Chapter 1: Rings and Ideals

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Exercise 1.1 *Let x be a nilpotent element of A. Show that $1 + x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit.*

Proof.

(1) Suppose $x^m = 0$ for some odd integer $m \geq 0$. Then
$$1 = 1 + x^m = (1 + x)(1 - x + x^2 - \cdots + (-1)^{m-1}x^{m-1}),$$
or $1 + x$ is a unit.

(2) If u is any unit and x is any nilpotent, $u + x = u \cdot (1 + u^{-1}x)$ is a product of two units (using that $u^{-1}x$ is nilpotent and applying (1)) and hence a unit again.

□

Proof (Proposition 1.9).

(1) *The nilradical is a subset of the Jacobson radical.*

  (a) The nilradical $\mathfrak{N}$ of A is the intersection of all the prime ideals of A by Proposition 1.8.

  (b) The Jacobson radical $\mathfrak{R}$ of A is the intersection of all the maximal ideals of A by definition.

(2) By Proposition 1.9, $x \in \mathfrak{R}$ if and only if $1 - xy$ is a unit in A for all $y \in A$. So $1 + x = 1 - (-x) \cdot 1$ is a unit in A since x is a nilpotent and $\mathfrak{R}$ is an ideal.

□

Exercise 1.2 *Let A be a ring and let $A[x]$ be the ring of polynomials in an indeterminate x, with coefficients in A. Let $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$. Prove that*

(i) *f is a unit in $A[x]$ if and only if a_0 is a unit in A and $a_1, ..., a_n$ are nilpotent. (Hint: If $b_0 + b_1x + \cdots + b_mx^m$ is the inverse of f, prove by induction on r that $a_n^{r+1}b_{m-r} = 0$. Hence show that a_n is nilpotent, and then use Exercise 1.1.)*

(ii) *f is nilpotent if and only if $a_0, a_1, ..., a_n$ are nilpotent.*

(iii) *f is a zero-divisor if and only if there exists $a \neq 0$ such that $af = 0$. (Hint: Choose a polynomial $g = b_0 + b_1x + \cdots + b_mx^m$ of least degree m such that $fg = 0$. Then $a_nb_m = 0$, hence $a_ng = 0$ (because a_ng annihilates f and has degree $< m$). Now show by induction that $a_{n-r}g = 0$ $(0 \leq r \leq n)$.)*

(iv) *f is said to be primitive if $(a_0, a_1, ..., a_n) = (1)$. Prove that if $f, g \in A[x]$, then fg is primitive if and only if f and g are primitive.*

Proof of (i).

(1) $(\Longleftarrow)$ holds by Exercise 1.1.

(2) $(\Longrightarrow)$ There exists the inverse g of f, say $g = b_0 + b_1x + \cdots + b_mx^m$ satisfying $1 = fg$. Clearly, $1 = a_0b_0$, or a_0 is a unit in A. Also,

$$\begin{aligned}
0 &= a_nb_m, \\
0 &= a_nb_{m-1} + a_{n-1}b_m, \\
0 &= a_nb_{m-2} + a_{n-1}b_{m-1} + a_{n-2}b_m, \\
&\cdots
\end{aligned}$$

A direct computing shows that

$$\begin{aligned}
0 &= a_n^1b_m, \\
0 &= a_n(a_nb_{m-1} + a_{n-1}b_m) \\
&= a_n^2b_{m-1} + a_{n-1}a_nb_m \\
&= a_n^2b_{m-1}, \\
0 &= a_n^2(a_nb_{m-2} + a_{n-1}b_{m-1} + a_{n-2}b_m) \\
&= a_n^3b_{m-2} + a_{n-1}a_n^2b_{m-1} + a_{n-2}a_n^2b_m \\
&= a_n^3b_{m-2}, \\
&\cdots
\end{aligned}$$

So we might have $a_n^{r+1}b_{m-r} = 0$ for $r = 0, 1, 2, ..., m$.

(3) *Show that $a_n^{r+1}b_{m-r} = 0$ for $r = 0, 1, 2, ..., m$ by induction on r.*

(a) As $r = 0$, $a_nb_m = 0$ by comparing the coefficient of $fg = 1$ at x^{n+m}.

(b) For any $r > 0$, comparing the coefficient of $fg = 1$ at x^{n+m-r},

$$0 = a_nb_{m-r} + a_{n-1}b_{m-r+1} + \cdots + a_{n-r}b_m.$$

Multiplying by a_n^r on the both sides,

$$\begin{aligned}
0 &= a_n^{r+1}b_{m-r} + a_{n-1}a_n^rb_{m-r+1} + \cdots + a_{n-r}a_n^rb_m \\
&= a_n^{r+1}b_{m-r}.
\end{aligned}$$

by the induction hypothesis.

