

Chapter 1: Curves

Author: Meng-Gen Tsai

Email: plover@gmail.com

Section 1-1: Introduction

Classical differential geometry: the study of local properties of curves and surfaces.

Global differential geometry: the study of the influence of the local properties on the behavior of the entire curve and surface.

No exercises.

Section 1-2: Parametrized Curves

Exercise 1-2.1. Find a parametrized curve $\alpha(t)$ whose trace is the circle $x^2 + y^2 = 1$ such that $\alpha(t)$ runs clockwise around the circle with $\alpha(0) = (0, 1)$.

Proof. $\alpha(t) = (\sin t, \cos t)$, $t \in \mathbb{R}$. \square

Exercise 1-2.2. Let $\alpha(t)$ be a parametrized curve which does not pass through the origin. If $\alpha(t_0)$ is the point of the trace of α closest to the origin and $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

Proof. Let $f(t) = |\alpha(t)|^2 = \alpha(t) \cdot \alpha(t)$. $f(t)$ is differentiable and $f(t)$ has a local minimum at a point $t = t_0 \in I$. So $f'(t_0) = 0$. [Theorem 5.8 in W. Rudin, Principles of Mathematical Analysis, 3rd edition.] Since

$$f'(t) = 2\alpha(t) \cdot \alpha'(t),$$

$f'(t_0) = 2\alpha(t_0) \cdot \alpha'(t_0) = 0$, or $\alpha(t_0) \cdot \alpha'(t_0) = 0$. Since $\alpha(t_0) \neq 0$ and $\alpha'(t_0) \neq 0$, $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$. \square

Exercise 1-2.3. A parametrized curve $\alpha(t)$ has a property that its second derivative $\alpha''(t)$ is identically zero. What can be said about α ?

$\alpha(t)$ is a straight line.

Proof. Since $\alpha''(t)$ is identically zero, $\alpha'(t) = a$ is a constant. [Theorem 5.11 in W. Rudin, Principles of Mathematical Analysis, 3rd edition.] Define

$f(t) = \alpha(t) - at$ (on I). Since $f'(t) = \alpha'(t) - a = 0$, $f(t) = \alpha(t) - at = b$ is a constant again. Therefore, $\alpha(t) = at + b$, which is a straight line (on I). \square

Exercise 1-2.4. Let $\alpha : I \rightarrow \mathbb{R}^3$ be a parametrized curve and let $v \in \mathbb{R}^3$ be a fixed vector. Assume that $\alpha'(t)$ is orthogonal to v for all $t \in I$ and that $\alpha(0)$ is orthogonal to v . Prove that $\alpha(t)$ is orthogonal to v for all $t \in I$.

Need to assume that $\alpha(t) \neq 0$ for all $t \in I$.

Proof. Given any $t \neq 0 \in I$. (Nothing to do at $t = 0$.) Define $f : I \rightarrow \mathbb{R}$ by $f(t) = \alpha(t) \cdot v$. By the mean value theorem, there exists a point ξ between 0 and t such that

$$f(t) - f(0) = f'(\xi)(t - 0),$$

where $f'(t) = \alpha'(t) \cdot v + \alpha(t) \cdot v' = \alpha'(t) \cdot v$. Note that $f(0) = 0$ since $\alpha(0)$ is orthogonal to v , and $f'(\xi) = 0$ since $\alpha'(\xi)$ is orthogonal to v . So the identity is reduced to

$$f(t) = 0,$$

or $\alpha(t) \cdot v = 0$, or $\alpha(t)$ is orthogonal to v . \square

Exercise 1-2.5. Let $\alpha : I \rightarrow \mathbb{R}^3$ be a parametrized curve, with $\alpha'(t) \neq 0$ for all $t \in I$. Show that $|\alpha(t)|$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.

The same trick in Exercise 1-2.2.

Proof. It is equivalent to show that $|\alpha(t)|^2$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$. Let

$$f(t) = |\alpha(t)|^2 = \alpha(t) \cdot \alpha(t).$$

Notice that $\alpha'(t) \neq 0$, and thus

$$\begin{aligned} & |\alpha(t)| \text{ is a nonzero constant} \\ \iff & f(t) = |\alpha(t)|^2 \text{ is a nonzero constant} \\ \iff & f'(t) = 0 \text{ and } f(t) \text{ is a nonzero constant} \\ \iff & \alpha(t) \cdot \alpha'(t) = 0 \text{ and } \alpha(t) \text{ is a nonzero constant} \\ \iff & \alpha(t) \text{ is orthogonal to } \alpha'(t) \text{ for all } t \in I. \end{aligned}$$

\square

Section 1-3: Regular Curves; Arc Length

Exercise 1-3.10. (*Straight Lines as Shortest.*) Let $\alpha : I \rightarrow \mathbb{R}^3$ be a parametrized curve. Let $[a, b] \subseteq I$ and set $\alpha(a) = p$, $\alpha(b) = q$.

