Chapter 10: Integration of Differential Forms

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Exercise 10.1. *...*

Proof.

(1)

(2)

□

Exercise 10.2. *For $i = 1, 2, 3, \ldots$, let $\varphi_i \in \mathscr{C}(\mathbb{R}^1)$ have support in $(2^{-i}, 2^{1-i})$, such that $\int \varphi_i = 1$. Put*

$$f(x, y) = \sum_{i=1}^{\infty} [\varphi_i(x) - \varphi_{i+1}(x)]\varphi_i(y)$$

Then f has compact support in $\mathbb{R}^2$, f is continuous except at $(0, 0)$, and

$$\int dy \int f(x, y) dx = 0 \qquad \text{but} \qquad \int dx \int f(x, y) dy = 1.$$

Observe that f is unbounded in every neighborhood of $(0, 0)$.

Proof.

(1)

(2)

□

Exercise 10.3. *...*

Proof.

(1)

(2)

$\square$

Exercise 10.4. *For* $(x, y) \in \mathbb{R}^2$, *define*

$$\mathbf{F}(x, y) = (e^x \cos y - 1, e^x \sin y)$$

Prove that $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$, *where*

$$\begin{aligned} \mathbf{G}_1(x, y) &= (e^x \cos y - 1, y) \\ \mathbf{G}_2(u, v) &= (u, (1 + u) \tan v) \end{aligned}$$

are primitive in some neighborhood of $(0, 0)$. *Compute the Jacobians of* $\mathbf{G}_1$, $\mathbf{G}_2$, $\mathbf{F}$ *at* $(0, 0)$. *Define*

$$\mathbf{H}_2(x, y) = (x, e^x \sin y)$$

and find

$$\mathbf{H}_1(u, v) = (h(u, v), v)$$

so that $\mathbf{F} = \mathbf{H}_1 \circ \mathbf{H}_2$ *is in some neighborhood of* $(0, 0)$.

Proof.

(1) By Definition 10.5,

$$\begin{aligned} \mathbf{G}_1(x, y) &= (e^x \cos y - 1)\mathbf{e}_1 + y\mathbf{e}_2, \\ \mathbf{G}_2(u, v) &= u\mathbf{e}_1 + ((1 + u) \tan v)\mathbf{e}_2 \end{aligned}$$

are primitive in some neighborhood of $(0, 0)$.

(2) *Show that* $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$. Given any $(x, y) \in \mathbb{R}^2$, we have

$$\begin{aligned} (\mathbf{G}_2 \circ \mathbf{G}_1)(x, y) &= \mathbf{G}_2(\mathbf{G}_1(x, y)) \\ &= \mathbf{G}_2(e^x \cos y - 1, y) \\ &= (e^x \cos y - 1, (1 + (e^x \cos y - 1)) \tan y) \\ &= (e^x \cos y - 1, e^x \sin y) \\ &= \mathbf{F}(x, y). \end{aligned}$$

(3) Since

$$\begin{aligned} J_{\mathbf{G}_1}(x, y) &= \begin{bmatrix} e^x \cos y & -e^x \sin y \\ 0 & 1 \end{bmatrix} \\ J_{\mathbf{G}_2}(x, y) &= \begin{bmatrix} 1 & 0 \\ \tan y & (1 + x) \sec^2 y \end{bmatrix} \\ J_{\mathbf{F}}(x, y) &= \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}, \end{aligned}$$

$$J_{\mathbf{G}_1}(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$J_{\mathbf{G}_2}(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$J_{\mathbf{F}}(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(4) Define $h(u,v) = \sqrt{e^{2u} - v^2} - 1$ on
$$B\left((0,0); \frac{1}{64}\right) \subseteq \mathbb{R}^2.$$
$h(u,v)$ is well-defined since $e^{2u} - v^2 > 0$ for all $(u,v) \in B\left((0,0); \frac{1}{64}\right)$.

