Solutions to the book:
Friedeberg, Insel and Spence, Linear Algebra, 3rd edition

Meng-Gen Tsai
plover@gmail.com

July 20, 2021

Contents

Chapter 1: Vector Spaces

Section 1.2: Vector Spaces

Exercise 1.2.2.

Write the zero vector of $\mathsf{M}_{3\times 4}(F)$.

Proof. $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathsf{M}_{3\times 4}(F)$. $\square$

Exercise 1.2.3.

If $M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ *what are* M_{13}, M_{21}, M_{22}*?*

Proof. Since $M_{ij} = 3(i-1)+j$, $M_{13} = 3$, $M_{21} = 4$ and $M_{22} = 5$. $\square$

Exercise 1.2.22.

How many elements are there in the vector space $\mathsf{M}_{m\times n}(\mathbb{Z}/2\mathbb{Z})$*?*

Proof. 2^{mn}. $\square$

Section 1.6: Bases and Dimension

Exercise 1.6.19.

Let V *be a vector space having dimension* n*, and let* S *be a subset of* V *that generates* V.

(a) *Prove that there is a subset of* S *that is a basis for* V*. (Be careful not to assume that* S *is finite.)*

(b) *Prove that* S *contains at least* n *elements.*

Proof of (a). Similar to the argument in Theorem 1.9.

(1) If $S = \varnothing$ or $S = \{0\}$, then $\mathsf{V} = \{0\}$ and $\varnothing$ is a subset of S that is a basis for V.

(2) Otherwise S contains a nonzero element u_1. $\{u_1\}$ is a linearly independent set. Continue, if possible, choosing elements $u_2, ..., u_k$ in S such that $\{u_1, u_2, ..., u_k\}$ is linearly independent. By the Replacement Theorem (Theorem 1.10), we must eventually reach a stage at which $\beta = \{u_1, u_2, ..., u_k\}$ is a linearly independent subset of S with $k \leq n$.

(3) β generates S by the construction of β, and S generates V. Therefore, β generates V (and thus $k = n$ by the definition of dimension).

Therefore, there is a subset of S that is a basis for V. $\square$

Proof of (b). By (a), there is a subset $\beta \subseteq S$ of size n that is a basis for V. So S contains at least n elements of β. $\square$

Chapter 2: Linear Transformations and Matrices

Section 2.4: Invertibility and Isomorphisms

Exercise 2.4.8.

Let A and B be $n \times n$ matrices such that $AB = I_n$. Prove

(a) *A and B are invertible.*

(b) *$A = B^{-1}$ (and hence $B = A^{-1}$). (We are in effect saying that for square matrices, a "one-sided" inverse is a "two-sided" inverse.)*

(c) *State and prove analogous results for linear transformations defined on finite-dimensional vector spaces.*

Proof of (a). Regard $\mathsf{V} = \mathsf{M}_{n\times n}(F)$ as a finite-dimensional vector space over F. Given $X \in \mathsf{M}_{n\times n}(F)$, consider the subset V_X of V defined by

$$\mathsf{V}_X = \{XY : Y \in \mathsf{M}_{n\times n}(F)\}.$$

(1) $\mathsf{V}_0 = 0$.

(2) $\mathsf{V}_{I_n} = \mathsf{V}$. In general, $\mathsf{V}_X = \mathsf{V}$ for any invertible matrix $X \in \mathsf{M}_{n\times n}(F)$.

(3) V_X is a subspace of V for any $X \in \mathsf{M}_{n\times n}(F)$.

(4) There is a descending sequence of subspaces

$$\mathsf{V} \supseteq \mathsf{V}_X \supseteq \cdots \supseteq \mathsf{V}_{X^k} \supseteq \cdots$$

This sequence must be stationary since V is finite-dimensional, that is,

$$\mathsf{V}_{X^k} = \mathsf{V}_{X^{k+1}} = \cdots$$

for some k. (Descending chain condition.) In particular, $B^k = B^{k+1}C$ for some $C \in \mathsf{V}$. Multiply with A^k on the left to get $I_n = BC$. ($A^kB^k = A^{k-1}(AB)B^{k-1} = A^{k-1}B^{k-1} = \cdots = I_n$.)

