Chapter 9: Functions of Several Variables

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Exercise 9.1. *If S is a nonempty subset of a vector space X, prove (as asserted in Section 9.1) that the span of S is a vector space.*

Denote the span of S by $\mathrm{span}(S)$.

Proof.

(1) Since $S \neq \varnothing$, there is $\mathbf{z} \in S$. So $1\mathbf{z} = \mathbf{z} \in \mathrm{span}(S) \neq \varnothing$. (In fact, $\mathrm{span}(S) \supseteq S$.)

(2) If $\mathbf{x}, \mathbf{y} \in \mathrm{span}(S)$, then there exist elements $\mathbf{x}_1, \ldots, \mathbf{x}_m$, $\mathbf{y}_1, \ldots, \mathbf{y}_n \in S$ and scalars $a_1, \ldots, a_m$, $b_1, \ldots, b_n$ such that

$$\mathbf{x} = a_1\mathbf{x}_1 + \cdots + a_m\mathbf{x}_m,$$
$$\mathbf{y} = b_1\mathbf{y}_1 + \cdots + b_n\mathbf{y}_n.$$

Then
$$\mathbf{x} + \mathbf{y} = a_1\mathbf{x}_1 + \cdots + a_m\mathbf{x}_m + b_1\mathbf{y}_1 + \cdots + b_n\mathbf{y}_n$$
is a linear combination of the elements of S. For any scalar c,

$$c\mathbf{x} = (ca_1)\mathbf{x}_1 + \cdots + (ca_m)\mathbf{x}_m$$

is again linear combination of the elements of S.

(3) By (1)(2), $\mathrm{span}(S)$ is a vector space.

□

Note. Any subspace of X that contains S must also contain $\mathrm{span}(S)$.

Exercise 9.2. *Prove (as asserted in Section 9.6) that BA is linear if A and B are linear transformations. Prove also that A^{-1} is linear and invertible if A is invertible.*

Proof. Use the notation in Definitions 9.6.

(1) *Show that BA is linear if A and B are linear transformations.* Let X, Y, Z be vector spaces, $A \in L(X, Y)$ and $B \in L(Y, Z)$.

(a) Given any $\mathbf{x}_1, \mathbf{x}_2 \in X$.

$$\begin{aligned}
(BA)(\mathbf{x}_1 + \mathbf{x}_2) &= B(A(\mathbf{x}_1 + \mathbf{x}_2)) \\
&= B(A\mathbf{x}_1 + A\mathbf{x}_2) && (A \text{ is a linear transformation}) \\
&= B(A\mathbf{x}_1) + B(A\mathbf{x}_2) && (B \text{ is a linear transformation}) \\
&= (BA)\mathbf{x}_1 + (BA)\mathbf{x}_2.
\end{aligned}$$

(b) For any $\mathbf{x} \in X$ and scalar c,

$$\begin{aligned}
(BA)(c\mathbf{x}) &= B(A(c\mathbf{x})) \\
&= B(cA\mathbf{x}) && (A \text{ is a linear transformation}) \\
&= cB(A\mathbf{x}) && (B \text{ is a linear transformation}) \\
&= c(BA)\mathbf{x}.
\end{aligned}$$

By (a)(b), $BA \in L(X, Z)$.

(2) *Show that A^{-1} is linear if A is invertible.*

(a) Given any $\mathbf{y}_1, \mathbf{y}_2 \in X$. Since A is surjective, there exist $\mathbf{x}_1, \mathbf{x}_2 \in X$ such that

$$\begin{aligned}
\mathbf{y}_1 &= A\mathbf{x}_1 \\
\mathbf{y}_2 &= A\mathbf{x}_2.
\end{aligned}$$

So

$$\begin{aligned}
A^{-1}\mathbf{y}_1 &= A^{-1}(A\mathbf{x}_1) = \mathbf{x}_1 \\
A^{-1}\mathbf{y}_2 &= A^{-1}(A\mathbf{x}_2) = \mathbf{x}_2
\end{aligned}$$

(by Definitions 9.4). Hence

$$\begin{aligned}
A^{-1}(\mathbf{y}_1 + \mathbf{y}_2) &= A^{-1}(A\mathbf{x}_1 + A\mathbf{x}_2) \\
&= A^{-1}(A(\mathbf{x}_1 + \mathbf{x}_2)) && (A \text{ is a linear transformation}) \\
&= \mathbf{x}_1 + \mathbf{x}_2 && (\text{Definitions 9.4}) \\
&= A^{-1}\mathbf{y}_1 + A^{-1}\mathbf{y}_2.
\end{aligned}$$

(b) For any $\mathbf{y} \in X$ and scalar c, there is a corresponding $\mathbf{x} \in X$ such that $\mathbf{y} = A\mathbf{x}$ since A is surjective. So $A^{-1}\mathbf{y} = \mathbf{x}$ by Definition 9.4. Hence

$$\begin{aligned}
A^{-1}(c\mathbf{y}) &= A^{-1}(cA\mathbf{x}) \\
&= A^{-1}(A(c\mathbf{x})) && (A \text{ is a linear transformation}) \\
&= c\mathbf{x} && (\text{Definitions 9.4}) \\
&= cA^{-1}\mathbf{y}.
\end{aligned}$$

By (a)(b), $A^{-1} \in L(X)$.

