

# Solutions to the book: *Fulton, Algebraic Curves*

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# Chapter 1: Affine Algebraic Sets

## 1.1. Algebraic Preliminaries

### Problem 1.1.\*

Let  $R$  be a domain.

- (a) If  $f, g$  are forms of degree  $r, s$  respectively in  $R[x_1, \dots, x_n]$ , show that  $fg$  is a form of degree  $r + s$ .
- (b) Show that any factor of a form in  $R[x_1, \dots, x_n]$  is also a form.

*Proof of (a).*

- (1) Write

$$f = \sum_{(i)} a_{(i)} x^{(i)},$$
$$g = \sum_{(j)} b_{(j)} x^{(j)},$$

where  $\sum_{(i)}$  is the summation over  $(i) = (i_1, \dots, i_n)$  with  $i_1 + \dots + i_n = r$  and  $\sum_{(j)}$  is the summation over  $(j) = (j_1, \dots, j_n)$  with  $j_1 + \dots + j_n = s$ .

- (2) Hence,

$$fg = \sum_{(i)} \sum_{(j)} a_{(i)} b_{(j)} x^{(i)} x^{(j)}$$
$$= \sum_{(i),(j)} a_{(i)} b_{(j)} x^{(k)}$$

where  $(k) = (i_1 + j_1, \dots, i_n + j_n)$  with  $(i_1 + j_1) + \dots + (i_n + j_n) = r + s$ . Each  $x^{(k)}$  is the form of degree  $r + s$  and  $a_{(i)} b_{(j)} \in R$ . Hence  $fg$  is a form of degree  $r + s$ .

□

*Proof of (b).*

- (1) Given any form  $f \in R[x_1, \dots, x_n]$ , and write  $f = gh$ . It suffices to show that  $g$  is a form as well. (So does  $h$ .)
- (2) Write

$$g = g_0 + \dots + g_r, \quad h = h_0 + \dots + h_s$$

where  $g_r \neq 0$  and  $h_s \neq 0$ . So

$$f = gh = g_0h_0 + \cdots + g_rh_s.$$

Since  $R$  is a domain,  $R[x_1, \dots, x_n]$  is a domain and thus  $g_rh_s \neq 0$ . The maximality of  $r$  and  $s$  implies that  $\deg f = r + s$ . Therefore, by the maximality of  $r + s$ ,  $f = g_rh_s$ , or  $g = g_r$ , or  $g$  is a form.

□

**Problem 1.2.\***

Let  $R$  be a UFD,  $K$  the quotient field of  $R$ . Show that every element  $z$  of  $K$  may be written  $z = a/b$ , where  $a, b \in R$  have no common factors; this representative is unique up to units of  $R$ .

*Proof.*

- (1) Show that every element  $z$  of  $K$  may be written  $z = a/b$ , where  $a, b \in R$  have no common factors. Given any  $z = a/b \in K$  where  $a, b \in R$ . Write

$$\begin{aligned} a &= p_1 \cdots p_n, \\ b &= q_1 \cdots q_m \end{aligned}$$

where all  $p_1, \dots, p_n, q_1, \dots, q_m$  are irreducible in  $R$ . (It is possible since  $R$  is a UFD.) For each  $i$ , suppose  $p_i \mid q_j$  for some  $i, j$ . Write  $q_j = p_i u$  for some  $u \in R$ . By the irreducibility of  $p_i$  and  $q_j$ ,  $u$  is a unit. So

$$z = \frac{a}{b} = \frac{p_1 \cdots \widehat{p_i} \cdots p_n}{q_1 \cdots \widehat{q_j} \cdots q_m} = \frac{p_1 \cdots \widehat{p_i} \cdots p_n}{u q_1 \cdots \widehat{q_j} \cdots q_m}.$$

Continue this method we can write  $z = \frac{a'}{b'}$  where  $a'$  and  $b'$  have no common factors.

- (2) Write  $z = a/b = a'/b'$  where

- (a)  $a, b, a', b' \in R$ ,
- (b)  $a$  and  $b$  have no common factors,
- (c)  $a'$  and  $b'$  have no common factors.

Write

$$\begin{aligned} a &= p_1 \cdots p_n, \\ b &= q_1 \cdots q_m, \\ a' &= p'_1 \cdots p'_{n'}, \\ b' &= q'_1 \cdots q'_{m'} \end{aligned}$$

where all  $p_i, q_j, p'_{i'}, q'_{j'}$  are irreducible in  $R$ . As  $z = a/b = a'/b'$ ,  $ab' = a'b$  or

$$p_1 \cdots p_n q'_1 \cdots q'_{m'} = p'_1 \cdots p'_{n'} q_1 \cdots q_m.$$

- (3) For  $i = 1$ ,  $p_1 = u_1 p'_{i'}$  for some unit  $u_1 \in R$  since  $a$  and  $b$  have no common factors and all  $p_1, q_j, p'_{i'}$  are irreducible. Hence

$$u_1 \widehat{p_1} p_2 \cdots p_n q'_1 \cdots q'_{m'} = p'_1 \cdots \widehat{p'_{i'}} \cdots p'_{n'} q_1 \cdots q_m.$$

Continue this method, we have  $n \leq n'$  and all  $p_1, \dots, p_n$  are canceled.

- (4) Conversely, we can apply the argument in (3) to  $i' = 1, \dots, n'$  to conclude that  $n' \leq n$ . Therefore,  $n = n'$  and

$$\underbrace{u_1 \cdots u_n}_{\text{a unit in } R} q'_1 \cdots q'_{m'} = q_1 \cdots q_m.$$

Hence,  $b = ub'$  where  $u = u_1 \cdots u_n$  is a unit in  $R$ . Similarly,  $a = va'$  where  $v$  is a unit in  $R$ . So the representative of  $z \in K$  is unique up to units of  $R$ .

□

### Problem 1.3.\*

Let  $R$  be a PID. Let  $\mathfrak{p}$  be a nonzero, proper, prime ideal in  $R$ .

- (a) Show that  $\mathfrak{p}$  is generated by an irreducible element.
- (b) Show that  $\mathfrak{p}$  is maximal.

*Proof of (a).*

- (1) Let  $\mathfrak{p} = (a)$  be a nonzero, proper, prime ideal in  $R$ . It suffices to show that  $a$  is irreducible.
- (2) Suppose  $a = bc$ . By the primality of  $\mathfrak{p}$ ,  $b \in \mathfrak{p}$  or  $c \in \mathfrak{p}$ . Suppose  $b \in \mathfrak{p} = (a)$ . (The case  $c \in \mathfrak{p}$  is similar.) Then there is a  $d \in R$  such that  $b = ad$ . Hence,  $a = bc = adc$  or  $(1 - dc)a = 0$ .
- (3) Since  $R$  is a domain,  $1 = dc$  or  $a = 0$ .  $a = 0$  implies that  $\mathfrak{p} = (0)$  is a zero ideal, contrary to the assumption. Therefore,  $1 = dc$ , or  $c$  is a unit, or  $a$  is irreducible.

□

*Proof of (b).*

- (1) Given any ideal  $I = (b)$  of  $R$  containing  $\mathfrak{p} = (a)$ . As the generator  $a$  of  $\mathfrak{p}$  is in  $\mathfrak{p} \subseteq I$ , there is some  $c \in R$  such that  $a = bc$ . By the irreducibility of  $a$  (in  $(a)$ ),  $b$  is a unit or  $c$  is a unit.
- (2)  $b$  is a unit implies that  $I = R$ .  $c$  is a unit implies that  $I = \mathfrak{p}$ . In any case, we conclude that  $\mathfrak{p}$  is maximal.

□

**Problem 1.4.\***

Let  $k$  be an infinite field,  $f \in k[x_1, \dots, x_n]$ . Suppose  $f(a_1, \dots, a_n) = 0$  for all  $a_1, \dots, a_n \in k$ . Show that  $f = 0$ . (Hint: Write

$$f = \sum f_i x_n^i, \quad f_i \in k[x_1, \dots, x_{n-1}].$$

Use induction on  $n$ , and the fact that  $f(a_1, \dots, a_{n-1}, x_n)$  has only a finite number of roots if any  $f_i(a_1, \dots, a_{n-1}) \neq 0$ .)

*Proof.*

- (1) Induction on  $n$ . The case  $n = 1$ . (Reductio ad absurdum) If there were a nonzero  $f \in k[x_1]$  such that  $f(a) = 0$  for all  $a \in k$ . Note that  $f$  has at most  $\deg f < \infty$  roots, contrary to the infinity of  $k$ .
- (2) Assume that the conclusion holds for  $n - 1$ , then for any  $f \in k[x_1, \dots, x_n]$  we can write

$$f = \sum f_i x_n^i, \quad f_i \in k[x_1, \dots, x_{n-1}]$$

as  $f \in (k[x_1, \dots, x_{n-1}])[x_n]$ . Suppose  $f(a_1, \dots, a_n) = 0$  for all  $a_1, \dots, a_n \in k$ . For fixed  $a_1, \dots, a_{n-1}$ , the polynomial  $f(a_1, \dots, a_{n-1}, x_n) \in k[x_n]$  has all distinct roots in an infinite field  $k$ . By (1),  $f(a_1, \dots, a_{n-1}, x_n) = 0 \in k[x_n]$ , or each  $f_i(a_1, \dots, a_{n-1}) = 0$ . As all  $a_1, \dots, a_{n-1}$  run over  $k$ , we can apply the induction hypothesis each  $f_i(x_1, \dots, x_{n-1}) = 0 \in k[x_1, \dots, x_{n-1}]$ . Hence,  $f = 0 \in k[x_1, \dots, x_n]$ .

□

*Note.* If  $k$  is a finite field of order  $q = p^k$ , then the polynomial  $f(x) = x^q - x$  has  $q$  distinct roots in  $k$ .



**Problem 1.5.\***

Let  $k$  be any field. Show that there are an infinitely number of irreducible monic polynomials in  $k[x]$ . (Hint: Suppose  $f_1, \dots, f_n$  were all of them, and factor  $f_1 \cdots f_n + 1$  into irreducible factors.)

*Proof (Due to Euclid).*

- (1) If  $f_1, \dots, f_n$  were all irreducible monic polynomials, then we consider

$$g = f_1 \cdots f_n + 1 \in k[x].$$

So there is an irreducible monic polynomial  $f = f_i$  dividing  $g$  for some  $i$  since

$$\deg g = \deg f_1 + \cdots + \deg f_n \geq 1$$

and  $k[x]$  is a UFD.

- (2) However,  $f$  would divide the difference

$$g - f_1 \cdots f_{i-1} f_{i+1} \cdots f_n = 1,$$

contrary to  $\deg f_i \geq 1$ .

□

**Problem 1.6.\***

Show that any algebraically closed field is infinite. (Hint: The irreducible monic polynomials are  $x - a$ ,  $a \in k$ .)

*Proof (Due to Euclid).*

- (1) Let  $k$  be an algebraically closed field. If  $a_1, \dots, a_n$  were all elements in  $k$ , then we consider a monic polynomials

$$f(x) = (x - a_1) \cdots (x - a_n) + 1 \in k[x].$$

- (2) Since  $k$  is algebraically closed, there is an element  $a \in k$  such that  $f(a) = 0$ . By assumption,  $a = a_i$  for some  $1 \leq i \leq n$ , and thus  $f(a) = f(a_i) = 1$ , contrary to the fact that a field is a commutative ring where  $0 \neq 1$  and all nonzero elements are invertible.

□

**Problem 1.7.\***

Let  $k$  be a field,  $f \in k[x_1, \dots, x_n]$ ,  $a_1, \dots, a_n \in k$ .

(a) Show that

$$f = \sum \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}, \quad \lambda_{(i)} \in k.$$

(b) If  $f(a_1, \dots, a_n) = 0$ , show that  $f = \sum_{i=1}^n (x_i - a_i)g_i$  for some (not unique)  $g_i$  in  $k[x_1, \dots, x_n]$ .

*Proof of (a).*

(1) Regard  $k[x_1, \dots, x_n]$  as  $(k[x_1, \dots, x_{n-1}])[x_n]$ . Since  $(k[x_1, \dots, x_{n-1}])[x_n]$  is a Euclidean domain with a function

$$f \in (k[x_1, \dots, x_{n-1}])[x_n] \mapsto \deg_{x_n} f \in \mathbb{Z}_{\geq 0}$$

satisfying the division-with-remainder property.

(2) Apply the division algorithm for  $f$  and nonzero  $x_n - a_n$  to produce a quotient  $q$  and remainder  $r$  with  $f = (x_n - a_n)q + r$  and either  $r = 0$  or  $\deg_{x_n}(r) < \deg_{x_n}(x_n - a_n) = 1$ . That is,  $r \in k[x_1, \dots, x_{n-1}]$  is a constant in  $(k[x_1, \dots, x_{n-1}])[x_n]$ . Continue this process to get that  $f$  is of the form

$$f = \sum_{i_n} f_{i_n} (x_n - a_n)^{i_n}$$

where  $f_{i_n} \in k[x_1, \dots, x_{n-1}]$ .

(3) Use the same argument in (2) for each  $f_{i_n} \in k[x_1, \dots, x_{n-1}]$ , we have

$$\begin{aligned} f_{i_n} &= \sum_{i_{n-1} \in k[x_1, \dots, x_{n-2}]} \underbrace{f_{i_n, i_{n-1}}}_{\in k[x_1, \dots, x_{n-2}]} (x_{n-1} - a_{n-1})^{i_{n-1}} \\ f_{i_n, i_{n-1}} &= \sum_{i_{n-2} \in k[x_1, \dots, x_{n-3}]} \underbrace{f_{i_n, i_{n-1}, i_{n-2}}}_{\in k[x_1, \dots, x_{n-3}]} (x_{n-2} - a_{n-2})^{i_{n-2}}, \\ &\dots \\ f_{i_n, \dots, i_2} &= \sum_{i_1 \in k} \underbrace{f_{i_n, \dots, i_1}}_{\in k} (x_1 - a_1)^{i_1}. \end{aligned}$$

Note that  $f_{i_n, \dots, i_1} \in k$ , we can write

$$f = \sum \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}, \quad \lambda_{(i)} \in k.$$

by replacing all  $f_{i_n, \dots, i_k}$  by  $f_{i_n, \dots, i_{k-1}}$  for  $k = n, n-1, \dots, 2$ .

(4) Or use the induction on  $n$ .

□

*Proof of (b).*

(1) Write

$$f = \sum \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}, \quad \lambda_{(i)} \in k$$

by (a).

(2) As  $f(a_1, \dots, a_n) = 0$ ,  $\lambda_{(i)} = 0$  if all  $i_1, \dots, i_n$  are zero, that is, there is no nonzero constant term in the representation of  $f$ . Hence, for each term

$$f_{(i)} := \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}$$

with  $\lambda_{(i)} \neq 0$ , there exists one  $i_k > 0$  for some  $1 \leq k \leq n$ . So we can write

$$f_{(i)} = (x_k - a_k) \underbrace{(\lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_k - a_k)^{i_k-1} \cdots (x_n - a_n)^{i_n})}_{:= g_{(i)} \in k[x_1, \dots, x_n]}.$$

Note that the expression of  $f_{(i)}$  is not unique since there may exist more than one  $i_k > 0$  as  $1 \leq k \leq n$ .

(3) Now we iterate each nonzero term in  $f$ , apply the factorization in (2), and then group by each  $x_k - a_k$ . Therefore, we can write

$$f = \sum_{i=1}^n (x_i - a_i) g_i$$

for some  $g_i \in k[x_1, \dots, x_n]$ .

(4) The expression of  $f$  is not unique. For example, take  $f(x, y) = x^2 + 2xy + y^2 \in k[x, y]$ . As  $f(0, 0) = 0$ , we can write

$$\begin{aligned} f(x, y) &= x \cdot \underbrace{(x + 2y)}_{g_1} + y \cdot \underbrace{y}_{g_2}, \text{ or} \\ &= x \cdot \underbrace{(x + y)}_{g_1} + y \cdot \underbrace{(x + y)}_{g_2}, \text{ or} \\ &= x \cdot \underbrace{x}_{g_1} + y \cdot \underbrace{(2x + y)}_{g_2}. \end{aligned}$$

□

## 1.2. Affine Space and Algebraic Sets

### Problem 1.8.\*

Show that the algebraic subsets of  $\mathbf{A}^1(k)$  are just the finite subsets, together with  $\mathbf{A}^1(k)$  itself.

*Proof.*

(1) Show that  $k[x]$  is a PID if  $k$  is a field.

- (a) Let  $I$  be an ideal of  $k[x]$ .
- (b) If  $I = \{0\}$  then  $I = (0)$  and  $I$  is principal.
- (c) If  $I \neq \{0\}$ , then take  $f$  to be a polynomial of minimal degree in  $I$ . It suffices to show that  $I = (f)$ . Clearly,  $(f) \subseteq I$  since  $I$  is an ideal. Conversely, for any  $g \in I$ ,

$$g(x) = f(x)h(x) + r(x)$$

for some  $h, r \in k[x]$  with  $r = 0$  or  $\deg r < \deg f$  (as  $k[x]$  is a Euclidean domain). Now as

$$r = g - fh \in I,$$

$r = 0$  (otherwise contrary to the minimality of  $f$ ), we have  $g = fh \in (f)$  for all  $g \in I$ .

(2) Let  $Y$  be an algebraic subset of  $\mathbf{A}^1(k)$ , say  $Y = V(I)$  for some ideal  $I$  of  $k[x]$ . Since  $k[x]$  is a PID,  $I = (f)$  for some  $f \in k[x]$ .

- (a) If  $f = 0$ , then  $I = (0)$  and  $Y = V(0) = \mathbf{A}^1(k)$ .
- (b) If  $f \neq 0$ , then  $f(x) = 0$  has finitely many roots in  $k$ , say  $a_1, \dots, a_m \in k$ . Hence,

$$Y = V(I) = V(f) = \{f(a) = 0 : a \in k\} = \{a_1, \dots, a_m\}$$

is a finite subsets of  $\mathbf{A}^1(k)$ .

By (a)(b), the result is established.

□

*Notes.*

- (1) By the Hilbert basis theorem,  $k[x]$  is Noetherian as  $k$  is Noetherian. Hence, for any algebraic subset  $Y = V(I)$  of  $\mathbf{A}^1(k)$ , we can write  $I = (f_1, \dots, f_m)$ . Note that

$$Y = V(I) = V(f_1) \cap \dots \cap V(f_m).$$

Now apply the same argument to get the same conclusion.

- (2) Suppose  $k = \bar{k}$ .  $\mathbf{A}^1(k)$  is irreducible, because its only proper closed subsets are finite, yet it is infinite (because  $k$  is algebraically closed, hence infinite).

**Problem 1.9.**

If  $k$  is a finite field, show that every subset of  $\mathbf{A}^n(k)$  is algebraic.

*Proof.*

- (1) Every subset of  $\mathbf{A}^n(k)$  is finite since  $|\mathbf{A}^n(k)| = |k|^n$  is finite.
- (2) Note that  $V(x_1 - a_1, \dots, x_n - a_n) = \{(a_1, \dots, a_n)\} \subseteq \mathbf{A}^n(k)$  (Property (5) in §1.2) and any finite union of algebraic sets is algebraic (Property (4) in §1.2). Thus, every subset of  $\mathbf{A}^n(k)$  is algebraic (by (1)).

□

**Problem 1.10.**

Give an example of a countable collection of algebraic sets whose union is not algebraic.

*Proof.*

- (1) Let  $k = \mathbb{Q}$  be an infinite field.  $V(x - a) = \{a\}$  is an algebraic sets for all  $a \in \mathbb{Q}$ . In particular,  $V(x - a) = \{a\}$  is algebraic for all  $a \in \mathbb{Z}$ .
- (2) Note that

$$Y := \bigcup_{a \in \mathbb{Z}} V(x - a) = \mathbb{Z}$$

is a countable union of algebraic sets. Since  $Y$  is a proper subset of  $k = \mathbb{Q}$ , it cannot be algebraic by Problem 1.8.

□

**Problem 1.11.**

Show that the following are algebraic sets:

- (a)  $\{(t, t^2, t^3) \in \mathbf{A}^3(k) : t \in k\}$ ;
- (b)  $\{(\cos(t), \sin(t)) \in \mathbf{A}^2(\mathbb{R}) : t \in \mathbb{R}\}$ ;
- (c) the set of points in  $\mathbf{A}^2(\mathbb{R})$  whose polar coordinates  $(r, \theta)$  satisfy the equation  $r = \sin(\theta)$ .

*Proof of (a).*

- (1) The twisted cubic curve

$$Y = \{(t, t^2, t^3) \in \mathbf{A}^3(k) : t \in k\} = V(x^2 - y) \cap V(x^3 - z)$$

is algebraic. We say that  $Y$  is given by the parametric representation  $x = t, y = t^2, z = t^3$ .

- (2) The generators for the ideal  $I(Y)$  are  $x^2 - y$  and  $x^3 - z$ .  
 (3)  $Y$  is an affine variety of dimension 1.  
 (4) The affine coordinate ring  $A(Y)$  is isomorphic to a polynomial ring in one variable over  $k$ .

□

*Proof of (b).* The circle

$$\{(\cos(t), \sin(t)) \in \mathbf{A}^2(\mathbb{R}) : t \in \mathbb{R}\} = V(x^2 + y^2 - 1)$$

is algebraic. □

*Proof of (c).* The circle

$$\{(r, \theta) : r = \sin(\theta)\} = V(x^2 + y^2 - y)$$

is algebraic again. □

### Problem 1.12.

Suppose  $C$  is an affine plane curve, and  $L$  is a line in  $\mathbf{A}^2(k)$ ,  $L \not\subseteq C$ . Suppose  $C = V(f)$ ,  $f \in k[x, y]$  a polynomial of degree  $n$ . Show that  $L \cap C$  is a finite set of no more than  $n$  points. (Hint: Suppose  $L = V(y - (ax + b))$ , and consider  $f(x, ax + b) \in k[x]$ .)

*Proof.*

- (1) Say  $L = V(y - (ax + b))$  be a line in  $\mathbf{A}^2(k)$ . (The case  $L = V(x - (ay + b))$  is similar.)  
 (2) Note that  $L \not\subseteq C$  implies that  $(y - (ax + b)) \nmid f$ . Hence, the polynomial

$$g : x \mapsto f(x, ax + b) \in k[x]$$

is nonzero and  $\deg g \leq n$ . Therefore, the number of roots of  $g$  in  $k$  is no more than  $n$ .

(3) Hence,

$$\begin{aligned}
L \cap C &= V(y - (ax + b)) \cap V(f) \\
&= \{(x, y) \in \mathbf{A}^2(k) : y = ax + b \text{ and } f(x, y) = 0\} \\
&= \{(x, y) \in \mathbf{A}^2(k) : f(x, ax + b) = 0\}
\end{aligned}$$

is finite of no more than  $n$  points.

□

**Problem 1.13.**

Show that each of the following sets is not algebraic:

- (a)  $\{(x, y) \in \mathbf{A}^2(\mathbb{R}) : y = \sin(x)\}$ .
- (b)  $\{(z, w) \in \mathbf{A}^2(\mathbb{C}) : |z|^2 + |w|^2 = 1\}$ , where  $|x + iy|^2 = x^2 + y^2$  for  $x, y \in \mathbb{R}$ .
- (c)  $\{(\cos(t), \sin(t), t) \in \mathbf{A}^3(\mathbb{R}) : t \in \mathbb{R}\}$ .

*Proof of (a).*

- (1) (Reductio ad absurdum) If

$$Y := \{(x, y) \in \mathbf{A}^2(\mathbb{R}) : y = \sin(x)\}$$

were algebraic, then there is a subset  $S$  of  $\mathbb{R}[x, y]$  such that

$$Y = V(S) = \bigcap_{f \in S} V(f).$$

- (2)  $S \neq \emptyset$  since  $Y \neq \mathbf{A}^2(\mathbb{R})$ . ( $(89, 64) \in \mathbf{A}^2(\mathbb{R}) - Y$ .)
- (3) Take a fixed line  $L = V(y)$  in  $\mathbf{A}^2(\mathbb{R})$ . For each affine curve  $f \in S$ , we have

$$V(f) \cap L \supseteq \bigcap_{f \in S} V(f) \cap L = Y \cap L = \{(n\pi, 0) \in \mathbf{A}^2(\mathbb{R}) : n \in \mathbb{Z}\},$$

which is infinite. By problem 1.12,  $y \mid f$ . As  $f$  runs over  $S$ ,  $Y \subseteq V(y) = L$ , contradicts that  $(0, \frac{\pi}{2}) \in L - Y$ .

□

*Proof of (b).*

- (1) Similar to (a). (Reductio ad absurdum) If

$$Y := \{(x, y) \in \mathbf{A}^2(\mathbb{C}) : |x|^2 + |y|^2 = 1\}$$

were algebraic, then there is a subset  $S$  of  $\mathbb{C}[x, y]$  such that

$$Y = V(S) = \bigcap_{f \in S} V(f).$$

- (2)  $S \neq \emptyset$  since  $Y \neq \mathbf{A}^2(\mathbb{C})$ .  $((89, 64) \in \mathbf{A}^2(\mathbb{C}) - Y)$   
 (3) Take a fixed line  $L = V(x)$  in  $\mathbf{A}^2(\mathbb{C})$ . For each affine curve  $f \in S$ , we have

$$V(f) \cap L \supseteq \bigcap_{f \in S} V(f) \cap L = Y \cap L = \{(0, y) \in \mathbf{A}^2(\mathbb{C}) : |y| = 1\},$$

which is infinite (since  $Y$  contains a unit circle in the complex plane). By problem 1.12,  $x \mid f$ . As  $f$  runs over  $S$ ,  $Y \subseteq V(x) = L$ , contradicts that the origin  $(0, 0) \in L - Y$ .

□

*Proof of (c).*

- (1) Similar to (a) and (b).  
 (2) Suppose  $C$  is an affine plane curve, and  $L$  is a line in  $\mathbf{A}^3(k)$ ,  $L \not\subseteq C$ . Suppose  $C = V(f)$ ,  $f \in k[x, y, z]$  a polynomial of degree  $n$ . Show that  $L \cap C$  is a finite set of no more than  $n$  points. The proof is similar to Problem 1.12.  
 (a) Say  $L = V(y - (ax + b), z - (cx + d))$  be a line in  $\mathbf{A}^3(k)$ .  
 (b) Note that  $L \not\subseteq C$  implies that  $(y - (ax + b)) \nmid f$  and  $(z - (cx + d)) \nmid f$ . Hence, the polynomial

$$g : x \mapsto f(x, ax + b, cx + d) \in k[x]$$

is nonzero and  $\deg g \leq n$ . Therefore, the number of roots of  $g$  in  $k$  is no more than  $n$ .

- (c) Hence,

$$\begin{aligned} L \cap C &= V(y - (ax + b), z - (cx + d)) \cap V(f) \\ &= \{(x, y) \in \mathbf{A}^2(k) : y = ax + b, z = cx + d \text{ and } f(x, y) = 0\} \\ &= \{(x, y) \in \mathbf{A}^2(k) : f(x, ax + b, cx + d) = 0\} \end{aligned}$$

is finite of no more than  $n$  points.



(3) (Reductio ad absurdum) If

$$Y := \{(\cos(t), \sin(t), t) \in \mathbf{A}^3(\mathbb{R}) : t \in \mathbb{R}\}$$

were algebraic, then there is a subset  $S$  of  $\mathbb{R}[x, y, z]$  such that

$$Y = V(S) = \bigcap_{f \in S} V(f).$$

(4)  $S \neq \emptyset$  since  $Y \neq \mathbf{A}^3(\mathbb{R})$ . ((1989, 6, 4)  $\in \mathbf{A}^3(\mathbb{R}) - Y$ .)

(5) Take a fixed line  $L = V(x - 1, y)$  in  $\mathbf{A}^3(\mathbb{R})$ . For each affine curve  $f \in S$ , we have

$$V(f) \cap L \supseteq \bigcap_{f \in S} V(f) \cap L = Y \cap L = \{(1, 0, 2n\pi) \in \mathbf{A}^3(\mathbb{R}) : n \in \mathbb{Z}\},$$

which is infinite. By (2),  $(x - 1) \mid f$  and  $y \mid f$ . As  $f$  runs over  $S$ ,  $Y \subseteq V(x - 1, y) = L$ , contradicts that  $(1, 0, \pi) \in L - Y$ .

□

**Supplement.** A circular disk of radius 1 in the plane  $xy$  rolls without slipping along the  $x$  axis. The figure described by a point of the circumference of the disk is called a **cycloid**. The parametrized curve  $\alpha : \mathbb{R} \rightarrow \mathbb{R}^2$  is

$$\begin{cases} x = t - \sin t \\ y = 1 - \cos t. \end{cases}$$

The cycloid is not algebraic (as (a)).

