

## Chapter 5: Differentiation

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**Exercise 5.1.** Let  $f$  be defined for all real  $x$ , and suppose that

$$|f(x) - f(y)| \leq (x - y)^2$$

for all real  $x$  and  $y$ . Prove that  $f$  is a constant.

*Proof.*

(1) Write

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq |x - y|$$

if  $x \neq y$ .

(2) Given any  $y \in \mathbb{R}$ ,

$$\left| \frac{f(x) - f(y)}{x - y} \right| \rightarrow 0 \quad \text{as } x \rightarrow y,$$

or  $|f'(y)| = 0$ .

(3) Or using  $\varepsilon$ - $\delta$  argument. Fix  $y \in \mathbb{R}$ . Given any  $\varepsilon > 0$ , there exists  $\delta = \varepsilon > 0$  such that

$$\left| \frac{f(x) - f(y)}{x - y} - 0 \right| \leq |x - y| < \delta = \varepsilon$$

whenever  $|x - y| < \delta$ . That is,  $|f'(y)| = 0$ .

(4) So  $f'(y) = 0$  for any  $y \in \mathbb{R}$ . By Theorem 5.11 (b),  $f$  is a constant.

□

**Exercise 5.2.** Suppose  $f'(x) > 0$  in  $(a, b)$ . Prove that  $f$  is strictly increasing in  $(a, b)$ , and let  $g$  be its inverse function. Prove that  $g$  is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)} \quad (a < x < b).$$

*Proof.* Let  $E = (a, b)$ .

- (1) Theorem 5.10 implies that for any  $a < p < q < b$  there exists  $\xi \in (p, q)$  such that

$$f(p) - f(q) = (p - q)f'(\xi).$$

Since  $\xi \in (p, q) \subseteq E$ , by assumption  $f'(\xi) > 0$ . Hence  $f(p) - f(q) = (p - q)f'(\xi) < 0$  (here  $p - q < 0$ ), or

$$f(p) < f(q)$$

if  $p < q$ . Therefore,  $f$  is strictly increasing in  $(a, b)$ .

- (2) Show that  $f$  is one-to-one in  $E$  if  $f$  is strictly increasing in  $E$ . If  $f(p) = f(q)$ , then it cannot be  $p > q$  or  $p < q$  ((1)). So that  $p = q$ , or  $f$  is injective.
- (3) Show that  $g$  is well-defined. Theorem 5.2 and Theorem 4.17.
- (4) Show that  $g'(f(x)) = \frac{1}{f'(x)}$ . Given  $y \in f(E)$ , say  $y = f(x)$  for some  $x \in E$ . Given any  $s \in f(E)$  with  $s \neq y$ . Here  $s = f(t)$  for some  $t \in E$  and  $t \neq x$ .

$$\begin{aligned} \lim_{s \rightarrow y} \frac{g(s) - g(y)}{s - y} &= \lim_{f(t) \rightarrow f(x)} \frac{g(f(t)) - g(f(x))}{f(t) - f(x)} \\ &= \lim_{t \rightarrow x} \frac{t - x}{f(t) - f(x)} \\ &= \lim_{t \rightarrow x} \frac{1}{\frac{f(t) - f(x)}{t - x}} \\ &= \frac{1}{f'(x)}. \end{aligned} \quad (f' > 0)$$

Here  $s \rightarrow y$  if and only if  $t \rightarrow x$  since both  $f$  and  $g$  are continuous and one-to-one. Hence  $g$  is differentiable and  $g'(f(x)) = \frac{1}{f'(x)}$ .

□

**Exercise 5.3.** Suppose  $g$  is a real function on  $\mathbb{R}^1$ , with bounded derivative (say  $|g'| \leq M$ ). Fix  $\varepsilon > 0$ , and define  $f(x) = x + \varepsilon g(x)$ . Prove that  $f$  is one-to-one if  $\varepsilon$  is small enough. (A set of admissible values of  $\varepsilon$  can be determined which depends only on  $M$ .)

*Proof.*

- (1) Note that  $f'(x) = 1 + \varepsilon g'(x)$  (Theorem 5.3). Since  $|g'| \leq M$ ,

$$1 - \varepsilon M \leq f'(x) \leq 1 + \varepsilon M.$$

- (2) Pick

$$\varepsilon = \frac{1}{M + 1} > 0.$$

Thus,

$$f'(x) \geq \frac{1}{M+1} > 0.$$

By Exercise 5.2,  $f(x)$  is strictly increasing in  $\mathbb{R}$  or one-to-one in  $\mathbb{R}$ .

□

**Exercise 5.4.** *If*

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$

where  $C_0, \dots, C_n$  are real constants, prove that the equation

$$C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n = 0$$

has at least one real root between 0 and 1.

*Proof.* Let

$$g(x) = C_0x + \frac{C_1}{2}x^2 + \cdots + \frac{C_{n-1}}{n}x^n + \frac{C_n}{n+1}x^{n+1} \in \mathbb{R}[x].$$

Then  $g(0) = g(1) = 0$ , and  $g'(x) = C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n$ . By the mean value theorem (Theorem 5.10), there exists a point  $\xi \in (0, 1)$  at which

$$g(1) - g(0) = g'(\xi)(1 - 0),$$

or  $g'(\xi) = 0$ . That is, there exists a real root  $x = \xi$  between 0 and 1 at which  $C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n = 0$ . □

**Exercise 5.5.** *Suppose  $f$  is defined and differentiable for every  $x > 0$ , and  $f'(x) \rightarrow 0$  as  $x \rightarrow +\infty$ . Put  $g(x) = f(x+1) - f(x)$ . Prove that  $g(x) \rightarrow 0$  as  $x \rightarrow +\infty$ .*

*Proof.* Given any  $x > 0$ . Since  $f$  is differentiable for every  $x > 0$ ,  $f$  is differentiable on  $[x, x+1]$ . By Theorem 5.2 and Theorem 5.10 (the mean value theorem), there is a point  $\xi \in (x, x+1)$  at which

$$f(x+1) - f(x) = [(x+1) - x]f'(\xi)$$

or

$$g(x) = f'(\xi).$$

As  $x \rightarrow +\infty$ ,  $\xi \rightarrow +\infty$ . Hence

$$\lim_{x \rightarrow +\infty} g(x) = \lim_{\xi \rightarrow +\infty} f'(\xi) = 0.$$

□

**Exercise 5.6.** Suppose

- (a)  $f$  is continuous for  $x \geq 0$ ,
- (b)  $f'(x)$  exists for  $x > 0$ ,
- (c)  $f(0) = 0$ ,
- (d)  $f'$  is monotonically increasing.

Put

$$g(x) = \frac{f(x)}{x} \quad (x > 0)$$

and prove that  $g$  is monotonically increasing.

*Proof.*

- (1) It suffices to show that  $g'(x) \geq 0$  for  $x > 0$  (Theorem 5.11(a)), that is, to show that

$$g'(x) = \frac{xf'(x) - f(x)}{x^2} \geq 0 \quad (x > 0),$$

or

$$xf'(x) - f(x) \geq 0 \quad (x > 0)$$

since  $x^2 > 0$  for all nonzero  $x$ .

- (2) Given  $x > 0$ . By (a)(b), we apply the mean value theorem (Theorem 5.10) on  $f$  to get

$$f(x) - f(0) = (x - 0)f'(\xi)$$

for some  $\xi \in (0, x)$ . By (c),

$$f(x) = xf'(\xi).$$

By (d),

$$f(x) = xf'(\xi) \leq xf'(x).$$

Hence  $xf'(x) - f(x) \geq 0$ , or  $g$  is monotonically increasing.

□

*Note.*  $g$  is increasing strictly if  $f$  is increasing strictly.

**Exercise 5.7.** Suppose  $f'(x)$ ,  $g'(x)$  exist,  $g'(x) \neq 0$ , and  $f(x) = g(x) = 0$ . Prove that

$$\lim_{t \rightarrow x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}.$$

(This holds also for complex functions.)