(4) Since $AB = I_n$ and $BC = I_n$, $A = AI_n = A(BC) = (AB)C = I_nC = C$, or $AB = BA = I_n$. By definition of invertibility, A and B are invertible.

$\square$

Proof of (b). By (a), $A = B^{-1}$ and $B = A^{-1}$. $\square$

Proof of (c). Let V *be a finite-dimensional vector space, and let* $\mathsf{S}, \mathsf{T} : \mathsf{V} \to \mathsf{V}$ *be linear such that* ST *is invertible. Show that* S *and* T *are invertible.* Let

$$\beta = \{\beta_1, ..., \beta_n\}$$

be an ordered basis for V where $n = \dim(\mathsf{V})$. Let $A = [\mathsf{S}]_\beta$ and $B = [\mathsf{T}]_\beta$. So

$$AB = [\mathsf{S}]_\beta [\mathsf{T}]_\beta = [\mathsf{ST}]_\beta = [\mathsf{I}_\mathsf{V}]_\beta = I_n$$

(Theorem 2.11). By (a), $A = [\mathsf{S}]_\beta$ and $B = [\mathsf{T}]_\beta$ are invertible, or S and T are invertible (Theorem 2.18). $\square$

Section 2.7: Homogeneous Linear Differential Equations with Constant Coefficients

Exercise 2.7.3.

Find a basis for the solution space of each of the following differential equations

(a) $y'' + 2y' + y = 0$

(b) $y''' = y'$

(c) $y^{(4)} - 2y^{(2)} + y = 0$

(d) $y'' + 2y' + y = 0$

(e) $y^{(3)} - y^{(2)} + 3y^{(1)} + 5y = 0$.

Use Theorem 2.35.

Proof of (a). The auxiliary polynomial is $t^2 + ty + 1 = (t + 1)^2$. $\{e^{-t}, te^{-t}\}$ is a basis for the solution space. $\square$

Proof of (b). The auxiliary polynomial is $t^3 - t = t(t - 1)(t + 1)$. $\{1, e^t, e^{-t}\}$ is a basis for the solution space. $\square$

Proof of (c). The auxiliary polynomial is $t^4 - 2t^2 + 1 = (t - 1)^2(t + 1)^2$. $\{e^t, te^t, e^{-t}, te^{-t}\}$ is a basis for the solution space. $\square$

Proof of (d). Same as (a). $\square$

Proof of (e). The auxiliary polynomial is

$$t^3 - t^2 + 3t + 5 = (t + 1)(t - 1 - 2i)(t - 1 + 2i).$$

$\{e^{-t}, e^{(1+2i)t}, e^{(1-2i)t}\}$, or $\{e^{-t}, e^t\cos(2t), e^t\sin(2t)\}$ is a basis for the solution space. $\square$

Exercise 2.7.4.

Find a basis for each of the following subspaces of C^∞.

(a) $\mathsf{N}(\mathsf{D}^2 - \mathsf{D} - \mathsf{I})$

(b) $\mathsf{N}(\mathsf{D}^3 - 3\mathsf{D}^2 + 3\mathsf{D} - \mathsf{I})$

(c) $\mathsf{N}(\mathsf{D}^3 - 6\mathsf{D}^2 - 8\mathsf{D})$

Use Theorem 2.35.

Proof of (a). The auxiliary polynomial is

$$t^2 - t - 1 = \left(t - \frac{1+\sqrt{5}}{2}\right)\left(t - \frac{1-\sqrt{5}}{2}\right).$$

$\left\{e^{\frac{1+\sqrt{5}}{2}t}, e^{\frac{1-\sqrt{5}}{2}t}\right\}$ is a basis for the solution space. $\square$

Proof of (b). The auxiliary polynomial is $t^3-3t^2+3t-1 = (t-1)^3$. $\{e^t, te^t, t^2e^t\}$ is a basis for the solution space. $\square$

Proof of (c). The auxiliary polynomial is $t^3 + 6t^2 + 8t = t(t+2)(t+4)$. $\{1, e^{-2t}, e^{-4t}\}$ is a basis for the solution space. $\square$

Exercise 2.7.5.

