

## Chapter 5: Differentiation

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**Exercise 5.1.** Let  $f$  be defined for all real  $x$ , and suppose that

$$|f(x) - f(y)| \leq (x - y)^2$$

for all real  $x$  and  $y$ . Prove that  $f$  is a constant.

*Proof.*

(1) Write

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq |x - y|$$

if  $x \neq y$ .

(2) Given any  $y \in \mathbb{R}$ ,

$$\left| \frac{f(x) - f(y)}{x - y} \right| \rightarrow 0 \quad \text{as } x \rightarrow y,$$

or  $|f'(y)| = 0$ .

(3) Or using  $\varepsilon$ - $\delta$  argument. Fix  $y \in \mathbb{R}$ . Given any  $\varepsilon > 0$ , there exists  $\delta = \varepsilon > 0$  such that

$$\left| \frac{f(x) - f(y)}{x - y} - 0 \right| \leq |x - y| < \delta = \varepsilon$$

whenever  $|x - y| < \delta$ . That is,  $|f'(y)| = 0$ .

(4) So  $f'(y) = 0$  for any  $y \in \mathbb{R}$ . By Theorem 5.11 (b),  $f$  is a constant.

□

**Exercise 5.2.** Suppose  $f'(x) > 0$  in  $(a, b)$ . Prove that  $f$  is strictly increasing in  $(a, b)$ , and let  $g$  be its inverse function. Prove that  $g$  is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)} \quad (a < x < b).$$

*Proof.* Let  $E = (a, b)$ .

- (1) Theorem 5.10 implies that for any  $a < p < q < b$  there exists  $\xi \in (p, q)$  such that

$$f(p) - f(q) = (p - q)f'(\xi).$$

Since  $\xi \in (p, q) \subseteq E$ , by assumption  $f'(\xi) > 0$ . Hence  $f(p) - f(q) = (p - q)f'(\xi) < 0$  (here  $p - q < 0$ ), or

$$f(p) < f(q)$$

if  $p < q$ . Therefore,  $f$  is strictly increasing in  $(a, b)$ .

- (2) Show that  $f$  is one-to-one in  $E$  if  $f$  is strictly increasing in  $E$ . If  $f(p) = f(q)$ , then it cannot be  $p > q$  or  $p < q$  ((1)). So that  $p = q$ , or  $f$  is injective.
- (3) Show that  $g$  is well-defined. Theorem 5.2 and Theorem 4.17.
- (4) Show that  $g'(f(x)) = \frac{1}{f'(x)}$ . Given  $y \in f(E)$ , say  $y = f(x)$  for some  $x \in E$ . Given any  $s \in f(E)$  with  $s \neq y$ . Here  $s = f(t)$  for some  $t \in E$  and  $t \neq x$ .

$$\begin{aligned} \lim_{s \rightarrow y} \frac{g(s) - g(y)}{s - y} &= \lim_{f(t) \rightarrow f(x)} \frac{g(f(t)) - g(f(x))}{f(t) - f(x)} \\ &= \lim_{t \rightarrow x} \frac{t - x}{f(t) - f(x)} \\ &= \lim_{t \rightarrow x} \frac{1}{\frac{f(t) - f(x)}{t - x}} \\ &= \frac{1}{f'(x)}. \end{aligned} \quad (f' > 0)$$

Here  $s \rightarrow y$  if and only if  $t \rightarrow x$  since both  $f$  and  $g$  are continuous and one-to-one. Hence  $g$  is differentiable and  $g'(f(x)) = \frac{1}{f'(x)}$ .

□

**Exercise 5.3.** Suppose  $g$  is a real function on  $\mathbb{R}^1$ , with bounded derivative (say  $|g'| \leq M$ ). Fix  $\varepsilon > 0$ , and define  $f(x) = x + \varepsilon g(x)$ . Prove that  $f$  is one-to-one if  $\varepsilon$  is small enough. (A set of admissible values of  $\varepsilon$  can be determined which depends only on  $M$ .)

