# Chapter 3: Numerical Sequences and Series

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**Exercise 3.1.** *Prove that the convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?*

*Proof.*

(1) Since $\{s_n\}$ is convergent, there is $s \in \mathbb{R}^1$ with the following property: given any $\varepsilon > 0$, there is $N$ such that $|s_n - s| < \varepsilon$ whenever $n \geq N$. So

$$||s_n| - |s|| \leq |s_n - s| < \varepsilon$$

(Exercise 1.13). That is, $\{|s_n|\}$ converges to $|s|$.

(2) The converse is not true by considering $s_n = (-1)^{n+1}$.

$\square$

**Exercise 3.2.** *Calculate* $\lim_{n\to\infty}(\sqrt{n^2+n} - n)$.

*Proof.*

$$\sqrt{n^2+n} - n = \frac{n}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+\frac{1}{n}}+1} \to \frac{1}{1+1} = \frac{1}{2}$$

as $n \to \infty$. $\square$

*Proof ($\varepsilon - N$ argument).* Let $s_n = \sqrt{n^2+n} - n$. *Show that the sequence $\{s_n\}$ converges to $s = \frac{1}{2}$.* Given any $\varepsilon > 0$, there is $N > \frac{1}{\varepsilon}$ such that

$$\begin{aligned}
|s_n - s| &= \left|(\sqrt{n^2+n} - n) - \frac{1}{2}\right| = \left|\frac{1}{\sqrt{1+\frac{1}{n}}+1} - \frac{1}{2}\right| \\
&= \left|\frac{2 - \left(\sqrt{1+\frac{1}{n}}+1\right)}{2\left(\sqrt{1+\frac{1}{n}}+1\right)}\right| = \left|\frac{1 - \sqrt{1+\frac{1}{n}}}{2\left(\sqrt{1+\frac{1}{n}}+1\right)}\right| \\
&= \left|\frac{1 - \left(1 - \frac{1}{n}\right)}{2\left(\sqrt{1+\frac{1}{n}}+1\right)^2}\right| = \left|\frac{-\frac{1}{n}}{2\left(\sqrt{1+\frac{1}{n}}+1\right)^2}\right| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon
\end{aligned}$$

wheneven $n \geq N$. $\square$

**Exercise 3.3.** *If* $s_1 = \sqrt{2}$ *and*

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} \;\; (n = 1, 2, 3, ...),$$

*prove that* $\{s_n\}$ *converges, and that* $s_n < 2$ *for* $n = 1, 2, 3, ....$

The convergence of $\{s_n\}$ implies there is $s \in \mathbb{R}$ such that $s_n \to s$ where $s = \sqrt{2 + \sqrt{s}}$ and $\sqrt{2} < s \leq 2$. WolframAlpha shows that

$$s = \frac{1}{3}\left(-1 + \sqrt[3]{\frac{1}{2}(79 - 3\sqrt{249})} + \sqrt[3]{\frac{1}{2}(79 + 3\sqrt{249})}\right).$$

*Proof (Theorem 3.14).*

(1) *Show that* $\{s_n\}$ *is increasing (by mathematical induction).*
   (a) *Show that* $s_2 > s_1$. In fact,

$$s_2 = \sqrt{2 + \sqrt{s_1}} = \sqrt{2 + \sqrt{\sqrt{2}}} < \sqrt{2} = s_1.$$

   (a) *Show that* $s_{n+1} > s_n$ *if* $s_n > s_{n-1}$.

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} > \sqrt{2 + \sqrt{s_{n-1}}} = s_n.$$

   By mathematical induction, $\{s_n\}$ is (strictly) increasing.

(2) *Show that* $\{s_n\}$ *is bounded (by mathematical induction).*
   (a) *Show that* $s_1 \leq 2$. $\sqrt{2} \leq 2$.
   (a) *Show that* $s_{n+1} \leq 2$ *if* $s_n \leq 2$.

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} \leq \sqrt{2 + \sqrt{2}} < 2.$$

   By mathematical induction, $\{s_n\}$ is bounded by 2.

Hence, $\{s_n\}$ converges since $\{s_n\}$ is increasing and bounded (Theorem 3.14). $\square$

**Exercise 3.4.** *Find the upper and lower limits of the sequences* $\{s_n\}$ *defined by*

$$s_1 = 0; s_{2m} = \frac{s_{2m-1}}{2}; s_{2m+1} = \frac{1}{2} + s_{2m}.$$

Write out the first few terms of $\{s_n\}$:

$$0, 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{3}{8}, \frac{7}{8}, \frac{7}{16}, \frac{15}{16}, ...$$

It suggests us

$$\begin{aligned} s_{2m+1} &= 1 - \frac{1}{2^m} \ \ (m = 0, 1, 2, ...), \\ s_{2m} &= \frac{1}{2} - \frac{1}{2^m} \ \ (m = 1, 2, 3, ...). \end{aligned}$$

*Proof.*

(1) *Show that*

$$\begin{aligned} s_{2m+1} &= 1 - \frac{1}{2^m} \ \ (m = 0, 1, 2, ...), \\ s_{2m} &= \frac{1}{2} - \frac{1}{2^m}. \ \ (m = 1, 2, 3, ...) \end{aligned}$$

Apply mathematical induction.

(2) The upper limit is 1.

(3) The lower limit is $\frac{1}{2}$.

□

**Exercise 3.5.** *For any two real sequences* $\{a_n\}$, $\{b_n\}$, *prove that*

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

*provided the sum of the right is not of the form* $\infty - \infty$.

*Proof.* Write $\alpha = \limsup_{n\to\infty} a_n$ and $\beta = \limsup_{n\to\infty} b_n$.

(1) $\alpha = \infty$ *and* $\beta = \infty$. Nothing to do.

(2) $\alpha = -\infty$ *and* $\beta = -\infty$. Since $\alpha = -\infty < \infty$, there exists $M'$ such that $a_n < M'$ for all $n$. For any real $M$, $a_n > M - M'$ for at most a finite number of values of $n$ (Theorem 3.17(a)). Hence $a_n + b_n > M$ for at most a finite number of values of $n$. Hence $\limsup_{n\to\infty}(a_n + b_n) = -\infty$, or

$$\limsup_{n\to\infty}(a_n + b_n) = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

in this case.

(3) $\alpha$ *and* $\beta$ *are finite.* (Similar to the argument in Theorem 3.37.) Choose $\alpha' > \alpha$ and $\beta' > \beta$. There is an integer $N$ such that

$$\alpha' \geq a_n \text{ and } \beta' \geq b_n$$

whenever $n \geq N$. Hence

$$a_n + b_n \leq \alpha' + \beta'$$

whenever $n \geq N$. Take lim sup to get Hence

$$\limsup_{n \to \infty}(a_n + b_n) \leq \alpha' + \beta'.$$

Since the inequality is true for every $\alpha' > \alpha$ and $\beta' > \beta$, we have

$$\limsup_{n \to \infty}(a_n + b_n) \leq \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n.$$

□

**Exercise 3.6.** *Investigate the behavior (convergence or divergence) of* $\sum a_n$ *if*

(a) $a_n = \sqrt{n+1} - \sqrt{n}$.

(b) $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}$.

(c) $a_n = (\sqrt[n]{n} - 1)^n$.

(d) $a_n = \frac{1}{1+z^n}$ *for complex values of* $z$.

*Proof of (a).*

(1) Divergence.

(2) $\sum_{n=1}^{k} a_n = \sqrt{k+1} - 1 \to \infty$ as $k \to \infty$.

□

*Proof of (b).*

(1) Convergence.

(2) Since

$$|a_n| = \frac{1}{n(\sqrt{n+1} + \sqrt{n})} < \frac{1}{2n^{\frac{3}{2}}}$$

holds for all $n$ and $\sum \frac{1}{2n^{\frac{3}{2}}}$ converges (Theorem 3.28 and Theorem 3.3), by the comparison test (Theorem 3.25), $\sum a_n$ converges.

□

*Proof of (c).*

(1) Convergence.

(2) Note that

$$\alpha = \limsup_{n \to \infty} \sqrt[n]{|a_n|} = \limsup_{n \to \infty} \sqrt[n]{n} - 1 = 0$$

(Theorem 3.20(c)). Since $\alpha < 1$, $\sum a_n$ converges by the root test (Theorem 3.33).

□

*Proof of (d).*

(1) Convergence if $|z| > 1$; divergence if $|z| \leq 1$.

(2) Note that $|z^n + 1| + |-1| \geq |z^n|$ (Theorem 1.33(e)), or

$$|z^n + 1| \geq |z|^n - 1.$$

(3) If $|z| > 1$, then there is an integer $N$ such that

$$|z|^n \geq 2 \text{ whenever } n \geq N.$$

Therefore, for $n \geq N$ we have

$$\begin{aligned} |a_n| &= \frac{1}{|z^n + 1|} \\ &\leq \frac{1}{|z|^n - 1} && ((2)) \\ &\leq \frac{1}{|z|^n - \frac{1}{2}|z|^n} \\ &= \frac{2}{|z|^n}. \end{aligned}$$

The geometric series $\sum \frac{2}{|z|^n}$ converges, by the comparison test (Theorem 3.25), $\sum a_n$ converges.

(4) If $|z| \leq 1$, then $|a_n| \geq \frac{1}{2}$, or $\lim a_n \neq 0$. By Theorem 3.23 ($\lim a_n = 0$ if $\sum a_n$ converges), $\sum a_n$ diverges.

□

**Exercise 3.7.** *Prove that the convergence of $\sum a_n$ implies the convergence of*

$$\sum \frac{\sqrt{a_n}}{n},$$

*if* $a_n \geq 0$.

*Proof (Cauchy's inequatity).*

(1) *Show that* $\sum \frac{\sqrt{a_n}}{n}$ *is bounded.* For any $k \in \mathbb{Z}^+$,

$$\begin{aligned}\left(\sum_{n=1}^{k} \frac{\sqrt{a_n}}{n}\right)^2 &\leq \left(\sum_{n=1}^{k} a_n\right)\left(\sum_{n=1}^{k} \frac{1}{n^2}\right) && \text{(Cauchy's inequatity)}\\ &\leq \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right). && \left(\sum a_n, \sum \frac{1}{n^2}\text{: convergent}\right)\end{aligned}$$

Thus, $\left(\sum_{n=1}^{k} \frac{\sqrt{a_n}}{n}\right)^2$ is bounded, or $\sum_{n=1}^{k} \frac{\sqrt{a_n}}{n}$ is bounded.

(2) *Show that* $\sum_{n=1}^{k} \frac{\sqrt{a_n}}{n}$ *is increasing.* It is clear due to $\frac{\sqrt{a_n}}{n} \geq 0$.

By Theorem 3.14, $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges. □

*Proof (AM-GM inequality). Show that* $\sum \frac{\sqrt{a_n}}{n}$ *is bounded.*

$$\begin{aligned}\frac{\sqrt{a_n}}{n} &\leq \frac{1}{2}\left(a_n + \frac{1}{n^2}\right) && \text{(AM-GM inequality)}\\ \sum_{n=1}^{k} \frac{\sqrt{a_n}}{n} &\leq \frac{1}{2}\left(\sum_{n=1}^{k} a_n + \sum_{n=1}^{k} \frac{1}{n^2}\right)\\ &\leq \frac{1}{2}\left(\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} \frac{1}{n^2}\right). && \left(\sum a_n, \sum \frac{1}{n^2}\text{: convergent}\right)\end{aligned}$$

Thus, $\sum_{n=1}^{k} \frac{\sqrt{a_n}}{n}$ is bounded. The rest proof is the same as previous. □

**Exercise 3.8.** *If* $\sum a_n$ *converges, and if* $\{b_n\}$ *is monotonic and bounded, prove that* $\sum a_n b_n$ *converges.*

*Proof (Theorem 3.42).* There are only two possible cases (might be overlapped).

