

Chapter 6: The Riemann-Stieltjes Integral

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Supplement. Another definition of Riemann-Stieltjes integral. (*Exercise 7.3, 7.4 of the book T. M. Apostol, Mathematical Analysis, Second Edition.*) Let P be a partition of $[a, b]$. The norm of a partition P is the length of the largest subinterval $[x_{i-1}, x_i]$ of P and is denoted by $\|P\|$.

We say $f \in \mathcal{R}(\alpha)$ if there exists $A \in \mathbb{R}$ having the property that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any partition P of $[a, b]$ with norm $\|P\| < \delta$ and for any choice of $t_i \in [x_{i-1}, x_i]$, we have $|\sum_{i=1}^n f(t_i)\Delta\alpha_i - A| < \varepsilon$.

Claim. $f \in \mathcal{R}$ in the sense of Definition 6.2 implies that $f \in \mathcal{R}$ in the sense of this another definition.

Proof of Claim. Let $A = \int f dx$, $M > 0$ be one upper bound of $|f|$ on $[a, b]$. Given $\varepsilon > 0$, there exists a partition $P_0 = \{a = x_0, x_1, \dots, x_{N-1}, x_N = b\}$ such that $U(P_0, f) \leq A + \frac{\varepsilon}{2}$. Let $\delta = \frac{\varepsilon}{2MN} > 0$. Then for any partition P with norm $\|P\| < \delta$, write

$$U(P, f) = \sum_{i=1}^n M_i \Delta x_i = S_1 + S_2,$$

where S_1 is the sum of terms arising from those subintervals of P containing no point of P_0 , S_2 is the sum of the remaining terms. Then

$$S_1 \leq U(P_0, f) < A + \frac{\varepsilon}{2},$$

$$S_2 \leq NM\|P\| < NM\delta < \frac{\varepsilon}{2}.$$

Therefore, $U(P, f) < A + \varepsilon$. Similarly, $L(P, f) > A - \varepsilon$ whenever $\|P\| < \delta'$. Hence, $|\sum_{i=1}^n f(t_i)\Delta x_i - A| < \varepsilon$ whenever $\|P\| < \min\{\delta, \delta'\}$. (Copy Apostol's hint and ensure $M > 0$. M in Apostol's hint might be zero if $f = 0$.) \square

This supplement will be used in computing $\int_0^\infty (\frac{\sin x}{x})^2 dx = \frac{\pi}{2}$ in Exercise 8.12.

Exercise 6.1. Suppose α increases on $[a, b]$, $a \leq x_0 \leq b$, α is continuous at x_0 , $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathcal{R}(\alpha)$ and that $\int f d\alpha = 0$.

Given any partition $P = \{a = p_0, p_1, \dots, p_{n-1}, p_n = b\}$, where $a = p_0 \leq p_1 \leq \dots \leq p_{n-1} \leq p_n = b$. We might compute $L(P, f, \alpha)$ and $U(P, f, \alpha)$ by using ε - δ

argument since we are hinted by the condition that α is continuous. A function which is continuous at x_0 has a nice property near x_0 and this property would help us estimate $U(P, f, \alpha)$ near x_0 . On the contrary, if both f and α are discontinuous at x_0 , it might be $f \notin \mathcal{R}(\alpha)$. Besides, if f has too many points of discontinuity ($f(x) = 0$ if $x \in \mathbb{Q}$ and $f(x) = 1$ otherwise, for example), then f might not be Riemann-integrable on $[0, 1]$.

Claim 1. $L(P, f, \alpha) = 0$.

Proof of Claim 1. $m_i = 0$ since $\inf f(x) = 0$ on any subinterval of $[a, b]$. So $L(P, f, \alpha) = \sum m_i \Delta \alpha_i = 0$. Here we don't need the condition that α is continuous at x_0 . \square

Claim 2. For any $\varepsilon > 0$, there exists a partition P such that $U(P, f, \alpha) < \varepsilon$.

Proof of Claim 2. Say $x_0 \in [p_{i_0-1}, p_{i_0}]$ for some i_0 . Then

$$M_i = \sup_{p_{i-1} \leq x \leq p_i} f(x) = \begin{cases} 0 & \text{if } i \neq i_0, \\ 1 & \text{if } i = i_0. \end{cases}$$

So

$$U(P, f, \alpha) = \sum M_i \Delta \alpha_i = \Delta \alpha_{i_0}.$$

It is not true for any arbitrary α . (For example, α has a jump on $x = x_0$.) In fact, Exercise 6.3 shows this. Luckily, α is continuous at x_0 . So for $\varepsilon > 0$, there exists $\delta > 0$ such that $|\alpha(x) - \alpha(x_0)| < \frac{\varepsilon}{2}$ whenever $|x - x_0| < \delta$ (and $x \in [a, b]$). Now we pick a nice partition

$$P = \{a, x_0 - \delta_1, x_0 + \delta_2, b\},$$

where $\delta_1 = \min\{\delta, x_0 - a\} \geq 0$ and $\delta_2 = \min\{\delta, b - x_0\} \geq 0$. (It is a trick about resizing “ δ ” to avoid considering the edge cases $x_0 = a$ or $x_0 = b$ or $a = b$.) Then $x_0 \in [x_0 - \delta_1, x_0 + \delta_2]$ and $\Delta \alpha$ on $[x_0 - \delta_1, x_0 + \delta_2]$ is

$$\begin{aligned} \alpha(x_0 + \delta_2) - \alpha(x_0 - \delta_1) &= (\alpha(x_0 + \delta_2) - \alpha(x_0)) + (\alpha(x_0) - \alpha(x_0 - \delta_1)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore, $U(P, f, \alpha) < \varepsilon$. \square

Proof (Definition 6.2). By Claim 1 and 2 and notice that $U(P, f, \alpha) \geq 0$ for any

partition P ,

$$\begin{aligned}\int_a^{\bar{b}} f d\alpha &= \inf U(P, f, \alpha) = 0, \\ \int_a^{\underline{b}} f d\alpha &= \sup L(P, f, \alpha) = 0,\end{aligned}$$

the inf and sup again being taken over all partitions. Hence $f \in \mathcal{R}(\alpha)$ and that $\int f d\alpha = 0$ by Definition 6.2. \square