*Proof.*

$$\begin{aligned}
 \frac{f'(t)}{g'(t)} &= \frac{\lim_{t \rightarrow x} \frac{f(t)-f(x)}{t-x}}{\lim_{t \rightarrow x} \frac{g(t)-g(x)}{t-x}} \\
 &= \lim_{t \rightarrow x} \frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}} && \text{(Both limits exist and } g' \neq 0) \\
 &= \lim_{t \rightarrow x} \frac{f(t)}{g(t)}. && (f(x) = g(x) = 0)
 \end{aligned}$$

This proof is also true for complex functions.  $\square$

**Exercise 5.8.** Suppose  $f'(x)$  is continuous on  $[a, b]$  and  $\varepsilon > 0$ . Prove that there exists  $\delta > 0$  such that

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \varepsilon$$

whenever  $0 < |t - x| < \delta$ ,  $a \leq x \leq b$ ,  $a \leq t \leq b$ . (This could be expressed by saying  $f$  is **uniformly differentiable** on  $[a, b]$  if  $f'$  is continuous on  $[a, b]$ .) Does this hold for vector-valued functions too?

*Proof.*

- (1) Since  $f'(x)$  is continuous on a compact set  $[a, b]$ ,  $f'(x)$  is uniformly continuous on  $[a, b]$ . So given any  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$|f'(t) - f'(x)| < \varepsilon$$

whenever  $0 < |t - x| < \delta$ ,  $a \leq x \leq b$ ,  $a \leq t \leq b$ .

- (2) For such  $t < x$  in (1), by the mean value theorem (Theorem 5.10), there exists a point  $\xi \in (t, x)$  at which

$$f'(\xi) = \frac{f(t) - f(x)}{t - x}.$$

Note that  $\xi$  is also satisfying  $0 < |t - \xi| < |t - x| < \delta$  and  $a \leq \xi \leq b$ . Hence by (1) we also have

$$|f'(\xi) - f'(x)| < \varepsilon,$$

or

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \varepsilon.$$

- (3) Suppose  $\mathbf{f}'(x)$  is continuous on  $[a, b]$  and  $\varepsilon > 0$ . Prove that there exists  $\delta > 0$  such that

$$\left| \frac{\mathbf{f}(t) - \mathbf{f}(x)}{t - x} - \mathbf{f}'(x) \right| < \varepsilon$$

whenever  $0 < |t - x| < \delta$ ,  $a \leq x \leq b$ ,  $a \leq t \leq b$ .

- (a) Write

$$\mathbf{f}(x) = (f_1(x), \dots, f_k(x)) \in \mathbb{R}^k.$$

By Remarks 5.16,  $\mathbf{f}(x)$  is differentiable at a point  $x$  if and only if each  $f_1, \dots, f_k$  is differentiable at  $x$ . So that

$$\mathbf{f}'(x) = (f'_1(x), \dots, f'_k(x)) \in \mathbb{R}^k.$$

By Theorem 4.10,  $\mathbf{f}'(x)$  is continuous if and only if each  $f_1, \dots, f_k$  is continuous.

- (b) Similar to (1)(2), Since  $f'_i(x)$  is continuous on a compact set  $[a, b]$  where  $1 \leq i \leq k$ ,  $f'_i(x)$  is uniformly continuous on  $[a, b]$ . So given any  $\varepsilon > 0$  there exists  $\delta_i > 0$  such that

$$|f'_i(t) - f'_i(x)| < \frac{\varepsilon}{\sqrt{k}}$$

whenever  $0 < |t - x| < \delta_i$ ,  $a \leq x \leq b$ ,  $a \leq t \leq b$ . Take  $\delta = \min_{1 \leq i \leq k} \delta_i > 0$ .

- (c) For such  $t < x$  in (1), by the mean value theorem (Theorem 5.10), there exists a point  $\xi_i \in (t, x)$  at which

$$f'_i(\xi_i) = \frac{f_i(t) - f_i(x)}{t - x}.$$

Note that  $\xi_i$  is also satisfying  $0 < |t - \xi_i| < |t - x| < \delta$  and  $a \leq \xi_i \leq b$ . Hence by (1) we also have

$$|f'_i(\xi_i) - f'_i(x)| < \frac{\varepsilon}{\sqrt{k}},$$

or

$$\left| \frac{f_i(t) - f_i(x)}{t - x} - f'_i(x) \right| < \frac{\varepsilon}{\sqrt{k}}.$$

- (d) Hence

$$\left| \frac{\mathbf{f}(t) - \mathbf{f}(x)}{t - x} - \mathbf{f}'(x) \right| = \left( \sum_{i=1}^k \left| \frac{f_i(t) - f_i(x)}{t - x} - f'_i(x) \right|^2 \right)^{\frac{1}{2}} < \varepsilon.$$

□

**Exercise 5.9.** Let  $f$  be a continuous real function on  $\mathbb{R}^1$ , of which it is known that  $f'(x)$  exists for all  $x \neq 0$  and that  $f'(x) \rightarrow 3$  as  $x \rightarrow 0$ . Does it follow that  $f'(0)$  exists?

*Proof.*

- (1) Show that  $f'(0) = 3$ . It is equivalent to show that

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = 3.$$

Write  $F(x) = f(x) - f(0)$  and  $G(x) = x - 0$  on  $\mathbb{R}^1$ . So that

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{F(x)}{G(x)} = 0.$$

- (2) Note that

$$\lim_{x \rightarrow 0} \frac{F'(x)}{G'(x)} = \lim_{x \rightarrow 0} \frac{f'(x)}{1} = 3.$$

- (3) Since  $f$  is continuous on  $\mathbb{R}^1$ ,  $F$  is continuous on  $\mathbb{R}^1$ . Hence

$$\lim_{x \rightarrow 0} F(x) = F(\lim_{x \rightarrow 0} x) = F(0) = 0.$$

Also,  $G$  is continuous on  $\mathbb{R}^1$  implies that

$$\lim_{x \rightarrow 0} G(x) = G(\lim_{x \rightarrow 0} x) = G(0) = 0.$$

- (4) Apply L'Hospital's rule (Theorem 5.13) to (2)(3), we have

$$\lim_{x \rightarrow 0} \frac{F(x)}{G(x)} = 3,$$

or  $f'(0) = 3$ .

□

**Exercise 5.10.**

**Exercise 5.11.** Suppose  $f$  is defined in a neighborhood of  $x$ , and suppose  $f''(x)$  exists. Show that

$$\lim_{h \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

Show by an example that the limit may exist even if  $f''(x)$  does not. (Hint: Use Theorem 5.13.)

*Proof (Theorem 5.13).*

- (1) Write  $F(h) = f(x+h) + f(x-h) - 2f(x)$  and  $G(h) = h^2$ . It is equivalent to show that

$$\lim_{h \rightarrow 0} \frac{F(h)}{G(h)} = f''(x).$$

We might apply Theorem 5.13 (L'Hospital rule) to get it.

- (2) Show that  $\lim_{h \rightarrow 0} F(h) = 0$  and  $\lim_{h \rightarrow 0} G(h) = 0$ . It is clear that  $\lim_{h \rightarrow 0} G(h) = \lim_{h \rightarrow 0} h^2 = 0$  since  $x \mapsto x^2$  is continuous on  $\mathbb{R}^1$ . Besides, since  $f$  is continuous at  $x$  (by applying Theorem 5.2 twice),

$$\lim_{h \rightarrow 0} F(h) = f(x) + f(x) - 2f(x) = 0.$$

- (3) Show that

$$\lim_{h \rightarrow 0} \frac{F'(h)}{G'(h)} = \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x-h)}{2h}$$

is well-defined. Since  $f''(x)$  exists in a neighborhood  $B(x; r)$  of  $x$  (where  $r > 0$ ),  $f'(x)$  exists and is continuous in  $B(x; r)$  (Theorem 5.2). As  $0 < |h| < \frac{r}{2}$ ,

$$x+h \in B\left(x+h; \frac{r}{2}\right) \subseteq B(x; r)$$

and

$$x-h \in B\left(x-h; \frac{r}{2}\right) \subseteq B(x; r).$$

So  $f'(x+h)$  and  $f'(x-h)$  exist in  $B(x; r)$  as  $0 < |h| < \frac{r}{2}$ . Hence

$$\lim_{h \rightarrow 0} \frac{F'(h)}{G'(h)} = \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x-h)}{2h}$$

is well-defined (Theorem 5.3 and Theorem 5.5 (the chain rule)).

