

## Chapter 6: The Riemann-Stieltjes Integral

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**Supplement.** Another definition of Riemann-Stieltjes integral. (*Exercise 7.3, 7.4 of the book T. M. Apostol, Mathematical Analysis, Second Edition.*) Let  $P$  be a partition of  $[a, b]$ . The norm of a partition  $P$  is the length of the largest subinterval  $[x_{i-1}, x_i]$  of  $P$  and is denoted by  $\|P\|$ .

We say  $f \in \mathcal{R}(\alpha)$  if there exists  $A \in \mathbb{R}$  having the property that for any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for any partition  $P$  of  $[a, b]$  with norm  $\|P\| < \delta$  and for any choice of  $t_i \in [x_{i-1}, x_i]$ , we have  $|\sum_{i=1}^n f(t_i)\Delta\alpha_i - A| < \varepsilon$ .

**Claim.**  $f \in \mathcal{R}$  in the sense of Definition 6.2 implies that  $f \in \mathcal{R}$  in the sense of this another definition.

*Proof of Claim.* Let  $A = \int f dx$ ,  $M > 0$  be one upper bound of  $|f|$  on  $[a, b]$ . Given  $\varepsilon > 0$ , there exists a partition  $P_0 = \{a = x_0, x_1, \dots, x_{N-1}, x_N = b\}$  such that  $U(P_0, f) \leq A + \frac{\varepsilon}{2}$ . Let  $\delta = \frac{\varepsilon}{2MN} > 0$ . Then for any partition  $P$  with norm  $\|P\| < \delta$ , write

$$U(P, f) = \sum_{i=1}^n M_i \Delta x_i = S_1 + S_2,$$

where  $S_1$  is the sum of terms arising from those subintervals of  $P$  containing no point of  $P_0$ ,  $S_2$  is the sum of the remaining terms. Then

$$S_1 \leq U(P_0, f) < A + \frac{\varepsilon}{2},$$

$$S_2 \leq NM\|P\| < NM\delta < \frac{\varepsilon}{2}.$$

Therefore,  $U(P, f) < A + \varepsilon$ . Similarly,  $L(P, f) > A - \varepsilon$  whenever  $\|P\| < \delta'$ . Hence,  $|\sum_{i=1}^n f(t_i)\Delta\alpha_i - A| < \varepsilon$  whenever  $\|P\| < \min\{\delta, \delta'\}$ . (Copy Apostol's hint and ensure  $M > 0$ .  $M$  in Apostol's hint might be zero if  $f = 0$ .)  $\square$

This supplement will be used in computing  $\int_0^\infty (\frac{\sin x}{x})^2 dx = \frac{\pi}{2}$  in Exercise 8.12.

**Exercise 6.1.** Suppose  $\alpha$  increases on  $[a, b]$ ,  $a \leq x_0 \leq b$ ,  $\alpha$  is continuous at  $x_0$ ,  $f(x_0) = 1$ , and  $f(x) = 0$  if  $x \neq x_0$ . Prove that  $f \in \mathcal{R}(\alpha)$  and that  $\int f d\alpha = 0$ .

Given any partition  $P = \{a = p_0, p_1, \dots, p_{n-1}, p_n = b\}$ , where  $a = p_0 \leq p_1 \leq \dots \leq p_{n-1} \leq p_n = b$ . We might compute  $L(P, f, \alpha)$  and  $U(P, f, \alpha)$  by using  $\varepsilon$ - $\delta$

argument since we are hinted by the condition that  $\alpha$  is continuous. A function which is continuous at  $x_0$  has a nice property near  $x_0$  and this property would help us estimate  $U(P, f, \alpha)$  near  $x_0$ . On the contrary, if both  $f$  and  $\alpha$  are discontinuous at  $x_0$ , it might be  $f \notin \mathcal{R}(\alpha)$ . Besides, if  $f$  has too many points of discontinuity ( $f(x) = 0$  if  $x \in \mathbb{Q}$  and  $f(x) = 1$  otherwise, for example), then  $f$  might not be Riemann-integrable on  $[0, 1]$ .

**Claim 1.**  $L(P, f, \alpha) = 0$ .

*Proof of Claim 1.*  $m_i = 0$  since  $\inf f(x) = 0$  on any subinterval of  $[a, b]$ . So  $L(P, f, \alpha) = \sum m_i \Delta \alpha_i = 0$ . Here we don't need the condition that  $\alpha$  is continuous at  $x_0$ .  $\square$

**Claim 2.** For any  $\varepsilon > 0$ , there exists a partition  $P$  such that  $U(P, f, \alpha) < \varepsilon$ .

*Proof of Claim 2.* Say  $x_0 \in [p_{i_0-1}, p_{i_0}]$  for some  $i_0$ . Then

$$M_i = \sup_{p_{i-1} \leq x \leq p_i} f(x) = \begin{cases} 0 & \text{if } i \neq i_0, \\ 1 & \text{if } i = i_0. \end{cases}$$

So

$$U(P, f, \alpha) = \sum M_i \Delta \alpha_i = \Delta \alpha_{i_0}.$$

It is not true for any arbitrary  $\alpha$ . (For example,  $\alpha$  has a jump on  $x = x_0$ .) In fact, Exercise 6.3 shows this. Luckily,  $\alpha$  is continuous at  $x_0$ . So for  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $|\alpha(x) - \alpha(x_0)| < \frac{\varepsilon}{2}$  whenever  $|x - x_0| < \delta$  (and  $x \in [a, b]$ ). Now we pick a nice partition

$$P = \{a, x_0 - \delta_1, x_0 + \delta_2, b\},$$

where  $\delta_1 = \min\{\delta, x_0 - a\} \geq 0$  and  $\delta_2 = \min\{\delta, b - x_0\} \geq 0$ . (It is a trick about resizing “ $\delta$ ” to avoid considering the edge cases  $x_0 = a$  or  $x_0 = b$  or  $a = b$ .) Then  $x_0 \in [x_0 - \delta_1, x_0 + \delta_2]$  and  $\Delta \alpha$  on  $[x_0 - \delta_1, x_0 + \delta_2]$  is

$$\begin{aligned} \alpha(x_0 + \delta_2) - \alpha(x_0 - \delta_1) &= (\alpha(x_0 + \delta_2) - \alpha(x_0)) + (\alpha(x_0) - \alpha(x_0 - \delta_1)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore,  $U(P, f, \alpha) < \varepsilon$ .  $\square$

*Proof (Definition 6.2).* By Claim 1 and 2 and notice that  $U(P, f, \alpha) \geq 0$  for any

partition  $P$ ,

$$\begin{aligned}\int_a^{\bar{b}} f d\alpha &= \inf U(P, f, \alpha) = 0, \\ \int_a^{\underline{b}} f d\alpha &= \sup L(P, f, \alpha) = 0,\end{aligned}$$

the inf and sup again being taken over all partitions. Hence  $f \in \mathcal{R}(\alpha)$  and that  $\int f d\alpha = 0$  by Definition 6.2.  $\square$

*Proof (Theorem 6.6).* By Claim 1 and 2,

$$0 \leq U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$$

Hence  $f \in \mathcal{R}(\alpha)$  by Theorem 6.6. Furthermore,

$$\int f d\alpha = \int_a^{\bar{b}} f d\alpha = \sup L(P, f, \alpha) = 0.$$

$\square$

*Proof (Theorem 6.10).*  $f \in \mathcal{R}(\alpha)$  by Theorem 6.10. Thus, by Claim 1

$$\int f d\alpha = \int_a^{\bar{b}} f d\alpha = \sup L(P, f, \alpha) = 0.$$

$\square$

**Exercise 6.2.** Suppose  $f \geq 0$ ,  $f$  is continuous on  $[a, b]$ , and  $\int_a^b f(x) dx = 0$ . Prove that  $f(x) = 0$  for all  $x \in [a, b]$ . (Compare with Exercise 6.1.)

For one application, see Exercise 7.20.

*Proof.* (Reductio ad absurdum) If there were  $p \in [a, b]$  such that  $f(p) > 0$ . Since  $f$  is continuous on  $[a, b]$ , given  $\varepsilon = \frac{1}{64}f(p) > 0$  there exists  $\delta > 0$  such that

$$|f(x) - f(p)| \leq \frac{1}{64}f(p) \text{ whenever } |x - p| \leq \delta, x \in [a, b].$$

Hence

$$f(x) \geq \frac{63}{64}f(p)$$

whenever  $x \in E = [\max\{a, p - \delta\}, \min\{b, p + \delta\}] \subseteq [a, b]$ . Note that the length of  $E$  is  $|E| > 0$ . So

$$0 = \int_a^b f(x) dx \geq \int_E f(x) dx \geq \int_E \frac{63}{64}f(p) dx = \frac{63}{64}f(p)|E| > 0,$$

which is absurd.  $\square$

*Note.* (Lebesgue integral) Let  $f$  be a nonnegative measurable function. Then  $\int f = 0$  implies  $f = 0$  a.e.

**Exercise 6.3.** Define three functions  $\beta_1, \beta_2, \beta_3$  as follows:  $\beta_j(x) = 0$  if  $x < 0$ ,  $\beta_j(x) = 1$  if  $x > 0$  for  $j = 1, 2, 3$ ; and  $\beta_1(0) = 0$ ,  $\beta_2(0) = 1$ ,  $\beta_3(0) = \frac{1}{2}$ . Let  $f$  be a bounded functions on  $[-1, 1]$ .

- (a) Prove that  $f \in \mathcal{R}(\beta_1)$  if and only if  $f(0+) = f(0)$  and that then

$$\int f d\beta_1 = f(0).$$

- (b) State and prove a similar result for  $\beta_2$ .  
(c) Prove that  $f \in \mathcal{R}(\beta_3)$  if and only if  $f$  is continuous at 0.  
(d) If  $f$  is continuous at 0 prove that

$$\int f d\beta_1 = \int f d\beta_2 = \int f d\beta_3 = f(0).$$

*Proof of (a).*

- (1) Given any  $\delta > 0$ , we have

$$|f(x) - f(0)| \leq \sup_{x \in [0, \delta]} f(x) - \inf_{x \in [0, \delta]} f(x)$$

if  $x \in [0, \delta]$ .

- (2) Given any  $\varepsilon > 0$  and  $\delta > 0$ . Show that if  $f$  is bounded and  $|f(x) - f(0)| < \varepsilon$  on  $[0, \delta]$  then

$$\sup_{x \in [0, \delta]} f(x) - \inf_{x \in [0, \delta]} f(x) < 2\varepsilon.$$

Since  $f$  is bounded, there exists  $x_1, x_2 \in [0, \delta]$  such that

$$f(x_1) = \sup_{x \in [0, \delta]} f(x) \quad \text{and} \quad f(x_2) = \inf_{x \in [0, \delta]} f(x).$$

By assumption,

$$f(x_1) - f(x_2) \leq |f(x_1) - f(0)| + |f(0) - f(x_2)| < 2\varepsilon.$$

(3) Show that  $f \in \mathcal{R}(\beta_1)$  iff  $f(0+) = f(0)$ .

$$f \in \mathcal{R}(\beta_1)$$

$$\iff \forall \varepsilon > 0 \text{ there is } P \text{ such that } U(P, f, \beta_1) - L(P, f, \beta_1) < \varepsilon \quad (\text{Theorem 6.6})$$

$$\iff \forall \varepsilon > 0 \text{ there is } P \text{ containing } 0 \text{ such that } U(P, f, \beta_1) - L(P, f, \beta_1) < \varepsilon \quad (\text{Theorem 6.4})$$

$$\text{where } P = \{-1 = x_0 < x_1 < \dots < x_k = 0 < \dots < x_n = 1\}$$

$$\iff \forall \varepsilon > 0 \text{ there is } P \text{ containing } 0 \text{ such that } M_{k+1} - m_{k+1} < \varepsilon$$

$$\iff \forall \varepsilon > 0 \text{ there is } P \text{ containing } 0 \text{ such that } \sup_{x \in [0, \delta]} f(x) - \inf_{x \in [0, \delta]} f(x) < \varepsilon$$

$$\text{where } [x_k, x_{k+1}] = [0, \delta], \delta > 0$$

$$(\text{Take } P = \{-1, 0, \delta, 1\} \text{ in “}\Leftarrow\text{” direction})$$

$$\iff \forall \varepsilon > 0 \text{ there is } \delta > 0 \text{ such that } |f(x) - f(0)| < \varepsilon \text{ whenever } x \in [0, \delta] \quad ((1)(2))$$

$$(\text{Replace } \varepsilon \text{ by } \frac{\varepsilon}{2} \text{ in “}\Leftarrow\text{” direction})$$

$$\iff \lim_{x \rightarrow 0+} f(x) = f(0).$$

(4) Show that  $\int f d\beta_1 = f(0)$  if  $f \in \mathcal{R}(\beta_1)$ . By (3) and Theorem 6.7,

$$\left| f(0) - \int_a^b f d\beta_1 \right| < \varepsilon.$$

Since  $\varepsilon$  is arbitrary,  $\int f d\beta_1 = f(0)$ .

