorem 5.2 and Theorem 4.17.

(4) *Show that $g'(f(x)) = \frac{1}{f'(x)}$.* Given $y \in f(E)$, say $y = f(x)$ for some $x \in E$. Given any $s \in f(E)$ with $s \neq y$. Here $s = f(t)$ for some $t \in E$ and $t \neq x$.

$$\begin{aligned}
\lim_{s \to y} \frac{g(s) - g(y)}{s - y} &= \lim_{f(t) \to f(x)} \frac{g(f(t)) - g(f(x))}{f(t) - f(x)} \\
&= \lim_{t \to x} \frac{t - x}{f(t) - f(x)} \\
&= \lim_{t \to x} \frac{1}{\frac{f(t) - f(x)}{t - x}} \\
&= \frac{1}{f'(x)}. && (f' > 0)
\end{aligned}$$

Here $s \to y$ if and only if $t \to x$ since both f and g are continuous and one-to-one. Hence g is differentiable and $g'(f(x)) = \frac{1}{f'(x)}$.

□

Exercise 5.3. *Suppose g is a real function on $\mathbb{R}^1$, with bounded derivative (say $|g'| \leq M$). Fix $\varepsilon > 0$, and define $f(x) = x + \varepsilon g(x)$. Prove that f is one-to-one if ε is small enough. (A set of admissible values of ε can be determined which depends only on M.)*

Proof.

(1) Note that $f'(x) = 1 + \varepsilon g'(x)$ (Theorem 5.3). Since $|g'| \leq M$,

$$1 - \varepsilon M \leq f'(x) \leq 1 + \varepsilon M.$$

(2) Pick

$$\varepsilon = \frac{1}{M + 1} > 0.$$

Thus,

$$f'(x) \geq \frac{1}{M+1} > 0.$$

By Exercise 5.2, $f(x)$ is strictly increasing in $\mathbb{R}$ or one-to-one in $\mathbb{R}$.

□

Exercise 5.4. *If*

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$

where $C_0, \ldots, C_n$ are real constants, prove that the equation

$$C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n = 0$$

has at least one real root between 0 *and* 1*.*

Proof. Let

$$g(x) = C_0 x + \frac{C_1}{2}x^2 + \cdots + \frac{C_{n-1}}{n}x^n + \frac{C_n}{n+1}x^{n+1} \in \mathbb{R}[x].$$

Then $g(0) = g(1) = 0$, and $g'(x) = C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n$. By the mean value theorem (Theorem 5.10), there exists a point $\xi \in (0,1)$ at which

$$g(1) - g(0) = g'(\xi)(1-0),$$

or $g'(\xi) = 0$. That is, there exists a real root $x = \xi$ between 0 and 1 at which $C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n = 0$. □

Exercise 5.5. *Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.*

Proof. Given any $x > 0$. Since f is differentiable for every $x > 0$, f is differentiable on $[x, x+1]$. By Theorem 5.2 and Theorem 5.10 (the mean value theorem), there is a point $\xi \in (x, x+1)$ at which

$$f(x+1) - f(x) = [(x+1) - x]f'(\xi)$$

or

$$g(x) = f'(\xi).$$

As $x \to +\infty$, $\xi \to +\infty$. Hence

$$\lim_{x \to +\infty} g(x) = \lim_{\xi \to +\infty} f'(\xi) = 0.$$

$\square$

Exercise 5.6. *Suppose*

(a) f *is continuous for* $x \geq 0$,

(b) $f'(x)$ *exists for* $x > 0$,

(c) $f(0) = 0$,

(d) f' *is monotonically increasing.*

Put

$$g(x) = \frac{f(x)}{x} \qquad (x > 0)$$

and prove that g *is monotonically increasing.*

Proof.

(1) It suffices to show that $g'(x) \geq 0$ for $x > 0$ (Theorem 5.11(a)), that is, to show that

$$g'(x) = \frac{xf'(x) - f(x)}{x^2} \geq 0 \qquad (x > 0),$$

or

$$xf'(x) - f(x) \geq 0 \qquad (x > 0)$$

since $x^2 > 0$ for all nonzero x.

(2) Given $x > 0$. By (a)(b), we apply the mean value theorem (Theorem 5.10) on f to get

$$f(x) - f(0) = (x - 0)f'(\xi)$$

for some $\xi \in (0, x)$. By (c),

$$f(x) = xf'(\xi).$$

By (d),

$$f(x) = xf'(\xi) \leq xf'(x).$$

Hence $xf'(x) - f(x) \geq 0$, or g is monotonically increasing.

$\square$

Note. g is increasing strictly if f is increasing strictly.

Exercise 5.7. *Suppose* $f'(x)$, $g'(x)$ *exist,* $g'(x) \neq 0$, *and* $f(x) = g(x) = 0$. *Prove that*

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}.$$

(This holds also for complex functions.)

Proof.

$$\begin{aligned}
\frac{f'(t)}{g'(t)} &= \frac{\lim_{t \to x} \frac{f(t)-f(x)}{t-x}}{\lim_{t \to x} \frac{g(t)-g(x)}{t-x}} \\
&= \lim_{t \to x} \frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}} && \text{(Both limits exist and } g' \neq 0) \\
&= \lim_{t \to x} \frac{f(t)}{g(t)}. && (f(x) = g(x) = 0)
\end{aligned}$$

This proof is also true for complex functions. $\square$

Exercise 5.8. *Suppose $f'(x)$ is continuous on $[a, b]$ and $\varepsilon > 0$. Prove that there exists $\delta > 0$ such that*

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \varepsilon$$

whenever $0 < |t - x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$. (This could be expressed by saying f is ***uniformly differentiable*** *on $[a, b]$ if f' is continuous on $[a, b]$.) Does this hold for vector-valued functions too?*

Proof.

(1) Since $f'(x)$ is continuous on a compact set $[a, b]$, $f'(x)$ is uniformly continuous on $[a, b]$. So given any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f'(t) - f'(x)| < \varepsilon$$

whenever $0 < |t - x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$.

(2) For such $t < x$ in (1), by the mean value theorem (Theorem 5.10), there exists a point $\xi \in (t, x)$ at which

$$f'(\xi) = \frac{f(t) - f(x)}{t - x}.$$

Note that ξ is also satisfying $0 < |t - \xi| < |t - x| < \delta$ and $a \leq \xi \leq b$. Hence by (1) we also have

$$|f'(\xi) - f'(x)| < \varepsilon,$$

or

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \varepsilon.$$

(3) *Suppose* $\mathbf{f}'(x)$ *is continuous on* $[a, b]$ *and* $\varepsilon > 0$. *Prove that there exists* $\delta > 0$ *such that*

$$\left| \frac{\mathbf{f}(t) - \mathbf{f}(x)}{t - x} - \mathbf{f}'(x) \right| < \varepsilon$$

whenever $0 < |t - x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$.

(a) Write

$$\mathbf{f}(x) = (f_1(x), \ldots, f_k(x)) \in \mathbb{R}^k.$$

By Remarks 5.16, $\mathbf{f}(x)$ is differentiable at a point x if and only if each $f_1, \ldots, f_k$ is differentiable at x. So that

$$\mathbf{f}'(x) = (f_1'(x), \ldots, f_k'(x)) \in \mathbb{R}^k.$$

By Theorem 4.10, $\mathbf{f}'(x)$ is continuous if and only if each $f_1, \ldots, f_k$ is continuous.

(b) Similar to (1)(2), Since $f_i'(x)$ is continuous on a compact set $[a, b]$ where $1 \leq i \leq k$, $f_i'(x)$ is uniformly continuous on $[a, b]$. So given any $\varepsilon > 0$ there exists $\delta_i > 0$ such that

$$|f_i'(t) - f_i'(x)| < \frac{\varepsilon}{\sqrt{k}}$$

whenever $0 < |t - x| < \delta_i$, $a \leq x \leq b$, $a \leq t \leq b$. Take $\delta = \min_{1 \leq i \leq k} \delta_i > 0$.

(c) For such $t < x$ in (1), by the mean value theorem (Theorem 5.10), there exists a point $\xi_i \in (t, x)$ at which

$$f_i'(\xi_i) = \frac{f_i(t) - f_i(x)}{t - x}.$$

Note that ξ_i is also satisfying $0 < |t - \xi_i| < |t - x| < \delta$ and $a \leq \xi_i \leq b$. Hence by (1) we also have

$$|f_i'(\xi_i) - f_i'(x)| < \frac{\varepsilon}{\sqrt{k}},$$

or

$$\left| \frac{f_i(t) - f_i(x)}{t - x} - f_i'(x) \right| < \frac{\varepsilon}{\sqrt{k}}.$$

(d) Hence

$$\left| \frac{\mathbf{f}(t) - \mathbf{f}(x)}{t - x} - \mathbf{f}'(x) \right| = \left(\sum_{i=1}^{k} \left| \frac{f_i(t) - f_i(x)}{t - x} - f_i'(x) \right|^2 \right)^{\frac{1}{2}} < \varepsilon.$$

$\square$

Exercise 5.9. *Let f be a continuous real function on $\mathbb{R}^1$, of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$. Dose it follow that $f'(0)$ exists?*

Proof.

(1) *Show that $f'(0) = 3$.* It is equivalent to show that

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 3.$$

Write $F(x) = f(x) - f(0)$ and $G(x) = x - 0$ on $\mathbb{R}^1$. So that

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{F(x)}{G(x)} = 0.$$

(2) Note that

$$\lim_{x \to 0} \frac{F'(x)}{G'(x)} = \lim_{x \to 0} \frac{f'(x)}{1} = 3.$$

(3) Since f is continuous on $\mathbb{R}^1$, F is continuous on $\mathbb{R}^1$. Hence

$$\lim_{x \to 0} F(x) = F(\lim_{x \to 0} x) = F(0) = 0.$$

Also, G is continuous on $\mathbb{R}^1$ implies that

$$\lim_{x \to 0} G(x) = G(\lim_{x \to 0} x) = G(0) = 0.$$

(4) Apply L'Hospital's rule (Theorem 5.13) to (2)(3), we have

$$\lim_{x \to 0} \frac{F(x)}{G(x)} = 3,$$

or $f'(0) = 3$.

□

Exercise 5.10. *Suppose f and g are complex differentiable functions on $(0, 1)$, $f(x) \to 0$, $g(x) \to 0$, $f'(x) \to A$, $g'(x) \to B$ as $x \to 0$, where A and B are complex numbers, $B \neq 0$. Prove that*

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{A}{B}.$$

Compare with Example 5.18. (Hint:

$$\frac{f(x)}{g(x)} = \left(\frac{f(x)}{x} - A \right) \frac{x}{g(x)} + A \frac{x}{g(x)}.$$

Apply Theorem 5.13 to the real and imaginary parts of $\frac{f(x)}{x}$ and $\frac{g(x)}{x}$.)

Proof (Hint).

(1) Write
$$f(x) = f_1(x) + if_2(x)$$
for $x \in (0,1)$, where both f_1 and f_2 are real functions. By Remarks 5.16, it is clear that
$$f'(x) = f_1'(x) + if_2'(x).$$

(2) Write
$$A = A_1 + iA_2$$
where both A_1 and A_2 are real numbers. Then as $x \to 0$, we have

(a) $f(x) \to 0$ if and only if $f_1(x) \to 0$ and $f_2(x) \to 0$.

(b) $f'(x) \to A$ if and only if $f_1'(x) \to A_1$ and $f_2'(x) \to A_2$.

Hence by L'Hospital's rule (Theorem 5.13),
$$\lim_{x\to 0} \frac{f_i(x)}{x} = \lim_{x\to 0} \frac{f_i'(x)}{1} = A_i$$
$(i = 1, 2)$ or
$$\begin{aligned}\lim_{x\to 0} \frac{f(x)}{x} &= \lim_{x\to 0} \frac{f_1(x) + if_2(x)}{x} \\ &= \lim_{x\to 0} \frac{f_1(x)}{x} + i \lim_{x\to 0} \frac{f_2(x)}{x} \\ &= A_1 + iA_2 \\ &= A.\end{aligned}$$

Similarly,
$$\lim_{x\to 0} \frac{g(x)}{x} = B.$$
Note that $B \neq 0$, and thus
$$\lim_{x\to 0} \frac{x}{g(x)} = \frac{1}{B}.$$

(3) Hence
$$\begin{aligned}\lim_{x\to 0} \frac{f(x)}{g(x)} &= \lim_{x\to 0} \left[\left(\frac{f(x)}{x} - A \right) \frac{x}{g(x)} + A \frac{x}{g(x)} \right] \\ &= \lim_{x\to 0} \left(\frac{f(x)}{x} - A \right) \cdot \lim_{x\to 0} \frac{x}{g(x)} + \lim_{x\to 0} A \frac{x}{g(x)} \\ &= 0 \cdot \frac{1}{B} + \frac{A}{B} \\ &= \frac{A}{B}.\end{aligned}$$

(4) *Compare with Example 5.18.* Define $f(x) = x$ and $g(x) = x + x^2 \exp\left(\frac{i}{x^2}\right)$ as in Example 5.18. Note that $f(x) \to 0$, $g(x) \to 0$, $f'(x) \to 1$ and $g'(x) \to \infty$ as $x \to 0$. By Example 5.18

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = 1 \neq 0 = \frac{1}{\infty} = \lim_{x \to 0} \frac{A}{B}.$$

$\square$

Exercise 5.11. *Suppose f is defined in a neighborhood of x, and suppose $f''(x)$ exists. Show that*

$$\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

Show by an example that the limit may exist even if $f''(x)$ dose not. (Hint: Use Theorem 5.13.)

