

Chapter 1: The Real and Complex Number Systems

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Unless the contrary is explicitly stated, all numbers that are mentioned in these exercise are understood to be real.

Exercise 1.1. *If r is a rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.*

Proof. Assume $r + x \in \mathbb{Q}$. \mathbb{Q} is a field, then $-r \in \mathbb{Q}$ for any $r \in \mathbb{Q}$. So $(-r) + (r + x) = (-r + r) + x = 0 + x = x \in \mathbb{Q}$, a contradiction.

Similarly, assume $rx \in \mathbb{Q}$. $r \in \mathbb{Q}$ with $r \neq 0$ implies that there exists an element $1/r \in \mathbb{Q}$ such that $r \cdot (1/r) = 1$. So $(1/r) \cdot (rx) = ((1/r) \cdot r) \cdot x = 1 \cdot x = x \in \mathbb{Q}$, a contradiction. \square

Exercise 1.2. *Prove that there is no rational number whose square is 12.*

Apply the argument in Example 1.1. Again we can examine this situation a little more closely. Let A be the set of all positive rational p such that $p^2 < 12$ and let B be the set of all positive rational p such that $p^2 > 12$. We might show that A contains no largest number and B contains no largest number again.

In fact, we can associate with each rational $p > 0$ the number

$$q = p - \frac{p^2 - 12}{p + 12} = \frac{12p + 12}{p + 12}.$$

Then

$$q^2 - 12 = \frac{132(p^2 - 12)}{(p + 12)^2}.$$

If $p \in A$ then $p^2 - 12 < 0$, $q > p$ and $q^2 < 12$. Thus $q \in A$. If $p \in B$ then $p^2 - 12 > 0$, $0 < q < p$ and $q^2 > 12$. Thus $q \in B$.

Proof (Example 1.1). We now show that the equation

$$p^2 = 12$$

is not satisfied by any rational p . If there were such a $p \in \mathbb{Q}$, we could write $p = \frac{m}{n}$ where $m, n \in \mathbb{Z}$ are relatively prime. Let us assume this is done. Then

$p^2 = 12$ implies

$$m^2 = 12n^2.$$

This shows that $3 \mid m^2$. Hence $3 \mid m$ (since 3 is a prime in \mathbb{Z}), and so m^2 is divisible by 9. It follows that $12n^2$ is divisible by 9, so that $4n^2$ is divisible by 3, so that n^2 is divisible by 3, which implies that $3 \mid n$. That is, both m and n have a common factor $3 > 1$, contrary to our choice of m and n . Hence $p^2 = 12$ is impossible for rational p . \square

Exercise 1.12. *If z_1, \dots, z_n are complex, prove that*

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$$

Proof. Use mathematical induction on n . $n = 2$ is established by Theorem 1.33 (e). Suppose the inequality holds on $n = k$, then $n = k + 1$ we again apply Theorem 1.33 (e) to get the result, say

$$\begin{aligned} |z_1 + z_2 + \cdots + z_k + z_{k+1}| &\leq |z_1 + z_2 + \cdots + z_k| + |z_{k+1}| \\ &\leq |z_1| + |z_2| + \cdots + |z_k| + |z_{k+1}| \end{aligned}$$

\square

Corollary. *If $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^k$, then*

$$|\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_n| \leq |\mathbf{x}_1| + |\mathbf{x}_2| + \cdots + |\mathbf{x}_n|.$$

Here we might use Theorem 1.37 (e) to prove it. Since the norm $|\cdot|$ on \mathbb{C} is the same as the norm on \mathbb{R}^2 , we might prove this corollary first and set $k = 2$ on $\mathbb{R}^k = \mathbb{R}^2$.