

Chapter 2: Number Fields and Number Rings

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Exercise 2.28. Let $f(x) = x^3 + ax + b$, a and $b \in \mathbb{Z}$, and assume f is irreducible over \mathbb{Q} . Let α be a root of f .

(a) Show that $f'(\alpha) = -\frac{2a\alpha+3b}{\alpha}$.

(b) Show that $2a\alpha + 3b$ is a root of

$$\left(\frac{x-3b}{2a}\right)^3 + a\left(\frac{x-3b}{2a}\right) + b.$$

Use this to find $N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(2a\alpha + 3b)$.

(c) Show that $\text{disc}(\alpha) = -(4a^3 + 27b^2)$.

(d) Suppose $\alpha^3 = \alpha + 1$. Prove that $\{1, \alpha, \alpha^2\}$ is an integral basis for $\mathbb{A} \cap \mathbb{Q}[\alpha]$. (See Exercise 2.27(e).) Do the same if $\alpha^3 + \alpha = 1$.

Proof of (a).

(1) Show that $\alpha \neq 0$. If α were 0, then $f(\alpha) = f(0) = b$. So $f(x) = x^3 + ax = x(x^2 + a)$ is reducible, contrary to the irreducibility of f .

(2) Since α be a root of f , $f(\alpha) = 0$, or $\alpha^3 + a\alpha + b = 0$, or $\alpha^3 = -a\alpha - b$.

(3)

$$\begin{aligned} f'(x) = 3x^2 + a &\implies f'(\alpha) = 3\alpha^2 + a \\ &\iff \alpha f'(\alpha) = 3\alpha^3 + a & (\alpha \neq 0) \\ &\iff \alpha f'(\alpha) = 3(-a\alpha - b) + a\alpha & (\alpha^3 = -a\alpha - b) \\ &\iff \alpha f'(\alpha) = -2a\alpha - 3b. \end{aligned}$$

$$\text{So } f'(\alpha) = -\frac{2a\alpha+3b}{\alpha}.$$

□

Proof of (b).

(1) Since $\alpha^3 + a\alpha + b = 0$,

$$\left(\frac{(2a\alpha + 3b) - 3b}{2a}\right)^3 + a\left(\frac{(2a\alpha + 3b) - 3b}{2a}\right) + b = 0.$$

That is, $2a\alpha + 3b$ is a root of $\left(\frac{x-3b}{2a}\right)^3 + a\left(\frac{x-3b}{2a}\right) + b$.

- (2) $N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(2a\alpha + 3b)$ is the product of three roots of $\left(\frac{x-3b}{2a}\right)^3 + a\left(\frac{x-3b}{2a}\right) + b$.
Hence,

$$\begin{aligned} N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(2a\alpha + 3b) &= (2a)^3 \left[\left(\frac{-3b}{2a}\right)^3 + a \cdot \frac{-3b}{2a} + b \right] \\ &= 8a^3 \left[\frac{-27b^3}{8a^3} - \frac{b}{2} \right] \\ &= -27b^3 - 4a^3b. \end{aligned}$$

□

Proof of (c).

$$\begin{aligned} \text{disc}(\alpha) &= (-1)^{\frac{n(n-1)}{2}} N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(f'(\alpha)) && \text{(Theorem 2.8)} \\ &= -N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]} \left(-\frac{2a\alpha + 3b}{\alpha} \right) && (n = 3 \text{ and (a)}) \\ &= \frac{N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(2a\alpha + 3b)}{N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(\alpha)} \\ &= \frac{-27b^3 - 4a^3b}{b} && ((b)) \\ &= -4a^3 - 27b^2. \end{aligned}$$

□

Proof of (d).

- (1) $\alpha^3 = \alpha + 1$. $\alpha^3 - \alpha - 1 = 0$. So $\text{disc}(\alpha) = -23$ (by (c)). Since $\text{disc}(\alpha)$ is squarefree, $\{1, \alpha, \alpha^2\}$ forms an integral basis for $\mathbb{A} \cap \mathbb{Q}[\alpha]$ (Exercise 2.27(e)).
- (2) $\alpha^3 + \alpha = 1$. $\alpha^3 + \alpha - 1 = 0$. So $\text{disc}(\alpha) = -31$ (by (c)). The rest is similar to (1).

□