

## Chapter 5: Differentiation

*Author: Meng-Gen Tsai  
Email: plover@gmail.com*

**Exercise 5.1.** Let  $f$  be defined for all real  $x$ , and suppose that

$$|f(x) - f(y)| \leq (x - y)^2$$

for all real  $x$  and  $y$ . Prove that  $f$  is a constant.

*Proof.*

(1) Write

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq |x - y|$$

if  $x \neq y$ .

(2) Given any  $y \in \mathbb{R}$ ,

$$\left| \frac{f(x) - f(y)}{x - y} \right| \rightarrow 0 \quad \text{as } x \rightarrow y,$$

or  $|f'(y)| = 0$ .

(3) Or using  $\varepsilon$ - $\delta$  argument. Fix  $y \in \mathbb{R}$ . Given any  $\varepsilon > 0$ , there exists  $\delta = \varepsilon > 0$  such that

$$\left| \frac{f(x) - f(y)}{x - y} - 0 \right| \leq |x - y| < \delta = \varepsilon$$

whenever  $|x - y| < \delta$ . That is,  $|f'(y)| = 0$ .

(4) So  $f'(y) = 0$  for any  $y \in \mathbb{R}$ . By Theorem 5.11 (b),  $f$  is a constant.

□

**Exercise 5.2.** Suppose  $f'(x) > 0$  in  $(a, b)$ . Prove that  $f$  is strictly increasing in  $(a, b)$ , and let  $g$  be its inverse function. Prove that  $g$  is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)} \quad (a < x < b).$$

*Proof.* Let  $E = (a, b)$ .

- (1) Theorem 5.10 implies that for any  $a < p < q < b$  there exists  $\xi \in (p, q)$  such that

$$f(p) - f(q) = (p - q)f'(\xi).$$

Since  $\xi \in (p, q) \subseteq E$ , by assumption  $f'(\xi) > 0$ . Hence  $f(p) - f(q) = (p - q)f'(\xi) < 0$  (here  $p - q < 0$ ), or

$$f(p) < f(q)$$

if  $p < q$ . Therefore,  $f$  is strictly increasing in  $(a, b)$ .

- (2) Show that  $f$  is one-to-one in  $E$  if  $f$  is strictly increasing in  $E$ . If  $f(p) = f(q)$ , then it cannot be  $p > q$  or  $p < q$  ((1)). So that  $p = q$ , or  $f$  is injective.
- (3) Show that  $g$  is well-defined. Theorem 5.2 and Theorem 4.17.
- (4) Show that  $g'(f(x)) = \frac{1}{f'(x)}$ . Given  $y \in f(E)$ , say  $y = f(x)$  for some  $x \in E$ . Given any  $s \in f(E)$  with  $s \neq y$ . Here  $s = f(t)$  for some  $t \in E$  and  $t \neq x$ .

$$\begin{aligned} \lim_{s \rightarrow y} \frac{g(s) - g(y)}{s - y} &= \lim_{f(t) \rightarrow f(x)} \frac{g(f(t)) - g(f(x))}{f(t) - f(x)} \\ &= \lim_{t \rightarrow x} \frac{t - x}{f(t) - f(x)} \\ &= \lim_{t \rightarrow x} \frac{1}{\frac{f(t) - f(x)}{t - x}} \\ &= \frac{1}{f'(x)}. \end{aligned} \quad (f' > 0)$$

Here  $s \rightarrow y$  if and only if  $t \rightarrow x$  since both  $f$  and  $g$  are continuous and one-to-one. Hence  $g$  is differentiable and  $g'(f(x)) = \frac{1}{f'(x)}$ .

□

**Exercise 5.3.** Suppose  $g$  is a real function on  $\mathbb{R}^1$ , with bounded derivative (say  $|g'| \leq M$ ). Fix  $\varepsilon > 0$ , and define  $f(x) = x + \varepsilon g(x)$ . Prove that  $f$  is one-to-one if  $\varepsilon$  is small enough. (A set of admissible values of  $\varepsilon$  can be determined which depends only on  $M$ .)

*Proof.*

- (1) Note that  $f'(x) = 1 + \varepsilon g'(x)$  (Theorem 5.3). Since  $|g'| \leq M$ ,

$$1 - \varepsilon M \leq f'(x) \leq 1 + \varepsilon M.$$

- (2) Pick

$$\varepsilon = \frac{1}{M + 1} > 0.$$

Thus,

$$f'(x) \geq \frac{1}{M+1} > 0.$$

By Exercise 5.2,  $f(x)$  is strictly increasing in  $\mathbb{R}$  or one-to-one in  $\mathbb{R}$ .

□

**Exercise 5.4.** *If*

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$

where  $C_0, \dots, C_n$  are real constants, prove that the equation

$$C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n = 0$$

has at least one real root between 0 and 1.