Proof (Theorem 5.13).

(1) Write $F(h) = f(x+h) + f(x-h) - 2f(x)$ and $G(h) = h^2$. It is equivalent to show that
$$\lim_{h \to 0} \frac{F(h)}{G(h)} = f''(x).$$
We might apply Theorem 5.13 (L'Hospital rule) to get it.

(2) *Show that* $\lim_{h \to 0} F(h) = 0$ *and* $\lim_{h \to 0} G(h) = 0$. It is clear that $\lim_{h \to 0} G(h) = \lim_{h \to 0} h^2 = 0$ since $x \mapsto x^2$ is continuous on $\mathbb{R}^1$. Besides, since f is continuous at x (by applying Theorem 5.2 twice),
$$\lim_{h \to 0} F(h) = f(x) + f(x) - 2f(x) = 0.$$

(3) *Show that*
$$\lim_{h \to 0} \frac{F'(h)}{G'(h)} = \lim_{h \to 0} \frac{f'(x+h) - f'(x-h)}{2h}$$
is well-defined. Since $f''(x)$ exists in a neighborhood $B(x; r)$ of x (where $r > 0$), $f'(x)$ exists and is continuous in $B(x; r)$ (Theorem 5.2). As $0 < |h| < \frac{r}{2}$,
$$x + h \in B\left(x + h; \frac{r}{2}\right) \subseteq B(x; r)$$
and
$$x - h \in B\left(x - h; \frac{r}{2}\right) \subseteq B(x; r).$$
So $f'(x+h)$ and $f'(x-h)$ exist in $B(x; r)$ as $0 < |h| < \frac{r}{2}$. Hence
$$\lim_{h \to 0} \frac{F'(h)}{G'(h)} = \lim_{h \to 0} \frac{f'(x+h) - f'(x-h)}{2h}$$
is well-defined (Theorem 5.3 and Theorem 5.5 (the chain rule)).

(4) *Show that*

$$\lim_{h \to 0} \frac{f'(x+h) - f'(x-h)}{2h} = f''(x).$$

Since $f''(x)$ exists, by definition

$$\lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = f''(x)$$

and

$$\lim_{h \to 0} \frac{f'(x-h) - f'(x)}{-h} = f''(x).$$

Sum up two expressions to get

$$2f''(x) = \lim_{h \to 0} \frac{f'(x-h) - f'(x-h)}{h}.$$

(5) By (2)(3)(4) and Theorem 5.13 (L'Hospital rule), the result is established.

(6) *Given* $f(x) = x|x|$ *on* $\mathbb{R}^1$. *Show that*

$$\lim_{h \to 0} \frac{f(h) + f(-h) - 2f(0)}{h^2} = 0$$

but $f''(x)$ *does not exist at* $x = 0$. Clearly,

$$\begin{aligned} \lim_{h \to 0} \frac{f(h) + f(-h) - 2f(0)}{h^2} &= \lim_{h \to 0} \frac{h|h| + (-h)| - h| - 2 \cdot 0}{h^2} \\ &= \lim_{h \to 0} \frac{h|h| - h|h| - 0}{h^2} \\ &= \lim_{h \to 0} 0 \\ &= 0. \end{aligned}$$

But $f''(x)$ does not exist by Exercise 5.12.

□

Exercise 5.12. *If* $f(x) = |x|^3$, *compute* $f'(x)$, $f''(x)$ *for all real* x, *and show that* $f^{(3)}(0)$ *does not exist.*

Proof.

(1) Write

$$f(x) = \begin{cases} x^3 & (x \geq 0), \\ -x^3 & (x \leq 0). \end{cases}$$

(2) *Show that* $f'(x) = 3x|x|$. It is trivial that

$$f'(x) = \begin{cases} 3x^2 & (x > 0), \\ -3x^2 & (x < 0). \end{cases}$$

Note that

$$\lim_{x \to 0} f'(x) = 0.$$

Apply the same argument in Exercise 5.9, we have

$$f'(0) = 0.$$

Hence f' exists and $f'(x) = 3x|x|$ for any $x \in \mathbb{R}$.

(3) *Show that* $f''(x) = 6|x|$. Similar to (2).

$$f''(x) = \begin{cases} 6x & (x > 0), \\ -6x & (x < 0). \end{cases}$$

Note that

$$\lim_{x \to 0} f''(x) = 0.$$

Apply the same argument in Exercise 5.9, we have

$$f''(0) = 0.$$

Hence f'' exists and $f''(x) = 6|x|$ for any $x \in \mathbb{R}$.

(4) *Show that* $f^{(3)}(0)$ *does not exist.*

$$f'''(x) = \begin{cases} 6 & (x > 0), \\ -6 & (x < 0). \end{cases}$$

There are some proofs for showing that $f^{(3)}(0)$ does not exist.

(a) Since

$$\lim_{t \to 0+} \frac{f''(t) - f''(0)}{t - 0} = \lim_{t \to 0+} \frac{6t}{t} = 6$$

and

$$\lim_{t \to 0-} \frac{f''(t) - f''(0)}{t - 0} = \lim_{t \to 0-} \frac{-6t}{t} = -6,$$

$f^{(3)}(0)$ does not exist.

(b) (Reductio ad absurdum) If f were differentiable on $\mathbb{R}^1$, then

$$\lim_{t \to 0+} f'''(t) = 6$$

and

$$\lim_{t \to 0-} f'''(t) = -6,$$

or f''' has a simple discontinuity at $x = 0$, contrary to Corollary to Theorem 5.12.

$\square$

Note. Given $k > 0$. We can construct one real function f on $\mathbb{R}^1$, say

$$f(x) = \begin{cases} |x|^k & (k \text{ is odd}), \\ x|x|^{k-1} & (k > 0 \text{ is even}), \end{cases}$$

such that all $f^{(0)}(0) = \cdots = f^{(k-1)}(0) = 0$ exist but $f^{(k)}(0)$ does not exist.

Exercise 5.13. *Suppose a and c are real numbers, $c > 0$, and f is defined on $[-1, 1]$ by*

$$f(x) = \begin{cases} x^a \sin(x^{-c}) & (\textit{if } x \neq 0), \\ 0 & (\textit{if } x = 0). \end{cases}$$

Prove the following statements:

(a) *f is continuous if and only if $a > 0$.*

(b) *$f'(0)$ exists if and only if $a > 1$.*

(c) *f' is bounded if and only if $a \geq 1 + c$.*

(d) *f' is continuous if and only if $a > 1 + c$.*

(e) *$f''(0)$ exists if and only if $a > 2 + c$.*

(f) *f'' is bounded if and only if $a \geq 2 + 2c$.*

(g) *f'' is continuous if and only if $a > 2 + 2c$.*

Note that f is not well-defined as a real function if $x < 0$. Hence we modify the definition of f for the case $x < 0$:

$$f(x) = \begin{cases} |x|^a \sin(|x|^{-c}) & (\text{if } x \neq 0), \\ 0 & (\text{if } x = 0). \end{cases}$$

Proof of (a).

(1) Since $|x|^a \sin(|x|^{-c})$ is continuous on $\mathbb{R}^1 - \{0\}$, f is continuous if and only if

$$\lim_{x \to 0} |x|^a \sin(|x|^{-c}) = 0.$$

(2) *Given $a > 0$. Show that*

$$\lim_{x \to 0} |x|^a \sin(|x|^{-c}) = 0.$$

Since $|x|^a \to 0$ as $x \to 0$ and $|\sin(|x|^{-c})|$ is bounded by 1, the limit $\lim |x|^a \sin(|x|^{-c})$ exists and is equal to 0.

(3) *Given* $a = 0$. *Show that*

$$\lim_{x \to 0} |x|^a \sin(|x|^{-c}) = \lim_{x \to 0} \sin(|x|^{-c})$$

does not exist although $|x|^a \sin(|x|^{-c}) = \sin(|x|^{-c})$ *is bounded on* $[-1,1] - \{0\}$.

(a) Take $x_n = \left(\frac{\pi}{2} + 2n\pi\right)^{-\frac{1}{c}} \neq 0$ for $n = 1, 2, 3, \ldots$. The sequence $\{x_n\}$ converges to 0, and

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} \sin(|x_n|^{-c}) = \lim_{n \to \infty} 1 = 1.$$

(b) Similarly, take $y_n = (2n\pi)^{-\frac{1}{c}} \neq 0$ for $n = 1, 2, 3, \ldots$. The sequence $\{y_n\}$ converges to 0, and

$$\lim_{n \to \infty} f(y_n) = 0.$$

(c) By (a)(b), $\lim_{x \to 0} |x|^a \sin(|x|^{-c})$ does not exist (Theorem 4.2).

(d) Clearly, $|\sin(|x|^{-c})| \leq 1$ as $\sin(|x|^{-c})$ is well-defined.

(4) *Given* $a < 0$. *Show that*

$$\lim_{x \to 0} |x|^a \sin(|x|^{-c})$$

does not exist. Similar to (3), we take the same $\{x_n\}$ and $\{y_n\}$ as (3) to get the similar result:

$$\lim_{n \to \infty} f(x_n) = \infty,$$
$$\lim_{n \to \infty} f(y_n) = 0.$$

By Theorem 4.2, $\lim_{x \to 0} |x|^a \sin(|x|^{-c})$ does not exist.

(5) By (2)(3)(4), f is continuous if and only if $a > 0$.

$\square$

Proof of (b).

(1) By definition,

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \operatorname{sgn}(x)|x|^{a-1} \sin(|x|^{-c}).$$

Here $\operatorname{sgn}(x)$ is the sign function defined by

$$\operatorname{sgn}(x) = \begin{cases} 1 & (x > 0), \\ 0 & (x = 0), \\ -1 & (x < 0). \end{cases}$$

(2) Similar to (2)(3)(4) in the proof of (a), $f'(0) = 0$ exists if and only if $a - 1 > 0$.

$\square$

Proof of (c).

(1) Write $E = [-1, 1] - \{0\}$. f' is bounded if and only if $f'(0)$ exists and f' is bounded on E.

(2) Given any $x \in E$,

$$\begin{aligned} f'(x) &= \operatorname{sgn}(x) \left(a|x|^{a-1} \sin\big(|x|^{-c}\big) + |x|^a \cos\big(|x|^{-c}\big)(-c)|x|^{-c-1}\right) \\ &= \operatorname{sgn}(x)|x|^{a-c-1} \left(a|x|^c \sin\big(|x|^{-c}\big) - c\cos\big(|x|^{-c}\big)\right). \end{aligned}$$

(3) *Given $a-c-1 \geq 0$. Show that f' is bounded on E.* Since $\operatorname{sgn}(x)$ is bounded by 1 on E, $|x|^{a-c-1}$ is bounded by 1 on E and $a|x|^c \sin(|x|^{-c}) - c\cos(|x|^{-c})$ is bounded by $|a| + |c|$ on E, f' is bounded on E.

(4) *Given $a - c - 1 < 0$. Show that f' is unbounded on E.* Take $x_n = (2n\pi)^{-\frac{1}{c}} \neq 0$ for $n = 1, 2, 3, \ldots$. The sequence $\{x_n\}$ converges to 0, and

$$\lim_{n\to\infty} f'(x_n) = \lim_{n\to\infty} -c(2n\pi)^{-\frac{a-c-1}{c}} = -\infty.$$

(5) By (b), $f'(0)$ exists if and only if $a > 1$. By (3)(4), f' is bounded on E if and only if $a - c - 1 \geq 0$. Since $c > 0$, f' is bounded on $[-1, 1]$ if and only if $a - c - 1 \geq 0$.

$\square$

Proof of (d). Similar to the proof of (a).

(1) Write $E = [-1, 1] - \{0\}$. By (b)(c),

$$f'(x) = \begin{cases} 0 & \text{if } x = 0, \\ \operatorname{sgn}(x)|x|^{a-c-1} \left(a|x|^c \sin(|x|^{-c}) - c\cos(|x|^{-c})\right) & \text{if } x \in E. \end{cases}$$

Clearly, f' is continuous on E. Hence, f' is continuous if and only if $\lim_{x\to 0} f'(x) = f'(0) = 0$.

(2) *Given $a - c - 1 > 0$. Show that* $\lim_{x\to 0} f'(x) = 0$. Since $|x|^{a-c-1} \to 0$ as $x \to 0$, $\operatorname{sgn}(x)$ is bounded by 1 on E, and $a|x|^c \sin(|x|^{-c}) - c\cos(|x|^{-c})$ is bounded by $|a| + |c|$ on E,

$$\operatorname{sgn}(x)|x|^{a-c-1} \left(a|x|^c \sin\big(|x|^{-c}\big) - c\cos\big(|x|^{-c}\big)\right) \to 0$$

as $x \to 0$. The result is established.