Show that C^∞ *is a subspace of* $\mathcal{F}(\mathbb{R}, \mathbb{C})$.

Proof.

(1) $0 \in \mathcal{F}(\mathbb{R}, \mathbb{C})$ clearly.

(2) Given any $f, g \in \mathsf{C}^\infty$. For any nonnegative k, $\mathsf{D}^k(f+g) = \mathsf{D}^k(f) + \mathsf{D}^k(g)$ holds. Thus $f + g \in \mathsf{C}^\infty$.

(3) Given any $f \in \mathcal{F}(\mathbb{R}, \mathbb{C})$, $r \in \mathbb{C}$. For any nonnegative k, $\mathsf{D}^k(cf) = c\mathsf{D}^k(f)$ holds. Thus $cf \in \mathsf{C}^\infty$.

By Theorem 1.3, C^∞ is a subspace. $\square$

Chapter 4: Determinants

Section 4.1: Determinants of Order 2

Exercise 4.1.1.

Label the following statements as being true or false.

(a) *The function* $\det : \mathsf{M}_{2\times 2}(F) \to F$ *is a linear transformation.*

(b) *The determinant of a* 2×2 *matrix is a linear function of each row of the matrix when the other row is held fixed.*

(c) *If* $A \in \mathsf{M}_{2\times 2}(F)$ *and* $\det(A) = 0$*, then* A *is invertible.*

(d) *If* u *and* v *are vectors in* $\mathbb{R}^2$ *emanating from the origin, then the area of the parallelogram having* u *and* v *as adjacent side is*

$$\det \begin{pmatrix} u \\ v \end{pmatrix}.$$

(e) *A coordinate system is right-handed if and only if its orientation equals* 1.

Proof of (a). False. Example 4.1.1, or take

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathsf{M}_{2\times 2}(F) \text{ and } B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \mathsf{M}_{2\times 2}(F).$$

Then $\det(A + B) = \det(I_2) = 1 \neq 0 = 0 + 0 = \det(A) + \det(B)$. □

Proof of (b). True. Proposition 4.1. □

Proof of (c). False. Proposition 4.2. □

Proof of (d). False. The area should be

$$O \begin{pmatrix} u \\ v \end{pmatrix} \cdot \det \begin{pmatrix} u \\ v \end{pmatrix} = \left| \det \begin{pmatrix} u \\ v \end{pmatrix} \right|.$$

□

Proof of (e). True. See Exercise 4.1.12. □

Exercise 4.1.2.

Compute the determinants of the following elements of $\mathsf{M}_{2\times 2}(\mathbb{R})$.

(a) $\begin{pmatrix} 6 & -3 \\ 2 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} -5 & 2 \\ 6 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 8 & 0 \\ 3 & -1 \end{pmatrix}$

Proof of (a).

$$\det \begin{pmatrix} 6 & -3 \\ 2 & 4 \end{pmatrix} = 6 \cdot 4 - (-3) \cdot 2 = 24 + 6 = 30.$$

□

Proof of (b).

$$\det \begin{pmatrix} -5 & 2 \\ 6 & 1 \end{pmatrix} = (-5) \cdot 1 - 2 \cdot 6 = -5 - 12 = -17.$$

□

Proof of (c).

$$\det \begin{pmatrix} 8 & 0 \\ 3 & -1 \end{pmatrix} = 8 \cdot (-1) - 0 \cdot 3 = -8.$$

□

Exercise 4.1.3.

Compute the determinants of the following elements of $\mathsf{M}_{2\times 2}(\mathbb{C})$.

(a) $\begin{pmatrix} -1+i & 1-4i \\ 3+2i & 2-3i \end{pmatrix}$

(b) $\begin{pmatrix} 5-2i & 6+4i \\ -3+i & 7i \end{pmatrix}$

(c) $\begin{pmatrix} 2i & 3 \\ 4 & 6i \end{pmatrix}$

Proof of (a).

$$\det\begin{pmatrix} -1+i & 1-4i \\ 3+2i & 2-3i \end{pmatrix} = (-1+i)\cdot(2-3i) - (1-4i)\cdot(3+2i)$$
$$= (1+5i) - (11-10i)$$
$$= -10+15i.$$

□

Proof of (b).

$$\det\begin{pmatrix} 5-2i & 6+4i \\ -3+i & 7i \end{pmatrix} = (5-2i)\cdot(7i) - (6+4i)\cdot(-3+i)$$
$$= (14+35i) - (-22-6i)$$
$$= 36+41i.$$

□

Proof of (c).

$$\det\begin{pmatrix} 2i & 3 \\ 4 & 6i \end{pmatrix} = (2i)\cdot(6i) - 3\cdot 4 = -12-12 = -24.$$

□

Exercise 4.1.4.

