Notes on the book: *Apostol, Introduction to Analytic Number Theory*

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Chapter 1: The Fundamental Theorem of Arithmetic

Exercise 1.15.

Prove that every $n \geq 12$ is the sum of two composite numbers.

Proof. Write $n = 2m$ (resp. $n = 2m + 1$) where $m \in \mathbb{Z}$, $m \geq 6$. Then $n = 8 + 2(m-4)$ (resp. $n = 9 + 2(m-4)$) is the sum of two composite numbers. $\square$

Exercise 1.30.

If $n > 1$ prove that the sum

$$\sum_{k=1}^{n} \frac{1}{k}$$

is not an integer.

Proof.

(1) (Reductio ad absurdum) Suppose

$$H := \sum_{k=1}^{n} \frac{1}{k}$$

were an integer.

(2) Let s be the largest integer such that $2^s \leq n$. So the integer number

$$\begin{aligned} 2^{s-1}H &= \sum_{k=1}^{n} \frac{2^{s-1}}{k} \\ &= 2^{s-1} + 2^{s-2} + \frac{2^{s-1}}{3} + 2^{s-3} + \frac{2^{s-1}}{5} + \frac{2^{s-2}}{3} + \cdots + \frac{1}{2} + \cdots . \end{aligned}$$

has only one term of even denominators (as $n > 1$) if we write all terms in irreducible fractions. That is,

$$2^{s-1}H = \frac{1}{2} + \frac{c}{d} \in \mathbb{Z}$$

where $\frac{c}{d}$ is an irreducible fraction with odd d. Hence it suffices to show that $2 \mid d$ to get a contradiction.

(3) By

$$\frac{1}{2} + \frac{c}{d} = \frac{d + 2c}{2d} \in \mathbb{Z}$$

we have $d + 2c = 2dd'$ for some $d' \in \mathbb{Z}$. Note that 2 is a prime. So $2 \mid (d + 2c)$ or $2 \mid d$, which is absurd.

$\square$

Chapter 2: Arithmetical functions and Dirichlet multiplication

Exercise 2.3.

Prove that

$$\frac{n}{\varphi(n)} = \sum_{d|n} \frac{\mu^2(d)}{\varphi(d)}.$$

Proof.

(1) Note that fg, f/g and $f * g$ are multiplicative if f and g are multiplicative (Example 5 on page 34 and Theorem 2.14). Hence $\frac{n}{\varphi(n)}$ and $\sum_{d|n} \frac{\mu^2(d)}{\varphi(d)}$ are multiplicative. Hence it might assume that $n = p^a$ for some prime p and integer $a \geq 1$. (The case $n = 1$ is trivial.)

(2)

$$\frac{p^a}{\varphi(p^a)} = \frac{p^a}{p^a - p^{a-1}} = \frac{p}{p-1}.$$

(3)

$$\begin{aligned}
\sum_{d|p^a} \frac{\mu^2(d)}{\varphi(d)} &= \frac{\mu^2(1)}{\varphi(1)} + \frac{\mu^2(p)}{\varphi(p)} + \overbrace{\frac{\mu^2(p^2)}{\varphi(p^2)}}^{=0} + \cdots + \overbrace{\frac{\mu^2(p^a)}{\varphi(p^a)}}^{=0} \\
&= 1 + \frac{1}{p-1} + 0 + \cdots + 0 \\
&= \frac{p}{p-1}.
\end{aligned}$$

□

Exercise 2.4.

Prove that $\varphi(n) > \frac{n}{6}$ *for all* n *with at most* 8 *distinct prime factors.*

Proof.

(1)

$$\begin{aligned}
\varphi(n) &= n\prod_{p|n}\left(1-\frac{1}{p}\right) && \text{(Theorem 2.4)}\\
&\geq n\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\\
&\qquad\left(1-\frac{1}{11}\right)\left(1-\frac{1}{13}\right)\left(1-\frac{1}{17}\right)\left(1-\frac{1}{19}\right)\\
&= \frac{55296}{323323}n\\
&> \frac{n}{6}.
\end{aligned}$$

(2) The conclusion does not hold if n has more than 9 distinct prime factors.

□