(3) *Given* $a - c - 1 = 0$. *Show that* $\lim_{x \to 0} f'(x)$ *does not exist.*

(a) Take $x_n = \left(\frac{\pi}{2} + 2n\pi\right)^{-\frac{1}{c}} \neq 0$ for $n = 1, 2, 3, \ldots$. The sequence $\{x_n\}$ converges to 0, and

$$\begin{aligned}\lim_{n \to \infty} f'(x_n) &= \lim_{n \to \infty} \mathrm{sgn}(x_n) \left(a|x_n|^c \sin\left(|x_n|^{-c}\right) - c\cos\left(|x_n|^{-c}\right)\right) \\ &= \lim_{n \to \infty} \frac{a}{\frac{\pi}{2} + 2n\pi} \\ &= 0.\end{aligned}$$

(b) Similarly, take $y_n = (2n\pi)^{-\frac{1}{c}} \neq 0$ for $n = 1, 2, 3, \ldots$. The sequence $\{y_n\}$ converges to 0, and

$$\begin{aligned}\lim_{n \to \infty} f'(y_n) &= \lim_{n \to \infty} \mathrm{sgn}(y_n) \left(a|y_n|^c \sin\left(|y_n|^{-c}\right) - c\cos\left(|y_n|^{-c}\right)\right) \\ &= \lim_{n \to \infty} -c \\ &= -c \neq 0.\end{aligned}$$

(c) By (a)(b), $\lim_{x \to 0} f'(x)$ does not exist (Theorem 4.2).

(4) *Given* $a - c - 1 < 0$. *Show that* $\lim_{x \to 0} f'(x)$ *does not exist.* It is the same as (4) in the proof of (c).

(5) By (2)(3)(4), f' is continuous if and only if $\lim_{x \to 0} f'(x) = 0$ if and only if $a - c - 1 > 0$.

□

Proof of (e). Similar to the proof of (b).

(1) Write $E = [-1, 1] - \{0\}$. By the proof of (d),

$$f'(x) = \begin{cases} 0 & \text{if } x = 0, \\ \mathrm{sgn}(x)|x|^{a-c-1} \left(a|x|^c \sin(|x|^{-c}) - c\cos(|x|^{-c})\right) & \text{if } x \in E. \end{cases}$$

By definition

$$\begin{aligned}f''(0) &= \lim_{x \to 0} \frac{f'(x) - f'(0)}{x - 0} \\ &= \lim_{x \to 0} |x|^{a-c-2} \left(a|x|^c \sin\left(|x|^{-c}\right) - c\cos\left(|x|^{-c}\right)\right).\end{aligned}$$

(Here $\mathrm{sgn}(x)^2 = 1$ if $x \neq 0$.)

(2) Similar to (2)(3)(4) in the proof of (d), $f''(0) = 0$ exists if and only if $(a - c - 1) - 1 = a - c - 2 > 0$.

□

Proof of (f). Similar to the proof of (c).

(1) Write $E = [-1,1] - \{0\}$. f'' is bounded if and only if $f''(0)$ exists and f'' is bounded on E.

(2) Given any $x \in E$,

$$\begin{aligned} f''(x) =& |x|^{a-2c-2} \\ &\cdot \left[(a(a-1)|x|^{2c} - c^2)\sin\big(|x|^{-c}\big) - c(2a-c-1)|x|^c \cos\big(|x|^{-c}\big)\right]. \end{aligned}$$

(3) *Given $a - 2c - 2 \geq 0$. Show that f'' is bounded on E.* Since $|x|^{a-2c-2}$ is bounded by 1 on E and

$$\begin{aligned} &\left|(a(a-1)|x|^{2c} - c^2)\sin\big(|x|^{-c}\big) - c(2a-c-1)|x|^c \cos\big(|x|^{-c}\big)\right| \\ \leq& |a(a-1)| + |c^2| + |c(2a-c-1)| \end{aligned}$$

is bounded on E, f'' is bounded on E.

(4) *Given $a - 2c - 2 < 0$. Show that f'' is unbounded on E.* Take $x_n = \left(\frac{\pi}{2} + 2n\pi\right)^{-\frac{1}{c}} \neq 0$ for $n = 1, 2, 3, \ldots$. The sequence $\{x_n\}$ converges to 0, and

$$\begin{aligned} &\lim_{n\to\infty} f''(x_n) \\ =& \lim_{n\to\infty} \underbrace{\left(a(a-1)\left(\frac{\pi}{2} + 2n\pi\right)^{-2} - c^2\right)}_{\to -c^2 \neq 0} \underbrace{\left(\frac{\pi}{2} + 2n\pi\right)^{-\frac{a-2c-2}{c}}}_{\to\infty} \\ =& -\infty. \end{aligned}$$

(5) By (e), $f''(0)$ exists if and only if $a - c - 2 > 0$. By (3)(4), f'' is bounded on E if and only if $a - 2c - 2 \geq 0$. Since $c > 0$, f'' is bounded on $[-1,1]$ if and only if $a - 2c - 2 \geq 0$.

□

Proof of (g). Similar to the proof of (a) or (d).

(1) Write $E = [-1,1] - \{0\}$. By (e)(f),

$$f''(x) = \begin{cases} 0 & \text{if } x = 0, \\ |x|^{a-2c-2}\left[(a(a-1)|x|^{2c} - c^2)\sin(|x|^{-c}) - c(2a-c-1)|x|^c \cos(|x|^{-c})\right]. & \text{if } x \in E. \end{cases}$$

Clearly, f'' is continuous on E. Hence, f'' is continuous if and only if $\lim_{x\to 0} f''(x) = f''(0) = 0$.

(2) *Given $a - 2c - 2 > 0$. Show that* $\lim_{x\to 0} f''(x) = 0$. Since $|x|^{a-2c-2} \to 0$ as $x \to 0$ and

$$(a(a-1)|x|^{2c} - c^2)\sin\big(|x|^{-c}\big) - c(2a-c-1)|x|^c \cos\big(|x|^{-c}\big)$$

is bounded by $|a(a-1)|+|c^2|+|c(2a-c-1)|$ on E,

$$|x|^{a-2c-2}$$
$$\cdot \left[(a(a-1)|x|^{2c}-c^2)\sin\left(|x|^{-c}\right)-c(2a-c-1)|x|^c\cos\left(|x|^{-c}\right)\right] \to 0$$

as $x \to 0$. The result is established.

(3) *Given $a-2c-2=0$. Show that $\lim_{x\to 0} f''(x)$ does not exist.*

(a) Take $x_n = \left(\frac{\pi}{2}+2n\pi\right)^{-\frac{1}{c}} \neq 0$ for $n=1,2,3,\ldots$. The sequence $\{x_n\}$ converges to 0, and

$$\begin{aligned}
&\lim_{n\to\infty} f''(x_n)\\
=&\lim_{n\to\infty}(a(a-1)|x_n|^{2c}-c^2)\sin\left(|x_n|^{-c}\right)-c(2a-c-1)|x_n|^c\cos\left(|x_n|^{-c}\right)\\
=&\lim_{n\to\infty}\frac{a(a-1)}{\left(\frac{\pi}{2}+2n\pi\right)^2}-c^2\\
=&-c^2
\end{aligned}$$

(b) Similarly, take $y_n = \left(\frac{3\pi}{2}+2n\pi\right)^{-\frac{1}{c}} \neq 0$ for $n=1,2,3,\ldots$. The sequence $\{y_n\}$ converges to 0, and

$$\begin{aligned}
&\lim_{n\to\infty} f''(y_n)\\
=&\lim_{n\to\infty}(a(a-1)|y_n|^{2c}-c^2)\sin\left(|y_n|^{-c}\right)-c(2a-c-1)|y_n|^c\cos\left(|y_n|^{-c}\right)\\
=&\lim_{n\to\infty}-\frac{a(a-1)}{\left(\frac{3\pi}{2}+2n\pi\right)^2}+c^2\\
=&c^2.
\end{aligned}$$

(c) By (a)(b), $\lim_{x\to 0} f''(x)$ does not exist (Theorem 4.2).

(4) *Given $a-2c-2<0$. Show that $\lim_{x\to 0} f''(x)$ does not exist.* It is the same as (4) in the proof of (f).

(5) By (2)(3)(4), f'' is continuous if and only if $\lim_{x\to 0} f''(x)=0$ if and only if $a-2c-2>0$.

□

Exercise 5.14. *Let f be a differentiable real function defined in (a,b). Prove that f is convex if and only if f' is monotonically increasing. Assume next $f''(x)$ exists for every $x \in (a,b)$, and prove that f is convex if and only if $f''(x) \geq 0$ for all $x \in (a,b)$.*

Proof.

(1) *Show that f' is monotonically increasing if f is convex.*

(a) Since f is convex, by definition (Exercise 4.23)

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

whenever $a < x < b$, $a < y < b$, $0 < \lambda < 1$.

(b) As $x \neq y$, we have

$$\begin{aligned} f(y) - f(x) &\geq \frac{f(x + \lambda(y-x)) - f(x)}{\lambda} \\ &= \frac{f(x + \lambda(y-x)) - f(x)}{\lambda(y-x)} \cdot (y-x) \end{aligned}$$

and let $\lambda \to 0$ to get

$$f(y) - f(x) \geq f'(x)(y-x)$$

(since $f'(x)$ exists). Similarly, we have

$$f(x) - f(y) \geq f'(y)(x-y).$$

(c) Given any $y > x$, we have

$$f'(y)(y-x) \geq f(y) - f(x) \geq f'(x)(y-x).$$

Hence $f'(y) \geq f'(x)$ whenever $y > x$, or f' is monotonically increasing.

(2) *Show that f is convex if f' is monotonically increasing.* Given any $y > x$ and any $0 < \lambda < 1$.

(a) By Theorem 5.10 (the mean value theorem), there is a point $x < \xi < y$ such that

$$f(y) - f(x) = f'(\xi)(y-x).$$

Since f' is monotonically increasing,

$$f'(y)(y-x) \geq f(y) - f(x) \geq f'(x)(y-x).$$

(b) Write $z = \lambda x + (1-\lambda)y$. Hence

$$\begin{aligned} f(y) - f(z) &\geq f'(z)(y-z), \\ f(z) - f(x) &\leq f'(z)(z-x), \end{aligned}$$

or

$$\begin{aligned} f(y) &\geq f(z) + f'(z)(y-z), \\ f(x) &\geq f(z) + f'(z)(x-z), \end{aligned}$$

or

$$\begin{aligned}\lambda f(x)+(1-\lambda)f(y) \geq &\lambda[f(z)+f'(z)(x-z)]\\ &+(1-\lambda)[f(z)+f'(z)(y-z)]\\ =&f(z)\\ =&f(\lambda x+(1-\lambda)y).\end{aligned}$$

Hence f is convex.

(3) *Show that $f''(x) \geq 0$ if f is convex and f'' exists.* By (1), f' is monotonically increasing since f is convex. Given any $x \neq y$, we have

$$\frac{f'(y)-f'(x)}{y-x} \geq 0.$$

Let $y \to x$, we have $f''(x) \geq 0$ if f'' exists.

(4) *Show that f is convex if f'' exists and $f''(x) \geq 0$.* By Theorem 5.11(a), f' is monotonically increasing. By (2), f is convex.

$\square$

Exercise 5.15 (Landau-Kolmogorov inequality on the half-line). *Suppose $a \in \mathbb{R}^1$, f is a twice-differentiable real function on (a,∞), and M_0, M_1, M_2 are the least upper bounds of $|f(x)|$, $|f'(x)|$, $|f''(x)|$, respectively, on (a,∞). Prove that*

$$M_1^2 \leq 4M_0M_2.$$

(Hint: If $h>0$, Taylor's theorem shows that

$$f'(x)=\frac{1}{2h}[f(x+2h)-f(x)]-hf''(\xi)$$

for some $\xi \in (x,x+2h)$. Hence

$$|f'(x)| \leq hM_2+\frac{M_0}{h}.)$$

To show that $M_1^2 = 4M_0M_2$ can actually happen, take $a=-1$, define

$$f(x)=\begin{cases}2x^2-1 & (-1<x<0),\\ \frac{x^2-1}{x^2+1} & (0 \leq x < \infty),\end{cases}$$

and show that $M_0=1$, $M_1=4$, $M_2=4$. Does $M_1^2 \leq 4M_0M_2$ hold for vector-valued functions too?

Note.

(1) Write
$$M_1 \leq 2M_0^{\frac{1}{2}} M_2^{\frac{1}{2}}.$$
2 is called the Landau-Kolmogorov constant, which is the best possible by the above example.

(2) In general, suppose $a \in \mathbb{R}^1$, f is a nth differentiable real function on (a, ∞), and M_0, M_k, M_n are the least upper bounds of $|f(x)|$, $\left|f^{(k)}(x)\right|$, $\left|f^{(n)}(x)\right|$, respectively, on (a, ∞) where $1 \leq k < n$. Then
$$M_k \leq C(n, k) M_0^{1-\frac{k}{n}} M_n^{\frac{k}{n}}.$$

Proof.

(1) Consider some trivial cases.

(a) If $M_0 = 0$, then $f(x) = 0$ on $(a, +\infty)$. So that $f'(x) = f''(x) = 0$ on $(a, +\infty)$, or $M_1 = M_2 = 0$. The inequality holds.

(b) If $M_2 = 0$, then $f''(x) = 0$ on $(a, +\infty)$. So that $f'(x) = \alpha$ for some constant $\alpha \in \mathbb{R}^1$ (Theorem 5.11(b)), and $f(x) = \alpha x + \beta$ for some constant $\beta \in \mathbb{R}^1$ (by applying Theorem 5.11(b) to $x \mapsto f(x) - \alpha x$). Hence $M_1 = |\alpha|$ and
$$M_0 = \begin{cases} +\infty & (\alpha \neq 0), \\ |\beta| & (\alpha = 0). \end{cases}$$
In any case, the inequality holds.