#### Problem 1.14.\*

Let  $f$  be a nonconstant polynomial in  $k[x_1, \dots, x_n]$ ,  $k$  algebraically closed. Show that  $\mathbf{A}^n(k) - V(f)$  is infinite if  $n \geq 1$ , and  $V(f)$  is infinite if  $n \geq 2$ . Conclude that the complement of any proper algebraic set is infinite. (Hint: See Problem 1.4.)

*Proof.*

(1) Show that  $\mathbf{A}^n(k) - V(f)$  is infinite if  $n \geq 1$ . Since  $f$  is a nonconstant polynomial in  $k[x_1, \dots, x_n]$ , we may assume that  $\deg_{x_n}(f) > 0$ . Hence

$$x_n \mapsto f(1, \dots, 1, x_n)$$

is a nonconstant polynomial of degree  $\deg_{x_n}(f) > 0$  in  $k[x_n]$ . So  $f$  has finitely many roots in  $k$ , say  $\xi_1, \dots, \xi_m$  ( $m \geq 0$ ). Hence,

$$(1, \dots, 1, x_n) \neq 0$$

whenever  $x_n \neq \xi_m$ . Such subset in  $\mathbf{A}^1(k)$  is infinite since  $k = \bar{k}$  (Problem 1.6). Therefore,

$$\begin{aligned}\mathbf{A}^n(k) - V(f) &= \{(a_1, \dots, a_n) \in \mathbf{A}^n(k) : f(a_1, \dots, a_n) \neq 0\} \\ &\supseteq \{a_n \in \mathbf{A}^1(k) : f(1, \dots, 1, x_n) \neq 0\}\end{aligned}$$

is infinite.

(2) Show that  $V(f)$  is infinite if  $n \geq 2$ .

(a) Similar to (1). Since  $f$  is a nonconstant polynomial in  $k[x_1, \dots, x_n]$ , we may assume that  $m := \deg_{x_n}(f) > 0$ . Write

$$f = \sum_{i=0}^m f_i(x_1, \dots, x_{n-1})x_n^i.$$

Note that each  $f_i$  is well-defined since  $n \geq 2$ .

(b) If  $f_n$  is constant in  $k[x_1, \dots, x_{n-1}]$ , then  $f_n$  is nonzero (since  $m > 0$ ) or  $V(f_n) = \emptyset$ . If  $f_n$  is nonconstant in  $k[x_1, \dots, x_{n-1}]$ , then the set  $\mathbf{A}^{n-1}(k) - V(f_n)$  is infinite by (1). In any case,

$$\mathbf{A}^{n-1}(k) - V(f_n)$$

is infinite.

(c) For each  $P = (a_1, \dots, a_{n-1}) \in \mathbf{A}^{n-1}(k) - V(f_n)$ ,

$$g_P : x_n \mapsto f(P, x_n) = f(a_1, \dots, a_{n-1}, x_n)$$

defines a polynomial in  $k[x_n]$  of degree  $m > 0$ . Since  $k = \bar{k}$ ,  $g_P$  has at least one root  $Q \in k$ . Hence

$$V(f) \supseteq \{(P, Q) \in \mathbf{A}^n(k) : P \in \mathbf{A}^{n-1}(k) - V(f_n), g_P(Q) = 0\}$$

is infinite since the set  $\mathbf{A}^{n-1}(k) - V(f_n)$  is infinite.

*Note.* It is not true if  $k \neq \bar{k}$ . For example,  $V(x^2 + y^2 + 1) = \emptyset$  in  $\mathbf{A}^2(\mathbb{R})$ .

(3) Note that

$$\mathbf{A}^n(k) - V(S) = \mathbf{A}^n(k) - \bigcap_{f \in S} V(f) = \bigcup_{f \in S} (\mathbf{A}^n(k) - V(f)).$$

Thus the complement of any proper algebraic set is infinite by (1).

□

**Problem 1.15.\***

Let  $V \subseteq \mathbf{A}^n(k)$ ,  $W \subseteq \mathbf{A}^m(k)$  be algebraic sets. Show that

$$V \times W = \{(a_1, \dots, a_n, b_1, \dots, b_m) : (a_1, \dots, a_n) \in V, (b_1, \dots, b_m) \in W\}$$

is an algebraic set in  $\mathbf{A}^{n+m}(k)$ . It is called the **product** of  $V$  and  $W$ .

*Proof.*

(1) Write

$$\begin{aligned} V &= V(S_V) = \{P \in \mathbf{A}^n(k) : f(P) = 0 \forall f \in S_V\} \\ W &= V(S_W) = \{Q \in \mathbf{A}^m(k) : g(Q) = 0 \forall g \in S_W\}, \end{aligned}$$

where  $S_V \subseteq k[x_1, \dots, x_n]$  and  $S_W \subseteq k[y_1, \dots, y_m]$ . It suffices to show that

$$V \times W = V(S),$$

where  $S \subseteq k[x_1, \dots, x_n, y_1, \dots, y_m]$  is the union of  $S_V$  and  $S_W$ .

(2) Here we can identify  $S_V$  with the subset of  $k[x_1, \dots, x_n, y_1, \dots, y_m]$  by noting that

$$k[x_1, \dots, x_n] \hookrightarrow (k[y_1, \dots, y_m])[x_1, \dots, x_n] = k[x_1, \dots, x_n, y_1, \dots, y_m].$$

Here we regard  $k$  as a subring of  $k[y_1, \dots, y_m]$ . Similar treatment to  $S_W$ .

(3) By construction,  $V \times W \subseteq V(S)$ . Conversely, given any  $(P, Q) \in V(S) \subseteq \mathbf{A}^{n+m}(k)$ , we have  $h(P, Q) = 0$  for all  $h \in S = S_V \cup S_W$  (by (2)). By construction,  $f(P) = 0$  for all  $f \in S_V$  since  $f$  only involve  $x_1, \dots, x_n$ . Hence,  $P \in V$ . Similarly,  $Q \in W$ . Therefore,  $(P, Q) \in V \times W$ .

□

### 1.3. The Ideal of a Set of Points

**Problem 1.16.\***

Let  $V, W$  be algebraic sets in  $\mathbf{A}^n(k)$ . Show that  $V = W$  if and only if  $I(V) = I(W)$ .

*Proof.*

(1) (Proof of Property (6) in §1.3.) Show that if  $X \subseteq Y$ , then  $I(X) \supseteq I(Y)$ . If  $f \in I(Y)$  then  $f(P) = 0$  for all  $P \in Y$ . So  $f(P) = 0$  for all  $P \in X \subseteq Y$  or  $f \in I(X)$ .

- (2) (Proof of Property (8) in §1.3.)  $I(V(S)) \supseteq S$  for any set  $S$  of polynomials;  $V(I(X)) \supseteq X$  for any set  $X$  of points.
- (a) If  $f \in S$  then  $f$  vanishes on  $V(S)$ , hence  $f \in IV(S)$ .
  - (b) If  $P \in X$  then every polynomial in  $I(X)$  vanishes at  $P$ , so  $P$  belongs to the zero set of  $I(X)$ .
- (3) (Proof of Property (9) in §1.3.)  $V(I(V(S))) = V(S)$  for any set  $S$  of polynomials, and  $I(V(I(X))) = I(X)$  for any set  $X$  of points. So if  $V$  is an algebraic set,  $V = V(I(V))$ , and if  $I$  is the ideal of an algebraic set,  $I = I(V(I))$ .
- (a) In each case, it suffices to show that the left side is a subset of the right side. (by Properties (6)(8) in §1.3).
  - (b) If  $P \in V(S)$  then  $f(P) = 0$  for all  $f \in I(V(S))$ , so  $P \in V(I(V(S)))$ .
  - (c) If  $f \in I(X)$  then  $f(P) = 0$  for all  $P \in V(I(X))$ . Thus  $f$  vanishes on  $V(I(X))$ , so  $f \in I(V(I(X)))$ .
- (4) Show that  $V = W$  if and only if  $I(V) = I(W)$ .
- (a) By Property (6) in §1.3,  $I(V) \supseteq I(W)$  if  $V \subseteq W$  and  $I(V) \subseteq I(W)$  if  $V \supseteq W$ . Thus,  $I(V) = I(W)$  if  $V = W$ .
  - (b) Conversely,  $I(V) = I(W)$  implies that  $V(I(V)) = V(I(W))$  by Property (3) in §1.2 and similar argument in (a). By Property (9) in §1.3,  $V(I(V)) = V$  and  $V(I(W)) = W$ . Thus,  $V = W$ .

□

**Problem 1.17.\***

- (a) Let  $V$  be an algebraic set in  $\mathbf{A}^n(k)$ ,  $P \in \mathbf{A}^n(k)$  a point not in  $V$ . Show that there is a polynomial  $f \in k[x_1, \dots, x_n]$  such that  $f(Q) = 0$  for all  $Q \in V$ , but  $f(P) = 1$ . (Hint:  $I(V) \neq I(V \cup \{P\})$ .)
- (b) Let  $P_1, \dots, P_r$  be distinct points in  $\mathbf{A}^n(k)$ , not in an algebraic set  $V$ . Show that there are polynomials  $f_1, \dots, f_r \in I(V)$  such that  $f_i(P_j) = 0$  if  $i \neq j$ , and  $f_i(P_i) = 1$ . (Hint: Apply (a) to the union of  $V$  and all but one point.)
- (c) With  $P_1, \dots, P_r$  and  $V$  as in (b), and  $a_{ij} \in k$  for  $1 \leq i, j \leq r$ , show that there are  $g_i \in I(V)$  with  $g_i(P_j) = a_{ij}$  for all  $i$  and  $j$ . (Hint: Consider  $\sum_j a_{ij} f_j$ .)

*Proof of (a).*

- (1) Since  $I(V) \subsetneq I(V \cup \{P\})$  (by Problem 1.16), there is a polynomial  $f \in k[x_1, \dots, x_n]$  such that  $f(Q) = 0$  for all  $Q \in V$ , but  $f(P) \neq 0$ .

- (2) Since  $k$  is a field,  $(f(P))^{-1} \in k$ . Consider the polynomial  $(f(P))^{-1}f \in k[x_1, \dots, x_n]$ . It is well-defined. Also,  $((f(P))^{-1}f)(Q) = (f(P))^{-1}f(Q) = 0$  for all  $Q \in V$ , but  $(f(P))^{-1}f(P) = (f(P))^{-1}f(P) = 1$ .

□

*Proof of (b).*

- (1) For  $1 \leq i \leq$ , define

$$W = V \cup \{P_1, \dots, P_r\}$$

$$W_i = V \cup \{P_1, \dots, \widehat{P_i}, \dots, P_r\}.$$

Here  $W = W_i \cup \{P_i\} \neq W_i$ .

- (2) By (a), there is a polynomial  $f_i \in k[x_1, \dots, x_n]$  such that  $f_i(Q) = 0$  for all  $Q \in W_i$ , but  $f_i(P_i) = 1$ . Here  $f_i \in I(V)$  and  $f_i(P_j) = \delta_{ij}$  where  $\delta_{ij}$  is the Kronecker delta.

□

*Proof of (c).*

- (1) For each  $1 \leq i \leq r$ , define

$$g_i = \sum_j a_{ij} f_j \in k[x_1, \dots, x_n].$$

- (2)  $g_i \in I(V)$  since  $g_i$  is a linear combination of  $f_j$  and  $I(V)$  is an ideal.

- (3) Also,

$$g_i(P_j) = \sum_{j'} a_{ij'} f_{j'}(P_j) = \sum_{j'} a_{ij'} \delta_{j'j} = a_{ij}.$$

□

### **Problem 1.18.\***

Let  $I$  be an ideal in a ring  $R$ . If  $a^n \in I$ ,  $b^m \in I$ , show that  $(a + b)^{n+m} \in I$ . Show that  $\text{rad}(I)$  is an ideal, in fact a radical ideal. Show that any prime ideal is radical.

*Proof.*

- (1) Show that  $(a + b)^{n+m} \in I$  if  $a^n \in I$ ,  $b^m \in I$ . By the binomial theorem,

$$(a + b)^{n+m} = \sum_{i=0}^{n+m} a^i b^{n+m-i}.$$

For each term  $a^i b^{n+m-i}$ , either  $i \geq n$  holds or  $n + m - i \geq m$  holds, and thus  $a^i b^{n+m-i} \in I$  (since  $a^n \in I$ ,  $b^m \in I$  and  $I$  is an ideal). Hence, the result is established.

- (2) Show that  $\text{rad}(I)$  is an ideal.

- (a)  $0 \in \text{rad}(I)$  since  $0 = 0^1 \in I$  for any ideal in  $R$ .
- (b)  $(a + b)^{n+m} \in I$  if  $a^n \in I$ ,  $b^m \in I$  by (1).
- (c)  $(-a)^{2n} = (a^n)^2 \in I$  if  $a^n \in I$  (since  $I$  is an ideal).
- (d)  $(ra)^n = r^n a^n \in I$  if  $a^n \in I$  and  $r \in R$  (since  $I$  is an ideal and  $R$  is commutative).

- (3) Show that  $\text{rad}(\text{rad}(I)) = \text{rad}(I)$ . It suffices to show  $\text{rad}(\text{rad}(I)) \subseteq \text{rad}(I)$ . Given any  $a \in \text{rad}(\text{rad}(I))$ . By definition  $a^n \in \text{rad}(I)$  for some positive integer  $n$ . Again by definition  $(a^n)^m = a^{nm} \in I$  for some positive integer  $m$ . As  $nm$  is a positive integer,  $a \in \text{rad}(I)$ .

- (4) Show that every prime ideal  $\mathfrak{p}$  is radical. Given any  $a \in \text{rad}(\mathfrak{p})$ , that is,  $a^n \in \mathfrak{p}$  for some positive integer. Write  $a^n = aa^{n-1}$  if  $n > 1$ . By the primality of  $\mathfrak{p}$ ,  $a \in \mathfrak{p}$  or  $a^{n-1} \in \mathfrak{p}$ . If  $a \in \mathfrak{p}$ , we are done. If  $a^{n-1} \in \mathfrak{p}$ , we continue this descending argument (or the mathematical induction) until the power of  $a$  is equal to 1. Hence  $\mathfrak{p}$  is radical.

□

### Problem 1.19.

Show that  $I = (x^2 + 1) \subseteq \mathbb{R}[x]$  is a radical (even a prime) ideal, but  $I$  is not the ideal of any set in  $\mathbf{A}^1(\mathbb{R})$ .

*Proof.*

- (1) Show that  $I = (x^2 + 1)$  is a prime ideal in  $\mathbb{R}[x]$ . Given any  $fg \in I$ . It suffices to show that  $f \in I$  or  $g \in I$ . By definition of  $I$ , there is a polynomial  $h \in \mathbb{R}[x]$  such that  $fg = (x^2 + 1)h$ . So  $(x^2 + 1) \mid f$  or  $(x^2 + 1) \mid g$  since  $x^2 + 1$  is irreducible in a unique factorization domain  $\mathbb{R}[x]$ . Therefore,  $f \in I$  or  $g \in I$ .
- (2) Show that  $I$  is not the ideal of any set in  $\mathbf{A}^1(\mathbb{R})$ . Since  $x^2 + 1$  has no roots in  $\mathbb{R}$ ,  $I$  cannot be the ideal of any nonempty set in  $\mathbf{A}^1(\mathbb{R})$ . Besides,  $I(\emptyset) = (1) \neq (x^2 + 1)$ .

□

**Problem 1.20.\***

Show that for any ideal  $I$  in  $k[x_1, \dots, x_n]$ ,  $V(I) = V(\text{rad}(I))$ , and  $\text{rad}(I) \subseteq I(V(I))$ .

*Proof.*

- (1) Show that  $V(I) = V(\text{rad}(I))$ . Since  $I \subseteq \text{rad}(I)$ , it suffices to show that  $V(I) \subseteq V(\text{rad}(I))$ . Given any  $P \in V(I)$ . For any  $f \in \text{rad}(I)$ ,  $f^n \in I$  for some positive integer  $n > 0$ . Note that

$$0 = (f^n)(P) = f(P)^n$$

since  $f^n \in I$  and  $P \in V(I)$ . As  $k$  is a domain,  $f(P)^n = 0$  implies  $f(P) = 0$ . So  $P \in V(\text{rad}(I))$ .

- (2) By Properties (6)(8) in §1.3,

$$I(V(I)) = I(V(\text{rad}(I))) \supseteq \text{rad}(I).$$

□

*Note.*

- (1) By the Hilbert's Nullstellensatz,  $I(V(I)) = \text{rad}(I)$  if  $k = \bar{k}$ .  
 (2) Take  $I = (x^2 + 1)$  as an ideal in  $\mathbb{R}[x]$ . Note that  $I(V(I)) = I(\emptyset) = (1)$  and  $\text{rad}(I) = I = (x^2 + 1)$ . So the equality in  $\text{rad}(I) \subsetneq I(V(I))$  might not hold if  $k \neq \bar{k}$ . (See Problem 1.19.)

**Problem 1.21.\***

Show that  $I = (x_1 - a_1, \dots, x_n - a_n) \subseteq k[x_1, \dots, x_n]$  is a maximal ideal, and that the natural homomorphism from  $k$  to  $k[x_1, \dots, x_n]/I$  is an isomorphism.

*Proof.*

- (1) Show that  $I$  is a maximal ideal. Suppose that  $J$  is an ideal such that  $J \supsetneq I$ . Take any  $f \in J - I$ . By Problem 1.7(a),

$$f = \sum \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}.$$

As  $f \notin I$ , there is a nonzero constant term in  $f$ , say  $\lambda \in k - \{0\}$ . Note that  $f - \lambda \in I \subsetneq J$ . Hence,

$$\lambda = f - (f - \lambda) \in J$$

since  $J$  is an ideal. As  $\lambda \neq 0$ ,  $J = k[x_1, \dots, x_n]$  is not a proper ideal containing  $I$ .

- (2) Let  $\varphi : k \rightarrow k[x_1, \dots, x_n]/I$  be the natural homomorphism. (That is,  $\varphi : \lambda \rightarrow \lambda + I \in k[x_1, \dots, x_n]/I$ .)
- (3) Show that  $\varphi$  is surjective. Given any  $f + I \in k[x_1, \dots, x_n]/I$ . By Problem 1.7(a),

$$f = \sum \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}.$$

So

$$\begin{aligned} f + I &= \sum \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n} + I \\ &= \left( f(a_1, \dots, a_n) + \sum_{\text{nonconstant}} \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n} \right) + I \\ &= f(a_1, \dots, a_n) + I. \end{aligned}$$

(Here the summation over all nonconstant terms is in  $I$ .) Hence

$$\varphi : f(a_1, \dots, a_n) \in k \mapsto f + I.$$

- (4) Show that  $\varphi$  is injective.  $\ker(\varphi) = \{\lambda \in k : \lambda \in I\} = k \cap I = \{0\}$  since  $I$  is a proper ideal.
- (5) By (2)(3)(4),  $\varphi : k \rightarrow k[x_1, \dots, x_n]/(x_1 - a_1, \dots, x_n - a_n)$  is an isomorphism.

□

## 1.4. The Hilbert Basis Theorem

### Problem 1.22.\* (Correspondence theorem for rings)

Let  $I$  be an ideal in a ring  $R$ ,  $\pi : R \rightarrow R/I$  the natural homomorphism.

- (a) Show that for every ideal  $J'$  of  $R/I$ ,  $\pi^{-1}(J') = J$  is an ideal of  $R$  containing  $I$ , and for every ideal  $J$  of  $R$  containing  $I$ ,  $\pi(J) = J'$  is an ideal of  $R/I$ . This sets up a natural one-to-one correspondence between  $\{\text{ideals of } R/I\}$  and  $\{\text{ideals of } R \text{ that contain } I\}$ .
- (b) Show that  $J'$  is a radical ideal if and only if  $J$  is radical. Similarly for prime and maximal ideals.



- (c) Show that  $J'$  is finitely generated if  $J$  is. Conclude that  $R/I$  is Noetherian if  $R$  is Noetherian. Any ring of the form  $k[x_1, \dots, x_n]/I$  is Noetherian.

*Proof of (a).*

- (1) Show that for every ideal  $J'$  of  $R/I$ ,  $\pi^{-1}(J') = J$  is an ideal of  $R$  containing  $I$ .

- (a) Show that  $J$  contains  $I$ . Note that  $\pi^{-1}(0) = I \subseteq \pi^{-1}(J') = J$ . So  $J$  contains  $I$ . In particular,  $J \neq \emptyset$  since  $I \neq \emptyset$ .  
(b) Show that  $J$  is a additive subgroup of  $R$ . It suffices to show that  $a - b \in J$  for any  $a \in J$  and  $b \in J$ . Actually,

$$\pi(a - b) = \pi(a) - \pi(b) \in J'$$

implies  $a - b \in \pi^{-1}(J') = J$ .

- (c) Show that for every  $r \in R$  and every  $a \in J$ , the product  $ra \in J$ . In fact,

$$\pi(ra) = \pi(r)\pi(a) \in J'$$

implies  $ra \in \pi^{-1}(J') = J$ .

- (2) Show that for every ideal  $J$  of  $R$  containing  $I$ ,  $\pi(J) = J'$  is an ideal of  $R/I$ .

- (a) Show that  $J'$  is nonempty. Note that  $\pi(a) = 0 \in \pi(I) \subseteq \pi(J) = J'$  for any  $a \in I$ . So  $J'$  is nonempty since  $J$  is nonempty.  
(b) Show that  $J'$  is a additive subgroup of  $R/I$ . It suffices to show that  $\pi(a) - \pi(b) \in J'$  for any  $\pi(a) \in J'$ ,  $\pi(b) \in J'$ ,  $a \in J$  and  $b \in J$ . It is trivial since

$$\pi(a) - \pi(b) = \pi(a - b) \in \pi(J) = J',$$

$\pi$  is a ring homomorphism and  $J$  is an ideal.

- (c) Show that for every  $\pi(r) \in R/I$  ( $r \in R$ ) and every  $\pi(a) \in J'$  ( $a \in J$ ), the product  $\pi(r)\pi(a) \in J'$ . It is trivial since

$$\pi(r)\pi(a) = \pi(ra) \in \pi(J) = J',$$

$\pi$  is a ring homomorphism and  $J$  is an ideal.

- (3) By (1)(2), we setup the correspondence between

$$\{\text{ideals of } R/I\} \longleftrightarrow \{\text{ideals of } R \text{ that contain } I\}.$$

Note that this correspondence preserves the subset relation, and thus this correspondence is one-to-one.

□

*Proof of (b).*

- (1) *Show that  $J'$  is radical if  $J$  is radical.* It suffices to show that  $(a + I)^n = a^n + I \in J'$  implies that  $a + I \in J'$ . Note that

$$(a + I)^n = a^n + I \in J'$$

implies that  $a^n \in J$  or  $a \in J$  since  $J$  is radical. Hence  $a + I \in J/I = J'$ .

- (2) *Show that  $J$  is radical if  $J'$  is radical.* It suffices to show that  $a^n \in J$  implies that  $a \in J$ . Note that

$$\pi(a^n) = \pi(a)^n \in J'$$

implies that  $\pi(a) \in J'$  since  $J'$  is radical.  $a \in \pi^{-1}(J') = J$ .

- (3) *Show that  $J'$  is prime if  $J$  is prime.* It suffices to show that  $(a + I)(b + I) = ab + I \in J'$  implies that  $a + I \in J'$  or  $b + I \in J'$ . Note that

$$(a + I)(b + I) = ab + I \in J'$$

implies that  $ab \in J$ . So  $a \in J$  or  $b \in J$  by the primality of  $J$ . Hence  $a + I \in J'$  or  $b + I \in J'$ .

- (4) *Show that  $J$  is prime if  $J'$  is prime.* It suffices to show that  $ab \in J$  implies that  $a \in J$  or  $b \in J$ . Note that

$$\pi(ab) = \pi(a)\pi(b) \in J'$$

implies that  $\pi(a) \in J'$  or  $\pi(b) \in J'$  by the primality of  $J'$ . So  $a \in \pi^{-1}(J') = J$  or  $b \in \pi^{-1}(J') = J$ .

- (5) *Show that  $J'$  is maximal if  $J$  is maximal.* Suppose  $\mathfrak{m}$  is an ideal containing  $J'$ . By (a),  $\pi^{-1}(\mathfrak{m})$  is an ideal containing  $J$ . So  $\pi^{-1}(\mathfrak{m}) = J$  or  $\pi^{-1}(\mathfrak{m}) = R$  by the maximality of  $J$ . Hence,  $\mathfrak{m} = \pi(J) = J'$  or  $\mathfrak{m} = \pi(R) = R/I$ .

- (6) *Show that  $J$  is maximal if  $J'$  is maximal.* Suppose  $\mathfrak{m}$  is an ideal containing  $J$ . By (a),  $\pi(\mathfrak{m})$  is an ideal containing  $J'$ . So  $\pi(\mathfrak{m}) = J'$  or  $\pi(\mathfrak{m}) = R/I$  by the maximality of  $J'$ . Hence,  $\mathfrak{m} = \pi^{-1}(J') = J$  or  $\mathfrak{m} = \pi^{-1}(R/I) = R$ .

□

*Note.*

- (1) Note that

$$R/J \cong (R/I)/(J/I)$$

if  $J$  is an ideal of  $R$  such that  $I \subseteq J$ .

- (2) Hence,  $J$  is prime iff  $R/J \cong (R/I)/(J/I)$  is a domain iff  $J/I$  is prime.  
(3) Also,  $J$  is maximal iff  $R/J \cong (R/I)/(J/I)$  is a field iff  $J/I$  is maximal.

*Proof of (c).*

- (1) *Show that  $J'$  is finitely generated if  $J$  is.* Suppose  $J$  is generated by  $a_1, \dots, a_m$ . It suffices to show that  $J'$  is generated by

$$a_1 + I, \dots, a_m + I \in J/I.$$

Given any  $a + I \in J'$  where  $a \in J$ . Write  $a = \sum_{1 \leq i \leq m} r_i a_i$  for some  $r_i \in R$ . Then

$$a + I = \sum r_i a_i + I = \sum (r_i + I)(a_i + I)$$

is generated by  $a_1 + I, \dots, a_m + I$ .

- (2) *Show that  $R/I$  is Noetherian if  $R$  is Noetherian.* Note that  $R$  is an ideal of itself.  
(3) *Show that any ring of the form  $k[x_1, \dots, x_n]/I$  is Noetherian.* By the corollary to the Hilbert basis theorem,  $k[x_1, \dots, x_n]$  is Noetherian. By (2), the ring  $k[x_1, \dots, x_n]/I$  is Noetherian.

□

## 1.5. Irreducible Components of an Algebraic Set

### Problem 1.23.

*Give an example of a collection of ideals  $\mathcal{S}$  ideals in a Noetherian ring such that no maximal member of  $\mathcal{S}$  is a maximal ideal.*

*Proof.*

- (1) Let  $R$  be any Noetherian ring. Let  $\mathcal{S}$  be any collection of ideals containing  $R$  itself. Then the only maximal member of  $\mathcal{S}$  is  $R$ , which is not a maximal ideal.  
(2) Or let  $R$  be any Noetherian ring and  $R$  is not a field. ( $R = k[x_1, \dots, x_n]$  where  $k$  is a field for example.) Let  $\mathcal{S} = \{(0)\}$ . Then the only maximal member of  $\mathcal{S}$  is  $(0)$ , which is not maximal since  $R$  is not a field.

□

**Problem 1.24.**

Show that every proper ideal in a Noetherian ring is contained in a maximal ideal. (Hint: If  $I$  is the ideal, apply the lemma to  $\{\text{proper ideals that contain } I\}$ .)

*Proof.*

- (1) Say  $I$  be any proper ideal in a Noetherian ring. Let

$$\mathcal{S} = \{\text{proper ideals that contain } I\}.$$

Apply the lemma to  $\mathcal{S}$  to get that  $\mathcal{S}$  has a maximal member  $\mathfrak{m} \in \mathcal{S}$ .

- (2) Show that  $\mathfrak{m}$  is maximal. Since  $\mathfrak{m} \in \mathcal{S}$ ,  $\mathfrak{m}$  is a proper ideal in  $R$ . Suppose  $\mathfrak{m}' \supsetneq \mathfrak{m}$  is a proper ideal containing  $\mathfrak{m}$ . As  $\mathfrak{m}$  contains  $I$ ,  $\mathfrak{m}'$  also contains  $I$  or  $\mathfrak{m}' \in \mathcal{S}$ . By the maximality of  $\mathfrak{m}$ ,  $\mathfrak{m}' \subseteq \mathfrak{m}$ . So  $\mathfrak{m}' = \mathfrak{m}$ .