(3) *Show that A^{-1} is invertible if A is invertible.* It suffices to show that A^{-1} is injective and surjective.

  (a) *Show that A^{-1} is injective.* Given any $\mathbf{y}_1, \mathbf{y}_2 \in X$. Since A is surjective, there exist $\mathbf{x}_1, \mathbf{x}_2 \in X$ such that

$$\mathbf{y}_1 = A\mathbf{x}_1$$
$$\mathbf{y}_2 = A\mathbf{x}_2.$$

  Suppose $A^{-1}\mathbf{y}_1 = A^{-1}\mathbf{y}_2$. So $A^{-1}(A\mathbf{x}_1) = A^{-1}(A\mathbf{x}_2)$, or $\mathbf{x}_1 = \mathbf{x}_2$, or $\mathbf{y}_1 = A\mathbf{x}_1 = A\mathbf{x}_2 = \mathbf{y}_2$.

  (b) *Show that A^{-1} is surjective.* For any $\mathbf{x} \in X$, there exists $A\mathbf{x} \in X$ such that $A^{-1}(A\mathbf{x}) = \mathbf{x}$ by Definitions 9.4.

□

Exercise 9.3. *Assume $A \in L(X, Y)$ and $A\mathbf{x} = \mathbf{0}$ only when $\mathbf{x} = \mathbf{0}$. Prove that A is then 1-1.*

Proof. Suppose $A\mathbf{x} = A\mathbf{y}$. Since A is a linear transformation, $A(\mathbf{x} - \mathbf{y}) = A\mathbf{x} - A\mathbf{y} = \mathbf{0}$. By assumption, $\mathbf{x} - \mathbf{y} = \mathbf{0}$ or $\mathbf{x} = \mathbf{y}$. □

Exercise 9.4. *Prove (as asserted in Section 9.30) that null spaces and ranges of linear transformations are vector spaces.*

Proof. Use the notation in Definitions 9.30. Suppose X, Y are vector spaces, and $A \in L(X, Y)$, as in Definition 9.6.

(1) *Show that $\mathscr{N}(A)$ is a vector space in X.*

  (a) Note that $\mathbf{0} \in X$. Since $A\mathbf{0} = \mathbf{0}$, $\mathbf{0} \in \mathscr{N}(A) \neq \varnothing$.

  (b) Suppose $\mathbf{x}_1, \mathbf{x}_2 \in \mathscr{N}(A)$. Then

$$\begin{aligned} A(\mathbf{x}_1 + \mathbf{x}_2) &= A\mathbf{x}_1 + A\mathbf{x}_2 && (A \text{ is a linear transformation}) \\ &= \mathbf{0} + \mathbf{0} && (\mathbf{x}_1, \mathbf{x}_2 \in \mathscr{N}(A)) \\ &= \mathbf{0}. \end{aligned}$$

  So $\mathbf{x}_1 + \mathbf{x}_2 \in \mathscr{N}(A)$.

  (c) Suppose $\mathbf{x} \in \mathscr{N}(A)$ and c is a scalar. Then

$$\begin{aligned} A(c\mathbf{x}) &= cA\mathbf{x} && (A \text{ is a linear transformation}) \\ &= c\mathbf{0} && (\mathbf{x} \in \mathscr{N}(A)) \\ &= \mathbf{0}. \end{aligned}$$

  So $c\mathbf{x} \in \mathscr{N}(A)$.

By (a)(b)(c), $\mathscr{N}(A)$ is a vector space.

(2) *Show that $\mathscr{R}(A)$ is a vector space in Y.*

(a) Note that $\mathbf{0} \in X$. So $A\mathbf{0} = \mathbf{0} \in \mathscr{R}(A) \neq \varnothing$.

(b) Suppose $\mathbf{y}_1, \mathbf{y}_2 \in \mathscr{R}(A)$. Then there exist $\mathbf{x}_1, \mathbf{x}_2 \in X$ such that $A\mathbf{x}_1 = \mathbf{y}_1$ and $A\mathbf{x}_2 = \mathbf{y}_2$. Hence

$$\begin{aligned} \mathbf{y}_1 + \mathbf{y}_2 &= A\mathbf{x}_1 + A\mathbf{x}_2 \\ &= A(\mathbf{x}_1 + \mathbf{x}_2) && (A \text{ is a linear transformation}). \end{aligned}$$

So $\mathbf{y}_1 + \mathbf{y}_2 \in \mathscr{R}(A)$.

(c) Suppose $\mathbf{y} \in \mathscr{R}(A)$ and c is a scalar. Then there exists $\mathbf{x} \in X$ such that $A\mathbf{x} = \mathbf{y}$. Hence

$$\begin{aligned} c\mathbf{y} &= cA\mathbf{x} \\ &= A(c\mathbf{x}) && (A \text{ is a linear transformation}). \end{aligned}$$

So $c\mathbf{y} \in \mathscr{R}(A)$.

By (a)(b)(c), $\mathscr{R}(A)$ is a vector space.

