

Notes on the book:
*P.J. Hilton and U. Stammbach, A
Course in Homological Algebra*

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Chapter I: Modules

§1. Modules

Exercise 1.1. (Diagram chasing)

Complete the proof of Lemma 1.1. Show moreover that α is surjective (resp. injective) if α' , α'' are surjective (resp. injective).

Lemma 1.1. Let $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ and $0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0$ be two short exact sequences. Suppose that in the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A' & \xrightarrow{\mu} & A & \xrightarrow{\varepsilon} & A'' \longrightarrow 0 \\ & & \downarrow \alpha' & & \downarrow \alpha & & \downarrow \alpha'' \\ 0 & \longrightarrow & B' & \xrightarrow{\mu'} & B & \xrightarrow{\varepsilon'} & B'' \longrightarrow 0 \end{array}$$

any two of the three homomorphisms α' , α , α'' are isomorphisms. Then the third is an isomorphism, too.

Proof (Diagram chasing).

(1) Show that α is surjective if α' , α'' are surjective.

- (a) Take any $b \in B$, it suffices to find $a \in A$ such that $\alpha a = b$.
- (b) Consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varepsilon} & A'' \\ \downarrow \alpha & & \downarrow \alpha'' \\ B & \xrightarrow{\varepsilon'} & B'' \end{array}$$

$\varepsilon' b \in B''$. By the surjectivity of α'' , $\exists a'' \in A''$ such that $\alpha'' a'' = \varepsilon' b$. By the surjectivity of ε , $\exists \bar{a} \in A$ such that $\varepsilon \bar{a} = a''$. Hence

$$\begin{aligned} \varepsilon'(b - \alpha \bar{a}) &= \varepsilon' b - \varepsilon' \alpha \bar{a} \\ &= \varepsilon' b - \alpha'' \varepsilon \bar{a} && \text{(The diagram commutes)} \\ &= \varepsilon' b - \alpha'' a'' \\ &= \varepsilon' b - \varepsilon' b \\ &= 0. \end{aligned}$$

- (c) Consider the short exact sequence

$$0 \longrightarrow B' \xrightarrow{\mu'} B \xrightarrow{\varepsilon'} B'' \longrightarrow 0$$

As $\varepsilon'(b - \alpha \bar{a}) = 0$, $\exists b' \in B'$ such that $\mu' b' = b - \alpha \bar{a}$.

(d) Consider the commutative diagram

$$\begin{array}{ccc} A' & \xrightarrow{\mu} & A \\ \downarrow \alpha' & & \downarrow \alpha \\ B' & \xrightarrow{\mu'} & B \end{array}$$

By the surjectivity of α' , $\exists a' \in A'$ such that $\alpha'a' = b'$. Hence

$$\begin{aligned} \alpha(\mu a' + \bar{a}) &= \alpha\mu a' + \alpha\bar{a} \\ &= \mu'\alpha'a' + \alpha\bar{a} && \text{(The diagram commutes)} \\ &= \mu'b' + \alpha\bar{a} \\ &= (b - \alpha\bar{a}) + \alpha\bar{a} \\ &= b. \end{aligned}$$

Therefore, there exists $a := \mu a' + \bar{a}$ such that $\alpha a = b$.

(2) Show that α is injective if α' , α'' are injective.

(a) It suffices to show that $\ker \alpha = 0$. Take $a \in \ker \alpha$. ($\alpha(a) = \alpha a = 0$.)

(b) Consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varepsilon} & A'' \\ \downarrow \alpha & & \downarrow \alpha'' \\ B & \xrightarrow{\varepsilon'} & B'' \end{array}$$

we have $0 = \varepsilon' \alpha a = \alpha'' \varepsilon a$. By the injectivity of α'' , $\varepsilon a = 0$.

(c) Consider the short exact sequence

$$0 \longrightarrow A' \xrightarrow{\mu} A \xrightarrow{\varepsilon} A'' \longrightarrow 0$$

As $\varepsilon a = 0$, $\exists a' \in A'$ such that $\mu a' = a$.