- (4) Show that

$$\lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x-h)}{2h} = f''(x).$$

Since  $f''(x)$  exists, by definition

$$\lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{h} = f''(x)$$

and

$$\lim_{h \rightarrow 0} \frac{f'(x-h) - f'(x)}{-h} = f''(x).$$

Sum up two expressions to get

$$2f''(x) = \lim_{h \rightarrow 0} \frac{f'(x-h) - f'(x-h)}{h}.$$



(5) By (2)(3)(4) and Theorem 5.13 (L'Hospital rule), the result is established.

(6) Given  $f(x) = x|x|$  on  $\mathbb{R}^1$ . Show that

$$\lim_{h \rightarrow 0} \frac{f(h) + f(-h) - 2f(0)}{h^2} = 0$$

but  $f''(x)$  does not exist at  $x = 0$ . Clearly,

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(h) + f(-h) - 2f(0)}{h^2} &= \lim_{h \rightarrow 0} \frac{h|h| + (-h)|-h| - 2 \cdot 0}{h^2} \\ &= \lim_{h \rightarrow 0} \frac{h|h| - h|h| - 0}{h^2} \\ &= \lim_{h \rightarrow 0} 0 \\ &= 0. \end{aligned}$$

But  $f''(x)$  does not exist by Exercise 5.12.

□

**Exercise 5.12.** If  $f(x) = |x|^3$ , compute  $f'(x)$ ,  $f''(x)$  for all real  $x$ , and show that  $f^{(3)}(0)$  does not exist.

*Proof.*

(1) Write

$$f(x) = \begin{cases} x^3 & (x \geq 0), \\ -x^3 & (x < 0). \end{cases}$$

(2) Show that  $f'(x) = 3x|x|$ . It is trivial that

$$f'(x) = \begin{cases} 3x^2 & (x > 0), \\ -3x^2 & (x < 0). \end{cases}$$

Note that

$$\lim_{x \rightarrow 0} f'(x) = 0.$$

Apply the same argument in Exercise 5.9, we have

$$f'(0) = 0.$$

Hence  $f'$  exists and  $f'(x) = 3x|x|$  for any  $x \in \mathbb{R}$ .

(3) Show that  $f''(x) = 6|x|$ . Similar to (2).

$$f''(x) = \begin{cases} 6x & (x > 0), \\ -6x & (x < 0). \end{cases}$$

Note that

$$\lim_{x \rightarrow 0} f''(x) = 0.$$

Apply the same argument in Exercise 5.9, we have

$$f''(0) = 0.$$

Hence  $f''$  exists and  $f''(x) = 6|x|$  for any  $x \in \mathbb{R}$ .

(4) Show that  $f^{(3)}(0)$  does not exist.

$$f'''(x) = \begin{cases} 6 & (x > 0), \\ -6 & (x < 0). \end{cases}$$

There are some proofs for showing that  $f^{(3)}(0)$  does not exist.

(a) Since

$$\lim_{t \rightarrow 0+} \frac{f''(t) - f''(0)}{t - 0} = \lim_{t \rightarrow 0+} \frac{6t}{t} = 6$$

and

$$\lim_{t \rightarrow 0-} \frac{f''(t) - f''(0)}{t - 0} = \lim_{t \rightarrow 0-} \frac{-6t}{t} = -6,$$

$f^{(3)}(0)$  does not exist.

(b) (Reductio ad absurdum) If  $f$  were differentiable on  $\mathbb{R}^1$ , then

$$\lim_{t \rightarrow 0+} f'''(t) = 6$$

and

$$\lim_{t \rightarrow 0-} f'''(t) = -6,$$

or  $f'''$  has a simple discontinuity at  $x = 0$ , contrary to Corollary to Theorem 5.12.

□

*Note.* Given  $k > 0$ . We can construct one real function  $f$  on  $\mathbb{R}^1$ , say

$$f(x) = \begin{cases} |x|^k & (k \text{ is odd}), \\ x|x|^{k-1} & (k > 0 \text{ is even}), \end{cases}$$

such that all  $f^{(0)}(0) = \dots = f^{(k-1)}(0) = 0$  exist but  $f^{(k)}(0)$  does not exist.

### Exercise 5.13.

**Exercise 5.14.** Let  $f$  be a differentiable real function defined in  $(a, b)$ . Prove that  $f$  is convex if and only if  $f'$  is monotonically increasing. Assume next  $f''(x)$  exists for every  $x \in (a, b)$ , and prove that  $f$  is convex if and only if  $f''(x) \geq 0$  for all  $x \in (a, b)$ .

*Proof.*

(1) Show that  $f'$  is monotonically increasing if  $f$  is convex.

(a) Since  $f$  is convex, by definition (Exercise 4.23)

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

whenever  $a < x < b$ ,  $a < y < b$ ,  $0 < \lambda < 1$ .

(b) As  $x \neq y$ , we have

$$\begin{aligned} f(y) - f(x) &\geq \frac{f(x + \lambda(y - x)) - f(x)}{\lambda} \\ &= \frac{f(x + \lambda(y - x)) - f(x)}{\lambda(y - x)} \cdot (y - x) \end{aligned}$$

and let  $\lambda \rightarrow 0$  to get

$$f(y) - f(x) \geq f'(x)(y - x)$$

(since  $f'(x)$  exists). Similarly, we have

$$f(x) - f(y) \geq f'(y)(x - y).$$

(c) Given any  $y > x$ , we have

$$f'(y)(y - x) \geq f(y) - f(x) \geq f'(x)(y - x).$$

Hence  $f'(y) \geq f'(x)$  whenever  $y > x$ , or  $f'$  is monotonically increasing.

(2) Show that  $f$  is convex if  $f'$  is monotonically increasing. Given any  $y > x$  and any  $0 < \lambda < 1$ .

(a) By Theorem 5.10 (the mean value theorem), there is a point  $x < \xi < y$  such that

$$f(y) - f(x) = f'(\xi)(y - x).$$

Since  $f'$  is monotonically increasing,

$$f'(y)(y - x) \geq f(y) - f(x) \geq f'(x)(y - x).$$

(b) Write  $z = \lambda x + (1 - \lambda)y$ . Hence

$$\begin{aligned} f(y) - f(z) &\geq f'(z)(y - z), \\ f(z) - f(x) &\leq f'(z)(z - x), \end{aligned}$$

or

$$\begin{aligned} f(y) &\geq f(z) + f'(z)(y - z), \\ f(x) &\geq f(z) + f'(z)(x - z), \end{aligned}$$

or

$$\begin{aligned}\lambda f(x) + (1 - \lambda)f(y) &\geq \lambda[f(z) + f'(z)(x - z)] \\ &\quad + (1 - \lambda)[f(z) + f'(z)(y - z)] \\ &= f(z) \\ &= f(\lambda x + (1 - \lambda)y).\end{aligned}$$

Hence  $f$  is convex.

- (3) Show that  $f''(x) \geq 0$  if  $f$  is convex and  $f''$  exists. By (1),  $f'$  is monotonically increasing since  $f$  is convex. Given any  $x \neq y$ , we have

$$\frac{f'(y) - f'(x)}{y - x} \geq 0.$$

Let  $y \rightarrow x$ , we have  $f''(x) \geq 0$  if  $f''$  exists.