For each of the following pairs of vectors u and v in $\mathbb{R}^2$, compute the area of the parallelogram determined by u and v.

(a) $u = (3, -2)$ *and* $v = (2, 5)$

(b) $u = (1, 3)$ *and* $v = (-3, 1)$

(c) $u = (4, -1)$ *and* $v = (-6, -2)$

(d) $u = (3, 4)$ *and* $v = (2, -6)$

Proof of (a).

$$\left|\det\begin{pmatrix} 3 & -2 \\ 2 & 5 \end{pmatrix}\right| = |19| = 19.$$

□

Proof of (b).

$$\left|\det\begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}\right| = |10| = 10.$$

$\square$

Proof of (c).
$$\left|\det\begin{pmatrix} 4 & -1 \\ -6 & -2 \end{pmatrix}\right| = |-14| = 14.$$

$\square$

Proof of (d).
$$\left|\det\begin{pmatrix} 3 & 4 \\ 2 & -6 \end{pmatrix}\right| = |-26| = 26.$$

$\square$

Exercise 4.1.5.

Prove that if B is the matrix obtained by interchanging the rows of a 2×2 matrix A, then $\det(B) = -\det(A)$.

Proof. Write
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{M}_{2\times 2}(F).$$
Then
$$B = \begin{pmatrix} c & d \\ a & b \end{pmatrix} \in \mathsf{M}_{2\times 2}(F).$$
Then $\det(B) = cb - ad = -(ad - bc) = -\det(A)$. $\square$

Exercise 4.1.6.

Prove that if the two columns of $A \in \mathsf{M}_{2\times 2}(F)$ are identical, then $\det(A) = 0$.

Proof. By assumption, write
$$A = \begin{pmatrix} a & a \\ c & c \end{pmatrix} \in \mathsf{M}_{2\times 2}(F).$$
Then $\det(A) = ac - ac = 0$. $\square$

Exercise 4.1.7.

Prove that for any $A \in \mathsf{M}_{2\times 2}(F)$, $\det(A^t) = \det(A)$.

Proof. Write
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{M}_{2\times 2}(F),$$
then
$$A^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathsf{M}_{2\times 2}(F).$$
So $\det(A) = ad - bc = ad - cb = \det(A^t)$. $\square$

Exercise 4.1.8.

Prove that if $A \in \mathsf{M}_{2\times 2}(F)$ is upper triangular, then $\det(A)$ equals the product of the diagonal entries of A.

Proof. Write
$$A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathsf{M}_{2\times 2}(F)$$
since A is upper triangular. Then $\det(A) = ad$, which is equal to the product of the diagonal entries, a and d, of A. $\square$

Exercise 4.1.9.

Prove that for any $A, B \in \mathsf{M}_{2\times 2}(F)$ we have $\det(AB) = \det(A) \cdot \det(B)$.

Proof. Write
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{M}_{2\times 2}(F),$$
$$B = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in \mathsf{M}_{2\times 2}(F).$$
Then
$$AB = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} \in \mathsf{M}_{2\times 2}(F).$$

A direct calculation shows

$$\begin{aligned}
\det(AB) &= (ae+bg)(cf+dh) - (af+bh)(ce+dg) \\
&= (acef + adeh + bcfg + bdgh) - (acef + adfg + bceh + bdgh) \\
&= adeh + bcfg - adfg - bceh \\
&= (ad-bc)(eh-fg) \\
&= \det(A)\det(B).
\end{aligned}$$

□

Exercise 4.1.10.

*The **classical adjoint** of a 2×2 matrix $A \in \mathsf{M}_{2\times 2}(F)$ is the matrix*

$$C = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}.$$

Prove

(a) $CA = AC = [\det(A)]I$.