(d) Consider the commutative diagram

$$\begin{array}{ccc} A' & \xrightarrow{\mu} & A \\ \downarrow \alpha' & & \downarrow \alpha \\ B' & \xrightarrow{\mu'} & B \end{array}$$

$0 = \alpha a = \alpha\mu a' = \mu'\alpha'a'$. By the injectivity of $\mu'\alpha'$, $a' = 0$. Therefore, $a = \mu a' = 0$.

(3) Suppose α is surjective. Show that α'' is surjective.

(a) Take any $b'' \in B''$, it suffices to find $a'' \in A''$ such that $\alpha'' a'' = b''$.

(b) Consider the commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{\varepsilon} & A'' \\
\downarrow \alpha & & \downarrow \alpha'' \\
B & \xrightarrow{\varepsilon'} & B''
\end{array}$$

By the surjectivity of ε' , $\exists b \in B$ such that $\varepsilon'b = b''$. By the surjectivity of α , $\exists a \in A$ such that $\alpha a = b$. Take $a'' := \varepsilon a \in A''$. Hence

$$\begin{aligned}
\alpha'' a'' &= \alpha'' \varepsilon a \\
&= \varepsilon' \alpha a && \text{(The diagram commutes)} \\
&= \varepsilon' b \\
&= b''.
\end{aligned}$$

(4) Suppose α' is surjective and α is injective. Show that α'' is injective.

(a) It suffices to show that $\ker \alpha'' = 0$. Take $a'' \in \ker \alpha''$. ($\alpha''(a'') = \alpha'' a'' = 0$.)

(b) Consider the commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{\varepsilon} & A'' \\
\downarrow \alpha & & \downarrow \alpha'' \\
B & \xrightarrow{\varepsilon'} & B''
\end{array}$$

By the surjectivity of ε , $\exists a \in A$ such that $\varepsilon a = a''$. So

$$\begin{aligned}
0 &= \alpha'' a'' \\
&= \alpha'' \varepsilon a \\
&= \varepsilon' \alpha a. && \text{(The diagram commutes)}
\end{aligned}$$

(c) Consider the short exact sequence

$$0 \longrightarrow B' \xrightarrow{\mu'} B \xrightarrow{\varepsilon'} B'' \longrightarrow 0$$

As $\varepsilon'(\alpha a) = 0$, $\exists b' \in B'$ such that $\mu' b' = \alpha a$.

(d) Consider the commutative diagram

$$\begin{array}{ccc}
A' & \xrightarrow{\mu} & A \\
\downarrow \alpha' & & \downarrow \alpha \\
B' & \xrightarrow{\mu'} & B
\end{array}$$

By surjectivity of α' , $\exists a' \in A'$ such that $\alpha' a' = b'$. So

$$\begin{aligned}
\alpha a &= \mu' b' \\
&= \mu' \alpha' a' \\
&= \alpha \mu a'. && \text{(The diagram commutes)}
\end{aligned}$$

By the injectivity of α , $a = \mu a'$. Hence

$$a'' = \varepsilon a = \varepsilon \mu a' = 0.$$

Therefore $\ker \alpha'' = 0$.

(5) By (3)(4), α'' is an isomorphism if both α' and α are isomorphisms.

(6) Suppose α is surjective and α'' is injective. Show that α' is surjective.

(a) Take any $b' \in B'$, it suffices to find $a' \in A'$ such that $\alpha' a' = b'$. Let $b := \mu' b' \in B$ and note that $\varepsilon' b = 0$ by the exactness of

$$0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0.$$

(b) Consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varepsilon} & A'' \\ \downarrow \alpha & & \downarrow \alpha'' \\ B & \xrightarrow{\varepsilon'} & B'' \end{array}$$

By the surjectivity of α , $\exists a \in A$ such that $\alpha a = b$. So

$$\begin{aligned} 0 &= \varepsilon' b \\ &= \varepsilon' \alpha a \\ &= \alpha'' \varepsilon a. \end{aligned} \quad (\text{The diagram commutes})$$

By the injectivity of α'' , $\varepsilon a = 0$.

(c) Consider the short exact sequence

$$0 \longrightarrow A' \xrightarrow{\mu} A \xrightarrow{\varepsilon} A'' \longrightarrow 0$$

As $\varepsilon a = 0$, $\exists a' \in A'$ such that $\mu a' = a$.

