Chapter 4: Continuity

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Exercise 4.1. *Suppose f is a real function define on $\mathbb{R}^1$ which satisfies*

$$\lim_{h \to 0} [f(x+h) - f(x-h)] = 0$$

for every $x \in \mathbb{R}^1$. Does this imply that f is continuous?

Proof. $\lim_{h \to 0}[f(x+h) - f(x-h)] = 0$ holds if f is continuous. But the converse of this statement and is not true. For example, define $f : \mathbb{R}^1 \to \mathbb{R}^1$ by

$$f(x) = \begin{cases} 1 & (x = 0), \\ 0 & (x \neq 0). \end{cases}$$

f is not continuous at $x = 0$ but

$$\lim_{h \to 0} [f(x+h) - f(x-h)] = 0$$

for any $x \in \mathbb{R}^1$. (The identity holds for $x \neq 0$ since f is continuous on $\mathbb{R}^1 - \{0\}$. Besides, $\lim_{h \to 0}[f(0+h) - f(0-h)] = \lim_{h \to 0}[0 - 0] = 0$.) $\square$

Exercise 4.2. *If f is a continuous mapping of a metric space X into a metric space Y, prove that $f(\overline{E}) \subseteq \overline{f(E)}$ for every set $E \subseteq X$. ($\overline{E}$ denotes the closure of E.) Show, by an example, that $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$.*

Proof.

(1) Since f is continuous and $\overline{f(E)}$ is closed, $f^{-1}(\overline{f(E)})$ is closed. Hence,

$$\begin{aligned}
f^{-1}(\overline{f(E)}) &\supseteq f^{-1}(f(E)) && \text{(Monotonicity of } f^{-1}\text{)} \\
&\supseteq E, && \text{(Note in Theorem 4.14)} \\
\overline{E} &\subseteq f^{-1}(\overline{f(E)}), && \text{(Monotonicity of closure)} \\
f(\overline{E}) &\subseteq f(f^{-1}(\overline{f(E)})) && \text{(Monotonicity of } f\text{)} \\
&\subseteq \overline{f(E)}. && \text{(Note in Theorem 4.14)}
\end{aligned}$$

(2) Let $f : (0, \infty) \to \mathbb{R}$ be a continuous function defined by

$$f(x) = \frac{1}{x}.$$

Consider $E = \mathbb{Z}^+ \subseteq (0, \infty)$. Then $f(E) = \left\{\frac{1}{n} : n \in \mathbb{Z}^+\right\}$, and thus

$$\begin{aligned} f(\overline{E}) &= \left\{\frac{1}{n} : n \in \mathbb{Z}^+\right\}. \\ \overline{f(E)} &= \left\{\frac{1}{n} : n \in \mathbb{Z}^+\right\} \bigcup \{0\}. \end{aligned}$$

$\square$

Supplement (Inverse image).

(1) *$E \subseteq f^{-1}[f(E)]$ for $E \subseteq X$.*

$$\begin{aligned} \forall\, x \in E &\Longrightarrow f(x) \in f(E) \\ &\Longleftrightarrow x \in f^{-1}[f(E)]. \qquad \text{(Definition of the inverse image)} \end{aligned}$$

$\square$

(2) *$f[f^{-1}(E)] \subseteq E$ for $E \subseteq Y$.*

$$\begin{aligned} \forall\, y \in f[f^{-1}(E)] &\Longleftrightarrow \exists\, x \in f^{-1}(E) \text{ such that } y = f(x) \\ &\Longleftrightarrow \exists\, x, f(x) \in E \text{ such that } y = f(x) \\ &\Longrightarrow \exists\, x, y = f(x) \in E. \end{aligned}$$

$\square$

Supplement (Continuity). *Let f be a map from a topological space on X to a topological space on Y. Then, the following statements are equivalent:*

(1) *f is continuous: For each $x \in X$ and every neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$.*

(2) *For every open set O in Y, the inverse image $f^{-1}(O)$ is open in X.*

(3) *For every closed set C in Y, the inverse image $f^{-1}(C)$ is closed in X.*

(4) *$f(A)^\circ \subseteq f(A^\circ)$ for every subset A of X.*

(5) *$f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ$ for every subset B of Y.*

(6) *$f(\overline{A}) \subseteq \overline{f(A)}$ for every subset A of X.*

(7) *$\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for every subset B of Y.*

Exercise 4.3. *Let f be a continuous real function on a metric space X. Let $Z(f)$ (the zero set of f) be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.*

Proof (Corollary to Theorem 4.8). Since f is continuous, $f^{-1}(\{0\}) = Z(f)$ is closed in X for a closed subset $\{0\}$ in $\mathbb{R}^1$. $\square$

Proof (Theorem 4.8). Consider the complement of $Z(f)$ in X,

$$\begin{aligned} \widetilde{Z(f)} &= \{x \in X : f(x) \neq 0\} \\ &= f^{-1}((-\infty, 0) \cup (0, \infty)). \end{aligned}$$

Since f is continuous, $f^{-1}((-\infty, 0) \cup (0, \infty)) = \widetilde{Z(f)}$ is open in X for a open subset $(-\infty, 0) \cup (0, \infty)$ in $\mathbb{R}^1$. $\square$

Proof (Definition 2.18(d)). Given any limit point p of $Z(f)$. *Show that $f(p) = 0$ or $p \in Z(f)$.* Since f is continuous, given any $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(p)| < \epsilon$ for all $x \in X$ for which $d_X(x, p) < \delta$. Since p is a limit point of $Z(f)$, for such $\delta > 0$ we have a point $q \neq p$ such that $q \in Z(f)$, or $f(q) = 0$. So $|f(p)| < \epsilon$ for any $\epsilon > 0$. $f(p) = 0$. $\square$

Proof (Definition 2.18(f)). Consider the complement of $Z(f)$ in X,

$$\widetilde{Z(f)} = \{x \in X : f(x) \neq 0\} = \{f > 0\} \cup \{f < 0\}$$

where $\{f > 0\} = \{x \in X : f(x) > 0\}$ and $\{f < 0\} = \{x \in X : f(x) < 0\}$. It suffices to show $\{f > 0\}$ is open. ($\{f < 0\}$ is similar.) Given any point p of $\{f > 0\}$ or $f(p) > 0$. *Want to show p is an interior point of $\{f > 0\}$.* Since f is continuous, given any $\epsilon = \frac{f(p)}{2} > 0$ there exists a $\delta > 0$ such that $|f(x) - f(p)| < \frac{f(p)}{2}$ for all $x \in X$ for which $d_X(x, p) < \delta$. For such x with $d_X(x, p) < \delta$ we have

$$\frac{1}{2} f(p) < f(x) < \frac{3}{2} f(p).$$

That is, $N = \{x : d_X(x, p) < \delta\}$ is a neighborhood p such that $N \subseteq \{f > 0\}$. $\square$