

## Chapter 8: Some Special Functions

**Exercise 8.1.** Define

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

Prove that  $f$  has derivatives of all orders at  $x = 0$ , and that  $f^{(n)}(0) = 0$  for  $n = 1, 2, 3, \dots$

**Claim 1.**

$$\lim_{x \rightarrow 0} g(x) e^{-\frac{1}{x^2}} = 0$$

for any rational function  $g(x) \in \mathbb{R}(x)$ .

*Proof.* Write  $g(x) = \frac{p(x)}{q(x)}$  for some  $p(x), q(x) \in \mathbb{R}[x]$ . Write  $q(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_0$ .  $q(x)$  is not identically zero, that is, there exists the unique coefficient of the least power of  $x$  in  $q(x)$  which is non-zero, say  $b_M \neq 0$ . Now write  $g(x)$  as  $g(x) = \frac{p(x)/x^M}{q(x)/x^M}$ . The denominator of  $g(x)$  tends to  $b_M \neq 0$  as  $x \rightarrow 0$ . By the similar argument of Theorem 8.6(f) ( $\lim_{x \rightarrow \infty} x^n e^{-x} = 0$  for any  $n \in \mathbb{Z}$ ),

$$\frac{p(x)}{x^M} e^{-\frac{1}{x^2}} \rightarrow 0 \text{ as } x \rightarrow 0.$$

Hence,  $\lim_{x \rightarrow 0} g(x) e^{-\frac{1}{x^2}} = 0$  for any  $g(x) \in \mathbb{R}(x)$ .  $\square$

**Claim 2.** Given any real  $x \neq 0$

$$f^{(n)}(x) = g_n(x) e^{-\frac{1}{x^2}}$$

for some rational function  $g(x) \in \mathbb{R}(x)$ .

*Proof.* Say  $g_0(x) = 1 \in \mathbb{R}(x)$ . Notice that  $\mathbb{R}(x)$  is a field and  $g'(x) \in \mathbb{R}(x)$  for any  $g(x) \in \mathbb{R}(x)$ . (Write  $g(x) = \frac{p(x)}{q(x)}$  for some  $p(x), q(x) \in \mathbb{R}[x]$ . Notice that  $p'(x) \in \mathbb{R}[x]$  for any  $p(x) \in \mathbb{R}[x]$ .) Now we prove by mathematical induction. For  $n = 1$ , we have

$$\begin{aligned} f'(x) &= g'_0(x) e^{-\frac{1}{x^2}} + g_0(x) \cdot \left( -\frac{1}{x^2} \right)' e^{-\frac{1}{x^2}} \\ &= \left( g'_0(x) + g_0(x) \cdot \left( -\frac{1}{x^2} \right)' \right) e^{-\frac{1}{x^2}} \\ &= g_1(x) e^{-\frac{1}{x^2}} \end{aligned}$$

where  $g_1(x) = g'_0(x) + g_0(x) \cdot \left( -\frac{1}{x^2} \right)' \in \mathbb{R}(x)$ . Now assume  $n = k$  holds. For  $n = k + 1$ , similar to  $n = 1$ ,  $f^{(k+1)}(x) = g_{k+1}(x) e^{-\frac{1}{x^2}}$  where  $g_{k+1}(x) =$

$$g'_k(x) + g_k(x) \cdot \left(-\frac{1}{x^2}\right)' \in \mathbb{R}(x). \quad \square$$

*Proof of Exercise 8.1.* Prove by mathematical induction. For  $n = 1$ ,

$$f'(0) = \lim_{t \rightarrow 0} \frac{e^{-\frac{1}{t^2}} - 0}{t} = 0.$$

(Use Claim 1.) Now assume  $n = k$  holds. For  $n = k + 1$ ,

$$f^{(k+1)}(0) = \lim_{t \rightarrow 0} \frac{f^{(k)}(t) - f^{(k)}(0)}{t} = \lim_{t \rightarrow 0} \frac{g_k(t)e^{-\frac{1}{t^2}} - 0}{t} = 0.$$

(Use Claim 1 and 2.) Thus,  $f^{(n)}(0) = 0$  for  $n \in \mathbb{Z}^+$ .  $\square$

**Exercise 8.6.** Suppose  $f(x)f(y) = f(x+y)$  for all real  $x$  and  $y$ .

(a) Assuming that  $f$  is differentiable and not zero, prove that

$$f(x) = e^{cx}$$

where  $c$  is a constant.

(b) Prove the same thing, assuming only that  $f$  is continuous.

(b) implies (a). We prove (b) directly.

*Proof of (b).* Since  $f(x)$  is not zero, there exists  $x_0 \in \mathbb{R}$  such that  $f(x_0) \neq 0$ . So  $f(0)f(x_0) = f(x_0)$ , or  $f(0) = 1$  by cancelling  $f(x_0) \neq 0$ .

