Chapter 2: Basic Topology

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Exercise 2.1. *Prove that the empty set is a subset of every set.*

Proof. By Definitions 1.3,

(1) The set which contains no element will be called the **empty set**,

(2) If A and B are sets, and if every element of A is an element of B, we say that A is a **subset** of B,

every element of the empty set (there are none) belongs to every set. That is, the empty set is a subset of every set. $\square$

Exercise 2.2. *A complex number z is said to be algebraic if there are integers $a_0, ..., a_n$, not all zero, such that*

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic numbers is countable. (Hint: For every positive integer N there are only finitely many equations with

$$n + |a_0| + |a_1| + \cdots + |a_n| = N.$$

Might assume $a_0 \neq 0$.

For example, all rational numbers are algebraic since $p = \frac{\alpha}{\beta}$ (where $\alpha, \beta \in \mathbb{Z}$) is a root of $\beta z - \alpha = 0$.

Besides, $z = \sqrt{2} + \sqrt{3}$ is algebraic since $z^4 - 10z^2 + 1 = 0$. In fact, $z = \pm\sqrt{2} \pm \sqrt{3}$ are also algebraic since $z^4 - 10z^2 + 1 = (z - \sqrt{2} - \sqrt{3})(z + \sqrt{2} - \sqrt{3})(z - \sqrt{2} + \sqrt{3})(z + \sqrt{2} + \sqrt{3})$.

Lemma. *The set of all polynomials over $\mathbb{Z}$ is countable implies that the set of algebraic numbers is countable.*

Proof of Lemma. By definition, we write the set of algebraic numbers as

$$S = \bigcup_{f(x) \in \mathbb{Z}[x]} \{z \in \mathbb{C} : f(z) = 0\}.$$

Since each polynomial of degree n has at most n roots, $\{z \in \mathbb{C} : f(z) = 0\}$ is finite for each given $f(x) \in \mathbb{Z}[x]$. So S is a countable union (by assumption) of finite sets, and hence at most countable. S is infinite since every integer α is a root of $f(z) = z - \alpha$. So S is countable. $\square$

Thus, it suffices to show that *the set of all polynomials over $\mathbb{Z}$ is countable.*

Proof (Hint). For every positive integer N there are only finitely many equations with $n + |a_0| + |a_1| + \cdots + |a_n| = N$. Write

$$P_N = \{f(x) \in \mathbb{Z}[x] : n + |a_0| + |a_1| + \cdots + |a_n| = N\}$$

where $f(x) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$ with $a_0 \neq 0$, and

$$P = \bigcup_{N=1}^{\infty} P_N.$$

P is the set of all polynomials over $\mathbb{Z}$.

Each P_N is finite for given N (since the equation $n + |a_0| + |a_1| + \cdots + |a_n| = N$ has finitely many solutions $(n, a_0, a_1, ..., a_n) \in \mathbb{Z}^{n+2}$). So P is a countable union of finite sets, and hence at most countable. P is infinite since $\mathbb{Z}$ is a subring of $\mathbb{Z}[x]$. So P is countable. $\square$

Proof (Theorem 2.13).

(1) *$\mathbb{Z}^N$ is countable for any integer $N > 0$.* Theorem 2.13.

(2) *The set of all polynomials over $\mathbb{Z}$ is countable.* Let

$$P_n = \{f \in \mathbb{Z}[x] : \deg f = n\},$$

and

$$P = \bigcup_{n=1}^{\infty} P_n = \mathbb{Z}[x].$$

Claim: P_n is countable. Define a 1-1 map $\varphi_n : P_n \to \mathbb{Z}^{n+1}$ by

$$\varphi_n(a_0 z^n + a_1 z^{n-1} + \cdots + a_n) = (a_0, a_1, ..., a_{n-1}, a_n).$$

By (1) and Theorem 2.8, P_n is countable. (P_n is infinite since $a_n \in \mathbb{Z}$.) Now P is a countable union of countable sets, and hence countable by Theorem 2.12.

$\square$

Proof (Unique factorization theorem).

(1) *The set of prime numbers is countable.* Write all primes in the ascending order as $p_1, p_2, ..., p_n, ...$ where $p_1 = 2, p_2 = 3, ..., p_{10001} = 104743, ...$ (See ProjectEuler 7: 10001st prime. Use sieve of Eratosthenes to get p_{10001}.)

