

## Chapter 10: Integration of Differential Forms

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### Exercise 10.1. ...

*Proof.*

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**Exercise 10.2.** For  $i = 1, 2, 3, \dots$ , let  $\varphi_i \in \mathcal{C}(\mathbb{R}^1)$  have support in  $(2^{-i}, 2^{1-i})$ , such that  $\int \varphi_i = 1$ . Put

$$f(x, y) = \sum_{i=1}^{\infty} [\varphi_i(x) - \varphi_{i+1}(x)] \varphi_i(y)$$

Then  $f$  has compact support in  $\mathbb{R}^2$ ,  $f$  is continuous except at  $(0, 0)$ , and

$$\int dy \int f(x, y) dx = 0 \quad \text{but} \quad \int dx \int f(x, y) dy = 1.$$

Observe that  $f$  is unbounded in every neighborhood of  $(0, 0)$ .

*Proof.*

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### Exercise 10.3. ...

*Proof.*

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**Exercise 10.4.** For  $(x, y) \in \mathbb{R}^2$ , define

$$\mathbf{F}(x, y) = (e^x \cos y - 1, e^x \sin y)$$

Prove that  $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$ , where

$$\mathbf{G}_1(x, y) = (e^x \cos y - 1, y)$$

$$\mathbf{G}_2(u, v) = (u, (1 + u) \tan v)$$

are primitive in some neighborhood of  $(0, 0)$ . Compute the Jacobians of  $\mathbf{G}_1$ ,  $\mathbf{G}_2$ ,  $\mathbf{F}$  at  $(0, 0)$ . Define

$$\mathbf{H}_2(x, y) = (x, e^x \sin y)$$

and find

$$\mathbf{H}_1(u, v) = (h(u, v), v)$$

so that  $\mathbf{F} = \mathbf{H}_1 \circ \mathbf{H}_2$  is in some neighborhood of  $(0, 0)$ .

*Proof.*

(1) By Definition 10.5,

$$\mathbf{G}_1(x, y) = (e^x \cos y - 1)\mathbf{e}_1 + y\mathbf{e}_2,$$

$$\mathbf{G}_2(u, v) = u\mathbf{e}_1 + ((1 + u) \tan v)\mathbf{e}_2$$

are primitive in some neighborhood of  $(0, 0)$ .

(2) Show that  $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$ . Given any  $(x, y) \in \mathbb{R}^2$ , we have

$$\begin{aligned} (\mathbf{G}_2 \circ \mathbf{G}_1)(x, y) &= \mathbf{G}_2(\mathbf{G}_1(x, y)) \\ &= \mathbf{G}_2(e^x \cos y - 1, y) \\ &= (e^x \cos y - 1, (1 + (e^x \cos y - 1)) \tan y) \\ &= (e^x \cos y - 1, e^x \sin y) \\ &= \mathbf{F}(x, y). \end{aligned}$$

(3) Since

$$\begin{aligned} J_{\mathbf{G}_1}(x, y) &= \begin{bmatrix} e^x \cos y & -e^x \sin y \\ 0 & 1 \end{bmatrix} \\ J_{\mathbf{G}_2}(x, y) &= \begin{bmatrix} 1 & 0 \\ \tan y & (1 + x) \sec^2 y \end{bmatrix} \\ J_{\mathbf{F}}(x, y) &= \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}, \end{aligned}$$

$$\begin{aligned}
J_{\mathbf{G}_1}(0,0) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
J_{\mathbf{G}_2}(0,0) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
J_{\mathbf{F}}(0,0) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

(4) Define  $h(u, v) = \sqrt{e^{2u} - v^2} - 1$  on

$$B\left((0,0); \frac{1}{64}\right) \subseteq \mathbb{R}^2.$$

$h(u, v)$  is well-defined since  $e^{2u} - v^2 > 0$  for all  $(u, v) \in B\left((0,0); \frac{1}{64}\right)$ .

