Chapter 6: The Riemann-Stieltjes Integral

Author: Meng-Gen Tsai
Email: plover@gmail.com

Supplement. Another definition of Riemann-Stieltjes integral. *(Exercise 7.3, 7.4 of the book T. M. Apostol, Mathematical Analysis, Second Edition.) Let P be a partition of $[a, b]$. The norm of a partition P is the length of the largest subinterval $[x_{i-1}, x_i]$ of P and is denoted by $\|P\|$.*

We say $f \in \mathscr{R}(\alpha)$ if there exists $A \in \mathbb{R}$ having the property that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any partition P of $[a, b]$ with norm $\|P\| < \delta$ and for any choice of $t_i \in [x_{i-1}, x_i]$, we have $|\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - A| < \varepsilon$.

Claim. *$f \in \mathscr{R}$ in the sense of Definition 6.2 implies that $f \in \mathscr{R}$ in the sense of this another definition.*

Proof of Claim. Let $A = \int f dx$, $M > 0$ be one upper bound of $|f|$ on $[a, b]$. Given $\varepsilon > 0$, there exists a partition $P_0 = \{a = x_0, x_1, \ldots, x_{N-1}, x_N = b\}$ such that $U(P_0, f) \leq A + \frac{\varepsilon}{2}$. Let $\delta = \frac{\varepsilon}{2MN} > 0$. Then for any partition P with norm $\|P\| < \delta$, write

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i = S_1 + S_2,$$

where S_1 is the sum of terms arising from those subintervals of P containing no point of P_0, S_2 is the sum of the remaining terms. Then

$$S_1 \leq U(P_0, f) < A + \frac{\varepsilon}{2},$$
$$S_2 \leq NM\|P\| < NM\delta < \frac{\varepsilon}{2}.$$

Therefore, $U(P, f) < A + \varepsilon$. Similarly, $L(P, f) > A - \varepsilon$ whenever $\|P\| < \delta'$. Hence, $|\sum_{i=1}^{n} f(t_i)\Delta x_i - A| < \varepsilon$ whenever $\|P\| < \min\{\delta, \delta'\}$. (Copy Apostol's hint and ensure $M > 0$. M in Apostol's hint might be zero if $f = 0$.) $\square$

This supplement will be used in computing $\int_0^\infty (\frac{\sin x}{x})^2 dx = \frac{\pi}{2}$ in Exercise 8.12.

Exercise 6.1. *Suppose α increases on $[a, b]$, $a \leq x_0 \leq b$, α is continuous at x_0, $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathscr{R}(\alpha)$ and that $\int f d\alpha = 0$.*

Given any partition $P = \{a = p_0, p_1, \ldots, p_{n-1}, p_n = b\}$, where $a = p_0 \leq p_1 \leq \cdots \leq p_{n-1} \leq p_n = b$. We might compute $L(P, f, \alpha)$ and $U(P, f, \alpha)$ by using ε-δ

argument since we are hinted by the condition that α is continuous. A function which is continuous at x_0 has a nice property near x_0 and this property would help us estimate $U(P, f, \alpha)$ near x_0. On the contrary, if both f and α are discontinuous at x_0, it might be $f \notin \mathscr{R}(\alpha)$. Besides, if f has too many points of discontinuity ($f(x) = 0$ if $x \in \mathbb{Q}$ and $f(x) = 1$ otherwise, for example), then f might not be Riemann-integrable on $[0, 1]$.

Claim 1. $L(P, f, \alpha) = 0$.

Proof of Claim 1. $m_i = 0$ since $\inf f(x) = 0$ on any subinterval of $[a, b]$. So $L(P, f, \alpha) = \sum m_i \Delta\alpha_i = 0$. Here we don't need the condition that α is continuous at x_0. $\square$

Claim 2. *For any $\varepsilon > 0$, there exists a partition P such that $U(P, f, \alpha) < \varepsilon$.*

Proof of Claim 2. Say $x_0 \in [p_{i_0-1}, p_{i_0}]$ for some i_0. Then

$$M_i = \sup_{p_{i-1} \leq x \leq p_i} f(x) = \begin{cases} 0 & \text{if } i \neq i_0, \\ 1 & \text{if } i = i_0. \end{cases}$$

So

$$U(P, f, \alpha) = \sum M_i \Delta\alpha_i = \Delta\alpha_{i_0}.$$

It is not true for any arbitrary α. (For example, α has a jump on $x = x_0$.) In fact, Exercise 6.3 shows this. Luckily, α is continuous at x_0. So for $\varepsilon > 0$, there exists $\delta > 0$ such that $|\alpha(x) - \alpha(x_0)| < \frac{\varepsilon}{2}$ whenever $|x - x_0| < \delta$ (and $x \in [a, b]$). Now we pick a nice partition

$$P = \{a, x_0 - \delta_1, x_0 + \delta_2, b\},$$

where $\delta_1 = \min\{\delta, x_0 - a\} \geq 0$ and $\delta_2 = \min\{\delta, b - x_0\} \geq 0$. (It is a trick about resizing "δ" to avoid considering the edge cases $x_0 = a$ or $x_0 = b$ or $a = b$.) Then $x_0 \in [x_0 - \delta_1, x_0 + \delta_2]$ and $\Delta\alpha$ on $[x_0 - \delta_1, x_0 + \delta_2]$ is

$$\begin{aligned} \alpha(x_0 + \delta_2) - \alpha(x_0 - \delta_1) &= (\alpha(x_0 + \delta_2) - \alpha(x_0)) + (\alpha(x_0) - \alpha(x_0 - \delta_1)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore, $U(P, f, \alpha) < \varepsilon$. $\square$

Proof (Definition 6.2). By Claim 1 and 2 and notice that $U(P, f, \alpha) \geq 0$ for any partition P,

$$\overline{\int_a^b} f d\alpha = \inf U(P, f, \alpha) = 0,$$
$$\underline{\int_a^b} f d\alpha = \sup L(P, f, \alpha) = 0,$$

the inf and sup again being taken over all partitions. Hence $f \in \mathscr{R}(\alpha)$ and that $\int f d\alpha = 0$ by Definition 6.2. □

Proof (Theorem 6.6). By Claim 1 and 2,

$$0 \leq U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$$

Hence $f \in \mathscr{R}(\alpha)$ by Theorem 6.6. Furthermore,

$$\int f d\alpha = \underline{\int_a^b} f d\alpha = \sup L(P, f, \alpha) = 0.$$

□

Proof (Theorem 6.10). $f \in \mathscr{R}(\alpha)$ by Theorem 6.10. Thus, by Claim 1

$$\int f d\alpha = \underline{\int_a^b} f d\alpha = \sup L(P, f, \alpha) = 0.$$

□

Exercise 6.2. *Suppose $f \geq 0$, f is continuous on $[a, b]$, and $\int_a^b f(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$. (Compare with Exercise 6.1.)*

For one application, see Exercise 7.20.

Proof. (Reductio ad absurdum) If there were $p \in [a, b]$ such that $f(p) > 0$. Since f is continuous on $[a, b]$, given $\varepsilon = \frac{1}{64} f(p) > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(p)| \leq \frac{1}{64} f(p) \text{ whenever } |x - p| \leq \delta, x \in [a, b].$$

Hence

$$f(x) \geq \frac{63}{64} f(p)$$

whenever $x \in E = [\max\{a, p - \delta\}, \min\{b, p + \delta\}] \subseteq [a, b]$. Note that the length of E is $|E| > 0$. So

$$0 = \int_a^b f(x)dx \geq \int_E f(x)dx \geq \int_E \frac{63}{64} f(p)dx = \frac{63}{64} f(p)|E| > 0,$$

which is absurd. $\square$

Note. (Lebesgue integral) *Let f be a nonnegative measurable function. Then $\int f = 0$ implies $f = 0$ a.e.*

Exercise 6.3. *Define three functions β_1, β_2, β_3 as follows: $\beta_j(x) = 0$ if $x < 0$, $\beta_j(x) = 1$ if $x > 0$ for $j = 1, 2, 3$; and $\beta_1(0) = 0$, $\beta_2(0) = 1$, $\beta_3(0) = \frac{1}{2}$. Let f be a bounded functions on $[-1, 1]$.*

(a) *Prove that $f \in \mathscr{R}(\beta_1)$ if and only if $f(0+) = f(0)$ and that then*
$$\int f d\beta_1 = f(0).$$

(b) *State and prove a similar result for β_2.*

(c) *Prove that $f \in \mathscr{R}(\beta_3)$ if and only if f is continuous at 0.*

(d) *If f is continuous at 0 prove that*
$$\int f d\beta_1 = \int f d\beta_2 = \int f d\beta_3 = f(0).$$

Proof of (a).

(1) Given any $\delta > 0$, we have
$$|f(x) - f(0)| \leq \sup_{x \in [0,\delta]} f(x) - \inf_{x \in [0,\delta]} f(x)$$
if $x \in [0, \delta]$.

(2) Given any $\varepsilon > 0$ and $\delta > 0$. *Show that if f is bounded and $|f(x) - f(0)| < \varepsilon$ on $[0, \delta]$ then*
$$\sup_{x \in [0,\delta]} f(x) - \inf_{x \in [0,\delta]} f(x) < 2\varepsilon.$$

Since f is bounded, there exists $x_1, x_2 \in [0, \delta]$ such that
$$f(x_1) = \sup_{x \in [0,\delta]} f(x) \quad \text{and} \quad f(x_2) = \inf_{x \in [0,\delta]} f(x).$$

By assumption,
$$f(x_1) - f(x_2) \leq |f(x_1) - f(0)| + |f(0) - f(x_2)| < 2\varepsilon.$$

(3) *Show that* $f \in \mathscr{R}(\beta_1)$ *iff* $f(0+) = f(0)$.

$$
\begin{aligned}
& f \in \mathscr{R}(\beta_1) \\
\Longleftrightarrow & \forall \varepsilon > 0 \text{ there is } P \text{ such that } U(P, f, \beta_1) - L(P, f, \beta_1) < \varepsilon && \text{(Theorem 6.6)} \\
\Longleftrightarrow & \forall \varepsilon > 0 \text{ there is } P \text{ containing } 0 \text{ such that } U(P, f, \beta_1) - L(P, f, \beta_1) < \varepsilon && \text{(Theorem 6.4)} \\
& \text{where } P = \{-1 = x_0 < x_1 < \ldots < x_k = 0 < \ldots < x_n = 1\} \\
\Longleftrightarrow & \forall \varepsilon > 0 \text{ there is } P \text{ containing } 0 \text{ such that } M_{k+1} - m_{k+1} < \varepsilon \\
\Longleftrightarrow & \forall \varepsilon > 0 \text{ there is } P \text{ containing } 0 \text{ such that } \sup_{x \in [0,\delta]} f(x) - \inf_{x \in [0,\delta]} f(x) < \varepsilon \\
& \text{where } [x_k, x_{k+1}] = [0, \delta],\ \delta > 0 \\
& \text{(Take } P = \{-1, 0, \delta, 1\} \text{ in "}\Longleftarrow\text{" direction)} \\
\Longleftrightarrow & \forall \varepsilon > 0 \text{ there is } \delta > 0 \text{ such that } |f(x) - f(0)| < \varepsilon \text{ whenever } x \in [0, \delta] && \text{((1)(2))} \\
& \text{(Replace } \varepsilon \text{ by } \frac{\varepsilon}{2} \text{ in "}\Longleftarrow\text{" direction)} \\
\Longleftrightarrow & \lim_{x \to 0+} f(x) = f(0).
\end{aligned}
$$

(4) *Show that* $\int f d\beta_1 = f(0)$ *if* $f \in \mathscr{R}(\beta_1)$. By (3) and Theorem 6.7,

$$\left| f(0) - \int_a^b f d\beta_1 \right| < \varepsilon.$$

Since ε is arbitrary, $\int f d\beta_1 = f(0)$.

$\square$

Proof of (b). Show that $f \in \mathscr{R}(\beta_2)$ *if and only if* $f(0-) = f(0)$ *and that then*

$$\int f d\beta_2 = f(0).$$

Similar to (a).

(1) Given any $\delta > 0$, we have

$$|f(x) - f(0)| \leq \sup_{x \in [-\delta,0]} f(x) - \inf_{x \in [-\delta,0]} f(x)$$

if $x \in [-\delta, 0]$.