Proof (Theorem 6.5). By Claim 1 and 2,

$$0 \leq U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$$

Hence $f \in \mathcal{R}(\alpha)$ by Theorem 6.5. Furthermore,

$$\int f d\alpha = \int_a^{\bar{b}} f d\alpha = \sup L(P, f, \alpha) = 0.$$

\square

Proof (Theorem 6.10). $f \in \mathcal{R}(\alpha)$ by Theorem 6.10. Thus, by Claim 1

$$\int f d\alpha = \int_a^{\bar{b}} f d\alpha = \sup L(P, f, \alpha) = 0.$$

\square

Exercise 6.2. Suppose $f \geq 0$, f is continuous on $[a, b]$, and $\int_a^b f(x) dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$. (Compare with Exercise 6.1.)

For one application, see Exercise 7.20.

Proof. (Reductio ad absurdum) If there were $p \in [a, b]$ such that $f(p) > 0$. Since f is continuous on $[a, b]$, given $\varepsilon = \frac{1}{64}f(p) > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(p)| \leq \frac{1}{64}f(p) \text{ whenever } |x - p| \leq \delta, x \in [a, b].$$

Hence

$$f(x) \geq \frac{63}{64}f(p)$$

whenever $x \in E = [\max\{a, p - \delta\}, \min\{b, p + \delta\}] \subseteq [a, b]$. Note that the length of E is $|E| > 0$. So

$$0 = \int_a^b f(x) dx \geq \int_E f(x) dx \geq \int_E \frac{63}{64}f(p) dx = \frac{63}{64}f(p)|E| > 0,$$

which is absurd. \square

Note. (Lebesgue integral) Let f be a nonnegative measurable function. Then $\int f = 0$ implies $f = 0$ a.e.

Exercise 6.3.
PLACEHOLDER

Exercise 6.4. If

$$f(x) = \begin{cases} 0 & \text{for all irrational } x, \\ 1 & \text{for all rational } x, \end{cases}$$

prove that $f \notin \mathcal{R}$ on $[a, b]$ for any $a < b$.

Proof. Given any partition

$$P = \{a = p_0, p_1, \dots, p_{n-1}, p_n = b\}$$

of $[a, b]$ where $a = p_0 \leq p_1 \leq \dots \leq p_{n-1} \leq p_n = b$. Since $a < b$, we might assume that $a = p_0 < p_1 < \dots < p_{n-1} < p_n = b$ by removing duplicated points. Since \mathbb{Q} and $\mathbb{R} - \mathbb{Q}$ are dense in \mathbb{R} , we have

$$\begin{aligned} M_i &= \sup_{p_{i-1} \leq x \leq p_i} f(x) = 1, \\ m_i &= \inf_{p_{i-1} \leq x \leq p_i} f(x) = 0, \\ U(P, f) &= \sum_{i=1}^n M_i \Delta x_i = \sum_{i=1}^n \Delta x_i = b - a, \\ L(P, f) &= \sum_{i=1}^n m_i \Delta x_i = \sum_{i=1}^n 0 = 0. \end{aligned}$$

Since P is arbitrary,

$$\begin{aligned} \int_a^b f dx &= \inf U(P, f) = b - a > 0, \\ \int_a^b f dx &= \sup L(P, f) = 0. \end{aligned}$$

Hence $f \notin \mathcal{R}$ on $[a, b]$ for any $a < b$. \square

Note.

- (1) (Lebesgue integral) f is Lebesgue integrable.
- (2) $f \in \mathcal{R}$ on $[a, b]$ iff $a = b$.

- (3) (Problem 4.1 in *H. L. Royden, Real Analysis, 3rd edition.*) Construct a sequence $\{f_n\}$ of nonnegative, Riemann integrable functions such that f_n increases monotonically to f . What does this imply about changing the order of integration and the limiting process? (Since \mathbb{Q} is countable, write

$$\mathbb{Q} = \{r_1, r_2, \dots\}.$$

Define

$$f_n(x) = \begin{cases} 0 & \text{if } x \notin \{r_1, \dots, r_n\}, \\ 1 & \text{if } x \in \{r_1, \dots, r_n\}. \end{cases}$$

By construction, f_n increases monotonically to f pointwise. Note that $f_n \rightarrow f$ not uniformly. Also, $\int_a^b f_n(x)dx = 0$ by using the same argument in Theorem 6.10. Therefore, $\lim_{n \rightarrow \infty} \int_a^b f_n(x)dx = 0$ but $\int_a^b \lim_{n \rightarrow \infty} f_n(x)dx = \int_a^b f(x)dx$ does not exist.)

Exercise 6.5. Suppose f is a bounded real function on $[a, b]$, and $f^2 \in \mathcal{R}$ on $[a, b]$. Does it follow that $f \in \mathcal{R}$? Does the answer change if we assume that $f^3 \in \mathcal{R}$?

Actually we can omit the boundedness assumption of f since $f^2 \in \mathcal{R}$ or $f^3 \in \mathcal{R}$.

Proof.

- (1) Show that $f^2 \in \mathcal{R}$ on $[a, b]$ does not imply that $f \in \mathcal{R}$ (unless $f \geq 0$ on $[a, b]$). Similar to Exercise 6.4, define

$$f(x) = \begin{cases} -1 & \text{for all irrational } x, \\ 1 & \text{for all rational } x. \end{cases}$$

$f^2 = 1 \in \mathcal{R}$ on $[a, b]$ but $f \notin \mathcal{R}$ on $[a, b]$ for any $a < b$. (The proof for the “unless” part is similar to (2).)

- (2) Show that $f^3 \in \mathcal{R}$ on $[a, b]$ implies that $f \in \mathcal{R}$. Let $\phi(x) = x^{\frac{1}{3}}$ on \mathbb{R} . By Theorem 6.11, $f(x) = \phi(f(x)^3) \in \mathcal{R}$. (The boundedness condition in Theorem 6.11 is unnecessary.)

