

## Chapter 2: Basic Topology

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**Exercise 2.1.** *Prove that the empty set is a subset of every set.*

*Proof.* By Definitions 1.3,

- (1) The set which contains no element will be called the **empty set**,
- (2) If  $A$  and  $B$  are sets, and if every element of  $A$  is an element of  $B$ , we say that  $A$  is a **subset** of  $B$ ,

every element of the empty set (there are none) belongs to every set. That is, the empty set is a subset of every set.  $\square$

**Exercise 2.2.** *A complex number  $z$  is said to be algebraic if there are integers  $a_0, \dots, a_n$ , not all zero, such that*

$$a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n = 0.$$

*Prove that the set of all algebraic numbers is countable. (Hint: For every positive integer  $N$  there are only finitely many equations with*

$$n + |a_0| + |a_1| + \dots + |a_n| = N.$$

Might assume  $a_0 \neq 0$ .

For example, all rational numbers are algebraic since  $p = \frac{\alpha}{\beta}$  (where  $\alpha, \beta \in \mathbb{Z}$ ) is a root of  $\beta z - \alpha = 0$ .

Besides,  $z = \sqrt{2} + \sqrt{3}$  is algebraic since  $z^4 - 10z^2 + 1 = 0$ . In fact,  $z = \pm\sqrt{2} + \pm\sqrt{3}$  are also algebraic since  $z^4 - 10z^2 + 1 = (z - \sqrt{2} - \sqrt{3})(z + \sqrt{2} - \sqrt{3})(z - \sqrt{2} + \sqrt{3})(z + \sqrt{2} + \sqrt{3})$ .

**Lemma.** *The set of all polynomials over  $\mathbb{Z}$  is countable implies that the set of algebraic numbers is countable.*

*Proof of Lemma.* By definition, we write the set of algebraic numbers as

$$S = \bigcup_{f(x) \in \mathbb{Z}[x]} \{z \in \mathbb{C} : f(z) = 0\}.$$

Since each polynomial of degree  $n$  has at most  $n$  roots,  $\{z \in \mathbb{C} : f(z) = 0\}$  is finite for each given  $f(x) \in \mathbb{Z}[x]$ . So  $S$  is a countable union (by assumption) of finite sets, and hence countable. ( $S$  is infinite since every integer  $\alpha$  is a root of  $f(z) = z - \alpha$ .)  $\square$

Thus, it suffices to show that *the set of all polynomials over  $\mathbb{Z}$  is countable*.

*Proof (Hint).* For every positive integer  $N$  there are only finitely many equations with  $n + |a_0| + |a_1| + \cdots + |a_n| = N$ . Write

$$P_N = \{f(x) \in \mathbb{Z}[x] : n + |a_0| + |a_1| + \cdots + |a_n| = N\}$$

where  $f(x) = a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n$  with  $a_0 \neq 0$ , and

$$P = \bigcup_{N=1}^{\infty} P_N.$$

$P$  is the set of all polynomials over  $\mathbb{Z}$ .

Each  $P_N$  is finite for given  $N$  (since the equation  $n + |a_0| + |a_1| + \cdots + |a_n| = N$  has finitely many solutions  $(n, a_0, a_1, \dots, a_n) \in \mathbb{Z}^{n+2}$ ). So  $P$  is a countable union of finite sets, and hence countable. ( $P$  is infinite since  $\mathbb{Z}$  is a subring of  $\mathbb{Z}[x]$ .)  $\square$

*Proof (Theorem 2.13).*

- (1)  $\mathbb{Z}^N$  is countable for any integer  $N > 0$ .
- (2) The set of all polynomials over  $\mathbb{Z}$  is countable. Let

$$P_n = \{f \in \mathbb{Z}[x] : \deg f = n\},$$

and

$$P = \bigcup_{n=1}^{\infty} P_n = \mathbb{Z}[x].$$

*Claim:*  $P_n$  is countable. Define a 1-1 correspondence  $\varphi_n : P_n \rightarrow \mathbb{Z}^{n+1}$  by

$$\varphi_n(a_0z^n + a_1z^{n-1} + \cdots + a_n) = (a_0, a_1, \dots, a_{n-1}, a_n).$$

By (1),  $P_n$  is countable. Now  $P$  is a countable union of countable sets, and hence countable by Theorem 2.12.

$\square$

*Proof (Unique factorization theorem).*

- (1) The set of prime numbers is countable. Write all primes in the ascending order as  $p_1, p_2, \dots, p_n, \dots$  where  $p_1 = 2, p_2 = 3, \dots, p_{10001} = 104743, \dots$  (See ProjectEuler 7: 10001st prime. Use sieve of Eratosthenes to get  $p_{10001}$ .)