*Proof.*

- (1) Note that  $f'(x) = 1 + \varepsilon g'(x)$  (Theorem 5.3). Since  $|g'| \leq M$ ,

$$1 - \varepsilon M \leq f'(x) \leq 1 + \varepsilon M.$$

- (2) Pick

$$\varepsilon = \frac{1}{M + 1} > 0.$$

Thus,

$$f'(x) \geq \frac{1}{M+1} > 0.$$

By Exercise 5.2,  $f(x)$  is strictly increasing in  $\mathbb{R}$  or one-to-one in  $\mathbb{R}$ .

□

**Exercise 5.4.** *If*

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$

where  $C_0, \dots, C_n$  are real constants, prove that the equation

$$C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n = 0$$

has at least one real root between 0 and 1.

*Proof.* Let

$$g(x) = C_0x + \frac{C_1}{2}x^2 + \cdots + \frac{C_{n-1}}{n}x^n + \frac{C_n}{n+1}x^{n+1} \in \mathbb{R}[x].$$

Then  $g(0) = g(1) = 0$ , and  $g'(x) = C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n$ . By the mean value theorem (Theorem 5.10), there exists a point  $\xi \in (0, 1)$  at which

$$g(1) - g(0) = g'(\xi)(1 - 0),$$

or  $g'(\xi) = 0$ . That is, there exists a real root  $x = \xi$  between 0 and 1 at which  $C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n = 0$ . □

**Exercise 5.5.** *Suppose  $f$  is defined and differentiable for every  $x > 0$ , and  $f'(x) \rightarrow 0$  as  $x \rightarrow +\infty$ . Put  $g(x) = f(x+1) - f(x)$ . Prove that  $g(x) \rightarrow 0$  as  $x \rightarrow +\infty$ .*

*Proof.* Given any  $x > 0$ . Since  $f$  is differentiable for every  $x > 0$ ,  $f$  is differentiable on  $[x, x+1]$ . By Theorem 5.2 and Theorem 5.10 (the mean value theorem), there is a point  $\xi \in (x, x+1)$  at which

$$f(x+1) - f(x) = [(x+1) - x]f'(\xi)$$

or

$$g(x) = f'(\xi).$$

As  $x \rightarrow +\infty$ ,  $\xi \rightarrow +\infty$ . Hence

$$\lim_{x \rightarrow +\infty} g(x) = \lim_{\xi \rightarrow +\infty} f'(\xi) = 0.$$

□

**Exercise 5.6.** Suppose

- (a)  $f$  is continuous for  $x \geq 0$ ,
- (b)  $f'(x)$  exists for  $x > 0$ ,
- (c)  $f(0) = 0$ ,
- (d)  $f'$  is monotonically increasing.

Put

$$g(x) = \frac{f(x)}{x} \quad (x > 0)$$

and prove that  $g$  is monotonically increasing.

*Proof.*

- (1) It suffices to show that  $g'(x) \geq 0$  for  $x > 0$  (Theorem 5.11(a)), that is, to show that

$$g'(x) = \frac{xf'(x) - f(x)}{x^2} \geq 0 \quad (x > 0),$$

or

$$xf'(x) - f(x) \geq 0 \quad (x > 0)$$

since  $x^2 > 0$  for all nonzero  $x$ .

- (2) Given  $x > 0$ . By (a)(b), we apply the mean value theorem (Theorem 5.10) on  $f$  to get

$$f(x) - f(0) = (x - 0)f'(\xi)$$

for some  $\xi \in (0, x)$ . By (c),

$$f(x) = xf'(\xi).$$

By (d),

$$f(x) = xf'(\xi) \leq xf'(x).$$

Hence  $xf'(x) - f(x) \geq 0$ , or  $g$  is monotonically increasing.

□

*Note.*  $g$  is increasing strictly if  $f$  is increasing strictly.

**Exercise 5.7.** Suppose  $f'(x)$ ,  $g'(x)$  exist,  $g'(x) \neq 0$ , and  $f(x) = g(x) = 0$ . Prove that

$$\lim_{t \rightarrow x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}.$$

(This holds also for complex functions.)

