

# Solutions to the book: *Fulton, Algebraic Curves*

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# Chapter 1: Affine Algebraic Sets

## 1.1. Algebraic Preliminaries

### Problem 1.1.\*

Let  $R$  be a domain.

- (a) If  $f, g$  are forms of degree  $r, s$  respectively in  $R[x_1, \dots, x_n]$ , show that  $fg$  is a form of degree  $r + s$ .
- (b) Show that any factor of a form in  $R[x_1, \dots, x_n]$  is also a form.

*Proof of (a).*

- (1) Write

$$f = \sum_{(i)} a_{(i)} x^{(i)},$$

$$g = \sum_{(j)} b_{(j)} x^{(j)},$$

where  $\sum_{(i)}$  is the summation over  $(i) = (i_1, \dots, i_n)$  with  $i_1 + \dots + i_n = r$  and  $\sum_{(j)}$  is the summation over  $(j) = (j_1, \dots, j_n)$  with  $j_1 + \dots + j_n = s$ .

- (2) Hence,

$$fg = \sum_{(i)} \sum_{(j)} a_{(i)} b_{(j)} x^{(i)} x^{(j)}$$

$$= \sum_{(i),(j)} a_{(i)} b_{(j)} x^{(k)}$$

where  $(k) = (i_1 + j_1, \dots, i_n + j_n)$  with  $(i_1 + j_1) + \dots + (i_n + j_n) = r + s$ . Each  $x^{(k)}$  is the form of degree  $r + s$  and  $a_{(i)} b_{(j)} \in R$ . Hence  $fg$  is a form of degree  $r + s$ .

□

*Proof of (b).*

- (1) Given any form  $f \in R[x_1, \dots, x_n]$ , and write  $f = gh$ . It suffices to show that  $g$  is a form as well. (So does  $h$ .)
- (2) Write

$$g = g_0 + \dots + g_r, \quad h = h_0 + \dots + h_s$$



where  $g_r \neq 0$  and  $h_s \neq 0$ . So

$$f = gh = g_0h_0 + \cdots + g_rh_s.$$

Since  $R$  is a domain,  $R[x_1, \dots, x_n]$  is a domain and thus  $g_rh_s \neq 0$ . The maximality of  $r$  and  $s$  implies that  $\deg f = r + s$ . Therefore, by the maximality of  $r + s$ ,  $f = g_rh_s$ , or  $g = g_r$ , or  $g$  is a form.

□

**Problem 1.2.\***

Let  $R$  be a UFD,  $K$  the quotient field of  $R$ . Show that every element  $z$  of  $K$  may be written  $z = a/b$ , where  $a, b \in R$  have no common factors; this representative is unique up to units of  $R$ .

*Proof.*

- (1) Show that every element  $z$  of  $K$  may be written  $z = a/b$ , where  $a, b \in R$  have no common factors. Given any  $z = a/b \in K$  where  $a, b \in R$ . Write

$$\begin{aligned} a &= p_1 \cdots p_n, \\ b &= q_1 \cdots q_m \end{aligned}$$

where all  $p_1, \dots, p_n, q_1, \dots, q_m$  are irreducible in  $R$ . (It is possible since  $R$  is a UFD.) For each  $i$ , suppose  $p_i \mid q_j$  for some  $i, j$ . Write  $q_j = p_i u$  for some  $u \in R$ . By the irreducibility of  $p_i$  and  $q_j$ ,  $u$  is a unit. So

$$z = \frac{a}{b} = \frac{p_1 \cdots \widehat{p_i} \cdots p_n}{q_1 \cdots \widehat{q_j} \cdots q_m} = \frac{p_1 \cdots \widehat{p_i} \cdots p_n}{u q_1 \cdots \widehat{q_j} \cdots q_m}.$$

Continue this method we can write  $z = \frac{a'}{b'}$  where  $a'$  and  $b'$  have no common factors.

- (2) Write  $z = a/b = a'/b'$  where

- (a)  $a, b, a', b' \in R$ ,
- (b)  $a$  and  $b$  have no common factors,
- (c)  $a'$  and  $b'$  have no common factors.

Write

$$\begin{aligned} a &= p_1 \cdots p_n, \\ b &= q_1 \cdots q_m, \\ a' &= p'_1 \cdots p'_{n'}, \\ b' &= q'_1 \cdots q'_{m'} \end{aligned}$$

where all  $p_i, q_j, p'_{i'}, q'_{j'}$  are irreducible in  $R$ . As  $z = a/b = a'/b'$ ,  $ab' = a'b$  or

$$p_1 \cdots p_n q'_1 \cdots q'_{m'} = p'_1 \cdots p'_{n'} q_1 \cdots q_m.$$

- (3) For  $i = 1$ ,  $p_1 = u_1 p'_{i'}$  for some unit  $u_1 \in R$  since  $a$  and  $b$  have no common factors and all  $p_1, q_j, p'_{i'}$  are irreducible. Hence

$$u_1 \widehat{p_1} p_2 \cdots p_n q'_1 \cdots q'_{m'} = p'_1 \cdots \widehat{p'_{i'}} \cdots p'_{n'} q_1 \cdots q_m.$$

Continue this method, we have  $n \leq n'$  and all  $p_1, \dots, p_n$  are canceled.

- (4) Conversely, we can apply the argument in (3) to  $i' = 1, \dots, n'$  to conclude that  $n' \leq n$ . Therefore,  $n = n'$  and

$$\underbrace{u_1 \cdots u_n}_{\text{a unit in } R} q'_1 \cdots q'_{m'} = q_1 \cdots q_m.$$

Hence,  $b = ub'$  where  $u = u_1 \cdots u_n$  is a unit in  $R$ . Similarly,  $a = va'$  where  $v$  is a unit in  $R$ . So the representative of  $z \in K$  is unique up to units of  $R$ .

□

### Problem 1.3.\*

Let  $R$  be a PID. Let  $\mathfrak{p}$  be a nonzero, proper, prime ideal in  $R$ .

- (a) Show that  $\mathfrak{p}$  is generated by an irreducible element.
- (b) Show that  $\mathfrak{p}$  is maximal.

*Proof of (a).*

- (1) Let  $\mathfrak{p} = (a)$  be a nonzero, proper, prime ideal in  $R$ . It suffices to show that  $a$  is irreducible.
- (2) Suppose  $a = bc$ . By the primality of  $\mathfrak{p}$ ,  $b \in \mathfrak{p}$  or  $c \in \mathfrak{p}$ . Suppose  $b \in \mathfrak{p} = (a)$ . (The case  $c \in \mathfrak{p}$  is similar.) Then there is a  $d \in R$  such that  $b = ad$ . Hence,  $a = bc = adc$  or  $(1 - dc)a = 0$ .
- (3) Since  $R$  is a domain,  $1 = dc$  or  $a = 0$ .  $a = 0$  implies that  $\mathfrak{p} = (0)$  is a zero ideal, contrary to the assumption. Therefore,  $1 = dc$ , or  $c$  is a unit, or  $a$  is irreducible.

□

*Proof of (b).*

- (1) Given any ideal  $I = (b)$  of  $R$  containing  $\mathfrak{p} = (a)$ . As the generator  $a$  of  $\mathfrak{p}$  is in  $\mathfrak{p} \subseteq I$ , there is some  $c \in R$  such that  $a = bc$ . By the irreducibility of  $a$  (in  $(a)$ ),  $b$  is a unit or  $c$  is a unit.
- (2)  $b$  is a unit implies that  $I = R$ .  $c$  is a unit implies that  $I = \mathfrak{p}$ . In any case, we conclude that  $\mathfrak{p}$  is maximal.

□

**Problem 1.4.\***

Let  $k$  be an infinite field,  $f \in k[x_1, \dots, x_n]$ . Suppose  $f(a_1, \dots, a_n) = 0$  for all  $a_1, \dots, a_n \in k$ . Show that  $f = 0$ . (Hint: Write

$$f = \sum f_i x_n^i, \quad f_i \in k[x_1, \dots, x_{n-1}].$$

Use induction on  $n$ , and the fact that  $f(a_1, \dots, a_{n-1}, x_n)$  has only a finite number of roots if any  $f_i(a_1, \dots, a_{n-1}) \neq 0$ .)

*Proof.*

- (1) Induction on  $n$ . The case  $n = 1$ . (Reductio ad absurdum) If there were a nonzero  $f \in k[x_1]$  such that  $f(a) = 0$  for all  $a \in k$ . Note that  $f$  has at most  $\deg f < \infty$  roots, contrary to the infinity of  $k$ .
- (2) Assume that the conclusion holds for  $n - 1$ , then for any  $f \in k[x_1, \dots, x_n]$  we can write

$$f = \sum f_i x_n^i, \quad f_i \in k[x_1, \dots, x_{n-1}]$$

as  $f \in (k[x_1, \dots, x_{n-1}])[x_n]$ . Suppose  $f(a_1, \dots, a_n) = 0$  for all  $a_1, \dots, a_n \in k$ . For fixed  $a_1, \dots, a_{n-1}$ , the polynomial  $f(a_1, \dots, a_{n-1}, x_n) \in k[x_n]$  has all distinct roots in an infinite field  $k$ . By (1),  $f(a_1, \dots, a_{n-1}, x_n) = 0 \in k[x_n]$ , or each  $f_i(a_1, \dots, a_{n-1}) = 0$ . As all  $a_1, \dots, a_{n-1}$  run over  $k$ , we can apply the induction hypothesis each  $f_i(x_1, \dots, x_{n-1}) = 0 \in k[x_1, \dots, x_{n-1}]$ . Hence,  $f = 0 \in k[x_1, \dots, x_n]$ .

□

*Note.* If  $k$  is a finite field of order  $q = p^k$ , then the polynomial  $f(x) = x^q - x$  has  $q$  distinct roots in  $k$ .

**Problem 1.5.\***

Let  $k$  be any field. Show that there are an infinitely number of irreducible monic polynomials in  $k[x]$ . (Hint: Suppose  $f_1, \dots, f_n$  were all of them, and factor  $f_1 \cdots f_n + 1$  into irreducible factors.)

*Proof (Due to Euclid).*

- (1) If  $f_1, \dots, f_n$  were all irreducible monic polynomials, then we consider

$$g = f_1 \cdots f_n + 1 \in k[x].$$

So there is an irreducible monic polynomial  $f = f_i$  dividing  $g$  for some  $i$  since

$$\deg g = \deg f_1 + \cdots + \deg f_n \geq 1$$

and  $k[x]$  is a UFD.

- (2) However,  $f$  would divide the difference

$$g - f_1 \cdots f_{i-1} f_{i+1} \cdots f_n = 1,$$

contrary to  $\deg f_i \geq 1$ .

□

**Problem 1.6.\***

Show that any algebraically closed field is infinite. (Hint: The irreducible monic polynomials are  $x - a$ ,  $a \in k$ .)

*Proof (Due to Euclid).*

- (1) Let  $k$  be an algebraically closed field. If  $a_1, \dots, a_n$  were all elements in  $k$ , then we consider a monic polynomials

$$f(x) = (x - a_1) \cdots (x - a_n) + 1 \in k[x].$$

- (2) Since  $k$  is algebraically closed, there is an element  $a \in k$  such that  $f(a) = 0$ . By assumption,  $a = a_i$  for some  $1 \leq i \leq n$ , and thus  $f(a) = f(a_i) = 1$ , contrary to the fact that a field is a commutative ring where  $0 \neq 1$  and all nonzero elements are invertible.

□

**Problem 1.7.\***

Let  $k$  be a field,  $f \in k[x_1, \dots, x_n]$ ,  $a_1, \dots, a_n \in k$ .

(a) Show that

$$f = \sum \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}, \quad \lambda_{(i)} \in k.$$

(b) If  $f(a_1, \dots, a_n) = 0$ , show that  $f = \sum_{i=1}^n (x_i - a_i)g_i$  for some (not unique)  $g_i$  in  $k[x_1, \dots, x_n]$ .

*Proof of (a).*

(1) Regard  $k[x_1, \dots, x_n]$  as  $(k[x_1, \dots, x_{n-1}])[x_n]$ . Since  $(k[x_1, \dots, x_{n-1}])[x_n]$  is a Euclidean domain with a function

$$f \in (k[x_1, \dots, x_{n-1}])[x_n] \mapsto \deg_{x_n} f \in \mathbb{Z}_{\geq 0}$$

satisfying the division-with-remainder property.

(2) Apply the division algorithm for  $f$  and nonzero  $x_n - a_n$  to produce a quotient  $q$  and remainder  $r$  with  $f = (x_n - a_n)q + r$  and either  $r = 0$  or  $\deg_{x_n}(r) < \deg_{x_n}(x_n - a_n) = 1$ . That is,  $r \in k[x_1, \dots, x_{n-1}]$  is a constant in  $(k[x_1, \dots, x_{n-1}])[x_n]$ . Continue this process to get that  $f$  is of the form

$$f = \sum_{i_n} f_{i_n} (x_n - a_n)^{i_n}$$

where  $f_{i_n} \in k[x_1, \dots, x_{n-1}]$ .

(3) Use the same argument in (2) for each  $f_{i_n} \in k[x_1, \dots, x_{n-1}]$ , we have

$$\begin{aligned} f_{i_n} &= \sum_{i_{n-1} \in k[x_1, \dots, x_{n-2}]} \underbrace{f_{i_n, i_{n-1}}}_{\in k[x_1, \dots, x_{n-2}]} (x_{n-1} - a_{n-1})^{i_{n-1}} \\ f_{i_n, i_{n-1}} &= \sum_{i_{n-2} \in k[x_1, \dots, x_{n-3}]} \underbrace{f_{i_n, i_{n-1}, i_{n-2}}}_{\in k[x_1, \dots, x_{n-3}]} (x_{n-2} - a_{n-2})^{i_{n-2}}, \\ &\dots \\ f_{i_n, \dots, i_2} &= \sum_{i_1 \in k} \underbrace{f_{i_n, \dots, i_1}}_{\in k} (x_1 - a_1)^{i_1}. \end{aligned}$$

Note that  $f_{i_n, \dots, i_1} \in k$ , we can write

$$f = \sum \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}, \quad \lambda_{(i)} \in k.$$

by replacing all  $f_{i_n, \dots, i_k}$  by  $f_{i_n, \dots, i_{k-1}}$  for  $k = n, n-1, \dots, 2$ .

(4) Or use the induction on  $n$ .

□

*Proof of (b).*

(1) Write

$$f = \sum \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}, \quad \lambda_{(i)} \in k$$

by (a).

(2) As  $f(a_1, \dots, a_n) = 0$ ,  $\lambda_{(i)} = 0$  if all  $i_1, \dots, i_n$  are zero, that is, there is no nonzero constant term in the representation of  $f$ . Hence, for each term

$$f_{(i)} := \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}$$

with  $\lambda_{(i)} \neq 0$ , there exists one  $i_k > 0$  for some  $1 \leq k \leq n$ . So we can write

$$f_{(i)} = (x_k - a_k) \underbrace{(\lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_k - a_k)^{i_k-1} \cdots (x_n - a_n)^{i_n})}_{:= g_{(i)} \in k[x_1, \dots, x_n]}.$$

Note that the expression of  $f_{(i)}$  is not unique since there may exist more than one  $i_k > 0$  as  $1 \leq k \leq n$ .

(3) Now we iterate each nonzero term in  $f$ , apply the factorization in (2), and then group by each  $x_k - a_k$ . Therefore, we can write

$$f = \sum_{i=1}^n (x_i - a_i) g_i$$

for some  $g_1 \in k[x_1, \dots, x_n]$ .

(4) The expression of  $f$  is not unique. For example, take  $f(x, y) = x^2 + 2xy + y^2 \in k[x, y]$ . As  $f(0, 0) = 0$ , we can write

$$\begin{aligned} f(x, y) &= x \cdot \underbrace{(x + 2y)}_{g_1} + y \cdot \underbrace{y}_{g_2}, \text{ or} \\ &= x \cdot \underbrace{(x + y)}_{g_1} + y \cdot \underbrace{(x + y)}_{g_2}, \text{ or} \\ &= x \cdot \underbrace{x}_{g_1} + y \cdot \underbrace{(2x + y)}_{g_2}. \end{aligned}$$

□

## 1.2. Affine Space and Algebraic Sets

### Problem 1.8.\*

Show that the algebraic subsets of  $\mathbf{A}^1(k)$  are just the finite subsets, together with  $\mathbf{A}^1(k)$  itself.

*Proof.*

(1) Show that  $k[x]$  is a PID if  $k$  is a field.

- (a) Let  $I$  be an ideal of  $k[x]$ .
- (b) If  $I = \{0\}$  then  $I = (0)$  and  $I$  is principal.
- (c) If  $I \neq \{0\}$ , then take  $f$  to be a polynomial of minimal degree in  $I$ . It suffices to show that  $I = (f)$ . Clearly,  $(f) \subseteq I$  since  $I$  is an ideal. Conversely, for any  $g \in I$ ,

$$g(x) = f(x)h(x) + r(x)$$

for some  $h, r \in k[x]$  with  $r = 0$  or  $\deg r < \deg f$  (as  $k[x]$  is a Euclidean domain). Now as

$$r = g - fh \in I,$$

$r = 0$  (otherwise contrary to the minimality of  $f$ ), we have  $g = fh \in (f)$  for all  $g \in I$ .

(2) Let  $Y$  be an algebraic subset of  $\mathbf{A}^1(k)$ , say  $Y = V(I)$  for some ideal  $I$  of  $k[x]$ . Since  $k[x]$  is a PID,  $I = (f)$  for some  $f \in k[x]$ .

- (a) If  $f = 0$ , then  $I = (0)$  and  $Y = V(0) = \mathbf{A}^1(k)$ .
- (b) If  $f \neq 0$ , then  $f(x) = 0$  has finitely many roots in  $k$ , say  $a_1, \dots, a_m \in k$ . Hence,

$$Y = V(I) = V(f) = \{f(a) = 0 : a \in k\} = \{a_1, \dots, a_m\}$$

is a finite subsets of  $\mathbf{A}^1(k)$ .

By (a)(b), the result is established.

□

*Notes.*

- (1) By the Hilbert basis theorem,  $k[x]$  is Noetherian as  $k$  is Noetherian. Hence, for any algebraic subset  $Y = V(I)$  of  $\mathbf{A}^1(k)$ , we can write  $I = (f_1, \dots, f_m)$ . Note that

$$Y = V(I) = V(f_1) \cap \dots \cap V(f_m).$$

Now apply the same argument to get the same conclusion.

- (2) Suppose  $k = \bar{k}$ .  $\mathbf{A}^1(k)$  is irreducible, because its only proper closed subsets are finite, yet it is infinite (because  $k$  is algebraically closed, hence infinite).

**Problem 1.9.**

If  $k$  is a finite field, show that every subset of  $\mathbf{A}^n(k)$  is algebraic.

*Proof.*

- (1) Every subset of  $\mathbf{A}^n(k)$  is finite since  $|\mathbf{A}^n(k)| = |k|^n$  is finite.
- (2) Note that  $V(x_1 - a_1, \dots, x_n - a_n) = \{(a_1, \dots, a_n)\} \subseteq \mathbf{A}^n(k)$  (Property (5) in §1.2) and any finite union of algebraic sets is algebraic (Property (4) in §1.2). Thus, every subset of  $\mathbf{A}^n(k)$  is algebraic (by (1)).

□

**Problem 1.10.**

Give an example of a countable collection of algebraic sets whose union is not algebraic.

*Proof.*

- (1) Let  $k = \mathbb{Q}$  be an infinite field.  $V(x - a) = \{a\}$  is an algebraic sets for all  $a \in \mathbb{Q}$ . In particular,  $V(x - a) = \{a\}$  is algebraic for all  $a \in \mathbb{Z}$ .
- (2) Note that

$$Y := \bigcup_{a \in \mathbb{Z}} V(x - a) = \mathbb{Z}$$

is a countable union of algebraic sets. Since  $Y$  is a proper subset of  $k = \mathbb{Q}$ , it cannot be algebraic by Problem 1.8.

□

**Problem 1.11.**

Show that the following are algebraic sets:

- (a)  $\{(t, t^2, t^3) \in \mathbf{A}^3(k) : t \in k\}$ ;
- (b)  $\{(\cos(t), \sin(t)) \in \mathbf{A}^2(\mathbb{R}) : t \in \mathbb{R}\}$ ;
- (c) the set of points in  $\mathbf{A}^2(\mathbb{R})$  whose polar coordinates  $(r, \theta)$  satisfy the equation  $r = \sin(\theta)$ .

*Proof of (a).*



- (1) The twisted cubic curve

$$Y = \{(t, t^2, t^3) \in \mathbf{A}^3(k) : t \in k\} = V(x^2 - y) \cap V(x^3 - z)$$

is algebraic. We say that  $Y$  is given by the parametric representation  $x = t, y = t^2, z = t^3$ .

- (2) The generators for the ideal  $I(Y)$  are  $x^2 - y$  and  $x^3 - z$ .  
 (3)  $Y$  is an affine variety of dimension 1.  
 (4) The affine coordinate ring  $A(Y)$  is isomorphic to a polynomial ring in one variable over  $k$ .

□

*Proof of (b).* The circle

$$\{(\cos(t), \sin(t)) \in \mathbf{A}^2(\mathbb{R}) : t \in \mathbb{R}\} = V(x^2 + y^2 - 1)$$

is algebraic. □

*Proof of (c).* The circle

$$\{(r, \theta) : r = \sin(\theta)\} = V(x^2 + y^2 - y)$$

is algebraic again. □

### Problem 1.12.

Suppose  $C$  is an affine plane curve, and  $L$  is a line in  $\mathbf{A}^2(k)$ ,  $L \not\subseteq C$ . Suppose  $C = V(f)$ ,  $f \in k[x, y]$  a polynomial of degree  $n$ . Show that  $L \cap C$  is a finite set of no more than  $n$  points. (Hint: Suppose  $L = V(y - (ax + b))$ , and consider  $f(x, ax + b) \in k[x]$ .)

*Proof.*

- (1) Say  $L = V(y - (ax + b))$  be a line in  $\mathbf{A}^2(k)$ . (The case  $L = V(x - (ay + b))$  is similar.)  
 (2) Note that  $L \not\subseteq C$  implies that  $(y - (ax + b)) \nmid f$ . Hence, the polynomial

$$g : x \mapsto f(x, ax + b) \in k[x]$$

is nonzero and  $\deg g \leq n$ . Therefore, the number of roots of  $g$  in  $k$  is no more than  $n$ .

(3) Hence,

$$\begin{aligned}
L \cap C &= V(y - (ax + b)) \cap V(f) \\
&= \{(x, y) \in \mathbf{A}^2(k) : y = ax + b \text{ and } f(x, y) = 0\} \\
&= \{(x, y) \in \mathbf{A}^2(k) : f(x, ax + b) = 0\}
\end{aligned}$$

is finite of no more than  $n$  points.

□

**Problem 1.13.**

Show that each of the following sets is not algebraic:

- (a)  $\{(x, y) \in \mathbf{A}^2(\mathbb{R}) : y = \sin(x)\}$ .
- (b)  $\{(z, w) \in \mathbf{A}^2(\mathbb{C}) : |z|^2 + |w|^2 = 1\}$ , where  $|x + iy|^2 = x^2 + y^2$  for  $x, y \in \mathbb{R}$ .
- (c)  $\{(\cos(t), \sin(t), t) \in \mathbf{A}^3(\mathbb{R}) : t \in \mathbb{R}\}$ .

*Proof of (a).*

- (1) (Reductio ad absurdum) If

$$Y := \{(x, y) \in \mathbf{A}^2(\mathbb{R}) : y = \sin(x)\}$$

were algebraic, then there is a subset  $S$  of  $\mathbb{R}[x, y]$  such that

$$Y = V(S) = \bigcap_{f \in S} V(f).$$

- (2)  $S \neq \emptyset$  since  $Y \neq \mathbf{A}^2(\mathbb{R})$ .  $((89, 64) \in \mathbf{A}^2(\mathbb{R}) - Y)$ .
- (3) Take a fixed line  $L = V(y)$  in  $\mathbf{A}^2(\mathbb{R})$ . For each affine curve  $f \in S$ , we have

$$V(f) \cap L \supseteq \bigcap_{f \in S} V(f) \cap L = Y \cap L = \{(n\pi, 0) \in \mathbf{A}^2(\mathbb{R}) : n \in \mathbb{Z}\},$$

which is infinite. By problem 1.12,  $y \mid f$ . As  $f$  runs over  $S$ ,  $Y \subseteq V(y) = L$ , contradicts that  $(0, \frac{\pi}{2}) \in L - Y$ .

□

*Proof of (b).*

- (1) Similar to (a). (Reductio ad absurdum) If

$$Y := \{(x, y) \in \mathbf{A}^2(\mathbb{C}) : |x|^2 + |y|^2 = 1\}$$

were algebraic, then there is a subset  $S$  of  $\mathbb{C}[x, y]$  such that

$$Y = V(S) = \bigcap_{f \in S} V(f).$$

- (2)  $S \neq \emptyset$  since  $Y \neq \mathbf{A}^2(\mathbb{C})$ .  $((89, 64) \in \mathbf{A}^2(\mathbb{C}) - Y)$   
(3) Take a fixed line  $L = V(x)$  in  $\mathbf{A}^2(\mathbb{C})$ . For each affine curve  $f \in S$ , we have

$$V(f) \cap L \supseteq \bigcap_{f \in S} V(f) \cap L = Y \cap L = \{(0, y) \in \mathbf{A}^2(\mathbb{C}) : |y| = 1\},$$

which is infinite (since  $Y$  contains a unit circle in the complex plane). By problem 1.12,  $x \mid f$ . As  $f$  runs over  $S$ ,  $Y \subseteq V(x) = L$ , contradicts that the origin  $(0, 0) \in L - Y$ .

□

*Proof of (c).*

- (1) Similar to (a) and (b).  
(2) Suppose  $C$  is an affine plane curve, and  $L$  is a line in  $\mathbf{A}^3(k)$ ,  $L \not\subseteq C$ . Suppose  $C = V(f)$ ,  $f \in k[x, y, z]$  a polynomial of degree  $n$ . Show that  $L \cap C$  is a finite set of no more than  $n$  points. The proof is similar to Problem 1.12.  
(a) Say  $L = V(y - (ax + b), z - (cx + d))$  be a line in  $\mathbf{A}^3(k)$ .  
(b) Note that  $L \not\subseteq C$  implies that  $(y - (ax + b)) \nmid f$  and  $(z - (cx + d)) \nmid f$ . Hence, the polynomial

$$g : x \mapsto f(x, ax + b, cx + d) \in k[x]$$

is nonzero and  $\deg g \leq n$ . Therefore, the number of roots of  $g$  in  $k$  is no more than  $n$ .

- (c) Hence,

$$\begin{aligned} L \cap C &= V(y - (ax + b), z - (cx + d)) \cap V(f) \\ &= \{(x, y) \in \mathbf{A}^2(k) : y = ax + b, z = cx + d \text{ and } f(x, y) = 0\} \\ &= \{(x, y) \in \mathbf{A}^2(k) : f(x, ax + b, cx + d) = 0\} \end{aligned}$$

is finite of no more than  $n$  points.

(3) (Reductio ad absurdum) If

$$Y := \{(\cos(t), \sin(t), t) \in \mathbf{A}^3(\mathbb{R}) : t \in \mathbb{R}\}$$

were algebraic, then there is a subset  $S$  of  $\mathbb{R}[x, y, z]$  such that

$$Y = V(S) = \bigcap_{f \in S} V(f).$$

(4)  $S \neq \emptyset$  since  $Y \neq \mathbf{A}^3(\mathbb{R})$ . ((1989, 6, 4)  $\in \mathbf{A}^3(\mathbb{R}) - Y$ .)

(5) Take a fixed line  $L = V(x - 1, y)$  in  $\mathbf{A}^3(\mathbb{R})$ . For each affine curve  $f \in S$ , we have

$$V(f) \cap L \supseteq \bigcap_{f \in S} V(f) \cap L = Y \cap L = \{(1, 0, 2n\pi) \in \mathbf{A}^3(\mathbb{R}) : n \in \mathbb{Z}\},$$

which is infinite. By (2),  $(x - 1) \mid f$  and  $y \mid f$ . As  $f$  runs over  $S$ ,  $Y \subseteq V(x - 1, y) = L$ , contradicts that  $(1, 0, \pi) \in L - Y$ .

□

**Supplement.** A circular disk of radius 1 in the plane  $xy$  rolls without slipping along the  $x$  axis. The figure described by a point of the circumference of the disk is called a **cycloid**. The parametrized curve  $\alpha : \mathbb{R} \rightarrow \mathbb{R}^2$  is

$$\begin{cases} x = t - \sin t \\ y = 1 - \cos t. \end{cases}$$

The cycloid is not algebraic (as (a)).

#### Problem 1.14.\*

Let  $f$  be a nonconstant polynomial in  $k[x_1, \dots, x_n]$ ,  $k$  algebraically closed. Show that  $\mathbf{A}^n(k) - V(f)$  is infinite if  $n \geq 1$ , and  $V(f)$  is infinite if  $n \geq 2$ . Conclude that the complement of any proper algebraic set is infinite. (Hint: See Problem 1.4.)

*Proof.*

(1) Show that  $\mathbf{A}^n(k) - V(f)$  is infinite if  $n \geq 1$ . Since  $f$  is a nonconstant polynomial in  $k[x_1, \dots, x_n]$ , we may assume that  $\deg_{x_n}(f) > 0$ . Hence

$$x_n \mapsto f(1, \dots, 1, x_n)$$

is a nonconstant polynomial of degree  $\deg_{x_n}(f) > 0$  in  $k[x_n]$ . So  $f$  has finitely many roots in  $k$ , say  $\xi_1, \dots, \xi_m$  ( $m \geq 0$ ). Hence,

$$(1, \dots, 1, x_n) \neq 0$$

whenever  $x_n \neq \xi_m$ . Such subset in  $\mathbf{A}^1(k)$  is infinite since  $k = \bar{k}$  (Problem 1.6). Therefore,

$$\begin{aligned}\mathbf{A}^n(k) - V(f) &= \{(a_1, \dots, a_n) \in \mathbf{A}^n(k) : f(a_1, \dots, a_n) \neq 0\} \\ &\supseteq \{a_n \in \mathbf{A}^1(k) : f(1, \dots, 1, x_n) \neq 0\}\end{aligned}$$

is infinite.

(2) Show that  $V(f)$  is infinite if  $n \geq 2$ .

(a) Similar to (1). Since  $f$  is a nonconstant polynomial in  $k[x_1, \dots, x_n]$ , we may assume that  $m := \deg_{x_n}(f) > 0$ . Write

$$f = \sum_{i=0}^m f_i(x_1, \dots, x_{n-1})x_n^i.$$

Note that each  $f_i$  is well-defined since  $n \geq 2$ .

(b) If  $f_n$  is constant in  $k[x_1, \dots, x_{n-1}]$ , then  $f_n$  is nonzero (since  $m > 0$ ) or  $V(f_n) = \emptyset$ . If  $f_n$  is nonconstant in  $k[x_1, \dots, x_{n-1}]$ , then the set  $\mathbf{A}^{n-1}(k) - V(f_n)$  is infinite by (1). In any case,

$$\mathbf{A}^{n-1}(k) - V(f_n)$$

is infinite.

(c) For each  $P = (a_1, \dots, a_{n-1}) \in \mathbf{A}^{n-1}(k) - V(f_n)$ ,

$$g_P : x_n \mapsto f(P, x_n) = f(a_1, \dots, a_{n-1}, x_n)$$

defines a polynomial in  $k[x_n]$  of degree  $m > 0$ . Since  $k = \bar{k}$ ,  $g_P$  has at least one root  $Q \in k$ . Hence

$$V(f) \supseteq \{(P, Q) \in \mathbf{A}^n(k) : P \in \mathbf{A}^{n-1}(k) - V(f_n), g_P(Q) = 0\}$$

is infinite since the set  $\mathbf{A}^{n-1}(k) - V(f_n)$  is infinite.

*Note.* It is not true if  $k \neq \bar{k}$ . For example,  $V(x^2 + y^2 + 1) = \emptyset$  in  $\mathbf{A}^2(\mathbb{R})$ .

(3) Note that

$$\mathbf{A}^n(k) - V(S) = \mathbf{A}^n(k) - \bigcap_{f \in S} V(f) = \bigcup_{f \in S} (\mathbf{A}^n(k) - V(f)).$$

Thus the complement of any proper algebraic set is infinite by (1).

□

**Problem 1.15.\***

Let  $V \subseteq \mathbf{A}^n(k)$ ,  $W \subseteq \mathbf{A}^m(k)$  be algebraic sets. Show that

$$V \times W = \{(a_1, \dots, a_n, b_1, \dots, b_m) : (a_1, \dots, a_n) \in V, (b_1, \dots, b_m) \in W\}$$

is an algebraic set in  $\mathbf{A}^{n+m}(k)$ . It is called the **product** of  $V$  and  $W$ .

*Proof.*

(1) Write

$$\begin{aligned} V &= V(S_V) = \{P \in \mathbf{A}^n(k) : f(P) = 0 \forall f \in S_V\} \\ W &= V(S_W) = \{Q \in \mathbf{A}^m(k) : g(Q) = 0 \forall g \in S_W\}, \end{aligned}$$

where  $S_V \subseteq k[x_1, \dots, x_n]$  and  $S_W \subseteq k[y_1, \dots, y_m]$ . It suffices to show that

$$V \times W = V(S),$$

where  $S \subseteq k[x_1, \dots, x_n, y_1, \dots, y_m]$  is the union of  $S_V$  and  $S_W$ .

(2) Here we can identify  $S_V$  with the subset of  $k[x_1, \dots, x_n, y_1, \dots, y_m]$  by noting that

$$k[x_1, \dots, x_n] \hookrightarrow (k[y_1, \dots, y_m])[x_1, \dots, x_n] = k[x_1, \dots, x_n, y_1, \dots, y_m].$$

Here we regard  $k$  as a subring of  $k[y_1, \dots, y_m]$ . Similar treatment to  $S_W$ .

(3) By construction,  $V \times W \subseteq V(S)$ . Conversely, given any  $(P, Q) \in V(S) \subseteq \mathbf{A}^{n+m}(k)$ , we have  $h(P, Q) = 0$  for all  $h \in S = S_V \cup S_W$  (by (2)). By construction,  $f(P) = 0$  for all  $f \in S_V$  since  $f$  only involve  $x_1, \dots, x_n$ . Hence,  $P \in V$ . Similarly,  $Q \in W$ . Therefore,  $(P, Q) \in V \times W$ .

□

### 1.3. The Ideal of a Set of Points

**Problem 1.16.\***

Let  $V, W$  be algebraic sets in  $\mathbf{A}^n(k)$ . Show that  $V = W$  if and only if  $I(V) = I(W)$ .

*Proof.*

(1) (Proof of Property (6) in §1.3.) Show that if  $X \subseteq Y$ , then  $I(X) \supseteq I(Y)$ . If  $f \in I(Y)$  then  $f(P) = 0$  for all  $P \in Y$ . So  $f(P) = 0$  for all  $P \in X \subseteq Y$  or  $f \in I(X)$ .

- (2) (Proof of Property (8) in §1.3.)  $I(V(S)) \supseteq S$  for any set  $S$  of polynomials;  $V(I(X)) \supseteq X$  for any set  $X$  of points.
- (a) If  $f \in S$  then  $f$  vanishes on  $V(S)$ , hence  $f \in IV(S)$ .
  - (b) If  $P \in X$  then every polynomial in  $I(X)$  vanishes at  $P$ , so  $P$  belongs to the zero set of  $I(X)$ .
- (3) (Proof of Property (9) in §1.3.)  $V(I(V(S))) = V(S)$  for any set  $S$  of polynomials, and  $I(V(I(X))) = I(X)$  for any set  $X$  of points. So if  $V$  is an algebraic set,  $V = V(I(V))$ , and if  $I$  is the ideal of an algebraic set,  $I = I(V(I))$ .
- (a) In each case, it suffices to show that the left side is a subset of the right side. (by Properties (6)(8) in §1.3).
  - (b) If  $P \in V(S)$  then  $f(P) = 0$  for all  $f \in I(V(S))$ , so  $P \in V(I(V(S)))$ .
  - (c) If  $f \in I(X)$  then  $f(P) = 0$  for all  $P \in V(I(X))$ . Thus  $f$  vanishes on  $V(I(X))$ , so  $f \in I(V(I(X)))$ .
- (4) Show that  $V = W$  if and only if  $I(V) = I(W)$ .
- (a) By Property (6) in §1.3,  $I(V) \supseteq I(W)$  if  $V \subseteq W$  and  $I(V) \subseteq I(W)$  if  $V \supseteq W$ . Thus,  $I(V) = I(W)$  if  $V = W$ .
  - (b) Conversely,  $I(V) = I(W)$  implies that  $V(I(V)) = V(I(W))$  by Property (3) in §1.2 and similar argument in (a). By Property (9) in §1.3,  $V(I(V)) = V$  and  $V(I(W)) = W$ . Thus,  $V = W$ .

□

**Problem 1.17.\***

- (a) Let  $V$  be an algebraic set in  $\mathbf{A}^n(k)$ ,  $P \in \mathbf{A}^n(k)$  a point not in  $V$ . Show that there is a polynomial  $f \in k[x_1, \dots, x_n]$  such that  $f(Q) = 0$  for all  $Q \in V$ , but  $f(P) = 1$ . (Hint:  $I(V) \neq I(V \cup \{P\})$ .)
- (b) Let  $P_1, \dots, P_r$  be distinct points in  $\mathbf{A}^n(k)$ , not in an algebraic set  $V$ . Show that there are polynomials  $f_1, \dots, f_r \in I(V)$  such that  $f_i(P_j) = 0$  if  $i \neq j$ , and  $f_i(P_i) = 1$ . (Hint: Apply (a) to the union of  $V$  and all but one point.)
- (c) With  $P_1, \dots, P_r$  and  $V$  as in (b), and  $a_{ij} \in k$  for  $1 \leq i, j \leq r$ , show that there are  $g_i \in I(V)$  with  $g_i(P_j) = a_{ij}$  for all  $i$  and  $j$ . (Hint: Consider  $\sum_j a_{ij} f_j$ .)

*Proof of (a).*

- (1) Since  $I(V) \subsetneq I(V \cup \{P\})$  (by Problem 1.16), there is a polynomial  $f \in k[x_1, \dots, x_n]$  such that  $f(Q) = 0$  for all  $Q \in V$ , but  $f(P) \neq 0$ .

- (2) Since  $k$  is a field,  $(f(P))^{-1} \in k$ . Consider the polynomial  $(f(P))^{-1}f \in k[x_1, \dots, x_n]$ . It is well-defined. Also,  $((f(P))^{-1}f)(Q) = (f(P))^{-1}f(Q) = 0$  for all  $Q \in V$ , but  $(f(P))^{-1}f(P) = (f(P))^{-1}f(P) = 1$ .

□

*Proof of (b).*

- (1) For  $1 \leq i \leq$ , define

$$W = V \cup \{P_1, \dots, P_r\}$$

$$W_i = V \cup \{P_1, \dots, \widehat{P_i}, \dots, P_r\}.$$

Here  $W = W_i \cup \{P_i\} \neq W_i$ .

- (2) By (a), there is a polynomial  $f_i \in k[x_1, \dots, x_n]$  such that  $f_i(Q) = 0$  for all  $Q \in W_i$ , but  $f_i(P_i) = 1$ . Here  $f_i \in I(V)$  and  $f_i(P_j) = \delta_{ij}$  where  $\delta_{ij}$  is the Kronecker delta.

□

*Proof of (c).*

- (1) For each  $1 \leq i \leq r$ , define

$$g_i = \sum_j a_{ij} f_j \in k[x_1, \dots, x_n].$$

- (2)  $g_i \in I(V)$  since  $g_i$  is a linear combination of  $f_j$  and  $I(V)$  is an ideal.

- (3) Also,

$$g_i(P_j) = \sum_{j'} a_{ij'} f_{j'}(P_j) = \sum_{j'} a_{ij'} \delta_{j'j} = a_{ij}.$$

□

### **Problem 1.18.\***

Let  $I$  be an ideal in a ring  $R$ . If  $a^n \in I$ ,  $b^m \in I$ , show that  $(a + b)^{n+m} \in I$ . Show that  $\text{rad}(I)$  is an ideal, in fact a radical ideal. Show that any prime ideal is radical.

*Proof.*



- (1) Show that  $(a + b)^{n+m} \in I$  if  $a^n \in I$ ,  $b^m \in I$ . By the binomial theorem,

$$(a + b)^{n+m} = \sum_{i=0}^{n+m} a^i b^{n+m-i}.$$

For each term  $a^i b^{n+m-i}$ , either  $i \geq n$  holds or  $n + m - i \geq m$  holds, and thus  $a^i b^{n+m-i} \in I$  (since  $a^n \in I$ ,  $b^m \in I$  and  $I$  is an ideal). Hence, the result is established.

- (2) Show that  $\text{rad}(I)$  is an ideal.

- (a)  $0 \in \text{rad}(I)$  since  $0 = 0^1 \in I$  for any ideal in  $R$ .
- (b)  $(a + b)^{n+m} \in I$  if  $a^n \in I$ ,  $b^m \in I$  by (1).
- (c)  $(-a)^{2n} = (a^n)^2 \in I$  if  $a^n \in I$  (since  $I$  is an ideal).
- (d)  $(ra)^n = r^n a^n \in I$  if  $a^n \in I$  and  $r \in R$  (since  $I$  is an ideal and  $R$  is commutative).

- (3) Show that  $\text{rad}(\text{rad}(I)) = \text{rad}(I)$ . It suffices to show  $\text{rad}(\text{rad}(I)) \subseteq \text{rad}(I)$ . Given any  $a \in \text{rad}(\text{rad}(I))$ . By definition  $a^n \in \text{rad}(I)$  for some positive integer  $n$ . Again by definition  $(a^n)^m = a^{nm} \in I$  for some positive integer  $m$ . As  $nm$  is a positive integer,  $a \in \text{rad}(I)$ .

- (4) Show that every prime ideal  $\mathfrak{p}$  is radical. Given any  $a \in \text{rad}(\mathfrak{p})$ , that is,  $a^n \in \mathfrak{p}$  for some positive integer. Write  $a^n = aa^{n-1}$  if  $n > 1$ . By the primality of  $\mathfrak{p}$ ,  $a \in \mathfrak{p}$  or  $a^{n-1} \in \mathfrak{p}$ . If  $a \in \mathfrak{p}$ , we are done. If  $a^{n-1} \in \mathfrak{p}$ , we continue this descending argument (or the mathematical induction) until the power of  $a$  is equal to 1. Hence  $\mathfrak{p}$  is radical.

□

### Problem 1.19.

Show that  $I = (x^2 + 1) \subseteq \mathbb{R}[x]$  is a radical (even a prime) ideal, but  $I$  is not the ideal of any set in  $\mathbf{A}^1(\mathbb{R})$ .

*Proof.*

- (1) Show that  $I = (x^2 + 1)$  is a prime ideal in  $\mathbb{R}[x]$ . Given any  $fg \in I$ . It suffices to show that  $f \in I$  or  $g \in I$ . By definition of  $I$ , there is a polynomial  $h \in \mathbb{R}[x]$  such that  $fg = (x^2 + 1)h$ . So  $(x^2 + 1) \mid f$  or  $(x^2 + 1) \mid g$  since  $x^2 + 1$  is irreducible in a unique factorization domain  $\mathbb{R}[x]$ . Therefore,  $f \in I$  or  $g \in I$ .
- (2) Show that  $I$  is not the ideal of any set in  $\mathbf{A}^1(\mathbb{R})$ . Since  $x^2 + 1$  has no roots in  $\mathbb{R}$ ,  $I$  cannot be the ideal of any nonempty set in  $\mathbf{A}^1(\mathbb{R})$ . Besides,  $I(\emptyset) = (1) \neq (x^2 + 1)$ .

□

**Problem 1.20.\***

Show that for any ideal  $I$  in  $k[x_1, \dots, x_n]$ ,  $V(I) = V(\text{rad}(I))$ , and  $\text{rad}(I) \subseteq I(V(I))$ .

*Proof.*

- (1) Show that  $V(I) = V(\text{rad}(I))$ . Since  $I \subseteq \text{rad}(I)$ , it suffices to show that  $V(I) \subseteq V(\text{rad}(I))$ . Given any  $P \in V(I)$ . For any  $f \in \text{rad}(I)$ ,  $f^n \in I$  for some positive integer  $n > 0$ . Note that

$$0 = (f^n)(P) = f(P)^n$$

since  $f^n \in I$  and  $P \in V(I)$ . As  $k$  is a domain,  $f(P)^n = 0$  implies  $f(P) = 0$ . So  $P \in V(\text{rad}(I))$ .

- (2) By Properties (6)(8) in §1.3,

$$I(V(I)) = I(V(\text{rad}(I))) \supseteq \text{rad}(I).$$

□

*Note.*

- (1) By the Hilbert's Nullstellensatz,  $I(V(I)) = \text{rad}(I)$  if  $k = \bar{k}$ .  
 (2) Take  $I = (x^2 + 1)$  as an ideal in  $\mathbb{R}[x]$ . Note that  $I(V(I)) = I(\emptyset) = (1)$  and  $\text{rad}(I) = I = (x^2 + 1)$ . So the equality in  $\text{rad}(I) \subsetneq I(V(I))$  might not hold if  $k \neq \bar{k}$ . (See Problem 1.19.)

**Problem 1.21.\***

Show that  $I = (x_1 - a_1, \dots, x_n - a_n) \subseteq k[x_1, \dots, x_n]$  is a maximal ideal, and that the natural homomorphism from  $k$  to  $k[x_1, \dots, x_n]/I$  is an isomorphism.

*Proof.*

- (1) Show that  $I$  is a maximal ideal. Suppose that  $J$  is an ideal such that  $J \supsetneq I$ . Take any  $f \in J - I$ . By Problem 1.7(a),

$$f = \sum \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}.$$

As  $f \notin I$ , there is a nonzero constant term in  $f$ , say  $\lambda \in k - \{0\}$ . Note that  $f - \lambda \in I \subsetneq J$ . Hence,

$$\lambda = f - (f - \lambda) \in J$$

since  $J$  is an ideal. As  $\lambda \neq 0$ ,  $J = k[x_1, \dots, x_n]$  is not a proper ideal containing  $I$ .

- (2) Let  $\varphi : k \rightarrow k[x_1, \dots, x_n]/I$  be the natural homomorphism. (That is,  $\varphi : \lambda \rightarrow \lambda + I \in k[x_1, \dots, x_n]/I$ .)
- (3) Show that  $\varphi$  is surjective. Given any  $f + I \in k[x_1, \dots, x_n]/I$ . By Problem 1.7(a),

$$f = \sum \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}.$$

So

$$\begin{aligned} f + I &= \sum \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n} + I \\ &= \left( f(a_1, \dots, a_n) + \sum_{\text{nonconstant}} \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n} \right) + I \\ &= f(a_1, \dots, a_n) + I. \end{aligned}$$

(Here the summation over all nonconstant terms is in  $I$ .) Hence

$$\varphi : f(a_1, \dots, a_n) \in k \mapsto f + I.$$

- (4) Show that  $\varphi$  is injective.  $\ker(\varphi) = \{\lambda \in k : \lambda \in I\} = k \cap I = \{0\}$  since  $I$  is a proper ideal.
- (5) By (2)(3)(4),  $\varphi : k \rightarrow k[x_1, \dots, x_n]/(x_1 - a_1, \dots, x_n - a_n)$  is an isomorphism.

□

## 1.4. The Hilbert Basis Theorem

### Problem 1.22.\* (Correspondence theorem for rings)

Let  $I$  be an ideal in a ring  $R$ ,  $\pi : R \rightarrow R/I$  the natural homomorphism.

- (a) Show that for every ideal  $J'$  of  $R/I$ ,  $\pi^{-1}(J') = J$  is an ideal of  $R$  containing  $I$ , and for every ideal  $J$  of  $R$  containing  $I$ ,  $\pi(J) = J'$  is an ideal of  $R/I$ . This sets up a natural one-to-one correspondence between  $\{\text{ideals of } R/I\}$  and  $\{\text{ideals of } R \text{ that contain } I\}$ .
- (b) Show that  $J'$  is a radical ideal if and only if  $J$  is radical. Similarly for prime and maximal ideals.

- (c) Show that  $J'$  is finitely generated if  $J$  is. Conclude that  $R/I$  is Noetherian if  $R$  is Noetherian. Any ring of the form  $k[x_1, \dots, x_n]/I$  is Noetherian.

*Proof of (a).*

- (1) Show that for every ideal  $J'$  of  $R/I$ ,  $\pi^{-1}(J') = J$  is an ideal of  $R$  containing  $I$ .

- (a) Show that  $J$  contains  $I$ . Note that  $\pi^{-1}(0) = I \subseteq \pi^{-1}(J') = J$ . So  $J$  contains  $I$ . In particular,  $J \neq \emptyset$  since  $I \neq \emptyset$ .  
(b) Show that  $J$  is a additive subgroup of  $R$ . It suffices to show that  $a - b \in J$  for any  $a \in J$  and  $b \in J$ . Actually,

$$\pi(a - b) = \pi(a) - \pi(b) \in J'$$

implies  $a - b \in \pi^{-1}(J') = J$ .

- (c) Show that for every  $r \in R$  and every  $a \in J$ , the product  $ra \in J$ . In fact,

$$\pi(ra) = \pi(r)\pi(a) \in J'$$

implies  $ra \in \pi^{-1}(J') = J$ .

- (2) Show that for every ideal  $J$  of  $R$  containing  $I$ ,  $\pi(J) = J'$  is an ideal of  $R/I$ .

- (a) Show that  $J'$  is nonempty. Note that  $\pi(a) = 0 \in \pi(I) \subseteq \pi(J) = J'$  for any  $a \in I$ . So  $J'$  is nonempty since  $J$  is nonempty.  
(b) Show that  $J'$  is a additive subgroup of  $R/I$ . It suffices to show that  $\pi(a) - \pi(b) \in J'$  for any  $\pi(a) \in J'$ ,  $\pi(b) \in J'$ ,  $a \in J$  and  $b \in J$ . It is trivial since

$$\pi(a) - \pi(b) = \pi(a - b) \in \pi(J) = J',$$

$\pi$  is a ring homomorphism and  $J$  is an ideal.

- (c) Show that for every  $\pi(r) \in R/I$  ( $r \in R$ ) and every  $\pi(a) \in J'$  ( $a \in J$ ), the product  $\pi(r)\pi(a) \in J'$ . It is trivial since

$$\pi(r)\pi(a) = \pi(ra) \in \pi(J) = J',$$

$\pi$  is a ring homomorphism and  $J$  is an ideal.

- (3) By (1)(2), we setup the correspondence between

$$\{\text{ideals of } R/I\} \longleftrightarrow \{\text{ideals of } R \text{ that contain } I\}.$$

Note that this correspondence preserves the subset relation, and thus this correspondence is one-to-one.

□

*Proof of (b).*

- (1) *Show that  $J'$  is radical if  $J$  is radical.* It suffices to show that  $(a + I)^n = a^n + I \in J'$  implies that  $a + I \in J'$ . Note that

$$(a + I)^n = a^n + I \in J'$$

implies that  $a^n \in J$  or  $a \in J$  since  $J$  is radical. Hence  $a + I \in J/I = J'$ .

- (2) *Show that  $J$  is radical if  $J'$  is radical.* It suffices to show that  $a^n \in J$  implies that  $a \in J$ . Note that

$$\pi(a^n) = \pi(a)^n \in J'$$

implies that  $\pi(a) \in J'$  since  $J'$  is radical.  $a \in \pi^{-1}(J') = J$ .

- (3) *Show that  $J'$  is prime if  $J$  is prime.* It suffices to show that  $(a + I)(b + I) = ab + I \in J'$  implies that  $a + I \in J'$  or  $b + I \in J'$ . Note that

$$(a + I)(b + I) = ab + I \in J'$$

implies that  $ab \in J$ . So  $a \in J$  or  $b \in J$  by the primality of  $J$ . Hence  $a + I \in J'$  or  $b + I \in J'$ .

- (4) *Show that  $J$  is prime if  $J'$  is prime.* It suffices to show that  $ab \in J$  implies that  $a \in J$  or  $b \in J$ . Note that

$$\pi(ab) = \pi(a)\pi(b) \in J'$$

implies that  $\pi(a) \in J'$  or  $\pi(b) \in J'$  by the primality of  $J'$ . So  $a \in \pi^{-1}(J') = J$  or  $b \in \pi^{-1}(J') = J$ .

- (5) *Show that  $J'$  is maximal if  $J$  is maximal.* Suppose  $\mathfrak{m}$  is an ideal containing  $J'$ . By (a),  $\pi^{-1}(\mathfrak{m})$  is an ideal containing  $J$ . So  $\pi^{-1}(\mathfrak{m}) = J$  or  $\pi^{-1}(\mathfrak{m}) = R$  by the maximality of  $J$ . Hence,  $\mathfrak{m} = \pi(J) = J'$  or  $\mathfrak{m} = \pi(R) = R/I$ .

- (6) *Show that  $J$  is maximal if  $J'$  is maximal.* Suppose  $\mathfrak{m}$  is an ideal containing  $J$ . By (a),  $\pi(\mathfrak{m})$  is an ideal containing  $J'$ . So  $\pi(\mathfrak{m}) = J'$  or  $\pi(\mathfrak{m}) = R/I$  by the maximality of  $J'$ . Hence,  $\mathfrak{m} = \pi^{-1}(J') = J$  or  $\mathfrak{m} = \pi^{-1}(R/I) = R$ .

□

*Note.*

- (1) Note that

$$R/J \cong (R/I)/(J/I)$$

if  $J$  is an ideal of  $R$  such that  $I \subseteq J$ .

- (2) Hence,  $J$  is prime iff  $R/J \cong (R/I)/(J/I)$  is a domain iff  $J/I$  is prime.  
(3) Also,  $J$  is maximal iff  $R/J \cong (R/I)/(J/I)$  is a field iff  $J/I$  is maximal.

*Proof of (c).*

- (1) *Show that  $J'$  is finitely generated if  $J$  is.* Suppose  $J$  is generated by  $a_1, \dots, a_m$ . It suffices to show that  $J'$  is generated by

$$a_1 + I, \dots, a_m + I \in J/I.$$

Given any  $a + I \in J'$  where  $a \in J$ . Write  $a = \sum_{1 \leq i \leq m} r_i a_i$  for some  $r_i \in R$ . Then

$$a + I = \sum r_i a_i + I = \sum (r_i + I)(a_i + I)$$

is generated by  $a_1 + I, \dots, a_m + I$ .

- (2) *Show that  $R/I$  is Noetherian if  $R$  is Noetherian.* Note that  $R$  is an ideal of itself.  
(3) *Show that any ring of the form  $k[x_1, \dots, x_n]/I$  is Noetherian.* By the corollary to the Hilbert basis theorem,  $k[x_1, \dots, x_n]$  is Noetherian. By (2), the ring  $k[x_1, \dots, x_n]/I$  is Noetherian.

□

## 1.5. Irreducible Components of an Algebraic Set

### Problem 1.23.

*Give an example of a collection of ideals  $\mathcal{S}$  in a Noetherian ring such that no maximal member of  $\mathcal{S}$  is a maximal ideal.*

*Proof.*

- (1) Let  $R$  be any Noetherian ring. Let  $\mathcal{S}$  be any collection of ideals containing  $R$  itself. Then the only maximal member of  $\mathcal{S}$  is  $R$ , which is not a maximal ideal.  
(2) Or let  $R$  be any Noetherian ring and  $R$  is not a field. ( $R = k[x_1, \dots, x_n]$  where  $k$  is a field for example.) Let  $\mathcal{S} = \{(0)\}$ . Then the only maximal member of  $\mathcal{S}$  is  $(0)$ , which is not maximal since  $R$  is not a field.

□

**Problem 1.24.**

Show that every proper ideal in a Noetherian ring is contained in a maximal ideal. (Hint: If  $I$  is the ideal, apply the lemma to  $\{\text{proper ideals that contain } I\}$ .)

*Proof.*

- (1) Say  $I$  be any proper ideal in a Noetherian ring. Let

$$\mathcal{S} = \{\text{proper ideals that contain } I\}.$$

Apply the lemma to  $\mathcal{S}$  to get that  $\mathcal{S}$  has a maximal member  $\mathfrak{m} \in \mathcal{S}$ .

- (2) Show that  $\mathfrak{m}$  is maximal. Since  $\mathfrak{m} \in \mathcal{S}$ ,  $\mathfrak{m}$  is a proper ideal in  $R$ . Suppose  $\mathfrak{m}' \supsetneq \mathfrak{m}$  is a proper ideal containing  $\mathfrak{m}$ . As  $\mathfrak{m}$  contains  $I$ ,  $\mathfrak{m}'$  also contains  $I$  or  $\mathfrak{m}' \in \mathcal{S}$ . By the maximality of  $\mathfrak{m}$ ,  $\mathfrak{m}' \subseteq \mathfrak{m}$ . So  $\mathfrak{m}' = \mathfrak{m}$ .

□

**Problem 1.25.**

- (a) Show that  $V(y - x^2) \subseteq \mathbf{A}^2(\mathbb{C})$  is irreducible, in fact,  $I(V(y - x^2)) = (y - x^2)$ .
- (b) Decompose  $V(y^4 - x^2, y^4 - x^2y^2 + xy^2 - x^3) \subseteq \mathbf{A}^2(\mathbb{C})$  into irreducible components.

*Proof of (a).*

- (1) Let  $I = (y - x^2)$  be an ideal of  $\mathbb{C}[x, y]$ . Since  $\mathbb{C}$  is algebraically closed,

$$I(V(I)) = \text{rad}(I)$$

by the Hilbert's Nullstellensatz. It suffices to show that  $I$  is prime, or to show that  $y - x^2$  is prime. Since  $\mathbb{C}[x, y]$  is a UFD, it suffices to show that  $y - x^2$  is irreducible.

- (2) Show that  $y - x^2$  is irreducible in  $\mathbb{C}[x, y]$ . Write

$$y - x^2 \in (\mathbb{C}[y])[x].$$

Note that  $\mathbb{C}[y]$  is a UFD and  $y$  is the constant term. If we can show that  $y$  is prime in  $\mathbb{C}[y]$ , then by the Eisenstein's criterion we can say  $y - x^2$  is irreducible in  $(\mathbb{C}[y])[x]$ .

- (3) As  $\mathbb{C}[y]/(y) \cong \mathbb{C}$  is a field or a domain,  $(y)$  is maximal or prime. Hence,  $y - x^2$  is irreducible.

(4) Or apply Corollary 1 to Proposition 2 in the next section to (2)(3).

□

*Proof of (b).*

(1) Write

$$\begin{aligned}
 Y &:= V(y^4 - x^2, y^4 - x^2y^2 + xy^2 - x^3) \\
 &= V((y^2 - x)(y^2 + x), (y^2 - x^2)(y^2 + x)) \\
 &= V(y^2 + x) \cup V(y^2 - x, y^2 - x^2) \\
 &= V(y^2 + x) \cup V(y^2 - x, x(x - 1)) \\
 &= V(y^2 + x) \cup V(x, y) \cup V(y + 1, x - 1) \cup V(y - 1, x - 1).
 \end{aligned}$$

(2) Here  $V(y^2 + x)$  is irreducible as (a). Besides,  $V(x, y)$ ,  $V(y + 1, x - 1)$  and  $V(y - 1, x - 1)$  are irreducible since all corresponding ideals are maximal (by the Hilbert's Nullstellensatz and Problem 1.21).

□

**Problem 1.26.**

Show that  $f = y^2 + x^2(x - 1)^2 \in \mathbb{R}[x, y]$  is an irreducible polynomial, but  $V(f)$  is reducible.

*Proof.*

(1) Show that  $f$  is an irreducible polynomial.

(a) Suppose

$$f = (f_2(x)y^2 + f_1(x)y + f_0(x)) \cdot g(x)$$

for some  $f_i(x), g(x) \in \mathbb{R}[x]$ . So

$$f_2(x)g(x) = 1, \quad f_1(x)g(x) = 0, \quad f_0(x)g(x) = x^2(x - 1)^2.$$

Hence,

$$f_2(x)y^2 + f_1(x)y + f_0(x) = uf, \quad g(x) = u^{-1},$$

where  $u$  is a unit in  $\mathbb{R}$ .

(b) Suppose

$$f = (f_1(x)y + f_0(x)) \cdot (g_1(x)y + g_0(x))$$



for some  $f_i(x), g_j(x) \in \mathbb{R}[x]$ . So

$$\begin{aligned} f_1(x)g_1(x) &= 1, \\ f_1(x)g_0(x) + f_0(x)g_1(x) &= 0, \\ f_0(x)g_0(x) &= x^2(x-1)^2. \end{aligned}$$

So  $f_1(x) = u$ ,  $g_1(x) = u^{-1}$  for some unit  $u \in \mathbb{R}$ . Hence,

$$u^2 g_0(x)^2 = -x^2(x-1)^2,$$

which is absurd since  $\mathbb{R}$  is not algebraically closed.

(c) By (a)(b),  $f$  is irreducible in  $\mathbb{R}[x, y]$ .

- (2) Show that  $V(f)$  is reducible.  $V(f) = \{(0, 0), (1, 0)\} = V(x, y) \cup V(x-1, y)$ .  
Here  $V(x, y)$  and  $V(x-1, y)$  are all proper algebraic sets in  $V(f)$ .

□

**Problem 1.27.**

Let  $V, W$  be algebraic sets in  $\mathbf{A}^n(k)$  with  $V \subseteq W$ . Show that each irreducible component of  $V$  is contained in some irreducible component of  $W$ .

*Proof.*

- (1) Write two decompositions of  $V, W$  into irreducible components as

$$\begin{aligned} V &= V_1 \cup \cdots \cup V_r, \\ W &= W_1 \cup \cdots \cup W_s, \end{aligned}$$

- (2) For each irreducible component  $V_i$  of  $V$ , consider  $V_i \cap W$ :

$$V_i \cap W = (V_i \cap W_1) \cup \cdots \cup (V_i \cap W_s).$$

By the irreducibility of  $V_i$ , there is only one  $j$  such that  $V_i \cap W_j = V_i$  and other intersections are empty. Therefore, each irreducible component  $V_i$  is contained in some irreducible component  $W_j$  of  $W$ .

□

**Problem 1.28.**

If  $V = V_1 \cup \cdots \cup V_r$  is the decomposition of an algebraic set into irreducible components, show that  $V_i \not\subseteq \bigcup_{j \neq i} V_j$ .

*Proof.*

- (1) (Reductio ad absurdum) If

$$V_i \subseteq \bigcup_{j \neq i} V_j$$

for some  $i$ , then

$$V = V_1 \cup \cdots \cup \widehat{V_i} \cup \cdots \cup V_r$$

is another decomposition of an algebraic set into irreducible components.

- (2) By Theorem 2 in §1.5, the number of irreducible components is unique determined, contrary to the assumption and (1).

□

**Problem 1.29.\***

Show that  $\mathbf{A}^n(k)$  is irreducible if  $k$  is infinite.

*Proof.*

- (1) (Reductio ad absurdum) If  $\mathbf{A}^n(k)$  were reducible, then  $\mathbf{A}^n(k) = V_1 \cup V_2$  where  $V_1, V_2$  are algebraic sets in  $\mathbf{A}^n(k)$ ,  $V_1$  and  $V_2$  are nonempty and proper in  $\mathbf{A}^n(k)$ .
- (2) Take  $P_i \in V_i$  for  $i = 1, 2$ . By Problem 1.17, there are two polynomials  $f_1, f_2 \in k[x_1, \dots, x_n]$  such that  $f_i(Q) = 0$  for all  $Q \in V_i$  and  $f_1(P_2) = f_2(P_1) = 1$ .
- (3) By construction,  $(f_1 f_2)(a_1, \dots, a_n) = 0$  for any  $a_1, \dots, a_n \in k$ . As  $k$  is infinite,  $f_1 f_2 = 0$  by Problem 1.4. Since  $k[x_1, \dots, x_n]$  is a domain,  $f_1 = 0$  or  $f_2 = 0$ , contrary to  $f_1(P_2) = f_2(P_1) \neq 0$ .

□

*Note.*  $\mathbf{A}^n(k)$  is reducible if  $k$  is finite.

## 1.6. Algebraic Subsets of the Plane

### Problem 1.30.

Let  $k = \mathbb{R}$ .

- (a) Show that  $I(V(x^2 + y^2 + 1)) = (1)$ .
- (b) Show that every algebraic subset of  $\mathbf{A}^2(\mathbb{R})$  is equal to  $V(f)$  for some  $f \in \mathbb{R}[x, y]$ .

This indicates why we usually require that  $k$  be algebraically closed.

*Proof of (a).*  $I(V(x^2 + y^2 + 1)) = I(\emptyset) = (1)$  since  $x^2 + y^2 + 1 \geq 1$  is never zero for any  $x, y \in \mathbb{R}$ .  $\square$

*Proof of (b).*

- (1) Given any algebraic subset  $V$  of  $\mathbf{A}^2(\mathbb{R})$ .  $V = V(1)$  if  $V = \emptyset$ .  $V = V(0)$  if  $V = \mathbf{A}^2(\mathbb{R})$ . Now suppose  $V$  is a nonempty proper algebraic subset of  $\mathbf{A}^2(\mathbb{R})$ . Write  $V = V_1 \cup \cdots \cup V_m$ , where each  $V_i$  is irreducible. Here  $V_i \neq \emptyset$  and  $V_i \neq \mathbf{A}^2(\mathbb{R})$  for all  $i$ .
- (2) As  $k = \mathbb{R}$  is infinite, Corollary 2 to Proposition 2 implies that each  $V_i$  is either a point or an irreducible plane curve  $V(f_i)$ , where  $f_i$  is an irreducible polynomial and  $V(f_i)$  is infinite.
- (3) If  $V_i = \{(a_i, b_i)\}$  is a point, then define

$$f_i(x, y) = (x - a_i)^2 + (y - b_i)^2.$$

By the property of  $\mathbb{R}$ ,  $V_i = V(f_i)$ .

- (4) Define  $f = f_1 \cdots f_m \in \mathbb{R}[x, y]$ . Hence,

$$\begin{aligned} V &= V_1 \cup \cdots \cup V_m \\ &= V(f_1) \cup \cdots \cup V(f_m) \\ &= V(f_1 \cdots f_m) \\ &= V(f). \end{aligned}$$

$\square$

### Problem 1.31.

- (a) Find the irreducible components of  $V(y^2 - xy - x^2y + x^3)$  in  $\mathbf{A}^2(\mathbb{R})$ , and also in  $\mathbf{A}^2(\mathbb{C})$ .

(b) Do the same for  $V(y^2 - x(x^2 - 1))$ , and for  $V(x^3 + x - x^2y - y)$ .

*Proof of (a).*

(1) Note that

$$\begin{aligned} V(y^2 - xy - x^2y + x^3) &= V((y - x^2)(y - x)) \\ &= V(y - x^2) \cup V(y - x). \end{aligned}$$

(2) Note that  $y - x^2$  and  $y - x$  are irreducible in  $\mathbb{C}[x, y]$  and thus also in  $\mathbb{R}[x, y]$  by the similar argument in Problem 1.25(a). Also,  $V(y - x^2)$  and  $V(y - x)$  are infinite in  $\mathbf{A}^2(\mathbb{R})$  and thus also in  $\mathbf{A}^2(\mathbb{C})$ .

(3) Therefore,  $V(y - x^2)$  and  $V(y - x)$  are the irreducible components of  $V(y^2 - xy - x^2y + x^3)$  in  $\mathbf{A}^2(\mathbb{R})$  and also in  $\mathbf{A}^2(\mathbb{C})$ .

□

*Outline of (b).*

- (1) The elliptic curve  $V(y^2 - x(x + 1)(x - 1))$  is irreducible over  $\mathbf{A}^2(\mathbb{R})$ .
- (2) The elliptic curve  $V(y^2 - x(x + 1)(x - 1))$  is irreducible over  $\mathbf{A}^2(\mathbb{C})$ .
- (3) The irreducible component of  $V(x^3 + x - x^2y - y)$  over  $\mathbf{A}^2(\mathbb{R})$  is  $V(x - y)$ .
- (4) The irreducible components of  $V(x^3 + x - x^2y - y)$  over  $\mathbf{A}^2(\mathbb{C})$  are  $V(x + i)$ ,  $V(x - i)$  and  $V(x - y)$ .

*Proof of (b).*

(1) Similar to Problem 1.25. To show  $y^2 - x(x + 1)(x - 1)$  is irreducible in  $\mathbb{C}[x, y]$ , we write

$$y^2 - x(x + 1)(x - 1) \in (\mathbb{C}[x])[y].$$

Note that  $\mathbb{C}[x]$  is a UFD and  $-x(x + 1)(x - 1)$  is the constant term. As  $\mathbb{C}[x]/(x) \cong \mathbb{C}$  is a domain,  $(x)$  is prime. Clearly,  $x \mid x(x + 1)(x - 1)$  but  $x^2 \nmid x(x + 1)(x - 1)$ . By the Eisenstein's criterion, we can say  $y^2 - x(x + 1)(x - 1)$  is irreducible over  $(\mathbb{C}[x])[y]$ .

- (2) Moreover,  $V(y^2 - x(x + 1)(x - 1))$  is infinite over  $\mathbf{A}^2(\mathbb{R})$  and thus also over  $\mathbf{A}^2(\mathbb{C})$ . ( $y = f(x) = \sqrt{x(x + 1)(x - 1)}$  is continuous and strictly increasing on  $[1, \infty)$  in the sense of calculus. As the measure of  $[1, \infty)$  is  $\infty$ , the set  $V(y^2 - x(x + 1)(x - 1))$  is infinite over  $\mathbf{A}^2(\mathbb{R})$ .)
- (3) By Corollary 1 to Proposition 2,  $V(y^2 - x(x^2 - 1))$  itself is irreducible over  $\mathbf{A}^2(\mathbb{R})$  or  $\mathbf{A}^2(\mathbb{C})$ .

- (4) Consider  $V(x^3 + x - x^2y - y) \subseteq \mathbf{A}^2(\mathbb{R})$ .

$$\begin{aligned} V(x^3 + x - x^2y - y) &= V((x^2 + 1)(x - y)) \\ &= V(x^2 + 1) \cup V(x - y) \\ &= \emptyset \cup V(x - y) \\ &= V(x - y). \end{aligned}$$

Here we use that fact that  $x^2 + 1 = 0$  has no real solution  $x \in \mathbb{R}$ . Similar to (a),  $V(x - y)$  is the only irreducible component of  $V(x^3 + x - x^2y - y)$  in  $\mathbf{A}^2(\mathbb{R})$ .

- (5) Consider  $V(x^3 + x - x^2y - y) \subseteq \mathbf{A}^2(\mathbb{C})$ .

$$\begin{aligned} V(x^3 + x - x^2y - y) &= V((x + i)(x - i)(x - y)) \\ &= V(x + i) \cup V(x - i) \cup V(x - y). \end{aligned}$$

Similar to (a),  $V(x \pm i)$  and  $V(x - y)$  are the irreducible components of  $V(x^3 + x - x^2y - y)$  in  $\mathbf{A}^2(\mathbb{C})$ .

□

## 1.7. Hilbert's Nullstellensatz

### Problem 1.32.

*Show that both theorems and all of the corollaries are false if  $k$  is not algebraically closed.*

*Proof.*

- (1) Weak Nullstellensatz:  $I = (x^2 + 1)$  is a proper ideal in  $\mathbb{R}[x]$  but  $V(I) = \emptyset$ .  
(2) Hilbert's Nullstellensatz: Let  $I = (y^2 + x^2(x - 1)^2)$  be an ideal in  $\mathbb{R}[x, y]$ . Hence,

$$\begin{aligned} I(V(I)) &= I(\{(0, 0), (1, 0)\}) && \text{(Problem 1.26.)} \\ &= (x(x - 1), y) \\ &\neq I \\ &= \text{rad}(I). \end{aligned}$$

The last equality holds since  $f$  is irreducible in a UFD  $\mathbb{R}[x, y]$  and thus  $I$  is a prime ideal.

- (3) Corollary 1: Same example in the case Hilbert's Nullstellensatz. If  $I = (y^2 + x^2(x - 1)^2)$  is a radical ideal in  $\mathbb{R}[x, y]$ . Then  $I(V(I)) \neq I$ .

- (4) Corollary 2: Same example in the case Hilbert's Nullstellensatz. If  $I = (y^2 + x^2(x-1)^2)$  is a prime ideal in  $\mathbb{R}[x, y]$ , then

$$V(I) = \{(0, 0), (1, 0)\} = V(x, y) \cup V(x-1, y)$$

is reducible. Next, consider a prime ideal  $J = (x^2 + y^2)$  in  $\mathbb{R}[x, y]$ . (Use the same argument in Problem 1.26 to get the irreducibility of  $x^2 + y^2$ .)  $V(J) = \{(0, 0)\}$  is a point but  $J$  is not a maximal ideal (since  $J \subsetneq (x^2 + y^2, x) \subsetneq (1)$ ).

- (5) Corollary 3: Same example in Corollary 2.
- (6) Corollary 4: Let  $I = (x^2 + y^2)$  be an ideal in  $\mathbb{R}[x, y]$ . Then  $V(I) = \{(0, 0)\}$  is a finite set. But  $\mathbb{R}[x, y]/(x^2 + y^2)$  is an infinite dimensional vector space over  $\mathbb{R}$ . In fact, the monomials

$$\{\overline{x^m}, \overline{x^m y} : m = 0, 1, 2, \dots\}$$

is a basis for  $\mathbb{R}[x, y]/(x^2 + y^2)$ .

□

**Problem 1.33.**

- (a) Decompose  $V(x^2 + y^2 - 1, x^2 - z^2 - 1) \subseteq \mathbf{A}^3(\mathbb{C})$  into irreducible components.
- (b) Let  $V = \{(t, t^2, t^3) \in \mathbf{A}^3(\mathbb{C}) : t \in \mathbb{C}\}$ . Find  $I(V)$ , and show that  $V$  is irreducible.

*Proof of (a).*

- (1) Write

$$\begin{aligned} & V(x^2 + y^2 - 1, x^2 - z^2 - 1) \\ &= V(x^2 + y^2 - 1, y^2 + z^2) \\ &= V(x^2 + y^2 - 1, (y + iz)(y - iz)) \\ &= V(x^2 + y^2 - 1, y + iz) \cup V(x^2 + y^2 - 1, y - iz). \end{aligned}$$

By the Hilbert's Nullstellensatz, it suffices to show that  $(x^2 + y^2 - 1, y + iz)$  and  $(x^2 + y^2 - 1, y - iz)$  are prime.

- (2) Show that  $I = (x^2 + y^2 - 1, y + iz)$  is prime in  $\mathbb{C}[x, y, z]$ . Note that

$$\mathbb{C}[x, y, z]/I \cong \mathbb{C}[x, y]/(x^2 + y^2 - 1)$$

is a ring isomorphism defined by

$$f(x, y, z) + I \mapsto f(x, y, -iy) + (x^2 + y^2 - 1).$$

(Use the similar argument in (b) to prove it is indeed an isomorphism.)  
So it suffices to show that

$$x^2 + y^2 - 1 \in \mathbb{C}[x, y]$$

is irreducible. (Thus,  $\mathbb{C}[x, y]/(x^2 + y^2 - 1) \cong \mathbb{C}[x, y, z]/I$  is a domain, or  $I$  is prime.) We can use the similar argument in Problem 1.31 (b) to show  $x^2 + y^2 - 1 = y^2 + (x+1)(x-1)$  is irreducible as showing the irreducibility of  $y^2 - x(x+1)(x-1)$ .

- (3) Similarly,  $I = (x^2 + y^2 - 1, y - iz)$  is prime. Therefore, the irreducible components of  $V(x^2 + y^2 - 1, x^2 - z^2 - 1)$  are  $V(x^2 + y^2 - 1, y + iz)$  and  $V(x^2 + y^2 - 1, y - iz)$ .

□

*Proof of (b).*

- (1) Write

$$V = \{(t, t^2, t^3) \in \mathbf{A}^3(\mathbb{C}) : t \in \mathbb{C}\} = V(x^2 - y, x^3 - z).$$

Let  $I = (x^2 - y, x^3 - z)$  in  $\mathbb{C}[x, y, z]$ . By the Hilbert's Nullstellensatz,  $I(V) = \text{rad}(I)$ . So it suffices to show that  $I = (x^2 - y, x^3 - z)$  is prime (and thus  $V$  is irreducible).

