Chapter 1: Affine Algebraic Sets

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Problem 1.1. *Let R be a domain.*

(a) *If F, G are forms of degree r, s respectively in $R[X_1, \ldots, X_n]$, show that FG is a form of degree $r+s$.*

(b) *Show that any factor of a form in $R[X_1, \ldots, X_n]$ is also a form.*

Proof of (a). Write

$$F = \sum_{(i)} a_{(i)} X^{(i)},$$
$$G = \sum_{(j)} b_{(j)} X^{(j)},$$

where $\sum_{(i)}$ is the summation over $(i) = (i_1, \ldots, i_n)$ with $i_1 + \cdots + i_n = r$ and $\sum_{(j)}$ is the summation over $(j) = (j_1, \ldots, j_n)$ with $j_1 + \cdots + j_n = s$. Hence,

$$\begin{aligned} FG &= \sum_{(i)} \sum_{(j)} a_{(i)} b_{(j)} X^{(i)} X^{(j)} \\ &= \sum_{(i),(j)} a_{(i)} b_{(j)} X^{(k)} \end{aligned}$$

where $(k) = (i_1+j_1, \ldots, i_n+j_n)$ with $(i_1+j_1)+\cdots+(i_n+j_n) = r+s$. Each $X^{(k)}$ is the form of degree $r+s$ and $a_{(i)}b_{(j)} \in R$. Hence FG is a form of degree $r+s$. $\square$

Proof of (b). Given any form $F \in R[X_1, \ldots, X_n]$, and write $F = GH$. *It suffices to show that G (or H) is a form as well.* Write

$$\begin{aligned} G &= G_0 + \cdots + G_r, \\ H &= H_0 + \cdots + H_s \end{aligned}$$

where $G_r \neq 0$ and $H_s \neq 0$. So

$$F = GH = G_0 H_0 + \cdots + G_r H_s.$$

Since R is a domain, $R[X_1, \ldots, X_n]$ is a domain and thus $G_r H_s \neq 0$. The maximality of r and s implies that $\deg(F) = r+s$. Therefore, by the maximality of $r+s$, $F = G_r H_s$, or $G = G_r$, or G is a form. $\square$