- (4) Show that  $f$  is convex if  $f''$  exists and  $f''(x) \geq 0$ . By Theorem 5.11(a),  $f'$  is monotonically increasing. By (2),  $f$  is convex.

□

**Exercise 5.15 (Landau-Kolmogorov inequality on the half-line).** Suppose  $a \in \mathbb{R}^1$ ,  $f$  is a twice-differentiable real function on  $(a, \infty)$ , and  $M_0, M_1, M_2$  are the least upper bounds of  $|f(x)|, |f'(x)|, |f''(x)|$ , respectively, on  $(a, \infty)$ . Prove that

$$M_1^2 \leq 4M_0M_2.$$

(Hint: If  $h > 0$ , Taylor's theorem shows that

$$f'(x) = \frac{1}{2h}[f(x + 2h) - f(x)] - hf''(\xi)$$

for some  $\xi \in (x, x + 2h)$ . Hence

$$|f'(x)| \leq hM_2 + \frac{M_0}{h}.)$$

To show that  $M_1^2 = 4M_0M_2$  can actually happen, take  $a = -1$ , define

$$f(x) = \begin{cases} 2x^2 - 1 & (-1 < x < 0), \\ \frac{x^2 - 1}{x^2 + 1} & (0 \leq x < \infty), \end{cases}$$

and show that  $M_0 = 1, M_1 = 4, M_2 = 4$ . Does  $M_1^2 \leq 4M_0M_2$  hold for vector-valued functions too?

Note.

(1) Write

$$M_1 \leq 2M_0^{\frac{1}{2}}M_2^{\frac{1}{2}}.$$

2 is called the Landau-Kolmogorov constant, which is the best possible by the above example.

(2) In general, suppose  $a \in \mathbb{R}^1$ ,  $f$  is a  $n$ th differentiable real function on  $(a, \infty)$ , and  $M_0, M_k, M_n$  are the least upper bounds of  $|f(x)|, |f^{(k)}(x)|, |f^{(n)}(x)|$ , respectively, on  $(a, \infty)$  where  $1 \leq k < n$ . Then

$$M_k \leq C(n, k)M_0^{1-\frac{k}{n}}M_n^{\frac{k}{n}}.$$

*Proof.*

(1) Consider some trivial cases.

- (a) If  $M_0 = 0$ , then  $f(x) = 0$  on  $(a, +\infty)$ . So that  $f'(x) = f''(x) = 0$  on  $(a, +\infty)$ , or  $M_1 = M_2 = 0$ . The inequality holds.
- (b) If  $M_2 = 0$ , then  $f''(x) = 0$  on  $(a, +\infty)$ . So that  $f'(x) = \alpha$  for some constant  $\alpha \in \mathbb{R}^1$  (Theorem 5.11(b)), and  $f(x) = \alpha x + \beta$  for some constant  $\beta \in \mathbb{R}^1$  (by applying Theorem 5.11(b) to  $x \mapsto f(x) - \alpha x$ ). Hence  $M_1 = |\alpha|$  and

$$M_0 = \begin{cases} +\infty & (\alpha \neq 0), \\ |\beta| & (\alpha = 0). \end{cases}$$

In any case, the inequality holds.

- (c) If  $M_0 = +\infty$  and  $M_2 \neq 0$ , there is nothing to do.
  - (d) If  $M_2 = +\infty$  and  $M_0 \neq 0$ , there is nothing to do.
- (2) By (1), we suppose that  $0 < M_0 < +\infty$  and  $0 < M_2 < +\infty$ . Given  $x \in (a, +\infty)$  and  $h > 0$ . By Taylor's theorem (Theorem 5.15):

$$f(x+2h) = f(x) + 2hf'(x) + 2h^2f''(\xi)$$

for some  $\xi \in (x, x+2h) \subseteq (a, +\infty)$ . Thus

$$\begin{aligned} 2h|f'(x)| &\leq |f(x+2h)| + |f(x)| + 2h^2|f''(\xi)| \\ &\leq 2M_0 + 2h^2M_2, \\ |f'(x)| &\leq \frac{M_0}{h} + hM_2 \end{aligned}$$

holds for all  $h > 0$ . In particular, take

$$h = \sqrt{\frac{M_0}{M_2}}$$

to get

$$|f'(x)| \leq 2\sqrt{M_0 M_2}.$$

Thus  $2\sqrt{M_0 M_2}$  is an upper bound of  $|f'(x)|$  for all  $x \in (a, +\infty)$ . Hence

$$M_1 \leq 2\sqrt{M_0 M_2}$$

or

$$M_1^2 \leq 4M_0 M_2.$$

(3) Define

$$f(x) = \begin{cases} 2x^2 - 1 & (-1 < x < 0), \\ \frac{x^2 - 1}{x^2 + 1} & (0 \leq x < \infty). \end{cases}$$

Show that  $M_0 = 1$ ,  $M_1 = 4$ ,  $M_2 = 4$ . Similar to Exercise 5.12,

$$f'(x) = \begin{cases} 4x & (-1 < x \leq 0), \\ \frac{4x}{(x^2 + 1)^2} & (0 \leq x < \infty). \end{cases}$$

(Here  $\lim_{x \rightarrow 0+} f'(x) = 0$  and  $\lim_{x \rightarrow 0-} f'(x) = 0$ . So  $f'(0) = 0$  by Exercise 5.9.) Also,

$$f''(x) = \begin{cases} 4 & (-1 < x \leq 0), \\ \frac{-12x^2 + 4}{(x^2 + 1)^3} & (0 \leq x < \infty). \end{cases}$$

(Here  $\lim_{x \rightarrow 0+} f''(x) = 4$  and  $\lim_{x \rightarrow 0-} f''(x) = 4$ . So  $f''(0) = 4$  by Exercise 5.9.) Hence,  $M_0 = 1$ ,  $M_1 = 4$ ,  $M_2 = 4$ .

(4) Given

$$\mathbf{f}(x) = (f_1(x), \dots, f_k(x))$$

be a twice-differentiable vector-valued function from  $(a, \infty)$  to  $\mathbb{R}^k$ . and  $M_0$ ,  $M_1$ ,  $M_2$  are the least upper bounds of  $|\mathbf{f}(x)|$ ,  $|\mathbf{f}'(x)|$ ,  $|\mathbf{f}''(x)|$ , respectively, on  $(a, \infty)$ . Show that

$$M_1^2 \leq 4M_0 M_2.$$

Similar to (1), we suppose that  $0 < M_0 < +\infty$  and  $0 < M_2 < +\infty$ . Given any  $\mathbf{v} = (v_1, \dots, v_k) \in \mathbb{R}^k$ ,  $\mathbf{v} \cdot \mathbf{f}$  is a twice-differentiable real function on  $(a, \infty)$ . Similar to (2), Given  $x \in (a, +\infty)$  and  $h > 0$ . By Taylor's theorem (Theorem 5.15):

$$(\mathbf{v} \cdot \mathbf{f})(x + 2h) = (\mathbf{v} \cdot \mathbf{f})(x) + 2h(\mathbf{v} \cdot \mathbf{f})'(x) + 2h^2(\mathbf{v} \cdot \mathbf{f})''(\xi)$$

for some  $\xi \in (x, x + 2h) \subseteq (a, +\infty)$ . Thus by the Schwarz inequality (Theorem 1.35)

$$\begin{aligned} 2h|(\mathbf{v} \cdot \mathbf{f})'(x)| &\leq |(\mathbf{v} \cdot \mathbf{f})(x + 2h)| + |(\mathbf{v} \cdot \mathbf{f})(x)| + 2h^2|(\mathbf{v} \cdot \mathbf{f})''(\xi)| \\ &\leq |\mathbf{v}||\mathbf{f}(x + 2h)| + |\mathbf{v}||\mathbf{f}(x)| + 2h^2|\mathbf{v}||\mathbf{f}''(\xi)| \\ &\leq (2M_0 + 2h^2 M_2)|\mathbf{v}|, \\ |(\mathbf{v} \cdot \mathbf{f})'(x)| &\leq \left( \frac{M_0}{h} + hM_2 \right) |\mathbf{v}| \end{aligned}$$

holds for any  $\mathbf{v}$  and  $h > 0$ . In particular, we take

$$\mathbf{v} = \mathbf{f}'(y)$$

and

$$h = \sqrt{\frac{M_0}{M_2}}$$

to get

$$|\mathbf{f}'(x) \cdot \mathbf{f}'(y)| \leq 2\sqrt{M_0 M_2} |\mathbf{f}'(y)| \leq 2M_1 \sqrt{M_0 M_2}.$$