(5) Given any $(x,y) \in \mathbb{R}^2$, we have
$$\begin{aligned}(\mathbf{H}_1 \circ \mathbf{H}_2)(x,y) &= \mathbf{H}_1(\mathbf{H}_2(x,y)) \\ &= \mathbf{H}_1(x, e^x \sin y) \\ &= (\sqrt{e^{2x} - (e^x \sin y)^2} - 1, e^x \sin y) \\ &= (e^x \cos y - 1, e^x \sin y) \\ &= \mathbf{F}(x,y).\end{aligned}$$

$\square$

Exercise 10.5. *Formulate and prove an analogue of Theorem 10.8, in which K is a compact subset of an arbitrary metric space. (Replace the functions φ_i that occur in the proof of Theorem 10.8 by functions of the type constructed in Exercise 4.22.)*

Proof (Theorem 10.8).

(1) *(Partitions of unity.) Suppose K is a compact subset of a metric space X, and $\{V_\alpha\}$ is an open cover of K. Then there exist functions $\psi_1, \ldots, \psi_s \in \mathscr{C}(X)$ such that*

  (a) $0 \leq \psi_i \leq 1$ *for* $1 \leq i \leq s$.

  (b) *each* ψ_i *has its support in some* V_α, *and*

  (c) $\psi_1(x) + \cdots + \psi_s(x) = 1$ *for every* $x \in K$.

(2) It is trivial that some $V_\alpha = X$ by taking $s = 1$ and $\psi_1(x) = 1 \in \mathscr{C}(X)$. Now we assume that all $V_\alpha \subsetneq X$.

(3) Associate with each $x \in K$ an index $\alpha(x)$ so that $x \in V_{\alpha(x)}$. Then there are open balls $B(x)$ and $W(x)$, centered at x, with
$$x \in B(x) \subseteq \overline{B(x)} \subseteq W(x) \subseteq \overline{W(x)} \subseteq V_{\alpha(x)}$$

(Since $V_{\alpha(x)}$ is open, there exists $r > 0$ such that $B(x; r) \subseteq V_{\alpha(x)}$. Take $B(x) = B\left(x; \frac{r}{89}\right)$ and $W(x) = B\left(x; \frac{r}{64}\right)$.)

(4) Since K is compact, there are finitely many points $x_1, \ldots, x_s \in K$ such that

$$K \subseteq B(x_1) \cup \cdots \cup B(x_s).$$

Note that

(a) $\overline{B(x_i)}$ is a nonempty closed set since $x_i \in B(x_i) \subseteq \overline{B(x_i)}$.

(b) $X - W(x_i) \supseteq X - V_{\alpha(x_i)}$ is a nonempty closed set by the assumption in (2).

(c) $\overline{B(x_i)} \cap (X - W(x_i)) \subseteq W(x_i) \cap (X - W(x_i)) = \varnothing$.

By Exercise 4.22, there is a function

$$\varphi_i(x) = \frac{\rho_{\overline{B(x_i)}}(x)}{\rho_{\overline{B(x_i)}}(x) + \rho_{X - W(x_i)}(x)} \in \mathscr{C}(X)$$

such that $\varphi_i(x) = 1$ on $\overline{B(x_i)}$, $\varphi_i(x) = 0$ outside $W(x_i)$, and $0 \leq \varphi_i(x) \leq 1$ on X for $1 \leq i \leq s$.

(5) Define $\psi_1 = \varphi_1 \in \mathscr{C}(X)$ and

$$\psi_{i+1} = (1 - \varphi_1) \cdots (1 - \varphi_i)\varphi_{i+1} \in \mathscr{C}(X)$$

for $1 \leq i \leq s - 1$. Properties (a) and (b) in (1) are clear. Also,

$$\psi_1(x) + \cdots + \psi_s(x) = 1 - (1 - \varphi_1(x)) \cdots (1 - \varphi_s(x))$$

by the construction of ψ_i. If $x \in K$, then $x \in B(x_i)$ for some i, hence $\varphi_i(x) = 1$, and the product $(1 - \varphi_1(x)) \cdots (1 - \varphi_s(x)) = 0$. This proves property (c) in (1).