□

Note. (Lebesgue integral) Suppose that f^2 is Lebesgue integrable. Does it follow that f is Lebesgue integrable? Does the answer change if we assume that f^3 is Lebesgue integrable? Both answers are no.

Exercise 6.6.

PLACEHOLDER

Exercise 6.7. Suppose f is a real function on $(0, 1]$ and $f \in \mathcal{R}$ on $[c, 1]$ for every $c > 0$. Define

$$\int_0^1 f(x)dx = \lim_{c \rightarrow 0} \int_c^1 f(x)dx$$

if this limit exists (and is finite).

- (a) If $f \in \mathcal{R}$ on $[0, 1]$, show that this definition of the integral agrees with the old one.
- (b) Construct a function such that the above limit exists, although it fails to exist with $|f|$ in place of f .

Proof of (a).

- (1) Since $f \in \mathcal{R}$ on $[0, 1]$, f is bounded or $|f| \leq M$ for some real M .
- (2) For any $0 < c < 1$, we have

$$\begin{aligned} \left| \int_0^1 f(x)dx - \int_c^1 f(x)dx \right| &= \left| \int_0^c f(x)dx \right| && \text{(Theorem 6.12(c))} \\ &\leq Mc. && \text{(Theorem 6.12(d))} \end{aligned}$$

- (3) Given any $\varepsilon > 0$, there exists $\delta = \frac{\varepsilon}{M+1} > 0$ such that

$$\left| \int_0^c f(x)dx - \int_0^1 f(x)dx \right| \leq Mc < M\delta = M \cdot \frac{\varepsilon}{M+1} < \varepsilon$$

whenever $0 < c < \delta$. Hence $\lim_{c \rightarrow 0} \int_c^1 f(x)dx = \int_0^1 f(x)dx$.

□

Proof of (b)(Construct by nonabsolutely convergent series).

- (1) Given any nonabsolutely (conditionally) convergent series $\sum_{k=1}^n a_k$ (take $\sum \frac{(-1)^n}{n}$ for example and then see Remark 3.46), we define f on $(0, 1]$ by

$$f(x) = 2^n a_n$$

if $\frac{1}{2^n} < x \leq \frac{1}{2^{n-1}}$ as $n = 1, 2, \dots$

- (2) By construction,

$$\int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} f(x)dx = \left(\frac{1}{2^{n-1}} - \frac{1}{2^n} \right) 2^n a_n = a_n.$$

and thus

$$\int_{\frac{1}{2^n}}^1 f(x)dx = \int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} f(x)dx + \cdots + \int_{\frac{1}{2}}^1 f(x)dx = \sum_{k=1}^n a_k.$$

- (3) Given any $\varepsilon > 0$. Since $\sum a_n$ is convergent, there exists a common integer N such that

$$|a_n| \leq \frac{\varepsilon}{89}$$

and

$$\left| \sum_{k=1}^n a_k - A \right| \leq \frac{\varepsilon}{64}$$

for some real A whenever $n \geq N$ (Definition 3.21 and Theorem 3.23). Therefore, for any $0 < c \leq \frac{1}{2^N}$, say $\frac{1}{2^{n+1}} < c \leq \frac{1}{2^n} \leq \frac{1}{2^N}$ for some $n \geq N$, we have

$$\begin{aligned} \left| \int_c^1 f(x)dx - A \right| &= \left| \int_c^{\frac{1}{2^n}} f(x)dx + \int_{\frac{1}{2^n}}^1 f(x)dx - A \right| \\ &\leq \left| \left(\frac{1}{2^n} - c \right) 2^{n+1} a_{n+1} \right| + \left| \sum_{k=1}^n a_k - A \right| \\ &\leq |a_{n+1}| + \left| \sum_{k=1}^n a_k - A \right| \\ &\leq \frac{\varepsilon}{89} + \frac{\varepsilon}{64} \\ &\leq \varepsilon. \end{aligned}$$

Hence, $\lim_{c \rightarrow 0} \int_c^1 f(x)dx = A$ exists.

- (4) Since

$$\int_{\frac{1}{2^n}}^1 |f(x)|dx = \int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} |f(x)|dx + \cdots + \int_{\frac{1}{2}}^1 |f(x)|dx = \sum_{k=1}^n |a_k| \rightarrow \infty$$

as $n \rightarrow \infty$, $\lim_{c \rightarrow 0} \int_c^1 f(x)dx$ does not exist.

□

Exercise 6.8.
PLACEHOLDER

Exercise 6.9.

PLACEHOLDER

Exercise 6.10. Let p and q be positive real integers such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Prove the following statements.

(a) If $u \geq 0$ and $v \geq 0$, then

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

Equality holds if and only if $u^p = v^q$.

(b) If $f \in \mathcal{R}(\alpha)$, $g \in \mathcal{R}(\alpha)$, $f \geq 0$, $g \geq 0$, and

$$\int_a^b f^p d\alpha = \int_a^b g^q d\alpha = 1,$$

then

$$\int_a^b fg d\alpha \leq 1.$$

(c) If f and g are complex functions in $\mathcal{R}(\alpha)$, then

$$\left| \int_a^b fg d\alpha \right| \leq \left\{ \int_a^b |f|^p d\alpha \right\}^{\frac{1}{p}} \left\{ \int_a^b |g|^q d\alpha \right\}^{\frac{1}{q}}.$$

This is **Hölder's inequality**. When $p = q = 2$ it is usually called the Schwarz inequality. (Note that Theorem 1.35 is a very special case of this.)

(d) Show that Hölder's inequality is also true for the "improper" integrals described in Exercise 6.7 and 6.8.

Proof of (a) (Young's inequality).

(1) $u = 0$ or $v = 0$ is nothing to do. For $u > 0$ and $v > 0$, we give some different proofs.

