Chapter 2: The Real Number System

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Problem 2.1. *Show that* $1 \in P$.

Proof. By the field axioms,

(1) $1 \in \mathbb{R}$ such that $1 \neq 0$.

(2) $-1 \in \mathbb{R}$.

(3) $(-1) \cdot (-1) = 1$.

By the axioms of order, $1 = 0$ or $1 \in P$ or $-1 \in P$. Consider three possible cases,

(1) $1 = 0$, contrary to the field axioms $1 \neq 0$.

(2) $1 \in P$.

(3) $-1 \in P$. By the axioms of order, $(-1)(-1) \in P$. Since $(-1)(-1) = 1$ by the field axioms, $1 \in P$. By the axioms of order, $-1 \notin P$, contrary to $-1 \in P$.

By (1)(2)(3), $1 \in P$. $\square$

Applying the similar argument to $\sqrt{-1}$, we get $\sqrt{-1} \notin \mathbb{R}$ as our expectation.