(a) Show that, for any constant vector v , $|v| = 1$,

$$(q - p) \cdot v = \int_a^b \alpha'(t) \cdot v dt \leq \int_a^b |\alpha'(t)| dt.$$

(b) Set

$$v = \frac{q - p}{|q - p|}$$

and show that

$$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)| dt;$$

that is, the curve of shortest length from $\alpha(a)$ to $\alpha(b)$ is the straight line joining these points.

Assume $p \neq q$ (otherwise $v = \frac{q-p}{|q-p|}$ is meaningless).

Proof of (a). Let $f(t) = \alpha(t) \cdot v$ defined on I . By the fundamental theorem of calculus,

$$\int_a^b f'(t) dt = f(b) - f(a).$$

Since $f'(t) = \alpha'(t) \cdot v$,

$$(\alpha(b) - \alpha(a)) \cdot v = \int_a^b \alpha'(t) \cdot v dt.$$

Therefore,

$$\begin{aligned} (q - p) \cdot v &= \int_a^b \alpha'(t) \cdot v dt \\ &\leq \int_a^b |\alpha'(t) \cdot v| dt \\ &\leq \int_a^b |\alpha'(t)| |v| dt \\ &= \int_a^b |\alpha'(t)| dt. \end{aligned}$$

□

Proof of (b). $|v| = \frac{|q-p|}{|q-p|} = 1$. So,

$$(q-p) \cdot \frac{q-p}{|q-p|} \leq \int_a^b |\alpha'(t)| dt,$$

$$|q-p| \leq \int_a^b |\alpha'(t)| dt.$$

□

Section 1-4: The Vector Product in \mathbb{R}^3

Exercise 1-4.13. Let $u(t) = (u_1(t), u_2(t), u_3(t))$ and $v(t) = (v_1(t), v_2(t), v_3(t))$ be differentiable maps from the interval (a, b) into \mathbb{R}^3 . If the derivatives $u'(t)$ and $v'(t)$ satisfy the conditions

$$u'(t) = au(t) + bv(t), v'(t) = cu(t) - av(t),$$

where a , b , and c are constants, show that $u(t) \wedge v(t)$ is a constant vector.

Proof. Since

$$\begin{aligned} \frac{d}{dt}(u(t) \wedge v(t)) &= u'(t) \wedge v(t) + u(t) \wedge v'(t) \\ &= (au(t) + bv(t)) \wedge v(t) + u(t) \wedge (cu(t) - av(t)) \\ &= au(t) \wedge v(t) + u(t) \wedge (-av(t)) \\ &= a(u(t) \wedge v(t)) + (-a)(u(t) \wedge v(t)) \\ &= (0, 0, 0), \end{aligned}$$

$u(t) \wedge v(t)$ is a constant vector. □

Section 1-5: The Local Theory of Curves Parametrized by Arc Length

Exercise 1-5.2. Show that the torsion τ of α is given by

$$\tau(s) = -\frac{\alpha'(s) \wedge \alpha''(s) \cdot \alpha'''(s)}{|\kappa(s)|^2}.$$

Proof.

- (1) Take inner product $n(s)$ to the definition of torsion $\tau(s)n(s) = b'(s)$, we have

$$\tau(s) = b'(s) \cdot n(s).$$

Since $b'(s) = t(s) \wedge n'(s)$, we have to compute $n'(s)$ first.

(2) Compute $n'(s)$.

$$n'(s) = \frac{d}{ds} \left(\frac{\alpha''(s)}{\kappa(s)} \right) = \frac{\alpha'''(s)}{\kappa(s)} - \frac{\alpha''(s)\kappa'(s)}{\kappa(s)^2}.$$

(3) By (1)(2),

$$\begin{aligned} \tau(s) &= b'(s) \cdot n(s) \\ &= (t(s) \wedge n'(s)) \cdot n(s) \\ &= \left(\alpha'(s) \wedge \left(\frac{\alpha'''(s)}{\kappa(s)} - \frac{\alpha''(s)\kappa'(s)}{\kappa(s)^2} \right) \right) \cdot \frac{\alpha''(s)}{\kappa(s)} \\ &= \left(\alpha'(s) \wedge \frac{\alpha'''(s)}{\kappa(s)} \right) \cdot \frac{\alpha''(s)}{\kappa(s)} \\ &= \frac{\alpha'(s) \wedge \alpha'''(s) \cdot \alpha''(s)}{|\kappa(s)|^2}, \end{aligned}$$

or

$$\tau(s) = \frac{\alpha'(s) \wedge \alpha'''(s) \cdot \alpha''(s)}{\alpha''(s)^2}.$$

□

Section 1-6: The Local Canonical Form

Section 1-7: Global Properties of Plane Curves