Chapter 7: Sequences and Series of Functions

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Exercise 7.1. *Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.*

Proof (Cauchy criterion). Let $\{f_n\}$ be a uniformly convergent sequence of bounded functions.

(1) Since f_n is bounded, there exists M_n such that $|f_n(x)| \leq M_n$.

(2) Since $\{f_n\}$ converges uniformly, given $1 > 0$ there exists an integer N such that
$$|f_n(x) - f_m(x)| \leq 1 \text{ whenever } n, m \geq N$$
(Theorem 7.8 (Cauchy criterion for uniformly convergence)). Especially,
$$|f_n(x)| \leq |f_n(x) - f_N(x)| + |f_N(x)| \leq 1 + M_N \text{ whenever } n \geq N.$$

(3) Thus, $\{f_n\}$ is uniformly bounded by $M = \max\{M_1, \ldots, M_{N-1}, M_N + 1\}$.

□

Exercise 7.2. *If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n+g_n\}$ converge uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E.*

Proof. Let $f_n \to f$ uniformly and $g_n \to g$ uniformly.

(1) *Show that $\{f_n + g_n\}$ converges uniformly.* Given $\varepsilon > 0$. Since $f_n \to f$ uniformly and $g_n \to g$ uniformly, there exist two integers N_1 and N_2 such that
$$|f_n(x) - f(x)| \leq \frac{\varepsilon}{2} \text{ whenever } n \geq N_1, x \in E$$
$$|g_n(x) - g(x)| \leq \frac{\varepsilon}{2} \text{ whenever } n \geq N_2, x \in E.$$
Take $N = \max\{N_1, N_2\}$, we have
$$\begin{aligned}
&|(f_n(x) + g_n(x)) - (f(x) + g(x))| \\
=&|(f_n(x) - f(x)) + (g_n(x) - g(x))| \\
\leq&|f_n(x) - f(x)| + |g_n(x) - g(x)| \\
\leq&\frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
=&\varepsilon
\end{aligned}$$

whenever $n \geq N$, $x \in E$. Hence $f_n + g_n \to f + g$ uniformly on E.

(2) *Show that $\{f_n g_n\}$ converges uniformly if, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions.* Given $\varepsilon > 0$.

(a) By Exercise 7.1, both $\{f_n\}$ and $\{g_n\}$ are uniformly bounded. So there exist M_1 and M_2 such that

$$|f_n(x)| \leq M_1 \text{ and } |g_n(x)| \leq M_2$$

for all n and $x \in E$. Also, $|f(x)| \leq M_1 + 1$ and $|g(x)| \leq M_2 + 1$.

(b) Since $f_n \to f$ uniformly and $g_n \to g$ uniformly, there exist two integers N_1 and N_2 such that

$$|f_n(x) - f(x)| \leq \frac{\varepsilon}{2(M_2+1)} \text{ whenever } n \geq N_1, x \in E$$
$$|g_n(x) - g(x)| \leq \frac{\varepsilon}{2(M_1+1)} \text{ whenever } n \geq N_2, x \in E.$$

(Note that each denominator of $\frac{\varepsilon}{2(M_j+1)}$ $(j = 1, 2)$ is well-defined and positive!) Take $N = \max\{N_1, N_2\}$, we have

$$\begin{aligned}
&|f_n(x)g_n(x) - f(x)g(x)| \\
=&|[f_n(x) - f(x)]g_n(x) + f(x)[g_n(x) - g(x)]| \\
\leq&|f_n(x) - f(x)||g_n(x)| + |f(x)||g_n(x) - g(x)| \\
\leq&\frac{\varepsilon}{2(M_2+1)} \cdot M_2 + (M_1+1) \cdot \frac{\varepsilon}{2(M_1+1)} \\
\leq&\varepsilon
\end{aligned}$$

whenever $n \geq N$, $x \in E$. Hence $f_n g_n \to fg$ uniformly on E.

□

Proof (Cauchy criterion).