□

Exercise 9.5. *Prove that to every $A \in L(\mathbb{R}^n, \mathbb{R}^1)$ corresponds a unique $\mathbf{y} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$. Prove also that $\|A\| = |\mathbf{y}|$. (Hint: Under certain conditions, equality holds in the Schwarz inequality.)*

Proof.

(1) Recall that $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$ (Definitions 9.1). Given any $\mathbf{x} \in \mathbb{R}^n$, write $\mathbf{x} = (x_1, \ldots, x_n)$ as $\mathbf{x} = \sum x_j \mathbf{e}_j$.

(2) *Show that $\mathbf{y}$ exists.* Since A is a linear transformation,

$$\begin{aligned} A\mathbf{x} &= A\left(\sum x_j \mathbf{e}_j\right) \\ &= \sum x_j A\mathbf{e}_j \\ &= (x_1, \ldots, x_n) \cdot (A\mathbf{e}_1, \ldots, A\mathbf{e}_n) \\ &= \mathbf{x} \cdot \sum (A\mathbf{e}_j)\mathbf{e}_j. \end{aligned}$$

Define $\mathbf{y} = \sum (A\mathbf{e}_j)\mathbf{e}_j \in \mathbb{R}^n$ so that $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$.

(3) *Show that $\mathbf{y}$ is unique.* Suppose there exists some $\mathbf{z} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{x} \cdot \mathbf{z}$. So

$$\begin{aligned} 0 &= A\mathbf{x} - A\mathbf{x} \\ &= \mathbf{x} \cdot \mathbf{y} - \mathbf{x} \cdot \mathbf{z} \\ &= \mathbf{x} \cdot (\mathbf{y} - \mathbf{z}) \end{aligned}$$

for any $\mathbf{x} \in \mathbb{R}^n$. In particular, take $\mathbf{x} = \mathbf{y} - \mathbf{z} \in \mathbb{R}^n$ to get

$$0 = (\mathbf{y} - \mathbf{z}) \cdot (\mathbf{y} - \mathbf{z}) = |\mathbf{y} - \mathbf{z}|^2$$

or $\mathbf{y} - \mathbf{z} = \mathbf{0}$ or $\mathbf{y} = \mathbf{z}$.

(4) *Show that* $\|A\| = |\mathbf{y}|$. By the Schwarz inequality (Theorem 1.37(d)),

$$|A\mathbf{x}| = |\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}| \leq |\mathbf{y}|$$

as $|\mathbf{x}| \leq 1$. Take the sup over all $|\mathbf{x}| \leq 1$ to get

$$\|A\| \leq |\mathbf{y}|.$$

If $\mathbf{y} = \mathbf{0}$, then $\|A\| = |\mathbf{y}| = 0$. If $\mathbf{y} \neq \mathbf{0}$, then the equality holds when $\mathbf{x} = \frac{\mathbf{y}}{|\mathbf{y}|} \in \mathbb{R}^n$. (Here $|\mathbf{x}| = 1$.)

$\square$

Exercise 9.6. *If* $f(0,0) = 0$ *and*

$$f(x,y) = \frac{xy}{x^2+y^2} \qquad \text{if } (x,y) \neq (0,0),$$

prove that $(D_1 f)(x,y)$ *and* $(D_2 f)(x,y)$ *exist at every point of* $\mathbb{R}^2$*, although* f *is not continuous at* $(0,0)$.

Proof.

(1) *Show that*

$$(D_1 f)(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0), \\ \frac{y(y^2-x^2)}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0). \end{cases}$$

Write

$$\begin{aligned}(D_1 f)(x,y) &= \lim_{t \to 0} \frac{f((x,y) + t(1,0)) - f(x,y)}{t} \\ &= \lim_{t \to 0} \frac{f(x+t,y) - f(x,y)}{t}.\end{aligned}$$

If $(x,y) = (0,0)$,

$$(D_1 f)(0,0) = \lim_{t \to 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t \to 0} \frac{0-0}{t} = 0.$$

If $(x, y) \neq (0, 0)$,

$$\begin{aligned}
(D_1 f)(x, y) &= \lim_{t \to 0} \frac{f(x+t, y) - f(x, y)}{t} \\
&= \lim_{t \to 0} \frac{\frac{(x+t)y}{(x+t)^2+y^2} - \frac{xy}{x^2+y^2}}{t} \\
&= \lim_{t \to 0} \frac{y(y^2 - x^2) - txy}{((x+t)^2 + y^2)(x^2 + y^2)} \\
&= \frac{y(y^2 - x^2)}{(x^2 + y^2)^2}.
\end{aligned}$$

(2) *Show that*

$$(D_2 f)(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0), \\ \frac{x(x^2-y^2)}{(x^2+y^2)^2} & \text{if } (x, y) \neq (0, 0). \end{cases}$$

Similar to (1).

(3) *Show that f is not continuous at $(0, 0)$.* Note that

$$\lim_{n \to \infty} f\left(\frac{1}{n}, \frac{1}{n}\right) = \lim_{n \to \infty} \frac{\frac{1}{n} \cdot \frac{1}{n}}{\frac{1}{n^2} + \frac{1}{n^2}} = \lim_{n \to \infty} \frac{1}{2} = \frac{1}{2}$$

and

$$\lim_{n \to \infty} f\left(\frac{1}{n}, 0\right) = \lim_{n \to \infty} \frac{0}{\frac{1}{n^2} + 0} = \lim_{n \to \infty} 0 = 0.$$

Hence the limit $\lim_{(x,y) \to (0,0)} f(x, y)$ does not exist.

