Chapter 1: The Real And Complex Number Systems

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Integers

Exercise 1.1 *Prove that there is no largest prime. (A proof was known to Euclid.)*

There are many proofs of this result. We provide some of them.

Proof (Due to Euclid). If $p_1, p_2, ..., p_t$ were all primes, then write

$$n = p_1 p_2 \cdots p_t + 1$$

and there were a prime number p dividing n. p can not be any of p_i for $1 \leq i \leq t$, otherwise p would divide the difference $n - p_1 p_2 \cdots p_t = 1$, that is, $p \neq p_i$ for $1 \leq i \leq t$, which is absurd. $\square$

Supplement (Due to Euclid). *Show that $k[x]$ has infinitely many irreducible polynomials.*

Proof (Due to Euclid). If $f_1, f_2, ..., f_t$ were all irreducible polynomials, then write

$$g = f_1 f_2 \cdots f_t + 1 \in k[x]$$

and there were a irreducible polynomial f dividing g (since $\deg g = \deg f_1 + \deg f_2 + \cdots + \deg f_t \geq 1$). f can not be any of $c_i f_i$ for $1 \leq i \leq t$ and $0 \neq c_i \in k$, otherwise f would divide the difference $g - f_1 f_2 \cdots f_t = 1$, that is, $f \neq c_i f_i$ for $1 \leq i \leq t$ and $0 \neq c_i \in k$, which is absurd. $\square$

Proof (Unique factorization theorem). Given N.

(1) *Show that $\sum_{n \leq N} \frac{1}{n} \leq \prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1}$.*
    By the unique factorization theorem on $n \leq N$,

$$\sum_{n \leq N} \frac{1}{n} \leq \prod_{p \leq N} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1}.$$

(2) By (1) and the fact that $\sum \frac{1}{n}$ diverges, there are infinitely many primes.

□

Proof (Due to Eckford Cohen).

(1) $\mathrm{ord}_p n! = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots$. For any $k = 1, 2, ..., n$, we can express k as $k = p^s t$ where $s = \mathrm{ord}_p k$ is a non-negative integer and $(t, p) = 1$. There are $\left[\frac{n}{p^a}\right]$ numbers such that $p^a \mid k$ for $a = 1, 2, ...$ Therefore, there are

$$\left[\frac{n}{p^a}\right] - \left[\frac{n}{p^{a+1}}\right]$$

numbers such that $\mathrm{ord}_p k = a$ for $a = 1, 2, ...$ Hence,

$$\mathrm{ord}_p n! = \left(\left[\frac{n}{p}\right] - \left[\frac{n}{p^2}\right]\right) + 2\left(\left[\frac{n}{p^2}\right] - \left[\frac{n}{p^3}\right]\right) + 3\left(\left[\frac{n}{p^3}\right] - \left[\frac{n}{p^4}\right]\right) + \cdots$$
$$= \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots .$$

(2) $\mathrm{ord}_p n! \leq \frac{n}{p-1}$ and that $n!^{\frac{1}{n}} \leq \prod_{p \mid n!} p^{\frac{1}{p-1}}$.

$$\mathrm{ord}_p n! = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots$$
$$\leq \frac{n}{p} + \frac{n}{p^2} + \frac{n}{p^3} + \cdots$$
$$= \frac{\frac{n}{p}}{1 - \frac{1}{p}}$$
$$= \frac{n}{p-1}.$$

Thus,

$$n! = \prod_{p \mid n!} p^{\mathrm{ord}_p n!} \leq \prod_{p \mid n!} p^{\frac{n}{p-1}} = \left(\prod_{p \mid n!} p^{\frac{1}{p-1}}\right)^n,$$

or

$$n!^{\frac{1}{n}} \leq \prod_{p \mid n!} p^{\frac{1}{p-1}}.$$

(3) $(n!)^2 \geq n^n$. Write $(n!)^2 = \prod_{k=1}^n k \prod_{k=1}^n (n+1-k) = \prod_{k=1}^n k(n+1-k)$, and $n^n = \prod_{k=1}^n n$. It suffices to show that $k(n+1-k) \geq n$ for each $1 \leq k \leq n$. Notice that $k(n+1-k) - n = (n-k)(k-1) \geq 0$ for $1 \leq k \leq n$. The inequality holds.

(4) By (3)(4), $\prod_{p \mid n!} p^{\frac{1}{p-1}} \geq \sqrt{n}$. Assume that there are finitely many primes, the value $\prod_{p \mid n!} p^{\frac{1}{p-1}}$ is a finite number whenever the value of n. However, $\sqrt{n} \to \infty$ as $n \to \infty$, which leads to a contradiction. Hence there are infinitely many primes.

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□

Proof (Formula for $\phi(n)$). If $p_1, p_2, ..., p_t$ were all primes, then let $n = p_1 p_2 \cdots p_t$ and all numbers between 2 and n are NOT relatively prime to n. Thus, $\phi(n) = 1$ by the definition of ϕ. By the formula for ϕ,

$$\phi(n) = n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_t}\right)$$

$$1 = (p_1 p_2 \cdots p_t) \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_t}\right)$$

$$= (p_1 - 1)(p_2 - 1) \cdots (p_t - 1) > 1,$$

which is a contradiction (since 3 is a prime). Hence there are infinitely many primes. □

Exercise 1.2 *If n is a positive integer, prove the algebraic identity*

$$a^n - b^n = (a - b) \sum_{k=0}^{n-1} a^k b^{n-1-k}.$$

Proof.