(d) Consider the commutative diagram

$$\begin{array}{ccc} A' & \xrightarrow{\mu} & A \\ \downarrow \alpha' & & \downarrow \alpha \\ B' & \xrightarrow{\mu'} & B \end{array}$$

Note that

$$\begin{aligned} \mu'(\alpha' a') &= \mu' \alpha' a' \\ &= \alpha \mu a' & (\text{The diagram commutes}) &= \alpha a \\ &= b \\ &= \mu' b'. \end{aligned}$$

By the injectivity of μ' , $b' = \alpha' a'$ for some $a' \in A'$.

(7) Suppose α is injective. Show that α' is injective.

(a) It suffices to show that $\ker \alpha' = 0$. Take $a' \in \ker \alpha'$. ($\alpha'(a') = \alpha'a' = 0$.)

(b) Consider the commutative diagram

$$\begin{array}{ccc} A' & \xrightarrow{\mu} & A \\ \downarrow \alpha' & & \downarrow \alpha \\ B' & \xrightarrow{\mu'} & B \end{array}$$

Note that

$$\begin{aligned} 0 &= \mu'0 \\ &= \mu'\alpha'a' \\ &= \alpha\mu a'. \end{aligned} \quad (\text{The diagram commutes})$$

The injectivity of $\alpha\mu$ shows that $a' = 0$.

(8) By (6)(7), α' is an isomorphism if both α and α'' are isomorphisms.

□

Exercise 1.2. (Five lemma)

Show that, given a commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 & \longrightarrow & \cdots \\ & & \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & \downarrow \varphi_4 & & \downarrow \varphi_5 & & \\ \cdots & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 & \longrightarrow & \cdots \end{array}$$

with exact rows, in which $\varphi_1, \varphi_2, \varphi_4, \varphi_5$ are isomorphisms, then φ_3 is also an isomorphism. Can we weaken the hypotheses in a reasonable way?

One reasonable hypotheses:

- (a) If φ_1 is surjective and φ_2, φ_4 is injective, then φ_3 is injective.
- (b) If φ_5 is injective and φ_2, φ_4 is surjective, then φ_3 is surjective.

Proof of (a).

(1) Write

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & A_5 & \longrightarrow & \cdots \\ & & \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & \downarrow \varphi_4 & & \downarrow \varphi_5 & & \\ \cdots & \longrightarrow & B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 & \xrightarrow{\beta_4} & B_5 & \longrightarrow & \cdots \end{array}$$

Take $a \in \ker(\varphi_3)$ and then we need to show $a = 0$.

(2) The commutative diagram

$$\begin{array}{ccc} A_3 & \xrightarrow{\alpha_3} & A_4 \\ \downarrow \varphi_3 & & \downarrow \varphi_4 \\ B_3 & \xrightarrow{\beta_3} & B_4 \end{array}$$

implies that $0 = \beta_3 0 = \beta_3 \varphi_3 a = \varphi_4 \alpha_3 a$. The injectivity of φ_4 implies that $\alpha_3 a = 0$.

(3) The exact sequence

$$\cdots \longrightarrow A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} A_4 \longrightarrow \cdots$$

shows that $a \in \ker(\alpha_3) = \text{im}(\alpha_2)$. So there exists $a_2 \in A_2$ such that $\alpha_2 a_2 = a$.

(4) The commutative diagram

$$\begin{array}{ccc} A_2 & \xrightarrow{\alpha_2} & A_3 \\ \downarrow \varphi_2 & & \downarrow \varphi_3 \\ B_2 & \xrightarrow{\beta_2} & B_3 \end{array}$$

implies that $0 = \varphi_3 a = \varphi_3 \alpha_2 a_2 = \beta_2 \varphi_2 a_2$.

(5) The exact sequence

$$\cdots \longrightarrow B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \longrightarrow \cdots$$

shows that $\varphi_2 a_2 \in \ker(\beta_2) = \text{im}(\beta_1)$. So there exists $b_1 \in B_1$ such that $\varphi_2 a_2 = \beta_1 b_1$.

