

# Solutions to the book: *Jürgen Neukirch, Algebraic Number Theory*

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# Chapter I: Algebraic Integers

## I.1. The Gaussian Integers

### Exercise I.1.1.

$\alpha \in \mathbb{Z}[i]$  is a unit if and only if  $N(\alpha) = 1$ .

*Proof.*

- (1) ( $\implies$ ) Since  $\alpha$  is a unit, there is  $\beta \in \mathbb{Z}[i]$  such that  $\alpha\beta = 1$ . So  $N(\alpha\beta) = N(1)$ , or  $N(\alpha)N(\beta) = 1$ . Since the image of  $N$  is nonnegative integers,  $N(\alpha) = 1$ .
- (2) ( $\impliedby$ )  $N(\alpha) = \alpha\bar{\alpha}$ , or  $1 = \alpha\bar{\alpha}$  since  $N(\alpha) = 1$ . That is,  $\bar{\alpha} \in \mathbb{Z}[i]$  is the inverse of  $\alpha \in \mathbb{Z}[i]$ . (Or we solve the equation  $N(\alpha) = a^2 + b^2 = 1$ , and show that all four solutions ( $\pm 1$  and  $\pm i$ ) are units.)
- (3) Conclusion: a unit  $\alpha = a + bi$  of  $\mathbb{Z}[i]$  is satisfying the equation  $N(\alpha) = a^2 + b^2 = 1$  by (1)(2). That is, the only unit of  $\mathbb{Z}[i]$  are  $\pm 1$  and  $\pm i$ .

□

### Exercise I.1.4.

Show that the ring  $\mathbb{Z}[i]$  cannot be ordered.

*Proof.* Similar to the fact that  $i$  cannot be ordered in  $\mathbb{C}$ . Thus  $i$  cannot be ordered in  $\mathbb{Z}[i]$  either. □

### Exercise I.1.5.

Show that the only units of the ring  $\mathbb{Z}[\sqrt{-d}] = \mathbb{Z} + \mathbb{Z}\sqrt{-d}$ , for any rational integer  $d > 1$ , are  $\pm 1$ .

*Proof.*

- (1) Define the norm  $N$  on  $\mathbb{Z}[\sqrt{-d}]$  by

$$N(x + y\sqrt{-d}) = (x + y\sqrt{-d})(x - y\sqrt{-d}) = x^2 + y^2d,$$

i.e., by  $N(z) = |z|^2$ . It is multiplicative.

(2) Similar to Exercise I.1.1,

$$\begin{aligned} x + y\sqrt{-d} \in \mathbb{Z}[\sqrt{-d}] \text{ is a unit} &\iff N(x + y\sqrt{-d}) = x^2 + y^2d = 1 \\ &\iff x^2 = 1 \text{ and } y = 0 \\ &\iff x = \pm 1 \text{ and } y = 0. \end{aligned}$$

Hence the only units of the ring  $\mathbb{Z}[\sqrt{-d}]$  are  $\pm 1$  ( $d > 1$ ).

□

## I.2. Integrality

### Exercise I.2.1.

Is  $\frac{3+2\sqrt{6}}{1-\sqrt{6}}$  an algebraic integer?

*Proof.*

- (1)  $\alpha := \frac{3+2\sqrt{6}}{1-\sqrt{6}} = -3 - \sqrt{6}$ . Since the set of all algebraic integers is a ring,  $\alpha$  is an algebraic integer.
- (2) Or show that  $\alpha$  satisfies a monic equation  $x^2 + 6x + 3 = 0 \in \mathbb{Z}[x]$ .

□

### Exercise I.2.2.

Show that, if the integral domain  $A$  is integrally closed, then so is the polynomial ring  $A[t]$ .