(2) *The set of all polynomials over $\mathbb{Z}$ is countable.* Let

$$P_n = \{f \in \mathbb{Z}[x] : \deg f = n\},$$

and

$$P = \bigcup_{n=1}^{\infty} P_n = \mathbb{Z}[x].$$

Claim: P_n is countable. Define a map $\varphi_n : P_n \to \mathbb{Z}^+$ by

$$\varphi_n(a_0 z^n + a_1 z^{n-1} + \cdots + a_n) = p_1^{\psi(a_0)} p_2^{\psi(a_1)} \cdots p_{n+1}^{\psi(a_n)},$$

where ψ is a 1-1 correspondence from $\mathbb{Z}$ to $\mathbb{Z}^+$ (Example 2.5). By the unique factorization theorem, φ_n is 1-1. So P_n is countable by Theorem 2.8. (P_n is infinite since $a_n \in \mathbb{Z}$.) Now P is a countable union of countable sets, and hence countable by Theorem 2.12.

$\square$

Exercise 2.3. *Prove that there exist real numbers which are not algebraic.*

Proof (Exercise 2.2). If all real numbers were algebraic, then $\mathbb{R}$ is countable by Exercise 2.2, contrary to the fact that $\mathbb{R}$ is uncountable (Corollary to Theorem 2.43). $\square$

Proof (Liouville, 1844).

(1) **Lemma.** *If ξ is a real algebraic number of degree $n > 1$, then there is a constant $A > 0$ (depending on ξ) such that*

$$\left| \xi - \frac{h}{k} \right| \geq \frac{A}{k^n}$$

for all rational numbers $\frac{h}{k}$.

(a) If $\left| \xi - \frac{h}{k} \right| \geq 1$, pick $A = 1 > 0$.

(b) If $\left| \xi - \frac{h}{k} \right| < 1$, let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ be an irreducible polynomial of degree $n > 1$ over $\mathbb{Z}$ such that $f(\xi) = 0$. By the mean value theorem,

$$f(\xi) - f\left(\frac{h}{k}\right) = \left(\xi - \frac{h}{k}\right) f'(c)$$

for some $c \in \left(\xi - \frac{h}{k}, \xi + \frac{h}{k}\right) \subseteq (\xi - 1, \xi + 1)$. Notice that

(i) $f(\xi) = 0$ by definition.

(ii) $f\left(\frac{h}{k}\right) \neq 0$ since $\frac{h}{k}$ cannot be a root of $f(x)$. Otherwise f is of degree 1, contrary to the assumption of f.

(iii) $\left|f\left(\frac{h}{k}\right)\right| \geq \frac{1}{k^n}$ since

$$\begin{aligned} f\left(\frac{h}{k}\right) &= a_0 + a_1\left(\frac{h}{k}\right) + \cdots + a_n\left(\frac{h}{k}\right)^n \neq 0, \\ k^n f\left(\frac{h}{k}\right) &= a_0 k^n + hk^{n-1}a_1 + \cdots + h^n a_n \neq 0, \\ k^n \left|f\left(\frac{h}{k}\right)\right| &\geq 1. \end{aligned}$$

(iv) $|f'(c)| \leq \sup_{x\in[\xi-1,\xi+1]} |f'(x)|$ since $c \in [\xi - 1, \xi + 1]$ and $f'(x)$ is continuous or bounded on a compact set $[\xi - 1, \xi + 1]$.

By (i)-(iv),

$$\begin{aligned} \left|f(\xi) - f\left(\frac{h}{k}\right)\right| &= \left|\left(\xi - \frac{h}{k}\right) f'(c)\right|, \\ \frac{1}{k^n} \leq \left|f\left(\frac{h}{k}\right)\right| &= \left|\xi - \frac{h}{k}\right| |f'(c)| \leq \left|\xi - \frac{h}{k}\right| \cdot \sup_{x\in[\xi-1,\xi+1]} |f'(x)|. \end{aligned}$$

Pick $A = (1 + \sup_{x\in[\xi-1,\xi+1]} |f'(x)|)^{-1} > 0$.

By (a)(b), we arrange $A = \min(1, (1+\sup_{x\in[\xi-1,\xi+1]} |f'(x)|)^{-1}) > 0$ to fit the inequality.

(2) $\xi = \sum_{n=0}^{\infty} 10^{-n!}$ *is transcendental.*

(a) Let $k_j = 10^{j!}$, $h_j = 10^{j!} \sum_{n=0}^{j} 10^{-n!}$. Then

$$\left|\xi - \frac{h_j}{k_j}\right| = \sum_{n=j+1}^{\infty} 10^{-n!} < \sum_{n=(j+1)!}^{\infty} 10^{-n} = \frac{A_j}{k_j^j}$$

where $A_j = \frac{10}{9} \cdot 10^{-j!}$.