(5) Given any  $(x, y) \in \mathbb{R}^2$ , we have

$$\begin{aligned}
(\mathbf{H}_1 \circ \mathbf{H}_2)(x, y) &= \mathbf{H}_1(\mathbf{H}_2(x, y)) \\
&= \mathbf{H}_1(x, e^x \sin y) \\
&= (\sqrt{e^{2x} - (e^x \sin y)^2} - 1, e^x \sin y) \\
&= (e^x \cos y - 1, e^x \sin y) \\
&= \mathbf{F}(x, y).
\end{aligned}$$

□

#### Exercise 10.5. ...

*Proof.*

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#### Exercise 10.6. ...

*Proof.*

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#### Exercise 10.7.

- (a) Show that the simplex  $Q^k$  is the smallest convex subset of  $\mathbb{R}^k$  such that contains  $\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_k$ .
- (b) Show that affine mappings take convex sets to convex sets.

*Proof of (a).*

- (1) Show that  $Q^k$  contains  $\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_k$ . Recall

$$Q^k = \{(x_1, \dots, x_k) \in \mathbb{R}^k : x_1 + \dots + x_k \leq 1 \text{ and } x_1, \dots, x_k \geq 0\}$$

(Example 10.14). Hence  $\mathbf{0} = (0, \dots, 0) \in Q^k$  and

$$\mathbf{e}_i = (0, \dots, \underbrace{1}_{i\text{th coordinate}}, \dots, 0) \in Q^k.$$

- (2) Show that  $Q^k$  is a convex subset of  $\mathbb{R}^k$ . Given any  $\mathbf{x} = (x_1, \dots, x_k) \in Q^k$ ,  $\mathbf{y} = (y_1, \dots, y_k) \in Q^k$  and  $0 < \lambda < 1$ . Hence

$$\lambda \mathbf{x} + (1 - \lambda) \mathbf{y} = (\lambda x_1 + (1 - \lambda) y_1, \dots, \lambda x_k + (1 - \lambda) y_k) \in Q^k$$

since each  $\lambda x_i + (1 - \lambda) y_i \geq 0$  and

$$\sum_{i=1}^k (\lambda x_i + (1 - \lambda) y_i) = \lambda \sum_{i=1}^k x_i + (1 - \lambda) \sum_{i=1}^k y_i \leq \lambda + (1 - \lambda) = 1.$$

- (3) Given any convex set  $E \subseteq \mathbb{R}^k$  containing  $\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_k$ . Show that  $E \supseteq Q^k$ .

- (a) Induction on  $k$ . Base case:  $k = 1$ . Given any  $\mathbf{x} = (x_1) \in Q^1$ . We have  $0 \leq x_1 \leq 1$  by the definition of  $Q^1$ . So that  $\mathbf{x} = x_1 \mathbf{e}_1 + (1 - x_1) \mathbf{0} \in E$  since  $\mathbf{0}, \mathbf{e}_1 \in E$  and  $E$  is convex.
- (b) Inductive step: suppose the statement holds for  $k = n$ . Given any  $\mathbf{x} = (x_1, \dots, x_n, x_{n+1}) \in Q^{n+1}$ . If  $x_{n+1} = 1$ , then  $x_1 = \dots = x_n = 0$  by the definition of  $Q^{n+1}$ . So  $\mathbf{x} = \mathbf{e}_{n+1} \in E$  by the assumption of  $E$ . If  $0 \leq x_{n+1} < 1$ , then  $x_1 + \dots + x_n \leq 1 - x_{n+1}$  or

$$\frac{x_1}{1 - x_{n+1}} + \dots + \frac{x_n}{1 - x_{n+1}} \leq 1.$$

So the point

$$\left( \frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}} \right) \in Q^n,$$

or

$$\left( \frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}}, 0 \right), \text{ say } \hat{\mathbf{x}} \in E$$

by the induction hypothesis. Note that  $\mathbf{e}_{n+1} \in E$ . Hence

$$\mathbf{x} = x_{n+1} \mathbf{e}_{n+1} + (1 - x_{n+1}) \hat{\mathbf{x}} \in E$$

by the convexity of  $E$ .

- (c) Conclusion: Since both the base case and the inductive step have been proved as true, by mathematical induction the statement holds.