(1) $\{b_n\}$ *is decreasing to* $b$. Define $\{\beta_n\}$ by $\beta_n = b_n - b$.

   (a) The partial sums of $\sum a_n$ form a bounded sequence since $\sum a_n$ converges.
   (b) $\{\beta_n\}$ is monotonically decreasing.
   (c) $\lim \beta_n = 0$.

   By (1)(2)(3), $\sum a_n \beta_n$ converges. Hence

$$\sum a_n b_n = \sum a_n \beta_n + \sum a_n b$$

   converges (Theorem 3.3(a)(b)).

(2) $\{b_n\}$ *is increasing to $b$.* Similar to (1). Define $\{\beta_n\}$ by $\beta_n = b - b_n$. Thus $\sum a_n\beta_n$ converges. Hence

$$\sum a_n b_n = -\sum a_n\beta_n + \sum a_n b$$

converges.

$\square$

**Exercise 3.9.** *Find the radius of convergence of each of the following power series:*

(a) $\sum n^3 z^n$,

(b) $\sum \frac{2^n}{n!} z^n$,

(c) $\sum \frac{2^n}{n^2} z^n$,

(d) $\sum \frac{n^3}{3^n} z^n$.

*Proof of (a).* Since

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{n^3} = \limsup_{n\to\infty} (\sqrt[n]{n})^3 = 1$$

(Theorem 3.20(c)), $R = \frac{1}{\alpha} = 1$. $\square$

*Proof of (b).*

(1) Note that $\sqrt[n]{n!} \leq \sqrt[n]{n^n} = n$. *Show that $\sqrt[n]{n!} \geq \sqrt{n}$.* Note that

$$(n!)^2 = \prod_{k=1}^{n} k(n+1-k).$$

For each term $k(n+1-k)$ (where $k = 1, \ldots, n$), we have

$$k(n+1-k) - n = (k-1)(n-k) \geq 0 \text{ or } k(n+1-k) > n.$$

or $k(n+1-k) > n$. Hence,

$$(n!)^2 = \prod_{k=1}^{n} k(n+1-k) \geq \prod_{k=1}^{n} n = n^n,$$

or $\sqrt[n]{n!} \geq \sqrt{n}$.

(2) Since

$$0 \leq \alpha = \limsup_{n\to\infty} \sqrt[n]{\frac{2^n}{n!}} = \limsup_{n\to\infty} \frac{2}{\sqrt[n]{n!}} \leq \limsup_{n\to\infty} \frac{2}{\sqrt{n}} = 0,$$

$\alpha = 0$ and $R = \frac{1}{\alpha} = \infty$.

$\square$

*Proof of (c).* Similar to (a). Since

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{\frac{2^n}{n^2}} = \limsup_{n\to\infty} \frac{2}{\sqrt[n]{n}^2} = 2$$

(Theorem 3.20(c)), $R = \frac{1}{\alpha} = \frac{1}{2}$. $\square$

*Proof of (d).* Similar to (a)(c). Since

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{\frac{n^3}{3^n}} = \limsup_{n\to\infty} \frac{\sqrt[n]{n}^3}{3} = \frac{1}{3}$$

(Theorem 3.20(c)), $R = \frac{1}{\alpha} = 3$. $\square$

**Exercise 3.10.** *Suppose that the coefficients of the power series $\sum a_n z^n$ are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.*

*Proof (Theorem 3.39).* $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|} \geq 1$ by assumption that $\{a_n\}$ has infinitely many nonzero integers. Hence the radius of convergence $R = \frac{1}{\alpha} \leq 1$. $\square$

**Exercise 3.11.** *Suppose $a_n > 0$, $s_n = a_1 + \cdots + a_n$, and $\sum a_n$ diverges.*

(a) *Prove that $\sum \frac{a_n}{1+a_n}$ diverges.*

(b) *Prove that*

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$$

*and deduce that $\sum \frac{a_n}{s_n}$ diverges.*

(c) *Prove that*

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

*and deduce that $\sum \frac{a_n}{s_n^2}$ converges.*

(d) *What can be said about*

$$\sum \frac{a_n}{1+na_n} \text{ and } \sum \frac{a_n}{1+n^2a_n}?$$

*Proof of (a).* (Reductio ad absurdum)

(1) If $\sum \frac{a_n}{1+a_n}$ were convergent, $\lim \frac{a_n}{1+a_n} = 0$ (Theorem 3.23). Note that $\frac{a_n}{1+a_n} = \frac{1}{1+\frac{1}{a_n}}$ implies $\lim a_n = 0$.

(2) Since $\lim a_n = 0$, there is an integer $N$ such that
$$0 < a_n < 1 \text{ whenever } n \geq N.$$
Hence
$$|a_n| = a_n \leq \frac{2a_n}{1+a_n} \text{ whenever } n \geq N.$$
By the comparison test (Theorem 3.25), $\sum a_n$ converges, contrary to the divergence of $\sum a_n$.

□

*Proof of (b).*

(1) Note that each $s_n > 0$ and $\{s_n\}$ is monotonic increasing. For $k \geq 1$,
$$\begin{aligned}\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} &\geq \frac{a_{N+1}}{s_{N+k}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \\ &= \frac{a_{N+1} + \cdots + a_{N+k}}{s_{N+k}} \\ &= \frac{s_{N+k} - s_N}{s_{N+k}} \\ &= 1 - \frac{s_N}{s_{N+k}}.\end{aligned}$$

(2) (Reductio ad absurdum) If $\sum \frac{a_n}{s_n}$ were convergent, by the Cauchy criterion (Theorem 3.22), for $\varepsilon = \frac{1}{64} > 0$, there is an integer $N$ such that
$$\left| \sum_{n=N+1}^{N+k} \frac{a_n}{s_n} \right| < \frac{1}{64} \quad \text{whenever} \quad k \geq 1.$$
So,
$$\frac{1}{64} > \sum_{n=N+1}^{N+k} \frac{a_n}{s_n} > 1 - \frac{s_N}{s_{N+k}} \quad \text{or} \quad s_{N+k} < \frac{64}{63} s_N,$$
contrary to divergence of $\sum a_n = \infty$ (as $k \to \infty$).

□

*Proof of (c).*

(1) For $n \geq 2$,
$$\frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{s_n - s_{n-1}}{s_{n-1}s_n} = \frac{a_n}{s_{n-1}s_n} \geq \frac{a_n}{s_n^2}.$$

(2) $\sum \frac{a_n}{s_n^2}$ is a series of nonnegative terms and its partial sums

$$\begin{aligned}\sum_{n=1}^{k} \frac{a_n}{s_n^2} &\leq \frac{a_1}{s_1^2} + \sum_{n=2}^{k} \left( \frac{1}{s_{n-1}} - \frac{1}{s_n} \right) \\ &= \frac{a_1}{s_1^2} + \frac{1}{s_1} - \frac{1}{s_k} \\ &= \frac{2}{a_1} - \frac{1}{s_k} \\ &< \frac{2}{a_1}\end{aligned}$$

is bounded (by $\frac{2}{a_1}$). Therefore, $\sum \frac{a_n}{s_n^2}$ converges (Theorem 3.24).

□

*Proof of (d).*

(1) *Show that there is a divergent series* $\sum a_n$ *with* $a_n > 0$ *such that* $\sum \frac{a_n}{1+na_n}$ *converges or diverges.*

(a) Take

$$a_n = \frac{1}{n(\log n)^p}$$

where $0 \leq p \leq 1$.

(b) Clearly,

$$\sum_{n=3}^{\infty} a_n = \sum_{n=3}^{\infty} \frac{1}{n(\log n)^p}$$

diverges (Theorem 3.29).

(c) Note that

$$\begin{aligned}\sum_{n=3}^{\infty} \frac{a_n}{1+na_n} &= \sum_{n=3}^{\infty} \frac{1}{n(\log n)^p} \cdot \frac{1}{1+(\log n)^p} \\ &= \sum_{n=3}^{\infty} \frac{1}{n(\log n)^p + n(\log n)^{2p}}.\end{aligned}$$

Hence,

$$\sum_{n=3}^{\infty} \frac{1}{2n(\log n)^{2p}} \leq \sum_{n=3}^{\infty} \frac{a_n}{1+na_n} < \sum_{n=3}^{\infty} \frac{1}{n(\log n)^{2p}}.$$

(Here we use the fact that $n(\log n)^p > 0$ and $(\log n)^p \geq 1$ if $n > e$.) Therefore,

$$\sum_{n=3}^{\infty} \frac{a_n}{1+na_n} = \begin{cases} \text{converges} & \text{if } 1 \geq p > \frac{1}{2} \\ \text{diverges} & \text{if } \frac{1}{2} \geq p \geq 0 \end{cases}$$

by Theorem 3.29 and the comparison test (Theorem 3.24).

*Note.* If a series $\sum a_n$ with $a_n > 0$ is convergent, then $\sum \frac{a_n}{1+na_n}$ is always convergent by the comparison test (Theorem 3.24).

(2) *Given any series $\sum a_n$ with $a_n > 0$. Show that*

$$\sum \frac{a_n}{1+n^2 a_n} < \infty$$

*converges.* Note that

$$\left| \frac{a_n}{1+n^2 a_n} \right| = \frac{1}{\frac{1}{a_n} + n^2} < \frac{1}{n^2}$$

for any $n$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (to $\frac{\pi^2}{6}$). By the comparison test (Theorem 3.25), $\sum \frac{a_n}{1+n^2 a_n}$ converges.

$\square$

*Note.* Similar to (d), *what can be said about*

$$\sum \frac{a_n}{1+n(\log n)a_n} \quad \text{and} \quad \sum \frac{a_n}{1+n(\log n)^2 a_n}?$$

**Exercise 3.12.** *Suppose $a_n > 0$ and $\sum a_n$ converges. Put*

$$r_n = \sum_{m=n}^{\infty} a_m.$$

(a) *Prove that*

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$

*if $m < n$, and deduce that $\sum \frac{a_n}{r_n}$ diverges.*

(b) *Prove that*

$$\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$$

*and deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.*

*Note.*

(1) Each $r_n$ is positive and finite (since $a_n > 0$ and $\sum a_n$ converges).

(2) $\{r_n\}$ is monotonic decreasing (since $a_n > 0$).

(3) $\{r_n\}$ converges to 0 (since $\sum a_n$ converges).

*Proof of (a).*

(1)

$$\begin{aligned}
\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} &> \frac{a_m}{r_m} + \cdots + \frac{a_n}{r_m} && (r_m > r_k \text{ for } k = m+1, \dots, n) \\
&= \frac{a_m + \cdots + a_n}{r_m} \\
&= \frac{r_m - r_{n+1}}{r_m} && (\text{Definition of } r_k) \\
&> \frac{r_m - r_n}{r_m} && (r_n > r_{n+1}) \\
&= 1 - \frac{r_n}{r_m}.
\end{aligned}$$

(2) (Reductio ad absurdum) If $\sum \frac{a_n}{r_n}$ were convergent, then given $\varepsilon = \frac{1}{64} > 0$ there is an integer $N$ such that

$$\left| \frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} \right| < \frac{1}{64} \text{ whenever } n \geq m \geq N$$

(Theorem 3.22). By (1), let $m = N$ to get

$$1 - \frac{r_n}{r_N} < \frac{1}{64} \text{ whenever } n \geq N,$$

or

$$r_n > \frac{63}{64} r_N,$$

contrary to the assumption that $\{r_n\}$ converges to 0 (since $\sum a_n$ converges).