(2) Given any $\varepsilon > 0$ and $\delta > 0$. *Show that if* f *is bounded and* $|f(x) - f(0)| < \varepsilon$ *on* $[-\delta, 0]$ *then*

$$\sup_{x \in [-\delta,0]} f(x) - \inf_{x \in [-\delta,0]} f(x) < 2\varepsilon.$$

Since f is bounded, there exists $x_1, x_2 \in [-\delta, 0]$ such that

$$f(x_1) = \sup_{x \in [-\delta,0]} f(x) \quad \text{and} \quad f(x_2) = \inf_{x \in [-\delta,0]} f(x).$$

By assumption,

$$f(x_1) - f(x_2) \leq |f(x_1) - f(0)| + |f(0) - f(x_2)| < 2\varepsilon.$$

(3) *Show that* $f \in \mathscr{R}(\beta_1)$ *iff* $f(0-) = f(0)$.

$$\begin{aligned}
&f \in \mathscr{R}(\beta_2) \\
\Longleftrightarrow &\forall\varepsilon > 0 \text{ there is } P \text{ such that } U(P, f, \beta_2) - L(P, f, \beta_2) < \varepsilon && \text{(Theorem 6.6)} \\
\Longleftrightarrow &\forall\varepsilon > 0 \text{ there is } P \text{ containing } 0 \text{ such that } U(P, f, \beta_2) - L(P, f, \beta_2) < \varepsilon && \text{(Theorem 6.4)} \\
&\text{where } P = \{-1 = x_0 < x_1 < \ldots < x_k = 0 < \ldots < x_n = 1\} \\
\Longleftrightarrow &\forall\varepsilon > 0 \text{ there is } P \text{ containing } 0 \text{ such that } M_k - m_k < \varepsilon \\
\Longleftrightarrow &\forall\varepsilon > 0 \text{ there is } P \text{ containing } 0 \text{ such that } \sup_{x \in [-\delta, 0]} f(x) - \inf_{x \in [-\delta, 0]} f(x) < \varepsilon \\
&\text{where } [x_{k-1}, x_k] = [-\delta, 0],\ \delta > 0 \\
&\text{(Take } P = \{-1, -\delta, 0, 1\} \text{ in ``}\Longleftarrow\text{'' direction)} \\
\Longleftrightarrow &\forall\varepsilon > 0 \text{ there is } \delta > 0 \text{ such that } |f(x) - f(0)| < \varepsilon \text{ whenever } x \in [-\delta, 0] && \text{((1)(2))} \\
&\text{(Replace } \varepsilon \text{ by } \frac{\varepsilon}{2} \text{ in ``}\Longleftarrow\text{'' direction)} \\
\Longleftrightarrow &\lim_{x \to 0-} f(x) = f(0).
\end{aligned}$$

(4) *Show that* $\int f d\beta_2 = f(0)$ *if* $f \in \mathscr{R}(\beta_2)$. By (3) and Theorem 6.7,

$$\left| f(0) - \int_a^b f d\beta_2 \right| < \varepsilon.$$

Since ε is arbitrary, $\int f d\beta_2 = f(0)$.

□

Proof of (c). Note that f is continuous at 0 iff $f(0+) = f(0-) = f(0)$. Apply the same argument in (a) and (b), we have $f \in \mathscr{R}(\beta_3)$ if and only if $f(0+) = f(0-) = f(0)$. □

Proof of (d). It suffices to show that

$$\int_a^b f d\beta_3 = f(0).$$

We can apply Theorem 6.12(d)(e) to $\beta_3 = \frac{1}{2}(\beta_1 + \beta_2)$. That is,

$$\int_a^b f d\beta_3 = \frac{1}{2}\left[\int_a^b f d\beta_1 + \int_a^b f d\beta_2\right] = \frac{1}{2}[f(0) + f(0)] = f(0).$$

Or apply the same argument in (a) and (b) to get

$$\left| f(0) - \int_a^b f d\beta_3 \right| < \varepsilon$$

for any $\varepsilon > 0$, or $\int_a^b f d\beta_3 = f(0)$. $\square$

Exercise 6.4. *If*

$$f(x) = \begin{cases} 0 & \text{for all irrational } x, \\ 1 & \text{for all rational } x, \end{cases}$$

prove that $f \notin \mathscr{R}$ on $[a, b]$ for any $a < b$.

Proof. Given any partition

$$P = \{a = p_0, p_1, \ldots, p_{n-1}, p_n = b\}$$

of $[a, b]$ where $a = p_0 \leq p_1 \leq \cdots \leq p_{n-1} \leq p_n = b$. Since $a < b$, we might assume that $a = p_0 < p_1 < \cdots < p_{n-1} < p_n = b$ by removing duplicated points. Since $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$ are dense in $\mathbb{R}$, we have

$$\begin{aligned} M_i &= \sup_{p_{i-1} \leq x \leq p_i} f(x) = 1, \\ m_i &= \inf_{p_{i-1} \leq x \leq p_i} f(x) = 0, \\ U(P, f) &= \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} \Delta x_i = b - a, \\ L(P, f) &= \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} 0 = 0. \end{aligned}$$

Since P is arbitrary,

$$\begin{aligned} \overline{\int_a^b} f dx &= \inf U(P, f) = b - a > 0, \\ \underline{\int_a^b} f dx &= \sup L(P, f) = 0. \end{aligned}$$

Hence $f \notin \mathscr{R}$ on $[a, b]$ for any $a < b$. $\square$

Note.

(1) (Lebesgue integral) f is Lebesgue integrable.

(2) $f \in \mathscr{R}$ on $[a, b]$ iff $a = b$.

(3) (Problem 4.1 in *H. L. Royden, Real Analysis, 3rd edition.*) *Construct a sequence $\{f_n\}$ of nonnegative, Riemann integrable functions such that f_n increases monotonically to f. What does this imply about changing the order of integration and the limiting process?* (Since $\mathbb{Q}$ is countable, write

$$\mathbb{Q} = \{r_1, r_2, \ldots\}.$$

Define

$$f_n(x) = \begin{cases} 0 & \text{if } x \not\in \{r_1, \ldots, r_n\}, \\ 1 & \text{if } x \in \{r_1, \ldots, r_n\}. \end{cases}$$

By construction, f_n increases monotonically to f pointwise. Note that $f_n \to f$ not uniformly. Also, $\int_a^b f_n(x)dx = 0$ by using the same argument in Theorem 6.10. Therefore, $\lim_{n\to\infty} \int_a^b f_n(x)dx = 0$ but $\int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx$ does not exist.)

Exercise 6.5. *Suppose f is a bounded real function on $[a, b]$, and $f^2 \in \mathscr{R}$ on $[a, b]$. Does it follow that $f \in \mathscr{R}$? Does the answer change if we assume that $f^3 \in \mathscr{R}$?*

Actually we can omit the boundedness assumption of f since $f^2 \in \mathscr{R}$ or $f^3 \in \mathscr{R}$.

Proof.

(1) *Show that $f^2 \in \mathscr{R}$ on $[a, b]$ does not imply that $f \in \mathscr{R}$ (unless $f \geq 0$ on $[a, b]$).* Similar to Exercise 6.4, define

$$f(x) = \begin{cases} -1 & \text{for all irrational } x, \\ 1 & \text{for all rational } x. \end{cases}$$

$f^2 = 1 \in \mathscr{R}$ on $[a, b]$ but $f \not\in \mathscr{R}$ on $[a, b]$ for any $a < b$. (The proof for the "unless" part is similar to (2).)

(2) *Show that $f^3 \in \mathscr{R}$ on $[a, b]$ implies that $f \in \mathscr{R}$.* Let $\phi(x) = x^{\frac{1}{3}}$ on $\mathbb{R}$. By Theorem 6.11, $f(x) = \phi(f(x)^3) \in \mathscr{R}$. (The boundedness condition in Theorem 6.11 is unnecessary.)

□

Note. (Lebesgue integral) *Suppose that f^2 is Lebesgue integrable. Does it follow that f is Lebesgue integrable? Does the answer change if we assume that f^3 is Lebesgue integrable?* Both answers are no.

Exercise 6.6. *Let P be the Cantor set constructed in Sec. 2.44. Let f be a bounded real function on $[0, 1]$ which is continuous at every point outside P.*

Prove that $f \in \mathscr{R}$ on $[0,1]$. (Hint: P can be covered by finitely many segments whose total length can be made as small as desired. Proceed as in Theorem 6.10.)

Proof (Theorem 6.10). Given any $\varepsilon > 0$.

(1) Note that in Section 2.44, we have

$$P = \bigcap_{n=1}^{\infty} E_n$$

and each E_n is the union of 2^n intervals, each of length $\frac{1}{3^n}$. For each interval $[u_j, v_j] \subseteq E_n$ of E_n $(1 \leq j \leq 2^n)$, we construct a slightly larger open set

$$(u_j - \lambda, v_j + \lambda) \supsetneq [u, v]$$

where $\lambda = \frac{1}{2}\left(\frac{1}{2.28^n} - \frac{1}{3^n}\right) > 0$. Each length of $(u_j - \lambda, v_j + \lambda)$ is $\frac{1}{2.28^n}$. Write

$$G_n = \bigcup_{1 \leq j \leq 2^n} (u_j - \lambda, v_j + \lambda).$$

Hence

$$G_n \supsetneq \bigcup_{1 \leq j \leq 2^n} [u_j, v_j] = E_n \supseteq P,$$

and the total length $|G_n|$ of G_n satisfies

$$|G_n| \leq \sum_{1 \leq j \leq 2^n} |(u_j - \lambda, v_j + \lambda)| = \left(\frac{2}{2.28}\right)^n.$$

(Two different subintervals might be overlapped.) As $n \to \infty$, P can be covered by finitely many open segments whose total length can be made as small as desired. Now we take an integer N such that $\left(\frac{2}{2.28}\right)^N < \frac{\varepsilon}{64(M+1)}$.

(2) Let $K = [0,1] - G_N$ be a compact set (Theorem 2.35). By construction, f is continuous on K and thus f is uniformly continuous. So there is $\delta > 0$ such that $|f(s) - f(t)| < \frac{\varepsilon}{89}$ if $s, t \in K$ and $|s - t| < \delta$.

(3) Now we construct a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$, as the following steps:

   (a) Put $\frac{0}{m}, \frac{1}{m}, \ldots, \frac{m}{m}$ in P for some integer $m \geq \frac{1}{\delta}$.

   (b) Put $u_j - \lambda$ and $v_j + \lambda$ in P.

   (c) Remove any points in the segment $(u_j - \lambda, v_j + \lambda)$ except 0 and 1.

(4) Note that $M_i - m_i \leq 2M$ $(1 \leq i \leq n)$ where $M = \sup |f(x)|$ is defined. Hence,

$$U(P, f) - L(P, f) \leq \frac{\varepsilon}{89} + 2M \cdot \frac{\varepsilon}{64(M+1)} \leq \varepsilon.$$

Since ε is arbitrary, Theorem 6.6 shows that $f \in \mathscr{R}$.

□

Supplement (Lebesgue's criterion for Riemann-integrability). *Let f be a bounded real function on $[a,b]$ and let D be the set of discontinuities of f in $[a,b]$. Then $f \in \mathscr{R}$ on $[a,b]$ if and only if D has measure zero.*

For a proof, see Theorem 7.48 in *Tom M. Apostol, Mathematical Analysis, 2nd edition.*

Exercise 6.7. *Suppose f is a real function on $(0,1]$ and $f \in \mathscr{R}$ on $[c,1]$ for every $c > 0$. Define*

$$\int_0^1 f(x)dx = \lim_{c \to 0} \int_c^1 f(x)dx$$

if this limit exists (and is finite).

(a) *If $f \in \mathscr{R}$ on $[0,1]$, show that this definition of the integral agrees with the old one.*

(b) *Construct a function such that the above limit exists, although it fails to exist with $|f|$ in place of f.*

Proof of (a).

(1) Since $f \in \mathscr{R}$ on $[0,1]$, f is bounded or $|f| \leq M$ for some real M.

(2) For any $0 < c < 1$, we have

$$\begin{aligned} \left| \int_0^1 f(x)dx - \int_c^1 f(x)dx \right| &= \left| \int_0^c f(x)dx \right| && \text{(Theorem 6.12(c))} \\ &\leq Mc. && \text{(Theorem 6.12(d))} \end{aligned}$$

(3) Given any $\varepsilon > 0$, there exists $\delta = \frac{\varepsilon}{M+1} > 0$ such that

$$\left| \int_0^c f(x)dx - \int_0^1 f(x)dx \right| \leq Mc < M\delta = M \cdot \frac{\varepsilon}{M+1} < \varepsilon$$

whenever $0 < c < \delta$. Hence $\lim_{c \to 0} \int_0^c f(x)dx = \int_0^1 f(x)dx$.