□

*Proof of (b).*

- (1) Let  $\mathbf{f}$  be an affine mapping that carries a vector space  $X$  into a vector space  $Y$  such that

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{0}) + A\mathbf{x}$$

for some  $A \in L(X, Y)$ .

- (2) Given any convex subset  $C$  of  $X$ . To show that  $\mathbf{f}(C)$  is convex, it suffices to show that

$$\lambda \mathbf{y}_1 + (1 - \lambda) \mathbf{y}_2 \in \mathbf{f}(C)$$

for any  $\mathbf{y}_1, \mathbf{y}_2 \in \mathbf{f}(C)$  and  $0 < \lambda < 1$ . Write  $\mathbf{y}_1 = \mathbf{f}(\mathbf{x}_1)$ ,  $\mathbf{y}_2 = \mathbf{f}(\mathbf{x}_2)$  for some  $\mathbf{x}_1, \mathbf{x}_2 \in C$ . Note that  $\lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2 \in C$  by the convexity of  $C$ . Hence

$$\begin{aligned} & \mathbf{f}(\lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2) \\ &= \mathbf{f}(\mathbf{0}) + A(\lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2) \\ &= \mathbf{f}(\mathbf{0}) + \lambda A\mathbf{x}_1 + (1 - \lambda) A\mathbf{x}_2 & (A \in L(X, Y)) \\ &= \lambda(\mathbf{f}(\mathbf{0}) + A\mathbf{x}_1) + (1 - \lambda)(\mathbf{f}(\mathbf{0}) + A\mathbf{x}_2) \\ &= \lambda \mathbf{f}(\mathbf{x}_1) + (1 - \lambda) \mathbf{f}(\mathbf{x}_2) \\ &= \lambda \mathbf{y}_1 + (1 - \lambda) \mathbf{y}_2 \in \mathbf{f}(C). \end{aligned}$$

□

**Exercise 10.8.** Let  $H$  be the parallelogram in  $\mathbb{R}^2$  whose vertices are  $(1, 1)$ ,  $(3, 2)$ ,  $(4, 5)$ ,  $(2, 4)$ . Find the affine map  $T$  which sends  $(0, 0)$  to  $(1, 1)$ ,  $(1, 0)$  to  $(3, 2)$ ,  $(1, 1)$  to  $(4, 5)$ ,  $(0, 1)$  to  $(2, 4)$ . Show that  $J_T = 5$ . Use  $T$  to convert the integral

$$\alpha = \int_H e^{x-y} dx dy$$

to an integral over  $I^2$  and thus compute  $\alpha$ .

*Proof.*

- (1) By Affine simplexes 10.26,

$$T(\mathbf{x}) = T(\mathbf{0}) + A\mathbf{x},$$

where  $A \in L(\mathbb{R}^2, \mathbb{R}^2)$ , say  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Note that  $T : \begin{bmatrix} 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . Thus

$$T : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 + ax + by \\ 1 + cx + dy \end{bmatrix}.$$

(2) By  $T : (1, 0) \mapsto (3, 2)$  and  $T : (0, 1) \mapsto (2, 4)$ , we can solve  $A$  as

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

It is easy to verify such

$$T : \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{x}} \mapsto \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{T(\mathbf{0})} + \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{x}} = \begin{bmatrix} 1 + 2x + y \\ 1 + x + 3y \end{bmatrix}$$

satisfying our requirement.

(3)

$$J_T = \det \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = 5.$$

(4)

$$\begin{aligned} \int_H e^{x-y} dx dy &= \int_{[0,1]^2} e^{(1+2u+v)-(1+u+3v)} |J_T| du dv \\ &= 5 \int_{[0,1]^2} e^{u-2v} du dv \\ &= 5 \left\{ \int_0^1 e^u du \right\} \left\{ \int_0^1 e^{-2v} dv \right\} \quad (\text{Theorem 10.2}) \\ &= \frac{5}{2} (e-1)(1-e^{-2}). \end{aligned}$$

□

#### Exercise 10.9. ...

*Proof.*

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(2)

