

## Chapter 15: Bernoulli Numbers

*Author: Meng-Gen Tsai*

*Email: plover@gmail.com*

**Supplement.** Exercise 6.73 in the book Graham, Knuth and Patashnik, Concrete Mathematics, Second Edition.

*Prove that*

$$x \cot x = 1 - 2 \sum_{n=1}^{\infty} \frac{x^2}{n^2 \pi^2 - x^2}.$$

*Proof.*

(1) *Show that*

$$\cot x = \frac{1}{2^n} \sum_{k=0}^{2^n-1} \cot \frac{x + k\pi}{2^n}$$

*for all integers  $n \geq 1$ . Notice that*

$$\begin{aligned} \cot(x + \pi) &= \cot x, \\ \cot\left(x + \frac{\pi}{2}\right) &= -\tan x, \\ \cot x &= \frac{1}{2} \left( \cot \frac{x}{2} - \tan \frac{x}{2} \right). \end{aligned}$$

Use mathematical induction. The case  $n = 1$  is the same as the note.

Assume the case  $n = m$  holds. For  $n = m + 1$ ,

$$\begin{aligned}
\sum_{k=0}^{2^{m+1}-1} \cot \frac{x+k\pi}{2^{m+1}} &= \sum_{k=0}^{2^m-1} \cot \frac{x+k\pi}{2^{m+1}} + \sum_{k=2^m}^{2^{m+1}-1} \cot \frac{x+k\pi}{2^{m+1}} \\
&= \sum_{k=0}^{2^m-1} \cot \frac{x+k\pi}{2^{m+1}} + \sum_{k=0}^{2^m-1} \cot \frac{x+(2^m+k)\pi}{2^{m+1}} \\
&= \sum_{k=0}^{2^m-1} \cot \frac{x+k\pi}{2^{m+1}} + \sum_{k=0}^{2^m-1} \cot \left( \frac{x+k\pi}{2^{m+1}} + \frac{\pi}{2} \right) \\
&= \sum_{k=0}^{2^m-1} \left( \cot \frac{x+k\pi}{2^{m+1}} - \tan \frac{x+k\pi}{2^{m+1}} \right) \\
&= \sum_{k=0}^{2^m-1} \left( \cot \frac{x+k\pi}{2^{m+1}} - \tan \frac{x+k\pi}{2^{m+1}} \right) \\
&= 2 \sum_{k=0}^{2^m-1} \cot \frac{x+k\pi}{2^m}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{2^{m+1}} \sum_{k=0}^{2^{m+1}-1} \cot \frac{x+k\pi}{2^{m+1}} &= \frac{1}{2^{m+1}} \cdot 2 \sum_{k=0}^{2^m-1} \cot \frac{x+k\pi}{2^m} \\
&= \frac{1}{2^m} \sum_{k=0}^{2^m-1} \cot \frac{x+k\pi}{2^m} \\
&= \cot x.
\end{aligned}$$

(2) By rearranging the index of summation of the identity in (1), we have

$$x \cot x = \frac{x}{2^n} \cot \frac{x}{2^n} - \frac{x}{2^n} \tan \frac{x}{2^n} + \sum_{k=1}^{2^{n-1}-1} \frac{x}{2^n} \left( \cot \frac{x+k\pi}{2^n} + \cot \frac{x-k\pi}{2^n} \right)$$

for all integers  $n \geq 1$ .

(3) Notice that  $\lim_{x \rightarrow 0} x \cot x = 1$ . Let  $n \rightarrow \infty$ , the result is established.

□

**Exercise 15.6.** For  $m \geq 3$ , show  $|B_{2m+2}| > |B_{2m}|$ . (Hint: Use Theorem 2.)

*Proof.* By Theorem 2,

$$2\zeta(2m) = (-1)^{m+1} \frac{(2\pi)^{2m}}{(2m)!} B_{2m}.$$

Thus,

$$\frac{|B_{2m+2}|}{|B_{2m}|} = \frac{\zeta(2m+2)(2m+2)(2m+1)}{\zeta(2m)(2\pi)^2} > \frac{1 \cdot 8 \cdot 7}{\zeta(6) \cdot (2\pi)^2} = \frac{13230}{\pi^8} > 1,$$

or  $|B_{2m+2}| > |B_{2m}|$ .  $\square$

**Exercise 15.8.** Consider the power series expansion of  $\tan x$  about the origin;

$$\sum_{k=1}^{\infty} T_k \frac{x^{2k-1}}{(2k-1)!}.$$

Show

$$T_k = (-1)^{k-1} \frac{B_{2k}}{2k} (2^{2k} - 1) 2^{2k}.$$

Note that  $T_k \in \mathbb{Z}$  for all  $k$  by Exercise 3.

Proof.

(1)

$$\cot x = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} x^{2k-1}.$$

(2) Note that  $\tan x = \cot x - 2 \cot(2x)$ .

$\square$