□

Proof of (b)(Construct by nonabsolutely convergent series).

(1) Given any nonabsolutely (conditionally) convergent series $\sum_{k=1}^n a_k$ (take $\sum \frac{(-1)^n}{n}$ for example and then see Remark 3.46), we define f on $(0,1]$ by

$$f(x) = 2^n a_n$$

if $\frac{1}{2^n} < x \leq \frac{1}{2^{n-1}}$ as $n = 1, 2, \ldots$.

(2) By construction,

$$\int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} f(x)dx = \left(\frac{1}{2^{n-1}} - \frac{1}{2^n}\right) 2^n a_n = a_n.$$

and thus

$$\int_{\frac{1}{2^n}}^{1} f(x)dx = \int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} f(x)dx + \cdots + \int_{\frac{1}{2}}^{1} f(x)dx = \sum_{k=1}^{n} a_k.$$

(3) Given any $\varepsilon > 0$. Since $\sum a_n$ is convergent, there exists a common integer N such that

$$|a_n| \leq \frac{\varepsilon}{89}$$

and

$$\left|\sum_{k=1}^{n} a_k - A\right| \leq \frac{\varepsilon}{64}$$

for some real A whenever $n \geq N$ (Definition 3.21 and Theorem 3.23). Therefore, for any $0 < c \leq \frac{1}{2^N}$, say $\frac{1}{2^{n+1}} < c \leq \frac{1}{2^n} \leq \frac{1}{2^N}$ for some $n \geq N$, we have

$$\begin{aligned}
\left|\int_c^1 f(x)dx - A\right| &= \left|\int_c^{\frac{1}{2^n}} f(x)dx + \int_{\frac{1}{2^n}}^{1} f(x)dx - A\right| \\
&\leq \left|\left(\frac{1}{2^n} - c\right) 2^{n+1} a_{n+1}\right| + \left|\sum_{k=1}^{n} a_k - A\right| \\
&\leq |a_{n+1}| + \left|\sum_{k=1}^{n} a_k - A\right| \\
&\leq \frac{\varepsilon}{89} + \frac{\varepsilon}{64} \\
&\leq \varepsilon.
\end{aligned}$$

Hence, $\lim_{c \to 0} \int_c^1 f(x)dx = A$ exists.

(4) Since

$$\int_{\frac{1}{2^n}}^{1} |f(x)|dx = \int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} |f(x)|dx + \cdots + \int_{\frac{1}{2}}^{1} |f(x)|dx = \sum_{k=1}^{n} |a_k| \to \infty$$

as $n \to \infty$, $\lim_{c \to 0} \int_c^1 f(x)dx$ does not exist. (Or show that $\lim_{c \to 0} \int_c^1 f(x)dx = \infty$ by definition directly.)

$\square$

Exercise 6.8. *Suppose $f \in \mathscr{R}$ on $[a, b]$ for every $b > a$ where a is fixed. Define*

$$\int_a^\infty f(x)dx = \lim_{b \to \infty} \int_a^b f(x)dx$$

if this limit exists (and is finite). In that case, we say that the integral on the left ***converges****. If it also converges after f has been replaced by $|f|$, it is said to converge* ***absolutely****. Assume that $f(x) \geq 0$ and that f decreases monotonically on $[1, \infty)$. Prove that*

$$\int_1^\infty f(x)dx$$

converges if and only if

$$\sum_{n=1}^\infty f(n)$$

converges. (This is the so-called "integral test" for convergence of series.)

Proof. Similar to Exercise 8.9.

(1) Define

$$\begin{aligned} a_n &= \int_1^n f(x)dx, \\ b_n &= \sum_{k=1}^n f(k), \\ c_n &= b_n - a_n \end{aligned}$$

for $n = 1, 2, 3, \ldots$.

(2) *Show that $\{c_n\}$ decreases.* Since f decreases monotonically on $[1, \infty)$, we have

$$\begin{aligned} c_n - c_{n+1} &= (b_n - a_n) - (b_{n+1} - a_{n+1}) \\ &= (a_{n+1} - a_n) - (b_{n+1} - b_n) \\ &= \int_n^{n+1} f(x)dx - f(n+1) \\ &\geq \int_n^{n+1} f(n+1)dx - f(n+1) \\ &= f(n+1) - f(n+1) \\ &= 0. \end{aligned}$$

(3) *Show that $\{c_n\}$ is bounded.* Since f decreases monotonically on $[1, \infty)$,

$$\begin{aligned}
c_n &= b_n - a_n \\
&= \sum_{k=1}^{n} f(k) - \int_1^n f(x)dx \\
&= \sum_{k=1}^{n} f(k) - \sum_{k=1}^{n-1} \int_k^{k+1} f(x)dx \\
&\geq \sum_{k=1}^{n} f(k) - \sum_{k=1}^{n-1} \int_k^{k+1} f(k)dx \\
&= \sum_{k=1}^{n} f(k) - \sum_{k=1}^{n-1} f(k) \\
&= f(n).
\end{aligned}$$

Since $f(n)$ is nonnegative, $c_n \geq 0$.

(4) By (2)(3), $\{c_n\}$ converges (Theorem 3.14).

(5) Since $c_n = b_n - a_n$ and $\{c_n\}$ converges, $\{a_n\}$ converges if and only if $\{b_n\}$ converges, or $\int_1^\infty f(x)dx$ converges if and only if $\sum_{n=1}^\infty f(n)$ converges.

□

Exercise 6.9. *Show that integration by parts can sometimes be applied to the "improper" integrals defined in Exercise 6.7 and 6.8. (State appropriate hypotheses, formulate a theorem, and prove it.) For instance show that*

$$\int_0^\infty \frac{\cos x}{1+x} dx = \int_0^\infty \frac{\sin x}{(1+x)^2} dx.$$

Show that one of these integrals converges ***absolutely****, but that the other does not.*

Proof.

(1) *Suppose F and G are differentiable functions on $(0, 1]$, $F' = f \in \mathscr{R}$ on $[c, 1]$ and $G' = g \in \mathscr{R}$ on $[c, 1]$ for every $c > 0$. Then*

$$\int_0^1 F(x)g(x)dx = F(1)G(1) - \lim_{c \to 0} F(c)G(c) - \int_0^1 f(x)G(x)dx$$

if any two of $\int_0^1 F(x)g(x)dx$, $\int_0^1 f(x)G(x)dx$ or $\lim_{c\to 0} F(c)G(c)$ exist. Theorem 6.22 (integration by parts) implies that

$$\int_c^1 F(x)g(x)dx = F(1)G(1) - F(c)G(c) - \int_c^1 f(x)G(x)dx.$$

Since any two of $\int_0^1 F(x)g(x)dx$ or $\int_0^1 f(x)G(x)dx$ or $\lim_{c\to 0} F(c)G(c)$ exist, the rest one exists and satisfies the identity

$$\int_0^1 F(x)g(x)dx = F(1)G(1) - \lim_{c\to 0} F(c)G(c) - \int_0^1 f(x)G(x)dx$$

by letting $c \to 0$.

(2) *Suppose F and G are differentiable functions on $[a,b]$ for every $b > a$ where a is fixed, $F' = f \in \mathscr{R}$ on $[a,b]$ and $G' = g \in \mathscr{R}$ on $[a,b]$. Then*

$$\int_a^\infty F(x)g(x)dx = \lim_{b\to\infty} F(b)G(b) - F(a)G(a) - \int_a^\infty f(x)G(x)dx$$

if any two of $\int_a^\infty F(x)g(x)dx$, $\int_a^\infty f(x)G(x)dx$ or $\lim_{b\to\infty} F(b)G(b)$ exist.
Theorem 6.22 (integration by parts) implies that

$$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx.$$

Since any two of $\int_a^\infty F(x)g(x)dx$ or $\int_a^\infty f(x)G(x)dx$ or $\lim_{b\to\infty} F(b)G(b)$ exist, the rest one exists and satisfies the identity

$$\int_a^\infty F(x)g(x)dx = \lim_{b\to\infty} F(b)G(b) - F(a)G(a) - \int_a^\infty f(x)G(x)dx$$

by letting $b \to \infty$.

(3) *Show that*

$$\int_0^\infty \frac{\cos x}{1+x}dx = \int_0^\infty \frac{\sin x}{(1+x)^2}dx.$$

Put $a = 0$, $F(x) = \frac{1}{1+x}$ and $G(x) = \sin x$ in

$$\int_a^\infty F(x)g(x)dx = \lim_{b\to\infty} F(b)G(b) - F(a)G(a) - \int_a^\infty f(x)G(x)dx$$

to get

$$\int_0^\infty \frac{(\sin x)'}{1+x}dx = \lim_{b\to\infty} \frac{\sin(b)}{1+b} - \frac{\sin(0)}{1+0} - \int_0^\infty \left(\frac{1}{1+x}\right)' \sin x dx$$

or

$$\int_0^\infty \frac{\cos x}{1+x}dx = \int_0^\infty \frac{\sin x}{(1+x)^2}dx.$$

(4) *Show that*

$$\int_0^\infty \frac{\sin x}{(1+x)^2}dx$$

converges absolutely. Notice that

$$\begin{aligned}\int_0^\infty \left|\frac{\sin x}{(1+x)^2}\right| dx &\leq \int_0^\infty \frac{1}{(1+x)^2} dx \\ &= \lim_{b\to\infty} \left[-\frac{1}{1+x}\right]_0^b - (-1) \\ &= 1.\end{aligned}$$

(5) *Show that*

$$\int_0^\infty \frac{\cos x}{1+x} dx$$

converges conditionally. By (3)(4), $\int_0^\infty \frac{\cos x}{1+x} dx$ converges. Note that

$$\cos x \geq \frac{1}{2}$$

if $x \in [-\frac{\pi}{3} + 2n\pi, \frac{\pi}{3} + 2n\pi]$ for $n = 1, 2, 3, \ldots$. Hence

$$\begin{aligned}\int_0^\infty \left|\frac{\cos x}{1+x}\right| dx &\geq \sum_{n=1}^\infty \int_{-\frac{\pi}{3}+2n\pi}^{\frac{\pi}{3}+2n\pi} \left|\frac{\cos x}{1+x}\right| dx \\ &\geq \sum_{n=1}^\infty \int_{-\frac{\pi}{3}+2n\pi}^{\frac{\pi}{3}+2n\pi} \frac{\frac{1}{2}}{1+\frac{\pi}{3}+2n\pi} dx \\ &= \sum_{n=1}^\infty \frac{2\pi}{3} \cdot \frac{\frac{1}{2}}{1+\frac{\pi}{3}+2n\pi} \\ &> \frac{\pi}{3} \sum_{n=1}^\infty \frac{1}{\pi+\pi+2n\pi} \\ &= \frac{1}{6} \sum_{n=1}^\infty \frac{1}{n+1}.\end{aligned}$$

By Theorem 3.28, $\sum_{n=1}^\infty \frac{1}{n+1} = \infty$ and thus $\int_0^\infty \frac{\cos x}{1+x} dx$ does not converge absolutely.

□

Exercise 6.10. *Let p and q be positive real integers such that*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Prove the following statements.

(a) *If $u \geq 0$ and $v \geq 0$, then*

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

Equality holds if and only if $u^p = v^q$.

(b) *If* $f \in \mathscr{R}(\alpha)$, $g \in \mathscr{R}(\alpha)$, $f \geq 0$, $g \geq 0$, *and*

$$\int_a^b f^p d\alpha = \int_a^b g^q d\alpha = 1,$$

then

$$\int_a^b fg d\alpha \leq 1.$$

(c) *If* f *and* g *are complex functions in* $\mathscr{R}(\alpha)$, *then*

$$\left| \int_a^b fg d\alpha \right| \leq \left\{ \int_a^b |f|^p d\alpha \right\}^{\frac{1}{p}} \left\{ \int_a^b |g|^q d\alpha \right\}^{\frac{1}{q}}.$$

This is ***Hölder's inequality****. When* $p = q = 2$ *it is usually called the Schwarz inequality. (Note that Theorem 1.35 is a very special case of this.)*

(d) *Show that Hölder's inequality is also true for the "improper" integrals described in Exercise 6.7 and 6.8.*

Proof of (a)(Young's inequality).

(1) $u = 0$ or $v = 0$ is nothing to do. For $u > 0$ and $v > 0$, we give some different proofs.

