Chapter I: Varieties

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Problem I.1.6. *Any nonempty open subset of an irreducible topological space is dense and irreducible. If Y is a subset of a topological space X, which is irreducible in its induced topology, then the closure $\overline{Y}$ is also irreducible.*

Proof.

(1) *Show that any nonempty open subset of an irreducible topological space is dense.* It suffices to show that $U_1 \cap U_2 \neq \varnothing$ for any nonempty open subsets of an irreducible topological space.

$$\begin{aligned}
&\forall \text{ nonempty open sets } U_1 \text{ and } U_2, U_1 \cap U_2 \neq \varnothing \\
\Longleftrightarrow &\forall \text{ nonempty open sets } U_1 \text{ and } U_2, X - (U_1 \cap U_2) \neq X \\
\Longleftrightarrow &\forall \text{ nonempty open sets } U_1 \text{ and } U_2, (X - U_1) \cup (X - U_2) \neq X \\
\Longleftrightarrow &\forall \text{ proper closed sets } Y_1 \text{ and } Y_2, Y_1 \cup Y_2 \neq X \\
\Longleftrightarrow &\nexists \text{ proper closed sets } Y_1 \text{ and } Y_2, Y_1 \cup Y_2 = X.
\end{aligned}$$

(2) *Show that any nonempty open subset of an irreducible topological space is irreducible.* Given any open subset U of an irreducible topological space X. Write $U \subseteq Y_1 \cup Y_2$ where Y_1 and Y_2 are closed in X.

$$\begin{aligned}
U \subseteq Y_1 \cup Y_2 \Longrightarrow &\overline{U} \subseteq \overline{Y_1 \cup Y_2} \\
\Longrightarrow &X \subseteq Y_1 \cup Y_2 && (U \text{ is dense, } Y_1 \cup Y_2 \text{ is closed}) \\
\Longrightarrow &Y_1 = X \supseteq U \text{ or } Y_2 = X \supseteq U && (X \text{ is irreducible}) \\
\Longrightarrow &U \text{ is irreducible.}
\end{aligned}$$

(3) *Show that if Y is a subset of a topological space X, which is irreducible (in its induced topology), then the closure $\overline{Y}$ is also irreducible.* (Reductio ad absurdum) If $\overline{Y}$ were reducible, there are two closed sets Y_1 and Y_2 such that

$$\overline{Y} \subseteq Y_1 \cup Y_2, \overline{Y} \not\subseteq Y_i (i = 1, 2).$$

(a) $Y \subseteq \overline{Y} \subseteq Y_1 \cup Y_2$.

(b) $Y \not\subseteq Y_i (i = 1, 2)$. If not, $Y \subseteq Y_i$ for some i. Take closure to get $\overline{Y} \subseteq \overline{Y_i} = Y_i$ (since Y_i is closed), contrary to the assumption.

By (a)(b), Y is reducible, which is absurd.

$\square$