*Proof.*

$$\begin{aligned}
 \frac{f'(t)}{g'(t)} &= \frac{\lim_{t \rightarrow x} \frac{f(t)-f(x)}{t-x}}{\lim_{t \rightarrow x} \frac{g(t)-g(x)}{t-x}} \\
 &= \lim_{t \rightarrow x} \frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}} && \text{(Both limits exist and } g' \neq 0) \\
 &= \lim_{t \rightarrow x} \frac{f(t)}{g(t)}. && (f(x) = g(x) = 0)
 \end{aligned}$$

This proof is also true for complex functions.  $\square$

**Exercise 5.8.**

**Exercise 5.9.** Let  $f$  be a continuous real function on  $\mathbb{R}^1$ , of which it is known that  $f'(x)$  exists for all  $x \neq 0$  and that  $f'(x) \rightarrow 3$  as  $x \rightarrow 0$ . Does it follow that  $f'(0)$  exists?

*Proof.*

- (1) Show that  $f'(0) = 3$ . It is equivalent to show that

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = 3.$$

Write  $F(x) = f(x) - f(0)$  and  $G(x) = x - 0$  on  $\mathbb{R}^1$ . So that

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{F(x)}{G(x)} = 0.$$

- (2) Note that

$$\lim_{x \rightarrow 0} \frac{F'(x)}{G'(x)} = \lim_{x \rightarrow 0} \frac{f'(x)}{1} = 3.$$

- (3) Since  $f$  is continuous on  $\mathbb{R}^1$ ,  $F$  is continuous on  $\mathbb{R}^1$ . Hence

$$\lim_{x \rightarrow 0} F(x) = F(\lim_{x \rightarrow 0} x) = F(0) = 0.$$

Also,  $G$  is continuous on  $\mathbb{R}^1$  implies that

$$\lim_{x \rightarrow 0} G(x) = G(\lim_{x \rightarrow 0} x) = G(0) = 0.$$

- (4) Apply L'Hospital's rule (Theorem 5.13) to (2)(3), we have

$$\lim_{x \rightarrow 0} \frac{F(x)}{G(x)} = 3,$$

or  $f'(0) = 3$ .

□

**Exercise 5.10.**

**Exercise 5.11.** Suppose  $f$  is defined in a neighborhood of  $x$ , and suppose  $f''(x)$  exists. Show that

$$\lim_{h \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

Show by an example that the limit may exist even if  $f''(x)$  does not. (Hint: Use Theorem 5.13.)

*Proof (Theorem 5.13).*

- (1) Write  $F(h) = f(x+h) + f(x-h) - 2f(x)$  and  $G(h) = h^2$ . It is equivalent to show that

$$\lim_{h \rightarrow 0} \frac{F(h)}{G(h)} = f''(x).$$

We might apply Theorem 5.13 (L'Hospital rule) to get it.

- (2) Show that  $\lim_{h \rightarrow 0} F(h) = 0$  and  $\lim_{h \rightarrow 0} G(h) = 0$ . It is clear that  $\lim_{h \rightarrow 0} G(h) = \lim_{h \rightarrow 0} h^2 = 0$  since  $x \mapsto x^2$  is continuous on  $\mathbb{R}^1$ . Besides, since  $f$  is continuous at  $x$  (by applying Theorem 5.2 twice),

$$\lim_{h \rightarrow 0} F(h) = f(x) + f(x) - 2f(x) = 0.$$

- (3) Show that

$$\lim_{h \rightarrow 0} \frac{F'(h)}{G'(h)} = \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x-h)}{2h}$$

is well-defined. Since  $f''(x)$  exists in a neighborhood  $B(x; r)$  of  $x$  (where  $r > 0$ ),  $f'(x)$  exists and is continuous in  $B(x; r)$  (Theorem 5.2). As  $0 < |h| < \frac{r}{2}$ ,

$$x+h \in B\left(x+h; \frac{r}{2}\right) \subseteq B(x; r)$$

and

$$x-h \in B\left(x-h; \frac{r}{2}\right) \subseteq B(x; r).$$

So  $f'(x+h)$  and  $f'(x-h)$  exist in  $B(x; r)$  as  $0 < |h| < \frac{r}{2}$ . Hence

$$\lim_{h \rightarrow 0} \frac{F'(h)}{G'(h)} = \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x-h)}{2h}$$

is well-defined (Theorem 5.3 and Theorem 5.5 (the chain rule)).