(1) *Show that $\{f_n + g_n\}$ converges uniformly.* Given $\varepsilon > 0$. Since $\{f_n\}$ and $\{g_n\}$ converge uniformly, there exist two integers N_1 and N_2 such that

$$|f_n(x) - f_m(x)| \leq \frac{\varepsilon}{2} \text{ whenever } n, m \geq N_1, x \in E$$
$$|g_n(x) - g_m(x)| \leq \frac{\varepsilon}{2} \text{ whenever } n, m \geq N_2, x \in E.$$

Take $N = \max\{N_1, N_2\}$, we have

$$\begin{aligned}
&|(f_n(x) + g_n(x)) - (f_m(x) + g_m(x))| \\
=&|(f_n(x) - f_n(x)) + (g_n(x) - g_m(x))| \\
\leq&|f_n(x) - f_n(x)| + |g_n(x) - g_m(x)| \\
\leq&\frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
=&\varepsilon
\end{aligned}$$

whenever $n, m \geq N$, $x \in E$. Hence $\{f_n + g_n\}$ converges uniformly on E.

(2) *Show that $\{f_n g_n\}$ converges uniformly if, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions.* Given $\varepsilon > 0$.

(a) By Exercise 7.1, both $\{f_n\}$ and $\{g_n\}$ are uniformly bounded. So there exist M_1 and M_2 such that

$$|f_n(x)| \leq M_1 \text{ and } |g_n(x)| \leq M_2$$

for all n and $x \in E$. Also, $|f(x)| \leq M_1 + 1$ and $|g(x)| \leq M_2 + 1$.

(b) Since $\{f_n\} \to f$ uniformly and $\{g_n\} \to g$ uniformly, there exist two integers N_1 and N_2 such that

$$|f_n(x) - f_m(x)| \leq \frac{\varepsilon}{2(M_2 + 1)} \text{ whenever } n, m \geq N_1, x \in E$$
$$|g_n(x) - g_m(x)| \leq \frac{\varepsilon}{2(M_1 + 1)} \text{ whenever } n, m \geq N_2, x \in E.$$

Take $N = \max\{N_1, N_2\}$, we have

$$\begin{aligned}
&|f_n(x)g_n(x) - f_m(x)g_m(x)| \\
=&|[f_n(x) - f_m(x)]g_n(x) + f_m(x)[g_n(x) - g_m(x)]| \\
\leq&|f_n(x) - f_m(x)||g_n(x)| + |f_m(x)||g_n(x) - g_m(x)| \\
\leq&\frac{\varepsilon}{2(M_2 + 1)} \cdot M_2 + M_1 \cdot \frac{\varepsilon}{2(M_1 + 1)} \\
\leq&\varepsilon
\end{aligned}$$

whenever $n \geq N$, $x \in E$. Hence $\{f_n g_n\}$ converges uniformly on E.

$\square$

Exercise 7.3. *Construct sequences $\{f_n\}$, $\{g_n\}$ which converge uniformly on some set E, but such that $\{f_n g_n\}$ does not converge uniformly on E (of course, $\{f_n g_n\}$ must converge on E).*

We provides some examples here.

Proof ($f_n(x) = x + \frac{1}{n}$).

(1) Define $\{f_n(x)\}$ on $E = \mathbb{R}$ by $f_n(x) = x + \frac{1}{n}$ and $f(x) = x$. Clearly, $\{f_n(x)\}$ converges to $f(x)$ pointwise.

(2) *Show that $\{f_n\}$ converges uniformly.* Given $\varepsilon > 0$. There exists an integer $N \geq \frac{1}{\varepsilon}$ such that

$$|f_n(x) - f(x)| = \frac{1}{n} \leq \frac{1}{N} \leq \varepsilon$$

whenever $n \geq N$ and $x \in E$. Hence $\{f_n\} \to f$ uniformly.

(3) *Show that $\{f_n^2\}$ does not converge uniformly.* Clearly, $\{f_n(x)^2\}$ converges to $f(x)^2$ pointwise. Hence

$$\sup_{x\in E} |f_n(x)^2 - f(x)^2| = \sup_{x\in E} \left| \frac{2x}{n} + \frac{1}{n^2} \right| \to \infty$$

as $n \to \infty$ (by considering $x = n^2 \in E$). Hence $\{f_n^2\}$ does not converge uniformly (Theorem 7.9).

□

Proof ($f_n(x) = \frac{1}{x}$, $g_n(x) = \frac{1}{n}$).

(1) Let $E = (0,1)$. Let $\{f_n(x)\}$ on E be $f_n(x) = \frac{1}{x}$ and $\{g_n(x)\}$ on E be $g_n(x) = \frac{1}{n}$. Clearly, $\{f_n(x)\}$ converges to $f(x) = \frac{1}{x}$ pointwise and $\{g_n(x)\}$ converges to $g(x) = 0$ pointwise.

