

Chapter 2: Basic Topology

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Notation.

- (1) E° or $\text{int}(E)$ is the interior of E .
- (2) \overline{E} is the closure of E .
- (3) \tilde{E} is the complement of E .
- (4) $B(p; r)$ or $B(p)$ is the set of all points q in a metric space (M, d) such that $d_M(p, q) < r$.

Exercise 2.1. *Prove that the empty set is a subset of every set.*

Proof. By Definitions 1.3,

- (1) The set which contains no element will be called the **empty set**,
- (2) If A and B are sets, and if every element of A is an element of B , we say that A is a **subset** of B ,

every element of the empty set (there are none) belongs to every set. That is, the empty set is a subset of every set. \square

Exercise 2.2. *A complex number z is said to be algebraic if there are integers a_0, \dots, a_n , not all zero, such that*

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic numbers is countable. (Hint: For every positive integer N there are only finitely many equations with

$$n + |a_0| + |a_1| + \cdots + |a_n| = N.$$

Might assume $a_0 \neq 0$.

For example, all rational numbers are algebraic since $p = \frac{\alpha}{\beta}$ (where $\alpha, \beta \in \mathbb{Z}$) is a root of $\beta z - \alpha = 0$.

Besides, $z = \sqrt{2} + \sqrt{3}$ is algebraic since $z^4 - 10z^2 + 1 = 0$. In fact, $z = \pm\sqrt{2} \pm \sqrt{3}$ are also algebraic since $z^4 - 10z^2 + 1 = (z - \sqrt{2} - \sqrt{3})(z + \sqrt{2} - \sqrt{3})(z - \sqrt{2} + \sqrt{3})(z + \sqrt{2} + \sqrt{3})$.

Lemma. *The set of all polynomials over \mathbb{Z} is countable implies that the set of algebraic numbers is countable.*

Proof of Lemma. By definition, we write the set of algebraic numbers as

$$S = \bigcup_{f(x) \in \mathbb{Z}[x]} \{z \in \mathbb{C} : f(z) = 0\}.$$

Since each polynomial of degree n has at most n roots, $\{z \in \mathbb{C} : f(z) = 0\}$ is finite for each given $f(x) \in \mathbb{Z}[x]$. So S is a countable union (by assumption) of finite sets, and hence at most countable. S is infinite since every integer α is a root of $f(z) = z - \alpha$. So S is countable. \square

Thus, it suffices to show that *the set of all polynomials over \mathbb{Z} is countable*.

Proof (Hint). For every positive integer N there are only finitely many equations with $n + |a_0| + |a_1| + \cdots + |a_n| = N$. Write

$$P_N = \{f(x) \in \mathbb{Z}[x] : n + |a_0| + |a_1| + \cdots + |a_n| = N\}$$

where $f(x) = a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n$ with $a_0 \neq 0$, and

$$P = \bigcup_{N=1}^{\infty} P_N.$$

P is the set of all polynomials over \mathbb{Z} .

Each P_N is finite for given N (since the equation $n + |a_0| + |a_1| + \cdots + |a_n| = N$ has finitely many solutions $(n, a_0, a_1, \dots, a_n) \in \mathbb{Z}^{n+2}$). So P is a countable union of finite sets, and hence at most countable. P is infinite since \mathbb{Z} is a subring of $\mathbb{Z}[x]$. So P is countable. \square

Proof (Theorem 2.13).

- (1) \mathbb{Z}^N is countable for any integer $N > 0$. Theorem 2.13.
- (2) The set of all polynomials over \mathbb{Z} is countable. Let

$$P_n = \{f \in \mathbb{Z}[x] : \deg f = n\},$$

and

$$P = \bigcup_{n=1}^{\infty} P_n = \mathbb{Z}[x].$$

Claim: P_n is countable. Define a 1-1 map $\varphi_n : P_n \rightarrow \mathbb{Z}^{n+1}$ by

$$\varphi_n(a_0z^n + a_1z^{n-1} + \cdots + a_n) = (a_0, a_1, \dots, a_{n-1}, a_n).$$

By (1) and Theorem 2.8, P_n is countable. (P_n is infinite since $a_n \in \mathbb{Z}$.) Now P is a countable union of countable sets, and hence countable by Theorem 2.12.

□

Proof (Unique factorization theorem).

- (1) *The set of prime numbers is countable.* Write all primes in the ascending order as $p_1, p_2, \dots, p_n, \dots$ where $p_1 = 2, p_2 = 3, \dots, p_{10001} = 104743, \dots$ (See ProjectEuler 7: 10001st prime. Use sieve of Eratosthenes to get p_{10001} .)
- (2) *The set of all polynomials over \mathbb{Z} is countable.* Let

$$P_n = \{f \in \mathbb{Z}[x] : \deg f = n\},$$

and

$$P = \bigcup_{n=1}^{\infty} P_n = \mathbb{Z}[x].$$

Claim: P_n is countable. Define a map $\varphi_n : P_n \rightarrow \mathbb{Z}^+$ by

$$\varphi_n(a_0z^n + a_1z^{n-1} + \cdots + a_n) = p_1^{\psi(a_0)} p_2^{\psi(a_1)} \cdots p_{n+1}^{\psi(a_n)},$$

where ψ is a 1-1 correspondence from \mathbb{Z} to \mathbb{Z}^+ (Example 2.5). By the unique factorization theorem, φ_n is 1-1. So P_n is countable by Theorem 2.8. (P_n is infinite since $a_n \in \mathbb{Z}$.) Now P is a countable union of countable sets, and hence countable by Theorem 2.12.

□

Exercise 2.3. *Prove that there exist real numbers which are not algebraic.*

Proof (Exercise 2.2). If all real numbers were algebraic, then \mathbb{R} is countable by Exercise 2.2, contrary to the fact that \mathbb{R} is uncountable (Corollary to Theorem 2.43). □

Proof (Liouville, 1844).

- (1) **Lemma.** *If ξ is a real algebraic number of degree $n > 1$, then there is a constant $A > 0$ (depending on ξ) such that*

$$\left| \xi - \frac{h}{k} \right| \geq \frac{A}{k^n}$$

for all rational numbers $\frac{h}{k}$.

- (a) If $\left|\xi - \frac{h}{k}\right| \geq 1$, pick $A = 1 > 0$.
- (b) If $\left|\xi - \frac{h}{k}\right| < 1$, let $f(x) = a_0 + a_1x + \cdots + a_nx^n$ be an irreducible polynomial of degree $n > 1$ over \mathbb{Z} such that $f(\xi) = 0$. By the mean value theorem,

$$f(\xi) - f\left(\frac{h}{k}\right) = \left(\xi - \frac{h}{k}\right) f'(c)$$

for some $c \in \left(\xi - \frac{h}{k}, \xi + \frac{h}{k}\right) \subseteq (\xi - 1, \xi + 1)$. Notice that

- (i) $f(\xi) = 0$ by definition.
- (ii) $f\left(\frac{h}{k}\right) \neq 0$ since $\frac{h}{k}$ cannot be a root of $f(x)$. Otherwise f is of degree 1, contrary to the assumption of f .
- (iii) $\left|f\left(\frac{h}{k}\right)\right| \geq \frac{1}{k^n}$ since

$$\begin{aligned} f\left(\frac{h}{k}\right) &= a_0 + a_1\left(\frac{h}{k}\right) + \cdots + a_n\left(\frac{h}{k}\right)^n \neq 0, \\ k^n f\left(\frac{h}{k}\right) &= a_0k^n + hk^{n-1}a_1 + \cdots + h^na_n \neq 0, \\ k^n \left|f\left(\frac{h}{k}\right)\right| &\geq 1. \end{aligned}$$

- (iv) $|f'(c)| \leq \sup_{x \in [\xi-1, \xi+1]} |f'(x)|$ since $c \in [\xi - 1, \xi + 1]$ and $f'(x)$ is continuous or bounded on a compact set $[\xi - 1, \xi + 1]$.

By (i)-(iv),

$$\begin{aligned} \left|f(\xi) - f\left(\frac{h}{k}\right)\right| &= \left|\left(\xi - \frac{h}{k}\right) f'(c)\right|, \\ \frac{1}{k^n} &\leq \left|f\left(\frac{h}{k}\right)\right| = \left|\xi - \frac{h}{k}\right| |f'(c)| \leq \left|\xi - \frac{h}{k}\right| \cdot \sup_{x \in [\xi-1, \xi+1]} |f'(x)|. \end{aligned}$$

Pick $A = (1 + \sup_{x \in [\xi-1, \xi+1]} |f'(x)|)^{-1} > 0$.

By (a)(b), we arrange $A = \min(1, (1 + \sup_{x \in [\xi-1, \xi+1]} |f'(x)|)^{-1}) > 0$ to fit the inequality.

- (2) $\xi = \sum_{n=0}^{\infty} 10^{-n!}$ is transcendental.

- (a) Let $k_j = 10^{j!}$, $h_j = 10^{j!} \sum_{n=0}^j 10^{-n!}$. Then

$$\left|\xi - \frac{h_j}{k_j}\right| = \sum_{n=j+1}^{\infty} 10^{-n!} < \sum_{n=(j+1)!}^{\infty} 10^{-n} = \frac{A_j}{k_j^j}$$

where $A_j = \frac{10}{9} \cdot 10^{-j!}$.

- (b) If ξ were a real algebraic number of degree $d > 1$, then by Lemma and (a),

$$\frac{A}{k_j^d} < \left| \xi - \frac{h_j}{k_j} \right| < \frac{A_j}{k_j^j} < \frac{A_j}{k_j^d}$$

for some $A > 0$ and $j \geq d$, or $0 < A < A_j$. Since j is arbitrary, $A_j \rightarrow 0$ as $j \rightarrow \infty$, contrary to $A > 0$.

- (c) If ξ were a real algebraic number of degree $d = 1$, $\xi = \frac{h}{k}$ is a rational number. So

$$\left| \xi - \frac{h_j}{k_j} \right| = \left| \frac{h}{k} - \frac{h_j}{k_j} \right| = \left| \frac{hk_j - kh_j}{kk_j} \right| \geq \left| \frac{1}{kk_j} \right| = \frac{|k|^{-1}}{k_j}$$

for all j . (It is impossible that $hk_j - kh_j = 0$ or $\frac{h}{k} = \frac{h_j}{k_j}$ since $|\frac{h}{k} - \frac{h_j}{k_j}| = \sum_{n=j+1}^{\infty} 10^{-n!} > 0$ for all j .) Again by (a),

$$\frac{|k|^{-1}}{k_j} \leq \left| \xi - \frac{h_j}{k_j} \right| < \frac{A_j}{k_j^j} < \frac{A_j}{k_j},$$

or $0 < |k|^{-1} < A_j$. (Similar to (b).) Since j is arbitrary, $A_j \rightarrow 0$ as $j \rightarrow \infty$, contrary to $|k|^{-1} > 0$.