(4) Show that

$$\lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x-h)}{2h} = f''(x).$$

Since  $f''(x)$  exists, by definition

$$\lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{h} = f''(x)$$

and

$$\lim_{h \rightarrow 0} \frac{f'(x-h) - f'(x)}{-h} = f''(x).$$

Sum up two expressions to get

$$2f''(x) = \lim_{h \rightarrow 0} \frac{f'(x-h) - f'(x-h)}{h}.$$

(5) By (2)(3)(4) and Theorem 5.13 (L'Hospital rule), the result is established.

(6) Given  $f(x) = x|x|$  on  $\mathbb{R}^1$ . Show that

$$\lim_{h \rightarrow 0} \frac{f(h) + f(-h) - 2f(0)}{h^2} = 0$$

but  $f''(x)$  does not exist at  $x = 0$ . Clearly,

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(h) + f(-h) - 2f(0)}{h^2} &= \lim_{h \rightarrow 0} \frac{h|h| + (-h)|-h| - 2 \cdot 0}{h^2} \\ &= \lim_{h \rightarrow 0} \frac{h|h| - h|h| - 0}{h^2} \\ &= \lim_{h \rightarrow 0} 0 \\ &= 0. \end{aligned}$$

But  $f''(x)$  does not exist by Exercise 5.12.

□

**Exercise 5.12.** If  $f(x) = |x|^3$ , compute  $f'(x)$ ,  $f''(x)$  for all real  $x$ , and show that  $f^{(3)}(0)$  does not exist.

*Proof.*

(1) Write

$$f(x) = \begin{cases} x^3 & (x \geq 0), \\ -x^3 & (x \leq 0). \end{cases}$$

(2) Show that  $f'(x) = 3x|x|$ . It is trivial that

$$f'(x) = \begin{cases} 3x^2 & (x > 0), \\ -3x^2 & (x < 0). \end{cases}$$

Note that

$$\lim_{x \rightarrow 0} f'(x) = 0.$$

Apply the same argument in Exercise 5.9, we have

$$f'(0) = 0.$$

Hence  $f'$  exists and  $f'(x) = 3x|x|$  for any  $x \in \mathbb{R}$ .

(3) Show that  $f''(x) = 6|x|$ . Similar to (2).

$$f''(x) = \begin{cases} 6x & (x > 0), \\ -6x & (x < 0). \end{cases}$$

Note that

$$\lim_{x \rightarrow 0} f''(x) = 0.$$

Apply the same argument in Exercise 5.9, we have

$$f''(0) = 0.$$

Hence  $f''$  exists and  $f''(x) = 6|x|$  for any  $x \in \mathbb{R}$ .

(4) Show that  $f^{(3)}(0)$  does not exist.

$$f'''(x) = \begin{cases} 6 & (x > 0), \\ -6 & (x < 0). \end{cases}$$

There are some proofs for showing that  $f^{(3)}(0)$  does not exist.

(a) Since

$$\lim_{t \rightarrow 0+} \frac{f''(t) - f''(0)}{t - 0} = \lim_{t \rightarrow 0+} \frac{6t}{t} = 6$$

and

$$\lim_{t \rightarrow 0-} \frac{f''(t) - f''(0)}{t - 0} = \lim_{t \rightarrow 0-} \frac{-6t}{t} = -6,$$

$f^{(3)}(0)$  does not exist.

(b) (Reductio ad absurdum) If  $f$  were differentiable on  $\mathbb{R}^1$ , then

$$\lim_{t \rightarrow 0+} f'''(t) = 6$$

and

$$\lim_{t \rightarrow 0-} f'''(t) = -6,$$

or  $f'''$  has a simple discontinuity at  $x = 0$ , contrary to Corollary to Theorem 5.12.



□

*Note.* Given  $k > 0$ . We can construct one real function  $f$  on  $\mathbb{R}^1$ , say

$$f(x) = \begin{cases} |x|^k & (k \text{ is odd}), \\ x|x|^{k-1} & (k > 0 \text{ is even}), \end{cases}$$

such that all  $f^{(0)}(0) = \dots = f^{(k-1)}(0) = 0$  exist but  $f^{(k)}(0)$  does not exist.

