Chapter 6: Inner Product Spaces

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Section 6.1: Inner Products and Norms

Exercise 6.1.6. *Complete the proof of Theorem 6.1.*

Theorem 6.1 *Let* V *be an inner product space. Then for* $x, y, z, \in \mathsf{V}$ *and* $c \in F$

(a) $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$,

(b) $\langle x, cy\rangle = \overline{c}\langle x, y\rangle$,

(c) $\langle x, x\rangle = 0$ *if and only if* $x = 0$,

(d) if $\langle x, y\rangle = \langle x, z\rangle$ *for all* $x \in \mathsf{V}$, *then* $y = z$.

Proof of (a).

$$\begin{aligned}
\langle x, y + z\rangle &= \overline{\langle y + z, x\rangle} \\
&= \overline{\langle y, x\rangle + \langle z, x\rangle} \\
&= \overline{\langle y, x\rangle} + \overline{\langle z, x\rangle} \\
&= \langle x, y\rangle + \langle x, z\rangle.
\end{aligned}$$

□

Proof of (b).

$$\begin{aligned}
\langle x, cy\rangle &= \overline{\langle cy, x\rangle} \\
&= \overline{c\langle y, x\rangle} \\
&= \overline{c}\overline{\langle y, x\rangle} \\
&= \overline{c}\langle x, y\rangle.
\end{aligned}$$

□

Proof of (c).

(1) ($\Longrightarrow$) If x were nonzero, by the definition of the inner product, $\langle x, x\rangle > 0$, contrary to the assumption. Hence $x = 0$.

(2) ($\Longleftarrow$) Since $0 = 0 + 0$, $\langle 0, 0\rangle = \langle 0 + 0, 0\rangle = \langle 0, 0\rangle + \langle 0, 0\rangle$. Thus $\langle 0, 0\rangle = 0$.

□

Proof of (d).

$$\begin{aligned}
\langle x, y\rangle = \langle x, z\rangle \ \forall x \in \mathsf{V} &\iff 0 = \langle x, y\rangle - \langle x, z\rangle \ \forall x \in \mathsf{V} \\
&\iff 0 = \langle x, y - z\rangle \ \forall x \in \mathsf{V} && \text{((a))} \\
&\implies 0 = \langle y - z, y - z\rangle && \text{(Take } x = y - z \in \mathsf{V}\text{)} \\
&\iff y - z = 0 && \text{((c))} \\
&\iff y = z.
\end{aligned}$$

□