□

*Proof of (b). Show that  $f \in \mathcal{R}(\beta_2)$  if and only if  $f(0-) = f(0)$  and that then*

$$\int f d\beta_2 = f(0).$$

Similar to (a).

(1) Given any  $\delta > 0$ , we have

$$|f(x) - f(0)| \leq \sup_{x \in [-\delta, 0]} f(x) - \inf_{x \in [-\delta, 0]} f(x)$$

$$\text{if } x \in [-\delta, 0].$$

(2) Given any  $\varepsilon > 0$  and  $\delta > 0$ . Show that if  $f$  is bounded and  $|f(x) - f(0)| < \varepsilon$  on  $[-\delta, 0]$  then

$$\sup_{x \in [-\delta, 0]} f(x) - \inf_{x \in [-\delta, 0]} f(x) < 2\varepsilon.$$

Since  $f$  is bounded, there exists  $x_1, x_2 \in [-\delta, 0]$  such that

$$f(x_1) = \sup_{x \in [-\delta, 0]} f(x) \text{ and } f(x_2) = \inf_{x \in [-\delta, 0]} f(x).$$

By assumption,

$$f(x_1) - f(x_2) \leq |f(x_1) - f(0)| + |f(0) - f(x_2)| < 2\varepsilon.$$

(3) Show that  $f \in \mathcal{R}(\beta_1)$  iff  $f(0-) = f(0)$ .

$$\begin{aligned}
& f \in \mathcal{R}(\beta_2) \\
& \iff \forall \varepsilon > 0 \text{ there is } P \text{ such that } U(P, f, \beta_2) - L(P, f, \beta_2) < \varepsilon & \text{(Theorem 6.6)} \\
& \iff \forall \varepsilon > 0 \text{ there is } P \text{ containing } 0 \text{ such that } U(P, f, \beta_2) - L(P, f, \beta_2) < \varepsilon & \text{(Theorem 6.4)} \\
& \quad \text{where } P = \{-1 = x_0 < x_1 < \dots < x_k = 0 < \dots < x_n = 1\} \\
& \iff \forall \varepsilon > 0 \text{ there is } P \text{ containing } 0 \text{ such that } M_k - m_k < \varepsilon \\
& \iff \forall \varepsilon > 0 \text{ there is } P \text{ containing } 0 \text{ such that } \sup_{x \in [-\delta, 0]} f(x) - \inf_{x \in [-\delta, 0]} f(x) < \varepsilon \\
& \quad \text{where } [x_{k-1}, x_k] = [-\delta, 0], \delta > 0 \\
& \quad \text{(Take } P = \{-1, -\delta, 0, 1\} \text{ in “} \Leftarrow \text{” direction)} \\
& \iff \forall \varepsilon > 0 \text{ there is } \delta > 0 \text{ such that } |f(x) - f(0)| < \varepsilon \text{ whenever } x \in [-\delta, 0] & \text{((1)(2))} \\
& \quad \text{(Replace } \varepsilon \text{ by } \frac{\varepsilon}{2} \text{ in “} \Leftarrow \text{” direction)} \\
& \iff \lim_{x \rightarrow 0-} f(x) = f(0).
\end{aligned}$$

(4) Show that  $\int f d\beta_2 = f(0)$  if  $f \in \mathcal{R}(\beta_2)$ . By (3) and Theorem 6.7,

$$\left| f(0) - \int_a^b f d\beta_2 \right| < \varepsilon.$$

Since  $\varepsilon$  is arbitrary,  $\int f d\beta_2 = f(0)$ .

□

*Proof of (c).* Note that  $f$  is continuous at 0 iff  $f(0+) = f(0-) = f(0)$ . Apply the same argument in (a) and (b), we have  $f \in \mathcal{R}(\beta_3)$  if and only if  $f(0+) = f(0-) = f(0)$ . □

*Proof of (d).* It suffices to show that

$$\int_a^b f d\beta_3 = f(0).$$

We can apply Theorem 6.12(d)(e) to  $\beta_3 = \frac{1}{2}(\beta_1 + \beta_2)$ . That is,

$$\int_a^b f d\beta_3 = \frac{1}{2} \left[ \int_a^b f d\beta_1 + \int_a^b f d\beta_2 \right] = \frac{1}{2}[f(0) + f(0)] = f(0).$$

Or apply the same argument in (a) and (b) to get

$$\left| f(0) - \int_a^b f d\beta_3 \right| < \varepsilon$$

for any  $\varepsilon > 0$ , or  $\int_a^b f d\beta_3 = f(0)$ .  $\square$

**Exercise 6.4.** *If*

$$f(x) = \begin{cases} 0 & \text{for all irrational } x, \\ 1 & \text{for all rational } x, \end{cases}$$

*prove that  $f \notin \mathcal{R}$  on  $[a, b]$  for any  $a < b$ .*

*Proof.* Given any partition

$$P = \{a = p_0, p_1, \dots, p_{n-1}, p_n = b\}$$

of  $[a, b]$  where  $a = p_0 \leq p_1 \leq \dots \leq p_{n-1} \leq p_n = b$ . Since  $a < b$ , we might assume that  $a = p_0 < p_1 < \dots < p_{n-1} < p_n = b$  by removing duplicated points. Since  $\mathbb{Q}$  and  $\mathbb{R} - \mathbb{Q}$  are dense in  $\mathbb{R}$ , we have

$$\begin{aligned} M_i &= \sup_{p_{i-1} \leq x \leq p_i} f(x) = 1, \\ m_i &= \inf_{p_{i-1} \leq x \leq p_i} f(x) = 0, \\ U(P, f) &= \sum_{i=1}^n M_i \Delta x_i = \sum_{i=1}^n \Delta x_i = b - a, \\ L(P, f) &= \sum_{i=1}^n m_i \Delta x_i = \sum_{i=1}^n 0 = 0. \end{aligned}$$

Since  $P$  is arbitrary,

$$\begin{aligned} \int_a^b f dx &= \inf U(P, f) = b - a > 0, \\ \int_a^b f dx &= \sup L(P, f) = 0. \end{aligned}$$

Hence  $f \notin \mathcal{R}$  on  $[a, b]$  for any  $a < b$ .  $\square$

*Note.*

- (1) (Lebesgue integral)  $f$  is Lebesgue integrable.
- (2)  $f \in \mathcal{R}$  on  $[a, b]$  iff  $a = b$ .

- (3) (Problem 4.1 in *H. L. Royden, Real Analysis, 3rd edition.*) Construct a sequence  $\{f_n\}$  of nonnegative, Riemann integrable functions such that  $f_n$  increases monotonically to  $f$ . What does this imply about changing the order of integration and the limiting process? (Since  $\mathbb{Q}$  is countable, write

$$\mathbb{Q} = \{r_1, r_2, \dots\}.$$

Define

$$f_n(x) = \begin{cases} 0 & \text{if } x \notin \{r_1, \dots, r_n\}, \\ 1 & \text{if } x \in \{r_1, \dots, r_n\}. \end{cases}$$

By construction,  $f_n$  increases monotonically to  $f$  pointwise. Note that  $f_n \rightarrow f$  not uniformly. Also,  $\int_a^b f_n(x)dx = 0$  by using the same argument in Theorem 6.10. Therefore,  $\lim_{n \rightarrow \infty} \int_a^b f_n(x)dx = 0$  but  $\int_a^b \lim_{n \rightarrow \infty} f_n(x)dx = \int_a^b f(x)dx$  does not exist.)

**Exercise 6.5.** Suppose  $f$  is a bounded real function on  $[a, b]$ , and  $f^2 \in \mathcal{R}$  on  $[a, b]$ . Does it follow that  $f \in \mathcal{R}$ ? Does the answer change if we assume that  $f^3 \in \mathcal{R}$ ?

Actually we can omit the boundedness assumption of  $f$  since  $f^2 \in \mathcal{R}$  or  $f^3 \in \mathcal{R}$ .

*Proof.*

- (1) Show that  $f^2 \in \mathcal{R}$  on  $[a, b]$  does not imply that  $f \in \mathcal{R}$  (unless  $f \geq 0$  on  $[a, b]$ ). Similar to Exercise 6.4, define

$$f(x) = \begin{cases} -1 & \text{for all irrational } x, \\ 1 & \text{for all rational } x. \end{cases}$$

$f^2 = 1 \in \mathcal{R}$  on  $[a, b]$  but  $f \notin \mathcal{R}$  on  $[a, b]$  for any  $a < b$ . (The proof for the “unless” part is similar to (2).)

- (2) Show that  $f^3 \in \mathcal{R}$  on  $[a, b]$  implies that  $f \in \mathcal{R}$ . Let  $\phi(x) = x^{\frac{1}{3}}$  on  $\mathbb{R}$ . By Theorem 6.11,  $f(x) = \phi(f(x)^3) \in \mathcal{R}$ . (The boundedness condition in Theorem 6.11 is unnecessary.)

□

*Note.* (Lebesgue integral) Suppose that  $f^2$  is Lebesgue integrable. Does it follow that  $f$  is Lebesgue integrable? Does the answer change if we assume that  $f^3$  is Lebesgue integrable? Both answers are no.

**Exercise 6.6.** Let  $P$  be the Cantor set constructed in Sec. 2.44. Let  $f$  be a bounded real function on  $[0, 1]$  which is continuous at every point outside  $P$ .



Prove that  $f \in \mathcal{R}$  on  $[0, 1]$ . (Hint:  $P$  can be covered by finitely many segments whose total length can be made as small as desired. Proceed as in Theorem 6.10.)

*Proof (Theorem 6.10).* Given any  $\varepsilon > 0$ .

- (1) Note that in Section 2.44, we have

$$P = \bigcap_{n=1}^{\infty} E_n$$

and each  $E_n$  is the union of  $2^n$  intervals, each of length  $\frac{1}{3^n}$ . For each interval  $[u_j, v_j] \subseteq E_n$  of  $E_n$  ( $1 \leq j \leq 2^n$ ), we construct a slightly larger open set

$$(u_j - \lambda, v_j + \lambda) \supsetneq [u_j, v_j]$$

where  $\lambda = \frac{1}{2} \left( \frac{1}{2 \cdot 28^n} - \frac{1}{3^n} \right) > 0$ . Each length of  $(u_j - \lambda, v_j + \lambda)$  is  $\frac{1}{2 \cdot 28^n}$ . Write

$$G_n = \bigcup_{1 \leq j \leq 2^n} (u_j - \lambda, v_j + \lambda).$$

Hence

$$G_n \supsetneq \bigcup_{1 \leq j \leq 2^n} [u_j, v_j] = E_n \supseteq P,$$

and the total length  $|G_n|$  of  $G_n$  satisfies

$$|G_n| \leq \sum_{1 \leq j \leq 2^n} |(u_j - \lambda, v_j + \lambda)| = \left( \frac{2}{2 \cdot 28} \right)^n.$$

(Two different subintervals might be overlapped.) As  $n \rightarrow \infty$ ,  $P$  can be covered by finitely many open segments whose total length can be made as small as desired. Now we take an integer  $N$  such that  $\left( \frac{2}{2 \cdot 28} \right)^N < \frac{\varepsilon}{64(M+1)}$ .