$\square$

Exercise 10.6. *...*

Proof.

(1)

(2)

$\square$

Exercise 10.7.

(a) *Show that the simplex Q^k is the smallest convex subset of $\mathbb{R}^k$ such that contains $\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_k$.*

(b) *Show that affine mappings take convex sets to convex sets.*

Proof of (a).

(1) *Show that Q^k contains $\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_k$.* Recall

$$Q^k = \{(x_1, \dots, x_k) \in \mathbb{R}^k : x_1 + \cdots + x_k \leq 1 \text{ and } x_1, \dots, x_k \geq 0\}$$

(Example 10.14). Hence $\mathbf{0} = (0, \dots, 0) \in Q^k$ and

$$\mathbf{e}_i = (0, \dots, \underbrace{1}_{i\text{th coordinate}}, \dots, 0) \in Q^k.$$

(2) *Show that Q^k is a convex subset of $\mathbb{R}^k$.* Given any $\mathbf{x} = (x_1, \dots, x_k) \in Q^k$, $\mathbf{y} = (y_1, \dots, y_k) \in Q^k$ and $0 < \lambda < 1$. Hence

$$\lambda\mathbf{x} + (1-\lambda)\mathbf{y} = (\lambda x_1 + (1-\lambda)y_1, \dots, \lambda x_k + (1-\lambda)y_k) \in Q^k$$

since each $\lambda x_i + (1-\lambda)y_i \geq 0$ and

$$\sum_{i=1}^{k}(\lambda x_i + (1-\lambda)y_i) = \lambda \sum_{i=1}^{k} x_i + (1-\lambda)\sum_{i=1}^{k} y_i \leq \lambda + (1-\lambda) = 1.$$

(3) *Given any convex set $E \subseteq \mathbb{R}^k$ containing $\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_k$. Show that $E \supseteq Q^k$.*

(a) Induction on k. Base case: $k = 1$. Given any $\mathbf{x} = (x_1) \in Q^1$. We have $0 \leq x_1 \leq 1$ by the definition of Q^1. So that $\mathbf{x} = x_1\mathbf{e}_1 + (1-x_1)\mathbf{0} \in E$ since $\mathbf{0}, \mathbf{e}_1 \in E$ and E is convex.

(b) Inductive step: suppose the statement holds for $k = n$. Given any $\mathbf{x} = (x_1, \dots, x_n, x_{n+1}) \in Q^{n+1}$. If $x_{n+1} = 1$, then $x_1 = \cdots = x_n = 0$ by the definition of Q^{n+1}. So $\mathbf{x} = \mathbf{e}_{n+1} \in E$ by the assumption of E. If $0 \leq x_{n+1} < 1$, then $x_1 + \cdots + x_n \leq 1 - x_{n+1}$ or

$$\frac{x_1}{1-x_{n+1}} + \cdots + \frac{x_n}{1-x_{n+1}} \leq 1.$$

So the point

$$\left(\frac{x_1}{1-x_{n+1}}, \dots, \frac{x_n}{1-x_{n+1}}\right) \in Q^n,$$

or

$$\left(\frac{x_1}{1-x_{n+1}}, \dots, \frac{x_n}{1-x_{n+1}}, 0\right), \text{ say } \widehat{\mathbf{x}}, \in E$$

by the induction hypothesis. Note that $\mathbf{e}_{n+1} \in E$. Hence

$$\mathbf{x} = x_{n+1}\mathbf{e}_{n+1} + (1-x_{n+1})\widehat{\mathbf{x}} \in E$$

by the convexity of E.

(c) Conclusion: Since both the base case and the inductive step have been proved as true, by mathematical induction the statement holds.

$\square$

Proof of (b).

(1) Let $\mathbf{f}$ be an affine mapping that carries a vector space X into a vector space Y such that

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{0}) + A\mathbf{x}$$

for some $A \in L(X, Y)$.