*Proof.* Let

$$g(x) = C_0x + \frac{C_1}{2}x^2 + \cdots + \frac{C_{n-1}}{n}x^n + \frac{C_n}{n+1}x^{n+1} \in \mathbb{R}[x].$$

Then  $g(0) = g(1) = 0$ , and  $g'(x) = C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n$ . By the mean value theorem (Theorem 5.10), there exists a point  $\xi \in (0, 1)$  at which

$$g(1) - g(0) = g'(\xi)(1 - 0),$$

or  $g'(\xi) = 0$ . That is, there exists a real root  $x = \xi$  between 0 and 1 at which  $C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n = 0$ . □

**Exercise 5.5.** *Suppose  $f$  is defined and differentiable for every  $x > 0$ , and  $f'(x) \rightarrow 0$  as  $x \rightarrow +\infty$ . Put  $g(x) = f(x+1) - f(x)$ . Prove that  $g(x) \rightarrow 0$  as  $x \rightarrow +\infty$ .*

*Proof.* Given any  $x > 0$ . Since  $f$  is differentiable for every  $x > 0$ ,  $f$  is differentiable on  $[x, x+1]$ . By Theorem 5.2 and Theorem 5.10 (the mean value theorem), there is a point  $\xi \in (x, x+1)$  at which

$$f(x+1) - f(x) = [(x+1) - x]f'(\xi)$$

or

$$g(x) = f'(\xi).$$

As  $x \rightarrow +\infty$ ,  $\xi \rightarrow +\infty$ . Hence

$$\lim_{x \rightarrow +\infty} g(x) = \lim_{\xi \rightarrow +\infty} f'(\xi) = 0.$$

□

**Exercise 5.6.** Suppose

- (a)  $f$  is continuous for  $x \geq 0$ ,
- (b)  $f'(x)$  exists for  $x > 0$ ,
- (c)  $f(0) = 0$ ,
- (d)  $f'$  is monotonically increasing.

Put

$$g(x) = \frac{f(x)}{x} \quad (x > 0)$$

and prove that  $g$  is monotonically increasing.

*Proof.*

- (1) It suffices to show that  $g'(x) \geq 0$  for  $x > 0$  (Theorem 5.11(a)), that is, to show that

$$g'(x) = \frac{xf'(x) - f(x)}{x^2} \geq 0 \quad (x > 0),$$

or

$$xf'(x) - f(x) \geq 0 \quad (x > 0)$$

since  $x^2 > 0$  for all nonzero  $x$ .

- (2) Given  $x > 0$ . By (a)(b), we apply the mean value theorem (Theorem 5.10) on  $f$  to get

$$f(x) - f(0) = (x - 0)f'(\xi)$$

for some  $\xi \in (0, x)$ . By (c),

$$f(x) = xf'(\xi).$$

By (d),

$$f(x) = xf'(\xi) \leq xf'(x).$$

Hence  $xf'(x) - f(x) \geq 0$ , or  $g$  is monotonically increasing.

□

*Note.*  $g$  is increasing strictly if  $f$  is increasing strictly.

**Exercise 5.14.** Let  $f$  be a differentiable real function defined in  $(a, b)$ . Prove that  $f$  is convex if and only if  $f'$  is monotonically increasing. Assume next  $f''(x)$  exists for every  $x \in (a, b)$ , and prove that  $f$  is convex if and only if  $f''(x) \geq 0$  for all  $x \in (a, b)$ .

*Proof.*

(1) Show that  $f'$  is monotonically increasing if  $f$  is convex.

(a) Since  $f$  is convex, by definition (Exercise 4.23)

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

whenever  $a < x < b$ ,  $a < y < b$ ,  $0 < \lambda < 1$ .

(b) As  $x \neq y$ , we have

$$\begin{aligned} f(y) - f(x) &\geq \frac{f(x + \lambda(y - x)) - f(x)}{\lambda} \\ &= \frac{f(x + \lambda(y - x)) - f(x)}{\lambda(y - x)} \cdot (y - x) \end{aligned}$$

and let  $\lambda \rightarrow 0$  to get

$$f(y) - f(x) \geq f'(x)(y - x)$$

(since  $f'(x)$  exists). Similarly, we have

$$f(x) - f(y) \geq f'(y)(x - y).$$

(c) Given any  $y > x$ , we have

$$f'(y)(y - x) \geq f(y) - f(x) \geq f'(x)(y - x).$$

Hence  $f'(y) \geq f'(x)$  whenever  $y > x$ , or  $f'$  is monotonically increasing.

(2) Show that  $f$  is convex if  $f'$  is monotonically increasing. Given any  $y > x$  and any  $0 < \lambda < 1$ .

(a) By Theorem 5.10 (the mean value theorem), there is a point  $x < \xi < y$  such that

$$f(y) - f(x) = f'(\xi)(y - x).$$

Since  $f'$  is monotonically increasing,

$$f'(y)(y - x) \geq f(y) - f(x) \geq f'(x)(y - x).$$

(b) Write  $z = \lambda x + (1 - \lambda)y$ . Hence

$$\begin{aligned} f(y) - f(z) &\geq f'(z)(y - z), \\ f(z) - f(x) &\leq f'(z)(z - x), \end{aligned}$$

or

$$\begin{aligned} f(y) &\geq f(z) + f'(z)(y - z), \\ f(x) &\geq f(z) + f'(z)(x - z), \end{aligned}$$

or

$$\begin{aligned}\lambda f(x) + (1 - \lambda)f(y) &\geq \lambda[f(z) + f'(z)(x - z)] \\ &\quad + (1 - \lambda)[f(z) + f'(z)(y - z)] \\ &= f(z) \\ &= f(\lambda x + (1 - \lambda)y).\end{aligned}$$

Hence  $f$  is convex.

- (3) *Show that  $f''(x) \geq 0$  if  $f$  is convex and  $f''$  exists.* By (1),  $f'$  is monotonically increasing since  $f$  is convex. Given any  $x \neq y$ , we have

$$\frac{f'(y) - f'(x)}{y - x} \geq 0.$$

Let  $y \rightarrow x$ , we have  $f''(x) \geq 0$  if  $f''$  exists.

- (4) *Show that  $f$  is convex if  $f''$  exists and  $f''(x) \geq 0$ .* By Theorem 5.11(a),  $f'$  is monotonically increasing. By (2),  $f$  is convex.

□