- (2) Let  $K = [0, 1] - G_N$  be a compact set (Theorem 2.35). By construction,  $f$  is continuous on  $K$  and thus  $f$  is uniformly continuous. So there is  $\delta > 0$  such that  $|f(s) - f(t)| < \frac{\varepsilon}{89}$  if  $s, t \in K$  and  $|s - t| < \delta$ .

- (3) Now we construct a partition  $P = \{x_0, x_1, \dots, x_n\}$  of  $[a, b]$ , as the following steps:

(a) Put  $\frac{0}{m}, \frac{1}{m}, \dots, \frac{m}{m}$  in  $P$  for some integer  $m \geq \frac{1}{\delta}$ .

(b) Put  $u_j - \lambda$  and  $v_j + \lambda$  in  $P$ .

(c) Remove any points in the segment  $(u_j - \lambda, v_j + \lambda)$  except 0 and 1.

- (4) Note that  $M_i - m_i \leq 2M$  ( $1 \leq i \leq n$ ) where  $M = \sup |f(x)|$  is defined. Hence,

$$U(P, f) - L(P, f) \leq \frac{\varepsilon}{89} + 2M \cdot \frac{\varepsilon}{64(M+1)} \leq \varepsilon.$$

Since  $\varepsilon$  is arbitrary, Theorem 6.6 shows that  $f \in \mathcal{R}$ .

□

**Supplement (Lebesgue's criterion for Riemann-integrability).** Let  $f$  be a bounded real function on  $[a, b]$  and let  $D$  be the set of discontinuities of  $f$  in  $[a, b]$ . Then  $f \in \mathcal{R}$  on  $[a, b]$  if and only if  $D$  has measure zero.

For a proof, see Theorem 7.48 in *Tom M. Apostol, Mathematical Analysis, 2nd edition*.

**Exercise 6.7.** Suppose  $f$  is a real function on  $(0, 1]$  and  $f \in \mathcal{R}$  on  $[c, 1]$  for every  $c > 0$ . Define

$$\int_0^1 f(x)dx = \lim_{c \rightarrow 0} \int_c^1 f(x)dx$$

if this limit exists (and is finite).

- (a) If  $f \in \mathcal{R}$  on  $[0, 1]$ , show that this definition of the integral agrees with the old one.
- (b) Construct a function such that the above limit exists, although it fails to exist with  $|f|$  in place of  $f$ .

*Proof of (a).*

- (1) Since  $f \in \mathcal{R}$  on  $[0, 1]$ ,  $f$  is bounded or  $|f| \leq M$  for some real  $M$ .
- (2) For any  $0 < c < 1$ , we have

$$\begin{aligned} \left| \int_0^1 f(x)dx - \int_c^1 f(x)dx \right| &= \left| \int_0^c f(x)dx \right| && \text{(Theorem 6.12(c))} \\ &\leq Mc. && \text{(Theorem 6.12(d))} \end{aligned}$$

- (3) Given any  $\varepsilon > 0$ , there exists  $\delta = \frac{\varepsilon}{M+1} > 0$  such that

$$\left| \int_0^c f(x)dx - \int_0^1 f(x)dx \right| \leq Mc < M\delta = M \cdot \frac{\varepsilon}{M+1} < \varepsilon$$

whenever  $0 < c < \delta$ . Hence  $\lim_{c \rightarrow 0} \int_c^1 f(x)dx = \int_0^1 f(x)dx$ .

□

*Proof of (b)(Construct by nonabsolutely convergent series).*

- (1) Given any nonabsolutely (conditionally) convergent series  $\sum_{k=1}^n a_k$  (take  $\sum \frac{(-1)^n}{n}$  for example and then see Remark 3.46), we define  $f$  on  $(0, 1]$  by

$$f(x) = 2^n a_n$$

if  $\frac{1}{2^n} < x \leq \frac{1}{2^{n-1}}$  as  $n = 1, 2, \dots$

(2) By construction,

$$\int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} f(x)dx = \left( \frac{1}{2^{n-1}} - \frac{1}{2^n} \right) 2^n a_n = a_n.$$

and thus

$$\int_{\frac{1}{2^n}}^1 f(x)dx = \int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} f(x)dx + \cdots + \int_{\frac{1}{2}}^1 f(x)dx = \sum_{k=1}^n a_k.$$

(3) Given any  $\varepsilon > 0$ . Since  $\sum a_n$  is convergent, there exists a common integer  $N$  such that

$$|a_n| \leq \frac{\varepsilon}{89}$$

and

$$\left| \sum_{k=1}^n a_k - A \right| \leq \frac{\varepsilon}{64}$$

for some real  $A$  whenever  $n \geq N$  (Definition 3.21 and Theorem 3.23). Therefore, for any  $0 < c \leq \frac{1}{2^N}$ , say  $\frac{1}{2^{n+1}} < c \leq \frac{1}{2^n} \leq \frac{1}{2^N}$  for some  $n \geq N$ , we have

$$\begin{aligned} \left| \int_c^1 f(x)dx - A \right| &= \left| \int_c^{\frac{1}{2^n}} f(x)dx + \int_{\frac{1}{2^n}}^1 f(x)dx - A \right| \\ &\leq \left| \left( \frac{1}{2^n} - c \right) 2^{n+1} a_{n+1} \right| + \left| \sum_{k=1}^n a_k - A \right| \\ &\leq |a_{n+1}| + \left| \sum_{k=1}^n a_k - A \right| \\ &\leq \frac{\varepsilon}{89} + \frac{\varepsilon}{64} \\ &\leq \varepsilon. \end{aligned}$$

Hence,  $\lim_{c \rightarrow 0} \int_c^1 f(x)dx = A$  exists.

(4) Since

$$\int_{\frac{1}{2^n}}^1 |f(x)|dx = \int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} |f(x)|dx + \cdots + \int_{\frac{1}{2}}^1 |f(x)|dx = \sum_{k=1}^n |a_k| \rightarrow \infty$$

as  $n \rightarrow \infty$ ,  $\lim_{c \rightarrow 0} \int_c^1 f(x)dx$  does not exist. (Or show that  $\lim_{c \rightarrow 0} \int_c^1 f(x)dx = \infty$  by definition directly.)

□

**Exercise 6.8.** Suppose  $f \in \mathcal{R}$  on  $[a, b]$  for every  $b > a$  where  $a$  is fixed. Define

$$\int_a^\infty f(x)dx = \lim_{b \rightarrow \infty} \int_a^b f(x)dx$$

if this limit exists (and is finite). In that case, we say that the integral on the left **converges**. If it also converges after  $f$  has been replaced by  $|f|$ , it is said to converge **absolutely**. Assume that  $f(x) \geq 0$  and that  $f$  decreases monotonically on  $[1, \infty)$ . Prove that

$$\int_1^\infty f(x)dx$$

converges if and only if

$$\sum_{n=1}^\infty f(n)$$

converges. (This is the so-called “integral test” for convergence of series.)

*Proof.* Similar to Exercise 8.9.

(1) Define

$$\begin{aligned} a_n &= \int_1^n f(x)dx, \\ b_n &= \sum_{k=1}^n f(k), \\ c_n &= b_n - a_n \end{aligned}$$

for  $n = 1, 2, 3, \dots$

(2) Show that  $\{c_n\}$  decreases. Since  $f$  decreases monotonically on  $[1, \infty)$ , we have

$$\begin{aligned} c_n - c_{n+1} &= (b_n - a_n) - (b_{n+1} - a_{n+1}) \\ &= (a_{n+1} - a_n) - (b_{n+1} - b_n) \\ &= \int_n^{n+1} f(x)dx - f(n+1) \\ &\geq \int_n^{n+1} f(n+1)dx - f(n+1) \\ &= f(n+1) - f(n+1) \\ &= 0. \end{aligned}$$

(3) Show that  $\{c_n\}$  is bounded. Since  $f$  decreases monotonically on  $[1, \infty)$ ,

$$\begin{aligned}
c_n &= b_n - a_n \\
&= \sum_{k=1}^n f(k) - \int_1^n f(x) dx \\
&= \sum_{k=1}^n f(k) - \sum_{k=1}^{n-1} \int_k^{k+1} f(x) dx \\
&\geq \sum_{k=1}^n f(k) - \sum_{k=1}^{n-1} \int_k^{k+1} f(k) dx \\
&= \sum_{k=1}^n f(k) - \sum_{k=1}^{n-1} f(k) \\
&= f(n).
\end{aligned}$$

Since  $f(n)$  is nonnegative,  $c_n \geq 0$ .

(4) By (2)(3),  $\{c_n\}$  converges (Theorem 3.14).

(5) Since  $c_n = b_n - a_n$  and  $\{c_n\}$  converges,  $\{a_n\}$  converges if and only if  $\{b_n\}$  converges, or  $\int_1^\infty f(x) dx$  converges if and only if  $\sum_{n=1}^\infty f(n)$  converges.

□

**Exercise 6.9.** Show that integration by parts can sometimes be applied to the “improper” integrals defined in Exercise 6.7 and 6.8. (State appropriate hypotheses, formulate a theorem, and prove it.) For instance show that

$$\int_0^\infty \frac{\cos x}{1+x} dx = \int_0^\infty \frac{\sin x}{(1+x)^2} dx.$$

Show that one of these integrals converges **absolutely**, but that the other does not.

*Proof.*

(1) Suppose  $F$  and  $G$  are differentiable functions on  $(0, 1]$ ,  $F' = f \in \mathcal{R}$  on  $[c, 1]$  and  $G' = g \in \mathcal{R}$  on  $[c, 1]$  for every  $c > 0$ . Then

$$\int_0^1 F(x)g(x)dx = F(1)G(1) - \lim_{c \rightarrow 0} F(c)G(c) - \int_0^1 f(x)G(x)dx$$

if any two of  $\int_0^1 F(x)g(x)dx$ ,  $\int_0^1 f(x)G(x)dx$  or  $\lim_{c \rightarrow 0} F(c)G(c)$  exist. Theorem 6.22 (integration by parts) implies that

$$\int_c^1 F(x)g(x)dx = F(1)G(1) - F(c)G(c) - \int_c^1 f(x)G(x)dx.$$

Since any two of  $\int_0^1 F(x)g(x)dx$  or  $\int_0^1 f(x)G(x)dx$  or  $\lim_{c \rightarrow 0} F(c)G(c)$  exist, the rest one exists and satisfies the identity

$$\int_0^1 F(x)g(x)dx = F(1)G(1) - \lim_{c \rightarrow 0} F(c)G(c) - \int_0^1 f(x)G(x)dx$$

by letting  $c \rightarrow 0$ .

- (2) Suppose  $F$  and  $G$  are differentiable functions on  $[a, b]$  for every  $b > a$  where  $a$  is fixed,  $F' = f \in \mathcal{R}$  on  $[a, b]$  and  $G' = g \in \mathcal{R}$  on  $[a, b]$ . Then

$$\int_a^\infty F(x)g(x)dx = \lim_{b \rightarrow \infty} F(b)G(b) - F(a)G(a) - \int_a^\infty f(x)G(x)dx$$

if any two of  $\int_a^\infty F(x)g(x)dx$ ,  $\int_a^\infty f(x)G(x)dx$  or  $\lim_{b \rightarrow \infty} F(b)G(b)$  exist. Theorem 6.22 (integration by parts) implies that

$$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx.$$

Since any two of  $\int_a^\infty F(x)g(x)dx$  or  $\int_a^\infty f(x)G(x)dx$  or  $\lim_{b \rightarrow \infty} F(b)G(b)$  exist, the rest one exists and satisfies the identity

$$\int_a^\infty F(x)g(x)dx = \lim_{b \rightarrow \infty} F(b)G(b) - F(a)G(a) - \int_a^\infty f(x)G(x)dx$$

by letting  $b \rightarrow \infty$ .