(2) First proof.

$$\begin{aligned} uv &= \exp(\log(uv)) \\ &= \exp\left(\frac{1}{p} \log(u^p) + \frac{1}{q} \log(v^q)\right) \\ &\leq \frac{1}{p} \exp(\log(u^p)) + \frac{1}{q} \exp(\log(v^q)) \quad (\text{Convexity of } \exp(x)) \\ &= \frac{u^p}{p} + \frac{v^q}{q}. \end{aligned}$$

Here the convexity of $\exp(x)$ can be derived by the fact that $(\exp(x))'' > 0$ and Exercise 5.14. The fact that the equality holds if and only if $u^p = v^q$ is derived from the strictly convexity of $\exp(x)$ additionally. (For the details about the exponential and logarithmic functions, might see Chapter 8.)

(3) Second proof.

$$\begin{aligned}\log\left(\frac{u^p}{p} + \frac{v^q}{q}\right) &\geq \frac{1}{p}\log(u^p) + \frac{1}{q}\log(v^q) && \text{(Concavity of } \log(x)\text{)} \\ &= \log(u) + \log(v) \\ &= \log(uv).\end{aligned}$$

Since $\log(x)$ increases monotonically ($(\log(x))' = \frac{1}{x} > 0$ if $x > 0$), $\frac{u^p}{p} + \frac{v^q}{q} \geq uv$ (or take the exponential function to get the same conclusion). Here the concavity of $\log(x)$ can be derived by the fact that $(\log(x))'' < 0$ and a statement that $f''(x) \leq 0$ if and only if f is concave. The fact that the equality holds if and only if $u^p = v^q$ is derived from the strictly concavity of $\log(x)$ additionally. (The proof is analogous to Exercise 5.14.)

(4) Third proof. Suppose that $f : [0, \infty) \rightarrow [0, \infty)$ is a strictly increasing continuous function such that $f(0) = 0$ and $\lim_{x \rightarrow \infty} f(x) = \infty$. Then

$$uv \leq \int_0^u f(x)dx + \int_0^v f^{-1}(x)dx$$

for every $u, v \geq 0$, and equality occurs if and only if $v = f(u)$. Define

$$F(x) = -xf(x) + \int_0^x f(t)dt + \int_0^{f(x)} f^{-1}(t)dt.$$

By Theorem 6.20 (the fundamental theorem of calculus) and Theorem 5.5 (chain rule),

$$F'(x) = -(f(x) + xf'(x)) + f(x) + f'(x)f^{-1}(f(x)) = 0.$$

Hence $F(x)$ is a constant on $(0, u)$ (Theorem 5.11(b)). Note that $F(x)$ is continuous on $[0, u]$ and $F(0) = 0$, so $F(x) = 0$ on $[0, u]$ or

$$\int_0^x f(t)dt + \int_0^{f(x)} f^{-1}(t)dt = xf(x).$$

Take $x = u$ to get

$$\int_0^u f(x)dx + \int_0^{f(u)} f^{-1}(x)dx = uf(u).$$

Hence

$$\begin{aligned}
& \int_0^u f(x)dx + \int_0^v f^{-1}(x)dx - uv \\
&= \int_0^u f(x)dx + \int_0^{f(u)} f^{-1}(x)dx + \int_{f(u)}^v f^{-1}(x)dx - uv \\
&= uf(u) + \int_{f(u)}^v f^{-1}(x)dx - uv \\
&= \int_{f(u)}^v [f^{-1}(x) - f^{-1}(f(u))]dx \\
&\geq 0.
\end{aligned}$$

The last inequality holds since f is strictly increasing and thus f^{-1} is strictly increasing too. Besides, the equality holds if and only if $f(u) = v$. Now the conclusion holds by taking $f(x) = x^{p-1}$ in

$$uv \leq \int_0^u f(x)dx + \int_0^v f^{-1}(x)dx$$

and the equality holds if and only if $u^p = v^q$.

□

Proof of (b). Every integral is well-defined (Theorem 6.11 and Theorem 6.13(a)). Let $u = f \geq 0$ and $v = g \geq 0$ in (a). Integrate both sides of the inequality

$$fg \leq \frac{f^p}{p} + \frac{g^q}{q}$$

to get

$$\begin{aligned}
\int_a^b fg d\alpha &\leq \int_a^b \left(\frac{f^p}{p} + \frac{g^q}{q} \right) d\alpha && \text{(Theorem 6.12(b))} \\
&= \int_a^b \frac{f^p}{p} d\alpha + \int_a^b \frac{g^q}{q} d\alpha && \text{(Theorem 6.12(a))} \\
&= \frac{1}{p} \int_a^b f^p d\alpha + \frac{1}{q} \int_a^b g^q d\alpha && \text{(Theorem 6.12(a))} \\
&= \frac{1}{p} + \frac{1}{q} && \text{(Assumption)} \\
&= 1.
\end{aligned}$$

The equality holds if $f^p = g^q$. Note that the equality does not hold only if $f^p = g^q$. (Consider α is constant on some subinterval $[c, d] \subsetneq [a, b]$.) Luckily, it is true for the additional assumption that $\alpha(x) = x$ and f, g are continuous on $[a, b]$. □

Proof of (c). There are three possible cases.

(1) The case $\left\{ \int_a^b |f|^p d\alpha \right\}^{\frac{1}{p}} = 0$. So $\int_a^b |f|^p d\alpha = 0$.

(a) Show that $\int_a^b |f| d\alpha = 0$ if $\int_a^b |f|^p d\alpha = 0$. (Reductio ad absurdum)
 If $\int_a^b |f| d\alpha = A > 0$, then given $\varepsilon = \frac{A}{2} > 0$, there exists a partition $P_0 = \{a = x_0 \leq \dots \leq x_n = b\}$ such that

$$\sum_{i=0}^n m_i \Delta\alpha_i > \frac{A}{2},$$

where $m_i = \inf_{x \in [x_{i-1}, x_i]} |f|$ and $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$. By the pigeonhole principle, there exists $1 \leq i_0 \leq n$ such that

$$L(P_0, |f|, \alpha) = m_{i_0} \Delta\alpha_{i_0} > \frac{A}{2n} > 0.$$

Especially, $m_{i_0} > 0$ and $\Delta\alpha_{i_0} > 0$. Now we consider $L(P, |f|^p, \alpha)$. Hence

$$L(P_0, |f|^p, \alpha) = \sum_{i=0}^n m_i^p \Delta\alpha_i \geq m_{i_0}^p \Delta\alpha_{i_0} > 0,$$

or

$$\int_a^b |f| d\alpha = \sup L(P, f, \alpha) \geq m_{i_0}^p \Delta\alpha_{i_0} > 0,$$

which is absurd.