□

Exercise 2.4. *Is the set of all irrational real numbers countable?*

Proof (Reductio ad absurdum). If $\mathbb{R} - \mathbb{Q}$ were countable, then $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} - \mathbb{Q})$ is countable (Theorem 2.12), contrary to the fact that \mathbb{R} is uncountable (Corollary to Theorem 2.43). □

Exercise 2.5. *Construct a bounded set of real numbers with exactly three limit points.*

Proof (Exercise 2.12). Let

$$K_p = \{p\} \cup \left\{ p + \frac{1}{n} : n \in \mathbb{Z}^+ \right\} \subseteq \mathbb{R}^1$$

be a compact set of \mathbb{R}^1 with exactly one limit point $p \in \mathbb{R}^1$ (Exercise 2.12). Then

$$K_{1989} \cup K_6 \cup K_4$$

is a compact set of \mathbb{R}^1 with exactly three limit points $1989, 6, 4 \in \mathbb{R}^1$. □

Exercise 2.6. Let E' be the set of all limit points of a set E . Prove that E' is closed. Prove that E and \overline{E} have the same limit points. (Recall that $\overline{E} = E \cup E'$.) Do E and E' always have the same limit points?

Proof.

(1) Show that E' is closed.

(a) Use Definition 2.18 (d).

- (i) It suffices to show every limit point of E' is a limit point of E . Given a limit point p of E' , so that every open neighborhood U of p contains a point $q_0 \neq p$ such that $q_0 \in E'$.
- (ii) Since q_0 is a limit point of E , there is an open neighborhood V of q_0 contains a point $q \neq q_0$ such that $q \in E$, where

$$V = U \cap B\left(q_0; \frac{1}{2}d_E(p, q_0)\right) \subseteq U$$

($B(x; r)$ is the open ball with center at x and radius r).

- (iii) By the construction of V , for such open neighborhood U of p , there is $q \neq p$ and $q \in V \subseteq U$ and $q \in E$. That is, p is a limit point of E .

(b) Use Definition 2.18 (e).

- (i) To show E' is closed or $X - E'$ is open, it suffices to show every point of $X - E'$ is an interior point of $X - E'$.
- (ii) Given a point $p \in X - E'$, or p is not a limit point of E . There is an open neighborhood U of p contains no point $q \neq p$ such that $q \in E$.
- (iii) To show U is an open neighborhood of p such that $U \subseteq X - E'$, it suffices to no point $q \neq p$ such that $q \in E'$. If there were a limit point q of E such that $q \neq p$ and $q \in U$, then

$$V = U \cap B\left(q; \frac{1}{2}d_E(p, q)\right) \subseteq U$$

is an open neighborhood of q contains no point of E , contrary to the assumption $q \in E'$. So $U \subseteq X - E'$ is an open neighborhood of $p \in X - E'$.

(2) Show that $E' = \overline{E}'$. It suffices to show $E' \supseteq \overline{E}'$. ($E' \subseteq \overline{E}'$ holds trivially since $E \subseteq \overline{E}$). Given a limit point p of $\overline{E} = E \cup E'$.

- (a) p is a limit point of E . Nothing to do.
- (b) p is a limit point of E' . Since p is a limit point of E' and E' is a closed set, $p \in E'$, or p is a limit point of E .

In any case, $E' \supseteq \overline{E}'$.

(3) E and E' might not have the same limit points. Let

$$E = \left\{ \frac{1}{n} : n \in \mathbb{Z}^+ \right\} \subseteq \mathbb{R}^1.$$

Then $E' = \{0\}$ and thus $(E')' = \emptyset$.

□

Exercise 2.7. Let A_1, A_2, A_3, \dots be subsets of a metric space.

(a) If $B_n = \bigcup_{i=1}^n A_i$, prove that $\overline{B_n} = \bigcup_{i=1}^n \overline{A_i}$, for $n = 1, 2, 3, \dots$

(b) If $B = \bigcup_{i=1}^\infty A_i$, prove that $\overline{B} \supseteq \bigcup_{i=1}^\infty \overline{A_i}$.

Show, by an example, that this inclusion can be proper.

Proof of (a).

(1) Show that $\overline{B_n} \subseteq \bigcup_{i=1}^n \overline{A_i}$. Since $A_i \subseteq \overline{A_i}$ for any i , we have

$$B_n = \bigcup_{i=1}^n A_i \subseteq \bigcup_{i=1}^n \overline{A_i}.$$

Since $\bigcup_{i=1}^n \overline{A_i}$ is a union of finitely many closed set $\overline{A_i}$, $\bigcup_{i=1}^n \overline{A_i}$ is closed (Theorem 2.24(d)). By Theorem 2.27(c), $\overline{B_n} \subseteq \bigcup_{i=1}^n \overline{A_i}$.

(2) Show that $\overline{B_n} \supseteq \bigcup_{i=1}^n \overline{A_i}$. Same argument in the proof of (b).

□

Proof of (b). Since $\bigcup_{j=1}^\infty A_j \supseteq A_i$ for any i , by the monotonicity of closure, we have $\overline{\bigcup_{j=1}^\infty A_j} \supseteq \overline{A_i}$ for any i , or $\overline{B} \supseteq \bigcup_{i=1}^\infty \overline{A_i}$. □

Proof of proper inclusion in (b). Let

$$A_n = \left(\frac{1}{n}, \infty \right) \subseteq \mathbb{R}^1$$

for any $n \in \mathbb{Z}^+$. Then

$$\begin{aligned} \bigcup_{n=1}^\infty A_n = (0, \infty) &\implies \overline{\bigcup_{n=1}^\infty A_n} = \overline{(0, \infty)} = [0, \infty), \\ \overline{A_n} = \left[\frac{1}{n}, \infty \right) &\implies \bigcup_{n=1}^\infty \overline{A_n} = \bigcup_{n=1}^\infty \left[\frac{1}{n}, \infty \right) = (0, \infty). \end{aligned}$$

□

Exercise 2.8. *Is every point of every open set $E \subseteq \mathbb{R}^2$ a limit point of E ? Answer the same question for closed sets in \mathbb{R}^2 .*

It is not true for all metric spaces X . The (discrete) metric in Exercise 2.10 implies no limit point exists in X .

Proof.

- (1) *Show that for every open set $E \subseteq \mathbb{R}^k$, $E \subseteq E'$. Given any point $\mathbf{p} \in E$, we shall show \mathbf{p} is a limit point of E .*

- (a) Since E is open, there is an open neighborhood $B(\mathbf{p}; r_0) \subseteq E$ for some $r_0 > 0$.
 (b) *In particular, given any $s \in \mathbb{R}$ such that $0 < s < r_0$, we can find*

$$\mathbf{q} \in B(\mathbf{p}; s) \subseteq B(\mathbf{p}; r_0) \subseteq E$$

such that $\mathbf{q} \neq \mathbf{p}$. Explicitly, write

$$\mathbf{p} = (p_1, \dots, p_k)$$

and choose

$$\mathbf{q} = \left(p_1 + \frac{s}{89}, p_2, \dots, p_k \right) \neq \mathbf{p}$$

(since $s > 0$). Clearly, \mathbf{q} is well-defined in \mathbb{R}^k and $|\mathbf{q} - \mathbf{p}| = \frac{s}{89} < s$ or $\mathbf{q} \in B(\mathbf{p}; s)$.

- (c) Now given every open neighborhood $B(\mathbf{p}, r)$ of \mathbf{p} . We can choose $s \in \mathbb{R}$ such that $0 < s < \min\{r_0, r\} \leq r_0$. (might pick $s = \frac{1}{64} \min\{r_0, r\}$.)
 By (b), there exists $\mathbf{q} \neq \mathbf{p}$ such that

$$\mathbf{q} \in B(\mathbf{p}; s) \subseteq B(\mathbf{p}; r) \subseteq E.$$

- (2) *Give an example of a closed set $E \subseteq \mathbb{R}^k$ such that $E \not\subseteq E'$. Pick $E = \{\mathbf{0}\}$. So $E' = \emptyset$ and thus $E \not\subseteq E'$.*

□

Exercise 2.9. *Let E° denote the set of all interior points of a set E . [See Definition 2.18(e); E° is called the interior of E .]*

- (a) *Prove that E° is always open.*
 (b) *Prove that E is open if and only if $E^\circ = E$.*
 (c) *If G is contained in E and G is open, prove that G is contained in E° .*

- (d) Prove that the complement of E° is the closure of the complement of E .
- (e) Do E and \overline{E} always have the same interiors?
- (f) Do E and E° always have the same closures?

Similar to Theorem 2.27.

Proof of (a). It is equivalent to show that $E^\circ \subseteq (E^\circ)^\circ$.

- (1) Given any point $x \in E^\circ$, there is $r > 0$ such that $B(x; r) \subseteq E$.
- (2) It suffices to show that $B(x; \frac{2}{r}) \subseteq E^\circ$. Given any point $y \in B(x; \frac{2}{r})$, we will show that there is an open neighborhood $B(y; \frac{2}{r})$ of y such that $B(y; \frac{2}{r}) \subseteq E$.
- (3) Given any point $z \in B(y; \frac{2}{r})$, we have

$$d(z, x) \leq d(z, y) + d(y, x) < \frac{2}{r} + \frac{2}{r} = r,$$

or $z \in B(x; r) \subseteq E$. Therefore, $B(y; \frac{2}{r}) \subseteq E$, or $y \in E^\circ$, or $B(x; \frac{2}{r}) \subseteq E^\circ$, or $x \in (E^\circ)^\circ$, or $E^\circ \subseteq (E^\circ)^\circ$.

□

Proof of (b).

- (1) (\implies)(Definition 2.18) Since E is open, every point of E is an interior point of E . Hence $E \subseteq E^\circ$. Note that $E^\circ \subseteq E$ is trivial, and thus $E^\circ = E$.
- (2) (\Leftarrow)(a) By (a), $E = E^\circ$ is always open.
- (3) (\Leftarrow)(Definition 2.18) Every point of E is an interior point of E since $E = E^\circ$. Hence E is open by Definition 2.18(f).

□

Proof of (c). $G \subseteq E$ implies $G^\circ \subseteq E^\circ$. $G = G^\circ$ since G is open ((b)). Hence $G = G^\circ \subseteq E^\circ$, that is, E° is the largest open set contained in E . (Similarly, \overline{E} is the smallest closed set containing E .) □

Proof of (d). Show that $X - E^\circ = \overline{X - E}$ and $(X - E)^\circ = X - \overline{E}$.