(2) Given any convex subset C of X. To show that $\mathbf{f}(C)$ is convex, it suffices to show that

$$\lambda\mathbf{y}_1 + (1-\lambda)\mathbf{y}_2 \in \mathbf{f}(C)$$

for any $\mathbf{y}_1, \mathbf{y}_2 \in \mathbf{f}(C)$ and $0 < \lambda < 1$. Write $\mathbf{y}_1 = \mathbf{f}(\mathbf{x}_1)$, $\mathbf{y}_2 = \mathbf{f}(\mathbf{x}_2)$ for some $\mathbf{x}_1, \mathbf{x}_2 \in C$. Note that $\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \in C$ by the convexity of C. Hence

$$\begin{aligned}
&\mathbf{f}(\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \\
=&\mathbf{f}(\mathbf{0}) + A(\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \\
=&\mathbf{f}(\mathbf{0}) + \lambda A\mathbf{x}_1 + (1-\lambda)A\mathbf{x}_2 && (A \in L(X,Y)) \\
=&\lambda(\mathbf{f}(\mathbf{0}) + A\mathbf{x}_1) + (1-\lambda)(\mathbf{f}(\mathbf{0}) + A\mathbf{x}_2) \\
=&\lambda\mathbf{f}(\mathbf{x}_1) + (1-\lambda)\mathbf{f}(\mathbf{x}_2) \\
=&\lambda\mathbf{y}_1 + (1-\lambda)\mathbf{y}_2 \in \mathbf{f}(C).
\end{aligned}$$

$\square$

Exercise 10.8. *Let H be the parallelogram in $\mathbb{R}^2$ whose vertices are $(1,1)$, $(3,2)$, $(4,5)$, $(2,4)$. Find the affine map T which sends $(0,0)$ to $(1,1)$, $(1,0)$ to $(3,2)$, $(1,1)$ to $(4,5)$, $(0,1)$ to $(2,4)$. Show that $J_T = 5$. Use T to convert the integral*

$$\alpha = \int_H e^{x-y}\,dxdy$$

to an integral over I^2 and thus compute α.

Proof.

(1) By Affine simplexes 10.26,

$$T(\mathbf{x}) = T(\mathbf{0}) + A\mathbf{x},$$

where $A \in L(\mathbb{R}^2, \mathbb{R}^2)$, say $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Note that $T : \begin{bmatrix} 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus

$$T : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 + ax + by \\ 1 + cx + dy \end{bmatrix}.$$

(2) By $T : (1,0) \mapsto (3,2)$ and $T : (0,1) \mapsto (2,4)$, we can solve A as

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

It is easy to verify such

$$T : \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{x}} \mapsto \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{T(\mathbf{0})} + \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{x}} = \begin{bmatrix} 1 + 2x + y \\ 1 + x + 3y \end{bmatrix}$$

satisfying our requirement.

(3)

$$J_T = \det \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = 5.$$

(4)

$$\begin{aligned} \int_H e^{x-y} dxdy &= \int_{[0,1]^2} e^{(1+2u+v)-(1+u+3v)} |J_T| dudv \\ &= 5 \int_{[0,1]^2} e^{u-2v} dudv \\ &= 5 \left\{ \int_0^1 e^u du \right\} \left\{ \int_0^1 e^{-2v} dv \right\} && \text{(Theorem 10.2)} \\ &= \frac{5}{2}(e-1)(1-e^{-2}). \end{aligned}$$

□

Exercise 10.9. ...

Proof.

(1)

(2)

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Exercise 10.10. ...

Proof.

(1)

(2)

□

Exercise 10.11. ...

Proof.

(1)

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Exercise 10.12. ...

Proof.

(1)

(2)

□

Exercise 10.13. ...

Proof.

