Chapter 1: The Real And Complex Number Systems

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Exercise 1.1 *Prove that there is no largest prime. (A proof was known to Euclid.)*

There are many proofs of this result. We provide some of them.

Proof (Due to Euclid). If $p_1, p_2, ..., p_t$ were all primes, then write

$$n = p_1 p_2 \cdots p_t + 1$$

and there were a prime number p dividing n.

(1) p can not be any of $p_i (1 \leq i \leq t)$, otherwise p would divide the difference $n - p_1 p_2 \cdots p_t = 1$.

(2) This prime p is another prime $\neq p_i$ for $1 \leq i \leq t$.

By (2), there exists a prime p other than all primes, which is absurd. □

Proof (Unique factorization theorem). Given N.

(1) *Show that* $\sum_{n \leq N} \frac{1}{n} \leq \prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1}$.
By the unique factorization theorem on $n \leq N$,

$$\sum_{n \leq N} \frac{1}{n} \leq \prod_{p \leq N} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1}.$$

(2) By (1) and the fact that $\sum \frac{1}{n}$ diverges, there are infinitely many primes.

□

Proof (Due to Eckford Cohen).

(1) *ord*$_p n! = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots$. For any $k = 1, 2, ..., n$, we can express k as $k = p^s t$ where $s = \text{ord}_p k$ is a non-negative integer and $(t, p) = 1$. There are $[\frac{n}{p^a}]$ numbers such that $p^a \mid k$ for $a = 1, 2, ...$ Therefore, there are

$$\left[\frac{n}{p^a}\right] - \left[\frac{n}{p^{a+1}}\right]$$

numbers such that $\text{ord}_p k = a$ for $a = 1, 2, ...$ Hence,

$$\begin{aligned}\text{ord}_p n! &= \left(\left[\frac{n}{p}\right] - \left[\frac{n}{p^2}\right]\right) + 2\left(\left[\frac{n}{p^2}\right] - \left[\frac{n}{p^3}\right]\right) + 3\left(\left[\frac{n}{p^3}\right] - \left[\frac{n}{p^4}\right]\right) + \cdots \\ &= \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots .\end{aligned}$$

(2) *$\text{ord}_p n! \leq \frac{n}{p-1}$ and that $n!^{\frac{1}{n}} \leq \prod_{p|n!} p^{\frac{1}{p-1}}$.*

$$\begin{aligned}\text{ord}_p n! &= \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots \\ &\leq \frac{n}{p} + \frac{n}{p^2} + \frac{n}{p^3} + \cdots \\ &= \frac{\frac{n}{p}}{1 - \frac{1}{p}} \\ &= \frac{n}{p-1}.\end{aligned}$$

Thus,

$$n! = \prod_{p|n!} p^{\text{ord}_p n!} \leq \prod_{p|n!} p^{\frac{n}{p-1}} = \left(\prod_{p|n!} p^{\frac{1}{p-1}}\right)^n,$$

or

$$n!^{\frac{1}{n}} \leq \prod_{p|n!} p^{\frac{1}{p-1}}.$$

(3) $(n!)^2 \geq n^n$. Write $(n!)^2 = \prod_{k=1}^n k \prod_{k=1}^n (n+1-k) = \prod_{k=1}^n k(n+1-k)$, and $n^n = \prod_{k=1}^n n$. It suffices to show that $k(n+1-k) \geq n$ for each $1 \leq k \leq n$. Notice that $k(n+1-k) - n = (n-k)(k-1) \geq 0$ for $1 \leq k \leq n$. The inequality holds.

(4) By (3)(4), $\prod_{p|n!} p^{\frac{1}{p-1}} \geq \sqrt{n}$. Assume that there are finitely many primes, the value $\prod_{p|n!} p^{\frac{1}{p-1}}$ is a finite number whenever the value of n. However, $\sqrt{n} \to \infty$ as $n \to \infty$, which leads to a contradiction. Hence there are infinitely many primes.

□

Proof (Formula for $\phi(n)$). If $p_1, p_2, ..., p_t$ were all primes, then let $n = p_1 p_2 \cdots p_t$ and all numbers between 2 and n are NOT relatively prime to n. Thus, $\phi(n) = 1$ by the definition of ϕ. By the formula for ϕ,

$$\begin{aligned}\phi(n) &= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_t}\right) \\ 1 &= (p_1 p_2 \cdots p_t)\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_t}\right) \\ &= (p_1 - 1)(p_2 - 1)\cdots(p_t - 1) > 1,\end{aligned}$$

which is a contradiction (since 3 is a prime). Hence there are infinitely many primes. $\square$