(1)

$$(a - b) \sum_{k=0}^{n-1} a^k b^{n-1-k} = a \sum_{k=0}^{n-1} a^k b^{n-1-k} - b \sum_{k=0}^{n-1} a^k b^{n-1-k}$$

$$= \sum_{k=0}^{n-1} a^{k+1} b^{n-1-k} - \sum_{k=0}^{n-1} a^k b^{n-k}.$$

(2) Arrange summation index:

$$\sum_{k=0}^{n-1} a^{k+1} b^{n-1-k} = \sum_{k=1}^{n} a^k b^{n-k} = a^n + \sum_{k=1}^{n-1} a^k b^{n-k},$$

$$\sum_{k=0}^{n-1} a^k b^{n-k} = b^n + \sum_{k=1}^{n-1} a^k b^{n-k}.$$

(3) By (1)(2),

$$(a - b) \sum_{k=0}^{n-1} a^k b^{n-1-k} = \left(a^n + \sum_{k=1}^{n-1} a^k b^{n-k}\right) - \left(b^n + \sum_{k=1}^{n-1} a^k b^{n-k}\right)$$

$$= a^n - b^n.$$

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□

Supplement. Some exercises without proof.

(1) *Let x be a nilpotent element of A. Show that $1 + x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit.* (Exercise 1.1 in Atiyah and Macdonald, Introduction to Commutative Algebra.)

(2) *Prove that $1^k + 2^k + \cdots + (p-1)^k \equiv 0 \ (p)$ if $p - 1 \nmid k$ and $-1(p)$ if $p-1 \mid k$.* (Exercise 4.11 in Kenneth Ireland and Michael Rosen, A Classical Introduction to Modern Number Theory, Second Edition)

(3) *Use the existence of a primitive root to give another proof of Wilson's theorem $(p - 1)! \equiv -1 \ (p)$.* (Exercise 4.12 in Kenneth Ireland and Michael Rosen, A Classical Introduction to Modern Number Theory, Second Edition)

(4) *Suppose n and F are integers and $n, F > 0$. Show that*

$$B_n(Fx) = F^{n-1} \sum_{a=0}^{F-1} B_n\left(x + \frac{a}{F}\right).$$

where $B_n(x)$ are Bernoulli polynomials. (Exercise 15.19 in Kenneth Ireland and Michael Rosen, A Classical Introduction to Modern Number Theory, Second Edition)

(5) Exercise 1.3.

(6) Exercise 1.4.

□

Exercise 1.3 *If $2^n - 1$ is a prime, prove that n is prime. A prime of the form $2^p - 1$, where p is prime, is called a Mersenne prime.*

It suffices to prove that: *If $a^n - 1$ is a prime, show that $a = 2$ and that n is a prime.* Primes of the form $2^p - 1$ are called Mersenne primes. For example, $2^3 - 1 = 7$ and $2^5 - 1 = 31$. It is not known if there are infinitely many Mersenne primes.

Proof.

(1) *n is a prime.* Assume n were not prime, say $n = rs$ for some $r, s > 1$. By Exercise 1.2, $a^{rs} - 1 = (a^s - 1)(\sum_{k=0}^{r-1} a^{sk})$. $a^s - 1 = 1$ since $a^s - 1 < a^{rs} - 1$ and $a^{rs} - 1$ is a prime. Hence $s = 1$ and $(a = 2)$, which is absurd.

(2) *$a = 2$.* If a is odd, then $a^p - 1 > 2$ is even, which is not a prime. If $a > 2$ is even, $a^p - 1 = (a - 1)(\sum_{k=0}^{p-1} a^k)$. Both $a - 1 > 1$ and $\sum_{k=0}^{p-1} a^k > 1$, which is absurd.

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By (1)(2), $a = 2$ and that n is a prime if $a^n - 1$ is a prime. $\square$

Rational and irrational numbers

Exercise 1.11 *Given any real $x > 0$, prove that there is an irrational number between 0 and x.*

Proof. There are only two possible cases: x is rational, or x is irrational.

(1) *x is rational.* Pick $y = \frac{x}{\sqrt{89}} \in (0, x) \subseteq \mathbb{R}$. y is irrational.

(2) *x is irrational.* Pick $y = \frac{x}{\sqrt{64}} \in (0, x) \subseteq \mathbb{R}$. y is irrational.

$\square$

Proof (Exercise 4.12). Pick

$$y = \lim_{m \to \infty} [\lim_{n \to \infty} \cos^{2n}(m!\pi x)] \cdot \frac{x}{\sqrt{89}} + (1 - \lim_{m \to \infty} [\lim_{n \to \infty} \cos^{2n}(m!\pi x)]) \cdot \frac{x}{\sqrt{64}}.$$

(1) *x is rational.* $y = \frac{x}{\sqrt{89}} \in (0, x) \subseteq \mathbb{R}$ is irrational.

(2) *x is irrational.* $y = \frac{x}{\sqrt{64}} \in (0, x) \subseteq \mathbb{R}$ is irrational.

$\square$