(1)

(2)

□

Exercise 10.14 (Levi-Civita symbol). *Prove* $\varepsilon(j_1, \dots, j_k) = s(j_1, \dots, j_k)$, *where*

$$s(j_1, \dots, j_k) = \prod_{p<q} \mathrm{sgn}(j_q - j_p).$$

It is usually to define the Levi-Civita symbol by

$$\varepsilon(j_1, \dots, j_k) = \begin{cases} 1 & \text{if } (j_1, \cdots, j_k) \text{ is an even permutation of } J, \\ -1 & \text{if } (j_1, \cdots, j_k) \text{ is an odd permutation of } J, \\ 0 & \text{otherwise} \end{cases}$$

(Basic k-forms 10.14). Thus, it is the sign of the permutation in the case of a permutation, and zero otherwise. So $\varepsilon(j_1, \ldots, j_k)$ is equivalent to an explicit expression $s(j_1, \ldots, j_k) = \prod_{p<q} \operatorname{sgn}(j_q - j_p)$.

Proof.

(1) Induction on k. Base case: *Show that* $\varepsilon(j_1, j_2) = s(j_1, j_2)$. Since

$$\varepsilon(j_1, j_2) = \begin{cases} 1 & \text{if } j_1 < j_2 \\ -1 & \text{if } j_1 > j_2, \end{cases}$$

$\varepsilon(j_1, j_2) = \operatorname{sgn}(j_2 - j_1) = s(j_1, j_2)$.

(2) Inductive step: *Show that for any* $s \geq 2$, *if* $\varepsilon(j_1, \ldots, j_s) = s(j_1, \ldots, j_s)$ *holds, then* $\varepsilon(j_1, \ldots, j_{s+1}) = s(j_1, \ldots, j_{s+1})$ *also holds.*

$$\begin{aligned}
\varepsilon(j_1, \ldots, j_{s+1}) &= \varepsilon(j_1, \ldots, j_s) \prod_{\substack{1 \leq p \leq s \\ q = s+1}} \operatorname{sgn}(j_q - j_p) \\
&= s(j_1, \ldots, j_s) \prod_{\substack{1 \leq p \leq s \\ q = s+1}} \operatorname{sgn}(j_q - j_p) \\
&= \prod_{1 \leq p < q \leq s} \operatorname{sgn}(j_q - j_p) \prod_{\substack{1 \leq p \leq s \\ q = s+1}} \operatorname{sgn}(j_q - j_p) \\
&= \prod_{1 \leq p < q \leq s+1} \operatorname{sgn}(j_q - j_p) \\
&= s(j_1, \ldots, j_{s+1}).
\end{aligned}$$

(3) Conclusion: Since both the base case and the inductive step have been proved as true, by mathematical induction the statement holds for every integer $k \geq 2$.

□

Exercise 10.15. *If ω and λ are k- and m-forms, respectively, prove that*

$$\omega \wedge \lambda = (-1)^{km} \lambda \wedge \omega.$$

Proof.

(1) Write

$$\omega = \sum_I b_I(\mathbf{x}) dx_I, \qquad \lambda = \sum_J c_J(\mathbf{x}) dx_J$$

in the stardard presentations, where I and J range over all increasing k-indices and over all increasing m-indices taken from the set $\{1, \ldots, n\}$.

(2) *Show that* $dx_I \wedge dx_J = (-1)^{km} dx_J \wedge dx_I$.

$$
\begin{aligned}
dx_I \wedge dx_J &= dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_J \\
&= (-1)^m dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}} \wedge dx_J \wedge dx_{i_k} \\
&= (-1)^{2m} dx_{i_1} \wedge \cdots \wedge dx_{i_{k-2}} \wedge dx_J \wedge dx_{i_{k-1}} \wedge dx_{i_k} \\
&\cdots \\
&= (-1)^{km} dx_J \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&= (-1)^{km} dx_J \wedge dx_I.
\end{aligned}
$$

(3)

$$
\begin{aligned}
\omega \wedge \lambda &= \sum_{I,J} b_I(\mathbf{x}) c_J(\mathbf{x}) dx_I \wedge dx_J \\
&= (-1)^{km} \sum_{J,I} c_J(\mathbf{x}) b_I(\mathbf{x}) dx_J \wedge dx_I \\
&= (-1)^{km} \lambda \wedge \omega.
\end{aligned}
$$