- (2) *Show that*

$$\mathbb{C}[x, y, z]/I \cong \mathbb{C}[t]$$

*is a domain, and thus  $I = (x^2 - y, x^3 - z)$  is a prime ideal.*

- (a) Define a ring homomorphism  $\alpha : \mathbb{C}[x, y, z]/I \rightarrow \mathbb{C}[t]$  by

$$\alpha : f(x, y, z) + I \mapsto f(t, t^2, t^3).$$

$\alpha$  is well-defined since  $\alpha((x^2 - y) + I) = 0$  and  $\alpha((x^3 - z) + I) = 0$ .

- (b) *Show that  $\alpha$  is surjective.*

$$\alpha : g(x) + I \in \mathbb{C}[x, y, z]/I \mapsto g(t) \in \mathbb{C}[t]$$

for any  $g(t)$ .

- (c) *Show that  $\alpha$  is injective.* Suppose  $\alpha(f(x, y, z) + I) = 0$ . Write

$$\begin{aligned} f(x, y, z) + I &= \sum_{(i)} \lambda_{(i)} x^{i_1} (y - x^2)^{i_2} (z - x^3)^{i_3} + I \\ &= \sum_i \lambda_i x^i + I. \end{aligned}$$

So

$$0 = \alpha(f(x, y, z) + I) = \alpha\left(\sum_i \lambda_i x^i + I\right) = \sum_i \lambda_i t^i.$$

Hence,  $\ker(\alpha) = I$ .

□

**Problem 1.34.**

Let  $R$  be a UFD.

- (a) Show that a monic polynomial of degree two or three in  $R[x]$  is irreducible if and only if it has no root in  $R$ .
- (b)  $x^2 - a \in R[x]$  is irreducible if and only if  $a$  is not a square in  $R$ .

*Proof of (a).*

- (1) It is equivalent to show that a monic polynomial of degree two or three in  $R[x]$  is reducible if and only if it has one root in  $R$ .
- (2) Suppose  $f$  is reducible of degree 2 or 3. Then there exist nonconstant monic polynomials  $g, h \in R[x]$  such that  $f = gh$ . By

$$\deg(g) + \deg(h) = \deg(f) = 2 \text{ or } 3,$$

we may assume that  $\deg(g) = 1$ . (Otherwise  $g$  or  $h$  will be a constant polynomial.) Say  $g(x) = x - a$  where  $a \in R$ . Now

$$f(a) = g(a)h(a) = 0$$

implies that  $a \in R$  is a root of  $f$ .

- (3) Conversely, if  $a \in R$  is a root of  $f$ , then apply the same argument in Problem 1.7 we can write

$$f = (x - a)g$$

for some  $g \in R[x]$ . Here  $\deg(g) \geq 1$  since  $\deg(f) = 1 + \deg(g) \geq 2$ . Therefore,  $f$  is reducible.

□

*Proof of (b).* By (a),  $x^2 - a \in R[x]$  is reducible  $\iff x^2 - a$  has one root  $\alpha \in R$   $\iff a = \alpha^2$  is a square in  $R$  for some  $\alpha \in R$ . □

**Problem 1.35.**

Show that  $V(y^2 - x(x - 1)(x - \lambda)) \subseteq \mathbf{A}^2(k)$  is an irreducible curve for any algebraically closed field  $k$ , and any  $\lambda \in k$ .

*Proof.*



- (1) By the Hilbert's Nullstellensatz, it suffices to show that

$$I = (y^2 - x(x-1)(x-\lambda))$$

is a prime ideal in  $k[x, y]$ , or show that

$$y^2 - x(x-1)(x-\lambda)$$

is irreducible (since  $k[x, y]$  is a UFD).

- (2) By Problem 1.34(b),  $y^2 - x(x-1)(x-\lambda) \in (\mathbb{C}[x])[y]$  is irreducible if  $x(x-1)(x-\lambda)$  is not a square in  $\mathbb{C}[x]$ . Note that every square in  $\mathbb{C}[x]$  is of even degree. So  $x(x-1)(x-\lambda)$  cannot be a square in  $\mathbb{C}[x]$  since  $\deg(x(x-1)(x-\lambda)) = 3$  is odd.

□

*Note.*  $V(y^2 - x(x-1)(x-\lambda))$  is the elliptic curve as Problem 1.31.

**Problem 1.36.**

Let  $I = (y^2 - x^2, y^2 + x^2) \subseteq \mathbb{C}[x, y]$ . Find  $V(I)$  and  $\dim_{\mathbb{C}}(\mathbb{C}[x, y]/I)$ .

*Proof.*

- (1) Clearly,  $V(I) = \{(0, 0)\}$  is a finite set. By Corollary 4 to the Hilbert's Nullstellensatz,

$$\dim_{\mathbb{C}}(\mathbb{C}[x, y]/I) < \infty.$$

In fact,  $\dim_{\mathbb{C}}(\mathbb{C}[x, y]/I) = 4$ .

- (2) Given any  $f + I \in \mathbb{C}[x, y]/I$  where  $f \in \mathbb{C}[x, y]$ . Write

$$f(x, y) = \sum_i f_i(x) y^i$$

where  $f_i(x) = \sum_j a_{ij} x^j \in \mathbb{C}[x]$ . Note that

$$\begin{aligned} x^2 &= \frac{1}{2}(y^2 + x^2) - \frac{1}{2}(y^2 - x^2) \in I, \\ y^2 &= \frac{1}{2}(y^2 + x^2) + \frac{1}{2}(y^2 - x^2) \in I. \end{aligned}$$

So

$$\begin{aligned}
f(x, y) + I &= \sum_i f_i(x) y^i + I \\
&= f_0(x) + f_1(x) y + I \\
&= \sum_j a_{0j} x^j + \left( \sum_j a_{1j} x^j \right) y + I \\
&= a_{00} + a_{01} x + a_{10} y + a_{11} xy + I
\end{aligned}$$

is generated by  $\mathcal{B} = \{\bar{1}, \bar{x}, \bar{y}, \bar{xy}\}$ .

- (3) Note that  $\mathcal{B}$  is a basis since any linear combination of elements in  $\mathcal{B}$  is not in  $I$ . Therefore,

$$\dim_{\mathbb{C}}(\mathbb{C}[x, y]/I) = |\mathcal{B}| = 4.$$

□

**Problem 1.37.\***

Let  $K$  be any field,  $f \in K[x]$  a polynomial of degree  $n > 0$ . Show that the residues  $\bar{1}, \bar{x}, \dots, \bar{x}^{n-1}$  form a basis for  $K[x]/(f)$  over  $K$ .

*Proof.*

- (1) Show that every element in  $K[x]/(f)$  is generated by  $\mathcal{B} = \{\bar{1}, \bar{x}, \dots, \bar{x}^{n-1}\}$ . Given any  $\bar{g} \in K[x]/(f)$  with  $g \in K[x]$ . By the division-with-remainder property of  $K[x]$ , there are some polynomials  $q, r \in K[x]$  such that

$$g = fq + r$$

where  $r = 0$  or  $\deg(r) < n$  if  $r \neq 0$ . Therefore,

$$g + (f) = fq + r + (f) = r + (f).$$

Note that  $r + (f)$  is generated by  $\mathcal{B}$ .

- (2) Show that  $\mathcal{B}$  is a basis for  $K[x]/(f)$  over  $K$ . Suppose

$$a_0 + a_1 x + \dots + a_{n-1} x^{n-1} \in (f)$$

for  $a_1, \dots, a_{n-1} \in K$ . We can regard any linear combination of  $\{1, x, \dots, x^{n-1}\}$  as a polynomial  $r(x)$  in  $K[x]$ .  $r \in (f)$  implies that there exists a polynomial  $g \in K[x]$  such that  $r = fg$ . If  $g \neq 0$ , then  $\deg(r) = \deg(f) + \deg(g) \geq n$ , which is impossible. So  $g = 0$  and thus  $r = fg = 0 \in K[x]$ . Therefore,  $a_0 = a_1 = \dots = a_{n-1} = 0 \in K$  and

$$\dim_K(K[x]/(f)) = \deg(f).$$

□

**Problem 1.38.\***

Let  $R = k[x_1, \dots, x_n]$ ,  $k$  algebraically closed,  $V = V(I)$ . Show that there is a natural one-to-one correspondence between algebraic subsets of  $V$  and radical ideals in  $k[x_1, \dots, x_n]/I$ , and that irreducible algebraic sets (resp. points) correspond to prime ideals (resp. maximal ideals). (See Problem 1.22.)

*Proof.*

- (1) Given any algebraic subset  $W$  of  $V$ . By the Hilbert's Nullstellensatz,

$$I(W) \supseteq I(V) = \text{rad}(I) \supseteq I.$$

- (2) By Corollary 1 to the Hilbert's Nullstellensatz and Problem 1.22(b), we have a one-to-one correspondence such that

$$\begin{aligned} & \{\text{algebraic subsets of } V\} \\ & \longleftrightarrow \{\text{radical ideals containing } I\} \\ & \longleftrightarrow \{\text{radical ideals of } k[x_1, \dots, x_n]/I\}. \end{aligned}$$

- (3) Again by Corollary 2 to the Hilbert's Nullstellensatz and Problem 1.22(b), we have a one-to-one correspondence such that

$$\begin{aligned} & \{\text{irreducible algebraic subsets (resp. points) of } V\} \\ & \longleftrightarrow \{\text{prime (resp. maximal) ideals containing } I\} \\ & \longleftrightarrow \{\text{prime (resp. maximal) ideals of } k[x_1, \dots, x_n]/I\}. \end{aligned}$$

□

**Problem 1.39.**

- (a) Let  $R$  be a UFD, and let  $\mathfrak{p} = (t)$  be a principal proper prime ideal. Show that there is no prime ideal  $\mathfrak{q}$  such that  $0 \subsetneq \mathfrak{q} \subsetneq \mathfrak{p}$ .
- (b) Let  $V = V(f)$  be irreducible hypersurface in  $\mathbf{A}^n$ . Show that there is no irreducible algebraic set  $W$  such that  $V \subsetneq W \subsetneq \mathbf{A}^n$ .

*Proof of (a).*

- (1) (Reductio ad absurdum) Suppose that  $\mathfrak{q}$  were a prime ideal in  $R$  such that  $0 \subsetneq \mathfrak{q} \subsetneq \mathfrak{p}$ .

- (2) Show that there is an irreducible element in  $\mathfrak{q}$ . Given any  $q \in \mathfrak{q}$ . Since  $\mathfrak{q}$  is proper, we can write

$$q = q_1 \cdots q_n$$

as a product of irreducible elements in a UFD. Since  $\mathfrak{q}$  is prime, there is one irreducible element  $q_i \in \mathfrak{q}$ .

- (3) Now  $q_i \in \mathfrak{q} \subseteq \mathfrak{p} = (t)$ . So  $q_i = ut$  for some  $u \in R$ . By the irreducibility of  $q_i$ ,  $u$  is a unit or  $t$  is a unit. If  $u$  is a unit, then

$$(t) = (q_i) \subseteq \mathfrak{q} \subseteq \mathfrak{p} = (t).$$

So  $\mathfrak{q} = \mathfrak{p}$ , which is absurd. If  $t$  is a unit, then  $\mathfrak{p} = (1)$ , contrary to the primality of  $\mathfrak{p}$ .

□

*Proof of (b).*

- (1) We might assume that  $k = \bar{k}$ . By Corollary 3 to the Hilbert's Nullstellensatz and the irreducibility of  $V(f)$ , there are an irreducible polynomial  $g \in k[x_1, \dots, x_n]$  and an integer  $m > 0$  such that

$$f = g^m,$$

and

$$I(V(f)) = (g).$$

- (2) (Reductio ad absurdum) Suppose that there were an irreducible algebraic set  $W$  such that  $V \subsetneq W \subsetneq \mathbf{A}^n$ . Then by Corollary 3 to the Hilbert's Nullstellensatz again,

$$(g) = I(V(f)) \supsetneq I(W) \supsetneq (1) \in k[x_1, \dots, x_n].$$

Here  $(g) = I(V(f))$  and  $I(W)$  are all prime.

- (3) Note that  $(g)$  is a principal proper prime ideal in a UFD  $k[x_1, \dots, x_n]$ . By (a), such ideal  $I(W)$  cannot be prime, which is absurd.

□

#### Problem 1.40.

Let  $I = (x^2 - y^3, y^2 - z^3) \subseteq k[x, y, z]$ . Define  $\alpha : k[x, y, z] \rightarrow k[t]$  by  $\alpha(x) = t^9$ ,  $\alpha(y) = t^6$ ,  $\alpha(z) = t^4$ .

- (a) Show that every element of  $k[x, y, z]/I$  is the residue of an element  $a + xb + yc + xzd$ , for some  $a, b, c, d \in k[z]$ .

- (b) If  $f = a + xb + yc + xyd$ ,  $a, b, c, d \in k[z]$  and  $\alpha(f) = 0$ , compare like powers of  $t$  to conclude that  $f = 0$ .
- (c) Show that  $\ker(\alpha) = I$ , so  $I$  is prime,  $V(I)$  is irreducible, and  $I(V(I)) = I$ .

*Proof of (a).*

- (1) Take any element  $\bar{f} \in k[x, y, z]/I$  where  $f \in k[x, y, z]$ . Regard  $f \in (k[y, z])[x]$ , By the division-with-remainder property of  $(k[y, z])[x]$ ,

$$f = (x^2 - y^3)q + r$$

where  $q, r \in (k[y, z])[x]$  and  $r = 0$  or  $\deg_x(r) < 2$ . In any case,  $r = xr_1 + r_0$  for some  $r_1, r_0 \in k[y, z]$ .

- (2) Apply the same argument to (1), we have

$$r_0 = (y^2 - z^3)q_0 + yc + a$$

$$r_1 = (y^2 - z^3)q_1 + yd + b$$

where  $q_0, q_1 \in k[y, z]$  and  $a, b, c, d \in k[z]$ .

- (3) By  $\bar{r}_0 = \overline{yc + a}$  and  $\bar{r}_1 = \overline{yd + b}$ ,

$$\begin{aligned} \bar{f} &= \bar{r} \\ &= \overline{xr_1} + \bar{r}_0 \\ &= \overline{x(yd + b)} + (\overline{yc + a}) \\ &= \bar{a} + \bar{b} \cdot \bar{x} + \bar{c} \cdot \bar{y} + \bar{d} \cdot \overline{xy}. \end{aligned}$$

□

*Proof of (b).* As  $0 = \alpha(f) = a + ct^6 + bt^9 + dt^{15} \in k[t]$ ,  $a = b = c = d = 0 \in k$ .

□

*Proof of (c).*

- (1)  $I \subseteq \ker(\alpha)$  is trivial.
- (2) Show that  $\ker(\alpha) \subseteq I$ . Take any  $f \in \ker(\alpha)$ , or  $\alpha(f) = 0$ . By (a),  $f = r + f_1$  where  $f_1 \in I$  and  $r = a + bx + cy + dxy \in k[x, y, z]$  for some  $a, b, c, d \in k[z]$ . Note that  $\alpha$  is a ring homomorphism. Therefore,

$$0 = \alpha(f) = \alpha(r + f_1) = \alpha(r) + \alpha(f_1) = \alpha(r).$$

By (b),  $r = 0 \in k[x, y, z]$  and thus  $f = f_1 \in I$ .

- (3) Therefore,

$$\alpha : k[x, y, z]/(x^2 - y^3, y^2 - z^3) \hookrightarrow k[t]$$

is injective.

□

## 1.8. Modules; Finiteness Conditions

### Problem 1.41.\*

If  $S$  is module-finite over  $R$ , then  $S$  is ring-finite over  $R$ .

*Proof.*

- (1) Write  $S = \sum Rs_i$  for some  $s_1, \dots, s_n \in S$  since  $S$  is module-finite over  $R$ .
- (2) Show that  $\sum Rs_i = R[s_1, \dots, s_n]$ .  $\sum Rs_i \subseteq R[s_1, \dots, s_n]$  is trivial. Conversely, take any  $v \in R[s_1, \dots, s_n]$ . Write

$$v = \sum_{(j)} \overbrace{a_{(j)} s_1^{j_1} \cdots s_n^{j_n}}^{\in \sum Rs_i}$$

$\in R \quad \in S = \sum Rs_i$

Here each term  $a_{(j)} s_1^{j_1} \cdots s_n^{j_n}$  is in  $\sum Rs_i$ . As  $\sum Rs_i$  is an  $R$ -module,

$$v = \sum_{(i)} a_{(i)} s_1^{i_1} \cdots s_n^{i_n} \in \sum Rs_i.$$

□

*Note.* The converse is not true (by Problem 1.42).

### Problem 1.42.

Show that  $S = R[x]$  (the ring of polynomials in one variable) is ring-finite over  $R$ , but not module-finite.

*Proof.*

- (1)  $S = R[x]$  is ring-finite over  $R$  by definition (as  $x \in S$ ).
- (2) (Reductio ad absurdum) If  $S = \sum Rs_i$  for some  $s_1, \dots, s_n \in S$  were module-finite over  $R$ . Any element  $s \in \sum Rs_i$  is of degree

$$\deg s \leq \max_{1 \leq i \leq n} \deg s_i := m.$$

So that  $x^{m+1} \in S = R[x]$  but not in  $\sum Rs_i$ , which is absurd.

□

**Problem 1.43.\***

If  $L$  is ring-finite over  $K$  ( $K, L$  fields) then  $L$  is a finitely generated field extension of  $K$ .

*Proof.*

- (1)  $L = K[v_1, \dots, v_n]$  for some  $v_i \in L$  since  $L$  is ring-finite over  $K$ .
- (2) Apply Proposition 4 in §1.10,  $L$  is module-finite (and hence algebraic) over  $K$ , that is,  $L = K[v_1, \dots, v_n] = K(v_1, \dots, v_n)$  is a finitely generated field extension of  $K$ .

□

**Problem 1.44.\***

Show that  $L = K(x)$  (the field of rational functions in one variable) is a finitely generated field extension of  $K$ , but  $L$  is not ring-finite over  $K$ . (Hint: If  $L$  were ring-finite over  $K$ , a common denominator of ring generators would be an element  $b \in K[x]$  such that for all  $z \in L$ ,  $b^n z \in K[x]$  for some  $n$ ; but let  $z = 1/c$ , where  $c$  doesn't divide  $b$  (Problem 1.5).)

*Proof.*

- (1) (Reductio ad absurdum) Suppose that  $L$  were ring-finite over  $K$ . Write  $L = K[v_1, \dots, v_m]$  where  $v_1, \dots, v_m \in L = K(x)$ . Let  $b \in K[x]$  be a common denominator of ring generators  $v_1, \dots, v_m$ . (So that all  $bv_i \in K[x]$ .) Therefore, for any  $z \in L = K[v_1, \dots, v_m]$ , there is an integer  $n > 0$  such that  $b^n z \in K[x]$ .
- (2) Consider  $z = 1/c \in K(x)$ , where  $c \in K[x]$  doesn't divide  $b$ . The existence of  $c$  is guaranteed by Problem 1.5. Hence, for any integer  $n > 0$

$$b^n z = b^n / c$$

is never in  $K[x]$  by the construction of  $c$ , which is absurd.

□

**Problem 1.45.\***

Let  $R$  be a subring of  $S$ ,  $S$  a subring of  $T$ .

- (a) If  $S = \sum Rv_i$ ,  $T = \sum Sw_j$ , show that  $T = \sum Rv_iw_j$ .
- (b) If  $S = R[v_1, \dots, v_n]$ ,  $T = S[w_1, \dots, w_m]$ , show that  $T = R[v_1, \dots, v_n, w_1, \dots, w_m]$ .
- (c) If  $R, S, T$  are fields, and  $S = R(v_1, \dots, v_n)$ ,  $T = S(w_1, \dots, w_m)$ , show that  $T = R(v_1, \dots, v_n, w_1, \dots, w_m)$ .

So each of the three finiteness conditions is a transitive relation.

*Proof of (a).*

- (1) Show that  $T \subseteq \sum Rv_iw_j$ . Given any  $t \in T = \sum Sw_j$ . There are some  $s_j \in S$  such that  $t = \sum_j s_j w_j$ . As  $s_j \in S = \sum Rv_i$ , there are some  $r_{ij} \in R$  such that  $s_j = \sum_i r_{ij} v_i$ . Hence,

$$t = \sum_j s_j w_j = \sum_j \left( \sum_i r_{ij} v_i \right) w_j = \sum_{i,j} r_{ij} v_i w_j \in \sum Rv_iw_j.$$

- (2) Show that  $T \supseteq \sum Rv_iw_j$ . Take any  $\sum r_{ij} v_i w_j \in \sum Rv_iw_j$ .

$$\sum r_{ij} v_i w_j = \sum_j \left( \sum_i r_{ij} v_i \right) w_j \in \sum_j Sw_j = T.$$

□

*Proof of (b).*

- (1) Note that  $R[x_1, \dots, x_m]$  is canonically isomorphic to  $R[x_1, \dots, x_{m-1}][x_m]$ . Hence  $R[x_1, \dots, x_m]$  is isomorphic to  $R[x_1][x_2] \cdots [x_m]$ .
- (2) Hence,

$$\begin{aligned} T &= S[w_1, \dots, w_m] \\ &= R[v_1, \dots, v_n][w_1, \dots, w_m] \\ &= R[v_1, \dots, v_n][w_1] \cdots [w_m] \\ &= R[v_1] \cdots [v_n][w_1] \cdots [w_m] \\ &= R[v_1, \dots, v_n, w_1, \dots, w_m]. \end{aligned}$$

□

*Proof of (c).*



- (1) By (b),  $R(v_1, \dots, v_n)$  is canonically isomorphic to  $R(v_1, \dots, v_{n-1})(v_n)$ . Hence  $R(v_1, \dots, v_n)$  is isomorphic to  $R(v_1) \cdots (v_n)$ . To see this, note that  $R[x_1, \dots, x_m] \cong R[x_1, \dots, x_{m-1}][x_m]$  implies that

$$R(x_1, \dots, x_m) \cong R[x_1, \dots, x_{m-1}](x_m) \hookrightarrow R(x_1, \dots, x_{m-1})(x_m).$$

Conversely, for any  $a/b \in R(x_1, \dots, x_{m-1})(x_m)$  where

$$\begin{aligned} a &= \sum_i a_i x_m^i \in R(x_1, \dots, x_{m-1})[x_m], \\ b &= \sum_j b_j x_m^j \in R(x_1, \dots, x_{m-1})[x_m] \end{aligned}$$

and  $b \neq 0$ , there is a nonzero polynomial  $c \in R[x_1, \dots, x_{m-1}]$  such that all  $ca_i$  and  $cb_j$  are in  $R[x_1, \dots, x_{m-1}]$ . Hence,

$$\begin{aligned} \frac{a}{b} &= \frac{\sum_i a_i x_m^i}{\sum_j b_j x_m^j} \\ &= \frac{c \sum_i a_i x_m^i}{c \sum_j b_j x_m^j} \\ &= \frac{\sum_i ca_i x_m^i}{\sum_j cb_j x_m^j} \\ &\in R[x_1, \dots, x_{m-1}](x_m). \end{aligned}$$

(2) Hence,

$$\begin{aligned} T &= S(w_1, \dots, w_m) \\ &= R(v_1, \dots, v_n)(w_1, \dots, w_m) \\ &= R(v_1, \dots, v_n)(w_1) \cdots (w_m) \\ &= R(v_1) \cdots (v_n)(w_1) \cdots (w_m) \\ &= R(v_1, \dots, v_n, w_1, \dots, w_m). \end{aligned}$$

□

## 1.9. Integral Elements

### Problem 1.46.\* (Transitivity of integral extensions)

Let  $R$  be a subring of  $S$ ,  $S$  a subring of (a domain)  $T$ . If  $S$  is integral over  $R$ , and  $T$  is integral over  $S$ , show that  $T$  is integral over  $R$ . (Hint: Let  $z \in T$ , so we have  $z^n + a_1 z^{n-1} + \cdots + a_n = 0$ ,  $a_i \in S$ . Then  $R[a_1, \dots, a_n, z]$  is module-finite

over  $R$ .)

*Proof (Hint).*

- (1) Let  $z \in T$ , so we have  $z^n + a_1 z^{n-1} + \cdots + a_n = 0$ ,  $a_i \in S$ . Therefore,  $z$  is integral over  $R[a_1, \dots, a_n]$ , or  $R[a_1, \dots, a_n, z]$  is module-finite over  $R[a_1, \dots, a_n]$ .
- (2) Show that  $R[a_1, \dots, a_n]$  is module-finite over  $R$  if all  $a_i \in S$ . Note that

$$\begin{aligned} a_1 &\text{ is integral over } R, \\ a_2 &\text{ is integral over } R[a_1] \supseteq R, \\ &\dots \\ a_n &\text{ is integral over } R[a_1, \dots, a_{n-1}]. \end{aligned}$$

By Proposition 3,

$$\begin{aligned} R[a_1] &\text{ is module-finite over } R, \\ R[a_1][a_2] &\text{ is module-finite over } R[a_1], \\ &\dots \\ R[a_1, \dots, a_{n-1}][a_n] &\text{ is module-finite over } R[a_1, \dots, a_{n-1}]. \end{aligned}$$

Also note that  $R[a_1, \dots, a_i] = R[a_1, \dots, a_{i-1}][a_i]$  if  $i > 1$ . By the transitive relation of the module-finiteness (Problem 1.45),  $R[a_1, \dots, a_n]$  is module-finite over  $R$ .

- (3) Again by the transitive relation of the module-finiteness (Problem 1.45),  $R[a_1, \dots, a_n, z]$  is module-finite over  $R$ . Hence,  $R[a_1, \dots, a_n, z]$  is a subring of  $T$  containing  $R[z]$  which is module-finite over  $R$ . By Proposition 3,  $z$  is integral over  $R$ .

□

#### **Problem 1.47.\***

Suppose (a domain)  $S$  is ring-finite over  $R$ . Show that  $S$  is module-finite over  $R$  if and only if  $S$  is integral over  $R$ .

*Proof.*

- (1) Write  $S = R[v_1, \dots, v_m]$  for some  $v_i \in S$ .
- (2) Suppose that  $S$  is integral over  $R$ . Then all  $v_i$  are integral over  $R$ . Use the same argument in Problem 1.46, we have

$$S = R[v_1, \dots, v_n]$$

is module-finite over  $R$ .

- (3) Conversely, suppose that  $S$  is module-finite over  $R$ . Take any  $v \in S$ . Write  $v = \sum_i r_i v_i \in S$  since  $S$  is module-finite over  $R$ . Note that  $S = R[v_1, \dots, v_m]$  is a subring of  $S$  itself containing  $R[v]$  which is module-finite over  $R$ . By Proposition 3,  $v$  is integral over  $R$ .

□

**Problem 1.48.\***

Let  $L$  be a field,  $k$  an algebraically closed subfield of  $L$ .

- (a) Show that any element of  $L$  that is algebraic over  $k$  is already in  $k$ .  
 (b) An algebraically closed field has no module-finite field extensions except itself.

*Proof of (a).*

- (1) Let  $\alpha \in L$  be algebraic over  $k$ . Then there is a nonzero polynomial  $f(x) \in k[x]$  with  $f(\alpha) = 0$ . Note that  $\deg f \geq 1$ .  
 (2) Since  $k$  is algebraically closed, every polynomial is a product of first degree polynomials, say

$$f(x) = c(x - \alpha_1) \cdots (x - \alpha_m)$$

where  $c \in k - \{0\}$  and  $\alpha_1, \dots, \alpha_m \in k$ . As  $f(\alpha) = 0$ ,  $\alpha = \alpha_i \in k$  for some  $1 \leq i \leq m$ . Hence,  $\alpha \in L$  is algebraic over  $k$  implies that  $\alpha \in k$ .

□

*Proof of (b).*

- (1) Suppose that  $L$  is module-finite field extensions of an algebraically closed field  $k$ .  
 (2) By Problem 1.41,  $L$  is ring-finite over  $k$ . By Problem 1.47,  $L$  is integral or algebraic over  $k$  (since  $k$  is a field). By (a),  $L = k$ .

□

**Problem 1.49.\***

Let  $K$  be a field,  $L = K(x)$  the field of rational functions in one variable over  $K$ .

- (a) Show that any element of  $L$  that is integral over  $K[x]$  is already in  $K[x]$ .  
(Hint: If  $z^n + a_1z^{n-1} + \cdots + a_n = 0$ , write  $z = f/g$ ,  $f, g$  relatively prime. Then  $f^n + a_1f^{n-1}g + \cdots + a_ng^n = 0$ , So  $g$  divides  $f$ .)
- (b) Show that there is no nonzero element  $f \in K[x]$  such that for every  $z \in L$ ,  $f^n z$  is integral over  $K[x]$  for some  $n > 0$ . (Hint: See Problem 1.44.)

*Proof of (a).*

- (1) Note that 0 is integral over  $K[x]$  and  $0 \in K[x]$  trivially.
- (2) Now we take any nonzero element  $z \in L = K(x)$  which is integral over  $K[x]$ . So  $z^n + a_1z^{n-1} + \cdots + a_n = 0$  for some  $a_1, \dots, a_n \in K[x]$  and  $a_n \neq 0$  (since  $z \neq 0$ ).
- (3) Write  $z = f/g$ ,  $f, g$  relatively prime in  $K[x]$ . Then

$$f^n + a_1f^{n-1}g + \cdots + a_ng^n = 0 \in K[x].$$

Since  $a_n \neq 0$ ,  $g \mid f^n$  or  $g \mid f$  or  $g = 1 \in K$ . Therefore,  $z = f \in K[x]$ .

□

*Proof of (b).*

- (1) (Reductio ad absurdum) Suppose there were a nonzero element  $f \in K[x]$  such that for every  $z \in L$ ,  $f^n z$  is integral over  $K[x]$  for some  $n > 0$ .
- (2) Let  $z = 1/g \in K(x)$ , where  $g$  is an irreducible polynomial not dividing  $f$ . The existence of  $g$  is guaranteed by Problem 1.5.
- (3) By the hypothesis in (1), there is an integer  $n > 0$  such that  $f^n z$  is integral over  $K[x]$ . By (a),  $f^n z = f^n/g$  is also in  $K[x]$ . So  $g \mid f^n$  or  $g \mid f$ , which is absurd.

□

### Problem 1.50.\*

Let  $K$  be a subfield of a field  $L$ .

- (a) Show that the set of elements of  $L$  that are algebraic over  $K$  is a subfield of  $L$  containing  $K$ . (Hint: If  $v^n + a_1v^{n-1} + \cdots + a_n = 0$ , and  $a_n \neq 0$ , then  $v(v^{n-1} + \cdots + a_{n-1}) = -a_n$ .)
- (b) Suppose  $L$  is module-finite over  $K$ , and  $K \subseteq R \subseteq L$ ,  $R$  a subring of  $L$ . Show that  $R$  is a field.

*Proof of (a).*

- (1) Let  $R$  be the set of elements of  $L$  that are algebraic over  $K$ . By Corollary to Proposition 3,  $R$  is a subring of  $L$  containing  $K$ . (Note that  $K$  is a field.) So it suffices to show that  $v^{-1} \in R$  if  $v \in R - \{0\}$ .
- (2) Since  $v$  is algebraic over  $K$ , we can write

$$v^n + a_1 v^{n-1} + \cdots + a_n = 0$$

for some  $a_1, \dots, a_n \in K$  and  $a_n \neq 0$ . So

$$(v^{-1})^n + \underbrace{\frac{a_{n-1}}{a_n}}_{\in K} (v^{-1})^{n-1} + \cdots + \underbrace{\frac{a_1}{a_n}}_{\in K} (v^{-1}) + \underbrace{\frac{1}{a_n}}_{\in K} = 0,$$

or  $v^{-1}$  is integral over  $K$ . Hence,  $v^{-1} \in R$ .

□

*Proof of (b).*

- (1) By Problem 1.47,  $L$  is algebraic over  $K$ . Hence,  $R$  is algebraic over  $K$ .
- (2) To show that  $R$  is a field, it suffices to show that  $v^{-1} \in R$  if  $v \in R - \{0\}$ . Since  $v$  is algebraic over  $K$ , we can write

$$v^n + a_1 v^{n-1} + \cdots + a_n = 0$$

for some  $a_1, \dots, a_n \in K$  and  $a_n \neq 0$ . So

$$v \left( - \underbrace{\frac{1}{a_n}}_{\in K \subseteq R} \underbrace{v^{n-1}}_{\in R} - \cdots - \underbrace{\frac{a_{n-1}}{a_n}}_{\in K \subseteq R} \right) = 1.$$

Here  $v^{-1} = \left( -\frac{1}{a_n} v^{n-1} - \cdots - \frac{a_{n-1}}{a_n} \right)$  is the inverse of  $v$  in  $R$  (since  $R$  is a ring containing  $K$ ).

□

## 1.10. Field Extensions

### Problem 1.51.\*

Let  $K$  be a field,  $f \in K[x]$  an irreducible monic polynomial of degree  $n > 0$ .

- (a) Show that  $L = K[x]/(f)$  is a field, and if  $\bar{x}$  is the residue of  $x$  in  $L$ , then  $f(\bar{x}) = 0$ .
- (b) Suppose  $L'$  is a field extension of  $K$ ,  $y \in L'$  such that  $f(y) = 0$ . Show that the homomorphism from  $K[x]$  to  $L'$  that takes  $x$  to  $y$  induces an isomorphism of  $L$  with  $K(y)$ .
- (c) With  $L'$ ,  $y$  as in (b), suppose  $g \in K[x]$  and  $g(y) = 0$ . Show that  $f$  divides  $g$ .
- (d) Show that  $f = (x - \bar{x})f_1$ ,  $f_1 \in L[x]$ .

*Proof of (a).*

- (1)  $(f)$  is a prime ideal in a UFD  $K[x]$  since  $f$  is irreducible. Note that  $K[x]$  is also a PID,  $(f)$  is maximal (Problem 1.3). Hence  $L = K[x]/(f)$  is a field.
- (2)  $f(\bar{x}) = f(x) + (f(x)) = (f(x)) = \bar{0}$ .

□

*Proof of (b).*

- (1) Let  $\alpha : K[x] \rightarrow L'$  be a homomorphism defined by

$$\alpha\left(\sum a_i x^i\right) = \sum a_i y^i$$

where  $a_i \in K$ .  $\text{im}(\alpha) = K(y)$  clearly.

- (2) Note that  $\ker(\alpha)$  is an ideal containing  $(f)$  since  $\alpha(f) = 0$ .  $\ker(\alpha)$  is proper since  $\alpha(1) = 1 \neq 0$ . By the maximality of  $(f)$ ,  $\ker(\alpha) = (f)$ .
- (3) Hence,  $\alpha$  induces an isomorphism of  $L$  with  $K(y)$ :

$$L = K[x]/(f) \cong K(y) \hookrightarrow L'.$$

□

*Proof of (c).* By (b),  $g \in \ker(\alpha) = (f)$ . So  $f \mid g$ . □

*Proof of (d).*

- (1) By (a),  $\bar{x} \in L$  is a root of  $f \in L[x]$  (by embedding  $K[x]$  in  $L[x]$ ).
- (2) Since  $L$  is a field, by Problem 1.7(b) we have

$$f = (x - \bar{x})f_1$$

for some  $f_1 \in L[x]$ .

□

**Problem 1.52.\* (Splitting fields)**

Let  $K$  be a field,  $f \in K[x]$ . Show that there is a field  $L$  containing  $K$  such that  $f = \prod_{i=1}^n (x - x_i) \in L[x]$ . (Hint: Use Problem 1.51(d) and induction on the degree.)  $L$  is called a **splitting field** of  $F$ .

*Proof.*

- (1) Let  $p(x) \in K[x]$  be an irreducible factor of  $f(x) \in K[x]$ , and let  $L'$  be the field  $K[x]/(p(x))$  (by Problem 1.51(a)).
- (2) Then we might regard  $K$  as a subfield of  $L'$  by sending  $a \in K$  to  $\bar{a} = a + (p(x)) \in L'$ .
- (3) By Problem 1.51(a),  $\bar{x}$  is a root of  $p \in L'$ ; therefore is a root of  $f$ .
- (4) Induction on  $n$ . By (1)(2)(3), there is a field  $L' \supseteq K$  such that  $L'$  contains a root  $\bar{x}$  of  $f(x)$ , say  $f(x) = (x - \bar{x})f_1(x)$  over  $L'[x]$  (by Problem 1.51(d)). By induction, there is a field  $L \supseteq L'$  such that  $f_1$  splits over  $L$ . Hence,  $f$  splits over  $L$ .

□

**Problem 1.53.\***

Suppose  $K$  is a field of characteristic zero,  $f$  an irreducible monic polynomial in  $K[x]$  of degree  $n > 0$ . Let  $L$  be a splitting field of  $f$ , so  $f = \prod_{i=1}^n (x - x_i)$ ,  $x_i \in L$ . Show that the  $x_i$  are distinct. (Hint: Apply Problem 1.51(c) to  $g = f_x$ ; if  $(x - \bar{x})^2$  divides  $f$ , then  $g(\bar{x}) = 0$ .)

*Proof.*

- (1) Since  $f \in K[x]$  is irreducible over  $K$ ,  $\gcd(f, f_x)$  is 1 or  $f$ . As  $\text{char}(K) = 0$ ,  $\deg(f_x) = \deg(f) - 1$ . So  $f$  does not divide  $f_x$  or  $\gcd(f, f_x) = 1$ . Hence, there are polynomials  $g, h \in K[x]$  such that

$$1 = fg + f_x h.$$

This equation is also true in  $L[x]$ .

- (2) Note that

$$f = \prod_{i=1}^n (x - x_i) \in L[x],$$

$$f_x = \sum_{i=1}^n (x - x_1) \cdots \widehat{(x - x_i)} \cdots (x - x_n) \in L[x].$$

If  $\bar{x}$  were a multiple root of  $f$ , then  $f(\bar{x}) = f_x(\bar{x}) = 0$ . By (1),

$$1 = f(\bar{x})g(\bar{x}) + f_x(\bar{x})h(\bar{x}) = 0,$$

which is absurd.

□

**Problem 1.54.\***

Let  $R$  be a domain with quotient field  $K$ , and let  $L$  be a finite algebraic extension of  $K$ .

- (a) For any  $v \in L$ , show that there is a nonzero  $a \in R$  such that  $av$  is integral over  $R$ .
- (b) Show that there is a basis  $v_1, \dots, v_n$  for  $L$  over  $K$  (as a vector space) such that each  $v_i$  is integral over  $R$ .

*Proof of (a).*

- (1) Take any  $v \in L$ , which is algebraic over  $K$ . Write

$$v^n + a_1v^{n-1} + \dots + a_n = 0$$

for some  $a_1, \dots, a_n \in K$  and  $a_n \neq 0$ . Since  $K$  is the quotient field of  $R$ , there is a common denominator  $a \in R$  of  $a_1, \dots, a_n$ . Here  $a \neq 0$  and  $aa_i \in R$  for all  $1 \leq i \leq n$ .

- (2) Hence,

$$\begin{aligned} & a^n v^n + a^n a_1 v^{n-1} + \dots + a^n a_n = 0 \\ \iff & (av)^n + \underbrace{(aa_1)}_{\in R} (av)^{n-1} + \underbrace{a(aa_2)}_{\in R} (av)^{n-2} + \dots + \underbrace{a^{n-1}(aa_n)}_{\in R} = 0. \end{aligned}$$

$av$  is integral over  $R$ .

□

*Proof of (b).*

- (1) Since  $L$  be a finite algebraic extension of  $K$ , there exists a basis

$$\{w_1, \dots, w_n\}$$

for  $L$  over  $K$  (as a vector space).



- (2) For each  $w_i \in L$ , there is a nonzero  $a_i \in R$  such that  $a_i w_i$  is integral over  $R$  (by (a)). So it suffices to show that

$$\{a_1 w_1, \dots, a_n w_n\}$$

is also a basis for  $L$  over  $K$ .

- (3) Suppose

$$0 = \sum_i \alpha_i (a_i w_i) = \sum_i (\alpha_i a_i) w_i$$

for some  $\alpha_1, \dots, \alpha_n \in K$ . Since  $\{w_1, \dots, w_n\}$  is a basis,  $\alpha_i a_i = 0$  for all  $i$ , or  $\alpha_i = 0$  for all  $i$  (since all  $a_i \neq 0$ ). Hence  $\{a_1 w_1, \dots, a_n w_n\}$  is linearly independent.

- (4) Also, for any  $w \in L$ , we can write

$$\begin{aligned} w &= \underbrace{\beta_1}_{\in K} w_1 + \dots + \underbrace{\beta_n}_{\in K} w_n \\ &= \underbrace{\frac{\beta_1}{a_1}}_{\in K} (a_1 w_1) + \dots + \underbrace{\frac{\beta_n}{a_n}}_{\in K} (a_n w_n) \end{aligned}$$

as a linear combination of  $\{a_1 w_1, \dots, a_n w_n\}$  over  $K$ .

□

## Chapter 2: Affine Varieties

### 2.1. Coordinate Rings

#### Problem 2.1.\*

Show that the map which associates to each  $f \in k[x_1, \dots, x_n]$  a polynomial function in  $\mathcal{F}(V, k)$  is a ring homomorphism whose kernel is  $I(V)$ .

*Proof.*

- (1) Define a map  $\alpha : k[x_1, \dots, x_n] \rightarrow \mathcal{F}(V, k)$ . Every polynomial  $f \in k[x_1, \dots, x_n]$  defines a function from  $V$  to  $k$  by

$$\alpha(f)(a_1, \dots, a_n) = f(a_1, \dots, a_n)$$

for all  $(a_1, \dots, a_n) \in V$ .

- (2)  $\alpha$  is a ring homomorphism by construction in (1).  
(3) Show that  $\ker(\alpha) = I(V)$ . In fact, given any  $f \in k[x_1, \dots, x_n]$ , we have  $\alpha(f) = 0$  (sending all  $a \in V$  to  $0 \in k$ ) if and only if  $f(a) = 0$  for all  $a \in V$  if and only if  $f \in I(V)$ .  
(4) Hence,

$$k[x_1, \dots, x_n]/I(V) = \Gamma(V) \cong \{\text{polynomial functions in } \mathcal{F}(V, k)\}$$

as a ring isomorphism.

□

#### Problem 2.2.\*

Let  $V \subseteq \mathbf{A}^n$  be a variety. A **subvariety** of  $V$  is a variety  $W \subseteq \mathbf{A}^n$  that is contained in  $V$ . Show that there is a natural one-to-one correspondence between algebraic subsets (resp. subvarieties, resp. points) of  $V$  and radical ideals (resp. prime ideals, resp. maximal ideals) of  $\Gamma(V)$ . (See Problems 1.22, 1.38.)

*Proof.* Repeat Problem 1.38 by replacing  $k[x_1, \dots, x_n]/I$  by  $\Gamma(V)$ . □

**Problem 2.3.\***

Let  $W$  be a subvariety of a variety  $V$ , and let  $I_V(W)$  be the ideal of  $\Gamma(V)$  corresponding to  $W$ .

- (a) Show that every polynomial function on  $V$  restricts to a polynomial function on  $W$ .
- (b) Show that the map from  $\Gamma(V)$  to  $\Gamma(W)$  defined in part (a) is a surjective homomorphism with kernel  $I_V(W)$ , so that  $\Gamma(W)$  is isomorphic to  $\Gamma(V)/I_V(W)$ .

*Proof of (a).*

- (1) Given any polynomial function  $f \in \mathcal{F}(V, k)$  on  $V$ . There is a polynomial  $g \in k[x_1, \dots, x_n]$  such that  $f(P) = g(P)$  for all  $P \in V \supseteq W$ ; thus  $f(P) = g(P)$  for all  $P \in W$ , or  $f|_W$  is a polynomial function on  $W$ .
- (2) The map  $\alpha : \{\text{polynomial functions in } \mathcal{F}(V, k)\} \rightarrow \{\text{polynomial functions in } \mathcal{F}(W, k)\}$  in (1) is defined by

$$\alpha(f) = f|_W.$$

It is a ring homomorphism.

□

*Proof of (b).*

- (1) Identify  $\Gamma(V)$  (resp.  $\Gamma(W)$ ) with the set of all polynomial functions in  $\mathcal{F}(V, k)$  (resp. in  $\mathcal{F}(W, k)$ ) by Problem 2.1. The map

$$\alpha : \Gamma(V) \rightarrow \Gamma(W)$$

is defined by

$$\alpha(f + I(V)) = f + I(W).$$

It is well-defined by (a).

- (2) Show that  $\alpha$  is surjective. For any  $f + I(W) \in \Gamma(W)$ , take  $f + I(V) \in \Gamma(V)$  and then  $\alpha(f + I(V)) = f + I(W)$ . (The choice of  $f + I(V)$  depends on the representation of  $f + I(W)$  and thus might not be unique.)
- (3) Show that  $\ker(\alpha) = I_V(W)$ , and thus  $\Gamma(W) \cong \Gamma(V)/I_V(W)$ . Since  $\alpha$  is a surjective homomorphism,

$$\begin{aligned} \ker(\alpha) &= \Gamma(V)/\Gamma(W) \\ &= (k[x_1, \dots, x_n]/I(V))/(k[x_1, \dots, x_n]/I(W)) \\ &= I(W)/I(V) \\ &= I_V(W). \end{aligned}$$

□

**Problem 2.4.\***

Let  $V \subseteq \mathbf{A}^n$  be a nonempty variety. Show that the following are equivalent:

- (i)  $V$  is a point.
- (ii)  $\Gamma(V) = k$ .
- (iii)  $\dim_k \Gamma(V) < \infty$ .

*Proof.*

- (1) (i)  $\implies$  (ii). By Corollary 2 to the Hilbert's Nullstellensatz in §1.7,  $V = \{(a_1, \dots, a_n)\}$  corresponds to the maximal ideal

$$I(V) = (x_1 - a_1, \dots, x_n - a_n)$$

in  $k[x_1, \dots, x_n]$ . Hence,

$$\Gamma(V) = k[x_1, \dots, x_n]/(x_1 - a_1, \dots, x_n - a_n) \cong k$$

(by Problem 1.24).

- (2) (ii)  $\implies$  (iii).  $\dim_k(\Gamma(V)) = \dim_k(k) = 1 < \infty$ .
- (3) (iii)  $\implies$  (i). By Corollary 4 to the Hilbert's Nullstellensatz in §1.7,  $V$  is a finite set of points in  $\mathbf{A}^n$ . Since  $V$  is a nonempty variety,  $V$  is exactly a point.

□

**Problem 2.5.**

Let  $f$  be an irreducible polynomial in  $k[x, y]$ , and suppose  $f$  is monic in  $y$ :  $f = y^n + a_1(x)y^{n-1} + \dots + a_n(x)$ , with  $n > 0$ . Let  $V = V(f) \subseteq \mathbf{A}^2$ . Show that the natural homomorphism from  $k[x]$  to  $\Gamma(V) = k[x, y]/(f)$  is one-to-one, so that  $k[x]$  may be regarded as a subring of  $\Gamma(V)$ ; show that the residues  $\bar{1}, \bar{y}, \dots, \bar{y}^{n-1}$  generate  $\Gamma(V)$  over  $k[x]$  as a module.

*Proof.*

- (1)  $\Gamma(V) = k[x, y]/(f)$  is well-defined since  $f$  is irreducible. Define a ring homomorphism  $\alpha : k[x] \rightarrow \Gamma(V) = k[x, y]/(f)$  by

$$\alpha : g(x) \mapsto g(x) + (f(x, y)).$$

- (2) *Show that  $\alpha$  is one-to-one.* If there were a nonzero polynomial  $g \in k[x]$  such that  $\alpha(g) = 0$ , then  $g = fh$  for some nonzero polynomial  $h \in k[x, y]$ . Hence

$$0 = \deg_y(g) = \deg_y(f) + \deg_y(h) \geq n > 0,$$

which is absurd. Therefore,  $\alpha$  is one-to-one. Hence  $k[x]$  may be regarded as a subring of  $\Gamma(V)$ , and thus the multiplication in  $\Gamma(V)$  makes  $\Gamma(V)$  a  $k[x]$ -module.

- (3) Given any  $g(x, y) + (f(x, y)) \in k[x, y]/(f)$  where  $g \in k[x, y] = (k[x])[y]$ . By the division-with-remainder property of  $(k[x])[y]$ ,

$$g = fq + r$$

for some  $q, r \in (k[x])[y]$  and

$$r = r_1(x)y^{n-1} + \cdots + r_n(x)$$

where  $r_1, \dots, r_n \in k[x]$ . Hence

$$\begin{aligned} g + (f) &= fq + r + (f) \\ &= r + (f) \\ &= r_1(x)y^{n-1} + \cdots + r_n(x) + (f) \\ &= \underbrace{r_1(x)}_{\in k[x]} \bar{y}^{n-1} + \cdots + \underbrace{r_n(x)}_{\in k[x]} \bar{1}, \end{aligned}$$

which means that the residues  $\bar{1}, \bar{y}, \dots, \bar{y}^{n-1}$  generate  $\Gamma(V)$  over  $k[x]$  as a module.

□

## 2.2. Polynomial Maps

### Problem 2.6.\*

Let  $\varphi : V \rightarrow W$ ,  $\psi : W \rightarrow Z$ . Show that  $\widetilde{\psi \circ \varphi} = \widetilde{\varphi} \circ \widetilde{\psi}$ . Show that the composition of polynomial maps is a polynomial map.

*Proof.*

- (1) *Show that  $\widetilde{\psi \circ \varphi} = \widetilde{\varphi} \circ \widetilde{\psi}$ .* It is equivalent to show that

$$(\widetilde{\psi \circ \varphi})(f) = (\widetilde{\varphi} \circ \widetilde{\psi})(f)$$

for all  $f \in \mathcal{F}(Z, k)$ . In fact,

$$\begin{aligned} (\widetilde{\psi \circ \varphi})(f) &= f \circ \psi \circ \varphi, \\ (\widetilde{\varphi} \circ \widetilde{\psi})(f) &= \widetilde{\varphi}(\widetilde{\psi}(f)) = \widetilde{\varphi}(f \circ \psi) = f \circ \psi \circ \varphi. \end{aligned}$$

- (2) Show that the composition of polynomial maps is a polynomial map. Say  $V \subseteq \mathbf{A}^n, W \subseteq \mathbf{A}^m, Z \subseteq \mathbf{A}^r$ . Since  $\varphi$  (resp.  $\psi$ ) is a polynomial map, there are polynomials  $t_1, \dots, t_m \in k[x_1, \dots, x_n]$  (resp.  $s_1, \dots, s_r \in k[x_1, \dots, x_m]$ ) such that

$$\begin{aligned}\varphi(P) &= (t_1(P), \dots, t_m(P)) \\ \psi(Q) &= (s_1(Q), \dots, s_r(Q))\end{aligned}$$

for all  $P \in V$  (resp.  $Q \in W$ ). Hence the composition  $\psi \circ \varphi$  is

$$\begin{aligned}(\psi \circ \varphi)(P) &= \psi(\varphi(P)) \\ &= \psi(t_1(P), \dots, t_m(P)) \\ &= (s_1(t_1(P), \dots, t_m(P)), \dots, s_r(t_1(P), \dots, t_m(P))).\end{aligned}$$

So there are polynomials  $y_1, \dots, y_r \in k[x_1, \dots, x_n]$  defined by

$$y_i(P) = s_i(t_1(P), \dots, t_m(P))$$

for all  $(a_1, \dots, a_n) \in \mathbf{A}^n$  such that

$$(\psi \circ \varphi)(P) = (y_1(P), \dots, y_r(P)).$$

(Note that the composition of polynomials is a polynomials.) Hence  $\psi \circ \varphi$  is a polynomial map.

□

### Problem 2.7.\*

If  $\varphi : V \rightarrow W$  is a polynomial map, and  $X$  is an algebraic subset of  $W$ , show that  $\varphi^{-1}(X)$  is an algebraic subset of  $V$ . If  $\varphi^{-1}(X)$  is irreducible, and  $X$  is contained in the image of  $\varphi$ , show that  $X$  is irreducible. This gives a useful test for irreducibility.

*Proof.*

- (1) Show that  $\varphi^{-1}(X) = V(\tilde{\varphi}(I(X)))$  is algebraic.

$$\begin{aligned}P \in \varphi^{-1}(X) &\iff \varphi(P) \in X \\ &\iff f(\varphi(P)) = 0 \forall f \in I(X) \\ &\iff \tilde{\varphi}(f)(P) = 0 \forall f \in I(X) \\ &\iff g(P) = 0 \forall g \in \tilde{\varphi}(I(X)) \\ &\iff P \in V(\tilde{\varphi}(I(X))).\end{aligned}$$

Also note that  $\tilde{\varphi}(I(X))$  is an ideal in  $k[x_1, \dots, x_n]$  since  $\varphi$  is a polynomial map.

- (2) If  $\varphi^{-1}(X)$  is irreducible, and  $X$  is contained in the image of  $\varphi$ , show that  $X$  is irreducible. (Reductio ad absurdum) Suppose that  $X$  were reducible or  $I(X)$  were not prime. So that there exist two polynomials  $f_1, f_2 \notin I(X)$  but  $f_1 f_2 \in I(X)$ . By definition of  $I(X)$ , there exist two points  $P_1, P_2 \in X$  such that  $f_i(P_i) \neq 0$  for  $i = 1, 2$ .
- (3) Since  $X$  is contained in the image of  $\varphi$ , there are two corresponding points  $Q_1, Q_2 \in \varphi^{-1}(X)$  such that  $\varphi(Q_i) = P_i$ . So  $\tilde{\varphi}(f_i)(Q_i) = f_i(P_i) \neq 0$ , or  $\tilde{\varphi}(f_i) \notin I(\varphi^{-1}(X))$ . However

$$\tilde{\varphi}(f_1)\tilde{\varphi}(f_2) = \tilde{\varphi}(f_1 f_2) \in I(\varphi^{-1}(X))$$

since  $f_1 f_2 \in I(X)$ , contrary to the primality of  $I(\varphi^{-1}(X))$ .

□

**Problem 2.8.**

- (a) Show that  $\{(t, t^2, t^3) \in \mathbf{A}^3(k) : t \in k\}$  is an affine variety.
- (b) Show that  $V(xz - y^2, yz - x^3, z^2 - x^2y) \subseteq \mathbf{A}^3(\mathbb{C})$  is a variety. (Hint:  $y^3 - x^4, z^3 - x^5, z^4 - y^5 \in I(V)$ . Find a polynomial map from  $\mathbf{A}^1(\mathbb{C})$  onto  $V$ .)

*Proof of (a).*

- (1) Let  $Y := \{(t, t^2, t^3) \in \mathbf{A}^3(k) : t \in k\}$  be the twisted cubic curve. By Problem 2.7, it suffices to show that there is a polynomial map from  $\mathbf{A}^1(k)$  onto  $Y$ . Here we use the fact that  $\mathbf{A}^1(k)$  is irreducible as  $k = \bar{k}$  is infinite (by Problem 1.29).
- (2) Define a mapping  $\varphi$  from  $\mathbf{A}^1(k)$  to  $Y$  by  $\varphi(t) = (t, t^2, t^3) \in Y$ .  $\varphi$  is a polynomial map. Also,  $\varphi$  is surjective.

□

*Note.* Also see Problems 1.11 and 1.33 (for the case  $k = \mathbb{C}$ ).

*Proof of (b).*

- (1) We prove for any algebraically closed field  $k$ .
- (2) Write

$$\begin{aligned} V &= V(xz - y^2, yz - x^3, z^2 - x^2y), \\ Y &= \{(t^3, t^4, t^5) \in \mathbf{A}^3(k) : t \in k\}. \end{aligned}$$

We want to show that  $Y = V$ .  $Y \subseteq V$  is trivial. Now given any  $(x, y, z) \in V$ . If  $x = 0$ , then  $y = z = 0$ . So  $(x, y, z) = (0, 0, 0) \in Y$ . If  $x \neq 0$ , define

$$t = \frac{y}{x} \in k.$$

Hence,

$$\begin{aligned} t^3 &= \frac{y^3}{x^3} = \frac{y(xz)}{x^3} = \frac{yz}{x^2} = \frac{x^3}{x^2} = x, \\ t^4 &= tx = y, \\ t^5 &= ty = \frac{y^2}{x} = \frac{xz}{x} = z. \end{aligned}$$

- (3) Same as (a). Define a mapping  $\varphi$  from  $\mathbf{A}^1(k)$  to  $Y = V$  by  $\varphi(t) = (t^3, t^4, t^5) \in Y = V$ .

□

*Note.*

- (1) We don't use the hint.
- (2) In fact, it is easy to show that

$$Y = V(y^3 - x^4, z^3 - x^5, z^4 - y^5).$$

- (3)  $I(V)$  is a prime ideal of height 2 in  $k[x, y, z]$  which cannot be generated by 2 elements. We say  $V$  is **not a local complete intersection**.

### Problem 2.9.\*

Let  $\varphi : V \rightarrow W$  be a polynomial map of affine varieties,  $V' \subseteq V$ ,  $W' \subseteq W$  subvarieties. Suppose  $\varphi(V') \subseteq W'$ .

- (a) Show that  $\tilde{\varphi}(I_W(W')) \subseteq I_V(V')$  (see Problems 2.3).
- (b) Show that the restriction of  $\varphi$  gives a polynomial map from  $V'$  to  $W'$ .

*Proof of (a).*

- (1) It suffices to show that  $f \in I_V(V')$  for any  $f = \tilde{\varphi}(g) \in \tilde{\varphi}(I_W(W'))$  for some  $g \in I_W(W')$ .
- (2) To show  $f \in I_V(V')$ , it suffices to show that  $f(P) = 0$  for all  $P \in \varphi(V')$ . In fact,

$$f(P) = \tilde{\varphi}(g)(P) = g(\varphi(P)) = 0$$

since  $\varphi(V') \subseteq W'$  and  $g \in I_W(W')$ .



□

*Proof of (b).*

(1) Similar to Problem 2.3.

(2) Since  $\varphi$  is a polynomial map, there are polynomials  $t_1, \dots, t_m \in k[x_1, \dots, x_n]$  such that

$$\varphi(P) = (t_1(P), \dots, t_m(P)) \in W$$

for all  $P \in V$ . So that  $\varphi|_{V'} : V' \rightarrow \varphi(V') \subseteq W'$  is also a polynomial map which is equipped with the same polynomials  $t_1, \dots, t_m$  such that

$$\varphi(P) = (t_1(P), \dots, t_m(P)) \in W' \subseteq W$$

for all  $P \in V' \subseteq V$ . (Note that both  $V'$  and  $W'$  are affine varieties.)

□

**Problem 2.10.\***

Show that the **projection map**  $\text{pr} : \mathbf{A}^n \rightarrow \mathbf{A}^r$ ,  $n \geq r$ , defined by  $\text{pr}(a_1, \dots, a_n) = (a_1, \dots, a_r)$  is a polynomial map.

*Proof.*

(1) Define  $t_i \in k[x_1, \dots, x_n]$  by  $t_i(x_1, \dots, x_n) = x_i$  for  $i = 1, \dots, r$ .

(2) Clearly,

$$\text{pr}(P) = (t_1(P), \dots, t_r(P))$$

for  $P = (a_1, \dots, a_n) \in \mathbf{A}^n$ , and thus  $\text{pr}$  is a polynomial map.

□

**Problem 2.11.**

Let  $f \in \Gamma(V)$ ,  $V$  a variety  $\subseteq \mathbf{A}^n$ . Define

$$G(f) = \{(a_1, \dots, a_n, a_{n+1}) \in \mathbf{A}^{n+1} : (a_1, \dots, a_n) \in V \text{ and } a_{n+1} = f(a_1, \dots, a_n)\},$$

the **graph** of  $f$ . Show that  $G(f)$  is an affine variety, and that the map  $(a_1, \dots, a_n) \mapsto (a_1, \dots, a_n, f(a_1, \dots, a_n))$  defines an isomorphism of  $V$  with  $G(f)$ . (Projection gives the inverse.)

*Proof.*

- (1) Define  $I = I(V)$  as an ideal in  $k[x_1, \dots, x_n]$ . Note that

$$G(f) = V(\underbrace{(I, x_{n+1} - f)}_{:=J}).$$

Here we can view  $I$  as an ideal of  $k[x_1, \dots, x_n, x_{n+1}]$ .

- (2) To show that  $G(f)$  is an affine variety, it suffices to show that

$$I(G(f)) = I(V(J)) = \text{rad}(J)$$

is prime (by Proposition 1 in §1.5 and the Hilbert's Nullstellensatz in §1.7). Suppose  $gh \in I(G(f)) = \text{rad}(J)$ . Write

$$\begin{aligned} g &= \sum_i g_i x_{n+1}^i = \sum_i g_i (\underbrace{(x_{n+1} - f) + f}_{\in J})^i, \\ h &= \sum_j h_j x_{n+1}^j = \sum_j h_j (\underbrace{(x_{n+1} - f) + f}_{\in J})^j \end{aligned}$$

where  $g_i, h_j \in k[x_1, \dots, x_n]$ .

- (3) Hence

$$\begin{aligned} \text{rad}(J) &= gh + \text{rad}(J) && (gh \in \text{rad}(J)) \\ &= (g + \text{rad}(J))(h + \text{rad}(J)) \\ &= \left( \sum_i g_i f^i + \text{rad}(J) \right) \left( \sum_j h_j f^j + \text{rad}(J) \right) && (x_{n+1} - f \in J) \\ &= \left( \sum_i g_i f^i \right) \left( \sum_j h_j f^j \right) + \text{rad}(J) \end{aligned}$$

or

$$\underbrace{\left( \sum_i g_i f^i \right)^N \left( \sum_j h_j f^j \right)^N}_{\in k[x_1, \dots, x_n]} \in J = (I, x_{n+1} - f)$$

for some positive integer  $N$ . So that  $(\sum_i g_i f^i)^N (\sum_j h_j f^j)^N \in I$ .

- (4) Since  $I = I(V)$  is a prime ideal, we might get  $\sum_i g_i f^i \in I \subseteq \text{rad}(J)$ . (The case  $\sum_j h_j f^j$  is similar.) Hence  $\text{rad}(J) = I(G(f))$  is a prime ideal, or  $G(f)$  is irreducible.

- (5) As  $G(f)$  is an affine variety, the map  $\alpha : V \rightarrow G(f)$  defined by

$$\alpha : (a_1, \dots, a_n) \mapsto (a_1, \dots, a_n, f(a_1, \dots, a_n))$$

is a polynomial map. (Here  $t_1 = x_1, \dots, t_n = x_n$  and  $t_{n+1} = f$ .)

- (6) By Problem 2.10, the projection map  $\text{pr}$  is a polynomial map. Also note that  $\text{pr} \circ \alpha = 1_V$  and  $\alpha \circ \text{pr} = 1_{G(f)}$ . Therefore,  $V \cong G(f)$  as an affine variety isomorphism.

□

**Problem 2.12.**

- (a) Let  $\varphi : \mathbf{A}^1 \rightarrow V = V(y^2 - x^3) \subseteq \mathbf{A}^2$  be defined by  $\varphi(t) = (t^2, t^3)$ . Show that although  $\varphi$  is a one-to-one, onto polynomial map,  $\varphi$  is not an isomorphism. (Hint:  $\tilde{\varphi}(\Gamma(V)) = k[t^2, t^3] \subsetneq k[t] = \Gamma(\mathbf{A}^1)$ .)
- (b) Let  $\varphi : \mathbf{A}^1 \rightarrow V = V(y^2 - x^2(x+1))$  be defined by  $\varphi(t) = (t^2 - 1, t(t^2 - 1))$ . Show that  $\varphi$  is one-to-one and onto, except that  $\varphi(\pm 1) = (0, 0)$ .

*Proof of (a).*

- (1) Similar to Problem 2.8(a),  $\varphi$  is a polynomial map.
- (2) Similar to Problem 2.8(a) again,

$$V = V(y^2 - x^3) = \{(t^2, t^3) \in \mathbf{A}^2(k) : t \in k\}.$$

Hence the map  $\varphi : t \mapsto (t^2, t^3)$  is surjective.

- (3) Show that  $\varphi$  is injective. Suppose  $(t^2, t^3) = (s^2, s^3)$  for some  $t, s \in k$ . If  $t = 0$ , then  $s = 0$ . If  $t \neq 0$ , then  $t = \frac{t^3}{t^2} = \frac{s^3}{s^2} = s$ . In any case,  $t = s$  whenever  $(t^2, t^3) = (s^2, s^3)$ .
- (4) Show that  $\varphi$  is not an isomorphism. It suffices to show that  $\tilde{\varphi}(\Gamma(V)) \subsetneq \Gamma(\mathbf{A}^1)$  by Proposition 1. For any  $f \in \Gamma(V)$ ,

$$\tilde{\varphi}(f)(t) = (f \circ \varphi)(t) = f(t^2, t^3) \in k[t^2, t^3].$$

Hence,

$$\tilde{\varphi}(\Gamma(V)) \subseteq k[t^2, t^3] \subsetneq k[t] = \Gamma(\mathbf{A}^1).$$

(Here note that  $t \notin k[t^2, t^3]$  but  $t \in k[t]$ .)

□

*Proof of (b).*

- (1) Write

$$Y = \{(t^2 - 1, t(t^2 - 1)) \in \mathbf{A}^2(k) : t \in k\}.$$

Show that  $Y = V$ . Similar to Problem 2.8(a). It suffices to show that  $(x, y) \in Y$  for any  $(x, y) \in V$ . If  $x = 0$ , then  $y = 0$  or  $(x, y) = (0, 0) \in Y$

whenever  $t = \pm 1$ . (In fact,  $(0, 0) = (t^2 - 1, t(t^2 - 1))$  iff  $t^2 - 1 = 0$  iff  $t = \pm 1$  in any field.) If  $x \neq 0$ , define

$$t = \frac{y}{x} \in k.$$

So  $y = tx$  and thus

$$0 = y^2 - x^2(x + 1) = t^2x^2 - x^2(x + 1) = x^2(t^2 - (x + 1)).$$

Since  $x \neq 0$  and  $k$  is a field, we have

$$t^2 - (x + 1) = 0 \iff x = t^2 - 1.$$

Hence,  $y = tx = t(t^2 - 1)$  and therefore  $(x, y) \in Y$ .

- (2) By (1),  $\varphi$  is surjective and  $\varphi(\pm 1) = (0, 0)$ .
- (3) Show that  $\varphi$  is injective except that  $\varphi(\pm 1) = (0, 0)$ . Given  $t, s \in k$ . It suffices to show that  $t = s$  whenever  $(t^2 - 1, t(t^2 - 1)) = (s^2 - 1, s(s^2 - 1)) \neq (0, 0)$ . In fact, by assumption we have  $t^2 - 1 = s^2 - 1 \neq 0$  by assumption. Therefore,

$$t = \frac{t(t^2 - 1)}{t^2 - 1} = \frac{s(s^2 - 1)}{s^2 - 1} = s.$$

□

### Problem 2.13.

Let  $V = V(x^2 - y^3, y^2 - z^3) \subseteq \mathbf{A}^3$  as in Problem 1.40,  $\bar{\alpha} : \Gamma(V) \rightarrow k[t]$  induced by the homomorphism  $\alpha$  of that problem.

- (a) What is the polynomial map  $f$  from  $\mathbf{A}^1$  to  $V$  such that  $\tilde{f} = \bar{\alpha}$ ?
- (b) Show that  $f$  is one-to-one and onto, but not an isomorphism.

*Proof of (a).*

- (1) Write

$$Y = \{(t^9, t^6, t^4) \in \mathbf{A}^3(k) : t \in k\}.$$

Show that  $Y = V$ . Similar to Problem 2.8(a). It suffices to show that  $(x, y, z) \in Y$  for any  $(x, y, z) \in V$ . If  $x = 0$ , then  $y = z = 0$  or  $(x, y, z) = (0, 0, 0) \in Y$  by taking  $t = 0$ . If  $x \neq 0$ , define

$$t = \frac{yz}{x} \in k.$$

Hence,

$$\begin{aligned} t^9 &= \frac{y^9 z^9}{x^9} = \frac{y^{15}}{x^9} = \frac{x^{10}}{x^9} = x, \\ t^6 &= \frac{y^6 z^6}{x^6} = \frac{y^5 z^6}{x^6} y = \frac{y^9}{x^6} y = \frac{x^6}{x^6} y = y, \\ t^4 &= \frac{y^4 z^4}{x^4} = \frac{y^4 z^3}{x^4} z = \frac{y^6}{x^4} z = \frac{x^4}{x^4} z = z. \end{aligned}$$

(2) Define a mapping  $f : \mathbf{A}^1 \rightarrow \mathbf{A}^3$  by

$$f : t \mapsto (t^9, t^6, t^4).$$

$f$  is a polynomial map by construction. By (1),  $f : \mathbf{A}^1 \rightarrow f(\mathbf{A}^1) = V$  and thus  $\tilde{f} = \bar{\alpha}$  by the definition of  $\alpha$ .

□

*Proof of (b).*

(1) Similar to Problem 2.12(a).

(2)  $f$  is surjective by the proof of (a).

(3) *Show that  $f$  is injective.* Suppose  $(t^9, t^6, t^4) = (s^9, s^6, s^4)$  for some  $t, s \in k$ . If  $t = 0$ , then  $s = 0$ . If  $t \neq 0$ , then  $t = \frac{t^6 t^4}{t^9} = \frac{s^6 s^4}{s^9} = s$ . In any case,  $t = s$  whenever  $(t^9, t^6, t^4) = (s^9, s^6, s^4)$ .

(4) *Show that  $f$  is not an isomorphism.* It suffices to show that  $\tilde{f}(\Gamma(V)) \subsetneq \Gamma(\mathbf{A}^1)$  by Proposition 1. For any  $g \in \Gamma(V)$ ,

$$\tilde{f}(g)(t) = (g \circ f)(t) = g(t^9, t^6, t^4) \in k[t^4, t^6, t^9].$$

Hence,

$$\tilde{\varphi}(\Gamma(V)) \subseteq k[t^4, t^6, t^9] \subsetneq k[t] = \Gamma(\mathbf{A}^1).$$

(Here note that  $t \notin k[t^4, t^6, t^9]$  but  $t \in k[t]$ .)

□

## 2.3. Coordinate Changes

### Problem 2.14.\* (Linear subvariety)

A set  $V \subseteq \mathbf{A}^n(k)$  is called a **linear subvariety** of  $\mathbf{A}^n(k)$  if  $V = V(f_1, \dots, f_r)$  for some polynomials  $f_i$  of degree 1.

- (a) Show that if  $t$  is an affine change of coordinates on  $\mathbf{A}^n(k)$ , then  $V^t$  is also a linear subvariety of  $\mathbf{A}^n(k)$ .
- (b) If  $V \neq \emptyset$ , show that there is an affine change of coordinates  $t$  of  $\mathbf{A}^n$  such that  $V^t = V(x_{m+1}, \dots, x_n)$ . (Hint: use induction on  $r$ .) So  $V$  is a variety.
- (c) Show that the  $m$  that appears in part (b) is independent of the choice of  $t$ . It is called the **dimension** of  $V$ . Then  $V$  is then isomorphic (as a variety) to  $\mathbf{A}^m(k)$ . (Hint: Suppose there were an affine change of coordinates  $t$  such that  $V(x_{m+1}, \dots, x_n)^t = V(x_{s+1}, \dots, x_n)$ ,  $m < s$ ; show that  $t_{m+1}, \dots, t_n$  would be dependent.)

*Proof of (a).*

- (1) Say  $t = (t_1, \dots, t_n)$  is an affine change of coordinates, and  $V = V(f_1, \dots, f_r)$  for some polynomials  $f_i$  of degree 1.
- (2) Show that  $V$  is a variety and thus  $I(V) = (f_1, \dots, f_r)$  by the Hilbert's Nullstellensatz.  $V(f_1, \dots, f_r)$  is the set of all solutions of the system of linear equations:

$$\begin{aligned} f_1 &= a_{11}x_1 + \dots + a_{1n}x_n - b_1 = 0, \\ &\dots \\ f_r &= a_{r1}x_1 + \dots + a_{rn}x_n - b_r = 0. \end{aligned}$$

Write  $Ax = b$  and  $V = V(Ax = b)$  where

$$A = \underbrace{\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{r1} & \dots & a_{rn} \end{pmatrix}}_{\in \mathbf{M}_{r \times n}(k)}, \quad x = \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{\in \mathbf{M}_{n \times 1}(k)}, \quad b = \underbrace{\begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix}}_{\in \mathbf{M}_{r \times 1}(k)}.$$

- (3) The Gaussian elimination in linear algebra says that  $(A|b)$  has the same solutions as its reduced row echelon form  $(A'|b')$ , that is,  $V(Ax = b) = V(A'x = b')$ .
- (4) If  $V(f_1, \dots, f_r) = \emptyset$ , nothing to do. If  $V(f_1, \dots, f_r) \neq \emptyset$ , then

$$V(f_1, \dots, f_r) = V(g_1, \dots, g_m)$$

where  $m = \text{rank}(A)$  is the number of nonzero rows in  $A'$  ( $m \leq r, n$ ) and  $g_i = a'_{i1}x_1 + \dots + a'_{in}x_n - b'_i$  for  $1 \leq i \leq m$ . ( $a'_{ij}$  is the entry of the matrix  $A'$ .)

- (5) Now given any  $f + I(V) \in k[x_1, \dots, x_n]/I(V)$ , we replace the leading term  $x_{i_1}$  of  $g_1$  by  $x_{i_1} - g_1$  to get

$$f + I(V) = f(x_1, \dots, \underbrace{x_{i_1} - g_1}_{i_1 \text{th position}}, \dots, x_n) + I(V) := f_1 + I(V)$$

where  $f_1 \in k[x_1, \dots, \widehat{x_{i_1}} \dots, x_n]$ . Continue this process to replace each leading term  $x_{i_j}$  of  $g_j$  by  $x_{i_j} - g_j$  to get one by one to get

$$f + I(V) = f_1 + I(V) \text{ where } f_1 \in k[x_1, \dots, \widehat{x_{i_1}} \dots, x_n].$$

...

$$f_{m-1} + I(V) = f_m + I(V) \text{ where } f_m \in k[x_1, \dots, \widehat{x_{i_1}} \dots, \widehat{x_{i_m}} \dots, x_n].$$

Hence, a routine shows that there is a ring isomorphism

$$\alpha : k[x_1, \dots, x_n]/I(V) \rightarrow \underbrace{k[x_1, \dots, \widehat{x_{i_1}} \dots, \widehat{x_{i_m}} \dots, x_n]}_{\text{a domain}}$$

sending  $f$  to  $f_m$ . Therefore,  $V$  is a variety.

- (6) As  $I(V) = (f_1, \dots, f_r)$ ,  $I(V)^t = (f_1^t, \dots, f_r^t)$  where each  $f_i^t$  is linear. Thus  $V^t = V(I(V)^t) = V(f_1^t, \dots, f_r^t)$  is also a linear subvariety of  $\mathbf{A}^n(k)$ .

□

*Proof of (b).*

- (1) Suppose  $A \in M_{r \times n}(k)$  is of rank  $n - m$ . Linear algebra says that there exist invertible matrices  $B \in M_{r \times r}(k)$  and  $C \in M_{n \times n}(k)$  such that  $D = BAC$ , where

$$D = BAC = \underbrace{\begin{pmatrix} O_1 & O_2 \\ O_3 & I_{n-m} \end{pmatrix}}_{\in M_{r \times n}(k)}$$

in which  $I_{n-m} \in M_{(n-m) \times (n-m)}(k)$  is the identity matrix and  $O_1, O_2$ , and  $O_3$  are zero matrices.

- (2) Let  $t'$  be the linear map corresponding to the matrix  $C$ . So

$$\begin{aligned} V^{t'} &= V(Ax = b)^{t'} \\ &= V(ACx = b) \\ &= V(BACx = Bb) && (B: \text{invertible}) \\ &= V(Dx = Bb) \\ &= V(-\beta_1, \dots, -\beta_m, x_{m+1} - \beta_{m+1}, \dots, x_n - \beta_n) && (V \neq \emptyset) \\ &= V(x_{m+1} - \beta_{m+1}, \dots, x_n - \beta_n) \end{aligned}$$

$$\text{where } Bb = \underbrace{\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}}_{\in M_{n \times 1}(k)}.$$

- (3) Let  $t''$  be the translation corresponding to the matrix  $Bb$ . Let  $t = t'' \circ t'$  be the desired affine change of coordinates. Therefore,

$$\begin{aligned} V^t &= (V^{t'})^{t''} \\ &= V(x_{m+1} - \beta_{m+1}, \dots, x_n - \beta_n)^{t''} \\ &= V(x_{m+1}, \dots, x_n). \end{aligned}$$

□

*Proof of (c).*

- (1) Linear algebra says that the rank of any matrix is uniquely determined. Therefore,  $\dim(V) = n - \text{rank}(A|b) = n - \text{rank}(A'|b')$  is uniquely determined.
- (2)  $V$  is then isomorphic to  $\mathbf{A}^m(k)$  as a variety.