(1) (Theorem 2.27 and (c))

$$\begin{aligned}
X - E^\circ &= X - \bigcup_{\text{Open } V \subseteq E} V \\
&= \bigcap_{\text{Open } V \subseteq E} (X - V) \\
&= \bigcap_{\text{Closed } W \supseteq X - E} W \\
&= \overline{X - E}. \\
X - \overline{E} &= X - \bigcap_{\text{Closed } W \supseteq E} W \\
&= \bigcup_{\text{Closed } W \supseteq E} (X - W) \\
&= \bigcup_{\text{Open } V \subseteq X - E} V \\
&= (X - E)^\circ.
\end{aligned}$$

(2) (Brute-force)

$$\begin{aligned}
x \in E^\circ &\iff \exists r > 0 \text{ such that } B(x; r) \subseteq E \\
&\iff \exists r > 0 \text{ such that } B(x; r) \cap (X - E) = \emptyset \\
&\iff x \notin \overline{X - E} \\
&\iff x \in X - \overline{X - E}. \\
x \in (X - E)^\circ &\iff \exists r > 0 \text{ such that } B(x; r) \subseteq (X - E) \\
&\iff \exists r > 0 \text{ such that } B(x; r) \cap E = \emptyset \\
&\iff x \notin \overline{E} \\
&\iff x \in X - \overline{E}.
\end{aligned}$$

Note that $X - E^\circ = \overline{X - E}$ is equivalent to $(X - E)^\circ = X - \overline{E}$ by mapping $E \mapsto X - E$. \square

Proof of (e). No.

(1) Let $X = \mathbb{R}^1$ equipped with the Euclidean metric, and $E = \mathbb{Q} \subseteq X$.

(2) $E^\circ = \emptyset$ since $\tilde{\mathbb{Q}}$ is dense in \mathbb{R} .

(3) $(\overline{E})^\circ = (\mathbb{R}^1)^\circ = \mathbb{R}^1$ since \mathbb{Q} is dense in \mathbb{R} and \mathbb{R}^1 is open.

\square

Proof of (f). No.

- (1) Let $X = \mathbb{R}^1$ equipped with the Euclidean metric, and $E = \mathbb{Q} \subseteq X$.
- (2) $\overline{E} = \mathbb{R}^1$ since \mathbb{Q} is dense in \mathbb{R} .
- (3) $\overline{E^\circ} = \overline{\emptyset} = \emptyset$ since $\widetilde{\mathbb{Q}}$ is dense in \mathbb{R} .

□

Exercise 2.10. Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p, q) = \begin{cases} 1 & (\text{if } p \neq q) \\ 0 & (\text{if } p = q). \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

(The statement holds even if X is finite.) We called d the discrete metric, and the corresponding topology on X induces the discrete topology. Conversely, if X has the discrete topology, X is always metrizable by the discrete metric.

Proof.

- (1) $d(p, q)$ is a metric.
 - (a) $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$. Trivial.
 - (b) $d(p, q) = d(q, p)$. Trivial.
 - (c) $d(p, q) \leq d(p, r) + d(r, q)$ for any $r \in X$. If $p = q$, nothing to do. If $p \neq q$, $r \neq p$ or $r \neq q$ for any $r \in X$. (Assume not true, $r = p$ and $r = q$ implies that $p = q$ which is a contradiction.) In any case $d(p, r) + d(r, q) \geq 1 = d(p, q)$.
- (2) Every subset is open. Let E be any subset of X . Then every point $p \in E$ is an interior point of E . In fact, we can pick one open neighborhood $U = B(p; \frac{1}{2})$ of p containing only one point $p \in E$ or $U = \{p\}$, and such open neighborhood U is a subset of E . So every subset of X is open.
- (3) Every subset is closed. Since every subset is open, every subset is closed by Theorem 2.23.

Supplement. Might use Definition 2.18 (d) to prove directly since there are no limit points in X if we consider one open neighborhood $U = B(p; \frac{1}{2})$ of p . Therefore, every subset is closed. Again we apply Theorem 2.23 to get that every subset is open without using Definition 2.18 (f).

- (4) A subset is compact iff it is finite.

- (a) *Any finite subset is compact.* Say $E = \{p_1, p_2, \dots, p_k\}$, and $\{G_\alpha\}$ be an open covering of E . From $\{G_\alpha\}$ we pick G_{α_1} containing p_1 , G_{α_2} containing p_2 , ..., and G_{α_k} containing p_k . This process can be done in the finitely many steps. Therefore,

$$\{G_{\alpha_1}, G_{\alpha_2}, \dots, G_{\alpha_k}\}$$

is a finite subcovering of $\{G_\alpha\}$ covering E .

- (b) *Any infinite subset is not compact.* Take a collection

$$\mathcal{G} = \{G_p = \{p\}\}$$

of open subsets where p runs all points in E . Clearly, $\{G_p\}$ is an open covering. Assume

$$\mathcal{G}' = \{G_{p_1}, G_{p_2}, \dots, G_{p_k}\}$$

is any finite subcovering of \mathcal{G} . Since E is infinite, there exist a point $p \in E$ such that $p \neq p_1, p \neq p_2, \dots, p \neq p_k$. Therefore, \mathcal{G}' does not cover p , or \mathcal{G} does not contain any finite subcovering \mathcal{G}' .

□

Notice that every subset is bounded. Therefore, every subset is closed and bounded, but only finite subset is compact, i.e., Heine-Borel theorem is not true in the infinite discrete topology.

Exercise 2.11. For $x \in \mathbb{R}^1$ and $y \in \mathbb{R}^1$, define

$$\begin{aligned} d_1(x, y) &= (x - y)^2, \\ d_2(x, y) &= \sqrt{|x - y|}, \\ d_3(x, y) &= |x^2 - y^2|, \\ d_4(x, y) &= |x - 2y|, \\ d_5(x, y) &= \frac{|x - y|}{1 + |x - y|}. \end{aligned}$$

Determine, for each of these, whether it is a metric or not.

Proof.

- (1) $d = d_1$ is not a metric. (Reductio ad absurdum) If d were a metric, then

$$d(0, 2) > d(0, 1) + d(1, 2),$$

contrary to Definition 2.15(c) that $d(p, q) \leq d(p, r) + d(r, q)$.

- (2) $d = d_2$ is a metric. It suffices to show that $d'(x, y) = \sqrt{d(x, y)}$ is a metric if $d(x, y)$ is a metric. For any $p, q, r \in \mathbb{R}^1$,

(a) $d'(p, q) = \sqrt{d(p, q)} > 0$ if $p \neq q$; $d'(p, p) = \sqrt{d(p, p)} = 0$.

(b) $d'(p, q) = \sqrt{d(p, q)} = \sqrt{d(q, p)} = d'(q, p)$.

(c)

$$\begin{aligned} \sqrt{d(p, r) + d(r, q)} &\leq \sqrt{d(p, r)} + \sqrt{d(r, q)} \\ \iff (\sqrt{d(p, r)} + \sqrt{d(r, q)})^2 &\leq (\sqrt{d(p, r)} + \sqrt{d(r, q)})^2 \\ \iff d(p, r) + d(r, q) &\leq d(p, r) + d(r, q) + 2\sqrt{d(p, r)}\sqrt{d(r, q)} \\ \iff 0 &\leq 2\sqrt{d(p, r)}\sqrt{d(r, q)}. \end{aligned}$$

(d)

$$\begin{aligned} d'(p, q) &= \sqrt{d(p, q)} \\ &\leq \sqrt{d(p, r) + d(r, q)} && \text{(Triangle inequality)} \\ &\leq \sqrt{d(p, r)} + \sqrt{d(r, q)} && ((c)) \\ &= d'(p, r) + d'(r, q). \end{aligned}$$

By Definition 2.15, d' is a metric.

- (3) $d = d_3$ is not a metric. (Reductio ad absurdum) If d were a metric, then

$$d(1, -1) = 0,$$

contrary to Definition 2.15(a): $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$.

- (4) $d = d_4$ is not a metric. (Reductio ad absurdum) If d were a metric, then

$$d(1, 1) = 1,$$

contrary to Definition 2.15(a): $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$.

- (5) $d = d_5$ is a metric. It suffices to show that $d'(x, y) = \frac{d(x, y)}{1+d(x, y)}$ is a metric if $d(x, y)$ is a metric. For any $p, q, r \in \mathbb{R}^1$,

(a) $d'(p, q) = \frac{d(p, q)}{1+d(p, q)} > 0$ if $p \neq q$; $d'(p, p) = \frac{d(p, p)}{1+d(p, p)} = 0$.

(b) $d'(p, q) = \frac{d(p, q)}{1+d(p, q)} = \frac{d(q, p)}{1+d(q, p)} = d'(q, p)$.

(c) Write $x = d(p, q)$, $y = d(p, r)$ and $z = d(r, q)$. So $x, y, z \geq 0$ and

$$\begin{aligned} x &\leq y + z \\ \iff x + x(y + z) &\leq y + z + x(y + z) \\ \iff x(1 + y + z) &\leq (1 + x)(y + z) \\ \iff \frac{x}{1 + x} &\leq \frac{y + z}{1 + y + z}. \end{aligned}$$

(d)

$$\begin{aligned}
d'(p, q) &= \frac{d(p, q)}{1 + d(p, q)} \\
&\leq \frac{d(p, r) + d(r, q)}{1 + d(p, r) + d(r, q)} && ((c)) \\
&= \frac{d(p, r)}{1 + d(p, r) + d(r, q)} + \frac{d(r, q)}{1 + d(p, r) + d(r, q)} \\
&= \frac{d(p, r)}{1 + d(p, r)} + \frac{d(r, q)}{1 + d(r, q)} \\
&= d'(p, r) + d'(r, q).
\end{aligned}$$

(e) Or we can show $d'(p, q) \leq d'(p, r) + d'(r, q)$ by

$$\begin{aligned}
\frac{x}{1+x} &\leq \frac{y}{1+y} + \frac{z}{1+z} \\
\iff x(1+y)(1+z) &\leq y(1+z)(1+x) + z(1+x)(1+y) \\
\iff x + xy + xz + xyz & \\
&\leq (y + xy + yz + xyz) + (z + xz + yz + xyz) \\
\iff x &\leq y + z + 2yz + xyz \\
\iff x &\leq y + z && (d \text{ is nonnegative})
\end{aligned}$$

By Definition 2.15, d' is a metric.

□

Exercise 2.12. Let $K \subseteq \mathbb{R}^1$ consist of 0 and the numbers $\frac{1}{n}$, for $n = 1, 2, 3, \dots$. Prove that K is compact directly from the definition (without using the Heine-Borel theorem).