□

Exercise 10.16. *If* $k \geq 2$ *and* $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_k]$ *is an oriented affine* k*-simplex, prove that* $\partial^2 \sigma = 0$*, directly from the definition of the boundary operator* ∂*. Deduce from this that* $\partial^2 \Psi = 0$ *for every chain* Ψ*. (Hint: For orientation, do it first for* $k = 2$*,* $k = 3$*. In general, if* $i < j$*, let* σ_{ij} *be the* $(k-2)$*-simplex obtained by deleting* $\mathbf{p}_i$ *and* $\mathbf{p}_j$ *from* σ*. Show that each* σ_{ij} *occurs twice in* $\partial^2 \sigma$*, with opposite sign.)*

Proof (Brute-force).

(1) Write the boundary of the oriented affine k-simplex $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_k]$ as

$$\partial \sigma = \sum_{i=0}^{k} (-1)^i [\mathbf{p}_0, \ldots, \widehat{\mathbf{p}}_i, \ldots, \mathbf{p}_k]$$

where where the oriented $(k-1)$-simplex $[\mathbf{p}_0, \ldots, \widehat{\mathbf{p}}_i, \ldots, \mathbf{p}_k]$ is obtained by deleting σ's i-th vertex (Boundaries 10.29).

(2)

$$
\begin{aligned}
\partial^2\sigma =& \partial\left(\sum_i (-1)^i[\mathbf{p}_0, \ldots, \widehat{\mathbf{p}_i}, \ldots, \mathbf{p}_k]\right) \\
=& \sum_i (-1)^i \partial[\mathbf{p}_0, \ldots, \widehat{\mathbf{p}_i}, \ldots, \mathbf{p}_k] \\
=& \sum_{j<i} (-1)^i(-1)^j[\mathbf{p}_0, \ldots, \widehat{\mathbf{p}_j}, \ldots, \widehat{\mathbf{p}_i}, \ldots, \mathbf{p}_k] \\
&+ \sum_{j>i} (-1)^i(-1)^{j-1}[\mathbf{p}_0, \ldots, \widehat{\mathbf{p}_i}, \ldots, \widehat{\mathbf{p}_j}, \ldots, \mathbf{p}_k] \\
=& \sum_{j<i} (-1)^{i+j}[\mathbf{p}_0, \ldots, \widehat{\mathbf{p}_j}, \ldots, \widehat{\mathbf{p}_i}, \ldots, \mathbf{p}_k] \\
&- \sum_{j>i} (-1)^{i+j}[\mathbf{p}_0, \ldots, \widehat{\mathbf{p}_i}, \ldots, \widehat{\mathbf{p}_j}, \ldots, \mathbf{p}_k].
\end{aligned}
$$

The latter two summations cancel since after switching i and j in the second sum. Therefore $\partial^2\sigma = 0$.

(3) The boundary of a chain is the linear combination of boundaries of the simplices in the chain. Write $\Psi = \sum_{i=1}^r \sigma_i$. where σ_i is an oriented affine simplex. Then

$$\partial^2\Psi = \partial\left(\partial \sum \sigma_i\right) = \partial\left(\sum \partial\sigma_i\right) = \sum \partial^2\sigma_i = \sum 0 = 0$$

for any affine chain Ψ.

□

Exercise 10.17. ...

Proof.

(1)

(2)

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Exercise 10.18. ...

Proof.

(1)

(2)

□

Exercise 10.19. ...

Proof.

(1)

(2)

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Exercise 10.20. ...

Proof.

(1)

(2)

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Exercise 10.21. ...

Proof.

(1)

(2)

□

Exercise 10.22. ...

Proof.

(1)

(2)

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Exercise 10.23. ...

Proof.

(1)

(2)

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Exercise 10.24. ...

Proof.

(1)

(2)

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Exercise 10.25. ...

Proof.

(1)

(2)

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Exercise 10.26. ...

Proof.

(1)

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Exercise 10.27. ...

Proof.

(1)

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Exercise 10.28. ...

Proof.

(1)

(2)

□

Exercise 10.29. ...

Proof.