**Exercise 5.13.**

**Exercise 5.14.** Let  $f$  be a differentiable real function defined in  $(a, b)$ . Prove that  $f$  is convex if and only if  $f'$  is monotonically increasing. Assume next  $f''(x)$  exists for every  $x \in (a, b)$ , and prove that  $f$  is convex if and only if  $f''(x) \geq 0$  for all  $x \in (a, b)$ .

*Proof.*

(1) Show that  $f'$  is monotonically increasing if  $f$  is convex.

(a) Since  $f$  is convex, by definition (Exercise 4.23)

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

whenever  $a < x < b$ ,  $a < y < b$ ,  $0 < \lambda < 1$ .

(b) As  $x \neq y$ , we have

$$\begin{aligned} f(y) - f(x) &\geq \frac{f(x + \lambda(y - x)) - f(x)}{\lambda} \\ &= \frac{f(x + \lambda(y - x)) - f(x)}{\lambda(y - x)} \cdot (y - x) \end{aligned}$$

and let  $\lambda \rightarrow 0$  to get

$$f(y) - f(x) \geq f'(x)(y - x)$$

(since  $f'(x)$  exists). Similarly, we have

$$f(x) - f(y) \geq f'(y)(x - y).$$

(c) Given any  $y > x$ , we have

$$f'(y)(y - x) \geq f(y) - f(x) \geq f'(x)(y - x).$$

Hence  $f'(y) \geq f'(x)$  whenever  $y > x$ , or  $f'$  is monotonically increasing.

(2) Show that  $f$  is convex if  $f'$  is monotonically increasing. Given any  $y > x$  and any  $0 < \lambda < 1$ .

- (a) By Theorem 5.10 (the mean value theorem), there is a point  $x < \xi < y$  such that

$$f(y) - f(x) = f'(\xi)(y - x).$$

Since  $f'$  is monotonically increasing,

$$f'(y)(y - x) \geq f(y) - f(x) \geq f'(x)(y - x).$$

- (b) Write  $z = \lambda x + (1 - \lambda)y$ . Hence

$$f(y) - f(z) \geq f'(z)(y - z),$$

$$f(z) - f(x) \leq f'(z)(z - x),$$

or

$$f(y) \geq f(z) + f'(z)(y - z),$$

$$f(x) \geq f(z) + f'(z)(x - z),$$

or

$$\begin{aligned} \lambda f(x) + (1 - \lambda)f(y) &\geq \lambda[f(z) + f'(z)(x - z)] \\ &\quad + (1 - \lambda)[f(z) + f'(z)(y - z)] \\ &= f(z) \\ &= f(\lambda x + (1 - \lambda)y). \end{aligned}$$

Hence  $f$  is convex.

- (3) Show that  $f''(x) \geq 0$  if  $f$  is convex and  $f''$  exists. By (1),  $f'$  is monotonically increasing since  $f$  is convex. Given any  $x \neq y$ , we have

$$\frac{f'(y) - f'(x)}{y - x} \geq 0.$$

Let  $y \rightarrow x$ , we have  $f''(x) \geq 0$  if  $f''$  exists.

- (4) Show that  $f$  is convex if  $f''$  exists and  $f''(x) \geq 0$ . By Theorem 5.11(a),  $f'$  is monotonically increasing. By (2),  $f$  is convex.

□

**Exercise 5.15 (Landau-Kolmogorov inequality on the half-line).** Suppose  $a \in \mathbb{R}^1$ ,  $f$  is a twice-differentiable real function on  $(a, \infty)$ , and  $M_0, M_1, M_2$  are the least upper bounds of  $|f(x)|, |f'(x)|, |f''(x)|$ , respectively, on  $(a, \infty)$ . Prove that

$$M_1^2 \leq 4M_0M_2.$$

(Hint: If  $h > 0$ , Taylor's theorem shows that

$$f'(x) = \frac{1}{2h}[f(x + 2h) - f(x)] - hf''(\xi)$$

for some  $\xi \in (x, x + 2h)$ . Hence

$$|f'(x)| \leq hM_2 + \frac{M_0}{h}.$$

To show that  $M_1^2 = 4M_0M_2$  can actually happen, take  $a = -1$ , define

$$f(x) = \begin{cases} 2x^2 - 1 & (-1 < x < 0), \\ \frac{x^2-1}{x^2+1} & (0 \leq x < \infty), \end{cases}$$

and show that  $M_0 = 1$ ,  $M_1 = 4$ ,  $M_2 = 4$ . Does  $M_1^2 \leq 4M_0M_2$  hold for vector-valued functions too?