(2) First proof.

$$\begin{aligned}
uv &= \exp(\log(uv)) \\
&= \exp\left(\frac{1}{p} \log(u^p) + \frac{1}{q} \log(v^q) \right) \\
&\leq \frac{1}{p} \exp(\log(u^p)) + \frac{1}{q} \exp(\log(v^q)) \qquad \text{(Convexity of } \exp(x)\text{)} \\
&= \frac{u^p}{p} + \frac{v^q}{q}.
\end{aligned}$$

Here the convexity of $\exp(x)$ can be derived by the fact that $(\exp(x))'' > 0$ and Exercise 5.14. The fact that the equality holds if and only if $u^p = v^q$ is derived from the strictly convexity of $\exp(x)$ additionally. (For the details about the exponential and logarithmic functions, might see Chapter 8.)

(3) Second proof.

$$\begin{aligned}
\log\left(\frac{u^p}{p} + \frac{v^q}{q} \right) &\geq \frac{1}{p} \log(u^p) + \frac{1}{q} \log(v^q) \qquad \text{(Concavity of } \log(x)\text{)} \\
&= \log(u) + \log(v) \\
&= \log(uv).
\end{aligned}$$

Since $\log(x)$ increases monotonically ($(\log(x))' = \frac{1}{x} > 0$ if $x > 0$), $\frac{u^p}{p} + \frac{v^q}{q} \geq uv$ (or take the exponential function to get the same conclusion). Here the concavity of $\log(x)$ can be derived by the fact that $(\log(x))'' < 0$ and a statement that $f''(x) \leq 0$ if and only if f is concave. The fact that the equality holds if and only if $u^p = v^q$ is derived from the strictly concavity of $\log(x)$ additionally. (The proof is analogous to Exercise 5.14.)

(4) Third proof. *Suppose that $f : [0, \infty) \to [0, \infty)$ is a strictly increasing continuous function such that $f(0) = 0$ and $\lim_{x\to\infty} f(x) = \infty$. Then*

$$uv \leq \int_0^u f(x)dx + \int_0^v f^{-1}(x)dx$$

for every $u, v \geq 0$, and equality occurs if and only if $v = f(u)$. Define

$$F(x) = -xf(x) + \int_0^x f(t)dt + \int_0^{f(x)} f^{-1}(t)dt.$$

By Theorem 6.20 (the fundamental theorem of calculus) and Theorem 5.5 (chain rule),

$$F'(x) = -(f(x) + xf'(x)) + f(x) + f'(x)f^{-1}(f(x)) = 0.$$

Hence $F(x)$ is a constant on $(0, u)$ (Theorem 5.11(b)). Note that $F(x)$ is continuous on $[0, u]$ and $F(0) = 0$, so $F(x) = 0$ on $[0, u]$ or

$$\int_0^x f(t)dt + \int_0^{f(x)} f^{-1}(t)dt = xf(x).$$

Take $x = u$ to get

$$\int_0^u f(x)dx + \int_0^{f(u)} f^{-1}(x)dx = uf(u).$$

Hence

$$\begin{aligned}
&\int_0^u f(x)dx + \int_0^v f^{-1}(x)dx - uv \\
=&\int_0^u f(x)dx + \int_0^{f(u)} f^{-1}(x)dx + \int_{f(u)}^v f^{-1}(x)dx - uv \\
=&uf(u) + \int_{f(u)}^v f^{-1}(x)dx - uv \\
=&\int_{f(u)}^v [f^{-1}(x) - f^{-1}(f(u))]dx \\
\geq&0.
\end{aligned}$$

The last inequality holds since f is strictly increasing and thus f^{-1} is strictly increasing too. Besides, the equality holds if and only if $f(u) = v$. Now the conclusion holds by taking $f(x) = x^{p-1}$ in

$$uv \leq \int_0^u f(x)dx + \int_0^v f^{-1}(x)dx$$

and the equality holds if and only if $u^p = v^q$.

$\square$

Proof of (b). Every integral is well-defined (Theorem 6.11 and Theorem 6.13(a)). Let $u = f \geq 0$ and $v = g \geq 0$ in (a). Integrate both sides of the inequality

$$fg \leq \frac{f^p}{p} + \frac{g^q}{q}$$

to get

$$\begin{aligned}
\int_a^b fg d\alpha &\leq \int_a^b \left(\frac{f^p}{p} + \frac{g^q}{q} \right) d\alpha && \text{(Theorem 6.12(b))} \\
&= \int_a^b \frac{f^p}{p} d\alpha + \int_a^b \frac{g^q}{q} d\alpha && \text{(Theorem 6.12(a))} \\
&= \frac{1}{p} \int_a^b f^p d\alpha + \frac{1}{q} \int_a^b g^q d\alpha && \text{(Theorem 6.12(a))} \\
&= \frac{1}{p} + \frac{1}{q} && \text{(Assumption)} \\
&= 1.
\end{aligned}$$

The equality holds if $f^p = g^q$. Note that the equality does not hold only if $f^p = g^q$. (Consider α is constant on some subinterval $[c, d] \subsetneq [a, b]$.) Luckily, it is true for the additional assumption that $\alpha(x) = x$ and f, g are continuous on $[a, b]$. $\square$

Proof of (c). There are three possible cases.

(1) The case $\left\{ \int_a^b |f|^p d\alpha \right\}^{\frac{1}{p}} = 0$. So $\int_a^b |f|^p d\alpha = 0$.

(a) *Show that* $\int_a^b |f| d\alpha = 0$ *if* $\int_a^b |f|^p d\alpha = 0$. (Reductio ad absurdum) If $\int_a^b |f| d\alpha = A > 0$, then given $\varepsilon = \frac{A}{2} > 0$, there exists a partition $P_0 = \{a = x_0 \leq \cdots \leq x_n = b\}$ such that

$$\sum_{i=0}^n m_i \Delta\alpha_i > \frac{A}{2},$$

where $m_i = \inf_{x \in [x_{i-1}, x_i]} |f|$ and $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$. By the pigeonhole principle, there exists $1 \leq i_0 \leq n$ such that

$$L(P_0, |f|, \alpha) = m_{i_0} \Delta\alpha_{i_0} > \frac{A}{2n} > 0.$$

Especially, $m_{i_0} > 0$ and $\Delta\alpha_{i_0} > 0$. Now we consider $L(P, |f|^p, \alpha)$. Hence

$$L(P_0, |f|^p, \alpha) = \sum_{i=0}^{n} m_i^p \Delta\alpha_i \geq m_{i_0}^p \Delta\alpha_{i_0} > 0,$$

or

$$\underline{\int_a^b} |f| d\alpha = \sup L(P, f, \alpha) \geq m_{i_0}^p \Delta\alpha_{i_0} > 0,$$

which is absurd.

(b) *Show that* $\int_a^b |fg| d\alpha = 0$ *if* $\int_a^b |f| d\alpha = 0$. Since $g \in \mathscr{R}(\alpha)$, $|g|$ is bounded by some real M on $[a, b]$, that is, $|g(x)| \leq M$. Hence

$$0 \leq \int_a^b |fg| d\alpha \leq \int_a^b M|f| d\alpha = M \int_a^b |f| d\alpha = 0.$$

Therefore $\int_a^b |fg| d\alpha = 0$.

By (a)(b), $\int_a^b |fg| d\alpha = 0$ and thus Hölder's inequality holds for this case.

(2) The case $\left\{ \int_a^b |g|^q d\alpha \right\}^{\frac{1}{q}} = 0$. Similar to (1).

(3) If both $\left\{ \int_a^b |f|^p d\alpha \right\}^{\frac{1}{p}} > 0$ and $\left\{ \int_a^b |g|^q d\alpha \right\}^{\frac{1}{q}} > 0$, then we apply (b) to

$$F(x) = \frac{|f(x)|}{\left\{ \int_a^b |f(x)|^p d\alpha \right\}^{\frac{1}{p}}} \qquad \text{and} \qquad G(x) = \frac{|g(x)|}{\left\{ \int_a^b |g(x)|^q d\alpha \right\}^{\frac{1}{q}}}.$$

Here $F(x) \geq 0$ and $G(x) \geq 0$ are well-defined and Riemann integrable. Thus the conclusion holds. The equality holds if $F(x)^p = G(x)^q$ or

$$\frac{|f|^p}{\int_a^b |f|^p d\alpha} = \frac{|g|^q}{\int_a^b |g|^q d\alpha}.$$

Note that the equality does not hold only if $\frac{|f|^p}{\int_a^b |f|^p d\alpha} = \frac{|g|^q}{\int_a^b |g|^q d\alpha}$. Luckily, it is true for the additional assumption that $\alpha(x) = x$ and f, g are continuous on $[a, b]$.

By (1)(2)(3), in any case the equality holds if

$$|f|^p \int_a^b |g|^q d\alpha = |g|^q \int_a^b |f|^p d\alpha.$$

In addition, if $\alpha(x) = x$ and f, g are continuous on $[a, b]$, then the equality holds if and only if

$$|f|^p \int_a^b |g|^q d\alpha = |g|^q \int_a^b |f|^p d\alpha.$$

$\square$

Proof of (d).

(1) *Suppose f and g are real functions on $(0, 1]$ and $f, g \in \mathscr{R}$ on $[c, 1]$ for every $c > 0$. Show that*

$$\left| \int_0^1 fg dx \right| \leq \left\{ \int_0^1 |f|^p dx \right\}^{\frac{1}{p}} \left\{ \int_0^1 |g|^q dx \right\}^{\frac{1}{q}}.$$

Here $\int_0^1$ is one improper integral defined in Exercise 6.7.

(a) By (c), we have

$$\left| \int_c^1 fg dx \right| \leq \left\{ \int_c^1 |f|^p dx \right\}^{\frac{1}{p}} \left\{ \int_c^1 |g|^q dx \right\}^{\frac{1}{q}}$$

for any $c \in (0, 1]$. Here every integral is well-defined (Theorem 6.11 and Theorem 6.13).

(b) Since every integral is ≥ 0, by taking the limit in the right hand side we have

$$\begin{aligned} \left| \int_c^1 fg dx \right| &\leq \left\{ \int_c^1 |f|^p dx \right\}^{\frac{1}{p}} \left\{ \int_c^1 |g|^q dx \right\}^{\frac{1}{q}} \\ &\leq \left\{ \int_0^1 |f|^p dx \right\}^{\frac{1}{p}} \left\{ \int_0^1 |g|^q dx \right\}^{\frac{1}{q}}. \end{aligned}$$

It is possible that $\left\{ \int_0^1 |f|^p dx \right\}^{\frac{1}{p}} = \infty$ or $\left\{ \int_0^1 |g|^q dx \right\}^{\frac{1}{q}} = \infty$.

(c) Now $\left| \int_c^1 fg dx \right|$ is bounded by $\left\{ \int_0^1 |f|^p dx \right\}^{\frac{1}{p}} \left\{ \int_0^1 |g|^q dx \right\}^{\frac{1}{q}}$. Take limit to get

$$\left| \int_0^1 fg dx \right| \leq \left\{ \int_0^1 |f|^p dx \right\}^{\frac{1}{p}} \left\{ \int_0^1 |g|^q dx \right\}^{\frac{1}{q}}$$

even if some limit is divergent.

(2) *Suppose f and g are real functions on $[a, b]$ and $f, g \in \mathscr{R}$ on $[a, b]$ for every $b > a$ where a is fixed. Show that*

$$\left| \int_a^\infty fg dx \right| \leq \left\{ \int_a^\infty |f|^p dx \right\}^{\frac{1}{p}} \left\{ \int_a^\infty |g|^q dx \right\}^{\frac{1}{q}}.$$

Here $\int_a^\infty$ is one improper integral defined in Exercise 6.8. Same as (1).

□

Exercise 6.11. *Let α be a fixed increasing function on $[a, b]$. For $u \in \mathscr{R}(\alpha)$, define*

$$\|u\|_2 = \left\{ \int_a^b |u|^2 d\alpha \right\}^{\frac{1}{2}}.$$

Suppose $f, g, h \in \mathscr{R}(\alpha)$, and prove the triangle inequality

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2$$

as a consequence of the Schwarz inequality, as in the proof of Theorem 1.37.

Proof.

(1) By Exercise 6.10(c) with $p = q = 2$, we have

$$\begin{aligned} \int_a^b |f - g||g - h| d\alpha &= \left| \int_a^b |f - g||g - h| d\alpha \right| \\ &\leq \left\{ \int_a^b |f - g|^2 dx \right\}^{\frac{1}{2}} \left\{ \int_a^b |g - h|^2 dx \right\}^{\frac{1}{2}} \\ &= \|f - g\|_2 \|g - h\|_2. \end{aligned}$$

Every integral is well-defined (Theorem 6.12 and Theorem 6.13 (or Theorem 6.11)).

