Notes on the book:
Robin Hartshorne, Algebraic Geometry

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Chapter I: Varieties

I.1 Affine Varieties

Exercise I.1.2. (Twisted cubic curve)

Let $Y \subseteq \mathbf{A}^3$ be the set $Y = \{(t, t^2, t^3) : t \in k\}$. Show that Y is an affine variety of dimension 1. Find generators for the ideal $I(Y)$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k. We say that Y is given by the ***parametric representation*** $x = t$, $y = t^2$, $z = t^3$.

Proof.

(1) Note that
$$Y = Z(x^2 - y, x^3 - z)$$
is an algebraic set. Hence $I(Y)$ is the radical of $\mathfrak{a} := (x^2 - y, x^3 - z)$. To show $I(Y) = \mathfrak{a}$, it suffices to show that $\mathfrak{a}$ is prime.

(2) *Show that $A/\mathfrak{a} \cong k[t]$ is a domain.*

  (a) Define a ring homomorphism $\alpha : A/\mathfrak{a} \to k[t]$ by
  $$\alpha : f(x, y, z) + \mathfrak{a} \mapsto f(t, t^2, t^3).$$
  α is well-defined since $\alpha((x^2 - y) + \mathfrak{a}) = 0$ and $\alpha((x^3 - z) + \mathfrak{a}) = 0$.

  (b) α is surjective since $\alpha(g(x) + \mathfrak{a}) = g(t)$ for any $g(t) \in k[t]$.

  (c) *Show that α is injective.* Suppose $\alpha(f(x, y, z) + \mathfrak{a}) = 0$. Write
  $$\begin{aligned} f(x, y, z) + \mathfrak{a} &= \sum_{(i)} \lambda_{(i)} x^{i_1} (y - x^2)^{i_2} (z - x^3)^{i_3} + \mathfrak{a} \\ &= \sum_i \lambda_i x^i + \mathfrak{a}. \end{aligned}$$
  So
  $$0 = \alpha(f(x, y, z) + \mathfrak{a}) = \alpha\left(\sum_i \lambda_i x^i + \mathfrak{a}\right) = \sum_i \lambda_i t^i.$$
  Hence $f(x, y, z) + \mathfrak{a} = \mathfrak{a}$.

(3) Hence Y is an affine variety of dimension 1 since $A(Y)$ is isomorphic to a polynomial ring in one variable t over k (Proposition I.1.7). Also, $I(Y) = \mathfrak{a} = (x^2 - y, x^3 - z)$ is generated by $x^2 - y$ and $x^3 - z$.

(4) Also see Problems 2.7 and 2.8 in the textbook: *William Fulton, Algebraic Curves.* If $\varphi : V \to W$ is a polynomial map, and X is an algebraic subset of W, show that $\varphi^{-1}(X)$ is an algebraic subset of V. If $\varphi^{-1}(X)$ is irreducible, and X is contained in the image of φ, show that X is irreducible. This gives a useful test for irreducibility.

$\square$

Exercise I.1.4.

If we identify $\mathbf{A}^2$ with $\mathbf{A}^1 \times \mathbf{A}^1$ in the natural way, show that the Zariski topology on $\mathbf{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\mathbf{A}^1$.

Proof.

(1) Let $\Delta = \{P \times P : P \in \mathbf{A}^1\}$ be the diagonal subset of $\mathbf{A}^2$.

(2) Δ is a proper closed subset $(= Z(x-y))$ of $\mathbf{A}^2$. However, Δ is not a finite union of horizontal and vertical lines and points (Example I.1.1.2). Hence Δ is not a closed set in the product topology of $\mathbf{A}^1 \times \mathbf{A}^1$.

$\square$

Exercise I.1.6.

Any nonempty open subset of an irreducible topological space is dense and irreducible. If Y is a subset of a topological space X, which is irreducible in its induced topology, then the closure $\overline{Y}$ is also irreducible.

Proof.