(c) If $M_0 = +\infty$ and $M_2 \neq 0$, there is nothing to do.

(d) If $M_2 = +\infty$ and $M_0 \neq 0$, there is nothing to do.

(2) By (1), we suppose that $0 < M_0 < +\infty$ and $0 < M_2 < +\infty$. Given $x \in (a, +\infty)$ and $h > 0$. By Taylor's theorem (Theorem 5.15):
$$f(x + 2h) = f(x) + 2hf'(x) + 2h^2 f''(\xi)$$
for some $\xi \in (x, x + 2h) \subseteq (a, +\infty)$. Thus
$$\begin{aligned} 2h|f'(x)| &\leq |f(x + 2h)| + |f(x)| + 2h^2|f''(\xi)| \\ &\leq 2M_0 + 2h^2 M_2, \\ |f'(x)| &\leq \frac{M_0}{h} + hM_2 \end{aligned}$$
holds for all $h > 0$. In particular, take
$$h = \sqrt{\frac{M_0}{M_2}}$$

to get

$$|f'(x)| \leq 2\sqrt{M_0 M_2}.$$

Thus $2\sqrt{M_0 M_2}$ is an upper bound of $|f'(x)|$ for all $x \in (a, +\infty)$. Hence

$$M_1 \leq 2\sqrt{M_0 M_2}$$

or

$$M_1^2 \leq 4M_0 M_2.$$

(3) *Define*

$$f(x) = \begin{cases} 2x^2 - 1 & (-1 < x < 0), \\ \frac{x^2-1}{x^2+1} & (0 \leq x < \infty). \end{cases}$$

Show that $M_0 = 1$, $M_1 = 4$, $M_2 = 4$. Similar to Exercise 5.12,

$$f'(x) = \begin{cases} 4x & (-1 < x \leq 0), \\ \frac{4x}{(x^2+1)^2} & (0 \leq x < \infty). \end{cases}$$

(Here $\lim_{x \to 0+} f'(x) = 0$ and $\lim_{x \to 0-} f'(x) = 0$. So $f'(0) = 0$ by Exercise 5.9.) Also,

$$f''(x) = \begin{cases} 4 & (-1 < x \leq 0), \\ \frac{-12x^2+4}{(x^2+1)^3} & (0 \leq x < \infty). \end{cases}$$

(Here $\lim_{x \to 0+} f''(x) = 4$ and $\lim_{x \to 0-} f''(x) = 4$. So $f''(0) = 4$ by Exercise 5.9.) Hence, $M_0 = 1$, $M_1 = 4$, $M_2 = 4$.

(4) *Given*

$$\mathbf{f}(x) = (f_1(x), \ldots, f_k(x))$$

be a twice-differentiable vector-valued function from (a, ∞) *to* $\mathbb{R}^k$. *and* M_0, M_1, M_2 *are the least upper bounds of* $|\mathbf{f}(x)|$, $|\mathbf{f}'(x)|$, $|\mathbf{f}''(x)|$, *respectively, on* (a, ∞). *Show that*

$$M_1^2 \leq 4M_0 M_2.$$

Similar to (1), we suppose that $0 < M_0 < +\infty$ and $0 < M_2 < +\infty$. Given any $\mathbf{v} = (v_1, \ldots, v_k) \in \mathbb{R}^k$, $\mathbf{v} \cdot \mathbf{f}$ is a twice-differentiable real function on (a, ∞). Similar to (2), Given $x \in (a, +\infty)$ and $h > 0$. By Taylor's theorem (Theorem 5.15):

$$(\mathbf{v} \cdot \mathbf{f})(x + 2h) = (\mathbf{v} \cdot \mathbf{f})(x) + 2h(\mathbf{v} \cdot \mathbf{f})'(x) + 2h^2(\mathbf{v} \cdot \mathbf{f})''(\xi)$$

for some $\xi \in (x, x + 2h) \subseteq (a, +\infty)$. Thus by the Schwarz inequality (Theorem 1.37(d))

$$\begin{aligned} 2h|(\mathbf{v} \cdot \mathbf{f})'(x)| &\leq |(\mathbf{v} \cdot \mathbf{f})(x + 2h)| + |(\mathbf{v} \cdot \mathbf{f})(x)| + 2h^2|(\mathbf{v} \cdot \mathbf{f})''(\xi)| \\ &\leq |\mathbf{v}||\mathbf{f}(x + 2h)| + |\mathbf{v}||\mathbf{f}(x)| + 2h^2|\mathbf{v}||\mathbf{f}''(\xi)| \\ &\leq (2M_0 + 2h^2 M_2)|\mathbf{v}|, \\ |(\mathbf{v} \cdot \mathbf{f})'(x)| &\leq \left(\frac{M_0}{h} + hM_2 \right) |\mathbf{v}| \end{aligned}$$

holds for any $\mathbf{v}$ and $h > 0$. In particular, we take

$$\mathbf{v} = \mathbf{f}'(y)$$

and

$$h = \sqrt{\frac{M_0}{M_2}}$$

to get

$$|\mathbf{f}'(x) \cdot \mathbf{f}'(y)| \leq 2\sqrt{M_0 M_2}|\mathbf{f}'(y)| \leq 2M_1\sqrt{M_0 M_2}.$$

Note that x and y are arbitrary (in $(a, +\infty)$). In particular, we take $x = y$ to get

$$|\mathbf{f}'(x)|^2 \leq 2M_1\sqrt{M_0 M_2}.$$

Thus $2M_1\sqrt{M_0 M_2}$ is an upper bound of $|\mathbf{f}'(x)|^2$ for all $x \in (a, +\infty)$. Hence

$$M_1^2 \leq 2M_1\sqrt{M_0 M_2}$$

or

$$M_1^2 \leq 4M_0 M_2.$$

□

Supplement (Landau-Kolmogorov inequality on the real line). *Suppose f is a twice-differentiable real function on $(-\infty, +\infty)$, and M_0, M_1, M_2 are the least upper bounds of $|f(x)|$, $|f'(x)|$, $|f''(x)|$, respectively, on $(-\infty, +\infty)$. Prove that*

$$M_1^2 \leq 2M_0 M_2.$$

Proof.

(1) Similar to (1) in Landau-Kolmogorov inequality on the half-line, we suppose that $0 < M_0 < +\infty$ and $0 < M_2 < +\infty$.

(2) Similar to (2) in Landau-Kolmogorov inequality on the half-line. Given $x \in \mathbb{R}^1$ and $h > 0$. By Taylor's theorem (Theorem 5.15):

$$f(x + 2h) = f(x) + 2hf'(x) + 2h^2 f''(\xi_1) \tag{I}$$
$$f(x - 2h) = f(x) - 2hf'(x) + 2h^2 f''(\xi_2) \tag{II}$$

for some $\xi_1 \in (x, x + 2h)$ and $\xi_2 \in (x, x - 2h)$. So (I) subtracts (II):

$$f(x + 2h) - f(x - 2h) = 4hf'(x) + 2h^2 f''(\xi_1) - 2h^2 f''(\xi_2).$$

Thus

$$\begin{aligned} 4h|f'(x)| &\leq |f(x + 2h)| + |f(x - 2h)| + 2h^2|f''(\xi_1)| + 2h^2|f''(\xi_2)| \\ &\leq 2M_0 + 4h^2 M_2, \\ |f'(x)| &\leq \frac{M_0}{2h} + hM_2 \end{aligned}$$

holds for all $h > 0$. In particular, take

$$h = \sqrt{\frac{M_0}{2M_2}}$$

to get

$$|f'(x)| \leq \sqrt{2M_0M_2}.$$

Thus $\sqrt{2M_0M_2}$ is an upper bound of $|f'(x)|$ for all $x \in \mathbb{R}^1$. Hence

$$M_1 \leq \sqrt{2M_0M_2}$$

or

$$M_1^2 \leq 2M_0M_2.$$

□

Note.

(1) Write

$$M_1 \leq \sqrt{2}M_0^{\frac{1}{2}}M_2^{\frac{1}{2}}.$$

$\sqrt{2}$ is called the Landau-Kolmogorov constant, which is the best possible.

(2) In general, suppose f is a nth differentiable real function on $\mathbb{R}^1$, and M_0, M_k, M_n are the least upper bounds of $|f(x)|$, $\left|f^{(k)}(x)\right|$, $\left|f^{(n)}(x)\right|$, respectively, on $\mathbb{R}^1$ where $1 \leq k < n$. Then

$$M_k \leq C(n,k)M_0^{1-\frac{k}{n}}M_n^{\frac{k}{n}}.$$

Exercise 5.16. *Suppose f is twice-differentiable on $(0, \infty)$, f'' is bounded on $(0, \infty)$, and $f(x) \to 0$ as $x \to \infty$. Prove that $f'(x) \to 0$ as $x \to \infty$. (Hint: Let $a \to \infty$ in Exercise 5.15.)*

Proof.

(1) Write $|f''| \leq M$ for some real M since f'' is bounded on $(0, \infty)$.

(2) Given any $a > 0$. As in Exercise 5.15, define M_0, M_1, M_2 are the least upper bounds of $|f(x)|$, $|f'(x)|$, $|f''(x)|$ on (a, ∞). Note that $M_2 \leq M$ for any $a > 0$ (by (1)). So that

$$M_1^2 \leq 4M_0M_2 \leq 4MM_0$$

for any $a > 0$.

(3) By assumption, $M_0 \to 0$ as $a \to \infty$. (So given any $\varepsilon > 0$, there exists a real A such that

$$0 \leq M_0 < \frac{\varepsilon}{4M+1}$$

whenever $a \geq A$. Hence

$$M_1^2 \leq 4MM_0 \leq 4M \cdot \frac{\varepsilon}{4M+1} < \varepsilon.$$

whenever $a \geq A$.) Therefore $M_1^2 \to 0$ as $a \to \infty$, or $f'(x) \to 0$ as $x \to \infty$.

□

Exercise 5.17. *Suppose f is a real, three times differentiable function on $[-1, 1]$, such that*

$$f(-1) = 0, \qquad f(0) = 0, \qquad f(1) = 1, \qquad f'(0) = 0.$$

Prove that $f^{(3)}(x) \geq 3$ for some $x \in (-1, 1)$. Note that equality holds for $\frac{1}{2}(x^3 + x^2)$. (Hint: Use Theorem 5.15, with $\alpha = 0$ and $\beta = \pm 1$, to show that there exist $s \in (0, 1)$ and $t \in (-1, 0)$ such that

$$f^{(3)}(s) + f^{(3)}(t) = 6.)$$

We can drop the assumption that $f(0) = 0$ actually.

Proof (Hint).

(1) Use Taylor's theorem (Theorem 5.15), with $\alpha = 0$ and $\beta = \pm 1$,

$$f(1) = f(0) + f'(0) + \frac{f''(0)}{2} + \frac{f'''(s)}{6} \tag{I}$$
$$f(-1) = f(0) - f'(0) + \frac{f''(0)}{2} - \frac{f'''(t)}{6} \tag{II}$$

for some $s \in (0, 1)$ and $t \in (-1, 0)$.

(2) (I) subtracts (II) implies that

$$f(1) - f(-1) = 2f'(0) + \frac{f'''(s)}{6} + \frac{f'''(t)}{6}.$$

By assumption, $f(-1) = 0$, $f(1) = 1$ and $f'(0) = 0$. Hence

$$f^{(3)}(s) + f^{(3)}(t) = 6$$

for some $s \in (0, 1)$ and $t \in (-1, 0)$. So either $f^{(3)}(s) \geq 3$ or $f^{(3)}(t) \geq 3$ for some $s, t \in (-1, 1)$.

□

Exercise 5.18. *Suppose f is a real function on $[a, b]$, n is a positive integer, and $f^{(n-1)}$ exists for every $t \in [a, b]$. Let α, β, and P be as in Taylor's theorem (Theorem 5.15). Define*

$$Q(t) = \frac{f(t) - f(\beta)}{t - \beta}$$

for $t \in [a, b]$, $t \neq \beta$, differentiate

$$f(t) - f(\beta) = (t - \beta)Q(t)$$

$n - 1$ times at $t = \alpha$, and derive the following version of Taylor's theorem:

$$f(\beta) = P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^n.$$

Proof.

(1) *Show that*

$$f^{(k)}(t) = kQ^{(k-1)}(t) + (t - \beta)Q^{(k)}(t)$$

for $k = 1, 2, \ldots, n$. Induction on k.

(a) If $k = 1$, then

$$f'(t) = Q(t) + (t - \beta)Q'(t)$$

(Theorem 5.3(b)).

(b) Assume the induction hypothesis that for the single case $k = m - 1$ holds. Apply Theorem 5.3(b) again to get

$$\begin{aligned}
f^{(m)}(t) &= (f^{(m-1)}(t))' \\
&= ((m-1)Q^{(m-2)}(t) + (t - \beta)Q^{(m-1)}(t))' \\
&= (m-1)Q^{(m-1)}(t) + Q^{(m-1)}(t) + (t - \beta)Q^{(m)}(t) \\
&= mQ^{(m-1)}(t) + (t - \beta)Q^{(m)}(t).
\end{aligned}$$

(c) Since both the base case in (a) and the inductive step in (b) have been proved as true, by mathematical induction the result holds.