$\square$

*Proof of (b).*

(1) Note that each $r_n$ is positive and finite, and thus

$$\begin{aligned}
\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}}) &\Longleftrightarrow \frac{r_n - r_{n+1}}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}}) \\
&\Longleftrightarrow \frac{\sqrt{r_n} + \sqrt{r_{n+1}}}{\sqrt{r_n}} < 2 \\
&\Longleftrightarrow \sqrt{r_n} + \sqrt{r_{n+1}} < 2\sqrt{r_n} \\
&\Longleftrightarrow \sqrt{r_{n+1}} < \sqrt{r_n} \\
&\Longleftrightarrow r_{n+1} < r_n.
\end{aligned}$$

The last statement holds since $\{r_n\}$ is monotonic decreasing.

(2) (a) Each term $\frac{a_n}{\sqrt{r_n}}$ of $\sum \frac{a_n}{\sqrt{r_n}}$ is nonnegative.

(b) The partial sum

$$\sum_{k=1}^{n} \frac{a_k}{\sqrt{r_k}} < \sum_{k=1}^{n} 2(\sqrt{r_k} - \sqrt{r_{k+1}}) = 2(\sqrt{r_1} - \sqrt{r_{n+1}}) < 2\sqrt{r_1}$$

is bounded by $2\sqrt{r_1}$.

By (a)(b), $\sum \frac{a_n}{\sqrt{r_n}}$ converges (Theorem 3.24).

□

**Exercise 3.13.** *Prove that the Cauchy product of two absolutely convergent series converges absolutely.*

*Proof.*

(1) Given two absolutely convergent series $\sum a_n$ and $\sum b_n$. The Cauchy product is $\sum c_n$ where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \ \ (n = 0, 1, 2, \ldots).$$

Let $\sum |a_n| = A < \infty$ and $\sum |b_n| = B < \infty$.

(2) Each term $|c_k|$ of $\sum_{k=0}^{n} |c_k|$ is nonnegative.

(3) Thus,

$$\begin{aligned}
\sum_{k=0}^{n} |c_k| &= \sum_{k=0}^{n} \left| \sum_{m=0}^{k} a_m b_{k-m} \right| \\
&\leq \sum_{k=0}^{n} \sum_{m=0}^{k} |a_m||b_{k-m}| \\
&= \sum_{k=0}^{n} |a_k| \sum_{m=0}^{n-k} |b_m| \\
&\leq \sum_{k=0}^{n} |a_k| B \\
&\leq AB \\
&< \infty.
\end{aligned}$$

(4) By (2)(3), $\sum_{k=0}^{n} |c_k|$ converges (Theorem 3.24), or $\sum_{k=0}^{n} c_k$ converges absolutely.

$\square$

**Exercise 3.14 (Cesàro convergence).** *If $\{s_n\}$ is a complex sequence, define its arithmetic means $\sigma_n$ by*

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1} \quad (n = 0, 1, 2, \ldots).$$

(a) *If $\lim s_n = s$, prove that $\lim \sigma_n = s$.*

(b) *Construct a sequence $\{s_n\}$ which does not converge, although $\lim \sigma_n = 0$.*

(c) *Can it happen that $s_n > 0$ for all $n$ and that $\limsup s_n = \infty$, although $\lim \sigma_n = 0$?*

(d) *Put $a_n = s_n - s_{n-1}$, for $n \geq 1$. Show that*

$$s_n - \sigma_n = \frac{1}{n+1} \sum_{k=1}^{n} k a_k.$$

*Assume that $\lim(na_n) = 0$ and that $\{\sigma_n\}$ converges. Prove that $\{s_n\}$ converges. [This gives a converse of (a), but under the additional assumption that $na_n \to 0$.]*

(e) *Derive the last conclusion from a weaker hypothesis: Assume $M \leq \infty$, $|na_n| < M$ for all $n$, and $\lim \sigma_n = \sigma$. Prove that $\lim s_n = \sigma$, by completing the following outline:*

*If $m < n$, then*

$$s_n - \sigma_n = \frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m} \sum_{i=m+1}^{n} (s_n - s_i).$$

*For these $i$,*

$$|s_n - s_i| \leq \frac{(n-i)M}{i+1} \leq \frac{(n-m-1)M}{m+2}.$$

*Fix $\varepsilon > 0$ and associate with each $n$ the integer $m$ that satisfies*

$$m \leq \frac{n-\varepsilon}{1+\varepsilon} < m+1.$$

*Then $\frac{m+1}{n-m} \leq \frac{1}{\varepsilon}$ and $|s_n - s_i| < M\varepsilon$. Hence*

$$\limsup_{n\to\infty} |s_n - \sigma| \leq M\varepsilon.$$

*Since $\varepsilon$ was arbitrary, $\lim s_n = \sigma$.*

*Proof of (a).* It is similar to Exercise 8.11. Given any $\varepsilon > 0$.

(1) For such $\varepsilon > 0$, there is an integer $N' \geq 1$ such that

$$|s_n - s| < \frac{\varepsilon}{64} \text{ whenever } n \geq N'.$$

(2) For such $N'$, $\sum_{n=0}^{N'} |s_n - s|$ is finite. Let $N''$ be an integer such that

$$\sum_{n=0}^{N'} |s_n - s| < \frac{N''\varepsilon}{89}$$

(by taking $N'' = \left\lfloor \frac{89}{\varepsilon} \sum_{n=0}^{N'} |s_n - s| \right\rfloor + 1$).

(3) Note that

$$\begin{aligned}
|\sigma_n - s| &= \left| \left( \frac{1}{n+1} \sum_{k=0}^{n} s_k \right) - s \right| \\
&= \left| \frac{1}{n+1} \sum_{k=0}^{n} (s_k - s) \right| \\
&\leq \frac{1}{n+1} \sum_{k=0}^{n} |s_k - s|
\end{aligned}$$

holds for each $n = 0, 1, 2, \ldots$. In particular, for $n \geq N = \max\{N', N''\} \geq 1$, we have

$$\begin{aligned}
|\sigma_n - s| &\leq \frac{1}{n+1} \sum_{k=0}^{n} |s_k - s| \\
&\leq \left( \frac{1}{n+1} \sum_{k=0}^{N'} |s_k - s| \right) + \left( \frac{1}{n+1} \sum_{k=N'+1}^{n} |s_k - s| \right) \\
&< \frac{1}{n+1} \cdot \frac{N''\varepsilon}{89} + \frac{1}{n+1} \cdot \frac{(n - N')\varepsilon}{64} \\
&< \frac{\varepsilon}{89} + \frac{\varepsilon}{64} \\
&< \varepsilon.
\end{aligned}$$

Therefore, $\lim \sigma_n = s$.

$\square$

*Proof of (b).* Define $\{s_n\}$ by $s_n = (-1)^{n+1}$. $\square$

*Proof of (c).* Yes. Define

$$s_n = \begin{cases} \frac{1}{n!} + m^{63} & \text{if } n = m^{89} \text{ for some } m \in \mathbb{Z}, \\ \frac{1}{n!} & \text{otherwise.} \end{cases}$$

(1) Clearly, $\limsup s_n = \infty$.

(2) Given any $n$, there is $m \in \mathbb{Z}$ satisfying $m^{89} \leq n < (m+1)^{89}$. So

$$\begin{aligned}
0 < \sigma_n &= \frac{1}{n+1} \sum_{k=0}^{n} s_k \\
&\leq \frac{1}{m^{89}+1} \sum_{k=0}^{n} s_k \\
&= \frac{1}{m^{89}+1} \left( \sum_{k=0}^{n} \frac{1}{n!} + \sum_{k=0}^{m} k^{63} \right) \\
&\leq \frac{1}{m^{89}+1} \left( \sum_{k=0}^{\infty} \frac{1}{n!} + \sum_{k=0}^{m} m^{63} \right) \\
&= \frac{e + m \cdot m^{63}}{m^{89}+1} \\
&= \frac{m^{64} + e}{m^{89}+1}.
\end{aligned}$$

Let $n \to \infty$, then $m \to \infty$ and thus $\lim \sigma_n = 0$.

□

*Proof of (d).*

(1)

$$\begin{aligned}
\frac{1}{n+1} \sum_{k=1}^{n} k a_k &= \frac{1}{n+1} \sum_{k=1}^{n} k(s_k - s_{k-1}) \\
&= \frac{1}{n+1} \left( \sum_{k=1}^{n} k s_k - \sum_{k=1}^{n} k s_{k-1} \right) \\
&= \frac{1}{n+1} \left( \sum_{k=1}^{n} k s_k - \sum_{k=1}^{n} (k-1) s_{k-1} - \sum_{k=1}^{n} s_{k-1} \right) \\
&= \frac{1}{n+1} \left( n s_n - \sum_{k=1}^{n} s_{k-1} \right) \\
&= \frac{1}{n+1} \left( (n+1) s_n - \sum_{k=1}^{n+1} s_{k-1} \right) \\
&= s_n - \sigma_n.
\end{aligned}$$

(2) Write

$$s_n = \sigma_n + \frac{1}{n+1} \sum_{k=1}^{n} k a_k.$$

Since $\lim_{n\to\infty}(na_n) = 0$, $\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=1}^{n} ka_k = 0$ ((a)). Since $\{\sigma_n\}$ converges,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sigma_n + \lim_{n\to\infty} \frac{1}{n+1}\sum_{k=1}^{n} ka_k = \lim_{n\to\infty} \sigma_n$$

(Theorem 3.3(a)).

□

*Proof of (e).*

(1) If $m < n$, then

$$\begin{aligned}
\sigma_n - \sigma_m &= \frac{1}{n+1}\sum_{k=0}^{n} s_k - \frac{1}{m+1}\sum_{k=0}^{m} s_k \\
&= \frac{1}{n+1}\sum_{k=0}^{n} s_k - \frac{1}{m+1}\sum_{k=0}^{n} s_k + \frac{1}{m+1}\sum_{i=m+1}^{n} s_i \\
&= \frac{m-n}{(m+1)(n+1)}\sum_{k=0}^{n} s_k + \frac{1}{m+1}\sum_{i=m+1}^{n} s_i \\
&= \frac{m-n}{m+1}\sigma_n + \frac{1}{m+1}\sum_{i=m+1}^{n} s_i, \\
\frac{m+1}{n-m}(\sigma_n - \sigma_m) &= -\sigma_n + \frac{1}{n-m}\sum_{i=m+1}^{n} s_i \\
&= -\sigma_n - \frac{1}{n-m}\sum_{i=m+1}^{n} (-s_i) \\
&= -\sigma_n - \left(\frac{1}{n-m}\sum_{i=m+1}^{n} (s_n - s_i)\right) + s_n, \\
s_n - \sigma_n &= \frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m}\sum_{i=m+1}^{n} (s_n - s_i).
\end{aligned}$$

(2) For these $i$,

$$\begin{aligned}
|s_n - s_i| &= \left| \sum_{k=i+1}^{n} a_k \right| && (s_n - s_i = \sum_{k=i+1}^{n} a_k) \\
&\leq \sum_{k=i+1}^{n} |a_k| && \text{(Triangle inequality)} \\
&< \sum_{k=i+1}^{n} \frac{M}{k} && (|ka_k| < M) \\
&\leq \sum_{k=i+1}^{n} \frac{M}{i+1} && (k \geq i+1) \\
&= \frac{(n-i)M}{i+1} \\
&= \left( \frac{n-1}{i+1} - 1 \right) M \\
&\leq \left( \frac{n-1}{m+2} - 1 \right) M && (i \geq m+1) \\
&= \frac{(n-m-1)M}{m+2}.
\end{aligned}$$

(3) Fix $1 > \varepsilon > 0$ and associate with each $n$ the integer $m$ that satisfies

$$m \leq \frac{n-\varepsilon}{1+\varepsilon} < m+1.$$

Clearly, $m \leq \frac{n-\varepsilon}{1+\varepsilon} < \frac{n}{1+\varepsilon} < n$. Then

$$\frac{m+1}{n-m} \leq \frac{1}{\varepsilon} \text{ and } \frac{n-m-1}{m+2} < \varepsilon.$$

Hence $|s_n - s_i| < M\varepsilon$ by (2).