Note that  $x$  and  $y$  are arbitrary (in  $(a, +\infty)$ ). In particular, we take  $x = y$  to get

$$|\mathbf{f}'(x)|^2 \leq 2M_1 \sqrt{M_0 M_2}.$$

Thus  $2M_1 \sqrt{M_0 M_2}$  is an upper bound of  $|\mathbf{f}'(x)|^2$  for all  $x \in (a, +\infty)$ . Hence

$$M_1^2 \leq 2M_1 \sqrt{M_0 M_2}$$

or

$$M_1^2 \leq 4M_0 M_2.$$

□

**Supplement (Landau-Kolmogorov inequality on the real line).** Suppose  $f$  is a twice-differentiable real function on  $(-\infty, +\infty)$ , and  $M_0, M_1, M_2$  are the least upper bounds of  $|f(x)|, |f'(x)|, |f''(x)|$ , respectively, on  $(-\infty, +\infty)$ . Prove that

$$M_1^2 \leq 2M_0 M_2.$$

*Proof.*

- (1) Similar to (1) in Landau-Kolmogorov inequality on the half-line, we suppose that  $0 < M_0 < +\infty$  and  $0 < M_2 < +\infty$ .
- (2) Similar to (2) in Landau-Kolmogorov inequality on the half-line. Given  $x \in \mathbb{R}^1$  and  $h > 0$ . By Taylor's theorem (Theorem 5.15):

$$f(x+2h) = f(x) + 2hf'(x) + 2h^2 f''(\xi_1) \quad (\text{I})$$

$$f(x-2h) = f(x) - 2hf'(x) + 2h^2 f''(\xi_2) \quad (\text{II})$$

for some  $\xi_1 \in (x, x+2h)$  and  $\xi_2 \in (x, x-2h)$ . So (I) subtracts (II):

$$f(x+2h) - f(x-2h) = 4hf'(x) + 2h^2 f''(\xi_1) - 2h^2 f''(\xi_2).$$

Thus

$$\begin{aligned} 4h|f'(x)| &\leq |f(x+2h)| + |f(x-2h)| + 2h^2|f''(\xi_1)| + 2h^2|f''(\xi_2)| \\ &\leq 2M_0 + 4h^2 M_2, \\ |f'(x)| &\leq \frac{M_0}{2h} + hM_2 \end{aligned}$$

holds for all  $h > 0$ . In particular, take

$$h = \sqrt{\frac{M_0}{2M_2}}$$

to get

$$|f'(x)| \leq \sqrt{2M_0M_2}.$$

Thus  $\sqrt{2M_0M_2}$  is an upper bound of  $|f'(x)|$  for all  $x \in \mathbb{R}^1$ . Hence

$$M_1 \leq \sqrt{2M_0M_2}$$

or

$$M_1^2 \leq 2M_0M_2.$$

□

*Note.*

- (1) Write

$$M_1 \leq \sqrt{2}M_0^{\frac{1}{2}}M_2^{\frac{1}{2}}.$$

$\sqrt{2}$  is called the Landau-Kolmogorov constant, which is the best possible.

- (2) In general, suppose  $f$  is a  $n$ th differentiable real function on  $\mathbb{R}^1$ , and  $M_0, M_k, M_n$  are the least upper bounds of  $|f(x)|, |f^{(k)}(x)|, |f^{(n)}(x)|$ , respectively, on  $\mathbb{R}^1$  where  $1 \leq k < n$ . Then

$$M_k \leq C(n, k)M_0^{1-\frac{k}{n}}M_n^{\frac{k}{n}}.$$

**Exercise 5.16.** Suppose  $f$  is twice-differentiable on  $(0, \infty)$ ,  $f''$  is bounded on  $(0, \infty)$ , and  $f(x) \rightarrow 0$  as  $x \rightarrow \infty$ . Prove that  $f'(x) \rightarrow 0$  as  $x \rightarrow \infty$ . (Hint: Let  $a \rightarrow \infty$  in Exercise 5.15.)

*Proof.*

- (1) Write  $|f''| \leq M$  for some real  $M$  since  $f''$  is bounded on  $(0, \infty)$ .  
 (2) Given any  $a > 0$ . As in Exercise 5.15, define  $M_0, M_1, M_2$  are the least upper bounds of  $|f(x)|, |f'(x)|, |f''(x)|$  on  $(a, \infty)$ . Note that  $M_2 \leq M$  for any  $a > 0$  (by (1)). So that

$$M_1^2 \leq 4M_0M_2 \leq 4MM_0$$

for any  $a > 0$ .



- (3) By assumption,  $M_0 \rightarrow 0$  as  $a \rightarrow \infty$ . (So given any  $\varepsilon > 0$ , there exists a real  $A$  such that

$$0 \leq M_0 < \frac{\varepsilon}{4M+1}$$

whenever  $a \geq A$ . Hence

$$M_1^2 \leq 4MM_0 \leq 4M \cdot \frac{\varepsilon}{4M+1} < \varepsilon.$$

whenever  $a \geq A$ .) Therefore  $M_1^2 \rightarrow 0$  as  $a \rightarrow \infty$ , or  $f'(x) \rightarrow 0$  as  $x \rightarrow \infty$ .

□

**Exercise 5.17.**

**Exercise 5.18.**

**Exercise 5.19.**

**Exercise 5.20.**

**Exercise 5.21.** Let  $E$  be a closed subset of  $\mathbb{R}^1$ . We saw in Exercise 4.22, that there is a real continuous function  $f$  on  $\mathbb{R}^1$  whose zero set is  $E$ . Is it possible, for each closed set  $E$ , to find such an  $f$  which is differentiable on  $\mathbb{R}^1$ , or one which is  $n$  times differentiable, or even one which has derivatives of all orders on  $\mathbb{R}^1$ ?

**Exercise 5.22.**

**Exercise 5.23.**

**Exercise 5.24.**

**Exercise 5.25.** Suppose  $f$  is twice differentiable on  $[a, b]$ ,  $f(a) < 0$ ,  $f(b) > 0$ ,  $f'(x) \geq \delta > 0$ , and  $0 \leq f''(x) \leq M$  for all  $x \in [a, b]$ . Let  $\xi$  be the unique point in  $(a, b)$  at which  $f(\xi) = 0$ . Complete the details in the following outline of **Newton's method** for computing  $\xi$ .

- (a) Choose  $x_1 \in (\xi, b)$ , and define  $\{x_n\}$  by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Interpret this geometrically, in terms of a tangent to the graph of  $f$ .

- (b) Prove that  $x_{n+1} < x_n$  and that

$$\lim_{n \rightarrow \infty} x_n = \xi.$$

- (c) Use Taylor's theorem to show that

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

for some  $t_n \in (\xi, x_n)$ .

(d) (Quadratic convergence) If  $A = \frac{M}{2\delta}$ , deduce that

$$0 \leq x_{n+1} - \xi \leq \frac{1}{A}[A(x_1 - \xi)]^{2^n}.$$

(Compare with Exercise 3.16 and 3.18.)

(e) Show that Newton's method amounts to finding a fixed point of the function  $g$  defined by

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

How does  $g'(x)$  behave for  $x$  near  $\xi$ ?

(f) Put  $f(x) = x^{\frac{1}{3}}$  on  $(-\infty, +\infty)$  and try Newton's method. What happens?