(b) Show that $\int_a^b |fg| d\alpha = 0$ if $\int_a^b |f| d\alpha = 0$. Since $g \in \mathcal{R}(\alpha)$, $|g|$ is bounded by some real M on $[a, b]$, that is, $|g(x)| \leq M$. Hence

$$0 \leq \int_a^b |fg| d\alpha \leq \int_a^b M |f| d\alpha = M \int_a^b |f| d\alpha = 0.$$

Therefore $\int_a^b |fg| d\alpha = 0$.

By (a)(b), $\int_a^b |fg| d\alpha = 0$ and thus Hölder's inequality holds for this case.

(2) The case $\left\{ \int_a^b |g|^q d\alpha \right\}^{\frac{1}{q}} = 0$. Similar to (1).

(3) If both $\left\{ \int_a^b |f|^p d\alpha \right\}^{\frac{1}{p}} > 0$ and $\left\{ \int_a^b |g|^q d\alpha \right\}^{\frac{1}{q}} > 0$, then we apply (b) to

$$F(x) = \frac{|f(x)|}{\left\{ \int_a^b |f(x)|^p d\alpha \right\}^{\frac{1}{p}}} \quad \text{and} \quad G(x) = \frac{|g(x)|}{\left\{ \int_a^b |g(x)|^q d\alpha \right\}^{\frac{1}{q}}}.$$

Here $F(x) \geq 0$ and $G(x) \geq 0$ are well-defined and Riemann integrable. Thus the conclusion holds. The equality holds if $F(x)^p = G(x)^q$ or

$$\frac{|f|^p}{\int_a^b |f|^p d\alpha} = \frac{|g|^q}{\int_a^b |g|^q d\alpha}.$$

Note that the equality does not hold only if $\frac{|f|^p}{\int_a^b |f|^p d\alpha} = \frac{|g|^q}{\int_a^b |g|^q d\alpha}$. Luckily, it is true for the additional assumption that $\alpha(x) = x$ and f, g are continuous on $[a, b]$.

By (1)(2)(3), in any case the equality holds if

$$|f|^p \int_a^b |g|^q d\alpha = |g|^q \int_a^b |f|^p d\alpha.$$

In addition, if $\alpha(x) = x$ and f, g are continuous on $[a, b]$, then the equality holds if and only if

$$|f|^p \int_a^b |g|^q d\alpha = |g|^q \int_a^b |f|^p d\alpha.$$

□

Proof of (d).

- (1) Suppose f and g are real functions on $(0, 1]$ and $f, g \in \mathcal{R}$ on $[c, 1]$ for every $c > 0$. Show that

$$\left| \int_0^1 f g dx \right| \leq \left\{ \int_0^1 |f|^p dx \right\}^{\frac{1}{p}} \left\{ \int_0^1 |g|^q dx \right\}^{\frac{1}{q}}.$$

Here \int_0^1 is one improper integral defined in Exercise 6.7.

- (a) By (c), we have

$$\left| \int_c^1 f g dx \right| \leq \left\{ \int_c^1 |f|^p dx \right\}^{\frac{1}{p}} \left\{ \int_c^1 |g|^q dx \right\}^{\frac{1}{q}}$$

for any $c \in (0, 1]$. Here every integral is well-defined (Theorem 6.11 and Theorem 6.13).

- (b) Since every integral is ≥ 0 , by taking the limit in the right hand side we have

$$\begin{aligned} \left| \int_c^1 f g dx \right| &\leq \left\{ \int_c^1 |f|^p dx \right\}^{\frac{1}{p}} \left\{ \int_c^1 |g|^q dx \right\}^{\frac{1}{q}} \\ &\leq \left\{ \int_0^1 |f|^p dx \right\}^{\frac{1}{p}} \left\{ \int_0^1 |g|^q dx \right\}^{\frac{1}{q}}. \end{aligned}$$

It is possible that $\left\{ \int_0^1 |f|^p dx \right\}^{\frac{1}{p}} = \infty$ or $\left\{ \int_0^1 |g|^q dx \right\}^{\frac{1}{q}} = \infty$.

- (c) Now $\left| \int_c^1 f g dx \right|$ is bounded by $\left\{ \int_0^1 |f|^p dx \right\}^{\frac{1}{p}} \left\{ \int_0^1 |g|^q dx \right\}^{\frac{1}{q}}$. Take limit to get

$$\left| \int_0^1 f g dx \right| \leq \left\{ \int_0^1 |f|^p dx \right\}^{\frac{1}{p}} \left\{ \int_0^1 |g|^q dx \right\}^{\frac{1}{q}}$$

even if some limit is divergent.

- (2) Suppose f and g are real functions on $[a, b]$ and $f, g \in \mathcal{R}$ on $[a, b]$ for every $b > a$ where a is fixed. Show that

$$\left| \int_a^\infty f g dx \right| \leq \left\{ \int_a^\infty |f|^p dx \right\}^{\frac{1}{p}} \left\{ \int_a^\infty |g|^q dx \right\}^{\frac{1}{q}}.$$

Here \int_a^∞ is one improper integral defined in Exercise 6.8. Same as (1).

□

Exercise 6.11. Let α be a fixed increasing function on $[a, b]$. For $u \in \mathcal{R}(\alpha)$, define

$$\|u\|_2 = \left\{ \int_a^b |u|^2 d\alpha \right\}^{\frac{1}{2}}.$$

Suppose $f, g, h \in \mathcal{R}(\alpha)$, and prove the triangle inequality

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2$$

as a consequence of the Schwarz inequality, as in the proof of Theorem 1.37.