(b) $\det(C) = \det(A)$.

(c) The classical adjoint of A^t is C^t.

(d) If A is invertible, then $A^{-1} = [\det(A)]^{-1}C$.

Note that

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Proof of (a).

$$\begin{aligned}
CA &= \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \\
&= \begin{pmatrix} A_{22}A_{11} - A_{12}A_{21} & A_{22}A_{12} - A_{12}A_{22} \\ -A_{21}A_{11} + A_{11}A_{21} & -A_{21}A_{12} + A_{11}A_{22} \end{pmatrix} \\
&= \begin{pmatrix} \det(A) & 0 \\ 0 & \det(A) \end{pmatrix} \\
&= [\det(A)]I.
\end{aligned}$$

$$
\begin{aligned}
AC &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \\
&= \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & -A_{11}A_{12} + A_{12}A_{11} \\ A_{21}A_{22} - A_{22}A_{21} & -A_{21}A_{12} + A_{22}A_{11} \end{pmatrix} \\
&= \begin{pmatrix} \det(A) & 0 \\ 0 & \det(A) \end{pmatrix} \\
&= [\det(A)]I.
\end{aligned}
$$

□

Proof of (b).

$$
\begin{aligned}
\det(C) &= A_{22}A_{11} - (-A_{12})(-A_{21}) \\
&= A_{11}A_{22} - A_{12}A_{21} \\
&= \det(A).
\end{aligned}
$$

□

Proof of (c).

$$
A^t = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix}.
$$

The classical adjoint of A^t is

$$
\begin{pmatrix} A_{22} & -A_{21} \\ -A_{12} & A_{11} \end{pmatrix} = C^t.
$$

□

Proof of (d). Proposition 4.2. □

Exercise 4.1.11.

Let $\delta : \mathsf{M}_{2\times 2}(F) \to F$ be a function with the following three properties.

(i) *δ is a linear function of each row of the matrix when the other row is held fixed.*

(ii) *If the two rows of $A \in \mathsf{M}_{2\times 2}(F)$ are identical, then $\delta(A) = 0$.*

(iii) *If I is the 2×2 identity matrix, then $\delta(I) = 1$.*

Prove that $\delta(A) = \det(A)$ for all $A \in \mathsf{M}_{2\times 2}(F)$. (This result is generalized in Section 4.5.)

Proof. Write

$$
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.
$$

(1) *If* u, v *are elements of* F^2 *and* k *is a scalar, then*

$$\delta \begin{pmatrix} u \\ v + ku \end{pmatrix} = \delta \begin{pmatrix} u + kv \\ v \end{pmatrix} = \delta \begin{pmatrix} u \\ v \end{pmatrix}.$$

In fact,

$$\begin{aligned} \delta \begin{pmatrix} u \\ v + ku \end{pmatrix} &= \delta \begin{pmatrix} u \\ v \end{pmatrix} + \delta \begin{pmatrix} u \\ ku \end{pmatrix} && \text{(Property (i))} \\ &= \delta \begin{pmatrix} u \\ v \end{pmatrix} + k\delta \begin{pmatrix} u \\ u \end{pmatrix} && \text{(Property (i))} \\ &= \delta \begin{pmatrix} u \\ v \end{pmatrix}. && \text{(Property (ii))} \end{aligned}$$

Similarly, $\delta \begin{pmatrix} u + kv \\ v \end{pmatrix} = \delta \begin{pmatrix} u \\ v \end{pmatrix}$.

(2) *If* u, v *are elements of* F^2*, then*

$$\delta \begin{pmatrix} u \\ v \end{pmatrix} = -\delta \begin{pmatrix} v \\ u \end{pmatrix}.$$

In fact,

$$\begin{aligned} 0 &= \delta \begin{pmatrix} u + v \\ u + v \end{pmatrix} && \text{(Property (ii))} \\ &= \delta \begin{pmatrix} u + v \\ u \end{pmatrix} + \delta \begin{pmatrix} u + v \\ v \end{pmatrix} && \text{(Property (i))} \\ &= \delta \begin{pmatrix} v \\ u \end{pmatrix} + \delta \begin{pmatrix} u \\ v \end{pmatrix}. && \text{((1))} \end{aligned}$$