(b) If ξ were a real algebraic number of degree $d > 1$, then by Lemma and (a),

$$\frac{A}{k_j^d} < \left|\xi - \frac{h_j}{k_j}\right| < \frac{A_j}{k_j^j} < \frac{A_j}{k_j^d}$$

for some $A > 0$ and $j \geq d$, or $0 < A < A_j$. Since j is arbitrary, $A_j \to 0$ as $j \to \infty$, contrary to $A > 0$.

(c) If ξ were a real algebraic number of degree $d = 1$, $\xi = \frac{h}{k}$ is a rational number. So

$$\left|\xi - \frac{h_j}{k_j}\right| = \left|\frac{h}{k} - \frac{h_j}{k_j}\right| = \left|\frac{hk_j - kh_j}{kk_j}\right| \geq \left|\frac{1}{kk_j}\right| = \frac{|k|^{-1}}{k_j}$$

for all j. (It is impossible that $hk_j - kh_j = 0$ or $\frac{h}{k} = \frac{h_i}{k_j}$ since $|\frac{h}{k} - \frac{h_j}{k_j}| = \sum_{n=j+1}^{\infty} 10^{-n!} > 0$ for all j.) Again by (a),

$$\frac{|k|^{-1}}{k_j} \leq \left|\xi - \frac{h_j}{k_j}\right| < \frac{A_j}{k_j^j} < \frac{A_j}{k_j},$$

or $0 < |k|^{-1} < A_j$. (Similar to (b).) Since j is arbitrary, $A_j \to 0$ as $j \to \infty$, contrary to $|k|^{-1} > 0$.

□

Exercise 2.4. *Is the set of all irrational real numbers countable?*

Proof (Reductio ad absurdum). If $\mathbb{R} - \mathbb{Q}$ were countable, then $\mathbb{R} = \mathbb{Q} \bigcup (\mathbb{R} - \mathbb{Q})$ is countable (Theorem 2.12), contrary to the fact that $\mathbb{R}$ is uncountable (Corollary to Theorem 2.43). □

Exercise 2.5. *Construct a bounded set of real numbers with exactly three limit points.*

Proof (Exercise 2.12). Let

$$K_p = \{p\} \bigcup \left\{ p + \frac{1}{n} : n \in \mathbb{Z}^+ \right\} \subseteq \mathbb{R}^1$$

be a compact set of $\mathbb{R}^1$ with exactly one limit point $p \in \mathbb{R}^1$ (Exercise 2.12). Then

$$K_{1989} \cup K_6 \cup K_4$$

is a compact set of $\mathbb{R}^1$ with exactly three limit points $1989, 6, 4 \in \mathbb{R}^1$. □

Exercise 2.6. *Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\overline{E}$ have the same limit points. (Recall that $\overline{E} = E \cup E'$.) Do E and E' always have the same limit points?*

Proof.

(1) *Show that E' is closed.*

  (a) *Use Definition 2.18 (d).*

    (i) It suffices to show every limit point of E' is a limit point of E. Given a limit point p of E', so that every open neighborhood U of p contains a point $q_0 \neq p$ such that $q_0 \in E'$.

(ii) Since q_0 is a limit point of E, there is an open neighborhood V of q_0 contains a point $q \neq q_0$ such that $q \in E$, where

$$V = U \cap B\left(q_0; \frac{1}{2}d_E(p, q_0)\right) \subseteq U$$

($B(x; r)$ is the open ball with center at x and radius r).

(iii) By the construction of V, for such open neighborhood U of p, there is $q \neq p$ and $q \in V \subseteq U$ and $q \in E$. That is, p is a limit point of E.

(b) *Use Definition 2.18 (e).*

(i) To show E' is closed or $X - E'$ is open, it suffices to show every point of $X - E'$ is an interior point of $X - E'$.

(ii) Given a point $p \in X - E'$, or p is not a limit point of E. There is an open neighborhood U of p contains no point $q \neq p$ such that $q \in E$.

(iii) To show U is an open neighborhood of p such that $U \subseteq X - E'$, it suffices to no point $q \neq p$ such that $q \in E'$. If there were a limit point q of E such that $q \neq p$ and $q \in U$, then

$$V = U \cap B\left(q; \frac{1}{2}d_E(p, q)\right) \subseteq U$$

is an open neighborhood of q contains no point of E, contrary to the assumption $q \in E'$. So $U \subseteq X - E'$ is an open neighborhood of $p \in X - E'$.

(2) *Show that* $E' = \overline{E}'$. It suffices to show $E' \supseteq \overline{E}'$. ($E' \subseteq \overline{E}'$ holds trivially since $E \subseteq \overline{E}$). Given a limit point p of $\overline{E} = E \cup E'$.