Proof.

- (1) By Exercise 6.10(c) with $p = q = 2$, we have

$$\begin{aligned} \int_a^b |f - g| |g - h| d\alpha &= \left| \int_a^b |f - g| |g - h| d\alpha \right| \\ &\leq \left\{ \int_a^b |f - g|^2 d\alpha \right\}^{\frac{1}{2}} \left\{ \int_a^b |g - h|^2 d\alpha \right\}^{\frac{1}{2}} \\ &= \|f - g\|_2 \|g - h\|_2. \end{aligned}$$

Every integral is well-defined (Theorem 6.12 and Theorem 6.13 (or Theorem 6.11)).

- (2) Since

$$\begin{aligned} \|f - h\|_2^2 &= \int_a^b |f - h|^2 d\alpha \\ &\leq \int_a^b (|f - g| + |g - h|)^2 d\alpha && \text{(Triangle inequality)} \\ &= \int_a^b (|f - g|^2 + 2|f - g||g - h| + |g - h|^2) d\alpha \\ &= \int_a^b |f - g|^2 d\alpha + 2 \int_a^b |f - g| |g - h| d\alpha + \int_a^b |g - h|^2 d\alpha \\ &\leq \|f - g\|_2^2 + 2\|f - g\|_2 \|g - h\|_2 + \|g - h\|_2^2 && ((1)) \\ &= (\|f - g\|_2 + \|g - h\|_2)^2, \end{aligned}$$

we have

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2.$$

Here every integral is well-defined (Theorem 6.12 and Theorem 6.13 (or Theorem 6.11)).

□

Exercise 6.12. With the notations of Exercise 6.11, suppose $f \in \mathcal{R}(\alpha)$ and $\varepsilon > 0$. Prove that there exists a continuous function g on $[a, b]$ such that $\|f - g\|_2 < \varepsilon$. (Hint: Let $P = \{a = x_0 \leq \cdots \leq x_n = b\}$ be a suitable partition of $[a, b]$, define

$$g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i)$$

if $x_{i-1} \leq t \leq x_i$.)

Proof. Given $\varepsilon > 0$.

- (1) There are some real numbers m and M such that $m \leq f(x) \leq M$ if $x \in [a, b]$ since $f \in \mathcal{R}(\alpha)$ or f is bounded on $[a, b]$. By Theorem 6.6, there exists a partition $P = \{a = x_0 \leq \cdots \leq x_n = b\}$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \frac{\varepsilon^2}{M - m + 1}.$$

Here

$$U(P, f, \alpha) = \sum_{i=1}^n M_i \Delta \alpha_i \text{ where } M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x)$$

$$L(P, f, \alpha) = \sum_{i=1}^n m_i \Delta \alpha_i \text{ where } m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x).$$

- (2) For such partition P , define g on $[a, b]$ by

$$g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i)$$

if $x_{i-1} \leq t \leq x_i$. So that

$$\begin{aligned} |f(t) - g(t)| &= \left| \left(\frac{x_i - t}{\Delta x_i} + \frac{t - x_{i-1}}{\Delta x_i} \right) f(t) - \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i) \right| \\ &= \left| \frac{x_i - t}{\Delta x_i} (f(t) - f(x_{i-1})) + \frac{t - x_{i-1}}{\Delta x_i} (f(t) - f(x_i)) \right| \\ &\leq \frac{x_i - t}{\Delta x_i} |f(t) - f(x_{i-1})| + \frac{t - x_{i-1}}{\Delta x_i} |f(t) - f(x_i)| \\ &\leq \frac{x_i - t}{\Delta x_i} (M_i - m_i) + \frac{t - x_{i-1}}{\Delta x_i} (M_i - m_i) \\ &= M_i - m_i \end{aligned}$$

if $x_{i-1} \leq t \leq x_i$. Especially,

$$|f(t) - g(t)| \leq M - m$$

if $a \leq t \leq b$.

- (3) Note that the integral $\int_a^b |f - g|^2 d\alpha$ is well-defined (Theorem 6.8, Theorem 6.11 and Theorem 6.12). So that

$$\begin{aligned} \int_a^b |f - g|^2 d\alpha &= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f - g|^2 d\alpha \\ &\leq \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (M - m)(M_i - m_i) d\alpha \\ &= (M - m) \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (M_i - m_i) \Delta\alpha_i \\ &= (M - m)[U(P, f, \alpha) - L(P, f, \alpha)] \\ &\leq (M - m) \cdot \frac{\varepsilon^2}{M - m + 1} \\ &< \varepsilon^2. \end{aligned}$$

Hence,

$$\|f - g\|_2 = \left\{ \int_a^b |f - g|^2 d\alpha \right\}^{\frac{1}{2}} < \varepsilon.$$

□

Note.

- (1) Apply the same argument we can prove the following statement:

Suppose $f \in \mathcal{R}(\alpha)$ and $\varepsilon > 0$. Prove that there exists a continuous function g on $[a, b]$ such that $\int_a^b |f - g| d\alpha < \varepsilon$.

- (2) (Lebesgue integral)

- (a) *Let f be Lebesgue integrable over E . Then, given $\varepsilon > 0$, there is a simple function φ such that*

$$\int_E |f - \varphi| < \varepsilon.$$

- (b) *Under the same hypothesis there is a step function ψ such that*

$$\int_E |f - \psi| < \varepsilon.$$

- (c) Under the same hypothesis there is a continuous function g vanishing outside a finite interval such that

$$\int_E |f - g| < \varepsilon.$$

Exercise 6.13.
PLACEHOLDER

Exercise 6.14.
PLACEHOLDER

Exercise 6.15. Suppose f is a real, continuously differentiable function on $[a, b]$, $f(a) = f(b) = 0$, and

$$\int_a^b f(x)^2 dx = 1.$$

Prove that

$$\int_a^b x f(x) f'(x) dx = -\frac{1}{2}$$

and that

$$\int_a^b [f'(x)]^2 dx \int_a^b x^2 f(x)^2 dx > \frac{1}{4}.$$

Proof. Every integral is well-defined (Theorem 4.9 and Theorem 6.8).

