Chapter 1: The Real and Complex Number Systems

Unless the contrary is explicitly stated, all numbers that are mentioned in these exercise are understood to be real.

Exercise 1.1. *If r is a rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.*

Proof. Assume $r + x \in \mathbb{Q}$. $\mathbb{Q}$ is a field, then $-r \in \mathbb{Q}$ for any $r \in \mathbb{Q}$. So $(-r) + (r + x) = (-r + r) + x = 0 + x = x \in \mathbb{Q}$, a contradiction.

Similarly, assume $rx \in \mathbb{Q}$. $r \in \mathbb{Q}$ with $r \neq 0$ implies that there exists an element $1/r \in \mathbb{Q}$ such that $r \cdot (1/r) = 1$. So $(1/r) \cdot (rx) = ((1/r) \cdot r) \cdot x = 1 \cdot x = x \in \mathbb{Q}$, a contradiction. □

Exercise 1.12. *If $z_1, ..., z_n$ are complex, prove that*

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$$

Proof. Use mathematical induction on n. $n = 2$ is established by Theorem 1.33 (e). Suppose the inequality holds on $n = k$, then $n = k + 1$ we again apply Theorem 1.33 (e) to get the result, say

$$\begin{aligned} |z_1 + z_2 + \cdots + z_k + z_{k+1}| &\leq |z_1 + z_2 + \cdots + z_k| + |z_{k+1}| \\ &\leq |z_1| + |z_2| + \cdots + |z_k| + |z_{k+1}| \end{aligned}$$

□

Corollary. *If $\mathbf{x}_1, ..., \mathbf{x_n} \in \mathbb{R}^k$, then*

$$|\mathbf{x_1} + \mathbf{x_2} + \cdots + \mathbf{x_n}| \leq |\mathbf{x_1}| + |\mathbf{x_2}| + \cdots + |\mathbf{x_n}|.$$

Here we might use Theorem 1.37 (e) to prove it. Since the norm $|\cdot|$ on $\mathbb{C}$ is the same as the norm on $\mathbb{R}^2$, we might prove this corollary first and set $k = 2$ on $\mathbb{R}^k = \mathbb{R}^2$.