□

Exercise 9.7. *Suppose that f is a real-valued function defined in an open set $E \subseteq \mathbb{R}^n$, and that the partial derivatives $D_1 f, \ldots, D_n f$ are bounded in E. Prove that f is continuous in E. (Hint: Proceed as in the proof of Theorem 9.21.)*

Proof.

(1) Since $D_j f$ is bounded in E, there is a real number M_j such that $|D_j f| \leq M_j$ in E. Take $M = \max_{1 \leq j \leq n} M_j$ so that $|D_j f| \leq M$ in E for all $1 \leq j \leq n$.

(2) Fix $\mathbf{x} \in E$ and $\varepsilon > 0$. Since E is open, there is an open neighborhood

$$B(\mathbf{x}; r) = \{\mathbf{x} + \mathbf{h} \in E : |\mathbf{h}| < r\} \subseteq E$$

with

$$0 < r < \frac{\varepsilon}{n(M + 1)}.$$

(3) Write $\mathbf{h} = \sum h_j\mathbf{e}_j$, $|\mathbf{h}| < r$, put $\mathbf{v}_0 = \mathbf{0}$, and $\mathbf{v}_k = h_1\mathbf{e}_1 + \cdots + h_k\mathbf{e}_k$ for $1 \leq k \leq n$. Then

$$f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) = \sum_{j=1}^{n}[f(\mathbf{x}+\mathbf{v}_j) - f(\mathbf{x}+\mathbf{v}_{j-1})].$$

Since $|\mathbf{v}_k| < r$ for $1 \leq k \leq n$ and since $B(\mathbf{x};r)$ is convex, the open interval with end points $\mathbf{x}+\mathbf{v}_{j-1}$ and $\mathbf{x}+\mathbf{v}_j$ lie in $B(\mathbf{x};r)$. Since $\mathbf{v}_j = \mathbf{v}_{j-1} - h_j\mathbf{e}_j$, the mean value theorem (Theorem 5.10) show that

$$f(\mathbf{x}+\mathbf{v}_j) - f(\mathbf{x}+\mathbf{v}_{j-1}) = h_j(D_jf)(\mathbf{x}+\mathbf{v}_{j-1}+\theta_jh_j\mathbf{e}_j)$$

for some $\theta_j \in (0,1)$.

(4) Note that $|h_j| \leq |\mathbf{h}| < r < \frac{\varepsilon}{n(M+1)}$. Hence

$$\begin{aligned}
|f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})| &\leq \sum_{j=1}^{n}|f(\mathbf{x}+\mathbf{v}_j) - f(\mathbf{x}+\mathbf{v}_{j-1})| \\
&= \sum_{j=1}^{n}|h_j||(D_jf)(\mathbf{x}+\mathbf{v}_{j-1}+\theta_jh_j\mathbf{e}_j)| \\
&\leq \sum_{j=1}^{n}\frac{\varepsilon}{n(M+1)} \cdot M \\
&< \varepsilon
\end{aligned}$$

as $|\mathbf{h}| < r < \frac{\varepsilon}{n(M+1)}$. Hence f is continuous at all $\mathbf{x} \in E$.

□

Exercise 9.8. *Suppose that f is a differentiable real function in an open set $E \subseteq \mathbb{R}^n$, and that f has a local maximum at a point $\mathbf{x} \in E$. Prove that $f'(\mathbf{x}) = 0$.*

Proof (Theorem 5.8).

(1) Apply Theorem 5.8 to each D_jf for $1 \leq j \leq n$. Since f has a local maximum at a point $\mathbf{x} \in E$, there is an open neighborhood $B(\mathbf{x};r)$ of $\mathbf{x}$ in E such that

$$f(\mathbf{y}) \leq f(\mathbf{x})$$

for all $\mathbf{y} \in B(\mathbf{x};r)$. Therefore,

$$f(\mathbf{x}+t\mathbf{e}_j) \leq f(\mathbf{x})$$

for all $|t| < r$ and $1 \leq j \leq n$, or $t \mapsto f(\mathbf{x}+t\mathbf{e}_j)$ has a local maximum at a point $t = 0 \in (-r,r)$.

(2) Since f is a differentiable in E, each partial derivatives $D_j f$ exist (Theorem 9.21). Hence Theorem 5.8 implies that $(D_j f)(\mathbf{x}) = 0$ for all $1 \leq j \leq n$. So

$$f'(\mathbf{x}) = [(D_1 f)(\mathbf{x}) \cdots (D_k f)(\mathbf{x})] = [0 \cdots 0] = 0$$

(as the zero matrix).

□

Exercise 9.9. *If* $\mathbf{f}$ *is a differentiable mapping of a connected open set* $E \subseteq \mathbb{R}^n$*, and if* $\mathbf{f}'(\mathbf{x}) = 0$ *for every* $\mathbf{x} \in E$*, prove that* $\mathbf{f}$ *is a constant in* E*.*

Proof.