(2) Since

$$\begin{aligned} \|f - h\|_2^2 &= \int_a^b |f - h|^2 d\alpha \\ &\leq \int_a^b (|f - g| + |g - h|)^2 d\alpha && \text{(Triangle inequality)} \\ &= \int_a^b (|f - g|^2 + 2|f - g||g - h| + |g - h|^2) d\alpha \\ &= \int_a^b |f - g|^2 d\alpha + 2 \int_a^b |f - g||g - h| d\alpha + \int_a^b |g - h|^2 d\alpha \\ &\leq \|f - g\|_2^2 + 2\|f - g\|_2 \|g - h\|_2 + \|g - h\|_2^2 && ((1)) \\ &= (\|f - g\|_2 + \|g - h\|_2)^2, \end{aligned}$$

we have

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2.$$

Here every integral is well-defined (Theorem 6.12 and Theorem 6.13 (or Theorem 6.11)).

□

Exercise 6.12. *With the notations of Exercise 6.11, suppose $f \in \mathscr{R}(\alpha)$ and $\varepsilon > 0$. Prove that there exists a continuous function g on $[a, b]$ such that $\|f - g\|_2 < \varepsilon$. (Hint: Let $P = \{a = x_0 \leq \cdots \leq x_n = b\}$ be a suitable partition of $[a, b]$, define*

$$g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i)$$

if $x_{i-1} \leq t \leq x_i$.)

Proof. Given $\varepsilon > 0$.

(1) There are some real numbers m and M such that $m \leq f(x) \leq M$ if $x \in [a, b]$ since $f \in \mathscr{R}(\alpha)$ or f is bounded on $[a, b]$. By Theorem 6.6, there exists a partition $P = \{a = x_0 \leq \cdots \leq x_n = b\}$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \frac{\varepsilon^2}{M - m + 1}.$$

Here

$$\begin{aligned} U(P, f, \alpha) &= \sum_{i=1}^{n} M_i \Delta\alpha_i \text{ where } M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x) \\ L(P, f, \alpha) &= \sum_{i=1}^{n} m_i \Delta\alpha_i \text{ where } m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x). \end{aligned}$$

(2) For such partition P, define g on $[a, b]$ by

$$g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i)$$

if $x_{i-1} \leq t \leq x_i$. So that

$$\begin{aligned} |f(t) - g(t)| &= \left| \left(\frac{x_i - t}{\Delta x_i} + \frac{t - x_{i-1}}{\Delta x_i} \right) f(t) - \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i) \right| \\ &= \left| \frac{x_i - t}{\Delta x_i} (f(t) - f(x_{i-1})) + \frac{t - x_{i-1}}{\Delta x_i} (f(t) - f(x_i)) \right| \\ &\leq \frac{x_i - t}{\Delta x_i} |f(t) - f(x_{i-1})| + \frac{t - x_{i-1}}{\Delta x_i} |f(t) - f(x_i)| \\ &\leq \frac{x_i - t}{\Delta x_i} (M_i - m_i) + \frac{t - x_{i-1}}{\Delta x_i} (M_i - m_i) \\ &= M_i - m_i \end{aligned}$$

if $x_{i-1} \leq t \leq x_i$. Especially,

$$|f(t) - g(t)| \leq M - m$$

if $a \leq t \leq b$.

(3) Note that the integral $\int_a^b |f-g|^2 d\alpha$ is well-defined (Theorem 6.8, Theorem 6.11 and Theorem 6.12). So that

$$\begin{aligned}\int_a^b |f-g|^2 d\alpha =& \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f-g|^2 d\alpha \\ \leq& \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (M-m)(M_i-m_i) d\alpha \\ =& (M-m) \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (M_i-m_i)\Delta\alpha_i \\ =& (M-m)[U(P,f,\alpha)-L(P,f,\alpha)] \\ \leq& (M-m)\cdot \frac{\varepsilon^2}{M-m+1} \\ <& \varepsilon^2.\end{aligned}$$

Hence,

$$\|f-g\|_2 = \left\{ \int_a^b |f-g|^2 d\alpha \right\}^{\frac{1}{2}} < \varepsilon.$$

$\square$

Note.

(1) Apply the same argument we can prove the following statement:

> *Suppose $f \in \mathscr{R}(\alpha)$ and $\varepsilon > 0$. Prove that there exists a continuous function g on $[a,b]$ such that $\int_a^b |f-g| d\alpha < \varepsilon$.*

(2) (Lebesgue integral)

(a) *Let f be Lebesgue integrable over E. Then, given $\varepsilon > 0$, there is a simple function φ such that*

$$\int_E |f-\varphi| < \varepsilon.$$

(b) *Under the same hypothesis there is a step function ψ such that*

$$\int_E |f-\psi| < \varepsilon.$$

(c) *Under the same hypothesis there is a continuous function g vanishing outside a finite interval such that*

$$\int_E |f-g| < \varepsilon.$$

Exercise 6.13. *Define*

$$f(x) = \int_x^{x+1} \sin(t^2) dt.$$

(a) *Prove that* $|f(x)| < \frac{1}{x}$ *if* $x > 0$. *(Hint: Put* $t^2 = u$ *and integrate by parts, to show that* $f(x)$ *is equal to*

$$\frac{\cos(x^2)}{2x} - \frac{\cos[(x+1)^2]}{2(x+1)} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{\frac{3}{2}}} du.$$

Replace $\cos u$ *by* -1.*)*

(b) *Prove that*

$$2xf(x) = \cos(x^2) - \cos[(x+1)^2] + r(x)$$

where $|r(x)| < \frac{c}{x}$ *and* c *is a constant.*

(c) *Find the upper and lower limits of* $xf(x)$, *as* $x \to \infty$.

(d) *Does* $\int_0^\infty \sin(t^2) dt$ *converges?*

Proof of (a).

(1) Put $t^2 = u$ and integrate by parts to get

$$\begin{aligned} f(x) &= \int_x^{x+1} \sin(t^2) dt \\ &= \int_{x^2}^{(x+1)^2} \frac{\sin u}{2u^{\frac{1}{2}}} du \\ &= -\frac{\cos[(x+1)^2]}{2(x+1)} + \frac{\cos(x^2)}{2x} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{\frac{3}{2}}} du. \end{aligned}$$

(2)

$$\begin{aligned} |f(x)| &\leq \left| \frac{\cos[(x+1)^2]}{2(x+1)} \right| + \left| \frac{\cos(x^2)}{2x} \right| + \left| \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{\frac{3}{2}}} du \right| \\ &\leq \left| \frac{\cos[(x+1)^2]}{2(x+1)} \right| + \left| \frac{\cos(x^2)}{2x} \right| + \int_{x^2}^{(x+1)^2} \frac{|\cos u|}{4u^{\frac{3}{2}}} du \\ &\leq \frac{1}{2(x+1)} + \frac{1}{2x} + \int_{x^2}^{(x+1)^2} \frac{1}{4u^{\frac{3}{2}}} du \\ &= \frac{1}{2(x+1)} + \frac{1}{2x} + \left[\frac{1}{2x} - \frac{1}{2(x+1)} \right] \\ &= \frac{1}{x}. \end{aligned}$$

(3) The equality in (2) holds only if $\left|\cos[(x+1)^2]\right| = 1$, $\left|\cos(x^2)\right| = 1$, and

$$\left|\int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{\frac{3}{2}}} du\right| = \int_{x^2}^{(x+1)^2} \frac{|\cos u|}{4u^{\frac{3}{2}}} du = \int_{x^2}^{(x+1)^2} \frac{1}{4u^{\frac{3}{2}}} du.$$

Since $\cos u$ has two absolute minimums or maximums at two different points $u = x^2$ and $u = (x+1)^2$, by the property of $\cos(u)$ there is some $u_0 \in [x^2, (x+1)^2]$ such that $\cos(u_0) = 0$. Hence given $\varepsilon = \frac{1}{2} > 0$ there exists $\delta > 0$ such that

$$|\cos(u)| \leq \frac{1}{2}$$

whenever

$$u \in E = [\max\{u_0 - \delta, x^2\}, \min\{u_0 + \delta, (x+1)^2\}] \subseteq [x^2, (x+1)^2].$$

Here $|E| > 0$. So that

$$\int_{x^2}^{(x+1)^2} \frac{|\cos u|}{4u^{\frac{3}{2}}} du \leq \int_{x^2}^{(x+1)^2} \frac{1}{4u^{\frac{3}{2}}} du - \frac{1}{2}\int_E \frac{1}{4u^{\frac{3}{2}}} du < \int_{x^2}^{(x+1)^2} \frac{1}{4u^{\frac{3}{2}}} du,$$

which is absurd. Hence the equality in (2) does not hold.

$\square$

Proof of (b).

(1) By (a),

$$2xf(x) = \cos(x^2) - \cos[(x+1)^2] + r(x)$$

where

$$r(x) = \frac{\cos[(x+1)^2]}{x+1} - 2x\int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{\frac{3}{2}}} du.$$

(2) Similar to (a),

$$\begin{aligned}
|r(x)| &\leq \frac{1}{x+1} + 2x\int_{x^2}^{(x+1)^2} \frac{1}{4u^{\frac{3}{2}}} du \\
&= \frac{1}{x+1} + 2x\left[\frac{1}{2x} - \frac{1}{2(x+1)}\right] \\
&= \frac{2}{x+1} \\
&< \frac{2}{x}.
\end{aligned}$$

$\square$

Proof of (c). Show that

$$\limsup_{x\to\infty} xf(x) = 1.$$

The case $\liminf_{x\to\infty} xf(x) = -1$ is similar.

(1) By (b), *it suffices to show that*

$$\limsup_{x\to\infty}\left[\cos\left(x^2\right)-\cos(x+1)^2\right]=2.$$

Take $x_n = 2n\sqrt{\pi}$ for $n = 1, 2, 3, \ldots$. So

$$\cos\left(x_n^2\right)-\cos(x_n+1)^2 = 1-\cos\left(4n\sqrt{\pi}+1\right).$$

It suffices to show that

$$\liminf_{n\to\infty}\cos\left(4n\sqrt{\pi}+1\right)=-1.$$

(2) $x \mapsto \cos(x)$ is uniformly continuous by the mean value theorem (Theorem 5.10) and $x \mapsto -\sin(x)$ is bounded by 1. So given any $\varepsilon > 0$, there exists $\delta > 0$ such that $|\cos(x) - \cos(y)| < \varepsilon$ whenever $|x - y| < \delta$

(3) Define $\alpha = \frac{1}{\sqrt{\pi}}$ and $x = \frac{\pi-1}{4\pi}$. Note that α is irrational. Exercise 4.25(b) implies that there exist integers $n > 0$ and m such that

$$|n\alpha - m - x| < \frac{\delta}{4\pi},$$

or

$$\left|(4n\sqrt{\pi}+1)-(4m\pi-\pi)\right| < \delta.$$

By (2),

$$\left|\cos\left(4n\sqrt{\pi}+1\right)-(-1)\right| < \varepsilon.$$

Hence $\liminf_{n\to\infty}\cos(4n\sqrt{\pi}+1) = -1$.

□

Proof of (d). Yes. $\int_0^\infty \sin\left(t^2\right)dt$ converges.

(1) Given any integer $N > 0$. Write

$$\begin{aligned}
\int_0^N \sin\left(t^2\right)dt &= \sum_{n=0}^{N-1}\int_n^{n+1}\sin\left(t^2\right)dt \\
&= \sum_{n=0}^{N-1} f(n) \\
&= f(0) + \sum_{n=1}^{N-1}\frac{\cos\left(n^2\right)}{2n} - \frac{\cos[(n+1)^2]}{2n} + \frac{r(n)}{2n} \\
&= f(0) + \sum_{n=1}^{N-1}\frac{\cos\left(n^2\right)}{2n} - \sum_{n=1}^{N-1}\frac{\cos[(n+1)^2]}{2n} + \sum_{n=1}^{N-1}\frac{r(n)}{2n} \\
&= f(0) + \sum_{n=1}^{N-1}\frac{\cos\left(n^2\right)}{2n} - \sum_{n=2}^{N}\frac{\cos\left(n^2\right)}{2(n-1)} + \sum_{n=1}^{N-1}\frac{r(n)}{2n} \\
&= f(0) + \frac{\cos(1)}{2} - \frac{\cos\left(N^2\right)}{2(N-1)} - \frac{1}{2}\sum_{n=2}^{N-1}\frac{\cos\left(n^2\right)}{n(n-1)} + \sum_{n=1}^{N-1}\frac{r(n)}{2n}
\end{aligned}$$

where $|r(n)| \leq \frac{2}{n}$ (by (b)).