□

**Problem 1.25.**

- (a) Show that  $V(y - x^2) \subseteq \mathbf{A}^2(\mathbb{C})$  is irreducible, in fact,  $I(V(y - x^2)) = (y - x^2)$ .
- (b) Decompose  $V(y^4 - x^2, y^4 - x^2y^2 + xy^2 - x^3) \subseteq \mathbf{A}^2(\mathbb{C})$  into irreducible components.

*Proof of (a).*

- (1) Let  $I = (y - x^2)$  be an ideal of  $\mathbb{C}[x, y]$ . Since  $\mathbb{C}$  is algebraically closed,

$$I(V(I)) = \text{rad}(I)$$

by the Hilbert's Nullstellensatz. It suffices to show that  $I$  is prime, or to show that  $y - x^2$  is prime. Since  $\mathbb{C}[x, y]$  is a UFD, it suffices to show that  $y - x^2$  is irreducible.

- (2) Show that  $y - x^2$  is irreducible in  $\mathbb{C}[x, y]$ . Write

$$y - x^2 \in (\mathbb{C}[y])[x].$$

Note that  $\mathbb{C}[y]$  is a UFD and  $y$  is the constant term. If we can show that  $y$  is prime in  $\mathbb{C}[y]$ , then by the Eisenstein's criterion we can say  $y - x^2$  is irreducible in  $(\mathbb{C}[y])[x]$ .

- (3) As  $\mathbb{C}[y]/(y) \cong \mathbb{C}$  is a field or a domain,  $(y)$  is maximal or prime. Hence,  $y - x^2$  is irreducible.

(4) Or apply Corollary 1 to Proposition 2 in the next section to (2)(3).

□

*Proof of (b).*

(1) Write

$$\begin{aligned}
 Y &:= V(y^4 - x^2, y^4 - x^2y^2 + xy^2 - x^3) \\
 &= V((y^2 - x)(y^2 + x), (y^2 - x^2)(y^2 + x)) \\
 &= V(y^2 + x) \cup V(y^2 - x, y^2 - x^2) \\
 &= V(y^2 + x) \cup V(y^2 - x, x(x - 1)) \\
 &= V(y^2 + x) \cup V(x, y) \cup V(y + 1, x - 1) \cup V(y - 1, x - 1).
 \end{aligned}$$

(2) Here  $V(y^2 + x)$  is irreducible as (a). Besides,  $V(x, y)$ ,  $V(y + 1, x - 1)$  and  $V(y - 1, x - 1)$  are irreducible since all corresponding ideals are maximal (by the Hilbert's Nullstellensatz and Problem 1.21).

□

**Problem 1.26.**

Show that  $f = y^2 + x^2(x - 1)^2 \in \mathbb{R}[x, y]$  is an irreducible polynomial, but  $V(f)$  is reducible.

*Proof.*

(1) Show that  $f$  is an irreducible polynomial.

(a) Suppose

$$f = (f_2(x)y^2 + f_1(x)y + f_0(x)) \cdot g(x)$$

for some  $f_i(x), g(x) \in \mathbb{R}[x]$ . So

$$f_2(x)g(x) = 1, \quad f_1(x)g(x) = 0, \quad f_0(x)g(x) = x^2(x - 1)^2.$$

Hence,

$$f_2(x)y^2 + f_1(x)y + f_0(x) = uf, \quad g(x) = u^{-1},$$

where  $u$  is a unit in  $\mathbb{R}$ .

(b) Suppose

$$f = (f_1(x)y + f_0(x)) \cdot (g_1(x)y + g_0(x))$$

for some  $f_i(x), g_j(x) \in \mathbb{R}[x]$ . So

$$\begin{aligned} f_1(x)g_1(x) &= 1, \\ f_1(x)g_0(x) + f_0(x)g_1(x) &= 0, \\ f_0(x)g_0(x) &= x^2(x-1)^2. \end{aligned}$$

So  $f_1(x) = u, g_1(x) = u^{-1}$  for some unit  $u \in \mathbb{R}$ . Hence,

$$u^2 g_0(x)^2 = -x^2(x-1)^2,$$

which is absurd since  $\mathbb{R}$  is not algebraically closed.

(c) By (a)(b),  $f$  is irreducible in  $\mathbb{R}[x, y]$ .

- (2) Show that  $V(f)$  is reducible.  $V(f) = \{(0, 0), (1, 0)\} = V(x, y) \cup V(x-1, y)$ .  
Here  $V(x, y)$  and  $V(x-1, y)$  are all proper algebraic sets in  $V(f)$ .

□

**Problem 1.27.**

Let  $V, W$  be algebraic sets in  $\mathbf{A}^n(k)$  with  $V \subseteq W$ . Show that each irreducible component of  $V$  is contained in some irreducible component of  $W$ .

*Proof.*

- (1) Write two decompositions of  $V, W$  into irreducible components as

$$\begin{aligned} V &= V_1 \cup \cdots \cup V_r, \\ W &= W_1 \cup \cdots \cup W_s, \end{aligned}$$

- (2) For each irreducible component  $V_i$  of  $V$ , consider  $V_i \cap W$ :

$$V_i \cap W = (V_i \cap W_1) \cup \cdots \cup (V_i \cap W_s).$$

By the irreducibility of  $V_i$ , there is only one  $j$  such that  $V_i \cap W_j = V_i$  and other intersections are empty. Therefore, each irreducible component  $V_i$  is contained in some irreducible component  $W_j$  of  $W$ .

□

**Problem 1.28.**

If  $V = V_1 \cup \cdots \cup V_r$  is the decomposition of an algebraic set into irreducible components, show that  $V_i \not\subseteq \bigcup_{j \neq i} V_j$ .

*Proof.*

- (1) (Reductio ad absurdum) If

$$V_i \subseteq \bigcup_{j \neq i} V_j$$

for some  $i$ , then

$$V = V_1 \cup \cdots \cup \widehat{V_i} \cup \cdots \cup V_r$$

is another decomposition of an algebraic set into irreducible components.

- (2) By Theorem 2 in §1.5, the number of irreducible components is unique determined, contrary to the assumption and (1).

□

**Problem 1.29.\***

Show that  $\mathbf{A}^n(k)$  is irreducible if  $k$  is infinite.

*Proof.*

- (1) (Reductio ad absurdum) If  $\mathbf{A}^n(k)$  were reducible, then  $\mathbf{A}^n(k) = V_1 \cup V_2$  where  $V_1, V_2$  are algebraic sets in  $\mathbf{A}^n(k)$ ,  $V_1$  and  $V_2$  are nonempty and proper in  $\mathbf{A}^n(k)$ .
- (2) Take  $P_i \in V_i$  for  $i = 1, 2$ . By Problem 1.17, there are two polynomials  $f_1, f_2 \in k[x_1, \dots, x_n]$  such that  $f_i(Q) = 0$  for all  $Q \in V_i$  and  $f_1(P_2) = f_2(P_1) = 1$ .
- (3) By construction,  $(f_1 f_2)(a_1, \dots, a_n) = 0$  for any  $a_1, \dots, a_n \in k$ . As  $k$  is infinite,  $f_1 f_2 = 0$  by Problem 1.4. Since  $k[x_1, \dots, x_n]$  is a domain,  $f_1 = 0$  or  $f_2 = 0$ , contrary to  $f_1(P_2) = f_2(P_1) \neq 0$ .

□

*Note.*  $\mathbf{A}^n(k)$  is reducible if  $k$  is finite.

## 1.6. Algebraic Subsets of the Plane

### Problem 1.30.

Let  $k = \mathbb{R}$ .

- (a) Show that  $I(V(x^2 + y^2 + 1)) = (1)$ .
- (b) Show that every algebraic subset of  $\mathbf{A}^2(\mathbb{R})$  is equal to  $V(f)$  for some  $f \in \mathbb{R}[x, y]$ .

This indicates why we usually require that  $k$  be algebraically closed.

*Proof of (a).*  $I(V(x^2 + y^2 + 1)) = I(\emptyset) = (1)$  since  $x^2 + y^2 + 1 \geq 1$  is never zero for any  $x, y \in \mathbb{R}$ .  $\square$

*Proof of (b).*

- (1) Given any algebraic subset  $V$  of  $\mathbf{A}^2(\mathbb{R})$ .  $V = V(1)$  if  $V = \emptyset$ .  $V = V(0)$  if  $V = \mathbf{A}^2(\mathbb{R})$ . Now suppose  $V$  is a nonempty proper algebraic subset  $V$  of  $\mathbf{A}^2(\mathbb{R})$ . Write  $V = V_1 \cup \dots \cup V_m$ , where each  $V_i$  is irreducible. Here  $V_i \neq \emptyset$  and  $V_i \neq \mathbf{A}^2(\mathbb{R})$  for all  $i$ .
- (2) As  $k = \mathbb{R}$  is infinite, Corollary 2 to Proposition 2 implies that each  $V_i$  is either a point or an irreducible plane curves  $V(f_i)$ , where  $f_i$  is an irreducible polynomial and  $V(f_i)$  is infinite.
- (3) If  $V_i = \{(a_i, b_i)\}$  is a point, then define

$$f_i(x, y) = (x - a_i)^2 + (y - b_i)^2.$$

By the property of  $\mathbb{R}$ ,  $V_i = V(f_i)$ .

- (4) Define  $f = f_1 \cdots f_m \in \mathbb{R}[x, y]$ . Hence,

$$\begin{aligned} V &= V_1 \cup \dots \cup V_m \\ &= V(f_1) \cup \dots \cup V(f_m) \\ &= V(f_1 \cdots f_m) \\ &= V(f). \end{aligned}$$

$\square$

### Problem 1.31.

- (a) Find the irreducible components of  $V(y^2 - xy - x^2y + x^3)$  in  $\mathbf{A}^2(\mathbb{R})$ , and also in  $\mathbf{A}^2(\mathbb{C})$ .



(b) Do the same for  $V(y^2 - x(x^2 - 1))$ , and for  $V(x^3 + x - x^2y - y)$ .

*Proof of (a).*

(1) Note that

$$\begin{aligned} V(y^2 - xy - x^2y + x^3) &= V((y - x^2)(y - x)) \\ &= V(y - x^2) \cup V(y - x). \end{aligned}$$

(2) Note that  $y - x^2$  and  $y - x$  are irreducible in  $\mathbb{C}[x, y]$  and thus also in  $\mathbb{R}[x, y]$  by the similar argument in Problem 1.25(a). Also,  $V(y - x^2)$  and  $V(y - x)$  are infinite in  $\mathbf{A}^2(\mathbb{R})$  and thus also in  $\mathbf{A}^2(\mathbb{C})$ .

(3) Therefore,  $V(y - x^2)$  and  $V(y - x)$  are the irreducible components of  $V(y^2 - xy - x^2y + x^3)$  in  $\mathbf{A}^2(\mathbb{R})$  and also in  $\mathbf{A}^2(\mathbb{C})$ .

□

*Outline of (b).*

- (1) The elliptic curve  $V(y^2 - x(x + 1)(x - 1))$  is irreducible over  $\mathbf{A}^2(\mathbb{R})$ .
- (2) The elliptic curve  $V(y^2 - x(x + 1)(x - 1))$  is irreducible over  $\mathbf{A}^2(\mathbb{C})$ .
- (3) The irreducible component of  $V(x^3 + x - x^2y - y)$  over  $\mathbf{A}^2(\mathbb{R})$  is  $V(x - y)$ .
- (4) The irreducible components of  $V(x^3 + x - x^2y - y)$  over  $\mathbf{A}^2(\mathbb{C})$  are  $V(x + i)$ ,  $V(x - i)$  and  $V(x - y)$ .

*Proof of (b).*

(1) Similar to Problem 1.25. To show  $y^2 - x(x + 1)(x - 1)$  is irreducible in  $\mathbb{C}[x, y]$ , we write

$$y^2 - x(x + 1)(x - 1) \in (\mathbb{C}[x])[y].$$

Note that  $\mathbb{C}[x]$  is a UFD and  $-x(x + 1)(x - 1)$  is the constant term. As  $\mathbb{C}[x]/(x) \cong \mathbb{C}$  is a domain,  $(x)$  is prime. Clearly,  $x \mid x(x + 1)(x - 1)$  but  $x^2 \nmid x(x + 1)(x - 1)$ . By the Eisenstein's criterion, we can say  $y^2 - x(x + 1)(x - 1)$  is irreducible over  $(\mathbb{C}[x])[y]$ .

- (2) Moreover,  $V(y^2 - x(x + 1)(x - 1))$  is infinite over  $\mathbf{A}^2(\mathbb{R})$  and thus also over  $\mathbf{A}^2(\mathbb{C})$ . ( $y = f(x) = \sqrt{x(x + 1)(x - 1)}$  is continuous and strictly increasing on  $[1, \infty)$  in the sense of calculus. As the measure of  $[1, \infty)$  is  $\infty$ , the set  $V(y^2 - x(x + 1)(x - 1))$  is infinite over  $\mathbf{A}^2(\mathbb{R})$ .)
- (3) By Corollary 1 to Proposition 2,  $V(y^2 - x(x^2 - 1))$  itself is irreducible over  $\mathbf{A}^2(\mathbb{R})$  or  $\mathbf{A}^2(\mathbb{C})$ .

- (4) Consider  $V(x^3 + x - x^2y - y) \subseteq \mathbf{A}^2(\mathbb{R})$ .

$$\begin{aligned} V(x^3 + x - x^2y - y) &= V((x^2 + 1)(x - y)) \\ &= V(x^2 + 1) \cup V(x - y) \\ &= \emptyset \cup V(x - y) \\ &= V(x - y). \end{aligned}$$

Here we use that fact that  $x^2 + 1 = 0$  has no real solution  $x \in \mathbb{R}$ . Similar to (a),  $V(x - y)$  is the only irreducible component of  $V(x^3 + x - x^2y - y)$  in  $\mathbf{A}^2(\mathbb{R})$ .

- (5) Consider  $V(x^3 + x - x^2y - y) \subseteq \mathbf{A}^2(\mathbb{C})$ .

$$\begin{aligned} V(x^3 + x - x^2y - y) &= V((x + i)(x - i)(x - y)) \\ &= V(x + i) \cup V(x - i) \cup V(x - y). \end{aligned}$$

Similar to (a),  $V(x \pm i)$  and  $V(x - y)$  are the irreducible components of  $V(x^3 + x - x^2y - y)$  in  $\mathbf{A}^2(\mathbb{C})$ .

□

## 1.7. Hilbert's Nullstellensatz

### Problem 1.32.

Show that both theorems and all of the corollaries are false if  $k$  is not algebraically closed.

*Proof.*

- (1) Weak Nullstellensatz:  $I = (x^2 + 1)$  is a proper ideal in  $\mathbb{R}[x]$  but  $V(I) = \emptyset$ .  
(2) Hilbert's Nullstellensatz: Let  $I = (y^2 + x^2(x - 1)^2)$  be an ideal in  $\mathbb{R}[x, y]$ . Hence,

$$\begin{aligned} I(V(I)) &= I(\{(0, 0), (1, 0)\}) && \text{(Problem 1.26.)} \\ &= (x(x - 1), y) \\ &\neq I \\ &= \text{rad}(I). \end{aligned}$$

The last equality holds since  $f$  is irreducible in a UFD  $\mathbb{R}[x, y]$  and thus  $I$  is a prime ideal.

- (3) Corollary 1: Same example in the case Hilbert's Nullstellensatz. If  $I = (y^2 + x^2(x - 1)^2)$  is a radical ideal in  $\mathbb{R}[x, y]$ . Then  $I(V(I)) \neq I$ .

- (4) Corollary 2: Same example in the case Hilbert's Nullstellensatz. If  $I = (y^2 + x^2(x-1)^2)$  is a prime ideal in  $\mathbb{R}[x, y]$ , then

$$V(I) = \{(0, 0), (1, 0)\} = V(x, y) \cup V(x-1, y)$$

is reducible. Next, consider a prime ideal  $J = (x^2 + y^2)$  in  $\mathbb{R}[x, y]$ . (Use the same argument in Problem 1.26 to get the irreducibility of  $x^2 + y^2$ .)  $V(J) = \{(0, 0)\}$  is a point but  $J$  is not a maximal ideal (since  $J \subsetneq (x^2 + y^2, x) \subsetneq (1)$ ).

- (5) Corollary 3: Same example in Corollary 2.
- (6) Corollary 4: Let  $I = (x^2 + y^2)$  be an ideal in  $\mathbb{R}[x, y]$ . Then  $V(I) = \{(0, 0)\}$  is a finite set. But  $\mathbb{R}[x, y]/(x^2 + y^2)$  is an infinite dimensional vector space over  $\mathbb{R}$ . In fact, the monomials

$$\{\overline{x^m}, \overline{x^m y} : m = 0, 1, 2, \dots\}$$

is a basis for  $\mathbb{R}[x, y]/(x^2 + y^2)$ .

□

**Problem 1.33.**

- (a) Decompose  $V(x^2 + y^2 - 1, x^2 - z^2 - 1) \subseteq \mathbf{A}^3(\mathbb{C})$  into irreducible components.
- (b) Let  $V = \{(t, t^2, t^3) \in \mathbf{A}^3(\mathbb{C}) : t \in \mathbb{C}\}$ . Find  $I(V)$ , and show that  $V$  is irreducible.

*Proof of (a).*

- (1) Write

$$\begin{aligned} & V(x^2 + y^2 - 1, x^2 - z^2 - 1) \\ &= V(x^2 + y^2 - 1, y^2 + z^2) \\ &= V(x^2 + y^2 - 1, (y + iz)(y - iz)) \\ &= V(x^2 + y^2 - 1, y + iz) \cup V(x^2 + y^2 - 1, y - iz). \end{aligned}$$

By the Hilbert's Nullstellensatz, it suffices to show that  $(x^2 + y^2 - 1, y + iz)$  and  $(x^2 + y^2 - 1, y - iz)$  are prime.

- (2) Show that  $I = (x^2 + y^2 - 1, y + iz)$  is prime in  $\mathbb{C}[x, y, z]$ . Note that

$$\mathbb{C}[x, y, z]/I \cong \mathbb{C}[x, y]/(x^2 + y^2 - 1)$$

is a ring isomorphism defined by

$$f(x, y, z) + I \mapsto f(x, y, -iy) + (x^2 + y^2 - 1).$$

(Use the similar argument in (b) to prove it is indeed an isomorphism.)  
So it suffices to show that

$$x^2 + y^2 - 1 \in \mathbb{C}[x, y]$$

is irreducible. (Thus,  $\mathbb{C}[x, y]/(x^2 + y^2 - 1) \cong \mathbb{C}[x, y, z]/I$  is a domain, or  $I$  is prime.) We can use the similar argument in Problem 1.31 (b) to show  $x^2 + y^2 - 1 = y^2 + (x+1)(x-1)$  is irreducible as showing the irreducibility of  $y^2 - x(x+1)(x-1)$ .

- (3) Similarly,  $I = (x^2 + y^2 - 1, y - iz)$  is prime. Therefore, the irreducible components of  $V(x^2 + y^2 - 1, x^2 - z^2 - 1)$  are  $V(x^2 + y^2 - 1, y + iz)$  and  $V(x^2 + y^2 - 1, y - iz)$ .

□

*Proof of (b).*

- (1) Write

$$V = \{(t, t^2, t^3) \in \mathbf{A}^3(\mathbb{C}) : t \in \mathbb{C}\} = V(x^2 - y, x^3 - z).$$

Let  $I = (x^2 - y, x^3 - z)$  in  $\mathbb{C}[x, y, z]$ . By the Hilbert's Nullstellensatz,  $I(V) = \text{rad}(I)$ . So it suffices to show that  $I = (x^2 - y, x^3 - z)$  is prime (and thus  $V$  is irreducible).

- (2) *Show that*

$$\mathbb{C}[x, y, z]/I \cong \mathbb{C}[t]$$

*is a domain, and thus  $I = (x^2 - y, x^3 - z)$  is a prime ideal.*

- (a) Define a ring homomorphism  $\alpha : \mathbb{C}[x, y, z]/I \rightarrow \mathbb{C}[t]$  by

$$\alpha : f(x, y, z) + I \mapsto f(t, t^2, t^3).$$

$\alpha$  is well-defined since  $\alpha((x^2 - y) + I) = 0$  and  $\alpha((x^3 - z) + I) = 0$ .

- (b) *Show that  $\alpha$  is surjective.*

$$\alpha : g(x) + I \in \mathbb{C}[x, y, z]/I \mapsto g(t) \in \mathbb{C}[t]$$

for any  $g(t)$ .

- (c) *Show that  $\alpha$  is injective.* Suppose  $\alpha(f(x, y, z) + I) = 0$ . Write

$$\begin{aligned} f(x, y, z) + I &= \sum_{(i)} \lambda_{(i)} x^{i_1} (y - x^2)^{i_2} (z - x^3)^{i_3} + I \\ &= \sum_i \lambda_i x^i + I. \end{aligned}$$

So

$$0 = \alpha(f(x, y, z) + I) = \alpha\left(\sum_i \lambda_i x^i + I\right) = \sum_i \lambda_i t^i.$$

Hence,  $\ker(\alpha) = I$ .

□

**Problem 1.34.**

Let  $R$  be a UFD.

- (a) Show that a monic polynomial of degree two or three in  $R[x]$  is irreducible if and only if it has no root in  $R$ .
- (b)  $x^2 - a \in R[x]$  is irreducible if and only if  $a$  is not a square in  $R$ .

*Proof of (a).*

- (1) It is equivalent to show that a monic polynomial of degree two or three in  $R[x]$  is reducible if and only if it has one root in  $R$ .
- (2) Suppose  $f$  is reducible of degree 2 or 3. Then there exist nonconstant monic polynomials  $g, h \in R[x]$  such that  $f = gh$ . By

$$\deg(g) + \deg(h) = \deg(f) = 2 \text{ or } 3,$$

we may assume that  $\deg(g) = 1$ . (Otherwise  $g$  or  $h$  will be a constant polynomial.) Say  $g(x) = x - a$  where  $a \in R$ . Now

$$f(a) = g(a)h(a) = 0$$

implies that  $a \in R$  is a root of  $f$ .

- (3) Conversely, if  $a \in R$  is a root of  $f$ , then apply the same argument in Problem 1.7 we can write

$$f = (x - a)g$$

for some  $g \in R[x]$ . Here  $\deg(g) \geq 1$  since  $\deg(f) = 1 + \deg(g) \geq 2$ . Therefore,  $f$  is reducible.

□

*Proof of (b).* By (a),  $x^2 - a \in R[x]$  is reducible  $\iff x^2 - a$  has one root  $\alpha \in R$   $\iff a = \alpha^2$  is a square in  $R$  for some  $\alpha \in R$ . □

**Problem 1.35.**

Show that  $V(y^2 - x(x - 1)(x - \lambda)) \subseteq \mathbf{A}^2(k)$  is an irreducible curve for any algebraically closed field  $k$ , and any  $\lambda \in k$ .

*Proof.*

- (1) By the Hilbert's Nullstellensatz, it suffices to show that

$$I = (y^2 - x(x-1)(x-\lambda))$$

is a prime ideal in  $k[x, y]$ , or show that

$$y^2 - x(x-1)(x-\lambda)$$

is irreducible (since  $k[x, y]$  is a UFD).

- (2) By Problem 1.34(b),  $y^2 - x(x-1)(x-\lambda) \in (\mathbb{C}[x])[y]$  is irreducible if  $x(x-1)(x-\lambda)$  is not a square in  $\mathbb{C}[x]$ . Note that every square in  $\mathbb{C}[x]$  is of even degree. So  $x(x-1)(x-\lambda)$  cannot be a square in  $\mathbb{C}[x]$  since  $\deg(x(x-1)(x-\lambda)) = 3$  is odd.

□

*Note.*  $V(y^2 - x(x-1)(x-\lambda))$  is the elliptic curve as Problem 1.31.

**Problem 1.36.**

Let  $I = (y^2 - x^2, y^2 + x^2) \subseteq \mathbb{C}[x, y]$ . Find  $V(I)$  and  $\dim_{\mathbb{C}}(\mathbb{C}[x, y]/I)$ .

*Proof.*

- (1) Clearly,  $V(I) = \{(0, 0)\}$  is a finite set. By Corollary 4 to the Hilbert's Nullstellensatz,

$$\dim_{\mathbb{C}}(\mathbb{C}[x, y]/I) < \infty.$$

In fact,  $\dim_{\mathbb{C}}(\mathbb{C}[x, y]/I) = 4$ .

- (2) Given any  $f + I \in \mathbb{C}[x, y]/I$  where  $f \in \mathbb{C}[x, y]$ . Write

$$f(x, y) = \sum_i f_i(x) y^i$$

where  $f_i(x) = \sum_j a_{ij} x^j \in \mathbb{C}[x]$ . Note that

$$\begin{aligned} x^2 &= \frac{1}{2}(y^2 + x^2) - \frac{1}{2}(y^2 - x^2) \in I, \\ y^2 &= \frac{1}{2}(y^2 + x^2) + \frac{1}{2}(y^2 - x^2) \in I. \end{aligned}$$

So

$$\begin{aligned}
f(x, y) + I &= \sum_i f_i(x) y^i + I \\
&= f_0(x) + f_1(x) y + I \\
&= \sum_j a_{0j} x^j + \left( \sum_j a_{1j} x^j \right) y + I \\
&= a_{00} + a_{01} x + a_{10} y + a_{11} xy + I
\end{aligned}$$

is generated by  $\mathcal{B} = \{\bar{1}, \bar{x}, \bar{y}, \bar{xy}\}$ .

- (3) Note that  $\mathcal{B}$  is a basis since any linear combination of elements in  $\mathcal{B}$  is not in  $I$ . Therefore,

$$\dim_{\mathbb{C}}(\mathbb{C}[x, y]/I) = |\mathcal{B}| = 4.$$

□

**Problem 1.37.\***

Let  $K$  be any field,  $f \in K[x]$  a polynomial of degree  $n > 0$ . Show that the residues  $\bar{1}, \bar{x}, \dots, \bar{x}^{n-1}$  form a basis for  $K[x]/(f)$  over  $K$ .

*Proof.*

- (1) Show that every element in  $K[x]/(f)$  is generated by  $\mathcal{B} = \{\bar{1}, \bar{x}, \dots, \bar{x}^{n-1}\}$ . Given any  $\bar{g} \in K[x]/(f)$  with  $g \in K[x]$ . By the division-with-remainder property of  $K[x]$ , there are some polynomials  $q, r \in K[x]$  such that

$$g = fq + r$$

where  $r = 0$  or  $\deg(r) < n$  if  $r \neq 0$ . Therefore,

$$g + (f) = fq + r + (f) = r + (f).$$

Note that  $r + (f)$  is generated by  $\mathcal{B}$ .

- (2) Show that  $\mathcal{B}$  is a basis for  $K[x]/(f)$  over  $K$ . Suppose

$$a_0 + a_1 x + \dots + a_{n-1} x^{n-1} \in (f)$$

for  $a_1, \dots, a_{n-1} \in K$ . We can regard any linear combination of  $\{1, x, \dots, x^{n-1}\}$  as a polynomial  $r(x)$  in  $K[x]$ .  $r \in (f)$  implies that there exists a polynomial  $g \in K[x]$  such that  $r = fg$ . If  $g \neq 0$ , then  $\deg(r) = \deg(f) + \deg(g) \geq n$ , which is impossible. So  $g = 0$  and thus  $r = fg = 0 \in K[x]$ . Therefore,  $a_0 = a_1 = \dots = a_{n-1} = 0 \in K$  and

$$\dim_K(K[x]/(f)) = \deg(f).$$

□

**Problem 1.38.\***

Let  $R = k[x_1, \dots, x_n]$ ,  $k$  algebraically closed,  $V = V(I)$ . Show that there is a natural one-to-one correspondence between algebraic subsets of  $V$  and radical ideals in  $k[x_1, \dots, x_n]/I$ , and that irreducible algebraic sets (resp. points) correspond to prime ideals (resp. maximal ideals). (See Problem 1.22.)

*Proof.*

- (1) Given any algebraic subset  $W$  of  $V$ . By the Hilbert's Nullstellensatz,

$$I(W) \supseteq I(V) = \text{rad}(I) \supseteq I.$$

- (2) By Corollary 1 to the Hilbert's Nullstellensatz and Problem 1.22(b), we have a one-to-one correspondence such that

$$\begin{aligned} & \{\text{algebraic subsets of } V\} \\ & \longleftrightarrow \{\text{radical ideals containing } I\} \\ & \longleftrightarrow \{\text{radical ideals of } k[x_1, \dots, x_n]/I\}. \end{aligned}$$

- (3) Again by Corollary 2 to the Hilbert's Nullstellensatz and Problem 1.22(b), we have a one-to-one correspondence such that

$$\begin{aligned} & \{\text{irreducible algebraic subsets (resp. points) of } V\} \\ & \longleftrightarrow \{\text{prime (resp. maximal) ideals containing } I\} \\ & \longleftrightarrow \{\text{prime (resp. maximal) ideals of } k[x_1, \dots, x_n]/I\}. \end{aligned}$$

□

**Problem 1.39.**

- (a) Let  $R$  be a UFD, and let  $\mathfrak{p} = (t)$  be a principal proper prime ideal. Show that there is no prime ideal  $\mathfrak{q}$  such that  $0 \subsetneq \mathfrak{q} \subsetneq \mathfrak{p}$ .
- (b) Let  $V = V(f)$  be irreducible hypersurface in  $\mathbf{A}^n$ . Show that there is no irreducible algebraic set  $W$  such that  $V \subsetneq W \subsetneq \mathbf{A}^n$ .