(2) *Show that*

$$f(\beta) = P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^n$$

where

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t - \alpha)^k.$$

Induction on n.

(a) If $n = 1$, then by the definition of $Q(t)$

$$f(\beta) = f(\alpha) + Q(\alpha)(\beta - \alpha).$$

(b) Assume the induction hypothesis that for the single case $n = m - 1$ holds. By (1), we have

$$Q^{(m-2)}(\alpha) = \frac{1}{m-1}(f^{(m-1)}(\alpha) + Q^{(m-1)}(\alpha)(\beta - \alpha)).$$

Hence

$$\begin{aligned} f(\beta) &= \sum_{k=0}^{m-2} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k + \frac{Q^{(m-2)}(\alpha)}{(m-2)!}(\beta - \alpha)^{m-1} \\ &= \sum_{k=0}^{m-2} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k \\ &\quad + \frac{f^{(m-1)}(\alpha)}{(m-1)!}(\beta - \alpha)^{m-1} + \frac{Q^{(m-1)}(\alpha)(\beta - \alpha)}{(m-1)!}(\beta - \alpha)^{m-1} \\ &= \sum_{k=0}^{m-1} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k + \frac{Q^{(m-1)}(\alpha)}{(m-1)!}(\beta - \alpha)^m. \end{aligned}$$

(c) Since both the base case in (a) and the inductive step in (b) have been proved as true, by mathematical induction the result holds.

□

Note. It is also true for vector-valued functions: *Suppose* $\mathbf{f}$ *is a function of* $[a, b]$ *into* $\mathbb{R}^m$*,* n *is a positive integer,* $\mathbf{f}^{(n-1)}$ *is continuous on* $[a, b]$*,* $\mathbf{f}^{(n)}(t)$ *exists for every* $t \in (a, b)$*. Let* α*,* β *be distinct points of* $[a, b]$*, and define*

$$\mathbf{P}(t) = \sum_{k=0}^{n-1} \frac{\mathbf{f}^{(k)}(\alpha)}{k!}(t - \alpha)^k$$

and

$$\mathbf{Q}(t) = \frac{\mathbf{f}(t) - \mathbf{f}(\beta)}{t - \beta}.$$

Then

$$\mathbf{f}(\beta) = \mathbf{P}(\beta) + \frac{\mathbf{Q}^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^n.$$

Exercise 5.19. *Suppose* f *is defined in* $(-1, 1)$ *and* $f'(0)$ *exists. Suppose* $-1 < \alpha_n < \beta_n < 1$*,* $\alpha_n \to 0$*, and* $\beta_n \to 0$ *as* $n \to \infty$*. Define the difference quotients*

$$D_n = \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}$$

Prove the following statements:

(a) *If $\alpha_n < 0 < \beta_n$, then $\lim D_n = f'(0)$.*

(b) *If $0 < \alpha_n < \beta_n$ and $\left\{ \frac{\beta_n}{\beta_n - \alpha_n} \right\}$ is bounded, then $\lim D_n = f'(0)$.*

(c) *If f' is continuous in $(-1, 1)$, then $\lim D_n = f'(0)$.*

Give an example in which f is differentiable in $(-1, 1)$ (but f' is not continuous at 0) and in which α_n, β_n tend to 0 in such a way that $\lim D_n$ exists but is different from $f'(0)$.

Proof of (a).

(1) Write

$$D_n = \frac{f(\beta_n) - f(0)}{\beta_n - 0} \cdot \frac{\beta_n}{\beta_n - \alpha_n} - \frac{f(\alpha_n) - f(0)}{\alpha_n - 0} \cdot \frac{\alpha_n}{\beta_n - \alpha_n}.$$

It is well-defined since $\alpha_n \neq 0$ and $\beta_n \neq 0$.

(2) Given any $\varepsilon > 0$. Since $f'(0)$ exists, there exists a common integer N such that

$$\left| \frac{f(\alpha_n) - f(0)}{\alpha_n - 0} - f'(0) \right| < \varepsilon \text{ and } \left| \frac{f(\beta_n) - f(0)}{\beta_n - 0} - f'(0) \right| < \varepsilon$$

whenever $n \geq N$.

(3) Thus

$$\begin{aligned}
&|D_n - f'(0)| \\
\leq &\frac{\beta_n}{\beta_n - \alpha_n} \cdot \left| \frac{f(\beta_n) - f(0)}{\beta_n - 0} - f'(0) \right| + \frac{-\alpha_n}{\beta_n - \alpha_n} \cdot \left| \frac{f(\alpha_n) - f(0)}{\alpha_n - 0} - f'(0) \right| \\
< &\frac{\beta_n}{\beta_n - \alpha_n}\varepsilon + \frac{-\alpha_n}{\beta_n - \alpha_n}\varepsilon \\
= &\varepsilon.
\end{aligned}$$

whenever $n \geq N$. Therefore, $\lim D_n = f'(0)$.

□

Proof of (b).

(1) Similar to (1) in the proof of (a). Write

$$D_n = \frac{f(\beta_n) - f(0)}{\beta_n - 0} \cdot \frac{\beta_n}{\beta_n - \alpha_n} - \frac{f(\alpha_n) - f(0)}{\alpha_n - 0} \cdot \frac{\alpha_n}{\beta_n - \alpha_n}.$$

It is well-defined since $\alpha_n \neq 0$ and $\beta_n \neq 0$.

(2) Write
$$\left|\frac{\beta_n}{\beta_n - \alpha_n}\right| \leq M$$
for some real $M \geq 0$. Hence $\left\{\frac{\alpha_n}{\beta_n - \alpha_n}\right\}$ is bounded too, say
$$\left|\frac{\alpha_n}{\beta_n - \alpha_n}\right| = \left|\frac{\beta_n}{\beta_n - \alpha_n} - 1\right| \leq M + 1.$$

(3) Given any $\varepsilon > 0$. Since $f'(0)$ exists, there exists a common integer N such that
$$\left|\frac{f(\alpha_n) - f(0)}{\alpha_n - 0} - f'(0)\right| < \frac{\varepsilon}{64(M+1)},$$
$$\left|\frac{f(\beta_n) - f(0)}{\beta_n - 0} - f'(0)\right| < \frac{\varepsilon}{89(M+1)}$$
whenever $n \geq N$.

(4) Thus
$$\begin{aligned}
&|D_n - f'(0)| \\
\leq &\left|\frac{\beta_n}{\beta_n - \alpha_n}\right| \cdot \left|\frac{f(\beta_n) - f(0)}{\beta_n - 0} - f'(0)\right| \\
&+ \left|\frac{-\alpha_n}{\beta_n - \alpha_n}\right| \cdot \left|\frac{f(\alpha_n) - f(0)}{\alpha_n - 0} - f'(0)\right| \\
< &\frac{M}{89(M+1)}\varepsilon + \frac{M+1}{64(M+1)}\varepsilon \\
< &\frac{\varepsilon}{89} + \frac{\varepsilon}{64} \\
< &\varepsilon
\end{aligned}$$
whenever $n \geq N$. Therefore, $\lim D_n = f'(0)$.

□

Proof of (c). By the mean value theorem (Theorem 5.10), there is point $\xi_n \in (\alpha_n, \beta_n)$ at which
$$f(\beta_n) - f(\alpha_n) = (\beta_n - \alpha_n) f'(\xi_n)$$
or
$$f'(\xi_n) = \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} = D_n.$$
Since $\xi_n \in (\alpha_n, \beta_n)$ and $\lim \alpha_n = \lim \beta_n = 0$, $\lim \xi_n = 0$. Since f' is continuous at $x = 0$,
$$\lim D_n = \lim f'(\xi_n) = f'(\lim \xi_n) = f'(0).$$

□

Note.

(1) *Give an example in which f is differentiable in $(-1, 1)$ (but f' is not continuous at 0) and in which α_n, β_n tend to 0 in such a way that $\lim D_n$ exists but is different from $f'(0)$.*

(2) Let f be defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & (x \neq 0), \\ 0 & (x = 0) \end{cases}$$

as in Examples 5.6(b). So

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

(3) Take $\alpha_n = (2n\pi)^{-1} \neq 0$ and $\beta_n = \left(\frac{\pi}{2} + 2n\pi\right)^{-1} \neq 0$ for $n = 1, 2, 3, \ldots$. Hence $\lim \alpha_n = \lim \beta_n = 0$, and

$$\begin{aligned} \lim D_n &= \lim \frac{\left(\frac{\pi}{2} + 2n\pi\right)^{-2}}{\left(\frac{\pi}{2} + 2n\pi\right)^{-1} - (2n\pi)^{-1}} \\ &= \lim \frac{2n\pi}{(2n\pi)\left(\frac{\pi}{2} + 2n\pi\right) - \left(\frac{\pi}{2} + 2n\pi\right)^2} \\ &= \lim \frac{2n\pi}{-\frac{\pi}{2}\left(\frac{\pi}{2} + 2n\pi\right)} \\ &= -\frac{2}{\pi} \neq f'(0). \end{aligned}$$

□

Exercise 5.20. *Formulate and prove an inequality which follows form Taylor's theorem and which remains valid for vector-valued function.*

Proof.

(1) *Suppose $\mathbf{f}$ is a function of $[a, b]$ into $\mathbb{R}^m$, n is a positive integer, $\mathbf{f}^{(n-1)}$ is continuous on $[a, b]$, $\mathbf{f}^{(n)}(t)$ exists for every $t \in (a, b)$. Let α, β be distinct points of $[a, b]$, and define*

$$\mathbf{P}(t) = \sum_{k=0}^{n-1} \frac{\mathbf{f}^{(k)}(\alpha)}{k!}(t - \alpha)^k.$$

Then there exists a point x between α and β such that

$$|\mathbf{f}(\beta) - \mathbf{P}(\beta)| \leq (\beta - \alpha)^n \left| \frac{\mathbf{f}^{(n)}(x)}{n!} \right|.$$

For $n = 1$, this is just Theorem 5.19.

(2) Similar to the proof of Theorem 5.19. Put

$$\mathbf{z} = \mathbf{f}(\beta) - \mathbf{P}(\beta).$$

Define

$$\varphi(t) = \mathbf{z} \cdot \mathbf{f}(t) \qquad (\alpha \leq t \leq \beta).$$

Then $\varphi(t)$ is a function of $[a, b]$ into $\mathbb{R}^1$, and

$$\varphi^{(k)}(t) = \mathbf{z} \cdot \mathbf{f}^{(k)}(t)$$

where $0 \leq k \leq n$. Also, $\varphi^{(n-1)}$ is continuous on $[\alpha, \beta]$, and $\varphi^{(n)}(t)$ exists for every $t \in (\alpha, \beta)$.

(3) By Taylor's theorem (Theorem 5.15), there exists $x \in (\alpha, \beta)$ such that

$$\varphi(\beta) = Q(\beta) + \frac{\varphi^{(n)}(x)}{n!}(\beta - \alpha)^n$$

where

$$Q(t) = \sum_{k=0}^{n-1} \frac{\varphi^{(k)}(\alpha)}{k!}(t - \alpha)^k.$$

By (2), we have $Q(t) = \mathbf{z} \cdot \mathbf{P}(t)$ and thus

$$\mathbf{z} \cdot (\mathbf{f}(\beta) - \mathbf{P}(\beta)) = \mathbf{z} \cdot \frac{\mathbf{f}^{(n)}(x)}{n!}(\beta - \alpha)^n.$$

Note that $\mathbf{z} = \mathbf{f}(\beta) - \mathbf{P}(\beta)$ and Schwarz inequality (Theorem 1.37(d)). Hence

$$\begin{aligned} |\mathbf{f}(\beta) - \mathbf{P}(\beta)|^2 &= \left| (\mathbf{f}(\beta) - \mathbf{P}(\beta)) \cdot \frac{\mathbf{f}^{(n)}(x)}{n!}(\beta - \alpha)^n \right| \\ &\leq |\mathbf{f}(\beta) - \mathbf{P}(\beta)| \left| \frac{\mathbf{f}^{(n)}(x)}{n!} \right| (\beta - \alpha)^n \end{aligned}$$

or

$$|\mathbf{f}(\beta) - \mathbf{P}(\beta)| \leq \left| \frac{\mathbf{f}^{(n)}(x)}{n!} \right| (\beta - \alpha)^n$$

(whether $\mathbf{f}(\beta) - \mathbf{P}(\beta)$ is zero nor not).

□

Exercise 5.21. *Let E be a closed subset of $\mathbb{R}^1$. We saw in Exercise 4.22, that there is a real continuous function f on $\mathbb{R}^1$ whose zero set is E. Is it possible, for each closed set E, to find such an f which is differentiable on $\mathbb{R}^1$, or one which is n times differentiable, or even one which has derivatives of all orders on $\mathbb{R}^1$?*

It is possible by leveraging Exercise 8.1.

Proof.

(1) Every open set in $\mathbb{R}^1$ is the union of an at most countable collection of disjoint segments (Exercise 2.29).

(2) We need to construct an infinitely differentiable real function f on $\mathbb{R}^1$ such that the zero set $Z(f)$ is E. By (1), write $\widetilde{E}$ as the union of an at most countable collection of disjoint segments, say

$$\widetilde{E} = \bigcup_{(a_i,b_i)\in\mathscr{C}} (a_i, b_i)$$

where $\mathscr{C}$ is at most countable and all (a_i, b_i) segments are disjoint.