*Proof of (a) (Wikipedia).* The equation of the tangent line to the curve  $y = f(x)$  at  $x = x_n$  is

$$y = f'(x_n)(x - x_n) + f(x_n).$$

The  $x$ -intercept of this line (the value of  $x$  which makes  $y = 0$ ) is taken as the next approximation,  $x_{n+1}$ , to the root, so that the equation of the tangent line is satisfied when  $(x, y) = (x_{n+1}, 0)$ :

$$0 = f'(x_n)(x - x_n) + f(x_n).$$

Solving for  $x_{n+1}$  gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

□

*Proof of (b).*

(1) Show that  $x_n \geq \xi$  for all  $n$ . Induction on  $n$ .

(a)  $n = 1$  is clearly true:  $x_1 > \xi$  by assumption.

(b) Assume the induction hypothesis that for the single case  $n = k$  holds. By the mean value theorem (Theorem 5.10), there is a point  $\xi_k \in (\xi, x_k)$

$$f(x_k) - f(\xi) = f'(\xi_k)(x_k - \xi),$$

or

$$f(x_k) = f'(\xi_k)(x_k - \xi)$$

(since  $f(\xi) = 0$ ). Since  $f'' \geq 0$ ,  $f'$  is monotonically increasing (Theorem 5.11(a)). Hence  $f'(\xi_k) \leq f'(x_k)$  and thus

$$f(x_k) = f'(\xi_k)(x_k - \xi) \leq f'(x_k)(x_k - \xi).$$

Since  $f'(x_k) > 0$  by assumption,

$$\xi \leq x_k - \frac{f(x_k)}{f'(x_k)} = x_{k+1}.$$

(c) Since both the base case in (a) and the inductive step in (b) have been proved as true, by mathematical induction  $x_n \geq \xi$  for all  $n$ .

(2) Show that  $x_{n+1} < x_n$  for all  $n$ .

(a) Since  $f' > 0$ ,  $f'(x_n) > 0$  for all  $n$ .

(b) Since  $f' > 0$ ,  $f$  is strictly increasing (Theorem 5.10). Hence  $f(x_n) > f(\xi) = 0$  for all  $n$  (by (1)).

(c) By (a)(b),  $\frac{f(x_n)}{f'(x_n)} > 0$  or

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} < x_n.$$

(3) By Theorem 3.14,  $\{x_n\}$  converges to some real number  $\zeta \geq \xi$ . Note that  $f$  and  $f'$  are continuous by the existence of  $f''$  (Theorem 5.2), we have

$$\lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} x_n - \frac{f(\lim_{n \rightarrow \infty} x_n)}{f'(\lim_{n \rightarrow \infty} x_n)}$$

provided  $f' \neq 0$  (Theorem 4.9 and Theorem 4.4). Hence

$$\zeta = \zeta - \frac{f(\zeta)}{f'(\zeta)}$$

or  $f(\zeta) = 0$ . By the uniqueness of  $\xi$ ,  $\zeta = \xi$  or  $\lim x_n = \xi$  as desired.

□

*Proof of (c).* By Taylor's theorem (Theorem 5.15),

$$f(\xi) = f(x_n) + f'(x_n)(\xi - x_n) + \frac{f''(t_n)}{2}(\xi - x_n)^2$$

for some  $t_n \in (\xi, x_n)$ . Note that  $f(\xi) = 0$ ,  $f'(x_n) \neq 0$  and  $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ , we have the desired result. □

*Proof of (d).* Clearly,  $0 \leq x_{n+1} - \xi$  for all  $n$  (by (b)). Besides, by (c)

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

Note that  $f'' \leq M$  and  $f' \geq \delta > 0$  by assumption, and thus

$$x_{n+1} - \xi \leq \frac{M}{2\delta}(x_n - \xi)^2 = A(x_n - \xi)^2.$$

By induction,

$$x_{n+1} - \xi \leq \frac{1}{A}(A(x_1 - \xi))^{2^n}.$$

□

*Note.* Compare with Exercise 3.16 and Exercise 3.18. Might assume that  $p > 1$ .

- (1) Fix a positive number  $\alpha$ . Let  $f(x) = x^p - \alpha$  on  $E = (a, b)$  where  $a = \frac{1}{2}\alpha^{\frac{1}{p}}$  and

$$b = \begin{cases} 2\alpha^{\frac{1}{p}} & (p = 2), \\ \left(\frac{2(p-1)}{p}\right)^{\frac{1}{p-2}} \alpha^{\frac{1}{p}} & (p > 2). \end{cases}$$

$E = (a, b)$  is well-defined since  $a < b$ . Besides,  $\xi = \alpha^{\frac{1}{p}} \in E = (a, b)$ .

- (2) By construction,

$$f(a) < 0 \text{ and } f(b) > 0.$$

By  $f'(x) = px^{p-1}$  and  $f''(x) = p(p-1)x^{p-2}$ ,

$$\begin{aligned} f'(x) &\geq pa^{p-1} > 0, \\ 0 &\leq f''(x) \leq p(p-1)b^{p-2}. \end{aligned}$$

on  $E$ . Write

$$\begin{aligned} \delta &= pa^{p-1} = \frac{p}{2^{p-1}} \alpha^{\frac{p-1}{p}}, \\ M &= p(p-1)b^{p-2} = 2(p-1)^2 \alpha^{\frac{p-2}{p}}. \end{aligned}$$

- (3) Hence the Newton's method works for  $f(x) = x^p - \alpha$ . That is, as we define  $\{x_n\}$  by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{p-1}{p}x_n + \frac{\alpha}{p}x_n^{-p+1},$$

we have  $\lim x_n = \xi = \alpha^{\frac{1}{p}}$ . And

$$0 \leq x_{n+1} - \xi \leq \frac{1}{A}(A(x_1 - \xi))^{2^n}.$$

Here

$$A = \frac{M}{2\delta} = \frac{2^{p-1}(p-1)^2}{p\alpha^{\frac{1}{p}}}.$$

- (4) Note that

$$\beta = \frac{p\alpha^{\frac{1}{p}}}{(p-1)^2} \neq \frac{p\alpha^{\frac{1}{p}}}{2^{p-1}(p-1)^2} = \frac{1}{A}.$$

where  $\beta$  is defined in the proof of Exercise 3.18. Note that  $f'(x_n) \geq f'(\xi)$  (since  $f'$  is monotonically increasing and all  $x_n \geq \xi$ ), and thus  $A$  can be chosen by a better estimation:

$$A = \frac{M}{2f'(\xi)} = \frac{(p-1)^2}{p\alpha^{\frac{1}{p}}} = \frac{1}{\beta}.$$

Now it is exactly the same as Exercise 3.16 and Exercise 3.18.

*Proof of (e).*

(1) Define  $g(x) = x - \frac{f(x)}{f'(x)}$  on  $[a, b]$ .  $g(\xi) = \xi$  if and only if  $f(\xi) = 0$ .

(2) By the construction of  $g$ ,  $g$  is differentiable and

$$g'(x) = 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2}.$$

(3) Hence

$$|g'(x)| \leq \left| \frac{f(x)f''(x)}{f'(x)^2} \right| = \frac{|f(x)||f''(x)|}{|f'(x)|^2} \leq \frac{M}{\delta^2}|f(x)|.$$

As  $x \rightarrow \xi$ ,  $|f(x)| \rightarrow 0$ . Therefore,  $|g'(x)| \rightarrow 0$  or  $g'(x) \rightarrow 0$  as  $x \rightarrow \xi$ .

□

*Proof of (f).*

(1) It is clearly that  $f(x) = 0$  if and only if  $x = 0$ . Write  $\xi = 0$ .

(2) Note that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = -2x_n,$$

or

$$x_n = (-2)^{n-1}x_1$$

for any  $x_1 \in (\xi, \infty)$  where  $n = 1, 2, 3, \dots$ . Hence, the sequence  $\{x_n\}$  does not converge for any choice of  $x_1 \in (\xi, \infty)$ . In this case we cannot find  $\xi$  satisfying  $f(\xi) = 0$  by Newton's method.