(4) By (1)(3),

$$s_n - \sigma = (\sigma_n - \sigma) + \frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m} \sum_{i=m+1}^{n} (s_n - s_i),$$

$$\begin{aligned}
|s_n - \sigma| &\leq |\sigma_n - \sigma| + \frac{m+1}{n-m}|\sigma_n - \sigma_m| + \frac{1}{n-m} \sum_{i=m+1}^{n} |s_n - s_i| \\
&< |\sigma_n - \sigma| + \frac{1}{\varepsilon}|\sigma_n - \sigma_m| + \frac{1}{n-m} \sum_{i=m+1}^{n} M\varepsilon \\
&= |\sigma_n - \sigma| + \frac{1}{\varepsilon}|\sigma_n - \sigma_m| + M\varepsilon
\end{aligned}$$

holds for any $n$ and $m$ satisfying $m \leq \frac{n-\varepsilon}{1+\varepsilon} < m+1$. Since $\{\sigma_n\}$ converges, there is an integer $N$ such that

$$|\sigma_n - \sigma_m| < \varepsilon^2 \text{ whenever } m, n \geq N,$$

$$|\sigma_n - \sigma| < \varepsilon \text{ whenever } n \geq N.$$

So,

$$|s_n - \sigma| < (M+2)\varepsilon$$

holds for any $n \geq 2N+3$ (and the corresponding $m$ satisfying $m \leq \frac{n-\varepsilon}{1+\varepsilon} < m+1$ (which implies $m > \frac{n-\varepsilon}{1+\varepsilon} - 1 \geq \frac{n-1}{2} - 1 \geq N$)). Take limit to get

$$\limsup_{n\to\infty} |s_n - \sigma| \leq (M+2)\varepsilon.$$

Since $\varepsilon$ was arbitrary, $\lim s_n = \sigma$.

□

**Exercise 3.15.** *Definition 3.21 can be extended to the case in which the* $a_n$ *lie in some fixed* $\mathbb{R}^k$. *Absolute convergence is defined as convergence of* $\sum |\mathbf{a}_n|$. *Show that Theorems 3.22, 3.23, 3.25(a), 3.33, 3.34, 3.42, 3.45, 3.47, and 3.55 are true in this more general settings. (Only slight modifications are required in any of the proofs.)*

**Definition 3.21.** *Given a sequence* $\{\mathbf{a}_n\} \subseteq \mathbb{R}^k$, *we use the notation*

$$\sum_{n=p}^{q} \mathbf{a}_n \;\; (p \leq q)$$

*to denote the sum* $\mathbf{a}_p + \mathbf{a}_{p+1} + \cdots + \mathbf{a}_q$. *With* $\{\mathbf{a}_n\}$ *we associate a sequence* $\{\mathbf{s}_n\}$, *where*

$$\mathbf{s}_n = \sum_{k=1}^{n} \mathbf{a}_k.$$

*For* $\{\mathbf{s}_n\}$ *we also use the symbolic expression*

$$\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \cdots$$

*or, more precisely,*

$$\sum_{n=1}^{\infty} \mathbf{a}_n. \tag{4}$$

*The symbol (4) we call an* ***infinite series****, or just a* ***series****. The number $\{\mathbf{s}_n\}$, are called the* ***partial sums*** *of the series. If $\{\mathbf{s}_n\}$ converges to $\mathbf{s}$, we say that the series* ***converges****, and write*

$$\sum_{n=1}^{\infty} \mathbf{a}_n = \mathbf{s}.$$

*The number $\mathbf{s}$ is called the sum of the series; but it should be clearly understood that $\mathbf{s}$* ***is the limit of a sequence of sums****, and is not obtained simply by addition.*

*If $\{\mathbf{s}_n\}$ diverges, the series said to be diverge.*

*Sometimes, for convenience of notation, we shall consider series of the form*

$$\sum_{n=0}^{\infty} \mathbf{a}_n. \tag{5}$$

*And frequently, when there is no possible ambiguity, or when the distinction is immaterial, we shall simply write $\sum \mathbf{a}_n$ in place of (4) or (5).*

*It is clear that every theorem about sequences can be stated in terms of series (putting $\mathbf{a}_1 = \mathbf{s}_1$ and $\mathbf{a}_n = \mathbf{s}_n - \mathbf{s}_{n-1}$ for $n > 1$), and vice versa. But it is nevertheless useful to consider both concepts.*

**Theorem 3.22 over $\mathbb{R}^k$.** *$\sum \mathbf{a}_n$ converges if and only if for every $\varepsilon > 0$ there is an integer $N$ such that*

$$\left| \sum_{k=n}^{m} \mathbf{a}_k \right| \leq \varepsilon$$

*if $m \geq n \geq N$.*

*Proof of Theorem 3.22 over $\mathbb{R}^k$.* The Cauchy criterion (Theorem 3.11) can be restated in this form. □

**Theorem 3.23 over $\mathbb{R}^k$.** *If $\sum \mathbf{a}_n$ converges, then $\lim_{n\to\infty} \mathbf{a}_n = \mathbf{0}$.*

*Proof of Theorem 3.23 over $\mathbb{R}^k$.* By taking $m = n$ in Theorem 3.22 over $\mathbb{R}^k$,

$$|\mathbf{a}_n| \leq \varepsilon \quad \text{whenever } n \geq N.$$

□

**Theorem 3.25(a) over $\mathbb{R}^k$ (Comparison Test).** *If $|\mathbf{a}_n| \le c_n$ for $n \ge N_0$, where $N_0$ is some fixed integer, and if $\sum c_n$ converges, then $\sum \mathbf{a}_n$ converges.*

*Proof of Theorem 3.25(a) over $\mathbb{R}^k$.* Given $\varepsilon > 0$, there exists $N \ge N_0$ such that $m \ge n \ge N$ implies

$$\sum_{k=n}^{m} c_k \le \varepsilon,$$

by the Cauchy criterion. Hence

$$\left| \sum_{k=n}^{m} \mathbf{a}_k \right| \le \sum_{k=n}^{m} |\mathbf{a}_k| \le \sum_{k=n}^{m} c_k \le \varepsilon,$$

and (a) follows. □

**Theorem 3.33 over $\mathbb{R}^k$ (Root Test).** *Given $\sum \mathbf{a}_n$, put $\alpha = \limsup_{n\to\infty} \sqrt[n]{|\mathbf{a}_n|}$. Then*

(a) *if $\alpha < 1$, $\sum \mathbf{a}_n$ converges;*

(b) *if $\alpha > 1$, $\sum \mathbf{a}_n$ diverges;*

(c) *if $\alpha = 1$, the test gives no information.*

*Proof of Theorem 3.33(a) over $\mathbb{R}^k$.* If $\alpha < 1$, we can choose $\beta$ so that $\alpha < \beta < 1$, and an integer $N$ such that

$$\sqrt[n]{|\mathbf{a}_n|} < \beta$$

for $n \ge N$ [by Theorem 3.17(b)]. That is, $n \ge N$ implies

$$|\mathbf{a}_n| < \beta^n.$$

Since $0 < \beta < 1$, $\sum \beta^n$ converges. Convergence of $\sum \mathbf{a}_n$ follows now from the comparison test. □

*Proof of Theorem 3.33(b) over $\mathbb{R}^k$.* If $\alpha > 1$, again by Theorem 3.17, there is a sequence $\{n_k\}$ such that

$$\sqrt[n_k]{|\mathbf{a}_{n_k}|} \to \alpha.$$

Hence $|\mathbf{a}_n| > 1$ for infinitely many values of $n$, so that the condition $\mathbf{a}_n \to \mathbf{0}$, necessary for convergence of $\sum \mathbf{a}_n$, does not hold (Theorem 3.23 over $\mathbb{R}^k$). □

*Proof of Theorem 3.33(c) over $\mathbb{R}^k$.* Same as the original proof. □

**Theorem 3.34 over $\mathbb{R}^k$ (Ratio Test).** *The series $\sum \mathbf{a}_n$*

(a) *converges if* $\limsup_{n\to\infty} \frac{|\mathbf{a}_{n+1}|}{|\mathbf{a}_n|} < 1$,

(b) *diverges if* $\frac{|\mathbf{a}_{n+1}|}{|\mathbf{a}_n|} \geq 1$ *for* $n \geq N_0$*, where* $N_0$ *is some fixed integer.*

*Proof of Theorem 3.34(a) over* $\mathbb{R}^k$. If condition (a) holds, we can find $\beta < 1$, and an integer $N$, such that

$$\frac{|\mathbf{a}_{n+1}|}{|\mathbf{a}_n|} < \beta$$

for $n \geq N$. In particular,

$$\begin{aligned} |\mathbf{a}_{N+1}| &< \beta|\mathbf{a}_N|, \\ |\mathbf{a}_{N+2}| &< \beta|\mathbf{a}_{N+1}| < \beta^2|\mathbf{a}_N|, \\ &\cdots \\ |\mathbf{a}_{N+p}| &< \beta^p|\mathbf{a}_N|. \end{aligned}$$

That is,

$$|\mathbf{a}_n| < |\mathbf{a}_N|\beta^{-N} \cdot \beta^n$$

for $n \geq N$, and (a) follows from the comparison test, since $\sum \beta^n$ converges. □

*Proof of Theorem 3.34(b) over* $\mathbb{R}^k$. If $|\mathbf{a}_{n+1}| \geq |\mathbf{a}_n|$ for $n \geq N_0$, it is easily seen that the condition $\mathbf{a}_n \to \mathbf{0}$ does not hold, and (b) follows. □

*Note.* The knowledge that $\lim \frac{|\mathbf{a}_{n+1}|}{|\mathbf{a}_n|} = 1$ implies nothing about the convergence of $\sum \mathbf{a}_n$. The series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$ demonstrate this.

**Theorem 3.42 over** $\mathbb{R}^k$**.** *Suppose*

(a) *the partial sums* $\mathbf{A}_n$ *of* $\sum \mathbf{a}_n$ *form a bounded sequence;*

(b) $b_0 \geq b_1 \geq b_2 \geq \cdots$*;*

(c) $\lim_{n\to\infty} b_n = 0$.