(a) p is a limit point of E. Nothing to do.

(b) p is a limit point of E'. Since p is a limit point of E' and E' is a closed set, $p \in E'$, or p is a limit point of E.

In any case, $E' \supseteq \overline{E}'$.

(3) *E and E' might not have the same limit points.* Let

$$E = \left\{\frac{1}{n} : n \in \mathbb{Z}^+\right\} \subseteq \mathbb{R}^1.$$

Then $E' = \{0\}$ and thus $(E')' = \varnothing$.

□

Exercise 2.7. *Let $A_1, A_2, A_3, ...$ be subsets of a metric space.*

(a) *If* $B_n = \bigcup_{i=1}^{n} A_i$, *prove that* $\overline{B_n} = \bigcup_{i=1}^{n} \overline{A_i}$, *for* $n = 1, 2, 3, ...$

(b) *If* $B = \bigcup_{i=1}^{\infty} A_i$, *prove that* $\overline{B} \supseteq \bigcup_{i=1}^{\infty} \overline{A_i}$.

Show, by an example, that this inclusion can be proper.

Proof of (a).

(1) *Show that* $\overline{B_n} \subseteq \bigcup_{i=1}^{n} \overline{A_i}$. Since $A_i \subseteq \overline{A_i}$ for any i, we have

$$B_n = \bigcup_{i=1}^{n} A_i \subseteq \bigcup_{i=1}^{n} \overline{A_i}.$$

Since $\bigcup_{i=1}^{n} \overline{A_i}$ is a union of finitely many closed set $\overline{A_i}$, $\bigcup_{i=1}^{n} \overline{A_i}$ is closed (Theorem 2.24(d)). By Theorem 2.27(c), $\overline{B_n} \subseteq \bigcup_{i=1}^{n} \overline{A_i}$.

(2) *Show that* $\overline{B_n} \supseteq \bigcup_{i=1}^{n} \overline{A_i}$. Same argument in the proof of (b).

$\Box$

Proof of (b). Since $\bigcup_{j=1}^{\infty} A_j \supseteq A_i$ for any i, by the monotonicity of closure, we have $\overline{\bigcup_{j=1}^{\infty} A_j} \supseteq \overline{A_i}$ for any i, or $\overline{B} \supseteq \bigcup_{i=1}^{\infty} \overline{A_i}$. $\Box$

Proof of proper inclusion in (b). Let

$$A_n = \left(\frac{1}{n}, \infty \right) \subseteq \mathbb{R}^1$$

for any $n \in \mathbb{Z}^+$. Then

$$\begin{aligned} \bigcup_{n=1}^{\infty} A_n = (0, \infty) &\Longrightarrow \overline{\bigcup_{n=1}^{\infty} A_n} = \overline{(0, \infty)} = [0, \infty), \\ \overline{A_n} = \left[\frac{1}{n}, \infty \right) &\Longrightarrow \bigcup_{n=1}^{\infty} \overline{A_n} = \bigcup_{n=1}^{\infty} \left[\frac{1}{n}, \infty \right) = (0, \infty). \end{aligned}$$

$\Box$

Exercise 2.8. *Is every point of every open set* $E \subseteq \mathbb{R}^2$ *a limit point of* E*? Answer the same question for closed sets in* $\mathbb{R}^2$*.*

It is not true for all metric spaces X. The (discrete) metric in Exercise 2.10 implies no limit point exists in X.

Proof.

(1) *Show that for every open set* $E \subseteq \mathbb{R}^k$, $E \subseteq E'$. Given any point $\mathbf{p} \in E$, we shall show $\mathbf{p}$ is a limit point of E.

(a) Since E is open, there is an open neighborhood $B(\mathbf{p}; r_0) \subseteq E$ for some $r_0 > 0$.

(b) *In particular, given any $s \in \mathbb{R}$ such that $0 < s < r_0$, we can find*

$$\mathbf{q} \in B(\mathbf{p}; s) \subseteq B(\mathbf{p}; r_0) \subseteq E$$

such that $\mathbf{q} \neq \mathbf{p}$. Explicitly, write

$$\mathbf{p} = (p_1, \ldots, p_k)$$

and choose

$$\mathbf{q} = \left(p_1 + \frac{s}{89}, p_2, \ldots, p_k\right) \neq \mathbf{p}$$

(since $s > 0$). Clearly, $\mathbf{q}$ is well-defined in $\mathbb{R}^k$ and $|\mathbf{q} - \mathbf{p}| = \frac{s}{89} < s$ or $\mathbf{q} \in B(\mathbf{p}; s)$.