Next,  $f(\frac{n}{m}) = f(\frac{1}{m})^n$  for  $m \in \mathbb{Z}$ ,  $n \in \mathbb{Z}^+$ . Since  $f$  is continuous at  $x = 0$ ,  $f$  is positive in the neighborhood of  $x = 0$ . That is, there exists  $N \in \mathbb{Z}^+$  such that  $f(\frac{1}{m}) > 0$  whenever  $|m| \geq N$ . So,  $f(\frac{n}{m}) = f(\frac{1}{m})^n > 0$ . (Since  $f(\frac{n}{m}) = f(\frac{kn}{km})$  for any  $k \in \mathbb{Z}^+$ , we can rescale  $m$  to  $km$  such that  $|km| \geq N$ .) That is,  $f$  is positive on  $\mathbb{Q}$ . Since  $\mathbb{Q}$  is dense in  $\mathbb{R}$  and  $f$  is continuous on  $\mathbb{R}$ ,  $f$  is positive on  $\mathbb{R}$ .

Now let  $c = \log f(1)$  (which is well-defined since  $f > 0$ ). We write  $f(1)$  in the two ways. Firstly,  $f(1) = f(\frac{n}{n}) = f(\frac{1}{n})^n$  where  $n \in \mathbb{Z}^+$ . Secondly,  $f(1) = e^c = (e^{\frac{c}{n}})^n$ . Since the positive  $n$ -th root is unique (Theorem 1.21),  $f(\frac{1}{n}) = e^{\frac{c}{n}}$  for  $n \in \mathbb{Z}^+$ . By  $f(x)f(-x) = f(0) = 1$  or  $f(-x) = \frac{1}{f(x)}$ ,  $f(-\frac{1}{n}) = \frac{1}{e^{\frac{c}{n}}} = e^{-\frac{c}{n}}$  for  $n \in \mathbb{Z}^+$ . Therefore,

$$f\left(\frac{1}{m}\right) = e^{\frac{c}{m}} \text{ where } m \in \mathbb{Z}.$$

By using  $f(\frac{n}{m}) = f(\frac{1}{m})^n$  for  $m \in \mathbb{Z}$ ,  $n \in \mathbb{Z}^+$  again,  $f(\frac{n}{m}) = e^{c\frac{n}{m}}$  where  $m \in \mathbb{Z}$ ,  $n \in \mathbb{Z}^+$ , or

$$f(x) = e^{cx} \text{ where } x \in \mathbb{Q}.$$

Since  $g(x) = f(x) - e^{cx}$  vanishes on a dense set of  $\mathbb{Q}$  and  $g$  is continuous on  $\mathbb{R}$ ,  $g$  vanishes on  $\mathbb{R}$ . Therefore,  $f(x) = e^{cx}$  for  $x \in \mathbb{R}$ .  $\square$

**Supplement.** Proof of (a).

*Proof of (a).* Since  $f(x)$  is not zero, there exists  $x_0 \in \mathbb{R}$  such that  $f(x_0) \neq 0$ . So  $f(0)f(x_0) = f(x_0)$ , or  $f(0) = 1$  by cancelling  $f(x_0) \neq 0$ .

Since  $f$  is differentiable, for any  $x \in \mathbb{R}$ ,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x)f(h) - f(x)}{h} \\ &= f(x) \lim_{h \rightarrow 0} \frac{f(h) - 1}{h} \\ &= f(x) \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} \\ &= f(x)f'(0). \end{aligned}$$

Let  $c = f'(0)$  be a constant. Then  $f'(x) = cf(x)$ . So  $f(x) = e^{cx}$  for  $x \in \mathbb{R}$ . (To see this, let  $g(x) = \frac{f(x)}{e^{cx}}$  be well-defined on  $\mathbb{R}$ .  $g(0) = 1$ .  $g'(x) = 0$  since  $f'(x) = cf(x)$ . So  $g(x)$  is a constant, or  $g(x) = 1$  since  $g(0) = 1$ . Therefore,  $f(x) = e^{cx}$  on  $\mathbb{R}$ .)  $\square$

**Supplement.** Cauchy's functional equation.

- (1) (*Cauchy's functional equation.*) Suppose  $f(x) + f(y) = f(x+y)$  for all real  $x$  and  $y$ . Assuming that  $f$  is continuous, prove that  $f(x) = cx$  where  $c$  is a constant.

Notice that we cannot let  $g(x) = \log f(x)$  and apply Cauchy's functional equation on  $g(x)$  to prove Exercise 8.6 since  $f(x)$  is not necessarily positive and thus  $g(x) = \log f(x)$  might be meaningless. However, this wrong approach gives you some useful ideas such as you need to prove that  $f(x)$  is positive first, and  $f(x)$  should be equal to  $e^{cx}$  where  $c = g(1) = \log f(1)$ .

- (2) Suppose  $f(xy) = f(x) + f(y)$  for all positive real  $x$  and  $y$ . Assuming that  $f$  is continuous, prove that  $f(x) = c \log x$  where  $c$  is a constant.
- (3) Suppose  $f(xy) = f(x)f(y)$  for all positive real  $x$  and  $y$ . Assuming that  $f$  is continuous and positive, prove that  $f(x) = x^c$  where  $c$  is a constant.
- (4) Suppose  $f(x+y) = f(x) + f(y) + xy$  for all real  $x$  and  $y$ . Assuming that  $f$  is continuous, prove that  $f(x) = \frac{1}{2}x^2 + cx$  where  $c$  is a constant.
- (5) (*USA 2002.*) Suppose  $f(x^2 - y^2) = xf(x) - yf(y)$  for all real  $x$  and  $y$ . Assuming that  $f$  is continuous, prove that  $f(x) = cx$  where  $c$  is a constant.