(6) Consider the commutative diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\alpha_1} & A_2 \\ \downarrow \varphi_1 & & \downarrow \varphi_2 \\ B_1 & \xrightarrow{\beta_1} & B_2 \end{array}$$

The surjectivity of φ_i implies that $\exists a_1 \in A_1$ such that $\varphi_1 a_1 = b_1$. Hence the commutative diagram implies that $\varphi_2(\alpha_1 a_1) = \varphi_2 \alpha_1 a_1 = \beta_1 \varphi_1 a_1 = \beta_1 b_1 = \varphi_2 a_2$. The injectivity of φ_2 implies that $\alpha_1 a_1 = a_2$.

(7) The exact sequence

$$\cdots \longrightarrow A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \longrightarrow \cdots$$

shows that $a = \alpha_2 a_2 = \alpha_2 \alpha_1 a_1 = 0$. Therefore φ_3 is injective.

□

Proof of (b).

- (1) Take any $b \in B_3$, it suffices to find $a \in A$ such that $\varphi_3 a = b$.
- (2) Let $b_4 := \beta_3 b \in B_4$. The exact sequence

$$\cdots \longrightarrow B_3 \xrightarrow{\beta_3} B_4 \xrightarrow{\beta_4} B_5 \longrightarrow \cdots$$

shows that $\beta_4 b_4 = \beta_4(\beta_3 b) = 0$.

- (3) Look at the commutative diagram

$$\begin{array}{ccc} A_4 & \xrightarrow{\alpha_4} & A_5 \\ \downarrow \varphi_4 & & \downarrow \varphi_5 \\ B_4 & \xrightarrow{\beta_4} & B_5 \end{array}$$

By the surjectivity of φ_4 , $\exists a_4 \in A_4$ such that $\varphi_4 a_4 = b_4$. So the commutative diagram says that $0 = \beta_4 b_4 = \beta_4 \varphi_4 a_4 = \varphi_5 \alpha_4 a_4$. By the injectivity of φ_5 , $\alpha_4 a_4 = 0$.

- (4) The exact sequence

$$\cdots \longrightarrow A_3 \xrightarrow{\alpha_3} A_4 \xrightarrow{\alpha_4} A_5 \longrightarrow \cdots$$

shows that $a_4 \in \ker(\alpha_4) = \text{im}(\alpha_3)$. So there exists $a_3 \in A_3$ such that $\alpha_3 a_3 = a_4$.

- (5) Let $\bar{b} = b - \varphi_3 a_3 \in B_3$. The commutative diagram

$$\begin{array}{ccc} A_3 & \xrightarrow{\alpha_3} & A_4 \\ \downarrow \varphi_3 & & \downarrow \varphi_4 \\ B_3 & \xrightarrow{\beta_3} & B_4 \end{array}$$

implies that $\beta_3 \bar{b} = \beta_3 b - \beta_3 \varphi_3 a_3 = \beta_3 b - \varphi_4 \alpha_3 a_3 = \beta_3 b - \varphi_4 a_4 = \beta_3 b - b_4 = \beta_3 b - \beta_3 b = 0$. So $\bar{b} \in \ker(\beta_3)$.

- (6) The exact sequence

$$\cdots \longrightarrow B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} B_4 \longrightarrow \cdots$$

shows that $\bar{b} \in \ker(\beta_3) = \text{im}(\beta_2)$. Hence $\exists b_2 \in B_2$ such that $\bar{b} = \beta_2 b_2$.

(7) Look at the commutative diagram

$$\begin{array}{ccc} A_2 & \xrightarrow{\alpha_2} & A_3 \\ \downarrow \varphi_2 & & \downarrow \varphi_3 \\ B_2 & \xrightarrow{\beta_2} & B_3 \end{array}$$

The surjectivity of φ_2 implies that $\exists a_2 \in A_2$ such that $b_2 = \varphi_2 a_2$. Let $a := \alpha_2 a_2 + a_3$. Hence

$$\begin{aligned} \varphi_3(a) &= \varphi_3 \alpha_2 a_2 + \varphi_3 a_3 \\ &= \beta_2 \varphi_2 a_2 + \varphi_3 a_3 && \text{(The diagram commutes)} \\ &= \beta_2 b_2 + \varphi_3 a_3 \\ &= \bar{b} + \varphi_3 a_3 \\ &= (b - \varphi_3 a_3) + \varphi_3 a_3 \\ &= b. \end{aligned}$$

□

Exercise 1.4.