(3) *If* v *is an element of* F^2*, then*

$$\delta \begin{pmatrix} 0 \\ v \end{pmatrix} = 0.$$

In fact,

$$\begin{aligned} \delta \begin{pmatrix} 0 \\ v \end{pmatrix} &= \delta \begin{pmatrix} 0 + 0 \\ v \end{pmatrix} \\ &= \delta \begin{pmatrix} 0 \\ v \end{pmatrix} + \delta \begin{pmatrix} 0 \\ v \end{pmatrix}. && \text{(Property (i))} \end{aligned}$$

In particular, $\delta \begin{pmatrix} 0 \\ v \end{pmatrix} = 0 = \det \begin{pmatrix} 0 \\ v \end{pmatrix}$.

(4) To show $\delta(A) = \det(A)$, we consider three possible cases about the first row: $A_{11} \neq 0$, $A_{12} \neq 0$, or $A_{11} = A_{12} = 0$. The case $A_{11} = A_{12} = 0$ is proved in (3). We prove the rest two cases in (5) and (6). Write

$$u = (A_{11}, A_{12}) \text{ and } v = (A_{21}, A_{22}).$$

(5) *Show that* $\delta(A) = \det(A)$ *if* $A_{11} \neq 0$. So

$$\begin{aligned}
\delta(A) &= \delta\begin{pmatrix} u \\ v \end{pmatrix} \\
&= \delta\begin{pmatrix} u \\ v - \frac{A_{21}}{A_{11}}u \end{pmatrix} && ((1)) \\
&= \delta\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} - \frac{A_{12}A_{21}}{A_{11}} \end{pmatrix} \\
&= \left(A_{22} - \frac{A_{12}A_{21}}{A_{11}}\right)\delta\begin{pmatrix} A_{11} & A_{12} \\ 0 & 1 \end{pmatrix} && \text{(Property (i))} \\
&= \left(A_{22} - \frac{A_{12}A_{21}}{A_{11}}\right)\delta\begin{pmatrix} A_{11} & 0 \\ 0 & 1 \end{pmatrix} && ((1)) \\
&= A_{11}\left(A_{22} - \frac{A_{12}A_{21}}{A_{11}}\right)\delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} && \text{(Property (i))} \\
&= \det(A)\delta(I) \\
&= \det(A). && \text{(Property (iii))}
\end{aligned}$$

(6) *Show that* $\delta(A) = \det(A)$ *if* $A_{12} \neq 0$. So

$$\begin{aligned}
\delta(A) &= \delta\begin{pmatrix} u \\ v \end{pmatrix} \\
&= \delta\begin{pmatrix} u \\ v - \frac{A_{22}}{A_{12}}u \end{pmatrix} && ((1)) \\
&= \delta\begin{pmatrix} A_{11} & A_{12} \\ A_{21} - \frac{A_{22}A_{11}}{A_{12}} & 0 \end{pmatrix} \\
&= \left(A_{21} - \frac{A_{22}A_{11}}{A_{12}}\right)\delta\begin{pmatrix} A_{11} & A_{12} \\ 1 & 0 \end{pmatrix} && \text{(Property (i))} \\
&= \left(A_{21} - \frac{A_{22}A_{11}}{A_{12}}\right)\delta\begin{pmatrix} 0 & A_{12} \\ 1 & 0 \end{pmatrix} && ((1)) \\
&= A_{12}\left(A_{21} - \frac{A_{22}A_{11}}{A_{12}}\right)\delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} && \text{(Property (i))} \\
&= -A_{12}\left(A_{21} - \frac{A_{22}A_{11}}{A_{12}}\right)\delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} && ((2)) \\
&= \det(A)\delta(I) \\
&= \det(A). && \text{(Property (iii))}
\end{aligned}$$

$\square$

Exercise 4.1.12.

Let $\{u, v\}$ be an ordered basis for $\mathbb{R}^2$. Prove that

$$O\begin{pmatrix} u \\ v \end{pmatrix} = 1$$

if and only if $\{u, v\}$ forms a right-handed coordinate system. (Hint: Recall the definition of a rotation given in Example 2.1.2.)