(3) In fact,

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} \rightarrow 0 \text{ as } x \rightarrow \pm\infty.$$

Hence such  $\delta > 0$  satisfying  $f'(x) \geq \delta > 0$  does not exist.

□

**Exercise 5.26.** Suppose  $f$  is differentiable on  $[a, b]$ ,  $f(a) = 0$ , and there is a real number  $A$  such that  $|f'(x)| \leq A|f(x)|$  on  $[a, b]$ . Prove that  $f(x) = 0$  for all  $x \in [a, b]$ . (Hint: Fix  $x_0 \in [a, b]$ , let

$$M_0 = \sup |f(x)|, \quad M_1 = \sup |f'(x)|$$

for  $a \leq x \leq x_0$ . For any such  $x$ ,

$$|f(x)| \leq M_1(x_0 - a) \leq A(x_0 - a)M_0.$$

Hence  $M_0 = 0$  if  $A(x_0 - a) < 1$ . That is,  $f = 0$  on  $[a, x_0]$ . Proceed.)

*Proof (Hint).*

- (1) If  $A = 0$ , then  $f'(x) = 0$  or  $f(x)$  is constant on  $[a, b]$  (Theorem 5.11(b)).  
Since  $f(a) = 0$ ,  $f(x) = 0$  on  $[a, b]$ .

- (2) Suppose that  $A > 0$ . Fix  $x_0 \in [a, b]$ , let

$$M_0 = \sup |f(x)|, \quad M_1 = \sup |f'(x)|$$

for  $a \leq x \leq x_0$ . Since  $|f'(x)| \leq A|f(x)|$  on  $[a, b]$ ,

$$|f'(x)| \leq A|f(x)| \leq AM_0.$$

Since  $AM_0$  is an upper bound for  $|f'(x)|$ ,

$$M_1 \leq AM_0.$$

- (3) Given any  $x \in [a, x_0]$ . Since  $f$  is differentiable on  $[a, x_0] \subseteq [a, b]$ , by the mean value theorem (Theorem 5.10), there is  $\xi \in (a, x)$  such that

$$f(x) - f(a) = f'(\xi)(x - a).$$

Note that  $f(a) = 0$  by assumption. So that

$$\begin{aligned} |f(x)| &= |f'(\xi)|(x - a) \\ &\leq M_1(x - a) && \text{(Definition of } M_1) \\ &\leq AM_0(x - a) && ((2)) \\ &\leq AM_0(x_0 - a). && (x \in [a, x_0]) \end{aligned}$$

Since  $AM_0(x_0 - a)$  is an upper bound for  $|f(x)|$ ,

$$M_0 \leq AM_0(x_0 - a).$$

Take

$$x_0 = \min \left\{ \frac{1}{2A} + a, b \right\}$$

so that  $M_0 \leq AM_0(x_0 - a) \leq \frac{M_0}{2}$ .  $M_0 = 0$  or  $f(x) = 0$  on  $[a, x_0]$ .

- (4) Take a partition

$$P = \{a = x_{-1}, x_0, \dots, x_n = b\}$$

of  $[a, b]$  such that each subinterval  $[x_{i-1}, x_i]$  satisfying  $\Delta x_i = x_i - x_{i-1} < \frac{1}{2A}$ . By (3),  $f(x) = 0$  on  $[x_{-1}, x_0]$ . Apply the same argument in (3),  $f(x) = 0$  on  $[x_0, x_1]$ . Continue this process,  $f(x) = 0$  on each subinterval and thus on the whole interval  $[a, b]$ .

□

*Note.* It holds for vector-valued functions too:

Suppose  $\mathbf{f}$  is a vector-valued differentiable function on  $[a, b]$ ,  $\mathbf{f}(a) = 0$ , and there is a real number  $A$  such that  $|\mathbf{f}'(x)| \leq A|\mathbf{f}(x)|$  on  $[a, b]$ . Prove that  $\mathbf{f}(x) = 0$  for all  $x \in [a, b]$ .

The proof is similar except using Theorem 5.19 ( $|\mathbf{f}(b) - \mathbf{f}(a)| \leq (b - a)|\mathbf{f}'(x)|$ ) in addition.

**Exercise 5.27.** Let  $\phi$  be a real function defined on a rectangle  $R$  in the plane, given by  $a \leq x \leq b$ ,  $\alpha \leq y \leq \beta$ . A **solution** of the initial-value problem

$$y' = \phi(x, y), \quad y(a) = c \quad (\alpha \leq c \leq \beta)$$

is, by definition, a differentiable function  $f$  on  $[a, b]$  such that  $f(a) = c$ ,  $\alpha \leq f(x) \leq \beta$ , and

$$f'(x) = \phi(x, f(x)) \quad (a \leq x \leq b)$$

Prove that such a problem has at most one solution if there is a constant  $A$  such that

$$|\phi(x, y_2) - \phi(x, y_1)| \leq A|y_2 - y_1|$$

whenever  $(x, y_1) \in R$  and  $(x, y_2) \in R$ . (Hint: Apply Exercise 26 to the difference of two solutions.) Note that this uniqueness theorem does not hold for the initial-value problem

$$y' = y^{\frac{1}{2}}, \quad y(0) = 0,$$

which has two solutions:  $f(x) = 0$  and  $f(x) = \frac{x^2}{4}$ . Find all other solutions.

*Proof (Hint).*

- (1) Suppose  $f_1$  and  $f_2$  are two solutions of that problem. Define  $f = f_1 - f_2$ .  $f$  is differentiable on  $[a, b]$ ,  $f(a) = f_1(a) - f_2(a) = c - c = 0$ . And

$$\begin{aligned} |f'(x)| &= |f_1'(x) - f_2'(x)| \\ &= |\phi(x, f_1(x)) - \phi(x, f_2(x))| \\ &\leq A|f_1(x) - f_2(x)| \end{aligned}$$

on  $[a, b]$ . By Exercise 5.26,  $f(x) = 0$  on  $[a, b]$ , or  $f_1(x) = f_2(x)$  on  $[a, b]$ .

- (2) *The initial-value problem*

$$y' = y^{\frac{1}{2}}, \quad y(0) = 0,$$

which has two solutions:  $f(x) = 0$  and  $f(x) = \frac{x^2}{4}$ . Find all other solutions.

*Note.* It does not exist a real  $A$  such that  $|\phi(x, y_2) - \phi(x, y_1)| \leq A|y_2 - y_1|$  in this initial-value problem.

- (a) Clearly,  $f(x) = 0$  and  $f(x) = \frac{x^2}{4}$  are two solutions for the initial-value problem.
- (b) Suppose  $f(x) \neq 0$  on  $[0, \infty)$ . Since  $f'(x) = f(x)^{\frac{1}{2}}$ ,  $f(x) \geq 0$ . Since  $f(x)$  is continuous (Theorem 5.2), the set

$$E = \{x \in [0, \infty) : f(x) > 0\}$$

is open in  $\mathbb{R}^1$  (Theorem 4.8). By Exercise 2.29 we write  $E$  as the union of an at most countable collection of disjoint segments, say

$$E = \bigcup_{(a_i, b_i) \subseteq [0, \infty)} (a_i, b_i)$$

where all  $(a_i, b_i)$  segments are disjoint. Note that  $E$  is nonempty.

- (c) For any segment  $(a_i, b_i)$ , define  $g(x) = f(x)^{\frac{1}{2}}$  on  $(a_i, b_i)$ . (Clearly,  $g(a_i) = f(a_i) = 0$  by the definition of  $E$ .) Thus

$$g'(x) = \frac{1}{2}f(x)^{-\frac{1}{2}}f'(x) = \frac{1}{2}.$$

Hence

$$g(x) = \frac{1}{2}x + c$$

for some constant  $c \in \mathbb{R}^1$ . So

$$f(x) = g(x)^2 = \left(\frac{1}{2}x + c\right)^2.$$

$f(a_i) = 0$  implies that  $c = -\frac{a_i}{2}$ . Hence

$$f(x) = \frac{1}{4}(x - a_i)^2$$

on  $(a_i, b_i)$ .