(1) *Show that any nonempty open subset of an irreducible topological space is dense.* It suffices to show that $U_1 \cap U_2 \neq \varnothing$ for any nonempty open subsets of an irreducible topological space.

$$\begin{aligned}
&\forall \text{ nonempty open sets } U_1 \text{ and } U_2, U_1 \cap U_2 \neq \varnothing \\
\Longleftrightarrow\ &\forall \text{ nonempty open sets } U_1 \text{ and } U_2, X - (U_1 \cap U_2) \neq X \\
\Longleftrightarrow\ &\forall \text{ nonempty open sets } U_1 \text{ and } U_2, (X - U_1) \cup (X - U_2) \neq X \\
\Longleftrightarrow\ &\forall \text{ proper closed sets } Y_1 \text{ and } Y_2, Y_1 \cup Y_2 \neq X \\
\Longleftrightarrow\ &\nexists \text{ proper closed sets } Y_1 \text{ and } Y_2, Y_1 \cup Y_2 = X.
\end{aligned}$$

(2) *Show that any nonempty open subset of an irreducible topological space is irreducible.* Given any open subset U of an irreducible topological space X. Write $U \subseteq Y_1 \cup Y_2$ where Y_1 and Y_2 are closed in X.

$$\begin{aligned}
&U \subseteq Y_1 \cup Y_2 \\
\Longrightarrow\ &\overline{U} \subseteq \overline{Y_1 \cup Y_2} \\
\Longrightarrow\ &X \subseteq Y_1 \cup Y_2 && (U \text{ is dense, } Y_1 \cup Y_2 \text{ is closed}) \\
\Longrightarrow\ &Y_1 = X \supseteq U \text{ or } Y_2 = X \supseteq U && (X \text{ is irreducible}) \\
\Longrightarrow\ &U \text{ is irreducible.}
\end{aligned}$$

(3) *Show that if Y is a subset of a topological space X, which is irreducible (in its induced topology), then the closure $\overline{Y}$ is also irreducible.* (Reductio ad absurdum) If $\overline{Y}$ were reducible, there are two closed sets Y_1 and Y_2 such that

$$\overline{Y} \subseteq Y_1 \cup Y_2, \overline{Y} \not\subseteq Y_i (i = 1, 2).$$

(a) $Y \subseteq \overline{Y} \subseteq Y_1 \cup Y_2$.

(b) $Y \not\subseteq Y_i (i = 1, 2)$. If not, $Y \subseteq Y_i$ for some i. Take closure to get $\overline{Y} \subseteq \overline{Y_i} = Y_i$ (since Y_i is closed), contrary to the assumption.

By (a)(b), Y is reducible, which is absurd.

□

I.8 What is Algebraic Geometry?

No exercises.

Chapter II: Schemes

II.1 Sheaves

Exercise II.1.1. (Constant presheaf)

Let A be an abelian group, and define the ***constant presheaf*** *associated to A on the topological space X to be the presheaf $U \mapsto A$ for all $U \neq \varnothing$, with restriction maps the identity. Show that the constant sheaf $\mathscr{A}$ defined in the text is the sheaf associated to this presheaf.*

Proof.

(1) Let $\mathscr{F}$ be the constant presheaf.

(2) Let $\theta : \mathscr{F} \to \mathscr{A}$ be a morphism consists of a morphism of abelian groups $\theta(U) : \mathscr{F}(U) = A \to \mathscr{A}(U)$ for each open set $U \subseteq X$ such that $\theta(U)(a) = f_a : x \mapsto a$ for each element $x \in U$. (It is well-defined.)

(3) Given any sheaf $\mathscr{G}$ and any morphism $\varphi : \mathscr{F} \to \mathscr{G}$, it suffices to find a morphism $\psi : \mathscr{A} \to \mathscr{G}$ such that $\varphi = \psi \circ \theta$.

(4) Given an open set $U \subseteq X$. Suppose $f \in \mathscr{A}(U)$ is a continuous maps of U into A. Since A is equipped with the discrete topology, f is locally constant, that is,
$$f(V_i) = a_i$$
where each V_i is a connected component of U. (In particular, $\{V_i\}$ is an open covering of U.)

(5) Now
$$s_i := \varphi(V_i)(a_i) \in \mathscr{G}(V_i)$$
is defined. Since $\mathscr{G}$ is a sheaf and all V_i are disjoint, there is a $s \in \mathscr{G}(U)$ such that $s|_{V_i} = s_i$ for each i. Now we define $\psi(U)$ by
$$\psi(U)(f) = s.$$
Thus ψ is a morphism and $\varphi = \psi \circ \theta$ by construction.

$\square$