Chapter 1: Roots of Commutative Algebra

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Exercise 1.3. *Let M' be a submodule of M. Show that M is Noetherian iff both M' and M/M' are Noetherian.*

Proof.

(1) ($\Longrightarrow$)

(a) *Show that M' is Noetherian if M is Noetherian.* This is an immediate consequence of the definition of a Noetherian module since a submodule of a submodule is a submodule.

(b) *Show that M/M' is Noetherian if M is Noetherian.* Every submodule of M/M' has the form M''/M' where M'' is a submodule of M with $M' \subseteq M'' \subseteq M$. Since M is Noetherian, M'' is finitely generated, and the reduction of those generators mod M' will generate M''/M' as a finitely generated module.

(2) ($\Longleftarrow$)

(a) Given any submodule M'' of M. Then the image of M'' in M/M' is finitely generated and $M'' \cap M'$ is finitely generated too.

(b) Say $x_1, \ldots, x_k \in M''$ generate the image of M'' in M/M' and say $y_1, \ldots, y_h \in M''$ generate $M'' \cap M'$.

(c) Given any $x \in M''$, we have

$$
\begin{aligned}
& x \equiv r_1 x_1 + \cdots + r_k x_k \pmod{M'} \text{ for some } r_i \in R \\
\Longrightarrow & x - \sum_{i=1}^{k} r_i x_k \equiv 0 \pmod{M'} \\
\Longrightarrow & x - \sum_{i=1}^{k} r_i x_k \in M' \\
\Longrightarrow & x - \sum_{i=1}^{k} r_i x_k \in M'' \cap M' \\
\Longrightarrow & x - \sum_{i=1}^{k} r_i x_k = \sum_{j=1}^{h} s_j y_j \text{ for some } s_j \in R \\
\Longrightarrow & x = \sum_{i=1}^{k} r_i x_k + \sum_{j=1}^{h} s_j y_j \\
\Longrightarrow & x \text{ is generated by } x_1, \ldots, x_k, y_1, \ldots, y_h
\end{aligned}
$$

Hence M'' is finitely generated for any submodule M'' of M, that is, M is Noetherian.

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