Notes on the book: *James R. Munkres, Elements of Algebraic Topology*

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August 26, 2021

Contents

Chapter 1: Homology Groups of a Simplicial Complex

§1. Simplices

Exercise 1.1.

Verify properties (1)-(3) of simplices:

(1) *The barycentric coordinates $t_i(x)$ of x with respect to $a_0, \dots, a_n$ are continuous functions of x.*

(2)

(3) *σ is compact, convex set in $\mathbb{R}^N$, which equals the intersection of all convex sets in $\mathbb{R}^N$ containing $a_0, \dots, a_n$.*

Proof of property (1).

(1) Let σ be the n-simplex spanned by $a_0, \dots, a_n$. It suffices to show that $t_i(x)$ is linear. Therefore $t_i(x)$ is automatically continuous (Theorem 9.7 in the textbook: *Rudin, Principles of Mathematical Analysis, 3rd edition*).

(2) Let
$$E = \left\{ x = \sum_{i=1}^{n} \widetilde{t_i}(x) a_i : \widetilde{t_i}(x) \in \mathbb{R} \right\} \supseteq \sigma$$
be the plane spanned by $a_0, \dots, a_n$. $\widetilde{t_i}(x)$ is well-defined on E and thus $\widetilde{t_i}|_\sigma = t_i$ (since $\{a_0, \dots, a_n\}$ is geometrically independent in $\mathbb{R}^N$). So it suffices to show that $\widetilde{t_i}$ is linear.

(3) Suppose $x = \sum_{i=1}^{n} \widetilde{t_i}(x) a_i \in E$ and $y = \sum_{i=1}^{n} \widetilde{t_i}(y) a_i \in E$. Then
$$x + y = \sum_{i=1}^{n} (\widetilde{t_i}(x) + \widetilde{t_i}(y)) a_i.$$
Note that the coefficient of a_i is uniquely determined by $x + y$. Thus $\widetilde{t_i}(x + y) = \widetilde{t_i}(x) + \widetilde{t_i}(y)$. Similarly, $\widetilde{t_i}(rx) = r\widetilde{t_i}(x)$ for $r \in \mathbb{R}$. Hence $\widetilde{t_i}$ is linear.

□

Proof of property (2).

(1)

□

Proof of property (3).

(1) *Show that* σ *is compact.*

(2) *Show that* σ *is convex.* Given any $x = \sum_i t_i a_i \in \sigma$ (with $\sum_i t_i = 1$), $y = \sum_i s_i a_i \in \sigma$ (with $\sum_i s_i = 1$) and $0 < \lambda < 1$, it suffices to show that

$$\lambda x + (1-\lambda)y \in \sigma.$$

In fact,

$$\begin{aligned}
\lambda x + (1-\lambda)y &= \lambda \sum_i t_i a_i \in \sigma + (1-\lambda) \sum_i s_i a_i \\
&= \sum_i (\lambda t_i + (1-\lambda)s_i) a_i,
\end{aligned}$$

where each $\lambda t_i + (1-\lambda)s_i \geq 0$ and

$$\sum_i (\lambda t_i + (1-\lambda)s_i) = \lambda \sum_i t_i + (1-\lambda) \sum_i s_i = \lambda + (1-\lambda) = 1.$$

So $\lambda x + (1-\lambda)y \in \sigma$.

(3) *Let* $\mathscr{C}$ *be the collection of all convex sets in* $\mathbb{R}^N$ *containing* $a_0, \ldots, a_n$. *Show that* $\sigma = \bigcap_{E \in \mathscr{C}} E$. By (2), $\sigma \in \mathscr{C}$ and thus $\sigma \supseteq \bigcap_{E \in \mathscr{C}} E$. Conversely, suppose $E \in \mathscr{C}$. The convexity of E implies that $\sum_i t_i a_i \in E$ whenever $\sum_i t_i = 1$ and $t_i \geq 0$. Hence $\sigma \subseteq E$ and thus $\sigma \subseteq \bigcap_{E \in \mathscr{C}} E$.

□