Chapter 2: Linear Transformations and Matrices

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Section 2.4: Invertibility and Isomorphisms

Exercise 2.4.8. *Let A and B be $n \times n$ matrices such that $AB = I_n$. Prove*

(a) *A and B are invertible.*

(b) *$A = B^{-1}$ (and hence $B = A^{-1}$). (We are in effect saying that for square matrices, a "one-sided" inverse is a "two-sided" inverse.)*

(c) *State and prove analogous results for linear transformations defined on finite-dimensional vector spaces.*

Proof of (a). Regard $\mathsf{V} = \mathsf{M}_{n\times n}(F)$ as a finite-dimensional vector space over F. Given $X \in \mathsf{M}_{n\times n}(F)$, consider the subset V_X of V defined by

$$\mathsf{V}_X = \{XY : Y \in \mathsf{M}_{n\times n}(F)\}.$$

(1) $\mathsf{V}_0 = 0$.

(2) $\mathsf{V}_{I_n} = \mathsf{V}$. In general, $\mathsf{V}_X = \mathsf{V}$ for any invertible matrix $X \in \mathsf{M}_{n\times n}(F)$.

(3) V_X is a subspace of V for any $X \in \mathsf{M}_{n\times n}(F)$.

(4) There is a descending sequence of subspaces

$$\mathsf{V} \supseteq \mathsf{V}_X \supseteq \cdots \supseteq \mathsf{V}_{X^k} \supseteq \cdots$$

This sequence must be stationary since V is finite-dimensional, that is,

$$\mathsf{V}_{X^k} = \mathsf{V}_{X^{k+1}} = \cdots$$

for some k. (Descending chain condition.) In particular, $B^k = B^{k+1}C$ for some $C \in \mathsf{V}$. Multiply with A^k on the left to get $I_n = BC$. ($A^kB^k = A^{k-1}(AB)B^{k-1} = A^{k-1}B^{k-1} = \cdots = I_n$.)

(4) Since $AB = I_n$ and $BC = I_n$, $A = AI_n = A(BC) = (AB)C = I_nC = C$, or $AB = BA = I_n$. By definition of invertibility, A and B are invertible.

□

Proof of (b). By (a), $A = B^{-1}$ and $B = A^{-1}$. □

Proof of (c). Let V *be a finite-dimensional vector space, and let* $\mathsf{S}, \mathsf{T} : \mathsf{V} \to \mathsf{V}$ *be linear such that* ST *is invertible. Show that* S *and* T *are invertible.* Let

$$\beta = \{\beta_1, ..., \beta_n\}$$

be an ordered basis for V where $n = \dim(\mathsf{V})$. Let $A = [\mathsf{S}]_\beta$ and $B = [\mathsf{T}]_\beta$. So

$$AB = [\mathsf{S}]_\beta[\mathsf{T}]_\beta = [\mathsf{ST}]_\beta = [\mathsf{I_V}]_\beta = I_n$$

(Theorem 2.11). By (a), $A = [\mathsf{S}]_\beta$ and $B = [\mathsf{T}]_\beta$ are invertible, or S and T are invertible (Theorem 2.18). □

Section 2.7: Homogeneous Linear Differential Equations with Constant Coefficients

Exercise 2.7.3. *Find a basis for the solution space of each of the following differential equations*

(a) $y'' + 2y' + y = 0$

(b) $y''' = y'$

(c) $y^{(4)} - 2y^{(2)} + y = 0$

(d) $y'' + 2y' + y = 0$

(e) $y^{(3)} - y^{(2)} + 3y^{(1)} + 5y = 0.$

Use Theorem 2.35.

Proof of (a). The auxiliary polynomial is $t^2 + ty + 1 = (t+1)^2$. $\{e^{-t}, te^{-t}\}$ is a basis for the solution space. □

Proof of (b). The auxiliary polynomial is $t^3 - t = t(t-1)(t+1)$. $\{1, e^t, e^{-t}\}$ is a basis for the solution space. □

Proof of (c). The auxiliary polynomial is $t^4 - 2t^2 + 1 = (t-1)^2(t+1)^2$. $\{e^t, te^t, e^{-t}, te^{-t}\}$ is a basis for the solution space. □

Proof of (d). Same as (a). □

Proof of (e). The auxiliary polynomial is

$$t^3 - t^2 + 3t + 5 = (t+1)(t-1-2i)(t-1+2i).$$

$\{e^{-t}, e^{(1+2i)t}, e^{(1-2i)t}\}$, or $\{e^{-t}, e^t\cos(2t), e^t\sin(2t)\}$ is a basis for the solution space. □

Exercise 2.7.4. *Find a basis for each of the following subspaces of* C^∞*.*

(a) $\mathsf{N}(\mathsf{D}^2 - \mathsf{D} - \mathsf{I})$

(b) $\mathsf{N}(\mathsf{D}^3 - 3\mathsf{D}^2 + 3\mathsf{D} - \mathsf{I})$

(c) $\mathsf{N}(\mathsf{D}^3 - 6\mathsf{D}^2 - 8\mathsf{D})$

Use Theorem 2.35.

Proof of (a). The auxiliary polynomial is

$$t^2 - t - 1 = \left(t - \frac{1+\sqrt{5}}{2}\right)\left(t - \frac{1-\sqrt{5}}{2}\right).$$

$\left\{e^{\frac{1+\sqrt{5}}{2}t}, e^{\frac{1-\sqrt{5}}{2}t}\right\}$ is a basis for the solution space. $\square$

Proof of (b). The auxiliary polynomial is $t^3-3t^2+3t-1 = (t-1)^3$. $\{e^t, te^t, t^2e^t\}$ is a basis for the solution space. $\square$

Proof of (c). The auxiliary polynomial is $t^3 + 6t^2 + 8t = t(t+2)(t+4)$. $\{1, e^{-2t}, e^{-4t}\}$ is a basis for the solution space. $\square$

Exercise 2.7.5. *Show that* C^∞ *is a subspace of* $\mathcal{F}(\mathbb{R}, \mathbb{C})$*.*

Proof.

(1) $0 \in \mathcal{F}(\mathbb{R}, \mathbb{C})$ clearly.

(2) Given any $f, g \in \mathsf{C}^\infty$. For any nonnegative k, $\mathsf{D}^k(f+g) = \mathsf{D}^k(f) + \mathsf{D}^k(g)$ holds. Thus $f+g \in \mathsf{C}^\infty$.

(3) Given any $f \in \mathcal{F}(\mathbb{R}, \mathbb{C})$, $r \in \mathbb{C}$. For any nonnegative k, $\mathsf{D}^k(cf) = c\mathsf{D}^k(f)$ holds. Thus $cf \in \mathsf{C}^\infty$.

By Theorem 1.3, C^∞ is a subspace. $\square$