*Proof of (a).*

- (1) (Reductio ad absurdum) Suppose that  $\mathfrak{q}$  were a prime ideal in  $R$  such that  $0 \subsetneq \mathfrak{q} \subsetneq \mathfrak{p}$ .



- (2) Show that there is an irreducible element in  $\mathfrak{q}$ . Given any  $q \in \mathfrak{q}$ . Since  $\mathfrak{q}$  is proper, we can write

$$q = q_1 \cdots q_n$$

as a product of irreducible elements in a UFD. Since  $\mathfrak{q}$  is prime, there is one irreducible element  $q_i \in \mathfrak{q}$ .

- (3) Now  $q_i \in \mathfrak{q} \subseteq \mathfrak{p} = (t)$ . So  $q_i = ut$  for some  $u \in R$ . By the irreducibility of  $q_i$ ,  $u$  is a unit or  $t$  is a unit. If  $u$  is a unit, then

$$(t) = (q_i) \subseteq \mathfrak{q} \subseteq \mathfrak{p} = (t).$$

So  $\mathfrak{q} = \mathfrak{p}$ , which is absurd. If  $t$  is a unit, then  $\mathfrak{p} = (1)$ , contrary to the primality of  $\mathfrak{p}$ .

□

*Proof of (b).*

- (1) We might assume that  $k = \bar{k}$ . By Corollary 3 to the Hilbert's Nullstellensatz and the irreducibility of  $V(f)$ , there are an irreducible polynomial  $g \in k[x_1, \dots, x_n]$  and an integer  $m > 0$  such that

$$f = g^m,$$

and

$$I(V(f)) = (g).$$

- (2) (Reductio ad absurdum) Suppose that there were an irreducible algebraic set  $W$  such that  $V \subsetneq W \subsetneq \mathbf{A}^n$ . Then by Corollary 3 to the Hilbert's Nullstellensatz again,

$$(g) = I(V(f)) \supsetneq I(W) \supsetneq (1) \in k[x_1, \dots, x_n].$$

Here  $(g) = I(V(f))$  and  $I(W)$  are all prime.

- (3) Note that  $(g)$  is a principal proper prime ideal in a UFD  $k[x_1, \dots, x_n]$ . By (a), such ideal  $I(W)$  cannot be prime, which is absurd.

□

#### Problem 1.40.

Let  $I = (x^2 - y^3, y^2 - z^3) \subseteq k[x, y, z]$ . Define  $\alpha : k[x, y, z] \rightarrow k[t]$  by  $\alpha(x) = t^9$ ,  $\alpha(y) = t^6$ ,  $\alpha(z) = t^4$ .

- (a) Show that every element of  $k[x, y, z]/I$  is the residue of an element  $a + xb + yc + xzd$ , for some  $a, b, c, d \in k[z]$ .

- (b) If  $f = a + xb + yc + xyd$ ,  $a, b, c, d \in k[z]$  and  $\alpha(f) = 0$ , compare like powers of  $t$  to conclude that  $f = 0$ .
- (c) Show that  $\ker(\alpha) = I$ , so  $I$  is prime,  $V(I)$  is irreducible, and  $I(V(I)) = I$ .

*Proof of (a).*

- (1) Take any element  $\bar{f} \in k[x, y, z]/I$  where  $f \in k[x, y, z]$ . Regard  $f \in (k[y, z])[x]$ , By the division-with-remainder property of  $(k[y, z])[x]$ ,

$$f = (x^2 - y^3)q + r$$

where  $q, r \in (k[y, z])[x]$  and  $r = 0$  or  $\deg_x(r) < 2$ . In any case,  $r = xr_1 + r_0$  for some  $r_1, r_0 \in k[y, z]$ .

- (2) Apply the same argument to (1), we have

$$r_0 = (y^2 - z^3)q_0 + yc + a$$

$$r_1 = (y^2 - z^3)q_1 + yd + b$$

where  $q_0, q_1 \in k[y, z]$  and  $a, b, c, d \in k[z]$ .

- (3) By  $\bar{r}_0 = \overline{yc + a}$  and  $\bar{r}_1 = \overline{yd + b}$ ,

$$\begin{aligned} \bar{f} &= \bar{r} \\ &= \overline{xr_1} + \bar{r}_0 \\ &= \overline{x(yd + b)} + (\overline{yc + a}) \\ &= \bar{a} + \bar{b} \cdot \bar{x} + \bar{c} \cdot \bar{y} + \bar{d} \cdot \overline{xy}. \end{aligned}$$

□

*Proof of (b).* As  $0 = \alpha(f) = a + ct^6 + bt^9 + dt^{15} \in k[t]$ ,  $a = b = c = d = 0 \in k$ .

□

*Proof of (c).*

- (1)  $I \subseteq \ker(\alpha)$  is trivial.
- (2) Show that  $\ker(\alpha) \subseteq I$ . Take any  $f \in \ker(\alpha)$ , or  $\alpha(f) = 0$ . By (a),  $f = r + f_1$  where  $f_1 \in I$  and  $r = a + bx + cy + dxy \in k[x, y, z]$  for some  $a, b, c, d \in k[z]$ . Note that  $\alpha$  is a ring homomorphism. Therefore,

$$0 = \alpha(f) = \alpha(r + f_1) = \alpha(r) + \alpha(f_1) = \alpha(r).$$

By (b),  $r = 0 \in k[x, y, z]$  and thus  $f = f_1 \in I$ .

- (3) Therefore,

$$\alpha : k[x, y, z]/(x^2 - y^3, y^2 - z^3) \hookrightarrow k[t]$$

is injective.

□

## 1.8. Modules; Finiteness Conditions

### Problem 1.41.\*

If  $S$  is module-finite over  $R$ , then  $S$  is ring-finite over  $R$ .

*Proof.*

- (1) Write  $S = \sum Rs_i$  for some  $s_1, \dots, s_n \in S$  since  $S$  is module-finite over  $R$ .
- (2) Show that  $\sum Rs_i = R[s_1, \dots, s_n]$ .  $\sum Rs_i \subseteq R[s_1, \dots, s_n]$  is trivial. Conversely, take any  $v \in R[s_1, \dots, s_n]$ . Write

$$v = \sum_{(j)} \overbrace{a_{(j)} s_1^{j_1} \cdots s_n^{j_n}}^{\in \sum Rs_i}$$

$\in R \quad \in S = \sum Rs_i$

Here each term  $a_{(j)} s_1^{j_1} \cdots s_n^{j_n}$  is in  $\sum Rs_i$ . As  $\sum Rs_i$  is an  $R$ -module,

$$v = \sum_{(i)} a_{(i)} s_1^{i_1} \cdots s_n^{i_n} \in \sum Rs_i.$$

□

*Note.* The converse is not true (by Problem 1.42).

### Problem 1.42.

Show that  $S = R[x]$  (the ring of polynomials in one variable) is ring-finite over  $R$ , but not module-finite.

*Proof.*

- (1)  $S = R[x]$  is ring-finite over  $R$  by definition (as  $x \in S$ ).
- (2) (Reductio ad absurdum) If  $S = \sum Rs_i$  for some  $s_1, \dots, s_n \in S$  were module-finite over  $R$ . Any element  $s \in \sum Rs_i$  is of degree

$$\deg s \leq \max_{1 \leq i \leq n} \deg s_i := m.$$

So that  $x^{m+1} \in S = R[x]$  but not in  $\sum Rs_i$ , which is absurd.

□

**Problem 1.43.\***

If  $L$  is ring-finite over  $K$  ( $K, L$  fields) then  $L$  is a finitely generated field extension of  $K$ .

*Proof.*

- (1)  $L = K[v_1, \dots, v_n]$  for some  $v_i \in L$  since  $L$  is ring-finite over  $K$ .
- (2) Apply Proposition 4 in §1.10,  $L$  is module-finite (and hence algebraic) over  $K$ , that is,  $L = K[v_1, \dots, v_n] = K(v_1, \dots, v_n)$  is a finitely generated field extension of  $K$ .

□

**Problem 1.44.\***

Show that  $L = K(x)$  (the field of rational functions in one variable) is a finitely generated field extension of  $K$ , but  $L$  is not ring-finite over  $K$ . (Hint: If  $L$  were ring-finite over  $K$ , a common denominator of ring generators would be an element  $b \in K[x]$  such that for all  $z \in L$ ,  $b^n z \in K[x]$  for some  $n$ ; but let  $z = 1/c$ , where  $c$  doesn't divide  $b$  (Problem 1.5).)

*Proof.*

- (1) (Reductio ad absurdum) Suppose that  $L$  were ring-finite over  $K$ . Write  $L = K[v_1, \dots, v_m]$  where  $v_1, \dots, v_m \in L = K(x)$ . Let  $b \in K[x]$  be a common denominator of ring generators  $v_1, \dots, v_m$ . (So that all  $bv_i \in K[x]$ .) Therefore, for any  $z \in L = K[v_1, \dots, v_m]$ , there is an integer  $n > 0$  such that  $b^n z \in K[x]$ .
- (2) Consider  $z = 1/c \in K(x)$ , where  $c \in K[x]$  doesn't divide  $b$ . The existence of  $c$  is guaranteed by Problem 1.5. Hence, for any integer  $n > 0$

$$b^n z = b^n / c$$

is never in  $K[x]$  by the construction of  $c$ , which is absurd.

□

**Problem 1.45.\***

Let  $R$  be a subring of  $S$ ,  $S$  a subring of  $T$ .

- (a) If  $S = \sum Rv_i$ ,  $T = \sum Sw_j$ , show that  $T = \sum Rv_iw_j$ .
- (b) If  $S = R[v_1, \dots, v_n]$ ,  $T = S[w_1, \dots, w_m]$ , show that  $T = R[v_1, \dots, v_n, w_1, \dots, w_m]$ .
- (c) If  $R, S, T$  are fields, and  $S = R(v_1, \dots, v_n)$ ,  $T = S(w_1, \dots, w_m)$ , show that  $T = R(v_1, \dots, v_n, w_1, \dots, w_m)$ .

So each of the three finiteness conditions is a transitive relation.

*Proof of (a).*

- (1) Show that  $T \subseteq \sum Rv_iw_j$ . Given any  $t \in T = \sum Sw_j$ . There are some  $s_j \in S$  such that  $t = \sum_j s_j w_j$ . As  $s_j \in S = \sum Rv_i$ , there are some  $r_{ij} \in R$  such that  $s_j = \sum_i r_{ij} v_i$ . Hence,

$$t = \sum_j s_j w_j = \sum_j \left( \sum_i r_{ij} v_i \right) w_j = \sum_{i,j} r_{ij} v_i w_j \in \sum Rv_iw_j.$$

- (2) Show that  $T \supseteq \sum Rv_iw_j$ . Take any  $\sum r_{ij} v_i w_j \in \sum Rv_iw_j$ .

$$\sum r_{ij} v_i w_j = \sum_j \left( \sum_i r_{ij} v_i \right) w_j \in \sum_j Sw_j = T.$$

□

*Proof of (b).*

- (1) Note that  $R[x_1, \dots, x_m]$  is canonically isomorphic to  $R[x_1, \dots, x_{m-1}][x_m]$ . Hence  $R[x_1, \dots, x_m]$  is isomorphic to  $R[x_1][x_2] \cdots [x_m]$ .
- (2) Hence,

$$\begin{aligned} T &= S[w_1, \dots, w_m] \\ &= R[v_1, \dots, v_n][w_1, \dots, w_m] \\ &= R[v_1, \dots, v_n][w_1] \cdots [w_m] \\ &= R[v_1] \cdots [v_n][w_1] \cdots [w_m] \\ &= R[v_1, \dots, v_n, w_1, \dots, w_m]. \end{aligned}$$

□

*Proof of (c).*

- (1) By (b),  $R(v_1, \dots, v_n)$  is canonically isomorphic to  $R(v_1, \dots, v_{n-1})(v_n)$ . Hence  $R(v_1, \dots, v_n)$  is isomorphic to  $R(v_1) \cdots (v_n)$ . To see this, note that  $R[x_1, \dots, x_m] \cong R[x_1, \dots, x_{m-1}][x_m]$  implies that

$$R(x_1, \dots, x_m) \cong R[x_1, \dots, x_{m-1}](x_m) \hookrightarrow R(x_1, \dots, x_{m-1})(x_m).$$

Conversely, for any  $a/b \in R(x_1, \dots, x_{m-1})(x_m)$  where

$$\begin{aligned} a &= \sum_i a_i x_m^i \in R(x_1, \dots, x_{m-1})[x_m], \\ b &= \sum_j b_j x_m^j \in R(x_1, \dots, x_{m-1})[x_m] \end{aligned}$$

and  $b \neq 0$ , there is a nonzero polynomial  $c \in R[x_1, \dots, x_{m-1}]$  such that all  $ca_i$  and  $cb_j$  are in  $R[x_1, \dots, x_{m-1}]$ . Hence,

$$\begin{aligned} \frac{a}{b} &= \frac{\sum_i a_i x_m^i}{\sum_j b_j x_m^j} \\ &= \frac{c \sum_i a_i x_m^i}{c \sum_j b_j x_m^j} \\ &= \frac{\sum_i ca_i x_m^i}{\sum_j cb_j x_m^j} \\ &\in R[x_1, \dots, x_{m-1}](x_m). \end{aligned}$$

(2) Hence,

$$\begin{aligned} T &= S(w_1, \dots, w_m) \\ &= R(v_1, \dots, v_n)(w_1, \dots, w_m) \\ &= R(v_1, \dots, v_n)(w_1) \cdots (w_m) \\ &= R(v_1) \cdots (v_n)(w_1) \cdots (w_m) \\ &= R(v_1, \dots, v_n, w_1, \dots, w_m). \end{aligned}$$

□

## 1.9. Integral Elements

### Problem 1.46.\* (Transitivity of integral extensions)

Let  $R$  be a subring of  $S$ ,  $S$  a subring of (a domain)  $T$ . If  $S$  is integral over  $R$ , and  $T$  is integral over  $S$ , show that  $T$  is integral over  $R$ . (Hint: Let  $z \in T$ , so we have  $z^n + a_1 z^{n-1} + \cdots + a_n = 0$ ,  $a_i \in S$ . Then  $R[a_1, \dots, a_n, z]$  is module-finite

over  $R$ .)

*Proof (Hint).*

- (1) Let  $z \in T$ , so we have  $z^n + a_1 z^{n-1} + \cdots + a_n = 0$ ,  $a_i \in S$ . Therefore,  $z$  is integral over  $R[a_1, \dots, a_n]$ , or  $R[a_1, \dots, a_n, z]$  is module-finite over  $R[a_1, \dots, a_n]$ .
- (2) Show that  $R[a_1, \dots, a_n]$  is module-finite over  $R$  if all  $a_i \in S$ . Note that

$$\begin{aligned} a_1 &\text{ is integral over } R, \\ a_2 &\text{ is integral over } R[a_1] \supseteq R, \\ &\dots \\ a_n &\text{ is integral over } R[a_1, \dots, a_{n-1}]. \end{aligned}$$

By Proposition 3,

$$\begin{aligned} R[a_1] &\text{ is module-finite over } R, \\ R[a_1][a_2] &\text{ is module-finite over } R[a_1], \\ &\dots \\ R[a_1, \dots, a_{n-1}][a_n] &\text{ is module-finite over } R[a_1, \dots, a_{n-1}]. \end{aligned}$$

Also note that  $R[a_1, \dots, a_i] = R[a_1, \dots, a_{i-1}][a_i]$  if  $i > 1$ . By the transitive relation of the module-finiteness (Problem 1.45),  $R[a_1, \dots, a_n]$  is module-finite over  $R$ .

- (3) Again by the transitive relation of the module-finiteness (Problem 1.45),  $R[a_1, \dots, a_n, z]$  is module-finite over  $R$ . Hence,  $R[a_1, \dots, a_n, z]$  is a subring of  $T$  containing  $R[z]$  which is module-finite over  $R$ . By Proposition 3,  $z$  is integral over  $R$ .

□

#### **Problem 1.47.\***

Suppose (a domain)  $S$  is ring-finite over  $R$ . Show that  $S$  is module-finite over  $R$  if and only if  $S$  is integral over  $R$ .

*Proof.*

- (1) Write  $S = R[v_1, \dots, v_m]$  for some  $v_i \in S$ .
- (2) Suppose that  $S$  is integral over  $R$ . Then all  $v_i$  are integral over  $R$ . Use the same argument in Problem 1.46, we have

$$S = R[v_1, \dots, v_n]$$

is module-finite over  $R$ .

- (3) Conversely, suppose that  $S$  is module-finite over  $R$ . Take any  $v \in S$ . Write  $v = \sum_i r_i v_i \in S$  since  $S$  is module-finite over  $R$ . Note that  $S = R[v_1, \dots, v_m]$  is a subring of  $S$  itself containing  $R[v]$  which is module-finite over  $R$ . By Proposition 3,  $v$  is integral over  $R$ .

□

**Problem 1.48.\***

Let  $L$  be a field,  $k$  an algebraically closed subfield of  $L$ .

- (a) Show that any element of  $L$  that is algebraic over  $k$  is already in  $k$ .  
 (b) An algebraically closed field has no module-finite field extensions except itself.

*Proof of (a).*

- (1) Let  $\alpha \in L$  be algebraic over  $k$ . Then there is a nonzero polynomial  $f(x) \in k[x]$  with  $f(\alpha) = 0$ . Note that  $\deg f \geq 1$ .  
 (2) Since  $k$  is algebraically closed, every polynomial is a product of first degree polynomials, say

$$f(x) = c(x - \alpha_1) \cdots (x - \alpha_m)$$

where  $c \in k - \{0\}$  and  $\alpha_1, \dots, \alpha_m \in k$ . As  $f(\alpha) = 0$ ,  $\alpha = \alpha_i \in k$  for some  $1 \leq i \leq m$ . Hence,  $\alpha \in L$  is algebraic over  $k$  implies that  $\alpha \in k$ .

□

*Proof of (b).*

- (1) Suppose that  $L$  is module-finite field extensions of an algebraically closed field  $k$ .  
 (2) By Problem 1.41,  $L$  is ring-finite over  $k$ . By Problem 1.47,  $L$  is integral or algebraic over  $k$  (since  $k$  is a field). By (a),  $L = k$ .

□

**Problem 1.49.\***

Let  $K$  be a field,  $L = K(x)$  the field of rational functions in one variable over  $K$ .



- (a) Show that any element of  $L$  that is integral over  $K[x]$  is already in  $K[x]$ .  
(Hint: If  $z^n + a_1z^{n-1} + \cdots + a_n = 0$ , write  $z = f/g$ ,  $f, g$  relatively prime.  
Then  $f^n + a_1f^{n-1}g + \cdots + a_ng^n = 0$ , So  $g$  divides  $f$ .)
- (b) Show that there is no nonzero element  $f \in K[x]$  such that for every  $z \in L$ ,  $f^n z$  is integral over  $K[x]$  for some  $n > 0$ . (Hint: See Problem 1.44.)

*Proof of (a).*

- (1) Note that 0 is integral over  $K[x]$  and  $0 \in K[x]$  trivially.
- (2) Now we take any nonzero element  $z \in L = K(x)$  which is integral over  $K[x]$ . So  $z^n + a_1z^{n-1} + \cdots + a_n = 0$  for some  $a_1, \dots, a_n \in K[x]$  and  $a_n \neq 0$  (since  $z \neq 0$ ).
- (3) Write  $z = f/g$ ,  $f, g$  relatively prime in  $K[x]$ . Then

$$f^n + a_1f^{n-1}g + \cdots + a_ng^n = 0 \in K[x].$$

Since  $a_n \neq 0$ ,  $g \mid f^n$  or  $g \mid f$  or  $g = 1 \in K$ . Therefore,  $z = f \in K[x]$ .

□

*Proof of (b).*

- (1) (Reductio ad absurdum) Suppose there were a nonzero element  $f \in K[x]$  such that for every  $z \in L$ ,  $f^n z$  is integral over  $K[x]$  for some  $n > 0$ .
- (2) Let  $z = 1/g \in K(x)$ , where  $g$  is an irreducible polynomial not dividing  $f$ . The existence of  $g$  is guaranteed by Problem 1.5.
- (3) By the hypothesis in (1), there is an integer  $n > 0$  such that  $f^n z$  is integral over  $K[x]$ . By (a),  $f^n z = f^n/g$  is also in  $K[x]$ . So  $g \mid f^n$  or  $g \mid f$ , which is absurd.

□

### Problem 1.50.\*

Let  $K$  be a subfield of a field  $L$ .

- (a) Show that the set of elements of  $L$  that are algebraic over  $K$  is a subfield of  $L$  containing  $K$ . (Hint: If  $v^n + a_1v^{n-1} + \cdots + a_n = 0$ , and  $a_n \neq 0$ , then  $v(v^{n-1} + \cdots + a_{n-1}) = -a_n$ .)
- (b) Suppose  $L$  is module-finite over  $K$ , and  $K \subseteq R \subseteq L$ ,  $R$  a subring of  $L$ . Show that  $R$  is a field.

*Proof of (a).*

- (1) Let  $R$  be the set of elements of  $L$  that are algebraic over  $K$ . By Corollary to Proposition 3,  $R$  is a subring of  $L$  containing  $K$ . (Note that  $K$  is a field.) So it suffices to show that  $v^{-1} \in R$  if  $v \in R - \{0\}$ .
- (2) Since  $v$  is algebraic over  $K$ , we can write

$$v^n + a_1 v^{n-1} + \cdots + a_n = 0$$

for some  $a_1, \dots, a_n \in K$  and  $a_n \neq 0$ . So

$$(v^{-1})^n + \underbrace{\frac{a_{n-1}}{a_n}}_{\in K} (v^{-1})^{n-1} + \cdots + \underbrace{\frac{a_1}{a_n}}_{\in K} (v^{-1}) + \underbrace{\frac{1}{a_n}}_{\in K} = 0,$$

or  $v^{-1}$  is integral over  $K$ . Hence,  $v^{-1} \in R$ .

□

*Proof of (b).*

- (1) By Problem 1.47,  $L$  is algebraic over  $K$ . Hence,  $R$  is algebraic over  $K$ .
- (2) To show that  $R$  is a field, it suffices to show that  $v^{-1} \in R$  if  $v \in R - \{0\}$ . Since  $v$  is algebraic over  $K$ , we can write

$$v^n + a_1 v^{n-1} + \cdots + a_n = 0$$

for some  $a_1, \dots, a_n \in K$  and  $a_n \neq 0$ . So

$$v \left( - \underbrace{\frac{1}{a_n}}_{\in K \subseteq R} \underbrace{v^{n-1}}_{\in R} - \cdots - \underbrace{\frac{a_{n-1}}{a_n}}_{\in K \subseteq R} \right) = 1.$$

Here  $v^{-1} = \left( -\frac{1}{a_n} v^{n-1} - \cdots - \frac{a_{n-1}}{a_n} \right)$  is the inverse of  $v$  in  $R$  (since  $R$  is a ring containing  $K$ ).

□

## 1.10. Field Extensions

### Problem 1.51.\*

Let  $K$  be a field,  $f \in K[x]$  an irreducible monic polynomial of degree  $n > 0$ .

- (a) Show that  $L = K[x]/(f)$  is a field, and if  $\bar{x}$  is the residue of  $x$  in  $L$ , then  $f(\bar{x}) = 0$ .
- (b) Suppose  $L'$  is a field extension of  $K$ ,  $y \in L'$  such that  $f(y) = 0$ . Show that the homomorphism from  $K[x]$  to  $L'$  that takes  $x$  to  $y$  induces an isomorphism of  $L$  with  $K(y)$ .
- (c) With  $L'$ ,  $y$  as in (b), suppose  $g \in K[x]$  and  $g(y) = 0$ . Show that  $f$  divides  $g$ .
- (d) Show that  $f = (x - \bar{x})f_1$ ,  $f_1 \in L[x]$ .

*Proof of (a).*

- (1)  $(f)$  is a prime ideal in a UFD  $K[x]$  since  $f$  is irreducible. Note that  $K[x]$  is also a PID,  $(f)$  is maximal (Problem 1.3). Hence  $L = K[x]/(f)$  is a field.
- (2)  $f(\bar{x}) = f(x) + (f(x)) = (f(x)) = \bar{0}$ .

□

*Proof of (b).*

- (1) Let  $\alpha : K[x] \rightarrow L'$  be a homomorphism defined by

$$\alpha\left(\sum a_i x^i\right) = \sum a_i y^i$$

where  $a_i \in K$ .  $\text{im}(\alpha) = K(y)$  clearly.

- (2) Note that  $\ker(\alpha)$  is an ideal containing  $(f)$  since  $\alpha(f) = 0$ .  $\ker(\alpha)$  is proper since  $\alpha(1) = 1 \neq 0$ . By the maximality of  $(f)$ ,  $\ker(\alpha) = (f)$ .
- (3) Hence,  $\alpha$  induces an isomorphism of  $L$  with  $K(y)$ :

$$L = K[x]/(f) \cong K(y) \hookrightarrow L'.$$

□

*Proof of (c).* By (b),  $g \in \ker(\alpha) = (f)$ . So  $f \mid g$ . □

*Proof of (d).*

- (1) By (a),  $\bar{x} \in L$  is a root of  $f \in L[x]$  (by embedding  $K[x]$  in  $L[x]$ ).
- (2) Since  $L$  is a field, by Problem 1.7(b) we have

$$f = (x - \bar{x})f_1$$

for some  $f_1 \in L[x]$ .

□

**Problem 1.52.\* (Splitting fields)**

Let  $K$  be a field,  $f \in K[x]$ . Show that there is a field  $L$  containing  $K$  such that  $f = \prod_{i=1}^n (x - x_i) \in L[x]$ . (Hint: Use Problem 1.51(d) and induction on the degree.)  $L$  is called a **splitting field** of  $F$ .

*Proof.*

- (1) Let  $p(x) \in K[x]$  be an irreducible factor of  $f(x) \in K[x]$ , and let  $L'$  be the field  $K[x]/(p(x))$  (by Problem 1.51(a)).
- (2) Then we might regard  $K$  as a subfield of  $L'$  by sending  $a \in K$  to  $\bar{a} = a + (p(x)) \in L'$ .
- (3) By Problem 1.51(a),  $\bar{x}$  is a root of  $p \in L'$ ; therefore is a root of  $f$ .
- (4) Induction on  $n$ . By (1)(2)(3), there is a field  $L' \supseteq K$  such that  $L'$  contains a root  $\bar{x}$  of  $f(x)$ , say  $f(x) = (x - \bar{x})f_1(x)$  over  $L'[x]$  (by Problem 1.51(d)). By induction, there is a field  $L \supseteq L'$  such that  $f_1$  splits over  $L$ . Hence,  $f$  splits over  $L$ .

□

**Problem 1.53.\***

Suppose  $K$  is a field of characteristic zero,  $f$  an irreducible monic polynomial in  $K[x]$  of degree  $n > 0$ . Let  $L$  be a splitting field of  $f$ , so  $f = \prod_{i=1}^n (x - x_i)$ ,  $x_i \in L$ . Show that the  $x_i$  are distinct. (Hint: Apply Problem 1.51(c) to  $g = f_x$ ; if  $(x - \bar{x})^2$  divides  $f$ , then  $g(\bar{x}) = 0$ .)

*Proof.*

- (1) Since  $f \in K[x]$  is irreducible over  $K$ ,  $\gcd(f, f_x)$  is 1 or  $f$ . As  $\text{char}(K) = 0$ ,  $\deg(f_x) = \deg(f) - 1$ . So  $f$  does not divide  $f_x$  or  $\gcd(f, f_x) = 1$ . Hence, there are polynomials  $g, h \in K[x]$  such that

$$1 = fg + f_x h.$$

This equation is also true in  $L[x]$ .

- (2) Note that

$$f = \prod_{i=1}^n (x - x_i) \in L[x],$$

$$f_x = \sum_{i=1}^n (x - x_1) \cdots \widehat{(x - x_i)} \cdots (x - x_n) \in L[x].$$

If  $\bar{x}$  were a multiple root of  $f$ , then  $f(\bar{x}) = f_x(\bar{x}) = 0$ . By (1),

$$1 = f(\bar{x})g(\bar{x}) + f_x(\bar{x})h(\bar{x}) = 0,$$

which is absurd.