*Note.*

(1) Write

$$M_1 \leq 2M_0^{\frac{1}{2}}M_2^{\frac{1}{2}}.$$

2 is called the Landau-Kolmogorov constant, which is the best possible by the above example.

(2) In general, suppose  $a \in \mathbb{R}^1$ ,  $f$  is a  $n$ th differentiable real function on  $(a, \infty)$ , and  $M_0$ ,  $M_k$ ,  $M_n$  are the least upper bounds of  $|f(x)|$ ,  $|f^{(k)}(x)|$ ,  $|f^{(n)}(x)|$ , respectively, on  $(a, \infty)$  where  $1 \leq k < n$ . Then

$$M_k \leq C(n, k)M_0^{1-\frac{k}{n}}M_n^{\frac{k}{n}}.$$

*Proof.*

(1) Consider some trivial cases.

- (a) If  $M_0 = 0$ , then  $f(x) = 0$  on  $(a, +\infty)$ . So that  $f'(x) = f''(x) = 0$  on  $(a, +\infty)$ , or  $M_1 = M_2 = 0$ . The inequality holds.
- (b) If  $M_2 = 0$ , then  $f''(x) = 0$  on  $(a, +\infty)$ . So that  $f'(x) = \alpha$  for some constant  $\alpha \in \mathbb{R}^1$  (Theorem 5.11(b)), and  $f(x) = \alpha x + \beta$  for some constant  $\beta \in \mathbb{R}^1$  (by applying Theorem 5.11(b) to  $x \mapsto f(x) - \alpha x$ ). Hence  $M_1 = |\alpha|$  and

$$M_0 = \begin{cases} +\infty & (\alpha \neq 0), \\ |\beta| & (\alpha = 0). \end{cases}$$

In any case, the inequality holds.

- (c) If  $M_0 = +\infty$  and  $M_2 \neq 0$ , there is nothing to do.
- (d) If  $M_2 = +\infty$  and  $M_0 \neq 0$ , there is nothing to do.

- (2) By (1), we suppose that  $0 < M_0 < +\infty$  and  $0 < M_2 < +\infty$ . Given  $x \in (a, +\infty)$  and  $h > 0$ . By Taylor's theorem (Theorem 5.15):

$$f(x+2h) = f(x) + 2hf'(x) + 2h^2f''(\xi)$$

for some  $\xi \in (x, x+2h) \subseteq (a, +\infty)$ . Thus

$$\begin{aligned} 2h|f'(x)| &\leq |f(x+2h)| - |f(x)| + 2h^2|f''(\xi)| \\ &\leq 2M_0 + 2h^2M_2, \\ |f'(x)| &\leq \frac{M_0}{h} + hM_2 \end{aligned}$$

holds for all  $h > 0$ . In particular, take

$$h = \sqrt{\frac{M_0}{M_2}}$$

to get

$$|f'(x)| \leq 2\sqrt{M_0M_2}.$$

Thus  $2\sqrt{M_0M_2}$  is an upper bound of  $|f'(x)|$  for all  $x \in (a, +\infty)$ . Hence

$$M_1 \leq 2\sqrt{M_0M_2}$$

or

$$M_1^2 \leq 4M_0M_2.$$

- (3) *Define*

$$f(x) = \begin{cases} 2x^2 - 1 & (-1 < x < 0), \\ \frac{x^2-1}{x^2+1} & (0 \leq x < \infty). \end{cases}$$

Show that  $M_0 = 1$ ,  $M_1 = 4$ ,  $M_2 = 4$ . Similar to Exercise 5.12,

$$f'(x) = \begin{cases} 4x & (-1 < x \leq 0), \\ \frac{4x}{(x^2+1)^2} & (0 \leq x < \infty). \end{cases}$$

(Here  $\lim_{x \rightarrow 0+} f'(x) = 0$  and  $\lim_{x \rightarrow 0-} f'(x) = 0$ . So  $f'(0) = 0$  by Exercise 5.9.) Also,

$$f''(x) = \begin{cases} 4 & (-1 < x \leq 0), \\ \frac{-12x^2+4}{(x^2+1)^3} & (0 \leq x < \infty). \end{cases}$$

(Here  $\lim_{x \rightarrow 0+} f''(x) = 4$  and  $\lim_{x \rightarrow 0-} f''(x) = 4$ . So  $f''(0) = 4$  by Exercise 5.9.) Hence,  $M_0 = 1$ ,  $M_1 = 4$ ,  $M_2 = 4$ .