*Proof.*

- (1) Suppose  $A$  is integrally closed in  $B$ . Show that  $A[t]$  is integrally closed in  $B[t]$ . Suppose  $f \in B[t]$  is integral over  $A[t]$ . Write

$$f^n + g_1 f^{n-1} + \cdots + g_{n-1} f + g_n = 0$$

where  $n > 0$  and  $g_i \in A[t]$ . Hence

$$\begin{aligned} f^n + g_1 f^{n-1} + \cdots + g_{n-1} f &= -g_n \in A[t] \\ \implies f(\underbrace{f^{n-1} + g_1 f^{n-2} + \cdots + g_{n-1}}_{:=g}) &\in A[t]. \end{aligned}$$

It is possible to show that  $fg \in A[t]$  implies that  $f \in A[t]$  and  $g \in A[t]$  by using the fact that  $A$  is integrally closed in  $B$ .

- (2) Suppose  $f, g$  are monic polynomials in  $B[t]$ . Show that  $fg \in A[t]$  implies that  $f \in A[t]$  and  $g \in A[t]$ . Write

$$f = \prod (t - \xi_i), \quad g = \prod (t - \eta_j)$$

in some splitting field  $F$  of  $f$  and  $g$  containing the quotient field of  $B$ . Note that each  $\xi_i$  and each  $\eta_j$  is a root of a monic equation  $fg$  in  $A[t]$ . Since  $A$  is integrally closed in  $B$ ,  $\xi_i, \eta_j \in A$ . Hence  $f, g \in A[t]$ .

- (3) To apply part (2), we need to remedy leading coefficients of  $f$  and  $g$ . Take an integer  $m > \max\{\deg(f), \deg(g_1), \dots, \deg(g_n)\}$ . Let  $f_0 = t^m + f$  be a monic polynomial in  $B[t]$ . Hence

$$\begin{aligned} (f_0 - t^m)^n + g_1(f_0 - t^m)^{n-1} + \dots + g_n &= 0 \\ \implies f_0^n + h_1 f_0^{n-1} + \dots + h_n &= 0 \end{aligned}$$

where

$$h_n = t^{mn} + (-1)^{n-1} g_1 t^{m(n-1)} + \dots + g_n \in A[t]$$

is also monic. So

$$\begin{aligned} f_0^n + h_1 f_0^{n-1} + \dots + h_{n-1} f &= -h_n \text{ is monic in } A[t] \\ \implies f_0 (\underbrace{f_0^{n-1} + h_1 f_0^{n-2} + \dots + h_{n-1}}_{:=h_0}) &\in A[t] \text{ where} \\ f_0 \text{ and } h_0 &\text{ both are monic in } B[t]. \end{aligned}$$

Now we can apply part (2) safely.

- (4) In part (1), we let  $B$  be the quotient field of  $A$  and thus the quotient field of  $A[t]$  is  $B(t)$ . Hence

$$\begin{aligned} f &\in B(t) \text{ integral over } A[t] \\ \implies f &\in B(t) \text{ integral over } B[t] & (A[t] \subseteq B[t]) \\ \implies f &\in B[t] & (B[t] \text{ is a UFD}) \\ \implies f &\in B[t] \text{ integral over } A[t] \\ \implies f &\in A[t]. & ((1)) \end{aligned}$$

□

### Exercise I.2.3.

In the polynomial ring  $A = \mathbb{Q}[x, y]$ , consider the principal ideal  $\mathfrak{p} = (x^2 - y^3)$ . Show that  $\mathfrak{p}$  is a prime ideal, but  $A/\mathfrak{p}$  is not integrally closed.

*Proof.*

- (1) It is easy to show that  $x^2 - y^3$  is irreducible in  $A$ . Hence  $\mathfrak{p} = (x^2 - y^3)$  is prime since  $A$  is a UFD.
- (2) By substituting  $x = t^3$ ,  $y = t^2$ ,  $A/\mathfrak{p} \cong \mathbb{Q}[t^3, t^2]$ , with quotient field  $\mathbb{Q}(t)$  (by noting  $t = \frac{x}{y}$ ). Note that  $\mathbb{Q}[t]$  is a UFD, thus is already integrally closed. So the integral closure will be  $\mathbb{Q}[t] \supsetneq \mathbb{Q}[t^3, t^2]$ . It suggests that  $A/\mathfrak{p}$  might not be integrally closed.
- (3) (Reductio ad absurdum) If not, then the element  $\frac{x}{y}$  satisfies a monic equation  $t^2 - y = 0 \in (A/\mathfrak{p})[t]$ . So  $\frac{x}{y} \in A/\mathfrak{p}$  or  $t \in \mathbb{Q}[t^3, t^2]$ , which is absurd.