(c) Now given every open neighborhood $B(\mathbf{p}, r)$ of $\mathbf{p}$. We can choose $s \in \mathbb{R}$ such that $0 < s < \min\{r_0, r\} \leq r_0$. (might pick $s = \frac{1}{64}\min\{r_0, r\}$.) By (b), there exists $\mathbf{q} \neq \mathbf{p}$ such that

$$\mathbf{q} \in B(\mathbf{p}; s) \subseteq B(\mathbf{p}; r) \subseteq E.$$

(2) *Give an example of a closed set $E \subseteq \mathbb{R}^k$ such that $E \not\subseteq E'$.* Pick $E = \{\mathbf{0}\}$. So $E' = \varnothing$ and thus $E \not\subseteq E'$.

□

Exercise 2.9. *Let E° denote the set of all interior points of a set E. [See Definition 2.18(e); E° is called the interior of E.]*

(a) *Prove that E° is always open.*

(b) *Prove that E is open if and only if $E^\circ = E$.*

(c) *If G is contained in E and G is open, prove that G is contained in E°.*

(d) *Prove that the complement of E° is the closure of the complement of E.*

(e) *Do E and $\overline{E}$ always have the same interiors?*

(f) *Do E and E° always have the same closures?*

Similar to Theorem 2.27.

Proof of (a). *It is equivalent to show that $E^\circ \subseteq (E^\circ)^\circ$.*

(1) Given any point $x \in E^\circ$, there is $r > 0$ such that $B(x; r) \subseteq E$.

(2) It suffices to show that $B\left(x; \frac{2}{r}\right) \subseteq E^\circ$. Given any point $y \in B\left(x; \frac{2}{r}\right)$, we will show that there is an open neighborhood $B\left(y; \frac{2}{r}\right)$ of y such that $B\left(y; \frac{2}{r}\right) \subseteq E$.

(3) Given any point $z \in B\left(y; \frac{2}{r}\right)$, we have

$$d(z,x) \leq d(z,y) + d(y,x) < \frac{2}{r} + \frac{2}{r} = r,$$

or $z \in B(x;r) \subseteq E$. Therefore, $B\left(y; \frac{2}{r}\right) \subseteq E$, or $y \in E^\circ$, or $B\left(x; \frac{2}{r}\right) \subseteq E^\circ$, or $x \in (E^\circ)^\circ$, or $E^\circ \subseteq (E^\circ)^\circ$.

□

Proof of (b).

(1) ($\Longrightarrow$)(Definition 2.18) Since E is open, every point of E is an interior point of E. Hence $E \subseteq E^\circ$. Note that $E^\circ \subseteq E$ is trivial, and thus $E^\circ = E$.

(2) ($\Longleftarrow$)((a)) By (a), $E = E^\circ$ is always open.

(3) ($\Longleftarrow$)(Definition 2.18) Every point of E is an interior point of E since $E = E^\circ$. Hence E is open by Definition 2.18(f).

□

Proof of (c). $G \subseteq E$ implies $G^\circ \subseteq E^\circ$. $G = G^\circ$ since G is open ((b)). Hence $G = G^\circ \subseteq E^\circ$, that is, E° is the largest open set contained in E. (Similarly, $\overline{E}$ is the smallest closed set containing E.) □

Proof of (d). Show that $X - E^\circ = \overline{X - E}$ and $(X-E)^\circ = X - \overline{E}$.

(1) (Theorem 2.27 and (c))

$$\begin{aligned}
X - E^\circ &= X - \bigcup_{\text{Open } V \subseteq E} V \\
&= \bigcap_{\text{Open } V \subseteq E} (X - V) \\
&= \bigcap_{\text{Closed } W \supseteq X - E} W \\
&= \overline{X - E}. \\
X - \overline{E} &= X - \bigcap_{\text{Closed } W \supseteq E} W \\
&= \bigcup_{\text{Closed } W \supseteq E} (X - W) \\
&= \bigcup_{\text{Open } V \subseteq X - E} V \\
&= (X - E)^\circ.
\end{aligned}$$

(2) (Brute-force)

$$\begin{aligned}
x \in E^\circ &\iff \exists r > 0 \text{ such that } B(x;r) \subseteq E \\
&\iff \exists r > 0 \text{ such that } B(x;r) \cap (X - E) = \varnothing \\
&\iff x \notin \overline{X - E} \\
&\iff x \in X - \overline{X - E}. \\
x \in (X - E)^\circ &\iff \exists r > 0 \text{ such that } B(x;r) \subseteq (X - E) \\
&\iff \exists r > 0 \text{ such that } B(x;r) \cap E = \varnothing \\
&\iff x \notin \overline{E} \\
&\iff x \in X - \overline{E}.
\end{aligned}$$

Note that $X - E^\circ = \overline{X - E}$ is equivalent to $(X - E)^\circ = X - \overline{E}$ by mapping $E \mapsto X - E$. $\square$

Proof of (e). No.