(1) *Show that* $\mathbf{f}$ *is* ***locally constant***. Given any $\mathbf{x} \in E$. Since E is open, there exists an open neighborhood $B(\mathbf{x}; r)$ of $\mathbf{x}$ such that $B(\mathbf{x}; r) \subseteq E$ and $r > 0$. Corollary to Theorem 9.19 implies that $\mathbf{f}$ is a constant on $B(\mathbf{x}; r)$, that is, $\mathbf{f}$ is locally constant.

(2) *Show that* $\mathbf{f}$ *is constant if* $\mathbf{f}$ *is locally constant in a connected set* $E \subseteq \mathbb{R}^n$. Might assume that $E \neq \varnothing$. (Otherwise there is nothing to do.) Take some $\mathbf{x}_0 \in E$.

(a) Let
$$U = \{\mathbf{y} \in E : \mathbf{f}(\mathbf{y}) = \mathbf{f}(\mathbf{x}_0)\}.$$

(b) U is open since $\mathbf{f}$ is locally constant (by (1)). (Take any $\mathbf{y} \in U$. Since $\mathbf{f}$ is locally constant, there is an open neighborhood $B(\mathbf{y}) \subseteq E$ of $\mathbf{y}$ such that $f(\mathbf{z}) = f(\mathbf{y}) = f(\mathbf{x}_0)$ whenever $\mathbf{z} \in B(\mathbf{y})$. So that $B(\mathbf{y}) \subseteq U$, or U is open.)

(c) Besides, since $\mathbf{f}$ is continuous (Remarks 9.13(c)), the set U is closed. (The proof is the same as Proof (Definition 2.18(d)) in Exercise 4.3.)

(d) So U is open and closed. Write $E = U \cup (E - U)$. Here U and $E - U$ are both open and closed. Hence $U \cap \overline{E - U} = U \cap (E - U) = \varnothing$ and $\overline{U} \cap (E - U) = U \cap (E - U) = \varnothing$. Note that $\mathbf{x}_0 \in U \neq \varnothing$. By the connectedness of E, $E - U = \varnothing$, or $E = U$, or $\mathbf{f}$ is constant on E.

Note. The only subsets of a connected set E *which are both open and closed are* E *and* $\varnothing$*.*

□

Exercise 9.10. *...*

Proof.

(1)

(2)

□

Exercise 9.11. *If f and g are differentiable real functions in $\mathbb{R}^n$, prove that*

$$\nabla(fg) = f\nabla g + g\nabla f$$

and that

$$\nabla\left(\frac{1}{f}\right) = -\frac{1}{f^2}\nabla f$$

whenever $f \neq 0$.

Proof. Recall Example 9.18:

$$(\nabla(f))(\mathbf{x}) = \sum_{i=1}^{n}(D_i f)(\mathbf{x})\mathbf{e}_i.$$

(1) *Show that $\nabla(fg) = f\nabla g + g\nabla f$.* For any $\mathbf{x} \in \mathbb{R}^n$,

$$\begin{aligned}
(\nabla(fg))(\mathbf{x}) &= \sum_{i=1}^{n}(D_i(fg))(\mathbf{x})\mathbf{e}_i \\
&= \sum_{i=1}^{n}(g(D_i f) + f(D_i g))(\mathbf{x})\mathbf{e}_i && \text{(Theorem 5.3(b))} \\
&= \sum_{i=1}^{n}\left[g(\mathbf{x})(D_i f)(\mathbf{x}) + f(\mathbf{x})(D_i g)(\mathbf{x})\right]\mathbf{e}_i \\
&= g(\mathbf{x})\sum_{i=1}^{n}(D_i f)(\mathbf{x})\mathbf{e}_i + f(\mathbf{x})\sum_{i=1}^{n}(D_i g)(\mathbf{x})\mathbf{e}_i \\
&= g(\mathbf{x})(\nabla f)(\mathbf{x}) + f(\mathbf{x})(\nabla g)(\mathbf{x}) \\
&= (f\nabla g + g\nabla f)(\mathbf{x}).
\end{aligned}$$

(2) *Show that*

$$\nabla\left(\frac{1}{f}\right) = -\frac{1}{f^2}\nabla f$$

whenever $f \neq 0$. Note that $\nabla(1) = 0$ since

$$\nabla(1)(\mathbf{x}) = \sum(D_i 1)(\mathbf{x})\mathbf{e}_i = \sum(0)(\mathbf{x})\mathbf{e}_i = \sum 0\mathbf{e}_i = 0.$$

Hence as $f \neq 0$, we have

$$\begin{aligned}
0 &= \nabla(1) \\
&= \nabla\left(f\frac{1}{f}\right) && (f \neq 0) \\
&= f\nabla\left(\frac{1}{f}\right) + \frac{1}{f}\nabla f && ((1)),
\end{aligned}$$

or $\nabla\left(\frac{1}{f}\right) = -\frac{1}{f^2}\nabla f$.

□

Exercise 9.12. *...*

Proof.