(2) $\frac{\cos(N^2)}{2(N-1)} \to 0$ as $N \to \infty$ since $\cos(N^2)$ is bounded by 1 and $\frac{1}{N-1} \to 0$ as $N \to \infty$.

(3) Since $\cos(n^2)$ is bounded by 1 and $\sum \frac{1}{n(n-1)} < \sum \frac{1}{(n-1)^2}$ converges,

$$\frac{1}{2} \sum_{n=2}^{\infty} \frac{\cos(n^2)}{n(n-1)}$$

converges absolutely.

(4) Since $|r(n)| \leq \frac{2}{n}$ and $\sum \frac{1}{n^2}$ converges,

$$\sum_{n=1}^{\infty} \frac{|r(n)|}{2n} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges. So $\sum_{n=1}^{\infty} \frac{r(n)}{2n}$ converges absolutely.

(5) By (1)(2)(3)(4),

$$\lim_{N \to \infty} \int_0^N \sin(t^2) dt$$

exists. Note that

$$\left| \int_x^y \sin(t^2) dt \right| < \frac{1}{x}$$

if $y \geq x > 0$ (by applying the same argument in (a)(2)). So

$$\lim_{\substack{x \to \infty \\ y \to \infty \\ y \geq x}} \int_x^y \sin(t^2) dt = 0.$$

Therefore,

$$\begin{aligned}
\int_0^{\infty} \sin(t^2) dt &= \lim_{b \to \infty} \int_0^b \sin(t^2) dt \\
&= \lim_{b \to \infty} \int_0^{[b]} \sin(t^2) dt + \int_{[b]}^b \sin(t^2) dt \\
&= \lim_{b \to \infty} \int_0^{[b]} \sin(t^2) dt + \lim_{b \to \infty} \int_{[b]}^b \sin(t^2) dt \\
&= \lim_{N \to \infty} \int_0^N \sin(t^2) dt + \lim_{\substack{[b] \to \infty \\ b \to \infty \\ b \geq [b]}} \int_{[b]}^b \sin(t^2) dt \\
&= \lim_{N \to \infty} \int_0^N \sin(t^2) dt
\end{aligned}$$

converges.

□

Note.

$$\int_0^\infty \sin\bigl(t^2\bigr)dt = \int_0^\infty \cos\bigl(t^2\bigr)dt = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

Exercise 6.14. *Deal similarly with*

$$f(x) = \int_x^{x+1} \sin\bigl(e^t\bigr)dt.$$

Show that

$$e^x|f(x)| < 2$$

and that

$$e^x f(x) = \cos(e^x) - e^{-1}\cos\bigl(e^{x+1}\bigr) + r(x)$$

where $|r(x)| < Ce^{-x}$ *for some constant* C.

Proof.

(1) Put $e^t = u$ and integrate by parts to get

$$\begin{aligned} f(x) &= \int_x^{x+1} \sin\bigl(e^t\bigr)dt \\ &= \int_{\exp(x)}^{\exp(x+1)} \frac{\sin u}{u} du \\ &= -\frac{\cos\bigl(e^{x+1}\bigr)}{e^{x+1}} + \frac{\cos(e^x)}{e^x} - \int_{\exp(x)}^{\exp(x+1)} \frac{\cos u}{u^2} du. \end{aligned}$$

(2) *Show that* $e^x|f(x)| \le 2$.

$$\begin{aligned} |f(x)| &\le \left|\frac{\cos\bigl(e^{x+1}\bigr)}{e^{x+1}}\right| + \left|\frac{\cos(e^x)}{e^x}\right| + \left|\int_{\exp(x)}^{\exp(x+1)} \frac{\cos u}{u^2} du\right| \\ &\le \left|\frac{\cos\bigl(e^{x+1}\bigr)}{e^{x+1}}\right| + \left|\frac{\cos(e^x)}{e^x}\right| + \int_{\exp(x)}^{\exp(x+1)} \frac{|\cos u|}{u^2} du \\ &\le \frac{1}{e^{x+1}} + \frac{1}{e^x} + \int_{\exp(x)}^{\exp(x+1)} \frac{1}{u^2} du \\ &= \frac{1}{e^{x+1}} + \frac{1}{e^x} + \left[\frac{1}{e^x} - \frac{1}{e^{x+1}}\right] \\ &= \frac{2}{e^x}. \end{aligned}$$

Hence $e^x|f(x)| \le 2$.

(3) *Show that* $e^x|f(x)| < 2$. Similar to (b)(3) in the proof of Exercise 6.13.

(4) *Show that*
$$e^x f(x) = \cos(e^x) - e^{-1}\cos\big(e^{x+1}\big) + r(x)$$
where $|r(x)| < Ce^{-x}$ *for some constant* C. By (1),
$$e^x f(x) = \cos(e^x) - e^{-1}\cos\big(e^{x+1}\big) - e^x \int_{\exp(x)}^{\exp(x+1)} \frac{\cos u}{u^2}du.$$
So that
$$r(x) = -e^x \int_{\exp(x)}^{\exp(x+1)} \frac{\cos u}{u^2}du.$$
By integration by parts (Theorem 6.22),
$$\begin{aligned}\int_{\exp(x)}^{\exp(x+1)} \frac{\cos u}{u^2}du &= \left[\frac{\sin u}{u^2}\right]_{u=\exp(x)}^{u=\exp(x+1)} - \int_{\exp(x)}^{\exp(x+1)} \frac{-2\sin u}{u^3}du \\ &= \frac{\sin e^{x+1}}{e^{2x+2}} - \frac{\sin e^x}{e^{2x}} + 2\int_{\exp(x)}^{\exp(x+1)} \frac{\sin u}{u^3}du.\end{aligned}$$
So that
$$\begin{aligned}|r(x)| &\leq \left|\frac{\sin e^{x+1}}{e^{x+2}}\right| + \left|\frac{\sin e^x}{e^x}\right| + 2e^x\int_{\exp(x)}^{\exp(x+1)} \frac{|\sin u|}{u^3}du \\ &\leq \frac{1}{e^{x+2}} + \frac{1}{e^x} + 2e^x\int_{\exp(x)}^{\exp(x+1)} \frac{du}{u^3} \\ &= \frac{1}{e^{x+2}} + \frac{1}{e^x} + 2e^x\left[-\frac{1}{2}u^{-2}\right]_{u=\exp(x)}^{u=\exp(x+1)} \\ &= \frac{2}{e^x}.\end{aligned}$$
The equality does not hold as in (3), or $|r(x)| < 2e^{-x}$.

(5) *Show that* $\int_0^\infty \sin(e^t)dt$ *converges.* Similar to (d) in Exercise 6.13. Given any integer $N > 0$, write
$$\int_0^N \sin\big(e^t\big)dt = f(0) + \frac{\cos(e)}{e} - \underbrace{\frac{\cos\big(e^N\big)}{e^N}}_{\to 0} + \underbrace{\sum_{n=1}^{N-1}\frac{r(n)}{e^n}}_{<\infty}$$
where $|r(n)| \leq \frac{2}{e^n}$ (by (4)). So $\lim \int_0^N \sin(e^t)dt$ exists. Also note that
$$\left|\int_x^y \sin\big(e^t\big)dt\right| < \frac{2}{e^x}$$

if $y \geq x > 0$ (by applying the same argument in (2)). Therefore

$$\int_0^\infty \sin\bigl(e^t\bigr)dt = \lim_{b\to\infty} \int_0^b \sin\bigl(e^t\bigr)dt = \lim_{N\to\infty} \int_0^N \sin\bigl(e^t\bigr)dt$$

converges.

□

Exercise 6.15. *Suppose f is a real, continuously differentiable function on $[a, b]$, $f(a) = f(b) = 0$, and*

$$\int_a^b f(x)^2 dx = 1.$$

Prove that

$$\int_a^b x f(x) f'(x) dx = -\frac{1}{2}$$

and that

$$\int_a^b [f'(x)]^2 dx \int_a^b x^2 f(x)^2 dx > \frac{1}{4}.$$

Proof. Every integral is well-defined (Theorem 4.9 and Theorem 6.8).

(1) By Theorem 6.22 (integration by parts),

$$\int_a^b x \left(\frac{f(x)^2}{2}\right)' dx = \left[x \cdot \frac{f(x)^2}{2}\right]_{x=a}^{x=b} - \int_a^b \frac{f(x)^2}{2} dx,$$

or

$$\int_a^b x f(x) f'(x) dx = \left[b \cdot \frac{f(b)^2}{2} - a \cdot \frac{f(a)^2}{2}\right] - \frac{1}{2} \int_a^b f(x)^2 dx = -\frac{1}{2}.$$

(2) By Exercise 6.10(c),

$$\int_a^b [f'(x)]^2 dx \int_a^b x^2 f(x)^2 dx \geq \left(\int_a^b x f(x) f'(x) dx\right)^2 = \frac{1}{4}.$$

(3) (Reductio ad absurdum) If the equality were holding, then by Exercise 6.10(c)

$$(f'(x))^2 \int_a^b x^2 f(x)^2 dx = x^2 f(x)^2 \int_a^b [f'(x)]^2 dx$$

on $[a, b]$ (since x, $f(x)$ and $f'(x)$ are continuous on $[a, b]$).

(a) *Show that both integrals are nonzero.* (Reductio ad absurdum) If $\int_a^b x^2 f(x)^2 dx = 0$, then $x^2 f(x)^2 = 0$ or $xf(x) = 0$ on $[a, b]$ (Exercise 6.2). So that

$$\int_a^b xf(x)f'(x)dx = 0 \neq -\frac{1}{2},$$

which is absure. Similarly, $\int_a^b [f'(x)]^2 dx \neq 0$.

(b) By (a), we write

$$C = \left\{ \frac{\int_a^b [f'(x)]^2 dx}{\int_a^b x^2 f(x)^2 dx} \right\}^{\frac{1}{2}} > 0$$

be a positive constant. Hence

$$f'(x) = \pm Cxf(x).$$

Here the sign "$\pm$" is not necessary unchanged on $[a, b]$. Luckily, we can show that the sign "$\pm$" is unchanged on some subinterval of $[a, b]$.

(c) To find such subinterval of $[a, b]$, we consider the zero set $Z(f')$ and $Z(xf)$ on $[a, b]$. Since $f'(x) = \pm Cxf(x)$ with $C > 0$, we have

$$Z(f') = Z(xf).$$

Note that $Z(f') = Z(xf)$ is closed (Exercise 4.3) and not equal to $[a, b]$ (by applying the same argument in (a)). Hence the complement of $Z(f') = Z(xf)$ is open and nonempty, which can be written as the union of an at most countable collection of disjoint segments (Exercise 2.29).

(d) Consider any nonempty open interval in (c), say

$$(c, d) \subseteq [a, b].$$

By construction, $f'(x) \neq 0$ for all $x \in (c, d)$. Since $f'(x)$ is continuous, by Theorem 4.23 there are only two mutually exclusive possible cases:

(i) $f'(x) > 0$ for all $x \in (c, d)$,

(ii) $f'(x) < 0$ for all $x \in (c, d)$.

Similar result for $xf(x)$. Therefore, the sign "$\pm$" of $f'(x) = \pm Cxf(x)$ are unchanged on (c, d), that is,

(i) $f'(x) = Cxf(x)$ for all $x \in (c, d)$,

(ii) $f'(x) = -Cxf(x)$ for all $x \in (c, d)$,

(e) Suppose $f'(x) = Cxf(x)$ on (c, d). Since $f'(x)$ and $xf(x)$ are both vanishing at $x = c$ and $x = d$, $f'(x) = Cxf(x)$ at $x = c$ and $x = d$. So

$$f'(x) = Cxf(x) \text{ if } x \in [c, d].$$

Define

$$\phi(x,y) = Cxy$$

be a real function on $R = [c,d] \times \mathbb{R}$. And consider the initial-value problem

$$y' = \phi(x,y) \qquad \text{with} \qquad y(c) = 0.$$

Then

$$|\phi(x,y_2) - \phi(x,y_1)| = Cx|y_2 - y_1| \leq A|y_2 - y_1|$$

where $A = C \cdot \max\{|c|,|d|\}$ is a constant. By Exercise 5.27, this initial-value problem has at most one solution. Clearly, $y = f(x) = 0$ on $[c,d]$ is one solution of this initial-value problem, contrary to the construction of $[c,d]$. Similar result for the case $f'(x) = -Cxf(x)$.