(1)

(2)

□

Exercise 10.30. *If* $\mathbf{N}$ *is the vector given by*

$$\mathbf{N} = (\alpha_2\beta_3 - \alpha_3\beta_2)\mathbf{e}_1 + (\alpha_3\beta_1 - \alpha_1\beta_3)\mathbf{e}_2 + (\alpha_1\beta_2 - \alpha_2\beta_1)\mathbf{e}_3$$

(Equation (135)), prove that

$$\det \begin{bmatrix} \alpha_1 & \beta_1 & \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_2 & \beta_2 & \alpha_3\beta_1 - \alpha_1\beta_3 \\ \alpha_3 & \beta_3 & \alpha_1\beta_2 - \alpha_2\beta_1 \end{bmatrix} = |\mathbf{N}|^2$$

Also, verify

$$\mathbf{N} \cdot (T\mathbf{e}_1) = \mathbf{N} \cdot (T\mathbf{e}_2)$$

(Equation (137)).

Proof.

(1) By Laplace's expansion along the third column,

$$\begin{aligned}
&\det \begin{bmatrix} \alpha_1 & \beta_1 & \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_2 & \beta_2 & \alpha_3\beta_1 - \alpha_1\beta_3 \\ \alpha_3 & \beta_3 & \alpha_1\beta_2 - \alpha_2\beta_1 \end{bmatrix} \\
=&(-1)^{1+3}(\alpha_2\beta_3 - \alpha_3\beta_2) \det \begin{bmatrix} \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} \\
&+ (-1)^{2+3}(\alpha_3\beta_1 - \alpha_1\beta_3) \det \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_3 & \beta_3 \end{bmatrix} \\
&+ (-1)^{3+3}(\alpha_1\beta_2 - \alpha_2\beta_1) \det \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix} \\
=&(\alpha_2\beta_3 - \alpha_3\beta_2)^2 + (\alpha_3\beta_1 - \alpha_1\beta_3)^2 + (\alpha_1\beta_2 - \alpha_2\beta_1)^2 \\
=&|\mathbf{N}|^2.
\end{aligned}$$

(2)

$$\begin{aligned}
\mathbf{N} \cdot (T\mathbf{e}_1) &= (\alpha_2\beta_3 - \alpha_3\beta_2, \alpha_3\beta_1 - \alpha_1\beta_3, \alpha_1\beta_2 - \alpha_2\beta_1) \cdot (\alpha_1, \alpha_2, \alpha_3) \\
&= (\alpha_2\beta_3 - \alpha_3\beta_2)\alpha_1 + (\alpha_3\beta_1 - \alpha_1\beta_3)\alpha_2 + (\alpha_1\beta_2 - \alpha_2\beta_1))\alpha_3 \\
&= (\alpha_3\alpha_2 - \alpha_2\alpha_3)\beta_1 + (\alpha_1\alpha_3 - \alpha_3\alpha_1)\beta_2 + (\alpha_2\alpha_1 - \alpha_1\alpha_2)\beta_3 \\
&= 0.
\end{aligned}$$

(3)

$$\begin{aligned}
\mathbf{N} \cdot (T\mathbf{e}_2) &= (\alpha_2\beta_3 - \alpha_3\beta_2, \alpha_3\beta_1 - \alpha_1\beta_3, \alpha_1\beta_2 - \alpha_2\beta_1) \cdot (\beta_1, \beta_2, \beta_3) \\
&= (\alpha_2\beta_3 - \alpha_3\beta_2)\beta_1 + (\alpha_3\beta_1 - \alpha_1\beta_3)\beta_2 + (\alpha_1\beta_2 - \alpha_2\beta_1))\beta_3 \\
&= (\beta_2\beta_3 - \beta_3\beta_2)\alpha_1 + (\beta_3\beta_1 - \beta_1\beta_3)\alpha_2 + (\beta_1\beta_2 - \beta_2\beta_1)\alpha_3 \\
&= 0.
\end{aligned}$$

□

Exercise 10.31. ...

Proof.

(1)

(2)

□

Exercise 10.32. ...

Proof.

(1)

(2)

□