- (d) By (c), if  $b_i < \infty$  is defined as a real number, then  $f(b_i) = 0$  by definition of  $E$ . Note that

$$\lim_{x \rightarrow b_i^-} f(x) = \frac{1}{4}(b_i - a_i)^2 > 0,$$

which is absurd. Hence  $b_i = \infty$  and thus  $E$  is of the form

$$E = (a, \infty) \quad (a \geq 0).$$

Therefore,

$$f(x) = \begin{cases} 0 & (0 \leq x \leq a), \\ \frac{1}{4}(x - a)^2 & (x > a \geq 0). \end{cases}$$



□

**Exercise 5.28.** Formulate and prove an analogous uniqueness theorem for systems of differential equations of the form

$$y'_j = \phi_j(x, y_1, \dots, y_k), \quad y_j(a) = c_j \quad (j = 1, \dots, k)$$

Note that this can be rewritten in the form

$$\mathbf{y}' = \phi(x, \mathbf{y}), \quad \mathbf{y}(a) = \mathbf{c}$$

where  $\mathbf{y} = (y_1, \dots, y_k)$  ranges over a  $k$ -cell,  $\phi$  is the mapping of a  $(k+1)$ -cell into the Euclidean  $k$ -space whose components are the function  $\phi_1, \dots, \phi_k$ , and  $\mathbf{c}$  is the vector  $(c_1, \dots, c_k)$ . Use Exercise 5.26, for vector-valued functions.

*Proof.*

- (1) A **solution** of the initial-value problem

$$\mathbf{y}' = \phi(x, \mathbf{y}), \quad \mathbf{y}(a) = \mathbf{c}$$

is, by definition, a differentiable function  $\mathbf{f}$  on  $[a, b]$  such that  $\mathbf{f}(a) = \mathbf{c}$ , and

$$\mathbf{f}'(x) = \phi(x, \mathbf{f}(x)) \quad (a \leq x \leq b).$$

Then this problem has at most one solution if there is a constant  $A$  such that

$$|\phi(x, \mathbf{y}_2) - \phi(x, \mathbf{y}_1)| \leq A|\mathbf{y}_2 - \mathbf{y}_1|$$

whenever  $(x, \mathbf{y}_1) \in R$  and  $(x, \mathbf{y}_2) \in R$  where  $R$  is a  $(k+1)$ -cell defined by

$$R = [a, b] \times [\alpha_1, \beta_1] \times \dots \times [\alpha_k, \beta_k].$$

- (2) Similar to Exercise 5.27, Suppose  $\mathbf{f}_1$  and  $\mathbf{f}_2$  are two solutions of that problem. Define  $\mathbf{f} = \mathbf{f}_1 - \mathbf{f}_2$ .  $\mathbf{f}$  is differentiable on  $[a, b]$ ,  $\mathbf{f}(a) = \mathbf{f}_1(a) - \mathbf{f}_2(a) = \mathbf{c} - \mathbf{c} = 0$ . And

$$\begin{aligned} |\mathbf{f}'(x)| &= |\mathbf{f}'_1(x) - \mathbf{f}'_2(x)| \\ &= |\phi(x, \mathbf{f}_1(x)) - \phi(x, \mathbf{f}_2(x))| \\ &\leq A|\mathbf{f}_1(x) - \mathbf{f}_2(x)| \end{aligned}$$

on  $[a, b]$ . By Note in Exercise 5.26,  $\mathbf{f}(x) = 0$  on  $[a, b]$ , or  $\mathbf{f}_1(x) = \mathbf{f}_2(x)$  on  $[a, b]$ .

□

**Exercise 5.29.** Specialize Exercise 5.28 by considering the system

$$\begin{aligned} y'_j &= y_{j+1} \quad (j = 1, \dots, k-1), \\ y'_k &= f(x) - \sum_{j=1}^k g_j(x)y_j \end{aligned}$$

where  $f, g_1, \dots, g_k$  are continuous real functions on  $[a, b]$ , and derive a uniqueness theorem for solutions of the equation

$$y^{(k)} + g_k(x)y^{(k-1)} + \dots + g_2(x)y' + g_1(x)y = f(x),$$

subject to initial conditions

$$y(a) = c_1, \quad y'(a) = c_1, \quad \dots, \quad y^{(k-1)}(a) = c_k.$$

*Proof.*

(1) Write

$$\begin{aligned} \mathbf{y} &= (y_1, \dots, y_k) \\ &= (y, y', y'', \dots, y^{(k-1)}), \\ \phi(x, \mathbf{y}) &= \left( y_2, y_3, \dots, y_{k-1}, f(x) - \sum_{j=1}^k g_j(x)y_j \right) \\ &= \left( y', y'', \dots, y^{(k-1)}, f(x) - \sum_{j=1}^k g_j(x)y^{(j-1)} \right), \\ \mathbf{c} &= (c_1, \dots, c_k). \end{aligned}$$

So that

$$\mathbf{y}' = \phi(x, \mathbf{y}), \quad \mathbf{y}(a) = \mathbf{c}$$

where  $\mathbf{y}$  ranges over a  $k$ -cell  $R$ .

(2) To show that the problem has at most one solution, by Exercise 5.28 it suffices to show that there is a constant  $A$  such that

$$|\phi(x, \mathbf{y}) - \phi(x, \mathbf{z})| \leq A|\mathbf{y} - \mathbf{z}|$$

whenever  $(x, \mathbf{y}) \in R$  and  $(x, \mathbf{z}) \in R$ .

(3) Since all  $g_j$  ( $1 \leq j \leq k$ ) are real continuous functions on a compact set  $[a, b]$ , all  $g_j$  are bounded (Theorem 4.15), say  $|g_j| \leq M$  on  $[a, b]$  for some  $M_j \in \mathbb{R}^1$  ( $1 \leq j \leq k$ ).

(4) Write  $\mathbf{y} = (y_1, \dots, y_k)$  and  $\mathbf{z} = (z_1, \dots, z_k)$ . So

$$\begin{aligned}
& |\phi(x, \mathbf{y}) - \phi(x, \mathbf{z})|^2 \\
&= \left| \left( y_2 - z_2, y_3 - z_3, \dots, y_{k-1} - z_{k-1}, -\sum_{j=1}^k g_j(x)(y_j - z_j) \right) \right|^2 \\
&= \sum_{j=2}^{k-1} (y_j - z_j)^2 + \left( -\sum_{j=1}^k g_j(x)(y_j - z_j) \right)^2 \\
&\leq \sum_{j=2}^{k-1} (y_j - z_j)^2 + \sum_{j=1}^k g_j(x)^2 \sum_{j=1}^k (y_j - z_j)^2 && \text{(Theorem 1.35)} \\
&\leq \sum_{j=2}^{k-1} (y_j - z_j)^2 + \sum_{j=1}^k M_j^2 \sum_{j=1}^k (y_j - z_j)^2 && ((3)) \\
&\leq \sum_{j=1}^k (y_j - z_j)^2 + \sum_{j=1}^k M_j^2 \sum_{j=1}^k (y_j - z_j)^2 && (x^2 \geq 0 \forall x \in \mathbb{R}^1) \\
&\leq \left( 1 + \sum_{j=1}^k M_j^2 \right) |\mathbf{y} - \mathbf{z}|^2.
\end{aligned}$$

Hence  $|\phi(x, \mathbf{y}) - \phi(x, \mathbf{z})| \leq A|\mathbf{y} - \mathbf{z}|$  for some  $A = \left( 1 + \sum_{j=1}^k M_j^2 \right)^{\frac{1}{2}}$ .

□