If $\beta = \{u, v\}$ is an ordered basis for $\mathbb{R}^2$, define the orientation of β as

$$O\begin{pmatrix} u \\ v \end{pmatrix} = \frac{\det\begin{pmatrix} u \\ v \end{pmatrix}}{\left|\det\begin{pmatrix} u \\ v \end{pmatrix}\right|}.$$

A coordinate system $\{u, v\}$ is called right-handed if u can be rotated in a counterclockwise direction through an angle θ $(0 < \theta < \pi)$ to coincide with v.

Example 2.1.2. For any angle θ, define $\mathsf{T}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\mathsf{T}_\theta(a_1, a_2) = (a_1 \cos\theta - a_2 \sin\theta, a_1 \sin\theta + a_2 \cos\theta).$$

T_θ is called the rotation by θ.

Proof.

(1) By Example 2.1.2, for any coordinate system $\{u, v\}$, there is $0 < \theta < 2\pi$ and $\alpha > 0$ such that $v = \alpha \mathsf{T}_\theta(u)$. Write $u = (u_1, u_2) \in \mathbb{R}^2, v = (v_1, v_2) \in \mathbb{R}^2$.

(2) Calculate $\det\begin{pmatrix} u \\ v \end{pmatrix}$.

$$\begin{aligned}
\det\begin{pmatrix} u \\ v \end{pmatrix} &= \det\begin{pmatrix} u \\ \alpha \mathsf{T}_\theta(u) \end{pmatrix} \\
&= \alpha \det\begin{pmatrix} u \\ \mathsf{T}_\theta(u) \end{pmatrix} \\
&= \alpha \det\begin{pmatrix} u_1 & u_2 \\ u_1 \cos\theta - u_2 \sin\theta & u_1 \sin\theta + u_2 \cos\theta \end{pmatrix} \\
&= \alpha(u_1^2 + u_2^2) \sin\theta.
\end{aligned}$$

(3)

$$
\begin{aligned}
O\begin{pmatrix} u \\ v \end{pmatrix} = 1 &\iff \det\begin{pmatrix} u \\ v \end{pmatrix} = \alpha(u_1^2 + u_2^2)\sin\theta > 0 \\
&\iff \sin\theta > 0 \\
&\iff 0 < \theta < \pi \\
&\iff \{u, v\} \text{ is a right-handed coordinate system.}
\end{aligned}
$$

□

Section 4.2: Determinants of Order n

Exercise 4.2.2.

Find the value of k that satisfies the following equation.

$$\det\begin{pmatrix} 3a_1 & 3a_2 & 3a_3 \\ 3b_1 & 3b_2 & 3b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{pmatrix} = k\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

Proof (Exercise 4.2.25). By Exercise 4.2.25, $\det(3A) = 3^3\det(A)$ for any $A \in \mathsf{M}_{3\times3}(F)$, or $k = 3^3 = 27$. □

Exercise 4.2.26.

Let $A \in \mathsf{M}_{n\times n}(F)$. Under what conditions is $\det(-A) = \det(A)$?

Proof (Exercise 4.2.25). By Exercise 4.2.25, $\det(-A) = (-1)^n\det(A)$ for any $A \in \mathsf{M}_{n\times n}(F)$. That is, n is even if and only if $\det(-A) = \det(A)$. □

Section 4.3: Properties of Determinants

Exercise 4.3.9.

A matrix $M \in \mathsf{M}_{n\times n}(\mathbb{C})$ is called nilpotent if, for some positive integer k, $M^k = O$, where O is the $n \times n$ zero matrix. Prove that if M is nilpotent, then $\det(M) = 0$.

Proof. Given any nilpotent matrix $M \in \mathsf{M}_{n\times n}(\mathbb{C})$ such that $M^k = O$ for some $k \in \mathbb{Z}^+$.

$$\begin{aligned} M^k = O \Longrightarrow \det\bigl(M^k\bigr) &= \det(O) \\ \Longleftrightarrow \det(M)^k &= 0 && \text{(Theorem 4.7)} \\ \Longleftrightarrow \det(M) &= 0. \end{aligned}$$

□

Exercise 4.3.11.