Show that the abelian group A admits the structure of a $\mathbb{Z}/(m)$ -module if and only if $mA = 0$.

Proof.

(1) (\implies) It suffices to show that $ma = 0$ for all $a \in A$. Let $\Lambda = \mathbb{Z}/(m)$.

$$\begin{aligned} ma &= \underbrace{a + \cdots + a}_{m \text{ times}} \\ &= \underbrace{1_\Lambda a + \cdots + 1_\Lambda a}_{m \text{ times}} && \text{(Axiom M3)} \\ &= \underbrace{(1_\Lambda + \cdots + 1_\Lambda)}_{m \text{ times}} a && \text{(Axiom M1)} \\ &= 0_\Lambda a && (\text{char}(\Lambda) = m) \\ &= 0. && \text{(Axiom M1)} \end{aligned}$$

(2) (\impliedby) Write $\bar{\lambda} \in \Lambda := \mathbb{Z}/(m)$ where $\lambda \in \mathbb{Z}$ and $\bar{\lambda}$ is the residue class of λ in Λ . Define $\omega : \Lambda \rightarrow \text{End}(A, A)$ by

$$\omega(\bar{\lambda})(a) = \lambda a$$

for all $a \in A$ and $\bar{\lambda} \in \Lambda$. ω is well-defined since $mA = 0$. Note that all four module axioms hold for A (as a Λ -module).

□

§2. The Group of Homomorphisms

Exercise 2.1.

Show that in the setting of Theorem 2.1 $\varepsilon_* = \text{Hom}(A, \varepsilon)$ is not, in general, surjective even if ε is. (Hint: Take $\Lambda = \mathbb{Z}$, $A = \mathbb{Z}/(n)$, the integers mod n , and the short exact sequence $\mathbb{Z} \xrightarrow{\mu} \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z}/(n)$ where μ is multiplication by n .)

Theorem 2.1. Let $B' \xrightarrow{\mu} B \xrightarrow{\varepsilon} B''$ be an exact sequence of Λ -modules. For every Λ -module A the induced sequence

$$0 \longrightarrow \text{Hom}_{\Lambda}(A, B') \xrightarrow{\mu_*} \text{Hom}_{\Lambda}(A, B) \xrightarrow{\varepsilon_*} \text{Hom}_{\Lambda}(A, B'')$$

is exact.

Proof.

(1) Consider

$$\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/(m), \mathbb{Z}) \xrightarrow{\varepsilon_*} \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/(m), \mathbb{Z}/(m)).$$

Note that $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/(m), \mathbb{Z}/(m))$ is not trivial. So to prove that ε_* is not surjective, it suffices to show that $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/(m), \mathbb{Z}) = 0$.

(2) Show that $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/(m), \mathbb{Z}) = 0$. Suppose $\alpha \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/(m), \mathbb{Z})$. Given any $a \in \mathbb{Z}/(m)$. So $ma = 0$ by the Lagrange's theorem in group theory. So

$$0 = \alpha(0) = \alpha(ma) = m\alpha(a) \in \mathbb{Z}.$$

So $\alpha(a) = 0 \in \mathbb{Z}$. Hence α is a zero map.

□

Exercise 2.2.

Prove Theorem 2.2.

Theorem 2.1. Let $A' \xrightarrow{\mu} A \xrightarrow{\varepsilon} A''$ be an exact sequence of Λ -modules. For every Λ -module B the induced sequence

$$0 \longrightarrow \operatorname{Hom}_{\Lambda}(A'', B) \xrightarrow{\varepsilon^*} \operatorname{Hom}_{\Lambda}(A, B) \xrightarrow{\mu^*} \operatorname{Hom}_{\Lambda}(A', B)$$

is exact.

Proof of Theorem 2.2.

(1)

□