*Then* $\sum \mathbf{a}_n b_n$ *converges.*

*Proof of Theorem 3.42 over* $\mathbb{R}^k$. Choose $M > 0$ such that $|\mathbf{A}_n| \leq M$ for all $n$.

Given $\varepsilon > 0$, there is an integer $N$ such that $b_N \leq \frac{\varepsilon}{2M}$. For $N \leq p \leq q$, we have

$$\begin{aligned}
\left|\sum_{n=p}^{q} \mathbf{a}_n b_n\right| &= \left|\sum_{n=p}^{q-1} \mathbf{A}_n(b_n - b_{n+1}) + \mathbf{A}_q b_q - \mathbf{A}_{p-1} b_p\right| \\
&\leq M\left|\sum_{n=p}^{q-1}(b_n - b_{n+1}) + b_q + b_p\right| && (b_n - b_{n+1} \geq 0) \\
&= 2Mb_p \\
&\leq 2Mb_N \\
&\leq \varepsilon.
\end{aligned}$$

Convergence now follows from the Cauchy criterion. □

The series $\sum \mathbf{a}_n$ is said to **converge absolutely** if the series $\sum |\mathbf{a}_n|$ converges.

**Theorem 3.45 over $\mathbb{R}^k$.** *If $\sum \mathbf{a}_n$ converges absolutely, then $\sum \mathbf{a}_n$ converges.*

*Proof of Theorem 3.45 over $\mathbb{R}^k$.* The assertion follows from the inequality

$$\left|\sum_{k=n}^{m} \mathbf{a}_k\right| \leq \sum_{k=n}^{m} |\mathbf{a}_k|$$

plus the Cauchy criterion. □

**Theorem 3.47 over $\mathbb{R}^k$.** *If $\sum \mathbf{a}_n = \mathbf{A}$, and $\sum \mathbf{b}_n = \mathbf{B}$, then $\sum(\mathbf{a}_n + \mathbf{b}_n) = \mathbf{A} + \mathbf{B}$, and $\sum c\mathbf{a}_n = c\mathbf{A}$ for any fixed $c \in \mathbb{R}$.*

*Proof of Theorem 3.47 over $\mathbb{R}^k$.* Let

$$\mathbf{A}_n = \sum_{k=0}^{n} \mathbf{a}_k, \qquad \mathbf{B}_n = \sum_{k=0}^{n} \mathbf{b}_k.$$

Then

$$\mathbf{A}_n + \mathbf{B}_n = \sum_{k=0}^{n} (\mathbf{a}_k + \mathbf{b}_k).$$

Since $\lim_{n\to\infty} \mathbf{A}_n = \mathbf{A}$ and $\lim_{n\to\infty} \mathbf{B}_n = \mathbf{B}$, we see that

$$\lim_{n\to\infty} (\mathbf{A}_n + \mathbf{B}_n) = \mathbf{A} + \mathbf{B}.$$

The proof of the second assertion is even simpler.

$$c\mathbf{A}_n = \sum_{k=0}^{n} (c\mathbf{a}_k).$$

Since $\lim_{n\to\infty} \mathbf{A}_n = \mathbf{A}$, we see that

$$\lim_{n\to\infty} (c\mathbf{A}_n) = c\mathbf{A}.$$

$\square$

**Theorem 3.55 over $\mathbb{R}^k$.** *If $\sum \mathbf{a}_n$ is a series in $\mathbb{R}^k$ which converges absolutely, then every rearrangement of $\sum \mathbf{a}_n$ converges, and they all converge to the same sum.*

*Proof of Theorem 3.55 over $\mathbb{R}^k$.* Let $\sum \mathbf{a}'_n$ be a rearrangement, with partial sums $\mathbf{s}'_n$. Given $\varepsilon > 0$, there exists an integer $N$ such that $m \ge n \ge N$ implies

$$\sum_{i=n}^{m} |\mathbf{a}_i| \le \varepsilon. \tag{26}$$

Now choose $p$ such that the integers $1, 2, \ldots, N$ are all contained in the set $k_1, k_2, \ldots, k_p$ (we use the notation of Definition 3.52). Then if $n > p$, the numbers $\mathbf{a}_1, \ldots, \mathbf{a}_N$ will cancel in the difference $\mathbf{s}_n - \mathbf{s}'_n$, so that $|\mathbf{s}_n - \mathbf{s}'_n| \le \varepsilon$, by (26). Hence $\{\mathbf{s}'_n\}$ converges to the same sum as $\{\mathbf{s}_n\}$. $\square$

**Exercise 3.16.** *Fix a positive number $\alpha$. Choose $x_1 > \sqrt{\alpha}$, and define $x_2, x_3, x_4, \ldots,$ by the recursion formula*

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right).$$

(a) *Prove that $\{x_n\}$ decreases monotonically and that $\lim x_n = \sqrt{\alpha}$.*

(b) *Put $\varepsilon_n = x_n - \sqrt{\alpha}$, and show that*

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$$

*so that, setting $\beta = 2\sqrt{\alpha}$,*

$$\varepsilon_{n+1} < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n} \quad (n = 1, 2, 3, \ldots).$$

(c) *This is a good algorithm for computing square roots, since the recursion formula is simple and the convergence is extremely rapid. For example, if $\alpha = 3$ and $x_1 = 2$, show that $\frac{\varepsilon_1}{\beta} < \frac{1}{10}$ and therefore*

$$\varepsilon_5 < 4 \cdot 10^{-16}, \;\; \varepsilon_6 < 4 \cdot 10^{-32}.$$

*Note.*

(1) It is the Newton's method described in Exercise 5.25. (Here $f(x) = x^2 - \alpha$.)

(2) It is a special case of Exercise 3.18 by letting $p = 2$.

*Proof of (a).*

(1) *Show that $x_n > 0$ for $n = 1, 2, \ldots$* It is trivial by induction on $n$.

(2) *Show that $x_n > \sqrt{\alpha}$ for $n = 1, 2, \ldots$.* Put $\varepsilon_n = x_n - \sqrt{\alpha}$ as in (b). *It is equivalent to show that $\varepsilon_n > 0$ for $n = 1, 2, \ldots$.* Since $x_1 > \sqrt{\alpha}$, $\varepsilon_1 = x_1 - \sqrt{\alpha} > 0$. For $n \geq 1$,

$$\begin{aligned}
\varepsilon_{n+1} &= x_{n+1} - \sqrt{\alpha} \\
&= \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) - \sqrt{\alpha} \\
&= \frac{x_n^2 + \alpha - 2\sqrt{\alpha}x_n}{2x_n} \\
&= \frac{(x_n - \sqrt{\alpha})^2}{2x_n} \\
&> 0
\end{aligned}$$

by (1). Therefore, $\varepsilon_n > 0$ or $x_n > \sqrt{\alpha}$.

(3) *Show that $\{x_n\}$ decreases monotonically.*

$$\begin{aligned}
x_{n+1} - x_n &= \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) - x_n \\
&= \frac{\alpha - x_n^2}{2x_n} \\
&< 0
\end{aligned}$$

for $n = 1, 2, \ldots$ ((1)(2)). Hence $\{x_n\}$ decreases monotonically.

(4) Since $\{x_n\}$ is monotonic and bounded by (2)(3), $\{x_n\}$ converges to $x > 0$ (Theorem 3.14). $x$ satisfies

$$x = \frac{1}{2}\left(x + \frac{\alpha}{x}\right)$$

(since $\lim x_{n+1} = \lim x_n = x$), or $x = \pm\sqrt{\alpha}$. Therefore, $\lim x_n = x = \sqrt{\alpha}$ since $x \geq 0$.

□

*Proof of (b).*

(1) By (a)(2), we have

$$\varepsilon_{n+1} = \frac{(x_n - \sqrt{\alpha})^2}{2x_n} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{3}}.$$

(2) *Show that*

$$\varepsilon_{n+1} < \beta \left( \frac{\varepsilon_1}{\beta} \right)^{2^n}.$$

Induction on $n$.

(a) $n = 1$.

$$\varepsilon_2 < \frac{\varepsilon_1^2}{2\sqrt{3}} = \frac{\varepsilon_1^2}{\beta} = \beta \left( \frac{\varepsilon_1}{\beta} \right)^{2^1}.$$

(b) Assume $n = k$ the statement holds. Then as $n = k + 1$, we have

$$\begin{aligned}
\varepsilon_{k+2} &< \frac{\varepsilon_{k+1}^2}{\beta} && ((1)) \\
&< \frac{1}{\beta} \left( \beta \left( \frac{\varepsilon_1}{\beta} \right)^{2^k} \right)^2 && \text{(Induction hypothesis)} \\
&= \beta \left( \frac{\varepsilon_1}{\beta} \right)^{2^{k+1}}.
\end{aligned}$$

By induction, the statement holds for all $n \in \mathbb{Z}^+$.

□

*Proof of (c).*

(1) Since $\varepsilon_1 = x_1 - \sqrt{\alpha} = 2 - \sqrt{3}$ and $\beta = 2\sqrt{\alpha} = 2\sqrt{3}$ and $\sqrt{3} < 1.8$,

$$\frac{\varepsilon_1}{\beta} = \frac{2 - \sqrt{3}}{2\sqrt{3}} = \frac{2\sqrt{3} - 3}{6} < \frac{2 \cdot 1.8 - 3}{6} = \frac{1}{10}.$$

(2) Since $\beta = 2\sqrt{\alpha} = 2\sqrt{3} < 4$, by (b) we have

$$\varepsilon_5 < \beta \left( \frac{\varepsilon_1}{\beta} \right)^{2^4} < 4 \cdot (10^{-1})^{16} = 4 \cdot 10^{-16},$$

$$\varepsilon_6 < \beta \left( \frac{\varepsilon_1}{\beta} \right)^{2^5} < 4 \cdot (10^{-1})^{32} = 4 \cdot 10^{-32}.$$

$\square$

**Exercise 3.17.** *Fix $\alpha > 1$. Take $x_1 > \sqrt{\alpha}$, and define*

$$x_{n+1} = \frac{\alpha + x_n}{1 + x_n} = x_n + \frac{\alpha - x_n^2}{1 + x_n}.$$

(a) *Prove that $x_1 > x_3 > x_5 > \cdots$.*

(b) *Prove that $x_2 < x_4 < x_6 < \cdots$.*

(c) *Prove that $\lim x_n = \sqrt{\alpha}$.*

(d) *Compare the rapidity of convergence of this process with the one described in Exercise 3.16.*

*Proof of (a).*

(1)

$$\begin{aligned} x_{n+1} - \sqrt{\alpha} &= \frac{\alpha + x_n}{1 + x_n} - \sqrt{\alpha} \\ &= -\frac{(\sqrt{\alpha} - 1)(x_n - \sqrt{\alpha})}{x_n + 1} \end{aligned}$$

holds for $n \geq 1$.

(2)

$$\begin{aligned} x_{n+2} - x_n &= \frac{\alpha + x_{n+1}}{1 + x_{n+1}} - x_n \\ &= \frac{\alpha + \frac{\alpha + x_n}{1 + x_n}}{1 + \frac{\alpha + x_n}{1 + x_n}} - x_n \\ &= \frac{\alpha x_n + x_n + 2\alpha}{2x_n + \alpha + 1} - x_n \\ &= -\frac{2(x_n^2 - \alpha)}{2x_n + \alpha + 1} \end{aligned}$$

holds for $n \geq 1$.

(3) Since $x_1, x_3, x_5, \ldots > \sqrt{\alpha}$ (by (1)), $x_1 > x_3 > x_5 > \cdots$ by (2).