□

#### Exercise 10.10. ...

*Proof.*

(1)

(2)

□

**Exercise 10.11. ...**

*Proof.*

(1)

(2)

□

**Exercise 10.12. ...**

*Proof.*

(1)

(2)

□

**Exercise 10.13. ...**

*Proof.*

(1)

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□

**Exercise 10.14 (Levi-Civita symbol).** *Prove  $\varepsilon(j_1, \dots, j_k) = s(j_1, \dots, j_k)$ , where*

$$s(j_1, \dots, j_k) = \prod_{p < q} \text{sgn}(j_q - j_p).$$

It is usually to define the Levi-Civita symbol by

$$\varepsilon(j_1, \dots, j_k) = \begin{cases} 1 & \text{if } (j_1, \dots, j_k) \text{ is an even permutation of } J, \\ -1 & \text{if } (j_1, \dots, j_k) \text{ is an odd permutation of } J, \\ 0 & \text{otherwise} \end{cases}$$

(Basic  $k$ -forms 10.14). Thus, it is the sign of the permutation in the case of a permutation, and zero otherwise. So  $\varepsilon(j_1, \dots, j_k)$  is equivalent to an explicit expression  $s(j_1, \dots, j_k) = \prod_{p < q} \text{sgn}(j_q - j_p)$ .

*Proof.*

(1) Induction on  $k$ . Base case: *Show that  $\varepsilon(j_1, j_2) = s(j_1, j_2)$ .* Since

$$\varepsilon(j_1, j_2) = \begin{cases} 1 & \text{if } j_1 < j_2 \\ -1 & \text{if } j_1 > j_2, \end{cases}$$

$$\varepsilon(j_1, j_2) = \text{sgn}(j_2 - j_1) = s(j_1, j_2).$$

(2) Inductive step: *Show that for any  $s \geq 2$ , if  $\varepsilon(j_1, \dots, j_s) = s(j_1, \dots, j_s)$  holds, then  $\varepsilon(j_1, \dots, j_{s+1}) = s(j_1, \dots, j_{s+1})$  also holds.*

$$\begin{aligned} \varepsilon(j_1, \dots, j_{s+1}) &= \varepsilon(j_1, \dots, j_s) \prod_{\substack{1 \leq p \leq s \\ q=s+1}} \text{sgn}(j_q - j_p) \\ &= s(j_1, \dots, j_s) \prod_{\substack{1 \leq p \leq s \\ q=s+1}} \text{sgn}(j_q - j_p) \\ &= \prod_{1 \leq p < q \leq s} \text{sgn}(j_q - j_p) \prod_{\substack{1 \leq p \leq s \\ q=s+1}} \text{sgn}(j_q - j_p) \\ &= \prod_{1 \leq p < q \leq s+1} \text{sgn}(j_q - j_p) \\ &= s(j_1, \dots, j_{s+1}). \end{aligned}$$

(3) Conclusion: Since both the base case and the inductive step have been proved as true, by mathematical induction the statement holds for every integer  $k \geq 2$ .

□

**Exercise 10.15.** *If  $\omega$  and  $\lambda$  are  $k$ - and  $m$ -forms, respectively, prove that*

$$\omega \wedge \lambda = (-1)^{km} \lambda \wedge \omega.$$

*Proof.*

(1) Write

$$\omega = \sum_I b_I(\mathbf{x}) dx_I, \quad \lambda = \sum_J c_J(\mathbf{x}) dx_J$$

in the standard presentations, where  $I$  and  $J$  range over all increasing  $k$ -indices and over all increasing  $m$ -indices taken from the set  $\{1, \dots, n\}$ .