- (1) By Theorem 6.22 (integration by parts),

$$\int_a^b x \left(\frac{f(x)^2}{2} \right)' dx = \left[x \cdot \frac{f(x)^2}{2} \right]_{x=a}^{x=b} - \int_a^b \frac{f(x)^2}{2} dx,$$

or

$$\int_a^b x f(x) f'(x) dx = \left[b \cdot \frac{f(b)^2}{2} - a \cdot \frac{f(a)^2}{2} \right] - \frac{1}{2} \int_a^b f(x)^2 dx = -\frac{1}{2}.$$

- (2) By Exercise 6.10(c),

$$\int_a^b [f'(x)]^2 dx \int_a^b x^2 f(x)^2 dx \geq \left(\int_a^b x f(x) f'(x) dx \right)^2 = \frac{1}{4}.$$

- (3) (Reductio ad absurdum) If the equality were holding, then by Exercise 6.10(c)

$$(f'(x))^2 \int_a^b x^2 f(x)^2 dx = x^2 f(x)^2 \int_a^b [f'(x)]^2 dx$$

on $[a, b]$ (since x , $f(x)$ and $f'(x)$ are continuous on $[a, b]$).

- (a) *Show that both integrals are nonzero.* (Reductio ad absurdum) If $\int_a^b x^2 f(x)^2 dx = 0$, then $x^2 f(x)^2 = 0$ or $xf(x) = 0$ on $[a, b]$ (Exercise 6.2). So that

$$\int_a^b xf(x)f'(x)dx = 0 \neq -\frac{1}{2},$$

which is absurd. Similarly, $\int_a^b [f'(x)]^2 dx \neq 0$.

- (b) By (a), we write

$$C = \left\{ \frac{\int_a^b [f'(x)]^2 dx}{\int_a^b x^2 f(x)^2 dx} \right\}^{\frac{1}{2}} > 0$$

be a positive constant. Hence

$$f'(x) = \pm Cxf(x).$$

Here the sign “ \pm ” is not necessary unchanged on $[a, b]$. Luckily, we can show that the sign “ \pm ” is unchanged on some subinterval of $[a, b]$.

- (c) To find such subinterval of $[a, b]$, we consider the zero set $Z(f')$ and $Z(xf)$ on $[a, b]$. Since $f'(x) = \pm Cxf(x)$ with $C > 0$, we have

$$Z(f') = Z(xf).$$

Note that $Z(f') = Z(xf)$ is closed (Exercise 4.3) and not equal to $[a, b]$ (by applying the same argument in (a)). Hence the complement of $Z(f') = Z(xf)$ is open and nonempty, which can be written as the union of an at most countable collection of disjoint segments (Exercise 2.29).

- (d) Consider any nonempty open interval in (c), say

$$(c, d) \subseteq [a, b].$$

By construction, $f'(x) \neq 0$ for all $x \in (c, d)$. Since $f'(x)$ is continuous, by Theorem 4.23 there are only two mutually exclusive possible cases:

- (i) $f'(x) > 0$ for all $x \in (c, d)$,
- (ii) $f'(x) < 0$ for all $x \in (c, d)$.

Similar result for $xf(x)$. Therefore, the sign “ \pm ” of $f'(x) = \pm Cxf(x)$ are unchanged on (c, d) , that is,

- (i) $f'(x) = Cxf(x)$ for all $x \in (c, d)$,
- (ii) $f'(x) = -Cxf(x)$ for all $x \in (c, d)$,
- (e) Suppose $f'(x) = Cxf(x)$ on (c, d) . Since $f'(x)$ and $xf(x)$ are both vanishing at $x = c$ and $x = d$, $f'(x) = Cxf(x)$ at $x = c$ and $x = d$. So

$$f'(x) = Cxf(x) \text{ if } x \in [c, d].$$

Define

$$\phi(x, y) = Cxy$$

be a real function on $R = [c, d] \times \mathbb{R}$. And consider the initial-value problem

$$y' = \phi(x, y) \quad \text{with} \quad y(c) = 0.$$

Then

$$|\phi(x, y_2) - \phi(x, y_1)| = Cx|y_2 - y_1| \leq A|y_2 - y_1|$$

where $A = C \cdot \max\{|c|, |d|\}$ is a constant. By Exercise 5.27, this initial-value problem has at most one solution. Clearly, $y = f(x) = 0$ on $[c, d]$ is one solution of this initial-value problem, contrary to the construction of $[c, d]$. Similar result for the case $f'(x) = -Cxf(x)$.

Therefore, the equality does not hold.

□

Exercise 6.16.
PLACEHOLDER

Exercise 6.17. Suppose α increases monotonically on $[a, b]$, g is continuous, and $g(x) = G'(x)$ for $a \leq x \leq b$. Prove that

$$\int_a^b \alpha(x)g(x)dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b Gd\alpha.$$

(Hint: Take g real, without loss of generality. Given $P = \{a = x_0, x_1, \dots, x_n = b\}$, choose $t_i \in (x_{i-1}, x_i)$ so that $g(t_i)\Delta x_i = G(x_i) - G(x_{i-1})$. Show that

$$\sum_{i=1}^n \alpha(x_i)g(t_i)\Delta x_i = G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^n G(x_{i-1})\Delta \alpha_i.)$$

Proof (Hint). Given $\varepsilon > 0$.

- (1) Take g real, without loss of generality. Given any partition

$$P = \{a = x_0 < x_1 < \dots < x_n = b\}$$

of $[a, b]$.