(3) For each disjoint segment (a_i, b_i) of

$$\widetilde{E} = \bigcup_{(a_i,b_i)\in\mathscr{C}} (a_i, b_i),$$

define $f(x)$ on $\mathbb{R}^1$ by

$$f(x) = \begin{cases} 1 & (x \in (-\infty,\infty)), \\ \exp\left(-\frac{1}{(x-a_i)^2}\right) & (x \in (a_i,\infty), a_i \neq -\infty), \\ \exp\left(-\frac{1}{(x-b_i)^2}\right) & (x \in (-\infty,b_i), b_i \neq \infty), \\ \exp\left(-\frac{1}{(x-a_i)^2(x-b_i)^2}\right) & (x \in (a_i,b_i), a_i \neq -\infty, b_i \neq \infty), \\ 0 & (x \in E). \end{cases}$$

By construction, $f(x) = 0$ if and only if $x \in E$ (Theorem 8.6(c)). By the same argument in the proof of Exercise 8.1, $f(x)$ is infinitely differentiable on $\mathbb{R}^1$.

□

Exercise 5.22 (Fixed-point iteration). *Suppose f is a real function on $(-\infty, +\infty)$. Call x a **fixed point** of f if $f(x) = x$.*

(a) *If f is differentiable and $f'(t) \neq 1$ for every real t, prove that f has at most one fixed point.*

(b) *Show that the function f defined by*

$$f(t) = t + (1 + e^t)^{-1}$$

has no fixed point, although $0 < f'(t) < 1$ for all real t.

(c) *However, if there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t, prove that a fixed point x of f exists, and that $x = \lim x_n$, where x_1 is an arbitrary real number and*

$$x_{n+1} = f(x_n)$$

for $n = 1, 2, 3, \ldots$.

(d) *Show that the process describe in (c) can be visualized by the zig-zag path*

$$(x_1, x_2) \to (x_2, x_2) \to (x_2, x_3) \to (x_3, x_3) \to (x_3, x_4) \to \dots.$$

Proof of (a). (Reductio ad absurdum)

(1) Suppose that there were two different fixed points $x_1 < x_2$. By the mean value theorem (Theorem 5.10), there exists $\xi \in (x_1, x_2)$ such that

$$f(x_1) - f(x_2) = (x_1 - x_2)f'(\xi).$$

(2) Since x_1 and x_2 are fixed points, $f(x_1) = x_1$ and $f(x_2) = x_2$ or

$$(x_1 - x_2)(f'(\xi) - 1) = 0.$$

Since $x_1 \neq x_2$, $f'(\xi) = 1$, contrary to the fact that $f'(t) \neq 1 \; \forall t \in \mathbb{R}^1$.

□

Proof of (b).

(1) *Show that f has no fixed point.*

$$\begin{aligned} f(t) = t &\Longleftrightarrow t + (1 + e^t)^{-1} = t \\ &\Longleftrightarrow (1 + e^t)^{-1} = 0, \end{aligned}$$

which is absurd since $1 + e^t > 1$ (Theorem 8.6(c)) and the multiplicative inverse of $(1 + e^t)^{-1}$ is never zero.

(2) *Show that $0 < f'(t) < 1$.*

$$f'(t) = 1 - \frac{e^t}{(1 + e^t)^2} = \frac{1 + e^t + e^{2t}}{1 + 2e^t + e^{2t}}.$$

Since $e^t > 0$ for all $t \in \mathbb{R}^1$, $0 < f'(t) < 1$ for all $t \in \mathbb{R}^1$.

□

Proof of (c)(Banach fixed point theorem). Might assume that $A > 0$. (If $A = 0$, then $f(x) = c$ for some constant c (Theorem 5.11(b)) and thus $x = c$ is the unique fixed point.)

(1) Given any integer $n > 1$. By the mean value theorem (Theorem 5.10), there exists ξ_{n-1} between x_{n-1} and x_n such that

$$f(x_n) - f(x_{n-1}) = (x_n - x_{n-1})f'(\xi_{n-1}).$$

By definition of $\{x_n\}$, $f(x_n) = x_{n+1}$ and $f(x_{n-1}) = x_n$. So that

$$\begin{aligned} |x_{n+1} - x_n| &= |f(x_n) - f(x_{n-1})| \\ &= |x_n - x_{n-1}||f'(\xi_{n-1})| \\ &\leq A|x_n - x_{n-1}|. \end{aligned}$$

(2) Hence by induction

$$|x_{n+1} - x_n| \leq A^{n-1}|x_2 - x_1|.$$

So if $m > n$ we have

$$\begin{aligned}
|x_m - x_n| &\leq \sum_{i=n}^{m-1} |x_{i+1} - x_i| \\
&\leq \sum_{i=n}^{m-1} A^{i-1}|x_2 - x_1| \\
&\leq \sum_{i=n}^{\infty} A^{i-1}|x_2 - x_1| \\
&= \frac{A^{n-1}}{1-A}|x_2 - x_1|.
\end{aligned}$$

(3) Given $\varepsilon > 0$. Take an integer N such that

$$\frac{A^{n-1}}{1-A}|x_2 - x_1| < \varepsilon$$

whenever $n \geq N$. For example,

$$N > 1 + \frac{\log \frac{(1-A)\varepsilon}{1+|x_2-x_1|}}{\log A}.$$

Hence as $m > n \geq N$, $|x_m - x_n| < \varepsilon$, or $\{x_n\}$ is a Cauchy sequence. Since $\mathbb{R}^1$ is complete (Theorem 3.11(c)), $\{x_n\}$ converges to $x \in \mathbb{R}^1$.

(4) Since f is differentiable, f is continuous (Theorem 5.2). Take $n \to \infty$ in $x_{n+1} = f(x_n)$ to get

$$x = \lim x_{n+1} = \lim f(x_n) = f(\lim x_n) = f(x).$$

So that $\lim x_n = x$ is a fixed point of f.

$\square$

Proof of (d). Write

$$(x_1, x_2) \to (x_2, x_2) \to (x_2, x_3) \to (x_3, x_3) \to \dots$$

as

$$\underbrace{(x_1, f(x_1))}_{\text{in } y=f(x)} \to \overbrace{(f(x_1), x_2)}^{\text{in } y=x} \to \underbrace{(x_2, f(x_2))}_{\text{in } y=f(x)} \to \overbrace{(f(x_2), x_3)}^{\text{in } y=x} \to \dots.$$

Hence the path is zig-zag in the visualization. $\square$

Exercise 5.23. *The function f defined by*

$$f(x) = \frac{x^3 + 1}{3}$$

has three fixed points, say α, β, γ, where

$$-2 < \alpha < -1, \qquad 0 < \beta < 1, \qquad 1 < \gamma < 2.$$

For arbitrarily chosen x_1, define $\{x_n\}$ by setting $x_{n+1} = f(x_n)$.

(a) *If $x_1 < \alpha$, prove that $x_n \to -\infty$ as $n \to \infty$.*

(b) *If $\alpha < x_1 < \gamma$, prove that $x_n \to \beta$ as $n \to \infty$.*

(c) *If $\gamma < x_1$, prove that $x_n \to +\infty$ as $n \to \infty$.*

Thus β can be located by this method, but α and γ cannot.

Note.

(1) $f'(x) = x^2$ is unbounded. So that it does not exist such $\sup |f'(x)| = A < 1$ in Exercise 5.22(c).

(2) Cardano's Formula implies that

$$\begin{aligned}
\alpha &= -2\cos\frac{\pi}{9}, \\
\beta &= 2\sin\frac{\pi}{18}, \\
\gamma &= \sqrt{3}\sin\frac{\pi}{9} + \cos\frac{\pi}{9}.
\end{aligned}$$

Proof of (a).

(1) Write

$$g(x) = f(x) - x = \frac{x^3}{3} - x + \frac{1}{3}.$$

$f(x) = x$ if and only if $g(x) = 0$.

(2) α, β and γ are the only three roots of $g(x)$ (Theorem 8.8). Hence α, β and γ are the only three fixed points of $f(x)$.

(3) *Show that $\{x_n\}$ is strictly decreasing, or $x_{n+1} < x_n < \alpha$ for $n = 1, 2, 3, \ldots$.* Induction on n.

(a) As $n = 1$, it suffices to show that

$$g(x_1) = f(x_1) - x_1 = x_2 - x_1 < 0.$$

$g'(x) = x^2 - 1$ implies that $g(x)$ is strictly increasing on $(-\infty, -1)$. Since $x_1 < \alpha < -1$, $g(x_1) < g(\alpha) = 0$.

(b) Assume the induction hypothesis that for the single case $n = k$ holds. So that $x_{k+1} < x_k < \alpha$. Apply the same argument in (a) to get

$$g(x_{k+1}) < g(\alpha) = 0.$$

Hence $x_{k+2} < x_{k+1} < \alpha$.

(c) Since both the base case in (a) and the inductive step in (b) have been proved as true, by mathematical induction $x_{n+1} < x_n$ for all n.

(4) *Show that $\{x_n\}$ is unbounded.* (Reductio ad absurdum) If $\{x_n\}$ were bounded, by (3) $\{x_n\}$ converges to some $\xi \in \mathbb{R}^1$ (Theorem 3.14). That is, ξ is a fixed point of f. Note that $\xi \leq x_1 < \alpha$, contrary to (2).

□

Proof of (b). Consider three possible cases.

(1) The case $x_1 = \beta$. There is nothing to prove since all $x_n = \beta$.

(2) The case $x_1 \in (\alpha, \beta)$.

(a) *Show that $g(x) > 0$ on (α, β).* $g'(x) = x^2 - 1 = (x-1)(x+1)$ implies that

(i) $g(x)$ is strictly increasing on $(-\infty, -1)$.
(ii) $g(x)$ is strictly decreasing on $(-1, 1)$.
(iii) $g(x)$ is strictly increasing on $(1, \infty)$.

As $x \in (\alpha, -1)$, $g(x) > g(\alpha) = 0$. As $x \in (-1, \beta)$, $g(x) > g(\beta) = 0$. As $x = -1$, $g(-1) = 1 > 0$. Hence $g(x) > 0$ on (α, β).

(b) *Show that $x_n \in (\alpha, \beta)$ for $n = 1, 2, 3, \ldots$.* Induction on n.

(i) $x_1 \in (\alpha, \beta)$ by assumption.
(ii) Assume the induction hypothesis that for the single case $n = k$ holds, that is, $\beta > x_k > \alpha$. Since $x \mapsto x^3$ is strictly increasing,

$$\begin{aligned}
\beta > x_k > \alpha &\Longrightarrow \beta^3 > x_k^3 > \alpha^3 \\
&\Longrightarrow \frac{\beta^3 + 1}{3} > \frac{x_k^3 + 1}{3} > \frac{\alpha^3 + 1}{3} \\
&\Longrightarrow \beta > x_{k+1} > \alpha.
\end{aligned}$$

By (a),

$$g(x_k) > 0.$$

Hence $x_{k+1} > x_k > \alpha$.

(iii) Since both the base case in (i) and the inductive step in (ii) have been proved as true, by mathematical induction $x_n \in (\alpha, \beta)$ for all n.

(c) *Show that $\{x_n\}$ is strictly increasing, or $\beta > x_{n+1} > x_n > \alpha$ for $n = 1, 2, 3, \ldots$.* Induction on n.

(i) As $n = 1$, by (a)

$$g(x_1) = f(x_1) - x_1 = x_2 - x_1 > 0.$$

Note that $x_2 \in (\alpha, \beta)$ by (b).

(ii) Assume the induction hypothesis that for the single case $n = k$ holds, that is, $\beta > x_{k+1} > x_k > \alpha$. By (a)

$$g(x_{k+1}) > 0.$$

Hence $x_{k+2} > x_{k+1}$. Note that $x_{k+2} \in (\alpha, \beta)$ by (b).

(iii) Since both the base case in (i) and the inductive step in (ii) have been proved as true, by mathematical induction $\beta > x_{n+1} > x_n > \alpha$ for all n.

(d) By (b)(c), $\{x_n\}$ converges to some $\xi \in \mathbb{R}^1$ (Theorem 3.14). That is, ξ is a fixed point of f. Note that $\beta \geq \xi \geq x_1 > \alpha$. By (2) in the proof of (a), $\lim x_n = \xi = \beta$.

(3) The case $x_1 \in (\beta, \gamma)$. Similar to (2).

(a) *Show that $g(x) < 0$ on (β, γ).* Similar to (2)(a).

(b) *Show that $x_n \in (\beta, \gamma)$ for $n = 1, 2, 3, \ldots$.* Similar to (2)(b).

(c) *Show that $\{x_n\}$ is strictly decreasing, or $\beta < x_{n+1} < x_n < \gamma$ for $n = 1, 2, 3, \ldots$.* Similar to (2)(c).

(d) By (b)(c), $\{x_n\}$ converges to some $\xi \in \mathbb{R}^1$ (Theorem 3.14). That is, ξ is a fixed point of f. Note that $\beta \leq \xi \leq x_1 < \gamma$. By (2) in the proof of (a), $\lim x_n = \xi = \beta$.

$\square$

Proof of (c). Similar to (a). Recall $g(x) = f(x) - x = \frac{x^3}{3} - x + \frac{1}{3}$.