- (3) Show that

$$\int_0^\infty \frac{\cos x}{1+x} dx = \int_0^\infty \frac{\sin x}{(1+x)^2} dx.$$

Put  $a = 0$ ,  $F(x) = \frac{1}{1+x}$  and  $G(x) = \sin x$  in

$$\int_a^\infty F(x)g(x)dx = \lim_{b \rightarrow \infty} F(b)G(b) - F(a)G(a) - \int_a^\infty f(x)G(x)dx$$

to get

$$\int_0^\infty \frac{(\sin x)'}{1+x} dx = \lim_{b \rightarrow \infty} \frac{\sin(b)}{1+b} - \frac{\sin(0)}{1+0} - \int_0^\infty \left( \frac{1}{1+x} \right)' \sin x dx$$

or

$$\int_0^\infty \frac{\cos x}{1+x} dx = \int_0^\infty \frac{\sin x}{(1+x)^2} dx.$$

- (4) Show that

$$\int_0^\infty \frac{\sin x}{(1+x)^2} dx$$

converges absolutely. Notice that

$$\begin{aligned}\int_0^\infty \left| \frac{\sin x}{(1+x)^2} \right| dx &\leq \int_0^\infty \frac{1}{(1+x)^2} dx \\ &= \lim_{b \rightarrow \infty} \left[ -\frac{1}{1+x} \right]_0^b - (-1) \\ &= 1.\end{aligned}$$

(5) Show that

$$\int_0^\infty \frac{\cos x}{1+x} dx$$

converges conditionally. By (3)(4),  $\int_0^\infty \frac{\cos x}{1+x} dx$  converges. Note that

$$\cos x \geq \frac{1}{2}$$

if  $x \in [-\frac{\pi}{3} + 2n\pi, \frac{\pi}{3} + 2n\pi]$  for  $n = 1, 2, 3, \dots$ . Hence

$$\begin{aligned}\int_0^\infty \left| \frac{\cos x}{1+x} \right| dx &\geq \sum_{n=1}^\infty \int_{-\frac{\pi}{3}+2n\pi}^{\frac{\pi}{3}+2n\pi} \left| \frac{\cos x}{1+x} \right| dx \\ &\geq \sum_{n=1}^\infty \int_{-\frac{\pi}{3}+2n\pi}^{\frac{\pi}{3}+2n\pi} \frac{\frac{1}{2}}{1 + \frac{\pi}{3} + 2n\pi} dx \\ &= \sum_{n=1}^\infty \frac{2\pi}{3} \cdot \frac{\frac{1}{2}}{1 + \frac{\pi}{3} + 2n\pi} \\ &> \frac{\pi}{3} \sum_{n=1}^\infty \frac{1}{\pi + \pi + 2n\pi} \\ &= \frac{1}{6} \sum_{n=1}^\infty \frac{1}{n+1}.\end{aligned}$$

By Theorem 3.28,  $\sum_{n=1}^\infty \frac{1}{n+1} = \infty$  and thus  $\int_0^\infty \frac{\cos x}{1+x} dx$  does not converge absolutely.

□

**Exercise 6.10.** Let  $p$  and  $q$  be positive real integers such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Prove the following statements.

(a) If  $u \geq 0$  and  $v \geq 0$ , then

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

Equality holds if and only if  $u^p = v^q$ .

(b) If  $f \in \mathcal{R}(\alpha)$ ,  $g \in \mathcal{R}(\alpha)$ ,  $f \geq 0$ ,  $g \geq 0$ , and

$$\int_a^b f^p d\alpha = \int_a^b g^q d\alpha = 1,$$

then

$$\int_a^b fg d\alpha \leq 1.$$

(c) If  $f$  and  $g$  are complex functions in  $\mathcal{R}(\alpha)$ , then

$$\left| \int_a^b fg d\alpha \right| \leq \left\{ \int_a^b |f|^p d\alpha \right\}^{\frac{1}{p}} \left\{ \int_a^b |g|^q d\alpha \right\}^{\frac{1}{q}}.$$

This is **Hölder's inequality**. When  $p = q = 2$  it is usually called the Schwarz inequality. (Note that Theorem 1.35 is a very special case of this.)

(d) Show that Hölder's inequality is also true for the “improper” integrals described in Exercise 6.7 and 6.8.

*Proof of (a) (Young's inequality).*

- (1)  $u = 0$  or  $v = 0$  is nothing to do. For  $u > 0$  and  $v > 0$ , we give some different proofs.
- (2) First proof.

$$\begin{aligned} uv &= \exp(\log(uv)) \\ &= \exp\left(\frac{1}{p} \log(u^p) + \frac{1}{q} \log(v^q)\right) \\ &\leq \frac{1}{p} \exp(\log(u^p)) + \frac{1}{q} \exp(\log(v^q)) \quad (\text{Convexity of } \exp(x)) \\ &= \frac{u^p}{p} + \frac{v^q}{q}. \end{aligned}$$

Here the convexity of  $\exp(x)$  can be derived by the fact that  $(\exp(x))'' > 0$  and Exercise 5.14. The fact that the equality holds if and only if  $u^p = v^q$  is derived from the strictly convexity of  $\exp(x)$  additionally. (For the details about the exponential and logarithmic functions, might see Chapter 8.)

- (3) Second proof.

$$\begin{aligned} \log\left(\frac{u^p}{p} + \frac{v^q}{q}\right) &\geq \frac{1}{p} \log(u^p) + \frac{1}{q} \log(v^q) \quad (\text{Concavity of } \log(x)) \\ &= \log(u) + \log(v) \\ &= \log(uv). \end{aligned}$$



Since  $\log(x)$  increases monotonically ( $(\log(x))' = \frac{1}{x} > 0$  if  $x > 0$ ),  $\frac{u^p}{p} + \frac{v^q}{q} \geq uv$  (or take the exponential function to get the same conclusion). Here the concavity of  $\log(x)$  can be derived by the fact that  $(\log(x))'' < 0$  and a statement that  $f''(x) \leq 0$  if and only if  $f$  is concave. The fact that the equality holds if and only if  $u^p = v^q$  is derived from the strictly concavity of  $\log(x)$  additionally. (The proof is analogous to Exercise 5.14.)

- (4) Third proof. Suppose that  $f : [0, \infty) \rightarrow [0, \infty)$  is a strictly increasing continuous function such that  $f(0) = 0$  and  $\lim_{x \rightarrow \infty} f(x) = \infty$ . Then

$$uv \leq \int_0^u f(x)dx + \int_0^v f^{-1}(x)dx$$

for every  $u, v \geq 0$ , and equality occurs if and only if  $v = f(u)$ . Define

$$F(x) = -xf(x) + \int_0^x f(t)dt + \int_0^{f(x)} f^{-1}(t)dt.$$

By Theorem 6.20 (the fundamental theorem of calculus) and Theorem 5.5 (chain rule),

$$F'(x) = -(f(x) + xf'(x)) + f(x) + f'(x)f^{-1}(f(x)) = 0.$$

Hence  $F(x)$  is a constant on  $(0, u)$  (Theorem 5.11(b)). Note that  $F(x)$  is continuous on  $[0, u]$  and  $F(0) = 0$ , so  $F(x) = 0$  on  $[0, u]$  or

$$\int_0^x f(t)dt + \int_0^{f(x)} f^{-1}(t)dt = xf(x).$$

Take  $x = u$  to get

$$\int_0^u f(x)dx + \int_0^{f(u)} f^{-1}(x)dx = uf(u).$$

Hence

$$\begin{aligned} & \int_0^u f(x)dx + \int_0^v f^{-1}(x)dx - uv \\ &= \int_0^u f(x)dx + \int_0^{f(u)} f^{-1}(x)dx + \int_{f(u)}^v f^{-1}(x)dx - uv \\ &= uf(u) + \int_{f(u)}^v f^{-1}(x)dx - uv \\ &= \int_{f(u)}^v [f^{-1}(x) - f^{-1}(f(u))]dx \\ &\geq 0. \end{aligned}$$

The last inequality holds since  $f$  is strictly increasing and thus  $f^{-1}$  is strictly increasing too. Besides, the equality holds if and only if  $f(u) = v$ . Now the conclusion holds by taking  $f(x) = x^{p-1}$  in

$$uv \leq \int_0^u f(x)dx + \int_0^v f^{-1}(x)dx$$

and the equality holds if and only if  $u^p = v^q$ .

□

*Proof of (b).* Every integral is well-defined (Theorem 6.11 and Theorem 6.13(a)). Let  $u = f \geq 0$  and  $v = g \geq 0$  in (a). Integrate both sides of the inequality

$$fg \leq \frac{f^p}{p} + \frac{g^q}{q}$$

to get

$$\begin{aligned} \int_a^b fg d\alpha &\leq \int_a^b \left( \frac{f^p}{p} + \frac{g^q}{q} \right) d\alpha && \text{(Theorem 6.12(b))} \\ &= \int_a^b \frac{f^p}{p} d\alpha + \int_a^b \frac{g^q}{q} d\alpha && \text{(Theorem 6.12(a))} \\ &= \frac{1}{p} \int_a^b f^p d\alpha + \frac{1}{q} \int_a^b g^q d\alpha && \text{(Theorem 6.12(a))} \\ &= \frac{1}{p} + \frac{1}{q} && \text{(Assumption)} \\ &= 1. \end{aligned}$$

The equality holds if  $f^p = g^q$ . Note that the equality does not hold only if  $f^p = g^q$ . (Consider  $\alpha$  is constant on some subinterval  $[c, d] \subsetneq [a, b]$ .) Luckily, it is true for the additional assumption that  $\alpha(x) = x$  and  $f, g$  are continuous on  $[a, b]$ . □

*Proof of (c).* There are three possible cases.

(1) The case  $\left\{ \int_a^b |f|^p d\alpha \right\}^{\frac{1}{p}} = 0$ . So  $\int_a^b |f|^p d\alpha = 0$ .

(a) Show that  $\int_a^b |f| d\alpha = 0$  if  $\int_a^b |f|^p d\alpha = 0$ . (Reductio ad absurdum)

If  $\int_a^b |f| d\alpha = A > 0$ , then given  $\varepsilon = \frac{A}{2} > 0$ , there exists a partition  $P_0 = \{a = x_0 \leq \cdots \leq x_n = b\}$  such that

$$\sum_{i=0}^n m_i \Delta \alpha_i > \frac{A}{2},$$

where  $m_i = \inf_{x \in [x_{i-1}, x_i]} |f|$  and  $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$ . By the pigeonhole principle, there exists  $1 \leq i_0 \leq n$  such that

$$L(P_0, |f|, \alpha) = m_{i_0} \Delta\alpha_{i_0} > \frac{A}{2n} > 0.$$

Especially,  $m_{i_0} > 0$  and  $\Delta\alpha_{i_0} > 0$ . Now we consider  $L(P, |f|^p, \alpha)$ . Hence

$$L(P_0, |f|^p, \alpha) = \sum_{i=0}^n m_i^p \Delta\alpha_i \geq m_{i_0}^p \Delta\alpha_{i_0} > 0,$$

or

$$\int_a^b |f| d\alpha = \sup L(P, f, \alpha) \geq m_{i_0}^p \Delta\alpha_{i_0} > 0,$$

which is absurd.