(2) *Show that $\{f_n\}$ converges uniformly.* Given $\varepsilon > 0$. There exists an integer $N = 1$ such that

$$|f_n(x) - f(x)| = 0 \leq \varepsilon$$

whenever $n \geq N$ and $x \in E$. Hence $\{f_n\} \to f$ uniformly.

(3) *Show that $\{g_n\}$ converges uniformly.* Given $\varepsilon > 0$. There exists an integer $N \geq \frac{1}{\varepsilon}$ such that

$$|g_n(x) - g(x)| = \frac{1}{n} \leq \frac{1}{N} \leq \varepsilon$$

whenever $n \geq N$ and $x \in E$. Hence $\{g_n\} \to g$ uniformly.

(4) *Show that $\{f_n g_n\}$ does not converge uniformly.* Clearly, $\{f_n(x)g_n(x)\}$ converges to $f(x)g(x) = 0$ pointwise. Hence

$$\sup_{x\in E} |f_n(x)g_n(x) - 0| = \sup_{x\in E} \left| \frac{1}{nx} \right| \to \infty$$

as $n \to \infty$ (by considering $x = \frac{1}{n^2} \in E$). Hence $\{f_n g_n\}$ does not converge uniformly (Theorem 7.9).

□

Proof (Exercise 9.2 in Tom M. Apostol, Mathematical Analysis, 2nd edition).

(1) Let $E = [\alpha, \beta] \subseteq \mathbb{R}$ be a bounded interval. Define two sequences $\{f_n\}$ and $\{g_n\}$ on E as follows:

$$f_n(x) = x\left(1 + \frac{1}{n}\right) \text{ if } x \in \mathbb{R},\ n = 1, 2, \cdots,$$

$$g_n(x) = \begin{cases} \frac{1}{n} & \text{if } x = 0 \text{ or if } x \text{ is irrational,} \\ b + \frac{1}{n} & \text{if } x \text{ is rational} \neq 0, \text{ say } x = \frac{a}{b},\ b > 0. \end{cases}$$

Here we assume that $\gcd(a,b)=1$. Clearly, $f(x)=x$ and

$$g(x)=\begin{cases} 0 & \text{if } x=0 \text{ or if } x \text{ is irrational,} \\ b & \text{if } x \text{ is rational} \neq 0, \text{ say } x=\frac{a}{b},\ b>0. \end{cases}$$

Let $M=\max\{|\alpha|,|\beta|\}\geq 0$.

(2) *Show that $\{f_n\}$ converges uniformly.* Given $\varepsilon>0$. There exists an integer $N\geq\frac{M}{\varepsilon}$ such that

$$|f_n(x)-f(x)|=\frac{|x|}{n}\leq\frac{M}{N}\leq\varepsilon$$

whenever $n\geq N$ and $x\in E$. Hence $\{f_n\}\to f$ uniformly.

(3) *Show that $\{g_n\}$ converges uniformly.* Given $\varepsilon>0$. There exists an integer $N\geq\frac{1}{\varepsilon}$ such that

$$|g_n(x)-g(x)|=\frac{1}{n}\leq\frac{1}{N}\leq\varepsilon$$

whenever $n\geq N$ and $x\in E$. Hence $\{g_n\}\to g$ uniformly.

(4) *Show that $\{f_ng_n\}$ does not converge uniformly.*

(a) Clearly, $\{f_n(x)g_n(x)\}$ converges to $f(x)g(x)$ pointwise where

$$f(x)g(x)=\begin{cases} 0 & \text{if } x=0 \text{ or if } x \text{ is irrational,} \\ a & \text{if } x=\frac{a}{b} \text{ is rational} \neq 0,\ b>0. \end{cases}$$

(b) Note that

$$f_n(x)g_n(x)=\begin{cases} \frac{x}{n}\left(1+\frac{1}{n}\right) & \text{if } x=0 \text{ or if } x \text{ is irrational,} \\ \left(a+\frac{x}{n}\right)\left(1+\frac{1}{n}\right) & \text{if } x=\frac{a}{b} \text{ is rational} \neq 0,\ b>0. \end{cases}$$