(2) The set of all polynomials over  $\mathbb{Z}$  is countable. Let

$$P_n = \{f \in \mathbb{Z}[x] : \deg f = n\},$$

and

$$P = \bigcup_{n=1}^{\infty} P_n = \mathbb{Z}[x].$$

*Claim:*  $P_n$  is countable. Define a map  $\varphi_n : P_n \rightarrow \mathbb{Z}^+$  by

$$\varphi_n(a_0 z^n + a_1 z^{n-1} + \cdots + a_n) = p_1^{\psi(a_0)} p_2^{\psi(a_1)} \cdots p_{n+1}^{\psi(a_n)},$$

where  $\psi$  is a 1-1 correspondence from  $\mathbb{Z}$  to  $\mathbb{Z}^+$  (Example 2.5). By the unique factorization theorem,  $\varphi_n$  is 1-1. (In fact,  $\varphi_n$  is a 1-1 correspondence from  $\mathbb{Z}^{n+1}$  to  $\mathbb{Z}^+$ .) So  $P_n$  is countable. Now  $P$  is a countable union of countable sets, and hence countable by Theorem 2.12.

□

**Exercise 2.10.** Let  $X$  be an infinite set. For  $p \in X$  and  $q \in X$ , define

$$d(p, q) = \begin{cases} 1 & (\text{if } p \neq q) \\ 0 & (\text{if } p = q). \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

(The statement holds even if  $X$  is finite.) We called  $d$  the discrete metric, and the corresponding topology on  $X$  induces the discrete topology. Conversely, if  $X$  has the discrete topology,  $X$  is always metrizable by the discrete metric.

*Proof.*

(1)  $d(p, q)$  is a metric.

(a)  $d(p, q) > 0$  if  $p \neq q$ ;  $d(p, p) = 0$ . Trivial.

(b)  $d(p, q) = d(q, p)$ . Trivial.

(c)  $d(p, q) \leq d(p, r) + d(r, q)$  for any  $r \in X$ . If  $p = q$ , nothing to do. If  $p \neq q$ ,  $r \neq p$  or  $r \neq q$  for any  $r \in X$ . (Assume not true,  $r = p$  and  $r = q$  implies that  $p = q$  which is a contradiction.) In any cases  $d(p, r) + d(r, q) \geq 1 = d(p, q)$ .

(2) Every subset is open. Let  $E$  be any subset of  $X$ . Then every point  $p \in E$  is an interior point of  $E$ . In fact, we can pick one neighborhood  $N_{\frac{1}{2}}(p)$  of  $p$  containing only one point  $p \in E$  or  $N_{\frac{1}{2}}(p) = \{p\}$ , and such neighborhood  $N_{\frac{1}{2}}(p)$  is a subset of  $E$ . So every subset of  $X$  is open.

- (3) *Every subset is closed.* Since every subset is open, every subset is closed by Theorem 2.23.

**Supplement.** Might use Definition 2.18 (d) to prove directly since there are no limit points in  $X$  if we consider one neighborhood  $N_{\frac{1}{2}}(p)$  of  $p$ . Therefore, every subset is closed. Again we apply Theorem 2.23 to get that every subset is open without using Definition 2.18 (f).

- (4) *A subset is compact iff it is finite.*

- (a) *Any finite subset is compact.* Say  $E = \{p_1, p_2, \dots, p_k\}$ , and  $\{G_\alpha\}$  be an open covering of  $E$ . From  $\{G_\alpha\}$  we pick  $G_{\alpha_1}$  containing  $p_1$ ,  $G_{\alpha_2}$  containing  $p_2$ , ..., and  $G_{\alpha_k}$  containing  $p_k$ . This process can be done in the finitely many steps. Therefore,

$$\{G_{\alpha_1}, G_{\alpha_2}, \dots, G_{\alpha_k}\}$$

is a finite subcovering of  $\{G_\alpha\}$  covering  $E$ .

- (b) *Any infinite subset is not compact.* Take a collection

$$\mathcal{G} = \{G_p = \{p\}\}$$

of open subsets where  $p$  runs all points in  $E$ . Clearly,  $\{G_p\}$  is an open covering. Assume

$$\mathcal{G}' = \{G_{p_1}, G_{p_2}, \dots, G_{p_k}\}$$

is any finite subcovering of  $\mathcal{G}$ . Since  $E$  is infinite, there exist a point  $p \in E$  such that  $p \neq p_1, p \neq p_2, \dots, p \neq p_k$ . Therefore,  $\mathcal{G}'$  does not cover  $p$ , or  $\mathcal{G}$  does not contain any finite subcovering  $\mathcal{G}'$ .