(2) Show that  $dx_I \wedge dx_J = (-1)^{km} dx_J \wedge dx_I$ .

$$\begin{aligned}
dx_I \wedge dx_J &= dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_J \\
&= (-1)^m dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}} \wedge dx_J \wedge dx_{i_k} \\
&= (-1)^{2m} dx_{i_1} \wedge \cdots \wedge dx_{i_{k-2}} \wedge dx_J \wedge dx_{i_{k-1}} \wedge dx_{i_k} \\
&\quad \dots \\
&= (-1)^{km} dx_J \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&= (-1)^{km} dx_J \wedge dx_I.
\end{aligned}$$

(3)

$$\begin{aligned}
\omega \wedge \lambda &= \sum_{I,J} b_I(\mathbf{x}) c_J(\mathbf{x}) dx_I \wedge dx_J \\
&= (-1)^{km} \sum_{J,I} c_J(\mathbf{x}) b_I(\mathbf{x}) dx_J \wedge dx_I \\
&= (-1)^{km} \lambda \wedge \omega.
\end{aligned}$$

□

**Exercise 10.16.** If  $k \geq 2$  and  $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_k]$  is an oriented affine  $k$ -simplex, prove that  $\partial^2 \sigma = 0$ , directly from the definition of the boundary operator  $\partial$ . Deduce from this that  $\partial^2 \Psi = 0$  for every chain  $\Psi$ . (Hint: For orientation, do it first for  $k = 2$ ,  $k = 3$ . In general, if  $i < j$ , let  $\sigma_{ij}$  be the  $(k-2)$ -simplex obtained by deleting  $\mathbf{p}_i$  and  $\mathbf{p}_j$  from  $\sigma$ . Show that each  $\sigma_{ij}$  occurs twice in  $\partial^2 \sigma$ , with opposite sign.)

*Proof (Brute-force).*

- (1) Write the boundary of the oriented affine  $k$ -simplex  $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_k]$  as

$$\partial \sigma = \sum_{i=0}^k (-1)^i [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k]$$

where where the oriented  $(k-1)$ -simplex  $[\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k]$  is obtained by deleting  $\sigma$ 's  $i$ -th vertex (Boundaries 10.29).

(2)

$$\begin{aligned}
\partial^2 \sigma &= \partial \left( \sum_i (-1)^i [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k] \right) \\
&= \sum_i (-1)^i \partial [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k] \\
&= \sum_{j < i} (-1)^i (-1)^j [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_j, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k] \\
&\quad + \sum_{j > i} (-1)^i (-1)^{j-1} [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \widehat{\mathbf{p}}_j, \dots, \mathbf{p}_k] \\
&= \sum_{j < i} (-1)^{i+j} [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_j, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k] \\
&\quad - \sum_{j > i} (-1)^{i+j} [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \widehat{\mathbf{p}}_j, \dots, \mathbf{p}_k].
\end{aligned}$$

The latter two summations cancel since after switching  $i$  and  $j$  in the second sum. Therefore  $\partial^2 \sigma = 0$ .

- (3) The boundary of a chain is the linear combination of boundaries of the simplices in the chain. Write  $\Psi = \sum_{i=1}^r \sigma_i$ . where  $\sigma_i$  is an oriented affine simplex. Then

$$\partial^2 \Psi = \partial \left( \partial \sum \sigma_i \right) = \partial \left( \sum \partial \sigma_i \right) = \sum \partial^2 \sigma_i = \sum 0 = 0$$

for any affine chain  $\Psi$ .

□

#### Exercise 10.17. ...

*Proof.*

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(2)

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#### Exercise 10.18. ...

*Proof.*

(1)

(2)

□

**Exercise 10.19. ...**

*Proof.*

(1)

(2)

□

**Exercise 10.20. ...**

*Proof.*

(1)

(2)

□

**Exercise 10.21. ...**

*Proof.*

(1)

(2)

□

**Exercise 10.22. ...**

*Proof.*

(1)

(2)

□

**Exercise 10.23. ...**

*Proof.*

(1)

(2)

□

**Exercise 10.24. ...**

*Proof.*

(1)

(2)

□

**Exercise 10.25. ...**

*Proof.*

(1)

(2)

□

**Exercise 10.26. ...**

*Proof.*

(1)

(2)

□

**Exercise 10.27. ...**

*Proof.*

(1)

(2)

□

**Exercise 10.28.** ...

*Proof.*

(1)

(2)

□

**Exercise 10.29.** ...

*Proof.*

(1)

(2)