Therefore,

$$f_n(x)g_n(x)-f(x)g(x)=\begin{cases} \frac{x}{n}\left(1+\frac{1}{n}\right) & \text{if } x=0 \text{ or if } x \text{ is irrational,} \\ \frac{x}{n}\left(1+b+\frac{1}{n}\right) & \text{if } x=\frac{a}{b} \text{ is rational} \neq 0,\ b>0. \end{cases}$$

(c) Hence

$$\begin{aligned}\sup_{x\in E}|f_n(x)g_n(x)-f(x)g(x)| &\geq \sup_{x\in E\cap\mathbb{Q}}|f_n(x)g_n(x)-f(x)g(x)| \\ &= \sup_{x\in E\cap\mathbb{Q}}|a|\left(\frac{1}{n}+\frac{1}{bn}+\frac{1}{bn^2}\right) \\ &\geq \sup_{x\in E\cap\mathbb{Q}}|a|\left(\frac{1}{n}\right) \\ &= \sup_{x\in E\cap\mathbb{Q}}\frac{|a|}{n}.\end{aligned}$$

(d) *Given any irrational number $\gamma \in E$, there exists a sequence*

$$\left\{ r_m = \frac{a_m}{b_m} \right\}$$

of nonzero rational numbers in E such that $\lim r_m = \gamma$. Show that $\{a_m\}$ is unbounded. If it is true, we can find $x_n = r_{m_n} = \frac{a_{m_n}}{b_{m_n}}$ such that $|a_{m_n}| \geq n^2$ and

$$\sup_{x \in E} |f_n(x) g_n(x) - f(x) g(x)| \geq \sup_{x \in E \cap \mathbb{Q}} \frac{|a|}{n} \geq \frac{n^2}{n} = n \to \infty$$

as $n \to \infty$.

(e) (Reductio ad absurdum) If $\{a_m\}$ were bounded, then there exists a **constant** subsequence of $\{a_{m_k}\}$ such that $\lim a_{m_k} = a \in \mathbb{Z}$. Since $\lim_{m \to \infty} r_m = \gamma$, $\lim_{k \to \infty} r_{m_k} = \gamma$ or

$$\lim_{k \to \infty} b_{m_k} = \lim_{k \to \infty} \frac{a_{m_k}}{r_{m_k}} = \frac{a}{\gamma}$$

(it is well-defined since r_{m_k} and γ cannot be zero). Since all b_{m_k} are positive integers, the limit $\lim b_{m_k} = b$ is a positive integer too, or $b = \frac{a}{\gamma} \in \mathbb{Z}^+$, or $\gamma = \frac{a}{b} \in \mathbb{Z}$, which is absurd.

Therefore, $\{f_n g_n\}$ does not converge uniformly.

$\square$

Dirichlet's test for convergence of a series
See Theorem 3.42. Suppose

(a) the partial sums A_n of $\sum a_n$ form a bounded sequence;

(b) $b_0 \geq b_1 \geq b_2 \geq \cdots$;

(c) $\lim_{n \to \infty} b_n = 0$.

Then $\sum a_n b_n$ converges.

Dirichlet's test for uniformly convergence of a function series. Suppose

(a) the partial sums $F_n(x)$ of $\sum f_n(x)$ form a uniformly bounded sequence;

(b) $g_1(x) \geq g_2(x) \geq \cdots$;

(c) $\lim_{n \to \infty} g_n(x) = 0$.

Then $\sum f_n(x)g_n(x)$ converges.

Proof (Theorem 3.42). Choose M such that $|F_n(x)| \leq M$ for all n. Given $\varepsilon > 0$, there is an integer N such that $g_N(x) \leq \frac{\varepsilon}{2(M+1)}$. For $N \leq p \leq q$, we have

$$\begin{aligned}
&\left|\sum_{n=p}^{q} f_n(x)g_n(x)\right| \\
=&\left|\sum_{n=p}^{q-1} F_n(x)(g_n(x) - g_{n+1}(x)) + F_q(x)g_q(x) - F_{p-1}(x)g_p(x)\right| \\
\leq& M\left|\sum_{n=p}^{q-1}(g_n(x) - g_{n+1}(x)) + g_q(x) + g_p(x)\right| \\
=& 2Mg_p(x) \\
\leq& 2Mg_N(x) \\
\leq& \varepsilon.
\end{aligned}$$

Uniformly convergence now follows from the Cauchy criterion (Theorem 7.8). Note that the first inequality in the above chain depends of course on the fact that $g_n(x) - g_{n+1}(x) \geq 0$. $\square$

Exercise 7.6. *Prove that the series*

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

Proof (Dirichlet's test). Given any bounded interval $E = [\alpha, \beta] \subseteq \mathbb{R}$. Write $f_n(x) = (-1)^n$ on E and $g_n(x) = \frac{x^2+n}{n^2}$ on E.