(1)

(2)

□

Exercise 9.13. *Suppose* $\mathbf{f}$ *is a differentiable mapping of* $\mathbb{R}^1$ *into* $\mathbb{R}^3$ *such that* $|\mathbf{f}(t)| = 1$ *for every* t. *Prove that* $\mathbf{f}'(t) \cdot \mathbf{f}(t) = 0$. *Interpret this result geometrically.*

Proof.

(1) Write $\mathbf{f} = (f_1, f_2, f_3)$ as a vector-valued function. By Remarks 5.16, $\mathbf{f}$ is differentiable if and only if each f_1, f_2, f_3 is differentiable. So $\mathbf{f}' = (f_1', f_2', f_3)'$. Hence

$$\begin{aligned}
&|\mathbf{f}(t)| = 1 \text{ for every } t \\
\Longleftrightarrow &\mathbf{f}(t) \cdot \mathbf{f}(t) = 1 \\
\Longleftrightarrow &f_1(t)^2 + f_2(t)^2 + f_3(t)^2 = 1 \\
\Longrightarrow &2f_1(t)f_1'(t) + 2f_2(t)f_2'(t) + 2f_3(t)f_3'(t) = 0 \\
\Longleftrightarrow &f_1(t)f_1'(t) + f_2(t)f_2'(t) + f_3(t)f_3'(t) = 0 \\
\Longleftrightarrow &(f_1(t), f_2(t), f_3(t)) \cdot (f_1'(t), f_2'(t), f_3'(t)) = 0 \\
\Longleftrightarrow &\mathbf{f}(t) \cdot \mathbf{f}'(t) = \mathbf{f}'(t) \cdot \mathbf{f}(t) = 0.
\end{aligned}$$

(2) The vector $\mathbf{f}'(t)$ is called the ***tangent vector*** (or ***velocity vector***) of $\mathbf{f}$ at t. Geometrically, given any mapping $\mathbf{f}$ lying on the sphere S^2, its tangent vector at t is lying on the tangent plane of S^2 at t.

□

Exercise 9.14. *Define* $f(0,0) = 0$ *and*

$$f(x, y) = \frac{x^3}{x^2 + y^2} \qquad \text{if } (x, y) \neq (0, 0).$$

(a) *Prove that $D_1 f$ and $D_2 f$ are bounded functions in $\mathbb{R}^2$. (Hence f is continuous.)*

(b) *Let $\mathbf{u}$ be any unit vector in $\mathbb{R}^2$. Show that the directional derivative $(D_{\mathbf{u}} f)(0,0)$ exists, and that its absolute value is at most 1.*

(c) *Let γ be a differentiable mapping of $\mathbb{R}^1$ into $\mathbb{R}^2$ (in other words, γ is a differentiable curve in $\mathbb{R}^2$), with $\gamma(t) = (0,0)$ and $\gamma'(t) \neq (0,0)$ for any $t \in \mathbb{R}^1$. Put $g(t) = f(\gamma(t))$ and prove that g is differentiable for every $t \in \mathbb{R}^1$. If $\gamma \in \mathscr{C}'$, prove that $g \in \mathscr{C}'$.*

(d) *In spite of this, prove that f is not differentiable at $(0,0)$.*

Proof of (a).

(1) *Show that*

$$(D_1 f)(x,y) = \begin{cases} 1 & \text{if } (x,y) = (0,0), \\ \frac{x^2(x^2+3y^2)}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0). \end{cases}$$

If $(x,y) = (0,0)$,

$$(D_1 f)(0,0) = \lim_{t \to 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t \to 0} \frac{t - 0}{t} = 1.$$

If $(x,y) \neq (0,0)$,

$$\begin{aligned}
(D_1 f)(x,y) &= \lim_{t \to 0} \frac{f(x+t,y) - f(x,y)}{t} \\
&= \lim_{t \to 0} \frac{\frac{(x+t)^3}{(x+t)^2+y^2} - \frac{x^3}{x^2+y^2}}{t} \\
&= \lim_{t \to 0} \frac{x^2(x^2+3y^2) + tx(2x^2+3y^2) + t^2(x^2+y^2)}{((x+t)^2+y^2)(x^2+y^2)} \\
&= \frac{x^2(x^2+3y^2)}{(x^2+y^2)^2}.
\end{aligned}$$

(Or differentiate directly.)

(2) *Show that $(D_1 f)(x,y)$ is bounded.* It suffices to show that $(D_1 f)(x,y)$ is bounded if $(x,y) \neq (0,0)$. Write $x = r\cos\theta$ and $y = r\sin\theta$ in the polar coordinates. (Here $r > 0$.) Hence

$$(D_1 f)(x,y) = \frac{x^2(x^2+3y^2)}{(x^2+y^2)^2} = \cos^2\theta(\cos^2\theta + 3\sin^2\theta)$$

is bounded by $1 \cdot (1+3) = 4$.