□

**Exercise 10.30.** If  $\mathbf{N}$  is the vector given by

$$\mathbf{N} = (\alpha_2\beta_3 - \alpha_3\beta_2)\mathbf{e}_1 + (\alpha_3\beta_1 - \alpha_1\beta_3)\mathbf{e}_2 + (\alpha_1\beta_2 - \alpha_2\beta_1)\mathbf{e}_3$$

(Equation (135)), prove that

$$\det \begin{bmatrix} \alpha_1 & \beta_1 & \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_2 & \beta_2 & \alpha_3\beta_1 - \alpha_1\beta_3 \\ \alpha_3 & \beta_3 & \alpha_1\beta_2 - \alpha_2\beta_1 \end{bmatrix} = |\mathbf{N}|^2$$

Also, verify

$$\mathbf{N} \cdot (T\mathbf{e}_1) = \mathbf{N} \cdot (T\mathbf{e}_2)$$

(Equation (137)).

*Proof.*

(1) By Laplace's expansion along the third column,

$$\begin{aligned}
& \det \begin{bmatrix} \alpha_1 & \beta_1 & \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_2 & \beta_2 & \alpha_3\beta_1 - \alpha_1\beta_3 \\ \alpha_3 & \beta_3 & \alpha_1\beta_2 - \alpha_2\beta_1 \end{bmatrix} \\
&= (-1)^{1+3}(\alpha_2\beta_3 - \alpha_3\beta_2) \det \begin{bmatrix} \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} \\
&\quad + (-1)^{2+3}(\alpha_3\beta_1 - \alpha_1\beta_3) \det \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_3 & \beta_3 \end{bmatrix} \\
&\quad + (-1)^{3+3}(\alpha_1\beta_2 - \alpha_2\beta_1) \det \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix} \\
&= (\alpha_2\beta_3 - \alpha_3\beta_2)^2 + (\alpha_3\beta_1 - \alpha_1\beta_3)^2 + (\alpha_1\beta_2 - \alpha_2\beta_1)^2 \\
&= |\mathbf{N}|^2.
\end{aligned}$$

(2)

$$\begin{aligned}
\mathbf{N} \cdot (T\mathbf{e}_1) &= (\alpha_2\beta_3 - \alpha_3\beta_2, \alpha_3\beta_1 - \alpha_1\beta_3, \alpha_1\beta_2 - \alpha_2\beta_1) \cdot (\alpha_1, \alpha_2, \alpha_3) \\
&= (\alpha_2\beta_3 - \alpha_3\beta_2)\alpha_1 + (\alpha_3\beta_1 - \alpha_1\beta_3)\alpha_2 + (\alpha_1\beta_2 - \alpha_2\beta_1)\alpha_3 \\
&= (\alpha_3\alpha_2 - \alpha_2\alpha_3)\beta_1 + (\alpha_1\alpha_3 - \alpha_3\alpha_1)\beta_2 + (\alpha_2\alpha_1 - \alpha_1\alpha_2)\beta_3 \\
&= 0.
\end{aligned}$$

(3)

$$\begin{aligned}
\mathbf{N} \cdot (T\mathbf{e}_2) &= (\alpha_2\beta_3 - \alpha_3\beta_2, \alpha_3\beta_1 - \alpha_1\beta_3, \alpha_1\beta_2 - \alpha_2\beta_1) \cdot (\beta_1, \beta_2, \beta_3) \\
&= (\alpha_2\beta_3 - \alpha_3\beta_2)\beta_1 + (\alpha_3\beta_1 - \alpha_1\beta_3)\beta_2 + (\alpha_1\beta_2 - \alpha_2\beta_1)\beta_3 \\
&= (\beta_2\beta_3 - \beta_3\beta_2)\alpha_1 + (\beta_3\beta_1 - \beta_1\beta_3)\alpha_2 + (\beta_1\beta_2 - \beta_2\beta_1)\alpha_3 \\
&= 0.
\end{aligned}$$

□

**Exercise 10.31.** ...

*Proof.*

(1)

(2)

□

**Exercise 10.32.** ...

*Proof.*

(1)

(2)

□