□

**Problem 1.54.\***

Let  $R$  be a domain with quotient field  $K$ , and let  $L$  be a finite algebraic extension of  $K$ .

- (a) For any  $v \in L$ , show that there is a nonzero  $a \in R$  such that  $av$  is integral over  $R$ .
- (b) Show that there is a basis  $v_1, \dots, v_n$  for  $L$  over  $K$  (as a vector space) such that each  $v_i$  is integral over  $R$ .

*Proof of (a).*

- (1) Take any  $v \in L$ , which is algebraic over  $K$ . Write

$$v^n + a_1v^{n-1} + \dots + a_n = 0$$

for some  $a_1, \dots, a_n \in K$  and  $a_n \neq 0$ . Since  $K$  is the quotient field of  $R$ , there is a common denominator  $a \in R$  of  $a_1, \dots, a_n$ . Here  $a \neq 0$  and  $aa_i \in R$  for all  $1 \leq i \leq n$ .

- (2) Hence,

$$\begin{aligned} & a^n v^n + a^n a_1 v^{n-1} + \dots + a^n a_n = 0 \\ \iff & (av)^n + \underbrace{(aa_1)}_{\in R} (av)^{n-1} + \underbrace{a(aa_2)}_{\in R} (av)^{n-2} + \dots + \underbrace{a^{n-1}(aa_n)}_{\in R} = 0. \end{aligned}$$

$av$  is integral over  $R$ .

□

*Proof of (b).*

- (1) Since  $L$  be a finite algebraic extension of  $K$ , there exists a basis

$$\{w_1, \dots, w_n\}$$

for  $L$  over  $K$  (as a vector space).

- (2) For each  $w_i \in L$ , there is a nonzero  $a_i \in R$  such that  $a_i w_i$  is integral over  $R$  (by (a)). So it suffices to show that

$$\{a_1 w_1, \dots, a_n w_n\}$$

is also a basis for  $L$  over  $K$ .

- (3) Suppose

$$0 = \sum_i \alpha_i (a_i w_i) = \sum_i (\alpha_i a_i) w_i$$

for some  $\alpha_1, \dots, \alpha_n \in K$ . Since  $\{w_1, \dots, w_n\}$  is a basis,  $\alpha_i a_i = 0$  for all  $i$ , or  $\alpha_i = 0$  for all  $i$  (since all  $a_i \neq 0$ ). Hence  $\{a_1 w_1, \dots, a_n w_n\}$  is linearly independent.

- (4) Also, for any  $w \in L$ , we can write

$$\begin{aligned} w &= \underbrace{\beta_1}_{\in K} w_1 + \dots + \underbrace{\beta_n}_{\in K} w_n \\ &= \underbrace{\frac{\beta_1}{a_1}}_{\in K} (a_1 w_1) + \dots + \underbrace{\frac{\beta_n}{a_n}}_{\in K} (a_n w_n) \end{aligned}$$

as a linear combination of  $\{a_1 w_1, \dots, a_n w_n\}$  over  $K$ .

□

## Chapter 2: Affine Varieties

### 2.1. Coordinate Rings

#### Problem 2.1.\*

Show that the map which associates to each  $f \in k[x_1, \dots, x_n]$  a polynomial function in  $\mathcal{F}(V, k)$  is a ring homomorphism whose kernel is  $I(V)$ .

*Proof.*

- (1) Define a map  $\alpha : k[x_1, \dots, x_n] \rightarrow \mathcal{F}(V, k)$ . Every polynomial  $f \in k[x_1, \dots, x_n]$  defines a function from  $V$  to  $k$  by

$$\alpha(f)(a_1, \dots, a_n) = f(a_1, \dots, a_n)$$

for all  $(a_1, \dots, a_n) \in V$ .

- (2)  $\alpha$  is a ring homomorphism by construction in (1).  
(3) Show that  $\ker(\alpha) = I(V)$ . In fact, given any  $f \in k[x_1, \dots, x_n]$ , we have  $\alpha(f) = 0$  (sending all  $a \in V$  to  $0 \in k$ ) if and only if  $f(a) = 0$  for all  $a \in V$  if and only if  $f \in I(V)$ .  
(4) Hence,

$$k[x_1, \dots, x_n]/I(V) = \Gamma(V) \cong \{\text{polynomial functions in } \mathcal{F}(V, k)\}$$

as a ring isomorphism.

□

#### Problem 2.2.\*

Let  $V \subseteq \mathbf{A}^n$  be a variety. A **subvariety** of  $V$  is a variety  $W \subseteq \mathbf{A}^n$  that is contained in  $V$ . Show that there is a natural one-to-one correspondence between algebraic subsets (resp. subvarieties, resp. points) of  $V$  and radical ideals (resp. prime ideals, resp. maximal ideals) of  $\Gamma(V)$ . (See Problems 1.22, 1.38.)

*Proof.* Repeat Problem 1.38 by replacing  $k[x_1, \dots, x_n]/I$  by  $\Gamma(V)$ . □

**Problem 2.3.\***

Let  $W$  be a subvariety of a variety  $V$ , and let  $I_V(W)$  be the ideal of  $\Gamma(V)$  corresponding to  $W$ .

- (a) Show that every polynomial function on  $V$  restricts to a polynomial function on  $W$ .
- (b) Show that the map from  $\Gamma(V)$  to  $\Gamma(W)$  defined in part (a) is a surjective homomorphism with kernel  $I_V(W)$ , so that  $\Gamma(W)$  is isomorphic to  $\Gamma(V)/I_V(W)$ .

*Proof of (a).*

- (1) Given any polynomial function  $f \in \mathcal{F}(V, k)$  on  $V$ . There is a polynomial  $g \in k[x_1, \dots, x_n]$  such that  $f(P) = g(P)$  for all  $P \in V \supseteq W$ ; thus  $f(P) = g(P)$  for all  $P \in W$ , or  $f|_W$  is a polynomial function on  $W$ .
- (2) The map  $\alpha : \{\text{polynomial functions in } \mathcal{F}(V, k)\} \rightarrow \{\text{polynomial functions in } \mathcal{F}(W, k)\}$  in (1) is defined by

$$\alpha(f) = f|_W.$$

It is a ring homomorphism.

□

*Proof of (b).*

- (1) Identify  $\Gamma(V)$  (resp.  $\Gamma(W)$ ) with the set of all polynomial functions in  $\mathcal{F}(V, k)$  (resp. in  $\mathcal{F}(W, k)$ ) by Problem 2.1. The map

$$\alpha : \Gamma(V) \rightarrow \Gamma(W)$$

is defined by

$$\alpha(f + I(V)) = f + I(W).$$

It is well-defined by (a).

- (2) Show that  $\alpha$  is surjective. For any  $f + I(W) \in \Gamma(W)$ , take  $f + I(V) \in \Gamma(V)$  and then  $\alpha(f + I(V)) = f + I(W)$ . (The choice of  $f + I(V)$  depends on the representation of  $f + I(W)$  and thus might not be unique.)
- (3) Show that  $\ker(\alpha) = I_V(W)$ , and thus  $\Gamma(W) \cong \Gamma(V)/I_V(W)$ . Since  $\alpha$  is a surjective homomorphism,

$$\begin{aligned} \ker(\alpha) &= \Gamma(V)/\Gamma(W) \\ &= (k[x_1, \dots, x_n]/I(V))/(k[x_1, \dots, x_n]/I(W)) \\ &= I(W)/I(V) \\ &= I_V(W). \end{aligned}$$

□



**Problem 2.4.\***

Let  $V \subseteq \mathbf{A}^n$  be a nonempty variety. Show that the following are equivalent:

- (i)  $V$  is a point.
- (ii)  $\Gamma(V) = k$ .
- (iii)  $\dim_k \Gamma(V) < \infty$ .

*Proof.*

- (1) (i)  $\implies$  (ii). By Corollary 2 to the Hilbert's Nullstellensatz in §1.7,  $V = \{(a_1, \dots, a_n)\}$  corresponds to the maximal ideal

$$I(V) = (x_1 - a_1, \dots, x_n - a_n)$$

in  $k[x_1, \dots, x_n]$ . Hence,

$$\Gamma(V) = k[x_1, \dots, x_n]/(x_1 - a_1, \dots, x_n - a_n) \cong k$$

(by Problem 1.24).

- (2) (ii)  $\implies$  (iii).  $\dim_k(\Gamma(V)) = \dim_k(k) = 1 < \infty$ .
- (3) (iii)  $\implies$  (i). By Corollary 4 to the Hilbert's Nullstellensatz in §1.7,  $V$  is a finite set of points in  $\mathbf{A}^n$ . Since  $V$  is a nonempty variety,  $V$  is exactly a point.

□

**Problem 2.5.**

Let  $f$  be an irreducible polynomial in  $k[x, y]$ , and suppose  $f$  is monic in  $y$ :  $f = y^n + a_1(x)y^{n-1} + \dots + a_n(x)$ , with  $n > 0$ . Let  $V = V(f) \subseteq \mathbf{A}^2$ . Show that the natural homomorphism from  $k[x]$  to  $\Gamma(V) = k[x, y]/(f)$  is one-to-one, so that  $k[x]$  may be regarded as a subring of  $\Gamma(V)$ ; show that the residues  $\bar{1}, \bar{y}, \dots, \bar{y}^{n-1}$  generate  $\Gamma(V)$  over  $k[x]$  as a module.

*Proof.*

- (1)  $\Gamma(V) = k[x, y]/(f)$  is well-defined since  $f$  is irreducible. Define a ring homomorphism  $\alpha : k[x] \rightarrow \Gamma(V) = k[x, y]/(f)$  by

$$\alpha : g(x) \mapsto g(x) + (f(x, y)).$$

- (2) *Show that  $\alpha$  is one-to-one.* If there were a nonzero polynomial  $g \in k[x]$  such that  $\alpha(g) = 0$ , then  $g = fh$  for some nonzero polynomial  $h \in k[x, y]$ . Hence

$$0 = \deg_y(g) = \deg_y(f) + \deg_y(h) \geq n > 0,$$

which is absurd. Therefore,  $\alpha$  is one-to-one. Hence  $k[x]$  may be regarded as a subring of  $\Gamma(V)$ , and thus the multiplication in  $\Gamma(V)$  makes  $\Gamma(V)$  a  $k[x]$ -module.

- (3) Given any  $g(x, y) + (f(x, y)) \in k[x, y]/(f)$  where  $g \in k[x, y] = (k[x])[y]$ . By the division-with-remainder property of  $(k[x])[y]$ ,

$$g = fq + r$$

for some  $q, r \in (k[x])[y]$  and

$$r = r_1(x)y^{n-1} + \cdots + r_n(x)$$

where  $r_1, \dots, r_n \in k[x]$ . Hence

$$\begin{aligned} g + (f) &= fq + r + (f) \\ &= r + (f) \\ &= r_1(x)y^{n-1} + \cdots + r_n(x) + (f) \\ &= \underbrace{r_1(x)}_{\in k[x]} \bar{y}^{n-1} + \cdots + \underbrace{r_n(x)}_{\in k[x]} \bar{1}, \end{aligned}$$

which means that the residues  $\bar{1}, \bar{y}, \dots, \bar{y}^{n-1}$  generate  $\Gamma(V)$  over  $k[x]$  as a module.

□

## 2.2. Polynomial Maps

### Problem 2.6.\*

Let  $\varphi : V \rightarrow W$ ,  $\psi : W \rightarrow Z$ . Show that  $\widetilde{\psi \circ \varphi} = \widetilde{\varphi} \circ \widetilde{\psi}$ . Show that the composition of polynomial maps is a polynomial map.

*Proof.*

- (1) *Show that  $\widetilde{\psi \circ \varphi} = \widetilde{\varphi} \circ \widetilde{\psi}$ .* It is equivalent to show that

$$(\widetilde{\psi \circ \varphi})(f) = (\widetilde{\varphi} \circ \widetilde{\psi})(f)$$

for all  $f \in \mathcal{F}(Z, k)$ . In fact,

$$\begin{aligned} (\widetilde{\psi \circ \varphi})(f) &= f \circ \psi \circ \varphi, \\ (\widetilde{\varphi} \circ \widetilde{\psi})(f) &= \widetilde{\varphi}(\widetilde{\psi}(f)) = \widetilde{\varphi}(f \circ \psi) = f \circ \psi \circ \varphi. \end{aligned}$$

- (2) Show that the composition of polynomial maps is a polynomial map. Say  $V \subseteq \mathbf{A}^n, W \subseteq \mathbf{A}^m, Z \subseteq \mathbf{A}^r$ . Since  $\varphi$  (resp.  $\psi$ ) is a polynomial map, there are polynomials  $t_1, \dots, t_m \in k[x_1, \dots, x_n]$  (resp.  $s_1, \dots, s_r \in k[x_1, \dots, x_m]$ ) such that

$$\begin{aligned}\varphi(P) &= (t_1(P), \dots, t_m(P)) \\ \psi(Q) &= (s_1(Q), \dots, s_r(Q))\end{aligned}$$

for all  $P \in V$  (resp.  $Q \in W$ ). Hence the composition  $\psi \circ \varphi$  is

$$\begin{aligned}(\psi \circ \varphi)(P) &= \psi(\varphi(P)) \\ &= \psi(t_1(P), \dots, t_m(P)) \\ &= (s_1(t_1(P), \dots, t_m(P)), \dots, s_r(t_1(P), \dots, t_m(P))).\end{aligned}$$

So there are polynomials  $y_1, \dots, y_r \in k[x_1, \dots, x_n]$  defined by

$$y_i(P) = s_i(t_1(P), \dots, t_m(P))$$

for all  $(a_1, \dots, a_n) \in \mathbf{A}^n$  such that

$$(\psi \circ \varphi)(P) = (y_1(P), \dots, y_r(P)).$$

(Note that the composition of polynomials is a polynomials.) Hence  $\psi \circ \varphi$  is a polynomial map.

□

### Problem 2.7.\*

If  $\varphi : V \rightarrow W$  is a polynomial map, and  $X$  is an algebraic subset of  $W$ , show that  $\varphi^{-1}(X)$  is an algebraic subset of  $V$ . If  $\varphi^{-1}(X)$  is irreducible, and  $X$  is contained in the image of  $\varphi$ , show that  $X$  is irreducible. This gives a useful test for irreducibility.

*Proof.*

- (1) Show that  $\varphi^{-1}(X) = V(\tilde{\varphi}(I(X)))$  is algebraic.

$$\begin{aligned}P \in \varphi^{-1}(X) &\iff \varphi(P) \in X \\ &\iff f(\varphi(P)) = 0 \forall f \in I(X) \\ &\iff \tilde{\varphi}(f)(P) = 0 \forall f \in I(X) \\ &\iff g(P) = 0 \forall g \in \tilde{\varphi}(I(X)) \\ &\iff P \in V(\tilde{\varphi}(I(X))).\end{aligned}$$

Also note that  $\tilde{\varphi}(I(X))$  is an ideal in  $k[x_1, \dots, x_n]$  since  $\varphi$  is a polynomial map.

- (2) If  $\varphi^{-1}(X)$  is irreducible, and  $X$  is contained in the image of  $\varphi$ , show that  $X$  is irreducible. (Reductio ad absurdum) Suppose that  $X$  were reducible or  $I(X)$  were not prime. So that there exist two polynomials  $f_1, f_2 \notin I(X)$  but  $f_1 f_2 \in I(X)$ . By definition of  $I(X)$ , there exist two points  $P_1, P_2 \in X$  such that  $f_i(P_i) \neq 0$  for  $i = 1, 2$ .
- (3) Since  $X$  is contained in the image of  $\varphi$ , there are two corresponding points  $Q_1, Q_2 \in \varphi^{-1}(X)$  such that  $\varphi(Q_i) = P_i$ . So  $\tilde{\varphi}(f_i)(Q_i) = f_i(P_i) \neq 0$ , or  $\tilde{\varphi}(f_i) \notin I(\varphi^{-1}(X))$ . However

$$\tilde{\varphi}(f_1)\tilde{\varphi}(f_2) = \tilde{\varphi}(f_1 f_2) \in I(\varphi^{-1}(X))$$

since  $f_1 f_2 \in I(X)$ , contrary to the primality of  $I(\varphi^{-1}(X))$ .

□

**Problem 2.8.**

- (a) Show that  $\{(t, t^2, t^3) \in \mathbf{A}^3(k) : t \in k\}$  is an affine variety.
- (b) Show that  $V(xz - y^2, yz - x^3, z^2 - x^2y) \subseteq \mathbf{A}^3(\mathbb{C})$  is a variety. (Hint:  $y^3 - x^4, z^3 - x^5, z^4 - y^5 \in I(V)$ . Find a polynomial map from  $\mathbf{A}^1(\mathbb{C})$  onto  $V$ .)

*Proof of (a).*

- (1) Let  $Y := \{(t, t^2, t^3) \in \mathbf{A}^3(k) : t \in k\}$  be the twisted cubic curve. By Problem 2.7, it suffices to show that there is a polynomial map from  $\mathbf{A}^1(k)$  onto  $Y$ . Here we use the fact that  $\mathbf{A}^1(k)$  is irreducible as  $k = \bar{k}$  is infinite (by Problem 1.29).
- (2) Define a mapping  $\varphi$  from  $\mathbf{A}^1(k)$  to  $Y$  by  $\varphi(t) = (t, t^2, t^3) \in Y$ .  $\varphi$  is a polynomial map. Also,  $\varphi$  is surjective.

□

*Note.* Also see Problems 1.11 and 1.33 (for the case  $k = \mathbb{C}$ ).

*Proof of (b).*

- (1) We prove for any algebraically closed field  $k$ .
- (2) Write

$$\begin{aligned} V &= V(xz - y^2, yz - x^3, z^2 - x^2y), \\ Y &= \{(t^3, t^4, t^5) \in \mathbf{A}^3(k) : t \in k\}. \end{aligned}$$

We want to show that  $Y = V$ .  $Y \subseteq V$  is trivial. Now given any  $(x, y, z) \in V$ . If  $x = 0$ , then  $y = z = 0$ . So  $(x, y, z) = (0, 0, 0) \in Y$ . If  $x \neq 0$ , define

$$t = \frac{y}{x} \in k.$$

Hence,

$$\begin{aligned} t^3 &= \frac{y^3}{x^3} = \frac{y(xz)}{x^3} = \frac{yz}{x^2} = \frac{x^3}{x^2} = x, \\ t^4 &= tx = y, \\ t^5 &= ty = \frac{y^2}{x} = \frac{xz}{x} = z. \end{aligned}$$

- (3) Same as (a). Define a mapping  $\varphi$  from  $\mathbf{A}^1(k)$  to  $Y = V$  by  $\varphi(t) = (t^3, t^4, t^5) \in Y = V$ .

□

*Note.*

- (1) We don't use the hint.
- (2) In fact, it is easy to show that

$$Y = V(y^3 - x^4, z^3 - x^5, z^4 - y^5).$$

- (3)  $I(V)$  is a prime ideal of height 2 in  $k[x, y, z]$  which cannot be generated by 2 elements. We say  $V$  is **not a local complete intersection**.

### Problem 2.9.\*

Let  $\varphi : V \rightarrow W$  be a polynomial map of affine varieties,  $V' \subseteq V$ ,  $W' \subseteq W$  subvarieties. Suppose  $\varphi(V') \subseteq W'$ .

- (a) Show that  $\tilde{\varphi}(I_W(W')) \subseteq I_V(V')$  (see Problems 2.3).
- (b) Show that the restriction of  $\varphi$  gives a polynomial map from  $V'$  to  $W'$ .

*Proof of (a).*

- (1) It suffices to show that  $f \in I_V(V')$  for any  $f = \tilde{\varphi}(g) \in \tilde{\varphi}(I_W(W'))$  for some  $g \in I_W(W')$ .
- (2) To show  $f \in I_V(V')$ , it suffices to show that  $f(P) = 0$  for all  $P \in \varphi(V')$ . In fact,

$$f(P) = \tilde{\varphi}(g)(P) = g(\varphi(P)) = 0$$

since  $\varphi(V') \subseteq W'$  and  $g \in I_W(W')$ .

□

*Proof of (b).*

(1) Similar to Problem 2.3.

(2) Since  $\varphi$  is a polynomial map, there are polynomials  $t_1, \dots, t_m \in k[x_1, \dots, x_n]$  such that

$$\varphi(P) = (t_1(P), \dots, t_m(P)) \in W$$

for all  $P \in V$ . So that  $\varphi|_{V'} : V' \rightarrow \varphi(V') \subseteq W'$  is also a polynomial map which is equipped with the same polynomials  $t_1, \dots, t_m$  such that

$$\varphi(P) = (t_1(P), \dots, t_m(P)) \in W' \subseteq W$$

for all  $P \in V' \subseteq V$ . (Note that both  $V'$  and  $W'$  are affine varieties.)

□

**Problem 2.10.\***

Show that the **projection map**  $\text{pr} : \mathbf{A}^n \rightarrow \mathbf{A}^r$ ,  $n \geq r$ , defined by  $\text{pr}(a_1, \dots, a_n) = (a_1, \dots, a_r)$  is a polynomial map.

*Proof.*

(1) Define  $t_i \in k[x_1, \dots, x_n]$  by  $t_i(x_1, \dots, x_n) = x_i$  for  $i = 1, \dots, r$ .

(2) Clearly,

$$\text{pr}(P) = (t_1(P), \dots, t_r(P))$$

for  $P = (a_1, \dots, a_n) \in \mathbf{A}^n$ , and thus  $\text{pr}$  is a polynomial map.

□

**Problem 2.11.**

Let  $f \in \Gamma(V)$ ,  $V$  a variety  $\subseteq \mathbf{A}^n$ . Define

$$\begin{aligned} G(f) = \{ & (a_1, \dots, a_n, a_{n+1}) \in \mathbf{A}^{n+1} \\ & : (a_1, \dots, a_n) \in V \text{ and } a_{n+1} = f(a_1, \dots, a_n) \}, \end{aligned}$$

the **graph** of  $f$ . Show that  $G(f)$  is an affine variety, and that the map  $(a_1, \dots, a_n) \mapsto (a_1, \dots, a_n, f(a_1, \dots, a_n))$  defines an isomorphism of  $V$  with  $G(f)$ . (Projection gives the inverse.)

*Proof.*

- (1) Define  $I = I(V)$  as an ideal in  $k[x_1, \dots, x_n]$ . Note that

$$G(f) = V(\underbrace{(I, x_{n+1} - f)}_{:=J}).$$

Here we can view  $I$  as an ideal of  $k[x_1, \dots, x_n, x_{n+1}]$ .

- (2) To show that  $G(f)$  is an affine variety, it suffices to show that

$$I(G(f)) = I(V(J)) = \text{rad}(J)$$

is prime (by Proposition 1 in §1.5 and the Hilbert's Nullstellensatz in §1.7). Suppose  $gh \in I(G(f)) = \text{rad}(J)$ . Write

$$\begin{aligned} g &= \sum_i g_i x_{n+1}^i = \sum_i g_i (\underbrace{(x_{n+1} - f) + f}_{\in J})^i, \\ h &= \sum_j h_j x_{n+1}^j = \sum_j h_j (\underbrace{(x_{n+1} - f) + f}_{\in J})^j \end{aligned}$$

where  $g_i, h_j \in k[x_1, \dots, x_n]$ .

- (3) Hence

$$\begin{aligned} \text{rad}(J) &= gh + \text{rad}(J) && (gh \in \text{rad}(J)) \\ &= (g + \text{rad}(J))(h + \text{rad}(J)) \\ &= \left( \sum_i g_i f^i + \text{rad}(J) \right) \left( \sum_j h_j f^j + \text{rad}(J) \right) && (x_{n+1} - f \in J) \\ &= \left( \sum_i g_i f^i \right) \left( \sum_j h_j f^j \right) + \text{rad}(J) \end{aligned}$$

or

$$\underbrace{\left( \sum_i g_i f^i \right)^N \left( \sum_j h_j f^j \right)^N}_{\in k[x_1, \dots, x_n]} \in J = (I, x_{n+1} - f)$$

for some positive integer  $N$ . So that  $(\sum_i g_i f^i)^N (\sum_j h_j f^j)^N \in I$ .

- (4) Since  $I = I(V)$  is a prime ideal, we might get  $\sum_i g_i f^i \in I \subseteq \text{rad}(J)$ . (The case  $\sum_j h_j f^j$  is similar.) Hence  $\text{rad}(J) = I(G(f))$  is a prime ideal, or  $G(f)$  is irreducible.

- (5) As  $G(f)$  is an affine variety, the map  $\alpha : V \rightarrow G(f)$  defined by

$$\alpha : (a_1, \dots, a_n) \mapsto (a_1, \dots, a_n, f(a_1, \dots, a_n))$$

is a polynomial map. (Here  $t_1 = x_1, \dots, t_n = x_n$  and  $t_{n+1} = f$ .)

- (6) By Problem 2.10, the projection map  $\text{pr}$  is a polynomial map. Also note that  $\text{pr} \circ \alpha = 1_V$  and  $\alpha \circ \text{pr} = 1_{G(f)}$ . Therefore,  $V \cong G(f)$  as an affine variety isomorphism.

□

**Problem 2.12.**

- (a) Let  $\varphi : \mathbf{A}^1 \rightarrow V = V(y^2 - x^3) \subseteq \mathbf{A}^2$  be defined by  $\varphi(t) = (t^2, t^3)$ . Show that although  $\varphi$  is a one-to-one, onto polynomial map,  $\varphi$  is not an isomorphism. (Hint:  $\tilde{\varphi}(\Gamma(V)) = k[t^2, t^3] \subsetneq k[t] = \Gamma(\mathbf{A}^1)$ .)
- (b) Let  $\varphi : \mathbf{A}^1 \rightarrow V = V(y^2 - x^2(x+1))$  be defined by  $\varphi(t) = (t^2 - 1, t(t^2 - 1))$ . Show that  $\varphi$  is one-to-one and onto, except that  $\varphi(\pm 1) = (0, 0)$ .

*Proof of (a).*

- (1) Similar to Problem 2.8(a),  $\varphi$  is a polynomial map.
- (2) Similar to Problem 2.8(a) again,

$$V = V(y^2 - x^3) = \{(t^2, t^3) \in \mathbf{A}^2(k) : t \in k\}.$$

Hence the map  $\varphi : t \mapsto (t^2, t^3)$  is surjective.

- (3) Show that  $\varphi$  is injective. Suppose  $(t^2, t^3) = (s^2, s^3)$  for some  $t, s \in k$ . If  $t = 0$ , then  $s = 0$ . If  $t \neq 0$ , then  $t = \frac{t^3}{t^2} = \frac{s^3}{s^2} = s$ . In any case,  $t = s$  whenever  $(t^2, t^3) = (s^2, s^3)$ .
- (4) Show that  $\varphi$  is not an isomorphism. It suffices to show that  $\tilde{\varphi}(\Gamma(V)) \subsetneq \Gamma(\mathbf{A}^1)$  by Proposition 1. For any  $f \in \Gamma(V)$ ,

$$\tilde{\varphi}(f)(t) = (f \circ \varphi)(t) = f(t^2, t^3) \in k[t^2, t^3].$$

Hence,

$$\tilde{\varphi}(\Gamma(V)) \subseteq k[t^2, t^3] \subsetneq k[t] = \Gamma(\mathbf{A}^1).$$

(Here note that  $t \notin k[t^2, t^3]$  but  $t \in k[t]$ .)

□

*Proof of (b).*

- (1) Write

$$Y = \{(t^2 - 1, t(t^2 - 1)) \in \mathbf{A}^2(k) : t \in k\}.$$

Show that  $Y = V$ . Similar to Problem 2.8(a). It suffices to show that  $(x, y) \in Y$  for any  $(x, y) \in V$ . If  $x = 0$ , then  $y = 0$  or  $(x, y) = (0, 0) \in Y$



whenever  $t = \pm 1$ . (In fact,  $(0, 0) = (t^2 - 1, t(t^2 - 1))$  iff  $t^2 - 1 = 0$  iff  $t = \pm 1$  in any field.) If  $x \neq 0$ , define

$$t = \frac{y}{x} \in k.$$

So  $y = tx$  and thus

$$0 = y^2 - x^2(x + 1) = t^2x^2 - x^2(x + 1) = x^2(t^2 - (x + 1)).$$

Since  $x \neq 0$  and  $k$  is a field, we have

$$t^2 - (x + 1) = 0 \iff x = t^2 - 1.$$

Hence,  $y = tx = t(t^2 - 1)$  and therefore  $(x, y) \in Y$ .

