# Notes on the book: *Apostol, Introduction to Analytic Number Theory*

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## Contents

# Chapter 1: The Fundamental Theorem of Arithmetic

**Exercise 1.15.**

*Prove that every $n \geq 12$ is the sum of two composite numbers.*

*Proof.* Write $n = 2m$ (resp. $n = 2m + 1$) where $m \in \mathbb{Z}$, $m \geq 6$. Then $n = 8 + 2(m - 4)$ (resp. $n = 9 + 2(m - 4)$) is the sum of two composite numbers. $\square$

**Exercise 1.30.**

*If $n > 1$ prove that the sum*

$$\sum_{k=1}^{n} \frac{1}{k}$$

*is not an integer.*

*Proof.*

(1) (Reductio ad absurdum) Suppose

$$H := \sum_{k=1}^{n} \frac{1}{k}$$

were an integer.

(2) Let $s$ be the largest integer such that $2^s \leq n$. So the integer number

$$\begin{aligned} 2^{s-1}H &= \sum_{k=1}^{n} \frac{2^{s-1}}{k} \\ &= 2^{s-1} + 2^{s-2} + \frac{2^{s-1}}{3} + 2^{s-3} + \frac{2^{s-1}}{5} + \frac{2^{s-2}}{3} + \cdots + \frac{1}{2} + \cdots . \end{aligned}$$

has only one term of even denominators (as $n > 1$) if we write all terms in irreducible fractions. That is,

$$2^{s-1}H = \frac{1}{2} + \frac{c}{d} \in \mathbb{Z}$$

where $\frac{c}{d}$ is an irreducible fraction with odd $d$. Hence it suffices to show that $2 \mid d$ to get a contradiction.

(3) By

$$\frac{1}{2} + \frac{c}{d} = \frac{d + 2c}{2d} \in \mathbb{Z}$$

we have $d + 2c = 2dd'$ for some $d' \in \mathbb{Z}$. Note that 2 is a prime. So $2 \mid (d + 2c)$ or $2 \mid d$, which is absurd.

$\square$

# Chapter 2: Arithmetical functions and Dirichlet multiplication

**Exercise 2.3.**

*Prove that*

$$\frac{n}{\varphi(n)} = \sum_{d|n} \frac{\mu(d)^2}{\varphi(d)}.$$

*Proof.*

(1) Note that $fg$, $f/g$ and $f * g$ are multiplicative if $f$ and $g$ are multiplicative (Example 5 on page 34 and Theorem 2.14). Hence $\frac{n}{\varphi(n)}$ and $\sum_{d|n} \frac{\mu^2(d)}{\varphi(d)}$ are multiplicative. Hence it might assume that $n = p^a$ for some prime $p$ and integer $a \geq 1$. (The case $n = 1$ is trivial.)

(2)

$$\frac{p^a}{\varphi(p^a)} = \frac{p^a}{p^a - p^{a-1}} = \frac{p}{p-1}.$$

(3)

$$\begin{aligned}
\sum_{d|p^a} \frac{\mu(d)^2}{\varphi(d)} &= \frac{\mu(1)^2}{\varphi(1)} + \frac{\mu(p)^2}{\varphi(p)} + \overbrace{\frac{\mu(p^2)^2}{\varphi(p^2)}}^{=0} + \cdots + \overbrace{\frac{\mu(p^a)^2}{\varphi(p^a)}}^{=0} \\
&= 1 + \frac{1}{p-1} + 0 + \cdots + 0 \\
&= \frac{p}{p-1}.
\end{aligned}$$

(4) Or apply Theorems 2.4 and 2.18 to get

$$\begin{aligned}
\sum_{d|n} \frac{\mu(d)^2}{\varphi(d)} &= \prod_{p|n} \left( 1 - \frac{\mu(p)}{\varphi(p)} \right) \\
&= \prod_{p|n} \left( 1 - \frac{-1}{p-1} \right) \\
&= \prod_{p|n} \frac{p}{p-1} \\
&= \frac{n}{\varphi(n)}.
\end{aligned}$$

□

**Supplement 2.3.1. (Chinese remainder theorem)**

(Exercise I.3.5 in the textbook: *Jörgen Neukirch, Algebraic Number Theory.*) *The quotient ring $\mathcal{O}/\mathfrak{a}$ of a Dedekind domain by an ideal $\mathfrak{a} \neq 0$ is a principal ideal domain. (Hint: For $\mathfrak{a} = \mathfrak{p}^n$ the only proper ideals of $\mathcal{O}/\mathfrak{a}$ are given by $\mathfrak{p}/\mathfrak{p}^n, \ldots, \mathfrak{p}^{n-1}/\mathfrak{p}^n$. Choose $\pi \in \mathfrak{p} \smallsetminus \mathfrak{p}^2$ and show that $\mathfrak{p}^\nu = \mathcal{O}\pi^\nu + \mathfrak{p}^n$.)*

*Proof.*

(1) By the Chinese remainder theorem, it suffices to show the case $\mathfrak{a} = \mathfrak{p}^n$ where $\mathfrak{p}$ is prime.

(2) There is a natural correspondence between

$$\{\text{ideals of } \mathcal{O}/\mathfrak{p}^n\} \longleftrightarrow \{\text{ideals of } \mathcal{O} \text{ containing } \mathfrak{p}^n\}.$$

Hence the proper ideals of $\mathcal{O}/\mathfrak{p}^n$ are given by $\mathfrak{p}/\mathfrak{p}^n, \ldots, \mathfrak{p}^{n-1}/\mathfrak{p}^n$.

(3) Similar to Exercise I.3.4, choose $\pi \in \mathfrak{p} \smallsetminus \mathfrak{p}^2$ and thus $\mathfrak{p}^\nu = \mathcal{O}\pi^\nu + \mathfrak{p}^n$ ($\nu = 1, \ldots, n-1$) since they have the same prime factorization. Hence $\mathfrak{p}^\nu/\mathfrak{p}^n = (\pi^\nu + \mathfrak{p}^n)$ is principal.

□

**Exercise 2.4.**

*Prove that $\varphi(n) > \frac{n}{6}$ for all $n$ with at most 8 distinct prime factors.*

*Proof.*

(1)

$$\begin{aligned}
\varphi(n) &= n \prod_{p \mid n} \left(1 - \frac{1}{p}\right) && \text{(Theorem 2.4)} \\
&\geq n \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{7}\right) \\
&\qquad \left(1 - \frac{1}{11}\right)\left(1 - \frac{1}{13}\right)\left(1 - \frac{1}{17}\right)\left(1 - \frac{1}{19}\right) \\
&= \frac{55296}{323323} n \\
&> \frac{n}{6}.
\end{aligned}$$

(2) The conclusion does not hold if $n$ has more than 9 distinct prime factors.

□