Proof. Let $\{G_\alpha\}$ be an open covering of K . There is an open set $G_0 \in \{G_\alpha\}$ containing 0. So there exists an open neighborhood $U = B(0; r)$ of 0 such that $U \subseteq G_0$. So U contains all points $q = \frac{1}{n}$ of K whenever $n > \frac{1}{r}$. To construct a finite subcovering of $\{G_\alpha\}$, we need to pick finitely many open sets from $\{G_\alpha\}$ to cover the remaining points $q = \frac{1}{n}$ where $n = 1, 2, \dots, [\frac{1}{r}]$, say G_1 contains $q = \frac{1}{1}$, G_2 contains $q = \frac{1}{2}$, ..., $G_{[\frac{1}{r}]}$ contains $q = \frac{1}{[\frac{1}{r}]}$. (Might be duplicated.) Hence,

$$\left\{ G_0, G_1, G_2, \dots, G_{[\frac{1}{r}]} \right\}$$

is a finite subcovering of $\{G_\alpha\}$ covering K . □

Proof (Heine-Borel theorem).

(1) K is closed. In fact, the only limit point of K is 0, which is in K .

- (a) $p = 0$ is a limit point. Given $r > 0$. There always exists $n \in \mathbb{Z}^+$ such that $r > \frac{1}{n}$. So any open neighborhood $B(0; r)$ of $p = 0$ contains at least one point $q = \frac{1}{n} \neq 0$ in K .
 - (b) $p < 0$ is not a limit point. Pick an open neighborhood $B(p; r)$ of p where $r = |p| > 0$. Then $B(p; r) \cap K = \emptyset$.
 - (c) $p > 0$ is not a limit point. There always exists $m \in \mathbb{Z}^+$ such that $p > \frac{1}{m}$ whenever $n \geq m$. Pick an open neighborhood $B(p; r)$ of p where $r = p - \frac{1}{m} > 0$. Then $B(p; r)$ does not have all points $q = \frac{1}{n} \in K$ whenever $n \geq m$. By Theorem 2.20, p cannot be a limit point of K .
- (2) K is bounded. There is a real number $M = 2$ and a point $q = 0 \in \mathbb{R}^1$ such that $|p - q| = |p| < 2$ for all $p \in K$.

By Heine-Borel theorem, K is compact in \mathbb{R}^1 . \square

Exercise 2.13. Construct a compact set of real numbers whose limit points form a countable set.

Proof (Exercise 2.12). Let $K(p; r) \subseteq \mathbb{R}^1$ be

$$K(p; r) = \left\{ p + \frac{r}{n} : n = 2, 3, \dots \right\} \cup \{p\}$$

and

$$K = \left(\bigcup_{i=0}^{\infty} K(2^{-i}; 2^{-i}) \right) \cup \{0\}.$$

- (1) The set of limit points of K is $K' = \{2^{-i} : i = 0, 1, 2, \dots\} \cup \{0\}$, which is (infinitely) countable.
 - (a) The unique limit point of $K(2^{-i}; 2^{-i})$ is 2^{-i} for each $i = 0, 1, 2, \dots$ (Exercise 2.12).
 - (b) 0 is a limit point of K .
 - (c) No other limit points of K . Similar to the argument of the proof of Exercise 2.12.
- (2) K is closed. All limit points are in K .
- (3) K is bounded. There is a real number $M = 2$ and a point $q = 0 \in \mathbb{R}^1$ such that $|p - q| = |p| < 2$ for all $p \in K$.

By Heine-Borel theorem, K is compact in \mathbb{R}^1 , and has infinitely countable limit points. \square

Exercise 2.14. Give an example of an open cover of the segment $(0, 1)$ which has no finite subcover.

Proof. In \mathbb{R}^1 , take a collection

$$\mathcal{G} = \left\{ G_n = \left(\frac{1}{n}, 1 \right) \right\}$$

of open subsets where $n \in \mathbb{Z}^+$.

- (1) \mathcal{G} is an open covering of $(0, 1) \subseteq \mathbb{R}^1$. Actually, given $x \in (0, 1)$, there exists a positive integer n such that $x > \frac{1}{n}$. That is, $x \in (\frac{1}{n}, 1) = G_n$.
- (2) There is no finite subcovering of \mathcal{G} . Assume

$$\mathcal{G}' = \{G_{n_1}, G_{n_2}, \dots, G_{n_k}\}$$

is any finite subcovering of \mathcal{G} where $n_1 < n_2 < \dots < n_k$. Take $x \in (0, \frac{1}{n_k}) \neq \emptyset$, $x = \frac{1}{2n_k}$ for example. Then $x \notin G_{n_1}, x \notin G_{n_2}, \dots, x \notin G_{n_k}$, which contradicts that \mathcal{G}' is a finite subcovering of \mathcal{G} covering $(0, 1)$.

□

Exercise 2.15. Show that Theorem 2.36 and its Corollary become false (in \mathbb{R}^1 , for example) if the word “compact” is replaced by “closed” or by “bounded.”

Recall:

- (1) Theorem 2.36: If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.
- (2) Corollary: If $\{K_n\}$ is a sequence of nonempty compact sets such that K_n contains K_{n+1} ($n = 1, 2, 3, \dots$), then $\bigcap K_n$ is not empty.

Proof. Let $X = \mathbb{R}^1$ with the usual Euclidean metric.

- (1) For the closeness, let $K_n = [n, \infty) \subseteq X$.
- (2) For the boundedness, let $K_n = (0, \frac{1}{n}) \subseteq X$.

In any case, $K_1 \supseteq K_2 \supseteq \dots$ and $\bigcap K_n = \emptyset$. □

Exercise 2.16. Regard \mathbb{Q} , the set of all rational numbers, as a metric space, with $d(p, q) = |p - q|$. Let E be the set of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Show that E is closed and bounded in \mathbb{Q} , but that E is not compact. Is E open in \mathbb{Q} ?

Lemma. Assume $S \subseteq T \subseteq M$. Then S is compact in (M, d) if, and only if, S is compact in the metric subspace (T, d) .

Proof of Lemma.

- (1) (\implies) Let \mathcal{F} be an open covering of S in (T, d) , say $S \subseteq \bigcup_{A \in \mathcal{F}} A$ where A is open in T . Then $A = B \cap T$ for some open set B in M (Theorem 3.33). Let \mathcal{G} be the collection of B . Then

$$S \subseteq \bigcup_{A \in \mathcal{F}} A = \bigcup_{B \in \mathcal{G}} (B \cap T) \subseteq \bigcup_{B \in \mathcal{G}} B,$$

or \mathcal{G} be an open covering of S in (M, d) . Since S is compact in (M, d) , \mathcal{G} contains a finite subcovering, say

$$S \subseteq B_1 \cap \cdots \cap B_p.$$

So

$$S \cap T \subseteq (B_1 \cap T) \cap \cdots \cap (B_p \cap T),$$

or

$$S \subseteq A_1 \cap \cdots \cap A_p$$

(since $S \subseteq T$ or $S \cap T = S$). So there is a finite subcovering of \mathcal{F} covering S , or S is compact in (T, d) .

- (2) (\impliedby) Let \mathcal{G} be an open covering of S in (M, d) , say $S \subseteq \bigcup_{B \in \mathcal{G}} B$ where B is open in M . Then $A = B \cap T$ is open in T . Let \mathcal{F} be the collection of A . Then

$$S \cap T \subseteq \bigcup_{B \in \mathcal{G}} (B \cap T) = \bigcup_{A \in \mathcal{F}} A,$$

or \mathcal{F} be an open covering of $S \cap T = S$ in (T, d) . Since S is compact in (T, d) , \mathcal{F} contains a finite subcovering, say

$$S \subseteq A_1 \cap \cdots \cap A_p.$$

Clearly, $S \subseteq B_1 \cap \cdots \cap B_p$ since $A = B \cap T \subseteq B$. So there is a finite subcovering of \mathcal{G} covering S , or S is compact in (M, d) .

□

Proof. Write $E_0 = (\sqrt{2}, \sqrt{3}) \cup (-\sqrt{3}, -\sqrt{2})$, and $E = E_0 \cap \mathbb{Q}$.

- (1) E is a subset of \mathbb{Q} .
- (2) Show that E is bounded in \mathbb{Q} . Since \mathbb{Q} is dense in \mathbb{R} , there is $p \in \mathbb{Q}$ such that $\sqrt{2} < p < \sqrt{3}$, or $p \in E$. Let $r = p + \sqrt{3} > 0$. Therefore, $E \subseteq B(p; r)$ for some $r > 0$ and $p \in E$, or E is bounded.

- (3) Show that E is closed in \mathbb{Q} . It suffices to show its complement is open in \mathbb{Q} . Given any

$$p \in \tilde{E} = ((-\infty, -\sqrt{3}] \cup [-\sqrt{2}, \sqrt{2}] \cup [\sqrt{3}, \infty)) \cap \mathbb{Q}.$$

$$p \leq -\sqrt{3} \text{ or } -\sqrt{2} \leq p \leq \sqrt{2} \text{ or } p \geq \sqrt{3}.$$

- (a) $p \leq -\sqrt{3}$. $p \neq -\sqrt{3}$ since $p \in \mathbb{Q}$ and $-\sqrt{3}$ is irrational. So $p < -\sqrt{3}$ and thus there exists $q \in \mathbb{Q}$ such that $p < q < -\sqrt{3}$ since \mathbb{Q} is dense in \mathbb{R} . Let $r = \max\{-\sqrt{3}-q, q-p\} > 0$. The ball $B(q; r)$ is contained in \tilde{E} .
- (b) $-\sqrt{2} \leq p \leq \sqrt{2}$. Similar to (a).
- (c) $p \geq \sqrt{3}$. Similar to (a).

By (a)(b), \tilde{E} is open in \mathbb{Q} , or E is closed in \mathbb{Q} .

- (4) Show that E is not compact in \mathbb{Q} . (Reductio ad absurdum) If E_0 were compact in the metric space \mathbb{Q} , E_0 is compact in the metric space \mathbb{R} (Lemma), which is absurd.

- (5) Show that E is open. Similar to (3).

□

Exercise 2.17. Let E be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in $[0, 1]$? Is E compact? Is E perfect?

Proof.

- (1) Show that E is uncountable. Same as Theorem 2.14. Or show that E is perfect and then apply Theorem 2.43.
- (2) Show that E is not dense in $[0, 1]$. Note that $E \subseteq [\frac{4}{9}, \frac{7}{9}]$. So

$$B\left(0; \frac{1}{64}\right) \cap E \subseteq B\left(0; \frac{1}{64}\right) \cap \left[\frac{4}{9}, \frac{7}{9}\right] = \emptyset$$

or 0 is not a limit point of E . Hence E is not dense in $[0, 1]$.

