Notes on the book:
James R. Munkres, Elements of Algebraic Topology

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Chapter 1: Homology Groups of a Simplicial Complex

§1. Simplices

Exercise 1.1.

Verify properties (1)-(3) of simplices:

(1) *The barycentric coordinates $t_i(x)$ of x with respect to $a_0, \ldots, a_n$ are continuous functions of x.*

(2) *σ equals the union of all line segments joining a_0 to points of the simplex s spanned by $a_1, \ldots, a_n$. Two such line segments intersect only in the point a_0.*

(3) *σ is compact, convex set in $\mathbb{R}^N$, which equals the intersection of all convex sets in $\mathbb{R}^N$ containing $a_0, \ldots, a_n$.*

Proof of property (1).

(1) Let σ be the n-simplex spanned by $a_0, \ldots, a_n$. It suffices to show that $t_i(x)$ is linear. Therefore $t_i(x)$ is automatically continuous (Theorem 9.7 in the textbook: *Rudin, Principles of Mathematical Analysis, 3rd edition*).

(2) Let
$$E = \left\{ x = \sum_{i=0}^{n} \widetilde{t_i}(x) a_i : \widetilde{t_i}(x) \in \mathbb{R} \right\} \supseteq \sigma$$
be the plane spanned by $a_0, \ldots, a_n$. $\widetilde{t_i}(x)$ is well-defined on E and thus $\widetilde{t_i}|_\sigma = t_i$ (since $\{a_0, \ldots, a_n\}$ is geometrically independent in $\mathbb{R}^N$). So it suffices to show that $\widetilde{t_i}$ is linear.

(3) Suppose $x = \sum_{i=0}^n \widetilde{t_i}(x) a_i \in E$ and $y = \sum_{i=0}^n \widetilde{t_i}(y) a_i \in E$. Then
$$x + y = \sum_{i=0}^{n} (\widetilde{t_i}(x) + \widetilde{t_i}(y)) a_i.$$
Note that the coefficient of a_i is uniquely determined by $x + y$. Thus $\widetilde{t_i}(x+y) = \widetilde{t_i}(x) + \widetilde{t_i}(y)$. Similarly, $\widetilde{t_i}(rx) = r\widetilde{t_i}(x)$ for $r \in \mathbb{R}$. Hence $\widetilde{t_i}$ is linear.

□

Proof of property (2).

(1) *Show that σ equals the union of all line segments joining a_0 to points of the simplex s spanned by $a_1, \dots, a_n$.* Nothing to do when $n = 0$. Assume $n \geq 1$. Let σ' be the union of all line segments joining a_0 to points of the simplex s spanned by $a_1, \dots, a_n$.

(2) Write one line segment L joining a_0 to a point $\sum_{i=1}^n t_i a_i \in s$ as

$$L = \left\{ t_0 a_0 + (1 - t_0) \sum_{i=1}^{n} t_i a_i : 0 \leq t_0 \leq 1 \right\}.$$

For each point x of L, each coefficient of a_i is ≥ 0 ($i = 0, \dots, n$) and the sum of them is $t_0 + (1 - t_0) \sum_{i=1}^n t_i = t_0 + (1 - t_0) = 1$. Hence $L \subseteq \sigma$ and thus $\sigma' \subseteq \sigma$.

(3) Conversely, given $x = \sum_{i=0}^n t_i a_i \in \sigma$. If $t_0 = 1$, then x is in the line segment joining a_0 to a_1. If $0 \leq t_0 < 1$, then write x as

$$x = t_0 a_0 + (1 - t_0) \sum_{i=1}^{n} \frac{t_i}{1 - t_0} a_i.$$

Note that $\sum_{i=1}^n \frac{t_i}{1-t_0} a_i \in s$. Hence x is in some line segment joining a_0 to points of the simplex s. Therefore $\sigma \subseteq \sigma'$.

(4) *Show that two such line segments intersect only in the point a_0.* Suppose L_1 (resp. L_2) is the line segment joining a_0 to $x \in s$ (resp. $y \in s$). If there is one point $z \neq a_0$ on $L_1 \cap L_2$, then

$$z = t_0 a_0 + (1 - t_0) x = s_0 a_0 + (1 - s_0) y$$

for some $0 \leq t_0, s_0 < 1$. $t_0 = s_0$ since $z \in \sigma$ is in a simplex. Hence $x = y$ or $L_1 = L_2$.

□

Proof of property (3).

(1) *Show that σ is compact.* Let Δ be the standard simplex defined by

$$\Delta = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} : \sum_i t_i = 1, t_i \geq 0 \right\}.$$

Δ is compact in $\mathbb{R}^{n+1}$ since Δ is closed and bounded. Consider the map $\alpha : \Delta \to \sigma$ defined by

$$(t_0, \dots, t_n) \mapsto \sum_{i=0}^{n} t_i a_i.$$

Similar to the proof of property (1), α is continuous. Hence the continuous image of a compact set is compact.

(2) *Show that σ is convex.* Given any $x = \sum_i t_i a_i \in \sigma$ (with $\sum_i t_i = 1$), $y = \sum_i s_i a_i \in \sigma$ (with $\sum_i s_i = 1$) and $0 < \lambda < 1$, it suffices to show that

$$\lambda x + (1 - \lambda) y \in \sigma.$$

In fact,

$$\begin{aligned} \lambda x + (1-\lambda)y &= \lambda \sum_i t_i a_i \in \sigma + (1-\lambda) \sum_i s_i a_i \\ &= \sum_i (\lambda t_i + (1-\lambda) s_i) a_i, \end{aligned}$$

where each $\lambda t_i + (1-\lambda)s_i \geq 0$ and

$$\sum_i (\lambda t_i + (1-\lambda)s_i) = \lambda \sum_i t_i + (1-\lambda) \sum_i s_i = \lambda + (1-\lambda) = 1.$$

So $\lambda x + (1-\lambda)y \in \sigma$.

(3) *Let $\mathscr{C}$ be the collection of all convex sets in $\mathbb{R}^N$ containing $a_0, \dots, a_n$. Show that $\sigma = \bigcap_{E \in \mathscr{C}} E$.* By (2), $\sigma \in \mathscr{C}$ and thus $\sigma \supseteq \bigcap_{E \in \mathscr{C}} E$. Conversely, suppose $E \in \mathscr{C}$. The convexity of E implies that $\sum_i t_i a_i \in E$ whenever $\sum_i t_i = 1$ and $t_i \geq 0$. Hence $\sigma \subseteq E$ and thus $\sigma \subseteq \bigcap_{E \in \mathscr{C}} E$.

$\square$