□

### Problem 2.15.\* (Line)

Let  $P = (a_1, \dots, a_n)$ ,  $Q = (b_1, \dots, b_n)$  be distinct points of  $\mathbf{A}^n$ . The **line** through  $P$  and  $Q$  is defined to be  $\{(a_1 + s(b_1 - a_1), \dots, a_n + s(b_n - a_n)) : s \in k\}$ .

- (a) Show that if  $L$  is the line through  $P$  and  $Q$ , and  $t$  is an affine change of coordinates, then  $t(L)$  is the line through  $t(P)$  and  $t(Q)$ .
- (b) Show that a line is a linear subvariety of dimension 1, and that a linear subvariety of dimension 1 is the line through any two of its points.
- (c) Show that, in  $\mathbf{A}^2$ , a line is the same thing as a hyperplane.
- (d) Let  $P, P' \in \mathbf{A}^2$ ,  $L_1, L_2$  two distinct lines through  $P$ ,  $L'_1, L'_2$  distinct lines through  $P'$ . Show that there is an affine change of coordinates  $t$  of  $\mathbf{A}^2$  such that  $t(P) = P'$  and  $t(L_i) = L'_i$ ,  $i = 1, 2$ .

*Proof of (a).*

- (1) Write  $t = (t_1, \dots, t_n)$  as

$$t_i = \sum_j c_{ij}x_j + c_{i0}.$$

Take any point  $P_s = (a_1 + s(b_1 - a_1), \dots, a_n + s(b_n - a_n)) \in L$  for some  $s \in k$ . (In particular,  $P_0 = P$  and  $P_1 = Q$ .)



(2) As

$$\begin{aligned}
t_i(P_s) &= \sum_j c_{ij}(a_j + s(b_j - a_j)) + c_{i0} \\
&= \left( \sum_j c_{ij}a_j + c_{i0} \right) \\
&\quad + s \left[ \left( \sum_j c_{ij}b_j + c_{i0} \right) - \left( \sum_j c_{ij}a_j + c_{i0} \right) \right] \\
&= t_i(P) + s(t_i(Q) - t_i(P)),
\end{aligned}$$

we have

$$\begin{aligned}
t(L) &= \{(t_1(P) + s(t_1(Q) - t_1(P)), \dots, t_n(P) + s(t_n(Q) - t_n(P))) \\
&\quad : s \in k\}.
\end{aligned}$$

Moreover,  $t(P) \in t(L)$  as  $s = 0$  and  $t(Q) \in t(L)$  as  $s = 1$ . Therefore,  $t(L)$  is the line through  $t(P)$  and  $t(Q)$ .

□

*Proof of (b).*

(1) Note that  $a_\alpha \neq b_\alpha$  for some  $1 \leq \alpha \leq n$  since  $P \neq Q$ . Write

$$L = V \left( x_i = a_i + \frac{x_\alpha - a_\alpha}{b_\alpha - a_\alpha} (b_i - a_i) : 1 \leq i \leq n \right).$$

(Here we solve  $s = \frac{x_\alpha - a_\alpha}{b_\alpha - a_\alpha}$  and then replace  $s$  in the equation  $x_i = a_i + t(b_i - a_i)$ .) By Problem 2.14,  $L$  is a linear subvariety.

(2) Note that

$$\begin{aligned}
n - \dim(L) &= \text{the rank of the corresponding augmented matrix } (A'|b') \\
&= \text{the maximal number of the linearly independent rows of } (A'|b') \\
&= n - 1,
\end{aligned}$$

which is uniquely determined. Therefore,  $\dim(V) = 1$ .

(3) Conversely,  $\dim(V) = 1$  implies that  $\text{rank}(A'|b') = n - 1$ . So all leading terms are all  $x_i$  except only one  $x_j$  for some  $j$ . Hence  $V$  is of the form

$$V = (x_i + a_{ij}x_j = b_i)$$

for  $1 \leq i \leq n$  and  $i \neq j$ . So

$$\begin{aligned} V &= \{(b_1 - a_{1j}s, \dots, \underbrace{s}_{j\text{th position}}, \dots, b_n - a_{nj}s) : s \in k\} \\ &= \{(b_1 + s((b_1 - a_{1j}) - b_1), \dots, \underbrace{0 + s(1 - 0)}_{j\text{th position}}, \dots, \\ &\quad (b_n + s((b_n - a_{nj}) - b_n)) : s \in k\} \end{aligned}$$

is a line passing

$$\begin{aligned} P &= (b_1, \dots, 0, \dots, b_n) \\ Q &= (b_1 - a_{1j}, \dots, 1, \dots, b_n - a_{nj}) \end{aligned}$$

with  $P \neq Q$  (since they are different in the  $j$ th position). (Here we can change  $P$  and  $Q$  to any two different points on  $V$ .)

□

*Proof of (c).*

- (1) A line  $L \subseteq \mathbf{A}^2$  is  $V(x + ay = b)$  or  $V(x + ay = b)$  by (b). In any case,  $L$  is a hyperplane in  $\mathbf{A}^2$ .
- (2) Conversely, given any hyperplane  $V = V(ax + by + c = 0) \subseteq \mathbf{A}^2$  where  $a$  and  $b$  are not all zero. Might assume that  $a \neq 0$ . (The case  $b \neq 0$  is similar.) So

$$V = \left\{ \left( -\frac{c}{a} - \frac{b}{a}s, s \right) : s \in k \right\}$$

is a line passing  $(-\frac{c}{a}, 0)$  and  $(-\frac{c+b}{a}, 1)$ .

□

*Proof of (d).*

- (1) It suffices to show that there is a bijective affine change of coordinates  $t$  of  $\mathbf{A}^2$  such that  $t(P) = (0, 0)$ ,  $t(L_1) = V(x = 0)$  and  $t(L_2) = V(y = 0)$ . Write  $P = (p_1, p_2)$  and  $L_i = a_ix + b_iy + c_i$  for  $i = 1, 2$ .
- (2) Let  $t'' = (t''_1, t''_2)$  be a translation defined by

$$\begin{pmatrix} t''_1 \\ t''_2 \end{pmatrix} = \begin{pmatrix} x - p_1 \\ y - p_2 \end{pmatrix}.$$

So  $L_1^{t''} = a_1x + b_1y$  and  $L_2^{t''} = a_2x + b_2y$ . Let

$$A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

and  $t' = (t'_1, t'_2)$  be a linear map defined by

$$\begin{pmatrix} t'_1 \\ t'_2 \end{pmatrix} = \frac{1}{\det(A)} \begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

( $t'$  is well-defined since  $L_1$  and  $L_2$  are distinct lines and thus  $\det(A) \neq 0$ .)  
Write  $t = (t_1, t_2) = t' \circ t''$ . So

$$\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \frac{1}{\det(A)} \begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix} \begin{pmatrix} x - p_1 \\ y - p_2 \end{pmatrix}$$

and

$$\begin{aligned} L_1^t &= (L_1^{t''})^{t'} = x \\ L_2^t &= (L_2^{t''})^{t'} = y. \end{aligned}$$

(3) Conversely, define an affine change of coordinates  $s = (s_1, s_2)$  of  $\mathbf{A}^2$  by

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} a_1x + b_1y + c_1 \\ a_2x + b_2y + c_2 \end{pmatrix}$$

so that  $x^s = L_1$  and  $y^s = L_2$ .

(4) By (2)(3), the statement in (1) is established.

□

### Problem 2.16.

Let  $k = \mathbb{C}$ . Give  $\mathbf{A}^n(\mathbb{C}) = \mathbb{C}^n$  the usual topology (obtained by identifying  $\mathbb{C}$  with  $\mathbb{R}^2$ , and hence  $\mathbb{C}^n$  with  $\mathbb{R}^{2n}$ ). Recall that a topological space  $X$  is path-connected if for any  $P, Q \in X$ , there is a continuous mapping  $\gamma : [0, 1] \rightarrow X$  such that  $\gamma(0) = P$ ,  $\gamma(1) = Q$ .

- (a) Show that  $\mathbb{C} - S$  is path-connected for any finite set  $S$ .
- (b) Let  $V$  be an algebraic set in  $\mathbf{A}^n(\mathbb{C})$ . Show that  $\mathbf{A}^n(\mathbb{C}) - V$  is path-connected. (Hint: If  $P, Q \in \mathbf{A}^n(\mathbb{C}) - V$ , let  $L$  be the line through  $P$  and  $Q$ . Then  $L \cap V$  is finite, and  $L$  is isomorphic to  $\mathbf{A}^1(\mathbb{C})$ .)

*Proof of (a).*

- (1) Regard  $\mathbb{C}^n$  as  $\mathbb{R}^{2n}$ . Given any  $P, Q \in \mathbb{C}^n - S$ . Write  $S := \{P_1, \dots, P_m\} \subseteq \mathbb{C}^n$ .
- (2) Let  $L_{P_i}$  (resp.  $L'_{P_i}$ ) be a line passing  $P$  (resp.  $Q$ ) and  $P_i$  for every  $P_i \in S$ . (It is well-defined since  $P, Q$  are not in  $S$ .) So  $\mathcal{C} := \{L_{P_i} : P_i \in S\}$  (resp.  $\mathcal{C}' := \{L'_{P_i} : P_i \in S\}$ ) is a collection of finitely many lines.

- (3) Consider a unit sphere  $\mathbb{S}^{2n-1}(P)$  centered at  $P$ .

$$\left( \bigcup_{L_i \in \mathcal{C}} L_i \right) \cap \mathbb{S}^{2n-1}(P)$$

is again a finite set (of order  $\leq 2|S| = 2m$ ). Since  $\mathbb{S}^{2n-1}$  is infinite, we can always take a line  $L$  passing  $P$  and some point in  $\mathbb{S}^{2n-1}(P)$  where  $L \cap S = \emptyset$ .

- (4) Similarly, we take a line  $L'$  passing  $Q$  and some point in  $\mathbb{S}^{2n-1}(Q)$  where  $L' \cap S = \emptyset$  and  $L' \cap L \neq \emptyset$ . (There are only two points in  $\mathbb{S}^{2n-1}(Q)$  such that  $L' \cap L = \emptyset$ . Note that  $\mathbb{S}^{2n-1}$  is infinite.)
- (5) Take any point  $A \in L' \cap L$ . So there is a path from  $P$  to  $A$  (on a segment contained in  $L$ ) and then to  $Q$  (on a segment contained in  $L'$ ). Therefore,  $\mathbb{C}^n - S$  is path-connected.

□

*Proof of (b).*

- (1) Given any  $P, Q \in \mathbf{A}^n(\mathbb{C}) - V$ . Let  $L$  be the line through  $P$  and  $Q$ . To show  $\mathbf{A}^n(\mathbb{C}) - V$  is path-connected, it suffices to show that

$$L - V = L - (V \cap L)$$

is path-connected.

- (2) Similar to Problem 1.12, we have  $V \cap L$  is finite. In fact, write  $V = (f_1, \dots, f_r)$  and  $L = \{(a_1 + t(b_1 - a_1), \dots, a_n + t(b_n - a_n)) : t \in k\}$  where  $P = (a_1, \dots, a_n)$  and  $Q = (b_1, \dots, b_n)$ . Then

$$\begin{aligned} V \cap L &= \bigcap_i (V(f_i) \cap L) \\ &= \bigcap_i \{f_i(a_1 + t(b_1 - a_1), \dots, a_n + t(b_n - a_n)) = 0 : t \in k\} \\ &= \bigcap_i \text{finite set} \\ &= \text{finite set.} \end{aligned}$$

Here  $f_i(a_1 + t(b_1 - a_1), \dots, a_n + t(b_n - a_n))$  is a nonzero polynomial since  $P, Q \notin V(f_i)$ .

- (3) Note that the path-connectedness is a topological invariant under homeomorphisms. Since any line is homeomorphic to  $\mathbb{C}^1$ ,  $L - V$  is homeomorphic to  $\mathbb{C}^1 - S$  for some finite set  $S$  (by (2)). By (a),  $L - V$  is path-connected and so is  $\mathbf{A}^n(\mathbb{C}) - V$ .

□

## 2.4. Rational Functions and Local Rings

### Problem 2.17.

Let  $V = V(y^2 - x^2(x + 1)) \subseteq \mathbf{A}^2$ , and  $\bar{x}, \bar{y}$  the residues of  $x, y$  in  $\Gamma(V)$ ; let  $z = \bar{y}/\bar{x} \in k(V)$ . Find the pole sets of  $z$  and of  $z^2$ .

*Proof.*

(1) Show that the pole set of  $z$  is  $\{(0, 0)\}$ .

(a) Since  $V$  is irreducible by Problem 2.12(b),  $V$  is a variety. Note that the pole set of  $z$  is

$$\bigcap_{z=\bar{f}/\bar{g}} V(\bar{g}).$$

(b) By (a),  $\{(0, 0)\}$  contains the pole set of  $z$ . (As the denominator  $x = 0$ , we solve the equation  $y^2 - x^2(x + 1) = 0$  to get  $y = 0$ .)

(c) (Reductio ad absurdum) If  $(0, 0)$  were not a pole, then there were  $\bar{f}, \bar{g} \in \Gamma(V)$  such that  $z = \bar{y}/\bar{x} = \bar{f}/\bar{g}$  where  $\bar{g}(0, 0) \neq 0$ . So

$$\begin{aligned} z &= \bar{y}/\bar{x} = \bar{f}/\bar{g} \\ \implies \overline{xf} &= \overline{yg} \\ \implies xf - yg &\in (y^2 - x^2(x + 1)) \\ \implies xf - yg &= h(y^2 - x^2(x + 1)) \text{ for some } h \\ \implies y(g + hy) &= x(f + hx(x + 1)) \in (x) \\ \implies g + hy &\in (x) \\ \implies g(0, 0) &= 0, \end{aligned}$$

which is absurd.

(2) Show that the pole set of  $z^2$  is empty. Note that  $z^2 = \overline{y^2}/\overline{x^2} = \overline{x+1}$ . So the pole set of  $z^2$  is

$$\bigcap_{z^2=\bar{f}/\bar{g}} V(\bar{g}) = \emptyset.$$

□

### Problem 2.18.

Let  $\mathcal{O}_P(V)$  be the local ring of a variety  $V$  at a point  $P$ . Show that there is a natural one-to-one correspondence between the prime ideals in  $\mathcal{O}_P(V)$  and the subvarieties of  $V$  that pass through  $P$ . (Hint: If  $I$  is prime in  $\mathcal{O}_P(V)$ ,  $I \cap \Gamma(V)$

is prime in  $\Gamma(V)$ , and  $I$  is generated by  $I \cap \Gamma(V)$ ; use Problem 2.2.)

*Proof.*

- (1) Write  $P = (a_1, \dots, a_n)$  and  $\mathfrak{m} := (x_1 - a_1, \dots, x_n - a_n)$ . It suffices to show that there is a natural one-to-one correspondence between the prime ideals in  $\mathcal{O}_P(V)$  and prime ideals in  $\Gamma(V)$  which is contained in  $I(V(P)) = \mathfrak{m}$  by Problem 2.2.
- (2) If  $\mathfrak{p}$  is prime in  $\mathcal{O}_P(V)$ ,  $\mathfrak{p} \cap \Gamma(V)$  is prime in  $\Gamma(V)$  since  $\Gamma(V)$  is a subring of  $\mathcal{O}_P(V)$ . Note that  $\mathfrak{p} \subseteq \mathfrak{m}_P(V)$  and thus

$$\mathfrak{p} \cap \Gamma(V) \subseteq \mathfrak{m}_P(V) \cap \Gamma(V) = (x_1 - a_1, \dots, x_n - a_n).$$

- (3) Conversely, if  $\mathfrak{q}$  is prime in  $\Gamma(V)$  which is contained in  $\mathfrak{m}$  then we need to show that  $\mathfrak{p} := \mathfrak{q}\mathcal{O}_P(V)$  is prime in  $\mathcal{O}_P(V)$ .
- (4) Note that  $\mathfrak{p}$  is proper (since  $\mathfrak{q} \subseteq \mathfrak{m}$ ). Suppose  $\frac{a}{b} \frac{c}{d} \in \mathfrak{p}$  with  $b(P) \neq 0$  and  $d(P) \neq 0$ . Hence

$$ac = \frac{a}{b} \frac{c}{d} \cdot (bd) \in \mathfrak{p} \cap \Gamma(V) = \mathfrak{q}.$$

By the primality of  $\mathfrak{q}$ , might assume that  $a \in \mathfrak{q}$ . (The case  $c \in \mathfrak{q}$  is the same.) So that  $\frac{a}{b} = a \cdot \frac{1}{b} \in \mathfrak{q}\mathcal{O}_P(V) = \mathfrak{p}$ . Therefore,  $\mathfrak{p}$  is prime.

□

### Problem 2.19.

Let  $f$  be a rational function on a variety  $V$ . Let

$$U = \{P \in V : f \text{ is defined at } P\}.$$

Then  $f$  defines a function from  $U$  to  $k$ . Show that this function determines  $f$  uniquely. So a rational function may be considered as a type of function, but only on the complement of an algebraic subset of  $V$ , not on  $V$  itself.

*Proof.*

- (1) Write  $f = a/b \in k(V)$  with  $b(P) \neq 0$ . Define  $f : U \rightarrow k$  by  $f : P \mapsto f(P) = a(P)/b(P)$ .
- (2) Show that this function is well-defined. Given any  $P \in U$ . Suppose that  $f = a/b = c/d$  with  $b(P) \neq 0$  and  $d(P) \neq 0$ . So,  $ad = bc \in \Gamma(V)$  implies that  $a(P)d(P) = b(P)c(P)$ . So,  $a(P)/b(P) = c(P)/d(P)$  (since  $b(P) \neq 0$  and  $d(P) \neq 0$ ). Therefore,  $f : U \rightarrow k$  is well-defined.

□

**Problem 2.20. (Quadric surface)**

Let

$$V = V(xw - yz) \subseteq \mathbf{A}^4(k),$$

and

$$\Gamma(V) = k[x, y, z, w]/(xw - yz).$$

Let  $\bar{x}, \bar{y}, \bar{z}, \bar{w}$  be the residues of  $x, y, z, w$  in  $\Gamma(V)$ . Then

$$\bar{x}/\bar{y} = \bar{z}/\bar{w} = f \in k(V)$$

is defined at  $P = (x, y, z, w) \in V$  if  $y \neq 0$  or  $w \neq 0$ . Show that it is impossible to write  $f = a/b$ , where  $a, b \in \Gamma(V)$ , and  $b(P) \neq 0$  for every  $P$  where  $f$  is defined. Show that the pole set of  $f$  is exactly  $\{(x, y, z, w) : y = 0 \text{ and } w = 0\}$ .

*Proof.*

- (1) Note that the pole set of  $f$  is

$$\bigcap_{f=\bar{a}/\bar{b}} V(\bar{b}) \subseteq \{(x, y, z, w) \in \mathbf{A}^4(k) : y = w = 0\}.$$

- (2) Show that  $f$  is not defined at the origin  $O = (0, 0, 0, 0)$ . (Reductio ad absurdum) Suppose  $f = \bar{x}/\bar{y} = \bar{a}/\bar{b}$  with  $\bar{b}(O) \neq 0$ . So  $\bar{b}x - \bar{a}y = 0$ , or

$$bx - ay = g(xw - yz)$$

for some  $g \in k[x, y, z, w]$ . Take 1-forms on the both sides to get

$$b(O)x - a(O)y = 0 \in k[x, y, z, w],$$

or  $a(O) = b(O) = 0$ , which is absurd.

- (3) Show that it is impossible to write  $f = \bar{a}/\bar{b}$ , where  $\bar{a}, \bar{b} \in \Gamma(V)$ , and  $\bar{b}(P) \neq 0$  for every  $P$  where  $f$  is defined. (Reductio ad absurdum) Consider the polynomial

$$\beta(y, w) := b(0, y, 0, w) \in k[y, w].$$

$\beta$  is not a constant polynomial since  $V(\beta) = \{(0, 0)\}$  by (1)(2). By Problem 1.14,  $V(\beta) = \{(0, 0)\}$  is infinite, which is absurd.

- (4) Show that the pole set of  $f$  is exactly  $\{(x, y, z, w) : y = 0 \text{ and } w = 0\}$ . (Reductio ad absurdum) Given any  $P = (x_0, 0, z_0, 0) \in \{(x, y, z, w) : y = 0 \text{ and } w = 0\}$ . Suppose  $f = \bar{x}/\bar{y} = \bar{a}/\bar{b}$  where  $\bar{b}(P) \neq 0$ . Similar to (2),

$$bx - ay = g(xw - yz)$$

for some  $g \in k[x, y, z, w]$ . So

$$b(P)x_0 = b(P)x_0 - a(P) \cdot 0 = g(P)(x_0 \cdot 0 - 0 \cdot z_0) = 0.$$

As  $b(P) \neq 0$ ,  $x_0 = 0$ . Similarly,  $z_0 = 0$  by noting that  $bz - aw = h(xw - yz)$  for some  $h \in k[x, y, z, w]$ . Hence  $P = (0, 0, 0, 0)$ , contrary to (2).

□

*Note.* It is equal to the Segre embedding of  $\mathbf{P}^1 \times \mathbf{P}^1$  in  $\mathbf{P}^3$ , for suitable choice of coordinates.

**Problem 2.21.\***

Let  $\varphi : V \rightarrow W$  be a polynomial map of affine varieties,  $\tilde{\varphi} : \Gamma(W) \rightarrow \Gamma(V)$  the induced map on coordinate rings. Suppose  $P \in V$ ,  $\varphi(P) = Q$ . Show that  $\tilde{\varphi}$  extends uniquely to a ring homomorphism (also written  $\tilde{\varphi}$ ) from  $\mathcal{O}_Q(W)$  to  $\mathcal{O}_P(V)$ . (Note that  $\tilde{\varphi}$  may not extend to all of  $k(W)$ .) Show that  $\tilde{\varphi}(\mathfrak{m}_Q(W)) \subseteq \mathfrak{m}_P(V)$ .

*Proof.*

- (1) Define  $\tilde{\varphi} : \mathcal{O}_Q(W) \rightarrow \mathcal{O}_P(V)$  by

$$\tilde{\varphi} : a/b \mapsto \tilde{\varphi}(a)/\tilde{\varphi}(b).$$

It is well-defined since  $b(Q) \neq 0$  implies that

$$\tilde{\varphi}(b)(P) = b(\varphi(P)) = b(Q) \neq 0.$$

- (2) Note that  $\tilde{\varphi}$  may not extend to all of  $k(W)$  since  $\tilde{\varphi} : k(W) \rightarrow k(V)$  might not be well-defined if  $\tilde{\varphi}(b) = 0$  for all  $b \in \Gamma(W)$ .
- (3) Show that  $\tilde{\varphi}(\mathfrak{m}_Q(W)) \subseteq \mathfrak{m}_P(V)$ . Take any  $a/b \in \mathfrak{m}_Q(W)$  with  $a(Q) = 0$  and  $b(Q) \neq 0$ . As

$$\tilde{\varphi}(a)(P) = a(\varphi(P)) = a(Q) = 0,$$

we have  $\tilde{\varphi}(a/b) \in \mathfrak{m}_P(V)$ .

□

**Problem 2.22.\***

Let  $t : \mathbf{A}^n \rightarrow \mathbf{A}^n$  be an affine change of coordinates,  $t(P) = Q$ . Show that  $\tilde{t} : \mathcal{O}_Q(\mathbf{A}^n) \rightarrow \mathcal{O}_P(\mathbf{A}^n)$  is an isomorphism. Show that  $\tilde{t}$  induces an isomorphism from  $\mathcal{O}_Q(V)$  to  $\mathcal{O}_P(V^t)$  if  $P \in V^t$ , for  $V$  a subvariety of  $\mathbf{A}^n$ .

*Proof.*

- (1) Since  $\tilde{t} : \Gamma(\mathbf{A}^n) \rightarrow \Gamma(\mathbf{A}^n)$  is a ring isomorphism, it extends uniquely to a ring isomorphism (also written  $\tilde{t}$ ) from  $\mathcal{O}_Q(\mathbf{A}^n)$  to  $\mathcal{O}_P(\mathbf{A}^n)$  by Problem 2.21.



- (2) Note that  $\mathcal{O}_Q(V) \hookrightarrow \mathcal{O}_Q(\mathbf{A}^n)$ ,  $\mathcal{O}_P(V^t) \hookrightarrow \mathcal{O}_P(\mathbf{A}^n)$ , and  $\tilde{t}(\mathcal{O}_Q(V)) = \mathcal{O}_P(V^t)$ ,  $\tilde{t}: \mathcal{O}_Q(V) \rightarrow \mathcal{O}_P(V^t)$  is an isomorphism.

□

## 2.5. Discrete Valuation Rings

### Problem 2.23.\*

Show that the order function on  $K$  is independent of the choice of uniformizing parameter.

*Proof.*

- (1) Show that a uniformizing parameter is unique up to a unit. Suppose  $t$  and  $t'$  are two uniformizing parameters for a discrete valuation ring  $R$  with the quotient field  $K$ . Since  $R$  is a DVR, the maximal ideal is

$$\mathfrak{m} = (t) = (s).$$

As  $s \in (t)$ , there is an element  $a \in R$  such that  $s = at$ . As  $s$  is irreducible (by the maximality of  $\mathfrak{m}$ ),  $a$  is a unit or  $t$  is a unit (which is impossible). Hence  $s = at$  for some unit  $a \in R$ .

- (2) For any  $z \in K$ , write

$$z = ut^n = vs^m$$

for some units  $u, v$  and integers  $n \geq m$ . (The case  $n \leq m$  is similar.) Replace  $s = at$  to get  $ut^n = va^mt^m$ . So  $t^{n-m} = u^{-1}va^m$  is a unit. Hence,  $m = n$ , or the order function on  $K$  is independent of the choice of uniformizing parameter.

□

### Problem 2.24.\*

Let  $V = \mathbf{A}^1$ ,  $\Gamma(V) = k[x]$ ,  $K = k(V) = k(x)$ .

- (a) For each  $a \in k = V$ , show that  $\mathcal{O}_a(V)$  is a DVR with uniformizing parameter  $t = x - a$ .
- (b) Show that  $\mathcal{O}_\infty = \{f/g \in k(x) : \deg(g) \geq \deg(f)\}$  is also a DVR, with uniformizing parameter  $t = 1/x$ .

*Proof of (a).*

- (1) By Proposition 7 in §2.4,  $\mathcal{O}_a(V)$  is a (Noetherian) local domain. It suffices to show that  $t = x - a$  is an irreducible element in  $\mathcal{O}_a(V)$  such that every nonzero  $z \in \mathcal{O}_a(V)$  might be written uniquely in the form  $z = ut^n$ ,  $u$  a unit in  $\mathcal{O}_a(V)$ ,  $n$  a nonnegative integer (by Proposition 4).
- (2) Write  $z = f/g \in \mathcal{O}_a(V)$  where  $g(a) \neq 0$ . By Problem 1.7,

$$f = \sum_{i=0}^{\deg(f)} \lambda_i (x - a)^i.$$

Let  $n$  be the smallest integer such that  $\lambda_n \neq 0$ . (Such  $n$  is existed since  $z$  or  $f$  is nonzero.) Hence,  $f = f_1(x - a)^n$  where  $f_1 = \sum_{i=n}^{\deg(f)} \lambda_i (x - a)^{i-n} \neq 0$  and  $f_1(a) = \lambda_n \neq 0$ . So

$$z = f/g = (f_1/g)(x - a)^n.$$

Here  $f_1/g$  is a unit in  $\mathcal{O}_a(V)$ . Besides, it is easy to show that  $n$  is unique by the similar argument in Problem 2.23. Hence,  $\mathcal{O}_a(V)$  is a DVR with uniformizing parameter  $t = x - a$ .

□

*Proof of (b).*

- (1) Show that  $\mathcal{O}_\infty$  is a subring of  $k(x)$ . Clearly,  $1 = 1/1 \in \mathcal{O}_\infty$ . Also, given any  $f = a/b, g = c/d \in \mathcal{O}_\infty$ . So

$$\begin{aligned} f - g &= a/b - c/d = \frac{ad - bc}{bd} \in \mathcal{O}_\infty \\ fg &= a/b \cdot c/d = \frac{ac}{bd} \in \mathcal{O}_\infty \end{aligned}$$

since

$$\begin{aligned} \deg(ad - bc) &\leq \max(\deg(ad), \deg(bc)) \\ &\leq \max(\deg(a) + \deg(d), \deg(b) + \deg(c)) \\ &\leq \max(\deg(b) + \deg(d), \deg(b) + \deg(d)) \\ &\leq \deg(b) + \deg(d) \\ &\leq \deg(bd) \end{aligned}$$

and

$$\deg(ac) = \deg(a) + \deg(c) \leq \deg(b) + \deg(d) = \deg(bd).$$

(Here we define  $\deg(0) = -\infty$  by convention.) By the subring test,  $\mathcal{O}_\infty$  is a subring of  $k(x)$ .

- (2) Show that  $\mathcal{O}_\infty$  is a DVR. Clearly  $\mathcal{O}_\infty$  is not a field since  $1/x \in \mathcal{O}_\infty$  but  $x = x/1 \notin \mathcal{O}_\infty$ . Let  $t = 1/x$  be an irreducible element of  $\mathcal{O}_\infty$ . ( $\deg(x) = 1$  implies the irreducibility of  $t$ .) Now for any nonzero  $f/g \in \mathcal{O}_\infty$ , write

$$f/g = ((fx^n)/g)(1/x^n) = ((fx^n)/g)t^n$$

where  $n := \deg(g) - \deg(f) \geq 0$ . Note that  $\deg(fx^n) = \deg(f) + n = \deg(g)$ . So  $(fx^n)/g$  is a unit since the inverse  $g/(fx^n)$  is also in  $\mathcal{O}_\infty$ . Besides, it is easy to show that  $n$  is unique by the similar argument in Problem 2.23. Hence,  $\mathcal{O}_\infty$  is a DVR.

□

*Note.*

- (1) The quotient field of  $\mathcal{O}_\infty$  is  $K = k(V) = k(x)$ .
- (2) The set of units in  $\mathcal{O}_\infty(V)$  is  $\{f/g \in k(x) : \deg(g) = \deg(f)\}$ .
- (3) The maximal ideal of  $\mathcal{O}_\infty(V)$  is  $\{f/g \in k(x) : \deg(g) > \deg(f)\}$ .

**Problem 2.25. ( $p$ -adic integers)**

Let  $p \in \mathbb{Z}$  be a prime number. Show that

$$\{r \in \mathbb{Q} : r = a/b, a, b \in \mathbb{Z}, p \text{ doesn't divide } b\}$$

is a DVR with quotient field  $\mathbb{Q}$ .

*Proof.*

- (1) Let

$$\mathbb{Z}_p = \{r \in \mathbb{Q} : r = a/b, a, b \in \mathbb{Z}, p \nmid b\}$$

be the set of all  $p$ -adic integers.

- (2) Show that  $\mathbb{Z}_p$  is a subring of  $\mathbb{Q}$ . Clearly,  $1 = 1/1 \in \mathbb{Z}_p$  (since  $p \nmid 1$ ). Also, given any  $r = a/b, s = c/d \in \mathbb{Z}_p$ . So

$$\begin{aligned} r - s &= a/b - c/d = \frac{ad - bc}{bd} \in \mathbb{Z}_p \\ rs &= a/b \cdot c/d = \frac{ac}{bd} \in \mathbb{Z}_p \end{aligned}$$

since  $p \nmid b, p \nmid d$  and  $p$  is a prime number. By the subring test,  $\mathbb{Z}_p$  is a subring of  $\mathbb{Q}$ .

- (3) Note that  $\mathbb{Z}_p \subseteq \mathbb{Q}$  is a domain and  $\mathbb{Z}_p$  is not a field (since  $p = p/1 \in \mathbb{Z}_p$  but  $p^{-1} = 1/p \notin \mathbb{Z}_p$ ).

- (4) Let  $t = p$  be an irreducible element in  $\mathbb{Z}_p$ . For the irreducibility of  $t = p$ , we write  $p = a/b \cdot c/d = \frac{ac}{bd}$  where  $p \nmid b, p \nmid d$ . So  $pbd = ac$  or

$$1 = \text{ord}_p(ac) = \text{ord}_p(a) + \text{ord}_p(c).$$

Here  $\text{ord}_p : \mathbb{Z} \rightarrow \mathbb{Z}_{\geq 0}$  is defined by  $\text{ord}_p(a) = n$  where  $n$  is the largest number such that  $p^n$  divides  $a$ , that is,  $p^n \mid a$  and  $p^{n+1} \nmid a$ . So  $(\text{ord}_p(a), \text{ord}_p(c)) = (0, 1)$  or  $(1, 0)$ . Hence,  $a/b$  or  $c/d$  is a unit in  $\mathbb{Z}_p$ , or  $p$  is irreducible in  $\mathbb{Z}_p$ .

- (5) For any nonzero  $r = a/b \in \mathbb{Z}_p$ ,  $a \neq 0$  can be written as  $a = p^n c$  for some nonnegative integer  $n$  and  $c \in \mathbb{Z}^+$  uniquely. Hence

$$r = a/b = (c/b)p^n = (c/b)t^n,$$

where  $c/b$  is a unit and  $n$  is a nonnegative integer. Besides, it is easy to show that  $n$  is unique by the similar argument in Problem 2.23. By Proposition 4,  $\mathbb{Z}_p$  is a DVR.

- (6) Show that the quotient field of  $\mathbb{Z}_p$  is  $\mathbb{Q}$ . It suffices to show that  $r$  is in the quotient field of  $\mathbb{Z}_p$  if  $r \in \mathbb{Q} - \mathbb{Z}_p$ . Note that  $r \neq 0$ . Write  $r = a/b$  with  $\gcd(a, b) = 1$ . As  $r \notin \mathbb{Z}_p$ ,  $p \mid b$  and  $p \nmid a$ . Therefore,  $1/r = b/a \in \mathbb{Z}_p$ , or  $r$  is in the quotient field of  $\mathbb{Z}_p$ .

□

*Note.*

- (1)  $p\mathbb{Z}_p$  is the maximal ideal of  $\mathbb{Z}_p$ .
- (2) The residue field  $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$ .

### Problem 2.26.\*

Let  $R$  be a DVR with quotient field  $K$ ; let  $\mathfrak{m}$  be the maximal ideal of  $R$ .

- (a) Show that if  $z \in K$ ,  $z \notin R$ , then  $z^{-1} \in \mathfrak{m}$ .
- (b) Suppose  $R \subseteq S \subseteq K$ , and  $S$  is also a DVR. Suppose the maximal ideal of  $S$  contains  $\mathfrak{m}$ . Show that  $S = R$ .

*Proof of (a).*

- (1) Suppose  $t$  is one uniformizing parameter for  $R$ . If  $z \in K - R$ , then we can write  $z = ut^{-n}$  for some unit  $u \in R$  and  $n \in \mathbb{Z}^+$ .
- (2) Hence,

$$z^{-1} = u^{-1}t^n.$$

Since  $u^{-1}$  is a unit in  $R$  and  $n > 0$ ,  $z^{-1} \in \mathfrak{m}$ .

□

*Proof of (b).*

- (1) (Reductio ad absurdum) Suppose  $z \in S - R \subseteq K - R$ . By (a),  $z^{-1} \in \mathfrak{m}$ . So  $z^{-1}$  is in the maximal ideal  $\mathfrak{m}'$  of  $S$  containing  $\mathfrak{m}$ .
- (2) As  $\mathfrak{m}'$  is an ideal,  $1 = z \cdot z^{-1} \in \mathfrak{m}'$ , which is absurd. Therefore,  $S = R$ .

□

**Problem 2.27.**

Show that the DVR's of Problem 2.24 are the only DVR's with quotient field  $k(x)$  that contain  $k$ . Show that those of Problem 2.25 are the only DVR's with quotient field  $\mathbb{Q}$ .

*Proof (Problem 2.26).*

- (1) Show that  $\mathcal{O}_a(V)$  and  $\mathcal{O}_\infty$  are the only DVR's with quotient field  $k(x)$  that contain  $k$ .
  - (a) Let  $k \subseteq R \subsetneq k(x)$  be a DVR with quotient field  $k(x)$ ,  $\mathfrak{m}$  be the unique maximal ideal of  $R$ .  $\mathfrak{m} \neq (0)$  and the set of units in  $R$  is  $R - \mathfrak{m}$ .
  - (b) There are two possible cases:  $x \in R$  or  $x \notin R$ .
  - (c) Suppose  $x \in R$ . So  $R$  contains  $k[x]$  as a subring. Consider the subset

$$S := \{x - a \in k[x] : a \in k\} \cap \mathfrak{m} \subseteq \mathfrak{m}.$$

Suppose there were two distinct elements  $x - a, x - b \in S$ . Then  $1 \in \mathfrak{m}$ , contrary to the maximality of  $\mathfrak{m}$ . Suppose  $S = \emptyset$ , then every  $x - a$  is a unit in  $R$ . Since  $k = \bar{k}$ ,  $R = k(x)$  is a field, which is absurd. Hence, there is only one  $x - a \in \mathfrak{m}$  for one unique  $a \in k$  and other  $x - b$  with  $b \neq a$  is a unit in  $R$ . Thus,  $R \supseteq \mathcal{O}_a(V)$  and  $\mathfrak{m}$  contains  $(x - a)\mathcal{O}_a(V)$ , which is the maximal ideal of  $\mathcal{O}_a(V)$ . By Problem 2.26,  $R = \mathcal{O}_a(V)$ .

- (d) If  $x \notin R$ , then  $x - a \notin R$  whenever  $a \in k \subseteq R$ . Hence  $(x - a)^{-1} \in \mathfrak{m}$  whenever  $a \in k$  by Problem 2.26(a). Next, given any  $f/g \in \mathcal{O}_\infty$ , by  $k = \bar{k}$  we have

$$f/g = \underbrace{u}_{\in k} \underbrace{\frac{x - \alpha_1}{x - \beta_1}}_{\in R} \cdots \underbrace{\frac{x - \alpha_n}{x - \beta_n}}_{\in R} \underbrace{\frac{1}{x - \beta_{n+1}}}_{\in \mathfrak{m}} \cdots \underbrace{\frac{1}{x - \beta_m}}_{\in \mathfrak{m}},$$

where  $n := \deg(f)$ ,  $m := \deg(g)$  and  $n \leq m$ . Here

$$\frac{x - \alpha_i}{x - \beta_i} = \underbrace{1}_{\in k} + \underbrace{\frac{\beta_i - \alpha_i}{x - \beta_i}}_{\in \mathfrak{m} \subseteq R} \in R.$$

Therefore,  $R \supseteq \mathcal{O}_\infty$  and  $\mathfrak{m}$  contains the maximal ideal  $x^{-1}\mathcal{O}_\infty$  of  $\mathcal{O}_\infty$ . By Problem 2.26,  $R = \mathcal{O}_\infty$ .

(2) Show that  $\mathbb{Z}_p$  are the only DVR's with quotient field  $\mathbb{Q}$ .

(a) Let  $R \subsetneq \mathbb{Q}$  be a DVR with quotient field  $\mathbb{Q}$ ,  $\mathfrak{m}$  be the unique maximal ideal of  $R$ .  $\mathfrak{m} \neq (0)$  and the set of units in  $R$  is  $R - \mathfrak{m}$ .

(b) Note that  $R \subseteq \mathbb{Q}$  contains  $\mathbb{Z}$  as a subring. Consider the subset

$$S := \{p \in \mathbb{Z} : p \text{ is a prime number}\} \cap \mathfrak{m} \subseteq \mathfrak{m}.$$

(c) Suppose there were two distinct prime integers  $p, q \in S$ . By the Bézout's identity, there exist integers  $a$  and  $b$  such that  $pa + qb = 1$ .  $1 \in \mathfrak{m}$ , contrary to the maximality of  $\mathfrak{m}$ .

(d) Suppose no prime integer were in  $S$ , then every prime integer is a unit in  $R$ . By the fundamental theorem of arithmetic,  $R = \mathbb{Q}$  is a field, which is absurd.

(e) By (c)(d),  $p \in \mathfrak{m}$  for one unique prime  $p \in \mathbb{Z}$ . Thus,  $R \supseteq \mathbb{Z}_p$  by the definition of  $\mathbb{Z}_p$  and  $\mathfrak{m}$  contains  $p\mathbb{Z}_p$ , which is the maximal ideal of  $\mathbb{Z}_p$ . By Problem 2.26,  $R = \mathbb{Z}_p$ .

□

### Problem 2.28.\*

An order function on a field  $K$  is a function  $\varphi$  from  $K$  onto  $\mathbb{Z} \cup \{\infty\}$ , satisfying:

(i)  $\varphi(a) = \infty$  if and only if  $a = 0$ .

(ii)  $\varphi(ab) = \varphi(a) + \varphi(b)$ .

(iii)  $\varphi(a + b) \geq \min(\varphi(a), \varphi(b))$ .

Show that  $R = \{z \in K : \varphi(z) \geq 0\}$  is a DVR with maximal ideal  $\mathfrak{m} = \{z \in K : \varphi(z) > 0\}$ , and quotient field  $K$ . Conversely, show that if  $R$  is a DVR with quotient field  $K$ , then the function  $\text{ord} : K \rightarrow \mathbb{Z} \cup \{\infty\}$  is an order function on  $K$ . Giving a DVR with quotient field  $K$  is equivalent to defining an order function on  $K$ .

*Proof.*

(1) Show that  $\varphi(1) = 0$ . Note that  $\varphi(1) = \varphi(1 \cdot 1) = \varphi(1) + \varphi(1)$  by (ii). By Property (i) of  $\varphi$ , we cancel  $\varphi(1) \in \mathbb{Z}$  on the both side to get  $\varphi(1) = 0$ .

- (2) Show that  $\varphi(-z) = \varphi(z)$  for all  $z \in K$ , and  $\varphi(z^{-1}) = -\varphi(z)$  for all  $z \in K - \{0\}$ . Note that  $\varphi(-1) = 0$  since  $0 = \varphi(1) = \varphi((-1) \cdot (-1)) = \varphi(-1) + \varphi(-1)$  (by (1)). Therefore,

$$\varphi(-z) = \varphi((-1) \cdot z) = \varphi(-1) + \varphi(z) = \varphi(z).$$

Besides,

$$0 = \varphi(1) = \varphi(z z^{-1}) = \varphi(z) + \varphi(z^{-1})$$

if  $z \neq 0$ . So  $\varphi(z^{-1}) = -\varphi(z)$  if  $z \neq 0$ .

- (3) Show that  $R = \{z \in K : \varphi(z) \geq 0\}$  is a ring.

(a)  $R \neq \emptyset$  since  $1 \in R$ .

(b) If  $a, b \in R$ , then

$$\varphi(a - b) \geq \min(\varphi(a), \varphi(-b)) = \min(\varphi(a), \varphi(b)) \geq 0$$

(by (2)), or  $a - b \in R$ .

(c) If  $a, b \in R$ , then  $\varphi(ab) = \varphi(a) + \varphi(b) \geq 0$ .

By the subring test,  $R$  is a subring of  $K$ .

- (4) Show that  $\{z \in K - \{0\} : \varphi(z) = 0\}$  is the set of all units in  $R$ . Given any  $z \in K - \{0\}$ , we have

$$0 = \varphi(z) + \varphi(z^{-1})$$

(by (2)). Hence  $z$  is a unit in  $R$  iff  $z, z^{-1} \in R$  iff  $\varphi(z) = \varphi(z^{-1}) = 0$ .

- (5) Show that  $\mathfrak{m} = \{z \in K : \varphi(z) > 0\}$  is a maximal ideal of  $R$ .

(a) If  $a, b \in \mathfrak{m}$ , then  $\varphi(a + b) \geq \min(\varphi(a), \varphi(b)) > 0$ .

(b) If  $a \in \mathfrak{m}$  and  $r \in R$ , then  $\varphi(ra) = \varphi(r) + \varphi(a) \geq \varphi(a) > 0$ .

(c) By (a)(b),  $\mathfrak{m}$  is an ideal of  $R$ .

(d) Note that each proper ideal in  $R$  does not have any unit, that is, such proper ideal is contained in  $\{z \in K : \varphi(z) > 0\} = \mathfrak{m}$  exactly (by (4)). Therefore,  $\mathfrak{m}$  is maximal. (Such maximal ideal  $\mathfrak{m}$  is unique and thus  $R$  is a local ring.)

- (6) Show that  $R$  is a DVR. It suffices to show that there is an irreducible element  $t \in R$  such that every nonzero  $z \in R$  may be written uniquely in the form  $z = ut^n$ ,  $u$  a unit in  $R$ ,  $n$  a nonnegative integer. Since  $\varphi$  is surjective, there is an element  $t \in R$  such that  $\varphi(t) = 1$ . Note that  $t \neq 0$  and irreducible (by using Property (ii) of  $\varphi$ ). Hence for any nonzero  $z \in R$  with  $n := \varphi(z) \in \mathbb{Z}$  and  $n \geq 0$ , the order of  $zt^{-n} \in K$  is

$$\varphi(zt^{-n}) = \varphi(z) - n\varphi(t) = n - n \cdot 1 = 0$$

(by (2)). That is,  $zt^{-n} = u$  is a unit in  $R$  (by (4)). Hence  $z = ut^n$  for some unit  $u \in R$  and nonnegative integer  $n$ . Note that  $n$  is uniquely determined by  $\varphi(z)$ . By Proposition 4,  $R$  is a DVR.

- (7) *Show that the quotient field of  $R$  is  $K$ .* Since  $R$  is a DVR, the quotient field of  $R$  is contained in  $K$ . Conversely, given any  $z \in K$ . If  $\varphi(z) \geq 0$ , then  $z \in R \subseteq K$ . If  $\varphi(z) < 0$ , then  $\varphi(z^{-1}) = -\varphi(z) > 0$  or  $z^{-1} \in R$ . Hence  $z = 1/z^{-1} \in K$  is in the quotient field of  $R$ .
- (8) *Show that giving a DVR with quotient field  $K$  is equivalent to defining an order function on  $K$ .* It suffices to show that  $\text{ord}(\cdot)$  on  $K$  defines an order function  $\varphi$  on  $K$ . By Problem 2.29, it suffices to show that

$$\text{ord}(a + b) \geq \min(\text{ord}(a), \text{ord}(b))$$

if  $\text{ord}(a) = \text{ord}(b) := n$ . Write  $a = ut^n, b = vt^n$  where  $u, v$  are units in  $R$ . Hence,

$$\begin{aligned} \text{ord}(a + b) &= \text{ord}(ut^n + vt^n) \\ &= \text{ord}((u + v)t^n) \\ &= \text{ord}(u + v) + n \\ &\geq n && (u + v \in R) \\ &= \min(\text{ord}(a), \text{ord}(b)). \end{aligned}$$

□

**Problem 2.29.\***

Let  $R$  be a DVR with quotient field  $K$ ,  $\text{ord}$  the order function on  $K$ .

- (a) *If  $\text{ord}(a) < \text{ord}(b)$ , show that  $\text{ord}(a + b) = \text{ord}(a)$ .*
- (b) *If  $a_1, \dots, a_n \in K$ , and for some  $i$ ,  $\text{ord}(a_i) < \text{ord}(a_j)$  (all  $j \neq i$ ), then  $a_1 + \dots + a_n \neq 0$ .*

*Proof of (a).*

- (1) Let  $t$  be a uniformizing parameter for  $R$ . Given any  $a, b \in K$ . Write  $a = ut^n, b = vt^m$  where  $u, v$  are units in  $R$  and  $n, m$  are integers.
- (2) Since  $\text{ord}(a) < \text{ord}(b)$ ,  $n < m$ . Hence,

$$a + b = (u + vt^{m-n})t^n.$$

To show that  $\text{ord}(a + b) = \text{ord}(a) = n$ , it suffices to show that  $u + vt^{m-n}$  is a unit in  $R$ .

- (3) (Reductio ad absurdum) Suppose that  $u + vt^{m-n}$  were not a unit. Since  $R$  is local, the maximal ideal  $(t)$  contains all nonunit elements in  $R$ . Hence,  $u + vt^{m-n} \in (t)$ . As  $m - n > 0$ ,  $vt^{m-n} \in (t)$  and thus a unit  $u \in (t)$ , contrary to the maximality of  $(t)$ .



□

*Proof of (b).*

- (1) Might assume that  $\text{ord}(a_1) < \text{ord}(a_j)$  (all  $j \neq 1$ ). In particular,  $\text{ord}(a_1) < \infty$ .
- (2) Similar to (a). Let  $t$  be a uniformizing parameter for  $R$ . Write  $a_i = u_i t^{m_i}$  where  $u_i$  are units in  $R$  and  $m_i$  are integers. ( $i = 1, \dots, n$ .) Since  $\text{ord}(a_1) < \text{ord}(a_j)$  (all  $j \neq 1$ ),  $m_1 < m_j$ . Hence,

$$a_1 + \dots + a_n = (u_1 + \underbrace{u_2 t^{m_2 - m_1} + \dots + u_n t^{m_n - m_1}}_{\in (t)}) t^{m_1}.$$

So  $u_1 + u_2 t^{m_2 - m_1} + \dots + u_n t^{m_n - m_1}$  is a unit in  $R$ .

- (3) By (1)(2),

$$\text{ord}(a_1 + \dots + a_n) = \text{ord}(a_1) < \infty,$$

or  $a_1 + \dots + a_n \neq 0$  (since  $\text{ord}$  is an order function on  $K$ ).

□

### Problem 2.30.\*

Let  $R$  be a DVR with maximal ideal  $\mathfrak{m}$ , and quotient field  $K$ , and suppose a field  $k$  is a subring of  $R$ , and that the composition  $k \rightarrow R \rightarrow R/\mathfrak{m}$  is an isomorphism of  $k$  with  $R/\mathfrak{m}$  (as for example in Problem 2.24). Verify the following assertions:

- (a) For any  $z \in R$ , there is a unique  $\lambda \in k$  such that  $z - \lambda \in \mathfrak{m}$ .
- (b) Let  $t$  be a uniformizing parameter for  $R$ ,  $z \in R$ . Then for any  $n \geq 0$  there are unique  $\lambda_0, \lambda_1, \dots, \lambda_n \in k$  and  $z_n \in R$  such that

$$z = \lambda_0 + \lambda_1 t + \lambda_2 t^2 + \dots + \lambda_n t^n + z_n t^{n+1}.$$

(Hint: For uniqueness use Problem 2.29; for existence use (a) and induction.)

*Proof of (a).*

- (1) Note that

$$k \xrightarrow{i} R \xrightarrow{\pi} R/\mathfrak{m}$$

is an isomorphism.

- (2) For  $z + \mathfrak{m} \in R/\mathfrak{m}$ , there exists the unique  $\lambda \in k$  such that

$$z + \mathfrak{m} = \pi(i(\lambda)) = \pi(\lambda) = \lambda + \mathfrak{m}.$$

So  $z - \lambda \in \mathfrak{m}$  for one unique  $\lambda \in k$ .

□

*Proof of (b).*

(1) Note that

$$\mathfrak{m} = \{z \in K : \text{ord}(z) > 0\}.$$

By (a),

$$z = \lambda_0 + \underbrace{tz_0}_{\in \mathfrak{m}}$$

for one unique  $\lambda_0 \in k$  and  $z_0 \in R$ . Continue this process or by induction, we have the expression

$$z = \lambda_0 + \lambda_1 t + \lambda_2 t^2 + \cdots + \lambda_n t^n + z_n t^{n+1}.$$

(2) For the uniqueness, suppose

$$0 = \lambda_0 + \lambda_1 t + \lambda_2 t^2 + \cdots + \lambda_n t^n + z_n t^{n+1}.$$

Note that

$$\text{ord}(\lambda_i t^i) = \begin{cases} \infty & (\lambda_i = 0) \\ i & (\lambda_i \neq 0) \end{cases}$$

since every nonzero element in  $k$  is a unit in  $k \subseteq R$ . Also,  $\text{ord}(z_n t^{n+1}) = \infty$  if  $z_n = 0$ ;  $\text{ord}(z_n t^{n+1}) \geq n+1$  if  $z_n \neq 0$ .

(3) Suppose  $i_0$  is the smallest integer such that  $\lambda_{i_0} \neq 0$ , then  $\text{ord}(\lambda_{i_0} t^{i_0}) = i_0 < \text{ord}(\lambda_j t^j)$  if  $i_0 \neq j$  and  $\text{ord}(\lambda_{i_0} t^{i_0}) = i_0 < n+1 \leq \text{ord}(z_n t^{n+1})$ . By Problem 2.29(b), such  $i_0$  does not exist. Hence all  $\lambda_i = 0$ . So as  $R$  is a domain,  $z_n$  is also equal to 0. Therefore, the uniqueness is established.

□

### Problem 2.31. (Formal power series)

Let  $k$  be a field. The ring of **formal power series** over  $k$ , written  $k[[x]]$ , is defined to be

$$\left\{ \sum_{i=0}^{\infty} a_i x^i : a_i \in k \right\}.$$

(As with polynomials, a rigorous definition is best given in terms of sequences  $(a_0, a_1, \dots)$  of elements in  $k$ ; here we allow an infinite number of nonzero terms.) Define the sum by

$$\sum a_i x^i + \sum b_i x^i = \sum (a_i + b_i) x^i,$$

and the (Cauchy) product by

$$\left(\sum a_i x^i\right) \left(\sum b_i x^i\right) = \sum c_i x^i,$$

where  $c_i = \sum_{j+k=i} a_j b_k$ . Show that  $k[[x]]$  is a ring containing  $k[x]$  as a subring. Show that  $k[[x]]$  is a DVR with uniformizing parameter  $x$ . Its quotient field is denoted  $k((x))$ .

*Proof.*

- (1) Two formal power series  $\sum a_i x^i$  and  $\sum b_i x^i$  in  $k[[x]]$  are considered equal if  $a_i = b_i$  for all integers  $i \geq 0$ .
- (2) The zero element in  $k[[x]]$  is  $0 = \sum_{i=0}^{\infty} 0x^i$ , and the multiplicative identity is

$$1 = 1 + 0x + \cdots + 0x^n + \cdots.$$

Hence,  $k[[x]]$  is a ring (by a tedious argument). Moreover,  $k[[x]]$  is a domain (again by a tedious argument).

- (3) Show that  $k[[x]] \supseteq k[x]$ . In fact, for any  $f = \sum_{i=0}^n a_i x^i \in k[x]$ , we can write

$$f = a_0 + a_1 x + \cdots + a_n x^n + 0x^{n+1} + \cdots \in k[[x]].$$

- (4) Show that  $f = \sum_{i=0}^{\infty} a_i x^i$  is a unit in  $k[[x]]$  if and only if  $a_0 \neq 0$ . Suppose  $g = \sum_{i=0}^{\infty} b_i x^i \in k[[x]]$  such that  $fg = 1$ . Then

$$\begin{aligned} 1 &= a_0 b_0, \\ 0 &= \sum_{j=0}^k a_j b_{k-j}. \end{aligned}$$

So  $f$  is not a unit in  $k[[x]]$  if  $a_0 = 0$ . Now if  $a_0 \neq 0$  then  $b_0 := a_0^{-1} \in k$ . Then by observing that

$$\begin{aligned} 0 &= \sum_{j=0}^k a_j b_{k-j} \iff a_0 b_k = - \sum_{j=1}^k a_j b_{k-j} \\ &\iff b_k = -b_0 \sum_{j=1}^k a_j b_{k-j}, \end{aligned}$$

we can solve  $b_1, b_2, \dots$  by induction, and  $(b_0, b_1, \dots)$  gives the existence of  $g \in k[[x]]$ .

- (5) By (4),  $k[[x]]$  is not a field since  $x \in k[[x]]$  but  $x^{-1} \notin k[[x]]$ . Let  $t = x$  be an irreducible element in  $k[[x]]$ . ( $\deg(x) = 1$  implies the irreducibility of  $t$ .) Hence every nonzero  $f \in k[[x]]$  can be written uniquely in the form

$$f = ux^n$$

where  $n$  is the smallest integer such that  $a_n \neq 0$ . By (4),

$$u = a_n + a_{n+1}x + \cdots$$

is a unit in  $k[[x]]$  as  $a_n \neq 0$ . Besides, it is easy to show that  $n$  is unique by the similar argument in Problem 2.23. Therefore,  $k[[x]]$  is a DVR with uniformizing parameter  $x$ .

□

**Problem 2.32. (Power series expansion)**

Let  $R$  be a DVR satisfying the conditions of Problem 2.30. Any  $z \in R$  then determines a power series  $\sum \lambda_i x^i$ , if  $\lambda_0, \lambda_1, \dots$  are determined as in Problem 2.30(b).

- (a) Show that the map  $z \rightarrow \sum \lambda_i x^i$  is a one-to-one ring homomorphism of  $R$  into  $k[[x]]$ . We often write  $z = \sum \lambda_i t^i$ , and call this the **power series expansion** of  $z$  in terms of  $t$ .
- (b) Show that the homomorphism extends to a homomorphism of  $K$  into  $k((x))$ , and that the order function on  $k((x))$  restricts to that on  $K$ .
- (c) Let  $a = 0$  in Problem 2.24,  $t = x$ . Find the power series expansion of  $z = (1 - x)^{-1}$  and of  $(1 - x)(1 + x^2)^{-1}$  in terms of  $t$ .

*Proof of (a).*

- (1) Define the map  $\alpha : R \rightarrow k[[x]]$  by

$$\alpha : z \mapsto \sum_{i=0}^{\infty} \lambda_i x^i$$

where  $\lambda_i$  are determined as in Problem 2.30(b).

- (2) Show that  $\alpha$  is well-defined and one-to-one. Write

$$\alpha(z) = \sum_{i=0}^{\infty} \lambda_i x^i = \sum_{i=0}^{\infty} \lambda'_i x^i.$$

If there were  $\lambda_n \neq \lambda'_n$  for some  $n$ , then Problem 2.30(b) implies that two expressions of  $z$

$$\begin{aligned} z &= \lambda_0 + \lambda_1 t + \cdots + \lambda_n t^n + z_n t^{n+1} \\ &= \lambda'_0 + \lambda'_1 t + \cdots + \lambda'_n t^n + z'_n t^{n+1} \end{aligned}$$

are the same. That is,  $\lambda_n = \lambda'_n$ , which is absurd. Hence,  $\alpha$  is well-defined. Also,  $0 = 0 + 0t + 0t^2 + \cdots + 0t^n + 0t^{n+1}$  implies that  $\alpha$  is one-to-one.

(3) *Show that  $\alpha$  is addition preserving.* Given  $a, b \in R$ . By Problem 2.30(b),

$$a + b = \lambda_0 + \lambda_1 t + \cdots + \lambda_n t^n + c_n t^{n+1}$$

and

$$\begin{aligned} a &= \mu_0 + \mu_1 t + \cdots + \mu_n t^n + a_n t^{n+1} \\ b &= \nu_0 + \nu_1 t + \cdots + \nu_n t^n + b_n t^{n+1} \end{aligned}$$

for any integer  $n \geq 0$ . So

$$a + b = \underbrace{(\mu_0 + \nu_0)}_{\in k} + \underbrace{(\mu_1 + \nu_1)}_{\in k} t + \cdots + \underbrace{(\mu_n + \nu_n)}_{\in k} t^n + \underbrace{(a_n + b_n)}_{\in R} t^{n+1}.$$

Since the expression of  $a + b$  is unique (by Problem 2.30(b)),

$$\lambda_i = \mu_i + \nu_i$$

for all  $i = 0, 1, \dots, n$ . Since  $n$  is arbitrary,  $\lambda_i = \mu_i + \nu_i$  is true for all nonnegative integers. Hence,  $\alpha(a + b) = \alpha(a) + \alpha(b)$ .

(4) *Show that  $\alpha$  is multiplication preserving.* Given  $a, b \in R$ . By Problem 2.30(b),

$$ab = \lambda_0 + \lambda_1 t + \cdots + \lambda_n t^n + c_n t^{n+1}$$

and

$$\begin{aligned} a &= \mu_0 + \mu_1 t + \cdots + \mu_n t^n + a_n t^{n+1} \\ b &= \nu_0 + \nu_1 t + \cdots + \nu_n t^n + b_n t^{n+1} \end{aligned}$$

for any integer  $n \geq 0$ . So

$$\begin{aligned} ab &= \underbrace{(\mu_0 \nu_0)}_{\in k} + \underbrace{(\mu_1 \nu_0 + \mu_0 \nu_1)}_{\in k} t + \cdots \\ &\quad + \underbrace{(\mu_n \nu_0 + \mu_{n-1} \nu_1 + \cdots + \mu_1 \nu_{n-1} + \mu_0 \nu_n)}_{\in k} t^n \\ &\quad + \underbrace{(\text{other terms})}_{\in R} t^{n+1}. \end{aligned}$$

Since the expression of  $a + b$  is unique (by Problem 2.30(b)),

$$\lambda_i = \sum_{j+k=i} \mu_j \nu_k$$

for all  $i = 0, 1, \dots, n$ . Since  $n$  is arbitrary,  $\lambda_i = \sum_{j+k=i} \mu_j \nu_k$  is true for all nonnegative integers. Hence,  $\alpha(ab) = \alpha(a)\alpha(b)$ .

(5) Show that  $\alpha$  is multiplicative identity preserving. Note that

$$1 = \underbrace{1}_{\in k} + \underbrace{0}_{\in k}t + \cdots + \underbrace{0}_{\in k}t^n + \underbrace{0}_{\in k}t^{n+1}$$

for every nonnegative integer  $n$ . Hence  $\alpha : 1 \mapsto 1 \in k[[x]]$ .

(6) By (3)(4)(5),  $\alpha$  is a ring homomorphism.

□

*Proof of (b).*

(1) Define the mapping  $\beta$  from  $K$  to  $k((x))$  by

$$\beta : a/b \mapsto \alpha(a)/\alpha(b)$$

where  $a, b \in R$  and  $b \neq 0$ .

(2)  $\beta$  is well-defined since:

(a)  $\alpha(b) \neq 0$  if  $b \neq 0$  by the injectivity of  $\alpha$ .

(b) The value of  $\beta(a/b)$  is independent of the choice of  $a/b \in K$  since  $\alpha$  is a ring homomorphism.

(3) Also,  $\beta$  is a ring homomorphism since  $\alpha$  is a ring homomorphism.

(4) To show that the order function on  $k((x))$  restricts to that on  $K$ , it suffices to show that

$$\text{ord}_R(z) = \text{ord}_{k[[x]]}(\alpha(z)).$$

In fact,

$$\begin{aligned} m := \text{ord}_R(z) &\iff z = \lambda_m t^m + \cdots + \lambda_n t^n + z_n t^{n+1} \text{ with } \lambda_m \neq 0 \\ &\iff \alpha(z) = \lambda_m x^m + \cdots \text{ with } \lambda_m \neq 0 \\ &\iff \text{ord}_{k[[x]]}(\alpha(z)) = m. \end{aligned}$$

□

*Proof of (c).*

(1) In calculus we have

$$(1 - x)^{-1} = 1 + x + x^2 + \cdots = \sum_{i=0}^{\infty} x^i$$

for  $|x| < 1$ . In the ring of formal power series  $k[[x]]$ ,  $1 - x$  is a unit (by (4) in the proof of Problem 2.31) and satisfies

$$(1 - x) \left( \sum_{i=0}^{\infty} x^i \right) = 1 \in k[[x]].$$

Hence, the power expansion of  $(1 - x)^{-1}$  is

$$(1 - x)^{-1} = \sum_{i=0}^{\infty} x^i \in k((x)).$$

(2) Note that  $1 + x^2$  is a unit in  $k[[x]]$  and satisfies

$$(1 + x^2) \left( \sum_{i=0}^{\infty} (-1)^i x^{2i} \right) = 1 \in k[[x]].$$

Hence, the power expansion of  $(1 - x)(1 + x^2)^{-1}$  is

$$\begin{aligned} (1 - x) \left( \sum_{i=0}^{\infty} (-1)^i x^{2i} \right) &= \left( \sum_{i=0}^{\infty} (-1)^i x^{2i} \right) - x \left( \sum_{i=0}^{\infty} (-1)^i x^{2i} \right) \\ &= \sum_{i=0}^{\infty} (-1)^i x^{2i} + \sum_{i=0}^{\infty} (-1)^{i+1} x^{2i+1} \\ &= \sum_{i=0}^{\infty} (-1)^i x^i \in k[[x]]. \end{aligned}$$

□

## 2.6. Forms

### Problem 2.33.

Factor  $y^3 - 2xy^2 + 2x^2y + x^3$  into linear factors in  $\mathbb{C}[x, y]$ .

*Proof.*

- (1) Let  $f(x, y) = y^3 - 2xy^2 + 2x^2y + x^3$ . Then  $f_*(x) = 1 - 2x + 2x^3 + x^3$ .
- (2) Solve  $f_*(x) = 0$  over  $\mathbb{C}$  by WolframAlpha (a computational knowledge engine) to get

$$\begin{aligned} \alpha_1 &= -\frac{2}{3} - \frac{10}{3} \sqrt[3]{\frac{2}{79 - 3\sqrt{249}}} - \frac{1}{3} \sqrt[3]{\frac{79 - 3\sqrt{249}}{2}} \\ \alpha_2 &= -\frac{2}{3} + \frac{5}{3}(1 - \sqrt{3}i) \sqrt[3]{\frac{2}{79 - 3\sqrt{249}}} + \frac{1}{6}(1 + \sqrt{3}i) \sqrt[3]{\frac{79 - 3\sqrt{249}}{2}} \\ \alpha_3 &= -\frac{2}{3} + \frac{5}{3}(1 + \sqrt{3}i) \sqrt[3]{\frac{2}{79 - 3\sqrt{249}}} + \frac{1}{6}(1 - \sqrt{3}i) \sqrt[3]{\frac{79 - 3\sqrt{249}}{2}}. \end{aligned}$$

So  $f_*(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$ .

(3) Hence,

$$\begin{aligned} f(x, y) &= (f_*)^* \\ &= ((x - \alpha_1)(x - \alpha_2)(x - \alpha_3))^* \\ &= (x - \alpha_1 y)(x - \alpha_2 y)(x - \alpha_3 y). \end{aligned}$$

□

*Note.* If  $f(x, y) = y^3 - 2xy^2 + 2x^2y + 4x^3$ , then

$$f(x, y) = (x - \alpha_1 y)(x - \alpha_2 y)(x - \alpha_3 y)$$

where

$$\begin{aligned} \alpha_1 &= -\frac{1}{6} - \frac{7}{6} \sqrt[3]{\frac{1}{37 - 3\sqrt{114}}} - \frac{1}{6} \sqrt[3]{37 - 3\sqrt{114}} \\ \alpha_2 &= -\frac{1}{6} + \frac{7}{12}(1 - \sqrt{3}i) \sqrt[3]{\frac{1}{37 - 3\sqrt{114}}} + \frac{1}{12}(1 + \sqrt{3}i) \sqrt[3]{37 - 3\sqrt{114}} \\ \alpha_3 &= -\frac{1}{6} + \frac{7}{12}(1 + \sqrt{3}i) \sqrt[3]{\frac{1}{37 - 3\sqrt{114}}} + \frac{1}{12}(1 - \sqrt{3}i) \sqrt[3]{37 - 3\sqrt{114}}. \end{aligned}$$

**Problem 2.34.**

Suppose  $f, g \in k[x_1, \dots, x_n]$  are forms of degree  $r, r + 1$  respectively, with no common factors ( $k$  a field). Show that  $f + g$  is irreducible.

*Proof.*

(1) Suppose  $f + g = rs \in k[x_1, \dots, x_n]$ . Proposition 5 implies that

$$(f + g)^* = (rs)^* \implies x_{n+1}f + g = r^*s^*.$$

Note that  $\deg_{x_{n+1}}(x_{n+1}f + g) = 1$ . So  $\deg_{x_{n+1}}(r^*) = 0$  or  $\deg_{x_{n+1}}(s^*) = 0$ . Might assume  $\deg_{x_{n+1}}(r^*) = 0$ . (The case  $\deg_{x_{n+1}}(s^*) = 0$  is similar.)

(2) Since  $\deg_{x_{n+1}}(r^*) = 0$ ,  $r^* \mid f$  and  $r^* \mid g$ . Note that  $\deg_{x_{n+1}}(r^*) = 0$  implies that  $r^* = r$  is a form in  $k[x_1, \dots, x_n]$ . Hence  $r$  is a common factor of  $f$  and  $g$ , or  $r$  is a constant in  $k[x_1, \dots, x_n]$ . So  $f + g$  is irreducible.

□



**Problem 2.35.\***

- (a) Show that there are  $d + 1$  monomials of degree  $d$  in  $R[x, y]$ , and  $1 + 2 + \cdots + (d + 1) = \frac{(d+1)(d+2)}{2}$  monomials of degree  $d$  in  $R[x, y, z]$ .
- (b) Let  $V(d, n) = \{\text{forms of degree } d \text{ in } k[x_1, \dots, x_n]\}$ ,  $k$  a field. Show that  $V(d, n)$  is a vector space over  $k$ , and that the monomials of degree  $d$  form a basis. So  $\dim V(d, 1) = 1$ ;  $\dim V(d, 2) = d + 1$ ;  $\dim V(d, 3) = \frac{(d+1)(d+2)}{2}$ .
- (c) Let  $\ell_1, \ell_2, \dots$  and  $m_1, m_2, \dots$  be sequences of nonzero linear forms in  $k[x, y]$ , and assume no  $\ell_i = \lambda m_j$ ,  $\lambda \in k$ . Let  $A_{ij} = \ell_1 \ell_2 \cdots \ell_i m_1 m_2 \cdots m_j$ ,  $i, j \geq 0$  ( $A_{00} = 1$ ). Show that  $\{A_{ij} : i + j = d\}$  forms a basis for  $V(d, 2)$ .

*Proof of (a).*

- (1) All monomials of degree  $d$  in  $R[x, y]$  are

$$x^d, x^{d-1}y, \dots, xy^{d-1}, y^d,$$

or of the form  $x^i y^j$  with  $i, j \geq 0$  and  $i + j = d$ . So there are  $d + 1$  monomials of degree  $d$  in  $R[x, y]$ .

- (2) Similar to (1), all monomials of degree  $d$  in  $R[x, y]$  are of the form  $x^i y^j z^k$  with  $i, j, k \geq 0$  and  $i + j + k = d$ . By the stars and bars (combinatorics) method, there are

$$\binom{d + 3 - 1}{3 - 1} = \frac{(d + 2)(d + 1)}{2}$$

monomials of degree  $d$  in  $R[x, y, z]$ .

□

*Proof of (b).*

- (1) To show  $V(d, n)$  is a vector space, it suffices to show that  $V(d, n)$  is a subspace of  $k[x_1, \dots, x_n]$  since  $k[x_1, \dots, x_n]$  is a vector space over  $k$ .
- (2) Note that  $0 \in V(d, n)$  is nonempty. For any  $f, g \in V(d, n)$  and  $a, b \in k$ , we have  $af + bg \in V(d, n)$ . Hence  $V(d, n)$  is subspace.
- (3) Let

$$\mathcal{B} = \{x_1^{i_1} \cdots x_n^{i_n} : i_1, \dots, i_n \geq 0, i_1 + \cdots + i_n = d\}.$$

$\mathcal{B}$  is an independent set, and  $\mathcal{B}$  generates  $V(d, n)$ . So  $\mathcal{B}$  is a basis for  $V(d, n)$ .

- (4) Similar to (a),

$$\dim_k V(d, n) = |\mathcal{B}| = \binom{d + n - 1}{n - 1}$$

by the stars and bars (combinatorics) method. In particular,  $\dim V(d, 1) = 1$ ;  $\dim V(d, 2) = d + 1$ ;  $\dim V(d, 3) = \frac{(d+1)(d+2)}{2}$ .

□

*Proof of (c).*

- (1) Show that  $\mathcal{B}' := \{A_{ij} : i + j = d\}$  is an independent set. (Reductio ad absurdum) Suppose that there were a nontrivial linear combination of  $A_{ij}$  such that

$$\sum_{i+j=d} c_{ij} A_{ij} = 0.$$

- (2) Let  $p$  be the smallest index  $i$  such that  $c_{ij} \neq 0$ . Write  $q := d - p$ . So

$$\begin{aligned} c_{pq} A_{pq} &= - \sum_{\substack{i+j=d \\ i \neq p, j \neq q}} c_{ij} A_{ij} = - \sum_{\substack{i+j=d \\ i > p, j < q}} c_{ij} A_{ij} \\ \iff A_{pq} &= - \sum_{\substack{i+j=d \\ i > p, j < q}} \frac{c_{ij}}{c_{pq}} A_{ij} \\ \iff \ell_1 \cdots \ell_p m_1 \cdots m_q &= - \sum_{\substack{i+j=d \\ i > p, j < q}} \frac{c_{ij}}{c_{pq}} \ell_1 \cdots \ell_p \ell_{p+1} \cdots \ell_i m_1 \cdots m_j \\ \iff m_1 \cdots m_q &= -\ell_{p+1} \sum_{\substack{i+j=d \\ i > p, j < q}} \frac{c_{ij}}{c_{pq}} \underbrace{\ell_{p+2} \cdots \ell_i}_{:=1 \text{ if } i=p+1} m_1 \cdots m_j \\ \iff \ell_{p+1} &| m_1 \cdots m_q. \end{aligned}$$

Since all  $\ell_i, m_j$  are linear forms,  $\ell_{p+1} | m_j$  for some  $1 \leq j \leq q$ , which is absurd since no  $\ell_i = \lambda m_j$ ,  $\lambda \in k$ . Therefore,  $\mathcal{B}'$  is an independent set.

- (3) Since

$$|\mathcal{B}'| = d + 1 = \dim_k V(d, 2),$$

$\mathcal{B}'$  is also a basis for  $V(d, 2)$ .

□

### Problem 2.36.

With the above notation, show that

$$\dim V(d, n) = \binom{d+n-1}{n-1},$$

the binomial coefficient.

*Proof.* See the proof of Problem 2.35(b). □

## 2.7. Direct Products of Rings

### Problem 2.37.

What are the additive and multiplicative identities in  $\times R_i$ ? Is the map from  $R_i$  to  $\times R_i$  taking  $a_i$  to  $(0, \dots, a_i, \dots, 0)$  a ring homomorphism?

*Proof.*

- (1)  $(0, \dots, 0)$  is the additive identity in  $\times R_i$ .
- (2)  $(1, \dots, 1)$  is the multiplicative identity in  $\times R_i$ .
- (3) The map  $\alpha : R_i \rightarrow \times R_i$  taking  $a_i$  to  $(0, \dots, a_i, \dots, 0)$  is not a ring homomorphism since

$$\alpha(1) = (0, \dots, 1, \dots, 0) \neq (1, \dots, 1),$$

or  $\alpha$  is not multiplicative identity preserving (if  $R_j$  is not the zero ring for some  $j \neq i$ ).

□

### Problem 2.38.\*

Show that if  $k \subseteq R_i$ , and each  $R_i$  is finite-dimensional over  $k$ , then  $\dim(\times R_i) = \sum \dim(R_i)$ .

*Proof.*

- (1) In the terminology of linear algebra,  $\times R_i$  is the direct sum  $\bigoplus R_i$  of  $R_i$ .
- (2) Hence,

$$\dim_k \left( \bigoplus R_i \right) = \sum \dim_k(R_i).$$

□

## 2.8. Operations with Ideals

### Problem 2.39.\*

Prove the following relations among ideals  $I_i, J$  in a ring  $R$ :

- (a)  $(I_1 + I_2)J = I_1J + I_2J$ .  
(b)  $(I_1 \cdots I_N)^n = I_1^n \cdots I_N^n$ .

*Proof of (a).*

- (1) Note that  $(I_1 + I_2)J$  and  $I_1J + I_2J$  are ideals.  
(2) Show that  $(I_1 + I_2)J \subseteq I_1J + I_2J$ . Given any

$$(x_1 + x_2)y \in (I_1 + I_2)J$$

where  $x_i \in I_i$  and  $y \in J$ . It suffices to show that  $(x_1 + x_2)y \in I_1J + I_2J$  (by (1)). In fact,

$$(x_1 + x_2)y = x_1y + x_2y \in I_1J + I_2J.$$

- (3) Show that  $(I_1 + I_2)J \supseteq I_1J + I_2J$ . Given any

$$x_1y_1 + x_2y_2 \in I_1J + I_2J$$

where  $x_i \in I_i$  and  $y_i \in J$ . It suffices to show that  $x_1y_1 + x_2y_2 \in (I_1 + I_2)J$  (by (1)). In fact,

$$x_1y_1 + x_2y_2 = (x_1 + \underbrace{0}_{\in I_2})y_1 + (\underbrace{0}_{\in I_1} + x_2)y_2 \in (I_1 + I_2)J$$

since  $(I_1 + I_2)J$  is an ideal.