- (3) Show that E is compact. It is equivalent to show that E is closed and bounded (Theorem 2.41). Let a decimal expansion of $x \in (0, 1)$ be $0.x_1x_2\cdots$.

- (a) Show that \tilde{E} is open. Since $E \subseteq [\frac{4}{9}, \frac{7}{9}]$, it suffices to show that every point $x \in (0, 1) \cap \tilde{E}$ is an interior point of \tilde{E} . Say a decimal expansion of x containing at least one digit $x_n \neq 4, 7$. Note that

$$|x - y| \geq 10^{-n} > 0$$

for any $y = 0.y_1y_2\cdots \in E$. Hence there is an open neighborhood $B(x; 10^{-n})$ of x such that $B(x; 10^{-n}) \cap E = \emptyset$, or $B(x; 10^{-n}) \subseteq \tilde{E}$, or x is an interior point of \tilde{E} .

(b) *Show that E is closed.* Given any limit point $x \in \mathbb{R}^1$ of E , we want to show that $x \in E$. (Reductio ad absurdum) Similar to (a).

(c) *Show that E is bounded.* $E \subseteq B(0; 1)$.

(4) *Show that E is perfect.*

(a) E is closed (by (3)).

(b) *Show that every point of E is a limit point of E .* Given any $x \in E$. Given any open neighborhood $B(x; r)$ of x , there is a positive integer n such that

$$\frac{3}{10^n} < r.$$

For such n , pick $y = 0.x_1x_2\cdots x_{n-1}y_n\cdots x_{n+1}\cdots \in E$ where

$$y_n = \begin{cases} 4 & (x_n = 7), \\ 7 & (x_n = 4). \end{cases}$$

$y \neq x$, and $|y - x| = \frac{3}{10^n} < r$. So that there is $y \neq x$ such that $y \in B(x; r)$, or x is a limit point of E .

□

Exercise 2.18. *Is there a nonempty perfect set in \mathbb{R}^1 which contains no rational number?*

Yes.

Lemma. $x \in \mathbb{Q}$ if and only if has repeating decimal expansion.

Proof of Lemma.

(1) (\Leftarrow) Given any repeating decimal

$$x = x_0.x_1x_2\cdots x_n\overline{x_{n+1}\cdots x_{n+m}}$$

where $x_0 \in \mathbb{Z}$ and $x_1, \dots, x_{n+m} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Thus $x = p/q$ where

$$p = (10^m - 1) \sum_{i=0}^n 10^{n-i} x_i + \sum_{j=1}^m 10^{m-j} x_{n+j} \in \mathbb{Z}$$

and

$$q = 10^n(10^m - 1) \in \mathbb{Z}.$$

(2) (\implies) (Euler's totient function) Given any $x = p/q$ where $p, q \in \mathbb{Z}$, $q > 0$.

- (a) Write $q = 2^a 5^b q_1$ where a, b are nonnegative integers and $(q_1, 10) = 1$ (Unique factorization theorem).
- (b) Let $n = \max\{a, b\}$. Then $2^{n-a} 5^{n-b} q = 10^n q_1$.
- (c) Since $(q_1, 10) = 1$, $10^m \equiv 1 \pmod{q_1}$ where $m = \varphi(q_1)$ is Euler's totient function of q_1 . Hence $10^m - 1 = q_1 q_2$ for some $q_2 \in \mathbb{Z}$, or

$$2^{n-a} 5^{n-b} q_2 q = 10^n (10^m - 1).$$

Here $2^{n-a} 5^{n-b} q_2$, n, m are nonnegative integers.

(d) Now write

$$x = \frac{p}{q} = \frac{2^{n-a} 5^{n-b} q_2 p}{10^n (10^m - 1)} = \frac{(10^m - 1)q_3 + r}{10^n (10^m - 1)} = \frac{q_3}{10^n} + \frac{r}{10^n (10^m - 1)}$$

where $q_3, r \in \mathbb{Z}$ with $0 \leq r < 10^m - 1$. Might assume $q_3 \geq 0$. (If $q_3 < 0$, apply the same argument to $-q_3$ and then add the minus symbol “-” in the front of a decimal expansion.) Hence

$$x = x_0.x_1x_2\cdots x_n\overline{x_{n+1}\cdots x_{n+m}}$$

where

$$\begin{aligned} x_0 &= \left\lfloor \frac{q_3}{10^n} \right\rfloor \\ x_i &= \text{last digit of } \left\lfloor \frac{q_3}{10^{n-i}} \right\rfloor & (1 \leq i \leq n) \\ x_{n+j} &= \text{last digit of } \left\lfloor \frac{r}{10^{m-j}} \right\rfloor & (1 \leq j \leq m) \end{aligned}$$

(3) (\implies) (Pigeonhole principle) Given any $x = p/q$ where $p, q \in \mathbb{Z}$, $q > 0$.

- (a) Might assume $p \geq 0$. (If $p < 0$, apply the same argument to $-p$ and then add the minus symbol “-” in the front of the decimal expansion.) Write

$$x = x_0.x_1x_2\cdots.$$

- (b) Apply Euclidean algorithm to get

$$p = x_0q + r_0 \quad \text{with } 0 \leq r_0 < q.$$

x_0 is the integer part of $x = p/q$. Continue Euclidean algorithm to get x_1 by

$$10r_0 = x_1q + r_1 \quad \text{with } 0 \leq r_1 < q.$$

In general, for $n \geq 1$, x_n is given by

$$10r_{i-1} = x_iq + r_i \quad \text{with } 0 \leq r_i < q.$$

- (c) The pigeonhole principle shows that there must be two equal remainders, that is,

$$r_n = r_{n+m} \quad \text{with} \quad m > 0.$$

By induction, $r_{n+k} = r_{n+m+k}$ for any $k \geq 0$. Thus $x_{n+k} = x_{n+m+k}$ holds for any $k > 0$, that is, x has a decimal expansion

$$x = x_0.x_1x_2 \cdots x_n \overline{x_{n+1} \cdots x_{n+m}}.$$

□

Proof (Exercise 2.17). Let A be the set of all $y \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7. Though $A \cap \mathbb{Q} \neq \emptyset$ since $\frac{4}{9} \in A$, we can shift A by a number $\xi = \sum_{n=0}^{\infty} 10^{-n!}$ (Exercise 2.3), that is, we construct

$$E = \{y + \xi : y \in A\}$$

and show that E is our desired nonempty perfect set in $\mathbb{R} - \mathbb{Q}$.

- (1) Any number $x \in E$ has decimal expansion $x = 0.x_1x_2 \cdots$ with $x_n \in \{5, 8\}$ if n is a factorial number; otherwise $x_n \in \{4, 7\}$.
- (2) E is a perfect set (Exercise 2.17).
- (3) $E \subseteq \mathbb{R} - \mathbb{Q}$. It suffices to show that each $x \in E$ has no repeating decimal expansions (Lemma). It is clear by the construction of $\xi = \sum_{n=0}^{\infty} 10^{-n!}$.

□

Proof (Exercise 2.3). Let E be a subset of Liouville numbers as

$$E = \left\{ \sum_{n=0}^{\infty} \frac{a_n}{10^{n!}} : a_n \in \{4, 7\} \right\}.$$

E is perfect. (The same argument of Exercise 2.17.) Besides, all numbers of E are transcendental. (Set $k_j = 10^{j!}$ and $h_j = 10^{j!} \sum_{n=0}^j \frac{a_n}{10^{n!}}$ and apply the same argument of Exercise 2.3.) □

Note. Or using Lemma to prove all numbers of E are irrational.

Proof (Theorem 3.32). Let

$$E = \left\{ \sum_{n=1989}^{\infty} \frac{a_n}{n!} : a_n \in \{6, 4\} \right\}.$$

E is perfect. (The same argument of Exercise 2.17.) Besides, all numbers of E are irrational (The same argument of Theorem 3.32.) □

Proof (Non constructive existence proof). By Cantor-Bendixson theorem (Exercise 2.28), it suffices to find a uncountable closed set in $\mathbb{R} - \mathbb{Q}$.

- (1) Write $\mathbb{Q} = \{r_1, r_2, \dots\}$ since \mathbb{Q} is countable. Let

$$I_n = B\left(r_n; \frac{1}{2^{n+1}}\right) \supseteq \{r_n\}$$

and

$$A = \bigcup_{n=1}^{\infty} I_n \supseteq \mathbb{Q}.$$

Hence A is an open subset in \mathbb{R} .

- (2) Let $E = \mathbb{R} - A$. By construction, E is closed and $E \cap \mathbb{Q} = \emptyset$.
 (3) *Show that E is uncountable. It suffices to show that $m^*(E) > 0$.* In fact, the outer measure of U is

$$m^*(A) \leq \sum_{n=1}^{\infty} m^*(I_n) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Thus,

$$m^*(E) \geq m^*(\mathbb{R}) - m^*(A) = \infty - 1 = \infty.$$

Hence, the set of all condensation points of E is our desired nonempty perfect set in $\mathbb{R} - \mathbb{Q}$. \square

Note. In fact, we can replace \mathbb{Q} by the set of all real algebraic numbers (Exercise 2.2).

Exercise 2.19.

- (a) *If A and B are disjoint closed sets in some metric space X , prove that they are separated.*
 (b) *Prove the same for disjoint open sets.*
 (c) *Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p, q) < \delta$, define B similarly, with $>$ in place of $<$. Prove that A and B are separated.*
 (d) *Prove that every connected metric space with at least two points is uncountable. Hint: Use (c).*

Proof of (a). Since

$$\begin{aligned} A \cap \overline{B} &= A \cap B && (B \text{ is closed}) \\ &= \emptyset, && (A \text{ and } B \text{ are disjoint}) \\ \overline{A} \cap B &= A \cap B && (A \text{ is closed}) \\ &= \emptyset. && (A \text{ and } B \text{ are disjoint}) \end{aligned}$$

A and B are separated. \square

Proof of (b)(Theorem 2.27(c)). Note that \tilde{A} is a closed set containing B . Since \bar{B} is the smallest closed set containing B , $\tilde{A} \supseteq \bar{B}$ (Theorem 2.27(c)). Hence

$$A \cap \bar{B} \subseteq A \cap \tilde{A} = \emptyset.$$

Similarly, $\bar{A} \cap B = \emptyset$. Hence A and B are separated. \square

Proof of (c). Since both

$$A = \{q \in X : d(p, q) < \delta\} \text{ and } B = \{q \in X : d(p, q) > \delta\}$$

are open in X , they are separated by (b). \square

Proof of (d). Let X be a connected metric space.

(1) Let $p, q \in X$ with $p \neq q$. Hence $d_X(p, q) = r > 0$ (Definition 2.15(a)).