- (4) *Given*

$$\mathbf{f}(x) = (f_1(x), \dots, f_k(x))$$

be a twice-differentiable vector-valued function from  $(a, \infty)$  to  $\mathbb{R}^k$ . and  $M_0, M_1, M_2$  are the least upper bounds of  $|\mathbf{f}(x)|, |\mathbf{f}'(x)|, |\mathbf{f}''(x)|$ , respectively, on  $(a, \infty)$ . Show that

$$M_1^2 \leq 4M_0M_2.$$

Similar to (1), we suppose that  $0 < M_0 < +\infty$  and  $0 < M_2 < +\infty$ . Given any  $\mathbf{v} = (v_1, \dots, v_k) \in \mathbb{R}^k$ ,  $\mathbf{v} \cdot \mathbf{f}$  is a twice-differentiable real function on  $(a, \infty)$ . Similar to (2), Given  $x \in (a, +\infty)$  and  $h > 0$ . By Taylor's theorem (Theorem 5.15):

$$(\mathbf{v} \cdot \mathbf{f})(x + 2h) = (\mathbf{v} \cdot \mathbf{f})(x) + 2h(\mathbf{v} \cdot \mathbf{f})'(x) + 2h^2(\mathbf{v} \cdot \mathbf{f})''(\xi)$$

for some  $\xi \in (x, x + 2h) \subseteq (a, +\infty)$ . Thus by the Schwarz inequality (Theorem 1.35)

$$\begin{aligned} 2h|(\mathbf{v} \cdot \mathbf{f})'(x)| &\leq |(\mathbf{v} \cdot \mathbf{f})(x + 2h)| + |(\mathbf{v} \cdot \mathbf{f})(x)| + 2h^2|(\mathbf{v} \cdot \mathbf{f})''(\xi)| \\ &\leq |\mathbf{v}||\mathbf{f}(x + 2h)| + |\mathbf{v}||\mathbf{f}(x)| + 2h^2|\mathbf{v}||\mathbf{f}''(\xi)| \\ &\leq (2M_0 + 2h^2M_2)|\mathbf{v}|, \\ |(\mathbf{v} \cdot \mathbf{f})'(x)| &\leq \left(\frac{M_0}{h} + hM_2\right)|\mathbf{v}| \end{aligned}$$

holds for any  $\mathbf{v}$  and  $h > 0$ . In particular, we take

$$\mathbf{v} = \mathbf{f}'(y)$$

and

$$h = \sqrt{\frac{M_0}{M_2}}$$

to get

$$|\mathbf{f}'(x) \cdot \mathbf{f}'(y)| \leq 2\sqrt{M_0M_2}|\mathbf{f}'(y)| \leq 2M_1\sqrt{M_0M_2}.$$

Note that  $x$  and  $y$  are arbitrary (in  $(a, +\infty)$ ). In particular, we take  $x = y$  to get

$$|\mathbf{f}'(x)|^2 \leq 2M_1\sqrt{M_0M_2}.$$

Thus  $2M_1\sqrt{M_0M_2}$  is an upper bound of  $|\mathbf{f}'(x)|^2$  for all  $x \in (a, +\infty)$ . Hence

$$M_1^2 \leq 2M_1\sqrt{M_0M_2}$$

or

$$M_1^2 \leq 4M_0M_2.$$

□

**Supplement (Landau-Kolmogorov inequality on the real line).** Suppose  $f$  is a twice-differentiable real function on  $(-\infty, +\infty)$ , and  $M_0, M_1, M_2$  are the

least upper bounds of  $|f(x)|$ ,  $|f'(x)|$ ,  $|f''(x)|$ , respectively, on  $(-\infty, +\infty)$ . Prove that