□

*Proof of (b).*

- (1) Note that  $(I_1 \cdots I_N)^n$  and  $I_1^n \cdots I_N^n$  are ideals.  
(2) Show that  $(I_1 \cdots I_N)^n \subseteq I_1^n \cdots I_N^n$ . Given any

$$x = x_1 \cdots x_n$$

where  $x_i \in I_1 \cdots I_N$ . It suffices to show that  $x \in I_1^n \cdots I_N^n$  (by (1)). For each  $x_i \in I_1 \cdots I_N$ , write

$$x_i = \sum_{j(i)} x_{j(i),1} \cdots x_{j(i),N}$$

where  $x_{j(i),k} \in I_k$  for  $1 \leq k \leq N$ . Hence

$$\begin{aligned}
x &= x_1 \cdots x_n \\
&= \left( \sum_{j(1)} x_{j(1),1} \cdots x_{j(1),N} \right) \cdots \left( \sum_{j(n)} x_{j(n),1} \cdots x_{j(n),N} \right) \\
&= \sum_{j(1), \dots, j(n)} (x_{j(1),1} \cdots x_{j(1),N}) \cdots (x_{j(n),1} \cdots x_{j(n),N}) \\
&= \sum_{j(1), \dots, j(n)} \underbrace{(x_{j(1),1} \cdots x_{j(n),1})}_{\in I_1^n} \cdots \underbrace{(x_{j(1),N} \cdots x_{j(n),N})}_{\in I_N^n} \\
&\in I_1^n \cdots I_N^n.
\end{aligned}$$

(3) Show that  $(I_1 \cdots I_N)^n \supseteq I_1^n \cdots I_N^n$ . Given any

$$x = x_1 \cdots x_N \in I_1^n \cdots I_N^n$$

where  $x_i \in I_i^n$  ( $1 \leq i \leq N$ ). It suffices to show that  $x \in (I_1 \cdots I_N)^n$  (by (1)). For each  $x_i \in I_i^n$ , write

$$x_i = \sum_{j(i)} x_{j(i),1} \cdots x_{j(i),n}$$

where  $x_{j(i),k} \in I_k$  for  $1 \leq k \leq n$ . Hence

$$\begin{aligned}
x &= x_1 \cdots x_N \\
&= \left( \sum_{j(1)} x_{j(1),1} \cdots x_{j(1),n} \right) \cdots \left( \sum_{j(N)} x_{j(N),1} \cdots x_{j(N),n} \right) \\
&= \sum_{j(1), \dots, j(N)} (x_{j(1),1} \cdots x_{j(1),n}) \cdots (x_{j(N),1} \cdots x_{j(N),n}) \\
&= \sum_{j(1), \dots, j(N)} \underbrace{(x_{j(1),1} \cdots x_{j(N),1})}_{\in I_1 \cdots I_N} \cdots \underbrace{(x_{j(1),n} \cdots x_{j(N),n})}_{\in I_1 \cdots I_N} \\
&\in (I_1 \cdots I_N)^n.
\end{aligned}$$

□

**Problem 2.40.\* (Chinese remainder theorem)**

- (a) Suppose  $I, J$  are comaximal ideals in  $R$ . Show that  $I + J^2 = R$ . Show that  $I^m$  and  $J^n$  are comaximal for all  $m, n$ .

- (b) Suppose  $I_1, \dots, I_N$  are ideals in  $R$ , and  $I_i$  and  $J_i = \bigcap_{j \neq i} I_j$  are comaximal for all  $i$ . Show that

$$I_1^n \cap \dots \cap I_N^n = (I_1 \cdots I_N)^n = (I_1 \cap \dots \cap I_N)^n$$

for all  $n$ .

*Proof of (a).*

- (1) It suffices to show that  $I^m + J^n = R$ .  
 (2) Since  $I^m + J^n \subseteq R$  is always true, it suffices to show that  $I^m + J^n \supseteq R$ .  
 In fact,

$$\begin{aligned} R &= R^{m+n-1} && (1 \in R) \\ &= (I + J)^{m+n-1} && (I, J \text{ are comaximal}) \\ &= \sum_{i=0}^{m+n-1} I^i J^{m+n-1-i} && (\text{Problem 2.39}) \\ &\subseteq I^m + J^n \end{aligned}$$

for all positive integers  $m, n$ . (If  $m = 0$  or  $n = 0$ , then nothing to prove.)

□

*Proof of (b).*

- (1) Show that  $I_i$  and  $I_j$  are comaximal if  $i \neq j$ . Note that

$$R = I_i + J_i \subseteq I_i + I_j \subseteq R$$

if  $i \neq j$ .

- (2) If  $I_i$  is comaximal to  $I_j$  and  $I_{j'}$ . Show that  $I_i$  is also comaximal to  $I_j I_{j'}$ .

$$\begin{aligned} R &= (I_i + I_j)(I_i + I_{j'}) \\ &= I_i(I_i + I_j + I_{j'}) + I_j I_{j'} && (\text{Problem 2.39(a)}) \\ &\subseteq I_i + I_j I_{j'} \subseteq R. \end{aligned}$$

- (3) By (2), it is easy to get that  $I_i$  and  $\prod_{j \neq i} I_j$  are comaximal by induction on the number of  $I_j$  for  $j \neq i$ .  
 (4) Show that  $I_1 \cdots I_N = I_1 \cap \dots \cap I_N$ . Induction on  $N$ .

$$\begin{aligned} I_1 \cap \dots \cap I_N &= I_1 \cap (I_2 \cap \dots \cap I_N) \\ &= I_1 \cap (I_2 \cdots I_N) && (\text{Induction hypothesis}) \\ &= I_1 \cdot (I_2 \cdots I_N) && ((3)) \\ &= I_1 \cdots I_N. \end{aligned}$$

- (5) Note that  $I_i^n$  and  $I_j^n$  are comaximal if  $i \neq j$  by (a). We can apply the same argument in (2)(3)(4) to show that

$$I_1^n \cdots I_N^n = I_1^n \cap \cdots \cap I_N^n.$$

- (6) Therefore,

$$\begin{aligned} (I_1 \cap \cdots \cap I_N)^n &= (I_1 \cdots I_N)^n && ((4)) \\ &= I_1^n \cdots I_N^n && (\text{Problem 2.39(b)}) \\ &= I_1^n \cap \cdots \cap I_N^n && ((5)). \end{aligned}$$

□

**Problem 2.41.\***

Let  $I, J$  be ideals in  $R$ . Suppose  $I$  is finitely generated and  $I \subseteq \text{rad}(J)$ . Show that  $I^n \subseteq J$  for some  $n$ .

*Proof.*

- (1) Let  $I$  be generated by  $x_1, \dots, x_m \in I$ . As  $I \subseteq \text{rad}(J)$ , there are integers  $n_i > 0$  such that  $x_i^{n_i} \in J$ .
- (2) Let  $N = n_1 + \cdots + n_m$ . Given any  $x = \sum_{i=1}^m r_i x_i \in I$ , so

$$\begin{aligned} x^N &= \left( \sum_{i=1}^m r_i x_i \right)^N \\ &= \sum_{k_1 + \cdots + k_m = N} \binom{N}{k_1, \dots, k_m} r_1^{k_1} x_1^{k_1} \cdots r_m^{k_m} x_m^{k_m}. \end{aligned}$$

- (3) Note that for each term there is some  $j$  such that  $k_j \geq n_j$ . Hence,

$$\begin{aligned} x_j^{k_j} &= x_j^{k_j - n_j} x_j^{n_j} \in J && (J \text{ is an ideal}) \\ \implies r_1^{k_1} x_1^{k_1} \cdots r_m^{k_m} x_m^{k_m} &\in J \text{ for each term} && (J \text{ is an ideal}) \\ \implies x^N &\in J. && (J \text{ is an ideal}) \\ \implies I^N &\subseteq J. \end{aligned}$$

□

**Supplement.** (Exercise 1.13 in the textbook: Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*.) Suppose that  $I$  is an ideal in a

commutative ring. Show that if  $\text{rad}(I)$  is finitely generated, then for some integer  $N$  we have  $(\text{rad}(I))^N \subseteq I$ . Conclude that in a Noetherian ring the ideals  $I$  and  $J$  have the same radical iff there is some integer  $N$  such that  $I^N \subseteq J$  and  $J^N \subseteq I$ . Use the Nullstellensatz to deduce that if  $I, J \subseteq S = k[x_1, \dots, x_n]$  are ideals and  $k$  is algebraically closed, then  $Z(I) = Z(J)$  iff  $I^N \subseteq J$  and  $J^N \subseteq I$  for some  $N$ .

*Proof.*

- (1) Show that if  $\text{rad}(I)$  is finitely generated, then for some integer  $N$  we have  $(\text{rad}(I))^N \subseteq I$ . Say  $x_1, \dots, x_m \in \text{rad}(I)$  generate  $\text{rad}(I)$ .

- (a) For each  $i$ , there exists an integer  $n_i > 0$  such that  $x_i^{n_i} \in I$  (since  $\text{rad}(I)$  is radical).  
(b) Let  $N = n_1 + \dots + n_m$ . Given any  $x = \sum_{i=1}^m r_i x_i \in \text{rad}(I)$ , so

$$\begin{aligned} x^N &= \left( \sum_{i=1}^m r_i x_i \right)^N \\ &= \sum_{k_1 + \dots + k_m = N} \binom{N}{k_1, \dots, k_m} r_1^{k_1} x_1^{k_1} \dots r_m^{k_m} x_m^{k_m}. \end{aligned}$$

- (c) Note that for each term there is some  $j$  such that  $k_j \geq n_j$ . Hence,

$$\begin{aligned} x_j^{k_j} &= x_j^{k_j - n_j} x_j^{n_j} \in I && (I \text{ is an ideal}) \\ \implies r_1^{k_1} x_1^{k_1} \dots r_m^{k_m} x_m^{k_m} &\in I \text{ for each term} && (I \text{ is an ideal}) \\ \implies x^N &\in I. && (I \text{ is an ideal}) \\ \implies (\text{rad}(I))^N &\subseteq I. \end{aligned}$$

- (2) Show that in a Noetherian ring the ideals  $I$  and  $J$  have the same radical iff there is some integer  $N$  such that  $I^N \subseteq J$  and  $J^N \subseteq I$ .

- (a) ( $\implies$ ) Since in a Noetherian ring every ideal is finitely generated,  $\text{rad}(I)$  and  $\text{rad}(J)$  are finitely generated. By (1), there is a common integer  $N$  such that

$$(\text{rad}(I))^N \subseteq I \quad \text{and} \quad (\text{rad}(J))^N \subseteq J.$$

Note that  $I^N \subseteq (\text{rad}(I))^N$  and  $J^N \subseteq (\text{rad}(J))^N$ . Since  $\text{rad}(I) = \text{rad}(J)$  by assumption,

$$\begin{aligned} I^N &\subseteq (\text{rad}(I))^N = (\text{rad}(J))^N \subseteq J, \\ J^N &\subseteq (\text{rad}(J))^N = (\text{rad}(I))^N \subseteq I. \end{aligned}$$

- (b) ( $\impliedby$ ) It suffices to show that  $\text{rad}(I) \subseteq \text{rad}(J)$ .  $\text{rad}(J) \subseteq \text{rad}(I)$  is similar. Given any  $x \in \text{rad}(I)$ , there is an integer  $M > 0$  such that  $x^M \in I$ . Hence  $x^{MN} \in I^N \subseteq J$ , or  $x \in \text{rad}(J)$ .



- (3) Show that if  $I, J \subseteq S = k[x_1, \dots, x_n]$  are ideals and  $k$  is algebraically closed, then  $Z(I) = Z(J)$  iff  $I^N \subseteq J$  and  $J^N \subseteq I$  for some  $N$ . Note that  $S$  is Noetherian and we can apply step (2). By the Nullstellensatz,  $Z(I) = Z(J)$  iff  $\text{rad}(I) = \text{rad}(J)$  iff  $I^N \subseteq J$  and  $J^N \subseteq I$  for some  $N$ .

□

**Problem 2.42.\* (Isomorphism theorems for rings)**

- (a) Let  $I \subseteq J$  be ideals in a ring  $R$ . Show that there is a natural ring homomorphism from  $R/I$  onto  $R/J$ .
- (b) Let  $I$  be an ideal in a ring  $R$ ,  $R$  a subring of a ring  $S$ . Show that there is a natural ring homomorphism from  $R/I$  to  $S/IS$ .

*Proof of (a).*

- (1) Define a map  $\alpha : R/I \rightarrow R/J$  by  $\alpha(r + I) = r + J$ .
- (2) Show that  $\alpha$  is well-defined. If  $a + I = b + I$ , then  $a - b \in I \subseteq J$  or  $a + J = b + J$ . Hence,  $\alpha(a + I) = a + J = b + J = \alpha(b + I)$ .
- (3) Show that  $\alpha$  is a surjective homomorphism.
- (a)  $\alpha$  is addition preserving.

$$\begin{aligned}\alpha((a + I) + (b + I)) &= \alpha(a + b + I) \\ &= a + b + J \\ &= (a + J) + (b + J) \\ &= \alpha(a + I) + \alpha(b + I).\end{aligned}$$

- (b)  $\alpha$  is multiplication preserving.

$$\begin{aligned}\alpha((a + I)(b + I)) &= \alpha(ab + I) \\ &= ab + J \\ &= (a + J)(b + J) \\ &= \alpha(a + I)\alpha(b + I).\end{aligned}$$

- (c)  $\alpha$  is multiplicative identity preserving.  $\alpha(1 + I) = 1 + J$ .

- (d)  $\alpha$  is surjective since for any  $a + J \in R/J$  there is an element  $a + I \in R/I$  such that  $\alpha(a + I) = a + J$ .

- (4) Note that  $\ker(\alpha) = J/I$ . So  $(R/I)/(J/I) \cong R/J$ .

□

*Proof of (b).*

- (1)  $I$  is not necessary an ideal of  $S$ ;  $IS$  an ideal of  $S$  (and thus  $S/IS$  is well-defined).
- (2) Define a map  $\alpha : R/I \rightarrow S/IS$  by  $\alpha(r + I) = r + IS$ . Note that  $I \subseteq IS$  as a subset in  $S$ . Apply the same argument in (a),  $\alpha$  is well-defined and  $\alpha$  is a surjective homomorphism.
- (3) Note that  $\ker(\alpha) = (R \cap SI)/I$ . So  $(R/I)/((R \cap SI)/I) \cong S/IS$ .

□

**Problem 2.43.\***

Let  $P = (0, \dots, 0) \in \mathbf{A}^n$ ,  $\mathcal{O} = \mathcal{O}_P(\mathbf{A}^n)$ ,  $\mathfrak{m} = \mathfrak{m}_P(\mathbf{A}^n)$ . Let  $I = (x_1, \dots, x_n) \subseteq k[x_1, \dots, x_n]$  be the ideal generated by  $x_1, \dots, x_n$ . Show that  $I\mathcal{O} = \mathfrak{m}$ , so  $I^r\mathcal{O} = \mathfrak{m}^r$  for all  $r$ .

*Proof.*

- (1) By the definition

$$\mathfrak{m} = \{f \in \mathcal{O} : f(P) = 0\},$$

$I\mathcal{O} \subseteq \mathfrak{m}$ . Conversely, by Problem 1.7(b) we have  $I\mathcal{O} \supseteq \mathfrak{m}$ .

- (2) By Problem 2.39(b),

$$\mathfrak{m}^r = (I\mathcal{O})^r = I^r\mathcal{O}^r = I^r\mathcal{O}.$$

Here  $\mathcal{O}^r = \mathcal{O}$  since  $1 \in \mathcal{O}$ .

□

**Problem 2.44.\***

Let  $V$  be a variety in  $\mathbf{A}^n$ ,  $I = I(V) \subseteq k[x_1, \dots, x_n]$ ,  $P \in V$ , and let  $J$  be an ideal of  $k[x_1, \dots, x_n]$  that contains  $I$ . Let  $J'$  be the image of  $J$  in  $\Gamma(V)$ . Show that there is a natural homomorphism  $\varphi$  from  $\mathcal{O}_P(\mathbf{A}^n)/J\mathcal{O}_P(\mathbf{A}^n)$  to  $\mathcal{O}_P(V)/J'\mathcal{O}_P(V)$ , and that  $\varphi$  is an isomorphism. In particular,  $\mathcal{O}_P(\mathbf{A}^n)/I\mathcal{O}_P(\mathbf{A}^n)$  is isomorphic to  $\mathcal{O}_P(V)$ .

*Proof.*

- (1) Define  $\varphi$  from  $\mathcal{O}_P(\mathbf{A}^n)/J\mathcal{O}_P(\mathbf{A}^n)$  to  $\mathcal{O}_P(V)/J'\mathcal{O}_P(V)$  by

$$\varphi : a/b + J\mathcal{O}_P(\mathbf{A}^n) \mapsto \bar{a}/\bar{b} + J'\mathcal{O}_P(V).$$

It is well-defined since  $\varphi(J\mathcal{O}_P(\mathbf{A}^n)) = J'\mathcal{O}_P(V)$  and  $b(P) \neq 0$  implies that  $\bar{b}(P) \neq 0$ .

- (2) Note that  $V$  is a subvariety of  $\mathbf{A}^n$ . So  $\varphi : \Gamma(\mathbf{A}^n) \rightarrow \Gamma(V)$  is a ring homomorphism by Problem 2.3 and then  $\varphi$  extends uniquely to a ring homomorphism by using the similar argument in Problem 2.21.
- (3)  $\varphi$  is surjective since  $\mathcal{O}_P(\mathbf{A}^n) \hookrightarrow \mathcal{O}_P(V)$  and  $\varphi(J\mathcal{O}_P(\mathbf{A}^n)) = J'\mathcal{O}_P(V)$ .  $\varphi$  is injective since  $\varphi(J\mathcal{O}_P(\mathbf{A}^n)) = J'\mathcal{O}_P(V)$ . Hence  $\varphi : \mathcal{O}_P(\mathbf{A}^n)/J \rightarrow \mathcal{O}_P(V)/J'$  is isomorphic. In particular,  $\mathcal{O}_P(\mathbf{A}^n)/I\mathcal{O}_P(\mathbf{A}^n) \cong \mathcal{O}_P(V)$  (by taking  $J = I$  and noting that  $J' = I' = 0$ ).

□

**Problem 2.45.\***

Show that ideals  $I, J \subseteq k[x_1, \dots, x_n]$  ( $k$  algebraically closed) are comaximal if and only if  $V(I) \cap V(J) = \emptyset$ .

*Proof.*

- (1) Show that  $V(I) \cap V(J) = V(I + J)$ .

$$\begin{aligned} P \in V(I) \cap V(J) &\iff f(P) = 0 \forall f \in I \text{ and } g(P) = 0 \forall g \in J \\ &\iff f(P) = 0 \forall f \in I + J \\ &\iff P \in V(I + J). \end{aligned}$$

- (2) Hence,

$$\begin{aligned} \emptyset = V(I) \cap V(J) &\iff \emptyset = V(I + J) && ((1)) \\ &\iff I + J = k[x_1, \dots, x_n] && (\text{Weak Nullstellensatz}) \\ &\iff I \text{ and } J \text{ are comaximal.} \end{aligned}$$

□

**Problem 2.46.\***

Let  $I = (x, y) \subseteq k[x, y]$ . Show that

$$\dim_k(k[x, y]/I^n) = 1 + 2 + \dots + n = \frac{n(n+1)}{2}.$$

*Proof.*

(1) The set

$$\mathcal{B} = \{x^i y^j + I^n : i, j \in \mathbb{Z}, i, j \geq 0, i + j < n\}$$

generates  $k[x, y]/I^n$  as a  $k$ -vector space. Besides, each nonzero element in  $I^n$  has the degree  $\geq n$ , and thus  $\mathcal{B}$  is an independent set. Therefore,  $\mathcal{B}$  is a basis for  $k[x, y]/I^n$ .

(2) Hence,

$$\dim_k(k[x, y]/I^n) = |\mathcal{B}| = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

□

## 2.9. Ideals with a Finite Number of Zeros

### Problem 2.47.

*Suppose  $R$  is a ring containing  $k$ , and  $R$  is finite dimensional over  $k$ . Show that  $R$  is isomorphic to a direct product of local rings.*

*Proof.*

- (1) Let  $\{v_1, \dots, v_n\}$  be a basis for  $R$  over  $k$  (as a vector space). Define a  $k$ -module homomorphism  $\alpha : k[x_1, \dots, x_n] \rightarrow R$  by  $\alpha(x_i) = v_i$ . Clearly,  $\alpha$  is surjective and thus

$$R \cong k[x_1, \dots, x_n] / \ker(\alpha)$$

as a  $k$ -module isomorphism. Note that  $\ker(\alpha)$  is an ideal of  $k[x_1, \dots, x_n]$ .

- (2) Write  $I := \ker(\alpha)$ . Hence,

$$\dim_k(k[x_1, \dots, x_n]/I) = \dim_k(R) < \infty.$$

By Corollary 4 to the Hilbert's Nullstellensatz in §1.7,  $V(I)$  is finite.

- (3) Write  $V(I) = \{P_1, \dots, P_N\}$  and  $\mathcal{O}_i = \mathcal{O}_{P_i}(\mathbf{A}^n)$ . By Proposition 6,

$$R \cong k[x_1, \dots, x_n]/I \cong \prod_{i=1}^N \mathcal{O}_i / I \mathcal{O}_i,$$

which is isomorphic to a direct product of local rings.

□

## 2.10. Quotient Modules and Exact Sequences

### Problem 2.48.\*

Verify that for any  $R$ -module homomorphism  $\varphi : M \rightarrow M'$ ,  $\ker(\varphi)$  and  $\text{im}(\varphi)$  are submodules of  $M$  and  $M'$  respectively. Show that

$$0 \rightarrow \ker(\varphi) \rightarrow M \xrightarrow{\varphi} \text{im}(\varphi) \rightarrow 0$$

is exact.

*Proof.*

- (1) Show that  $\ker(\varphi)$  is a subgroup of  $M$ . It suffices to show that  $a - b \in \ker(\varphi)$  for all  $a, b \in \ker(\varphi)$ . In fact,  $\varphi(a - b) = \varphi(a) - \varphi(b) = 0 - 0 = 0$ , or  $a - b \in \ker(\varphi)$ .
- (2) Show that  $\ker(\varphi)$  is a submodule of  $M$ . By (1), it suffices to show that  $ra \in \ker(\varphi)$  for all  $r \in R$  and  $a \in \ker(\varphi)$ . In fact,  $\varphi(ra) = r \cdot \varphi(a) = r \cdot 0 = 0$ , or  $ra \in \ker(\varphi)$ .
- (3) Show that  $\text{im}(\varphi)$  is a subgroup of  $M'$ . It suffices to show that  $a - b \in \text{im}(\varphi)$  for all  $a, b \in \text{im}(\varphi)$ . As  $a, b \in \text{im}(\varphi)$ , there are two elements  $a', b' \in M$  such that  $\varphi(a') = a$  and  $\varphi(b') = b$ . So  $\varphi(a' - b') = \varphi(a') - \varphi(b') = a - b$ , or  $a - b \in \text{im}(\varphi)$ .
- (4) Show that  $\text{im}(\varphi)$  is a submodule of  $M'$ . By (3), it suffices to show that  $ra \in \text{im}(\varphi)$  for all  $r \in R$  and  $a \in \text{im}(\varphi)$ . As  $a \in \text{im}(\varphi)$ , there is one element  $a' \in M$  such that  $\varphi(a') = a$ . So  $\varphi(ra') = r\varphi(a') = ra$ , or  $ra \in \text{im}(\varphi)$ .
- (5) Show that

$$0 \rightarrow \ker(\varphi) \xrightarrow{i} M \xrightarrow{\varphi} \text{im}(\varphi) \rightarrow 0$$

is exact. Note that  $\ker(\varphi) \xrightarrow{i} M$  is the natural inclusion and  $M \xrightarrow{\varphi} \text{im}(\varphi)$  is surjective. Also, it is trivial that  $\text{im}(i) = \ker(\varphi)$ .

□

### Problem 2.49.\*

- (a) (Factor theorem for modules) Let  $N$  be a submodule of  $M$ ,  $\pi : M \rightarrow M/N$  the natural homomorphism. Suppose  $\varphi : M \rightarrow M'$  is a homomorphism of  $R$ -modules, and  $\varphi(N) = 0$ . Show that there is a unique homomorphism  $\bar{\varphi} : M/N \rightarrow M'$  such that  $\bar{\varphi} \circ \pi = \varphi$ .

- (b) (Isomorphism theorems for modules) If  $N$  and  $P$  are submodules of a module  $M$ , with  $P \subseteq N$ , then there are natural homomorphisms from  $M/P$  onto  $M/N$  and from  $N/P$  into  $M/P$ . Show that the resulting sequence

$$0 \rightarrow N/P \rightarrow M/P \rightarrow M/N \rightarrow 0$$

is exact.

- (c) Let  $U \subseteq W \subseteq V$  be vector spaces, with  $V/U$  finite-dimensional. Then  $\dim V/U = \dim V/W + \dim W/U$ .
- (d) If  $J \subseteq I$  are ideals in a ring  $R$ , there is a natural exact sequence of  $R$ -modules:

$$0 \rightarrow I/J \rightarrow R/J \rightarrow R/I \rightarrow 0.$$

- (e) If  $\mathcal{O}$  is a local ring with maximal ideal  $\mathfrak{m}$ , there is a natural exact sequence of  $\mathcal{O}$ -modules

$$0 \rightarrow \mathfrak{m}^n/\mathfrak{m}^{n+1} \rightarrow \mathcal{O}/\mathfrak{m}^{n+1} \rightarrow \mathcal{O}/\mathfrak{m}^n \rightarrow 0.$$

*Proof of (a).*

- (1) Define  $\bar{\varphi} : M/N \rightarrow M'$  by

$$\bar{\varphi}(m + N) = \varphi(m).$$

$\bar{\varphi}$  is well-defined since  $m + N = n + N$  implies that  $m - n \in N \subseteq \ker(\varphi)$ .

- (2)  $\bar{\varphi}$  is a homomorphism of  $R$ -modules since  $\varphi$  is a homomorphism of  $R$ -modules.
- (3)  $\bar{\varphi} \circ \pi = \varphi$  by construction.
- (4) Suppose there is a homomorphism  $\psi : M/N \rightarrow M'$  such that  $\psi \circ \pi = \varphi$ . For any  $m + N \in M/N$ , we have

$$\bar{\varphi}(m + N) = \varphi(m) = (\psi \circ \pi)(m) = \psi(\pi(m)) = \psi(m + N).$$

That is,  $\psi = \bar{\varphi}$ .

□

*Proof of (b).*

- (1) Define  $\pi : M/P \rightarrow M/N$  by

$$\pi : \underbrace{m + P}_{\in M/P} \mapsto \underbrace{m + N}_{\in M/N}.$$

- (a) *Show that  $\pi$  is well-defined.* If  $m + P = n + P \in M/P$ , then  $m - n \in P \subseteq N$  or  $m + N = n + N \in M/N$ .
- (b)  $\pi$  is a module homomorphism since  $M/N$  is a module.
- (c)  $\pi$  is surjective by construction.

(2) Define  $i : N/P \hookrightarrow M/P$  by

$$i : \underbrace{m + P}_{\in N/P} \mapsto \underbrace{m + P}_{\in M/P}.$$

- (a) *Show that  $i$  is well-defined.* If  $m + P = n + P \in N/P$ , then  $m, n \in N \subseteq M$  and  $m - n \in P$ . So  $m + P = n + P \in M/P$ .
  - (b)  $i$  is a module homomorphism since  $M/P$  is a module.
  - (c)  $i$  is injective by construction.
- (3) To show that  $0 \rightarrow N/P \rightarrow M/P \rightarrow M/N \rightarrow 0$  is exact, it suffices to show that  $\ker(\pi) = \text{im}(i) = N/P$  (by the injectivity of  $i$ ). It is trivial since

$$m + P \in \ker(\pi) \iff m \in N \iff m + P \in N/P.$$

□

*Proof of (c).*

- (1) By (b),

$$0 \rightarrow W/U \rightarrow V/U \xrightarrow{\varphi} V/W \rightarrow 0$$

is exact.

- (2) By the rank-nullity theorem for a linear transformation,

$$\dim V/U = \dim \text{im}(\varphi) + \dim \ker(\varphi) = \dim V/W + \dim W/U.$$

□

*Proof of (d).*

- (1) Regard  $R$  as a  $R$ -module and  $I, J$  as submodules of a  $R$ -module  $R$ .
- (2) As  $J \subseteq I$ , by (b) we have

$$0 \rightarrow I/J \rightarrow R/J \rightarrow R/I \rightarrow 0.$$

□

*Proof of (e).*

- (1) Note that  $\mathfrak{m}^{n+1} \subseteq \mathfrak{m}^n$  are ideals in a local ring  $\mathcal{O}$ .

(2) By (d), there is a natural exact sequence of  $\mathcal{O}$ -modules:

$$0 \rightarrow \mathfrak{m}^n/\mathfrak{m}^{n+1} \rightarrow \mathcal{O}/\mathfrak{m}^{n+1} \rightarrow \mathcal{O}/\mathfrak{m}^n \rightarrow 0.$$

□

**Problem 2.50.\***

Let  $R$  be a DVR satisfying the conditions of Problem 2.30. Then  $\mathfrak{m}^n/\mathfrak{m}^{n+1}$  is an  $R$ -module, and so also a  $k$ -module, since  $k \subseteq R$ .

- (a) Show that  $\dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = 1$  for all  $n \geq 0$ .
- (b) Show that  $\dim_k(R/\mathfrak{m}^n) = n$  for all  $n > 0$ .
- (c) Let  $z \in R$ . Show that  $\text{ord}(z) = n$  if  $(z) = \mathfrak{m}^n$ , and hence that  $\text{ord}(z) = \dim_k(R/(z))$ .

*Proof of (a).*

- (1) By Problem 2.49(e),

$$0 \rightarrow \mathfrak{m}^n/\mathfrak{m}^{n+1} \rightarrow R/\mathfrak{m}^{n+1} \rightarrow R/\mathfrak{m}^n \rightarrow 0$$

is exact.

- (2) By the rank-nullity theorem (Proposition 3),

$$\begin{aligned} \dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1}) &= \dim_k(R/\mathfrak{m}^{n+1}) - \dim_k(R/\mathfrak{m}^n) \\ &= (n+1) - n \\ &= 1. \end{aligned} \tag{((b))}$$

□

*Proof of (b).*

- (1) Let  $t$  be a uniformizing parameter for  $R$ ,  $z \in R$ . By Problem 2.30(b), there are unique  $\lambda_0, \dots, \lambda_{n-1} \in k$  and  $z_{n-1} \in R$  such that

$$z = \lambda_0 + \lambda_1 t + \dots + \lambda_{n-1} t^{n-1} + z_{n-1} t^n.$$

Hence we can define a map  $\varphi : R/\mathfrak{m}^n \rightarrow k^n$  by

$$\varphi : z + \mathfrak{m}^n \mapsto (\lambda_0, \dots, \lambda_{n-1}).$$

- (2)  $\varphi$  is well-defined by the uniqueness of the expression of  $z$  in Problem 2.30(b).  $\varphi$  is a  $k$ -module homomorphism and  $\varphi$  is surjective (since  $k \subseteq R$ ).  $\varphi$  is injective by the uniqueness of the expression of  $z$  in Problem 2.30(b).



- (3) Hence,  $R/\mathfrak{m}^n \cong k^n$  or  $\dim_k(R/\mathfrak{m}^n) = n$  for  $n > 0$ . (It is also true for  $n = 0$  since  $\dim_k(\{0\}) = 0$ .)

□

*Proof of (c).*

- (1) Note that  $\mathfrak{m}^n = (t^n)$  as  $\mathfrak{m} = (t)$  where  $t$  is a uniformizing parameter for  $R$ .
- (2) Since  $(z) = (t^n) = \mathfrak{m}^n$ ,  $\text{ord}(z) = n$  by Problem 2.28. (Here  $\text{ord}(z) \geq n$  by  $z \in (t^n)$  and  $n \geq \text{ord}(z)$  by  $t^n \in (z)$ .)

- (3) Hence,

$$\text{ord}(z) = n = \dim_k(R/\mathfrak{m}^n) = \dim_k(R/(z))$$

by (b).

□

### Problem 2.51.

Let

$$0 \longrightarrow V_1 \longrightarrow \cdots \longrightarrow V_n \longrightarrow 0$$

be an exact sequence of finite-dimensional vector spaces. Show that  $\sum (-1)^i \dim(V_i) = 0$ .

*Proof (Proposition 7 in §2.10).*

- (1) For  $i = 0, \dots, n$ , by the rank-nullity theorem for a linear transformation  $\varphi_i : V_i \rightarrow V_{i+1}$ , we have

$$\dim V_i = \dim \text{im}(\varphi_i) + \dim \ker(\varphi_i).$$

(Here  $V_0 = V_{n+1} := 0$  by convention.)

- (2) By the exactness of the sequence, we have

$$(a) \quad \text{im}(\varphi_i) = \ker(\varphi_{i+1}) \text{ for } i = 0, \dots, n-1. \text{ In particular, } \ker(\varphi_1) = \text{im}(\varphi_0) = 0.$$

$$(b) \quad \ker(\varphi_n) = V_n.$$

Hence,

$$\begin{aligned}
\sum_{i=1}^{n-1} (-1)^i \dim(V_i) &= \sum_{i=1}^{n-1} (-1)^i \dim \operatorname{im}(\varphi_i) + \sum_{i=1}^{n-1} (-1)^i \dim \ker(\varphi_i) \\
&= \sum_{i=1}^{n-1} (-1)^i \dim \ker(\varphi_{i+1}) + \sum_{i=1}^{n-1} (-1)^i \dim \ker(\varphi_i) \\
&= (-1)^{n-1} \underbrace{\dim \ker(\varphi_n)}_{=V_n} + (-1)^1 \underbrace{\dim \ker(\varphi_1)}_{=0} \\
&= -(-1)^n \dim V_n,
\end{aligned}$$

$$\text{or } \sum (-1)^i \dim(V_i) = 0.$$

□

**Problem 2.52.\* (Isomorphism theorems for modules)**

Let  $N, P$  be submodules of a module  $M$ . Show that the subgroup

$$N + P = \{n + p : n \in N, p \in P\}$$

is a submodule of  $M$ . Show that there is a natural  $R$ -module isomorphism of  $N/(N \cap P)$  onto  $(N + P)/P$ .

*Proof.*

- (1) Show that  $N + P$  is a submodule of  $M$ . Given any  $n_1 + p_1, n_2 + p_2 \in N + P$ ,

$$(n_1 + p_1) + (n_2 + p_2) = \underbrace{n_1 + n_2}_{\in N} + \underbrace{p_1 + p_2}_{\in P} \in N + P.$$

Given any  $n + p \in N + P$  and  $r \in R$ ,

$$r(n + p) = \underbrace{rn}_{\in N} + \underbrace{rp}_{\in P} \in N + P.$$

Here we use the fact that  $N$  and  $P$  are modules.

- (2) Define a module homomorphism  $\varphi : N \rightarrow M/P$  by

$$\varphi : m \mapsto m + P.$$

$\ker(\varphi) = N \cap P$  and  $\operatorname{im}(\varphi) = \{m + P : m \in N\} = (N + P)/P$ . By Problem 2.48,  $\varphi$  induces a natural  $R$ -module isomorphism of  $N/\ker(\varphi) = N/(N \cap P)$  onto  $\operatorname{im}(\varphi) = (N + P)/P$  (which is sending  $m + (N \cap P)$  to  $m + P$ ).

□

**Problem 2.53.\***

Let  $V$  be a vector space,  $W$  a subspace,  $T : V \rightarrow V$  a one-to-one linear map such that  $T(W) \subseteq W$ , and assume  $V/W$  and  $W/T(W)$  are finite-dimensional.

- (a) Show that  $T$  induces an isomorphism of  $V/W$  with  $T(V)/T(W)$ .
- (b) Construct an isomorphism between  $T(V)/(W \cap T(V))$  and  $(W + T(V))/W$ , and an isomorphism between  $W/(W \cap T(V))$  and  $(W + T(V))/T(V)$ .
- (c) Use Problem 2.49(c) to show that  $\dim V/(W + T(V)) = \dim(W \cap T(V))/T(W)$ .
- (d) Conclude finally that  $\dim V/T(V) = \dim W/T(W)$ .

*Proof of (a).*

- (1) Define a map  $\bar{T} : V/W \rightarrow T(V)/T(W)$  by

$$\bar{T} : v + W \mapsto T(v) + T(W).$$

- (2) Show that  $\bar{T}$  is well-defined. Suppose  $u + W = v + W \in V/W$ . So  $u - v \in W$ . So  $T(u - v) = T(u) - T(v) \in T(W)$  (since  $T$  is a linear map). Hence,  $T(u) + T(W) = T(v) + T(W)$ .
- (3)  $\bar{T}$  is a linear map since  $T$  is a linear map.
- (4)  $\bar{T}$  is surjective by construction. Also,  $\bar{T}$  is injective since  $T$  is injective. Therefore,  $\bar{T} : V/W \xrightarrow{\sim} T(V)/T(W)$  is isomorphic.

□

*Proof of (b).*

- (1) Put  $N = T(V)$  and  $P = W$  in Problem 2.52 to get

$$T(V)/(W \cap T(V)) \cong (W + T(V))/W.$$

- (2) Put  $N = W$  and  $P = T(V)$  in Problem 2.52 to get

$$W/(W \cap T(V)) \cong (W + T(V))/T(V).$$

□

*Proof of (c).*

- (1) Note that  $W \subseteq W + T(V) \subseteq V$  as vector spaces and  $V/W$  is finite-dimensional. By Problem 2.49(c),

$$\dim V/W = \dim V/(W + T(V)) + \dim(W + T(V))/W.$$

- (2) Similarly,  $T(V) \subseteq W \cap T(V) \subseteq T(W)$  as vector spaces and  $T(V)/T(W)$  is finite-dimensional (since  $V/T(W)$  is finite-dimensional and  $T(V)/T(W)$  is a subspace of  $V/T(W)$ ). Again by Problem 2.49(c),

$$\dim T(V)/T(W) = \dim T(V)/(W \cap T(V)) + \dim(W \cap T(V))/T(W).$$

- (3) By (a),

$$\dim V/W = \dim T(V)/T(W).$$

By (b),

$$\dim(W + T(V))/W = \dim T(V)/(W \cap T(V)).$$

Hence, the result is established by (1)(2).

□

*Proof of (d).*

- (1) Note that  $V/T(V)$  is finite-dimensional. By Problem 2.49(c),

$$\dim V/T(V) = \dim V/(W + T(V)) + \dim(W + T(V))/T(V).$$

- (2) Similarly,

$$\dim W/T(W) = \dim W/(W \cap T(V)) + \dim(W \cap T(V))/T(W).$$

- (3) By (b),

$$\dim(W + T(V))/T(V) = \dim W/(W \cap T(V)).$$

By (c),

$$\dim V/(W + T(V)) = \dim(W \cap T(V))/T(W).$$

Hence, the result is established by (1)(2).

□

## 2.11. Free Modules

### Problem 2.54.

What does  $M$  being free on  $m_1, \dots, m_n$  say in terms of the elements of  $M$ ?

*Proof.*

- (1) Any element  $m \in M$  can be written uniquely as

$$m = \sum_{i=1}^n r_i m_i$$

for some  $r_i \in R$  (which is analogous to the vector space).

- (2) The number of members in a basis for  $M$  is called the **rank** of  $M$ . That is,  $n = \text{rank}(M)$ .

□

**Problem 2.55.**

Let  $f = x^n + a_1x^{n-1} + \cdots + a_n$  be a monic polynomial in  $R[x]$ . Show that  $R[x]/(f)$  is a free  $R$ -module with basis  $\bar{1}, \bar{x}, \dots, \bar{x}^{n-1}$ , where  $\bar{x}$  is the residue of  $x$ .

*Proof.*

- (1) Given any  $\bar{g} \in R[x]/(f)$  where

$$g = b_0x^m + b_1x^{m-1} + \cdots + b_mx \in R[x],$$

it suffices to show that  $\bar{g}$  is a linear combination of

$$\mathcal{B} := \{\bar{1}, \bar{x}, \dots, \bar{x}^{n-1}\}.$$

- (2) By the division-with-remainder property of  $R[x]$ ,

$$g = fq + r$$

where  $q, r \in R[x]$  with  $r = c_0x^{n-1} + \cdots + c_{n-1}$ . Hence,

$$\bar{g} = \bar{f}\bar{q} + \bar{r} = \bar{r} = c_0\bar{x}^{n-1} + \cdots + c_{n-1}\bar{1}$$

is a linear combination of  $\mathcal{B}$ .

□

**Problem 2.56.**

Show that a subset  $X$  of a module  $M$  generates  $M$  if and only if the homomorphism  $M_X \rightarrow M$  is onto. Every module is isomorphic to a quotient of a free module.

*Proof.*

- (1) If  $X$  generates  $M$ , then for any  $m \in M$  we can write

$$m = \sum_{x \in X} a_x x$$

as a finite sum where  $a_x \in R$  and  $x \in X \subseteq M$ . Define  $\varphi_x \in M_X$  by  $\varphi_x(y) = \delta_{xy}$  where  $\delta_{xy}$  is the Kronecker delta. Hence, the homomorphism  $\alpha : M_X \rightarrow M$  maps the finite sum  $\varphi := \sum_{x \in X} a_x \varphi_x$  to  $\sum_{x \in X} a_x x = m$ .

- (2) Conversely, if the homomorphism  $\alpha : M_X \rightarrow M$  is onto, then for any  $m \in M$  there is a finite sum  $\varphi := \sum_{x \in X} a_x \varphi_x$  such that  $\alpha(\varphi) = m$ . Hence,

$$m = \alpha(\varphi) = \alpha \left( \sum_{x \in X} a_x \varphi_x \right) = \sum_{x \in X} a_x x$$

is generated by  $X$ .

- (3) Let

$$F = \bigoplus_{m \in M} R$$

be a free module. Define a map  $\varphi : F \rightarrow M$  by

$$\varphi : (0, \dots, 0, \underbrace{1}_{m\text{th position}}, 0, \dots, 0) \mapsto m.$$

$\varphi$  is well-defined.  $\varphi$  is a module homomorphism.  $\varphi$  is surjective. Hence

$$M \cong F/\ker(\varphi)$$

is isomorphic to a quotient of a free module.

□

## Chapter 3: Local Properties of Plane Curves

### 3.1. Multiple Points and Tangent Lines

#### Problem 3.1.

Prove that in the above examples  $P = (0, 0)$  is the only multiple point on the curves  $c = y^2 - x^3$ ,  $d = y^2 - x^3 - x^2$ ,  $e = (x^2 + y^2)^2 + 3x^2y - y^3$ , and  $f = (x^2 + y^2)^3 - 4x^2y^2$ .

*Proof.*

(1)

$$\begin{aligned}\frac{\partial c}{\partial x} &= -3x^2 = 0 \\ \frac{\partial c}{\partial y} &= 2y = 0\end{aligned}$$

implies that  $(x, y) = (0, 0)$ . Note that  $c(0, 0) = \frac{\partial f}{\partial c}(0, 0) = \frac{\partial c}{\partial y}(0, 0) = 0$ . So  $(x, y) = (0, 0)$  is the only multiple point on  $c$ .

(2)

$$\begin{aligned}\frac{\partial d}{\partial x} &= -3x^2 - 2x = 0 \\ \frac{\partial d}{\partial y} &= 2y = 0\end{aligned}$$

implies that  $(x, y) = (0, 0) \in d$  is the only multiple point on  $d$ . (Note that  $(x, y) = (-\frac{2}{3}, 0) \notin d$ .)

(3)

$$\begin{aligned}\frac{\partial e}{\partial x} &= 4x(x^2 + y^2) + 6xy = 0 \\ \frac{\partial e}{\partial y} &= 4y(x^2 + y^2) + 3x^2 - 3y^2 = 0\end{aligned}$$

implies that  $x = 0$  or  $4(x^2 + y^2) + 6y = 0$ .

(a)  $x = 0$  implies that  $(x, y) = (0, 0)$  or  $(0, 1)$ . Note that  $(0, 1)$  is a simple point (since  $\frac{\partial e}{\partial y}(0, 1) = 1$ ).

(b)  $4(x^2 + y^2) + 6y = 0$  implies that  $x^2 + y^2 = -\frac{3y}{2}$  and thus

$$\begin{aligned} 0 &= 4y(x^2 + y^2) + 3x^2 - 3y^2 \\ &= 4y\left(-\frac{3y}{2}\right) + 3x^2 - 3y^2 \\ &= 3(x^2 - 3y^2). \end{aligned}$$

$(x, y) \in e$  implies that

$$\begin{aligned} 0 &= (x^2 + y^2)^2 + 3x^2y - y^3 \\ &= (3y^2 + y^2)^2 + 3(3y^2)y - y^3 \\ &= 8y^3(2y + 1). \end{aligned}$$

So  $(x, y) = (0, 0)$  or  $\left(\pm\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ . Note that  $\frac{\partial e}{\partial x}\left(\pm\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) \neq 0$ .

Therefore,  $(x, y) = (0, 0)$  is the only multiple point on  $e$ .

(4)

$$\begin{aligned} \frac{\partial f}{\partial x} &= 6x(x^2 + y^2)^2 - 8xy^2 = x(6(x^2 + y^2)^2 - 8y^2) = 0 \\ \frac{\partial f}{\partial y} &= 6y(x^2 + y^2)^2 - 8x^2y = y(6(x^2 + y^2)^2 - 8x^2) = 0 \end{aligned}$$

implies that  $(x, y) = (0, 0)$  or  $6(x^2 + y^2)^2 = 8x^2 = 8y^2$ .  $6(x^2 + y^2)^2 = 8x^2 = 8y^2$  implies that  $x^2 = y^2$ . So  $(x, y) \in f$  implies that  $x^2 = y^2 = \frac{1}{2}$ , contrary that  $6 = 6(x^2 + y^2)^2 \neq 8x^2 = 4$ . Therefore,  $(x, y) = (0, 0)$  is the only multiple point on  $f$ .

□

### Problem 3.2.

Find the multiple points, and the tangent lines at the multiple points, for each of the following curves:

(a)  $y^3 - y^2 + x^3 - x^2 + 3xy^2 + 3x^2y + 2xy$ .

(b)  $x^4 + y^4 - x^2y^2$ .

(c)  $x^3 + y^3 - 3x^2 - 3y^2 + 3xy + 1$ .

(d)  $y^2 + (x^2 - 5)(4x^4 - 20x^2 + 25)$ .

Sketch the part of the curve in (d) that is contained in  $\mathbf{A}^2(\mathbb{R}) \subseteq \mathbf{A}^2(\mathbb{C})$ .

Proof of (a).



- (1) Let  $f = y^3 - y^2 + x^3 - x^2 + 3xy^2 + 3x^2y + 2xy \in k[x, y]$ . So

$$\begin{aligned}\frac{\partial f}{\partial x} &= 3x^2 + 6xy + 3y^2 - 2x + 2y = 0 \\ \frac{\partial f}{\partial y} &= 3x^2 + 6xy + 3y^2 + 2x - 2y = 0\end{aligned}$$

implies that

$$\begin{aligned}6(x + y)^2 &= 0 \\ -4(x - y) &= 0\end{aligned}$$

Note that  $f(0, 0) = \frac{\partial f}{\partial x}(0, 0) = \frac{\partial f}{\partial y}(0, 0) = 0$ . Hence,  $(x, y) = (0, 0)$  is the only multiple point on  $f$ .

- (2) Write  $f = (y^3 + 3xy^2 + 3x^2y + x^3) + (-x^2 + 2xy - y^2)$ . The tangent lines at  $(x, y) = (0, 0)$  is the linear factors of  $-x^2 + 2xy - y^2 = -(x - y)^2$ . Hence, the line  $x - y = 0$  is the only tangent line at  $(x, y) = (0, 0)$  of the multiplicity = 2.

□

*Proof of (b).*

- (1) Let  $f = x^4 + y^4 - x^2y^2 \in k[x, y]$ . So

$$\begin{aligned}\frac{\partial f}{\partial x} &= 4x^3 - 2xy^2 = 0 \\ \frac{\partial f}{\partial y} &= 4y^3 - 2x^2y = 0\end{aligned}$$

implies that  $(x, y) = (0, 0)$ . Note that  $f(0, 0) = \frac{\partial f}{\partial x}(0, 0) = \frac{\partial f}{\partial y}(0, 0) = 0$ . Hence,  $(x, y) = (0, 0)$  is the only multiple point on  $f$ .

- (2) The tangent lines at  $(x, y) = (0, 0)$  is the linear factors of  $x^4 + y^4 - x^2y^2$ . Hence, there are four distinct tangent lines

$$x \pm \sqrt{\frac{1 \pm \sqrt{-3}}{2}}y$$

at  $(x, y) = (0, 0)$ . Each tangent line is simple.

□

*Proof of (c).*

- (1) Let  $f = x^3 + y^3 - 3x^2 - 3y^2 + 3xy + 1 \in k[x, y]$ . So

$$\begin{aligned}\frac{\partial f}{\partial x} &= 3(x^2 - 2x + y) = 0 \\ \frac{\partial f}{\partial y} &= 3(y^2 - 2y + x) = 0\end{aligned}$$

implies that  $(x - y)(x + y - 3) = 0$ .

- (2) The case  $x - y = 0$ . Take  $x = y$  in  $3x^2 - 6x + 3y = 0$  to get  $(x, y) = (1, 1), (0, 0)$ .  $(x, y) = (1, 1)$  is a multiple point on  $f$  since  $f(1, 1) = \frac{\partial f}{\partial x}(1, 1) = \frac{\partial f}{\partial y}(1, 1) = 0$ .  $(x, y) = (0, 0)$  is impossible since  $f(0, 0) = 1 \neq 0$ .
- (3) The case  $x + y - 3 = 0$ . Take  $x = -y + 3$  in  $f$  to get  $1 = 0$ , which is absurd.
- (4) By (2)(3), the only multiple point on  $f$  is  $(x, y) = (1, 1)$ .
- (5) Let  $t(x, y) = (x + 1, y + 1)$ . Then

$$f^t = f(x + 1, y + 1) = x^3 + y^3 + 3xy.$$

The tangent lines at  $(x, y) = (1, 1)$  is the linear factors of  $x^3 + y^3 + 3xy$ . Hence, there are two distinct simple tangent lines  $x$  and  $y$  at  $(x, y) = (1, 1)$ .

□

*Proof of (d).*

- (1) Let  $f = y^2 + (x^2 - 5)(4x^4 - 20x^2 + 25) \in k[x, y]$ . So

$$\begin{aligned}\frac{\partial f}{\partial x} &= 2x(2x^2 - 5)(6x^2 - 25) = 0 \\ \frac{\partial f}{\partial y} &= 2y = 0\end{aligned}$$

implies that there are only two multiple points

$$(x, y) = \left( \pm \sqrt{\frac{5}{2}}, 0 \right)$$

on  $f$ .

- (2) Let  $t(x, y) = \left( x + \sqrt{\frac{5}{2}}, y \right)$ . Then

$$\begin{aligned}f^t &= f\left(x + \sqrt{\frac{5}{2}}, y\right) \\ &= 4x^6 + 12\sqrt{10}x^5 + 110x^4 + 20\sqrt{10}x^3 - 100x^2 + y^2.\end{aligned}$$

The tangent lines at  $(x, y) = \left( \sqrt{\frac{5}{2}}, 0 \right)$  is the linear factors of  $-100x^2 + y^2 = -(10x + y)(10x - y)$ . Hence, there are two distinct simple tangent lines

$$10x \pm y$$

at  $(x, y) = \left( \sqrt{\frac{5}{2}}, 0 \right)$ .

(3) Similarly, there are also two distinct simple tangent lines

$$10x \pm y$$

$$\text{at } (x, y) = \left(-\sqrt{\frac{5}{2}}, 0\right).$$

□

### Problem 3.3.

If a curve  $f$  of degree  $n$  has a point  $P$  of multiplicity  $n$ , show that  $f$  consists of  $n$  lines through  $P$  (not necessarily distinct).

*Proof.*

(1) Might assume that  $P = (0, 0)$ . (Note that any translation of  $f$  preserves the degree of  $f$ .)

(2) Write

$$f = f_m + f_{m+1} + \cdots + f_n$$

where  $f_i$  is a form in  $k[x, y]$ . Since  $m$  is the multiplicity of  $f$  at  $P$ ,  $m = n$ . Hence,  $f$  is a form in two variables of degree  $n$ , or  $f$  consists of  $n$  lines through  $P$ .

□

### Problem 3.4.

Let  $P$  be a double point on a curve  $f$ . Show that  $P$  is a node if and only if

$$\frac{\partial^2 f}{\partial x \partial y}(P)^2 \neq \frac{\partial^2 f}{\partial x^2}(P) \cdot \frac{\partial^2 f}{\partial y^2}(P).$$

*Proof.*

(1) Might assume that  $P = (0, 0)$  is a double point on  $f$ . Write

$$f = f_2 + f_3 + \cdots + f_n \in k[x, y]$$

(where  $f_i$  is a form in  $k[x, y]$ ), and

$$f_2 = ax^2 + bxy + cy^2 \in k[x, y].$$

(2)  $P$  is a node if and only if the discriminant

$$b^2 - 4ac \neq 0.$$

Note that

$$\begin{aligned}\frac{\partial^2 f}{\partial x \partial y}(P) &= b, \\ \frac{\partial^2 f}{\partial x^2}(P) &= 2a, \\ \frac{\partial^2 f}{\partial y^2}(P) &= 2c.\end{aligned}$$

Hence,  $P$  is a node if and only if

$$b^2 - 4ac = b^2 - (2a)(2c) = \frac{\partial^2 f}{\partial x \partial y}(P)^2 - \frac{\partial^2 f}{\partial x^2}(P) \cdot \frac{\partial^2 f}{\partial y^2}(P) \neq 0.$$

□

**Problem 3.5.**

( $\text{char}(k) = 0$ ) Show that  $m_P(f)$  is the smallest integer  $m$  such that for some  $i + j = m$ ,

$$\frac{\partial^m f}{\partial x^i \partial y^j}(P) \neq 0.$$

Find an explicit description for the leading form for  $f$  at  $P$  in terms of these derivatives.

*Proof.*

(1) Might assume that  $P = (0, 0)$ . Write  $f = f_0 + f_1 + \cdots + f_n$  where  $f_i$  is a form in  $k[x, y]$ . Consider any form  $f_m$  of  $f$ . Write

$$f_m = c_m x^m + c_{m-1} x^{m-1} y + \cdots + c_0 y^m$$

where  $c_i \in k$  and not all  $c_i$  are zero.

(2) Similar to Problem 3.4,  $\frac{\partial^m f}{\partial x^i \partial y^j}(P) = c_i i! j!$ . Here  $i + j = m$ . Hence,

$$c_i = \frac{1}{i! j!} \frac{\partial^m f}{\partial x^i \partial y^j}(P) = \frac{1}{m!} \binom{m}{i} \frac{\partial^m f}{\partial x^i \partial y^j}(P)$$

and thus

$$f_m = \frac{1}{m!} \sum_{i=0}^m \binom{m}{i} \frac{\partial^m f}{\partial x^i \partial y^j}(P) x^i y^j.$$

- (3) Suppose  $m$  is the smallest integer such that for some  $i + j = m$ ,

$$\frac{\partial^m f}{\partial x^i \partial y^j}(P) \neq 0.$$

Then  $f_0 = f_1 = \cdots = f_{m-1} = 0$  and  $f_m \neq 0$  in  $k[x, y]$  by (2). Therefore,  $m = m_P(f)$ . The explicit description for the leading form  $f_m$  for  $f$  at  $P$  is already stated in (2).

□

### Problem 3.6.

*Irreducible curves with given tangent lines  $\ell_i$  of multiplicity  $r_i$  may be constructed as follows: if  $\sum r_i = m$ , let  $f = \prod \ell_i^{r_i} + f_{m+1}$ , where  $f_{m+1}$  is chosen to make  $f$  irreducible (see Problem 2.34).*

*Proof.*

- (1) Let  $f_m = \prod \ell_i^{r_i} \in k[x, y]$ . Problem 1.4 implies that there exists a point  $P = (a, b)$  such that  $f_m(P) \neq 0$  since  $f_m \neq 0 \in k[x, y]$ .
- (2) Let  $\ell : bx - ay = 0$  and  $f_{m+1} = \ell^{m+1}$ . Since  $(a, b) \neq (0, 0)$ ,  $\deg(\ell) = 1$ . Also,  $\ell_i \nmid \ell$  by (1). Hence,  $f_m$  and  $f_{m+1}$  have no common factors. By Problem 2.34,  $f = f_m + f_{m+1}$  is irreducible.

□

### Problem 3.7.

- (a) Show that the real part of the curve  $e$  of the examples is the set of points in  $\mathbf{A}^2(\mathbb{R})$  whose polar coordinates  $(r, \theta)$  satisfy the equation  $r = -\sin(3\theta)$ . Find the polar equation for the curve  $f$ .
- (b) If  $n$  is an odd integer  $\geq 1$ , show that the equation  $r = \sin(n\theta)$  defines the real part of a curve of degree  $n + 1$  with an ordinary  $n$ -tuple point at  $(0, 0)$ . (Use the fact that  $\sin(n\theta) = \operatorname{im}(e^{in\theta})$  to get the equation; note that rotation by  $\frac{\pi}{n}$  is a linear transformation that takes the curve into itself.)
- (c) Analyze the singularities that arise by looking at  $r^2 = \sin^2(n\theta)$ ,  $n$  even.
- (d) Show that the curves constructed in (b) and (c) are all irreducible in  $\mathbf{A}^2(\mathbb{C})$ . (Hint: Make the polynomials homogeneous with respect to a variable  $z$ , and use §2.6.)

*Proof of (a).*

(1) De Moivre's theorem implies that

$$\begin{aligned}
 \cos(n\theta) + i \sin(n\theta) &= (\cos \theta + i \sin \theta)^n \\
 &= \sum_{k=0}^n \binom{n}{k} (\cos \theta)^{n-k} i^k (\sin \theta)^k \\
 &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{2k} (\cos \theta)^{n-2k} (\sin \theta)^{2k} \\
 &\quad + i \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{2k+1} (\cos \theta)^{n-2k-1} (\sin \theta)^{2k+1}.
 \end{aligned}$$

Hence,

$$\sin(n\theta) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{2k+1} (\cos \theta)^{n-2k-1} (\sin \theta)^{2k+1}.$$

In particular,

$$\sin(3\theta) = 3 \cos^2 \theta \sin \theta - \sin^3 \theta.$$

(2)  $r = -\sin(3\theta)$  implies that

$$\begin{aligned}
 r^4 &= r^3(-\sin(3\theta)) \\
 &= r^3(-3 \cos^2 \theta \sin \theta + \sin^3 \theta) \\
 &= -3(r \cos \theta)^2(r \sin \theta) + (r \sin \theta)^3.
 \end{aligned}$$

Hence,  $r = -\sin(3\theta)$  implies that

$$(x^2 + y^2)^2 = -3x^2y + y^3$$

in  $\mathbf{A}^2(\mathbb{R})$ .

(3) As  $f(x, y) = (x^2 + y^2)^3 - 4x^2y^2$ ,  $f(r \cos \theta, r \sin \theta) = 0$  implies that

$$r^6 = 4r^4 \cos^2 \theta \sin^2 \theta = r^4 \sin^2 2\theta$$

or

$$r^2 = \sin^2 2\theta.$$

□

*Proof of (b).*

(1) By (a),  $r = \sin(n\theta)$  with odd  $n \geq 1$  implies that

$$\begin{aligned} r &= \sin(n\theta) = \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{2k+1} (\cos \theta)^{n-2k-1} (\sin \theta)^{2k+1} \\ \implies r^{n+1} &= \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{2k+1} (r \cos \theta)^{n-2k-1} (r \sin \theta)^{2k+1} \\ \implies (x^2 + y^2)^{\frac{n+1}{2}} &= \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1}. \end{aligned}$$

Hence,  $r = \sin(n\theta)$  defines the real part of a curve

$$\alpha : (x^2 + y^2)^{\frac{n+1}{2}} - \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1}$$

of degree  $n+1$ .

(2) Note that  $(0,0) \in \alpha$ . Since

$$\begin{aligned} \frac{\partial \alpha}{\partial x} &= (n+1)(x^2 + y^2)^{\frac{n-1}{2}} x \\ &\quad - \sum_{k=0}^{\frac{n-3}{2}} (-1)^k (n-2k-1) \binom{n}{2k+1} x^{n-2k-2} y^{2k+1} \\ \frac{\partial \alpha}{\partial y} &= (n+1)(x^2 + y^2)^{\frac{n-1}{2}} y \\ &\quad - \sum_{k=0}^{\frac{n-1}{2}} (-1)^k (2k+1) \binom{n}{2k+1} x^{n-2k-1} y^{2k}, \end{aligned}$$

$\frac{\partial \alpha}{\partial x}(0,0) = \frac{\partial \alpha}{\partial y}(0,0) = 0$ .  $(0,0)$  is a multiple point.

(3) The tangents at  $(0,0)$  are the linear factors of

$$\sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1}.$$

Clearly,  $y$  is a tangent line at  $(0,0)$ . Note that rotation by  $\frac{\pi}{n}$  is a linear transformation that takes the curve into itself. Hence, all tangents at  $(0,0)$  are

$$\ell_k : \sin \frac{k\pi}{n} x - \cos \frac{k\pi}{n} y$$

for  $k = 0, 1, \dots, n-1$ . All  $\ell_k$  are pairwise distinct, and thus  $(0,0)$  is an ordinary  $n$ -tuple point.

□

*Proof of (c).*

- (1) Similar to (b),  $r^2 = \sin^2(n\theta)$  defines the real part of a curve

$$\beta : (x^2 + y^2)^{n+1} - \left( \sum_{k=0}^{\frac{n-2}{2}} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1} \right)^2$$

of degree  $2n + 2$ .

- (2) Note that

$$\beta(0, 0) = \frac{\partial \beta}{\partial x}(0, 0) = \frac{\partial \beta}{\partial y}(0, 0) = 0.$$

Hence,  $(0, 0)$  is a multiple point.

- (3) Similar to (b), all tangents at  $(0, 0)$  are

$$\ell_k : \sin \frac{k\pi}{n} x - \cos \frac{k\pi}{n} y$$

of multiplicity  $= 2$  for  $k = 0, 1, \dots, n-1$ .

□

*Proof of (d).*

- (1) The case  $n$  is odd.

- (a) Consider

$$\alpha : (x^2 + y^2)^{\frac{n+1}{2}} - \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1}.$$

- (b) Since

$$\begin{aligned} \alpha_{n+1} &:= (x^2 + y^2)^{\frac{n+1}{2}} \\ &= (x + iy)^{\frac{n+1}{2}} (x - iy)^{\frac{n+1}{2}} \\ \alpha_n &= \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1} \\ &= c \prod_{k=0}^{n-1} \left( \sin \frac{k\pi}{n} x - \cos \frac{k\pi}{n} y \right) \end{aligned}$$

(with some  $c \in \mathbb{C}$ ),  $\alpha_{n+1}$  and  $\alpha_n$  have no common factors in  $\mathbb{C}[x, y]$ .  
By Problem 2.34,  $\alpha$  is irreducible.



(2) The case  $n$  is even.

(a) Consider

$$\beta : \underbrace{(x^2 + y^2)^{n+1}}_{:=\beta_{2n+2}} - \underbrace{\left( \sum_{k=0}^{\frac{n-2}{2}} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1} \right)^2}_{:=\beta_{2n}}.$$

(b) Similar to the proof of Problem 2.34, suppose  $\beta = \beta_{2n} + \beta_{2n+2} = rs \in \mathbb{C}[x, y]$ . So

$$(\beta_{2n} + \beta_{2n+2})^* = (rs)^* \implies z^2 \beta_{2n} + \beta_{2n+2} = r^* s^*.$$

Note that  $\deg_z(z^2 \beta_{2n} + \beta_{2n+2}) = 2$ . So  $\deg_z(r^*) = 0, 1, 2$ .

(c) The case  $\deg_z(r^*) = 0, 2$  is similar to the proof of Problem 2.34 because  $\beta_{2n}$  and  $\beta_{2n+2}$  have no common factors in  $\mathbb{C}[x, y]$ .

(d) The case  $\deg_z(r^*) = 1$ . (So  $\deg_z(s^*) = 1$ .) Write  $r = r_p + r_{p+1}$  and  $s = s_q + s_{q+1}$ . Hence,  $\beta = rs$  implies that

$$\begin{aligned} r_p s_q &= - \left( \sum_{k=0}^{\frac{n-2}{2}} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1} \right)^2 \\ r_p s_{q+1} + r_{p+1} s_q &= 0 \\ r_{p+1} s_{q+1} &= (x^2 + y^2)^{n+1} = (x + iy)^{n+1} (x - iy)^{n+1}. \end{aligned}$$

Since  $n+1$  is odd and  $x \pm iy \nmid \left( \sum_{k=0}^{\frac{n-2}{2}} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1} \right)^2$ ,  $r_p s_{q+1} + r_{p+1} s_q = 0$  implies that  $r_p = s_q = 0$ , which is absurd.

(e) By (b)(c)(d),  $\beta$  is irreducible over  $\mathbb{C}$ .

□

### Problem 3.8.

Let  $t : \mathbf{A}^2 \rightarrow \mathbf{A}^2$  be a polynomial map,  $t(Q) = P$ .

(a) Show that  $m_Q(f^t) \geq m_P(f)$ .

(b) Let  $t = (t_1, t_2)$ , and define

$$J_Q t = \left( \frac{\partial t_i}{\partial x_j}(Q) \right)$$

to be the **Jacobian matrix** of  $t$  at  $Q$ . Show that  $m_Q(f^t) = m_P(f)$  if  $J_Q t$  is invertible.

- (c) Show that the converse of (b) is false: let  $t = (x^2, y)$ ,  $f = y - x^2$ ,  $P = Q = (0, 0)$ .

*Proof of (a).*

- (1) Might assume that  $P = Q = (0, 0)$ . Write  $t = (t_1, t_2)$  and thus  $(0, 0) = t(0, 0) = (t_1(0, 0), t_2(0, 0))$ . So there is no nonzero constant term in  $t_i$  ( $i = 1, 2$ ).
- (2) Might assume that  $f \neq 0$  (since there is nothing to prove when  $f = 0$ ). Write  $f = f_m + f_{m+1} + \cdots + f_n$  where  $f_i$  is a form in  $k[x, y]$  and  $f_m \neq 0$ . So,

$$f(t_1(x, y), t_2(x, y)) = f_m(t_1(x, y), t_2(x, y)) + \cdots + f_n(t_1(x, y), t_2(x, y))$$

has the multiplicity  $= \infty$  or  $\geq m = m_P(f)$ . In any case,  $m_Q(f^t) \geq m_P(f)$ .

□

*Proof of (b).*

- (1) Might assume that  $P = Q = (0, 0)$ . Since  $J_Q t$  is invertible,

$$\begin{pmatrix} \frac{\partial t_1}{\partial x}(Q) & \frac{\partial t_1}{\partial y}(Q) \\ \frac{\partial t_2}{\partial x}(Q) & \frac{\partial t_2}{\partial y}(Q) \end{pmatrix} \neq 0$$

(as vectors). Hence  $t_1$  (resp.  $t_2$ ) has the multiplicity  $= 1$ .

- (2) Define

$$s = (s_1, s_2) : \mathbf{A}^2 \rightarrow \mathbf{A}^2$$

be a polynomial map such that  $s_i$  is the linear term of  $t_i$ . Note that  $J_Q s = J_Q t$  is invertible and  $m_Q(f^s) = m_Q(f^t)$  for any  $f$ .

- (3) Show that  $m_Q(f^s) = m_Q(f^t)$  for any  $f \in k[x, y]$ . Might assume that  $f \neq 0$  (since there is nothing to prove when  $f = 0$ ). Write  $f = f_m + f_{m+1} + \cdots + f_n$  where  $f_i$  is a form in  $k[x, y]$  and  $f_m \neq 0$ . Since

$$m_Q(f_i^t) = m_Q(f_i^s) = i \text{ or } \infty,$$

$$m_Q(f^s) = m_Q(f^t).$$

- (4) Since  $J_Q s$  is invertible,  $s^{-1}$  is also a polynomial map with an invertible Jacobian matrix  $J_Q s^{-1}$ . By (a),

$$m_Q(f^s) \geq m_P(f) = m_P((f^s)^{s^{-1}}) = m_Q(f^s)$$

or  $m_Q(f^s) = m_P(f)$ . Therefore,  $m_Q(f^t) = m_Q(f^s) = m_P(f)$ .

□

*Proof of (c).*  $m_P(f) = 1$  and  $m_Q(f^t) = 1$  since  $f^t = y - x^4$ . However,

$$J_Q t = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

is not invertible. □

### Problem 3.9.

Let  $f \in k[x_1, \dots, x_n]$  define a hypersurface  $V(f) \subseteq \mathbf{A}^n$ . Let  $P \in \mathbf{A}^n$ .

- (a) Define the multiplicity  $m_P(f)$  of  $f$  at  $P$ .
- (b) If  $m_P(f) = 1$ , define the tangent hyperplane to  $f$  at  $P$ .
- (c) Examine  $f = x^2 + y^2 - z^2$ ,  $P = (0, 0, 0)$ . Is it possible to define tangent hyperplanes at multiple points?

*Proof of (a).*

- (1) Let  $P = (0, \dots, 0)$ . Write  $f = f_m + f_{m+1} + \dots + f_n$ , where  $f_i$  is a form in  $k[x_1, \dots, x_n]$  of degree  $i$ ,  $f_m \neq 0$ . We define  $m$  to be the multiplicity of  $f$  at  $P = (0, \dots, 0)$ , write  $m = m_P(f)$ .
- (2) To extend these definitions to a point  $P = (a_1, \dots, a_n) \neq (0, \dots, 0)$ , let  $t$  be the translation that takes  $(0, \dots, 0)$  to  $P$ , i.e.,

$$t(x_1, \dots, x_n) = (x_1 + a_1, \dots, x_n + a_n).$$

Then

$$f^t = f(x_1 + a_1, \dots, x_n + a_n).$$

Define  $m_P(f)$  to be  $m_{(0, \dots, 0)}(f^t)$ , i.e., write  $f^t = g_m + g_{m+1} + \dots$ ,  $g_i$  forms,  $g_m \neq 0$ , and let  $m = m_P(f)$ .

□

*Proof of (b).*

- (1) Let  $P = (0, \dots, 0)$ . Write  $f = f_m + f_{m+1} + \dots + f_n$ , where  $f_i$  is a form in  $k[x_1, \dots, x_n]$  of degree  $i$ ,  $f_m \neq 0$ . If  $m = 1$ , then  $f_m = f_1$  is a hyperplane. Hence, we can define the tangent hyperplane to  $f$  at  $P$  to be  $f_1$ .
- (2) Similar to (a), the tangent hyperplane to  $f$  at  $P = (a_1, \dots, a_n) \neq (0, \dots, 0)$  is  $g_1$  where

$$f^t = g_1 + g_2 + \dots$$

and  $t(x_1, \dots, x_n) = (x_1 + a_1, \dots, x_n + a_n)$ .

□

*Proof of (c).*

(1) No.

(2) *Show that  $x^2 + y^2 - z^2$  is irreducible over  $k$ .* (Reductio ad absurdum)  
 Suppose  $x^2 + y^2 - z^2$  were reducible. By Problem 1.1, we can write

$$x^2 + y^2 - z^2 = (a_1x + a_2y + a_3z)(b_1x + b_2y + b_3z)$$

for some  $a_i, b_i \in k$  ( $i = 1, 2, 3$ ). Expanding out the right hand side and comparing coefficients to get

$$\begin{aligned} a_1b_1 &= a_2b_2 = -a_3b_3 = 1 \\ a_1b_2 + a_2b_1 &= a_2b_3 + a_3b_2 = a_3b_1 + a_1b_3 = 0. \end{aligned}$$

So  $a_i, b_i \neq 0$  for all  $i$  and

$$b_1 = \frac{-a_1}{a_3}b_3 = \frac{-a_1}{a_3} \cdot \frac{-a_3}{a_2}b_2 = \frac{-a_1}{a_3} \cdot \frac{-a_3}{a_2} \cdot \frac{-a_2}{a_1}b_1 = -b_1.$$

Hence,  $b_1 = 0$ , which is absurd.

(3) Since  $x^2 + y^2 - z^2$  is irreducible over any field  $k$  with  $\text{char}(k) = 0$ , it is impossible to define tangent hyperplanes at  $(0, 0, 0)$ .

□

### Problem 3.10.

*Show that an irreducible plane curve has only a finite number of multiple points. Is this true for hypersurfaces?*

*Proof.*

(1) Let  $f \in k[x, y]$  be an irreducible plane curve, and

$$V = V\left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

be the set of the multiple (singular) points of  $f$ . Moreover,  $V$  is an algebraic set.

(2) Suppose  $\frac{\partial f}{\partial x} \neq 0$  (since not all  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$  are zero). It is nothing to do if  $\frac{\partial f}{\partial x}$  is a nonzero constant. Suppose  $\deg(\frac{\partial f}{\partial x}) \geq 1$ . Since  $f$  is irreducible and  $\deg(\frac{\partial f}{\partial x}) = \deg f - 1$ ,  $f$  and  $\frac{\partial f}{\partial x}$  have no common factors. By Proposition 2 in §1.6,  $V\left(f, \frac{\partial f}{\partial x}\right)$  is a finite set. Hence,  $V \subseteq V\left(f, \frac{\partial f}{\partial x}\right)$  is finite as a subset of a finite set.

- (3) The conclusion is not true for hypersurfaces when  $n \geq 3$ . Consider  $f = x_2^2 - x_1^3 \in k[x_1, \dots, x_n]$ . The set of the multiple (singular) points of  $f$  is

$$\{(0, 0, a_3, \dots, a_n) : a_3, \dots, a_n \in k\},$$

which is infinite as  $k$  is infinite.

□

**Problem 3.11. (Tangent space)**

Let  $V \subseteq \mathbf{A}^n$  be an affine variety,  $P \in V$ . The **tangent space**  $T_P(V)$  is defined to be

$$\left\{ (v_1, \dots, v_n) \in \mathbf{A}^n : \sum_i \frac{\partial g}{\partial x_i}(P) v_i = 0 \ \forall g \in I(V) \right\}.$$

If  $V = V(f)$  is a hypersurface,  $f$  irreducible, show that

$$T_P(V) = \left\{ (v_1, \dots, v_n) \in \mathbf{A}^n : \sum_i \frac{\partial f}{\partial x_i}(P) v_i = 0 \right\}.$$

How does the dimension of  $T_P(V)$  relate to the multiplicity of  $f$  at  $P$ ?

*Proof.*

- (1) By the Hilbert's Nullstellensatz, the irreducibility of  $f$  implies that  $I(V) = I(V(f)) = (f)$ .

- (2) Let

$$W = \left\{ (v_1, \dots, v_n) \in \mathbf{A}^n : \sum_i \frac{\partial f}{\partial x_i}(P) v_i = 0 \right\}.$$

$W \supseteq T_P(V)$  is true since  $f \in I(V) = (f)$ .

- (3) Show that  $W \subseteq T_P(V)$ . Given any  $(v_1, \dots, v_n) \in W$ . Now for any  $g \in I(V) = (f)$ , there exists a  $h \in k[x_1, \dots, x_n]$  such that  $g = fh$ . Hence,

$$\begin{aligned} \sum_i \frac{\partial g}{\partial x_i}(P) v_i &= \sum_i \frac{\partial(fh)}{\partial x_i}(P) v_i \\ &= \sum_i \left( \frac{\partial(f)}{\partial x_i}(P) h(P) + \underbrace{f(P)}_{=0} \frac{\partial h}{\partial x_i}(P) \right) v_i \\ &= h(P) \underbrace{\sum_i \frac{\partial f}{\partial x_i}(P) v_i}_{=0} \\ &= 0 \end{aligned}$$

implies that  $(v_1, \dots, v_n) \in T_P(V)$ .

(4) By definition of  $T_P(V)$ ,

$$\dim_k(T_P(V)) = \begin{cases} n-1 & \text{if } m_P(f) = 1 \\ n & \text{if } m_P(f) > 1. \end{cases}$$

□

### 3.2. Multiplicities and Local Rings

#### Problem 3.12. (Flex)

A simple point  $P$  on a curve  $f$  is called a **flex** if  $\text{ord}_P^f(L) \geq 3$ , where  $L$  is the tangent to  $f$  at  $P$ . The flex is called **ordinary** if  $\text{ord}_P(L) = 3$ , a **higher flex** otherwise.