- (2) By (1),  $\varphi$  is surjective and  $\varphi(\pm 1) = (0, 0)$ .
- (3) Show that  $\varphi$  is injective except that  $\varphi(\pm 1) = (0, 0)$ . Given  $t, s \in k$ . It suffices to show that  $t = s$  whenever  $(t^2 - 1, t(t^2 - 1)) = (s^2 - 1, s(s^2 - 1)) \neq (0, 0)$ . In fact, by assumption we have  $t^2 - 1 = s^2 - 1 \neq 0$  by assumption. Therefore,

$$t = \frac{t(t^2 - 1)}{t^2 - 1} = \frac{s(s^2 - 1)}{s^2 - 1} = s.$$

□

### Problem 2.13.

Let  $V = V(x^2 - y^3, y^2 - z^3) \subseteq \mathbf{A}^3$  as in Problem 1.40,  $\bar{\alpha} : \Gamma(V) \rightarrow k[t]$  induced by the homomorphism  $\alpha$  of that problem.

- (a) What is the polynomial map  $f$  from  $\mathbf{A}^1$  to  $V$  such that  $\tilde{f} = \bar{\alpha}$ ?
- (b) Show that  $f$  is one-to-one and onto, but not an isomorphism.

*Proof of (a).*

- (1) Write

$$Y = \{(t^9, t^6, t^4) \in \mathbf{A}^3(k) : t \in k\}.$$

Show that  $Y = V$ . Similar to Problem 2.8(a). It suffices to show that  $(x, y, z) \in Y$  for any  $(x, y, z) \in V$ . If  $x = 0$ , then  $y = z = 0$  or  $(x, y, z) = (0, 0, 0) \in Y$  by taking  $t = 0$ . If  $x \neq 0$ , define

$$t = \frac{yz}{x} \in k.$$

Hence,

$$\begin{aligned} t^9 &= \frac{y^9 z^9}{x^9} = \frac{y^{15}}{x^9} = \frac{x^{10}}{x^9} = x, \\ t^6 &= \frac{y^6 z^6}{x^6} = \frac{y^5 z^6}{x^6} y = \frac{y^9}{x^6} y = \frac{x^6}{x^6} y = y, \\ t^4 &= \frac{y^4 z^4}{x^4} = \frac{y^4 z^3}{x^4} z = \frac{y^6}{x^4} z = \frac{x^4}{x^4} z = z. \end{aligned}$$

(2) Define a mapping  $f : \mathbf{A}^1 \rightarrow \mathbf{A}^3$  by

$$f : t \mapsto (t^9, t^6, t^4).$$

$f$  is a polynomial map by construction. By (1),  $f : \mathbf{A}^1 \rightarrow f(\mathbf{A}^1) = V$  and thus  $\tilde{f} = \bar{\alpha}$  by the definition of  $\alpha$ .

□

*Proof of (b).*

(1) Similar to Problem 2.12(a).

(2)  $f$  is surjective by the proof of (a).

(3) *Show that  $f$  is injective.* Suppose  $(t^9, t^6, t^4) = (s^9, s^6, s^4)$  for some  $t, s \in k$ . If  $t = 0$ , then  $s = 0$ . If  $t \neq 0$ , then  $t = \frac{t^6 t^4}{t^9} = \frac{s^6 s^4}{s^9} = s$ . In any case,  $t = s$  whenever  $(t^9, t^6, t^4) = (s^9, s^6, s^4)$ .

(4) *Show that  $f$  is not an isomorphism.* It suffices to show that  $\tilde{f}(\Gamma(V)) \subsetneq \Gamma(\mathbf{A}^1)$  by Proposition 1. For any  $g \in \Gamma(V)$ ,

$$\tilde{f}(g)(t) = (g \circ f)(t) = g(t^9, t^6, t^4) \in k[t^4, t^6, t^9].$$

Hence,

$$\tilde{\varphi}(\Gamma(V)) \subseteq k[t^4, t^6, t^9] \subsetneq k[t] = \Gamma(\mathbf{A}^1).$$

(Here note that  $t \notin k[t^4, t^6, t^9]$  but  $t \in k[t]$ .)

□

## 2.3. Coordinate Changes

### Problem 2.14.\* (Linear subvariety)

A set  $V \subseteq \mathbf{A}^n(k)$  is called a **linear subvariety** of  $\mathbf{A}^n(k)$  if  $V = V(f_1, \dots, f_r)$  for some polynomials  $f_i$  of degree 1.

- (a) Show that if  $t$  is an affine change of coordinates on  $\mathbf{A}^n(k)$ , then  $V^t$  is also a linear subvariety of  $\mathbf{A}^n(k)$ .
- (b) If  $V \neq \emptyset$ , show that there is an affine change of coordinates  $t$  of  $\mathbf{A}^n$  such that  $V^t = V(x_{m+1}, \dots, x_n)$ . (Hint: use induction on  $r$ .) So  $V$  is a variety.
- (c) Show that the  $m$  that appears in part (b) is independent of the choice of  $t$ . It is called the dimension of  $V$ . Then  $V$  is then isomorphic (as a variety) to  $\mathbf{A}^m(k)$ . (Hint: Suppose there were an affine change of coordinates  $t$  such that  $V(x_{m+1}, \dots, x_n)^t = V(x_{s+1}, \dots, x_n)$ ,  $m < s$ ; show that  $t_{m+1}, \dots, t_n$  would be dependent.)

*Proof of (a).*

- (1) Say  $t = (t_1, \dots, t_n)$  is an affine change of coordinates, and  $V = V(f_1, \dots, f_r)$  for some polynomials  $f_i$  of degree 1.
- (2)  $V(f_1, \dots, f_r)$  is the set of all solutions of the system of linear equations:

$$\begin{aligned} f_1 &= a_{11}x_1 + \dots + a_{1n}x_n - b_1 = 0, \\ &\dots \\ f_r &= a_{r1}x_1 + \dots + a_{rn}x_n - b_r = 0. \end{aligned}$$

Write  $Ax = b$  and  $V = V(Ax = b)$  where

$$A = \underbrace{\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{r1} & \dots & a_{rn} \end{pmatrix}}_{\in \mathbf{M}_{r \times n}(k)}, \quad x = \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{\in \mathbf{M}_{n \times 1}(k)}, \quad b = \underbrace{\begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix}}_{\in \mathbf{M}_{r \times 1}(k)}.$$

- (3) Linear algebra says that Gaussian elimination transforms  $(A|b)$  into its reduced row echelon form  $(A'|b')$ . Therefore,  $V(Ax = b) = V(A'x = b')$ .
- (4) If  $V(f_1, \dots, f_r) = \emptyset$ , nothing to do. If  $V(f_1, \dots, f_r) \neq \emptyset$ , then

$$V(f_1, \dots, f_r) = V(g_1, \dots, g_m)$$

where  $m = \text{rank}(A)$  is the number of nonzero rows in  $A'$  ( $m \leq r, n$ ) and  $g_i = a'_{i1}x_1 + \dots + a'_{in}x_n - b'_i$  for  $1 \leq i \leq m$ . ( $a'_{ij}$  is the entry of the matrix  $A'$ .)

- (5) Now given any  $f + I(V) \in k[x_1, \dots, x_n]/I(V)$ , we replace the leading term  $x_{i_1}$  of  $g_1$  by  $x_{i_1} - g_1$  to get

$$f + I(V) = f(x_1, \dots, \underbrace{x_{i_1} - g_1}_{i_1 \text{th position}}, \dots, x_n) + I(V) := f_1 + I(V)$$

where  $f_1 \in k[x_1, \dots, \widehat{x_{i_1}} \dots, x_n]$ . Continue this process to replace each leading term  $x_{i_j}$  of  $g_j$  by  $x_{i_j} - g_j$  to get one by one to get

$$f + I(V) = f_1 + I(V) \text{ where } f_1 \in k[x_1, \dots, \widehat{x_{i_1}} \dots, x_n].$$

...

$$f_{m-1} + I(V) = f_m + I(V) \text{ where } f_m \in k[x_1, \dots, \widehat{x_{i_1}} \dots, \widehat{x_{i_m}} \dots, x_n].$$

Hence, a routine shows that there is a ring isomorphism

$$\alpha : k[x_1, \dots, x_n]/I(V) \rightarrow \underbrace{k[x_1, \dots, \widehat{x_{i_1}} \dots, \widehat{x_{i_m}} \dots, x_n]}_{\text{a domain}}$$

sending  $f$  to  $f_m$ . Therefore,  $V$  is a variety.

- (6) As  $I(V) = (f_1, \dots, f_r)$ ,  $I(V)^t = (f_1^t, \dots, f_r^t)$  where each  $f_i^t$  is linear. Thus  $V^t = V(I(V)^t) = V(f_1^t, \dots, f_r^t)$  is also a linear subvariety of  $\mathbf{A}^n(k)$ .

□

*Proof of (b).*

- (1) Suppose  $A \in M_{r \times n}(k)$  is of rank  $n-m$ . Linear algebra says that there exist invertible matrices  $B \in M_{r \times r}(k)$  and  $C \in M_{n \times n}(k)$  such that  $D = BAC$ , where

$$D = BAC = \underbrace{\begin{pmatrix} O_1 & O_2 \\ O_3 & I_{n-m} \end{pmatrix}}_{\in M_{r \times n}(k)}$$

in which  $I_{n-m} \in M_{(n-m) \times (n-m)}(k)$  is the identity matrix and  $O_1, O_2$ , and  $O_3$  are zero matrices.

- (2) Let  $t'$  be the linear map corresponding to the matrix  $C$ . So

$$\begin{aligned} V^{t'} &= V(Ax = b)^{t'} \\ &= V(ACx = b) \\ &= V(BACx = Bb) && (B: \text{invertible}) \\ &= V(Dx = Bb) \\ &= V(-\beta_1, \dots, -\beta_m, x_{m+1} - \beta_{m+1}, \dots, x_n - \beta_n) && (V \neq \emptyset) \\ &= V(x_{m+1} - \beta_{m+1}, \dots, x_n - \beta_n) \end{aligned}$$

$$\text{where } Bb = \underbrace{\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}}_{\in M_{n \times 1}(k)}.$$

- (3) Let  $t''$  be the translation corresponding to the matrix  $Bb$ . Let  $t = t'' \circ t'$  be the desired affine change of coordinates. Therefore,

$$\begin{aligned} V^t &= (V^{t'})^{t''} \\ &= V(x_{m+1} - \beta_{m+1}, \dots, x_n - \beta_n)^{t''} \\ &= V(x_{m+1}, \dots, x_n). \end{aligned}$$

□

*Proof of (c).*

- (1) Linear algebra says that the rank of any matrix is the maximal number of its linearly independent rows, which is uniquely determined.
- (2)  $V$  is then isomorphic to  $\mathbf{A}^m(k)$  as a variety.

□

## 2.4. Rational Functions and Local Rings

## 2.5. Discrete Valuation Rings

### Problem 2.23.\*

*Show that the order function on  $K$  is independent of the choice of uniformizing parameter.*

*Proof.*

- (1) *Show that a uniformizing parameter is unique up to a unit.* Suppose  $t$  and  $t'$  are two uniformizing parameters for a discrete valuation ring  $R$  with the quotient field  $K$ . Since  $R$  is a DVR, the maximal ideal is

$$\mathfrak{m} = (t) = (s).$$

As  $s \in (t)$ , there is an element  $a \in R$  such that  $s = at$ . As  $s$  is irreducible (by the maximality of  $\mathfrak{m}$ ),  $a$  is a unit or  $t$  is a unit (which is impossible). Hence  $s = at$  for some unit  $a \in R$ .

- (2) For any  $z \in K$ , write

$$z = ut^n = vs^m$$

for some units  $u, v$  and integers  $n \geq m$ . (The case  $n \leq m$  is similar.) Replace  $s = at$  to get  $ut^n = va^mt^m$ . So  $t^{n-m} = u^{-1}va^m$  is a unit. Hence,  $m = n$ , or the order function on  $K$  is independent of the choice of uniformizing parameter.

□

**Problem 2.24.\***

Let  $V = \mathbf{A}^1$ ,  $\Gamma(V) = k[x]$ ,  $K = k(V) = k(x)$ .

- (a) For each  $a \in k = V$ , show that  $\mathcal{O}_a(V)$  is a DVR with uniformizing parameter  $t = x - a$ .
- (b) Show that  $\mathcal{O}_\infty = \{f/g \in k(x) : \deg(g) \geq \deg(f)\}$  is also a DVR, with uniformizing parameter  $t = 1/x$ .

*Proof of (a).*

- (1) By Proposition 7 in §2.4,  $\mathcal{O}_a(V)$  is a (Noetherian) local domain. It suffices to show that  $t = x - a$  is an irreducible element in  $\mathcal{O}_a(V)$  such that every nonzero  $z \in \mathcal{O}_a(V)$  might be written uniquely in the form  $z = ut^n$ ,  $u$  a unit in  $\mathcal{O}_a(V)$ ,  $n$  a nonnegative integer (by Proposition 4).
- (2) Write  $z = f/g \in \mathcal{O}_a(V)$  where  $g(a) \neq 0$ . By Problem 1.7,

$$f = \sum_{i=0}^{\deg(f)} \lambda_i (x - a)^i.$$

Let  $n$  be the smallest integer such that  $\lambda_n \neq 0$ . (Such  $n$  is existed since  $z$  or  $f$  is nonzero.) Hence,  $f = f_1(x - a)^n$  where  $f_1 = \sum_{i=n}^{\deg(f)} \lambda_i (x - a)^{i-n} \neq 0$  and  $f_1(a) = \lambda_n \neq 0$ . So

$$z = f/g = (f_1/g)(x - a)^n.$$

Here  $f_1/g$  is a unit in  $\mathcal{O}_a(V)$ . Besides, it is easy to show that  $n$  is unique by the similar argument in Problem 2.23. Hence,  $\mathcal{O}_a(V)$  is a DVR with uniformizing parameter  $t = x - a$ .

□

*Proof of (b).*

- (1) Show that  $\mathcal{O}_\infty$  is a subring of  $k(x)$ . Clearly,  $1 = 1/1 \in \mathcal{O}_\infty$ . Also, given any  $f = a/b, g = c/d \in \mathcal{O}_\infty$ . So

$$\begin{aligned} f - g &= a/b - c/d = \frac{ad - bc}{bd} \in \mathcal{O}_\infty \\ fg &= a/b \cdot c/d = \frac{ac}{bd} \in \mathcal{O}_\infty \end{aligned}$$

since

$$\begin{aligned}
\deg(ad - bc) &\leq \max(\deg(ad), \deg(bc)) \\
&\leq \max(\deg(a) + \deg(d), \deg(b) + \deg(c)) \\
&\leq \max(\deg(b) + \deg(d), \deg(b) + \deg(d)) \\
&\leq \deg(b) + \deg(d) \\
&\leq \deg(bd)
\end{aligned}$$

and

$$\deg(ac) = \deg(a) + \deg(c) \leq \deg(b) + \deg(d) = \deg(bd).$$

(Here we define  $\deg(0) = -\infty$  by convention.) By the subring test,  $\mathcal{O}_\infty$  is a subring of  $k(x)$ .

- (2) *Show that  $\mathcal{O}_\infty$  is a DVR.* Clearly  $\mathcal{O}_\infty$  is not a field since  $1/x \in \mathcal{O}_\infty$  but  $x = x/1 \notin \mathcal{O}_\infty$ . Let  $t = 1/x$  be an irreducible element of  $\mathcal{O}_\infty$ . ( $\deg(x) = 1$  implies the irreducibility of  $t$ .) Now for any nonzero  $f/g \in \mathcal{O}_\infty$ , write

$$f/g = ((fx^n)/g)(1/x^n) = ((fx^n)/g)t^n$$

where  $n := \deg(g) - \deg(f) \geq 0$ . Note that  $\deg(fx^n) = \deg(f) + n = \deg(g)$ . So  $(fx^n)/g$  is a unit since the inverse  $g/(fx^n)$  is also in  $\mathcal{O}_\infty$ . Besides, it is easy to show that  $n$  is unique by the similar argument in Problem 2.23. Hence,  $\mathcal{O}_\infty$  is a DVR.

□

*Note.*

- (1) The quotient field of  $\mathcal{O}_\infty$  is  $K = k(V) = k(x)$ .
- (2) The set of units in  $\mathcal{O}_\infty(V)$  is  $\{f/g \in k(x) : \deg(g) = \deg(f)\}$ .
- (3) The maximal ideal of  $\mathcal{O}_\infty(V)$  is  $\{f/g \in k(x) : \deg(g) > \deg(f)\}$ .

**Problem 2.25. ( $p$ -adic integers)**

Let  $p \in \mathbb{Z}$  be a prime number. Show that

$$\{r \in \mathbb{Q} : r = a/b, a, b \in \mathbb{Z}, p \text{ doesn't divide } b\}$$

is a DVR with quotient field  $\mathbb{Q}$ .

*Proof.*

- (1) Let

$$\mathbb{Z}_p = \{r \in \mathbb{Q} : r = a/b, a, b \in \mathbb{Z}, p \nmid b\}$$

be the set of all  $p$ -adic integers.

- (2) Show that  $\mathbb{Z}_p$  is a subring of  $\mathbb{Q}$ . Clearly,  $1 = 1/1 \in \mathbb{Z}_p$  (since  $p \nmid 1$ ). Also, given any  $r = a/b, s = c/d \in \mathbb{Z}_p$ . So

$$r - s = a/b - c/d = \frac{ad - bc}{bd} \in \mathbb{Z}_p$$

$$rs = a/b \cdot c/d = \frac{ac}{bd} \in \mathbb{Z}_p$$

since  $p \nmid b, p \nmid d$  and  $p$  is a prime number. By the subring test,  $\mathbb{Z}_p$  is a subring of  $\mathbb{Q}$ .

- (3) Note that  $\mathbb{Z}_p \subseteq \mathbb{Q}$  is a domain and  $\mathbb{Z}_p$  is not a field (since  $p = p/1 \in \mathbb{Z}_p$  but  $p^{-1} = 1/p \notin \mathbb{Z}_p$ ).
- (4) Let  $t = p$  be an irreducible element in  $\mathbb{Z}_p$ . For the irreducibility of  $t = p$ , we write  $p = a/b \cdot c/d = \frac{ac}{bd}$  where  $p \nmid b, p \nmid d$ . So  $pbd = ac$  or

$$1 = \text{ord}_p(ac) = \text{ord}_p(a) + \text{ord}_p(c).$$

Here  $\text{ord}_p : \mathbb{Z} \rightarrow \mathbb{Z}_{\geq 0}$  is defined by  $\text{ord}_p(a) = n$  where  $n$  is the largest number such that  $p^n$  divides  $a$ , that is,  $p^n \mid a$  and  $p^{n+1} \nmid a$ . So  $(\text{ord}_p(a), \text{ord}_p(c)) = (0, 1)$  or  $(1, 0)$ . Hence,  $a/b$  or  $c/d$  is a unit in  $\mathbb{Z}_p$ , or  $p$  is irreducible in  $\mathbb{Z}_p$ .

- (5) For any nonzero  $r = a/b \in \mathbb{Z}_p$ ,  $a \neq 0$  can be written as  $a = p^n c$  for some nonnegative integer  $n$  and  $c \in \mathbb{Z}^+$  uniquely. Hence

$$r = a/b = (c/b)p^n = (c/b)t^n,$$

where  $c/b$  is a unit and  $n$  is a nonnegative integer. Besides, it is easy to show that  $n$  is unique by the similar argument in Problem 2.23. By Proposition 4,  $\mathbb{Z}_p$  is a DVR.

- (6) Show that the quotient field of  $\mathbb{Z}_p$  is  $\mathbb{Q}$ . It suffices to show that  $r$  is in the quotient field of  $\mathbb{Z}_p$  if  $r \in \mathbb{Q} - \mathbb{Z}_p$ . Note that  $r \neq 0$ . Write  $r = a/b$  with  $\gcd(a, b) = 1$ . As  $r \notin \mathbb{Z}_p$ ,  $p \mid b$  and  $p \nmid a$ . Therefore,  $1/r = b/a \in \mathbb{Z}_p$ , or  $r$  is in the quotient field of  $\mathbb{Z}_p$ .

□

*Note.*

- (1)  $p\mathbb{Z}_p$  is the maximal ideal of  $\mathbb{Z}_p$ .
- (2) The residue field  $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$ .



**Problem 2.26.\***

Let  $R$  be a DVR with quotient field  $K$ ; let  $\mathfrak{m}$  be the maximal ideal of  $R$ .

- (a) Show that if  $z \in K$ ,  $z \notin R$ , then  $z^{-1} \in \mathfrak{m}$ .
- (b) Suppose  $R \subseteq S \subseteq K$ , and  $S$  is also a DVR. Suppose the maximal ideal of  $S$  contains  $\mathfrak{m}$ . Show that  $S = R$ .

*Proof of (a).*

- (1) Suppose  $t$  is one uniformizing parameter for  $R$ . If  $z \in K - R$ , then we can write  $z = ut^{-n}$  for some unit  $u \in R$  and  $n \in \mathbb{Z}^+$ .

- (2) Hence,

$$z^{-1} = u^{-1}t^n.$$

Since  $u^{-1}$  is a unit in  $R$  and  $n > 0$ ,  $z^{-1} \in \mathfrak{m}$ .

□

*Proof of (b).*

- (1) (Reductio ad absurdum) Suppose  $z \in S - R \subseteq K - R$ . By (a),  $z^{-1} \in \mathfrak{m}$ . So  $z^{-1}$  is in the maximal ideal  $\mathfrak{m}'$  of  $S$  containing  $\mathfrak{m}$ .

- (2) As  $\mathfrak{m}'$  is an ideal,  $1 = z \cdot z^{-1} \in \mathfrak{m}'$ , which is absurd. Therefore,  $S = R$ .

□

**Problem 2.27.**

Show that the DVR's of Problem 2.24 are the only DVR's with quotient field  $k(x)$  that contain  $k$ . Show that those of Problem 2.25 are the only DVR's with quotient field  $\mathbb{Q}$ .

*Proof (Problem 2.26).*

- (1) Show that  $\mathcal{O}_a(V)$  and  $\mathcal{O}_\infty$  are the only DVR's with quotient field  $k(x)$  that contain  $k$ .

- (a) Let  $k \subseteq R \subsetneq k(x)$  be a DVR with quotient field  $k(x)$ ,  $\mathfrak{m}$  be the unique maximal ideal of  $R$ .  $\mathfrak{m} \neq (0)$  and the set of units in  $R$  is  $R - \mathfrak{m}$ .

- (b) There are two possible cases:  $x \in R$  or  $x \notin R$ .

- (c) Suppose  $x \in R$ . So  $R$  contains  $k[x]$  as a subring. Consider the subset

$$S := \{x - a \in k[x] : a \in k\} \cap \mathfrak{m} \subseteq \mathfrak{m}.$$

Suppose there were two distinct elements  $x - a, x - b \in S$ . Then  $1 \in \mathfrak{m}$ , contrary to the maximality of  $\mathfrak{m}$ . Suppose  $S = \emptyset$ , then every  $x - a$  is a unit in  $R$ . Since  $k = \bar{k}$ ,  $R = k(x)$  is a field, which is absurd. Hence, there is only one  $x - a \in \mathfrak{m}$  for one unique  $a \in k$  and other  $x - b$  with  $b \neq a$  is a unit in  $R$ . Thus,  $R \supseteq \mathcal{O}_a(V)$  and  $\mathfrak{m}$  contains  $(x - a)\mathcal{O}_a(V)$ , which is the maximal ideal of  $\mathcal{O}_a(V)$ . By Problem 2.26,  $R = \mathcal{O}_a(V)$ .

- (d) If  $x \notin R$ , then  $x - a \notin R$  whenever  $a \in k \subseteq R$ . Hence  $(x - a)^{-1} \in \mathfrak{m}$  whenever  $a \in k$  by Problem 2.26(a). Next, given any  $f/g \in \mathcal{O}_\infty$ , by  $k = \bar{k}$  we have

$$f/g = \underbrace{u}_{\in k} \underbrace{\frac{x - \alpha_1}{x - \beta_1}}_{\in R} \cdots \underbrace{\frac{x - \alpha_n}{x - \beta_n}}_{\in R} \underbrace{\frac{1}{x - \beta_{n+1}}}_{\in \mathfrak{m}} \cdots \underbrace{\frac{1}{x - \beta_m}}_{\in \mathfrak{m}},$$

where  $n := \deg(f)$ ,  $m := \deg(g)$  and  $n \leq m$ . Here

$$\frac{x - \alpha_i}{x - \beta_i} = \underbrace{1}_{\in k} + \underbrace{\frac{\beta_i - \alpha_i}{x - \beta_i}}_{\in \mathfrak{m} \subseteq R} \in R.$$

Therefore,  $R \supseteq \mathcal{O}_\infty$  and  $\mathfrak{m}$  contains the maximal ideal  $x^{-1}\mathcal{O}_\infty$  of  $\mathcal{O}_\infty$ . By Problem 2.26,  $R = \mathcal{O}_\infty$ .

- (2) Show that  $\mathbb{Z}_p$  are the only DVR's with quotient field  $\mathbb{Q}$ .

- (a) Let  $R \subsetneq \mathbb{Q}$  be a DVR with quotient field  $\mathbb{Q}$ ,  $\mathfrak{m}$  be the unique maximal ideal of  $R$ .  $\mathfrak{m} \neq (0)$  and the set of units in  $R$  is  $R - \mathfrak{m}$ .  
(b) Note that  $R \subseteq \mathbb{Q}$  contains  $\mathbb{Z}$  as a subring. Consider the subset

$$S := \{p \in \mathbb{Z} : p \text{ is a prime number}\} \cap \mathfrak{m} \subseteq \mathfrak{m}.$$

- (c) Suppose there were two distinct prime integers  $p, q \in S$ . By the Bézout's identity, there exist integers  $a$  and  $b$  such that  $pa + qb = 1$ .  $1 \in \mathfrak{m}$ , contrary to the maximality of  $\mathfrak{m}$ .  
(d) Suppose no prime integer were in  $S$ , then every prime integer is a unit in  $R$ . By the fundamental theorem of arithmetic,  $R = \mathbb{Q}$  is a field, which is absurd.  
(e) By (c)(d),  $p \in \mathfrak{m}$  for one unique prime  $p \in \mathbb{Z}$ . Thus,  $R \supseteq \mathbb{Z}_p$  by the definition of  $\mathbb{Z}_p$  and  $\mathfrak{m}$  contains  $p\mathbb{Z}_p$ , which is the maximal ideal of  $\mathbb{Z}_p$ . By Problem 2.26,  $R = \mathbb{Z}_p$ .

□

**Problem 2.28.\***

An order function on a field  $K$  is a function  $\varphi$  from  $K$  onto  $\mathbb{Z} \cup \{\infty\}$ , satisfying:

- (i)  $\varphi(a) = \infty$  if and only if  $a = 0$ .
- (ii)  $\varphi(ab) = \varphi(a) + \varphi(b)$ .
- (iii)  $\varphi(a + b) \geq \min(\varphi(a), \varphi(b))$ .

Show that  $R = \{z \in K : \varphi(z) \geq 0\}$  is a DVR with maximal ideal  $\mathfrak{m} = \{z \in K : \varphi(z) > 0\}$ , and quotient field  $K$ . Conversely, show that if  $R$  is a DVR with quotient field  $K$ , then the function  $\text{ord} : K \rightarrow \mathbb{Z} \cup \{\infty\}$  is an order function on  $K$ . Giving a DVR with quotient field  $K$  is equivalent to defining an order function on  $K$ .

*Proof.*

- (1) Show that  $\varphi(1) = 0$ . Note that  $\varphi(1) = \varphi(1 \cdot 1) = \varphi(1) + \varphi(1)$  by (ii). By Property (i) of  $\varphi$ , we cancel  $\varphi(1) \in \mathbb{Z}$  on the both side to get  $\varphi(1) = 0$ .
- (2) Show that  $\varphi(-z) = \varphi(z)$  for all  $z \in K$ , and  $\varphi(z^{-1}) = -\varphi(z)$  for all  $z \in K - \{0\}$ . Note that  $\varphi(-1) = 0$  since  $0 = \varphi(1) = \varphi((-1) \cdot (-1)) = \varphi(-1) + \varphi(-1)$  (by (1)). Therefore,

$$\varphi(-z) = \varphi((-1) \cdot z) = \varphi(-1) + \varphi(z) = \varphi(z).$$

Besides,

$$0 = \varphi(1) = \varphi(z z^{-1}) = \varphi(z) + \varphi(z^{-1})$$

if  $z \neq 0$ . So  $\varphi(z^{-1}) = -\varphi(z)$  if  $z \neq 0$ .

- (3) Show that  $R = \{z \in K : \varphi(z) \geq 0\}$  is a ring.