(2) Given any $\delta \in (0, r)$. Define

$$A = \{x \in X : d(p, x) < \delta\} \text{ and } B = \{x \in X : d(p, x) > \delta\}.$$

$$p \in A \neq \emptyset \text{ and } q \in B \neq \emptyset.$$

(3) If there were no $y_\delta \in X$ such that $d(p, y_\delta) = \delta$, we can write $X = A \cup B$ as a union of two nonempty separated sets ((c)), contrary to the connectedness of X .

(4) Collect these y as E . Since d is a function, there is a one-to-one map from $(0, r)$ to E defined by $\delta \mapsto y_\delta$ in (3). Since $(0, r)$ is uncountable, $X \supseteq E$ is uncountable.

\square

Exercise 2.20. *Are closures and interiors of connected sets always connected? (Look at subsets of \mathbb{R}^2 .)*

Proof.

(1) *Interiors of connected sets are not always connected.* Let $X = \mathbb{R}^2$ with the usual Euclidean metric be a metric space. Take

$$E = B(89; 1) \bigcup B(64; 1) \bigcup \{(x, 0) \in \mathbb{R}^2 : 64 \leq x \leq 89\}.$$

E is connected and

$$E^\circ = B(89; 1) \bigcup B(64; 1)$$

is disconnected.

(2) *Closures of connected sets are always connected. It suffices to show that E is disconnected if \overline{E} is disconnected.*

(a) Write $\overline{E} = A \cup B$ as a union of two nonempty separated sets. Here $A \neq \emptyset$, $B \neq \emptyset$, $A \cap \overline{B} = \emptyset$ and $\overline{A} \cap B = \emptyset$.

(b) Write

$$E = (A \cap E) \cup (B \cap E)$$

and we will show that E is disconnected.

(c) *Show that $A \cap E$ and $B \cap E$ are separated.* In fact,

$$(A \cap E) \cap \overline{B \cap E} \subseteq A \cap \overline{B} = \emptyset,$$

$$\overline{A \cap E} \cap (B \cap E) \subseteq \overline{A} \cap B = \emptyset.$$

(d) *Show that $A \cap E$ and $B \cap E$ are nonempty.* (Reductio ad absurdum)
If $A \cap E = \emptyset$, then

$$E = (A \cap E) \cup (B \cap E) = B \cap E \implies E \subseteq B.$$

So

$$\begin{aligned} A &= (A \cup B) \cap A && (A \subseteq A \cup B) \\ &= \overline{E} \cap A \\ &\subseteq \overline{B} \cap A && (E \subseteq B) \\ &= \emptyset \end{aligned}$$

which contradicts $A \neq \emptyset$ in (a). Therefore, $A \cap E \neq \emptyset$. Similarly, $B \cap E \neq \emptyset$.

Hence, E is disconnected if \overline{E} is disconnected, or closures of connected sets are always connected.

□

Exercise 2.21. Let A and B be separated subsets of some \mathbb{R}^k , suppose $\mathbf{a} \in A$, $\mathbf{b} \in B$, and define

$$\mathbf{p}(t) = (1 - t)\mathbf{a} + t\mathbf{b}$$

for $t \in \mathbb{R}^1$. Put $A_0 = \mathbf{p}^{-1}(A)$, $B_0 = \mathbf{p}^{-1}(B)$. [Thus $t \in A_0$ if and only if $\mathbf{p}(t) \in A$.]

(a) Prove that A_0 and B_0 are separated subsets of \mathbb{R}^1 .

(b) Prove that there exists $t_0 \in (0, 1)$ such that $\mathbf{p}(t_0) \notin A \cup B$.

(c) Prove that every convex subset of \mathbb{R}^k is connected.

Proof of (a).

(1) Note that

- (a) $\mathbf{a} \neq \mathbf{b}$ or $|\mathbf{a} - \mathbf{b}| > 0$ since $A \cap B = \emptyset$.
- (b) $|\mathbf{p}(t) - \mathbf{p}(s)| = |t - s||\mathbf{a} - \mathbf{b}|$ by a direct calculation.
- (c) $\mathbf{p}(t) = \mathbf{p}(s)$ if and only if $t = s$ by (a)(b).

(2) Show that $A_0 \cap \overline{B_0} = \emptyset$. (Reductio ad absurdum) If there were $t \in A_0 \cap \overline{B_0}$, then $t \in A_0$ and t is a limit point of B_0 .

- (a) $t \in A_0$ implies that $\mathbf{p}(t) \in A$.
- (b) Show that t is a limit point of $B_0 \implies \mathbf{p}(t)$ is a limit point of B .
Given any $\varepsilon > 0$, there is $s \in B_0$ such that

$$|t - s| < \frac{\varepsilon}{|\mathbf{a} - \mathbf{b}|} \quad \text{with } s \neq t$$

since t is a limit point of B_0 . So by (1),

$$|\mathbf{p}(t) - \mathbf{p}(s)| = |t - s||\mathbf{a} - \mathbf{b}| < \varepsilon.$$

Here $\mathbf{p}(s) \in B$ and $\mathbf{p}(s) \neq \mathbf{p}(t)$. So $\mathbf{p}(t)$ is a limit point of B .

By (a)(b), $\mathbf{p}(t) \in A \cap \overline{B} = \emptyset$, contrary to the assumption that A and B are separated.

(3) Show that $\overline{A_0} \cap B_0 = \emptyset$. Similar to (2).

By (2)(3), A_0 and B_0 are separated. \square

Proof of (b). (Reductio ad absurdum) If $\mathbf{p}(t)$ were in $A \cup B$ for all $t \in (0, 1)$, we will show that $[0, 1]$ is separated by $A_0 \cap [0, 1]$ and $B_0 \cap [0, 1]$ to get a contradiction.

(1) $\mathbf{p}(t)$ were in $A \cup B$ for all $t \in [0, 1]$ since $\mathbf{p}(0) = \mathbf{a} \in A \cup B$ and $\mathbf{p}(1) = \mathbf{b} \in A \cup B$. Therefore,

$$[0, 1] \subseteq \mathbf{p}^{-1}(A \cup B) = \mathbf{p}^{-1}(A) \cup \mathbf{p}^{-1}(B) = A_0 \cup B_0.$$

(2) Let $A_1 = A_0 \cap [0, 1]$ and $B_1 = B_0 \cap [0, 1]$. So $[0, 1] = A_1 \cup B_1$.

(3) Show that $A_1 \neq \emptyset$ and $B_1 \neq \emptyset$.

$$\begin{aligned} \mathbf{p}(0) \in A &\iff 0 \in \mathbf{p}^{-1}(A) = A_0 \\ &\iff 0 \in A_0 \text{ and } 0 \in [0, 1] \\ &\iff 0 \in A_0 \cap [0, 1] = A_1. \end{aligned}$$

Similarly, $1 \in B_1$.

Note. That's why we consider $[0, 1]$ instead of $(0, 1)$.

- (4) Show that $A_1 \cap \overline{B_1} = \emptyset$ and $\overline{A_1} \cap B_1 = \emptyset$. Since $A_1 \subseteq A_0$ and $B_1 \subseteq B_0$, $A_1 \cap \overline{B_1} \subseteq A_0 \cap \overline{B_0} = \emptyset$ or $A_1 \cap \overline{B_1} = \emptyset$. Similarly, $\overline{A_1} \cap B_1 = \emptyset$.

By (2)(3)(4), $[0, 1]$ is separated, contrary to the connectedness of $[0, 1]$ (Theorem 2.47). \square

Proof of (c).

- (1) Let E be a convex subset of \mathbb{R}^k . Recall

$$\mathbf{p}(t) = (1 - t)\mathbf{a} + t\mathbf{b} \in E$$

whenever $\mathbf{a}, \mathbf{b} \in E$ and $t \in (0, 1)$.

- (2) (Reductio ad absurdum) If E were separated by A and B , pick $\mathbf{a} \in A \subseteq E$ and $\mathbf{b} \in B \subseteq E$.
- (3) By (b), there exists $t_0 \in (0, 1)$ such that $\mathbf{p}(t_0) \notin A \cup B = E$, contrary to the convexity of E .

\square

Exercise 2.22. A metric space is called *separable* if it contains a countable dense subset. Show that \mathbb{R}^k is separable. (Hint: Consider the set of points which have only rational coordinates.)

Proof. Let E be the set of points which have only rational coordinates.

- (1) Show that E is countable. \mathbb{Q} is countable and thus $E = \mathbb{Q}^k$ is countable (Theorem 2.13).
- (2) Show that E is dense. Given any $\mathbf{p} = (p_1, \dots, p_k) \in \mathbb{R}^k$. We want to show that \mathbf{p} is a limit point of E .
- (a) Given any open neighborhood $B(\mathbf{p}; r)$ of \mathbf{p} , $r > 0$.
- (b) Since \mathbb{Q} is dense in \mathbb{R} (Theorem 1.20), every coordinate of \mathbf{p} is a limit point of \mathbb{Q} . In particular, for every $i = 1, 2, \dots, k$, the open neighborhood $B\left(p_i, \frac{r}{\sqrt{k}}\right)$ of p_i contains a point $q_i \neq p_i$ and $q_i \in \mathbb{Q}$.
- (c) Collect all q_i in (b) and define $\mathbf{q} = (q_1, \dots, q_k) \in \mathbb{Q}^k = E$. By construction $\mathbf{q} \neq \mathbf{p}$ and

$$\begin{aligned} |\mathbf{p} - \mathbf{q}| &= \sqrt{(p_1 - q_1)^2 + \dots + (p_k - q_k)^2} \\ &< \sqrt{\left(\frac{r}{\sqrt{k}}\right)^2 + \dots + \left(\frac{r}{\sqrt{k}}\right)^2} \\ &= \sqrt{k \cdot \frac{r^2}{k}} \\ &= r \end{aligned}$$

or $\mathbf{q} \in B(\mathbf{p}; r)$.

By (a)(b)(c), E is dense in \mathbb{R}^k .

By (1)(2), \mathbb{R}^k is separable. \square

Exercise 2.23. A collection $\{V_\alpha\}$ of open subsets of X is said to be a base for X if the following is true: For every $x \in X$ and every open set $G \subseteq X$ such that $x \in G$, we have $x \in V_\alpha \subseteq G$ for some α . In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$.

Prove that every separable metric space has a countable base. (Hint: Take all neighborhoods with rational radius and center in some countable dense subset of X .)

Note. \mathbb{R}^k has a countable base (Exercise 2.22).

Proof (Hint). Let X be a separable metric space, and E be a countable dense subset of X . Let \mathcal{B} be a collection of all neighborhoods with rational radius and center in E .