- (b) Show that  $\int_a^b |fg| d\alpha = 0$  if  $\int_a^b |f| d\alpha = 0$ . Since  $g \in \mathcal{R}(\alpha)$ ,  $|g|$  is bounded by some real  $M$  on  $[a, b]$ , that is,  $|g(x)| \leq M$ . Hence

$$0 \leq \int_a^b |fg| d\alpha \leq \int_a^b M|f| d\alpha = M \int_a^b |f| d\alpha = 0.$$

Therefore  $\int_a^b |fg| d\alpha = 0$ .

By (a)(b),  $\int_a^b |fg| d\alpha = 0$  and thus Hölder's inequality holds for this case.

- (2) The case  $\left\{ \int_a^b |g|^q d\alpha \right\}^{\frac{1}{q}} = 0$ . Similar to (1).

- (3) If both  $\left\{ \int_a^b |f|^p d\alpha \right\}^{\frac{1}{p}} > 0$  and  $\left\{ \int_a^b |g|^q d\alpha \right\}^{\frac{1}{q}} > 0$ , then we apply (b) to

$$F(x) = \frac{|f(x)|}{\left\{ \int_a^b |f(x)|^p d\alpha \right\}^{\frac{1}{p}}} \quad \text{and} \quad G(x) = \frac{|g(x)|}{\left\{ \int_a^b |g(x)|^q d\alpha \right\}^{\frac{1}{q}}}.$$

Here  $F(x) \geq 0$  and  $G(x) \geq 0$  are well-defined and Riemann integrable. Thus the conclusion holds. The equality holds if  $F(x)^p = G(x)^q$  or

$$\frac{|f|^p}{\int_a^b |f|^p d\alpha} = \frac{|g|^q}{\int_a^b |g|^q d\alpha}.$$

Note that the equality does not hold only if  $\frac{|f|^p}{\int_a^b |f|^p d\alpha} = \frac{|g|^q}{\int_a^b |g|^q d\alpha}$ . Luckily, it is true for the additional assumption that  $\alpha(x) = x$  and  $f, g$  are continuous on  $[a, b]$ .

By (1)(2)(3), in any case the equality holds if

$$|f|^p \int_a^b |g|^q d\alpha = |g|^q \int_a^b |f|^p d\alpha.$$

In addition, if  $\alpha(x) = x$  and  $f, g$  are continuous on  $[a, b]$ , then the equality holds if and only if

$$|f|^p \int_a^b |g|^q d\alpha = |g|^q \int_a^b |f|^p d\alpha.$$

□

*Proof of (d).*

- (1) Suppose  $f$  and  $g$  are real functions on  $(0, 1]$  and  $f, g \in \mathcal{R}$  on  $[c, 1]$  for every  $c > 0$ . Show that

$$\left| \int_0^1 fg dx \right| \leq \left\{ \int_0^1 |f|^p dx \right\}^{\frac{1}{p}} \left\{ \int_0^1 |g|^q dx \right\}^{\frac{1}{q}}.$$

Here  $\int_0^1$  is one improper integral defined in Exercise 6.7.

- (a) By (c), we have

$$\left| \int_c^1 fg dx \right| \leq \left\{ \int_c^1 |f|^p dx \right\}^{\frac{1}{p}} \left\{ \int_c^1 |g|^q dx \right\}^{\frac{1}{q}}$$

for any  $c \in (0, 1]$ . Here every integral is well-defined (Theorem 6.11 and Theorem 6.13).

- (b) Since every integral is  $\geq 0$ , by taking the limit in the right hand side we have

$$\begin{aligned} \left| \int_c^1 fg dx \right| &\leq \left\{ \int_c^1 |f|^p dx \right\}^{\frac{1}{p}} \left\{ \int_c^1 |g|^q dx \right\}^{\frac{1}{q}} \\ &\leq \left\{ \int_0^1 |f|^p dx \right\}^{\frac{1}{p}} \left\{ \int_0^1 |g|^q dx \right\}^{\frac{1}{q}}. \end{aligned}$$

It is possible that  $\left\{ \int_0^1 |f|^p dx \right\}^{\frac{1}{p}} = \infty$  or  $\left\{ \int_0^1 |g|^q dx \right\}^{\frac{1}{q}} = \infty$ .

- (c) Now  $\left| \int_c^1 fg dx \right|$  is bounded by  $\left\{ \int_0^1 |f|^p dx \right\}^{\frac{1}{p}} \left\{ \int_0^1 |g|^q dx \right\}^{\frac{1}{q}}$ . Take limit to get

$$\left| \int_0^1 fg dx \right| \leq \left\{ \int_0^1 |f|^p dx \right\}^{\frac{1}{p}} \left\{ \int_0^1 |g|^q dx \right\}^{\frac{1}{q}}$$

even if some limit is divergent.

- (2) Suppose  $f$  and  $g$  are real functions on  $[a, b]$  and  $f, g \in \mathcal{R}$  on  $[a, b]$  for every  $b > a$  where  $a$  is fixed. Show that

$$\left| \int_a^\infty fg dx \right| \leq \left\{ \int_a^\infty |f|^p dx \right\}^{\frac{1}{p}} \left\{ \int_a^\infty |g|^q dx \right\}^{\frac{1}{q}}.$$

Here  $\int_a^\infty$  is one improper integral defined in Exercise 6.8. Same as (1).

□

**Exercise 6.11.** Let  $\alpha$  be a fixed increasing function on  $[a, b]$ . For  $u \in \mathcal{R}(\alpha)$ , define

$$\|u\|_2 = \left\{ \int_a^b |u|^2 d\alpha \right\}^{\frac{1}{2}}.$$

Suppose  $f, g, h \in \mathcal{R}(\alpha)$ , and prove the triangle inequality

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2$$

as a consequence of the Schwarz inequality, as in the proof of Theorem 1.37.

*Proof.*

(1) By Exercise 6.10(c) with  $p = q = 2$ , we have

$$\begin{aligned} \int_a^b |f - g||g - h| d\alpha &= \left| \int_a^b |f - g||g - h| d\alpha \right| \\ &\leq \left\{ \int_a^b |f - g|^2 d\alpha \right\}^{\frac{1}{2}} \left\{ \int_a^b |g - h|^2 d\alpha \right\}^{\frac{1}{2}} \\ &= \|f - g\|_2 \|g - h\|_2. \end{aligned}$$

Every integral is well-defined (Theorem 6.12 and Theorem 6.13 (or Theorem 6.11)).

(2) Since

$$\begin{aligned} \|f - h\|_2^2 &= \int_a^b |f - h|^2 d\alpha \\ &\leq \int_a^b (|f - g| + |g - h|)^2 d\alpha && \text{(Triangle inequality)} \\ &= \int_a^b (|f - g|^2 + 2|f - g||g - h| + |g - h|^2) d\alpha \\ &= \int_a^b |f - g|^2 d\alpha + 2 \int_a^b |f - g||g - h| d\alpha + \int_a^b |g - h|^2 d\alpha \\ &\leq \|f - g\|_2^2 + 2\|f - g\|_2 \|g - h\|_2 + \|g - h\|_2^2 && ((1)) \\ &= (\|f - g\|_2 + \|g - h\|_2)^2, \end{aligned}$$

we have

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2.$$

Here every integral is well-defined (Theorem 6.12 and Theorem 6.13 (or Theorem 6.11)).

□

**Exercise 6.12.** With the notations of Exercise 6.11, suppose  $f \in \mathcal{R}(\alpha)$  and  $\varepsilon > 0$ . Prove that there exists a continuous function  $g$  on  $[a, b]$  such that  $\|f - g\|_2 < \varepsilon$ . (Hint: Let  $P = \{a = x_0 \leq \dots \leq x_n = b\}$  be a suitable partition of  $[a, b]$ , define

$$g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i)$$

if  $x_{i-1} \leq t \leq x_i$ .)

*Proof.* Given  $\varepsilon > 0$ .

- (1) There are some real numbers  $m$  and  $M$  such that  $m \leq f(x) \leq M$  if  $x \in [a, b]$  since  $f \in \mathcal{R}(\alpha)$  or  $f$  is bounded on  $[a, b]$ . By Theorem 6.6, there exists a partition  $P = \{a = x_0 \leq \dots \leq x_n = b\}$  such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \frac{\varepsilon^2}{M - m + 1}.$$

Here

$$U(P, f, \alpha) = \sum_{i=1}^n M_i \Delta \alpha_i \text{ where } M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x)$$

$$L(P, f, \alpha) = \sum_{i=1}^n m_i \Delta \alpha_i \text{ where } m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x).$$

- (2) For such partition  $P$ , define  $g$  on  $[a, b]$  by

$$g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i)$$

if  $x_{i-1} \leq t \leq x_i$ . So that

$$\begin{aligned} |f(t) - g(t)| &= \left| \left( \frac{x_i - t}{\Delta x_i} + \frac{t - x_{i-1}}{\Delta x_i} \right) f(t) - \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i) \right| \\ &= \left| \frac{x_i - t}{\Delta x_i} (f(t) - f(x_{i-1})) + \frac{t - x_{i-1}}{\Delta x_i} (f(t) - f(x_i)) \right| \\ &\leq \frac{x_i - t}{\Delta x_i} |f(t) - f(x_{i-1})| + \frac{t - x_{i-1}}{\Delta x_i} |f(t) - f(x_i)| \\ &\leq \frac{x_i - t}{\Delta x_i} (M_i - m_i) + \frac{t - x_{i-1}}{\Delta x_i} (M_i - m_i) \\ &= M_i - m_i \end{aligned}$$

if  $x_{i-1} \leq t \leq x_i$ . Especially,

$$|f(t) - g(t)| \leq M - m$$

if  $a \leq t \leq b$ .

- (3) Note that the integral  $\int_a^b |f - g|^2 d\alpha$  is well-defined (Theorem 6.8, Theorem 6.11 and Theorem 6.12). So that

$$\begin{aligned}
\int_a^b |f - g|^2 d\alpha &= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f - g|^2 d\alpha \\
&\leq \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (M - m)(M_i - m_i) d\alpha \\
&= (M - m) \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (M_i - m_i) \Delta\alpha_i \\
&= (M - m) [U(P, f, \alpha) - L(P, f, \alpha)] \\
&\leq (M - m) \cdot \frac{\varepsilon^2}{M - m + 1} \\
&< \varepsilon^2.
\end{aligned}$$

Hence,

$$\|f - g\|_2 = \left\{ \int_a^b |f - g|^2 d\alpha \right\}^{\frac{1}{2}} < \varepsilon.$$

□

*Note.*

- (1) Apply the same argument we can prove the following statement:

*Suppose  $f \in \mathcal{R}(\alpha)$  and  $\varepsilon > 0$ . Prove that there exists a continuous function  $g$  on  $[a, b]$  such that  $\int_a^b |f - g| d\alpha < \varepsilon$ .*

- (2) (Lebesgue integral)

- (a) *Let  $f$  be Lebesgue integrable over  $E$ . Then, given  $\varepsilon > 0$ , there is a simple function  $\varphi$  such that*

$$\int_E |f - \varphi| < \varepsilon.$$

- (b) *Under the same hypothesis there is a step function  $\psi$  such that*

$$\int_E |f - \psi| < \varepsilon.$$

- (c) *Under the same hypothesis there is a continuous function  $g$  vanishing outside a finite interval such that*

$$\int_E |f - g| < \varepsilon.$$

**Exercise 6.13.** Define

$$f(x) = \int_x^{x+1} \sin(t^2) dt.$$

- (a) Prove that  $|f(x)| < \frac{1}{x}$  if  $x > 0$ . (Hint: Put  $t^2 = u$  and integrate by parts, to show that  $f(x)$  is equal to

$$\frac{\cos(x^2)}{2x} - \frac{\cos[(x+1)^2]}{2(x+1)} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{\frac{3}{2}}} du.$$

Replace  $\cos u$  by  $-1$ .)

- (b) Prove that

$$2xf(x) = \cos(x^2) - \cos[(x+1)^2] + r(x)$$

where  $|r(x)| < \frac{c}{x}$  and  $c$  is a constant.