$$M_1^2 \leq 2M_0M_2.$$

*Proof.*

- (1) Similar to (1) in Landau-Kolmogorov inequality on the half-line, we suppose that  $0 < M_0 < +\infty$  and  $0 < M_2 < +\infty$ .
- (2) Similar to (2) in Landau-Kolmogorov inequality on the half-line. Given  $x \in \mathbb{R}^1$  and  $h > 0$ . By Taylor's theorem (Theorem 5.15):

$$f(x+2h) = f(x) + 2hf'(x) + 2h^2f''(\xi_1) \quad (1)$$

$$f(x-2h) = f(x) - 2hf'(x) + 2h^2f''(\xi_2) \quad (2)$$

for some  $\xi_1 \in (x, x+2h)$  and  $\xi_2 \in (x, x-2h)$ . So (1) subtracts (2):

$$f(x+2h) - f(x-2h) = 4hf'(x) + 2h^2f''(\xi_1) - 2h^2f''(\xi_2).$$

Thus

$$\begin{aligned} 4h|f'(x)| &\leq |f(x+2h)| + |f(x-2h)| + 2h^2|f''(\xi_1)| + 2h^2|f''(\xi_2)| \\ &\leq 2M_0 + 4h^2M_2, \\ |f'(x)| &\leq \frac{M_0}{2h} + hM_2 \end{aligned}$$

holds for all  $h > 0$ . In particular, take

$$h = \sqrt{\frac{M_0}{2M_2}}$$

to get

$$|f'(x)| \leq \sqrt{2M_0M_2}.$$

Thus  $\sqrt{2M_0M_2}$  is an upper bound of  $|f'(x)|$  for all  $x \in \mathbb{R}^1$ . Hence

$$M_1 \leq \sqrt{2M_0M_2}$$

or

$$M_1^2 \leq 2M_0M_2.$$

□

*Note.*

- (1) Write

$$M_1 \leq \sqrt{2}M_0^{\frac{1}{2}}M_2^{\frac{1}{2}}.$$

$\sqrt{2}$  is called the Landau-Kolmogorov constant, which is the best possible.

- (2) In general, suppose  $f$  is a  $n$ th differentiable real function on  $\mathbb{R}^1$ , and  $M_0, M_k, M_n$  are the least upper bounds of  $|f(x)|, |f^{(k)}(x)|, |f^{(n)}(x)|$ , respectively, on  $\mathbb{R}^1$  where  $1 \leq k < n$ . Then

$$M_k \leq C(n, k) M_0^{1-\frac{k}{n}} M_n^{\frac{k}{n}}.$$

**Exercise 5.16.** Suppose  $f$  is twice-differentiable on  $(0, \infty)$ ,  $f''$  is bounded on  $(0, \infty)$ , and  $f(x) \rightarrow 0$  as  $x \rightarrow \infty$ . Prove that  $f'(x) \rightarrow 0$  as  $x \rightarrow \infty$ . (Hint: Let  $a \rightarrow \infty$  in Exercise 5.15.)

*Proof.*

- (1) Write  $|f''| \leq M$  for some real  $M$  since  $f''$  is bounded on  $(0, \infty)$ .  
 (2) Given any  $a > 0$ . As in Exercise 5.15, define  $M_0, M_1, M_2$  are the least upper bounds of  $|f(x)|, |f'(x)|, |f''(x)|$  on  $(a, \infty)$ . Note that  $M_2 \leq M$  for any  $a > 0$  (by (1)). So that

$$M_1^2 \leq 4M_0M_2 \leq 4MM_0$$

for any  $a > 0$ .

- (3) By assumption,  $M_0 \rightarrow 0$  as  $a \rightarrow \infty$ . (So given any  $\varepsilon > 0$ , there exists a real  $A$  such that

$$0 \leq M_0 < \frac{\varepsilon}{4M+1}$$

whenever  $a \geq A$ . Hence

$$M_1^2 \leq 4MM_0 \leq 4M \cdot \frac{\varepsilon}{4M+1} < \varepsilon.$$

whenever  $a \geq A$ .) Therefore  $M_1^2 \rightarrow 0$  as  $a \rightarrow \infty$ , or  $f'(x) \rightarrow 0$  as  $x \rightarrow \infty$ .

□