□

Notice that every subset is bounded. Therefore, every subset is closed and bounded, but only finite subset is compact, i.e., Heine-Borel theorem is not true in the infinite discrete topology.

**Exercise 2.12.** Let  $K \subseteq \mathbb{R}^1$  consist of 0 and the numbers  $\frac{1}{n}$ , for  $n = 1, 2, 3, \dots$ . Prove that  $K$  is compact directly from the definition (without using the Heine-Borel theorem).

*Proof.* Let  $\{G_\alpha\}$  be an open covering of  $K$ . There is an open set  $G_0 \in \{G_\alpha\}$  containing 0. So there exists a neighborhood  $N_r(0)$  of 0 such that  $N_r(0) \subseteq G_0$ . So  $N_r(0)$  contains all points  $q = \frac{1}{n}$  of  $K$  whenever  $n > \frac{1}{r}$ . To construct a finite subcovering of  $\{G_\alpha\}$ , we need to pick finitely many open sets from  $\{G_\alpha\}$  to

cover the remaining points  $q = \frac{1}{n}$  where  $n = 1, 2, \dots, \lceil \frac{1}{r} \rceil$ , say  $G_1$  contains  $q = \frac{1}{1}$ ,  $G_2$  contains  $q = \frac{1}{2}$ , ...,  $G_{\lceil \frac{1}{r} \rceil}$  contains  $q = \frac{1}{\lceil \frac{1}{r} \rceil}$ . (Might be duplicated.) Hence,

$$\left\{ G_0, G_1, G_2, \dots, G_{\lceil \frac{1}{r} \rceil} \right\}$$

is a finite subcovering of  $\{G_\alpha\}$  covering  $K$ .  $\square$

*Proof (Heine-Borel theorem).*

- (1)  $K$  is closed. In fact, the only limit point of  $K$  is 0, which is in  $K$ .
  - (a)  $p = 0$  is a limit point. Given  $r > 0$ . There always exists  $n \in \mathbb{Z}^+$  such that  $r > \frac{1}{n}$ . So any neighborhood  $N_r(0)$  of  $p = 0$  contains at least one point  $q = \frac{1}{n} \neq 0$  in  $K$ .
  - (b)  $p < 0$  is not a limit point. Pick a neighborhood  $N_r(p)$  of  $p$  where  $r = |p| > 0$ . Then  $N_r(p) \cap K = \emptyset$ .
  - (c)  $p > 0$  is not a limit point. There always exists  $m \in \mathbb{Z}^+$  such that  $p > \frac{1}{m}$  whenever  $n \geq m$ . Pick a neighborhood  $N_r(p)$  of  $p$  where  $r = p - \frac{1}{m} > 0$ . Then  $N_r(p)$  does not have all points  $q = \frac{1}{n} \in K$  whenever  $n \geq m$ . By Theorem 2.20,  $p$  cannot be a limit point of  $K$ .
- (2)  $K$  is bounded. There is a real number  $M = 2$  and a point  $q = 0 \in \mathbb{R}^1$  such that  $|p - q| = |p| < 2$  for all  $p \in K$ .

By Heine-Borel theorem,  $K$  is compact in  $\mathbb{R}^1$ .  $\square$

**Exercise 2.14.** Give an example of an open cover of the segment  $(0, 1)$  which has no finite subcover.

*Proof.* In  $\mathbb{R}^1$ , take a collection

$$\mathcal{G} = \left\{ G_n = \left( \frac{1}{n}, 1 \right) \right\}$$

of open subsets where  $n \in \mathbb{Z}^+$ .

- (1)  $\mathcal{G}$  is an open covering of  $(0, 1) \subseteq \mathbb{R}^1$ . Actually, given  $x \in (0, 1)$ , there exists a positive integer  $n$  such that  $x > \frac{1}{n}$ . That is,  $x \in (\frac{1}{n}, 1) = G_n$ .
- (2) There is no finite subcovering of  $\mathcal{G}$ . Assume

$$\mathcal{G}' = \{G_{n_1}, G_{n_2}, \dots, G_{n_k}\}$$

is any finite subcovering of  $\mathcal{G}$  where  $n_1 < n_2 < \dots < n_k$ . Take  $x \in \left(0, \frac{1}{n_k}\right) \neq \emptyset$ ,  $x = \frac{1}{2n_k}$  for example. Then  $x \notin G_{n_1}, x \notin G_{n_2}, \dots, x \notin G_{n_k}$ , which contradicts that  $\mathcal{G}'$  is a finite subcovering of  $\mathcal{G}$  covering  $(0, 1)$ .

$\square$