Therefore, the equality does not hold.

$\square$

Exercise 6.16. *For $1 < s < \infty$, define*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

(This is Riemann's zeta function, of great importance in the study of the distribution of prime numbers.) Prove that

(a)

$$\zeta(s) = s \int_1^{\infty} \frac{[x]}{x^{s+1}} dx$$

and that

(b)

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{\infty} \frac{x - [x]}{x^{s+1}} dx,$$

where $[x]$ denotes the greatest integer $\leq x$. Prove that the integral in (b) converges for all $s > 0$. (Hint: To prove (a), compute the difference between the integral over $[1,N]$ and the Nth partial sum of the series that defines $\zeta(s)$.)

Proof of (a) (Hint).

(1) Define

$$a_N = s \int_1^{N} \frac{[x]}{x^{s+1}} dx - \sum_{n=1}^{N} \frac{1}{n^s}.$$

Hence

$$
\begin{aligned}
s\int_1^N \frac{[x]}{x^{s+1}}dx &= \sum_{n=1}^{N-1} s\int_n^{n+1} \frac{[x]}{x^{s+1}}dx \\
&= \sum_{n=1}^{N-1} s\int_n^{n+1} \frac{n}{x^{s+1}}dx \\
&= \sum_{n=1}^{N-1} n\left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) \\
&= \left(\sum_{n=1}^{N} \frac{1}{n^s}\right) - \frac{1}{N^{s-1}},
\end{aligned}
$$

or

$$
a_N = -\frac{1}{N^{s-1}}.
$$

(2) So

$$
\lim_{N\to\infty} a_N = 0
$$

(since $s-1>0$). By Theorem 3.28, $\zeta(s)$ converges if $s>1$. Hence

$$
\lim_{N\to\infty} s\int_1^N \frac{[x]}{x^{s+1}}dx = \zeta(s)
$$

converges.

(3) Hence given any real $b>1$, there exists an integer N such that $N \leq b < N+1$. Since $x \mapsto \frac{[x]}{x^{s+1}} \geq 0$ on $[1,\infty)$,

$$
s\int_1^N \frac{[x]}{x^{s+1}}dx \leq s\int_1^b \frac{[x]}{x^{s+1}}dx \leq s\int_1^{N+1} \frac{[x]}{x^{s+1}}dx.
$$

Since $b\to\infty$ if and only if $N\to\infty$,

$$
\begin{aligned}
&\lim_{N\to\infty} s\int_1^N \frac{[x]}{x^{s+1}}dx \leq \lim_{b\to\infty} s\int_1^b \frac{[x]}{x^{s+1}}dx \leq \lim_{N\to\infty} s\int_1^{N+1} \frac{[x]}{x^{s+1}}dx \\
\Longrightarrow &\zeta(s) \leq \lim_{b\to\infty} s\int_1^b \frac{[x]}{x^{s+1}}dx \leq \zeta(s).
\end{aligned}
$$

Hence

$$
\lim_{b\to\infty} s\int_1^b \frac{[x]}{x^{s+1}}dx = s\int_1^\infty \frac{[x]}{x^{s+1}}dx = \zeta(s)
$$

(in the sense of Exercise 6.8).

□

Proof of (b).

(1) *Show that*
$$s\int_1^\infty \frac{1}{x^s}dx = \frac{s}{s-1}.$$
Given any real $b > 1$. By the fundamental theorem of calculus (Theorem 6.21),
$$s\int_1^b \frac{1}{x^s}dx = \frac{s}{s-1} - \frac{s}{(s-1)b^{s-1}}.$$
Hence
$$\lim_{b\to\infty} s\int_1^b \frac{1}{x^s}dx = \frac{s}{s-1}$$
since $\frac{1}{b^{s-1}} \to 0$ as $b \to \infty$ (in the sense of Exercise 6.8).

(2) By (a) and (1), $s\int_1^\infty \frac{x-[x]}{x^{s+1}}dx$ exists and equal to
$$s\int_1^\infty \frac{x-[x]}{x^{s+1}}dx = s\int_1^\infty \frac{1}{x^s}dx - s\int_1^\infty \frac{[x]}{x^{s+1}}dx = \frac{s}{s-1} - \zeta(s).$$
The result is established.

(3) *Show that*
$$\int_1^\infty \frac{x-[x]}{x^{s+1}}dx$$
converges for all $s > 0$. Note that $0 \leq x - [x] < 1$ on $[1, \infty)$. So
$$\int_1^b \frac{x-[x]}{x^{s+1}}dx \leq \int_1^b \frac{1}{x^{s+1}}dx = \frac{1}{s} - \frac{1}{sb^s}.$$
Since $\frac{1}{sb^s} \to 0$ as $b \to \infty$,
$$\int_1^\infty \frac{x-[x]}{x^{s+1}}dx = \lim_{b\to\infty}\int_1^b \frac{x-[x]}{x^{s+1}}dx \leq \lim_{b\to\infty} \frac{1}{s} - \frac{1}{sb^s} = \frac{1}{s}.$$
Note that $\frac{1}{s}$ is finite, and thus the integral $\int_1^\infty \frac{x-[x]}{x^{s+1}}dx$ converges.

$\square$

Note.

(1) The integral $\int_1^\infty \frac{[x]}{x^{s+1}}dx$ does not converge for all $1 \geq s > 0$.

(2) Compare to Exercise 8.9.

(3) **Euler's summation formula.** (Theorem 7.13 in the textbook: *Tom. M. Apostol, Mathematical Analysis, 2nd edition.*) *If f has a continuous derivative f' on $[a, b]$, then we have*
$$\sum_{a<n\leq b} f(n) = \int_a^b f(x)dx + \int_a^b f'(x)\{x\}dx + f(a)\{a\} - f(b)\{b\},$$

where $\sum_{a<n\leq b}$ *means the sum from* $n = [a]+1$ *to* $n = [b]$*. When* a *and* b *are integers, this becomes*

$$\sum_{n=a}^{b} f(n) = \int_a^b f(x)dx + \int_a^b f'(x)\left(\{x\} - \frac{1}{2}\right)dx + \frac{f(a)+f(b)}{2}.$$

By taking $f(x) = \frac{1}{x^s}$ we can get (a) as well.

Exercise 6.17. *Suppose* α *increases monotonically on* $[a,b]$*,* g *is continuous, and* $g(x) = G'(x)$ *for* $a \leq x \leq b$*. Prove that*

$$\int_a^b \alpha(x)g(x)dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G d\alpha.$$

(Hint: Take g *real, without loss of generality. Given* $P = \{a = x_0, x_1, \ldots, x_n = b\}$*, choose* $t_i \in (x_{i-1}, x_i)$ *so that* $g(t_i)\Delta x_i = G(x_i) - G(x_{i-1})$*. Show that*

$$\sum_{i=1}^{n} \alpha(x_i)g(t_i)\Delta x_i = G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^{n} G(x_{i-1})\Delta\alpha_i.)$$

Proof (Hint). Given $\varepsilon > 0$.

(1) Take g real, without loss of generality. Given any partition

$$P = \{a = x_0 < x_1 < \ldots < x_n = b\}$$

of $[a,b]$.

(2) By the mean value theorem (Theorem 5.10), there is $t_i \in (x_{i-1}, x_i)$ such that

$$G(x_i) - G(x_{i-1}) = (x_i - x_{i-1})G'(t_i) = g(t_i)\Delta x_i.$$

(3) Hence,

$$\begin{aligned}
\sum_{i=1}^{n} \alpha(x_i)g(t_i)\Delta x_i &= \sum_{i=1}^{n} \alpha(x_i)(G(x_i) - G(x_{i-1})) \\
&= \sum_{i=1}^{n} \alpha(x_i)G(x_i) - \sum_{i=1}^{n} \alpha(x_i)G(x_{i-1}) \\
&= \underbrace{G(b)\alpha(b) - G(a)\alpha(a) + \sum_{i=1}^{n} \alpha(x_{i-1})G(x_{i-1})}_{\text{adjust the index of } \sum_{i=1}^{n} \alpha(x_i)G(x_i)} - \sum_{i=1}^{n} \alpha(x_i)G(x_{i-1}) \\
&= G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^{n} G(x_{i-1})\Delta\alpha_i.
\end{aligned}$$

(4) Since $G(x)$ is differentiable on $[a,b]$, $G(x)$ is continuous (Theorem 5.2) and thus $G \in \mathscr{R}(\alpha)$ (Theorem 6.8). So there is a partition P_1 such that

$$\left| \sum_{j=1}^{n} G(t_j)\Delta\alpha_j - \int_a^b G d\alpha \right| < \varepsilon$$

whenever $t_j \in [x_{j-1}, x_j]$ (Theorem 6.7). In particular, we pick $t_j = x_{j-1} \in [x_{j-1}, x_j]$ for all j, that is,

$$\left| \sum_{j=1}^{n} G(x_{j-1})\Delta\alpha_j - \int_a^b G d\alpha \right| < \varepsilon.$$

Note that if P^* is a refinement of P, the result is true too (Theorem 6.4).

(5) Since α increases monotonically, $\alpha \in \mathscr{R}$ (Theorem 6.9). Since g is continuous, $g \in \mathscr{R}$ (Theorem 6.8). Hence $\alpha g \in \mathscr{R}$ (Theorem 6.13). So there is a partition P_2 such that

$$\left| \sum_{k=1}^{m} \alpha(t_k)g(t_k)\Delta x_k - \int_a^b \alpha g dx \right| < \varepsilon$$

whenever $t_k \in [x_{k-1}, x_k]$ (Theorem 6.7). In particular, we pick $t_k = x_k \in [x_{k-1}, x_k]$ for all k, that is,

$$\left| \sum_{k=1}^{m} \alpha(x_k)g(x_k)\Delta x_k - \int_a^b \alpha g dx \right| < \varepsilon.$$

Note that if P^* is a refinement of P, the result is true too (Theorem 6.4).

(6) Since g is continuous on a compact set $[a,b]$, g is uniformly continuous. Hence there exists $\delta > 0$ such that

$$|g(y) - g(x)| < \varepsilon$$

whenever $|y - x| < \delta$ and $x, y \in [a,b]$. For such δ, we construct a partition P_3 such that

$$|g(t_l) - g(x_l)| < \varepsilon$$

whenever $t_l \in [x_{l-1}, x_l]$. (For example, we might take

$$P_3 = \left\{ a, a + \frac{1}{N}(b-a), a + \frac{2}{N}(b-a), \dots, a + \frac{N-1}{N}(b-a), b \right\}$$

where N is an integer $\geq \frac{b-a}{\delta}$.) Hence

$$\begin{aligned}
&\left|\sum_{l=1}^{N} \alpha(x_l)g(t_l)\Delta x_l - \sum_{l=1}^{N} \alpha(x_l)g(x_l)\Delta x_l\right| \\
=&\left|\sum_{l=1}^{N} \alpha(x_l)[g(t_l) - g(x_l)]\Delta x_l\right| \\
\leq&\sum_{l=1}^{N} |\alpha(x_l)| \cdot |g(t_l) - g(x_l)| \cdot \Delta x_l \\
\leq&M\varepsilon \sum_{l=1}^{N} \Delta x_l \\
=&M(b-a)\varepsilon.
\end{aligned}$$

Note that if P^* is a refinement of P, the result is true too (by the uniformly convergence of g).