- (a) Let  $f = y - x^n$ . For which  $n$  does  $f$  have a flex at  $P = (0, 0)$ , and what kind of flex?
- (b) Suppose  $P = (0, 0)$ ,  $L = y$  is the tangent line,  $f = y + ax^2 + \cdots$ . Show that  $P$  is a flex on  $f$  if and only if  $a = 0$ . Give a simple criterion for calculating  $\text{ord}_P^f(y)$ , and therefore for determining if  $P$  is a higher flex.

*Proof of (a).*

- (1) When  $n = 0$  or  $1$ , the tangent line  $L$  to  $f$  at any point is  $L = f$  itself. So

$$\text{ord}_P^f(L) = \text{ord}_P^f(f) = \text{ord}_P^f(0) = \infty.$$

$P$  is a higher flex.

- (2) When  $n > 1$ , the tangent line  $L$  to  $f$  at  $P = (0, 0)$  is  $L = y$ . So

$$\text{ord}_P^f(L) = \text{ord}_P^f(y) = \text{ord}_P^f(x^n) = n.$$

Here  $x$  is a uniformizing parameter for  $\mathcal{O}_P(f)$  since the line  $x$  is not tangent to  $f$  (Theorem 1). Hence,  $P$  is a flex if  $n \geq 3$ ,  $P$  is an ordinary flex if  $n = 3$ , and  $P$  is a higher flex if  $n > 3$ .

□

*Proof of (b).*

- (1) Since  $y$  is the tangent line,  $\text{ord}_P^f(y) \geq 2$ . By Problem 2.29(a),

$$\text{ord}_P^f(y) = \text{ord}_P^f(ax^2 + \cdots) = 2$$

if and only if  $a \neq 0$ . Hence,  $P$  is flex iff  $\text{ord}_P^f(y) \geq 3$  iff  $a = 0$ .

(2) In general,

$$\text{ord}_P^f(y) = \text{ord}_P^f(ax^2 + \cdots) = m_P(ax^2 + \cdots) = m_P(f - y).$$

Hence,  $P$  is a higher flex if  $f - y$  has no nonzero form of degree 3.

□

**Problem 3.13.\***

With the notation of Theorem 2, and  $\mathfrak{m} = \mathfrak{m}_P(f)$ , show that  $\dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = n + 1$  for  $0 \leq n < m_P(f)$ . In particular,  $P$  is a simple point if and only if  $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 1$ ; otherwise  $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 2$ .

*Proof.*

(1) From the exact sequence

$$0 \rightarrow \mathfrak{m}^n/\mathfrak{m}^{n+1} \rightarrow \mathcal{O}/\mathfrak{m}^{n+1} \rightarrow \mathcal{O}/\mathfrak{m}^n \rightarrow 0,$$

it suffices to show that

$$\dim_k(\mathcal{O}/\mathfrak{m}^n) = \frac{n(n+1)}{2}$$

as  $0 \leq n < m_P(f)$ . (Problem 2.49 and Proposition 7 in §2.10.)

(2) We may assume that  $P = (0, 0)$ . Similar to the proof of Theorem 2, we are reduced to calculating the dimension of  $k[x, y]/(I^n, f)$ . Let  $m = m_P(f)$ . By the definition of  $m$ ,

$$f \in \underbrace{(x, y)}_{=I}^m = I^m.$$

So if  $0 \leq n < m_P(f)$ , then  $f \in I^m \subseteq I^n$  and thus  $(I^n, f) = I^n$ . Therefore,

$$\begin{aligned} \dim_k(\mathcal{O}/\mathfrak{m}^n) &= \dim_k(k[x, y]/(I^n, f)) \\ &= \dim_k(k[x, y]/I^n) \\ &= \frac{n(n+1)}{2}. \end{aligned} \quad (\text{Problem 2.46})$$

So

$$\dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = \dim_k(\mathcal{O}/\mathfrak{m}^{n+1}) - \dim_k(\mathcal{O}/\mathfrak{m}^n) = n + 1.$$

(3)  $P$  is a simple point if  $m = m_P(f) = 1$  by definition. Note that

$$\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \begin{cases} 1 & \text{if } m = 1 \text{ (Theorem 1)} \\ 2 & \text{if } m > 1 \text{ ((2))}. \end{cases}$$

Therefore,  $P$  is a simple iff  $m = 1$  iff  $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 1$ .

□

**Problem 3.14.**

Let  $V = V(x^2 - y^3, y^2 - z^3) \subseteq \mathbf{A}^3$ ,  $P = (0, 0, 0)$ ,  $\mathfrak{m} = \mathfrak{m}_P(V)$ . Find  $\dim_k(\mathfrak{m}/\mathfrak{m}^2)$ . (See Problem 1.40.)

*Proof.*

(1)  $\mathfrak{m} = (x, y, z)$ .

(2) Write  $\mathcal{O} = \mathcal{O}_P(V)$ . By Problem 1.40(a), every element of  $\mathcal{O}$  is of the form

$$\bar{a} + \bar{x}\bar{b} + \bar{y}\bar{c} + \bar{x}\bar{y}\bar{d}$$

for some  $a, b, c, d \in k[z]$ .

(3) By (1)(2),  $\{\bar{1}\}$  (resp.  $\{\bar{1}, \bar{z}, \bar{y}, \bar{z}\}$ ) is a basis for  $\mathcal{O}/\mathfrak{m}$  (resp.  $\mathcal{O}/\mathfrak{m}^2$ ). Hence,  $\dim_k(\mathcal{O}/\mathfrak{m}) = 1$  (resp.  $\dim_k(\mathcal{O}/\mathfrak{m}^2) = 4$ ). Therefore,

$$\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \dim_k(\mathcal{O}/\mathfrak{m}^2) - \dim_k(\mathcal{O}/\mathfrak{m}) = 3$$

by Proposition 7 in §2.10.

(4) By Theorem I.5.1 in *Robin Hartshorne, Algebraic Geometry*,

$$3 = \dim_k(\mathfrak{m}/\mathfrak{m}^2) + \text{rank} J = \dim_k(\mathfrak{m}/\mathfrak{m}^2).$$

Here the Jacobian matrix of  $V$  at  $P$

$$J = \left[ \frac{\partial f_i}{\partial x_j}(P) \right] = \begin{pmatrix} 2x & -3y^2 & 0 \\ 0 & 2y & -3z^2 \end{pmatrix}_{P=(0,0,0)} = 0$$

has rank zero.

□

**Problem 3.15.**

(a) Let  $\mathcal{O} = \mathcal{O}_P(\mathbf{A}^2)$  for some  $P \in \mathbf{A}^2$ ,  $\mathfrak{m} = \mathfrak{m}_P(\mathbf{A}^2)$ . Calculate  $\chi(n) = \dim_k(\mathcal{O}/\mathfrak{m}^n)$ .

(b) Let  $\mathcal{O} = \mathcal{O}_P(\mathbf{A}^r(k))$ . Show that  $\chi(n)$  is a polynomial of degree  $r$  in  $n$ , with leading coefficient  $\frac{1}{r!}$  (see Problem 2.36).

*Proof of (a).* Might assume that  $P = (0, 0)$ . By Problem 2.46, the Hilbert-Samuel polynomial is

$$\chi(n) = \dim_k(\mathcal{O}/\mathfrak{m}^n) = \dim_k(k[x, y]/(x, y)^n) = \frac{n(n+1)}{2}.$$

□

*Proof of (b).*



(1) Might assume that  $P = (0, \dots, 0)$ . Similar to (a),

$$\chi(n) = \dim_k(\mathcal{O}/\mathfrak{m}^n) = \dim_k(k[x_1, \dots, x_r]/(x_1, \dots, x_r)^n).$$

(2) Since

$$\mathcal{B} = \{x_1^{i_1} x_2^{i_2} \cdots x_r^{i_r} : i_1 + \cdots + i_r < n\}$$

is a basis for  $k[x_1, \dots, x_r]/(x_1, \dots, x_r)^n$ ,

$$\dim_k(k[x_1, \dots, x_r]/(x_1, \dots, x_r)^n) = |\mathcal{B}|.$$

(3) By the stars and bars (combinatorics) method (as Problem 2.35(b)),

$$\begin{aligned} |\mathcal{B}| &= |\{(i_1, \dots, i_r) : i_1 + \cdots + i_r \leq n\}| \\ &\quad - |\{(i_1, \dots, i_r) : i_1 + \cdots + i_r = n\}| \\ &= |\{(i_1, \dots, i_r, j) : i_1 + \cdots + i_r + j = n\}| \\ &\quad - |\{(i_1, \dots, i_r) : i_1 + \cdots + i_r = n\}| \\ &= \binom{n+r}{r} - \binom{n+r-1}{r-1} \\ &= \binom{n+r-1}{r} \\ &= \frac{1}{r!} (n+r-1)(n+r-2) \cdots (n+1)(n). \end{aligned}$$

So

$$\chi(n) = \frac{1}{r!} (n+r-1)(n+r-2) \cdots (n+1)(n)$$

is a polynomial of degree  $r$  in  $n$ , with leading coefficient  $\frac{1}{r!}$ .

(4) By Problem 2.36, we can also deduce that

$$\begin{aligned} \dim_k(k[x_1, \dots, x_r]/(x_1, \dots, x_r)^n) &= \sum_{i=0}^{n-1} \dim_k V(i, r) \\ &= \sum_{i=0}^{n-1} \binom{i+r-1}{r-1} \\ &= \binom{n+r-1}{r} \end{aligned}$$

by the Pascal's identity.

□

**Problem 3.16.**

Let  $f \in k[x_1, \dots, x_r]$  define a hypersurface in  $\mathbf{A}^r$ . Write  $f = f_m + f_{m+1} + \dots$ , and let  $m = m_P(f)$  where  $P = (0, \dots, 0)$ . Suppose  $f$  is irreducible, and let  $\mathcal{O} = \mathcal{O}_P(V(f))$ ,  $\mathfrak{m}$  its maximal ideal. Show that  $\chi(n) = \dim_k(\mathcal{O}/\mathfrak{m}^n)$  is a polynomial of degree  $r-1$  for sufficiently large  $n$ , and that the leading coefficient of  $\chi$  is  $\frac{m_P(f)}{(r-1)!}$ . Can you find a definition for the multiplicity of a local ring that makes sense in all the cases you know?

*Proof.*

- (1) Similar to the proof of Theorem 2. By Problem 2.43,

$$\mathfrak{m}^n = I^n \mathcal{O}$$

where  $I = (x_1, \dots, x_r) \subseteq k[x_1, \dots, x_r]$ . Since  $V(I^n) = \{P\}$ ,

$$k[x_1, \dots, x_r]/(I^n, f) \cong \mathcal{O}_P(\mathbf{A}^r)/(I^n, f) \mathcal{O}_P(\mathbf{A}^r) \cong \mathcal{O}/I^n \mathcal{O} \cong \mathcal{O}/\mathfrak{m}^n$$

(Corollary 2 to Proposition 6 in §2.9 and Problem 2.44).

- (2) So we are reduced to calculating the dimension of  $k[x_1, \dots, x_r]/(I^n, f)$ . As  $n \geq m = m_P(f)$ , there is a natural ring homomorphism

$$\varphi : k[x_1, \dots, x_r]/I^n \rightarrow k[x_1, \dots, x_r]/(I^n, f)$$

and a  $k$ -linear map

$$\psi : k[x_1, \dots, x_r]/I^{n-m} \rightarrow k[x_1, \dots, x_r]/I^n$$

defined by  $\bar{g} \mapsto \overline{fg}$ . It is easy to verify that the sequence

$$0 \rightarrow k[x_1, \dots, x_r]/I^{n-m} \xrightarrow{\psi} k[x_1, \dots, x_r]/I^n \xrightarrow{\varphi} k[x_1, \dots, x_r]/(I^n, f) \rightarrow 0$$

is exact.

- (3) By Problem 3.15,

$$\begin{aligned} & \dim_k(k[x_1, \dots, x_r]/(I^n, f)) \\ &= \binom{n+r-1}{r} - \binom{n-m+r-1}{r} \\ &= \frac{1}{r!} ((n+r-1) \cdots n - (n-m+r-1) \cdots (n-m)) \\ &= \frac{1}{r!} (n^{r-1}(rm) + \dots) \\ &= \frac{m}{(r-1)!} n^{r-1} + \dots \end{aligned}$$

Therefore,  $\chi(n) = \dim_k(\mathcal{O}/\mathfrak{m}^n)$  is a polynomial of degree  $r-1$  for  $n \geq m$ , and that the leading coefficient of  $\chi$  is  $\frac{m_P(f)}{(r-1)!}$ .

(4) It is reasonable to define the multiplicity of a Noetherian local ring  $\mathcal{O}$  by

$$(d!) \cdot (\text{leading coefficient of } \chi(n))$$

for sufficiently large  $n$ , where  $d$  is the dimension (or Krull dimension) of  $\mathcal{O}$ . (Note that the dimension of a hypersurface in  $\mathbf{A}^r$  is  $r - 1$ .)

□

### 3.3. Intersection Numbers

#### Problem 3.17.

*Find the intersection numbers of various pairs of curves*

(a)  $a = y - x^2$

(b)  $b = y^2 - x^3 + x$

(c)  $c = y^2 - x^3$

(d)  $d = y^2 - x^3 - x^2$

(e)  $e = (x^2 + y^2)^2 + 3x^2y - y^3$

(f)  $f = (x^2 + y^2)^3 - 4x^2y^2$

at the point  $P = (0, 0)$ .

*Proof.*

(1) Note that Example in §3.3 shows that  $I(P, e \cap f) = 14$ . Also,

$$I(P, a \cap b) = m_P(a)m_P(b) = 1 \cdot 1 = 1$$

$$I(P, a \cap d) = m_P(a)m_P(d) = 1 \cdot 2 = 2$$

$$I(P, b \cap c) = m_P(b)m_P(c) = 1 \cdot 2 = 2$$

$$I(P, b \cap d) = m_P(b)m_P(d) = 1 \cdot 2 = 2$$

$$I(P, b \cap e) = m_P(b)m_P(e) = 1 \cdot 3 = 3$$

$$I(P, c \cap d) = m_P(c)m_P(d) = 2 \cdot 2 = 4$$

$$I(P, d \cap e) = m_P(d)m_P(e) = 2 \cdot 3 = 6$$

$$I(P, d \cap f) = m_P(d)m_P(f) = 2 \cdot 4 = 8$$

by Property (5).

(2) Show that  $I(P, a \cap c) = 3$ .

$$\begin{aligned}
I(P, a \cap c) &= I(P, a \cap (c + (-x^2 - y)a)) && \text{(Property (7))} \\
&= I(P, a \cap x^3(x - 1)) \\
&= 3I(P, a \cap x) + I(P, a \cap (x - 1)) && \text{(Property (6))} \\
&= 3I(P, a \cap x) && \text{(Property (2))} \\
&= 3. && \text{(Property (5))}
\end{aligned}$$

(3) Show that  $I(P, a \cap e) = 4$ .

$$\begin{aligned}
I(P, a \cap e) &= I(P, a \cap (e + (x^2 + 2y^2 + 4y)a)) && \text{(Property (7))} \\
&= I(P, a \cap y^2(y^2 + y + 4)) \\
&= 2I(P, a \cap y) + I(P, a \cap (y^2 + y + 4)) && \text{(Property (6))} \\
&= 2I(P, a \cap y) && \text{(Property (2))} \\
&= 2I(P, (a - y) \cap y) && \text{(Property (7))} \\
&= 2I(P, (-x^2) \cap y) \\
&= 4. && \text{(Property (5))}
\end{aligned}$$

(4) Show that  $I(P, a \cap f) = 6$ . Similar to (3). Let

$$q_4 = x^4 + 3x^2y^2 + 3y^4 + x^2y + 3y^3 - 3y^2.$$

So

$$\begin{aligned}
I(P, a \cap f) &= I(P, a \cap \underbrace{(f + q_4a)}_{\text{(Property (7))}}) \\
&= I(P, a \cap y^3(y^3 + 3y^2 + 3y - 3)) \\
&\quad \text{(Property (6))} \\
&= \underbrace{3I(P, a \cap y)}_{=2 \text{ (by (3))}} + \underbrace{I(P, a \cap (y^3 + 3y^2 + 3y - 3))}_{=0 \text{ (Property (2))}} \\
&= 6.
\end{aligned}$$

(5) Show that  $I(P, b \cap f) = 6$ . Similar to (3). Let

$$q_5 = -x^6 - 3x^5 - x^3y^2 - x^4 - 3x^2y^2 - y^4 + 3x^3 + xy^2 + 3x^2.$$

So

$$\begin{aligned}
& I(P, b \cap f) \\
&= I(P, b \cap \underbrace{(f + q_5 b)}_{\text{(Property (7))}}) \\
&= I(P, b \cap x^3(x^6 + 3x^5 - 5x^3 - 4x^2 + 3x + 3)) \\
&\quad \text{(Property (6))} \\
&= \underbrace{3I(P, b \cap x)}_{=2} + \underbrace{I(P, b \cap (x^6 + 3x^5 - 5x^3 - 4x^2 + 3x + 3))}_{=0 \text{ (Property (2))}} \\
&= 6.
\end{aligned}$$

Here

$$I(P, b \cap x) = I(P, (b + (x^2 - 1)x) \cap x) = I(P, y^2 \cap x) = 2.$$

(6) Show that  $I(P, c \cap e) = 7$ . Similar to (3).

$$\begin{aligned}
I(P, c \cap e) &= I(P, c \cap \underbrace{(e + (-x^3 - 2x^2 - y^2 + y)c)}_{\text{(Property (7))}}) \\
&= I(P, c \cap x^2(x^4 + 2x^3 + x^2 - xy + 3y)) \\
&\quad \text{(Property (6))} \\
&= 2 \underbrace{I(P, c \cap x)}_{=2 \text{ (Property (5))}} + \underbrace{I(P, c \cap (x^4 + 2x^3 + x^2 - xy + 3y))}_{:=h_6} \\
&= 4 + I(P, c \cap h_6).
\end{aligned}$$

Here

$$\begin{aligned}
& I(P, c \cap h_6) \\
&= I(P, (3c) \cap h_6) \\
&= I(P, (3c - yh_6) \cap h_6) \\
&= I(P, (-x^4y - 2x^3y - \underbrace{3x^3 - x^2y + xy^2}_{\text{}}) \cap (x^4 + 2x^3 + x^2 - xy + \underbrace{3y}_{\text{}})) \\
&= 3 \cdot 1 \\
&= 3
\end{aligned}$$

by Properties (5) and (7). Therefore,  $I(P, c \cap e) = 4 + 3 = 7$ .

(7) Show that  $I(P, c \cap f) = 10$ . Similar to (5). Let

$$q_7 = -x^6 - 3x^5 - x^3y^2 - 3x^4 - 3x^2y^2 - y^4 + 4x^2.$$

So

$$\begin{aligned}
& I(P, c \cap f) \\
&= I(P, c \cap \underbrace{(f + q_7 c)}_{\text{(Property (7))}}) \\
&= I(P, c \cap x^5(x^4 + 3x^3 + 3x^2 + x - 4)) \\
&\quad \text{(Property (6))} \\
&= \underbrace{5 I(P, c \cap x)}_{=2} + \underbrace{I(P, c \cap (x^4 + 3x^3 + 3x^2 + x - 4))}_{=0 \text{ (Property (2))}} \\
&= 10.
\end{aligned}$$

□

**Problem 3.18.**

Give a proof of Property (8) that uses only Properties (1)-(7).

Recall Properties (1)-(8):

- (1)  $I(P, f \cap g)$  is a nonnegative integer for any  $f, g$ , and  $P$  such that  $f$  and  $g$  intersect properly at  $P$ .  $I(P, f \cap g) = \infty$  if  $f$  and  $g$  do not intersect properly at  $P$ .
- (2)  $I(P, f \cap g) = 0$  if and only if  $P \notin f \cap g$ .  $I(P, f \cap g)$  depends only on the components of  $f$  and  $g$  that pass through  $P$ .
- (3) If  $t$  is an affine change of coordinates on  $\mathbf{A}^2$ , and  $t(Q) = P$ , then

$$I(P, f \cap g) = I(Q, f^t \cap g^t).$$

- (4)  $I(P, f \cap g) = I(P, g \cap f)$ .
- (5)  $I(P, f \cap g) \geq m_P(f)m_P(g)$ , with equality occurring if and only if  $f$  and  $g$  have not tangent lines in common at  $P$ .
- (6) If  $f = \prod f_i^{r_i}$ , and  $g = \prod g_j^{s_j}$ , then

$$I(P, f \cap g) = \sum_{i,j} r_i s_j I(P, f_i \cap g_j).$$

- (7)  $I(P, f \cap g) = I(P, f \cap (g + af))$  for any  $a \in k[x, y]$ .
- (8) If  $P$  is a simple point on  $f$ , then  $I(P, f \cap g) = \text{ord}_P^f(g)$ .

*Proof.*

- (1) Might assume that  $f$  is irreducible. There is nothing to prove if  $\text{ord}_P^f(g) = \infty$ . Might assume that  $n = \text{ord}_P^f(g) < \infty$ .
- (2) Similar to the proof of Theorem 1 in §3.2, Property (3) implies that we might assume that  $P = (0, 0)$ , that  $y$  is the tangent line, and that  $x$  is one line through  $P$  which is not tangent to  $f$  at  $P$ . Here  $x$  is a uniformizing parameter for  $\mathcal{O}_P(f)$  (Theorem 1 in §3.2).
- (3) By definition,

$$g = ux^n$$

for some unit in  $\mathcal{O}_P(f)$ . Write  $u = \frac{a}{b}$  where  $a, b \in k[x, y]$ ,  $a(P) \neq 0$  and  $b(P) \neq 0$ . Hence,

$$bg = ax^n + cf$$

in  $k[x, y]$  for some  $c \in k[x, y]$ .

- (4) Therefore,

$$\begin{aligned}
 I(P, f \cap g) &= I(P, f \cap bg) - I(P, f \cap b) && \text{(Property (6))} \\
 &= I(P, f \cap bg) - 0 && \text{(Property (2))} \\
 &= I(P, f \cap (ax^n + cf)) && \text{(Step (3))} \\
 &= I(P, f \cap ax^n) && \text{(Property (7))} \\
 &= I(P, f \cap a) + nI(P, f \cap x) && \text{(Property (6))} \\
 &= 0 + nI(P, f \cap x) && \text{(Property (2))} \\
 &= n. && \text{(Property (5))}
 \end{aligned}$$

□

### **Problem 3.19.\***

*A line  $L$  is tangent to a curve  $f$  at a point  $P$  if and only if  $I(P, f \cap L) > m_P(f)$ .*

*Proof.*

- (1) Note that  $m_P(L) = 1$  and the only tangent line of  $L$  is itself.
- (2) By Property (5),

$$\begin{aligned}
 I(P, f \cap L) &> m_P(f) = m_P(f)m_P(L) \\
 \iff f \text{ and } L &\text{ have one common tangent line at } P \\
 \iff L \text{ is tangent to a curve } f &\text{ at } P. && \text{(By (1))}
 \end{aligned}$$

□

**Problem 3.20.**

If  $P$  is a simple point on  $f$ , then  $I(P, f \cap (g+h)) \geq \min\{I(P, f \cap g), I(P, f \cap h)\}$ . Give an example to show that this may be false if  $P$  is not simple on  $f$ .

*Proof.*

(1)

$$\begin{aligned} I(P, f \cap (g+h)) &= \text{ord}_P^f(g+h) && \text{(Property (8))} \\ &\geq \min\{\text{ord}_P^f(g), \text{ord}_P^f(h)\} && \text{(Problem 2.28)} \\ &= \min\{I(P, f \cap g), I(P, f \cap h)\}. && \text{(Property (8))} \end{aligned}$$

(2) Pick  $P = (0, 0)$ ,  $f = (x^2 + y^2)^3 - 4x^2y^2$ ,  $g = x$  and  $h = y$ . By Property (5),  $I(P, f \cap (g+h)) = 4$ ,  $I(P, f \cap g) > 4$  and  $I(P, f \cap h) > 4$ . So

$$I(P, f \cap (g+h)) < \min\{I(P, f \cap g), I(P, f \cap h)\}$$

for such example.

□

**Problem 3.21.**

Let  $f$  be an affine plane curve. Let  $L$  be a line that is not a component of  $f$ . Suppose  $L = \{(a+tb, c+td) : t \in k\}$ . Define  $g(t) = f(a+tb, c+td)$ . Factor  $g(t) = \prod (t - \lambda_i)^{e_i}$ ,  $\lambda_i$  distinct. Show that there is a natural one-to-one correspondence between the  $\lambda_i$  and the points  $P_i \in L \cap f$ . Show that under this correspondence,  $I(P_i, L \cap f) = e_i$ . In particular,  $\sum I(P, L \cap f) \leq \deg(f)$ .

*Proof.*

(1) Show that there is a natural one-to-one correspondence between the  $\lambda_i$  and the points  $P_i \in L \cap f$ .

$$\begin{aligned} P_i \in L \cap f &\iff P_i \in L \text{ and } P \in f \\ &\iff \exists \lambda \in k \text{ such that } 0 = f(P_i) = f(a + \lambda b, c + \lambda d) = g(\lambda) \\ &\iff \lambda \in k \text{ is a root of } g(t) = \prod (t - \lambda_i)^{e_i} \\ &\iff \lambda = \lambda_i \in k \text{ for some } i. \end{aligned}$$

(2) Show that  $I(P_i, L \cap f) = e_i$ . By Property (3), we may suppose  $P_i = (0, 0)$  and  $L = y$ . Write

$$f(x, y) = f_0(x) + f_1(x)y + f_2(x)y^2 + \cdots \in (k[x])[y]$$



where  $f_j \in k[x]$ . Note that  $f_0(x) = f(x, 0) = g(x)$ . So

$$\begin{aligned}
I(P_i, L \cap f) &= I(P_i, y \cap (f_0(x) + f_1(x)y + \cdots)) \\
&= I(P_i, y \cap f_0(x)) && \text{(Property (7))} \\
&= I(P_i, y \cap g(x)) \\
&= I\left(P_i, y \cap \prod (x - \lambda_i)^{e_i}\right) \\
&= \sum_j e_j I(P_j, y \cap (x - \lambda_j)) && \text{(Property (6))} \\
&= e_i I(P_i, y \cap x) && \text{(Property (2))} \\
&= e_i. && \text{(Property (5))}
\end{aligned}$$

Here  $\lambda_i = 0$  by the correspondence of (1).

(3) In particular,

$$\sum_i I(P_i, L \cap f) = \sum_i e_i = \deg(g(x)) = \deg(f(x, 0)) \leq \deg(f(x, y)).$$

□

### Problem 3.22. (Cusp)

Suppose  $P$  is a double point on a curve  $f$ , and suppose  $f$  has only one tangent  $L$  at  $P$ .

- (a) Show that  $I(P, f \cap L) \geq 3$ . The curve  $f$  is said to have an (ordinary) **cusp** at  $P$  if  $I(P, f \cap L) = 3$ .
- (b) Suppose  $P = (0, 0)$ , and  $L = y$ . Show that  $P$  is a cusp if and only if  $\frac{\partial^3 f}{\partial x^3}(P) \neq 0$ . Give some examples.
- (c) Show that if  $P$  is a cusp on  $f$ , then  $f$  has only one component passing through  $P$ .

Might assume that  $\text{char}(k) = 0$ .

*Proof of (a).* Since  $I(P, f \cap L) > m_P(f) = 2$  (Problem 3.19),  $I(P, f \cap L) \geq 3$  (Property (1)). □

*Proof of (b).*

(1) By assumption,

$$f = y^2 + f_3 + f_4 + \cdots$$

where  $f_i$  is a form in  $k[x, y]$ .

(2) Hence,  $P$  is a cusp of  $f$  if and only if

$$\begin{aligned}
3 &= I(P, f \cap y) \\
&= I(P, (y^2 + f_3 + f_4 + \cdots) \cap y) \\
&= I(P, (f_3 + f_4 + \cdots) \cap y) \\
&\geq m_P(f_3 + f_4 + \cdots)m_P(y) \\
&\geq 3.
\end{aligned}$$

Here the equality is occurring if and only if  $y$  is not a tangent line of  $f_3 + f_4 + \cdots$  (Property (5)).

(3) Note that

$$\begin{aligned}
&y \text{ is not a tangent line of } f_3 + f_4 + \cdots \\
&\iff y \nmid f_3 \\
&\iff \frac{\partial^3 f}{\partial x^3}(P) \neq 0.
\end{aligned} \tag{Problem 3.5}$$

(4) Examples:  $y^2 = x^3$ ,  $y^2 = -x^2y^2 + x^3$  and so on.

□

*Proof of (c).*

(1) Might assume  $P = (0, 0)$  and  $L = y$  by Property (3).

(2) Given  $f = gh$ . Write

$$\begin{aligned}
f &= y^2 + \text{higher terms} \\
g &= g_r + \text{higher terms} \\
h &= h_s + \text{higher terms}
\end{aligned}$$

where  $g_r$  (resp.  $h_s$ ) is a form of degree  $r$  (resp.  $s$ ) in  $k[x, y]$ . So  $f = gh$  implies that

$$\begin{aligned}
y^2 + \text{higher terms} &= (g_r + \text{higher terms})(h_s + \text{higher terms}) \\
&= (g_r h_s) + \text{higher terms}.
\end{aligned}$$

Hence  $y^2 = g_r h_s$ . In particular,  $2 = r + s$ .

(3) If  $y \mid g_r$  and  $y \mid h_s$ , then

$$\begin{aligned}
I(P, g \cap L) &> m_P(g)m_P(L) = r \implies I(P, g \cap L) \geq r + 1 \\
I(P, h \cap L) &> m_P(h)m_P(L) = s \implies I(P, h \cap L) \geq s + 1.
\end{aligned}$$

So,

$$\begin{aligned}
3 &= I(P, f \cap L) \\
&= I(P, g \cap L) + I(P, h \cap L) \\
&\geq (r+1) + (s+1) \\
&= 4,
\end{aligned}$$

which is absurd. So we might assume that  $g_r = 1$  and  $h_s = y^2$ .  $f = gh$  implies that  $g = 1 + g_1 + \cdots$  is not passing through  $P$ . Hence the conclusion is established.

□

**Problem 3.23. (Hypercusp)**

A point  $P$  on a curve  $f$  is called a **hypercusp** if  $m_P(f) > 1$ ,  $f$  has only one tangent line  $L$  at  $P$ , and  $I(P, L \cap f) = m_P(f) + 1$ . Generalize the results of the preceding problem to this case.

*Generalization.*

- (a)  $I(P, f \cap L) \geq m_P(f) + 1$ .
- (b) Suppose  $P = (0, 0)$ ,  $L = y$  and  $m = m_P(f)$ . Then  $P$  is a hypercusp if and only if  $\frac{\partial^{m+1} f}{\partial x^{m+1}}(P) \neq 0$ . Give some examples.
- (c) If  $P$  is a hypercusp on  $f$ , then  $f$  has only one component passing through  $P$ .

The proof is almost the same as Problem 3.22 by replacing 2 by  $m_P(f)$ . □

**Problem 3.24.\***

The object of this problem is to find a property of the local ring  $\mathcal{O}_P(f)$  that determines whether or not  $P$  is an ordinary multiple point on  $f$ . Let  $f$  be an irreducible plane curve,  $P = (0, 0)$ ,  $\mathfrak{m} = \mathfrak{m}_P(f) > 1$ . Let  $m = m_P(f)$ . For  $g \in k[x, y]$  or  $\in \Gamma(f)$ , denote its residue in  $\mathfrak{m}/\mathfrak{m}^2$  by  $\bar{g}$ .

- (a) Show that the map from  $V = \{\text{forms of degree 1 in } k[x, y]\}$  to  $\mathfrak{m}/\mathfrak{m}^2$  taking  $ax + by$  to  $\bar{ax + by}$  is an isomorphism of vector spaces (see Problem 3.13).
- (b) Suppose  $P$  is an ordinary multiple point, with tangents  $L_1, \dots, L_m$ . Show that  $I(P, f \cap L_i) > m$  and  $\bar{L}_i \neq \lambda \bar{L}_j$  for all  $i \neq j$ , and all  $\lambda \in k$ .

- (c) Suppose there are  $g_1, \dots, g_m \in k[x, y]$  such that  $I(P, f \cap g_i) > m$  and  $\overline{g_i} \neq \lambda \overline{g_j}$  for all  $i \neq j$ , and all  $\lambda \in k$ . Show that  $P$  is an ordinary multiple point on  $f$ . (Hint: Write  $g_i = L_i + \text{higher terms} \in k[x, y]$ .  $\overline{L_i} = \overline{g_i} \neq 0$ , and  $L_i$  is the tangent to  $g_i$ , so  $L_i$  is tangent to  $f$  by Property (5) of intersection numbers. Thus  $f$  has  $m$  tangents at  $P$ .)
- (d) Show that  $P$  is an ordinary multiple point on  $f$  if and only if there are  $g_1, \dots, g_m \in \mathfrak{m}$  such that  $\overline{g_i} \neq \lambda \overline{g_j}$  for all  $i \neq j$ ,  $\lambda \in k$ , and

$$\dim_k \mathcal{O}_P(f)/(g_i) > m.$$

*Proof of (a).*

- (1)  $\mathcal{B} = \{x, y\}$  is a basis for  $V$  as a  $k$ -vector space.
- (2)  $\mathcal{B}' = \{\overline{x}, \overline{y}\}$  is a basis for  $\mathfrak{m}/\mathfrak{m}^2$  as a  $k$ -vector space.
- (3) By (1)(2), we can define a canonical isomorphism

$$\alpha : V \rightarrow \mathfrak{m}/\mathfrak{m}^2$$

by sending  $\mathcal{B}$  to  $\mathcal{B}'$ , that is,  $\alpha(x) = \overline{x}$  and  $\alpha(y) = \overline{y}$ .

□

*Proof of (b).*

- (1) Write

$$f = \prod_{i=1}^m L_i + \text{higher terms}.$$

Problem 3.19 says that  $I(P, f \cap L_i) > m_P(f) = m$ .

- (2) Since  $P$  is an ordinary multiple point on  $f$ ,  $L_i$  and  $L_j$  are linearly independent in  $V$  in the sense of (a). Hence,  $\alpha(L_i) = \overline{L_i}$  and  $\alpha(L_j) = \overline{L_j}$  are linearly independent in  $\mathfrak{m}/\mathfrak{m}^2$  (since  $\alpha$  is an isomorphism). The conclusion holds.

□

*Proof of (c).*

- (1) Write  $g_i = L_i + \text{higher terms} \in k[x, y]$ . Here  $g_i$  has no constant term since  $P \in g_i$  by the definition of intersection numbers.
- (2) Pick  $\lambda = 0 \in k$  and  $j \neq i$ . ( $m > 1$  implies the existence of  $j$ .) So

$$\alpha(L_i) = \overline{L_i} = \overline{g_i} \neq \lambda \overline{g_j} = 0$$

or  $L_i \neq 0$  (since  $\alpha$  is an isomorphism).

- (3) Hence,  $L_i$  is the tangent to  $g_i$ . So Property (5) implies that

$$I(P, f \cap g_i) \geq m_P(f)m_P(g_i) = m.$$

Note that  $L_i$  is the only tangent line of  $g_i$ . By Property (5), the assumption  $I(P, f \cap g_i) > m$  implies that  $L_i$  is tangent to  $f$ .

- (4) Note that  $\overline{g_i} \neq \lambda \overline{g_j}$  for all  $i \neq j$ , and all  $\lambda \in k$ . So  $\overline{L_i} \neq \lambda \overline{L_j}$  for all  $i \neq j$ , and all  $\lambda \in k$ . Since  $\alpha$  is an isomorphism, all  $L_i$  are linearly independent and thus  $f$  has  $m$  tangents  $L_1, \dots, L_m$  at  $P$ . Therefore,  $P$  is an ordinary multiple point.

□

*Proof of (d).*

- (1) Note that

$$\dim_k \mathcal{O}_P(f)/(g_i) = \dim_k \mathcal{O}_P(\mathbf{A}^2)/(f, g_i) = I(P, f \cap g_i)$$

(by Problem 2.44).

- (2) ( $\implies$ ) Suppose that  $P$  is an ordinary multiple point, with tangents  $L_1, \dots, L_m$ . By (b),

$$\dim_k \mathcal{O}_P(f)/(L_i) = I(P, f \cap L_i) > m$$

and  $\overline{L_i} \neq \lambda \overline{L_j}$  for all  $i \neq j$ , and all  $\lambda \in k$ . Take

$$g_i = L_i + I(f) \in \Gamma(f) \subseteq \mathcal{O}_P(f).$$

Since  $g_i \in \mathfrak{m}$  (by  $L_i(P) = 0$ ) and  $\overline{g_i} = \overline{L_i}$ , the conclusion is proved.

- (3) ( $\impliedby$ ) Suppose that there are  $g_1, \dots, g_m \in \mathfrak{m}$  such that  $\overline{g_i} \neq \lambda \overline{g_j}$ , and  $\dim_k \mathcal{O}_P(f)/(g_i) > m$ . For each  $i = 1, \dots, m$ , we take  $g'_i + I(f) = \underline{g_i} \in \mathfrak{m}$  for some  $g'_i \in k[x, y]$ . Although  $g'_i$  is not uniquely determined by  $g_i$ ,  $\overline{g'_i} = \overline{g_i}$  and thus  $\overline{g'_i} \neq \lambda \overline{g'_j}$ . By (1),

$$\dim_k \mathcal{O}_P(f)/(g_i) = \dim_k \mathcal{O}_P(f)/(g'_i) = I(P, f \cap g'_i) > m$$

Hence, by (c)  $f$  has  $m$  tangents at  $P$ .

□

## Chapter 4: Projective Varieties

### 4.1. Projective Space

#### Problem 4.1.

What points in  $\mathbf{P}^2$  do not belong to two of the three sets  $U_1, U_2, U_3$ ?

*Proof.*

- (1) The point  $[1 : 0 : 0]$  does not belong to  $U_2$  and  $U_3$ .
- (2) The point  $[0 : 1 : 0]$  does not belong to  $U_3$  and  $U_1$ .
- (3) The point  $[0 : 0 : 1]$  does not belong to  $U_1$  and  $U_2$ .

□

#### Problem 4.2.\*

Let  $f \in k[x_1, \dots, x_{n+1}]$  ( $k$  infinite). Write  $f = \sum f_i$ ,  $f_i$  a form of degree  $i$ . Let  $P \in \mathbf{P}^n(k)$ , and suppose  $f(x_1, \dots, x_{n+1}) = 0$  for every choice of homogeneous coordinates  $(x_1, \dots, x_{n+1})$  for  $P$ . Show that each  $f_i(x_1, \dots, x_{n+1}) = 0$  for all homogeneous coordinates for  $P$ . (Hint: consider

$$g(\lambda) = f(\lambda x_1, \dots, \lambda x_{n+1}) = \sum \lambda^i f_i(x_1, \dots, x_{n+1})$$

for fixed  $(x_1, \dots, x_{n+1})$ .)

*Proof.*

- (1) Consider

$$g(\lambda) = f(\lambda x_1, \dots, \lambda x_{n+1}) = \sum \lambda^i f_i(x_1, \dots, x_{n+1})$$

for fixed  $(x_1, \dots, x_{n+1})$ .  $g(\lambda)$  is a polynomial in  $k[\lambda]$ .

- (2) Since  $g(\lambda) = f(\lambda x_1, \dots, \lambda x_{n+1}) = 0$  for all  $\lambda \in k - \{0\}$ ,  $g(\lambda) = 0$  has infinitely many solutions in  $k$ . Similar to Problem 1.4,  $g = 0 \in k[\lambda]$ , that is, each  $f_i(x_1, \dots, x_{n+1}) = 0$  for all homogeneous coordinates for  $P$ .

□

**Problem 4.3.**

- (a) *Show that the definitions of this section carry over without change to the case where  $k$  is an arbitrary field.*
- (b) *If  $k_0$  is a subfield of  $k$ , show that  $\mathbf{P}^n(k_0)$  may be identified with a subset of  $\mathbf{P}^n(k)$ .*

*Proof of (a).* Note that a field is a commutative ring where  $0 \neq 1$  and all nonzero elements are invertible. Hence the definitions in this section are well-defined for any field  $k$ .  $\square$

*Proof of (b).* Note that  $0 \in k_0$  and  $0 \in k$ . So any point  $P \in \mathbf{P}^n(k_0)$  is also in  $\mathbf{P}^n(k)$  since  $P \neq (0, \dots, 0)$  and

$$\{(\lambda x_1, \dots, \lambda x_{n+1}) : \lambda \in k_0\} \subseteq \{(\lambda x_1, \dots, \lambda x_{n+1}) : \lambda \in k\}$$

as a subset.  $\square$

**4.2. Projective Algebraic Sets****Problem 4.4.\***

*Let  $I$  be a homogeneous ideal in  $k[x_1, \dots, x_{n+1}]$ . Show that  $I$  is prime if and only if the following condition is satisfied: for any forms  $f, g \in k[x_1, \dots, x_{n+1}]$ , if  $fg \in I$ , then  $f \in I$  or  $g \in I$ .*

*Proof.*

- (1)  $(\implies)$  Trivial.
- (2)  $(\impliedby)$  Suppose that  $f, g \in k[x_1, \dots, x_{n+1}]$  and  $fg \in I$ . Write  $f = \sum_{i=0}^r f_i$  (resp.  $g = \sum_{j=0}^s g_j$ ),  $f_i$  a form of degree  $i$  (resp.  $g_j$  a form of degree  $j$ ). Induction on the  $\deg(fg) = r + s$ .
- (3) When  $r + s = 0$ , nothing to do.
- (4) Assume that the result is true for smaller values of  $r + s$ . Then the highest homogeneous component  $f_r g_s$  of  $fg$  is also in  $I$  since  $I$  is homogeneous. By assumption,  $f_r \in I$  or  $g_s \in I$ , and might say that  $f_r \in I$ . Therefore,

$$(f - f_r)g \in I.$$

By the induction hypothesis,  $f - f_r \in I$  or  $g \in I$ . Hence,  $f = (f - f_r) + f_r \in I$  or  $g \in I$ .

(5) Therefore, (3)(4) implies that  $I$  is prime.

□

**Problem 4.5.**

*If  $I$  is a homogeneous ideal, show that  $\text{rad}(I)$  is also homogeneous.*

*Proof.*

- (1) Given any  $f = \sum_{i=0}^r f_i \in \text{rad}(I)$ ,  $f_i$  a form of degree  $i$ . It suffices to show that each  $f_i \in \text{rad}(I)$ . Note that  $f^m \in I$  for some  $m > 0$ .
- (2) The highest homogeneous component  $f_r^m$  of  $f^m$  is also in  $I$  since  $I$  is homogeneous. Hence,  $f_r \in \text{rad}(I)$ . Again note that  $f - f_r \in \text{rad}(I)$  and  $\deg(f - f_r) < r$ . Continue this process (or by induction), and we have  $f_{r-1}, \dots, f_0 \in \text{rad}(I)$ .

□

**Problem 4.6.**

*State and prove the projective analogues of properties (1)-(10) of Chapter 1, Sections 2 and 3.*

*Statements.*

- (1) *If  $I$  is the homogeneous ideal in  $k[x_1, \dots, x_{n+1}]$  generated by a set of forms  $S$ , then  $V(S) = V(I)$ ; so every algebraic set is equal to  $V(I)$  for some homogeneous ideal  $I$ .*
- (2) *If  $\{I_\alpha\}$  is any collection of homogeneous ideals, then  $V(\cup_\alpha I_\alpha) = \cap_\alpha V(I_\alpha)$ ; so the intersection of any collection of algebraic sets is an algebraic set.*
- (3) *If  $I \subseteq J$ , then  $V(I) \supseteq V(J)$ .*
- (4)  *$V(fg) = V(f) \cup V(g)$  for any forms  $f, g$ ;  $V(I) \cup V(J) = V(\{fg : f \in I, g \in J\})$ ; so any finite union of algebraic sets is an algebraic set.*
- (5)  *$V(0) = \mathbf{P}^n(k)$ ;  $V(I) = \emptyset$  if  $I$  contains all forms of degree  $\geq N$  for some  $N$ ;  $V(a_i x_j - a_j x_i) = \{[a_1 : \dots : a_{n+1}]\}$  for  $[a_1 : \dots : a_{n+1}] \in \mathbf{P}^n(k)$ . So any finite subset of  $\mathbf{P}^n(k)$  is an algebraic set.*
- (6) *If  $X \subseteq Y$  are nonempty, then  $I(X) \supseteq I(Y)$ .*
- (7)  *$I(\emptyset) = k[x_1, \dots, x_{n+1}]$ ;  $I(\mathbf{P}^n(k)) = (0)$  if  $k$  is an infinite field;*



- (8)  $I(V(S)) \supseteq S$  for any set  $S$  of forms;  $V(I(X)) \supseteq X$  for any set  $X$  of points.
- (9)  $V(I(V(S))) = V(S)$  for any set  $S$  of forms, and  $I(V(I(X))) = I(X)$  for any set  $X$  of points. So if  $V$  is an algebraic set,  $V = V(I(V))$ , and if  $I$  is the homogeneous ideal of an algebraic set,  $I = I(V(I))$ .
- (10)  $I(X)$  is a radical ideal for any nonempty  $X \subseteq \mathbf{P}^n(k)$ .

*Proof.* Proposition 1 and the projective Nullstellensatz give all.  $\square$

**Problem 4.7.**

*Show that each irreducible component of a cone is also a cone.*

*Proof.*

- (1) Let  $V$  is an algebraic set in  $\mathbf{P}^n$ , and  $C(V)$  be the cone over  $V$ . Might assume that  $V \neq \emptyset$ .
- (2) *Show that  $V$  is irreducible if and only if  $C(V)$  is irreducible.*  $V$  is irreducible if and only if  $I_p(V) = I_a(C(V))$  is prime if and only if  $C(V)$  is irreducible.
- (3) Let  $V = V_1 \cup \cdots \cup V_r$  be the decomposition of an algebraic set into irreducible components. Note that

$$C(V_1 \cup \cdots \cup V_r) = C(V_1) \cup \cdots \cup C(V_r)$$

(by the definition of the cone). Here each  $C(V_i)$  is irreducible (by (2)). By Theorem 2 in §1.5, each irreducible component of  $C(V)$  must be one of  $C(V_i)$ , which is also a cone.

$\square$

**Problem 4.8.**

*Let  $V = \mathbf{P}^1$ ,  $\Gamma_h(V) = k[x, y]$ . Let  $t = x/y \in k(V)$ , and show that  $k(V) = k(t)$ . Show that there is a natural one-to-one correspondence between the points of  $\mathbf{P}^1$  and the DVRs with quotient field  $k(V)$  that contain  $k$  (see Problem 2.27); which DVR corresponds to the point at infinity?*

*Proof.*

- (1) *Show that  $k(V) = k(t)$ . Given any  $f/g \in k(V)$  where  $f, g \in \Gamma_h(V)$  are of the same degree. Then*

$$f(x, y)/g(x, y) = f(t, 1)/g(t, 1) \in k(t).$$

Conversely, given any  $f/g \in k(t)$ ,

$$\frac{f(t)}{g(t)} = \frac{f(x/y)}{g(x/y)} = \frac{y^d f(x/y)}{y^d g(x/y)} \in k(V)$$

where  $d = \max\{\deg(f), \deg(g)\}$ .

- (2) Note that  $k = \bar{k}$ . By Problem 2.27, the DVR's with quotient field  $k(V) = k(t)$  are

$$\mathcal{O}_a(\mathbf{A}^1) \text{ where } a \in \mathbf{A}^1 = k \text{ and } \mathcal{O}_\infty(\mathbf{A}^1),$$

which correspond to

$$\mathbf{P}^1(k) = \{[a : 1] : a \in k\} \cup \{[1 : 0]\}.$$

In particular, the DVR  $\mathcal{O}_\infty(\mathbf{A}^1)$  corresponds to the point at infinity ( $= [1 : 0]$ ).

□

**Problem 4.9.\***

Let  $I$  be a homogeneous ideal in  $k[x_1, \dots, x_{n+1}]$ , and

$$\Gamma = k[x_1, \dots, x_{n+1}]/I.$$

Show that the forms of degree  $d$  in  $\Gamma$  form a finite-dimensional vector space over  $k$ .

*Proof.*

- (1) Write  $R = k[x_1, \dots, x_{n+1}]$ . For  $R$  (resp.  $\Gamma$ ), define  $R_{(d)}$  (resp.  $\Gamma_{(d)}$ ) to be the corresponding homogeneous component of degree  $d$ . Consider a natural homomorphism

$$\alpha : R_{(d)} \rightarrow R_{(d)}/I \cong \Gamma_{(d)}$$

by  $\alpha(h) = \bar{h}$  for any form  $h$  of degree  $d$ .

- (2)  $\Gamma_{(d)}$  can be regarded as a subspace of  $R_{(d)}$  since  $\alpha$  is surjective. Since  $R_{(d)}$  is finite-dimensional with  $\dim R_{(d)} = \binom{d+n-1}{n-1}$ ,  $\Gamma_{(d)}$  is finite-dimensional by linear algebra.

□

**Problem 4.10.**

Let  $R = k[x, y, z]$ ,  $f \in R$  an irreducible form of degree  $n$ ,  $V = V(f) \subseteq \mathbf{P}^2$ , and  $\Gamma = \Gamma_h(V)$ .

(a) Construct an exact sequence

$$0 \rightarrow R \xrightarrow{\psi} R \xrightarrow{\varphi} \Gamma \rightarrow 0$$

where  $\psi$  is multiplication by  $f$ .

(b) Show that

$$\dim_k \{\text{forms of degree } d \text{ in } \Gamma\} = dn - \frac{n(n-3)}{2}$$

if  $d > n$ .

*Proof of (a).*

- (1)  $\psi$  is defined by  $\psi(g) = fg$  and  $\varphi$  is naturally defined by  $\varphi(h) = \bar{h}$ .  $\psi$  and  $\varphi$  are well-defined and homomorphisms of vector space.
- (2)  $\text{im}(\psi) = \ker(\varphi) = (f) = I(f)$  (since  $f$  is irreducible).  $\psi$  is injective since  $R = k[x, y, z]$  is a domain.  $\varphi$  is surjective trivially. Hence, the sequence of vector spaces over  $k$  is exact.

□

*Proof of (b).*

(1) The exact sequence

$$0 \rightarrow R \xrightarrow{\psi} R \xrightarrow{\varphi} \Gamma \rightarrow 0$$

induces the exact sequence

$$0 \rightarrow R_{(d-n)} \xrightarrow{\psi} R_{(d)} \xrightarrow{\varphi} \Gamma_{(d)} \rightarrow 0,$$

where  $*_{(d)}$  (resp.  $*_{(d-n)}$ ) denotes the corresponding homogeneous component of degree  $d$  (resp.  $d-n$ ).

(2) By Problem 2.36,

$$\dim_k R_{(d-n)} = \binom{d-n+2}{2}, \quad \dim_k R_{(d)} = \binom{d+2}{2}.$$

(3) Since  $R_{(d)}$  is finite-dimensional,  $\Gamma_{(d)}$  is also finite-dimensional (by regarding  $\Gamma_{(d)}$  as a subspace of  $R_{(d)}$ ). Proposition 7 in §2.9 shows that

$$\begin{aligned} \dim_k \Gamma_{(d)} &= \dim_k R_{(d)} - \dim_k R_{(d-n)} \\ &= \binom{d+2}{2} - \binom{d-n+2}{2} \\ &= dn - \frac{n^2}{2} + \frac{3n}{2}. \end{aligned}$$

□

**Problem 4.11.\* (Linear subvariety)**

A set  $V \subseteq \mathbf{P}^n(k)$  is called a **linear subvariety** of  $\mathbf{P}^n(k)$  if  $V = V(h_1, \dots, h_r)$ , where each  $h_i$  is a form of degree 1.

- (a) Show that if  $t$  is a projective change of coordinates, then  $V^t = t^{-1}(V)$  is also a linear subvariety.
- (b) Show that there is a projective change of coordinates  $t$  of  $\mathbf{P}^n$  such that  $V^t = V(x_{m+2}, \dots, x_{n+1})$ , so  $V$  is a variety.
- (c) Show that the  $m$  that appears in part (b) is independent of the choice of  $t$ . It is called the **dimension** of  $V$  ( $m = -1$  if  $V = \emptyset$ ).

*Proof of (a).*

- (1) Say  $t = (t_1, \dots, t_{n+1})$  is a projective change of coordinates, and  $V = V(h_1, \dots, h_r)$ , where each  $h_i$  is a form of degree 1.
- (2) Show that  $V$  is a variety and thus  $I(V) = (h_1, \dots, h_r)$  by the projective Nullstellensatz.  $V$  is the set of all non-trivial solutions of the system of linear equations:

$$\begin{aligned} h_1 &= a_{1,1}x_1 + \dots + a_{1,n+1}x_{n+1} = 0, \\ &\dots \\ h_r &= a_{r,1}x_1 + \dots + a_{r,n+1}x_{n+1} = 0. \end{aligned}$$

(Here we identify  $[x_1 : \dots : x_{n+1}] \in \mathbf{P}^n$ .) Write  $Ax = 0$  and thus  $V = V(Ax = 0)$ , where

$$A = \underbrace{\begin{pmatrix} a_{1,1} & \dots & a_{1,n+1} \\ \vdots & \ddots & \vdots \\ a_{r,1} & \dots & a_{r,n+1} \end{pmatrix}}_{\in \mathbf{M}_{r \times (n+1)}(k)}, \quad x = \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix}}_{\in \mathbf{M}_{(n+1) \times 1}(k)}.$$

- (3) The Gaussian elimination in linear algebra says that  $(A|0)$  has the same solutions as its reduced row echelon form  $(A'|0)$ , that is,  $V(Ax = 0) = V(A'x = 0)$ .
- (4) If  $V(h_1, \dots, h_r) = \emptyset$ , nothing to do. If  $V(h_1, \dots, h_r) \neq \emptyset$ , then

$$V(h_1, \dots, h_r) = V(g_1, \dots, g_{m+1})$$

where  $m+1 = \text{rank}(A)$  is the number of nonzero rows in  $A'$  ( $m+1 \leq r, n+1$ ) and  $g_i = a'_{i,1}x_1 + \dots + a'_{i,n+1}x_{n+1}$  for  $1 \leq i \leq m+1$ . ( $a'_{i,j}$  is the entry of the matrix  $A'$ .)

- (5) Now given any  $f + I(V) \in k[x_1, \dots, x_{n+1}]/I(V)$ , we replace the leading term  $x_{i_1}$  of  $g_1$  by  $x_{i_1} - g_1$  to get

$$f + I(V) = f(x_1, \dots, \underbrace{x_{i_1} - g_1}_{i_1 \text{th position}}, \dots, x_{n+1}) + I(V) := f_1 + I(V)$$

where  $f_1 \in k[x_1, \dots, \widehat{x_{i_1}} \dots, x_{n+1}]$ . Continue this process to replace each leading term  $x_{i_j}$  of  $g_j$  by  $x_{i_j} - g_j$  to get one by one to get

$$f + I(V) = f_1 + I(V), f_1 \in k[x_1, \dots, \widehat{x_{i_1}} \dots, x_{n+1}].$$

...

$$f_m + I(V) = f_{m+1} + I(V), f_{m+1} \in k[x_1, \dots, \widehat{x_{i_1}}, \dots, \widehat{x_{i_{m+1}}} \dots, x_{n+1}].$$

Hence, a routine shows that there is a ring isomorphism

$$\alpha : k[x_1, \dots, x_{n+1}]/I(V) \rightarrow \underbrace{k[x_1, \dots, \widehat{x_{i_1}}, \dots, \widehat{x_{i_{m+1}}} \dots, x_{n+1}]}_{\text{a domain}}$$

sending  $f$  to  $f_{m+1}$ . Therefore,  $V$  is a variety.

- (6) As  $I(V) = (h_1, \dots, h_r)$ ,  $I(V)^t = (h_1^t, \dots, h_r^t)$  where each  $h_i^t$  is a form of degree 1. Thus  $V^t = V(I(V)^t) = V(h_1^t, \dots, h_r^t)$  is also a linear subvariety of  $\mathbf{P}^n(k)$ .

□

*Proof of (b).*

- (1) Suppose  $A \in \mathbf{M}_{r \times (n+1)}(k)$  is of rank  $(n+1) - (m+1) = n - m$ . Linear algebra says that there exist invertible matrices  $B \in \mathbf{M}_{r \times r}(k)$  and  $C \in \mathbf{M}_{(n+1) \times (n+1)}(k)$  such that  $D = BAC$ , where

$$D = BAC = \underbrace{\begin{pmatrix} O_1 & O_2 \\ O_3 & I_{n-m} \end{pmatrix}}_{\in \mathbf{M}_{r \times (n+1)}(k)}$$

in which  $I_{n-m} \in \mathbf{M}_{(n-m) \times (n-m)}(k)$  is the identity matrix and  $O_1, O_2$ , and  $O_3$  are zero matrices.

- (2) Let  $t'$  be the linear map corresponding to the matrix  $C$ . So

$$\begin{aligned} V^{t'} &= V(Ax = 0)^{t'} \\ &= V(ACx = 0) \\ &= V(BACx = 0) && (B: \text{invertible}) \\ &= V(Dx = 0) \\ &= V(0, \dots, 0, x_{m+2}, \dots, x_{n+1}) && (V \neq \emptyset) \\ &= V(x_{m+2}, \dots, x_{n+1}). \end{aligned}$$

□

*Proof of (c).* Linear algebra says that the rank of any matrix is uniquely determined. Therefore,  $m = n - \text{rank}(A)$  is uniquely determined. □

**Problem 4.12.\***

Let  $H_1, \dots, H_m$  be hyperplanes in  $\mathbf{P}^n$ ,  $m \leq n$ . Show that

$$H_1 \cap H_2 \cap \cdots \cap H_m \neq \emptyset.$$

*Proof.*

(1) Let

$$H_i : a_{i,1}x_1 + \cdots + a_{i,n+1}x_{n+1} = 0$$

for  $i = 1, 2, \dots, m$ .

(2) View (1) as the system of linear equations. Let the coefficient matrix  $A$  be

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n+1} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n+1} \end{pmatrix}.$$

Note that  $\text{rank}(A) \leq \min\{m, n+1\} = m$ . The rank-nullity theorem shows that

$$\dim_k \ker(A) = (n+1) - \text{rank}(A) \geq (n+1) - m \geq 1.$$

Hence, there is a nonzero solution of  $\bigcap_{i=1}^m H_i$ , or  $\bigcap_{i=1}^m H_i \neq \emptyset \in \mathbf{P}^n$ .

□

**Problem 4.13.\* (Line)**

Let  $P = [a_1 : \cdots : a_{n+1}]$ ,  $Q = [b_1 : \cdots : b_{n+1}]$  be distinct points of  $\mathbf{P}^n$ . The **line**  $L$  through  $P$  and  $Q$  is defined by

$$L = \{[\lambda a_1 + \mu b_1 : \cdots : \lambda a_{n+1} + \mu b_{n+1}] : \lambda, \mu \in k, \lambda \neq 0 \text{ or } \mu \neq 0\}.$$

Prove the projective analogue of Problem 2.15.

- (a) Show that if  $L$  is the line through  $P$  and  $Q$ , and  $t$  is a projective change of coordinates, then  $t(L)$  is the line through  $t(P)$  and  $t(Q)$ .
- (b) Show that a line is a linear subvariety of dimension 1, and that a linear subvariety of dimension 1 is the line through any two of its points.

- (c) Show that, in  $\mathbf{P}^2$ , a line is the same thing as a hyperplane.
- (d) Let  $P, P' \in \mathbf{P}^2$ ,  $L_1, L_2$  two distinct lines through  $P$ ,  $L'_1, L'_2$  distinct lines through  $P'$ . Show that there is a projective change of coordinates  $t$  of  $\mathbf{P}^2$  such that  $t(P) = P'$  and  $t(L_i) = L'_i$ ,  $i = 1, 2$ .

*Proof of (a).*

- (1) Write  $t = (t_1, \dots, t_{n+1})$  as

$$t_i = \sum_j c_{ij} x_j.$$

Given any point  $P_{\lambda, \mu} = [\lambda a_1 + \mu b_1 : \dots : \lambda a_{n+1} + \mu b_{n+1}] \in L$  for some not all zeros  $\lambda, \mu \in k$ . (In particular,  $P_{1,0} = P$  and  $P_{0,1} = Q$ .)

- (2) As

$$\begin{aligned} t_i(P_{\lambda, \mu}) &= \sum_j c_{ij} (\lambda a_j + \mu b_j) \\ &= \lambda \sum_j c_{ij} a_j + \mu \sum_j c_{ij} b_j \\ &= \lambda t_i(P) + \mu t_i(Q), \end{aligned}$$

we have

$$\begin{aligned} t(L) &= \{[\lambda t_1(P) + \mu t_1(Q) : \dots : \lambda t_{n+1}(P) + \mu t_{n+1}(Q)] \\ &\quad : \lambda, \mu \in k, \lambda \neq 0 \text{ or } \mu \neq 0\}. \end{aligned}$$

Moreover,  $t(P) \in t(L)$  as  $(\lambda, \mu) = (1, 0)$ ,  $t(Q) \in t(L)$  as  $(\lambda, \mu) = (0, 1)$ , and  $t(P) \neq t(Q)$  (since  $P \neq Q$  and  $t$  is a projective change of coordinates.) Therefore,  $t(L)$  is the line through  $t(P)$  and  $t(Q)$ .

□

*Proof of (b).*

- (1) First, write  $L$  as the system of equations

$$x_i = \lambda a_i + \mu b_i$$

( $i = 1, \dots, n+1$ ) where  $\lambda, \mu \in k, \lambda \neq 0$  or  $\mu \neq 0$ . Since  $P \neq Q \in \mathbf{P}^n$ , there exist  $1 \leq \alpha, \beta \leq n$  such that  $a_\alpha b_\beta - a_\beta b_\alpha \neq 0$ . So we can solve  $\lambda$  and  $\mu$  in terms of  $x_\alpha$  and  $x_\beta$  by Cramer's rule, say

$$\lambda = \frac{x_\alpha b_\beta - x_\beta b_\alpha}{a_\alpha b_\beta - a_\beta b_\alpha}, \quad \mu = \frac{a_\alpha x_\beta - a_\beta x_\alpha}{a_\alpha b_\beta - a_\beta b_\alpha}.$$

(2) Define

$$\begin{aligned} V &= V \left( x_i = \frac{x_\alpha b_\beta - x_\beta b_\alpha}{a_\alpha b_\beta - a_\beta b_\alpha} a_i + \frac{a_\alpha x_\beta - a_\beta x_\alpha}{a_\alpha b_\beta - a_\beta b_\alpha} b_i : 1 \leq i \leq n+1 \right) \\ &= V \left( \begin{vmatrix} x_i & a_i & b_i \\ x_\alpha & a_\alpha & b_\alpha \\ x_\beta & a_\beta & b_\beta \end{vmatrix} = 0 : 1 \leq i \leq n+1 \right). \end{aligned}$$

By construction,  $L = V$  is a linear subvariety in  $\mathbf{P}^n$ . (See Problem 4.11.)

(3) Might assume that  $\alpha = n$  and  $\beta = n+1$ . View

$$\begin{vmatrix} x_i & a_i & b_i \\ x_\alpha & a_\alpha & b_\alpha \\ x_\beta & a_\beta & b_\beta \end{vmatrix} = 0, i = 1, \dots, n+1$$

as the system of linear equations. Write  $A$  as the corresponding coefficient matrix.  $A$  is a reduced row echelon form of rank  $(n+1) - 2 = n-1$ . So  $\dim(V) = n - \text{rank}(A) = n - (n-1) = 1$ .

(4) Conversely,  $\dim(V) = 1$  implies that  $\text{rank}(A'|0) = n-1$ . So all leading terms are all  $x_i$  except two  $x_\alpha, x_\beta$  for some  $\alpha \neq \beta$ , might say  $\alpha = n$  and  $\beta = n+1$ . Hence  $V$  is of the form

$$V = (x_i + a_i x_n + b_i x_{n+1} = 0)$$

for  $1 \leq i \leq n-1$ . So

$$\begin{aligned} V &= \{[-a_1 \lambda - b_1 \mu : \dots : -a_{n-1} \lambda - b_{n-1} \mu : \lambda : \mu] : \\ &\quad \lambda, \mu \in k, \lambda \neq 0 \text{ or } \mu \neq 0\} \end{aligned}$$

is a line passing two different points

$$\begin{aligned} P &= [-a_1 : \dots : -a_{n-1} : 1 : 0] \\ Q &= [-b_1 : \dots : -b_{n-1} : 0 : 1]. \end{aligned}$$

□

*Proof of (c).*

(1) By part (b), a line  $L \subseteq \mathbf{P}^2$  is

$$\begin{aligned} &V((a_2 b_3 - b_2 a_3)x + (a_3 b_1 - a_1 b_3)y + (a_1 b_2 - a_2 b_1)z = 0) \\ &= V \left( \begin{vmatrix} x & a_1 & b_1 \\ y & a_2 & b_2 \\ z & a_3 & b_3 \end{vmatrix} = 0 \right), \end{aligned}$$

which is also a plane in  $\mathbf{P}^2$ .



(2) Conversely, given any plane

$$V = V(ax + by + cz = 0) \subseteq \mathbf{P}^2$$

where  $a, b, c$  are not all zero. Might assume that  $a \neq 0$ . (Other cases are similar.) So

$$V = \left\{ \left[ -\frac{b}{a}\lambda - \frac{c}{a}\mu : \lambda : \mu \right] \in \mathbf{P}^2 : \lambda, \mu \in k, \lambda \neq 0 \text{ or } \mu \neq 0 \right\}$$

is a line passing  $P = \left[ -\frac{b}{a} : 1 : 0 \right] \in \mathbf{P}^2$  and  $Q = \left[ -\frac{c}{a} : 0 : 1 \right] \in \mathbf{P}^2$ .

□

*Proof of (d).*

- (1) Take one point  $P_i \in L_i$  (resp.  $P'_i \in L'_i$ ) other than  $P$  (resp.  $P'$ ) for  $i = 1, 2$ . It is possible since every line is passing two distinct points.
- (2) By Problem 4.15,  $P_1 \notin L_2$  (resp.  $P'_1 \notin L'_2$ ) and  $P_2 \notin L_1$  (resp.  $P'_2 \notin L'_1$ ).
- (3) By Problem 4.14, there is a unique projective change of coordinates  $t : \mathbf{P}^2 \rightarrow \mathbf{P}^2$  such that  $t(P) = P'$ ,  $t(P_1) = P'_1$  and  $t(P_2) = P'_2$ .
- (4) Hence, the line  $t(L_i)$  (by part (a)) and the line  $L'_i$  are both passing  $P'$  and  $P'_i$  for  $i = 1, 2$ . Since  $P' \neq P'_i$  by construction, Problem 4.15 implies that  $t(L_i) = L'_i$ .

□

#### **Problem 4.14.\***

Let  $P_1, P_2, P_3$  (resp.  $Q_1, Q_2, Q_3$ ) be three points in  $\mathbf{P}^2$  not lying on a line. Show that there is a projective change of coordinates  $t : \mathbf{P}^2 \rightarrow \mathbf{P}^2$  such that  $t(P_i) = Q_i$ ,  $i = 1, 2, 3$ . Extend this to  $n + 1$  points in  $\mathbf{P}^n$ , not lying on a hyperplane.

*Proof.*

- (1) Write

$$P_i = [a_{i1} : a_{i2} : a_{i3}] \in \mathbf{P}^2(k)$$

$$Q_i = [b_{i1} : b_{i2} : b_{i3}] \in \mathbf{P}^2(k)$$

for  $i = 1, 2, 3$ .

(2) Define

$$A = \begin{pmatrix} P_1 & P_2 & P_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

$$B = \begin{pmatrix} Q_1 & Q_2 & Q_3 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \end{pmatrix}.$$

Note that  $A$  (resp.  $B$ ) is depending on the representations of  $P_i$  (resp.  $Q_i$ ) up to a nonzero constant in  $k$ .

(3) Here  $A$  (resp.  $B$ ) is invertible since  $P_1, P_2, P_3$  (resp.  $Q_1, Q_2, Q_3$ ) are not lying on a line. Define  $t : \mathbf{P}^2 \rightarrow \mathbf{P}^2$  by sending  $P = [x : y : z] \in \mathbf{P}^2$  to

$$t(P) = BA^{-1}P = \begin{pmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Note that the matrix  $BA^{-1} \in \mathrm{GL}_3(k)$  depends on the representations of  $P_i$  (resp.  $Q_i$ ) up to a nonzero constant in  $k$ . Hence,  $t$  is a well-defined map from  $\mathbf{P}^2$  to  $\mathbf{P}^2$ . Besides,  $t$  is a projective change of coordinates mapping  $P_i$  to  $Q_i$  ( $i = 1, 2, 3$ ).

(4) *Generalization.* Let  $P_i$  (resp.  $Q_i$ ) be  $n+1$  points in  $\mathbf{P}^n$  ( $i = 1, \dots, n+1$ ) not lying on a hyperplane. Then there is a projective change of coordinates  $t : \mathbf{P}^n \rightarrow \mathbf{P}^n$  such that  $t(P_i) = Q_i$  ( $i = 1, \dots, n+1$ ). The proof is the same except replacing 2 by  $n$ .

□

#### Problem 4.15.\*

Show that any two distinct lines in  $\mathbf{P}^2$  intersect in one point.

*Proof.*

(1) Let

$$L_1 : a_1x + b_1y + c_1z = 0$$

$$L_2 : a_2x + b_2y + c_2z = 0$$

be two distinct lines in  $\mathbf{P}^2$ .

(2) View (1) as the system of linear equations. Let the coefficient matrix  $A$  be

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}.$$

Since  $L_1$  and  $L_2$  are distinct,  $\text{rank}(A) = 2$ . The rank-nullity theorem shows that

$$\dim_k \ker(A) = 3 - \text{rank}(A) = 1.$$

- (3) Might take a basis  $\{(x_0, y_0, z_0)\}$  for  $\ker(A)$ . Here  $(x_0, y_0, z_0) \neq 0$  and any other nonzero solutions of  $L_1 \cap L_2$  is of the form  $(\lambda x_0, \lambda y_0, \lambda z_0)$  ( $\lambda \neq 0$ ). Therefore,

$$L_1 \cap L_2 = \{[x_0 : y_0 : z_0] \in \mathbf{P}^2\}.$$

□

### Problem 4.16.\*

Let  $L_1, L_2, L_3$  (resp.  $M_1, M_2, M_3$ ) be lines in  $\mathbf{P}^2(k)$  that do not all pass through a point. Show that there is a projective change of coordinates:  $t : \mathbf{P}^2 \rightarrow \mathbf{P}^2$  such that  $t(L_i) = M_i$ . (Hint: Let  $P_{ij} = L_i \cap L_j$ ,  $Q_{ij} = M_i \cap M_j$ ,  $i \neq j$ , and apply Problem 4.14.) Extend this to  $n + 1$  hyperplanes in  $\mathbf{P}^n$ , not passing through a point.

*Proof.*

- (1) Let  $P_{ij} = L_i \cap L_j$  (resp.  $Q_{ij} = M_i \cap M_j$ ),  $i \neq j$ .  $P_{ij}$  (resp.  $Q_{ij}$ ) is uniquely determined by  $L_i$  and  $L_j$  (resp.  $M_i$  and  $M_j$ ) (Problem 4.15). Also,  $P_{ij} \neq P_{i'j'}$  (resp.  $Q_{ij} \neq Q_{i'j'}$ ) if  $\{i, j\} \neq \{i', j'\}$  (as sets) by assumption.
- (2) Problem 4.14 shows that there is a projective change of coordinates  $t : \mathbf{P}^2 \rightarrow \mathbf{P}^2$  such that  $t(P_{ij}) = Q_{ij}$ ,  $i \neq j$ . Similar to the argument in Problem 4.13(d), we conclude that  $t(L_i) = M_i$ .
- (3) Show that we can extend this to  $n + 1$  hyperplanes in  $\mathbf{P}^n$ , not passing through a point. We cannot apply steps (1)(2) to the generalized case  $\mathbf{P}^n$ . Instead, we can apply the proof of Problem 4.14.
- (4) Let  $E_i$  (resp.  $F_i$ ) ( $i = 1, \dots, n + 1$ ) be  $(n + 1)$  hyperplanes in  $\mathbf{P}^n$  that do not passing through a point. Write

$$E_i : a_{i,1}x_1 + \dots + a_{i,n+1}x_{n+1} = 0 \in \mathbf{P}^n(k)$$

$$F_i : b_{i,1}x_1 + \dots + b_{i,n+1}x_{n+1} = 0 \in \mathbf{P}^n(k)$$

for  $i = 1, \dots, n + 1$ .

- (5) View  $E_i$  (resp.  $F_i$ ) as the system of linear equations. Let the coefficient

matrix  $A$  (resp.  $B$ ) be

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{n+1,1} \\ \vdots & \ddots & \vdots \\ a_{1,n+1} & \cdots & a_{n+1,n+1} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{1,1} & \cdots & b_{n+1,1} \\ \vdots & \ddots & \vdots \\ b_{1,n+1} & \cdots & b_{n+1,n+1} \end{pmatrix}.$$

Note that  $A$  (resp.  $B$ ) is depending on the representations of  $E_i$  (resp.  $F_i$ ) up to a nonzero constant in  $k$ .

- (6) Here  $A$  (resp.  $B$ ) is invertible since  $E_i$  (resp.  $F_i$ ) are not passing through a point. Define  $t : \mathbf{P}^n \rightarrow \mathbf{P}^n$  by sending  $P = [x_1 : \cdots : x_{n+1}] \in \mathbf{P}^n$  to

$$t(P) = BA^{-1}P$$

$$= \begin{pmatrix} b_{1,1} & \cdots & b_{n+1,1} \\ \vdots & \ddots & \vdots \\ b_{1,n+1} & \cdots & b_{n+1,n+1} \end{pmatrix} \begin{pmatrix} a_{1,1} & \cdots & a_{n+1,1} \\ \vdots & \ddots & \vdots \\ a_{1,n+1} & \cdots & a_{n+1,n+1} \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix}.$$

Note that the matrix  $BA^{-1} \in \mathrm{GL}_{n+1}(k)$  depends on the representations of  $E_i$  (resp.  $F_i$ ) up to a nonzero constant in  $k$ . Hence,  $t$  is a well-defined map from  $\mathbf{P}^n$  to  $\mathbf{P}^n$ . Besides,  $t$  is a projective change of coordinates mapping  $E_i$  to  $F_i$  ( $i = 1, \dots, n+1$ ).

□

*Note.* It is the duality of Problem 4.14. See Problem 4.18 for more details.

#### **Problem 4.17.\***

Let  $z$  be a rational function on a projective variety  $V$ . Show that the pole set of  $z$  is an algebraic subset of  $V$ .

*Proof.*

- (1) Similar to the proof of Proposition 2 in §2.4. For  $g \in k[x_1, \dots, x_{n+1}]$ , denote the residue of  $g$  in  $\Gamma_h(V)$  by  $\bar{g}$ .

- (2) Let

$$J_z = \{g \in k[x_1, \dots, x_{n+1}] : \bar{g}z \in \Gamma_h(V)\}.$$

$J_z$  is an ideal in  $k[x_1, \dots, x_{n+1}]$  containing  $I(V)$ , and the points of  $V(J_z)$  are exactly those points where  $z$  is not defined if  $J_z$  is homogeneous.

- (3) Show that  $J_z$  is homogeneous by the homogeneous property of  $I(V)$ . Given any  $g = g_r + g_{r+1} + \cdots \in J_z$ , where  $g_i$  is a form of  $k[x_1, \dots, x_{n+1}]$ . Write  $z = a/b$  for some form  $a, b \in \Gamma_h(V)$  of the same degree. So  $\bar{g}z = \bar{g}a/b \in \Gamma_h(V)$ . So there is  $f = f_s + f_{s+1} + \cdots \in k[x_1, \dots, x_{n+1}]$  (where  $f_i$  is a form of  $k[x_1, \dots, x_{n+1}]$ ) such that  $ga - fb \in I(V)$ . Since  $I(V)$  is homogeneous, each form  $g_i a - f_i b$  is in  $I(V)$ . So  $\bar{g}_i z = \bar{f}_i \in \Gamma_h(V)$  for each  $i$  (since  $\Gamma_h(V)$  is homogeneous), that is,  $g_i \in J_z$  for each  $i$ .

