

Chapter 10: Integration of Differential Forms

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Exercise 10.1. ...

Proof.

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Exercise 10.2. For $i = 1, 2, 3, \dots$, let $\varphi_i \in \mathcal{C}(\mathbb{R}^1)$ have support in $(2^{-i}, 2^{1-i})$, such that $\int \varphi_i = 1$. Put

$$f(x, y) = \sum_{i=1}^{\infty} [\varphi_i(x) - \varphi_{i+1}(x)] \varphi_i(y)$$

Then f has compact support in \mathbb{R}^2 , f is continuous except at $(0, 0)$, and

$$\int dy \int f(x, y) dx = 0 \quad \text{but} \quad \int dx \int f(x, y) dy = 1.$$

Observe that f is unbounded in every neighborhood of $(0, 0)$.

Proof.

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Exercise 10.3. ...

Proof.

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Exercise 10.4. For $(x, y) \in \mathbb{R}^2$, define

$$\mathbf{F}(x, y) = (e^x \cos y - 1, e^x \sin y)$$

Prove that $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$, where

$$\mathbf{G}_1(x, y) = (e^x \cos y - 1, y)$$

$$\mathbf{G}_2(u, v) = (u, (1 + u) \tan v)$$

are primitive in some neighborhood of $(0, 0)$. Compute the Jacobians of \mathbf{G}_1 , \mathbf{G}_2 , \mathbf{F} at $(0, 0)$. Define

$$\mathbf{H}_2(x, y) = (x, e^x \sin y)$$

and find

$$\mathbf{H}_1(u, v) = (h(u, v), v)$$

so that $\mathbf{F} = \mathbf{H}_1 \circ \mathbf{H}_2$ is in some neighborhood of $(0, 0)$.

Proof.

(1) By Definition 10.5,

$$\mathbf{G}_1(x, y) = (e^x \cos y - 1)\mathbf{e}_1 + y\mathbf{e}_2,$$

$$\mathbf{G}_2(u, v) = u\mathbf{e}_1 + ((1 + u) \tan v)\mathbf{e}_2$$

are primitive in some neighborhood of $(0, 0)$.

(2) Show that $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$. Given any $(x, y) \in \mathbb{R}^2$, we have

$$\begin{aligned} (\mathbf{G}_2 \circ \mathbf{G}_1)(x, y) &= \mathbf{G}_2(\mathbf{G}_1(x, y)) \\ &= \mathbf{G}_2(e^x \cos y - 1, y) \\ &= (e^x \cos y - 1, (1 + (e^x \cos y - 1)) \tan y) \\ &= (e^x \cos y - 1, e^x \sin y) \\ &= \mathbf{F}(x, y). \end{aligned}$$

(3) Since

$$\begin{aligned} J_{\mathbf{G}_1}(x, y) &= \begin{bmatrix} e^x \cos y & -e^x \sin y \\ 0 & 1 \end{bmatrix} \\ J_{\mathbf{G}_2}(x, y) &= \begin{bmatrix} 1 & 0 \\ \tan y & (1 + x) \sec^2 y \end{bmatrix} \\ J_{\mathbf{F}}(x, y) &= \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}, \end{aligned}$$

$$\begin{aligned}
J_{\mathbf{G}_1}(0,0) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
J_{\mathbf{G}_2}(0,0) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
J_{\mathbf{F}}(0,0) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

(4) Define $h(u, v) = \sqrt{e^{2u} - v^2} - 1$ on

$$B\left((0,0); \frac{1}{64}\right) \subseteq \mathbb{R}^2.$$

$h(u, v)$ is well-defined since $e^{2u} - v^2 > 0$ for all $(u, v) \in B\left((0,0); \frac{1}{64}\right)$.

(5) Given any $(x, y) \in \mathbb{R}^2$, we have

$$\begin{aligned}
(\mathbf{H}_1 \circ \mathbf{H}_2)(x, y) &= \mathbf{H}_1(\mathbf{H}_2(x, y)) \\
&= \mathbf{H}_1(x, e^x \sin y) \\
&= (\sqrt{e^{2x} - (e^x \sin y)^2} - 1, e^x \sin y) \\
&= (e^x \cos y - 1, e^x \sin y) \\
&= \mathbf{F}(x, y).
\end{aligned}$$

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Exercise 10.5. ...

Proof.

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Exercise 10.6. ...

Proof.

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Exercise 10.7. ...

Proof.

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Exercise 10.8. Let H be the parallelogram in \mathbb{R}^2 whose vertices are $(1, 1)$, $(3, 2)$, $(4, 5)$, $(2, 4)$. Find the affine map T which sends $(0, 0)$ to $(1, 1)$, $(1, 0)$ to $(3, 2)$, $(1, 1)$ to $(4, 5)$, $(0, 1)$ to $(2, 4)$. Show that $J_T = 5$. Use T to convert the integral

$$\alpha = \int_H e^{x-y} dx dy$$

to an integral over I^2 and thus compute α .

Proof.

(1) By Affine simplexes 10.26,

$$T(\mathbf{x}) = T(\mathbf{0}) + A\mathbf{x},$$

where $A \in L(\mathbb{R}^2, \mathbb{R}^2)$, say $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Note that $T : \begin{bmatrix} 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus

$$T : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 + ax + by \\ 1 + cx + dy \end{bmatrix}.$$

(2) By $T : (1, 0) \mapsto (3, 2)$ and $T : (0, 1) \mapsto (2, 4)$, we can solve A as

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

It is easy to verify such

$$T : \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{x}} \mapsto \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{T(\mathbf{0})} + \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{x}} = \begin{bmatrix} 1 + 2x + y \\ 1 + x + 3y \end{bmatrix}$$

satisfying our requirement.