(3) *Show that*

$$(D_2 f)(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0), \\ \frac{-2x^3 y}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0). \end{cases}$$

If $(x,y) = (0,0)$,

$$(D_2 f)(0,0) = \lim_{t \to 0} \frac{f(0,t) - f(0,0)}{t} = \lim_{t \to 0} \frac{0-0}{t} = 0.$$

If $(x,y) \neq (0,0)$,

$$\begin{aligned} (D_2 f)(x,y) &= \lim_{t \to 0} \frac{f(x,y+t) - f(x,y)}{t} \\ &= \lim_{t \to 0} \frac{\frac{x^3}{x^2+(y+t)^2} - \frac{x^3}{x^2+y^2}}{t} \\ &= \lim_{t \to 0} \frac{-2x^3 y - tx^3}{(x^2+(y+t)^2)(x^2+y^2)} \\ &= \frac{-2x^3 y}{(x^2+y^2)^2}. \end{aligned}$$

(Or differentiate directly.)

(4) *Show that $(D_2 f)(x,y)$ is bounded.* Similar to (2).

(5) *Show that f is continuous.* Apply Exercise 9.7 to (2)(4).

□

Proof of (b).

(1) Write $\mathbf{u} = (u_1, u_2)$. The formula

$$(D_{\mathbf{u}} f)(0,0) = (D_1 f)(0,0) u_1 + (D_2 f)(0,0) u_2 = u_1$$

might be false since we don't know if f is differentiable or not. Actually, we will show that $(D_{\mathbf{u}} f)(0,0) = u_1^3 \neq u_1$.

(2)

$$\begin{aligned} (D_{\mathbf{u}} f)(0,0) &= \lim_{t \to 0} \frac{f(tu_1, tu_2) - f(0,0)}{t} \\ &= \lim_{t \to 0} \frac{\frac{t^3 u_1^3}{t^2 u_1^2 + t^2 u_2^2} - 0}{t} \\ &= \lim_{t \to 0} u_1^3 && (|\mathbf{u}| = 1) \\ &= u_1^3. \end{aligned}$$

Also $|(D_{\mathbf{u}} f)(0,0)| = |u_1|^3 \leq 1$ since $|\mathbf{u}| = 1$.

$\square$

Proof of (c).

(1) Given any $t \in \mathbb{R}^1$.

$$g'(t) = \lim_{x \to t} \frac{g(x) - g(t)}{x - t} = \lim_{x \to t} \frac{f(\gamma(x)) - f(\gamma(t))}{x - t}.$$

Write $\gamma(t) = (\gamma_1(t), \gamma_2(t))$.

(2) Suppose that $\gamma(t) \neq (0,0)$. Since γ is differentiable, γ is continuous. So there exists an open neighborhood $B(t) \subseteq \mathbb{R}^1$ of t such that $\gamma(x) \neq (0,0)$ whenever $x \in B(t)$. Hence

$$\begin{aligned} g'(t) &= \lim_{x \to t} \frac{\frac{\gamma_1(x)^3}{\gamma_1(x)^2 + \gamma_2(x)^2} - \frac{\gamma_1(t)^3}{\gamma_1(t)^2 + \gamma_2(t)^2}}{x - t} \\ &= \frac{d}{dt}\left(\frac{\gamma_1(t)^3}{\gamma_1(t)^2 + \gamma_2(t)^2} \right) \\ &= \frac{3\gamma_1(t)^2 \gamma_1'(t)}{\gamma_1(t)^2 + \gamma_2(t)^2} - \frac{\gamma_1(t)^3 (2\gamma_1(t)\gamma_1'(t) + 2\gamma_2(t)\gamma_2'(t))}{(\gamma_1(t)^2 + \gamma_2(t)^2)^2}. \end{aligned}$$

exists since γ_1 and γ_2 are differentiable.

(3) Suppose that $\gamma(t) = (0,0)$ and thus $\gamma'(t) \neq (0,0)$. So

$$g'(t) = \lim_{x \to t} \frac{f(\gamma(x))}{x - t}$$

Note that $\gamma(x) \neq (0,0)$ in some open neighborhood of t since

$$\lim_{\substack{x \to t \\ \gamma(x) = (0,0)}} \frac{\gamma(x) - \gamma(t)}{x - t} = (0,0),$$

contrary to the assumption that $\gamma'(t) \neq (0,0)$. Note that $\gamma_1(t) = \gamma_2(t) = 0$. So

$$\begin{aligned} g'(t) &= \lim_{x \to t} \frac{f(\gamma(x))}{x - t} \\ &= \lim_{x \to t} \frac{\gamma_1(x)^3}{\gamma_1(x)^2 + \gamma_2(x)^2} \cdot \frac{1}{x - t} \\ &= \lim_{x \to t} \frac{(\gamma_1(x) - \gamma_1(t))^3}{(\gamma_1(x) - \gamma_1(t))^2 + (\gamma_2(x) - \gamma_2(t))^2} \cdot \frac{1}{x - t} \\ &= \lim_{x \to t} \frac{\left(\frac{\gamma_1(x) - \gamma_1(t)}{x - t} \right)^3}{\left(\frac{\gamma_1(x) - \gamma_1(t)}{x - t} \right)^2 + \left(\frac{\gamma_2(x) - \gamma_2(t)}{x - t} \right)^2} \\ &= \frac{\gamma_1'(t)^3}{\gamma_1'(t)^2 + \gamma_2'(t)^2} \end{aligned}$$

since $\gamma'(t) \neq (0,0)$.