(1) Let $X = \mathbb{R}^1$ equipped with the Euclidean metric, and $E = \mathbb{Q} \subseteq X$.

(2) $E^\circ = \varnothing$ since $\widetilde{\mathbb{Q}}$ is dense in $\mathbb{R}$.

(3) $(\overline{E})^\circ = (\mathbb{R}^1)^\circ = \mathbb{R}^1$ since $\mathbb{Q}$ is dense in $\mathbb{R}$ and $\mathbb{R}^1$ is open.

$\square$

Proof of (f). No.

(1) Let $X = \mathbb{R}^1$ equipped with the Euclidean metric, and $E = \mathbb{Q} \subseteq X$.

(2) $\overline{E} = \mathbb{R}^1$ since $\mathbb{Q}$ is dense in $\mathbb{R}$.

(3) $\overline{E^\circ} = \overline{\varnothing} = \varnothing$ since $\widetilde{\mathbb{Q}}$ is dense in $\mathbb{R}$.

$\square$

Exercise 2.10. *Let X be an infinite set. For $p \in X$ and $q \in X$, define*

$$d(p,q) = \begin{cases} 1 & (\text{if } p \neq q) \\ 0 & (\text{if } p = q). \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

(The statement holds even if X is finite.) We called d the discrete metric, and the corresponding topology on X induces the discrete topology. Conversely, if X has the discrete topology, X is always metrizable by the discrete metric.

Proof.

(1) *$d(p,q)$ is a metric.*

   (a) *$d(p,q) > 0$ if $p \neq q$; $d(p,p) = 0$.* Trivial.

   (b) *$d(p,q) = d(q,p)$.* Trivial.

   (c) *$d(p,q) \leq d(p,r) + d(r,q)$ for any $r \in X$.* If $p = q$, nothing to do. If $p \neq q$, $r \neq p$ or $r \neq q$ for any $r \in X$. (Assume not true, $r = p$ and $r = q$ implies that $p = q$ which is a contradiction.) In any case $d(p,r) + d(r,q) \geq 1 = d(p,q)$.

(2) *Every subset is open.* Let E be any subset of X. Then every point $p \in E$ is an interior point of E. In fact, we can pick one open neighborhood $U = B\left(p; \frac{1}{2}\right)$ of p containing only one point $p \in E$ or $U = \{p\}$, and such open neighborhood U is a subset of E. So every subset of X is open.

(3) *Every subset is closed.* Since every subset is open, every subset is closed by Theorem 2.23.

   Supplement. Might use Definition 2.18 (d) to prove directly since there are no limit points in X if we consider one open neighborhood $U = B\left(p; \frac{1}{2}\right)$ of p. Therefore, every subset is closed. Again we apply Theorem 2.23 to get that every subset is open without using Definition 2.18 (f).

(4) *A subset is compact iff it is finite.*

   (a) *Any finite subset is compact.* Say $E = \{p_1, p_2, ..., p_k\}$, and $\{G_\alpha\}$ be an open covering of E. From $\{G_\alpha\}$ we pick G_{α_1} containing p_1, G_{α_2} containing p_2, ..., and G_{α_k} containing p_k. This process can be done in the finitely many steps. Therefore,

$$\{G_{\alpha_1}, G_{\alpha_2}, ..., G_{\alpha_k}\}$$

   is a finite subcovering of $\{G_\alpha\}$ covering E.

   (b) *Any infinite subset is not compact.* Take a collection

$$\mathscr{G} = \{G_p = \{p\}\}$$

   of open subsets where p runs all points in E. Clearly, $\{G_p\}$ is an open covering. Assume

$$\mathscr{G}' = \{G_{p_1}, G_{p_2}, ..., G_{p_k}\}$$

   is any finite subcovering of $\mathscr{G}$. Since E is infinite, there exist a point $p \in E$ such that $p \neq p_1$, $p \neq p_2$, ..., $p \neq p_k$. Therefore, $\mathscr{G}'$ does not cover p, or $\mathscr{G}$ does not contains any finite subcovering $\mathscr{G}'$.

□

Notice that every subset is bounded. Therefore, every subset is closed and bounded, but only finite subset is compact, i.e., Heine-Borel theorem is not true in the infinite discrete topology.