□

*Note.*

- (1) Serre's criterion for normality.
- (2) Hence smoothness is the same as normality for affine curves in  $\mathbb{Q}[x, y]$ . Note that  $x^2 - y^3$  is an irreducible cubic with a cusp at the origin  $(0, 0)$ .
- (3) There is an affine variety  $X \in \mathbb{Q}[x, y, z]$  such that  $X$  is normal but not smooth. ( $X = V(x^2 + y^2 - z^2)$  for example.)

#### Exercise I.2.4.

Let  $D$  be a squarefree rational integer  $\neq 0, 1$  and  $d$  the discriminant of the quadratic number field  $K = \mathbb{Q}(\sqrt{D})$ . Show that

$$d = \begin{cases} D & \text{if } D \equiv 1 \pmod{4}, \\ 4D & \text{if } D \equiv 2, 3 \pmod{4}. \end{cases}$$

and that an integral basis of  $K$  is given by  $\{1, \sqrt{D}\}$  in the second case, by  $\left\{1, \frac{1+\sqrt{D}}{2}\right\}$  in the first case, and by  $\left\{1, \frac{d+\sqrt{d}}{2}\right\}$  in both case.

*Proof.*

- (1) The Galois group of  $K|\mathbb{Q}$  has two elements, the identity and an automorphism sending  $\sqrt{D}$  to  $-\sqrt{D}$ .
- (2) Note that  $\alpha \in \mathcal{O}_K$  iff  $\text{Tr}_{K|\mathbb{Q}}(\alpha), N_{K|\mathbb{Q}}(\alpha) \in \mathbb{Z}$  (by noting that the equation  $x^2 - \text{Tr}_{K|\mathbb{Q}}(\alpha)x + N_{K|\mathbb{Q}}(\alpha) = 0$  has a root  $x = \alpha$ ). So given  $\alpha = x + y\sqrt{D} \in \mathcal{O}_K$ , we have

$$\begin{aligned} \text{Tr}_{K|\mathbb{Q}}(\alpha) &= 2x \in \mathbb{Z}, \\ N_{K|\mathbb{Q}}(\alpha) &= x^2 - Dy^2 \in \mathbb{Z}. \end{aligned}$$

- (3) So  $4(x^2 - Dy^2) = (2x)^2 - D(2y)^2 \in \mathbb{Z}$ . So  $D(2y)^2 \in \mathbb{Z}$  since  $2x \in \mathbb{Z}$ . So  $2y \in \mathbb{Z}$  since  $D$  is squarefree  $\neq 0, 1$ . Let  $r = 2x, s = 2y$ . Then  $r^2 -Ds^2 \equiv 0 \pmod{4}$ . Note that a square  $\equiv 0, 1 \pmod{4}$ .
- (4) If  $D \equiv 1 \pmod{4}$ , then

$$\begin{aligned}
r^2 - Ds^2 &\equiv r^2 - s^2 \pmod{4} \\
&\implies r \text{ and } s \text{ has the same parity} \\
&\implies \mathcal{O}_K = \left\{ \frac{r + s\sqrt{D}}{2} : r \equiv s \pmod{2} \right\} \\
&\implies \mathcal{O}_K = \left\{ \frac{r-s}{2} + s \cdot \frac{1+\sqrt{D}}{2} : r \equiv s \pmod{2} \right\} \\
&\implies \mathcal{O}_K = \mathbb{Z} + \mathbb{Z} \frac{1+\sqrt{D}}{2}.
\end{aligned}$$

