Notes on the book: *P.J. Hilton and U. Stammbach, A Course in Homological Algebra*

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Chapter I: Modules

§1. Modules

Exercise 1.1. (Diagram chasing)

Complete the proof of Lemma 1.1. Show moreover that α is surjective (resp. injective) if α', α'' are surjective (resp. injective).

Lemma 1.1. Let $0 \to A' \to A \to A'' \to 0$ and $0 \to B' \to B \to B'' \to 0$ be two short exact sequences. Suppose that in the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \xrightarrow{\mu} & A & \xrightarrow{\varepsilon} & A'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha''} & & \\
0 & \longrightarrow & B' & \xrightarrow{\mu'} & B & \xrightarrow{\varepsilon'} & B'' & \longrightarrow & 0
\end{array}$$

any two of the three homomorphisms α', α, α'' are isomorphisms. Then the third is an isomorphism, too.

Proof (Diagram chasing).

(1) *Show that α is surjective if α', α'' are surjective.*

(a) Take any $b \in B$, it suffices to find $a \in A$ such that $\alpha a = b$.

(b) Consider the commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{\varepsilon} \!\!\!\!\rightarrow & A'' \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha''} \\
B & \xrightarrow{\varepsilon'} & B''
\end{array}$$

$\varepsilon' b \in B'$. By the surjectivity of α'', $\exists\, a'' \in A''$ such that $\alpha'' a'' = \varepsilon' b$. By the surjectivity of ε, $\exists\, \overline{a} \in A$ such that $\varepsilon \overline{a} = a''$. Hence

$$\begin{aligned}
\varepsilon'(b - \alpha\overline{a}) &= \varepsilon' b - \varepsilon' \alpha \overline{a} \\
&= \varepsilon' b - \alpha'' \varepsilon \overline{a} && \text{(The diagram commutes)} \\
&= \varepsilon' b - \alpha'' a'' \\
&= \varepsilon' b - \varepsilon' b \\
&= 0.
\end{aligned}$$

(c) Consider the short exact sequence

$$0 \longrightarrow B' \xrightarrow{\mu'} B \xrightarrow{\varepsilon'} B'' \longrightarrow 0$$

As $\varepsilon'(b - \alpha\overline{a}) = 0$, $\exists\, b' \in B'$ such that $\mu' b' = b - \alpha\overline{a}$.

(d) Consider the commutative diagram

$$\begin{tikzcd} A' \arrow[r, "\mu"] \arrow[d, "\alpha'"] & A \arrow[d, "\alpha"] \\ B' \arrow[r, "\mu'"] & B \end{tikzcd}$$

By the surjectivity of α', $\exists\, a' \in A'$ such that $\alpha' a' = b'$. Hence

$$\begin{aligned} \alpha(\mu a' + \overline{a}) &= \alpha\mu a' + \alpha\overline{a} \\ &= \mu'\alpha' a' + \alpha\overline{a} && \text{(The diagram commutes)} \\ &= \mu' b' + \alpha\overline{a} \\ &= (b - \alpha\overline{a}) + \alpha\overline{a} \\ &= b. \end{aligned}$$

Therefore, there exists $a := \mu a' + \overline{a}$ such that $\alpha a = b$.

(2) *Show that α is injective if α', α'' are injective.*

(a) It suffices to show that $\ker\alpha = 0$. Take $a \in \ker\alpha$. ($\alpha(a) = \alpha a = 0$.)

(b) Consider the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\varepsilon"] \arrow[d, "\alpha"] & A'' \arrow[d, "\alpha''"] \\ B \arrow[r, "\varepsilon'"] & B'' \end{tikzcd}$$

we have $0 = \varepsilon'\alpha a = \alpha''\varepsilon a$. By the injectivity of α'', $\varepsilon a = 0$.

(c) Consider the short exact sequence

$$0 \longrightarrow A' \xrightarrow{\mu} A \xrightarrow{\varepsilon} A'' \longrightarrow 0$$

As $\varepsilon a = 0$, $\exists\, a' \in A'$ such that $\mu a' = a$.

(d) Consider the commutative diagram

$$\begin{tikzcd} A' \arrow[r, "\mu"] \arrow[d, "\alpha'"] & A \arrow[d, "\alpha"] \\ B' \arrow[r, tail, "\mu'"] & B \end{tikzcd}$$

$0 = \alpha a = \alpha\mu a' = \mu'\alpha' a'$. By the injectivity of $\mu'\alpha'$, $a' = 0$. Therefore, $a = \mu a' = 0$.

(3) *Suppose α is surjective. Show that α'' is surjective.*

(a) Take any $b'' \in B''$, it suffices to find $a'' \in A''$ such that $\alpha'' a'' = b''$.

