Chapter 2: Four Important Linear PDE

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Notes.

(1) (Equation (7))

$$|D\Phi(x)| \leq \frac{C}{|x|^{n-1}}, \qquad |D^2\Phi(x)| \leq \frac{C}{|x|^n} \qquad (x \neq 0)$$

for some constant $C > 0$. In fact,

$$\begin{aligned}
\frac{\partial}{\partial x_i}\Phi(x) &= -\frac{1}{n\alpha(n)} x_i |x|^{-n}, \\
\frac{\partial^2}{\partial x_i \partial x_j}\Phi(x) &= \frac{1}{n\alpha(n)}(n x_i x_j - |x|^2 \delta_{ij})|x|^{-n-2}.
\end{aligned}$$

Problem 2.1. *Write down an explicit formula for a function u solving the initial-value problem*

$$\begin{cases} u_t + b \cdot Du + cu = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Here $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ are constants.

Proof (Transport equation). Define

$$z(s) = u(x + sb, t + s) \qquad (s \in \mathbb{R}).$$

So

$$\begin{aligned}
\dot{z}(s) &= Du(x + sb, t + s) \cdot b + u_t(x + sb, t + s) \\
&= -cu(x + sb, t + s) \\
&= -cz(s).
\end{aligned}$$

Solve this ODE to get

$$\begin{aligned}
z(s) = z(0)e^{-cs} &\Longrightarrow u(x + sb, t + s) = u(x, t)e^{-cs} \\
&\Longrightarrow u(x - tb, 0) = u(x, t)e^{ct} \qquad\qquad (\text{Let } s = -t) \\
&\Longrightarrow g(x - tb) = u(x, t)e^{ct} \\
&\Longrightarrow u(x, t) = g(x - tb)e^{-ct}.
\end{aligned}$$

$\square$

Problem 2.2. *Prove that Laplace's equation $\Delta u = 0$ is rotation invariant; that is, if O is an orthogonal $n \times n$ matrix and we define*

$$v(x) := u(Ox) \qquad (x \in \mathbb{R}^n),$$

then $\Delta v = 0$.

Proof.

(1) Let $O = [O_{ij}]$. O is orthogonal if $OO^t = O^tO = I$, or

$$\sum_{i=1}^{n} O_{pi}O_{qi} = \delta_{pq}$$

where δ_{pq} is the Kronecker delta.

(2) Let $y = Ox$. So that

$$\begin{aligned}
D_i v(x) &= \sum_{p=1}^{n} D_p u(y) O_{pi}, \\
D_{ij} v(x) &= \sum_{q=1}^{n} \sum_{p=1}^{n} D_{pq} u(y) O_{pi} O_{qj}, \\
\Delta v(x) &= \sum_{i=1}^{n} D_{ii} v(x) \\
&= \sum_{i=1}^{n} \sum_{q=1}^{n} \sum_{p=1}^{n} D_{pq} u(y) O_{pi} O_{qi} \\
&= \sum_{q=1}^{n} \sum_{p=1}^{n} D_{pq} u(y) \left(\sum_{i=1}^{n} O_{pi} O_{qi} \right) \\
&= \sum_{q=1}^{n} \sum_{p=1}^{n} D_{pq} \delta_{pq} \\
&= \sum_{q=1}^{n} D_{qq} u(y) \\
&= \Delta u(y).
\end{aligned}$$

(3) As $\Delta u(y) = 0$, $\Delta v(x) = 0$.

$\square$

Problem 2.3. *Modify the proof of the mean value formulas to show for $n \geq 3$ that*

$$u(0) = \fint_{\partial B(0,r)} g dS + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f dx,$$

provided

$$\begin{cases} -\Delta u = f & \text{in } B^0(0,r) \\ u = g & \text{on } \partial B(0,r). \end{cases}$$

Proof.

(1) ...

(2) ...

□

Problem 2.4. ...

Proof.

(1) ...

(2) ...

□

Problem 2.5. ...

Proof.

(1) ...

(2) ...

□

Problem 2.6. ...

Proof.

(1) ...

(2) ...

$\square$

Problem 2.7. ...

Proof.

(1) ...

(2) ...

$\square$

Problem 2.8. ...

Proof.

(1) ...

(2) ...

$\square$

Problem 2.9. ...

Proof.

(1) ...

(2) ...

$\square$

Problem 2.10. ...

Proof.

(1) ...

(2) ...

$\square$

Problem 2.11. ...

Proof.

(1) ...

(2) ...

□

Problem 2.12. *...*

Proof.

(1) ...

(2) ...

□

Problem 2.13. *...*

Proof.

(1) ...

(2) ...

□

Problem 2.14. *...*

Proof.

(1) ...

(2) ...

□

Problem 2.15. *...*

Proof.

(1) ...

(2) ...

□

Problem 2.16. *...*

Proof.

(1) ...

(2) ...

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Problem 2.17. *...*

Proof.

(1) ...

(2) ...

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Problem 2.18. *...*

Proof.

(1) ...

(2) ...

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