□

**Problem 4.18. (Duality)**

Let  $H = V(\sum a_i x_i)$  be a hyperplane in  $\mathbf{P}^n$ . Note that  $(a_1, \dots, a_{n+1})$  is determined by  $H$  up to a constant.

- (a) Show that assigning  $[a_1 : \cdots : a_{n+1}] \in \mathbf{P}^n$  to  $H$  sets up a natural one-to-one correspondence between  $\{\text{hyperplanes in } \mathbf{P}^n\}$  and  $\mathbf{P}^n$ . If  $P \in \mathbf{P}^n$ , let  $P^*$  be the corresponding hyperplane; if  $H$  is a hyperplane,  $H^*$  denotes the corresponding point.
- (b) Show that  $P^{**} = P$ ,  $H^{**} = H$ . Show that  $P \in H$  if and only if  $H^* \in P^*$ .

This is the well-known **duality** of the projective space.

*Proof of (a).*

- (1) Define  $\alpha : \{\text{hyperplanes}\} \rightarrow \mathbf{P}^n$  (resp.  $\beta : \mathbf{P}^n \rightarrow \{\text{hyperplanes}\}$ ) by

$$\begin{aligned}\alpha : V\left(\sum a_i x_i\right) &\mapsto [a_1 : \cdots : a_{n+1}], \\ \beta : [a_1 : \cdots : a_{n+1}] &\mapsto V\left(\sum a_i x_i\right).\end{aligned}$$

- (2) As  $H = V(\sum a_i x_i)$  is a hyperplane, the corresponding

$$\alpha(H) = [a_1 : \cdots : a_{n+1}] \in \mathbf{P}^n$$

and thus  $\alpha$  is well-defined. Similarly,  $\beta$  is well-defined.

- (3) Note that both  $\alpha \circ \beta$  and  $\beta \circ \alpha$  are identity maps.  $\alpha$  (resp.  $\beta$ ) is an isomorphism, that is, there is a natural one-to-one correspondence between

$$\{\text{hyperplanes in } \mathbf{P}^n\} \longleftrightarrow \mathbf{P}^n.$$

□

*Proof of (b).*

- (1) We've showed that  $P^{**} = P$ ,  $H^{**} = H$  in (a). It suffices to show that  $P \in H$  iff  $H^* \in P^*$ .
- (2) Write  $H = V(\sum a_i x_i) \subseteq \mathbf{P}^n$  and  $P = [b_1 : \cdots : b_{n+1}] \in \mathbf{P}^n$ . Hence,

$$\begin{aligned} P \in H &\iff a_1 b_1 + \cdots + a_{n+1} b_{n+1} = 0 \\ &\iff b_1 a_1 + \cdots + b_{n+1} a_{n+1} = 0 \\ &\iff H^* \in P^*. \end{aligned}$$

□

### 4.3. Affine and Projective Varieties

#### Problem 4.19.\*

If  $I = (f)$  is the ideal of an affine hypersurface, show that  $I^* = (f^*)$ .

*Proof.*

- (1) Note that  $I^*$  is the ideal in  $k[x_1, \dots, x_{n+1}]$  generated by  $\{g^* : g \in I = (f)\}$ . In particular,  $f^* \in I^*$ . Thus  $(f^*) \subseteq I^*$ .
- (2) Conversely, given any  $g = \sum a_i (h_i f)^* \in I^*$ , where  $a_i, h_i \in k[x_1, \dots, x_{n+1}]$ . Thus,

$$g = \sum_i a_i (h_i f)^* = \sum_i a_i h_i^* f^* = \left( \sum_i a_i h_i^* \right) f^* \in (f^*)$$

(by Proposition 5 in §2.6).

□

#### Problem 4.20.

Let  $V = V(y - x^2, z - x^3) \subseteq \mathbf{A}^3$ . Prove:

- (a)  $I(V) = (y - x^2, z - x^3)$ .
- (b)  $zw - xy \in I(V)^* \subseteq k[x, y, z, w]$ , but  $zw - xy \notin I((y - x^2)^*, (z - x^3)^*)$ . So if  $I(V) = (f_1, \dots, f_r)$ , it does not follow that  $I(V)^* = (f_1^*, \dots, f_r^*)$ .

*Proof of (a).* By Problems 1.11 and 2.8,  $V$  is an affine variety. Thus  $I(V) = (y - x^2, z - x^3)$  is a prime ideal.  $\square$

*Proof of (b).*

- (1) Since  $z - xy = (z - x^3) - x(y - x^2) \in I(V)$ ,  $zw - xy = (z - xy)^* \in I(V)^*$ .
- (2) Suppose  $zw - xy \in I((y - x^2)^*, (z - x^3)^*) = I(yw - x^2, zw^2 - x^3)$ . Write  $zw - xy = f \cdot (yw - x^2) + g \cdot (zw^2 - x^3)$  for some  $f, g \in k[x, y, z, w]$ . Then  $\deg_x(zw - xy) = 1$  but  $\deg_x(f \cdot (yw - x^2) + g \cdot (zw^2 - x^3))$  cannot be 1, which is absurd.

$\square$

**Problem 4.21.**

Show that if  $V \subseteq W \subseteq \mathbf{P}^n$  are varieties, and  $V$  is a hypersurface, then  $W = V$  or  $W = \mathbf{P}^n$  (see Problem 1.39).

*Proof.*

- (1) Write  $V = V(f)$  for some irreducible form  $f \in k[x_1, \dots, x_{n+1}]$ . So  $I(V) = (f)$  by the projective Nullstellensatz.
- (2) Note that  $I(W)$  is a prime ideal such that

$$(f) \supseteq I(W) \supseteq (0).$$

By Problem 1.39,  $I(W) = (f)$  or  $(0)$ . Thus,  $W = V$  or  $W = \mathbf{P}^n$ .

$\square$

**Problem 4.22.\***

Suppose  $V$  is a variety in  $\mathbf{P}^n$  and  $V \supseteq H_\infty$ . Show that  $V = \mathbf{P}^n$  or  $V = H_\infty$ . If  $V = \mathbf{P}^n$ ,  $V_* = \mathbf{A}^n$ , while if  $V = H_\infty$ ,  $V_* = \emptyset$ .

*Proof.*

- (1) Note that  $H_\infty = V(x_{n+1})$  is a hypersurface. By Problem 4.21,  $V = \mathbf{P}^n$  or  $V = H_\infty$ .
- (2) If  $V = \mathbf{P}^n$ , then  $I = I(V) = (0)$ . So  $V_* = V(I_*) = V(0) = \mathbf{A}^n$ .
- (3) If  $V = H_\infty$ , then  $I = I(V) = (x_{n+1})$ . So  $V_* = V(I_*) = V(1) = \emptyset$ .

$\square$

**Problem 4.23.\***

Describe all subvarieties in  $\mathbf{P}^1$  and in  $\mathbf{P}^2$ .

*Proof.*

- (1) Show that all subvarieties in  $\mathbf{P}^1$  are  $\emptyset$ ,  $\mathbf{P}^1$ , and single points. (Also compare to Problem 1.8.)
  - (a)  $\emptyset$ , single points and  $\mathbf{P}^1$  are all varieties.
  - (b) Let  $V$  be a nonempty subvariety in  $\mathbf{P}^1$ . Write  $\{[a : b]\} = V(bx - ay) \subseteq V$  for some point  $[a : b] \in \mathbf{P}^1$ . So  $V(bx - ay) \subseteq V \subseteq \mathbf{P}^1$ .
  - (c) Since  $V(bx - ay)$  is a hypersurface and  $bx - ay$  is irreducible, Problem 4.21 implies that  $V = V(bx - ay)$  (a single point) or  $V = \mathbf{P}^1$  itself.
- (2) Show that all subvarieties in  $\mathbf{P}^2$  are  $\emptyset$ ,  $\mathbf{P}^2$ , single points, and hypersurfaces  $V(f)$ , where  $f$  is an irreducible form. Similar to Corollary 2 of Proposition 2 in §1.6.
  - (a) Let  $V$  be a subvariety in  $\mathbf{P}^2$ . If  $V$  is finite or  $I(V) = (0)$ ,  $V$  is of one required type.
  - (b) Otherwise  $I(V)$  contains a non-constant form  $f$ ; since  $I(V)$  is prime, we may assume  $f$  is irreducible.
  - (c) Hence,  $I(V) = (f)$ ; for if  $g \in I(V)$ ,  $g \notin (f)$ , then  $V \subseteq V(f, g)$  is finite by Problem 5.7, which is a projective analogue of Proposition 2 in §1.6.

□

**Problem 4.24.\***

Let  $P = [0 : 1 : 0] \in \mathbf{P}^2(k)$ . Show that the lines through  $P$  consist of the following:

- (a) The vertical lines  $L_\lambda = V(x - \lambda z) = \{[\lambda : t : 1] : t \in k\} \cup \{P\}$ .
- (b) The line at infinity  $L_\infty = V(z) = \{[x : y : 0] : x, y \in k, x \neq 0 \text{ or } y \neq 0\}$ .

*Proof.*

- (1) Let  $L$  be one line passing through  $P$  and  $Q = [b_1 : b_2 : b_3] \neq P$ . Thus,

$$\begin{aligned}
 L &= \{[\mu b_1 : \lambda + \mu b_2 : \mu b_3] : \lambda, \mu \in k, \lambda \neq 0 \text{ or } \mu \neq 0\} \\
 &= \{[\mu b_1 : \lambda + \mu b_2 : \mu b_3] : \lambda, \mu \in k, \mu \neq 0\} \cup \{P\} \\
 &= \{[b_1 : t + b_2 : b_3] : t \in k\} \cup \{P\} \qquad (t := \lambda/\mu)
 \end{aligned}$$

Consider two cases:  $b_3 \neq 0$  and  $b_3 = 0$ .



(2)  $b_3 \neq 0$ . So

$$\begin{aligned}
L &= \{[b_1 : t + b_2 : b_3] : t \in k\} \cup \{P\} \\
&= \left\{ \left[ \frac{b_1}{b_3} : \frac{t}{b_3} + \frac{b_2}{b_3} : 1 \right] : t \in k \right\} \cup \{P\} \\
&= \{[\lambda : t : 1] : t \in k\} \cup \{P\} & \left( \frac{b_1}{b_3} \mapsto \lambda, \frac{t + b_2}{b_3} \mapsto t \right) \\
&= L_\lambda.
\end{aligned}$$

(3)  $b_3 = 0$ . So

$$\begin{aligned}
L &= \{[b_1 : t + b_2 : 0] : t \in k\} \cup \{P\} \\
&= \{[x : y : 0] : y \in k\} \cup \{P\} & (b_1 \mapsto x, t + b_2 \mapsto y) \\
&= \{[x : y : 0] : y \in k, x \neq 0\} \cup \{P\} \\
&= \{[x : y : 0] : y \in k, x \neq 0 \text{ or } y \neq 0\} \\
&= L_\infty.
\end{aligned}$$

□

**Problem 4.25.\***

Let  $P = [x : y : z] \in \mathbf{P}^2$ .

- (a) Show that  $\{(a, b, c) \in \mathbf{A}^3 : ax + by + cz = 0\}$  is a hyperplane in  $\mathbf{A}^3$ .
- (b) Show that for any finite set of points in  $\mathbf{P}^2$ , there is a line not passing through any of them.

*Proof of (a).*

- (1) Let  $V = \{(a, b, c) \in \mathbf{A}^3 : ax + by + cz = 0\}$ .
- (2)  $V$  is well-defined and the form  $ax + by + cz$  is of degree one since not all  $a, b, c$  are zero. Hence  $V$  is a hyperplane (plane) in  $\mathbf{A}^3$ .

□

*Proof of (b).*

- (1) Let  $S$  be any finite set of points in  $\mathbf{P}^2(k)$ . Write  $S = (S \cap H_\infty) \cup (S - H_\infty)$ .
- (2)  $S \cap H_\infty$  is a finite subset of  $H_\infty = \mathbf{P}^1(k) \supseteq \mathbf{A}^1(k) = k$ . Since  $k = \bar{k}$  is infinite (Problem 1.6), there is one point  $[\beta : 1 : 0] \in H_\infty - S$ . Here  $\beta \in k$  is uniquely determined.

- (3)  $S - H_\infty$  is finite too. Write  $S - H_\infty = \{[\lambda_1 : \mu_1 : 1], \dots, [\lambda_r : \mu_r : 1]\}$ . Here  $\lambda_i$  and  $\mu_i$  are all uniquely determined ( $i = 1, \dots, r$ ). Take any  $\gamma \in k$  such that  $\gamma \notin \{\lambda_1 - \beta\mu_1, \dots, \lambda_r - \beta\mu_r\}$ . (It is possible since  $k$  is infinite.)

- (4) Let

$$L = V(x - \beta y - \gamma z)$$

be a line in  $\mathbf{P}^2$ . By construction,  $L \cap S = \emptyset$ .

□

## 4.4. Multiprojective Space

### Problem 4.26.\*

- (a) Define maps  $\varphi_{i,j} : \mathbf{A}^{n+m} \rightarrow U_i \times U_j \subseteq \mathbf{P}^n \times \mathbf{P}^m$ . Using  $\varphi_{n+1,m+1}$ , define the “biprojective closure” of an algebraic set in  $\mathbf{A}^{n+m}$ . Prove an analogue of Proposition 3 of §4.3.
- (b) Generalize part (a) to maps

$$\varphi : \mathbf{A}^{n_1} \times \dots \times \mathbf{A}^{n_r} \times \mathbf{A}^m \rightarrow \mathbf{P}^{n_1} \times \dots \times \mathbf{P}^{n_r} \times \mathbf{A}^m.$$

Show that this sets up a correspondence between

$$\{\text{nonempty affine varieties in } \mathbf{A}^{n_1+\dots+n_r+m}\}$$

and

$$\{\text{varieties in } \mathbf{P}^{n_1} \times \dots \times \mathbf{P}^{n_r} \times \mathbf{A}^m \\ \text{that intersect } U_{n_1+1} \times \dots \times U_{n_r+1} \times \mathbf{A}^m\}.$$

Show that this correspondence preserves function fields and local rings.

*Proof of (a).*

- (1) Define maps  $\varphi_{i,j} : \mathbf{A}^{n+m} \rightarrow U_i \times U_j \subseteq \mathbf{P}^n \times \mathbf{P}^m$  by

$$\varphi_{i,j} : (x_1, \dots, x_n, y_1, \dots, y_m) \mapsto \\ [x_1 : \dots : x_{i-1} : 1 : x_i : \dots : x_n] \times [y_1 : \dots : y_{j-1} : 1 : y_j : \dots : y_m].$$

Actually, we can write  $\varphi_{i,j}(x, y) = (\varphi_i(x), \varphi_j(y))$  for  $x = (x_1, \dots, x_n) \in \mathbf{A}^n$  and  $y = (y_1, \dots, y_m) \in \mathbf{A}^m$ .

- (2) Given any algebraic set  $V$  in  $\mathbf{A}^{n+m}$ . Let  $I^*$  be the bihomogeneous ideal in  $k[x, y] = k[x_1, \dots, x_{n+1}, y_1, \dots, y_{m+1}]$  generated by  $\{f^* : f \in I\}$ . Here

$$f^* = x_{n+1}^r y_{m+1}^s f\left(\frac{x_1}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}}, \frac{y_1}{y_{m+1}}, \dots, \frac{y_m}{y_{m+1}}\right)$$

where  $r$  (resp.  $s$ ) is the degree of  $f$  in  $x$  (resp.  $y$ ). Define  $V^* = V(I^*)$  as the “biprojective closure.”

- (3) Proposition 3 for multiprojective space.

- (i) If  $V \subseteq \mathbf{A}^{n+m}$ ,  $\varphi_{n+1, m+1}(V) = V^* \cap (U_{n+1} \times U_{m+1})$  and  $(V^*)_* = V$ .
- (ii) If  $V \subseteq W \subseteq \mathbf{A}^{n+m}$ , then  $V^* \subseteq W^* \subseteq \mathbf{P}^n \times \mathbf{P}^m$ . If  $V \subseteq W \subseteq \mathbf{P}^n \times \mathbf{P}^m$ , then  $V_* \subseteq W_* \subseteq \mathbf{A}^{n+m}$ .
- (iii) If  $V$  is irreducible in  $\mathbf{A}^{n+m}$ , then  $V^*$  is irreducible in  $\mathbf{P}^n \times \mathbf{P}^m$ .
- (iv) If  $V = \bigcup_i V_i$  is the irreducible decomposition of  $V$  in  $\mathbf{A}^{n+m}$ , then  $V^* = \bigcup_i V_i^*$  is the irreducible decomposition of  $V^*$  in  $\mathbf{P}^n \times \mathbf{P}^m$ .
- (v) If  $V \subseteq \mathbf{A}^{n+m}$ , then  $V^*$  is the smallest algebraic set in  $\mathbf{P}^n \times \mathbf{P}^m$  that contains  $\varphi_{n+1, m+1}(V)$ .
- (vi) If  $V \subsetneq \mathbf{A}^{n+m}$ , then no component of  $V^*$  lies in or contains  $H_\infty \times H_\infty$ .
- (vii) If  $V \subseteq \mathbf{P}^n \times \mathbf{P}^m$ , and no component of  $V$  lies in or contains  $(H_\infty \times \mathbf{P}^m) \cup (\mathbf{P}^n \times H_\infty)$ , then  $V_* \subsetneq \mathbf{A}^{n+m}$  and  $(V_*)^* = V$ .

- (4) Proof of Proposition 3 for multiprojective space.

- (i) It follows from Proposition 5 in §2.6.
- (ii) It is obvious.
- (iii) If  $V \subseteq \mathbf{A}^{n+m}$ ,  $I = I(V)$ , then a biform  $f$  belongs to  $I^*$  if and only if  $f_* \in I$ . If  $I$  is prime, it follows readily that  $I^*$  is also prime.
- (iv) It follows from (ii)(iii)(v).
- (v) Suppose  $W$  is an algebraic set in  $\mathbf{P}^n \times \mathbf{P}^m$  which contains  $\varphi_{n+1, m+1}(V)$ . If  $f \in I(W)$ , then  $f_* \in I(V)$ , so  $f = x_{n+1}^r y_{m+1}^s (f_*)^* \in I(V)^*$ . Therefore  $I(W) \subseteq I(V)^*$ , so  $W \supseteq V^*$ , as desired.
- (vi) We may assume  $V$  is irreducible.  $V^* \not\subseteq H_\infty \times H_\infty$  by (i). If  $V \subseteq H_\infty \times H_\infty$ , then  $I(V)^* \subseteq I(V^*) \subseteq I(H_\infty \times H_\infty) = (x_{n+1}, y_{m+1})$ . But if  $0 \neq f \in I(V)$ , then  $f^* \in I(V)^*$ , but  $f^* \notin (x_{n+1}, y_{m+1})$ . So  $V^* \not\subseteq H_\infty \times H_\infty$ .
- (vii) We may assume  $V$  is irreducible. Since  $\varphi_{n+1, m+1}(V_*) \subseteq V$ , it suffices to show that  $V \subseteq (V_*)^*$ , or that  $I(V_*)^* \subseteq I(V)$ . Let  $f \in I(V_*)$ . Then  $f^N \in I(V)_*$  for some  $N$  (Nullstellensatz), so  $x_{n+1}^t y_{m+1}^s (f^N)^* \in I(V)$  for some  $t, s$  (Proposition 5 in §2.6). But  $I(V)$  is prime, and  $x_{n+1}, y_{m+1} \notin I(V)$  since  $V \not\subseteq (H_\infty \times \mathbf{P}^m) \cup (\mathbf{P}^n \times H_\infty)$ , so  $f^* \in I(V)$ , as desired.

□

*Proof of (b).*

- (1) Define maps  $\varphi_{i_1, \dots, i_r} : \mathbf{A}^{n_1} \times \dots \times \mathbf{A}^{n_r} \times \mathbf{A}^m \rightarrow U_{i_1} \times \dots \times U_{i_r} \times \mathbf{A}^m$  by

$$\varphi_{i_1, \dots, i_r} : (\mathbf{x}_1, \dots, \mathbf{x}_r, \mathbf{y}) \mapsto (\varphi_{i_1}(\mathbf{x}_1), \dots, \varphi_{i_r}(\mathbf{x}_r), \mathbf{y}),$$

where  $\mathbf{x}_i = (x_{i,1}, \dots, x_{i,n_i}) \in \mathbf{A}^{n_i}$  and  $\mathbf{y} = (y_1, \dots, y_m) \in \mathbf{A}^m$ .

- (2) Applying the same argument in (a), we have that there is a correspondence between

$$\{\text{nonempty affine varieties in } \mathbf{A}^{n_1 + \dots + n_r + m}\}$$

and

$$\{\text{varieties in } \mathbf{P}^{n_1} \times \dots \times \mathbf{P}^{n_r} \times \mathbf{A}^m \\ \text{that intersect } U_{n_1+1} \times \dots \times U_{n_r+1} \times \mathbf{A}^m\}.$$

- (3) Let  $V$  be a nonempty affine varieties in  $\mathbf{A}^{n_1 + \dots + n_r + m}$ . Let  $V^* \subseteq \mathbf{P}^{n_1} \times \dots \times \mathbf{P}^{n_r} \times \mathbf{A}^m$  be the multiprojective closure of  $V$  intersecting  $U_{n_1+1} \times \dots \times U_{n_r+1} \times \mathbf{A}^m$ .
- (4) If  $\bar{f} \in \Gamma(V^*)$  is a multi-form, we may define  $\bar{f}_*$  as follows: take a multi-form  $f \in k[\mathbf{x}_1, \dots, \mathbf{x}_r, \mathbf{y}]$  such whose  $I(V^*)$ -residue in  $\bar{f}$ , and let  $\bar{f}_*$  to be  $I(V)$ -residue of  $f_*$  (one checks that this is independent of the choice of  $f$ ).
- (5) Define a natural isomorphism  $\alpha : k(V^*) \rightarrow k(V)$  by  $\alpha(f/g) = f_*/g_*$ , where  $f, g$  are multi-forms of the same multi-degree on  $V^*$ .
- (6) If  $P \in V$ , we may consider  $P \in V^*$  (by means of  $\varphi_{n_1+1, \dots, n_r+1}$ ) and then  $\alpha$  induces an isomorphism of  $\mathcal{O}_P(V^*)$  with  $\mathcal{O}_P(V)$ .

□

#### **Problem 4.27.\***

*Show that the pole set of a rational function on a variety in any multispace is an algebraic subset.*

*Proof.*

- (1) Similar to Problem 4.17. Let  $V$  be a variety in one multispace  $\mathbf{P}^{n_1} \times \dots \times \mathbf{P}^{n_r} \times \mathbf{A}^m$ . Let  $z$  be a rational function on  $V$ .

(2) Let

$$J_z = \{g \in k[\mathbf{x}_1, \dots, \mathbf{x}_r, \mathbf{y}] : \bar{g}z \in \Gamma(V)\},$$

where  $\mathbf{x}_i = [x_{i,1} : \dots : x_{i,n_i+1}] \in \mathbf{P}^{n_i}$  for  $1 \leq i \leq r$ ,  $\mathbf{y} = (y_1, \dots, y_m) \in \mathbf{A}^m$ , and  $\bar{g}$  is the residue of  $g$  in the multi-homogeneous coordinate ring  $\Gamma(V)$ .

- (3)  $J_z$  is an ideal in  $k[\mathbf{x}_1, \dots, \mathbf{x}_r, \mathbf{y}]$  containing the multi-homogeneous ideal  $I(V)$ , and the points of  $V(J_z)$  are exactly those points where  $z$  is not defined if  $J_z$  is multi-homogeneous.
- (4) Show that  $J_z$  is multi-homogeneous by the multi-homogeneous property of  $I(V)$ . Induction on  $r$  and then apply the proof of Problem 4.17.

□

#### Problem 4.28.\* (Segre embedding)

For simplicity of notation, in this problem we let  $x_0, \dots, x_n$  be coordinates for  $\mathbf{P}^n$ ,  $y_0, \dots, y_m$  coordinates for  $\mathbf{P}^m$ , and  $t_{00}, t_{01}, \dots, t_{0m}, t_{10}, \dots, t_{nm}$  coordinates for  $\mathbf{P}^N$ , where  $N = (n+1)(m+1) - 1 = n + m + nm$ . Define  $S : \mathbf{P}^n \times \mathbf{P}^m \rightarrow \mathbf{P}^N$  by the formula:  $S([x_0 : \dots : x_n], [y_0 : \dots : y_m]) = [x_0y_0 : x_0y_1 : \dots : x_ny_m]$ .  $S$  is called the **Segre embedding** of  $\mathbf{P}^n \times \mathbf{P}^m$  in  $\mathbf{P}^{n+m+nm}$ .

- (a) Show that  $S$  is a well-defined, one-to-one mapping.
- (b) Show that if  $W$  is an algebraic subset of  $\mathbf{P}^N$ , then  $S^{-1}(W)$  is an algebraic subset of  $\mathbf{P}^n \times \mathbf{P}^m$ .
- (c) Let  $V = V(\{t_{ij}t_{k\ell} - t_{i\ell}t_{kj} : i, k = 0, \dots, n; j, \ell = 0, \dots, m\}) \subseteq \mathbf{P}^N$ . Show that  $S(\mathbf{P}^n \times \mathbf{P}^m) = V$ . In fact,  $S(U_i \times U_j) = V \cap U_{ij}$ , where  $U_{ij} = \{[t] : t_{ij} \neq 0\}$ .
- (d) Show that  $V$  is a variety.

*Proof of (a).*

- (1) Show that  $S$  is well-defined. Given any non-zero  $\lambda, \mu \in k$ .

$$\begin{aligned} & S([\lambda x_0 : \dots : \lambda x_n], [\mu y_0 : \dots : \mu y_m]) \\ &= [(\lambda x_0)(\mu y_0) : (\lambda x_0)(\mu y_1) : \dots : (\lambda x_n)(\mu y_m)] \\ &= [(\lambda \mu)x_0y_0 : (\lambda \mu)x_0y_1 : \dots : (\lambda \mu)x_ny_m] \\ &= [x_0y_0 : x_0y_1 : \dots : x_ny_m] \quad (\lambda \mu \neq 0) \\ &= S([x_0 : \dots : x_n], [y_0 : \dots : y_m]). \end{aligned}$$

So  $S$  is independent of the choices of  $[x_0 : \dots : x_n]$  and  $[y_0 : \dots : y_m]$ . Since there is an index  $i$  (resp.  $j$ ) such that  $x_i \neq 0$  (resp.  $y_j \neq 0$ ),  $x_iy_j \neq 0$  and thus  $\text{im}(S) \subseteq \mathbf{P}^N$ .

- (2) *Show that  $S$  is one-to-one.* Given any  $P = [x_0y_0 : x_0y_1 : \dots : x_ny_m] \in \text{im}(S) \subseteq \mathbf{P}^N$ . Might assume that  $x_0 = y_0 = 1$ . So

$$\begin{aligned} P &= [x_0y_0 : x_0y_1 : \dots : x_ny_m] \\ &= [1 : y_1 : y_2 : \dots : y_m : x_1 : \dots : x_2 : \dots : x_n : \dots x_ny_m]. \end{aligned}$$

By the expression of  $P$ ,  $x_1, \dots, x_n$  and  $y_1, \dots, y_m$  are uniquely determined. Hence  $[1 : x_1 : \dots : x_n]$  and  $[1 : y_1 : \dots : y_m]$  are uniquely determined by  $P$ .  $S$  is injective.

□

*Proof of (b).*

- (1)  $S^{-1}(W) = \{P \in \mathbf{P}^n \times \mathbf{P}^m : S(P) \in W\}$  is the inverse image of  $W$  under  $S$ . Write

$$W = V(h_1, \dots, h_r)$$

where each  $h_i \in k[t_{00}, \dots, t_{nm}]$  is a form of degree  $d_i$ .

- (2) Define the pullback  $\tilde{h}_i \in k[x_0, \dots, x_n, y_0, \dots, y_m]$  as

$$\tilde{h}_i(x_0, \dots, x_n, y_0, \dots, y_m) = h_i(x_0y_0, \dots, x_ny_m)$$

for each  $i$ . Clearly,  $\tilde{h}_i$  is a biform of bidegree  $(d_i, d_i)$ .

- (3) As

$$\begin{aligned} P \in S^{-1}(W) &\iff S(P) \in W \\ &\iff h_i(S(P)) = 0 \forall h_i \\ &\iff \tilde{h}_i(P) = 0 \forall \tilde{h}_i \\ &\iff P \in V(\tilde{h}_1, \dots, \tilde{h}_r), \end{aligned}$$

$S^{-1}(W) = V(\tilde{h}_1, \dots, \tilde{h}_r)$  is algebraic.

□

*Proof of (c).*

- (1) *Show that  $\text{im}(S) = V$  is algebraic.*  $\text{im}(S) \subseteq V$  by the construction of  $S$ . Conversely, take  $P = [a_{ij}]_{\substack{i=0, \dots, n \\ j=0, \dots, m}} \in V \subseteq \mathbf{P}^N$ . So there is one entry  $a_{k\ell} \neq 0$ . So

$$\begin{aligned} P &= [a_{ij}]_{\substack{i=0, \dots, n \\ j=0, \dots, m}} \\ &= [a_{ij}a_{k\ell}]_{\substack{i=0, \dots, n \\ j=0, \dots, m}} \\ &= [a_{i\ell}a_{kj}]_{\substack{i=0, \dots, n \\ j=0, \dots, m}} \\ &= S([a_{i\ell}]_{i=0, \dots, n}, [a_{kj}]_{j=0, \dots, m}) \end{aligned}$$

is in  $\text{im}(S)$ .

- (2) In the proof of (1), we have  $S(U_i \times U_j) = V \cap U_{ij}$  where  $U_{ij} = \{[t] : t_{ij} \neq 0\}$  (by setting  $(k, \ell) \rightarrow (i, j)$ ).

□

*Proof of (d).*

- (1) Write  $V = V_1 \cup V_2$  where  $V_1, V_2$  are algebraic sets in  $\mathbf{P}^N$ . By (b)(c),

$$\mathbf{P}^n \times \mathbf{P}^m = S^{-1}(V) = S^{-1}(V_1) \cup S^{-1}(V_2).$$

- (2) Since  $\mathbf{P}^n \times \mathbf{P}^m$  is irreducible (Problem 4.26), we might assume that  $\mathbf{P}^n \times \mathbf{P}^m = S^{-1}(V_1)$ . Note that  $S(S^{-1}(W)) = W$  holds for any set  $W \subseteq V$ . Hence

$$V_1 = S(S^{-1}(V_1)) = S(\mathbf{P}^n \times \mathbf{P}^m) = V,$$

that is,  $V$  is irreducible.

□

## Chapter 5: Projective Plane Curves

### 5.1. Definitions

#### Problem 5.1.\*

Let  $f$  be a projective plane curve. Show that a point  $P$  is a multiple point of  $f$  if and only if  $f(P) = \frac{\partial f}{\partial x}(P) = \frac{\partial f}{\partial y}(P) = \frac{\partial f}{\partial z}(P) = 0$ .

*Proof.*

- (1) Let  $P$  be a point in  $f$ . Might assume  $P \in U_3$ . By dehomogenizing  $f$  with respect to  $z$ ,  $m_P(f) = m_P(f_*)$ . Thus,

$$\begin{aligned} m_P(f) &= m_P(f_*) > 1 \\ \iff f_*(P) &= \frac{\partial f_*}{\partial x}(P) = \frac{\partial f_*}{\partial y}(P) = 0 \\ \iff f(P) &= \frac{\partial f}{\partial x}(P) = \frac{\partial f}{\partial y}(P) = 0. \quad (f \text{ is a form}) \end{aligned}$$

- (2) By Euler's Theorem in §1.1,  $f(P) = \frac{\partial f}{\partial x}(P) = \frac{\partial f}{\partial y}(P) = 0$  implies that  $\frac{\partial f}{\partial z}(P) = 0$ .

□

#### Problem 5.2.

Show that the following curves are irreducible; find their multiple points, and the multiplicities and tangents at the multiple points.

- (a)  $xy^4 + yz^4 + xz^4$ .
- (b)  $x^2y^3 + x^2z^3 + y^2z^3$ .
- (c)  $y^2z - x(x - z)(x - \lambda z)$ ,  $\lambda \in k$ .
- (d)  $x^n + y^n + z^n$ ,  $n > 0$ .

*Proof of (a).*

- (1) Let  $f = xy^4 + yz^4 + xz^4$ . It suffices to show that  $f_* = x(y^4 + 1) + y$  is irreducible. View  $f_* \in (k[y])[x]$  as a linear function in  $x$ . Since  $y^4 + 1$  and  $y$  have no common factors in  $k[x]$ ,  $f_*$  is irreducible in  $(k[y])[x] = k[x, y]$ .



(2) By solving the system of equations  $f(P) = \frac{\partial f}{\partial x}(P) = \frac{\partial f}{\partial y}(P) = \frac{\partial f}{\partial z}(P) = 0$ , there is only one multiple point  $[1 : 0 : 0]$  of  $f$ .

(3) Since  $P = [1 : 0 : 0]$ , we dehomogenize  $f$  w.r.t.  $x$  to get

$$m_P(f) = m_P(f_*) = m_{(0,0)}(yz^4 + y^4 + z^4) = 4.$$

$P$  is an ordinary multiple point of 4 distinct tangents  $y \pm (-1)^{\frac{1}{4}}z$ ,  $y \pm (-1)^{\frac{3}{4}}z$ .

□

*Proof of (b).*

(1) Let  $f = x^2y^3 + x^2z^3 + y^2z^3$ . It suffices to show that  $f_* = x^2(y^3 + 1) + y^2$  is irreducible. View  $f_* \in (k[y])[x]$  as a quadratic function in  $x$ . Note that  $y^3 + 1$  and  $y^2$  have no common factors in  $k[x]$ . We may write

$$f_*(x, y) = (a(y)x + b(y))(c(y)x + d(y))$$

if  $f_*$  were reducible. So

$$\begin{aligned} ac &= y^3 + 1 \\ ad + bc &= 0 \\ bd &= y^2. \end{aligned}$$

There is no solutions for  $a, b, c, d \in k[y]$ . Hence,  $f_*$  is irreducible in  $(k[y])[x] = k[x, y]$ .

(2) By solving the system of equations  $f(P) = \frac{\partial f}{\partial x}(P) = \frac{\partial f}{\partial y}(P) = \frac{\partial f}{\partial z}(P) = 0$ , we have  $P = [1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$ .

(3) Suppose  $P = [1 : 0 : 0]$ . We dehomogenize  $f$  w.r.t.  $x$  to get

$$m_P(f) = m_P(f_*) = m_{(0,0)}(y^2z^3 + y^3 + z^3) = 3.$$

$P$  is an ordinary multiple point of 3 distinct tangents  $y + z$ ,  $y + (-1)^{\frac{1}{3}}z$ ,  $y - (-1)^{\frac{2}{3}}z$ .

(4) Suppose  $P = [0 : 1 : 0]$ . We dehomogenize  $f$  w.r.t.  $y$  to get

$$m_P(f) = m_P(f_*) = m_{(0,0)}(x^2z^3 + z^3 + x^2) = 2.$$

$x$  is the tangent to  $f$  at  $P$  of the multiplicity = 2.

(5) Suppose  $P = [0 : 0 : 1]$ . We dehomogenize  $f$  w.r.t.  $z$  to get

$$m_P(f) = m_P(f_*) = m_{(0,0)}(x^2y^3 + x^2 + y^2) = 2.$$

$P$  is an ordinary multiple point of 2 distinct tangents  $x \pm (-1)^{\frac{1}{2}}y$ .

□

*Proof of (c).*

- (1) Note that  $y^2 - x(x-1)(x-\lambda)$  is irreducible (Problem 1.35). Hence, the homogenizing form  $f = y^2z - x(x-z)(x-\lambda z)$  of  $y^2 - x(x-1)(x-\lambda)$  w.r.t.  $z$  is also irreducible.
- (2) By solving the system of equations  $f(P) = \frac{\partial f}{\partial x}(P) = \frac{\partial f}{\partial y}(P) = \frac{\partial f}{\partial z}(P) = 0$ , we have

$$\begin{aligned} P &= [0 : 0 : 1] \text{ if } \lambda = 0 \\ P &= [1 : 0 : 1] \text{ if } \lambda = 1 \end{aligned}$$

are all multiple points of  $f$ .

- (3) Suppose  $P = [0 : 0 : 1]$  with  $\lambda = 0$ . Then

$$m_P(f) = m_P(f_*) = m_{(0,0)}(y^2 - x^2(x-1)) = 2.$$

$y$  is the tangent to  $f$  at  $P$  of the multiplicity = 2. (The projective closure of  $y$  is  $y$ .)

- (4) Similarly, suppose  $P = [1 : 0 : 1]$  with  $\lambda = 1$ . Then

$$\begin{aligned} m_P(f) &= m_P(f_*) \\ &= m_{(1,0)}(y^2 - x(x-1)^2) \\ &= m_{(0,0)}(y^2 - x^2(x+1)) \\ &= 2. \end{aligned}$$

$y$  is the tangent to  $f$  at  $P$  of the multiplicity = 2.

□

*Proof of (d).*

- (1) Let  $f = x^n + y^n + z^n$ . It suffices to show that  $f_* = x^n + y^n + 1$  is irreducible. Write

$$f_* = x^n + (y^n + 1) \in (k[y])[x].$$

Note that  $k[y]$  is a UFD and  $y^n + 1$  is separable. Thus by the Eisenstein's criterion,  $f_*$  irreducible in  $(k[y])[x] = k[x, y]$ .

- (2) By solving the system of equations  $f(P) = \frac{\partial f}{\partial x}(P) = \frac{\partial f}{\partial y}(P) = \frac{\partial f}{\partial z}(P) = 0$ , there are no multiple points on  $f$  for  $n \geq 1$ .

□

**Problem 5.3.**

Find all points of intersection of the following pairs of curves, and the intersection numbers at these points:

- (a)  $y^2z - x(x - 2z)(x + z)$  and  $y^2 + x^2 - 2xz$ .
- (b)  $(x^2 + y^2)z + x^3 + y^3$  and  $x^3 + y^3 - 2xyz$ .
- (c)  $y^5 - x(y^2 - xz)^2$  and  $y^4 + y^3z - x^2z^2$ .
- (d)  $(x^2 + y^2)^2 + 3x^2yz - y^3z$  and  $(x^2 + y^2)^3 - 4x^2y^2z^2$ .

*Proof of (a).*

- (1) Say  $f = y^2z - x(x - 2z)(x + z)$  and  $g = y^2 + x^2 - 2xz$ . By  $f - zg = 0$ ,  $x(x - 2z)(x + 2z) = 0$ . So the intersection points in  $\mathbf{P}^2$  are

$$\underbrace{[0 : 0 : 1]}_{x=0}, \underbrace{[2 : 0 : 1]}_{x-2z=0}, \underbrace{[-2, \pm\sqrt{-8} : 1]}_{x+2z=0}.$$

- (2) Suppose  $P = [0 : 0 : 1]$ . We dehomogenize  $f$  and  $g$  w.r.t.  $z$  to get

$$\begin{aligned} I(P, f \cap g) &= I((0, 0), f_* \cap g_*) \\ &= I((0, 0), (-x(x - 2)(x + 1) + y^2) \cap (y^2 + x^2 - 2x)) \\ &= 2 \end{aligned}$$

(since they have no common tangents).

- (3) Suppose  $P = [2 : 0 : 1]$ . We dehomogenize  $f$  and  $g$  w.r.t.  $z$  to get

$$\begin{aligned} I(P, f \cap g) &= I(P, f_* \cap g_*) \\ &= I(P, (-x(x - 2)(x + 1) + y^2) \cap (y^2 + x^2 - 2x)) \\ &= I((0, 0), (-x(x + 2)(x + 3) + y^2) \cap (y^2 + x^2 + 2x)) \\ &= 2. \end{aligned}$$

- (4) Suppose  $P = [-2, \pm\sqrt{-8} : 1]$ . We dehomogenize  $f$  and  $g$  w.r.t.  $z$  to get

$$\begin{aligned} I(P, f \cap g) &= I(P, f_* \cap g_*) \\ &= I(P, (-x(x - 2)(x + 1) + y^2) \cap (y^2 + x^2 - 2x)) \\ &= I((0, 0), (-x^3 + 7x^2 + y^2 - 14x \pm 2\sqrt{-8}y) \cap \\ &\quad (y^2 + x^2 - 6x \pm 2\sqrt{-8}y)) \\ &= 1. \end{aligned}$$

(Or by Bézout's theorem, each intersection point has the intersection number 1 in this case.)

□

*Proof of (b).*

- (1) Say  $f = (x^2 + y^2)z + x^3 + y^3$  and  $g = x^3 + y^3 - 2xyz$ . By  $f - g = 0$ ,  $(x + y)^2 z = 0$ . So the intersection points in  $\mathbf{P}^2$  are

$$\underbrace{[0 : 0 : 1]}_{x+y=0}, \underbrace{[1 : -1 : 0], [1 : \omega : 0], [1 : 1 - \omega : 0]}_{z=0}.$$

where  $\omega = (-1)^{\frac{1}{3}} \in k$ .

- (2) Suppose  $P = [0 : 0 : 1]$ . We dehomogenize  $f$  and  $g$  w.r.t.  $z$  to get

$$\begin{aligned} I(P, f \cap g) &= I((0, 0), f_* \cap g_*) \\ &= I((0, 0), (x^3 + y^3 + x^2 + y^2) \cap (x^3 + y^3 - 2xy)) \\ &= 4. \end{aligned}$$

- (3) Suppose  $P = [1 : -1 : 0]$ . We dehomogenize  $f$  and  $g$  w.r.t.  $x$  to get

$$\begin{aligned} I(P, f \cap g) &= I(P, f_* \cap g_*) \\ &= I(P, (f_* - g_*) \cap g_*) \\ &= I(P, (y + 1)^2 z \cap (1 + y^3 - 2yz)) \\ &= I((0, 0), y^2 z \cap (y^3 - 3y^2 - 2yz + 3y + 2z)) \\ &= 3. \end{aligned}$$

- (4) Suppose  $P = [1 : \omega : 0]$ . Similar to (3), we dehomogenize  $f$  and  $g$  w.r.t.  $x$  to get

$$\begin{aligned} I(P, f \cap g) &= I(P, (y + 1)^2 z \cap (1 + y^3 - 2yz)) \\ &= I((0, 0), (y^2 z + 2(1 + \omega)yz + (1 + \omega)^2 z) \cap \\ &\quad (y^3 + 3\omega y^2 - 2yz + 3\omega^2 y - 2\omega z)) \\ &= I((0, 0), (y^2 z + 2(1 + \omega)yz + 3\omega z) \cap \\ &\quad (y^3 + 3\omega y^2 - 2yz + 3\omega^2 y - 2\omega z)) \\ &= 1. \end{aligned}$$

- (5) Suppose  $P = [1 : 1 - \omega : 0]$ . We dehomogenize  $f$  and  $g$  w.r.t.  $x$  to get

$$\begin{aligned} I(P, f \cap g) &= I(P, (y + 1)^2 z \cap (1 + y^3 - 2yz)) \\ &= I((0, 0), (y^2 z + 2(2 - \omega)yz + (2 - \omega)^2 z) \cap \\ &\quad (y^3 + 3(1 - \omega)y^2 - 2yz + 3(1 - \omega)^2 y - 2(1 - \omega)z)) \\ &= I((0, 0), (y^2 z + 2(2 - \omega)yz + 3(1 - \omega)z) \cap \\ &\quad (y^3 + 3(1 - \omega)y^2 - 2yz - 3\omega y - 2(1 - \omega)z)) \\ &= 1. \end{aligned}$$

□

*Proof of (c).*

- (1) Say  $f = y^5 - x(y^2 - xz)^2$  and  $g = y^4 + y^3z - x^2z^2$ . If  $y = 0$ , the intersection points in  $\mathbf{P}^2 - U_2$  are

$$[0 : 0 : 1], [1 : 0 : 0].$$

If  $y \neq 0$ , the intersection points in  $2_3 \cong \mathbf{A}^2$  are the solutions of  $f_* = 1 - x(1 - xz)^2$  and  $g_* = 1 + z - x^2z^2$ :

$$\left[ \frac{1}{2} : 1 : 2 \pm 2\sqrt{2} \right].$$

Note that there are only 4 different intersection points (Bézout's theorem).

- (2) Suppose  $P = [0 : 0 : 1]$ . We dehomogenize  $f$  and  $g$  w.r.t.  $z$  to get

$$\begin{aligned} & I(P, f \cap g) \\ &= I((0, 0), f_* \cap g_*) \\ &= I((0, 0), (y^5 - x(y^2 - x)^2) \cap (y^4 + y^3 - x^2)) \\ &= I((0, 0), ((y^5 - x(y^2 - x)^2) - x(y^4 + y^3 - x^2)) \cap (y^4 + y^3 - x^2)) \\ &= I((0, 0), y^2(y^2 - x)(2x - y) \cap (y^4 + y^3 - x^2)) \\ &= I((0, 0), y^2 \cap (y^4 + y^3 - x^2)) \\ &\quad + I((0, 0), (y^2 - x) \cap (y^4 + y^3 - x^2)) \\ &\quad + I((0, 0), (2x - y) \cap (y^4 + y^3 - x^2)) \\ &= 4 + I((0, 0), (y^2 - x) \cap (y^4 + y^3 - x^2)) + 2 \\ &= 9. \end{aligned}$$

Here

$$\begin{aligned} & I((0, 0), (y^2 - x) \cap (y^4 + y^3 - x^2)) \\ &= I((0, 0), (y^2 - x) \cap ((y^4 + y^3 - x^2) - x(y^2 - x))) \\ &= I((0, 0), (y^2 - x) \cap (y^4 - xy^2 + y^3)) \\ &= 3. \end{aligned}$$

(3) Suppose  $P = [1 : 0 : 0]$ . We dehomogenize  $f$  and  $g$  w.r.t.  $x$  to get

$$\begin{aligned}
& I(P, f \cap g) \\
&= I((0, 0), f_* \cap g_*) \\
&= I((0, 0), (y^5 - (y^2 - z)^2) \cap (y^4 + y^3z - z^2)) \\
&= I((0, 0), ((y^5 - (y^2 - z)^2) - (y^4 + y^3z - z^2)) \cap (y^4 + y^3z - z^2)) \\
&= I((0, 0), y^2(y^2 - z)(y - 2) \cap (y^4 + y^3z - z^2)) \\
&= I((0, 0), y^2 \cap (y^4 + y^3z - z^2)) \\
&\quad + I((0, 0), (y^2 - z) \cap (y^4 + y^3z - z^2)) \\
&\quad + I((0, 0), (y - 2) \cap (y^4 + y^3z - z^2)) \\
&= 4 + I((0, 0), (y^2 - z) \cap (y^4 + y^3z - z^2)) + 0 \\
&= 4 + 5 + 0 \\
&= 9.
\end{aligned}$$

Here

$$\begin{aligned}
& I((0, 0), (y^2 - z) \cap (y^4 + y^3z - z^2)) \\
&= I((0, 0), (y^2 - z) \cap ((y^4 + y^3z - z^2) - z(y^2 - z))) \\
&= I((0, 0), (y^2 - z) \cap y^2(y^2 + yz - z)) \\
&= I((0, 0), (y^2 - z) \cap y^2) + I((0, 0), (y^2 - z) \cap (y^2 + yz - z)) \\
&= 2 + I((0, 0), (y^2 - z) \cap yz) \\
&= 2 + I((0, 0), (y^2 - z) \cap y) + I((0, 0), (y^2 - z) \cap z) \\
&= 2 + 1 + I((0, 0), y^2 \cap z) \\
&= 2 + 1 + 2 \\
&= 5.
\end{aligned}$$

(4) Suppose  $P = [\frac{1}{2} : 1 : 2 + 2\sqrt{2}]$ . We dehomogenize  $f$  and  $g$  w.r.t.  $y$  to get

$$\begin{aligned}
& I(P, f \cap g) \\
&= I(P, f_* \cap g_*) \\
&= I(P, (1 - x(1 - xz)^2) \cap (1 + z - x^2z^2)) \\
&= I\left((0, 0), \left(-(6 + 2\sqrt{2})x - \sqrt{2}/2 \cdot z + \text{higher terms}\right) \cap \right. \\
&\quad \left.(-(12 + 8\sqrt{2})x - \sqrt{2}z + \text{higher terms})\right) \\
&= 1.
\end{aligned}$$

Similarly,  $I(P, f \cap g) = 1$  as  $P = [\frac{1}{2} : 1 : 2 - 2\sqrt{2}]$ .

□

*Proof of (d).*

- (1) Say  $f = (x^2 + y^2)^2 + 3x^2yz - y^3z$  and  $g = (x^2 + y^2)^3 - 4x^2y^2z^2$ . If  $z = 0$ , the intersection points in  $\mathbf{P}^2 - U_3$  are

$$[1 : \pm\sqrt{-1} : 0].$$

If  $z \neq 0$ , the intersection points in  $U_3 \cong \mathbf{A}^2$  are the solutions of  $f_* = (x^2 + y^2)^2 + 3x^2y - y^3$  and  $g_* = (x^2 + y^2)^3 - 4x^2y^2$ :

$$[0 : 0 : 1], \left[ \pm \frac{\sqrt{5+2\sqrt{5}}}{4} : \frac{-\sqrt{5}}{4} : 1 \right], \left[ \pm \frac{\sqrt{5-2\sqrt{5}}}{4} : \frac{\sqrt{5}}{4} : 1 \right].$$

(To find these solutions, we might solve  $f_*(r \cos \theta, r \sin \theta) = g_*(r \cos \theta, r \sin \theta) = 0$  over  $k = \mathbb{C}$ . As  $r = 0$ , nothing to do. As  $r > 0$ , we get  $\theta = \frac{m\pi}{5}$  for all integers  $m$ .) Note that there are only 7 different intersection points (Bézout's theorem).

- (2) Suppose  $P = [1 : \sqrt{-1} : 0]$ . We dehomogenize  $f$  and  $g$  w.r.t.  $x$  to get

$$\begin{aligned} & I(P, f \cap g) \\ &= I(P, f_* \cap g_*) \\ &= I(P, ((1 + y^2)^2 + 3yz - y^3z) \cap ((1 + y^2)^3 - 4y^2z^2)) \\ &= I(P, ((1 + y^2)^2 + 3yz - y^3z) \cap (yz(y^4 - 2y^2 - 4yz - 3))) \\ &= I(P, ((1 + y^2)^2 + 3yz - y^3z) \cap y) \\ &\quad + I(P, ((1 + y^2)^2 + 3yz - y^3z) \cap z) \\ &\quad + I(P, ((1 + y^2)^2 + 3yz - y^3z) \cap (y^4 - 2y^2 - 4yz - 3)) \\ &= 2I(P, (1 + y^2) \cap z) \\ &\quad + I(P, ((1 + y^2)^2 + 3yz - y^3z) \cap (y^4 - 2y^2 - 4yz - 3)) \\ &= 3. \end{aligned}$$

Here

$$I(P, (1 + y^2) \cap z) = I((0, 0), (y^2 + 2\sqrt{-1}y) \cap z) = 1$$

and

$$\begin{aligned} & I(P, ((1 + y^2)^2 + 3yz - y^3z) \cap (y^4 - 2y^2 - 4yz - 3)) \\ &= I((0, 0), (y^4 - y^3z + 4\sqrt{-1}y^3 - 3\sqrt{-1}y^2z - 4y^2 + 6yz + 4\sqrt{-1}z) \cap \\ &\quad (y^4 + 4\sqrt{-1}y^3 - 8y^2 - 4yz - 8\sqrt{-1}y - 4\sqrt{-1}z)) \\ &= 1. \end{aligned}$$

Similarly,  $I(P, f \cap g) = 3$  if  $P = [1 : -\sqrt{-1} : 0]$ .

- (3) Suppose  $P = [0 : 0 : 1]$ . We dehomogenize  $f$  and  $g$  w.r.t.  $z$  to get

$$\begin{aligned} I(P, f \cap g) &= I((0, 0), f_* \cap g_*) \\ &= I((0, 0), ((x^2 + y^2)^2 + 3x^2y - y^3) \cap ((x^2 + y^2)^3 - 4x^2y^2)) \\ &= 14 \end{aligned}$$

(Example in §3.3).

- (4) Suppose  $P = \left[ \frac{\sqrt{5+2\sqrt{5}}}{4} : \frac{-\sqrt{5}}{4} : 1 \right]$ . We dehomogenize  $f$  and  $g$  w.r.t.  $z$  to get

$$\begin{aligned}
& I(P, f \cap g) \\
&= I(P, f_* \cap g_*) \\
&= I(P, ((x^2 + y^2)^2 + 3x^2y - y^3) \cap ((x^2 + y^2)^3 - 4x^2y^2)) \\
&= I(P, ((x^2 + y^2)^2 + 3x^2y - y^3) \cap y(-3x^4 - 2x^2y^2 + y^4 - 4x^2y)) \\
&= 4I(P, x \cap y) \\
&\quad + I(P, ((x^2 + y^2)^2 + 3x^2y - y^3) \cap (-3x^4 - 2x^2y^2 + y^4 - 4x^2y)) \\
&= 1.
\end{aligned}$$

Here

$$I(P, x \cap y) = I\left((0, 0), \left(x + \frac{\sqrt{5+2\sqrt{5}}}{4}\right) \cap \left(y + \frac{-\sqrt{5}}{4}\right)\right) = 0$$

and

$$\begin{aligned}
& I(P, ((x^2 + y^2)^2 + 3x^2y - y^3) \cap (-3x^4 - 2x^2y^2 + y^4 - 4x^2y)) \\
&= I\left((0, 0), \left(\frac{(5+2\sqrt{5})\sqrt{5+2\sqrt{5}}}{8}x - \frac{5+2\sqrt{5}}{8}y + \text{higher terms}\right) \cap \right. \\
&\quad \left.\left(\frac{(-10+\sqrt{5})\sqrt{5+2\sqrt{5}}}{8}x - \frac{5+4\sqrt{5}}{8}y + \text{higher terms}\right)\right) \\
&= 1.
\end{aligned}$$

Similarly,  $I(P, f \cap g) = 1$  for rest three cases.

□

#### Problem 5.4.\*

Let  $P$  be a simple point on  $f$ . Show that the tangent line to  $f$  at  $P$  has the equation  $\frac{\partial f}{\partial x}(P)x + \frac{\partial f}{\partial y}(P)y + \frac{\partial f}{\partial z}(P)z = 0$ .

*Proof.*

- (1) Similar to Problem 5.1. Might assume  $P = [x_0 : y_0 : 1] \in U_3$ . By



dehomogenizing  $f$  with respect to  $z$ , the tangent line to  $f_*$  at  $P$  is

$$\begin{aligned} L : 0 &= \frac{\partial f}{\partial x}(P)(x - x_0) + \frac{\partial f}{\partial y}(P)(y - y_0) \\ &= \frac{\partial f}{\partial x}(P)x + \frac{\partial f}{\partial y}(P)y - \left( \frac{\partial f}{\partial x}(P)x_0 + \frac{\partial f}{\partial y}(P)y_0 \right) \\ &= \frac{\partial f}{\partial x}(P)x + \frac{\partial f}{\partial y}(P)y + \frac{\partial f}{\partial z}(P). \end{aligned}$$

(The last equation is due to Euler's Theorem in §1.1.)

(2) Hence, the projective closure of  $L$ , say

$$L^* : 0 = \frac{\partial f}{\partial x}(P)x + \frac{\partial f}{\partial y}(P)y + \frac{\partial f}{\partial z}(P)z,$$

is the tangent line to  $f$  at  $P$ .

□

### Problem 5.5.\*

Let  $P = [0 : 1 : 0]$ ,  $f$  a curve of degree  $n$ ,  $f = \sum f_i(x, z)y^{n-i}$ ,  $f_i$  a form of degree  $i$ . Show that  $m_P(f)$  is the smallest  $m$  such that  $f_m \neq 0$ , and the factors of  $f_m$  determine the tangents to  $f$  at  $P$ .

*Proof.*

(1) By dehomogenizing  $f$  with respect to  $y$ , we have

$$m_P(f) = m_P(f_*) = m_{(0,0)}\left(\sum f_i(x, z)\right) = m$$

where  $m$  is the smallest integer such that  $f_m \neq 0$ .

(2) The factors of  $f_m$  determine the tangents to  $f_*$  at  $P$ . Note that each factor of  $f_m$  is a form. Therefore, these factors of  $f_m$  also determine the tangents to  $f$  at  $P$ .

□

### Problem 5.6.\*

For any  $f$ ,  $P \in f$ , show that  $m_P\left(\frac{\partial f}{\partial x}\right) \geq m_P(f) - 1$ .

*Proof.*

- (1) If  $P = [1 : 0 : 0]$ , by dehomogenizing  $f$  with respect to  $x$  we get

$$m_P \left( \frac{\partial f}{\partial x} \right) = m_P \left( \frac{\partial f_*}{\partial x} \right) = m_P(0) = \infty.$$

The result is established.

- (2) Might assume  $P \in U_3$ . By dehomogenizing  $f$  with respect to  $z$ , it suffices to show that

$$m_P \left( \frac{\partial f_*}{\partial x} \right) \geq m_P(f_*) - 1.$$

- (3) Write  $m = m_P(f_*)$  and  $f_* = g_m + g_{m+1} + \cdots \in k[x, y]$  where  $g_i \neq 0$  is a form of degree  $m$ . Note that

$$\frac{\partial f_*}{\partial x} = \frac{\partial g_m}{\partial x} + \frac{\partial g_{m+1}}{\partial x} + \cdots.$$

So that  $m_P \left( \frac{\partial f_*}{\partial x} \right) \geq m - 1 = m_P(f_*) - 1.$

□

### Problem 5.7.\*

Show that two plane curves with no common components intersect in a finite number of points.

*Proof.*

- (1) Similar to Proposition 2 in §1.6. Let  $f$  and  $g$  be forms in  $k[x, y, z]$  with no common components. In particular,  $f \neq 0$  and  $g \neq 0$ .
- (2) Show that the conclusion is true for a line  $g$ . Similar to Problem 1.12. Let  $\alpha : \mathbf{P}^1 \rightarrow \mathbf{P}^2$  be the map

$$\alpha([\lambda : \mu]) = [\lambda a_1 + \mu b_1 : \lambda a_2 + \mu b_2 : \lambda a_3 + \mu b_3]$$

and  $g = \text{im}(\alpha)$  be a line. Note that the points of  $V(f, g)$  correspond to the zero of

$$\tilde{f}(\lambda, \mu) = f(\lambda a_1 + \mu b_1, \lambda a_2 + \mu b_2, \lambda a_3 + \mu b_3) = 0 \text{ in } \mathbf{P}^1.$$

$\tilde{f}$  is a form in  $k[\lambda, \mu]$  of degree  $m := \deg(f)$  since  $f$  and  $g$  have no common components. Hence,  $\tilde{f}$  has exactly  $m$  zeros in  $\mathbf{P}^1$  (with multiplicity).  $V(f, g)$  has exactly  $m$  zeros (with multiplicity).

*Note.* It is the same as Problem 5.12(b).

- (3) Show that the conclusion is true for any  $f, g$ . Note that  $f$  and  $g$  have no common factors in  $k(x, y)[z]$ . Since  $k(x, y)[z]$  is a PID,  $(f, g) = (1) \in k(x, y)[z]$ . So  $\alpha f + \beta g = 1$  for some  $\alpha, \beta \in k(x, y)[z]$ . Clearing denominators from  $\alpha, \beta$  gives us

$$af + bg = d$$

where  $d$  is a non-zero polynomial in  $k[x, y]$ ,  $a = d\alpha \in k[x, y, z]$  and  $b = d\beta \in k[x, y, z]$ . Taking suitable homogeneous parts of  $a, b$  and  $d$ , we might assume  $a, b$  and  $d$  are all homogeneous.

- (4) By Corollary to Proposition 5 in §2.6,  $d$  is a product of linear forms, say  $d = L_1 \cdots L_r$ . A common zero of  $f$  and  $g$  is also a zero of  $d$ , and therefore it is a zero of  $L_j$  for some  $j$ . So it suffices to show that  $V = (f, g) = V(f, g, L_j)$  is finite for each  $j$ .
- (5) (Reductio ad absurdum) If not, there were one  $j$  such that both  $V(f, L_j)$  and  $V(g, L_j)$  were infinite. By (2),  $L_j \mid f$  and  $L_j \mid g$ .  $f$  and  $g$  have one common factor, which is absurd.

□

### Problem 5.8.\*

Let  $f$  be an irreducible curve.

- (a) Show that  $\frac{\partial f}{\partial x}$ ,  $\frac{\partial f}{\partial y}$ , or  $\frac{\partial f}{\partial z} \neq 0$ .
- (b) Show that  $f$  has only a finite number of multiple points.

*Proof of (a).* If  $\frac{\partial f}{\partial x}$ ,  $\frac{\partial f}{\partial y}$ , and  $\frac{\partial f}{\partial z}$  are all zero,  $f$  is a constant, which is not a projective plane curve. □

*Proof of (b).*

- (1) Let

$$V = V\left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

be the set of the multiple points of  $f$  (Problem 5.1). Moreover,  $V$  is an algebraic set.

- (2) Suppose  $\frac{\partial f}{\partial x} \neq 0$  (since not all  $\frac{\partial f}{\partial x}$ ,  $\frac{\partial f}{\partial y}$  and  $\frac{\partial f}{\partial z}$  are zero). It is nothing to do if  $\frac{\partial f}{\partial x}$  is a nonzero constant. Suppose  $\deg\left(\frac{\partial f}{\partial x}\right) \geq 1$ . Since  $f$  is irreducible and  $\deg\left(\frac{\partial f}{\partial x}\right) = \deg f - 1$ ,  $f$  and  $\frac{\partial f}{\partial x}$  have no common factors. By Problem 5.7,  $V\left(f, \frac{\partial f}{\partial x}\right)$  is a finite set. Hence,  $V$  is finite as a subset of a finite set  $V\left(f, \frac{\partial f}{\partial x}\right)$ .

□

**Problem 5.9. (Conic)**

- (a) Let  $f$  be an irreducible conic,  $P = [0 : 1 : 0]$  a simple point on  $f$ , and  $z = 0$  the tangent line to  $f$  at  $P$ . Show that  $f = ayz - bx^2 - cxz - dz^2$ ,  $a, b \neq 0$ . Find a projective change of coordinates  $t$  so that  $f^t = yz - x^2 - c'xz - d'z^2$ . Find  $t'$  so that  $(f^t)^{t'} = yz - x^2$ . ( $t' = (x, y + c'x + d'z, z)$ .)
- (b) Show that, up to projective equivalence, there is only one irreducible conic:  $yz = x^2$ . Any irreducible conic is nonsingular.

*Proof of (a).*

- (1) Given any conic

$$f = ayz - bx^2 - cxz - dz^2 + \alpha y^2 + \beta xy.$$

- (2)  $\alpha = 0$  since  $P \in f$ .

- (3) Show that  $a \neq 0$  and  $\beta = 0$ . We dehomogenize  $f$  w.r.t.  $y$  to get  $f_* = -bx^2 - cxz - dz^2 + az + \beta x$ . Thus, the tangent line of  $f_*$  at  $(0, 0)$  is  $az + \beta x$ . So  $a \neq 0$  and  $\beta = 0$  since  $z$  is the tangent line to  $f$  at  $P$ .

- (4) Show that  $b \neq 0$ . If  $b = 0$ ,  $f = ayz - cxz - dz^2 = z(ay - cx - dz)$  is reducible, which is absurd.

- (5) Write

$$\begin{aligned} f &= ayz - bx^2 - cxz - dz^2 \\ &= (\sqrt{ay})(\sqrt{az}) - (\sqrt{bx})^2 - \frac{c}{\sqrt{ab}}(\sqrt{bx})(\sqrt{az}) - \frac{d}{a}(\sqrt{az})^2. \end{aligned}$$

Apply a projective change of coordinates  $t = \left(\frac{x}{\sqrt{b}}, \frac{y}{\sqrt{a}}, \frac{z}{\sqrt{a}}\right)$  to  $f$ :

$$f^t = yz - x^2 - c'xz - d'z^2,$$

where  $c' = \frac{c}{\sqrt{ab}}$  and  $d' = \frac{d}{a}$ .

- (6) Write

$$f^t = yz - x^2 - c'xz - d'z^2 = (y - c'x - d'z)z - x^2.$$

Apply a projective change of coordinates  $t' = (x, y + c'x + d'z, z)$  to  $f^t$ :

$$(f^t)^{t'} = yz - x^2.$$

□

*Proof of (b).*

- (1) Take one simple point  $P_1 \in f$  and the corresponding tangent line  $L$ . (It is possible since Problem 5.8 and  $|f| = \infty$ .) Take  $P_2 \in L - f$  and  $P_3 \in f - L$ . (It is possible since Problem 5.7 or 5.12 and both  $|f|$  and  $|L|$  are  $\infty$ .)
- (2) Let  $Q_1 = [0 : 1 : 0]$ ,  $Q_2 = [1 : 0 : 0]$ ,  $Q_3 = [0 : 0 : 1]$ . Since  $P_1, P_2, P_3$  (resp.  $Q_1, Q_2, Q_3$ ) are not lying on a common line, there is a unique projective change of coordinates  $t$  such that  $t(P_i) = Q_i$  ( $i = 1, 2, 3$ ) by Problem 4.14.
- (3) Hence,  $f$  is projectively equivalent to an irreducible conic  $g$  such that  $P = [0 : 1 : 0]$  is a simple point on  $g$  and  $z = 0$  is the tangent line to  $g$  at  $P$ . By (a),

$$f \sim g \sim (yz - x^2).$$

- (4) Nonsingularity is preserved under projectively equivalence. Might assume that  $f = yz - x^2$ .  $\frac{\partial f}{\partial x}(P) = \frac{\partial f}{\partial y}(P) = \frac{\partial f}{\partial z}(P) = 0$  implies that  $P = [0 : 0 : 0]$ , which is not in  $\mathbf{P}^2$ . Hence  $f$  is nonsingular (Problem 5.1).

□

**Problem 5.10. (Cubic with a cusp)**

Let  $f$  be an irreducible cubic,  $P = [0 : 0 : 1]$  a cusp on  $f$ ,  $y = 0$  the tangent line to  $f$  at  $P$ . Show that  $f = ay^2z - bx^3 - cx^2y - dxy^2 - ey^3$ . Find projective changes of coordinates

- (i) to make  $a = b = 1$ ;
- (ii) to make  $c = 0$  (change  $x$  to  $x - \frac{c}{3}y$ );
- (iii) to make  $d = e = 0$  ( $z$  to  $z + dx + ey$ ).

Up to projective equivalence, there is only one irreducible cubic with a cusp:  $y^2z = x^3$ . It has no other singularities.

*Proof.*

- (1) Given any cubic

$$\begin{aligned} f = & ay^2z - bx^3 - cx^2y - dxy^2 - ey^3 \\ & + \alpha_1z^3 + \alpha_2x^2z + \alpha_3z^2x + \alpha_4z^2y + \alpha_5xyz. \end{aligned}$$

- (2)  $\alpha_1 = 0$  since  $P \in f$ .

- (3) Show that  $b \neq 0$  and  $\alpha_2 = \alpha_3 = 0$ . Since  $P = [0 : 0 : 1]$  is a cusp on  $f$  and  $y = 0$  is the tangent line to  $f$  at  $P$ ,  $I(f \cap y) = 3$ . So

$$\begin{aligned} 3 &= I(P, f \cap y) \\ &= I(P, (-bx^3 + \alpha_2 x^2 z + \alpha_3 z^2 x) \cap y) \\ &= I(P, (-bx^3 + \alpha_2 x^2 + \alpha_3 x) \cap y) \quad (\text{Dehomogenize w.r.t. } z) \end{aligned}$$

if and only if  $b \neq 0$  and  $\alpha_2 = \alpha_3 = 0$ .

- (4) Show that  $a \neq 0$  and  $\alpha_4 = \alpha_5 = 0$ . We dehomogenize  $f$  w.r.t.  $z$  to get

$$f_* = -bx^3 - cx^2y - dxy^2 - ey^3 + ay^2 + \alpha_5xy + \alpha_4y.$$

Since  $(0, 0)$  is a double point,  $\alpha_4 = 0$ . Since  $f_*$  has only one tangent  $y$  on  $(0, 0)$ ,

$$ay^2 + \alpha_5xy = y(ay + \alpha_5x)$$

implies that  $a \neq 0$  and  $\alpha_5 = 0$ .

- (5) Therefore,  $f = ay^2z - bx^3 - cx^2y - dxy^2 - ey^3$  with  $a \neq 0$  and  $b \neq 0$ . Similar to Problem 5.9,

- (i) Write

$$\begin{aligned} f &= ay^2z - bx^3 - cx^2y - dxy^2 - ey^3 \\ &= (\sqrt[3]{ay})^2(\sqrt[3]{az}) - (\sqrt[3]{bx})^3 - \frac{c}{\sqrt[3]{ab^2}}(\sqrt[3]{bx})^2(\sqrt[3]{ay}) \\ &\quad - \frac{d}{\sqrt[3]{a^2b}}(\sqrt[3]{bx})(\sqrt[3]{ay})^2 - \frac{e}{a}(\sqrt[3]{ay})^3. \end{aligned}$$

Apply a projective change of coordinates  $t_1 = \left(\frac{x}{\sqrt[3]{b}}, \frac{y}{\sqrt[3]{a}}, \frac{z}{\sqrt[3]{a}}\right)$  to  $f$ :

$$f^{t_1} = y^2z - x^3 - c_1x^2y - d_1xy^2 - e_1y^3,$$

where  $c_1 = \frac{c}{\sqrt[3]{ab^2}}$  and  $d_1 = \frac{d}{\sqrt[3]{a^2b}}$  and  $e_1 = \frac{e}{a}$ .

- (ii) Write

$$\begin{aligned} f^{t_1} &= y^2z - x^3 - c_1x^2y - d_1xy^2 - e_1y^3 \\ &= y^2z - \left(x + \frac{c_1}{3}y\right)^3 - \left(d_1 - \frac{c_1^2}{3}\right)\left(x + \frac{c_1}{3}y\right)y^2 \\ &\quad - \left(e_1 - \frac{c_1d_1}{3} + \frac{2c_1^3}{27}\right)y^3. \end{aligned}$$

Apply a projective change of coordinates  $t_2 = \left(x - \frac{c_1}{3}y, y, z\right)$  to  $f^{t_1}$ :

$$(f^{t_1})^{t_2} = y^2z - x^3 - d_2xy^2 - e_2y^3$$

where  $d_2 = d_1 - \frac{c_1^2}{3}$  and  $e_2 = e_1 - \frac{c_1d_1}{3} + \frac{2c_1^3}{27}$ . (Compare to Cardano's formula.)

(iii) Write

$$\begin{aligned}(f^{t_1})^{t_2} &= y^2z - x^3 - d_2xy^2 - e_2y^3 \\ &= y^2(z - d_2x - e_2y) - x^3.\end{aligned}$$

Apply a projective change of coordinates  $t_3 = (x, y, z + d_2x + e_2y)$  to  $(f^{t_1})^{t_2}$ :

$$((f^{t_1})^{t_2})^{t_3} = y^2z - x^3.$$

- (6) Given any irreducible cubic  $f$  with a cusp. Apply the proof of Problem 5.9(b),  $f$  is projectively equivalent to an irreducible cubic  $g$  such that  $P = [0 : 0 : 1]$  is a cusp on  $g$  and  $y = 0$  is the tangent line to  $g$  at  $P$ . By (5),

$$f \sim g \sim (y^2z - x^3).$$

Similar to Problem 5.9(b),  $y^2z - x^3$  has no other singularities except one point  $P = [0 : 0 : 1]$ .

□

### Problem 5.11. (Cubic with a node)

*Up to projective equivalence, there is only one irreducible cubic with a node:  $xyz = x^3 + y^3$ . It has no other singularities.*

*Proof.*

- (1) Let  $f$  be an irreducible cubic,  $P = [0 : 0 : 1]$  a node on  $f$ ,  $x = 0$ ,  $y = 0$  the tangent lines to  $f$  at  $P$ . Show that  $f = ax^3 + by^3 - cxyz - dx^2y - exy^2$ ,  $a \neq 0$ ,  $b \neq 0$ ,  $c \neq 0$ .

(a) Given any cubic

$$\begin{aligned}f &= ax^3 + by^3 - cxyz - dx^2y - exy^2 \\ &\quad + \alpha_1z^3 + \alpha_2x^2z + \alpha_3y^2z + \alpha_4z^2x + \alpha_5z^2y.\end{aligned}$$

(b)  $\alpha_1 = 0$  since  $P \in f$ .

(c) We dehomogenize  $f$  w.r.t.  $z$  to get

$$f_* = ax^3 + by^3 - cxy - dx^2y - exy^2 + \alpha_2x^2 + \alpha_3y^2 + \alpha_4x + \alpha_5y.$$

Since  $(0, 0)$  is a double point,  $\alpha_4 = \alpha_5 = 0$ . Since  $f_*$  have only two tangents  $x, y$  on  $(0, 0)$ ,  $-cxy + \alpha_2x^2 + \alpha_3y^2$  implies that  $c \neq 0$  and  $\alpha_2 = \alpha_3 = 0$ .

- (d) Now  $f = ax^3 + by^3 - cxyz - dx^2y - exy^2$  with  $c \neq 0$ . If  $a = 0$  or  $b = 0$ ,  $f$  is reducible, which is absurd. So  $a \neq 0$  and  $b \neq 0$ .

(2) Hence,

$$f \sim (x^3 + y^3 - xyz - dx^2y - exy^2) \sim (x^3 + y^3 - xyz)$$

by  $(x, y, z) \mapsto \left(\frac{1}{\sqrt[3]{a}}x, \frac{1}{\sqrt[3]{b}}y, \frac{\sqrt[3]{ab}}{c}z\right)$  and  $(x, y, z) \mapsto (x, y, z - dx - ey)$ .

(3) Given any irreducible cubic  $f$  with a node. Apply the proof of Problem 5.9(b),  $f$  is projectively equivalent to an irreducible cubic  $g$  such that  $P = [0 : 0 : 1]$  is a node on  $g$  and  $x, y$  are the tangent lines to  $g$  at  $P$ . By (2),

$$f \sim g \sim (x^3 + y^3 - xyz).$$

Similar to Problem 5.9(b),  $x^3 + y^3 - xyz$  has no other singularities except one point  $P = [0 : 0 : 1]$ .

□

**Problem 5.12.\***

(a) Assume  $P = [0 : 1 : 0] \notin f$ ,  $f$  a curve of degree  $n$ . Show that

$$\sum_P I(P, f \cap x) = n.$$

(b) Show that if  $f$  is a curve of degree  $n$ ,  $L$  a line not contained in  $f$ , then

$$\sum_P I(P, f \cap L) = n.$$

*Proof of (a).*

(1) Similar to Problem 5.7. The intersection points of  $f$  and  $x$  are the solutions of

$$f(0, y, z) = 0.$$

Write

$$f(0, y, z) = cy^n + \cdots$$

as a form in  $k[y, z]$  of degree  $n$ .

(2) Note that  $c \neq 0$  since  $P = [0 : 1 : 0] \notin f$ . By Corollary to Proposition 5 in §2.6,

$$f(0, y, z) = c \prod_i (y - \alpha_i z)^{r_i}$$

where  $\alpha_i \in k$  and  $\sum r_i = n$ . So all intersection points are  $P_i = [0 : \alpha_i : 1]$  with multiplicity  $r_i$ .



(3) Hence,

$$\begin{aligned}
& \sum_P I(P, f \cap x) \\
&= \sum_i I(P_i, f \cap x) \\
&= \sum_i I(P_i, f(0, y, z) \cap x) && \text{(Property (7) in §3.3)} \\
&= \sum_i \sum_j r_j I(P_i, (y - \alpha_j z) \cap x) && \text{(Property (6) in §3.3)} \\
&= \sum_i \sum_j r_j I((0, \alpha_i), (y - \alpha_j) \cap x) && \text{(Dehomogenize w.r.t. } z) \\
&= \sum_i \sum_j r_j I((0, 0), (y + \alpha_i - \alpha_j) \cap x) \\
&= \sum_i \sum_j r_j \delta_{ij} \\
&= \sum_i r_i \\
&= n.
\end{aligned}$$

□

*Proof of (b).*

- (1) Take any point  $Q_1 \notin f \cup L$  (since  $\mathbf{P}^2$  is irreducible). Suppose  $L$  is a line passing  $Q_2$  and  $Q_3$ ,  $Q_2 \neq Q_3$ .
- (2) By Problem 4.14, there is a projective change of coordinates  $t : \mathbf{P}^2 \rightarrow \mathbf{P}^2$  such that  $t(Q_1) = [0 : 1 : 0]$  and  $t(L) = x$ . In particular,  $t(Q_1) = [0 : 1 : 0] \notin f^t$  (by the construction in (1)).
- (3) Hence,

$$\sum_P I(P, f \cap L) = \sum_P I(P, f^t \cap L^t) = \sum_P I(P, f^t \cap x) = n.$$

□

### Problem 5.13.

*Prove that an irreducible cubic is either nonsingular or has at most one double point (a node or a cusp). (Hint: Use Problem 5.12, where  $L$  is a line through*

two multiple points; or use Problems 5.10 and 5.11.)