(7) Let $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ be a common refinement of P_1, P_2 and P_3. By (3)(4)(5)(6) we have

$$\begin{aligned}
&\left|\int_a^b \alpha(x)g(x)dx - G(b)\alpha(b) + G(a)\alpha(a) + \int_a^b Gd\alpha\right| \\
=&\left|\int_a^b \alpha(x)g(x)dx - \sum_{i=1}^{n} \alpha(x_i)g(t_i)\Delta x_i + \int_a^b Gd\alpha - \sum_{i=1}^{n} G(x_{i-1})\Delta\alpha_i\right| \\
\leq&\left|\int_a^b \alpha(x)g(x)dx - \sum_{i=1}^{n} \alpha(x_i)g(t_i)\Delta x_i\right| + \left|\int_a^b Gd\alpha - \sum_{i=1}^{n} G(x_{i-1})\Delta\alpha_i\right| \\
\leq&\left|\int_a^b \alpha(x)g(x)dx - \sum_{i=1}^{n} \alpha(x_i)g(x_i)\Delta x_i\right| + \left|\sum_{i=1}^{n} \alpha(x_i)g(x_i)\Delta x_i - \sum_{i=1}^{n} \alpha(x_i)g(t_i)\Delta x_i\right| \\
&+ \left|\int_a^b Gd\alpha - \sum_{i=1}^{n} G(x_{i-1})\Delta\alpha_i\right| \\
\leq&\varepsilon + M(b-a)\varepsilon + \varepsilon \\
=&(M(b-a) + 2)\varepsilon.
\end{aligned}$$

Since ε is arbitrary,

$$\left|\int_a^b \alpha(x)g(x)dx - G(b)\alpha(b) + G(a)\alpha(a) + \int_a^b Gd\alpha\right| = 0,$$

or

$$\int_a^b \alpha(x)g(x)dx - G(b)\alpha(b) + G(a)\alpha(a) + \int_a^b Gd\alpha = 0,$$

or

$$\int_a^b \alpha(x)g(x)dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b Gd\alpha.$$

□

Exercise 6.18. *Let γ_1, γ_2, γ_3 be curves in the complex plane, defined on $[0, 2\pi]$ by*

$$\begin{aligned}
\gamma_1 &= \exp(it), \\
\gamma_2 &= \exp(2it), \\
\gamma_3 &= \exp\left(2\pi it \sin\left(\frac{1}{t}\right)\right).
\end{aligned}$$

Show that these three curves have the same range, that γ_1 and γ_2 are rectifiable, that the length of γ_1 is 2π, that the length of γ_2 is 4π, and that γ_3 is not rectifiable.

Might assume that $\gamma_3(0) = 1$.

Proof. Write $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

(1) *Show that γ_1 has the range S^1.* Given any $z \in S^1$. Theorem 8.7(d) implies that there is a unique $t \in [0, 2\pi)$ such that $\exp(it) = z$.

(2) *Show that γ_1 is rectifiable and its length is 2π.* By the definition of $\exp(z)$,

$$\gamma_1'(t) = i\exp(it),$$

which is continuous on $[0, 2\pi]$. Hence γ_1 is rectifiable is rectifiable, and its length is

$$\Lambda(\gamma_1) = \int_0^{2\pi} |\gamma_1'(t)|dt = \int_0^{2\pi} dt = 2\pi$$

(Theorem 6.27).

(3) *Show that γ_2 has the range S^1.* Similar to (1). Given any $z \in S^1$. Theorem 8.7(d) implies that there is a unique $t \in [0, 2\pi)$ such that $\exp(it) = z$. Write $\exp(it) = \exp\big(2i\left(\frac{t}{2}\right)\big)$ where $\frac{t}{2} \in [0, \pi) \subseteq [0, 2\pi)$.

(4) *Show that γ_2 is rectifiable and its length is 4π.* Similar to (2).

$$\gamma_2'(t) = 2i\exp(2it),$$

and

$$\Lambda(\gamma_2) = \int_0^{2\pi} |\gamma_2'(t)|dt = \int_0^{2\pi} 2dt = 4\pi.$$

(5) *Show that γ_3 has the range S^1.* Define

$$f(t) = \begin{cases} 0 & (t = 0), \\ t \sin \frac{1}{t} & (t \neq 0). \end{cases}$$

It suffices to show that $f(I) \supseteq J$ for for some segment $I \subseteq [0, 2\pi]$ and some segment J in $\mathbb{R}$ of the length ≥ 1 (Theorem 8.7(a)). Define $I = \left[\frac{6}{7\pi}, \frac{6}{\pi}\right] \subseteq [0, 2\pi]$ and $J = \left[-\frac{3}{7\pi}, \frac{3}{\pi}\right]$ of the length $\frac{24}{7\pi} > 1$. Hence $f(I)$ is connected since I is connected (Theorem 4.22). Since

$$f\left(\frac{6}{7\pi}\right) = \frac{6}{7\pi} \sin \frac{7\pi}{6} = -\frac{3}{7\pi},$$
$$f\left(\frac{6}{\pi}\right) = \frac{6}{\pi} \sin \frac{\pi}{6} = \frac{3}{\pi},$$

$f(I) \supseteq J$ (Theorem 2.47). The result is established.

(6) *Show that γ_3 is not rectifiable.*

(a) Since
$$\gamma_3' = 2\pi i \left(\sin \frac{1}{t} - \frac{1}{t} \cos \frac{1}{t} \right) \exp\left(2\pi i t \sin\left(\frac{1}{t}\right)\right)$$
is continuous on $[c, 2\pi]$ for any $c > 0$, γ_3 is rectifiable on $[c, 2\pi]$ (not on $[0, 2\pi]$), and
$$\Lambda_{[c,2\pi]}(\gamma_3) = \int_c^{2\pi} |\gamma_3'(t)| dt$$
on $[c, 2\pi]$.

(b)
$$\begin{aligned} \int_c^{2\pi} |\gamma_3'(t)| dt &= 2\pi \int_c^{2\pi} \left| \sin \frac{1}{t} - \frac{1}{t} \cos \frac{1}{t} \right| dt \\ &\geq 2\pi \int_c^{2\pi} \left| \frac{1}{t} \cos \frac{1}{t} \right| - 1 dt \\ &= 2\pi \int_c^{2\pi} \left| \frac{1}{t} \cos \frac{1}{t} \right| dt - 4\pi^2. \end{aligned}$$

(c) For any integer $n > 0$, we have

$$\begin{aligned}
&\int_{\left(2n\pi+\frac{\pi}{3}\right)^{-1}}^{\left(2n\pi-\frac{\pi}{3}\right)^{-1}} \left|\frac{1}{t}\cos\frac{1}{t}\right| dt \\
\geq &\int_{\left(2n\pi+\frac{\pi}{3}\right)^{-1}}^{\left(2n\pi-\frac{\pi}{3}\right)^{-1}} \left(2n\pi - \frac{\pi}{3}\right)\cdot\frac{1}{2} dt \\
= &\left[\left(2n\pi - \frac{\pi}{3}\right)^{-1} - \left(2n\pi + \frac{\pi}{3}\right)^{-1}\right]\cdot\left(2n\pi - \frac{\pi}{3}\right)\cdot\frac{1}{2} \\
= &\frac{\frac{\pi}{3}}{2n\pi + \frac{\pi}{3}} \\
> &\frac{1}{6}\cdot\frac{1}{n+1}
\end{aligned}$$

since both $t \mapsto \frac{1}{t} \geq 2n\pi - \frac{\pi}{3}$ and $t \mapsto \cos t \geq \frac{1}{2}$ on $\left[\left(2n\pi+\frac{\pi}{3}\right)^{-1}, \left(2n\pi-\frac{\pi}{3}\right)^{-1}\right]$.

(d) As $c \geq \frac{1}{2N\pi - \frac{\pi}{3}}$ for some integer N, by (b)(c) we have

$$\begin{aligned}
\int_c^{2\pi} |\gamma_3'(t)| dt &\geq 2\pi \int_c^{2\pi} \left|\frac{1}{t}\cos\frac{1}{t}\right| dt - 4\pi^2 \\
&\geq 2\pi \sum_{n=1}^{n=N} \int_{\left(2n\pi+\frac{\pi}{3}\right)^{-1}}^{\left(2n\pi-\frac{\pi}{3}\right)^{-1}} \left|\frac{1}{t}\cos\frac{1}{t}\right| dt - 4\pi^2 \\
&\geq 2\pi \sum_{n=1}^{n=N} \frac{1}{6}\cdot\frac{1}{n+1} - 4\pi^2 \\
&= \frac{\pi}{3} \sum_{n=1}^{n=N} \frac{1}{n+1} - 4\pi^2.
\end{aligned}$$

(e) Hence

$$\Lambda(\gamma_3) \geq \Lambda_{\left[\left(2N\pi-\frac{\pi}{3}\right)^{-1}, 2\pi\right]}(\gamma_3) \geq \frac{\pi}{3}\sum_{n=1}^{n=N}\frac{1}{n+1} - 4\pi^2.$$

Let $N \to \infty$, and thus $\Lambda(\gamma_3)$ cannot be bounded (Theorem 3.28).

□

Exercise 6.19. *Let γ_1 be a curve in $\mathbb{R}^k$, defined on $[a, b]$; let ϕ be a continuous 1-1 mapping of $[c, d]$ onto $[a, b]$, such that $\phi(c) = a$; and define $\gamma_2(s) = \gamma_1(\phi(s))$. Prove that γ_2 is an arc, a closed curve, or a rectifiable curve if and only if the same is true of γ_1. Prove that γ_2 and γ_1 have the same length.*

Proof.

(1) *Show that ϕ is strictly monotonic.* Similar to Exercise 4.15.

   (a) (Reductio ad absurdum) If ϕ were not strictly monotonic, then there exist $a < c < b \in \mathbb{R}^1$ such that
   $$\phi(a) \leq \phi(c) \geq \phi(b)$$
   or
   $$\phi(a) \geq \phi(c) \leq \phi(b).$$
   Since ϕ is one-to-one, all equalities does not hold. Hence
   $$\phi(a) < \phi(c) > \phi(b)$$
   or
   $$\phi(a) > \phi(c) < \phi(b).$$

   (b) The case $\phi(a) < \phi(c) > \phi(b)$. Take
   $$t = \frac{\max\{\phi(a), \phi(b)\} + \phi(c)}{2}$$
   so that $\phi(c) > t > \phi(a)$ and $\phi(c) > t > \phi(b)$. By Theorem 4.23 there exist $\xi_1 \in (a, c)$ and $\xi_2 \in (c, b)$ such that $\phi(\xi_1) = \phi(\xi_2) = t$. Here $\xi_1 \neq \xi_2$, contrary to the injectivity of ϕ.

   (c) The case $\phi(a) > \phi(c) < \phi(b)$. The proof is similar to (b).

   (d) By (b)(c), ϕ is strictly monotonic.

(2) $\phi(d) = b$ since ϕ is strictly monotonic (by (1)), surjective and $\phi(c) = a$.

(3) The inverse mapping ϕ^{-1} is a continuous and injective mapping of $[a, b]$ onto $[c, d]$ since ϕ is continuous and injective on a compact set $[c, d]$ (Theorem 4.17).

(4) *Show that γ_2 is an arc if and only if γ_1 is an arc.* Note the the composition of two injective maps is injective. Hence the result is established since $\gamma_2 = \gamma_1 \circ \phi$ and $\gamma_1 = \gamma_2 \circ \phi^{-1}$.

(5) *Show that γ_2 is a closed curve if and only if γ_1 is a closed curve.* Since $\gamma_2 = \gamma_1 \circ \phi$ and $\gamma_1 = \gamma_2 \circ \phi^{-1}$ (as in (4)), $\gamma_1(a) = \gamma_1(b)$ if and only if $\gamma_2(c) = \gamma_2(d)$.

(6) *Show that γ_2 is a rectifiable curve if and only if γ_1 is a rectifiable curve.* Given any partition $P_1 = \{x_0, \dots, x_n\}$ of $[a, b]$, there is a corresponding partition $P_2 = \{\phi^{-1}(x_0), \dots, \phi^{-1}(x_n)\}$ of $[c, d]$, and vice versa. (Given a partition $P_2 = \{x_0, \dots, x_n\}$ of $[c, d]$, there is a corresponding partition $P_1 = \{\phi(x_0), \dots, \phi(x_n)\}$ of $[a, b]$.) Again, since $\gamma_2 = \gamma_1 \circ \phi$ and $\gamma_1 = \gamma_2 \circ \phi^{-1}$ (as in (4)),
$$\Lambda(P_1, \gamma_1) = \Lambda(P_2, \gamma_2).$$
Hence γ_2 is rectifiable if and only if γ_1 is rectifiable.

(7) *Show that* γ_2 *and* γ_1 *have the same length.* Take the supremum over all partitions P_1 of $[a, b]$ to get

$$\Lambda(P_1, \gamma_1) = \Lambda(P_2, \gamma_2) \leq \Lambda(\gamma_1).$$

Hence $\Lambda(\gamma_1)$ is an upper of $\Lambda(P_2, \gamma_2)$. So

$$\Lambda(\gamma_2) \leq \Lambda(\gamma_1).$$

Similarly, $\Lambda(\gamma_1) \leq \Lambda(\gamma_2)$. Therefore $\Lambda(\gamma_1) = \Lambda(\gamma_2)$ (whether $\Lambda(\gamma_1)$ or $\Lambda(\gamma_2)$ is finite or not).

$\square$