Exercise 2.11. *For $x \in \mathbb{R}^1$ and $y \in \mathbb{R}^1$, define*

$$\begin{aligned}
d_1(x,y) &= (x-y)^2, \\
d_2(x,y) &= \sqrt{|x-y|}, \\
d_3(x,y) &= |x^2-y^2|, \\
d_4(x,y) &= |x-2y|, \\
d_5(x,y) &= \frac{|x-y|}{1+|x-y|}.
\end{aligned}$$

Determine, for each of these, whether it is a metric or not.

Proof.

(1) *$d = d_1$ is not a metric.* (Reductio ad absurdum) If d were a metric, then

$$d(0,2) > d(0,1) + d(1,2),$$

contrary to Definition 2.15(c) that $d(p,q) \leq d(p,r) + d(r,q)$.

(2) *$d = d_2$ is a metric. It suffices to show that $d'(x,y) = \sqrt{d(x,y)}$ is a metric if $d(x,y)$ is a metric.* For any $p, q, r \in \mathbb{R}^1$,

(a) $d'(p,q) = \sqrt{d(p,q)} > 0$ if $p \neq q$; $d'(p,p) = \sqrt{d(p,p)} = 0$.

(b) $d'(p,q) = \sqrt{d(p,q)} = \sqrt{d(q,p)} = d'(q,p)$.

(c)

$$\begin{aligned}
&\sqrt{d(p,r)+d(r,q)} \leq \sqrt{d(p,r)} + \sqrt{d(r,q)} \\
\Longleftrightarrow &(\sqrt{d(p,r)+d(r,q)})^2 \leq (\sqrt{d(p,r)} + \sqrt{d(r,q)})^2 \\
\Longleftrightarrow &d(p,r)+d(r,q) \leq d(p,r)+d(r,q)+2\sqrt{d(p,r)}\sqrt{d(r,q)} \\
\Longleftrightarrow &0 \leq 2\sqrt{d(p,r)}\sqrt{d(r,q)}.
\end{aligned}$$

(d)

$$\begin{aligned}
d'(p,q) &= \sqrt{d(p,q)} \\
&\leq \sqrt{d(p,r)+d(r,q)} && \text{(Triangle inequality)} \\
&\leq \sqrt{d(p,r)} + \sqrt{d(r,q)} && \text{((c))} \\
&= d'(p,r) + d'(r,q).
\end{aligned}$$

By Definition 2.15, d' is a metric.

(3) $d = d_3$ *is not a metric.* (Reductio ad absurdum) If d were a metric, then

$$d(1, -1) = 0,$$

contrary to Definition 2.15(a): $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$.

(4) $d = d_4$ *is not a metric.* (Reductio ad absurdum) If d were a metric, then

$$d(1, 1) = 1,$$

contrary to Definition 2.15(a): $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$.

(5) $d = d_5$ *is a metric. It suffices to show that* $d'(x, y) = \frac{d(x,y)}{1+d(x,y)}$ *is a metric if* $d(x, y)$ *is a metric.* For any $p, q, r \in \mathbb{R}^1$,

(a) $d'(p, q) = \frac{d(p,q)}{1+d(p,q)} > 0$ if $p \neq q$; $d'(p, p) = \frac{d(p,p)}{1+d(p,p)} = 0$.

(b) $d'(p, q) = \frac{d(p,q)}{1+d(p,q)} = \frac{d(q,p)}{1+d(q,p)} = d'(q, p)$.

(c) Write $x = d(p, q)$, $y = d(p, r)$ and $z = d(r, q)$. So $x, y, z \geq 0$ and

$$\begin{aligned}
& x \leq y + z \\
\Longleftrightarrow & x + x(y + z) \leq y + z + x(y + z) \\
\Longleftrightarrow & x(1 + y + z) \leq (1 + x)(y + z) \\
\Longleftrightarrow & \frac{x}{1 + x} \leq \frac{y + z}{1 + y + z}.
\end{aligned}$$

(d)

$$\begin{aligned}
d'(p, q) &= \frac{d(p, q)}{1 + d(p, q)} \\
&\leq \frac{d(p, r) + d(r, q)}{1 + d(p, r) + d(r, q)} && \text{((c))} \\
&= \frac{d(p, r)}{1 + d(p, r) + d(r, q)} + \frac{d(r, q)}{1 + d(p, r) + d(r, q)} \\
&= \frac{d(p, r)}{1 + d(p, r)} + \frac{d(r, q)}{1 + d(r, q)} \\
&= d'(p, r) + d'(r, q).
\end{aligned}$$

(e) Or we can show $d'(p, q) \leq d'(p, r) + d'(r, q)$ by

$$\begin{aligned}
& \frac{x}{1 + x} \leq \frac{y}{1 + y} + \frac{z}{1 + z} \\
\Longleftrightarrow & x(1 + y)(1 + z) \leq y(1 + z)(1 + x) + z(1 + x)(1 + y) \\
\Longleftrightarrow & x + xy + xz + xyz \\
& \quad \leq (y + xy + yz + xyz) + (z + xz + yz + xyz) \\
\Longleftrightarrow & x \leq y + z + 2yz + xyz \\
\Longleftarrow & x \leq y + z && (d \text{ is nonnegative})
\end{aligned}$$

By Definition 2.15, d' is a metric.