$\square$

*Proof of (b).* Since $x_1 > \sqrt{\alpha}$, $x_2 < \sqrt{\alpha}$ by (a)(1). Hence $x_2 < x_4 < x_6 < \cdots$ by (a)(2). $\square$

*Proof of (c).*

(1) Since $\{x_{2n+1}\}$ is monotonic and bounded by (a), $\{x_{2n+1}\}$ converges to $x_1 \geq \sqrt{\alpha}$ (Theorem 3.14).

(2) Since $\{x_{2n}\}$ is monotonic and bounded by (a), $\{x_{2n}\}$ converges to $x_2 \leq \sqrt{\alpha}$ (Theorem 3.14).

(3) In any case, $x = x_1$ or $x = x_2$ satisfy

$$0 = -\frac{2(x^2 - \alpha)}{2x + \alpha + 1}$$

by (a)(2) (since $\lim x_{n+2} = \lim x_n = x$), or $x = \pm\sqrt{\alpha}$. Therefore, $\lim x_{2n+1} = \lim x_{2n} = x = \sqrt{\alpha}$ since $x \geq 0$. Hence $\lim x_n = x = \sqrt{\alpha}$.

$\square$

*Proof of (d).* Put $\varepsilon_n = |x_n - \sqrt{\alpha}|$, and by (a)(1) we have

$$\varepsilon_{n+1} \leq \frac{\sqrt{\alpha} - 1}{x_1 + 1}\varepsilon_n$$

for $n \geq 1$. (Here $0 < x_n \leq x_1$.) Therefore, the convergence is geometric, not quadratically geometric in Exercise 3.16, that is, the rate of convergence is slower than one in Exercise 3.16. $\square$

**Exercise 3.18.** *Replace the recursion formula of Exercise 3.16 by*

$$x_{n+1} = \frac{p-1}{p}x_n + \frac{\alpha}{p}x_n^{-p+1}$$

*where $p$ is a fixed positive integer, and describe the behavior of the resulting sequences $\{x_n\}$.*

Might assume that $p > 1$ since the case $p = 1$ is nothing to do.

*Outline.* Let $\xi = \alpha^{\frac{1}{p}}$.

(a) *Prove that $\{x_n\}$ decreases monotonically and that* $\lim x_n = \xi$.

(b) *Put $\varepsilon_n = x_n - \xi$, and show that*

$$\varepsilon_{n+1} < \frac{(p-1)^2\varepsilon_n^2}{px_n} < \frac{(p-1)^2\varepsilon_n^2}{p\alpha^{\frac{1}{p}}}$$

*so that, setting* $\beta = \frac{p\alpha^{\frac{1}{p}}}{(p-1)^2}$,

$$\varepsilon_{n+1} < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n} \quad (n = 1, 2, 3, \ldots).$$

*Proof of (a).*

(1) *Show that $x_n > 0$ for $n = 1, 2, \ldots$* It is trivial by induction on $n$.

(2) *Show that $x_n > \xi$ for $n = 1, 2, \ldots$.* Put $\varepsilon_n = x_n - \xi$ as in (b). *It is equivalent to show that $\varepsilon_n > 0$ for $n = 1, 2, \ldots$.* Since $x_1 > \xi$, $\varepsilon_1 = x_1 - \xi > 0$. For $n \geq 1$,

$$\begin{aligned}
\varepsilon_{n+1} &= x_{n+1} - \xi \\
&= \frac{p-1}{p} x_n + \frac{\alpha}{p} x_n^{-p+1} - \xi \\
&= \frac{p-1}{p}(x_n - \xi) - \frac{1}{p}\left(\xi - \xi^p x_n^{-p+1}\right) \\
&= \frac{p-1}{p}(x_n - \xi) - \frac{\xi}{p x_n^{p-1}}(x_n^{p-1} - \xi^{p-1}) \\
&= \frac{p-1}{p}(x_n - \xi) - \frac{\xi}{p x_n^{p-1}}(x_n - \xi)(x_n^{p-2} + \cdots + \xi^{p-2}) \\
&> \frac{p-1}{p}(x_n - \xi) - \frac{\xi}{p x_n^{p-1}}(x_n - \xi)(p-1)x_n^{p-2} \\
&= \frac{p-1}{p}(x_n - \xi)\left(1 - \frac{\xi}{x_n}\right) \\
&= \frac{(p-1)(x_n - \xi)^2}{p x_n} \\
&> 0
\end{aligned}$$

by (1). Therefore, $\varepsilon_n > 0$ or $x_n > \sqrt{\alpha}$.

(3) *Show that $\{x_n\}$ decreases monotonically.*

$$\begin{aligned}
x_{n+1} - x_n &= \frac{p-1}{p} x_n + \frac{\alpha}{p} x_n^{-p+1} - x_n \\
&= \frac{\xi^p - x_n^p}{p x_n^{p-1}} \\
&< 0
\end{aligned}$$

for $n = 1, 2, \ldots$ ((1)(2)). Hence $\{x_n\}$ decreases monotonically.

(4) Since $\{x_n\}$ is monotonic and bounded by (2)(3), $\{x_n\}$ converges to $x > 0$ (Theorem 3.14). $x$ satisfies

$$x = \frac{p-1}{p} x + \frac{\alpha}{p} x^{-p+1}$$

(since $\lim x_{n+1} = \lim x_n = x$), or $x^p = \alpha$. Therefore, $\lim x_n = x = \alpha^{\frac{1}{p}}$ since $x \geq 0$.

□

*Proof of (b).*

(1) By (a)(2), we have

$$\begin{aligned}
\varepsilon_{n+1} &= \frac{p-1}{p}(x_n - \xi) - \frac{\xi}{p x_n^{p-1}}(x_n - \xi)(x_n^{p-2} + \cdots + \xi^{p-2}) \\
&< \frac{p-1}{p}(x_n - \xi) - \frac{\xi}{p x_n^{p-1}}(x_n - \xi)(p-1)\xi^{p-2} \\
&= \frac{(p-1)\varepsilon_n}{p x_n^{p-1}}(x_n^{p-1} - \xi^{p-1}) \\
&= \frac{(p-1)\varepsilon_n}{p x_n^{p-1}}(x_n - \xi)(x_n^{p-2} + \cdots + \xi^{p-2}) \\
&< \frac{(p-1)\varepsilon_n}{p x_n^{p-1}}(x_n - \xi)(p-1)x_n^{p-2} \\
&= \frac{(p-1)^2\varepsilon_n^2}{p x_n} \\
&< \frac{(p-1)^2\varepsilon_n^2}{p \alpha^{\frac{1}{p}}}.
\end{aligned}$$

(2) *Show that*

$$\varepsilon_{n+1} < \beta \left( \frac{\varepsilon_1}{\beta} \right)^{2^n}.$$

Induction on $n$.

(a) $n = 1$.

$$\varepsilon_2 < \frac{(p-1)^2\varepsilon_1^2}{p\alpha^{\frac{1}{p}}} = \frac{\varepsilon_1^2}{\beta} = \beta \left( \frac{\varepsilon_1}{\beta} \right)^{2^1}.$$

(b) Assume $n = k$ the statement holds. Then as $n = k+1$, we have

$$\begin{aligned}
\varepsilon_{k+2} &< \frac{\varepsilon_{k+1}^2}{\beta} && ((1)) \\
&< \frac{1}{\beta}\left( \beta \left( \frac{\varepsilon_1}{\beta} \right)^{2^k} \right)^2 && \text{(Induction hypothesis)} \\
&= \beta \left( \frac{\varepsilon_1}{\beta} \right)^{2^{k+1}}.
\end{aligned}$$

By induction, the statement holds for all $n \in \mathbb{Z}^+$.

$\square$

**Exercise 3.19.** *Associate to each sequence $a = \{\alpha_n\}$, in which $\alpha_n$ is 0 or 2, the real number*

$$x(a) = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}.$$

*Prove that the set of all $x(a)$ is precisely the Cantor set described in Sec. 2.44.*

*Cantor set.* Let $E_0$ be the interval $[0, 1]$. Remote the segment $(\frac{1}{3}, \frac{2}{3})$, and let $E_1$ be the union of the intervals

$$\left[0, \frac{1}{3}\right] \text{ and } \left[\frac{2}{3}, 1\right].$$

Remote the middle thirds of these intervals, and let $E_2$ be the union of the intervals

$$\left[0, \frac{1}{9}\right], \left[\frac{2}{9}, \frac{3}{9}\right], \left[\frac{6}{9}, \frac{7}{9}\right] \text{ and } \left[\frac{8}{9}, 1\right].$$

Continuing in this way, we obtain a sequence of compact set $E_n$, such that

(a) $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$;

(b) $E_n$ is the union of $2^n$ intervals, each of length $3^{-n}$.

The set

$$P = \bigcap_{n=1}^{\infty} E_n$$

is called the Cantor set. $P$ is compact, non empty, perfect, uncountable and measure zero.

*Proof.* Let

$$C = \{x(a) : a = \{\alpha_n\}, \text{ in which } \alpha_n \text{ is 0 or 2 }\}.$$

(1) ($P \subseteq C$). Given any

$$x \in P = \bigcap_{n=1}^{\infty} E_n.$$

Hence $x \in E_n$ for all $n \geq 1$. Write $x = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}$ where $\alpha_n \in \{0, 1, 2\}$ for $n \geq 1$. (It is possible since $0 \leq x \leq 1$ and every point in the $[0, 1]$ has the ternary notation.)

(a) $x \in E_1$. So

$$\begin{aligned} & x \in \left[0, \frac{1}{3}\right] \bigcup \left[\frac{2}{3}, 1\right] \\ \Longleftrightarrow & x \in \left[0, \frac{1}{3}\right], \left[\frac{2}{3}, 1\right] \\ \Longleftrightarrow & \alpha_1 \in \{0, 2\}. \end{aligned}$$

Here we express $\frac{1}{3}$ as $(0.0\overline{2})_3$ instead of $(0.1)_3$.

(b) $x \in E_2$. So

$$\begin{aligned} & x \in \left[0, \frac{1}{9}\right] \bigcup \left[\frac{2}{9}, \frac{3}{9}\right] \bigcup \left[\frac{6}{9}, \frac{7}{9}\right] \bigcup \left[\frac{8}{9}, 1\right] \\ \Longleftrightarrow & x \in \left[0, \frac{1}{9}\right], \left[\frac{2}{9}, \frac{3}{9}\right], \left[\frac{6}{9}, \frac{7}{9}\right], \left[\frac{8}{9}, 1\right] \\ \Longleftrightarrow & \alpha_1 \in \{0, 2\}, \alpha_2 \in \{0, 2\}. \end{aligned}$$

(c) Continuing in this way, we obtain a sequence of $\alpha_n$ such that $\alpha_n \in \{0, 2\}$ for $n \geq 1$. Therefore, $x \in C$.

(2) ($C \subseteq P$). Given any

$$x = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n} \in C.$$

Applying the same argument in (1), we have $x \in E_n$ for all $n \geq 1$. Therefore, $x \in \bigcap E_n = P$.

□

**Exercise 3.20.** *Suppose $\{p_n\}$ is a Cauchy sequence in a metric space $X$, and some subsequence $\{p_{n_i}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to $p$.*

*Proof.* Given any $\varepsilon > 0$.