A matrix $Q \in \mathsf{M}_{n\times n}(\mathbb{R})$ is called orthogonal if $QQ^t = I$. Prove that if Q is orthogonal, then $\det(Q) = \pm 1$.

Proof. By the orthogonality of Q, $QQ^t = I$. So

$$\begin{aligned} QQ^t = I &\Longrightarrow \det\bigl(QQ^t\bigr) = \det(I) \\ &\Longleftrightarrow \det(Q)\det\bigl(Q^t\bigr) = \det(I) && \text{(Theorem 4.7)} \\ &\Longleftrightarrow \det(Q)\det(Q) = \det(I) && \text{(Theorem 4.8)} \\ &\Longleftrightarrow \det(Q)^2 = 1 && \text{(Example 4.2.4)} \\ &\Longleftrightarrow \det(Q) = \pm 1. \end{aligned}$$

□

Exercise 4.3.14.

Prove that if $A, B \in \mathsf{M}_{n\times n}(F)$ are similar, then $\det(A) = \det(B)$.

Proof. Since A, B are similar, there exists an invertible matrix Q such that $B = Q^{-1}AQ$. So

$$\begin{aligned} \det(B) &= \det\bigl(Q^{-1}AQ\bigr) \\ &= \det\bigl(Q^{-1}\bigr)\det(A)\det(Q) && \text{(Theorem 4.7)} \\ &= \det(Q)\det\bigl(Q^{-1}\bigr)\det(A) && (F \text{ is field}) \\ &= \det\bigl(QQ^{-1}\bigr)\det(A) && \text{(Theorem 4.7)} \\ &= \det(I)\det(A) \\ &= 1 \cdot \det(A) && \text{(Example 4.2.4)} \\ &= \det(A). \end{aligned}$$

□

Chapter 6: Inner Product Spaces

Section 6.1: Inner Products and Norms

Exercise 6.1.6.

Complete the proof of Theorem 6.1.

Theorem 6.1 *Let* V *be an inner product space. Then for* $x, y, z, \in \mathsf{V}$ *and* $c \in F$

(a) $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$,

(b) $\langle x, cy \rangle = \bar{c}\langle x, y \rangle$,

(c) $\langle x, x \rangle = 0$ *if and only if* $x = 0$,

(d) if $\langle x, y \rangle = \langle x, z \rangle$ *for all* $x \in \mathsf{V}$, *then* $y = z$.

Proof of (a).

$$\begin{aligned} \langle x, y+z \rangle &= \overline{\langle y+z, x \rangle} \\ &= \overline{\langle y, x \rangle + \langle z, x \rangle} \\ &= \overline{\langle y, x \rangle} + \overline{\langle z, x \rangle} \\ &= \langle x, y \rangle + \langle x, z \rangle. \end{aligned}$$

□

Proof of (b).

$$\begin{aligned} \langle x, cy \rangle &= \overline{\langle cy, x \rangle} \\ &= \overline{c\langle y, x \rangle} \\ &= \bar{c}\overline{\langle y, x \rangle} \\ &= \bar{c}\langle x, y \rangle. \end{aligned}$$

□

Proof of (c).

(1) ($\Longrightarrow$) If x were nonzero, by the definition of the inner product, $\langle x, x \rangle > 0$, contrary to the assumption. Hence $x = 0$.

(2) ($\Longleftarrow$) Since $0 = 0 + 0$, $\langle 0, 0 \rangle = \langle 0 + 0, 0 \rangle = \langle 0, 0 \rangle + \langle 0, 0 \rangle$. Thus $\langle 0, 0 \rangle = 0$.

□

Proof of (d).

$$
\begin{aligned}
\langle x, y\rangle = \langle x, z\rangle \ \forall x \in \mathsf{V} &\iff 0 = \langle x, y\rangle - \langle x, z\rangle \ \forall x \in \mathsf{V} \\
&\iff 0 = \langle x, y - z\rangle \ \forall x \in \mathsf{V} && \text{((a))} \\
&\implies 0 = \langle y - z, y - z\rangle && (\text{Take } x = y - z \in \mathsf{V}) \\
&\iff y - z = 0 && \text{((c))} \\
&\iff y = z.
\end{aligned}
$$

□