(3)

$$J_T = \det \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = 5.$$

(4)

$$\begin{aligned}\int_H e^{x-y} dx dy &= \int_{[0,1]^2} e^{(1+2u+v)-(1+u+3v)} |J_T| du dv \\ &= 5 \int_{[0,1]^2} e^{u-2v} du dv \\ &= 5 \left\{ \int_0^1 e^u du \right\} \left\{ \int_0^1 e^{-2v} dv \right\} \quad (\text{Theorem 10.2}) \\ &= \frac{5}{2} (e-1)(1-e^{-2}).\end{aligned}$$

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Exercise 10.9. ...

Proof.

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Exercise 10.10. ...

Proof.

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Exercise 10.11. ...

Proof.

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Exercise 10.12. ...

Proof.

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Exercise 10.13. ...

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Exercise 10.14. ...

Proof.

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Exercise 10.15. If ω and λ are k - and m -forms, respectively, prove that

$$\omega \wedge \lambda = (-1)^{km} \lambda \wedge \omega.$$

Proof.

(1) Write

$$\omega = \sum_I b_I(\mathbf{x}) dx_I, \quad \lambda = \sum_J c_J(\mathbf{x}) dx_J$$

in the standard presentations, where I and J range over all increasing k -indices and over all increasing m -indices taken from the set $\{1, \dots, n\}$.

(2) Show that $dx_I \wedge dx_J = (-1)^{km} dx_J \wedge dx_I$.

$$\begin{aligned} dx_I \wedge dx_J &= dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_J \\ &= (-1)^m dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}} \wedge dx_J \wedge dx_{i_k} \\ &= (-1)^{2m} dx_{i_1} \wedge \cdots \wedge dx_{i_{k-2}} \wedge dx_J \wedge dx_{i_{k-1}} \wedge dx_{i_k} \\ &\quad \dots \\ &= (-1)^{km} dx_J \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= (-1)^{km} dx_J \wedge dx_I. \end{aligned}$$

(3)

$$\begin{aligned}
\omega \wedge \lambda &= \sum_{I,J} b_I(\mathbf{x}) c_J(\mathbf{x}) dx_I \wedge dx_J \\
&= (-1)^{km} \sum_{J,I} c_J(\mathbf{x}) b_I(\mathbf{x}) dx_J \wedge dx_I \\
&= (-1)^{km} \lambda \wedge \omega.
\end{aligned}$$

□

Exercise 10.16. If $k \geq 2$ and $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_k]$ is an oriented affine k -simplex, prove that $\partial^2 \sigma = 0$, directly from the definition of the boundary operator ∂ . Deduce from this that $\partial^2 \Psi = 0$ for every chain Ψ . (Hint: For orientation, do it first for $k = 2$, $k = 3$. In general, if $i < j$, let σ_{ij} be the $(k-2)$ -simplex obtained by deleting \mathbf{p}_i and \mathbf{p}_j from σ . Show that each σ_{ij} occurs twice in $\partial^2 \sigma$, with opposite sign.)

Proof (Brute-force).

- (1) Write the boundary of the oriented affine k -simplex $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_k]$ as

$$\partial \sigma = \sum_{i=0}^k (-1)^i [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k]$$

where where the oriented $(k-1)$ -simplex $[\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k]$ is obtained by deleting σ 's i -th vertex (Boundaries 10.29).

- (2)

$$\begin{aligned}
\partial^2 \sigma &= \partial \left(\sum_i (-1)^i [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k] \right) \\
&= \sum_i (-1)^i \partial [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k] \\
&= \sum_{j < i} (-1)^i (-1)^j [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_j, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k] \\
&\quad + \sum_{j > i} (-1)^i (-1)^{j-1} [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \widehat{\mathbf{p}}_j, \dots, \mathbf{p}_k] \\
&= \sum_{j < i} (-1)^{i+j} [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_j, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k] \\
&\quad - \sum_{j > i} (-1)^{i+j} [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \widehat{\mathbf{p}}_j, \dots, \mathbf{p}_k].
\end{aligned}$$

The latter two summations cancel since after switching i and j in the second sum. Therefore $\partial^2 \sigma = 0$.

- (3) The boundary of a chain is the linear combination of boundaries of the simplices in the chain. Write $\Psi = \sum_{i=1}^r \sigma_i$, where σ_i is an oriented affine simplex. Then

$$\partial^2 \Psi = \partial \left(\partial \sum \sigma_i \right) = \partial \left(\sum \partial \sigma_i \right) = \sum \partial^2 \sigma_i = \sum 0 = 0$$

for any affine chain Ψ .

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Exercise 10.17. ...

Proof.

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Exercise 10.18. ...

Proof.

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Exercise 10.19. ...

Proof.

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Exercise 10.20. ...

Proof.

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Exercise 10.21. ...

Proof.

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Exercise 10.22. ...

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Exercise 10.23. ...

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Exercise 10.24. ...

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Exercise 10.25. ...

Proof.

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Exercise 10.26. ...

Proof.

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Exercise 10.27. ...

Proof.

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Exercise 10.28. ...

Proof.

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Exercise 10.29. ...

Proof.

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Exercise 10.30. ...

Proof.

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Exercise 10.31. ...

Proof.

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Exercise 10.32. ...

Proof.

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