- (c) Find the upper and lower limits of  $xf(x)$ , as  $x \rightarrow \infty$ .

- (d) Does  $\int_0^\infty \sin(t^2) dt$  converges?

*Proof of (a).*

- (1) Put  $t^2 = u$  and integrate by parts to get

$$\begin{aligned} f(x) &= \int_x^{x+1} \sin(t^2) dt \\ &= \int_{x^2}^{(x+1)^2} \frac{\sin u}{2u^{\frac{1}{2}}} du \\ &= -\frac{\cos[(x+1)^2]}{2(x+1)} + \frac{\cos(x^2)}{2x} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{\frac{3}{2}}} du. \end{aligned}$$

- (2)

$$\begin{aligned} |f(x)| &\leq \left| \frac{\cos[(x+1)^2]}{2(x+1)} \right| + \left| \frac{\cos(x^2)}{2x} \right| + \left| \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{\frac{3}{2}}} du \right| \\ &\leq \left| \frac{\cos[(x+1)^2]}{2(x+1)} \right| + \left| \frac{\cos(x^2)}{2x} \right| + \int_{x^2}^{(x+1)^2} \frac{|\cos u|}{4u^{\frac{3}{2}}} du \\ &\leq \frac{1}{2(x+1)} + \frac{1}{2x} + \int_{x^2}^{(x+1)^2} \frac{1}{4u^{\frac{3}{2}}} du \\ &= \frac{1}{2(x+1)} + \frac{1}{2x} + \left[ \frac{1}{2x} - \frac{1}{2(x+1)} \right] \\ &= \frac{1}{x}. \end{aligned}$$



(3) The equality in (2) holds only if  $|\cos[(x+1)^2]| = 1$ ,  $|\cos(x^2)| = 1$ , and

$$\left| \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{\frac{3}{2}}} du \right| = \int_{x^2}^{(x+1)^2} \frac{|\cos u|}{4u^{\frac{3}{2}}} du = \int_{x^2}^{(x+1)^2} \frac{1}{4u^{\frac{3}{2}}} du.$$

Since  $\cos u$  has two absolute minimums or maximums at two different points  $u = x^2$  and  $u = (x+1)^2$ , by the property of  $\cos(u)$  there is some  $u_0 \in [x^2, (x+1)^2]$  such that  $\cos(u_0) = 0$ . Hence given  $\varepsilon = \frac{1}{2} > 0$  there exists  $\delta > 0$  such that

$$|\cos(u)| \leq \frac{1}{2}$$

whenever

$$u \in E = [\max\{u_0 - \delta, x^2\}, \min\{u_0 + \delta, (x+1)^2\}] \subseteq [x^2, (x+1)^2].$$

Here  $|E| > 0$ . So that

$$\int_{x^2}^{(x+1)^2} \frac{|\cos u|}{4u^{\frac{3}{2}}} du = \int_{x^2}^{(x+1)^2} \frac{1}{4u^{\frac{3}{2}}} du - \frac{1}{2} \int_E \frac{1}{4u^{\frac{3}{2}}} du < \int_{x^2}^{(x+1)^2} \frac{1}{4u^{\frac{3}{2}}} du,$$

which is absurd. Hence the equality in (2) does not hold.

□

*Proof of (b).*

(1) By (a),

$$2xf(x) = \cos(x^2) - \cos[(x+1)^2] + r(x)$$

where

$$r(x) = \frac{\cos[(x+1)^2]}{x+1} - 2x \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{\frac{3}{2}}} du.$$

(2) Similar to (a),

$$\begin{aligned} |2xf(x)| &\leq \frac{1}{x+1} + 2x \int_{x^2}^{(x+1)^2} \frac{1}{4u^{\frac{3}{2}}} du \\ &= \frac{1}{x+1} + 2x \left[ \frac{1}{2x} - \frac{1}{2(x+1)} \right] \\ &= \frac{2}{x+1} \\ &< \frac{2}{x}. \end{aligned}$$

□

*Proof of (c).* □

Note.

$$\int_0^\infty \sin(t^2) dt = \int_0^\infty \cos(t^2) dt = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

**Exercise 6.14.** Deal similarly with

$$f(x) = \int_x^{x+1} \sin(e^t) dt.$$

Show that

$$e^x |f(x)| < 2$$

and that

$$e^x f(x) = \cos(e^x) - e^{-1} \cos(e^{x+1}) + r(x)$$

where  $|r(x)| < Ce^{-x}$  for some constant  $C$ .

PLACEHOLDER

**Exercise 6.15.** Suppose  $f$  is a real, continuously differentiable function on  $[a, b]$ ,  $f(a) = f(b) = 0$ , and

$$\int_a^b f(x)^2 dx = 1.$$

Prove that

$$\int_a^b x f(x) f'(x) dx = -\frac{1}{2}$$

and that

$$\int_a^b [f'(x)]^2 dx \int_a^b x^2 f(x)^2 dx > \frac{1}{4}.$$

*Proof.* Every integral is well-defined (Theorem 4.9 and Theorem 6.8).

(1) By Theorem 6.22 (integration by parts),

$$\int_a^b x \left( \frac{f(x)^2}{2} \right)' dx = \left[ x \cdot \frac{f(x)^2}{2} \right]_{x=a}^{x=b} - \int_a^b \frac{f(x)^2}{2} dx,$$

or

$$\int_a^b x f(x) f'(x) dx = \left[ b \cdot \frac{f(b)^2}{2} - a \cdot \frac{f(a)^2}{2} \right] - \frac{1}{2} \int_a^b f(x)^2 dx = -\frac{1}{2}.$$

(2) By Exercise 6.10(c),

$$\int_a^b [f'(x)]^2 dx \int_a^b x^2 f(x)^2 dx \geq \left( \int_a^b x f(x) f'(x) dx \right)^2 = \frac{1}{4}.$$

(3) (Reductio ad absurdum) If the equality were holding, then by Exercise 6.10(c)

$$(f'(x))^2 \int_a^b x^2 f(x)^2 dx = x^2 f(x)^2 \int_a^b [f'(x)]^2 dx$$

on  $[a, b]$  (since  $x$ ,  $f(x)$  and  $f'(x)$  are continuous on  $[a, b]$ ).

(a) *Show that both integrals are nonzero.* (Reductio ad absurdum) If  $\int_a^b x^2 f(x)^2 dx = 0$ , then  $x^2 f(x)^2 = 0$  or  $xf(x) = 0$  on  $[a, b]$  (Exercise 6.2). So that

$$\int_a^b x f(x) f'(x) dx = 0 \neq -\frac{1}{2},$$

which is absurd. Similarly,  $\int_a^b [f'(x)]^2 dx \neq 0$ .

(b) By (a), we write

$$C = \left\{ \frac{\int_a^b [f'(x)]^2 dx}{\int_a^b x^2 f(x)^2 dx} \right\}^{\frac{1}{2}} > 0$$

be a positive constant. Hence

$$f'(x) = \pm C x f(x).$$

Here the sign “ $\pm$ ” is not necessary unchanged on  $[a, b]$ . Luckily, we can show that the sign “ $\pm$ ” is unchanged on some subinterval of  $[a, b]$ .

(c) To find such subinterval of  $[a, b]$ , we consider the zero set  $Z(f')$  and  $Z(xf)$  on  $[a, b]$ . Since  $f'(x) = \pm C x f(x)$  with  $C > 0$ , we have

$$Z(f') = Z(xf).$$

Note that  $Z(f') = Z(xf)$  is closed (Exercise 4.3) and not equal to  $[a, b]$  (by applying the same argument in (a)). Hence the complement of  $Z(f') = Z(xf)$  is open and nonempty, which can be written as the union of an at most countable collection of disjoint segments (Exercise 2.29).

(d) Consider any nonempty open interval in (c), say

$$(c, d) \subseteq [a, b].$$

By construction,  $f'(x) \neq 0$  for all  $x \in (c, d)$ . Since  $f'(x)$  is continuous, by Theorem 4.23 there are only two mutually exclusive possible cases:

- (i)  $f'(x) > 0$  for all  $x \in (c, d)$ ,
- (ii)  $f'(x) < 0$  for all  $x \in (c, d)$ .

Similar result for  $xf(x)$ . Therefore, the sign “ $\pm$ ” of  $f'(x) = \pm Cxf(x)$  are unchanged on  $(c, d)$ , that is,

- (i)  $f'(x) = Cxf(x)$  for all  $x \in (c, d)$ ,
- (ii)  $f'(x) = -Cxf(x)$  for all  $x \in (c, d)$ ,
- (e) Suppose  $f'(x) = Cxf(x)$  on  $(c, d)$ . Since  $f'(x)$  and  $xf(x)$  are both vanishing at  $x = c$  and  $x = d$ ,  $f'(x) = Cxf(x)$  at  $x = c$  and  $x = d$ . So

$$f'(x) = Cxf(x) \text{ if } x \in [c, d].$$

Define

$$\phi(x, y) = Cxy$$

be a real function on  $R = [c, d] \times \mathbb{R}$ . And consider the initial-value problem

$$y' = \phi(x, y) \quad \text{with} \quad y(c) = 0.$$

Then

$$|\phi(x, y_2) - \phi(x, y_1)| = Cx|y_2 - y_1| \leq A|y_2 - y_1|$$

where  $A = C \cdot \max\{|c|, |d|\}$  is a constant. By Exercise 5.27, this initial-value problem has at most one solution. Clearly,  $y = f(x) = 0$  on  $[c, d]$  is one solution of this initial-value problem, contrary to the construction of  $[c, d]$ . Similar result for the case  $f'(x) = -Cxf(x)$ .

Therefore, the equality does not hold.

□

**Exercise 6.16.** For  $1 < s < \infty$ , define

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

(This is Riemann's zeta function, of great importance in the study of the distribution of prime numbers.) Prove that

(a)

$$\zeta(s) = s \int_1^{\infty} \frac{[x]}{x^{s+1}} dx$$

and that

(b)

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{\infty} \frac{x - [x]}{x^{s+1}} dx,$$

where  $[x]$  denotes the greatest integer  $\leq x$ . Prove that the integral in (b) converges for all  $s > 0$ . (Hint: To prove (a), compute the difference between the integral over  $[1, N]$  and the  $N$ th partial sum of the series that defines  $\zeta(s)$ .)

*Proof of (a) (Hint).*

(1) Define

$$a_N = s \int_1^N \frac{[x]}{x^{s+1}} dx - \sum_{n=1}^N \frac{1}{n^s}.$$

Hence

$$\begin{aligned} s \int_1^N \frac{[x]}{x^{s+1}} dx &= \sum_{n=1}^{N-1} s \int_n^{n+1} \frac{[x]}{x^{s+1}} dx \\ &= \sum_{n=1}^{N-1} s \int_n^{n+1} \frac{n}{x^{s+1}} dx \\ &= \sum_{n=1}^{N-1} n \left( \frac{1}{n^s} - \frac{1}{(n+1)^s} \right) \\ &= \left( \sum_{n=1}^N \frac{1}{n^s} \right) - \frac{1}{N^{s-1}}, \end{aligned}$$

or

$$a_N = -\frac{1}{N^{s-1}}.$$

(2) So

$$\lim_{N \rightarrow \infty} a_N = 0$$

(since  $s - 1 > 0$ ). By Theorem 3.28,  $\zeta(s)$  converges if  $s > 1$ . Hence

$$\lim_{N \rightarrow \infty} s \int_1^N \frac{[x]}{x^{s+1}} dx = \zeta(s)$$

converges.