$\square$

Exercise 2.12. *Let $K \subseteq \mathbb{R}^1$ consist of 0 and the numbers $\frac{1}{n}$, for $n = 1, 2, 3, ...$ Prove that K is compact directly from the definition (without using the Heine-Borel theorem).*

Proof. Let $\{G_\alpha\}$ be an open covering of K. There is an open set $G_0 \in \{G_\alpha\}$ containing 0. So there exists an open neighborhood $U = B(0; r)$ of 0 such that $U \subseteq G_0$. So U contains all points $q = \frac{1}{n}$ of K whenever $n > \frac{1}{r}$. To construct a finite subcovering of $\{G_\alpha\}$, we need to pick finitely many open sets from $\{G_\alpha\}$ to cover the remaining points $q = \frac{1}{n}$ where $n = 1, 2, ..., \left[\frac{1}{r}\right]$, say G_1 contains $q = \frac{1}{1}$, G_2 contains $q = \frac{1}{2}$, ..., $G_{\left[\frac{1}{r}\right]}$ contains $q = \frac{1}{\left[\frac{1}{r}\right]}$. (Might be duplicated.) Hence,

$$\left\{G_0, G_1, G_2, ..., G_{\left[\frac{1}{r}\right]}\right\}$$

is a finite subcovering of $\{G_\alpha\}$ covering K. $\square$

Proof (Heine-Borel theorem).

(1) K is closed. In fact, the only limit point of K is 0, which is in K.

   (a) *$p = 0$ is a limit point.* Given $r > 0$. There always exists $n \in \mathbb{Z}^+$ such that $r > \frac{1}{n}$. So any open neighborhood $B(0; r)$ of $p = 0$ contains at least one point $q = \frac{1}{n} \neq 0$ in K.

   (b) *$p < 0$ is not a limit point.* Pick an open neighborhood $B(p; r)$ of p where $r = |p| > 0$. Then $B(p; r) \cap K = \varnothing$.

   (c) *$p > 0$ is not a limit point.* There always exists $m \in \mathbb{Z}^+$ such that $p > \frac{1}{n}$ whenever $n \geq m$. Pick an open neighborhood $B(p; r)$ of p where $r = p - \frac{1}{m} > 0$. Then $B(p; r)$ does not have all points $q = \frac{1}{n} \in K$ whenever $n \geq m$. By Theorem 2.20, p cannot be a limit point of K.

(2) K is bounded. There is a real number $M = 2$ and a point $q = 0 \in \mathbb{R}^1$ such that $|p - q| = |p| < 2$ for all $p \in K$.

By Heine-Borel theorem, K is compact in $\mathbb{R}^1$. $\square$

Exercise 2.14. *Give an example of an open cover of the segment $(0, 1)$ which has no finite subcover.*

Proof. In $\mathbb{R}^1$, take a collection

$$\mathscr{G} = \left\{G_n = \left(\frac{1}{n}, 1\right)\right\}$$

of open subsets where $n \in \mathbb{Z}^+$.

(1) *$\mathscr{G}$ is an open covering of* $(0,1) \subseteq \mathbb{R}^1$. Actually, given $x \in (0,1)$, there exists an positive integer n such that $x > \frac{1}{n}$. That is, $x \in \left(\frac{1}{n}, 1\right) = G_n$.

(2) *There is no finite subcovering of $\mathscr{G}$.* Assume

$$\mathscr{G}' = \{G_{n_1}, G_{n_2}, ..., G_{n_k}\}$$

is any finite subcovering of $\mathscr{G}$ where $n_1 < n_2 < ... < n_k$. Take $x \in \left(0, \frac{1}{n_k}\right) \neq \varnothing$, $x = \frac{1}{2n_k}$ for example. Then $x \notin G_{n_1}$, $x \notin G_{n_1}$, ..., $x \notin G_{n_k}$, which contradicts that $\mathscr{G}'$ is a finite subcovering of $\mathscr{G}$ covering $(0,1)$.

$\square$