(1) *Show that $\{x_n\}$ is strictly increasing, or $x_{n+1} > x_n > \gamma$ for $n = 1, 2, 3, \ldots$.* Induction on n.

(a) As $n = 1$, it suffices to show that

$$g(x_1) = f(x_1) - x_1 = x_2 - x_1 > 0.$$

$g'(x) = x^2 - 1$ implies that $g(x)$ is strictly increasing on $(1, \infty)$. Since $x_1 > \gamma > 1$, $g(x_1) > g(\gamma) = 0$.

(b) Assume the induction hypothesis that for the single case $n = k$ holds. So that $x_{k+1} > x_k > \gamma$. Apply the same argument in (a) to get

$$g(x_{k+1}) > g(\gamma) = 0.$$

Hence $x_{k+2} > x_{k+1} > \gamma$.

(c) Since both the base case in (a) and the inductive step in (b) have been proved as true, by mathematical induction $x_{n+1} > x_n$ for all n.

(2) *Show that $\{x_n\}$ is unbounded.* (Reductio ad absurdum) If $\{x_n\}$ were bounded, by (1) $\{x_n\}$ converges to some $\xi \in \mathbb{R}^1$ (Theorem 3.14). That is, ξ is a fixed point of f. Note that $\xi \geq x_1 > \gamma$, contrary to (2) in the proof of (a).

□

Exercise 5.24. *The process described in part (c) of Exercise 5.22 can of course also be applied to functions that map $(0, \infty)$ to $(0, \infty)$. Fix some $\alpha > 1$, and put*

$$f(x) = \frac{1}{2}\left(x + \frac{\alpha}{x}\right), \qquad g(x) = \frac{\alpha + x}{1 + x}.$$

Both f and g have $\sqrt{\alpha}$ as their fixed point in $(0, \infty)$. Try to explain, on the basis of properties of f and g, why the convergence in Exercise 3.16, is so much more rapid than it is in Exercise 3.17. (Compare f' and g', draw the zig-zags suggested in Exercise 5.22.)

Proof.

(1) Note that

$$\begin{aligned} f'(x) &= \frac{1}{2}\left(1 - \frac{\alpha}{x^2}\right) \to 0 \quad \text{and} \\ g'(x) &= \frac{1 - \alpha}{(1 + x)^2} \to \frac{1 - \sqrt{\alpha}}{1 + \sqrt{\alpha}} \neq 0 \end{aligned}$$

as $x \to \alpha$.

(2) The rate of convergence of $f(x)$ is at least quadratically geometric (since $A \to 0$ in the sense of Exercise 5.22(c)).

(3) The rate of convergence of $g(x)$ is geometric of the ratio $A = \frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}}$ in the sense of Exercise 5.22(c).

(4) Hence the rate of convergence of $f(x)$ is much more rapid than of $g(x)$. (Omit drawing two zig-zag paths.)

$\square$

Exercise 5.25. *Suppose f is twice differentiable on $[a,b]$, $f(a) < 0$, $f(b) > 0$, $f'(x) \geq \delta > 0$, and $0 \leq f''(x) \leq M$ for all $x \in [a,b]$. Let ξ be the unique point in (a,b) at which $f(\xi) = 0$. Complete the details in the following outline of* ***Newton's method*** *for computing ξ.*

(a) *Choose $x_1 \in (\xi, b)$, and define $\{x_n\}$ by*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Interpret this geometrically, in terms of a tangent to the graph of f.

(b) *Prove that $x_{n+1} < x_n$ and that*

$$\lim_{n\to\infty} x_n = \xi.$$

(c) *Use Taylor's theorem to show that*

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

for some $t_n \in (\xi, x_n)$.

(d) *(Quadratic convergence) If $A = \frac{M}{2\delta}$, deduce that*

$$0 \leq x_{n+1} - \xi \leq \frac{1}{A}[A(x_1 - \xi)]^{2^n}.$$

(Compare with Exercise 3.16 and 3.18.)

(e) *Show that Newton's method amounts to finding a fixed point of the function g defined by*

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

How does $g'(x)$ behave for x near ξ?

(f) *Put $f(x) = x^{\frac{1}{3}}$ on $(-\infty, +\infty)$ and try Newton's method. What happens?*

Proof of (a) (Wikipedia). The equation of the tangent line to the curve $y = f(x)$ at $x = x_n$ is

$$y = f'(x_n)(x - x_n) + f(x_n).$$

The x-intercept of this line (the value of x which makes $y = 0$) is taken as the next approximation, x_{n+1}, to the root, so that the equation of the tangent line is satisfied when $(x,y) = (x_{n+1}, 0)$:

$$0 = f'(x_n)(x - x_n) + f(x_n).$$

Solving for x_{n+1} gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

$\square$

Proof of (b).

(1) *Show that $x_n \geq \xi$ for all n.* Induction on n.

   (a) $n = 1$ is clearly true: $x_1 > \xi$ by assumption.

   (b) Assume the induction hypothesis that for the single case $n = k$ holds. By the mean value theorem (Theorem 5.10), there is a point $\xi_k \in (\xi, x_k)$

   $$f(x_k) - f(\xi) = f'(\xi_k)(x_k - \xi),$$

   or

   $$f(x_k) = f'(\xi_k)(x_k - \xi)$$

   (since $f(\xi) = 0$). Since $f'' \geq 0$, f' is monotonically increasing (Theorem 5.11(a)). Hence $f'(\xi_k) \leq f'(x_k)$ and thus

   $$f(x_k) = f'(\xi_k)(x_k - \xi) \leq f'(x_k)(x_k - \xi).$$

   Since $f'(x_k) > 0$ by assumption,

   $$\xi \leq x_k - \frac{f(x_k)}{f'(x_k)} = x_{k+1}.$$

   (c) Since both the base case in (a) and the inductive step in (b) have been proved as true, by mathematical induction $x_n \geq \xi$ for all n.

(2) *Show that $x_{n+1} < x_n$ for all n.*

   (a) Since $f' > 0$, $f'(x_n) > 0$ for all n.

   (b) Since $f' > 0$, f is strictly increasing (Theorem 5.10). Hence $f(x_n) > f(\xi) = 0$ for all n (by (1)).

   (c) By (a)(b), $\frac{f(x_n)}{f'(x_n)} > 0$ or

   $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} < x_n.$$

(3) By Theorem 3.14, $\{x_n\}$ converges to some real number $\zeta \geq \xi$. Note that f and f' are continuous by the existence of f'' (Theorem 5.2), we have

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n - \frac{f(\lim_{n\to\infty} x_n)}{f'(\lim_{n\to\infty} x_n)}$$

provided $f' \neq 0$ (Theorem 4.9 and Theorem 4.4). Hence

$$\zeta = \zeta - \frac{f(\zeta)}{f'(\zeta)}$$

or $f(\zeta) = 0$. By the uniqueness of ξ, $\zeta = \xi$ or $\lim x_n = \xi$ as desired.

$\square$

Proof of (c). By Taylor's theorem (Theorem 5.15),

$$f(\xi) = f(x_n) + f'(x_n)(\xi - x_n) + \frac{f''(t_n)}{2}(\xi - x_n)^2$$

for some $t_n \in (\xi, x_n)$. Note that $f(\xi) = 0$, $f'(x_n) \neq 0$ and $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, we have the desired result. $\square$

Proof of (d). Clearly, $0 \leq x_{n+1} - \xi$ for all n (by (b)). Besides, by (c)

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

Note that $f'' \leq M$ and $f' \geq \delta > 0$ by assumption, and thus

$$x_{n+1} - \xi \leq \frac{M}{2\delta}(x_n - \xi)^2 = A(x_n - \xi)^2.$$

By induction,

$$x_{n+1} - \xi \leq \frac{1}{A}(A(x_1 - \xi))^{2^n}.$$

$\square$

Note. Compare with Exercise 3.16 and Exercise 3.18. Might assume that $p > 1$.

(1) Fix a positive number α. Let $f(x) = x^p - \alpha$ on $E = (a, b)$ where $a = \frac{1}{2}\alpha^{\frac{1}{p}}$ and

$$b = \begin{cases} 2\alpha^{\frac{1}{p}} & (p = 2), \\ \left(\frac{2(p-1)}{p}\right)^{\frac{1}{p-2}} \alpha^{\frac{1}{p}} & (p > 2). \end{cases}$$

$E = (a, b)$ is well-defined since $a < b$. Besides, $\xi = \alpha^{\frac{1}{p}} \in E = (a, b)$.

(2) By construction,

$$f(a) < 0 \text{ and } f(b) > 0.$$

By $f'(x) = px^{p-1}$ and $f''(x) = p(p-1)x^{p-2}$,

$$f'(x) \geq pa^{p-1} > 0,$$
$$0 \leq f''(x) \leq p(p-1)b^{p-2}.$$

on E. Write

$$\delta = pa^{p-1} = \frac{p}{2^{p-1}}\alpha^{\frac{p-1}{p}},$$
$$M = p(p-1)b^{p-2} = 2(p-1)^2\alpha^{\frac{p-2}{p}}.$$

(3) Hence the Newton's method works for $f(x) = x^p - \alpha$. That is, as we define $\{x_n\}$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{p-1}{p}x_n + \frac{\alpha}{p}x_n^{-p+1},$$

we have $\lim x_n = \xi = \alpha^{\frac{1}{p}}$. And

$$0 \leq x_{n+1} - \xi \leq \frac{1}{A}(A(x_1 - \xi))^{2^n}.$$

Here

$$A = \frac{M}{2\delta} = \frac{2^{p-1}(p-1)^2}{p\alpha^{\frac{1}{p}}}.$$

(4) Note that

$$\beta = \frac{p\alpha^{\frac{1}{p}}}{(p-1)^2} \neq \frac{p\alpha^{\frac{1}{p}}}{2^{p-1}(p-1)^2} = \frac{1}{A}.$$

where β is defined in the proof of Exercise 3.18. Note that $f'(x_n) \geq f'(\xi)$ (since f' is monotonically increasing and all $x_n \geq \xi$), and thus A can be chosen by a better estimation:

$$A = \frac{M}{2f'(\xi)} = \frac{(p-1)^2}{p\alpha^{\frac{1}{p}}} = \frac{1}{\beta}.$$

Now it is exactly the same as Exercise 3.16 and Exercise 3.18.

Proof of (e).

(1) Define $g(x) = x - \frac{f(x)}{f'(x)}$ on $[a, b]$. $g(\xi) = \xi$ if and only if $f(\xi) = 0$.

(2) By the construction of g, g is differentiable and

$$g'(x) = 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2}.$$

(3) Hence

$$|g'(x)| \leq \left|\frac{f(x)f''(x)}{f'(x)^2}\right| = \frac{|f(x)||f''(x)|}{|f'(x)|^2} \leq \frac{M}{\delta^2}|f(x)|.$$

As $x \to \xi$, $|f(x)| \to 0$. Therefore, $|g'(x)| \to 0$ or $g'(x) \to 0$ as $x \to \xi$.

$\square$

Proof of (f).

(1) It is clearly that $f(x) = 0$ if and only if $x = 0$. Write $\xi = 0$.

(2) Note that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = -2x_n,$$

or

$$x_n = (-2)^{n-1} x_1$$

for any $x_1 \in (\xi, \infty)$ where $n = 1, 2, 3, \ldots$. Hence, the sequence $\{x_n\}$ does not converge for any choice of $x_1 \in (\xi, \infty)$. In this case we cannot find ξ satisfying $f(\xi) = 0$ by Newton's method.

(3) In fact,

$$f'(x) = \frac{1}{3} x^{-\frac{2}{3}} \to 0 \text{ as } x \to \pm\infty.$$

Hence such $\delta > 0$ satisfying $f'(x) \geq \delta > 0$ does not exist.

$\square$

Exercise 5.26. *Suppose f is differentiable on $[a, b]$, $f(a) = 0$, and there is a real number A such that $|f'(x)| \leq A|f(x)|$ on $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$. (Hint: Fix $x_0 \in [a, b]$, let*

$$M_0 = \sup |f(x)|, \qquad M_1 = \sup |f'(x)|$$

for $a \leq x \leq x_0$. For any such x,

$$|f(x)| \leq M_1(x_0 - a) \leq A(x_0 - a)M_0.$$

Hence $M_0 = 0$ if $A(x_0 - a) < 1$. That is, $f = 0$ on $[a, x_0]$. Proceed.)

Proof (Hint).

(1) If $A = 0$, then $f'(x) = 0$ or $f(x)$ is constant on $[a, b]$ (Theorem 5.11(b)). Since $f(a) = 0$, $f(x) = 0$ on $[a, b]$.

(2) Suppose that $A > 0$. Fix $x_0 \in [a, b]$, let

$$M_0 = \sup |f(x)|, \qquad M_1 = \sup |f'(x)|$$

for $a \leq x \leq x_0$. Since $|f'(x)| \leq A|f(x)|$ on $[a, b]$,

$$|f'(x)| \leq A|f(x)| \leq AM_0.$$

Since AM_0 is an upper bound for $|f'(x)|$,

$$M_1 \leq AM_0.$$

(3) Given any $x \in [a, x_0]$. Since f is differentiable on $[a, x_0] \subseteq [a, b]$, by the mean value theorem (Theorem 5.10), there is $\xi \in (a, x)$ such that

$$f(x) - f(a) = f'(\xi)(x - a).$$

Note that $f(a) = 0$ by assumption. So that

$$\begin{aligned}
|f(x)| &= |f'(\xi)|(x-a) & \\
&\leq M_1(x-a) & \text{(Definition of } M_1) \\
&\leq AM_0(x-a) & ((2)) \\
&\leq AM_0(x_0-a). & (x \in [a, x_0])
\end{aligned}$$

Since $AM_0(x_0 - a)$ is an upper bound for $|f(x)|$,

$$M_0 \leq AM_0(x_0 - a).$$

Take

$$x_0 = \min \left\{ \frac{1}{2A} + a, b \right\}$$

so that $M_0 \leq AM_0(x_0 - a) \leq \frac{M_0}{2}$. $M_0 = 0$ or $f(x) = 0$ on $[a, x_0]$.