(1) Since $\{p_n\}$ is a Cauchy sequence, there exists a positive integer $N_1$ such that

$$d(p_n, p_m) < \frac{\varepsilon}{2} \text{ whenever } n, m \geq N_1.$$

(2) Since the subsequence $\{p_{n_i}\}$ converges to a point $p \in X$, there exists a positive integer $N_2$ such that

$$d(p_{n_i}, p) < \frac{\varepsilon}{2} \text{ whenever } n_i \geq N_2.$$

(3) Let $N = \max\{N_1, N_2\}$ be a positive integer. So

$$\begin{aligned} d(p_n, p) &\leq d(p_n, p_{n_i}) + d(p_{n_i}, p) && \text{(Definition 2.15(c))} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \text{ whenever } n, n_i \geq N && \text{((1)(2))} \\ &= \varepsilon \text{ whenever } n \geq N. \end{aligned}$$

Hence the full sequence $\{p_n\}$ converges to $p$.

□

**Exercise 3.21.** *Prove the following analogue of Theorem 3.10(b): If $\{E_n\}$ is a sequence of closed and bounded sets in a complete metric space $X$, if $E_n \supseteq E_{n+1}$, and if*

$$\lim_{n\to\infty} \operatorname{diam}(E_n) = 0,$$

*then $\bigcap_{n=1}^{\infty} E_n$ consists of exactly one point.*

Assume $E_n \neq \varnothing$. It is unnecessary to assume that $E_n$ is bounded since we have the condition that $\lim_{n\to\infty} \operatorname{diam}(E_n) = 0$.

*Note.* Every compact metric space is complete, but complete spaces need not be compact. In fact, a metric space is compact if and only if it is complete and totally bounded.

*Proof.*

(1) Pick $p_n \in E_n$ for $n = 1, 2, \ldots$.

(2) *Show that $\{p_n\}$ is a Cauchy sequence.* Given any $\varepsilon > 0$. There is a positive integer $N$ such that $\operatorname{diam}(E_n) < \varepsilon$ whenever $n \geq N$. Especially,

$$\operatorname{diam}(E_N) < \varepsilon.$$

As $m, n \geq N$, $p_m \in E_m \subseteq E_N$ and $p_n \in E_n \subseteq E_N$. By the definition of the diameter of $E_N$,

$$d(p_m, p_n) \leq \operatorname{diam}(E_N) < \varepsilon \text{ whenever } m, n \geq N.$$

(3) Since $X$ is complete, $\{p_n\}$ converges to a point $p \in X$.

(4) *Show that $p \in \bigcap_{n=1}^{\infty} E_n$.* (Reductio ad absurdum) If there were some $n$ such that $p \notin E_n$. Consider the subsequence

$$p_n, p_{n+1}, p_{n+2}, \ldots.$$

Note that all $p_n, p_{n+1}, \ldots$ are in $E_n$. By (3), it converges to $p$. Thus $p$ is a limit point of $E_n$. Since $E_n$ is closed, $p \in E_n$, which is absurd.

(5) *Show that $\bigcap_{n=1}^{\infty} E_n = \{p\}$.* (Reductio ad absurdum) If there were $q \in \bigcap_{n=1}^{\infty} E_n$ with $q \neq p$, then $d(p, q) > 0$ (Definition 2.15(a)). It implies that

$$\operatorname{diam}(E_n) \geq d(p, q) > 0 \text{ for all } n,$$

contrary to $\lim_{n\to\infty} \operatorname{diam}(E_n) = 0$.

$\square$

**Exercise 3.22 (Baire category theorem).** *Suppose $X$ is a complete metric space, and $\{G_n\}$ is a sequence of dense open subsets of $X$. Prove Baire's theorem, namely, that $\bigcap_1^\infty G_n$ is not empty. (In fact, it is dense in $X$.) (Hint: Find a shrinking sequence of neighborhoods $E_n$ such that $\overline{E_n} \subseteq G_n$, and apply Exercise 3.21.)*

*Proof.* Given any open set $G_0$ in $X$, will show that

$$\bigcap_{n=0}^{\infty} G_n \neq \varnothing.$$

(1) Since $G_1$ is dense, $G_0 \cap G_1$ is nonempty. Take any one point $p_1$ in the open set $G_0 \cap G_1$, then there exists a closed neighborhood

$$V_1 = \{q \in X : d(q, p_1) < r_1\}$$

of $p_1$ with $r_1 < 1$ such that

$$V_1 \subseteq G_0 \cap G_1.$$

Take $U_1 \subseteq E_1 \subseteq V_1$ such that

$$E_1 = \left\{q \in X : d(q, p_1) \leq \frac{r_1}{64}\right\} \subseteq V_1,$$
$$U_1 = \left\{q \in X : d(q, p_1) < \frac{r_1}{89}\right\} \subseteq E_1.$$

(2) Suppose $V_n, E_n, U_n$ have been constructed, take any one point $p_{n+1}$ in the open set $U_n \cap G_{n+1}$, there exists an open neighborhood

$$V_{n+1} = \{q \in X : d(q, p_{n+1}) < r_{n+1}\}$$

of $p_{n+1}$ with $r_{n+1}$ with $r_{n+1} < \frac{1}{n+1}$ such that

$$V_{n+1} \subseteq U_n \cap G_{n+1}.$$

Take $U_1 \subseteq E_1 \subseteq V_1$ such that

$$E_{n+1} = \left\{q \in X : d(q, p_{n+1}) \leq \frac{r_{n+1}}{64}\right\} \subseteq V_{n+1},$$
$$U_{n+1} = \left\{q \in X : d(q, p_{n+1}) < \frac{r_{n+1}}{89}\right\} \subseteq E_{n+1}.$$

(3) Note that

(a) $E_n$ is closed and nonempty (since $p_n \in E_n$).

(b) $\lim_{n\to\infty} \operatorname{diam}(E_n) = 0$ (since $\operatorname{diam}(E_n) \leq 2 \cdot \frac{r_n}{64} < r_n < \frac{1}{n}$.)

(c) $E_1 \supseteq E_2 \supseteq \cdots$ (since $E_{n+1} \subseteq V_{n+1} \subseteq U_n \cap G_{n+1} \subseteq U_n \subseteq E_n$).

Since $X$ is complete, by Exercise 3.21,

$$\bigcap_{n=1}^{\infty} E_n = \{p\}$$

for some $p \in X$.

(4) Hence

$$\begin{aligned}
p \in \bigcap_{n=1}^{\infty} E_n &\Longleftrightarrow p \in E_n \text{ for all } n = 1, 2, 3, \ldots \\
&\Longrightarrow p \in E_1 \subseteq G_0 \cap G_1 \text{ and } p \in E_{n+1} \subseteq U_n \cap G_{n+1} \subseteq G_{n+1} \\
&\Longrightarrow p \in G_0 \cap G_1 \cap \cdots = \bigcap_{n=0}^{\infty} G_n \\
&\Longrightarrow \bigcap_{n=0}^{\infty} G_n \neq \varnothing.
\end{aligned}$$

□

**Exercise 3.23.** *Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space $X$. Show that the sequence $\{d(p_n, q_n)\}$ converges. (Hint: For any $m, n$,*

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n);$$

*it follows that*

$$|d(p_n, q_n) - d(p_m, q_m)|$$

*is small if $m$ and $n$ are large.)*

*Proof.* Given any $\varepsilon > 0$.

(1) Since $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences, there exists $N$ such that

$$d(p_n, p_m) < \frac{\varepsilon}{2} \text{ and } d(q_m, q_n) < \frac{\varepsilon}{2}$$

whenever $m, n \geq N$.

(2) Note that

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n).$$

It follows that

$$|d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_m, q_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $\{d(p_n, q_n)\}$ is a Cauchy sequence in $\mathbb{R}^1$ (not in $X$).

(3) Since $\mathbb{R}^1$ is a complete metric space, $\{d(p_n, q_n)\}$ converges.

$\square$

**Exercise 3.24.** *Let $X$ be a metric space.*

(a) *Call two Cauchy sequences $\{p_n\}$, $\{q_n\}$ in $X$ equivalent if*

$$\lim_{n \to \infty} d(p_n, q_n) = 0.$$

*Prove that this is an equivalence relation.*

(b) *Let $X^*$ be the set of all equivalence classes so obtained. If $P \in X^*$, $Q \in X^*$, $\{p_n\} \in P$, $\{q_n\} \in Q$, define*

$$\Delta(P, Q) = \lim_{n \to \infty} d(p_n, q_n);$$

*by Exercise 3.23, this limit exists. Show that the number $\Delta(P, Q)$ is unchanged if $\{p_n\}$ and $\{q_n\}$ are replaced by equivalent sequences, and hence that $\Delta$ is a distance function in $X^*$.*

(c) *Prove that the resulting metric space $X^*$ is complete.*

(d) *For each $p \in X$, there is a Cauchy sequence all of whose terms are $p$; let $P_p$ be the element of $X^*$ which contains this sequence. Prove that*

$$\Delta(P_p, P_q) = d(p, q)$$

*for all $p, q \in X$. In other words, the mapping $\varphi$ defined by $\varphi(p) = P_p$ is an isometry (i.e., a distance-preserving mapping) of $X$ into $X^*$.*

(e) *Prove that $\varphi(X)$ is dense in $X^*$, and that $\varphi(X) = X^*$ if $X$ is complete. By (d), we may identify $X$ and $\varphi(X)$ and thus regard $X$ as embedded in the complete metric space $X^*$. We call $X^*$ the* ***completion*** *of $X$.*

*Proof of (a).* Given Cauchy sequences $\{p_n\}$, $\{q_n\}$, $\{r_n\}$ in $X$.

(1) *(Reflexivity)*

$$\lim_{n \to \infty} d(p_n, q_n) = \lim_{n \to \infty} 0 = 0$$

by the reflexivity of the metric function $d$.

(2) *(Symmetry)*

$$\lim_{n \to \infty} d(p_n, q_n) = \lim_{n \to \infty} d(q_n, p_n) = 0$$

by the symmetry of the metric function $d$.

(3) *(Transitivity)* Suppose that $\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(q_n, r_n) = 0$. By the triangle inequality of the metric function $d$, we have

$$0 \leq d(p_n, r_n) \leq d(p_n, q_n) + d(q_n, r_n).$$

Take limit to get

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} d(p_n, r_n) \\ &\leq \lim_{n\to\infty} (d(p_n, q_n) + d(q_n, r_n)) \\ &= \lim_{n\to\infty} d(p_n, q_n) + \lim_{n\to\infty} d(q_n, r_n) \\ &= 0 \end{aligned}$$

or $\lim_{n\to\infty} d(p_n, r_n) = 0$.

□

*Proof of (b).*

(1) *Show that* $\Delta$ *is well-defined.* Given any $\{p_n\}, \{p'_n\} \in P$ and $\{q_n\}, \{q'_n\} \in Q$.

(a) $\lim_{n\to\infty} d(p_n, p'_n) = 0$ since $\{p_n\}$ and $\{p'_n\}$ are in the same equivalence class.

(b) $\lim_{n\to\infty} d(q_n, q'_n) = 0$ (similar to (a)).