(b) Consider the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\varepsilon"] \arrow[d, "\alpha"] & A'' \arrow[d, "\alpha''"] \\
B \arrow[r, two heads, "\varepsilon'"] & B''
\end{tikzcd}$$

By the surjectivity of ε', $\exists\, b \in B$ such that $\varepsilon' b = b''$. By the surjectivity of α, $\exists\, a \in A$ such that $\alpha a = b$. Take $a'' := \varepsilon a \in A''$. Hence

$$\begin{aligned}
\alpha'' a'' &= \alpha'' \varepsilon a \\
&= \varepsilon' \alpha a && \text{(The diagram commutes)} \\
&= \varepsilon' b \\
&= b''.
\end{aligned}$$

(4) *Suppose α' is surjective and α is injective. Show that α'' is injective.*

(a) It suffices to show that $\ker \alpha'' = 0$. Take $a'' \in \ker \alpha''$. ($\alpha''(a'') = \alpha'' a'' = 0$.)

(b) Consider the commutative diagram

$$\begin{tikzcd}
A \arrow[r, two heads, "\varepsilon"] \arrow[d, "\alpha"] & A'' \arrow[d, "\alpha''"] \\
B \arrow[r, "\varepsilon'"] & B''
\end{tikzcd}$$

By the surjectivity of ε, $\exists\, a \in A$ such that $\varepsilon a = a''$. So

$$\begin{aligned}
0 &= \alpha'' a'' \\
&= \alpha'' \varepsilon a \\
&= \varepsilon' \alpha a. && \text{(The diagram commutes)}
\end{aligned}$$

(c) Consider the short exact sequence

$$0 \longrightarrow B' \xrightarrow{\mu'} B \xrightarrow{\varepsilon'} B'' \longrightarrow 0$$

As $\varepsilon'(\alpha a) = 0$, $\exists\, b' \in B'$ such that $\mu' b' = \alpha a$.

(d) Consider the commutative diagram

$$\begin{tikzcd}
A' \arrow[r, "\mu"] \arrow[d, "\alpha'"] & A \arrow[d, "\alpha"] \\
B' \arrow[r, "\mu'"] & B
\end{tikzcd}$$

By surjectivity of α', $\exists\, a' \in A'$ such that $\alpha' a' = b'$. So

$$\begin{aligned}
\alpha a &= \mu' b' \\
&= \mu' \alpha' a' \\
&= \alpha \mu a'. && \text{(The diagram commutes)}
\end{aligned}$$

By the injectivity of α, $a = \mu a'$. Hence

$$a'' = \varepsilon a = \varepsilon \mu a' = 0.$$

Therefore $\ker \alpha'' = 0$.

(5) By (3)(4), α'' is an isomorphism if both α' and α are isomorphisms.

(6) *Suppose α is surjective and α'' is injective. Show that α' is surjective.*

(a) Take any $b' \in B'$, it suffices to find $a' \in A'$ such that $\alpha' a' = b'$. Let $b := \mu' b' \in B$ and note that $\varepsilon' b = 0$ by the exactness of

$$0 \to B' \to B \to B'' \to 0.$$

(b) Consider the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\varepsilon"] \arrow[d, "\alpha"] & A'' \arrow[d, "\alpha''"] \\ B \arrow[r, "\varepsilon'"] & B'' \end{tikzcd}$$

By the surjectivity of α, $\exists\, a \in A$ such that $\alpha a = b$. So

$$\begin{aligned} 0 &= \varepsilon' b \\ &= \varepsilon' \alpha a \\ &= \alpha'' \varepsilon a. && \text{(The diagram commutes)} \end{aligned}$$

By the injectivity of α'', $\varepsilon a = 0$.

(c) Consider the short exact sequence

$$0 \longrightarrow A' \xrightarrow{\mu} A \xrightarrow{\varepsilon} A'' \longrightarrow 0$$

As $\varepsilon a = 0$, $\exists\, a' \in A'$ such that $\mu a' = a$.

(d) Consider the commutative diagram

$$\begin{tikzcd} A' \arrow[r, "\mu"] \arrow[d, "\alpha'"] & A \arrow[d, "\alpha"] \\ B' \arrow[r, tail, "\mu'"] & B \end{tikzcd}$$

Note that

$$\begin{aligned} \mu'(\alpha' a') &= \mu' \alpha' a' \\ &= \alpha \mu a' && \text{(The diagram commutes)} = \alpha a \\ &= b \\ &= \mu' b'. \end{aligned}$$

By the injectivity of μ', $b' = \alpha' a'$ for some $a' \in A'$.

(7) *Suppose α is injective. Show that α' is injective.*

(a) It suffices to show that $\ker \alpha' = 0$. Take $a' \in \ker \alpha'$. ($\alpha'(a') = \alpha' a' = 0$.)

(b) Consider the commutative diagram

$$\begin{tikzcd} A' \arrow[r, tail, "\mu"] \arrow[d, "\alpha'"] & A \arrow[d, "\alpha"] \\ B' \arrow[r, "\mu'"] & B \end{tikzcd}$$

Note that

$$\begin{aligned} 0 &= \mu' 0 \\ &= \mu' \alpha' a' \\ &= \alpha \mu a'. && \text{(The diagram commutes)} \end{aligned}$$

The injectivity of $\alpha\mu$ shows that $a' = 0$.

(8) By (6)(7), α' is an isomorphism if both α and α'' are isomorphisms.

$\square$