*Proof (Problem 5.12).*

- (1) (Reductio ad absurdum) Suppose there were a line  $L$  passing through two multiple points  $P, Q$  of an irreducible cubic  $f$ .
- (2) Note that

$$\begin{aligned}
 \sum I(P, f \cap L) &\geq I(P, f \cap L) + I(Q, f \cap L) \\
 &\geq m_P(f)m_P(L) + m_Q(f)m_Q(L) \\
 &\geq 2 \cdot 1 + 2 \cdot 1 \\
 &= 4.
 \end{aligned}$$

By Problem 5.12,  $\sum I(P, f \cap L) = 3$ , a contradiction. Hence  $f$  has at most one multiple point. In particular,  $f$  has at most one double point.

□

*Proof (Problems 5.10 and 5.11).*

- (1) Suppose one irreducible cubic  $f$  has one multiple point  $P$ . It suffices to show that  $P$  is a cusp or a node, or it suffices to show that  $P$  is a double point.
- (2) Up to projective equivalence, might assume that  $P = [0 : 0 : 1]$ . If  $m_P(f) = m_P(f_*) = 3$ , then  $f_*$  is a form in  $k[x, y]$  of degree 3. Thus  $f$  is in  $k[x, y] \hookrightarrow k[x, y, z]$ , which is reducible. Hence  $P$  is a double point. By Problems 5.10 and 5.11,  $f$  has no other singularities.

□

#### **Problem 5.14.\***

Let  $P_1, \dots, P_n \in \mathbf{P}^2$ . Show that there are an infinite number of lines passing through  $P_1$ , but not through  $P_2, \dots, P_n$ . If  $P_1$  is a simple point on  $f$ , we may take these lines transversal to  $f$  at  $P_1$ .

*Proof.*

- (1) Might assume that all  $P_1, \dots, P_n$  are distinct. Write a line  $L_{ij}$  passing through  $P_i$  and  $P_j$ .
- (2) (Reductio ad absurdum) Suppose there were finite number of lines passing through  $P_1$ , but not through  $P_2, \dots, P_n$ , say  $M_1, \dots, M_r$ . Hence, we can write  $\mathbf{P}^2$  as

$$\mathbf{P}^2 = L_{12} \cup \dots \cup L_{1n} \cup M_1 \cup \dots \cup M_r,$$

contrary to the fact that  $\mathbf{P}^2$  is irreducible.

□

**Problem 5.15.\***

Let  $C$  be an irreducible projective plane curve,  $P_1, \dots, P_n$  simple points on  $C$ ,  $m_1, \dots, m_n$  integers. Show that there is a  $z \in k(C)$  with  $\text{ord}_{P_i}(z) = m_i$  for  $i = 1, \dots, n$ . (Hint: Take lines  $L_i$  as in Problem 5.14 for  $P_i$ , and a line  $L_0$  not through any  $P_j$ , and let  $z = \prod L_i^{m_i} L_0^{-\sum m_i}$ .)

*Proof.*

- (1) Take lines  $L_i$  not tangent to  $C$  at  $P_i$  as in Problem 5.14 for  $P_i$ . Since  $\mathbf{P}^2 = V(0) \supsetneq V(C)$ , we can pick a point  $P$  outside  $C$ . Take one line  $L_0$  through  $P$  but not through any  $P_j$  as in Problem 5.14.
- (2) Define  $z = \prod L_i^{m_i} / L_0^{\sum m_i} \in k(C)$  since  $\prod L_i^{m_i}$  and  $L_0^{\sum m_i}$  are in  $\Gamma(C)$  of the same degree  $\sum m_i$ . Note that  $L_0(P_j) \neq 0$  for any  $P_j$ . So  $\text{ord}_{P_i}(z)$  is well-defined. Moreover,

$$\text{ord}_{P_i}(z) = \sum_j m_j \text{ord}_{P_i}(L_j) - \left( \sum m_j \right) \text{ord}_{P_i}(L_0) = m_i$$

since  $L_j$  ( $j \neq i$ ) and  $L_0$  are units in  $k(C)$  and  $L_i \ni P_i$  is not tangent to  $C$ .

□

**Problem 5.16.\***

Let  $f$  be an irreducible curve in  $\mathbf{P}^2$ . Suppose  $I(P, f \cap z) = 1$ , and  $P \neq [1 : 0 : 0]$ . Show that  $\frac{\partial f}{\partial x}(P) \neq 0$ . (Hint: If not, use Euler's Theorem to show that  $\frac{\partial f}{\partial y}(P) = 0$ ; but  $z$  is not tangent to  $f$  at  $P$ .)

*Proof.*

- (1) Might write  $P = [\alpha : 1 : 0] \in \mathbf{P}^2$  since  $z(P) = 0$  (by  $I(P, f \cap z) > 0$ ) and  $P \neq [1 : 0 : 0]$ .
- (2) Since  $I(P, f \cap z) = 1$ ,  $m_P(f) = 1$  and  $z$  is not tangent to  $f$  at  $P$ . Thus,  $P$  is a simple point of  $f$ . By Problem 5.4, the tangent line  $L$  to  $f$  at  $P$  is

$$\frac{\partial f}{\partial x}(P)x + \frac{\partial f}{\partial y}(P)y + \frac{\partial f}{\partial z}(P)z = 0.$$

In particular,  $P = [\alpha : 1 : 0] \in L$ , that is,

$$\frac{\partial f}{\partial x}(P)\alpha + \frac{\partial f}{\partial y}(P) = 0.$$

- (3) (Reductio ad absurdum) If  $\frac{\partial f}{\partial x}(P) = 0$ ,  $\frac{\partial f}{\partial y}(P) = 0$  by (2). Hence, the tangent line  $L = z$ , which is absurd.

□

## 5.2. Linear Systems of Curves

### Problem 5.17.

Let  $P_1, P_2, P_3, P_4$  are distinct points in  $\mathbf{P}^2$ . Let  $V$  be the linear system of conics passing through these points. Show that  $\dim(V) = 2$  if  $P_1, P_2, P_3, P_4$  lie on a line, and  $\dim(V) = 1$  otherwise.

*Proof.*

- (1) Let  $V_i = V(2; P_1, \dots, P_i)$ . (So  $V = V_4$ .) By Theorem 1 in §5.2, we have

$$\dim V_4 \geq \frac{2(2+3)}{2} - \sum_{i=1}^4 \frac{1(1+1)}{2} = 1,$$

$$\dim V_3 = \frac{2(2+3)}{2} - \sum_{i=1}^3 \frac{1(1+1)}{2} = 2.$$

Also note that  $V_4 \subseteq V_3$ . Hence

$$\dim V_4 = 1 \text{ or } 2.$$

Therefore, it suffices to show that  $P_1, P_2, P_3, P_4$  lie on a line if and only if

$$V_4 = V_3.$$

- (2) The case  $P_1, P_2, P_3, P_4$  lie on a line  $L$ . Take any conic  $C \in V_3$ . By Bézout's theorem or Problem 5.12,  $L \mid C$ . Hence  $P_4 \in L \subseteq C$  and thus  $C \in V_4$ .
- (3) The case  $P_1, P_2, P_3, P_4$  are not collinear.
- (a) Suppose there are some three points, say  $P_1, P_2, P_3$ , are in a line  $L$ . Since  $P_4 \notin L$  by assumption, the conic  $L^2$  is in  $V_3$  but not in  $V_4$ .
- (b) Suppose there are no three points lying on a line. Let  $L_{ij}$  be the line passing through  $P_i$  and  $P_j$ . Then the conic  $L_{12}L_{13} \in V_3$  but not in  $V_4$ .

By (a)(b),  $V_4 \subsetneq V_3$  if  $P_1, P_2, P_3, P_4$  are not collinear.

□

**Problem 5.18.**

Show that there is only one conic passing through the five points  $[0 : 0 : 1]$ ,  $[0 : 1 : 0]$ ,  $[1 : 0 : 0]$ ,  $[1 : 1 : 1]$ , and  $[1 : 2 : 3]$ ; show that it is nonsingular.

*Proof.*

- (1) Let  $C : ax^2 + bxy + cxz + dy^2 + eyz + fz^2$  be the conic passing these five points. That is,

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 6 & 9 \end{pmatrix}}_{\text{say } A \in \mathbf{M}_{5 \times 6}(k)} \underbrace{\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}}_{\in \mathbf{P}^5(k)} = 0.$$

- (2) The row echelon form  $A'$  of  $A$  is

$$A' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \mathbf{M}_{5 \times 6}(k).$$

$\text{rank}(A') = 5$  and thus there is a unique solution in  $\mathbf{P}^5$ , say  $[0 : 3 : -4 : 0 : 1 : 0] \in \mathbf{P}^5$ . Hence,

$$C : 3xy - 4xz + yz = 0$$

is only one conic passing through these five points.

- (3) By solving the system of equations  $C(P) = \frac{\partial C}{\partial x}(P) = \frac{\partial C}{\partial y}(P) = \frac{\partial C}{\partial z}(P) = 0$ ,  $P = [0 : 0 : 0]$ , which is absurd. Hence  $C$  is nonsingular.

□

**Problem 5.19.**

Consider the nine points  $[0 : 0 : 1]$ ,  $[0 : 1 : 1]$ ,  $[1 : 0 : 1]$ ,  $[1 : 1 : 1]$ ,  $[0 : 2 : 1]$ ,  $[2 : 0 : 1]$ ,  $[1 : 2 : 1]$ ,  $[2 : 1 : 1]$ , and  $[2 : 2 : 1] \in \mathbf{P}^2$  (Sketch). Show that there are an infinite number of cubics passing through these points.

*Proof.*

- (1) Let  $C : a_1x^3 + a_2y^3 + a_3z^3 + a_4x^2y + a_5x^2z + a_6y^2x + a_7y^2z + a_8z^2x + a_9z^2y + a_{10}xyz$  be the cubic passing these five points. That is,

$$\underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 8 & 1 & 0 & 0 & 0 & 4 & 0 & 2 & 0 \\ 8 & 0 & 1 & 0 & 4 & 0 & 0 & 2 & 0 & 0 \\ 1 & 8 & 1 & 2 & 1 & 4 & 4 & 1 & 2 & 2 \\ 8 & 1 & 1 & 4 & 4 & 2 & 1 & 2 & 1 & 2 \\ 8 & 8 & 1 & 8 & 4 & 8 & 4 & 2 & 2 & 4 \end{pmatrix}}_{\text{say } A \in M_{9 \times 10}(k)} \underbrace{\begin{pmatrix} a_1 \\ \vdots \\ a_{10} \end{pmatrix}}_{\in \mathbf{P}^9(k)} = 0.$$

- (2) The row echelon form  $A'$  of  $A$  is

$$A' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in M_{9 \times 10}(k).$$

$\text{rank}(A') = 8 < 9$  and thus there are an infinite number of cubics passing through these points, say

$$C : \lambda x(x - z)(x - 2z) + \mu y(y - z)(y - 2z),$$

where  $[\lambda : \mu] \in \mathbf{P}^1(k)$ .

□

### 5.3. Bézout's Theorem

#### Problem 5.20.

*Check your answers of Problem 5.3 with Bézout's theorem*

*Proof.*

$$(1) \quad f = y^2z - x(x - 2z)(x + z) \text{ and } g = y^2 + x^2 - 2xz.$$

$$I([0 : 0 : 1], f \cap g) = 2,$$

$$I([2 : 0 : 1], f \cap g) = 2,$$

$$I([-2, \sqrt{-8} : 1], f \cap g) = 1,$$

$$I([-2, -\sqrt{-8} : 1], f \cap g) = 1.$$

$$\text{Hence, } \sum_P I(P, f \cap g) = 2 + 2 + 1 + 1 = 6 = \deg(f) \deg(g).$$

$$(2) \quad f = (x^2 + y^2)z + x^3 + y^3 \text{ and } g = x^3 + y^3 - 2xyz.$$

$$I([0 : 0 : 1], f \cap g) = 4,$$

$$I([1 : -1 : 0], f \cap g) = 3,$$

$$I([1 : \sqrt[3]{-1} : 0], f \cap g) = 1,$$

$$I([1 : 1 - \sqrt[3]{-1} : 0], f \cap g) = 1.$$

$$\text{Hence, } \sum_P I(P, f \cap g) = 4 + 3 + 1 + 1 = 9 = \deg(f) \deg(g).$$

$$(3) \quad f = y^5 - x(y^2 - xz)^2 \text{ and } g = y^4 + y^3z - x^2z^2.$$

$$I([0 : 0 : 1], f \cap g) = 9,$$

$$I([1 : 0 : 0], f \cap g) = 9,$$

$$I\left(\left[\frac{1}{2} : 1 : 2 + 2\sqrt{2}\right], f \cap g\right) = 1,$$

$$I\left(\left[\frac{1}{2} : 1 : 2 - 2\sqrt{2}\right], f \cap g\right) = 1.$$

$$\text{Hence, } \sum_P I(P, f \cap g) = 9 + 9 + 1 + 1 = 20 = \deg(f) \deg(g).$$

$$(4) \quad f = (x^2 + y^2)^2 + 3x^2yz - y^3z \text{ and } g = (x^2 + y^2)^3 - 4x^2y^2z^2.$$

$$I([1 : \sqrt{-1} : 0], f \cap g) = 3,$$

$$I([1 : -\sqrt{-1} : 0], f \cap g) = 3,$$

$$I([0 : 0 : 1], f \cap g) = 14,$$

$$I\left(\left[\frac{\sqrt{5} + 2\sqrt{5}}{4} : \frac{-\sqrt{5}}{4} : 1\right], f \cap g\right) = 1,$$

$$I\left(\left[-\frac{\sqrt{5} + 2\sqrt{5}}{4} : \frac{-\sqrt{5}}{4} : 1\right], f \cap g\right) = 1,$$

$$I\left(\left[\frac{\sqrt{5} - 2\sqrt{5}}{4} : \frac{\sqrt{5}}{4} : 1\right], f \cap g\right) = 1,$$

$$I\left(\left[-\frac{\sqrt{5} - 2\sqrt{5}}{4} : \frac{\sqrt{5}}{4} : 1\right], f \cap g\right) = 1.$$

$$\text{Hence, } \sum_P I(P, f \cap g) = 3 + 3 + 14 + 1 + 1 + 1 + 1 = 24 = \deg(f) \deg(g).$$

□

**Problem 5.21.\***

*Show that every nonsingular projective plane curve is irreducible. Is this true for affine curves?*

*Proof.*

- (1) Let  $f$  be a nonsingular projective plane curve. (Reductio ad absurdum) Suppose  $f = gh$  were reducible. Then  $g$  and  $h$  intersect in  $\mathbf{P}^2$  (Bézout's theorem).
- (2) Show that each intersection point  $P$  is a singularity of  $f$ . Note that  $g(P) = h(P) = 0$  and the chain rule for derivatives. Thus,

$$\frac{\partial f}{\partial x_i}(P) = g(P) \frac{\partial h}{\partial x_i}(P) + \frac{\partial g}{\partial x_i}(P) h(P) = 0$$

for  $i = 1, 2, 3$ .  $P$  is a singularity of  $f$  (Problem 5.1).

- (3) Show that it is not true for affine curves.  $V(x^2 - x) \subseteq \mathbf{A}^2$  is nonsingular but not irreducible.

□

**Problem 5.22.\***

*Let  $f$  be an irreducible curve of degree  $n$ . Assume  $\frac{\partial f}{\partial x} \neq 0$ . Apply Corollary 1 to  $f$  and  $\frac{\partial f}{\partial x}$ , and conclude that  $\sum m_P(f)(m_P(f) - 1) \leq n(n - 1)$ . In particular,  $f$  has at most  $\frac{n(n-1)}{2}$  multiple points. (See Problems 5.6, 5.8.)*

*Proof.*

- (1) There is nothing to do for  $n = 1$ . Might assume that  $n > 1$ .
- (2) Since  $f$  and  $\frac{\partial f}{\partial x}$  have no common component (Problem 5.8), Corollary 1 to Bézout's theorem gives us

$$\sum m_P(f)m_P\left(\frac{\partial f}{\partial x}\right) \leq \deg(f) \deg\left(\frac{\partial f}{\partial x}\right) \leq n(n - 1).$$

By Problem 5.6, the conclusion is established.



(3) Then

$$\begin{aligned}
 n(n-1) &\geq \sum_P m_P(f)(m_P(f)-1) \\
 &\geq \sum_{m_P(f)>1} 2(2-1) \\
 &\geq \sum_{m_P(f)>1} 2 \\
 &= 2 \cdot (\text{number of multiple points}).
 \end{aligned}$$

□

**Problem 5.23. (Hessian matrix)**

A problem about flexes (see Problem 3.12): Let  $f$  be a projective plane curve of degree  $n$ , and assume  $f$  contains no lines.

Let  $D_i f = \frac{\partial f}{\partial x_i}$  and  $D_{ij} f = \frac{\partial^2 f}{\partial x_i \partial x_j}$ , forms of degree  $n-1$  and  $n-2$  respectively. Form a  $3 \times 3$  matrix with the entry in the  $(i, j)$ th place being  $D_{ij} f$  ( $i, j = 1, 2, 3$ ). Let  $H$  be the determinant of this matrix, a form of degree  $3(n-2)$ . This  $H$  is called the **Hessian** of  $f$ . Problems 5.22 and 6.47 show that  $H \neq 0$ , for  $f$  irreducible. The following theorem shows the relationship between flexes and the Hessian.

**Theorem.** ( $\text{char}(k) = 0$ )

- (1)  $P \in H \cap f$  if and only if  $P$  is either a flex or a multiple point of  $f$ .
- (2)  $I(P, H \cap f) = 1$  if and only if  $P$  is an ordinary flex.

*Outline of proof.*

- (a) Let  $t$  be a projective change of coordinates. Then the Hessian of  $f^t$  is  $\det(t)^2(H^t)$ . So we can assume  $P = [0 : 0 : 1]$ ; write  $f_*(x, y) = f(x, y, 1)$  and  $H_*(x, y) = H(x, y, 1)$ .
- (b)  $(n-1)D_j f = \sum_i x_i D_{ij} f$ . (Use Euler's Theorem.)
- (c)  $I(P, f_* \cap H_*) = I(P, f_* \cap g)$  where

$$g = (D_2 f_*)^2 (D_{11} f_*) + (D_1 f_*)^2 (D_{22} f_*) - 2(D_1 f_*)(D_2 f_*)(D_{12} f_*).$$

(Hint: Perform row and column operations on the matrix for  $H_*$ . Add  $x$  times the first row plus  $y$  times the second row to the third row, then apply part (b). Do the same with the columns. Then calculate the determinant.)

(d) If  $P$  is a multiple point on  $f$ , then  $I(P, f_* \cap g) > 1$ .

(e) Suppose  $P$  is a simple point,  $y = 0$  is the tangent line to  $f$  at  $P$ , so

$$f_* = y + ax^2 + bxy + cy^2 + dx^3 + ex^2y + \cdots.$$

Then  $P$  is a flex if and only if  $a = 0$ , and  $P$  is an ordinary flex if and only if  $a = 0$  and  $d \neq 0$ . A short calculation shows that

$$g = 2a + 6dx + (8ac - 2b^2 + 2e)y + \text{higher terms},$$

which concludes the proof.

**Corollary.**

- (1) A nonsingular curve of degree  $> 2$  always has a flex.
- (2) A nonsingular cubic has nine flexes, all ordinary.

*Proof of (a).*

- (1) Write  $t = (t_1, t_2, t_3)$  where  $t_1, t_2, t_3 \in k[x, y, z]$ .
- (2) By the chain rule for differentiation,

$$\frac{\partial f^t}{\partial x_j} = \sum_{\beta=1}^3 \frac{\partial f^t}{\partial t_\beta} \frac{\partial t_\beta}{\partial x_j}$$

and thus

$$\begin{aligned} \frac{\partial^2 f^t}{\partial x_i \partial x_j} &= \frac{\partial}{\partial x_i} \left( \frac{\partial f^t}{\partial x_j} \right) \\ &= \frac{\partial}{\partial x_i} \left( \sum_{\beta=1}^3 \frac{\partial f^t}{\partial t_\beta} \frac{\partial t_\beta}{\partial x_j} \right) \\ &= \sum_{\beta=1}^3 \frac{\partial}{\partial x_i} \left( \frac{\partial f^t}{\partial t_\beta} \right) \cdot \frac{\partial t_\beta}{\partial x_j} + \frac{\partial f^t}{\partial t_\beta} \cdot \underbrace{\frac{\partial^2 t_\beta}{\partial x_i \partial x_j}}_{=0} \\ &= \sum_{\beta=1}^3 \sum_{\alpha=1}^3 \frac{\partial t_\alpha}{\partial x_i} \left( \frac{\partial f^t}{\partial t_\alpha \partial t_\beta} \right) \cdot \frac{\partial t_\beta}{\partial x_j} \\ &= \sum_{\alpha=1}^3 \sum_{\beta=1}^3 \frac{\partial t_\alpha}{\partial x_i} \left( \frac{\partial f^t}{\partial t_\alpha \partial t_\beta} \right) \frac{\partial t_\beta}{\partial x_j}. \end{aligned}$$

(3) Hence, the Hessian of  $f^t$  is

$$\begin{aligned}
H(f^t) &= \det \left( \left( \frac{\partial^2 f^t}{\partial x_i \partial x_j} \right)_{i,j} \right) \\
&= \det \left( \left( \sum_{\alpha=1}^3 \sum_{\beta=1}^3 \frac{\partial t_\alpha}{\partial x_i} \left( \frac{\partial f^t}{\partial t_\alpha \partial t_\beta} \right) \frac{\partial t_\beta}{\partial x_j} \right)_{i,j} \right) \\
&= \det \left( \left( \frac{\partial t_\alpha}{\partial x_i} \right)_{\alpha,i}^T \left( \frac{\partial f^t}{\partial t_\alpha \partial t_\beta} \right)_{\alpha,\beta} \left( \frac{\partial t_\beta}{\partial x_j} \right)_{\beta,j} \right) \\
&= \det \left( \left( \frac{\partial t_\alpha}{\partial x_i} \right)_{\alpha,i}^T \right) \det \left( \left( \frac{\partial f^t}{\partial t_\alpha \partial t_\beta} \right)_{\alpha,\beta} \right) \det \left( \left( \frac{\partial t_\beta}{\partial x_j} \right)_{\beta,j} \right) \\
&= \det(t^T) \cdot H(f)^t \cdot \det(t) \\
&= \det(t)^2 H(f)^t.
\end{aligned}$$

(4) Note that  $t$  is invertible. So we can assume  $P = [0 : 0 : 1]$ .

□

*Proof of (b).* Since  $D_j f$  is a form of degree  $n - 1$ , the result is established by Euler's theorem in §1.1. □

*Proof of (c).*

(1) Note that

$$\begin{aligned}
& z^2 H \\
&= z^2 \begin{vmatrix} D_{11}f & D_{12}f & D_{13}f \\ D_{21}f & D_{22}f & D_{23}f \\ D_{31}f & D_{32}f & D_{33}f \end{vmatrix} \\
&= z \begin{vmatrix} D_{11}f & D_{12}f & D_{13}f \\ D_{21}f & D_{22}f & D_{23}f \\ xD_{11}f + yD_{21}f + zD_{31}f & xD_{12}f + yD_{22}f + zD_{32}f & xD_{13}f + yD_{23}f + zD_{33}f \end{vmatrix} \\
&= z \begin{vmatrix} D_{11}f & D_{12}f & D_{13}f \\ D_{21}f & D_{22}f & D_{23}f \\ (n-1)D_1f & (n-1)D_2f & (n-1)D_3f \end{vmatrix} \\
&= \begin{vmatrix} D_{11}f & D_{12}f & xD_{11}f + yD_{12}f + zD_{13}f \\ D_{21}f & D_{22}f & xD_{21}f + yD_{22}f + zD_{23}f \\ (n-1)D_1f & (n-1)D_2f & (n-1)xD_1f + (n-1)yD_2f + (n-1)zD_3f \end{vmatrix} \\
&= \begin{vmatrix} D_{11}f & D_{12}f & (n-1)D_1f \\ D_{21}f & D_{22}f & (n-1)D_2f \\ (n-1)D_1f & (n-1)D_2f & n(n-1)f \end{vmatrix} \\
&= (n-1)^2(D_1f)(D_{21}fD_2f - D_{22}fD_1f) \\
&\quad - (n-1)^2(D_2f)(D_{11}fD_2f - D_{12}fD_1f) \\
&\quad + n(n-1)f(D_{11}fD_{22}f - D_{12}fD_{21}f) \\
&= (n-1)^2(-(D_1f)^2D_{22}f - (D_2f)^2D_{11}f + 2(D_1f)(D_2f)(D_{12}f)) \\
&\quad + n(n-1)f(D_{11}fD_{22}f - D_{12}fD_{21}f).
\end{aligned}$$

(2) Dehomogenize  $z^2H$  w.r.t.  $z$ :

$$H_* = -(n-1)^2g + n(n-1)f_*[(D_{11}f_*)(D_{22}f_*) - (D_{12}f_*)^2].$$

Hence if  $n > 1$ , then

$$I(P, f_* \cap H_*) = I(P, f_* \cap (-(n-1)^2g)) = I(P, f_* \cap g).$$

(If  $n \leq 1$ , then  $I(P, f_* \cap H_*) = I(P, f_* \cap g) = \infty$ .)

□

*Proof of (d).*

(1) Since  $I(P, f_* \cap g) \geq m_P(f_*)m_P(g) \geq 2m_P(g)$ , it suffices to show that  $m_P(g) > 0$  or  $P \in g$ .

(2) Problem 5.1 shows that

$$\begin{aligned}
g(P) &= (D_1 f_*(P))^2 D_{22} f_*(P) \\
&\quad + (D_2 f_*(P))^2 D_{11} f_*(P) \\
&\quad - 2(D_1 f_*(P))(D_2 f_*(P))(D_{12} f_*(P)) \\
&= 0^2 \cdot D_{22} f_*(P) + 0^2 \cdot D_{11} f_*(P) - 2 \cdot 0 \cdot 0 \cdot D_{12} f_*(P) \\
&= 0.
\end{aligned}$$

□

*Proof of (e).*

(1) Suppose  $P$  is a simple point of  $f$ ,  $y = 0$  is the tangent line to  $f$  at  $P$ . Write

$$f_* = y + ax^2 + bxy + cy^2 + dx^3 + ex^2y + \cdots.$$

Then  $P$  is a flex if and only if  $a = 0$ , and  $P$  is an ordinary flex if and only if  $a = 0$  and  $d \neq 0$  (Problem 3.12).

(2) A short calculation shows that

$$\begin{aligned}
g &= (D_2 f_*)^2 (D_{11} f_*) + (D_1 f_*)^2 (D_{22} f_*) - 2(D_1 f_*)(D_2 f_*)(D_{12} f_*) \\
&= [2a + (6d + 4ab)x + (8ac + 2e)y] + 0 - 2[2abx + b^2y] + \text{higher terms} \\
&= 2a + 6dx + (8ac - 2b^2 + 2e)y + \text{higher terms}.
\end{aligned}$$

(3) Hence,

$$\begin{aligned}
P &\in f_* \cap g \\
\iff I(P, f_* \cap g) &\geq 1 \\
\iff I(P, (y + \cdots) \cap (2a + 6dx + (8ac - 2b^2 + 2e)y + \cdots)) &\geq 1 \\
\iff a &= 0 \\
\iff P &\text{ is a flex}
\end{aligned}$$

if  $P$  is a simple point of  $f$ .

(4) Also note that

$$\begin{aligned}
I(P, f_* \cap g) &= 1 \\
\iff I(P, (y + \cdots) \cap (2a + 6dx + (8ac - 2b^2 + 2e)y + \cdots)) &= 1 \\
\iff a = 0 \text{ and } d \neq 0 & \\
\iff P &\text{ is an ordinary flex}
\end{aligned}$$

if  $P$  is a simple point of  $f$ .

□

*Proof of Theorem (1).*

- (1) ( $\implies$ ) Suppose  $P \in H \cap f$  and  $P$  is not a multiple point of  $f$ . So  $P$  is a simple point of  $f$ . Note that

$$I(P, H \cap f) = I(P, H_* \cap f_*) = I(P, g \cap f_*) \geq 1.$$

So  $P$  is a flex by part (e).

- (2) ( $\impliedby$ ) If  $P$  is a flex of  $f$ , then part (e) shows that

$$I(P, H \cap f) = I(P, H_* \cap f_*) = I(P, g \cap f_*) \geq 1.$$

If  $P$  is a multiple point of  $f$ , then part (d) shows that

$$I(P, H \cap f) = I(P, H_* \cap f_*) \geq 2.$$

In any case,  $P \in H \cap f$ .

□

*Proof of Theorem (2).*

- (1) ( $\implies$ ) By the proof of Theorem (1),  $P$  is a simple point of  $f$ . Therefore, the conclusion is established by part (e).  
 (2) ( $\impliedby$ ) Part (e).

□

*Proof of Corollary (1).*

- (1) Let  $f$  be a nonsingular curve of degree  $> 2$ . So  $\deg(H) = 3(n-2) > 0$ .  $H \cap f \neq \emptyset$  (Bézout's theorem).  
 (2) Since  $f$  is nonsingular, every point in  $H \cap f$  is a simple point. So every point in  $H \cap f$  is a flex (Theorem (1)).

□

*Proof of Corollary (2).*

- (1) *Show that every flex of a nonsingular cubic  $f$  is ordinary.* Given any flex  $P \in f$  and the corresponding tangent line  $L$  to  $f$  at  $P$ . By definition,

$$\text{ord}_P^f(L) = I(P, f \cap L) \geq 3.$$

Bézout's theorem or Problem 5.12 shows that

$$\sum_P I(P, f \cap L) = I(P, f \cap L) = 3.$$

That is,  $\text{ord}_P^f(L) = 3$  or  $P$  is an ordinary flex.

(2) Since  $f$  is nonsingular, we have

$$\begin{aligned} H \cap f &= \{\text{all flexes of } f\} && \text{(Theorem (1))} \\ &= \{\text{all ordinary flexes of } f\} && ((1)) \\ &= \{P \in f : I(P, H \cap f) = 1\}. && \text{(Theorem (2))} \end{aligned}$$

(3) Hence

$$\sum_P I(P, H \cap f) = \deg(H) \deg(f) = 3 \cdot 3 = 9$$

(Bézout's theorem) shows that there are exactly 9 (ordinary) flexes of  $f$ .  
(Also see Problem 5.37.)

□

### Problem 5.24.

(char( $k$ ) = 0)

- (a) Let  $P := [0 : 1 : 0]$  be a flex on an irreducible cubic  $f$ ,  $z = 0$  the tangent line to  $f$  at  $[0 : 1 : 0]$ . Show that  $f = zy^2 + byz^2 + cyxz + \text{terms in } x, z$ . Find a projective change of coordinates (using  $y \mapsto y - \frac{b}{2}z - \frac{c}{2}x$ ) to get  $f$  to the form  $zy^2 = \text{cubic in } x, z$ .
- (b) Show that any irreducible cubic is projectively equivalent to one of the following:  $y^2z = x^3$ ,  $y^2z = x^2(x + z)$ , or  $y^2z = x(x - z)(x - \lambda z)$ ,  $\lambda \in k$ ,  $\lambda \neq 0, 1$ . (See Problems 5.10, 5.11.)

*Proof of (a).*

- (1) As  $z = 0$  is the tangent line to  $f$  at  $[0 : 1 : 0]$ , we can write

$$f(x, 1, z) = z + ax^2 + bz^2 + cxz - \alpha_0x^3 - \alpha_1x^2z - \alpha_2xz^2 - \alpha_3z^3.$$

By assumption,

$$I(P, f \cap z) = I(P, (ax^2 - \alpha_0x^3) \cap z) \geq 3$$

implies that  $a = 0$ . Also,  $\alpha_0 \neq 0$  since  $f$  is irreducible. Hence,

$$f(x, y, z) = zy^2 + byz^2 + cyxz - \underbrace{\alpha_0}_{\neq 0} x^3 - \alpha_1x^2z - \alpha_2xz^2 - \alpha_3z^3.$$

- (2) So

$$\begin{aligned} f &\sim y^2z - \underbrace{\alpha_0}_{:=\beta_0} x^3 - \underbrace{\left(\alpha_1 - \frac{c^2}{4}\right)}_{:=\beta_1} x^2z - \underbrace{\left(\alpha_2 - \frac{bc}{2}\right)}_{:=\beta_2} xz^2 - \underbrace{\left(\alpha_3 - \frac{b^2}{4}\right)}_{:=\beta_3} z^3 \\ &= y^2z - \beta_0x^3 - \beta_1x^2z - \beta_2xz^2 - \beta_3z^3 \end{aligned}$$

by  $y \mapsto y - \frac{b}{2}z - \frac{c}{2}x$ . Hence  $f$  is of the form  $zy^2 = \text{cubic in } x, z$  (up to projective equivalence).

□

*Proof of (b).*

(1) Since  $\beta_0 \neq 0$  and  $k = \bar{k}$ , by  $x \mapsto \beta_0^{-\frac{1}{3}}x$  we might write

$$f \sim y^2z - x^3 - \beta_1x^2z - \beta_2xz^2 - \beta_3z^3.$$

(2) Since  $k = \bar{k}$ ,

$$x^3 + \beta_1x^2z + \beta_2xz^2 + \beta_3z^3 = (x - \gamma_1z)(x - \gamma_2z)(x - \gamma_3z)$$

for some  $\gamma_1, \gamma_2, \gamma_3 \in k$ . Consider the cases about  $\gamma_1, \gamma_2, \gamma_3$ .

(3) The case  $\gamma_1 = \gamma_2 = \gamma_3$ .

$$\begin{aligned} f &\sim y^2z - (x - \gamma_1z)(x - \gamma_2z)(x - \gamma_3z) \\ &\sim y^2z - x^3. \end{aligned} \quad (x \mapsto x + \gamma_1z)$$

(4) The case  $\gamma_1 = \gamma_2 \neq \gamma_3$ .

$$\begin{aligned} f &\sim y^2z - (x - \gamma_1z)(x - \gamma_2z)(x - \gamma_3z) \\ &\sim y^2z - x^2(x + (\gamma_1 - \gamma_3)z) && (x \mapsto x + \gamma_1z) \\ &\sim (\gamma_1 - \gamma_3)^{-1}y^2z - x^2(x + z) && (z \mapsto (\gamma_1 - \gamma_3)^{-1}z) \\ &\sim y^2z - x^2(x + z). && (y \mapsto (\gamma_1 - \gamma_3)^{\frac{1}{2}}y) \end{aligned}$$

(5) The case all  $\gamma_1, \gamma_2, \gamma_3$  are different.

$$\begin{aligned} f &\sim y^2z - (x - \gamma_1z)(x - \gamma_2z)(x - \gamma_3z) \\ &\sim y^2z - x(x - (\gamma_2 - \gamma_1)z)(x - (\gamma_3 - \gamma_1)z) && (x \mapsto x + \gamma_1z) \\ &\sim (\gamma_2 - \gamma_1)^{-1}y^2z - x^2(x - z) \left( x - \frac{\gamma_3 - \gamma_1}{\gamma_2 - \gamma_1}z \right) && (z \mapsto (\gamma_2 - \gamma_1)^{-1}z) \\ &\sim y^2z - x^2(x - z) \left( x - \frac{\gamma_3 - \gamma_1}{\gamma_2 - \gamma_1}z \right) && (y \mapsto (\gamma_2 - \gamma_1)^{\frac{1}{2}}y) \\ &\sim y^2z - x^2(x - z)(x - \lambda z). && \left( \lambda := \frac{\gamma_3 - \gamma_1}{\gamma_2 - \gamma_1}, \lambda \neq 0, 1 \right) \end{aligned}$$

□



## 5.4. Multiple Points

### Problem 5.25.

Let  $f$  be a projective plane curve of degree  $n$  with no multiple components, and  $c$  simple components. Show that

$$\sum \frac{m_P(m_P - 1)}{2} \leq \frac{(n-1)(n-2)}{2} + c - 1 \leq \frac{n(n-1)}{2}.$$

(Hint: Let  $f = f_1 f_2$ ; consider separately the points on one  $f_i$  or on both.)

*Proof.*

(1) Write

$$f = f_1 \cdots f_c,$$

where each  $f_i$  is irreducible. By assumption,  $f_i$  and  $f_j$  have no common factor whenever  $i \neq j$ . By Bézout's theorem, we might consider separately the points on one  $f_i$  or on many components.

(2) Hence

$$\begin{aligned} & \sum \frac{m_P(m_P - 1)}{2} \\ & \leq \sum_{P \in f_1} \frac{m_P(m_P - 1)}{2} + \sum_{P \notin f_1} \frac{m_P(m_P - 1)}{2} \\ & \leq \underbrace{\frac{(d-1)(d-2)}{2}}_{\text{Theorem 2 in §5.4}} + \underbrace{\frac{(n-d-1)(n-d-2)}{2}}_{\text{Mathematical induction}} + (c-1) - 1 \\ & = \frac{(n-1)(n-2)}{2} + c - 1 + \underbrace{1 - d(n-d)}_{\leq 0} \\ & \leq \frac{(n-1)(n-2)}{2} + c - 1 \end{aligned}$$

where  $d = \deg(f_1) \geq 1$ .

(3) Note that  $c \leq n$ . So

$$\frac{(n-1)(n-2)}{2} + c - 1 \leq \frac{(n-1)(n-2)}{2} + n - 1 = \frac{n(n-1)}{2}.$$

□

**Problem 5.26.\***

(char( $k$ ) = 0) Let  $f$  be an irreducible curve of degree  $n$  in  $\mathbf{P}^2$ . Suppose  $P \in \mathbf{P}^2$ , with  $m_P(f) = r \geq 0$ . Then for all but a finite number of lines  $L$  through  $P$ ,  $L$  intersects  $f$  in  $n - r$  distinct points other than  $P$ . We outline a proof:

- (a) We may assume  $P = [0 : 1 : 0]$ . If  $L_\lambda = \{[\lambda : t : 1] : t \in k\} \cup \{P\}$ , we need only consider the  $L_\lambda$ . Then  $f = a_r(x, z)y^{n-r} + \cdots + a_n(x, z)$ ,  $a_r \neq 0$ . (See Problems 4.24, 5.5).
- (b) Let  $g_\lambda(t) = f(\lambda, t, 1)$ . It is enough to show that for all but a finite number of  $\lambda$ ,  $g_\lambda$  has  $n - r$  distinct points.
- (c) Show that  $g_\lambda$  has  $n - r$  distinct roots if  $a_r(\lambda, 1) \neq 0$ , and  $f \cap \frac{\partial f}{\partial y} \cap L_\lambda = \{P\}$  if  $P \in f$  (see Problem 1.53).

*Proof of (a).*

- (1) Might assume  $P = [0 : 1 : 0]$  (up to projective equivalence).
- (2) By Problem 4.24, we might consider the lines  $L_\lambda = \{[\lambda : t : 1] : t \in k\} \cup \{P\}$  except finitely one line  $L_\infty$ .
- (3) Write  $f = a_r(x, z)y^{n-r} + \cdots + a_n(x, z)$ ,  $a_r \neq 0$  (Problem 5.5).

□

*Proof of (b).* Let  $g_\lambda(t) = f(\lambda, t, 1) = a_r(\lambda, 1)t^{n-r} + \cdots + a_n(\lambda, 1)$ . It suffices to show that all but a finite number of  $\lambda$ ,  $g_\lambda(t)$  has  $n - r$  distinct roots in  $k$ . □

*Proof of (c).*

- (1) There are finitely number of  $\lambda \in k$  such that  $a_r(\lambda, 1) = 0$  since  $k$  is a field.
- (2)  $g_\lambda(t)$  is irreducible over  $(k[\lambda])[t]$  since  $f$  is irreducible. Hence  $a_r(\lambda, 1) \neq 0$  and char( $k$ ) = 0 implies that  $g_\lambda(t)$  has  $n - r$  distinct roots in  $k$  (Problem 1.53).
- (3) By (1)(2), the result is established.
- (4) Suppose  $P \in f$ . Show that  $f \cap \frac{\partial f}{\partial y} \cap L_\lambda = \{P\}$  (for all but finitely number of  $\lambda$ ). Since  $f \cap L_\lambda = \{P\}$ , it suffices to show that  $P \in \frac{\partial f}{\partial y}$ .
  - (a) It is nothing to do if  $P$  is a multiple point of  $f$  (Problem 5.1).
  - (b) If  $P$  is a simple point of  $f$ , then the tangent line to  $f$  at  $P$  is  $L = \frac{\partial f}{\partial x}(P)x + \frac{\partial f}{\partial y}(P)y + \frac{\partial f}{\partial z}(P)z$  (Problem 5.4). Since  $P \in L$ ,  $\frac{\partial f}{\partial y}(P) = 0$ .

In any case,  $\frac{\partial f}{\partial y}(P) = 0$ .

□

**Problem 5.27.**

Show that Problem 5.26 remains true if  $f$  is reducible, provided it has no multiple components.

*Proof.*

- (1) Write  $f = f_1 \cdots f_s$  where each  $f_i$  is irreducible and  $f_i, f_j$  have no common factor whenever  $i \neq j$ .
- (2) Apply Problem 5.26 to each  $f_i$  to get that for all but a finite number of lines  $L_i$  through  $P$ ,  $L_i$  intersects  $f_i$  in  $n_i - r_i$  distinct points other than  $P$ , where  $n_i = \deg(f_i)$  and  $r_i = m_P(f_i)$ . Collect these lines as  $\mathcal{L}_i$ .
- (3) Since  $f_i \cap f_j$  is finite whenever  $i \neq j$  (Problem 5.7), there are finitely many lines through  $P$  also passing through  $f_i \cap f_j$  for any  $i \neq j$ . Collect these lines as  $\mathcal{L}'$ .
- (4) Hence any line from

$$(\mathcal{L}_1 \cup \cdots \cup \mathcal{L}_s) - \mathcal{L}'$$

passes through  $P$  and intersects  $f$  in

$$\sum (n_i - r_i) = \sum n_i - \sum r_i = n - r$$

distinct points other than  $P$  (by part (2) and our construction). The result is established.

□

**Problem 5.28. (Terrible point)**

( $\text{char}(k) = p > 0$ )  $f = x^{p+1} - y^p z$ ,  $P = [0 : 1 : 0]$ . Find  $L \cap f$  for all lines  $L$  passing through  $P$ . Show that every line that is tangent to  $f$  at a simple point passes through  $P$ !

*Proof.*

- (1) The case  $\text{char}(k) = p > 0$  for Problem 4.24 shows that  $L = L_\lambda = V(x - \lambda z)$  for any  $\lambda \in k$  or  $L = L_\infty = V(z)$ . Note that  $L_0 \cap f = \{P\}$  and  $L_\infty \cap f = \{P\}$ . It suffices to consider the case  $L_\lambda \cap f$  for  $\lambda \neq 0$ .

(2) So

$$\begin{aligned}
0 &= x^{p+1} - y^p z \\
&= x^{p+1} - y^p (\lambda^{-1} x) && (x - \lambda z = 0) \\
&= x(x^p - \lambda^{-1} y^p) \\
&= x \left( x^p - \left( \lambda^{-\frac{1}{p}} y \right)^p \right) && (k = \bar{k}) \\
&= x \left( x - \lambda^{-\frac{1}{p}} y \right)^p. && (\text{char}(k) = p > 0)
\end{aligned}$$

Here  $\lambda^{-\frac{1}{p}}$  is the unique  $p^{\text{th}}$  root of  $\lambda^{-1}$  (since  $\text{char}(k) = p$ ). Hence

$$L_\lambda \cap f = \left\{ P, [\lambda^{-\frac{1}{p}} : 1 : \lambda^{-\frac{p+1}{p}}] \right\}.$$

(3) Show that every line that is tangent to  $f$  at a simple point passes through  $P$ . Let  $L$  be the tangent line to  $f$  at a simple point  $Q = [a : b : c]$  passing through  $P$ . Problem 5.4 shows that

$$\begin{aligned}
L &= \frac{\partial f}{\partial x}(Q)x + \frac{\partial f}{\partial y}(Q)y + \frac{\partial f}{\partial z}(Q)z \\
&= \frac{\partial f}{\partial x}(Q)x + \frac{\partial f}{\partial z}(Q)z.
\end{aligned}$$

Here  $\frac{\partial f}{\partial y} = -py^{p-1} = 0 \in k[x, y, z]$  since  $\text{char}(k) = p > 0$ . So  $P = [0 : 1 : 0] \in L$  for any tangent line to  $f$  at a simple point of  $f$ .

□

## 5.5. Max Noether's Fundamental Theorem

### Problem 5.29.

Fix  $f$ ,  $g$ , and  $P$ . Show that in cases (1) and (2) - but not (3) - of Proposition 1 the conditions on  $h$  are equivalent to Noether's conditions.

Recall Proposition 1 in §5.5:

- (1)  $f$  and  $g$  meet transversally at  $P$ , and  $P \in h$ .
- (2)  $P$  is a simple point on  $f$ , and  $I(P, h \cap f) \geq I(P, g \cap f)$ .
- (3)  $f$  and  $g$  have distinct tangents at  $P$ , and  $m_P(h) \geq m_P(f) + m_P(g) - 1$ .

*Proof.*

- (1) (Noether's condition  $\implies$  (1) of Proposition 1) Since  $h_* = af_* + bg_*$  for some  $a, b \in \mathcal{O}_P(\mathbf{P}^2)$ ,

$$\begin{aligned}
 I(P, h \cap f) &= I(P, h_* \cap f_*) \\
 &= I(P, (af_* + bg_*) \cap f_*) \\
 &= I(P, (bg_*) \cap f_*) \\
 &= I(P, b \cap f_*) + I(P, g_* \cap f_*) \\
 &\geq I(P, g_* \cap f_*) \\
 &= 1.
 \end{aligned}$$

Hence  $P \in h$ .

- (2) (Noether's condition  $\implies$  (2) of Proposition 1) As  $h_* \in (f_*, g_*) \subseteq \mathcal{O}_P(\mathbf{P}^2)$ ,  $(f_*, h_*) \subseteq (f_*, g_*)$ . So

$$\begin{aligned}
 I(P, h \cap f) &= I(P, h_* \cap f_*) \\
 &= \dim_k(\mathcal{O}_P(\mathbf{P}^2)/(f_*, h_*)) \\
 &\geq \dim_k(\mathcal{O}_P(\mathbf{P}^2)/(f_*, g_*)) \\
 &= I(P, g_* \cap f_*) \\
 &= I(P, g \cap f).
 \end{aligned}$$

- (3) (Noether's condition  $\not\Rightarrow$  (3) of Proposition 1) Consider  $h = y^2z - x^3$ ,  $f = y^2$ ,  $g = x^3$  and  $P = [0 : 0 : 1]$ . Noether's condition holds for  $h_*$ ,  $f_*$ ,  $g_*$  at the point  $P = [0 : 0 : 1]$ , and  $f, g$  have distinct tangents at  $P$ . However,

$$\underbrace{m_P(h)}_{=2} < \underbrace{m_P(f)}_{=2} + \underbrace{m_P(g)}_{=3} - 1.$$

□

### Problem 5.30.

Let  $f$  be an irreducible projective plane curve. Suppose  $z \in k(f)$  is defined at every  $P \in f$ . Show that  $z \in k$ . (Hint: Write  $z = \frac{h}{g}$ , and use max Noether's fundamental theorem).

*Proof.*

- (1) Write  $z = \frac{h}{g}$  where  $h, g$  are forms of the same degree in  $k[x, y, z]$  and  $g(P) \neq 0$  for any  $P \in V(f)$ . Given any  $P \in \mathbf{P}^2$  and consider two cases about  $g(P)$ .
- (2) If  $g(P) \neq 0$ , then  $h - zg = 0 \in \mathcal{O}_P(f) = \mathcal{O}_P(\mathbf{P}^2)/(f)\mathcal{O}_P(\mathbf{P}^2)$ . So

$$h = \alpha f + zg \in \mathcal{O}_P(\mathbf{P}^2)$$

for some  $\alpha \in \mathcal{O}_P(\mathbf{P}^2)$ .

- (3) If  $g(P) = 0$ , then  $f(P) \neq 0$  and thus  $f$  is a unit in  $\mathcal{O}_P(\mathbf{P}^2)$ . Hence  $h_* \in \mathcal{O}_P(\mathbf{P}^2) = (f_*, g_*)$  is always true.
- (4) By (2)(3), Noether's conditions are satisfied at  $P \in \mathbf{P}^2$ . Hence there is an equation  $h = af + bg$  with  $a, b$  forms of degree  $\deg(h) - \deg(f)$ ,  $\deg(h) - \deg(g)$  (Max Noether's fundamental theorem in §5.5). Note that  $b \in k$  since  $\deg(b) = \deg(h) - \deg(g) = 0$ . Therefore,

$$z = h/g = \underbrace{(a/g)f}_{= 0 \text{ in } k(f)} + b = b \in k.$$

□

## 5.6. Applications of Noether's Theorem

### Problem 5.31.

*If in Pascal's Theorem we let some adjacent vertices coincide (the side being a tangent), we get many new theorems:*

- (a) *State and sketch what happens if  $P_1 = P_2$ ,  $P_3 = P_4$ ,  $P_5 = P_6$ .*
- (b) *Let  $P_1 = P_2$ , the other four distinct.*
- (c) *From (b) deduce a rule for constructing the tangent to a given conic at a given point, using only a straight-edge.*

*Proof of (a).*

- (1) Suppose a triangle  $\triangle$  is inscribed in an irreducible conic. Then tangent lines to the conic at the vertices of  $\triangle$  with the unique opposite sides meet in collinear points.
- (2) See Pascal's theorem: degenerations for a sketch.

□

*Proof of (b).*

- (1) Suppose a pentagon  $P_1P_3P_4P_5P_6$  is inscribed in an irreducible conic. The tangent line to the conic at  $P_1$  with any opposite side, and the rest opposite sides meet in collinear points.
- (2) See Pascal's theorem: degenerations for a sketch.

□

*Proof of (c).*

- (1) By (b), we can draw the collinear line  $L$  passing through two points  $Q_1, Q_2$  whose are determined by opposite sides.
- (2) The fifth unused side of the pentagon meets the collinear line  $L$  at  $Q_3$ . Hence the line passing through  $P_1$  and  $Q_3$  is the tangent line to the conic at  $P_1$ .

□

**Problem 5.32. (Braikenridge-Maclaurin theorem)**

*Suppose the intersections of the opposite sides of a hexagon lie on a straight line. Show that the vertices lie on a conic.*

*Proof.*

- (1) Let  $C$  be three sides,  $C'$  the three opposite sides,  $L$  the line.
- (2) Note that all points of  $C' \cap L$  are simple points of  $C'$  (Corollary 2 to Bézout's theorem), and  $C \bullet C' \geq L \bullet C'$ . Max Noether's fundamental theorem shows that there is a conic  $Q$  such that  $Q \bullet C' = C \bullet C' - L \bullet C'$  where  $Q$  contains the hexagon.

□

**Problem 5.33.**

*Let  $C$  be an irreducible cubic,  $L$  a line such that  $L \bullet C = P_1 + P_2 + P_3$ ,  $P_i$  distinct. Let  $L_i$  be the tangent line to  $C$  at  $P_i$ :  $L_i \bullet C = 2P_i + Q_i$  for some  $Q_i$ . Show that  $Q_1, Q_2, Q_3$  lie on a line. ( $L^2$  is a conic!)*

*Proof.*

- (1) Similar to the proof of Proposition 2 in §5.6. Corollary 2 to Bézout's theorem shows that each  $P_i$  is simple. So all the points of  $L^2 \cap C$  are simple points of  $C$ .
- (2) Note that

$$\begin{aligned}
 (L_1 L_2 L_3) \bullet C &= L_1 \bullet C + L_2 \bullet C + L_3 \bullet C \\
 &= (2P_1 + Q_1) + (2P_2 + Q_2) + (2P_3 + Q_3) \\
 &= (2P_1 + 2P_2 + 2P_3) + (Q_1 + Q_2 + Q_3)
 \end{aligned}$$

and

$$L^2 \bullet C = 2(L \bullet C) = 2(P_1 + P_2 + P_3) = 2P_1 + 2P_2 + 2P_3.$$

So  $(L_1 L_2 L_3) \bullet C \geq L^2 \bullet C$ .

- (3) By Corollary (2) to max Noether's fundamental theorem, there is a curve  $M$  such that  $M \bullet C = Q_1 + Q_2 + Q_3$ . Bézout's theorem shows that  $\deg(M) = 1$  or  $M$  is a line. Therefore,  $M$  is a line passing  $Q_1, Q_2, Q_3$ .

□

**Problem 5.34.**

*Show that a line through two flexes on a cubic passes through a third flex.*

*Proof.*

- (1) Let  $C$  be a cubic,  $L$  a line such that  $L \bullet C = P_1 + P_2 + P_3$ .  
 (2) Let  $P_i$  be a simple point in  $C$ ,  $L_i$  the tangent line to  $C$  at  $P_i$ . Show that  $P_i$  is a flex in  $C$  if and only if  $L_i \bullet C = 3P_i$ .

$$\begin{aligned} &P_i \text{ is a flex in } C \\ \iff &I(P_i, L_i \cap C) = \text{ord}_{P_i}^C(L_i) \geq 3 \\ \iff &I(P_i, L_i \cap C) = 3 \text{ and } L_i \cap C = \{P_i\} \\ \iff &L_i \bullet C = 3P_i. \end{aligned} \quad (\text{Bézout's theorem})$$

- (3) Suppose  $P_1$  and  $P_2$  are flexes in  $C$ .  $I(P_3, L_3 \cap C) > 1$  since  $L_3$  is one tangent line to  $C$  at  $P_3$ . So  $L_3 \bullet C = 2P_3 + P'_3$  for some  $P'_3 \in C$ . Now it suffices to show that  $P'_3 = P_3$ . ( $P_3$  is simple since  $P_1$  and  $P_2$  are simple points by Corollary 2 to Bézout's theorem. So  $P_3$  is a flex if  $L_3 \bullet C = 3P_3$  by (2).)

- (4) Consider  $C' = L^3$  and  $C'' = L_1 L_2 L_3$ . So

$$\begin{aligned} C' \bullet C &= 3(L \bullet C) = 3(P_1 + P_2 + P_3) = 3P_1 + 3P_2 + 3P_3 \\ C'' \bullet C &= L_1 \bullet C + L_2 \bullet C + L_3 \bullet C = 3P_1 + 3P_2 + 2P_3 + P'_3. \end{aligned}$$

By Proposition 3 in §5.6,  $P_3 = P'_3$ .

□



**Problem 5.35.**

Let  $C$  be any irreducible cubic, or any cubic without multiple components,  $C^\circ$  the set of simple points of  $C$ ,  $O \in C^\circ$ . Show that the same definition as in the nonsingular case makes  $C^\circ$  into an abelian group.

*Proof.*

- (1) It suffices to show that  $\varphi(C^\circ \times C^\circ) \subseteq C^\circ$ .
- (2) Let  $C$  be any irreducible cubic. Given any  $P_1, P_2 \in C^\circ$ . Then there is a unique line  $L$  such that  $L \bullet C = P_1 + P_2 + P_3$  for some  $P_3 \in C$ . (If  $P_1 = P_2$ ,  $L$  is the tangent to  $C$  at  $P$ .) So it suffices to show that  $P_3 \in C^\circ$ .
- (3) Might assume  $P_3 \neq P_1, P_3 \neq P_2$ . Bézout's theorem shows that

$$\sum_P I(P, L \cap C) = 3.$$

So  $I(P_3, L \cap C) = 1$  whenever  $P_1 = P_2$  or  $P_1 \neq P_2$ . Therefore,  $P_3 \in C^\circ$  (Property (5) in §3.3).

□

**Problem 5.36.**

Let  $C$  be an irreducible cubic,  $O$  a simple point on  $C$  giving rise to the addition  $\oplus$  on the set  $C^\circ$  of simple points. Suppose another  $O'$  gives rise to an addition  $\oplus'$ . Define  $\alpha : (C, O, \oplus) \rightarrow (C, O', \oplus')$  by  $\alpha(P) = \varphi(\varphi(O, O'), P)$ . Show that  $\alpha$  is a group isomorphism. So the structure of the group is independent of the choice of  $O$ .

*Proof.*

- (1) Show that  $\alpha$  is a group homomorphism, that is,  $\alpha(P \oplus Q) = \alpha(P) \oplus' \alpha(Q)$  for all  $P, Q \in C^\circ$ . The definition is well-defined (Problem 5.35).
- (2) Similar to Proposition 4 in §5.6. Let

$$\begin{aligned} L_1 \bullet C &= P + Q + R', \\ M_1 \bullet C &= R' + O + R, & (R = P \oplus Q) \\ L_2 \bullet C &= O + O' + O'' \\ L_3 \bullet C &= R + O'' + T. & (T = \alpha(P \oplus Q)) \end{aligned}$$

Let

$$\begin{aligned}
M_2 \bullet C &= P + O'' + P' & (P' = \alpha(P)) \\
M_3 \bullet C &= Q + O'' + Q' & (Q' = \alpha(Q)) \\
L_4 \bullet C &= P' + Q' + S \\
M_4 \bullet C &= S + O' + T'. & (T' = \alpha(P) \oplus' \alpha(Q))
\end{aligned}$$

Let  $f = L_1 L_2 L_3 L_4$ ,  $f' = M_1 M_2 M_3 M_4$ . Apply Problem 5.42 on  $C^\circ$  to get  $T = T'$  or  $\alpha(P \oplus Q) = \alpha(P) \oplus' \alpha(Q)$ .

- (3) Define  $\beta : (C, O', \oplus') \rightarrow (C, O, \oplus)$  by  $\beta(P) = \varphi(\varphi(O', O), P)$ .  $\beta$  is a group homomorphism too. By step (1) in the proof of Problem 5.37(b),  $\alpha \circ \beta = 1_{(C, O', \oplus')}$  and  $\beta \circ \alpha = 1_{(C, O, \oplus)}$ . Hence  $\alpha$  is a group isomorphism.

□

### Problem 5.37.

In Proposition 4, suppose  $O$  is a flex on  $C$ .

- Show that the flexes form a subgroup of  $C$ ; as an abelian group, this subgroup is isomorphic to  $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ .
- Show that the flexes are exactly the elements of order three in the group. (i.e., exactly those elements  $P$  such that  $P \oplus P \oplus P = O$ ).
- Show that a point  $P$  is of order two in the group if and only if the tangent to  $C$  at  $P$  passes through  $O$ .
- Let  $C = y^2z - x(x - z)(x - \lambda z)$ ,  $\lambda \neq 0, 1$ ,  $O = [0 : 1 : 0]$ . Find the points of order two.
- Show that the points of order two on a nonsingular cubic form a group isomorphic to  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ .
- Let  $C$  be a nonsingular cubic,  $P \in C$ . How many lines through  $P$  are tangent to  $C$  at some point  $Q \neq P$ ? (The answer depends on whether  $P$  is a flex.)

*Proof of (a).*

- (1) Let  $G$  be the set of all flexes of  $C$ . Show that  $G$  is a subgroup of  $C$ .  $G \neq \emptyset$  since  $O \in G$ . Given any two flexes  $P, Q \in C$ , we have

$$\begin{aligned}
-P &= \varphi(O, P) \in G \\
P \oplus Q &= \varphi(O, \varphi(P, Q)) \in G
\end{aligned}$$

by using Problem 5.34 three times. Thus  $G$  is a subgroup of  $C$ .

- (2) Show that  $P \oplus P \oplus P = O$  for all flexes of  $C$ . Let  $L$  be the tangent line to  $C$  at  $P$ . By step (2) in the proof of Problem 5.34,  $P$  is a flex in  $C$  if and only if  $L \bullet C = 3P$ . Hence  $\varphi(P, P) = P$ ,

$$P \oplus P = \varphi(O, \varphi(P, P)) = \varphi(O, P) = -P.$$

- (3) Note that  $|G| = 9$  since a nonsingular cubic  $C$  has 9 ordinary flexes (Corollary (2) to Problem 5.23). So  $G$  is an abelian group of order 9 ((1)). So  $G \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$  (the fundamental theorem of finite abelian groups and (2)).

□

*Proof of (b).*

- (1) Let  $L$  be one line such that  $L \bullet C = P_1 + P_2 + P_3$ . Then  $L$  is uniquely determined by  $P_1$  and  $P_2$ . (If  $P_1 \neq P_2$ , nothing to do. If  $P_1 = P_2$ , the tangent line is unique since  $P_1 = P_2$  is simple.)
- (2) Might assume  $P \neq O$ . ( $O$  is of order 1.) So

$$\begin{aligned}
 & P \oplus P \oplus P = O \\
 \iff & P \oplus P = -P \\
 \iff & \varphi(O, \varphi(P, P)) = \varphi(O, P) \\
 \iff & \varphi(O, \varphi(O, P)) = \varphi(P, P) & ((1)) \\
 \iff & P = \varphi(P, P) \\
 \iff & L \bullet C = 3P \text{ where } L \text{ is the tangent line to } C \text{ at } P \\
 \iff & P \text{ is a flex in } C. & (\text{Problem 5.34})
 \end{aligned}$$

□

*Proof of (c).*

- (1)  $P \neq O$  since  $O$  is of order 1.
- (2) So

$$\begin{aligned}
 & P \oplus P = O \\
 \iff & \varphi(O, \varphi(P, P)) = O \\
 \iff & \varphi(O, O) = \varphi(P, P) & ((b)(1)) \\
 \iff & O = \varphi(P, P) & (O: \text{flex}) \\
 \iff & \text{The tangent line to } C \text{ at } P \text{ passing through } O.
 \end{aligned}$$

□

*Proof of (d).*

- (1) Problem 5.4 shows that the tangent line  $L$  to  $C$  at  $P := [a : b : c] \in C$  is

$$L = \frac{\partial C}{\partial x}(P)x + \frac{\partial C}{\partial y}(P)y + \frac{\partial C}{\partial z}(P)z.$$

Might assume  $P \neq O$ . By (c),  $O \in L$  and thus  $\frac{\partial C}{\partial y}(P) = 2bc = 0$ .  $b = 0$  or  $c = 0$ .

- (2) The case  $b = 0$ .  $P \in C$  implies that  $a(a - c)(a - \lambda c) = 0$ . So  $P = [0 : 0 : 1], [1 : 0 : 1], [\lambda : 0 : 1]$ . It is easy to verify that  $P \oplus P = O$  and  $P \neq O$ .
- (3) The case  $c = 0$ .  $P = [0 : 1 : 0] = O$ .
- (4) By (2)(3), the points of order two (including  $O$ ) are

$$[0 : 0 : 1], [1 : 0 : 1], [\lambda : 0 : 1], [0 : 1 : 0].$$

□

*Proof of (e).*

- (1) Let  $H$  be the set of all points  $P \in C$  with  $P \oplus P = O$ . Show that  $H$  is a subgroup of  $C$ .  $H \neq \emptyset$  since  $O \in H$ . Given any two  $P, Q \in H$ , we have

$$\begin{aligned} (-P) \oplus (-P) &= P \oplus P = O, \\ (P \oplus Q) \oplus (P \oplus Q) &= (P \oplus P) \oplus (Q \oplus Q) = O \oplus O = O. \end{aligned}$$

So  $H$  is a subgroup of  $C$ .

- (2) Note that the nonsingularity is preserved under projectively equivalence. By Problems 5.21, 5.24, 5.10 and 5.11, it suffices to assume that  $C = y^2z - x(x - z)(x - \lambda z)$ .
- (3) By (d) and the fundamental theorem of finite abelian groups,  $H \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ .

□

*Proof of (f).*

- (1) To find lines through  $P$  are tangent to  $C$  at some point  $Q \in C$ , it suffices to find  $Q \in C$  satisfying  $\varphi(Q, Q) = P$ . Construct an abelian group  $(C, P, \oplus)$  on  $C$ .
- (2) The case  $P \in C$  is a flex. (c)(e) implies that there are 4 lines through  $P$  are tangent to  $C$  at some point  $Q \in C$ . Since  $\varphi(P, P) = P$ , there are 3 lines through  $P$  are tangent to  $C$  at some point  $Q \neq P$ .

- (3) The case  $P \in C$  is not a flex. Construct another abelian group  $(C, P', \oplus')$  on  $C$  where  $P' \in C$  is any flex. (The existence of  $P'$  is guaranteed by (a).) By Problem 5.36, the map

$$\alpha : (C, P, \oplus) \rightarrow (C, P', \oplus')$$

defined by  $\alpha(Q) = \varphi(\varphi(P, P'), Q)$  is an isomorphism. Here  $\alpha^{-1}(Q') = \varphi(\varphi(P, P'), Q')$ .

- (4) By (2), there are 4 points  $Q'_i \in C$  ( $i = 1, 2, 3, 4$ ) such that

$$Q'_i \oplus' Q'_i = P'.$$

Take  $\alpha^{-1}$  on the both sides to get

$$Q_i \oplus Q_i = P,$$

where  $Q_i := \alpha^{-1}(Q'_i)$ . Note that each  $Q_i \neq P$  since  $P$  is not a flex. Therefore, there are 4 lines through  $P$  are tangent to  $C$  at some point  $Q \in C$ .

□

### Problem 5.38.

Let  $C$  be a nonsingular cubic given by the equation

$$y^2z = x^3 + ax^2z + bxz^2 + cz^3,$$

$O = [0 : 1 : 0]$ . Let  $P_i = [x_i : y_i : 1]$ ,  $i = 1, 2, 3$ , and suppose  $P_1 \oplus P_2 = P_3$ . Let

$$\lambda = \begin{cases} \frac{y_1 - y_2}{x_1 - x_2} & \text{if } x_1 \neq x_2 \\ \frac{3x_1^2 + 2ax_1 + b}{2y_1} & \text{if } P_1 = P_2 \text{ and } y_1 \neq 0. \end{cases}$$

Let  $\mu = y_i - \lambda x_i$ ,  $i = 1, 2$ . Show that

$$\begin{cases} x_3 = \lambda^2 - a - x_1 - x_2 \\ y_3 = -\lambda x_3 - \mu. \end{cases}$$

This gives an explicit method for calculating in the group.

*Proof.*

- (1) Suppose  $x_1 \neq x_2$ . Let the line  $L_1$  passing through  $P_1$  and  $P_2$ . Since  $L_1 \subseteq U_3 = \{[x : y : z] : z \neq 0\}$ , we dehomogenize  $C$  (resp.  $L_1$ ) to get

$$\begin{aligned} C_* : y^2 &= x^3 + ax^2 + bx + c, \\ (L_1)_* : y - y_1 &= \lambda(x - x_1). \end{aligned}$$

Here  $\lambda = \frac{y_2 - y_1}{x_2 - x_1}$ . Write  $y = \lambda(x - x_1) + y_1 = \lambda x + \mu$  where  $\mu = y_1 - \lambda x_1$ . ( $\mu$  is also equal to  $y_2 - \lambda x_2$ .) So

$$\begin{aligned} (\lambda x + \mu)^2 &= x^3 + ax^2 + \text{lower terms} \\ \iff x^3 + (a - \lambda^2)x^2 + \text{lower terms} &= 0 \\ \iff x_1 + x_2 + x'_3 &= \lambda^2 - a & (\text{Vieta's formulas}) \\ \iff \alpha = \lambda^2 - a - x_1 - x_2, \beta &= \lambda\alpha + \mu. \end{aligned}$$

Here  $P'_3 = [\alpha : \beta : 1]$  is the third point on  $L_1 \cap C$ . ( $L_1 \bullet C = P_1 + P_2 + P'_3$ .)

- (2) Suppose  $P_1 = P_2$  and  $y_1 \neq 0$ . Write  $C = y^2z - x^3 - ax^2z - bxz^2 - cz^3$ . By Problem 5.4, the tangent line  $L_1$  at  $P_1 = P_2$  is

$$L_1 : y = \lambda x + \frac{bx_1 + 2c}{2y_1}z.$$

(Note that  $\lambda = \frac{3x_1^2 + 2ax_1 + b}{2y_1}$  and  $P_1 \in C$ .) In particular,  $O \notin L_1$ . By dehomogenizing  $C$  and replacing  $y$  by  $\lambda x + \frac{bx_1 + 2c}{2y_1}$ , we have

$$\begin{aligned} \left( \lambda x + \frac{bx_1 + 2c}{2y_1} \right)^2 &= x^3 + ax^2 + \text{lower terms} \\ \iff x^3 + (a - \lambda^2)x^2 + \text{lower terms} &= 0 \\ \iff x_1 + x_2 + x'_3 &= \lambda^2 - a & (\text{Vieta's formulas}) \\ \iff \alpha = \lambda^2 - a - x_1 - x_2, \beta &= \lambda\alpha + \mu. \end{aligned}$$

Here  $\mu = y_1 - \lambda x_1$  and  $P'_3 = [\alpha : \beta : 1]$  is the third point on  $L_1 \cap C$ . ( $L_1 \bullet C = P_1 + P_2 + P'_3$ .)

- (3) In any case of (1) or (2), we let the line  $L_2$  passing through  $P'_3$  and  $O$ .  $L_2 = x - \alpha z$ . So

$$\begin{aligned} y^2z &= \alpha^3z^3 + a\alpha^2z^3 + b\alpha z^3 + cz^3 = \beta^2z \\ \iff (y + \beta)(y - \beta)z &= 0 \\ \iff P_3 = [\alpha : -\beta : 1] &= [\lambda^2 - a - x_1 - x_2 : -\lambda x_3 - \mu : 1]. \end{aligned}$$

Here  $P_3$  is the third point on  $L_2 \cap C$ . ( $L_2 \bullet C = P'_3 + O + P_3$ .)

□

### Problem 5.39.

- (a) Let  $C = y^2z - x^3 - 4xz^2$ ,  $O = P_0 = [0 : 1 : 0]$ ,  $P_1 = [2 : 4 : 1]$ ,  $P_2 = [0 : 0 : 1]$  and  $P_3 = [2 : -4 : 1]$ . Show that  $G = \{P_0, P_1, P_2, P_3\}$  form a subgroup of  $C$  that is cyclic of order 4.

- (b) Let  $C = y^2z - x^3 + 43xz^2 - 166z^3$ . Let  $O = [0 : 1 : 0]$ ,  $P = [3 : 8 : 1]$ . Show that  $P$  is an element of order 7 in  $C$ .

*Proof of (a).*

- (1) A direct calculation shows that

$$P_i \oplus P_j = P_{i+j}$$

for all  $i, j \in \mathbb{Z}/4\mathbb{Z}$  (Problem 5.38). Hence  $G$  is a subgroup of  $C$ .

- (2) Note that  $G \xrightarrow{\sim} \mathbb{Z}/4\mathbb{Z}$  (by  $P_i \mapsto i$ ) is an isomorphism. So  $G$  is a cyclic group of order 4.

□

*Proof of (b).*

- (1) A direct calculation shows that  $2P = P \oplus P = [-5 : -16 : 1]$ ,  $4P = 2P \oplus 2P = [11 : 32 : 1]$ , and  $8P = 4P \oplus 4P = [3 : 8 : 1] = P$  (Problem 5.38).
- (2) Since  $8P = P$ ,  $7P = O$ . The order of  $P$  is 1 or 7 (since 7 is a prime integer). Note that  $P \neq O$ . Therefore,  $P$  is of order 7.

□

#### Problem 5.40. (Rational point)

Let  $k_0$  be a subfield of  $k$ . If  $V$  is an affine variety,  $V \subseteq \mathbf{A}^n(k)$ , a point  $P = (a_1, \dots, a_n) \in V$  is **rational** over  $k_0$ , if each  $a_i \in k_0$ . If  $V \subseteq \mathbf{P}^n(k)$  is projective, a point  $P \in V$  is rational over  $k_0$  if for some homogeneous coordinates  $(a_1, \dots, a_{n+1})$  for  $P$ , each  $a_i \in k_0$ .

A curve  $f$  of degree  $d$  is said to be *emphrational* over  $k_0$  if the corresponding point in  $\mathbf{P}^{\frac{d(d+3)}{2}}$  is rational over  $k_0$ .

Suppose a nonsingular cubic  $C$  is rational over  $k_0$ . Let  $C(k_0)$  be the set of points of  $C$  that are rational over  $k_0$ .

- (a) If  $P, Q \in C(k_0)$ , show that  $\varphi(P, Q)$  is in  $C(k_0)$ .
- (b) If  $O \in C(k_0)$ , show that  $C(k_0)$  forms a subgroup of  $C$ .

(If  $k_0 = \mathbb{Q}$ ,  $k = \mathbb{C}$ , this has important applications to number theory.)

*Proof of (a).*

- (1) If  $P \neq Q$ , then the line  $L$  passing through  $P = [a_1 : a_2 : a_3]$  and  $Q = [b_1 : b_2 : b_3]$  is

$$L : \{[\lambda a_1 + \mu b_1 : \lambda a_2 + \mu b_2 : \lambda a_3 + \mu b_3] : [\lambda : \mu] \in \mathbf{P}^2(k)\}.$$

Hence

$$C \cap L = C(\lambda a_1 + \mu b_1, \lambda a_2 + \mu b_2, \lambda a_3 + \mu b_3) := \tilde{C}(\lambda, \mu).$$

- (2) Note that  $\deg(\tilde{C}) = 3$  since  $C \neq 0$  is nonsingular (and thus is irreducible by Problem 5.21). By Problem 5.7 or Bézout's theorem,  $\tilde{C}$  has exactly 3 zeros in  $\mathbf{P}^1(k)$ .
- (3) Note that  $\tilde{C} \in k_0[\lambda, \mu]$  since  $C \in k_0[x, y, z]$  and  $P, Q \in \mathbf{P}^2(k_0)$ . Since  $P, Q \in \mathbf{P}^2(k_0)$ , we apply Vieta's formulas on  $\tilde{C}$  to get the third point  $\varphi(P, Q) \in \mathbf{P}^2(k_0)$  too.
- (4) If  $P = Q$ , then the line  $L$  tangent to  $C$  at  $P$  is  $\frac{\partial C}{\partial x}(P)x + \frac{\partial C}{\partial y}(P)y + \frac{\partial C}{\partial z}(P)z = 0$  (Problem 5.4). Define

$$[b_1 : b_2 : b_3] = \begin{cases} [a_1 : a_2 : a_3 + 1] & \text{if } \frac{\partial C}{\partial z}(P) = 0 \\ \left[ a_1 + \frac{\partial C}{\partial y}(P) : a_2 - \frac{\partial C}{\partial x}(P) : a_3 \right] & \text{otherwise.} \end{cases}$$

Note that  $L \in k_0[x, y, z]$  and  $[b_1 : b_2 : b_3] \in \mathbf{P}^2(k_0)$ . Hence  $L$  has the same form as (1). The rest proof is the same as (2)(3).

□

*Proof of (b).* Since  $O \in C(k_0)$  and  $\varphi(C(k_0) \times C(k_0)) \subseteq C(k_0)$ ,  $C(k_0)$  is a subgroup of  $C$  (as Problem 5.35). □

#### Problem 5.41.

Let  $C$  be a nonsingular cubic,  $O$  a flex on  $C$ . Let  $P_1, \dots, P_{3m} \in C$ . Show that  $P_1 \oplus \dots \oplus P_{3m} = O$  if and only if there is a curve  $f$  of degree  $m$  such that  $f \bullet C = \sum_{i=1}^{3m} P_i$ . (Hint: Use induction on  $m$ . Let  $L \bullet C = P_1 + P_2 + Q$ ,  $L' \bullet C = P_3 + P_4 + R$ ,  $L'' \bullet C = Q + R + S$ , and apply induction to  $S, P_5, \dots, P_{3m}$ ; use Noether's theorem.)