(3) Hence given any real  $b > 1$ , there exists an integer  $N$  such that  $N \leq b < N + 1$ . Since  $x \mapsto \frac{[x]}{x^{s+1}} \geq 0$  on  $[1, \infty)$ ,

$$s \int_1^N \frac{[x]}{x^{s+1}} dx \leq s \int_1^b \frac{[x]}{x^{s+1}} dx \leq s \int_1^{N+1} \frac{[x]}{x^{s+1}} dx.$$

Since  $b \rightarrow \infty$  if and only if  $N \rightarrow \infty$ ,

$$\begin{aligned} \lim_{N \rightarrow \infty} s \int_1^N \frac{[x]}{x^{s+1}} dx &\leq \lim_{b \rightarrow \infty} s \int_1^b \frac{[x]}{x^{s+1}} dx \leq \lim_{N \rightarrow \infty} s \int_1^{N+1} \frac{[x]}{x^{s+1}} dx \\ \implies \zeta(s) &\leq \lim_{b \rightarrow \infty} s \int_1^b \frac{[x]}{x^{s+1}} dx \leq \zeta(s). \end{aligned}$$

Hence

$$\lim_{b \rightarrow \infty} s \int_1^b \frac{[x]}{x^{s+1}} dx = s \int_1^\infty \frac{[x]}{x^{s+1}} dx = \zeta(s)$$

(in the sense of Exercise 6.8).

□

*Proof of (b).*

(1) *Show that*

$$s \int_1^\infty \frac{1}{x^s} dx = \frac{s}{s-1}.$$

Given any real  $b > 1$ . By the fundamental theorem of calculus (Theorem 6.21),

$$s \int_1^b \frac{1}{x^s} dx = \frac{s}{s-1} - \frac{s}{(s-1)b^{s-1}}.$$

Hence

$$\lim_{b \rightarrow \infty} s \int_1^b \frac{1}{x^s} dx = \frac{s}{s-1}$$

since  $\frac{1}{b^{s-1}} \rightarrow 0$  as  $b \rightarrow \infty$  (in the sense of Exercise 6.8).

(2) By (a) and (1),  $s \int_1^\infty \frac{x-[x]}{x^{s+1}} dx$  exists and equal to

$$s \int_1^\infty \frac{x-[x]}{x^{s+1}} dx = s \int_1^\infty \frac{1}{x^s} dx - s \int_1^\infty \frac{[x]}{x^{s+1}} dx = \frac{s}{s-1} - \zeta(s).$$

The result is established.

(3) *Show that*

$$\int_1^\infty \frac{x-[x]}{x^{s+1}} dx$$

converges for all  $s > 0$ . Note that  $0 \leq x - [x] < 1$  on  $[1, \infty)$ . So

$$\int_1^b \frac{x-[x]}{x^{s+1}} dx \leq \int_1^b \frac{1}{x^{s+1}} dx = \frac{1}{s} - \frac{1}{sb^s}.$$

Since  $\frac{1}{sb^s} \rightarrow 0$  as  $b \rightarrow \infty$ ,

$$\int_1^\infty \frac{x-[x]}{x^{s+1}} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{x-[x]}{x^{s+1}} dx \leq \lim_{b \rightarrow \infty} \frac{1}{s} - \frac{1}{sb^s} = \frac{1}{s}.$$

Note that  $\frac{1}{s}$  is finite, and thus the integral  $\int_1^\infty \frac{x-[x]}{x^{s+1}} dx$  converges.

□

*Note.*

- (1) The integral  $\int_1^\infty \frac{[x]}{x^{s+1}} dx$  does not converge for all  $1 \geq s > 0$ .
- (2) Compare to Exercise 8.9.
- (3) **Euler's summation formula.** (Theorem 7.13 in the textbook: *Tom. M. Apostol, Mathematical Analysis, 2nd edition.*) If  $f$  has a continuous derivative  $f'$  on  $[a, b]$ , then we have

$$\sum_{a < n \leq b} f(n) = \int_a^b f(x) dx + \int_a^b f'(x) \{x\} dx + f(a)\{a\} - f(b)\{b\},$$

where  $\sum_{a < n \leq b}$  means the sum from  $n = [a] + 1$  to  $n = [b]$ . When  $a$  and  $b$  are integers, this becomes

$$\sum_{n=a}^b f(n) = \int_a^b f(x) dx + \int_a^b f'(x) \left( \{x\} - \frac{1}{2} \right) dx + \frac{f(a) + f(b)}{2}.$$

By taking  $f(x) = \frac{1}{x^s}$  we can get (a) as well.

**Exercise 6.17.** Suppose  $\alpha$  increases monotonically on  $[a, b]$ ,  $g$  is continuous, and  $g(x) = G'(x)$  for  $a \leq x \leq b$ . Prove that

$$\int_a^b \alpha(x) g(x) dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G d\alpha.$$

(Hint: Take  $g$  real, without loss of generality. Given  $P = \{a = x_0, x_1, \dots, x_n = b\}$ , choose  $t_i \in (x_{i-1}, x_i)$  so that  $g(t_i)\Delta x_i = G(x_i) - G(x_{i-1})$ . Show that

$$\sum_{i=1}^n \alpha(x_i) g(t_i) \Delta x_i = G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^n G(x_{i-1}) \Delta \alpha_i.$$

*Proof (Hint).* Given  $\varepsilon > 0$ .

- (1) Take  $g$  real, without loss of generality. Given any partition

$$P = \{a = x_0 < x_1 < \dots < x_n = b\}$$

of  $[a, b]$ .

- (2) By the mean value theorem (Theorem 5.10), there is  $t_i \in (x_{i-1}, x_i)$  such that

$$G(x_i) - G(x_{i-1}) = (x_i - x_{i-1})G'(t_i) = g(t_i)\Delta x_i.$$

(3) Hence,

$$\begin{aligned}
\sum_{i=1}^n \alpha(x_i)g(t_i)\Delta x_i &= \sum_{i=1}^n \alpha(x_i)(G(x_i) - G(x_{i-1})) \\
&= \sum_{i=1}^n \alpha(x_i)G(x_i) - \sum_{i=1}^n \alpha(x_i)G(x_{i-1}) \\
&= \underbrace{G(b)\alpha(b) - G(a)\alpha(a) + \sum_{i=1}^n \alpha(x_{i-1})G(x_{i-1}) - \sum_{i=1}^n \alpha(x_i)G(x_{i-1})}_{\text{adjust the index of } \sum_{i=1}^n \alpha(x_i)G(x_i)} \\
&= G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^n G(x_{i-1})\Delta \alpha_i.
\end{aligned}$$

(4) Since  $G(x)$  is differentiable on  $[a, b]$ ,  $G(x)$  is continuous (Theorem 5.2) and thus  $G \in \mathcal{R}(\alpha)$  (Theorem 6.8). So there is a partition  $P_1$  such that

$$\left| \sum_{j=1}^n G(t_j)\Delta \alpha_j - \int_a^b G d\alpha \right| < \varepsilon$$

whenever  $t_j \in [x_{j-1}, x_j]$  (Theorem 6.7). In particular, we pick  $t_j = x_{j-1} \in [x_{j-1}, x_j]$  for all  $j$ , that is,

$$\left| \sum_{j=1}^n G(x_{j-1})\Delta \alpha_j - \int_a^b G d\alpha \right| < \varepsilon.$$

Note that if  $P^*$  is a refinement of  $P$ , the result is true too (Theorem 6.4).

(5) Since  $\alpha$  increases monotonically,  $\alpha \in \mathcal{R}$  (Theorem 6.9). Since  $g$  is continuous,  $g \in \mathcal{R}$  (Theorem 6.8). Hence  $\alpha g \in \mathcal{R}$  (Theorem 6.13). So there is a partition  $P_2$  such that

$$\left| \sum_{k=1}^m \alpha(t_k)g(t_k)\Delta x_k - \int_a^b \alpha g dx \right| < \varepsilon$$

whenever  $t_k \in [x_{k-1}, x_k]$  (Theorem 6.7). In particular, we pick  $t_k = x_k \in [x_{k-1}, x_k]$  for all  $k$ , that is,

$$\left| \sum_{k=1}^m \alpha(x_k)g(x_k)\Delta x_k - \int_a^b \alpha g dx \right| < \varepsilon.$$

Note that if  $P^*$  is a refinement of  $P$ , the result is true too (Theorem 6.4).



- (6) Since  $g$  is continuous on a compact set  $[a, b]$ ,  $g$  is uniformly continuous. Hence there exists  $\delta > 0$  such that

$$|g(y) - g(x)| < \varepsilon$$

whenever  $|y - x| < \delta$  and  $x, y \in [a, b]$ . For such  $\delta$ , we construct a partition  $P_3$  such that

$$|g(t_l) - g(x_l)| < \varepsilon$$

whenever  $t_l \in [x_{l-1}, x_l]$ . (For example, we might take

$$P_3 = \left\{ a, a + \frac{1}{N}(b-a), a + \frac{2}{N}(b-a), \dots, a + \frac{N-1}{N}(b-a), b \right\}$$

where  $N$  is an integer  $\geq \frac{b-a}{\delta}$ .) Hence

$$\begin{aligned} & \left| \sum_{l=1}^N \alpha(x_l)g(t_l)\Delta x_l - \sum_{l=1}^N \alpha(x_l)g(x_l)\Delta x_l \right| \\ &= \left| \sum_{l=1}^N \alpha(x_l)[g(t_l) - g(x_l)]\Delta x_l \right| \\ &\leq \sum_{l=1}^N |\alpha(x_l)| \cdot |g(t_l) - g(x_l)| \cdot \Delta x_l \\ &\leq M\varepsilon \sum_{l=1}^N \Delta x_l \\ &= M(b-a)\varepsilon. \end{aligned}$$

Note that if  $P^*$  is a refinement of  $P$ , the result is true too (by the uniform convergence of  $g$ ).

- (7) Let  $P = \{a = x_0 < x_1 < \dots < x_n = b\}$  be a common refinement of  $P_1, P_2$

and  $P_3$ . By (3)(4)(5)(6) we have

$$\begin{aligned}
& \left| \int_a^b \alpha(x)g(x)dx - G(b)\alpha(b) + G(a)\alpha(a) + \int_a^b Gd\alpha \right| \\
&= \left| \int_a^b \alpha(x)g(x)dx - \sum_{i=1}^n \alpha(x_i)g(t_i)\Delta x_i + \int_a^b Gd\alpha - \sum_{i=1}^n G(x_{i-1})\Delta\alpha_i \right| \\
&\leq \left| \int_a^b \alpha(x)g(x)dx - \sum_{i=1}^n \alpha(x_i)g(t_i)\Delta x_i \right| + \left| \int_a^b Gd\alpha - \sum_{i=1}^n G(x_{i-1})\Delta\alpha_i \right| \\
&\leq \left| \int_a^b \alpha(x)g(x)dx - \sum_{i=1}^n \alpha(x_i)g(x_i)\Delta x_i \right| + \left| \sum_{i=1}^n \alpha(x_i)g(x_i)\Delta x_i - \sum_{i=1}^n \alpha(x_i)g(t_i)\Delta x_i \right| \\
&\quad + \left| \int_a^b Gd\alpha - \sum_{i=1}^n G(x_{i-1})\Delta\alpha_i \right| \\
&\leq \varepsilon + M(b-a)\varepsilon + \varepsilon \\
&= (M(b-a) + 2)\varepsilon.
\end{aligned}$$

Since  $\varepsilon$  is arbitrary,

$$\left| \int_a^b \alpha(x)g(x)dx - G(b)\alpha(b) + G(a)\alpha(a) + \int_a^b Gd\alpha \right| = 0,$$

or

$$\int_a^b \alpha(x)g(x)dx - G(b)\alpha(b) + G(a)\alpha(a) + \int_a^b Gd\alpha = 0,$$

or

$$\int_a^b \alpha(x)g(x)dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b Gd\alpha.$$

□

**Exercise 6.18.** Let  $\gamma_1, \gamma_2, \gamma_3$  be curves in the complex plane, defined on  $[0, 2\pi]$  by

$$\begin{aligned}
\gamma_1 &= \exp(it), \\
\gamma_2 &= \exp(2it), \\
\gamma_3 &= \exp\left(2\pi it \sin\left(\frac{1}{t}\right)\right).
\end{aligned}$$

Show that these three curves have the same range, that  $\gamma_1$  and  $\gamma_2$  are rectifiable, that the length of  $\gamma_1$  is  $2\pi$ , that the length of  $\gamma_2$  is  $4\pi$ , and that  $\gamma_3$  is not rectifiable.