- (1) \mathcal{B} is countable (Theorem 2.12).
- (2) \mathcal{B} is a base for X . Similar to Exercise 2.9(a). Given any $p \in X$ and every open set $G \subseteq X$ such that $p \in G$. Since p is in an open set G , there exists an open neighborhood $B(p; r)$ of p such that $B(p; r) \subseteq G$.
- (3) Let r_0 be rational such that $0 < r_0 < \frac{r}{2}$ (Theorem 1.20(b)). Since E is dense in X , there is $q \in E$ such that $d_X(p, q) < r_0$. For such $r_0 \in \mathbb{Q}$ we pick an open neighborhood $B(q; r_0)$ of q . Clearly, $B(q; r_0) \in \mathcal{B}$.
- (4) $p \in B(q; r_0)$ since $d_X(p, q) < r_0$.
- (5) Show that $B(q; r_0) \subseteq B(p; r) \subseteq G$. For any $z \in B(q; r_0)$, $d_X(z, p) \leq d_X(z, q) + d_X(q, p) < r_0 + r_0 < \frac{r}{2} + \frac{r}{2} = r$. That is, $z \in B(p; r)$.

By (3)(4)(5), (2) is established. By (1)(2), \mathcal{B} is a countable base for X . \square

Supplement.

- (1) In topology, a second-countable space, also called a completely separable space, is a topological space whose topology has a countable base.
- (2) Every second-countable space is separable.
- (3) The reverse implication of (2) does not hold in general. However, for metric spaces the properties of being second-countable and separable are equivalent.
- (4) Show that every second-countable metric space X is separable.

- (a) Let $\mathcal{B} = \{B_n : n \in \mathbb{Z}^+\}$ be a countable base of X .
- (b) For every $B_n \in \mathcal{B}$, pick any point p_n of B_n and collect them as

$$E = \{p_n : p_n \in B_n \text{ for } n \in \mathbb{Z}^+\}.$$

- (c) E is countable.
- (d) *Show that E is dense.* Given any $x \in X$. For any open neighborhood $B(x)$ of x , $B(x)$ is a union of subcollection of \mathcal{B} . That is, there is always a point in E by the construction of E .

□

Exercise 2.24. Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable.

(Hint: Fix $\delta > 0$, and pick $x_1 \in X$. Having chosen $x_1, \dots, x_j \in X$, choose x_{j+1} , if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \dots, j$. Show that this process must stop after finite number of steps, and that X can therefore be covered by finite many neighborhoods of radius δ . Take $\delta = \frac{1}{n}$ ($n = 1, 2, 3, \dots$) and consider the centers of the corresponding neighborhoods.)

Note. The reverse implication does not hold (Exercise 2.10).

Proof (Hint).

- (1) Fix $\delta > 0$, and pick $x_1 \in X$. Show that every limit point compact metric space X is totally bounded.
 - (a) Having chosen $x_1, \dots, x_j \in X$, choose x_{j+1} , if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \dots, j$. Let E_δ be the set of these x_i .
 - (b) *Show that this process must stop after finite number of steps, and that X can therefore be covered by finite many neighborhoods of radius δ .* (Reductio ad absurdum)
 - (i) If not, E_δ is an infinite subset of X . By assumption there is a limit point of E_δ , say $p \in X$.
 - (ii) In particular, an open neighborhood $B(p; \frac{\delta}{64})$ of p contains a point $x_n \in E_\delta$ with $p \neq x_n$.
 - (iii) The neighborhood $B(p; \frac{\delta}{64})$ contains no other point $x_m \in E_\delta$ with $m \neq n$. If so,

$$d_X(x_n, x_m) \leq d_X(x_n, p) + d_X(p, x_m) < \frac{\delta}{64} + \frac{\delta}{64} < \delta,$$

contrary to the construction of E_δ .

- (iv) Note that $p \notin E_\delta$ as a corollary to (iii).

- (v) So another open neighborhood $B(p; r)$ of p with $r = d_X(p, x_n) > 0$ contains no points $x_m \in E_\delta$ with $p \neq x_m$, contrary to the assumption that p is a limit point of E_δ .
- (2) Show that every totally bounded metric space X is separable. Take $\delta = \frac{1}{n}$ ($n = 1, 2, 3, \dots$) in (1), and union all $E_{\frac{1}{n}}$ as

$$E = \bigcup_{n=1}^{\infty} E_{\frac{1}{n}} \subseteq X.$$

Show that E is a countable dense subset of X .

- (a) Show that E is countable. Since E is the countable union of finite set $E_{\frac{1}{n}}$, E is countable (Theorem 2.12).
- (b) Show that E is dense in X . Given any $p \in X$. It suffices to show that given any open neighborhood $B(p; r)$ of $p \in X - E$, there exists $q \in E$ such that $q \in B(p; r)$. Pick any $n \in \mathbb{Z}^+$ such that $\frac{1}{n} < r$ (Theorem 1.20(a)). By the construction of $E_{\frac{1}{n}}$, there is $q \in E_{\frac{1}{n}}$ such that $p \in B(q; \frac{1}{n})$, or $d_X(p, q) < \frac{1}{n} < r$, or $q \in B(p; r)$.

□

Supplement.

- (1) A topological space X is said to be limit point compact or weakly countably compact if every infinite subset of X has a limit point in X .
- (2) In a metric space, limit point compactness, compactness, and sequential compactness are all equivalent. For general topological spaces, however, these three notions of compactness are not equivalent.
- (3) A metric space X is totally bounded if and only if for every real number $\delta > 0$, there exists a finite collection of open balls in X of radius δ whose union contains X .

Exercise 2.25. Prove that every compact metric space K has a countable base, and that K is therefore separable. (Hint: For every positive integer n , there are finitely many neighborhood of radius $\frac{1}{n}$ whose union covers K .)

Proof (Exercise 2.24(a)).

- (1) Show that every compact metric space K is limit point compact. Given any subset $E \subseteq K$. It suffices to show that if E has no limit point, then E must be finite.
 - (a) Since E has no limit point, E is closed.

- (b) For any point $p \in E$. Since p is not a limit point, there is an open neighborhood $B(p)$ such that $B(p)$ contains no point other than p .
- (c) Similar to the proof of Theorem 2.35, let

$$\mathcal{F} = \{B(p) : p \in E \text{ with } B(p) \cap E = \{p\}\} \bigcup \tilde{E}.$$

Hence \mathcal{F} is an open covering of K .

- (d) Since K is compact by assumption, there is a finite subcovering \mathcal{F}' of K . Since \tilde{E} does not intersect E , each $B(p) \in \mathcal{F}'$ contains only one point of E and so E is finite.

- (2) Since K is limit point compact, K is separable (Theorem 2.24).

□

Proof (Exercise 2.24(b)).

- (1) Show that every compact metric space K is totally bounded. Given any real number $\delta > 0$, define an open covering \mathcal{F} of K by

$$\mathcal{F} = \{B(p; \delta) : p \in K\}.$$

Since K is compact, there exists a finite subcovering \mathcal{F}' of K . \mathcal{F}' is our desired finite collection of open balls in X of radius δ whose union contains X .

- (2) Since K is totally bounded, K is separable (Theorem 2.24).

□

Proof (Hint).

- (1) Given any positive integer $n > 0$, define an open covering \mathcal{F}_n of K by

$$\mathcal{F}_n = \left\{ B\left(p; \frac{1}{n}\right) : p \in K \right\}.$$

Since K is compact, there exists a finite subcovering \mathcal{G}_n of K .

- (2) Show that every compact metric space K is second-countable.

- (a) Define

$$\mathcal{B} = \bigcup_{n \geq 1} \mathcal{G}_n$$

be a collection. Since \mathcal{B} is a countable union of finite set \mathcal{G}_n , \mathcal{B} is countable. Hence it suffices to show that for every $p \in K$ and every open set $G \subseteq K$ such that $p \in G$, there is $B \in \mathcal{B}$ such that $x \in B \subseteq G$.

- (b) Since G is open, there is an open neighborhood $B(p; r)$ of p such that $B(p; r) \subseteq G$.
- (c) For such $r > 0$, there is $n \in \mathbb{Z}^+$ with $0 < \frac{1}{n} < \frac{r}{2}$ (Theorem 1.20(a)). So p is in some $B(q; \frac{1}{n}) \in \mathcal{G}_n \subseteq \mathcal{B}$ since \mathcal{G}_n is a subcovering of K .
- (d) Show that $B(q; \frac{1}{n}) \subseteq B(p; r) \subseteq G$. For any $z \in B(q; \frac{1}{n})$,

$$d_K(z, p) \leq d_K(z, q) + d_K(q, p) < \frac{1}{n} + \frac{1}{n} < \frac{r}{2} + \frac{r}{2} = r.$$

That is, $z \in B(p; r)$, or $B(q; \frac{1}{n}) \subseteq B(p; r) \subseteq G$.

By (a)(b)(c)(d), K is second-countable.

- (3) Show that every second-countable metric space is separable. Supplement (4) to Exercise 2.23.

□

Exercise 2.26. Let X be a metric space in which every infinite subsets has a limit point. Prove that X is compact.

By Exercises 2.23 and 2.24, X has a countable base. It follows that every open cover of X has a countable subcovering $\{G_n\}$, $n = 1, 2, 3, \dots$. If no finite subcollection of $\{G_n\}$ covers X , then the complement F_n of $G_1 \cup \dots \cup G_n$ is nonempty for each n , but $\bigcap F_n$ is empty. If E is a set contains a point from each F_n , consider a limit point of E , and obtain a contradiction.

Note. In every metric space, we have

$$\begin{aligned} \{\text{compact}\} &\iff \{\text{limit point compact}\} \\ &\implies \{\text{totally bounded}\} \\ &\implies \{\text{separable}\} \\ &\iff \{\text{second-countable}\} \\ &\iff \{\text{Lindelof}\}. \end{aligned}$$

Proof (Hint).

- (1) Since X is limit point compact, X is separable (Exercise 2.24). Since X is separable, X is second-countable (Exercise 2.23).
- (2) Show that X is Lindelof if X is second-countable. Let X be a second-countable metric space. Let $\mathcal{B} = \{B_n\}$ be a countable base of X . Given any open covering \mathcal{F} of X .

- (a) Iterate each $B_n \in \mathcal{B}$, pick one $G_n \in \mathcal{F}$ containing B_n , and collect them as

$$\mathcal{G} = \{G_n : G_n \supseteq B_n \text{ for } n \in \mathbb{Z}^+\}.$$

(G_n might be duplicated.)