(1) The partial sums $F_n(x)$ of $\sum f_n(x)$ form a uniformly bounded sequence.

(2) $g_1(x) \geq g_2(x) \geq \cdots$ since

$$g_{n+1}(x) = \frac{x^2}{(n+1)^2} + \frac{1}{n+1} < \frac{x^2}{n^2} + \frac{1}{n} = g_n(x).$$

(3) Write $M = \max\{|\alpha|, |\beta|\}$. Since

$$|g_n(x)| = \frac{x^2}{n^2} + \frac{1}{n} \leq \frac{M^2}{n^2} + \frac{1}{n} \to \infty$$

as $n \to \infty$, $\lim_{n\to\infty} g_n(x) = 0$.

By Dirichlet's test, $\sum_{n=1}^{\infty} f_n(x)g_n(x) = \sum_{n=1}^{\infty}(-1)^n \frac{x^2+n}{n^2}$ converges. $\square$

Exercise 7.7. *For $n = 1, 2, 3, \ldots$, x real, put*

$$f_n(x) = \frac{x}{1+nx^2}.$$

Show that $\{f_n\}$ converges uniformly to a function f, and that the equation

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

is correct if $x \neq 0$, but false if $x = 0$.

$f_n(x)$ is defined on $\mathbb{R}$.

Proof.

(1) Since

$$|f_n(x)| = \left|\frac{x}{1+nx^2}\right| \leq \frac{|x|}{\sqrt{n}|x|} = \frac{1}{\sqrt{n}} \to \infty$$

as $n \to \infty$, $f_n \to 0$ uniformly (Theorem 7.9).

(2) Clearly, $f'(x) = 0$. Since

$$f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2},$$

$$\lim_{n\to\infty} f_n'(x) = \begin{cases} 1 & (x = 0), \\ 0 & (x \neq 0). \end{cases}$$

So that the equation

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

is correct if $x \neq 0$, but false if $x = 0$.

$\square$

Note. $f_n'(x)$ does not converge uniformly by considering

$$f_n'\left(\frac{1}{n}\right) = \frac{1-\frac{1}{n}}{(1+\frac{1}{n})^2} \to 1$$

as $n \to \infty$.

Exercise 7.8. *If*

$$I(x) = \begin{cases} 0 & (x \leq 0), \\ 1 & (x > 0), \end{cases}$$

if $\{x_n\}$ is a sequence of distinct points of (a, b), and if $\sum |c_n|$ converges, prove that the series

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n) \qquad (a \leq x \leq b)$$

converges uniformly, and that f is continuous for every $x \neq x_n$.

Proof.

(1) Define $f_n(x) = c_n I(x - x_n)$ on (a, b). So

$$|f_n(x)| = |c_n||I(x - x_n)| \leq |c_n| \qquad (x \in (a, b), n = 1, 2, 3, \ldots).$$

Since $\sum |c_n|$ converges, $f = \sum f_n$ converges uniformly (Theorem 7.10).

(2) Given any $p \in (a, b)$ with $p \neq x_n$ for all $n = 1, 2, 3, \ldots$. So each $I(x - x_n)$ is continuous at $x = p$, and thus each partial sum $\sum_{n=1}^{N} f_n(x)$ is continuous.

(3) By Theorem 7.11

$$\begin{aligned} \lim_{x \to p} f(x) &= \lim_{x \to p} \sum_{n=1}^{\infty} f_n(x) \\ &= \lim_{N \to \infty} \left(\lim_{x \to p} \sum_{n=1}^{N} f_n(x) \right) \\ &= \lim_{N \to \infty} \sum_{n=1}^{N} f_n(p) \\ &= \sum_{n=1}^{\infty} f_n(p) \\ &= f(p). \end{aligned}$$

$f(x)$ is continuous at $x = p$ too.

□