- (a)  $R \neq \emptyset$  since  $1 \in R$ .
- (b) If  $a, b \in R$ , then

$$\varphi(a - b) \geq \min(\varphi(a), \varphi(-b)) = \min(\varphi(a), \varphi(b)) \geq 0$$

(by (2)), or  $a - b \in R$ .

- (c) If  $a, b \in R$ , then  $\varphi(ab) = \varphi(a) + \varphi(b) \geq 0$ .

By the subring test,  $R$  is a subring of  $K$ .

- (4) Show that  $\{z \in K - \{0\} : \varphi(z) = 0\}$  is the set of all units in  $R$ . Given any  $z \in K - \{0\}$ , we have

$$0 = \varphi(z) + \varphi(z^{-1})$$

(by (2)). Hence  $z$  is a unit in  $R$  iff  $z, z^{-1} \in R$  iff  $\varphi(z) = \varphi(z^{-1}) = 0$ .

- (5) Show that  $\mathfrak{m} = \{z \in K : \varphi(z) > 0\}$  is a maximal ideal of  $R$ .
- (a) If  $a, b \in \mathfrak{m}$ , then  $\varphi(a + b) \geq \min(\varphi(a), \varphi(b)) > 0$ .
  - (b) If  $a \in \mathfrak{m}$  and  $r \in R$ , then  $\varphi(ra) = \varphi(r) + \varphi(a) \geq \varphi(a) > 0$ .
  - (c) By (a)(b),  $\mathfrak{m}$  is an ideal of  $R$ .
  - (d) Note that each proper ideal in  $R$  does not have any unit, that is, such proper ideal is contained in  $\{z \in K : \varphi(z) > 0\} = \mathfrak{m}$  exactly (by (4)). Therefore,  $\mathfrak{m}$  is maximal. (Such maximal ideal  $\mathfrak{m}$  is unique and thus  $R$  is a local ring.)
- (6) Show that  $R$  is a DVR. It suffices to show that there is an irreducible element  $t \in R$  such that every nonzero  $z \in R$  may be written uniquely in the form  $z = ut^n$ ,  $u$  a unit in  $R$ ,  $n$  a nonnegative integer. Since  $\varphi$  is surjective, there is an element  $t \in R$  such that  $\varphi(t) = 1$ . Note that  $t \neq 0$  and irreducible (by using Property (ii) of  $\varphi$ ). Hence for any nonzero  $z \in R$  with  $n := \varphi(z) \in \mathbb{Z}$  and  $n \geq 0$ , the order of  $zt^{-n} \in K$  is

$$\varphi(zt^{-n}) = \varphi(z) - n\varphi(t) = n - n \cdot 1 = 0$$

(by (2)). That is,  $zt^{-n} = u$  is a unit in  $R$  (by (4)). Hence  $z = ut^n$  for some unit  $u \in R$  and nonnegative integer  $n$ . Note that  $n$  is uniquely determined by  $\varphi(z)$ . By Proposition 4,  $R$  is a DVR.

- (7) Show that the quotient field of  $R$  is  $K$ . Since  $R$  is a DVR, the quotient field of  $R$  is contained in  $K$ . Conversely, given any  $z \in K$ . If  $\varphi(z) \geq 0$ , then  $z \in R \subseteq K$ . If  $\varphi(z) < 0$ , then  $\varphi(z^{-1}) = -\varphi(z) > 0$  or  $z^{-1} \in R$ . Hence  $z = 1/z^{-1} \in K$  is in the quotient field of  $R$ .
- (8) Show that giving a DVR with quotient field  $K$  is equivalent to defining an order function on  $K$ . It suffices to show that  $\text{ord}(\cdot)$  on  $K$  defines an order function  $\varphi$  on  $K$ . By Problem 2.29, it suffices to show that

$$\text{ord}(a + b) \geq \min(\text{ord}(a), \text{ord}(b))$$

if  $\text{ord}(a) = \text{ord}(b) := n$ . Write  $a = ut^n, b = vt^n$  where  $u, v$  are units in  $R$ . Hence,

$$\begin{aligned} \text{ord}(a + b) &= \text{ord}(ut^n + vt^n) \\ &= \text{ord}((u + v)t^n) \\ &= \text{ord}(u + v) + n \\ &\geq n && (u + v \in R) \\ &= \min(\text{ord}(a), \text{ord}(b)). \end{aligned}$$

□

**Problem 2.29.\***

Let  $R$  be a DVR with quotient field  $K$ , ord the order function on  $K$ .

- (a) If  $\text{ord}(a) < \text{ord}(b)$ , show that  $\text{ord}(a + b) = \text{ord}(a)$ .
- (b) If  $a_1, \dots, a_n \in K$ , and for some  $i$ ,  $\text{ord}(a_i) < \text{ord}(a_j)$  (all  $j \neq i$ ), then  $a_1 + \dots + a_n \neq 0$ .

*Proof of (a).*

- (1) Let  $t$  be a uniformizing parameter for  $R$ . Given any  $a, b \in K$ . Write  $a = ut^n, b = vt^m$  where  $u, v$  are units in  $R$  and  $n, m$  are integers.
- (2) Since  $\text{ord}(a) < \text{ord}(b)$ ,  $n < m$ . Hence,

$$a + b = (u + vt^{m-n})t^n.$$

To show that  $\text{ord}(a + b) = \text{ord}(a) = n$ , it suffices to show that  $u + vt^{m-n}$  is a unit in  $R$ .

- (3) (Reductio ad absurdum) Suppose that  $u + vt^{m-n}$  were not a unit. Since  $R$  is local, the maximal ideal  $(t)$  contains all nonunit elements in  $R$ . Hence,  $u + vt^{m-n} \in (t)$ . As  $m - n > 0$ ,  $vt^{m-n} \in (t)$  and thus a unit  $u \in (t)$ , contrary to the maximality of  $(t)$ .

□

*Proof of (b).*

- (1) Might assume that  $\text{ord}(a_1) < \text{ord}(a_j)$  (all  $j \neq 1$ ). In particular,  $\text{ord}(a_1) < \infty$ .
- (2) Similar to (a). Let  $t$  be a uniformizing parameter for  $R$ . Write  $a_i = u_i t^{m_i}$  where  $u_i$  are units in  $R$  and  $m_i$  are integers. ( $i = 1, \dots, n$ .) Since  $\text{ord}(a_1) < \text{ord}(a_j)$  (all  $j \neq 1$ ),  $m_1 < m_j$ . Hence,

$$a_1 + \dots + a_n = (u_1 + \underbrace{u_2 t^{m_2 - m_1} + \dots + u_n t^{m_n - m_1}}_{\in (t)}) t^{m_1}.$$

So  $u_1 + u_2 t^{m_2 - m_1} + \dots + u_n t^{m_n - m_1}$  is a unit in  $R$ .

- (3) By (1)(2),
$$\text{ord}(a_1 + \dots + a_n) = \text{ord}(a_1) < \infty,$$
or  $a_1 + \dots + a_n \neq 0$  (since ord is an order function on  $K$ ).

□

**Problem 2.30.\***

Let  $R$  be a DVR with maximal ideal  $\mathfrak{m}$ , and quotient field  $K$ , and suppose a field  $k$  is a subring of  $R$ , and that the composition  $k \rightarrow R \rightarrow R/\mathfrak{m}$  is an isomorphism of  $k$  with  $R/\mathfrak{m}$  (as for example in Problem 2.24). Verify the following assertions:

- (a) For any  $z \in R$ , there is a unique  $\lambda \in k$  such that  $z - \lambda \in \mathfrak{m}$ .
- (b) Let  $t$  be a uniformizing parameter for  $R$ ,  $z \in R$ . Then for any  $n \geq 0$  there are unique  $\lambda_0, \lambda_1, \dots, \lambda_n \in k$  and  $z_n \in R$  such that

$$z = \lambda_0 + \lambda_1 t + \lambda_2 t^2 + \cdots + \lambda_n t^n + z_n t^{n+1}.$$

(Hint: For uniqueness use Problem 2.29; for existence use (a) and induction.)

*Proof of (a).*

- (1) Note that

$$k \xrightarrow{i} R \xrightarrow{\pi} R/\mathfrak{m}$$

is an isomorphism.

- (2) For  $z + \mathfrak{m} \in R/\mathfrak{m}$ , there exists the unique  $\lambda \in k$  such that

$$z + \mathfrak{m} = \pi(i(\lambda)) = \pi(\lambda) = \lambda + \mathfrak{m}.$$

So  $z - \lambda \in \mathfrak{m}$  for one unique  $\lambda \in k$ .

□

*Proof of (b).*

- (1) Note that

$$\mathfrak{m} = \{z \in K : \text{ord}(z) > 0\}.$$

By (a),

$$z = \lambda_0 + \underbrace{tz_0}_{\in \mathfrak{m}}$$

for one unique  $\lambda_0 \in k$  and  $z_0 \in R$ . Continue this process or by induction, we have the expression

$$z = \lambda_0 + \lambda_1 t + \lambda_2 t^2 + \cdots + \lambda_n t^n + z_n t^{n+1}.$$

- (2) For the uniqueness, suppose

$$0 = \lambda_0 + \lambda_1 t + \lambda_2 t^2 + \cdots + \lambda_n t^n + z_n t^{n+1}.$$

Note that

$$\text{ord}(\lambda_i t^i) = \begin{cases} \infty & (\lambda_i = 0) \\ i & (\lambda_i \neq 0) \end{cases}$$

since every nonzero element in  $k$  is a unit in  $k \subseteq R$ . Also,  $\text{ord}(z_n t^{n+1}) = \infty$  if  $z_n = 0$ ;  $\text{ord}(z_n t^{n+1}) \geq n+1$  if  $z_n \neq 0$ .

- (3) Suppose  $i_0$  is the smallest integer such that  $\lambda_{i_0} \neq 0$ , then  $\text{ord}(\lambda_{i_0} t^{i_0}) = i_0 < \text{ord}(\lambda_j t^j)$  if  $i_0 \neq j$  and  $\text{ord}(\lambda_{i_0} t^{i_0}) = i_0 < n+1 \leq \text{ord}(z_n t^{n+1})$ . By Problem 2.29(b), such  $i_0$  does not exist. Hence all  $\lambda_i = 0$ . So as  $R$  is a domain,  $z_n$  is also equal to 0. Therefore, the uniqueness is established.

□

**Problem 2.31. (Formal power series)**

Let  $k$  be a field. The ring of **formal power series** over  $k$ , written  $k[[x]]$ , is defined to be

$$\left\{ \sum_{i=0}^{\infty} a_i x^i : a_i \in k \right\}.$$

(As with polynomials, a rigorous definition is best given in terms of sequences  $(a_0, a_1, \dots)$  of elements in  $k$ ; here we allow an infinite number of nonzero terms.) Define the sum by

$$\sum a_i x^i + \sum b_i x^i = \sum (a_i + b_i) x^i,$$

and the (Cauchy) product by

$$\left( \sum a_i x^i \right) \left( \sum b_i x^i \right) = \sum c_i x^i,$$

where  $c_i = \sum_{j+k=i} a_j b_k$ . Show that  $k[[x]]$  is a ring containing  $k[x]$  as a subring. Show that  $k[[x]]$  is a DVR with uniformizing parameter  $x$ . Its quotient field is denoted  $k((x))$ .

*Proof.*

- (1) Two formal power series  $\sum a_i x^i$  and  $\sum b_i x^i$  in  $k[[x]]$  are considered equal if  $a_i = b_i$  for all integers  $i \geq 0$ .
- (2) The zero element in  $k[[x]]$  is  $0 = \sum_{i=0}^{\infty} 0x^i$ , and the multiplicative identity is

$$1 = 1 + 0x + \cdots + 0x^n + \cdots.$$

Hence,  $k[[x]]$  is a ring (by a tedious argument). Moreover,  $k[[x]]$  is a domain (again by a tedious argument).

- (3) Show that  $k[[x]] \supseteq k[x]$ . In fact, for any  $f = \sum_{i=0}^n a_i x^i \in k[x]$ , we can write

$$f = a_0 + a_1 x + \cdots + a_n x^n + 0x^{n+1} + \cdots \in k[[x]].$$

- (4) Show that  $f = \sum_{i=0}^{\infty} a_i x^i$  is a unit in  $k[[x]]$  if and only if  $a_0 \neq 0$ . Suppose  $g = \sum_{i=0}^{\infty} b_i x^i \in k[[x]]$  such that  $fg = 1$ . Then

$$\begin{aligned} 1 &= a_0 b_0, \\ 0 &= \sum_{j=0}^k a_j b_{k-j}. \end{aligned}$$

So  $f$  is not a unit in  $k[[x]]$  if  $a_0 = 0$ . Now if  $a_0 \neq 0$  then  $b_0 := a_0^{-1} \in k$ . Then by observing that

$$\begin{aligned} 0 &= \sum_{j=0}^k a_j b_{k-j} \iff a_0 b_k = - \sum_{j=1}^k a_j b_{k-j} \\ &\iff b_k = -b_0 \sum_{j=1}^k a_j b_{k-j}, \end{aligned}$$

we can solve  $b_1, b_2, \dots$  by induction, and  $(b_0, b_1, \dots)$  gives the existence of  $g \in k[[x]]$ .

- (5) By (4),  $k[[x]]$  is not a field since  $x \in k[[x]]$  but  $x^{-1} \notin k[[x]]$ . Let  $t = x$  be an irreducible element in  $k[[x]]$ . ( $\deg(x) = 1$  implies the irreducibility of  $t$ .) Hence every nonzero  $f \in k[[x]]$  can be written uniquely in the form

$$f = ux^n$$

where  $n$  is the smallest integer such that  $a_n \neq 0$ . By (4),

$$u = a_n + a_{n+1}x + \dots$$

is a unit in  $k[[x]]$  as  $a_n \neq 0$ . Besides, it is easy to show that  $n$  is unique by the similar argument in Problem 2.23. Therefore,  $k[[x]]$  is a DVR with uniformizing parameter  $x$ .

□

### Problem 2.32. (Power series expansion)

Let  $R$  be a DVR satisfying the conditions of Problem 2.30. Any  $z \in R$  then determines a power series  $\sum \lambda_i x^i$ , if  $\lambda_0, \lambda_1, \dots$  are determined as in Problem 2.30(b).

- (a) Show that the map  $z \rightarrow \sum \lambda_i x^i$  is a one-to-one ring homomorphism of  $R$  into  $k[[x]]$ . We often write  $z = \sum \lambda_i t^i$ , and call this the **power series expansion** of  $z$  in terms of  $t$ .



- (b) Show that the homomorphism extends to a homomorphism of  $K$  into  $k((x))$ , and that the order function on  $k((x))$  restricts to that on  $K$ .
- (c) Let  $a = 0$  in Problem 2.24,  $t = x$ . Find the power series expansion of  $z = (1 - x)^{-1}$  and of  $(1 - x)(1 + x^2)^{-1}$  in terms of  $t$ .

*Proof of (a).*

- (1) Define the map  $\alpha : R \rightarrow k[[x]]$  by

$$\alpha : z \mapsto \sum_{i=0}^{\infty} \lambda_i x^i$$

where  $\lambda_i$  are determined as in Problem 2.30(b).

- (2) Show that  $\alpha$  is well-defined and one-to-one. Write

$$\alpha(z) = \sum_{i=0}^{\infty} \lambda_i x^i = \sum_{i=0}^{\infty} \lambda'_i x^i.$$

If there were  $\lambda_n \neq \lambda'_n$  for some  $n$ , then Problem 2.30(b) implies that two expressions of  $z$

$$\begin{aligned} z &= \lambda_0 + \lambda_1 t + \cdots + \lambda_n t^n + z_n t^{n+1} \\ &= \lambda'_0 + \lambda'_1 t + \cdots + \lambda'_n t^n + z'_n t^{n+1} \end{aligned}$$

are the same. That is,  $\lambda_n = \lambda'_n$ , which is absurd. Hence,  $\alpha$  is well-defined. Also,  $0 = 0 + 0t + 0t^2 + \cdots + 0t^n + 0t^{n+1}$  implies that  $\alpha$  is one-to-one.

- (3) Show that  $\alpha$  is addition preserving. Given  $a, b \in R$ . By Problem 2.30(b),

$$a + b = \lambda_0 + \lambda_1 t + \cdots + \lambda_n t^n + c_n t^{n+1}$$

and

$$\begin{aligned} a &= \mu_0 + \mu_1 t + \cdots + \mu_n t^n + a_n t^{n+1} \\ b &= \nu_0 + \nu_1 t + \cdots + \nu_n t^n + b_n t^{n+1} \end{aligned}$$

for any integer  $n \geq 0$ . So

$$a + b = \underbrace{(\mu_0 + \nu_0)}_{\in k} + \underbrace{(\mu_1 + \nu_1)}_{\in k} t + \cdots + \underbrace{(\mu_n + \nu_n)}_{\in k} t^n + \underbrace{(a_n + b_n)}_{\in R} t^{n+1}.$$

Since the expression of  $a + b$  is unique (by Problem 2.30(b)),

$$\lambda_i = \mu_i + \nu_i$$

for all  $i = 0, 1, \dots, n$ . Since  $n$  is arbitrary,  $\lambda_i = \mu_i + \nu_i$  is true for all nonnegative integers. Hence,  $\alpha(a + b) = \alpha(a) + \alpha(b)$ .

- (4) *Show that  $\alpha$  is multiplication preserving.* Given  $a, b \in R$ . By Problem 2.30(b),

$$ab = \lambda_0 + \lambda_1 t + \cdots + \lambda_n t^n + c_n t^{n+1}$$

and

$$a = \mu_0 + \mu_1 t + \cdots + \mu_n t^n + a_n t^{n+1}$$

$$b = \nu_0 + \nu_1 t + \cdots + \nu_n t^n + b_n t^{n+1}$$

for any integer  $n \geq 0$ . So

$$\begin{aligned} ab &= \underbrace{(\mu_0 \nu_0)}_{\in k} + \underbrace{(\mu_1 \nu_0 + \mu_0 \nu_1)}_{\in k} t + \cdots \\ &\quad + \underbrace{(\mu_n \nu_0 + \mu_{n-1} \nu_1 + \cdots + \mu_1 \nu_{n-1} + \mu_0 \nu_n)}_{\in k} t^n \\ &\quad + \underbrace{(\text{other terms})}_{\in R} t^{n+1}. \end{aligned}$$

Since the expression of  $a + b$  is unique (by Problem 2.30(b)),

$$\lambda_i = \sum_{j+k=i} \mu_j \nu_k$$

for all  $i = 0, 1, \dots, n$ . Since  $n$  is arbitrary,  $\lambda_i = \sum_{j+k=i} \mu_j + \nu_k$  is true for all nonnegative integers. Hence,  $\alpha(ab) = \alpha(a)\alpha(b)$ .

- (5) *Show that  $\alpha$  is multiplicative identity preserving.* Note that

$$1 = \underbrace{1}_{\in k} + \underbrace{0}_{\in k} t + \cdots + \underbrace{0}_{\in k} t^n + \underbrace{0}_{\in k} t^{n+1}$$

for every nonnegative integer  $n$ . Hence  $\alpha : 1 \mapsto 1 \in k[[x]]$ .

- (6) By (3)(4)(5),  $\alpha$  is a ring homomorphism.

□

*Proof of (b).*

- (1) Define the mapping  $\beta$  from  $K$  to  $k((x))$  by

$$\beta : a/b \mapsto \alpha(a)/\alpha(b)$$

where  $a, b \in R$  and  $b \neq 0$ .

- (2)  $\beta$  is well-defined since:

- (a)  $\alpha(b) \neq 0$  if  $b \neq 0$  by the injectivity of  $\alpha$ .

- (b) The value of  $\beta(a/b)$  is independent of the choice of  $a/b \in K$  since  $\alpha$  is a ring homomorphism.
- (3) Also,  $\beta$  is a ring homomorphism since  $\alpha$  is a ring homomorphism.
- (4) To show that the order function on  $k((x))$  restricts to that on  $K$ , it suffices to show that

$$\text{ord}_R(z) = \text{ord}_{k[[x]]}(\alpha(z)).$$

In fact,

$$\begin{aligned} m := \text{ord}_R(z) &\iff z = \lambda_m t^m + \cdots + \lambda_n t^n + z_n t^{n+1} \text{ with } \lambda_m \neq 0 \\ &\iff \alpha(z) = \lambda_m x^m + \cdots \text{ with } \lambda_m \neq 0 \\ &\iff \text{ord}_{k[[x]]}(\alpha(z)) = m. \end{aligned}$$

□

*Proof of (c).*

- (1) In calculus we have

$$(1 - x)^{-1} = 1 + x + x^2 + \cdots = \sum_{i=0}^{\infty} x^i$$

for  $|x| < 1$ . In the ring of formal power series  $k[[x]]$ ,  $1 - x$  is a unit (by (4) in the proof of Problem 2.31) and satisfies

$$(1 - x) \left( \sum_{i=0}^{\infty} x^i \right) = 1 \in k[[x]].$$

Hence, the power expansion of  $(1 - x)^{-1}$  is

$$(1 - x)^{-1} = \sum_{i=0}^{\infty} x^i \in k((x)).$$

- (2) Note that  $1 + x^2$  is a unit in  $k[[x]]$  and satisfies

$$(1 + x^2) \left( \sum_{i=0}^{\infty} (-1)^i x^{2i} \right) = 1 \in k[[x]].$$

Hence, the power expansion of  $(1 - x)(1 + x^2)^{-1}$  is

$$\begin{aligned} (1 - x) \left( \sum_{i=0}^{\infty} (-1)^i x^{2i} \right) &= \left( \sum_{i=0}^{\infty} (-1)^i x^{2i} \right) - x \left( \sum_{i=0}^{\infty} (-1)^i x^{2i} \right) \\ &= \sum_{i=0}^{\infty} (-1)^i x^{2i} + \sum_{i=0}^{\infty} (-1)^{i+1} x^{2i+1} \\ &= \sum_{i=0}^{\infty} (-1)^i x^i \in k[[x]]. \end{aligned}$$

□

## 2.6. Forms

### Problem 2.33.

Factor  $y^3 - 2xy^2 + 2x^2y + x^3$  into linear factors in  $\mathbb{C}[x, y]$ .

*Proof.*

- (1) Let  $f(x, y) = y^3 - 2xy^2 + 2x^2y + x^3$ . Then  $f_*(x) = 1 - 2x + 2x^3 + x^3$ .
- (2) Solve  $f_*(x) = 0$  over  $\mathbb{C}$  by WolframAlpha (a computational knowledge engine) to get

$$\begin{aligned}\alpha_1 &= -\frac{2}{3} - \frac{10}{3} \sqrt[3]{\frac{2}{79 - 3\sqrt{249}}} - \frac{1}{3} \sqrt[3]{\frac{79 - 3\sqrt{249}}{2}} \\ \alpha_2 &= -\frac{2}{3} + \frac{5}{3}(1 - \sqrt{3}i) \sqrt[3]{\frac{2}{79 - 3\sqrt{249}}} + \frac{1}{6}(1 + \sqrt{3}i) \sqrt[3]{\frac{79 - 3\sqrt{249}}{2}} \\ \alpha_3 &= -\frac{2}{3} + \frac{5}{3}(1 + \sqrt{3}i) \sqrt[3]{\frac{2}{79 - 3\sqrt{249}}} + \frac{1}{6}(1 - \sqrt{3}i) \sqrt[3]{\frac{79 - 3\sqrt{249}}{2}}.\end{aligned}$$

So  $f_*(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$ .

- (3) Hence,

$$\begin{aligned}f(x, y) &= (f_*)^* \\ &= ((x - \alpha_1)(x - \alpha_2)(x - \alpha_3))^* \\ &= (x - \alpha_1y)(x - \alpha_2y)(x - \alpha_3y).\end{aligned}$$

□

*Note.* If  $f(x, y) = y^3 - 2xy^2 + 2x^2y + 4x^3$ , then

$$f(x, y) = (x - \alpha_1y)(x - \alpha_2y)(x - \alpha_3y)$$

where

$$\begin{aligned}\alpha_1 &= -\frac{1}{6} - \frac{7}{6} \sqrt[3]{\frac{1}{37-3\sqrt{114}}} - \frac{1}{6} \sqrt[3]{37-3\sqrt{114}} \\ \alpha_2 &= -\frac{1}{6} + \frac{7}{12}(1-\sqrt{3}i) \sqrt[3]{\frac{1}{37-3\sqrt{114}}} + \frac{1}{12}(1+\sqrt{3}i) \sqrt[3]{37-3\sqrt{114}} \\ \alpha_3 &= -\frac{1}{6} + \frac{7}{12}(1+\sqrt{3}i) \sqrt[3]{\frac{1}{37-3\sqrt{114}}} + \frac{1}{12}(1-\sqrt{3}i) \sqrt[3]{37-3\sqrt{114}}.\end{aligned}$$

**Problem 2.34.**

Suppose  $f, g \in k[x_1, \dots, x_n]$  are forms of degree  $r, r+1$  respectively, with no common factors ( $k$  a field). Show that  $f+g$  is irreducible.

*Proof.*

- (1) Suppose  $f+g = rs \in k[x_1, \dots, x_n]$ . Proposition 5 implies that

$$(f+g)^* = (rs)^* \implies x_{n+1}f + g = r^*s^*.$$

Note that  $\deg_{x_{n+1}}(x_{n+1}f + g) = 1$ . So  $\deg_{x_{n+1}}(r^*) = 0$  or  $\deg_{x_{n+1}}(s^*) = 0$ . Might assume  $\deg_{x_{n+1}}(r^*) = 0$ . (The case  $\deg_{x_{n+1}}(s^*) = 0$  is similar.)

- (2) Since  $\deg_{x_{n+1}}(r^*) = 0$ ,  $r^* \mid f$  and  $r^* \mid g$ . Note that  $\deg_{x_{n+1}}(r^*) = 0$  implies that  $r^* = r$  is a form in  $k[x_1, \dots, x_n]$ . Hence  $r$  is a common factor of  $f$  and  $g$ , or  $r$  is a constant in  $k[x_1, \dots, x_n]$ . So  $f+g$  is irreducible.

□

**Problem 2.35.\***

- (a) Show that there are  $d+1$  monomials of degree  $d$  in  $R[x, y]$ , and  $1+2+\dots+(d+1) = \frac{(d+1)(d+2)}{2}$  monomials of degree  $d$  in  $R[x, y, z]$ .
- (b) Let  $V(d, n) = \{\text{forms of degree } d \text{ in } k[x_1, \dots, x_n]\}$ ,  $k$  a field. Show that  $V(d, n)$  is a vector space over  $k$ , and that the monomials of degree  $d$  form a basis. So  $\dim V(d, 1) = 1$ ;  $\dim V(d, 2) = d+1$ ;  $\dim V(d, 3) = \frac{(d+1)(d+2)}{2}$ .
- (c) Let  $\ell_1, \ell_2, \dots$  and  $m_1, m_2, \dots$  be sequences of nonzero linear forms in  $k[x, y]$ , and assume no  $\ell_i = \lambda m_j$ ,  $\lambda \in k$ . Let  $A_{ij} = \ell_1 \ell_2 \cdots \ell_i m_1 m_2 \cdots m_j$ ,  $i, j \geq 0$  ( $A_{00} = 1$ ). Show that  $\{A_{ij} : i+j = d\}$  forms a basis for  $V(d, 2)$ .

*Proof of (a).*

- (1) All monomials of degree  $d$  in  $R[x, y]$  are

$$x^d, x^{d-1}y, \dots, xy^{d-1}, y^d,$$

or of the form  $x^i y^j$  with  $i, j \geq 0$  and  $i + j = d$ . So there are  $d + 1$  monomials of degree  $d$  in  $R[x, y]$ .

- (2) Similar to (1), all monomials of degree  $d$  in  $R[x, y, z]$  are of the form  $x^i y^j z^k$  with  $i, j, k \geq 0$  and  $i + j + k = d$ . By the stars and bars (combinatorics) method, there are

$$\binom{d+3-1}{3-1} = \frac{(d+2)(d+1)}{2}$$

monomials of degree  $d$  in  $R[x, y, z]$ .

□

*Proof of (b).*

- (1) To show  $V(d, n)$  is a vector space, it suffices to show that  $V(d, n)$  is a subspace of  $k[x_1, \dots, x_n]$  since  $k[x_1, \dots, x_n]$  is a vector space over  $k$ .
- (2) Note that  $0 \in V(d, n)$  is nonempty. For any  $f, g \in V(d, n)$  and  $a, b \in k$ , we have  $af + bg \in V(d, n)$ . Hence  $V(d, n)$  is subspace.
- (3) Let

$$\mathcal{B} = \{x_1^{i_1} \cdots x_n^{i_n} : i_1, \dots, i_n \geq 0, i_1 + \cdots + i_n = d\}.$$

$\mathcal{B}$  is an independent set, and  $\mathcal{B}$  generates  $V(d, n)$ . So  $\mathcal{B}$  is a basis for  $V(d, n)$ .