- (2) By the mean value theorem (Theorem 5.10), there is $t_i \in (x_{i-1}, x_i)$ such that

$$G(x_i) - G(x_{i-1}) = (x_i - x_{i-1})G'(t_i) = g(t_i)\Delta x_i.$$

- (3) Hence,

$$\begin{aligned} \sum_{i=1}^n \alpha(x_i)g(t_i)\Delta x_i &= \sum_{i=1}^n \alpha(x_i)(G(x_i) - G(x_{i-1})) \\ &= \sum_{i=1}^n \alpha(x_i)G(x_i) - \sum_{i=1}^n \alpha(x_i)G(x_{i-1}) \\ &= \underbrace{G(b)\alpha(b) - G(a)\alpha(a) + \sum_{i=1}^n \alpha(x_{i-1})G(x_{i-1}) - \sum_{i=1}^n \alpha(x_i)G(x_{i-1})}_{\text{adjust the index of } \sum_{i=1}^n \alpha(x_i)G(x_i)} \\ &= G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^n G(x_{i-1})\Delta \alpha_i. \end{aligned}$$

- (4) Since $G(x)$ is differentiable on $[a, b]$, $G(x)$ is continuous (Theorem 5.2) and thus $G \in \mathcal{R}(\alpha)$ (Theorem 6.8). So there is a partition P_1 such that

$$\left| \sum_{j=1}^n G(t_j)\Delta \alpha_j - \int_a^b G d\alpha \right| < \varepsilon$$

whenever $t_j \in [x_{j-1}, x_j]$ (Theorem 6.7). In particular, we pick $t_j = x_{j-1} \in [x_{j-1}, x_j]$ for all j , that is,

$$\left| \sum_{j=1}^n G(x_{j-1})\Delta \alpha_j - \int_a^b G d\alpha \right| < \varepsilon.$$

Note that if P^* is a refinement of P , the result is true too (Theorem 6.4).

- (5) Since α increases monotonically, $\alpha \in \mathcal{R}$ (Theorem 6.9). Since g is continuous, $g \in \mathcal{R}$ (Theorem 6.8). Hence $\alpha g \in \mathcal{R}$ (Theorem 6.13). So there is a partition P_2 such that

$$\left| \sum_{k=1}^m \alpha(t_k)g(t_k)\Delta x_k - \int_a^b \alpha g dx \right| < \varepsilon$$

whenever $t_k \in [x_{k-1}, x_k]$ (Theorem 6.7). In particular, we pick $t_k = x_k \in [x_{k-1}, x_k]$ for all k , that is,

$$\left| \sum_{k=1}^m \alpha(x_k)g(x_k)\Delta x_k - \int_a^b \alpha g dx \right| < \varepsilon.$$

Note that if P^* is a refinement of P , the result is true too (Theorem 6.4).

- (6) Since g is continuous on a compact set $[a, b]$, g is uniformly continuous. Hence there exists $\delta > 0$ such that

$$|g(y) - g(x)| < \varepsilon$$

whenever $|y - x| < \delta$ and $x, y \in [a, b]$. For such δ , we construct a partition P_3 such that

$$|g(t_l) - g(x_l)| < \varepsilon$$

whenever $t_l \in [x_{l-1}, x_l]$. (For example, we might take

$$P_3 = \left\{ a, a + \frac{1}{N}(b-a), a + \frac{2}{N}(b-a), \dots, a + \frac{N-1}{N}(b-a), b \right\}$$

where N is an integer $\geq \frac{b-a}{\delta}$.) Hence

$$\begin{aligned} & \left| \sum_{l=1}^N \alpha(x_l)g(t_l)\Delta x_l - \sum_{l=1}^N \alpha(x_l)g(x_l)\Delta x_l \right| \\ &= \left| \sum_{l=1}^N \alpha(x_l)[g(t_l) - g(x_l)]\Delta x_l \right| \\ &\leq \sum_{l=1}^N |\alpha(x_l)| \cdot |g(t_l) - g(x_l)| \cdot \Delta x_l \\ &\leq M\varepsilon \sum_{l=1}^N \Delta x_l \\ &= M(b-a)\varepsilon. \end{aligned}$$

Note that if P^* is a refinement of P , the result is true too (by the uniform convergence of g).

- (7) Let $P = \{a = x_0 < x_1 < \dots < x_n = b\}$ be a common refinement of P_1, P_2

and P_3 . By (3)(4)(5)(6) we have

$$\begin{aligned}
& \left| \int_a^b \alpha(x)g(x)dx - G(b)\alpha(b) + G(a)\alpha(a) + \int_a^b Gd\alpha \right| \\
&= \left| \int_a^b \alpha(x)g(x)dx - \sum_{i=1}^n \alpha(x_i)g(t_i)\Delta x_i + \int_a^b Gd\alpha - \sum_{i=1}^n G(x_{i-1})\Delta\alpha_i \right| \\
&\leq \left| \int_a^b \alpha(x)g(x)dx - \sum_{i=1}^n \alpha(x_i)g(t_i)\Delta x_i \right| + \left| \int_a^b Gd\alpha - \sum_{i=1}^n G(x_{i-1})\Delta\alpha_i \right| \\
&\leq \left| \int_a^b \alpha(x)g(x)dx - \sum_{i=1}^n \alpha(x_i)g(x_i)\Delta x_i \right| + \left| \sum_{i=1}^n \alpha(x_i)g(x_i)\Delta x_i - \sum_{i=1}^n \alpha(x_i)g(t_i)\Delta x_i \right| \\
&\quad + \left| \int_a^b Gd\alpha - \sum_{i=1}^n G(x_{i-1})\Delta\alpha_i \right| \\
&\leq \varepsilon + M(b-a)\varepsilon + \varepsilon \\
&= (M(b-a) + 2)\varepsilon.
\end{aligned}$$

Since ε is arbitrary,

$$\left| \int_a^b \alpha(x)g(x)dx - G(b)\alpha(b) + G(a)\alpha(a) + \int_a^b Gd\alpha \right| = 0,$$

or

$$\int_a^b \alpha(x)g(x)dx - G(b)\alpha(b) + G(a)\alpha(a) + \int_a^b Gd\alpha = 0,$$

or

$$\int_a^b \alpha(x)g(x)dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b Gd\alpha.$$

□

Exercise 6.18.
PLACEHOLDER

Exercise 6.19.
PLACEHOLDER