(4) By (2)(3), $g'(t)$ exists and

$$g'(t) = \begin{cases} \frac{3\gamma_1(t)^2\gamma_1'(t)}{\gamma_1(t)^2+\gamma_2(t)^2} - \frac{\gamma_1(t)^3(2\gamma_1(t)\gamma_1'(t)+2\gamma_2(t)\gamma_2'(t))}{(\gamma_1(t)^2+\gamma_2(t)^2)^2} & \text{if } \gamma(t) \neq (0,0), \\ \frac{\gamma_1'(t)^3}{\gamma_1'(t)^2+\gamma_2'(t)^2} & \text{if } \gamma(t) = (0,0). \end{cases}$$

(5) Now suppose $\gamma \in \mathscr{C}'$. To show $g' \in \mathscr{C}'$, it suffices to show that

$$\lim_{x \to t} g'(x) = g'(t)$$

if $\gamma(t) = (0,0)$ since $g'(t)$ is always continuous if $\gamma(t) \neq (0,0)$. Here all $\gamma_1, \gamma_2, \gamma_1', \gamma_2'$ are continuous and $\gamma_1(t)^2 + \gamma_2(t)^2 \neq 0$ by assumption. So

$$\begin{aligned} &\lim_{x \to t} \frac{3\gamma_1(x)^2\gamma_1'(x)}{\gamma_1(x)^2+\gamma_2(x)^2} \\ =&\lim_{x \to t} \frac{3\left(\frac{\gamma_1(x)-\gamma_1(t)}{x-t}\right)^2 \gamma_1'(x)}{\left(\frac{\gamma_1(x)-\gamma_1(t)}{x-t}\right)^2 + \left(\frac{\gamma_2(x)-\gamma_2(t)}{x-t}\right)^2} \\ =&\frac{3\gamma_1'(t)^2 \cdot \gamma_1'(t)}{\gamma_1'(t)^2+\gamma_2'(t)^2} \\ =&\frac{3\gamma_1'(t)^3}{\gamma_1'(t)^2+\gamma_2'(t)^2} \end{aligned}$$

and similarly

$$\begin{aligned} &\lim_{x \to t} \frac{\gamma_1(t)^3(2\gamma_1(t)\gamma_1'(t)+2\gamma_2(t)\gamma_2'(t))}{(\gamma_1(t)^2+\gamma_2(t)^2)^2} \\ =&\lim_{x \to t} \frac{\left(\frac{\gamma_1(x)-\gamma_1(t)}{x-t}\right)^3 \left(2\frac{\gamma_1(x)-\gamma_1(t)}{x-t}\gamma_1'(t) + 2\frac{\gamma_2(x)-\gamma_2(t)}{x-t}\gamma_2'(t)\right)}{\left(\left(\frac{\gamma_1(x)-\gamma_1(t)}{x-t}\right)^2 + \left(\frac{\gamma_2(x)-\gamma_2(t)}{x-t}\right)^2\right)^2} \\ =&\frac{\gamma_1'(t)^3 \cdot (2\gamma_1'(t)\gamma_1'(t) + 2\gamma_2'(t)\gamma_2'(t))}{(\gamma_1'(t)^2+\gamma_2'(t)^2)^2} \\ =&\frac{2\gamma_1'(t)^3}{\gamma_1'(t)^2+\gamma_2'(t)^2}. \end{aligned}$$

Hence

$$\lim_{x \to t} g'(x) = \frac{3\gamma_1'(t)^3}{\gamma_1'(t)^2+\gamma_2'(t)^2} - \frac{2\gamma_1'(t)^3}{\gamma_1'(t)^2+\gamma_2'(t)^2} = g'(t).$$

□

Proof of (d). (Reductio ad absurdum) If f were differentiable, then

$$(D_{\mathbf{u}}f)(0,0) = (D_1 f)(0,0)u_1 + (D_2 f)(0,0)u_2 = u_1$$

(Formula (40) in Chapter 9), contrary to (b) if we take $\mathbf{u} = \left(\frac{1}{64}, \frac{\sqrt{4095}}{64}\right)$. □

Exercise 9.15. ...

Proof.

(1)

(2)

□

Exercise 9.16. ...

Proof.

(1)

(2)

□

Exercise 9.17. ...

Proof.

(1)

(2)

□

Exercise 9.18. ...

Proof.

(1)

(2)

□

Exercise 9.19. ...

Proof.

(1)

(2)

□

Exercise 9.20. ...

Proof.

(1)

(2)

□

Exercise 9.21. ...

Proof.

(1)

(2)

□

Exercise 9.22. ...

Proof.

(1)

(2)

□

Exercise 9.23. ...

Proof.

(1)

(2)

□

Exercise 9.24. ...

Proof.

(1)

(2)

□

Exercise 9.25. ...

Proof.

(1)

(2)

□

Exercise 9.26. ...

Proof.

(1)

(2)

□

Exercise 9.27. ...

Proof.

(1)

(2)

□

Exercise 9.28. ...

Proof.

(1)

(2)

□

Exercise 9.29. ...

Proof.

(1)

(2)

□

Exercise 9.30. ...

Proof.

(1)

(2)

□

Exercise 9.31. ...

Proof.

(1)

(2)

□