*Proof.*

- (1) Induction on  $m$ .
- (2) Base case.

$$\begin{aligned} P_1 \oplus P_2 \oplus P_3 = O &\iff P_1, P_2, P_3 \text{ are collinear} \\ &\iff \exists \text{ a line } L \text{ such that } L \bullet C = P_1 + P_2 + P_3. \end{aligned}$$



- (3) Inductive step. Let  $L \bullet C = P_1 + P_2 + Q$ ,  $L' \bullet C = P_3 + P_4 + R$ ,  $L'' \bullet C = Q + R + S$ . By (2),  $O = P_1 \oplus P_2 \oplus Q = P_3 \oplus P_4 \oplus R = Q \oplus R \oplus S$ . Hence

$$S = P_1 \oplus P_2 \oplus P_3 \oplus P_4.$$

Hence

$$\begin{aligned} P_1 \oplus \cdots \oplus P_{3m} &= O \\ \iff S \oplus P_5 \oplus P_6 \cdots \oplus P_{3m} &= O \\ \iff \exists f' \text{ of degree } m-1 \text{ such that } f' \bullet C &= S + \sum_{i=5}^{3m} P_i \\ \implies (LL'f') \bullet C &\geq L'' \bullet C \\ \implies \exists f \text{ of degree } m \text{ such that } f \bullet C &= \sum_{i=1}^{3m} P_i \\ \implies (L''f) \bullet C &\geq (LL') \bullet C \\ \implies \exists f' \text{ of degree } m-1 \text{ such that } f' \bullet C &= S + \sum_{i=5}^{3m} P_i. \end{aligned}$$

- (4) Since both the base case and the inductive step have been proved as true, by mathematical induction the statement holds for any  $m$ .

□

**Problem 5.42.**

Let  $C$  be a nonsingular cubic,  $f, f'$  curves of degree  $m$  such that  $f \bullet C = \sum_{i=1}^{3m} P_i$ ,  $f' \bullet C = \sum_{i=1}^{3m-1} P_i + Q$ . Show that  $P_{3m} = Q$ .

*Proof.*

- (1) Similar to Proposition 3 in §5.6.  
(2) Let  $L$  be a line through  $P_{3m}$  that doesn't pass through  $Q$ ;  $L \bullet C = P_{3m} + R + S$ . Then

$$\begin{aligned} (Lf') \bullet C &= L \bullet C + f' \bullet C \\ &= (P_{3m} + R + S) + \left( \sum_{i=1}^{3m-1} P_i + Q \right) \\ &= \sum_{i=1}^{3m} P_i + Q + R + S \\ &= f \bullet C + Q + R + S, \end{aligned}$$

so there is a line  $L'$  such that  $L' \bullet C = Q + R + S$  (max Noether's fundamental theorem). But then  $L' = L$  and so  $P_{3m} = Q$ .

□

**Problem 5.43. (Sextatic point)**

*For which points  $P$  on a nonsingular cubic  $C$  does there exist a nonsingular conic that intersects  $C$  only at  $P$ ?*

*Proof.*

- (1) *We say  $P$  a sextatic point of  $C$  if there is a nonsingular conic  $K$  that intersects  $C$  six times at  $P$ . Suppose  $O$  is a flex on  $C$ . Show that the followings are equivalent:*

- (i)  *$P$  is a sextatic point of  $C$ .*
- (ii)  *$6P = O$  and  $3P \neq O$ .*
- (iii) *The tangent at  $P$  contains a flex of  $C$  not equal to  $P$ .*
- (iv)  *$P$  has order 2 or 6.*

- (2) (ii)  $\iff$  (iii)  $\iff$  (iv) is followed by Problem 5.37. To prove (i)  $\iff$  (ii), it suffices to show that

(\*)  $6P = O \iff$  *there is a conic  $K$  that intersects  $C$  six times at  $P$ , which is followed by Problem 5.41.*

(\*\*)  $3P = 0 \iff K$  *is reducible.*

- (3) Note that

$$\begin{aligned}
 & K = L_1 L_2 \text{ for some lines } L_1, L_2 \\
 \implies & C \bullet K = C \bullet (L_1 L_2) = C \bullet L_1 + C \bullet L_2 = 6P \\
 \implies & C \bullet L_1 = C \bullet L_2 = 3P \\
 \implies & 3P = O.
 \end{aligned}
 \tag{Problem 5.41}$$

and

$$\begin{aligned}
 & 3P = O \\
 \implies & C \bullet L = 3P \text{ for some line } L \tag{Problem 5.41} \\
 \implies & C \bullet K = C \bullet L^2 = 6P \\
 \implies & K = L^2 \text{ is reducible (up to a nonzero constant).}
 \end{aligned}$$

Therefore, the statement (\*\*) is established (and thus part (a) holds).

- (4) There are 27 sextatic points of  $C$  by Problem 5.37(a)(c)(e) and (iii). Note that  $G = \{ \text{flex points} + \text{sextatic points} \}$  is a subgroup of  $C$  with  $|G| = 27 + 9 = 36$ . So

$$G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}.$$

□

## Chapter 6: Varieties, Morphisms, and Rational Maps

### 6.1. The Zariski Topology

#### Problem 6.1.\*

Let  $Z \subseteq Y \subseteq X$ ,  $X$  a topological space. Give  $Y$  the induced topology. Show that the topology induced by  $Y$  on  $Z$  is the same as that induced by  $X$  on  $Z$ .

*Proof.*

- (1) Take any open set  $U$  in  $Z$  whose topology is induced by  $Y$ .  $U = Z \cap B$  for some open subset  $B$  in  $Y$ . Since  $Y$  is induced by  $X$ ,  $B = Y \cap A$  for some open subset  $A$  in  $X$ . Hence,

$$U = Z \cap B = Z \cap (Y \cap A) = Z \cap A,$$

that is,  $U$  is also an open set in  $Z$  whose topology is induced by  $X$ .

- (2) Conversely, take any open set  $U$  in  $Z$  whose topology is induced by  $X$ .  $U = Z \cap A$  for some open subset  $A$  in  $X$ . Hence

$$U = Z \cap A = Z \cap (Y \cap A),$$

where  $Y \cap A$  is an open set in  $Y$  whose topology is induced by  $X$ . So  $U$  is also an open set in  $Z$  whose topology is induced by  $Y$ .

□

#### Problem 6.2.\*

- (a) Let  $X$  be a topological space,  $X = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ ,  $U_\alpha$  open in  $X$ . Show that a subset  $W$  of  $X$  is closed if and only if each  $W \cap U_\alpha$  is closed (in the induced topology) in  $U_\alpha$ .
- (b) Suppose similarly  $Y = \bigcup_{\alpha \in \mathcal{A}} V_\alpha$ ,  $V_\alpha$  open in  $Y$ , and suppose  $f : X \rightarrow Y$  is a mapping such that  $f(U_\alpha) \subseteq V_\alpha$ . Show that  $f$  is continuous if and only if the restriction of  $f$  to each  $U_\alpha$  is a continuous mapping from  $U_\alpha$  to  $V_\alpha$ .

*Proof of (a).*

- (1) Show that a subset  $E$  of  $X$  is open if and only if each  $E \cap U_\alpha$  is open (in the induced topology) in  $U_\alpha$ . One way is nothing to do. Conversely,

$$E = E \cap X = E \cap \left( \bigcup_{\alpha \in \mathcal{A}} U_\alpha \right) = \bigcup_{\alpha \in \mathcal{A}} (E \cap U_\alpha)$$

is a union of open sets, which is open again.

- (2) Let  $E = X - W$  be an open set in  $X$ . Note that

$$E \cap U_\alpha = (X - W) \cap U_\alpha = X \cap U_\alpha - W \cap U_\alpha = U_\alpha - W \cap U_\alpha.$$

Hence the conclusion holds.

- (3) It motivates the local study of topologies.

□

*Proof of (b).*

- (1) ( $\implies$ ) Suppose  $f$  is continuous. It suffices to show that  $(f|_{U_\alpha})^{-1}(W)$  is closed in  $U_\alpha$  for every closed set  $W$  in  $V_\alpha$ . Write  $W = W_\alpha \cap V_\alpha$  where  $W_\alpha$  is closed in  $Y$ . Note that

$$\begin{aligned} (f|_{U_\alpha})^{-1}(W) &= f^{-1}(W) \cap U_\alpha \\ &= f^{-1}(W_\alpha \cap V_\alpha) \cap U_\alpha \\ &= f^{-1}(W_\alpha) \cap f^{-1}(V_\alpha) \cap U_\alpha \\ &= f^{-1}(W_\alpha) \cap U_\alpha. \end{aligned} \quad (f^{-1}(V_\alpha) \supseteq U_\alpha)$$

Since  $f^{-1}(W_\alpha)$  is closed in  $X$ ,  $(f|_{U_\alpha})^{-1}(W)$  is closed in  $U_\alpha$ .

- (2) ( $\impliedby$ ) Conversely, suppose  $f|_{U_\alpha}$  is a continuous mapping from  $U_\alpha$  to  $V_\alpha$ . It suffices to show that  $f^{-1}(W)$  is closed for every closed set  $W$  in  $Y$ . Note that  $W \cap V_\alpha$  is closed in each  $V_\alpha$ . By the continuity of  $f|_{U_\alpha}$ ,

$$\begin{aligned} (f|_{U_\alpha})^{-1}(W \cap V_\alpha) &= f^{-1}(W \cap V_\alpha) \cap U_\alpha \\ &= f^{-1}(W) \cap f^{-1}(V_\alpha) \cap U_\alpha \\ &= f^{-1}(W) \cap U_\alpha \end{aligned}$$

is closed in  $U_\alpha$ . Since  $\alpha \in \mathcal{A}$  is arbitrary and  $X = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ ,  $f^{-1}(W)$  is closed in  $X$  by (a).

□

**Problem 6.3.\***

- (a) Let  $V$  be an affine variety,  $f \in \Gamma(V)$ . Considering  $f$  as a mapping from  $V$  to  $k = \mathbf{A}^1$ , show that  $f$  is continuous.
- (b) Show that any polynomial map of affine varieties is continuous.

*Proof of (a).*

- (1) It is a special case of (b) since  $f$  itself is a polynomial map from  $V$  to  $\mathbf{A}^1$ .
- (2) By Problem 6.6, it suffices to show that

$$f^{-1}(\lambda) = \{P \in V : f(P) = \lambda\}$$

is closed for any  $\lambda \in k$ . Note that  $f^{-1}(\lambda) = V \cap V(f - \lambda = 0)$  is closed.

□

*Proof of (b).*

- (1) Let  $V \subseteq \mathbf{A}^n$ ,  $W \subseteq \mathbf{A}^m$  be affine varieties. Let  $\varphi$  be a polynomial map from  $V$  to  $W$ .
- (2) Problem 2.7 shows that  $\varphi^{-1}(X)$  is closed for every closed subset  $X$  of  $W$ . Hence,  $\varphi$  is continuous.

□

**Problem 6.4.\***

Let  $U_i \subseteq \mathbf{P}^n$ ,  $\varphi_i : \mathbf{A}^n \rightarrow U_i$  as in Chapter 4. Give  $U_i$  the topology induced from  $\mathbf{P}^n$ .

- (a) Show that  $\varphi_i$  is a homeomorphism.
- (b) Show that a set  $W \subseteq \mathbf{P}^n$  is closed if and only if each  $\varphi_i^{-1}(W)$  is closed in  $\mathbf{A}^n$ ,  $i = 1, \dots, n+1$ .
- (c) Show that if  $V \subseteq \mathbf{A}^n$  is an affine variety, then the projective closure  $V^*$  of  $V$  is the closure of  $\varphi_{n+1}(V)$  in  $\mathbf{P}^n$ .

*Proof of (a).*

- (1) Note that  $\varphi_i$  is one-to-one and onto. It suffices to show that  $\varphi_i$  and  $\varphi_i^{-1}$  are continuous.

- (2) *Show that  $\varphi_i$  is continuous.* Suppose  $W \subseteq \mathbf{P}^n$  is closed in  $U_i$ . Let  $\overline{W}$  be its closure in  $\mathbf{P}^n$ .  $\overline{W} = V(I)$  for some homogeneous ideal  $I$  in  $k[x_1, \dots, x_{n+1}]$ . Thus  $\varphi_i^{-1}(W) = V(I_*)$  (w.r.t.  $x_i$ ) is closed.
- (3) *Show that  $\varphi_i^{-1}$  is continuous.* Suppose  $V \subseteq \mathbf{A}^n$  is closed. Proposition 3 in §4.3 gives us  $\varphi_i(V) = V^* \cap U_i$  is closed in  $U_i$ .

□

*Proof of (b).*

$$\begin{aligned}
 & W \subseteq \mathbf{P}^n \text{ is closed} \\
 \iff & W \text{ is closed in } U_i \text{ for each } i && \text{(Problem 6.2.)} \\
 \iff & \varphi_i^{-1}(W) \text{ is closed in } \mathbf{A}^n \text{ for each } i. && (\varphi_i \text{ is a homeomorphism})
 \end{aligned}$$

□

*Proof of (c).* By Proposition 3 in §4.3, the projective closure  $V^*$  of  $V$  is the smallest algebraic set in  $\mathbf{P}^n$  containing  $V = \varphi_{n+1}(V)$ . That is the same as the definition of the closure of  $V$ . □

### Problem 6.5.

*Any infinite subset of an irreducible plane curve  $V$  is dense in  $V$ . Any one-to-one mapping from one irreducible plane curve onto another is a homeomorphism.*

*Proof.*

- (1) *Show that any infinite subset of an irreducible plane curve  $V$  is dense in  $V$ . It is not true if  $V$  is reducible.*
- (a) Let  $W$  be the closure of an infinite subset of a plane curve  $V$ .  $W \subseteq V$ .
  - (b) By Corollary 2 to Proposition 2 in §1.6 or Problem 4.23,  $W$  is a finite union of points and irreducible plane curves. By Proposition 2 in §1.6 or Problem 5.7, each of these plane curves intersects  $V$  in finite number of points, or is equal to  $V$ .
  - (c) Since  $W$  is infinite,  $W = V$ . That is, any infinite subset of an irreducible plane curve is dense in  $V$ .
- (2) *Show that any one-to-one mapping from one irreducible plane curve onto another is a homeomorphism.*
- (a) Let  $\varphi : f \rightarrow g$  be a bijective mapping. It suffices to show that  $\varphi$  is continuous.

- (b) The closed sets in  $V(g)$  are  $V(g)$  itself and points (by the same argument in (1)).  $\varphi^{-1}(V(g)) = V(f)$  is closed. Given any point  $Q \in V(g)$ ,  $\varphi^{-1}(Q) = P$  is closed for some point  $P \in V(f)$  since  $\varphi$  is bijective. So,  $\varphi$  is continuous.
- (c) Similarly,  $\varphi^{-1}$  is continuous too. Hence,  $\varphi$  is homeomorphic.

□

**Problem 6.6.\***

Let  $X$  be a topological space,  $f : X \rightarrow \mathbf{A}^n$  a mapping. Then  $f$  is continuous if and only if for each hypersurface  $V = V(f)$  of  $\mathbf{A}^n$ ,  $f^{-1}(V)$  is closed in  $X$ . A mapping  $f : X \rightarrow k = \mathbf{A}^1$  is continuous if and only if  $f^{-1}(\lambda)$  is closed for any  $\lambda \in k$ .

*Proof.*

- (1) We give a stronger version of the statement. Show that  $f$  is continuous iff  $f^{-1}(V)$  is closed for each irreducible hypersurface  $V = V(f) \subseteq \mathbf{A}^n$ . One way is nothing to do.
- (2) Conversely, note that  $f^{-1}(\mathbf{A}^n) = X$  and  $f^{-1}(\emptyset) = \emptyset$  are closed. Now given any proper closed set  $V$  in  $\mathbf{A}^n$ ,  $V$  is of the form

$$V = V(f_1, \dots, f_r) = V(f_1) \cap \dots \cap V(f_r),$$

where each  $f_i$  is irreducible polynomial. Hence,

$$f^{-1}(V) = f^{-1}(V(f_1)) \cap \dots \cap f^{-1}(V(f_r))$$

is a union of closed sets in a topological space  $X$ , which is closed again.

- (3)  $f$  is continuous iff  $f^{-1}(V)$  is closed for each irreducible hypersurface  $V = V(f) \subseteq \mathbf{A}^1$  (by (1)). Note that the irreducible hypersurface  $f$  in  $\mathbf{A}^1$  is  $f = x - \lambda$  for some  $\lambda \in k$ . So  $f$  is continuous iff

$$f^{-1}(V) = f^{-1}(x - \lambda) = f^{-1}(\lambda)$$

is closed for each  $\lambda \in k$ .

□



**Problem 6.7.\***

Let  $V$  be an affine variety,  $f \in \Gamma(V)$ .

- (a) Show that  $V(f) = \{P \in V : f(P) = 0\}$  is a closed subset of  $V$ , and  $V(f) \neq V$  unless  $f = 0$ .
- (b) Suppose  $U$  is a dense subset of  $V$  and  $f(P) = 0$  for all  $P \in U$ . Then  $f = 0$ .

*Proof of (a).*

- (1) Write  $f = \tilde{f} + I(V) \in \Gamma(V)$  for some  $\tilde{f} \in k[x_1, \dots, x_n]$ . Note that  $f(P) = \tilde{f}(P)$  is well-defined for any  $P \in V$ . Thus,

$$\begin{aligned} V(f) &= \{P \in V : f(P) = 0\} \\ &= \{P \in V : \tilde{f}(P) = 0\} \\ &= V \cap \{P \in \mathbf{A}^n : \tilde{f}(P) = 0\} \\ &= V \cap V(\tilde{f}) \end{aligned}$$

is closed in  $V$ .

- (2) Also,

$$\begin{aligned} V(f) = V &\iff V \cap V(\tilde{f}) = V \\ &\iff V(\tilde{f}) \supseteq V \\ &\iff \tilde{f} \in I(V) \\ &\iff f = 0 \in \Gamma(V). \end{aligned}$$

□

*Proof of (b).*

- (1) Since  $V(f) \supseteq U$  (by assumption) and  $V(f)$  is closed (by (a)),  $V(f) \supseteq \overline{U}$  by the definition of the closure of  $U$ .
- (2) Since  $U$  is dense in  $V$ ,  $\overline{U} = V$ . Thus  $V \supseteq V(f) \supseteq \overline{U} = V$  or  $V(f) = V$ . By (a),  $f = 0$ .

□

**Problem 6.8.\***

Let  $U$  be an open subset of a variety  $V$ ,  $z \in k(V)$ . Suppose  $z \in \mathcal{O}_P(V)$  for all  $P \in U$ . Show that  $U_z = \{P \in U : z(P) \neq 0\}$  is open, and that the mapping from  $U$  to  $k = \mathbf{A}^1$  defined by  $P \mapsto z(P)$  is continuous.

*Proof.*

- (1) Show that  $U_z$  is open. By Problem 6.1, it suffices to show that  $U_z$  is open in  $U$ , or show that

$$W_z := U - U_z = \{P \in U : z(P) = 0\}$$

is closed in  $U$ .

- (2) Write  $z = f/g$  where  $f, g \in \Gamma(V)$ ,  $g(P) \neq 0$  for all  $P \in V$ .  $z(P) = 0$  if and only if  $f(P) = 0$  on  $V$ . Hence,

$$W_z = \{P \in U : f(P) = 0\} = U \cap \{P \in V : f(P) = 0\}$$

is closed in  $U$  (Problem 6.7(a)).

- (3) Show that the mapping  $\varphi : U \rightarrow k = \mathbf{A}^1$  defined by  $\varphi : P \mapsto z(P)$  is continuous. By Problem 6.6, it suffices to show that  $\varphi^{-1}(\lambda)$  is closed in  $U$  for any  $\lambda \in k$ .

- (4) Note that  $z - \lambda \in \mathcal{O}_P(V)$  for all  $P \in V$ . Hence,

$$\varphi^{-1}(\lambda) = \{P \in U : z(P) = \lambda\} = \{P \in U : (z - \lambda)(P) = 0\} = W_{z-\lambda}$$

is closed in  $U$  by (2).

□

**6.2. Varieties****Problem 6.9.**

Let  $X = \mathbf{A}^2 - \{(0, 0)\}$ , an open subvariety of  $\mathbf{A}^2$ . Show that  $\Gamma(X) = \Gamma(\mathbf{A}^2) = k[x, y]$ .

*Proof.*

- (1) Let  $z = f/g \in \Gamma(X)$  be a regular function on  $X$ . So  $f = 0$  (nothing to do) or  $g(P) \neq 0$  for all  $P \in X = \mathbf{A}^2 - \{(0, 0)\}$ .
- (2) As  $V(g)$  is contained in a finite set  $\{(0, 0)\}$ ,  $g$  is a nonzero constant. Therefore,  $z \in k[x, y] = \Gamma(\mathbf{A}^2)$  (Problem 1.4).

□

**Problem 6.10.\***

Let  $U$  be an open subvariety of a variety  $X$ ,  $Y$  a closed subvariety of  $U$ . Let  $\overline{Y}$  be the closure of  $Y$  in  $X$ . Show that

- (a)  $\overline{Y}$  is a closed subvariety of  $X$ .
- (b)  $Y$  is an open subvariety of  $\overline{Y}$ .

*Proof of (a).*

- (1) It suffices to show that  $\overline{Y}$  is irreducible if  $Y$  is irreducible (since  $\overline{Y}$  is closed in  $X$  already).
- (2) (Reductio ad absurdum) If  $\overline{Y}$  were reducible, there are two closed sets  $Y_1$  and  $Y_2$  such that  $\overline{Y} \subseteq Y_1 \cup Y_2$  and  $\overline{Y} \not\subseteq Y_1, Y_2$ .
- (3) Show that  $Y \subseteq \overline{Y} \subseteq Y_1 \cup Y_2$  and  $Y \not\subseteq Y_1, Y_2$  if  $\overline{Y}$  were reducible. If not, might assume that  $Y \subseteq Y_1$ . Take closure to get  $\overline{Y} \subseteq \overline{Y_1} = Y_1$  (since  $Y_1$  is closed), contrary to the assumption.
- (4) Therefore,  $Y$  is reducible, a contradiction.

□

*Proof of (b).*

- (1) It suffices to show that  $Y = \overline{Y} \cap X$ .
- (2)  $Y \subseteq \overline{Y} \cap X$  is trivial.
- (3) Conversely, write  $Y = \tilde{Y} \cap X$  for some closed set  $\tilde{Y}$  in  $X$ .  $\tilde{Y} \supseteq \overline{Y}$  since  $\overline{Y}$  is the smallest closed set containing  $Y$ . So  $Y = \tilde{Y} \cap X \supseteq \overline{Y} \cap X$ .

□

**Problem 6.11. (Noetherian topological space)**

- (a) Show that every family of closed subsets of a variety has a minimal member.
- (b) Show that if a variety is a union of a collection of open subsets, it is a union of a finite number of these subsets. (All varieties are **quasi-compact**.)

*Proof of (a).*

- (1) Similar to the proof of Proposition 1 in §6.2, we may assume that  $X$  is an affine variety in  $\mathbf{A}^N$ .

- (2) By Problem 1.38, there is a natural one-to-one correspondence between closed subsets of  $X$  and radical ideals in  $k[x_1, \dots, x_N]/I(X)$ .
- (3) Note that  $I(X)$  is finitely generated (by the Hilbert basis theorem in §1.4). By Problem 1.22,  $k[x_1, \dots, x_N]/I(X)$  is Noetherian. Hence, Lemma in §1.5 shows that every family of closed subsets of  $X$  has a minimal member.

□

*Proof of (b).*

- (1) Let  $\{U_\alpha\}$  be an open covering of  $X$ . Let  $\Sigma$  be the family of closed sets being finite intersections of complements of members of  $\{U_\alpha\}$ , say

$$\bigcap_{\text{finite } \alpha} (X - U_\alpha) = X - \bigcup_{\text{finite } \alpha} U_\alpha.$$

By (a),  $\Sigma$  has a minimal member  $S$ . Now it suffices to show that  $S = \emptyset$ .

- (2) By the minimality of  $S$ ,  $S \cap (X - U_\alpha) = S$  for all  $\alpha$ . So  $S \subseteq X - U_\alpha$ , or  $X - S \supseteq U_\alpha$  for all  $\alpha$ . Hence,

$$X \supseteq X - S \supseteq \bigcup_{\text{all } \alpha} U_\alpha \supseteq X.$$

- (3) Besides, we can show that  $X$  is a Noetherian topological space if every open subset of  $X$  is quasi-compact.

□

### **Problem 6.12.\***

*Let  $X$  be a variety,  $z \in k(X)$ . Show that the pole set of  $z$  is closed. If  $z \in \mathcal{O}_P(X)$ , there is a neighborhood  $U$  of  $z$  such that  $z \in \Gamma(U)$ ; so  $\mathcal{O}_P(X)$  is the union of all  $\Gamma(U)$ , where  $U$  runs through all neighborhoods of  $P$ .*

*Proof.*

- (1) Similar to the proof of Proposition 1 in §6.2, we may assume that  $X$  is an affine variety in  $\mathbf{A}^N$ .
- (2) Proposition 2 in §2.4 shows that the pole set  $V(J_z) \subseteq X$  of  $z$  is algebraic or closed.
- (3) Given  $z \in \mathcal{O}_P(X)$  ( $P \notin V(J_z)$ ), there is an open neighborhood  $U := X - V(J_z)$  of  $z$  such that  $z \in \Gamma(U)$ . So

$$\mathcal{O}_P(X) = \bigcup_U \Gamma(U)$$

where  $U$  runs through all neighborhoods of  $P$ .

□

### 6.3. Morphisms of Varieties

#### Problem 6.13.\*

Let  $R$  be a domain with quotient field  $K$ ,  $f \neq 0$  in  $R$ . Let

$$R[1/f] = \{a/f^n : a \in R, n \in \mathbb{Z}\},$$

a subring of  $K$ .

- (a) Show that if  $\varphi : R \rightarrow S$  is any ring homomorphism such that  $\varphi(f)$  is a unit in  $S$ , then  $\varphi$  extends uniquely to a ring homomorphism from  $R[1/f]$  to  $S$ .
- (b) Show that the ring homomorphism from  $R[x]/(xf-1)$  to  $R[1/f]$  that takes  $x$  to  $1/f$  is an isomorphism.

*Proof of (a).*

- (1) Suppose  $\psi : R[1/f] \rightarrow S$  is a ring homomorphism such that  $\psi = \varphi$  on  $R$ .
- (2) To see  $\psi$  is uniquely determined, it suffices to show that  $\psi(1/f)$  is uniquely determined (since  $\psi$  is a homomorphism).
- (3) By  $1 = f/f$ , we have  $\psi(1) = \psi(f)\psi(1/f)$ .  $\varphi(1) = \varphi(f)\psi(1/f)$  since  $\psi = \varphi$  on  $R$ . Since  $\varphi(1) = 1$  and  $\varphi(f)$  is a unit in  $S$ ,  $\psi(1/f) = \varphi(f)^{-1}$  is uniquely determined.
- (4) Lastly, it is easy to check  $\psi(a/f^n) = \varphi(a)\varphi(f)^{-n}$  is a ring homomorphism.

□

*Proof of (b).*

- (1) Define a ring homomorphism  $\varphi : R \hookrightarrow R[x] \rightarrow S := R[x]/(xf-1)$  by

$$\varphi : r \mapsto r \mapsto \bar{r} = r + (xf-1).$$

- (2) Note that  $\varphi(f) = \bar{f}$  is a unit in  $S$  since  $\bar{x}\bar{f} = \bar{1}$  in  $S$ . By (a),  $\varphi$  extends uniquely to a ring homomorphism from  $R[1/f]$  to  $S$ . In particular,  $\varphi(1/f) = \varphi(f)^{-1} = \bar{x}$ .
- (3) Show that  $\varphi$  is surjective. Note that  $\varphi(r) = \bar{r}$  for any  $r \in R$  and  $\varphi(1/f) = \bar{x}$ . So,  $\varphi$  is surjective since  $\varphi$  is a ring homomorphism.

- (4) Show that  $\varphi$  is injective. Suppose  $\varphi(r/f^n) = 0$  for some  $r/f^n \in R[1/f]$ . May assume that  $n > 0$ . Write

$$rx^n = (xf - 1)g(x)$$

for some  $g(x) \in R[x]$ . It suffices to show that  $g(x) = 0$ . (Reductio ad absurdum) If  $g(x) \neq 0$ , then  $r \neq 0$  and  $\deg(g) = n - 1$ . Take  $x = 0$  in  $rx^n = (xf - 1)g(x)$  to get  $g(0) = 0$ . So  $g = xg_1$  for some  $g_1 \in R[x]$  if  $n > 1$ . Hence,

$$rx^{n-1} = (xf - 1)g_1(x)$$

where  $\deg g_1 = n - 2$  if  $n > 1$ . Continue this process to get  $g(x) = cx^{n-1} \in R[x]$  for some nonzero  $c \in R$ . So  $rx = cfx - c$ , which is absurd.

□

**Problem 6.14.\***

Let  $X, Y$  be varieties,  $f : X \rightarrow Y$  a mapping. Let  $X = \bigcup_{\alpha} U_{\alpha}$ ,  $Y = \bigcup_{\alpha} V_{\alpha}$ , with  $U_{\alpha}, V_{\alpha}$  open subvarieties, and suppose  $f(U_{\alpha}) \subseteq V_{\alpha}$  for all  $\alpha$ .

- (a) Show that  $f$  is a morphism if and only if each restriction  $f_{\alpha} : U_{\alpha} \rightarrow V_{\alpha}$  of  $f$  is a morphism.
- (b) If each  $U_{\alpha}, V_{\alpha}$  is affine,  $f$  is a morphism if and only if each  $\tilde{f}(\Gamma(V_{\alpha})) \subseteq \Gamma(U_{\alpha})$ .

*Proof of (a).*

- (1) Problem 6.2 shows that  $f$  is continuous if and only if each restriction  $f_{\alpha} : U_{\alpha} \rightarrow V_{\alpha}$  of  $f$  is continuous.
- (2) Suppose  $f : X \rightarrow Y$  is a morphism. The diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i_X \uparrow & & \uparrow i_Y \\ U_{\alpha} & \xrightarrow{f_{\alpha}} & V_{\alpha} \end{array}$$

is commutative. By Problem 6.15(a), two inclusion maps  $i_X$  and  $i_Y$  are morphisms. By Problem 6.15(b),  $f \circ i_X$  is a morphism. So it suffices to show that  $f_{\alpha}$  is a morphism if  $i_Y \circ f_{\alpha}$  is a morphism.

- (3) Given a rational function  $g \in \Gamma(V, \mathcal{O}_{V_{\alpha}})$  on an open set  $V$  of  $V_{\alpha}$ . So  $g \circ i_Y$  is a rational function in  $\Gamma(V, \mathcal{O}_Y)$  on an open set  $V$  of  $Y$  (Problem 6.1).

Therefore, given any  $y \in V$  we have

$$\begin{aligned}(\widetilde{f_\alpha}(g))(y) &= (g \circ f_\alpha)(y) \\ &= g(f_\alpha(y)) \\ &= g(i_Y(f_\alpha(y))) \\ &= (\widetilde{i_Y \circ f_\alpha}(g))(y).\end{aligned}$$

Hence  $\widetilde{f_\alpha}$  pulls back  $g$  to a rational function  $\widetilde{f_\alpha}(g)$  on  $U_\alpha$ . Therefore,  $f_\alpha$  is a morphism if  $f$  is a morphism.

- (4) Suppose  $f_\alpha : U_\alpha \rightarrow V_\alpha$  of  $f$  is a morphism for all  $\alpha$ . Given a rational function  $g \in \Gamma(V, \mathcal{O}_Y)$  on an open set  $V$  of  $Y$ .  $g$  is a rational function in  $\Gamma(V \cap V_\alpha, \mathcal{O}_{V_\alpha})$  on an open set  $V \cap V_\alpha$  of  $V_\alpha \subseteq Y$  for all  $\alpha$ . Therefore,

$$\widetilde{f_\alpha}(g) = g \circ f_\alpha \in \Gamma(f_\alpha^{-1}(V \cap V_\alpha), \mathcal{O}_{U_\alpha}) = \Gamma(f_\alpha^{-1}(V), \mathcal{O}_{U_\alpha})$$

for all  $\alpha$ . So each restriction  $g \circ f_\alpha = (g \circ f)|_{U_\alpha}$  is a rational function on  $U_\alpha$ . Hence,  $\widetilde{f}(g) = g \circ f$  is a rational function on  $X = \bigcup_\alpha U_\alpha$ .

□

*Proof of (b).*

- (1) Part (a) and Proposition 2 in §6.2 show that  $f$  is a morphism if and only if each  $\widetilde{f_\alpha}(\Gamma(V_\alpha)) \subseteq \Gamma(U_\alpha)$ .
- (2) Since  $f$  and  $f_\alpha$  are the same on  $U_\alpha$ ,  $\widetilde{f_\alpha} = \widetilde{f}$  on  $\Gamma(V_\alpha)$ .

□

### Problem 6.15.\*

- (a) If  $Y$  is an open or closed subvariety of  $X$ , the inclusion  $i : Y \rightarrow X$  is a morphism.
- (b) The composition of morphisms is a morphism.

*Proof of (a).*

- (1) The inclusion map is continuous in any topological space. ( $i^{-1}(U) = U \cap Y$  is open in  $Y$  for every open set  $U$  of  $X$ .)
- (2) For every open set  $U$  of  $X$ , if  $f \in \Gamma(U, \mathcal{O}_X)$ , then  $\widetilde{i}(f)$  is a rational function defined on  $U \cap Y = i^{-1}(U)$ . Thus,  $\widetilde{i}(f) = f \circ i \in \Gamma(i^{-1}(U), \mathcal{O}_Y)$ . (The ring  $\Gamma(i^{-1}(U), \mathcal{O}_Y)$  is well-defined since  $Y$  is a variety.)

□

*Proof of (b).*

- (1) Given two morphisms  $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$ . The composition continuous functions  $\psi \circ \varphi$  is continuous in any topological space.
- (2) For every open set  $U$  of  $Z$ , if  $f \in \Gamma(U, \mathcal{O}_Z)$ , then  $\tilde{\psi}(f) = f \circ \psi \in \Gamma(\psi^{-1}(U), \mathcal{O}_Y)$ . Note that  $\psi^{-1}(U)$  is open since  $\psi$  is continuous. So

$$\tilde{\varphi}(\tilde{\psi}(f)) = f \circ \psi \circ \varphi \in \Gamma(\varphi^{-1}(\psi^{-1}(U)), \mathcal{O}_X).$$

Problem 2.6 gives that  $\tilde{\varphi} \circ \tilde{\psi} = \widetilde{\psi \circ \varphi}$ . Besides,  $\varphi^{-1} \circ \psi^{-1} = (\psi \circ \varphi)^{-1}$ . Therefore,  $\widetilde{\psi \circ \varphi}$  pulls back rational functions to rational functions on every open set.

□

**Problem 6.17.**

- (a) Show that  $\mathbf{A}^2 - \{(0, 0)\}$  is not an affine variety (see Problem 6.9).
- (b) The union of two open affine subvarieties of a variety may not be affine.

Note that  $\mathbf{A}^1 - \{0\}$  is an affine variety which is isomorphic to  $V(1 - xy) \subseteq \mathbf{A}^2$  (Proposition 5 in §6.3).

*Proof of (a).*

- (1) (Reductio ad absurdum) Suppose  $X := \mathbf{A}^2 - \{(0, 0)\}$  were an affine variety. By Problem 6.15(a), the inclusion map  $i : X \hookrightarrow \mathbf{A}^2$  is a morphism. By Proposition 2 in §6.2,  $\tilde{i} : \Gamma(\mathbf{A}^2) \hookrightarrow \Gamma(X)$  is a ring homomorphism.
- (2) Problem 6.9 shows that  $\Gamma(\mathbf{A}^2) = \Gamma(X) = k[x, y]$ . Hence  $\tilde{i}$  is an isomorphism, so  $i$  is an isomorphism. However,  $i$  is not surjective on the point  $(0, 0)$ , which is absurd.

□

*Proof of (b).*

- (1) Write  $\mathbf{A}^2 - \{(0, 0)\} = (\mathbf{A}^2 - V(x)) \cup (\mathbf{A}^2 - V(y))$ . It suffices to show that  $\mathbf{A}^2 - V(x)$  is an affine variety.
- (2) Note that  $\mathbf{A}^2 - V(x)$  is isomorphic to a closed subvariety  $V(1 - xz)$  of  $\mathbf{A}^3$ .

□



**Problem 6.18.**

Show that the natural map  $\pi$  from  $\mathbf{A}^{n+1} - \{(0, \dots, 0)\}$  to  $\mathbf{P}^n$  is a morphism of varieties, and that a subset  $U$  of  $\mathbf{P}^n$  is open if and only if  $\pi^{-1}(U)$  is open.

*Proof.*

- (1) Show that  $\pi$  is the morphism by Problem 6.14. Write

$$X := \mathbf{A}^{n+1} - \{(0, \dots, 0)\} = \bigcup_{i=1}^{n+1} U_i$$

where  $U_i = \{(a_1, \dots, a_{n+1}) \in X : a_i \neq 0\}$  are open subvarieties in  $X$ . Similarly, write

$$Y := \mathbf{P}^n = \bigcup_{i=1}^{n+1} V_i$$

where  $V_i = \{[a_1 : \dots : a_{n+1}] \in Y : a_i \neq 0\}$  are open subvarieties. Hence  $\pi(U_i) = V_i$  for all  $i$ .

- (2) Now it suffices to show  $\pi|_{U_i} : U_i \rightarrow V_i$  is a morphism for all  $i$  (Problem 6.14). Given any closed subset  $V(f) \cap V_i$  of  $V_i$ , where  $f \in k[x_1, \dots, x_{n+1}]$ .  $(\pi|_{U_i})^{-1}(V(f) \cap V_i) = V(f) \cap U_i$  is also closed in  $U_i$ . Hence  $\pi|_{U_i}$  is continuous. Also,  $\pi|_{U_i}$  pulls back rational functions to a rational functions.
- (3) Show that  $U \subseteq \mathbf{P}^n$  is open if and only if  $\pi^{-1}(U)$  is open.  $\pi^{-1}(U)$  is open if  $U$  is open since  $\pi$  is continuous. It suffices to show that  $\pi(E)$  is open if  $E$  is open in  $X$ .
- (4) Let  $\tilde{E}$  be the union of all the lines through the origin that meet  $E$ , that is,  $\tilde{E} = \pi^{-1}(\pi(E))$ . Note that

$$\tilde{E} = \bigcup_{a \in k - \{0\}} (aE)$$

is open in  $X$  since each  $aE$  is open in  $X$ . (Write  $E = X - V(f_1, \dots, f_r)$ . Then  $aE = X - V(g_1, \dots, g_r)$ , where

$$g_i(x_1, \dots, x_{n+1}) := f_i(a^{-1}x_1, \dots, a^{-1}x_{n+1}) \in k[x_1, \dots, x_{n+1}]$$

for each  $i$ .) By the continuity of  $\pi$ ,  $\pi(E)$  is open since  $\tilde{E} = \pi^{-1}(\pi(E))$  is open.

□

**Problem 6.21.**

Any variety is the union of a finite number of open affine subvarieties.

*Proof.*

- (1) Propositions 4 and 3 in §6.2 give all.
- (2) *Proof of Proposition 3 in §6.2.* By Problem 6.4,  $V_i = \varphi_i^{-1}(V)$  is a closed subvariety of  $\mathbf{A}^{n+1}$ . Since  $\varphi : \varphi_i^{-1}(V) \rightarrow V \cap U_i$  is an isomorphism,  $V \cap U_i$  is an open affine subvariety. Therefore,  $V = \bigcup_{i=1}^{n+1} (V \cap U_i)$  is the union of a finite number of open affine subvarieties.

□

**6.4. Products and Graphs****Problem 6.26.\***

- (a) Let  $f : X \rightarrow Y$  be a morphism of varieties such that  $f(X)$  is dense in  $Y$ . Show that the homomorphism  $\tilde{f} : \Gamma(Y) \rightarrow \Gamma(X)$  is one-to-one.
- (b) If  $X$  and  $Y$  are affine, show that  $f(X)$  is dense in  $Y$  if and only if  $\tilde{f} : \Gamma(Y) \rightarrow \Gamma(X)$  is one-to-one. Is this true if  $Y$  is not affine?

*Proof of (a).*

- (1) Given  $\varphi \in \Gamma(Y)$  such that  $\tilde{f}(\varphi) = \varphi \circ f = 0$ .
- (2) So  $\varphi = 0$  on  $f(X)$ . Since  $f(X)$  is dense in  $Y$ ,  $\varphi = 0$  on  $Y$  by Problem 6.7(b). Hence  $\tilde{f}$  is injective.

□

*Proof of (b).*

- (1) If  $X$  and  $Y$  are affine, show that  $f(X)$  is dense in  $Y$  if and only if  $\tilde{f} : \Gamma(Y) \rightarrow \Gamma(X)$  is one-to-one. One direction is followed by (a).
- (2) If  $f(X)$  is not dense in  $Y \subseteq \mathbf{A}^N$ , then there is one open set  $O = \bigcup_{i=1}^r V_{\varphi_i}$  such that

$$O \cap Y = \bigcup (V_{\varphi_i} \cap Y) \neq \emptyset$$

and

$$O \cap f(X) = \bigcup (V_{\varphi_i} \cap f(X)) = \emptyset.$$

(Note that  $Y$  is affine.) Hence  $V_{\varphi_i} \cap Y \neq \emptyset$  for some  $i = i_0$  and  $V_{\varphi_i} \cap f(X) = \emptyset$  for all  $i$ . Take  $\varphi = \varphi_{i_0}$ . So there is  $\varphi \in \Gamma(Y)$  such that  $\tilde{f}(\varphi) = 0$  and  $\varphi \neq 0$ . So  $\tilde{f}$  is not injective. Hence part (1) is established.

(3) If  $Y$  is not affine, then  $\tilde{f} : \Gamma(Y) \rightarrow \Gamma(X)$  is one-to-one  $\not\Rightarrow f(X)$  is dense in  $Y$ . Let  $f : \mathbf{P}^1 \rightarrow \mathbf{P}^2$  be a morphism as in Problem 6.25. Hence the image of  $f$  is not dense in  $\mathbf{P}^2$  (since  $f(\mathbf{P}^1) \cap \{z \neq 0\} = \emptyset$ ).

(4) Note that  $\Gamma(\mathbf{P}^n(k)) = k$  for any projective  $n$ -space over  $k$  (since  $k = \bar{k}$ .) Therefore, the  $k$ -homomorphism

$$\tilde{f} : \underbrace{\Gamma(\mathbf{P}^2)}_{=k} \rightarrow \underbrace{\Gamma(\mathbf{P}^1)}_{=k}$$

is an isomorphism (and thus  $\tilde{f}$  is injective).

□

### Problem 6.29. (Algebraic group)

Let  $V$  be a variety, and suppose  $V$  is also a group, i.e., there are mappings  $\varphi : V \times V \rightarrow V$  (multiplication or addition), and  $\psi : V \rightarrow V$  (inverse) satisfying the group axioms. If  $\varphi$  and  $\psi$  are morphisms,  $V$  is said to be an **algebraic group**. Show that each of the following is an algebraic group:

- (a)  $\mathbf{A}^1 = k$ , with the usual addition on  $k$ ; this group is often denoted  $\mathbb{G}_a$ .
- (b)  $\mathbf{A}^1 - \{(0)\} = k - \{0\}$ , with the usual multiplication on  $k$ ; this is denoted  $\mathbb{G}_m$ .
- (c)  $\mathbf{A}^n(k)$  with addition; likewise  $\mathbf{M}_n(k) = \{n \text{ by } n \text{ matrices}\}$  under addition may be identified with  $A^{n^2}(k)$ .
- (d)  $\mathbf{GL}_n(k) = \{\text{invertible } n \times n \text{ matrices}\}$  is an affine open subvariety of  $\mathbf{M}_n(k)$ , and a group under multiplication.
- (e)  $C$  a nonsingular plane cubic,  $O \in C$ ,  $\oplus$  the resulting addition (see Problem 5.38).

*Proof of (a).*

- (1)  $\mathbf{A}^1 = k$  is an affine variety.
- (2)  $k$  is an abelian group under addition with 0 as the additive identity.
- (3)  $\varphi : (a, b) \mapsto a + b$  is continuous (by identifying  $\mathbf{A}^1 \times \mathbf{A}^1$  to  $\mathbf{A}^2$ ) and pulls back rational functions to a rational functions.

- (4)  $\psi : a \mapsto -a$  is continuous and pulls back rational functions to a rational functions.

□

*Proof of (b).*

- (1)  $\mathbf{A}^1 - \{0\}$  is an affine variety which is isomorphic to  $V(1 - xy) \subseteq \mathbf{A}^2$ . It is the same as Proposition 5 in §6.3.
- (2)  $k$  is an abelian group under multiplication with 1 as the multiplicative identity.
- (3)  $\varphi : (a, b) \mapsto ab$  is continuous (by identifying  $\mathbf{A}^1 \times \mathbf{A}^1$  to  $\mathbf{A}^2$ ) and pulls back rational functions to a rational functions.
- (4)  $\psi : a \mapsto a^{-1}$  is continuous and pulls back rational functions to a rational functions.

□

*Proof of (c).*

- (1)  $\mathbf{A}^n = k^n$  is an affine variety (or apply Proposition 6 in §6.4).
- (2)  $k^n$  is a direct sum of  $n$  copies of abelian groups  $k$  under addition.
- (3) By Proposition 7 in §6.4 and (a), both  $\varphi$  and  $\psi$  are morphisms.

□

*Proof of (d).*

- (1) Similar to (b).  $\mathrm{GL}_n(k)$  is an affine open subvariety of  $\mathbf{M}_n(k)$  which is isomorphic to  $V(1 - \det(x_{ij})y) \subseteq \mathbf{A}^{n^2+1}$  (where  $(x_{ij}) \in \mathbf{M}_n(k)$ ).
- (2)  $\mathrm{GL}_n(k)$  is a group under multiplication with  $I_n$  as the multiplicative identity.
- (3) Similar to (b). Both  $\varphi$  and  $\psi$  are morphisms.

□

*Proof of (e).*

- (1) Similar to Problem 5.10, we might assume that  $C = y^2z - x^3 - ax^2z - bxz^2 - cz^3$  and  $O = [0 : 1 : 0] \in C$ .  $C$  is an affine variety in  $\mathbf{P}^2$ .
- (2) By Proposition 4 in §5.6,  $C$  is an abelian group with the addition  $\oplus$ . Now we can use Problem 5.38 to compute  $P \oplus Q$  for any  $P, Q \in C$ .

- (3) Define the addition  $\varphi$  as in Problem 5.38. Let

$$\begin{aligned} U_1 &= \{x_1 \neq x_2\}, \\ U_2 &= \{x_1 = x_2, y_1 \neq 0\}, \\ U_3 &= \{x_1 = x_2, y_1 = 0, z_1 \neq 0\} \end{aligned}$$

be an open covering of  $C \times C$  (Proposition 7 in §6.4). It suffices to show that  $\varphi$  is a morphism on each  $U_\alpha$  ( $\alpha = 1, 2, 3$ ) (Problem 6.14). Since  $\varphi$  is a polynomial map on each  $U_\alpha$ ,  $\varphi$  is continuous on each  $U_\alpha$ . Also,  $\varphi$  pulls back rational functions to a rational functions on each  $U_\alpha$ . Hence  $\varphi$  is a morphism.

- (4) Define the inverse  $\psi$  as  $\psi(O) = O$  and  $\psi([x : y : 1]) = [x : -y : 1]$  (Problem 5.38).  $\psi$  is continuous and pulls back rational functions to a rational functions. So  $\psi$  is a morphism.

□

## 6.5. Algebraic Function Fields and Dimension of Varieties

### Problem 6.31.\* (Theorem of the primitive element)

Let  $K$  be a field of a characteristic zero,  $L$  a finite (algebraic) extension of  $K$ . Then there is a  $z \in L$  such that  $L = K(z)$ .

*Outline of Proof.*

- (1) Suppose  $L = K(x, y)$ . Let  $f$  and  $g$  be monic irreducible polynomials in  $K[t]$  such that  $f(x) = 0$ ,  $g(y) = 0$ . Let  $L'$  be a field in which  $f = \prod_{i=1}^n (t - x_i)$ ,  $g = \prod_{j=1}^m (t - y_j)$ ,  $x = x_1$ ,  $y = y_1$ ,  $L' \supseteq L$  (see Problems 1.52, 1.53).
- (2) Choose  $\lambda \neq 0$  in  $K$  so that  $\lambda x + y \neq \lambda x_i + y_j$  for all  $i \neq 1, j \neq 1$ . Let  $z = \lambda x + y$ ,  $K' = K(z)$ . Set  $h(t) = g(z - \lambda t) \in K'[t]$ . Then  $h(x) = 0$ ,  $h(x_i) \neq 0$  if  $i \neq 1$ . Therefore  $(h, f) = (t - x) \in K'[t]$ . Then  $x \in K'$ , so  $y \in K'$ , so  $L = K'$ .
- (3) If  $L = K(x_1, \dots, x_n)$ , use induction on  $n$  to find  $\lambda_1, \dots, \lambda_n \in K$  such that  $L = K(\sum \lambda_i x_i)$ .

*Proof.*

- (1) Suppose  $L = K(x, y)$ . Let  $f$  and  $g$  be monic irreducible polynomials in  $K[t]$  such that  $f(x) = 0$ ,  $g(y) = 0$ . Let  $L'$  be a splitting field of  $L$  in which  $f = \prod_{i=1}^n (t - x_i)$ ,  $g = \prod_{j=1}^m (t - y_j)$ ,  $x = x_1$ ,  $y = y_1$ ,  $L' \supseteq L$  (Problems 1.52). Note that the  $x_i$  (resp.  $y_j$ ) are distinct by the irreducibility of  $f$  (resp.  $g$ ) (Problem 1.53).

- (2) Choose  $\lambda \neq 0$  in  $K$  so that  $\lambda x + y \neq \lambda x_i + y_j$  for all  $i \neq 1, j \neq 1$ . (Note that  $\mathbb{Q} \hookrightarrow K$  since  $\text{char}(K) = 0$ . Hence, such one  $\lambda$  exists since  $K$  contains an infinite set  $\mathbb{Q}$ .) Let  $z = \lambda x + y$  and  $K' = K(z)$ .
- (3) Show that  $L = K'$ .  $K' = K(z) \subseteq L$  since  $z \in L$ . Conversely, it suffices to show that  $x \in K'$ . Define  $h(t) = g(z - \lambda t) \in K'[t]$ . Then  $h(x) = 0$  and  $h(x_i) \neq 0$  if  $i \neq 1$  by the construction of  $\lambda$ . Therefore  $(h, f) = (t - x) \in K'[t]$ . Then  $x \in K'$ , so  $y \in K'$ , so  $L = K'$ .
- (4) If  $L = K(x_1, \dots, x_n)$  is a finite (algebraic) extension of  $K$ , use induction on  $n$  to find  $\lambda_1, \dots, \lambda_n \in K$  such that

$$\begin{aligned} L &= K(x_1, \dots, x_{n-1})(x_n) \\ &= K(\lambda_1 x_1 + \dots + \lambda_{n-1} x_{n-1})(x_n) && \text{(Induction hypothesis)} \\ &= K(\lambda_1 x_1 + \dots + \lambda_n x_n) && \text{(Base step)} \end{aligned}$$

where  $\lambda_1 := 1$ .

□

### Problem 6.32.\*

Let  $L = K(x_1, \dots, x_n)$  as in Problem 6.31. Suppose  $k \subseteq K$  is an algebraically closed subfield, and  $V \subsetneq \mathbf{A}^n(k)$  is an algebraic set. Show that  $L = K(\sum \lambda_i x_i)$  for some  $(\lambda_1, \dots, \lambda_n) \in \mathbf{A}^n - V$ .

*Proof.*

- (1) Since  $V \subsetneq \mathbf{A}^n$ , there is one index  $1 \leq i \leq n$  such that the projection map  $\pi_i : V \rightarrow \mathbf{A}^1$  defined by  $\pi_i(a_1, \dots, a_n) = a_i$  satisfies  $\pi_i(V) \subsetneq \mathbf{A}^1$  (Problem 2.10). Here  $\pi_i(V)$  is a finite subset of  $\mathbf{A}^1$  (Problem 1.8).
- (2) In the proof of Problem 6.31, we also take  $\lambda_i \notin \pi_i(V)$ . It is possible since  $k$  is infinite. Therefore,  $(\lambda_1, \dots, \lambda_n) \in \mathbf{A}^n - V$ .

□

### Problem 6.33. (Transcendence degree)

The notion of transcendence degree is analogous to the idea of the dimension of a vector space. If  $k \subseteq K$ , we say that  $x_1, \dots, x_n \in K$  are **algebraically independent** if there is no nonzero polynomial  $f \in k[x_1, \dots, x_n]$  such that  $f(x_1, \dots, x_n) = 0$ . By methods entirely analogous to those for bases of vector spaces, one can prove:

- (a) Let  $x_1, \dots, x_n \in K$ ,  $K$  a finitely generated extension of  $k$ . Then  $x_1, \dots, x_n$  is a minimal set such that  $K$  is algebraic over  $k(x_1, \dots, x_n)$  if and only if  $x_1, \dots, x_n$  is a maximal set of algebraically independent elements of  $K$ . Such  $\{x_1, \dots, x_n\}$  is called a **transcendence basis** of  $K$  over  $k$ .
- (b) Any algebraically independent set may be completed to a transcendence basis. Any set  $\{x_1, \dots, x_n\}$  such that  $K$  is algebraic over  $k(x_1, \dots, x_n)$  contains a transcendence basis.
- (c)  $\text{tr. deg}_k K$  is the number of elements in any transcendence basis of  $K$  over  $k$ .

*Proof of (a).*

- (1) Let  $S = \{s_1, \dots, s_n\} \subseteq K$ . Show that the following statements are equivalent:

- (i)  $S$  is algebraically independent over  $k$ .
- (ii) For each  $i$ ,  $s_i$  is transcendental over  $k(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$ .
- (iii) For each  $i$ ,  $s_i$  is transcendental over  $k(s_1, \dots, s_{i-1})$ .

(Lemma 19.7 in the textbook: *Patrick Morandi, Field and Galois Theory*.)

The result is established if we can show that (i)  $\iff$  (ii).

- (2) ((i)  $\implies$  (ii)) Suppose there are  $a_j \in k(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$  such that

$$s_i^m + \dots + a_1 s_i + a_0 = 0.$$

Clearing denominators, might assume  $a_j \in k[s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n]$ . ( $a_m = 1$ .) Then

$$f = \sum_j a_j(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n) s_i^j \in k[s_1, \dots, s_n]$$

has a root  $(s_1, \dots, s_n) \in \mathbf{A}^n(k)$ . Since  $S$  is algebraically independent over  $k$ ,  $f = 0$ . So  $a_j = 0$ . So  $s_i$  is transcendental over  $k(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$ .

- (3) ((ii)  $\implies$  (iii)) Clearly.

- (4) ((iii)  $\implies$  (i)) Suppose  $S$  were not algebraically independent over  $k$ . Choose  $m$  minimal such that there exists a nonzero  $f \in k[x_1, \dots, x_m]$  having a root  $(s_1, \dots, s_m) \in \mathbf{A}^n$ . Write  $f = \sum_j a_j x_m^j$  where  $a_j \in k[x_1, \dots, x_{m-1}]$ . So

$$\sum_j a_j(s_1, \dots, s_{m-1}) s_m^j = 0.$$

Since  $s_m$  is not algebraic over  $k(s_1, \dots, s_{m-1})$  by assumption, all  $a_j(s_1, \dots, s_{m-1}) = 0$  in  $k$ . By the minimality of  $m$ , all  $a_j = 0$  in  $k[x_1, \dots, x_{m-1}]$ . So  $f = 0$ , which is absurd.

□

*Proof of (b).*

- (1) *Show that any algebraically independent set may be completed to a transcendence basis.* Let  $A$  be any algebraically independent subset of  $K$ . We claim we can find a transcendence basis  $\mathcal{B}$  such that  $A \subseteq \mathcal{B} \subseteq K$ .
- (2) (Zorn's lemma) Let

$$\Sigma = \{S \text{ is algebraically independent} : A \subseteq S \subseteq K\}.$$

Order  $\Sigma$  by inclusion.  $\Sigma$  is non empty since  $A \in \Sigma$ . To apply Zorn's lemma we must show that every chain in  $\Sigma$  has an upper bound in  $\Sigma$ .

- (3) Let  $(S_\alpha)$  be a chain in  $\Sigma$ , so that for each pair of indices  $\alpha, \beta$  we have either  $S_\alpha \subseteq S_\beta$ , or  $S_\beta \subseteq S_\alpha$ . Let  $B = \bigcup_\alpha S_\alpha$ . Clearly,  $B \in \Sigma$  since  $A \subseteq B \subseteq K$  and polynomials only involve finitely many variables. By Zorn's lemma,  $\Sigma$  has a maximal element  $\mathcal{B}$ .
- (4) *Show that  $K$  is algebraic over  $k(\mathcal{B})$ .* (Reductio ad absurdum) Suppose there were an element  $s \in K$  transcendental over  $k(\mathcal{B})$ . Then  $\mathcal{B} \cup \{s\}$  is algebraically independent, which contradicts the maximality of  $\mathcal{B}$ .
- (5) *Show that any set  $A = \{x_1, \dots, x_n\}$  such that  $K$  is algebraic over  $k(x_1, \dots, x_n)$  contains a transcendence basis.* If  $A$  is algebraically independent, nothing to do. If  $A$  is algebraically dependent, then there is some  $x_i$  such that  $x_i$  algebraic over  $k(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ . So  $K$  is algebraic over  $k(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ . Continue this process, we can find a minimal subset  $\mathcal{B}$  of  $A$  such that  $K$  is algebraic over  $k(x_1, \dots, x_r)$  (with suitable relabeling). Here  $\mathcal{B} \subseteq A$  is a transcendence basis.

□

*Proof of (c).*

- (1) It suffices to show that  $\text{tr. deg}_k K$  is well-defined. *Let  $K$  be a field extension of  $k$ . If  $S$  and  $T$  are transcendence bases for  $K/k$ , then  $|S| = |T|$ .* (Theorem 19.15 in the textbook: *Patrick Morandi, Field and Galois Theory*.)
- (2) Suppose  $S = \{s_1, \dots, s_n\}$  is finite. In the proof of (b),  $K/k(S - \{s_1\})$  is not algebraic. Since  $K/k(T)$  is algebraic, some  $t \in T$ , say  $t_1 \in T$  (by relabeling), must be transcendental over  $k(S - \{s_1\})$ . By the proof of (a),

$$S_1 := \{t_1, s_2, \dots, s_n\}$$

is algebraically independent. Note that  $s_1$  is algebraic over  $k(S_1)$ , or else  $\{t_1, s_1, \dots, s_n\}$  is algebraically independent, contrary to the assumption of  $S$ . Therefore,  $K/k(S_1)$  is algebraic.



- (3) Continue this process to replace each element in  $S$  to get  $S_n = \{t_1, \dots, t_n\} \subseteq T$ . So  $|S| \leq |T|$ . Similarly,  $|T| \leq |S|$ . Therefore,  $|S| = |T|$  if  $S$  is finite.
- (4) Suppose  $S$  (and thus  $T$ ) are infinite. Since  $t$  is algebraic over  $k(S)$  for each  $t \in T$ , there is a corresponding finite subset  $S_t \subseteq S$  such that  $t$  is algebraic over  $k(S_t)$ .
- (5) Let

$$S' = \bigcup_{t \in T} S_t.$$

Then each  $t \in T$  is algebraic over  $k(S')$ . Since  $K/k(T)$  is algebraic,  $K/k(S')$  is algebraic. Thus  $S' = S$  since  $S' \subseteq S$  and  $S$  is a transcendental basis for  $K/k$ .

- (6) Hence,

$$|S| = |S'| = \left| \bigcup_{t \in T} S_t \right| \leq |T \times \mathbb{N}| = |T|,$$

where the last equality holds since  $|T| = \infty$ . Similarly,  $|T| \leq |S|$ . Therefore,  $|S| = |T|$  if  $S$  is infinite.

□

### Problem 6.34.

Show that  $\dim \mathbf{A}^n = \dim \mathbf{P}^n = n$ .

*Proof.*

- (1) By Proposition 10 in §6.5,

$$\dim \mathbf{P}^n = \dim \mathbf{A}^n$$

since  $\mathbf{P}^n$  is the projective closure of  $\mathbf{A}^n$ .

- (2) A direct calculation shows

$$\dim \mathbf{A}^n = \text{tr. deg}_k k(\mathbf{A}^n) = \text{tr. deg}_k k(x_1, \dots, x_n) = n.$$

□

## 6.6. Rational Maps

## Chapter 7: Resolution of Singularities

### 7.1. Rational Maps of Curves

#### Problem 7.1.

*Show that any curve has only a finite number of multiple points.*

*Proof.*

- (1) Write  $C = f_1^{\alpha_1} \cdots f_r^{\alpha_r}$  where each  $f_i$  is irreducible and  $\alpha_i$  is positive integer. Define  $D = f_1 \cdots f_r$ . Hence  $\mathcal{O}_P(C) = \mathcal{O}_P(D)$  and thus

$$\{\text{simple points of } C\} = \{\text{simple points of } D\}.$$

- (2) Note that the simple points of  $D$  agrees with our original definition. Hence  $D$  has only a finite number of multiple points (Problem 5.25). Therefore  $C$  has only a finite number of multiple points.

□

### 7.2. Blowing up a Point in $\mathbb{A}^2$

### 7.3. Blowing up a Point in $\mathbb{P}^2$

### 7.4. Quadratic Transformations

### 7.5. Nonsingular Models of Curves

## Chapter 8: Riemann-Roch Theorem

### 8.1. Divisors

#### Problem 8.1.

Let  $X = C = \mathbf{P}^1$ ,  $k(X) = k(t)$ , where  $t = \frac{x_1}{x_2}$ ,  $x_1, x_2$  homogeneous coordinates on  $\mathbf{P}^1$ .

- (a) Calculate  $\operatorname{div}(t)$ .
- (b) Calculate  $\operatorname{div}\left(\frac{f}{g}\right)$ ,  $f, g$  in  $k[t]$ .
- (c) Prove Proposition 1 in §8.1 directly in this case.

*Proof of (a).*

- (1) By definition,

$$\operatorname{div}(t) = \operatorname{ord}_{[0:1]}(x_1) \cdot [0 : 1] - \operatorname{ord}_{[1:0]}(x_2) \cdot [1 : 0] = [0 : 1] - [1 : 0].$$

- (2) Note that  $\deg(\operatorname{div}(t)) = 1 - 1 = 0$ .

□

*Proof of (b).*

- (1) Write

$$f = \prod_i (t - \lambda_i)^{\alpha_i} \text{ and } g = \prod_i (t - \lambda_i)^{\beta_i}$$

where  $\lambda_i \in k$  and  $\alpha_i, \beta_i$  are nonnegative integers (since  $k = \bar{k}$ ).

- (2) Hence

$$\operatorname{ord}_{[\lambda:1]}\left(\frac{f}{g}\right) = \begin{cases} \alpha_i - \beta_i & \text{if } \lambda_i = \lambda \\ 0 & \text{otherwise} \end{cases}$$

- (3) It remains to find  $\operatorname{ord}_{[1:0]}\left(\frac{f}{g}\right)$ . Write

$$\begin{aligned} \frac{f}{g} &= \prod_i (t - \lambda_i)^{\alpha_i - \beta_i} \\ &= \prod_i \left(\frac{x_1}{x_2} - \lambda_i\right)^{\alpha_i - \beta_i} \\ &= \left(\frac{x_2}{x_1}\right)^{-\sum_i (\alpha_i - \beta_i)} \prod_i \left(1 - \lambda_i \frac{x_2}{x_1}\right)^{\alpha_i - \beta_i}. \end{aligned}$$

So

$$\text{ord}_{[1:0]} \left( \frac{f}{g} \right) = - \sum_i (\alpha_i - \beta_i).$$

(4)

$$\text{div} \left( \frac{f}{g} \right) = \sum_i (\alpha_i - \beta_i) \cdot [\lambda_i : 1] - \left( \sum_i (\alpha_i - \beta_i) \right) \cdot [1 : 0].$$

□

*Proof of (c).* It suffices to show that  $\deg \left( \text{div} \left( \frac{f}{g} \right) \right) = 0$ . In fact,

$$\deg \left( \text{div} \left( \frac{f}{g} \right) \right) = \sum_i (\alpha_i - \beta_i) - \left( \sum_i (\alpha_i - \beta_i) \right) = 0.$$

□

**Problem 8.2.**

Let  $X = C = V(y^2z - x(x - z)(x - \lambda z)) \subseteq \mathbf{P}^2$ ,  $\lambda \in k$ ,  $\lambda \neq 0, 1$ . Let  $\bar{x} = \frac{x}{z}$ ,  $\bar{y} = \frac{y}{z} \in K$ ;  $K = k(\bar{x}, \bar{y})$ . Calculate  $\text{div}(\bar{x})$  and  $\text{div}(\bar{y})$ .

*Proof.*

- (1) Note that  $V(x) \cap C = \{[0 : 1 : 0], [0 : 0 : 1]\}$  and  $V(z) \cap C = \{[0 : 1 : 0]\}$ .  
Hence

$$\begin{aligned} \text{div} \left( \frac{x}{z} \right) &= \text{ord}_{[0:1:0]}(x)[0 : 1 : 0] + \text{ord}_{[0:0:1]}(x)[0 : 0 : 1] \\ &\quad - \text{ord}_{[0:1:0]}(z)[0 : 1 : 0]. \end{aligned}$$

(2) Since

$$\begin{aligned}
\text{ord}_{[0:1:0]}(x) &= I((0, 0), x \cap C_*) \\
&= I((0, 0), x \cap (z - x(x - z)(x - \lambda z))) \\
&= I((0, 0), x \cap z) \\
&= 1, \\
\text{ord}_{[0:0:1]}(x) &= I((0, 0), x \cap C_*) \\
&= I((0, 0), x \cap (y^2 - x(x - 1)(x - \lambda))) \\
&= I((0, 0), x \cap y^2) \\
&= 2, \\
\text{ord}_{[0:1:0]}(z) &= I((0, 0), z \cap C_*) \\
&= I((0, 0), z \cap (z - x(x - z)(x - \lambda z))) \\
&= I((0, 0), z \cap x^3) \\
&= 3,
\end{aligned}$$

we have

$$\begin{aligned}
\text{div} \left( \frac{x}{z} \right) &= [0 : 1 : 0] + 2[0 : 0 : 1] - 3[0 : 1 : 0] \\
&= 2[0 : 0 : 1] - 2[0 : 1 : 0].
\end{aligned}$$

(3) Note that  $V(y) \cap C = \{[0 : 0 : 1], [1 : 0 : 1], [\lambda : 0 : 1]\}$ . Hence

$$\begin{aligned}
\text{div} \left( \frac{y}{z} \right) &= \text{ord}_{[0:0:1]}(y)[0 : 0 : 1] + \text{ord}_{[1:0:1]}(y)[1 : 0 : 1] \\
&\quad + \text{ord}_{[\lambda:0:1]}(y)[\lambda : 0 : 1] - \text{ord}_{[0:1:0]}(z)[0 : 1 : 0].
\end{aligned}$$

(4) Since

$$\begin{aligned}
\text{ord}_{[0:0:1]}(y) &= I((0, 0), y \cap C_*) \\
&= I((0, 0), y \cap (y^2 - x(x - 1)(x - \lambda))) \\
&= I((0, 0), y \cap x(x - 1)(x - \lambda)) \\
&= 1, \\
\text{ord}_{[1:0:1]}(y) &= I((1, 0), y \cap C_*) \\
&= I((1, 0), y \cap x(x - 1)(x - \lambda)) \\
&= I((0, 0), y \cap x(x + 1)(x + 1 - \lambda)) \\
&= 1, \\
\text{ord}_{[\lambda:0:1]}(y) &= I((\lambda, 0), y \cap C_*) \\
&= I((\lambda, 0), y \cap x(x - 1)(x - \lambda)) \\
&= I((0, 0), y \cap x(x - 1 + \lambda)(x + \lambda)) \\
&= 1,
\end{aligned}$$

we have

$$\operatorname{div} \left( \frac{y}{z} \right) = [0 : 0 : 1] + [1 : 0 : 1] + [\lambda : 0 : 1] - 3[0 : 1 : 0].$$

□

**Problem 8.3.**

Let  $C = X$  be a nonsingular cubic.

- (a) Let  $P, Q \in C$ . Show that  $P \equiv Q$  if and only if  $P = Q$ . (Hint: Lines are adjoints of degree 1.)
- (b) Let  $P, Q, R, S \in C$ . Show that  $P + Q \equiv R + S$  if and only if the line through  $P$  and  $Q$  intersects the line through  $R$  and  $S$  in a point on  $C$  (if  $P = Q$  use the tangent line).
- (c) Let  $P_0$  be a fixed point on  $C$ , thus defining an addition  $\oplus$  on  $C$  (§5.6). Show that  $P \oplus Q = R$  if and only if  $P + Q \equiv R + P_0$ . Use this to give another proof of Proposition 4 of §5.6.

*Proof of (a).*

- (1) One direction is trivial by taking  $z \in k - \{0\}$  in the definition of linearly equivalence. For the opposite direction, we need the residue theorem in §8.1.
- (2) Take  $E = 0$  (since  $C$  is nonsingular),  $D = P$ ,  $D' = Q$ , and  $G = L$  be any line passing through  $D$ , that is,

$$\operatorname{div}(L) = P + E + A = P + A$$

for some effective divisor  $A$ . The existence of  $A$  is guaranteed by Bézout theorem. Note that we can write  $A = P' + P''$  where  $P'$  and  $P''$  are points in  $C$ .

- (3) By residue theorem in §8.1, there is an adjoint  $L'$  of degree 1 such that

$$\operatorname{div}(L') = Q + E + A = Q + A.$$

Therefore,

$$P = \varphi(P', P'') = Q.$$

□

*Proof of (b).*

- (1) Let  $L$  (resp.  $L'$ ) be a line passing through  $P, Q$  (resp.  $R, S$ ). So

$$\begin{aligned}\operatorname{div}(L) &= P + Q + \varphi(P, Q) \\ \operatorname{div}(L') &= R + S + \varphi(R, S).\end{aligned}$$

Hence

$$P + Q + \varphi(P, Q) \equiv R + S + \varphi(R, S)$$

(by Proposition 2(5) in §8.1).

- (2) Therefore,

$$\begin{aligned}P + Q &\equiv R + S \\ \iff \varphi(P, Q) &\equiv \varphi(R, S) && \text{(Proposition 2(4))} \\ \iff \varphi(P, Q) &= \varphi(R, S) && ((a)) \\ \iff \text{the line through } P &\text{ and } Q \text{ intersects} \\ &\text{the line through } R \text{ and } S \text{ in a point on } C.\end{aligned}$$

□

*Proof of (c).*

- (1) By taking  $S = P_0$  in (b), we have  $P \oplus Q = R \iff P + Q \equiv R + P_0$ .  
(2)  $P + Q \equiv R + P_0$  implies that  $R \in C$ . That is,  $\oplus$  defines an addition on  $C$ .  
 $P \oplus Q = Q \oplus P$  since  $P + Q = Q + P \equiv R + P_0$ .  $P_0$  is the identity since  $P + P_0 \equiv P + P_0$  is always true for any  $P \in C$ . Hence the rest is to prove the associativity.  
(3) Suppose  $P, Q, R \in C$ . Hence

$$\begin{aligned}&\underbrace{(P \oplus Q)}_{:=S} \oplus R := T \\ \iff S + R &\equiv T + P_0 \text{ and } P + Q \equiv S + P_0 \\ \implies P + Q + R &\equiv T + 2P_0 && \text{(Proposition 2(4))}\end{aligned}$$

and

$$\begin{aligned}&P \oplus \underbrace{(Q \oplus R)}_{:=S'} := T' \\ \iff P + S' &\equiv T' + P_0 \text{ and } Q + R \equiv S' + P_0 \\ \implies P + Q + R &\equiv T' + 2P_0. && \text{(Proposition 2(4))}\end{aligned}$$

Proposition 2(1) in §8.1 implies that  $T + 2P_0 \equiv T' + 2P_0$ . Hence  $T \equiv T'$  (Proposition 2(4)). Therefore  $T = T'$  by (a).

□

**Problem 8.4.**

Let  $C$  be a cubic with a node. Show that for any two simple points  $P, Q$  on  $C$ ,  $P \equiv Q$ .

*Proof.*

- (1) Let  $E$  be the node of  $C$ . Consider a line  $L$  passing through  $P$  and  $E$ . So

$$\operatorname{div}(L) = P + E + \varphi(P, E).$$

Since  $P$  is simple,  $\varphi(P, E) = E$  (Problems 5.35, 5.11). So

$$\operatorname{div}(L) = P + 2E.$$

- (2) Similarly, if  $L'$  is a line passing through  $Q$  and  $E$ , then

$$\operatorname{div}(L') = Q + 2E.$$

- (3) Hence  $\operatorname{div}(L') + P + 2E = \operatorname{div}(L) + Q + 2E$  or  $\operatorname{div}(L') + P = \operatorname{div}(L) + Q$  (Proposition 2(4) in §8.1). Therefore  $P \equiv Q$  (Proposition 2(5) in §8.1).

- (4) Note that the conclusion also holds for an irreducible cubic with a cusp.

□

**Problem 8.6. (Divisor class group)**

Let  $D(X)$  be the group of divisors on  $X$ ,  $D_0(X)$  the subgroup consisting of divisors of degree zero, and  $P(X)$  the subgroup of  $D_0(X)$  consisting of divisors of rational functions. Let  $C_0(X) = D_0(X)/P(X)$  be the quotient group. It is the divisor class group on  $X$ .

- (a) If  $X = \mathbf{P}^1$ , then  $C_0(X) = 0$ .
- (b) Let  $X = C$  be a nonsingular cubic. Pick  $P_0 \in C$ , defining  $\oplus$  on  $C$ . Show that the map from  $C$  to  $C_0(X)$  that sends  $P$  to the residue class of the divisor  $P - P_0$  is an isomorphism from  $(C, \oplus)$  onto  $C_0(X)$ .

*Proof of (a).*

- (1) Given a divisor

$$D = \sum_{P \in X} n_P P \in D_0(X)$$

where  $n_P \in \mathbb{Z}$  and  $\sum_P n_P = 0$ .



(2) Note that  $\sum_P n_P = 0$ . We can define a rational function  $z \in k(X)$  by

$$z = \prod_{P=[a_P:b_P] \in X} (b_P x - a_P y)^{n_P}.$$

Hence  $\operatorname{div}(z) = D \in P(X)$ . Therefore  $C_0(X) = D_0(X)/P(X) = 0$ .

□

*Proof of (b).*

(1) Define  $\alpha : (C, \oplus) \rightarrow C_0(X)$  by  $P \mapsto [P - P_0]$ .

(2) Show that  $\alpha$  is a group homomorphism. If  $P \oplus Q = R$ , then

$$\begin{aligned} P \oplus Q &= R \\ \iff [P + Q] &= [R + P_0] && \text{(Problem 8.3(c))} \\ \iff [P - P_0] + [Q - P_0] &= [R - P_0] && \text{(Proposition 2)} \\ \iff \alpha(P) + \alpha(Q) &= \alpha(R) = \alpha(P \oplus Q). \end{aligned}$$

(3) Show that  $\alpha$  is injective.

$$\begin{aligned} \alpha(P) = 0 &\iff [P - P_0] = 0 \\ &\iff [P] = [P_0] && \text{(Proposition 2)} \\ &\iff P = P_0. && \text{(Problem 8.3(a))} \end{aligned}$$

(4) Show that  $\alpha$  is surjective. Given  $[D] \in C_0(X)$  and we want to find a point  $P \in C$  such that  $\alpha(P) = [D]$ . Write

$$D = (P_1 + \cdots + P_r) - (Q_1 + \cdots + Q_r)$$

for some  $P_i, Q_i \in C$ . So

$$\begin{aligned} [D] &= [P_1 - P_0] + \cdots + [P_r - P_0] - [Q_1 - P_0] - \cdots - [Q_r - P_0] \\ &= \alpha(P_1) + \cdots + \alpha(P_r) - \alpha(Q_1) - \cdots - \alpha(Q_r) \\ &= \alpha(P_1) + \cdots + \alpha(P_r) + \alpha(Q'_1) + \cdots + \alpha(Q'_r) \\ &= \alpha(P_1 \oplus \cdots \oplus P_r \oplus Q'_1 \oplus \cdots \oplus Q'_r). \end{aligned}$$

where  $Q'_i$  is the inverse of  $Q_i$  in  $(C, \oplus)$ . Hence there is a point  $P := P_1 \oplus \cdots \oplus P_r \oplus Q'_1 \oplus \cdots \oplus Q'_r \in C$  such that  $\alpha(P) = [D]$ .

□

## 8.2. The Vector Spaces $L(D)$

## 8.3. Riemann's Theorem

## 8.4. Derivations and Differentials

## 8.5. Canonical Divisors

## 8.6. Riemann-Roch Theorem