So  $\left\{1, \frac{1+\sqrt{D}}{2}\right\}$  is an integral basis of  $K$ . Hence

$$d = \begin{vmatrix} 1 & \frac{1+\sqrt{D}}{2} \\ 1 & \frac{1-\sqrt{D}}{2} \end{vmatrix}^2 = D.$$

- (5) If  $D \equiv 2, 3 \pmod{4}$ , then

$$\begin{aligned}
r^2 - Ds^2 &\equiv r^2 + 2s^2 \text{ or } r^2 + s^2 \pmod{4} \\
&\implies \text{both } r \text{ and } s \text{ are even} \\
&\implies \text{both } x \text{ and } y \text{ are rational integers} \\
&\implies \mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\sqrt{D}.
\end{aligned}$$

So  $\{1, \sqrt{D}\}$  is an integral basis of  $K$ . Hence

$$d = \begin{vmatrix} 1 & \sqrt{D} \\ 1 & -\sqrt{D} \end{vmatrix}^2 = 4D.$$

- (6) By (4)(5),  $\left\{1, \frac{d+\sqrt{d}}{2}\right\}$  is an integral basis of  $K$  for any case.

□

### Exercise I.2.7. (Stickelberger's discriminant relation)

The discriminant  $d_K$  of an algebraic number field  $K$  is always  $\equiv 0 \pmod{4}$  or  $\equiv 1 \pmod{4}$ . (Hint: The discriminant  $\det(\sigma_i \omega_j)$  of an integral basis  $\omega_j$

is a sum of terms, each prefixed by a positive or a negative sign. Writing  $P$  (resp.  $N$ ) for the sum of the positive (resp. negative) terms, one find  $d_K = (P - N)^2 = (P + N)^2 - 4PN$ .)

*Proof (Hint).*

- (1) Let  $S_n$  be the symmetric group of degree  $n$ , and  $A_n$  be the alternating group of degree  $n$ . So

$$\begin{aligned} \det(\sigma_i \omega_j) &= \sum_{\pi \in S_n} \left( \operatorname{sgn}(\pi) \prod_{i=1}^n \sigma_i \omega_{\pi(i)} \right) \\ &= \underbrace{\sum_{\pi \in A_n} \prod_{i=1}^n \sigma_i \omega_{\pi(i)}}_{:=P} - \underbrace{\sum_{\pi \in S_n - A_n} \prod_{i=1}^n \sigma_i \omega_{\pi(i)}}_{:=N}. \end{aligned}$$

- (2) Note that  $\sigma_i(P + N) = P + N$  and  $\sigma_i(PN) = PN$  for all  $\sigma_i$ . Hence  $P + N, PN \in \mathbb{Q}$ . Therefore  $P + N, PN \in \mathbb{Q} \cap \mathcal{O}_K = \mathbb{Z}$ .

- (3) By (1)(2),

$$\begin{aligned} d_K &= \det(\sigma_i \omega_j)^2 \\ &= (P - N)^2 \\ &= (P + N)^2 - 4PN \\ &\equiv 0, 1 \pmod{4}. \end{aligned}$$

□

## Chapter VII: Zeta Functions and $L$ -series

### VII.1. The Riemann Zeta Function

#### Exercise VII.1.4.

For the power sum

$$s_k(n) = 1^k + 2^k + 3^k + \cdots + n^k$$

one has

$$s_k(n) = \frac{1}{k+1}(B_{k+1}(n) - B_{k+1}(0)).$$

*Proof.* By Exercise VII.1.3,

$$x^k = \frac{1}{k+1}(B_{k+1}(x) - B_{k+1}(x-1)).$$

Hence the telescoping sum is

$$\begin{aligned} s_k(n) &= \sum_{x=1}^n x^k \\ &= \sum_{x=1}^n \frac{1}{k+1}(B_{k+1}(x) - B_{k+1}(x-1)) \\ &= \frac{1}{k+1}(B_{k+1}(n) - B_{k+1}(0)). \end{aligned}$$

□