Chapter 2: Modules

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Exercise 2.1 *Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$ if m, n are coprime.*

It suffices to show that

$$(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/(m,n)\mathbb{Z}$$

where (m,n) the greatest common divisor of m and n.

Outlines.

(1) Define $\widetilde{\varphi}$ by

$$\begin{array}{rccc} \widetilde{\varphi}: & (\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z}) & \longrightarrow & \mathbb{Z}/g\mathbb{Z} \\ & \cup & & \cup \\ & (x+m\mathbb{Z}, y+n\mathbb{Z}) & \longmapsto & xy+(m,n)\mathbb{Z}. \end{array}$$

$\widetilde{\varphi}$ is well-defined and $\mathbb{Z}$-bilinear.

(2) By the universal property, $\widetilde{\varphi}$ factors through a $\mathbb{Z}$-linear map

$$\varphi : (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/(m,n)\mathbb{Z}$$

(such that $\varphi(x \otimes y) = \widetilde{\varphi}(x,y)$).

(3) To show that φ is isomorphic, might find the inverse map $\psi : \mathbb{Z}/(m,n)\mathbb{Z} \to (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$ of φ. Define ψ by

$$\begin{array}{rccc} \psi: & \mathbb{Z}/g\mathbb{Z} & \longrightarrow & (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \\ & \cup & & \cup \\ & z+(m,n)\mathbb{Z} & \longmapsto & (z+m\mathbb{Z}) \otimes (1+n\mathbb{Z}). \end{array}$$

ψ is well-defined and $\mathbb{Z}$-linear.

(4) $\psi \circ \varphi = \mathrm{id}$.

(5) $\varphi \circ \psi = \mathrm{id}$.

Proof of (1).

(a) *$\widetilde{\varphi}$ is well-defined.* Say $x' = x + am$ for some $a \in \mathbb{Z}$ and $y' = y + bn$ for some $b \in \mathbb{Z}$. Then $x'y' - xy = yam + xbn + abmn \in (m, n)\mathbb{Z}$. That is, $\widetilde{\varphi}$ is independent of coset representative.

(b) *$\widetilde{\varphi}$ is $\mathbb{Z}$-bilinear.*

(i) *For any $\lambda \in \mathbb{Z}$, $\widetilde{\varphi}(\lambda x, y) = \widetilde{\varphi}(x, \lambda y) = \lambda\widetilde{\varphi}(x, y)$.* In fact,

$$\begin{aligned}
\widetilde{\varphi}(\lambda(x + m\mathbb{Z}), y + n\mathbb{Z}) &= \widetilde{\varphi}(\lambda x + m\mathbb{Z}, y + n\mathbb{Z}) = \lambda xy + (m, n)\mathbb{Z}, \\
\widetilde{\varphi}(x + m\mathbb{Z}, \lambda(y + n\mathbb{Z})) &= \widetilde{\varphi}(x + m\mathbb{Z}, \lambda y + n\mathbb{Z}) = \lambda xy + (m, n)\mathbb{Z}, \\
\widetilde{\varphi}(x_1 + m\mathbb{Z}, y + n\mathbb{Z}) &= \lambda(xy + (m, n)\mathbb{Z}) = \lambda xy + (m, n)\mathbb{Z}.
\end{aligned}$$

(ii) $\widetilde{\varphi}(x_1 + x_2, y) = \widetilde{\varphi}(x_1, y) + \widetilde{\varphi}(x_2, y)$. In fact,

$$\begin{aligned}
\widetilde{\varphi}((x_1 + x_2) + m\mathbb{Z}, y + n\mathbb{Z}) &= (x_1 + x_2)y + (m, n)\mathbb{Z}, \\
\widetilde{\varphi}(x_1 + m\mathbb{Z}, y + n\mathbb{Z}) + \widetilde{\varphi}(x_2 + m\mathbb{Z}, y + n\mathbb{Z}) &= (x_1 y + (m, n)\mathbb{Z}) + (x_2 y + (m, n)\mathbb{Z}) \\
&= (x_1 + x_2)y + (m, n)\mathbb{Z}.
\end{aligned}$$

(iii) $\widetilde{\varphi}(x, y_1 + y_2) = \widetilde{\varphi}(x, y_1) + \widetilde{\varphi}(x, y_2)$. Similar to (ii).

□

Proof of (3).

(a) *ψ is well-defined.* Say $z' = z + c(m, n)$ for some $c \in \mathbb{Z}$. Note that $(m, n) = \alpha m + \beta n$ for some $\alpha, \beta \in \mathbb{Z}$. Thus

$$\begin{aligned}
\psi(z' + (m, n)\mathbb{Z}) &= \psi(z + c(m, n) + (m, n)\mathbb{Z}) \\
&= \psi(z + c(\alpha m + \beta n) + (m, n)\mathbb{Z}) \\
&= (z + c(\alpha m + \beta n) + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\
&= (z + c\beta n + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\
&= (z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) + (c\beta n + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\
&= \psi(z + (m, n)\mathbb{Z}) + (1 + m\mathbb{Z}) \otimes (c\beta n + n\mathbb{Z}) \\
&= \psi(z + (m, n)\mathbb{Z}).
\end{aligned}$$

(b) *ψ is $\mathbb{Z}$-linear.* For any $\lambda \in \mathbb{Z}$,

$$\begin{aligned}
\psi(\lambda(z + (m, n)\mathbb{Z})) &= \psi(\lambda z + (m, n)\mathbb{Z})) = (\lambda z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}), \\
\lambda\psi(z + (m, n)\mathbb{Z}) &= \lambda((z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z})) = (\lambda z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}).
\end{aligned}$$

□

Proof of (4). For any $(x + m\mathbb{Z}) \otimes (y + n\mathbb{Z}) \in (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$,

$$\begin{aligned}
\psi(\varphi((x + m\mathbb{Z}) \otimes (y + n\mathbb{Z}))) &= \psi(xy + (m, n)\mathbb{Z}) \\
&= (xy + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\
&= (x + m\mathbb{Z}) \otimes (y + n\mathbb{Z}).
\end{aligned}$$

□

Proof of (5). For any $z + (m,n)\mathbb{Z} \in (m,n)\mathbb{Z}$,

$$\begin{aligned}\varphi(\psi(z + (m,n)\mathbb{Z}) &= \varphi((z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z})) \\ &= z + (m,n)\mathbb{Z}.\end{aligned}$$

□