- (b) \mathcal{G} is a countable subset of \mathcal{F} .
- (c) \mathcal{G} covers X since \mathcal{B} is a countable base of X .
- (3) Hence, given any open covering \mathcal{F} of X , there is a countable subcovering $\mathcal{G} = \{G_n\}$ of X . (Reductio ad absurdum) If there were no finite subcovering of \mathcal{G} , then the complement F_n of $G_1 \cup \cdots \cup G_n$ is nonempty for each n , but $\cap F_n$ is empty.
- (4) Let E be a set contains a point from each F_n . E is infinite and thus E has a limit point, say p . $p \in G_n$ for some n since $\mathcal{G} = \{G_n\}$ is an open covering of X . Since G_n is open, there is an open neighborhood $B(p)$ of p such that $B(p) \subseteq G_n$. By the construction of F_n ,

$$B(p) \cap F_m = \emptyset$$

whenever $m \geq n$, contrary to the assumption that p is a limit point of E .

Hence, X is compact if X is limit point compact. \square

Supplement.

- (1) Lindelof space is a topological space in which every open covering has a countable subcovering.
- (2) Show that X is second-countable if X is Lindelof. Same as the Proof (Hint) of Exercise 2.25 except changing the word “compact” to “Lindelof” and “finite” to “countable.” \square
- (3) In every metric space, we have

$$\{\text{compact}\} \iff \{\text{limit point compact}\} \iff \{\text{sequentially compact}\}.$$

Exercise 2.27. Define a point p in a metric space X to be a condensation point of a set $E \subseteq X$ if every neighborhood of p contains uncountably many points of E .

Suppose $E \subseteq \mathbb{R}^k$, E is uncountable, and let P be the set of all condensation points of E . Prove that P is perfect and that at most countably many points of E are not in P . In other words, show that $\widetilde{P} \cap E$ is at most countable.

(Hint: Let $\{V_n\}$ be a countable base of \mathbb{R}^k , let W be the union of those V_n for which $E \cap V_n$ is at most countable, and show that $P = \widetilde{W}$.)

Note. The statement is also true for separable metric space.

Proof.

(1) Let $\{V_n\}$ be a countable base of \mathbb{R}^k (Exercise 2.22 and 2.23). Let W be the union of those V_n for which $E \cap V_n$ is at most countable.

(2) Show that $P = \widetilde{W}$.

(a) ($P \subseteq \widetilde{W}$) Given any $x \in P$.

$$\begin{aligned} x \in P &\implies x \text{ is a condensation point of } E \\ &\implies \forall V_n \ni x, \exists B(x) \subseteq V_n \text{ such that } E \cap B(x) \text{ is uncountable} \\ &\implies E \cap V_n \text{ is uncountable} \\ &\implies x \notin W. \end{aligned}$$

(b) ($P \supseteq \widetilde{W}$) Given any $x \in \widetilde{W}$. Let $P(V_n)$ be the proposition that $E \cap V_n$ is at most countable.

$$\begin{aligned} x \in \widetilde{W} &\implies x \notin W = \bigcup_{P(V_n)} V_n \\ &\implies x \notin V_n \text{ for which } E \cap V_n \text{ is at most countable} \\ &\implies \forall B(x) \text{ of } x, x \in V_m \subseteq B(x) \text{ for some } V_m \quad (\{V_n\}: \text{ base of } X) \\ &\implies E \cap V_m \text{ is uncountable} \\ &\implies E \cap B(x) \supseteq E \cap V_m \text{ is uncountable} \\ &\implies x \text{ is a condensation point of } E \\ &\implies x \in P. \end{aligned}$$

(3) Show that P is closed. P is the complement of an open subset W .

(4) Show that $P \subseteq P'$. (Reductio ad absurdum)

(a) If there were an isolated point $x \in P$, then there exists an open neighborhood $B(x)$ of x such that $B(x) \cap P = \{x\}$.

(b) Since x is a condensation point of E , there are uncountably many points of E in $B(x)$, and such points y are not a condensation points of E except $y = x$.

(c) Given any point $y \in E \cap B(x)$ with $y \neq x$. Since y is not a condensation point, there exists a neighborhood $B(y)$ of y such that $B(y) \cap E$ is at most countable. Since $\{V_n\}$ is a base, for each $B(y)$ there exists $V_{n(y)}$ such that $y \in V_{n(y)} \subseteq B(y)$. Hence

$$V_{n(y)} \cap E \subseteq B(y) \cap E$$

is at most countable.

(d) Hence,

$$\begin{aligned} E \cap B(x) - \{x\} &\subseteq \bigcup_{y \in E \cap B(x) - \{x\}} V_{n(y)} \\ &= \bigcup_{n(y)} V_{n(y)} \end{aligned}$$

is a countable union of at most countable sets, which is countable. Hence $E \cap B(x) - \{x\}$ or $E \cap B(x)$ is countable, contrary to the assumption that $E \cap B(x)$ is uncountable.

(5) Show that $E \cap \tilde{P}$ is at most countable.

$$E \cap \tilde{P} = E \cap \left(\bigcup_{P(V_n)} V_n \right) = \bigcup_{P(V_n)} (E \cap V_n)$$

is at most countable.

□

Exercise 2.28. Prove that every closed set in a separable metric space is the union of a (possible empty) perfect set and a set which is at most countable. (Corollary: Every countable closed set in \mathbb{R}^k has isolated points.) (Hint: Use Exercise 2.27.)

Proof (Exercise 2.27). Let E be a closed set in a separable metric space.

- (1) E contains all limit points of E , especially contains all condensation points of E . So we can write

$$E = P \cup (E - P)$$

where P is the set of all condensation points of E .

- (2) By Exercise 2.27, P is perfect and $E - P = E \cap \tilde{P}$ is at most countable.

□

Cantor-Bendixson theorem.

- (1) Closed sets of a Polish space X have the perfect set property in a particularly strong form: any closed subset of X may be written uniquely as the disjoint union of a perfect set and a countable set.
- (2) A Polish space is a separable completely metrizable topological space; that is, a space homeomorphic to a complete metric space that has a countable dense subset.

Exercise 2.29. Prove that every open set in \mathbb{R}^1 is the union of an at most countable collection of disjoint segments. (Hint: Use Exercise 2.22.)

Proof. Let E be an open subset of \mathbb{R}^1 .

- (1) For each $x \in E$, let I_x denote the largest open interval containing x and contained in E . More precisely, since E is open, x is contained in some small (non-trivial) interval, and therefore if

$$a_x = \inf\{a < x : (a, x) \subseteq E\} \text{ and } b_x = \sup\{b > x : (x, b) \subseteq E\}$$

we must have $a_x < x < b_x$ (with possibly infinite values for a_x and b_x).

- (2) Let $I_x = (a_x, b_x)$, then by construction we have $x \in I_x$ as well as $I_x \subseteq E$. Hence

$$E = \bigcup_{I_x \in \mathcal{F}} I_x,$$

where $\mathcal{F} = \{I_x\}_{x \in E}$.

- (3) Suppose that two intervals I_x and I_y intersect. Then their union (which is also an open interval) is contained in E and contains x (and y). Since I_x is maximal, $I_x \cup I_y \subseteq I_x$, and similarly $I_x \cup I_y \subseteq I_y$. This can happen only if $I_x = I_y$.
- (4) Therefore, any two distinct intervals in \mathcal{F} must be disjoint. Hence \mathcal{F} is countable since each open interval $I_x \in \mathcal{F}$ contains a rational number.

□

Exercise 2.30. Imitate the proof of Theorem 2.43 to obtain the following result:

If $\mathbb{R}^k = \bigcup_{n=1}^{\infty} F_n$, where each F_n is a closed subset of \mathbb{R}^k , then at least one F_n has a nonempty interior.

Equivalent statement: If G_n is a dense open subset of \mathbb{R}^k , for $n = 1, 2, 3, \dots$, then $\bigcap_{n=1}^{\infty} G_n$ is not empty (in fact, it is dense in \mathbb{R}^k).

(This is a special case of Baire's theorem; see Exercise 3.22 for the general case.)

Baire category theorem. If G_n is a dense open subset of \mathbb{R}^k , for $n = 1, 2, \dots$, then

$$\bigcap_{n=1}^{\infty} G_n$$

is dense in \mathbb{R}^k .

Proof of Baire category theorem. Given any open set G_0 in \mathbb{R}^k , will show that

$$\bigcap_{n=0}^{\infty} G_n \neq \emptyset.$$

- (1) Since G_1 is dense, $G_0 \cap G_1$ is nonempty. Take any one point \mathbf{x}_1 in the open set $G_0 \cap G_1$, there exists open neighborhood

$$V_1 = \{\mathbf{y} \in \mathbb{R}^k : |\mathbf{y} - \mathbf{x}_1| < r_1\}$$

of \mathbf{x}_1 such that

$$\overline{V_1} = \{\mathbf{y} \in \mathbb{R}^k : |\mathbf{y} - \mathbf{x}_1| \leq r_1\} \subseteq G_0 \cap G_1.$$

- (2) Suppose V_n has been constructed, take any one point \mathbf{x}_{n+1} in the open set $V_n \cap G_{n+1}$, there exists open neighborhood

$$V_{n+1} = \{\mathbf{y} \in \mathbb{R}^k : |\mathbf{y} - \mathbf{x}_{n+1}| < r_{n+1}\}$$

of \mathbf{x}_{n+1} with r_{n+1} such that

$$\overline{V_{n+1}} = \{\mathbf{y} \in \mathbb{R}^k : |\mathbf{y} - \mathbf{x}_{n+1}| \leq r_{n+1}\} \subseteq V_n \cap G_{n+1}.$$

- (3) Note that

- (a) each $\overline{V_n}$ is nonempty (containing \mathbf{x}_n) and compact.
- (b) $\overline{V_1} \supseteq \overline{V_2} \supseteq \cdots$ (since $\overline{V_{n+1}} \subseteq V_n \cap G_{n+1} \subseteq V_n \subseteq \overline{V_n}$).

By Corollary to Theorem 2.36,

$$\bigcap_{n=1}^{\infty} \overline{V_n} \neq \emptyset.$$

- (4) Pick $\mathbf{x} \in \bigcap_{n=1}^{\infty} \overline{V_n}$. Hence

$$\begin{aligned} \mathbf{x} \in \bigcap_{n=1}^{\infty} \overline{V_n} &\iff \mathbf{x} \in \overline{V_n} \text{ for all } n = 1, 2, 3, \dots \\ &\implies \mathbf{x} \in \overline{V_1} \subseteq G_0 \cap G_1 \text{ and } \mathbf{x} \in \overline{V_{n+1}} \subseteq V_n \cap G_{n+1} \subseteq G_{n+1} \\ &\implies \mathbf{x} \in G_0 \cap G_1 \cap \cdots = \bigcap_{n=0}^{\infty} G_n \\ &\implies \bigcap_{n=0}^{\infty} G_n \neq \emptyset. \end{aligned}$$

□