(c) *Show that* $\lim_{n\to\infty} d(p_n, q_n) \leq \lim_{n\to\infty} d(p'_n, q'_n)$. Since $d(p_n, q_n) \leq d(p_n, p'_n) + d(p'_n, q'_n) + d(q'_n, q_n)$, take limit to get

$$\begin{aligned} \lim_{n\to\infty} d(p_n, q_n) &\leq \lim_{n\to\infty} (d(p_n, p'_n) + d(p'_n, q'_n) + d(q'_n, q_n)) \\ &= \lim_{n\to\infty} d(p_n, p'_n) + \lim_{n\to\infty} d(p'_n, q'_n) + \lim_{n\to\infty} d(q'_n, q_n) \\ &= 0 + \lim_{n\to\infty} d(p'_n, q'_n) + 0 \\ &= \lim_{n\to\infty} d(p'_n, q'_n) \end{aligned}$$

since (a)(b).

(d) *Show that* $\lim_{n\to\infty} d(p_n, q_n) \geq \lim_{n\to\infty} d(p'_n, q'_n)$. Similar to (c).

By (c)(d), $\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(p'_n, q'_n)$, or $\Delta(P, Q)$ is well-defined.

(2) *Show that* $\Delta$ *is a metric.*

(a) *Show that* $\Delta(P, Q) > 0$ *if* $P \neq Q$; $\Delta(P, P) = 0$. It is the definition of $\Delta$.

(b) *Show that* $\Delta(P, Q) = \Delta(Q, P)$. Similar to the argument in (a)(2).

(c) *Show that* $\Delta(P, Q) \leq \Delta(P, R) + \Delta(R, Q)$. Similar to the argument in (a)(3).

$\square$

*Proof of (c). Show that $\{P_k\}_{k=1}^{\infty}$ converges to $P$ in $(X^*, \Delta)$ for any given Cauchy sequence $\{P_k\}$.*

(1) Take a Cauchy sequence $\{p_n^{(k)}\}_{n=1}^{\infty}$ to represent $P_k$ for each $k$. *We will construct a Cauchy sequence $\{p_k\}$ in $(X, d)$ such that $\{P_k\}$ converges to $P$ which is the equivalent class of $\{p_k\}$.*

(2) For each $k$, there exists $N_k$ such that

$$d\left(p_m^{(k)}, p_n^{(k)}\right) < \frac{1}{k} \text{ whenever } m, n \geq N_k.$$

Especially,

$$d\left(p_m^{(k)}, p_{N_k}^{(k)}\right) < \frac{1}{k} \text{ whenever } m \geq N_k.$$

Let $p_k = p_{N_k}^{(k)}$ and collect all $p_k$ as $\{p_k\}_{k=1}^{\infty}$.

(3) *Show that $\{p_k\}$ is a Cauchy sequence in $(X, d)$.* Note that for any $k$, we have

$$\begin{aligned} d(p_m, p_n) &= d\left(p_{N_m}^{(m)}, p_{N_n}^{(n)}\right) \\ &\leq d\left(p_{N_m}^{(m)}, p_k^{(m)}\right) + d\left(p_k^{(m)}, p_k^{(n)}\right) + d\left(p_k^{(n)}, p_{N_n}^{(n)}\right). \end{aligned}$$

Let $k \to \infty$, we have

$$\begin{aligned} d(p_m, p_n) &\leq \limsup_{k \to \infty} \left[d\left(p_{N_m}^{(m)}, p_k^{(m)}\right) + d\left(p_k^{(m)}, p_k^{(n)}\right) + d\left(p_k^{(n)}, p_{N_n}^{(n)}\right)\right] \\ &\leq \frac{1}{m} + \Delta(P_m, P_n) + \frac{1}{n} \end{aligned}$$

for any $m, n$ (by (2)). Let $m, n \to \infty$, we establish the result (since $\{P_k\}$ is Cauchy).

(4) *Show that $\{P_k\}$ converges to $P \ni \{p_k\}$.* Given any $\varepsilon > 0$. Since $\{p_k\}$ is Cauchy (3), there is $N > \frac{2}{\varepsilon}$ such that

$$d(p_m, p_n) < \frac{\varepsilon}{2} \text{ whenever } m, n \geq N.$$

Note that

$$\begin{aligned} d\left(p_n^{(k)}, p_n\right) &= d\left(p_n^{(k)}, p_{N_n}^{(n)}\right) \\ &\leq d\left(p_n^{(k)}, p_{N_k}^{(k)}\right) + d\left(p_{N_k}^{(k)}, p_{N_n}^{(n)}\right). \end{aligned}$$

For any $k \geq N$, let $n \to \infty$ to get

$$\begin{aligned}
\Delta(P_k, P) &= \lim_{n\to\infty} d\left(p_n^{(k)}, p_n\right) \\
&\leq \limsup_{n\to\infty} d\left(p_n^{(k)}, p_{N_k}^{(k)}\right) + \limsup_{n\to\infty} d\left(p_{N_k}^{(k)}, p_{N_n}^{(n)}\right) \\
&< \frac{1}{k} + \frac{\varepsilon}{2} \\
&\leq \frac{1}{N} + \frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&< \varepsilon.
\end{aligned}$$

Hence, $(X^*, \Delta)$ is complete. $\square$

*Proof of (d).*

(1) Define $\{p_n\}$ by $p_n = p$ $(n = 1, 2, \ldots)$ for any $p \in X$.

(2) *Show that $\{p_n\}$ is a Cauchy sequence.* $d(p_m, p_n) = d(p, p) = 0$.

(3) Take $\{p\} \in P_p$ and $\{q\} \in P_q$. Then

$$\Delta(P_p, P_q) = \lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(p, q) = d(p, q).$$

$\square$

*Proof of (e).*

(1) *Show that $\varphi(X)$ is dense in $X^*$.* Given any $P \in X^*$, any $\{p_n\} \in P$ and any $\varepsilon > 0$. Since $\{p_n\}$ is Cauchy, there is $N$ such that

$$d(p_m, p_n) < \frac{\varepsilon}{64} \text{ whenever } m, n \geq N.$$

Note that $p_N \in X$. Pick $\{p_N\} \in P_{p_N} = \varphi(p_N) \in \varphi(X)$. So

$$\Delta(P, P_{p_N}) = \lim_{n\to\infty} d(p_n, p_N) \leq \frac{\varepsilon}{64} < \varepsilon.$$

Hence $\varphi(X)$ is dense in $X^*$.

(2) *Show that $\varphi(X) = X^*$ if $X$ is complete.* Given any $P \in X^* \ni \{p_n\}$. Since $X$ is complete, a Cauchy sequence $\{p_n\}$ converges to $p \in X$. Pick $\{p\} \in P_p = \varphi(p) \in \varphi(X)$. So

$$\Delta(P, P_p) = \lim_{n\to\infty} d(p_n, p) = 0,$$

or $P = P_p$, or $\varphi(X) = X^*$.

□

**Exercise 3.25.** *Let $X$ be the metric space whose points are rational numbers, with the metric $d(x, y) = |x-y|$. What is the completion of this space? (Compare Exercise 3.24.)*

*Proof.* By Exercise 3.24, we can identify one completion $(X^*, \Delta)$ with $(\mathbb{R}, |\cdot|)$ (Theorem 3.11(c) and Theorem 1.20(b)). □

**Supplement (Uniqueness of completion).** *Show that a completion of a metric space is unique up to isometry.*

*Outline.* Suppose there are two completions $\{\varphi_i, (X_i^*, d_i^*)\}$ $(i = 1, 2)$ of $(X, d)$. Let

$$\psi = \varphi_2 \circ \varphi_1^{-1} : \varphi_1(X) \to \varphi_2(X)$$

be an isometry from $\varphi_1(X)$ into $\varphi_2(X)$ The sets $\varphi_i(X)$ $(i = 1, 2)$ are dense in $X_i^*$. So we can extend $\psi$ (continuously) to a map $\psi : X_1^* \to X_2^*$.

*Proof.*

(1) Given any $P \in X_1^*$, there is a Cauchy sequence $\{P_{p_n}\} = \{\varphi_1(p_n)\}$ in $\varphi_1(X)$ converging to $P$. Define $\psi(P)$ by

$$\psi(P) = \lim_{n\to\infty} \psi(P_{p_n}).$$

(2) *Show that $\psi$ is well-defined.* Note that

$$\begin{aligned}
\Delta_2(\psi(P_{p_m}), \psi(P_{p_n})) &= \Delta_2(\psi(\varphi_1(p_m)), \psi(\varphi_1(p_n))) \\
&= \Delta_2(\varphi_2(p_m), \varphi_2(p_n)) \\
&= d(p_n, p_m) && (\varphi_2 \text{ is isometric}) \\
&= \Delta_1(\varphi_1(p_m), \varphi_1(p_n)) && (\varphi_1 \text{ is isometric}) \\
&= \Delta_1(P_{p_m}, P_{p_n}).
\end{aligned}$$

So $\{\psi(P_{p_n})\}$ is a Cauchy sequence in $\varphi_2(X)$ if (and only if) $\{P_{p_n}\}$ is a Cauchy sequence in $\varphi_1(X)$. Since $X_2^*$ is complete, $\{\psi(P_{p_n})\}$ converges to $\psi(P)$. The limit $\psi(P)$ is uniquely determined since $\Delta_2$ is a metric function.

(3) Since $\psi$ is an isometry from $\varphi_1(X)$ into $\varphi_2(X)$,

$$\psi^{-1} = \varphi_1 \circ \varphi_2^{-1} : \varphi_2(X) \to \varphi_1(X)$$

is an isometry from $\varphi_2(X)$ into $\varphi_1(X)$. Besides, $\psi^{-1} \circ \psi = 1_{\varphi_1(X)}$ and $\psi \circ \psi^{-1} = 1_{\varphi_2(X)}$.

(4) *Show that $\psi$ is surjective.* Given any $Q \in X_2^*$, there is a Cauchy sequence $\{P_{q_n}\} = \{\varphi_2(q_n)\}$ in $\varphi_2(X)$ converging to $Q$. Define

$$P_{p_n} = \psi^{-1}(P_{q_n}) \in \varphi_1(X).$$

$\psi(P_{p_n}) = 1_{\varphi_2(X)}(P_{q_n}) = P_{q_n}$. Besides, similar to argument in (2), $\{P_{p_n}\}$ is a Cauchy sequence in $\varphi_1(X)$. Since $X_1^*$ is complete, $\{P_{p_n}\}$ converges to $P \in X_1^*$. It is easy to verify that $\psi(P) = Q$.

(5) *Show that $\psi$ is injective.* Given any $P \in X_1^*$ and $Q \in X_1^*$, there are Cauchy sequences

$$\{P_{p_n}\} = \{\varphi_1(p_n)\} \to P \text{ and } \{P_{q_n}\} = \{\varphi_1(q_n)\} \to Q.$$

So

$$\begin{aligned}
\psi(P) = \psi(Q) &\Longrightarrow \lim_{n\to\infty} \psi(P_{p_n}) = \lim_{n\to\infty} \psi(P_{q_n}) \\
&\Longrightarrow 0 = \lim_{n\to\infty} \Delta_2(\psi(P_{p_n}), \psi(P_{q_n})) \\
&\Longrightarrow 0 = \lim_{n\to\infty} \Delta_2(\psi(\varphi_1(p_n)), \psi(\varphi_1(q_n))) \\
&\Longrightarrow 0 = \lim_{n\to\infty} \Delta_2(\varphi_2(p_n), \varphi_2(q_n)) \\
&\Longrightarrow 0 = \lim_{n\to\infty} d(p_n, q_n). && (\varphi_2 \text{ is isometric})
\end{aligned}$$

Thus $\{p_n\} \in P$ and $\{q_n\} \in Q$ in the same equivalence class. Thus $P = Q$.

□