- (4) Similar to (a),

$$\dim_k V(d, n) = |\mathcal{B}| = \binom{d+n-1}{n-1}$$

by the stars and bars (combinatorics) method. In particular,  $\dim V(d, 1) = 1$ ;  $\dim V(d, 2) = d + 1$ ;  $\dim V(d, 3) = \frac{(d+1)(d+2)}{2}$ .

□

*Proof of (c).*

- (1) Show that  $\mathcal{B}' := \{A_{ij} : i + j = d\}$  is an independent set. (Reductio ad absurdum) Suppose that there were a nontrivial linear combination of  $A_{ij}$  such that

$$\sum_{i+j=d} c_{ij} A_{ij} = 0.$$

(2) Let  $p$  be the smallest index  $i$  such that  $c_{ij} \neq 0$ . Write  $q := d - p$ . So

$$\begin{aligned}
c_{pq}A_{pq} &= - \sum_{\substack{i+j=d \\ i \neq p, j \neq q}} c_{ij}A_{ij} = - \sum_{\substack{i+j=d \\ i > p, j < q}} c_{ij}A_{ij} \\
\iff A_{pq} &= - \sum_{\substack{i+j=d \\ i > p, j < q}} \frac{c_{ij}}{c_{pq}} A_{ij} \\
\iff \ell_1 \cdots \ell_p m_1 \cdots m_q &= - \sum_{\substack{i+j=d \\ i > p, j < q}} \frac{c_{ij}}{c_{pq}} \ell_1 \cdots \ell_p \ell_{p+1} \cdots \ell_i m_1 \cdots m_j \\
\iff m_1 \cdots m_q &= -\ell_{p+1} \sum_{\substack{i+j=d \\ i > p, j < q}} \frac{c_{ij}}{c_{pq}} \underbrace{\ell_{p+2} \cdots \ell_i}_{:=1 \text{ if } i=p+1} m_1 \cdots m_j \\
\iff \ell_{p+1} &\mid m_1 \cdots m_q.
\end{aligned}$$

Since all  $\ell_i, m_j$  are linear forms,  $\ell_{p+1} \mid m_j$  for some  $1 \leq j \leq q$ , which is absurd since no  $\ell_i = \lambda m_j$ ,  $\lambda \in k$ . Therefore,  $\mathcal{B}'$  is an independent set.

(3) Since

$$|\mathcal{B}'| = d + 1 = \dim_k V(d, 2),$$

$\mathcal{B}'$  is also a basis for  $V(d, 2)$ .

□

### Problem 2.36.

With the above notation, show that

$$\dim V(d, n) = \binom{d+n-1}{n-1},$$

the binomial coefficient.

*Proof.* See the proof of Problem 2.35(b). □

## 2.7. Direct Products of Rings

### Problem 2.37.

What are the additive and multiplicative identities in  $\times R_i$ ? Is the map from  $R_i$  to  $\times R_i$  taking  $a_i$  to  $(0, \dots, a_i, \dots, 0)$  a ring homomorphism?

*Proof.*

- (1)  $(0, \dots, 0)$  is the additive identity in  $\times R_i$ .
- (2)  $(1, \dots, 1)$  is the multiplicative identity in  $\times R_i$ .
- (3) The map  $\alpha : R_i \rightarrow \times R_i$  taking  $a_i$  to  $(0, \dots, a_i, \dots, 0)$  is not a ring homomorphism since

$$\alpha(1) = (0, \dots, 1, \dots, 0) \neq (1, \dots, 1),$$

or  $\alpha$  is not multiplicative identity preserving (if  $R_j$  is not the zero ring for some  $j \neq i$ ).

□

**Problem 2.38.\***

Show that if  $k \subseteq R_i$ , and each  $R_i$  is finite-dimensional over  $k$ , then  $\dim(\times R_i) = \sum \dim(R_i)$ .

*Proof.*

- (1) In the terminology of linear algebra,  $\times R_i$  is the direct sum  $\oplus R_i$  of  $R_i$ .
- (2) Hence,

$$\dim_k \left( \bigoplus R_i \right) = \sum \dim_k(R_i).$$

□

## 2.8. Operations with Ideals

**Problem 2.39.\***

Prove the following relations among ideals  $I_i$ ,  $J$  in a ring  $R$ :

- (a)  $(I_1 + I_2)J = I_1J + I_2J$ .
- (b)  $(I_1 \cdots I_N)^n = I_1^n \cdots I_N^n$ .

*Proof of (a).*

- (1) Note that  $(I_1 + I_2)J$  and  $I_1J + I_2J$  are ideals.



(2) Show that  $(I_1 + I_2)J \subseteq I_1J + I_2J$ . Given any

$$(x_1 + x_2)y \in (I_1 + I_2)J$$

where  $x_i \in I_i$  and  $y \in J$ . It suffices to show that  $(x_1 + x_2)y \in I_1J + I_2J$  (by (1)). In fact,

$$(x_1 + x_2)y = x_1y + x_2y \in I_1J + I_2J.$$

(3) Show that  $(I_1 + I_2)J \supseteq I_1J + I_2J$ . Given any

$$x_1y_1 + x_2y_2 \in I_1J + I_2J$$

where  $x_i \in I_i$  and  $y_i \in J$ . It suffices to show that  $x_1y_1 + x_2y_2 \in (I_1 + I_2)J$  (by (1)). In fact,

$$x_1y_1 + x_2y_2 = (x_1 + \underbrace{0}_{\in I_2})y_1 + (\underbrace{0}_{\in I_1} + x_2)y_2 \in (I_1 + I_2)J$$

since  $(I_1 + I_2)J$  is an ideal.

□

*Proof of (b).*

(1) Note that  $(I_1 \cdots I_N)^n$  and  $I_1^n \cdots I_N^n$  are ideals.

(2) Show that  $(I_1 \cdots I_N)^n \subseteq I_1^n \cdots I_N^n$ . Given any

$$x = x_1 \cdots x_n$$

where  $x_i \in I_1 \cdots I_N$ . It suffices to show that  $x \in I_1^n \cdots I_N^n$  (by (1)). For each  $x_i \in I_1 \cdots I_N$ , write

$$x_i = \sum_{j(i)} x_{j(i),1} \cdots x_{j(i),N}$$

where  $x_{j(i),k} \in I_k$  for  $1 \leq k \leq N$ . Hence

$$\begin{aligned} x &= x_1 \cdots x_n \\ &= \left( \sum_{j(1)} x_{j(1),1} \cdots x_{j(1),N} \right) \cdots \left( \sum_{j(n)} x_{j(n),1} \cdots x_{j(n),N} \right) \\ &= \sum_{j(1), \dots, j(n)} (x_{j(1),1} \cdots x_{j(1),N}) \cdots (x_{j(n),1} \cdots x_{j(n),N}) \\ &= \sum_{j(1), \dots, j(n)} \underbrace{(x_{j(1),1} \cdots x_{j(n),1})}_{\in I_1^n} \cdots \underbrace{(x_{j(1),N} \cdots x_{j(n),N})}_{\in I_N^n} \\ &\in I_1^n \cdots I_N^n. \end{aligned}$$

(3) Show that  $(I_1 \cdots I_N)^n \supseteq I_1^n \cdots I_N^n$ . Given any

$$x = x_1 \cdots x_N \in I_1^n \cdots I_N^n$$

where  $x_i \in I_i^n$  ( $1 \leq i \leq N$ ). It suffices to show that  $x \in (I_1 \cdots I_N)^n$  (by (1)). For each  $x_i \in I_i^n$ , write

$$x_i = \sum_{j(i)} x_{j(i),1} \cdots x_{j(i),n}$$

where  $x_{j(i),k} \in I_i$  for  $1 \leq k \leq n$ . Hence

$$\begin{aligned} x &= x_1 \cdots x_N \\ &= \left( \sum_{j(1)} x_{j(1),1} \cdots x_{j(1),n} \right) \cdots \left( \sum_{j(N)} x_{j(N),1} \cdots x_{j(N),n} \right) \\ &= \sum_{j(1), \dots, j(N)} (x_{j(1),1} \cdots x_{j(1),n}) \cdots (x_{j(N),1} \cdots x_{j(N),n}) \\ &= \sum_{j(1), \dots, j(N)} \underbrace{(x_{j(1),1} \cdots x_{j(N),1})}_{\in I_1 \cdots I_N} \cdots \underbrace{(x_{j(1),n} \cdots x_{j(N),n})}_{\in I_1 \cdots I_N} \\ &\in (I_1 \cdots I_N)^n. \end{aligned}$$

□

**Problem 2.40.\* (Chinese remainder theorem)**

- (a) Suppose  $I, J$  are comaximal ideals in  $R$ . Show that  $I + J^2 = R$ . Show that  $I^m$  and  $J^n$  are comaximal for all  $m, n$ .
- (b) Suppose  $I_1, \dots, I_N$  are ideals in  $R$ , and  $I_i$  and  $J_i = \bigcap_{j \neq i} I_j$  are comaximal for all  $i$ . Show that

$$I_1^n \cap \cdots \cap I_N^n = (I_1 \cdots I_N)^n = (I_1 \cap \cdots \cap I_N)^n$$

for all  $n$ .

*Proof of (a).*

- (1) It suffices to show that  $I^m + J^n = R$ .

- (2) Since  $I^m + J^n \subseteq R$  is always true, it suffices to show that  $I^m + J^n \supseteq R$ .  
In fact,

$$\begin{aligned}
 R &= R^{m+n-1} && (1 \in R) \\
 &= (I + J)^{m+n-1} && (I, J \text{ are comaximal}) \\
 &= \sum_{i=0}^{m+n-1} I^i J^{m+n-1-i} && (\text{Problem 2.39}) \\
 &\subseteq I^m + J^n
 \end{aligned}$$

for all positive integers  $m, n$ . (If  $m = 0$  or  $n = 0$ , then nothing to prove.)

□

*Proof of (b).*

- (1) Show that  $I_i$  and  $I_j$  are comaximal if  $i \neq j$ . Note that

$$R = I_i + J_i \subseteq I_i + I_j \subseteq R$$

if  $i \neq j$ .

- (2) If  $I_i$  is comaximal to  $I_j$  and  $I_{j'}$ . Show that  $I_i$  is also comaximal to  $I_j I_{j'}$ .

$$\begin{aligned}
 R &= (I_i + I_j)(I_i + I_{j'}) \\
 &= I_i(I_i + I_j + I_{j'}) + I_j I_{j'} && (\text{Problem 2.39(a)}) \\
 &\subseteq I_i + I_j I_{j'} \subseteq R.
 \end{aligned}$$

- (3) By (2), it is easy to get that  $I_i$  and  $\prod_{j \neq i} I_j$  are comaximal by induction on the number of  $I_j$  for  $j \neq i$ .

- (4) Show that  $I_1 \cdots I_N = I_1 \cap \cdots \cap I_N$ . Induction on  $N$ .

$$\begin{aligned}
 I_1 \cap \cdots \cap I_N &= I_1 \cap (I_2 \cap \cdots \cap I_N) \\
 &= I_1 \cap (I_2 \cdots I_N) && (\text{Induction hypothesis}) \\
 &= I_1 \cdot (I_2 \cdots I_N) && ((3)) \\
 &= I_1 \cdots I_N.
 \end{aligned}$$

- (5) Note that  $I_i^n$  and  $I_j^n$  are comaximal if  $i \neq j$  by (a). We can apply the same argument in (2)(3)(4) to show that

$$I_1^n \cdots I_N^n = I_1^n \cap \cdots \cap I_N^n.$$

- (6) Therefore,

$$\begin{aligned}
 (I_1 \cap \cdots \cap I_N)^n &= (I_1 \cdots I_N)^n && ((4)) \\
 &= I_1^n \cdots I_N^n && (\text{Problem 2.39(b)}) \\
 &= I_1^n \cap \cdots \cap I_N^n && ((5)).
 \end{aligned}$$

□

**Problem 2.41.\***

Let  $I, J$  be ideals in  $R$ . Suppose  $I$  is finitely generated and  $I \subseteq \text{rad}(J)$ . Show that  $I^n \subseteq J$  for some  $n$ .

*Proof.*

- (1) Let  $I$  be generated by  $x_1, \dots, x_m \in I$ . As  $I \subseteq \text{rad}(J)$ , there are integers  $n_i > 0$  such that  $x_i^{n_i} \in J$ .
- (2) Let  $N = n_1 + \dots + n_m$ . Given any  $x = \sum_{i=1}^m r_i x_i \in I$ , so

$$\begin{aligned} x^N &= \left( \sum_{i=1}^m r_i x_i \right)^N \\ &= \sum_{k_1 + \dots + k_m = N} \binom{N}{k_1, \dots, k_m} r_1^{k_1} x_1^{k_1} \dots r_m^{k_m} x_m^{k_m}. \end{aligned}$$

- (3) Note that for each term there is some  $j$  such that  $k_j \geq n_j$ . Hence,

$$\begin{aligned} x_j^{k_j} &= x_j^{k_j - n_j} x_j^{n_j} \in J && (J \text{ is an ideal}) \\ \implies r_1^{k_1} x_1^{k_1} \dots r_m^{k_m} x_m^{k_m} &\in J \text{ for each term} && (J \text{ is an ideal}) \\ \implies x^N &\in J. && (J \text{ is an ideal}) \\ \implies I^N &\subseteq J. \end{aligned}$$

□

**Supplement.** (Exercise 1.13 in the textbook: Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*.) Suppose that  $I$  is an ideal in a commutative ring. Show that if  $\text{rad}(I)$  is finitely generated, then for some integer  $N$  we have  $(\text{rad}(I))^N \subseteq I$ . Conclude that in a Noetherian ring the ideals  $I$  and  $J$  have the same radical iff there is some integer  $N$  such that  $I^N \subseteq J$  and  $J^N \subseteq I$ . Use the Nullstellensatz to deduce that if  $I, J \subseteq S = k[x_1, \dots, x_n]$  are ideals and  $k$  is algebraically closed, then  $Z(I) = Z(J)$  iff  $I^N \subseteq J$  and  $J^N \subseteq I$  for some  $N$ .

*Proof.*

- (1) Show that if  $\text{rad}(I)$  is finitely generated, then for some integer  $N$  we have  $(\text{rad}(I))^N \subseteq I$ . Say  $x_1, \dots, x_m \in \text{rad}(I)$  generate  $\text{rad}(I)$ .
  - (a) For each  $i$ , there exists an integer  $n_i > 0$  such that  $x_i^{n_i} \in I$  (since  $\text{rad}(I)$  is radical).

(b) Let  $N = n_1 + \cdots + n_m$ . Given any  $x = \sum_{i=1}^m r_i x_i \in \text{rad}(I)$ , so

$$\begin{aligned} x^N &= \left( \sum_{i=1}^m r_i x_i \right)^N \\ &= \sum_{k_1 + \cdots + k_m = N} \binom{N}{k_1, \dots, k_m} r_1^{k_1} x_1^{k_1} \cdots r_m^{k_m} x_m^{k_m}. \end{aligned}$$

(c) Note that for each term there is some  $j$  such that  $k_j \geq n_j$ . Hence,

$$\begin{aligned} x_j^{k_j} &= x_j^{k_j - n_j} x_j^{n_j} \in I && (I \text{ is an ideal}) \\ \implies r_1^{k_1} x_1^{k_1} \cdots r_m^{k_m} x_m^{k_m} &\in I \text{ for each term} && (I \text{ is an ideal}) \\ \implies x^N &\in I. && (I \text{ is an ideal}) \\ \implies (\text{rad}(I))^N &\subseteq I. \end{aligned}$$

(2) Show that in a Noetherian ring the ideals  $I$  and  $J$  have the same radical iff there is some integer  $N$  such that  $I^N \subseteq J$  and  $J^N \subseteq I$ .

(a) ( $\implies$ ) Since in a Noetherian ring every ideal is finitely generated,  $\text{rad}(I)$  and  $\text{rad}(J)$  are finitely generated. By (1), there is a common integer  $N$  such that

$$(\text{rad}(I))^N \subseteq I \quad \text{and} \quad (\text{rad}(J))^N \subseteq J.$$

Note that  $I^N \subseteq (\text{rad}(I))^N$  and  $J^N \subseteq (\text{rad}(J))^N$ . Since  $\text{rad}(I) = \text{rad}(J)$  by assumption,

$$\begin{aligned} I^N &\subseteq (\text{rad}(I))^N = (\text{rad}(J))^N \subseteq J, \\ J^N &\subseteq (\text{rad}(J))^N = (\text{rad}(I))^N \subseteq I. \end{aligned}$$

(b) ( $\impliedby$ ) It suffices to show that  $\text{rad}(I) \subseteq \text{rad}(J)$ .  $\text{rad}(J) \subseteq \text{rad}(I)$  is similar. Given any  $x \in \text{rad}(I)$ , there is an integer  $M > 0$  such that  $x^M \in I$ . Hence  $x^{MN} \in I^N \subseteq J$ , or  $x \in \text{rad}(J)$ .

(3) Show that if  $I, J \subseteq S = k[x_1, \dots, x_n]$  are ideals and  $k$  is algebraically closed, then  $Z(I) = Z(J)$  iff  $I^N \subseteq J$  and  $J^N \subseteq I$  for some  $N$ . Note that  $S$  is Noetherian and we can apply part (2). By the Nullstellensatz,  $Z(I) = Z(J)$  iff  $\text{rad}(I) = \text{rad}(J)$  iff  $I^N \subseteq J$  and  $J^N \subseteq I$  for some  $N$ .

□

#### Problem 2.42.\* (Isomorphism theorems for rings)

(a) Let  $I \subseteq J$  be ideals in a ring  $R$ . Show that there is a natural ring homomorphism from  $R/I$  onto  $R/J$ .

- (b) Let  $I$  be an ideal in a ring  $R$ ,  $R$  a subring of a ring  $S$ . Show that there is a natural ring homomorphism from  $R/I$  to  $S/IS$ .

*Proof of (a).*

- (1) Define a map  $\alpha : R/I \rightarrow R/J$  by  $\alpha(r + I) = r + J$ .
- (2) Show that  $\alpha$  is well-defined. If  $a + I = b + I$ , then  $a - b \in I \subseteq J$  or  $a + J = b + J$ . Hence,  $\alpha(a + I) = a + J = b + J = \alpha(b + I)$ .
- (3) Show that  $\alpha$  is a surjective homomorphism.
  - (a)  $\alpha$  is addition preserving.

$$\begin{aligned}
 \alpha((a + I) + (b + I)) &= \alpha(a + b + I) \\
 &= a + b + J \\
 &= (a + J) + (b + J) \\
 &= \alpha(a + I) + \alpha(b + I).
 \end{aligned}$$

- (b)  $\alpha$  is multiplication preserving.

$$\begin{aligned}
 \alpha((a + I)(b + I)) &= \alpha(ab + I) \\
 &= ab + J \\
 &= (a + J)(b + J) \\
 &= \alpha(a + I)\alpha(b + I).
 \end{aligned}$$

- (c)  $\alpha$  is multiplicative identity preserving.  $\alpha(1 + I) = 1 + J$ .

- (d)  $\alpha$  is surjective since for any  $a + J \in R/J$  there is an element  $a + I \in R/I$  such that  $\alpha(a + I) = a + J$ .

- (4) Note that  $\ker(\alpha) = J/I$ . So  $(R/I)/(J/I) \cong R/J$ .

□

*Proof of (b).*

- (1)  $I$  is not necessary an ideal of  $S$ ;  $IS$  an ideal of  $S$  (and thus  $S/IS$  is well-defined).
- (2) Define a map  $\alpha : R/I \rightarrow S/IS$  by  $\alpha(r + I) = r + IS$ . Note that  $I \subseteq IS$  as a subset in  $S$ . Apply the same argument in (a),  $\alpha$  is well-defined and  $\alpha$  is a surjective homomorphism.
- (3) Note that  $\ker(\alpha) = (R \cap SI)/I$ . So  $(R/I)/((R \cap SI)/I) \cong S/IS$ .

□

**Problem 2.45.\***

Show that ideals  $I, J \subseteq k[x_1, \dots, x_n]$  ( $k$  algebraically closed) are comaximal if and only if  $V(I) \cap V(J) = \emptyset$ .

*Proof.*

(1) Show that  $V(I) \cap V(J) = V(I + J)$ .

$$\begin{aligned} P \in V(I) \cap V(J) &\iff f(P) = 0 \forall f \in I \text{ and } g(P) = 0 \forall g \in J \\ &\iff f(P) = 0 \forall f \in I + J \\ &\iff P \in V(I + J). \end{aligned}$$

(2) Hence,

$$\begin{aligned} \emptyset = V(I) \cap V(J) &\iff \emptyset = V(I + J) && ((1)) \\ &\iff I + J = k[x_1, \dots, x_n] && \text{(Weak Nullstellensatz)} \\ &\iff I \text{ and } J \text{ are comaximal.} \end{aligned}$$

□

**Problem 2.46.\***

Let  $I = (x, y) \subseteq k[x, y]$ . Show that

$$\dim_k(k[x, y]/I^n) = 1 + 2 + \dots + n = \frac{n(n+1)}{2}.$$

*Proof.*

(1) The set

$$\mathcal{B} = \{x^i y^j + I^n : i, j \in \mathbb{Z}, i, j \geq 0, i + j < n\}$$

generates  $k[x, y]/I^n$  as a  $k$ -vector space. Besides, each nonzero element in  $I^n$  has the degree  $\geq n$ , and thus  $\mathcal{B}$  is an independent set. Therefore,  $\mathcal{B}$  is a basis for  $k[x, y]/I^n$ .

(2) Hence,

$$\dim_k(k[x, y]/I^n) = |\mathcal{B}| = 1 + 2 + \dots + n = \frac{n(n+1)}{2}.$$

□

## 2.9. Ideals with a Finite Number of Zeros

### Problem 2.47.

Suppose  $R$  is a ring containing  $k$ , and  $R$  is finite dimensional over  $k$ . Show that  $R$  is isomorphic to a direct product of local rings.

*Proof.*

- (1) Let  $\{v_1, \dots, v_n\}$  be a basis for  $R$  over  $k$  (as a vector space). Define a  $k$ -module homomorphism  $\alpha : k[x_1, \dots, x_n] \rightarrow R$  by  $\alpha(x_i) = v_i$ . Clearly,  $\alpha$  is surjective and thus

$$R \cong k[x_1, \dots, x_n] / \ker(\alpha)$$

as a  $k$ -module isomorphism. Note that  $\ker(\alpha)$  is an ideal of  $k[x_1, \dots, x_n]$ .

- (2) Write  $I := \ker(\alpha)$ . Hence,

$$\dim_k(k[x_1, \dots, x_n]/I) = \dim_k(R) < \infty.$$

By Corollary 4 to the Hilbert's Nullstellensatz in §1.7,  $V(I)$  is finite.

- (3) Write  $V(I) = \{P_1, \dots, P_N\}$  and  $\mathcal{O}_i = \mathcal{O}_{P_i}(\mathbf{A}^n)$ . By Proposition 6,

$$R \cong k[x_1, \dots, x_n]/I \cong \prod_{i=1}^N \mathcal{O}_i / I\mathcal{O}_i,$$

which is isomorphic to a direct product of local rings.

□

## 2.10. Quotient Modules and Exact Sequences

### Problem 2.48.\*

Verify that for any  $R$ -module homomorphism  $\varphi : M \rightarrow M'$ ,  $\ker(\varphi)$  and  $\operatorname{im}(\varphi)$  are submodules of  $M$  and  $M'$  respectively. Show that

$$0 \rightarrow \ker(\varphi) \rightarrow M \xrightarrow{\varphi} \operatorname{im}(\varphi) \rightarrow 0$$

is exact.

*Proof.*



- (1) *Show that  $\ker(\varphi)$  is a subgroup of  $M$ .* It suffices to show that  $a - b \in \ker(\varphi)$  for all  $a, b \in \ker(\varphi)$ . In fact,  $\varphi(a - b) = \varphi(a) - \varphi(b) = 0 - 0 = 0$ , or  $a - b \in \ker(\varphi)$ .
- (2) *Show that  $\ker(\varphi)$  is a submodule of  $M$ .* By (1), it suffices to show that  $ra \in \ker(\varphi)$  for all  $r \in R$  and  $a \in \ker(\varphi)$ . In fact,  $\varphi(ra) = r \cdot \varphi(a) = r \cdot 0 = 0$ , or  $ra \in \ker(\varphi)$ .
- (3) *Show that  $\operatorname{im}(\varphi)$  is a subgroup of  $M'$ .* It suffices to show that  $a - b \in \operatorname{im}(\varphi)$  for all  $a, b \in \operatorname{im}(\varphi)$ . As  $a, b \in \operatorname{im}(\varphi)$ , there are two elements  $a', b' \in M$  such that  $\varphi(a') = a$  and  $\varphi(b') = b$ . So  $\varphi(a' - b') = \varphi(a') - \varphi(b') = a - b$ , or  $a - b \in \operatorname{im}(\varphi)$ .
- (4) *Show that  $\operatorname{im}(\varphi)$  is a submodule of  $M$ .* By (3), it suffices to show that  $ra \in \operatorname{im}(\varphi)$  for all  $r \in R$  and  $a \in \operatorname{im}(\varphi)$ . As  $a \in \operatorname{im}(\varphi)$ , there is one element  $a' \in M$  such that  $\varphi(a') = a$ . So  $\varphi(ra') = r\varphi(a') = ra$ , or  $ra \in \operatorname{im}(\varphi)$ .
- (5) *Show that*

$$0 \rightarrow \ker(\varphi) \xrightarrow{i} M \xrightarrow{\varphi} \operatorname{im}(\varphi) \rightarrow 0$$

*is exact.* Note that  $\ker(\varphi) \xrightarrow{i} M$  is the natural inclusion and  $M \xrightarrow{\varphi} \operatorname{im}(\varphi)$  is surjective. Also, it is trivial that  $\operatorname{im}(i) = \ker(\varphi)$ .

□

### Problem 2.51.

Let

$$0 \longrightarrow V_1 \longrightarrow \cdots \longrightarrow V_n \longrightarrow 0$$

be an exact sequence of finite-dimensional vector spaces. Show that  $\sum (-1)^i \dim(V_i) = 0$ .

*Proof (Proposition 7 in §2.10).*

- (1) For  $i = 0, \dots, n$ , by the rank-nullity theorem for a linear transformation  $\varphi_i : V_i \rightarrow V_{i+1}$ , we have

$$\dim V_i = \dim \operatorname{im}(\varphi_i) + \dim \ker(\varphi_i).$$

(Here  $V_0 = V_{n+1} := 0$  by convention.)

- (2) By the exactness of the sequence, we have

$$(a) \operatorname{im}(\varphi_i) = \ker(\varphi_{i+1}) \text{ for } i = 0, \dots, n-1. \text{ In particular, } \ker(\varphi_1) = \operatorname{im}(\varphi_0) = 0.$$

$$(b) \ker(\varphi_n) = V_n.$$

Hence,

$$\begin{aligned}
\sum_{i=1}^{n-1} (-1)^i \dim(V_i) &= \sum_{i=1}^{n-1} (-1)^i \dim \operatorname{im}(\varphi_i) + \sum_{i=1}^{n-1} (-1)^i \dim \operatorname{ker}(\varphi_i) \\
&= \sum_{i=1}^{n-1} (-1)^i \dim \operatorname{ker}(\varphi_{i+1}) + \sum_{i=1}^{n-1} (-1)^i \dim \operatorname{ker}(\varphi_i) \\
&= (-1)^{n-1} \underbrace{\dim \operatorname{ker}(\varphi_n)}_{=V_n} + (-1)^1 \underbrace{\dim \operatorname{ker}(\varphi_1)}_{=0} \\
&= -(-1)^n \dim V_n,
\end{aligned}$$

or  $\sum (-1)^i \dim(V_i) = 0$ .

□

## 2.11. Free Modules

## Chapter 3: Local Properties of Plane Curves

### 3.1. Multiple Points and Tangent Lines

### 3.2. Multiplicities and Local Rings

### 3.3. Intersection Numbers

## Chapter 4: Projective Varieties

### 4.1. Projective Space

### 4.2. Projective Algebraic Sets

### 4.3. Affine and Projective Varieties

### 4.4. Multiprojective Space

## Chapter 5: Projective Plane Curves

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### 6.1. The Zariski Topology

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### 6.3. Morphisms of Varieties

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### 7.2. Blowing up a Point in $A^2$

### 7.3. Blowing up a Point in $P^2$

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## Chapter 8: Riemann-Roch Theorem

### 8.1. Divisors

### 8.2. The Vector Spaces $L(D)$

### 8.3. Riemann's Theorem

### 8.4. Derivations and Differentials

### 8.5. Canonical Divisors

### 8.6. Riemann-Roch Theorem