Might assume that  $\gamma_3(0) = 1$ .

*Proof.* Write  $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ .

- (1) *Show that  $\gamma_1$  has the range  $S^1$ .* Given any  $z \in S^1$ . Theorem 8.7(d) implies that there is a unique  $t \in [0, 2\pi)$  such that  $\exp(it) = z$ .
- (2) *Show that  $\gamma_1$  is rectifiable and its length is  $2\pi$ .* By the definition of  $\exp(z)$ ,

$$\gamma_1'(t) = i \exp(it),$$

which is continuous on  $[0, 2\pi]$ . Hence  $\gamma_1$  is rectifiable, and its length is

$$\Lambda(\gamma_1) = \int_0^{2\pi} |\gamma_1'(t)| dt = \int_0^{2\pi} dt = 2\pi$$

(Theorem 6.27).

- (3) *Show that  $\gamma_2$  has the range  $S^1$ .* Similar to (1). Given any  $z \in S^1$ . Theorem 8.7(d) implies that there is a unique  $t \in [0, 2\pi)$  such that  $\exp(it) = z$ . Write  $\exp(it) = \exp(2i(\frac{t}{2}))$  where  $\frac{t}{2} \in [0, \pi) \subseteq [0, 2\pi)$ .
- (4) *Show that  $\gamma_2$  is rectifiable and its length is  $4\pi$ .* Similar to (2).

$$\gamma_2'(t) = 2i \exp(2it),$$

and

$$\Lambda(\gamma_2) = \int_0^{2\pi} |\gamma_2'(t)| dt = \int_0^{2\pi} 2 dt = 4\pi.$$

- (5) *Show that  $\gamma_3$  has the range  $S^1$ .* Define

$$f(t) = \begin{cases} 0 & (t = 0), \\ t \sin \frac{1}{t} & (t \neq 0). \end{cases}$$

It suffices to show that  $f(I) \supseteq J$  for some segment  $I \subseteq [0, 2\pi]$  and some segment  $J$  in  $\mathbb{R}$  of the length  $\geq 1$  (Theorem 8.7(a)). Define  $I = [\frac{6}{7\pi}, \frac{6}{\pi}] \subseteq [0, 2\pi]$  and  $J = [-\frac{3}{7\pi}, \frac{3}{\pi}]$  of the length  $\frac{24}{7\pi} > 1$ . Hence  $f(I)$  is connected since  $I$  is connected (Theorem 4.22). Since

$$\begin{aligned} f\left(\frac{6}{7\pi}\right) &= \frac{6}{7\pi} \sin \frac{7\pi}{6} = -\frac{3}{7\pi}, \\ f\left(\frac{6}{\pi}\right) &= \frac{6}{\pi} \sin \frac{\pi}{6} = \frac{3}{\pi}, \end{aligned}$$

$f(I) \supseteq J$  (Theorem 2.47). The result is established.

- (6) *Show that  $\gamma_3$  is not rectifiable.*

(a) Since

$$\gamma'_3 = 2\pi i \left( \sin \frac{1}{t} - \frac{1}{t} \cos \frac{1}{t} \right) \exp \left( 2\pi i t \sin \left( \frac{1}{t} \right) \right)$$

is continuous on  $[c, 2\pi]$  for any  $c > 0$ ,  $\gamma_3$  is rectifiable on  $[c, 2\pi]$  (not on  $[0, 2\pi]$ ), and

$$\Lambda_{[c, 2\pi]}(\gamma_3) = \int_c^{2\pi} |\gamma'_3(t)| dt$$

on  $[c, 2\pi]$ .

(b)

$$\begin{aligned} \int_c^{2\pi} |\gamma'_3(t)| dt &= 2\pi \int_c^{2\pi} \left| \sin \frac{1}{t} - \frac{1}{t} \cos \frac{1}{t} \right| dt \\ &\geq 2\pi \int_c^{2\pi} \left| \frac{1}{t} \cos \frac{1}{t} \right| - 1 dt \\ &= 2\pi \int_c^{2\pi} \left| \frac{1}{t} \cos \frac{1}{t} \right| dt - 4\pi^2. \end{aligned}$$

(c) For any integer  $n > 0$ , we have

$$\begin{aligned} &\int_{(2n\pi + \frac{\pi}{3})^{-1}}^{(2n\pi - \frac{\pi}{3})^{-1}} \left| \frac{1}{t} \cos \frac{1}{t} \right| dt \\ &\geq \int_{(2n\pi + \frac{\pi}{3})^{-1}}^{(2n\pi - \frac{\pi}{3})^{-1}} \left( 2n\pi - \frac{\pi}{3} \right) \cdot \frac{1}{2} dt \\ &= \left[ \left( 2n\pi - \frac{\pi}{3} \right)^{-1} - \left( 2n\pi + \frac{\pi}{3} \right)^{-1} \right] \cdot \left( 2n\pi - \frac{\pi}{3} \right) \cdot \frac{1}{2} \\ &= \frac{\frac{\pi}{3}}{2n\pi + \frac{\pi}{3}} \\ &> \frac{1}{6} \cdot \frac{1}{n+1} \end{aligned}$$

since both  $t \mapsto \frac{1}{t} \geq 2n\pi - \frac{\pi}{3}$  and  $t \mapsto \cos t \geq \frac{1}{2}$  on  $\left[ \left( 2n\pi + \frac{\pi}{3} \right)^{-1}, \left( 2n\pi - \frac{\pi}{3} \right)^{-1} \right]$ .

(d) As  $c \geq \frac{1}{2N\pi - \frac{\pi}{3}}$  for some integer  $N$ , by (b)(c) we have

$$\begin{aligned}
\int_c^{2\pi} |\gamma_3'(t)| dt &\geq 2\pi \int_c^{2\pi} \left| \frac{1}{t} \cos \frac{1}{t} \right| dt - 4\pi^2 \\
&\geq 2\pi \sum_{n=1}^{n=N} \int_{(2n\pi + \frac{\pi}{3})^{-1}}^{(2n\pi - \frac{\pi}{3})^{-1}} \left| \frac{1}{t} \cos \frac{1}{t} \right| dt - 4\pi^2 \\
&\geq 2\pi \sum_{n=1}^{n=N} \frac{1}{6} \cdot \frac{1}{n+1} - 4\pi^2 \\
&= \frac{\pi}{3} \sum_{n=1}^{n=N} \frac{1}{n+1} - 4\pi^2.
\end{aligned}$$

(e) Hence

$$\Lambda(\gamma_3) \geq \Lambda_{[(2N\pi - \frac{\pi}{3})^{-1}, 2\pi]}(\gamma_3) \geq \frac{\pi}{3} \sum_{n=1}^{n=N} \frac{1}{n+1} - 4\pi^2.$$

Let  $N \rightarrow \infty$ , and thus  $\Lambda(\gamma_3)$  cannot be bounded (Theorem 3.28).

□

**Exercise 6.19.** Let  $\gamma_1$  be a curve in  $\mathbb{R}^k$ , defined on  $[a, b]$ ; let  $\phi$  be a continuous 1-1 mapping of  $[c, d]$  onto  $[a, b]$ , such that  $\phi(c) = a$ ; and define  $\gamma_2(s) = \gamma_1(\phi(s))$ . Prove that  $\gamma_2$  is an arc, a closed curve, or a rectifiable curve if and only if the same is true of  $\gamma_1$ . Prove that  $\gamma_2$  and  $\gamma_1$  have the same length.

*Proof.*

(1) Show that  $\phi$  is strictly monotonic. Similar to Exercise 4.15.

(a) (Reductio ad absurdum) If  $\phi$  were not strictly monotonic, then there exist  $a < c < b \in \mathbb{R}^1$  such that

$$\phi(a) \leq \phi(c) \geq \phi(b)$$

or

$$\phi(a) \geq \phi(c) \leq \phi(b).$$

Since  $\phi$  is one-to-one, all equalities does not hold. Hence

$$\phi(a) < \phi(c) > \phi(b)$$

or

$$\phi(a) > \phi(c) < \phi(b).$$

(b) The case  $\phi(a) < \phi(c) > \phi(b)$ . Take

$$t = \frac{\max\{\phi(a), \phi(b)\} + \phi(c)}{2}$$

so that  $\phi(c) > t > \phi(a)$  and  $\phi(c) > t > \phi(b)$ . By Theorem 4.23 there exist  $\xi_1 \in (a, c)$  and  $\xi_2 \in (c, b)$  such that  $\phi(\xi_1) = \phi(\xi_2) = t$ . Here  $\xi_1 \neq \xi_2$ , contrary to the injectivity of  $\phi$ .

(c) The case  $\phi(a) > \phi(c) < \phi(b)$ . The proof is similar to (b).

(d) By (b)(c),  $\phi$  is strictly monotonic.

- (2)  $\phi(d) = b$  since  $\phi$  is strictly monotonic (by (1)), surjective and  $\phi(c) = a$ .
- (3) The inverse mapping  $\phi^{-1}$  is a continuous and injective mapping of  $[a, b]$  onto  $[c, d]$  since  $\phi$  is continuous and injective on a compact set  $[c, d]$  (Theorem 4.17).
- (4) Show that  $\gamma_2$  is an arc if and only if  $\gamma_1$  is an arc. Note the the composition of two injective maps is injective. Hence the result is established since  $\gamma_2 = \gamma_1 \circ \phi$  and  $\gamma_1 = \gamma_2 \circ \phi^{-1}$ .
- (5) Show that  $\gamma_2$  is a closed curve if and only if  $\gamma_1$  is a closed curve. Since  $\gamma_2 = \gamma_1 \circ \phi$  and  $\gamma_1 = \gamma_2 \circ \phi^{-1}$  (as in (4)),  $\gamma_1(a) = \gamma_1(b)$  if and only if  $\gamma_2(c) = \gamma_2(d)$ .
- (6) Show that  $\gamma_2$  is a rectifiable curve if and only if  $\gamma_1$  is a rectifiable curve. Given any partition  $P_1 = \{x_0, \dots, x_n\}$  of  $[a, b]$ , there is a corresponding partition  $P_2 = \{\phi^{-1}(x_0), \dots, \phi^{-1}(x_n)\}$  of  $[c, d]$ , and vice versa. (Given a partition  $P_2 = \{x_0, \dots, x_n\}$  of  $[c, d]$ , there is a corresponding partition  $P_1 = \{\phi(x_0), \dots, \phi(x_n)\}$  of  $[a, b]$ .) Again, since  $\gamma_2 = \gamma_1 \circ \phi$  and  $\gamma_1 = \gamma_2 \circ \phi^{-1}$  (as in (4)),

$$\Lambda(P_1, \gamma_1) = \Lambda(P_2, \gamma_2).$$

Hence  $\gamma_2$  is rectifiable if and only if  $\gamma_1$  is rectifiable.

- (7) Show that  $\gamma_2$  and  $\gamma_1$  have the same length. Take the supremum over all partitions  $P_1$  of  $[a, b]$  to get

$$\Lambda(P_1, \gamma_1) = \Lambda(P_2, \gamma_2) \leq \Lambda(\gamma_1).$$

Hence  $\Lambda(\gamma_1)$  is an upper of  $\Lambda(P_2, \gamma_2)$ . So

$$\Lambda(\gamma_2) \leq \Lambda(\gamma_1).$$

Similarly,  $\Lambda(\gamma_1) \leq \Lambda(\gamma_2)$ . Therefore  $\Lambda(\gamma_1) = \Lambda(\gamma_2)$  (whether  $\Lambda(\gamma_1)$  or  $\Lambda(\gamma_2)$  is finite or not).

□