(4) Take a partition

$$P = \{a = x_{-1}, x_0, \ldots, x_n = b\}$$

of $[a, b]$ such that each subinterval $[x_{i-1}, x_i]$ satisfying $\Delta x_i = x_i - x_{i-1} < \frac{1}{2A}$. By (3), $f(x) = 0$ on $[x_{-1}, x_0]$. Apply the same argument in (3), $f(x) = 0$ on $[x_0, x_1]$. Continue this process, $f(x) = 0$ on each subinterval and thus on the whole interval $[a, b]$.

□

Note. It holds for vector-valued functions too:

> *Suppose* $\mathbf{f}$ *is a vector-valued differentiable function on* $[a, b]$, $f(a) = 0$, *and there is a real number* A *such that* $|\mathbf{f}'(x)| \leq A|\mathbf{f}(x)|$ *on* $[a, b]$. *Prove that* $\mathbf{f}(x) = 0$ *for all* $x \in [a, b]$.

The proof is similar except using Theorem 5.19 ($|\mathbf{f}(b) - \mathbf{f}(a)| \leq (b - a)|\mathbf{f}'(x)|$) in addition.

Exercise 5.27. *Let* ϕ *be a real function defined on a rectangle* R *in the plane, given by* $a \leq x \leq b$, $\alpha \leq y \leq \beta$. *A* ***solution*** *of the initial-value problem*

$$y' = \phi(x, y), \qquad y(a) = c \qquad (\alpha \leq c \leq \beta)$$

is, by definition, a differentiable function f *on* $[a, b]$ *such that* $f(a) = c$, $\alpha \leq f(x) \leq \beta$, *and*

$$f'(x) = \phi(x, f(x)) \qquad (a \leq x \leq b)$$

Prove that such a problem has at most one solution if there is a constant A such that

$$|\phi(x, y_2) - \phi(x, y_1)| \leq A|y_2 - y_1|$$

whenever $(x, y_1) \in R$ and $(x, y_2) \in R$. (Hint: Apply Exercise 26 to the difference of two solutions.) Note that this uniqueness theorem does not hold for the initial-value problem

$$y' = y^{\frac{1}{2}}, \qquad y(0) = 0,$$

which has two solutions: $f(x) = 0$ and $f(x) = \frac{x^2}{4}$. Find all other solutions.

Proof (Hint).

(1) Suppose f_1 and f_2 are two solutions of that problem. Define $f = f_1 - f_2$. f is differentiable on $[a, b]$, $f(a) = f_1(a) - f_2(a) = c - c = 0$. And

$$\begin{aligned} |f'(x)| &= |f_1'(x) - f_2'(x)| \\ &= |\phi(x, f_1(x)) - \phi(x, f_2(x))| \\ &\leq A|f_1(x) - f_2(x)| \end{aligned}$$

on $[a, b]$. By Exercise 5.26, $f(x) = 0$ on $[a, b]$, or $f_1(x) = f_2(x)$ on $[a, b]$.

(2) *The initial-value problem*

$$y' = y^{\frac{1}{2}}, \qquad y(0) = 0,$$

which has two solutions: $f(x) = 0$ and $f(x) = \frac{x^2}{4}$. Find all other solutions.

Note. It does not exist a real A such that $|\phi(x, y_2) - \phi(x, y_1)| \leq A|y_2 - y_1|$ in this initial-value problem.

(a) Clearly, $f(x) = 0$ and $f(x) = \frac{x^2}{4}$ are two solutions for the initial-value problem.

(b) Suppose $f(x) \neq 0$ on $[0, \infty)$. Since $f'(x) = f(x)^{\frac{1}{2}}$, $f(x) \geq 0$. Since $f(x)$ is continuous (Theorem 5.2), the set

$$E = \{x \in [0, \infty) : f(x) > 0\}$$

is open in $\mathbb{R}^1$ (Theorem 4.8). By Exercise 2.29 we write E as the union of an at most countable collection of disjoint segments, say

$$E = \bigcup_{(a_i, b_i) \in \mathscr{C}} (a_i, b_i)$$

where $\mathscr{C}$ is at most countable and all (a_i, b_i) segments are disjoint. Note that E (or $\mathscr{C}$) is nonempty.

(c) For any segment (a_i, b_i), define $g(x) = f(x)^{\frac{1}{2}}$ on (a_i, b_i). (Clearly, $g(a_i) = f(a_i) = 0$ by the definition of E.) Thus

$$g'(x) = \frac{1}{2} f(x)^{-\frac{1}{2}} f'(x) = \frac{1}{2}.$$

Hence

$$g(x) = \frac{1}{2}x + c$$

for some constant $c \in \mathbb{R}^1$. So

$$f(x) = g(x)^2 = \left(\frac{1}{2}x + c\right)^2.$$

$f(a_i) = 0$ implies that $c = -\frac{a_i}{2}$. Hence

$$f(x) = \frac{1}{4}(x - a_i)^2$$

on (a_i, b_i).

(d) By (c), if $b_i < 0$ is defined as a real number, then $f(b_i) = 0$ by definition of E. Note that

$$\lim_{x \to b_i -} f(x) = \frac{1}{4}(b_i - a_i)^2 > 0,$$

which is absurd. Hence $b_i = \infty$ and thus E is of the form

$$E = (a, \infty) \qquad (a \geq 0).$$

Therefore,

$$f(x) = \begin{cases} 0 & (0 \leq x \leq a), \\ \frac{1}{4}(x - a)^2 & (x > a \geq 0). \end{cases}$$

□

Exercise 5.28. *Formulate and prove an analogous uniqueness theorem for systems of differential equations of the form*

$$y_j' = \phi_j(x, y_1, \ldots, y_k), \qquad y_j(a) = c_j \qquad (j = 1, \ldots, k)$$

Note that this can be rewritten in the form

$$\mathbf{y}' = \boldsymbol{\phi}(x, \mathbf{y}), \qquad \mathbf{y}(a) = \mathbf{c}$$

where $\mathbf{y} = (y_1, \ldots, y_k)$ *ranges over a* k*-cell,* $\boldsymbol{\phi}$ *is the mapping of a* $(k+1)$*-cell into the Euclidean* k*-space whose components are the function* $\phi_1, \ldots, \phi_k$*, and* $\mathbf{c}$ *is the vector* $(c_1, \ldots, c_k)$*. Use Exercise 5.26, for vector-valued functions.*

Proof.

(1) *A **solution** of the initial-value problem*

$$\mathbf{y}' = \boldsymbol{\phi}(x, \mathbf{y}), \qquad \mathbf{y}(a) = \mathbf{c}$$

is, by definition, a differentiable function $\mathbf{f}$ *on* $[a, b]$ *such that* $\mathbf{f}(a) = \mathbf{c}$*, and*

$$\mathbf{f}'(x) = \boldsymbol{\phi}(x, \mathbf{f}(x)) \qquad (a \leq x \leq b).$$

Then this problem has at most one solution if there is a constant A *such that*

$$|\boldsymbol{\phi}(x, \mathbf{y}_2) - \boldsymbol{\phi}(x, \mathbf{y}_1)| \leq A|\mathbf{y}_2 - \mathbf{y}_1|$$

whenever $(x, \mathbf{y}_1) \in R$ *and* $(x, \mathbf{y}_2) \in R$ *where* R *is a* $(k+1)$*-cell defined by*

$$R = [a, b] \times [\alpha_1, \beta_1] \times \cdots \times [\alpha_k, \beta_k].$$

(2) Similar to Exercise 5.27, Suppose $\mathbf{f}_1$ and $\mathbf{f}_2$ are two solutions of that problem. Define $\mathbf{f} = \mathbf{f}_1 - \mathbf{f}_2$. $\mathbf{f}$ is differentiable on $[a, b]$, $\mathbf{f}(a) = \mathbf{f}_1(a) - \mathbf{f}_2(a) = \mathbf{c} - \mathbf{c} = 0$. And

$$\begin{aligned} |\mathbf{f}'(x)| &= |\mathbf{f}_1'(x) - \mathbf{f}_2'(x)| \\ &= |\boldsymbol{\phi}(x, \mathbf{f}_1(x)) - \boldsymbol{\phi}(x, \mathbf{f}_2(x))| \\ &\leq A|\mathbf{f}_1(x) - \mathbf{f}_2(x)| \end{aligned}$$

on $[a, b]$. By Note in Exercise 5.26, $\mathbf{f}(x) = 0$ on $[a, b]$, or $\mathbf{f}_1(x) = \mathbf{f}_2(x)$ on $[a, b]$.

□

Exercise 5.29. *Specialize Exercise 5.28 by considering the system*

$$\begin{aligned} y_j' &= y_{j+1} \qquad (j = 1, \ldots, k-1), \\ y_k' &= f(x) - \sum_{j=1}^{k} g_j(x) y_j \end{aligned}$$

where $f, g_1, \ldots, g_k$ *are continuous real functions on* $[a, b]$*, and derive a uniqueness theorem for solutions of the equation*

$$y^{(k)} + g_k(x) y^{(k-1)} + \cdots + g_2(x) y' + g_1(x) y = f(x),$$

subject to initial conditions

$$y(a) = c_1, \qquad y'(a) = c_1, \qquad \ldots, \qquad y^{(k-1)}(a) = c_k.$$

Proof.

(1) Write

$$
\begin{aligned}
\mathbf{y} &= (y_1, \ldots, y_k) \\
&= \left(y, y', y'', \ldots, y^{(k-1)}\right), \\
\boldsymbol{\phi}(x, \mathbf{y}) &= \left(y_2, y_3, \ldots, y_{k-1}, f(x) - \sum_{j=1}^{k} g_j(x) y_j\right) \\
&= \left(y', y'', \ldots, y^{(k-1)}, f(x) - \sum_{j=1}^{k} g_j(x) y^{(j-1)}\right), \\
\mathbf{c} &= (c_1, \ldots, c_k).
\end{aligned}
$$

So that

$$\mathbf{y}' = \boldsymbol{\phi}(x, \mathbf{y}), \qquad \mathbf{y}(a) = \mathbf{c}$$

where $\mathbf{y}$ ranges over a k-cell R.

(2) *To show that the problem has at most one solution, by Exercise 5.28 it suffices to show that there is a constant A such that*

$$|\boldsymbol{\phi}(x, \mathbf{y}) - \boldsymbol{\phi}(x, \mathbf{z})| \leq A|\mathbf{y} - \mathbf{z}|$$

whenever $(x, \mathbf{y}) \in R$ and $(x, \mathbf{z}) \in R$.

(3) Since all g_j $(1 \leq j \leq k)$ are real continuous functions on a compact set $[a, b]$, all g_j are bounded (Theorem 4.15), say $|g_j| \leq M$ on $[a, b]$ for some $M_j \in \mathbb{R}^1$ $(1 \leq j \leq k)$.

(4) Write $\mathbf{y} = (y_1, \dots, y_k)$ and $\mathbf{z} = (z_1, \dots, z_k)$. So

$$\begin{aligned}
&|\boldsymbol{\phi}(x,\mathbf{y}) - \boldsymbol{\phi}(x,\mathbf{z})|^2 \\
=&\left|\left(y_2 - z_2, y_3 - z_3, \dots, y_{k-1} - z_{k-1}, -\sum_{j=1}^{k} g_j(x)(y_j - z_j)\right)\right|^2 \\
=&\sum_{j=2}^{k-1}(y_j - z_j)^2 + \left(-\sum_{j=1}^{k} g_j(x)(y_j - z_j)\right)^2 \\
\leq&\sum_{j=2}^{k-1}(y_j - z_j)^2 + \sum_{j=1}^{k} g_j(x)^2 \sum_{j=1}^{k}(y_j - z_j)^2 && \text{(Theorem 1.35)} \\
\leq&\sum_{j=2}^{k-1}(y_j - z_j)^2 + \sum_{j=1}^{k} M_j^2 \sum_{j=1}^{k}(y_j - z_j)^2 && \text{((3))} \\
\leq&\sum_{j=1}^{k}(y_j - z_j)^2 + \sum_{j=1}^{k} M_j^2 \sum_{j=1}^{k}(y_j - z_j)^2 && (x^2 \geq 0\ \forall x \in \mathbb{R}^1) \\
\leq&\left(1 + \sum_{j=1}^{k} M_j^2\right)|\mathbf{y} - \mathbf{z}|^2.
\end{aligned}$$

Hence $|\boldsymbol{\phi}(x,\mathbf{y}) - \boldsymbol{\phi}(x,\mathbf{z})| \leq A|\mathbf{y} - \mathbf{z}|$ for some $A = \left(1 + \sum_{j=1}^{k} M_j^2\right)^{\frac{1}{2}}$.

$\square$