(4) a_n *is a nilpotent.* Putting $r = m$ in $a_n^{r+1}b_{m-r} = 0$ and get $a_n^{m+1}b_0 = 0$. Notice that b_0 is a unit, $a_n^{m+1} = 0$, or a_n is a nilpotent.

(5) Consider $f - a_nx^n = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$, a polynomial $\in A[x]$ of degree $n-1$. Note that f is a unit and a_nx^n is a nilpotent. By Exercise 1.1, $f - a_nx^n$ is a unit too. Applying the (2)(3)(4) again, a_{n-1} is a nilpotent as $n - 1 > 0$, that is, applying descending induction on n then yields the desired property.

□

Proof of (ii).

(1) ($\Longleftarrow$) holds since the nilradical of any ring is an ideal.

(2) ($\Longrightarrow$) $f^N = 0$ for some $N > 0$. So $0 = f^N = a_n^N x^{nN} + \cdots + a_0^N$. Comparing the coefficient in the leading term x^{nN} leads to $a_n^N = 0$, or a_n is a nilpotent.

(3) Consider $f - a_nx^n = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$, a polynomial $\in A[x]$ of degree $n - 1$. Note that f and a_nx^n are nilpotent. $f - a_nx^n$ is a nilpotent too. Similar to step (5) in the proof of (i), applying descending induction on n then yields the desired property.

□

Proof of (iii).

(1) ($\Longleftarrow$) holds trivially.

(2) ($\Longrightarrow$) Pick a polynomial $g = b_0 + b_1x + \cdots + b_mx^m$ of least degree m such that $fg = 0$. Especially, $a_nb_m = 0$.

(3) Consider

$$\begin{aligned} a_ng &= a_nb_0 + \cdots + a_nb_{m-1}x^{m-1} + a_nb_mx^m \\ &= a_nb_0 + \cdots + a_nb_{m-1}x^{m-1} \end{aligned}$$

(since $a_nb_m = 0$). a_ng is a polynomial over A of having degree strictly less than m. Notice that $f \cdot (a_ng) = a_n \cdot (fg) = 0$. By minimality of m, $a_ng = 0$.

(4) Induction on the degree n of f.

(a) As $n = 0$, $f = a_0$. There exists $b_m \neq 0$ such that $b_mf = b_ma_0 = 0$ by (2).

(b) For any zero-divisor f of degree n, there is a polynomial $g = b_0 + b_1x + \cdots + b_mx^m$ of least degree m such that $fg = 0$. By (2)(3),

$$\begin{aligned} (f - a_nx^n) \cdot g &= fg - a_nx^ng \\ &= 0 - 0 \\ &= 0. \end{aligned}$$

That is, $f - a_n x^n$ is a zero-divisor of degree $n-1$. By the induction hypothesis, there exists $b_m \neq 0$ such that $b_m(f - a_n x^n) = 0$. So $b_m f = b_m(f - a_n x^n) + b_m a_n x^n = 0 + 0 = 0$.

(c) By (a)(b), ($\Longrightarrow$) holds by mathematical induction.

$\square$

Proof of (iv). Note that

(1) $f \notin \mathfrak{m}[x]$ for any maximal ideal $\mathfrak{m}$ of A if and only if f is primitive.

(2) For any maximal ideal $\mathfrak{m}$ of A, $A/\mathfrak{m}$ is a field (or an integral domain).

(3) $A[x]$ is an integral domain if A is an integral domain.

(4) $A[x]/\mathfrak{m}[x] \cong (A/\mathfrak{m})[x]$ as a ring isomorphism.

Hence,

$$\begin{aligned} f, g : \text{ primitive} &\Longleftrightarrow f, g \notin \mathfrak{m}[x] \text{ for any maximal ideal } \mathfrak{m} \\ &\Longleftrightarrow f, g \neq 0 \text{ in } (A/\mathfrak{m})[x] \text{ for any maximal ideal } \mathfrak{m} \\ &\Longleftrightarrow fg \neq 0 \text{ in } (A/\mathfrak{m})[x] \text{ for any maximal ideal } \mathfrak{m} \\ &\Longleftrightarrow fg \notin \mathfrak{m}[x] \text{ for any maximal ideal } \mathfrak{m} \\ &\Longleftrightarrow fg : \text{ primitive.} \end{aligned}$$

$\square$

Exercise 1.4 *In the ring $A[x]$, the Jacobson radical is equal to the nilradical.*

Proof.

(1) The nilradical $\mathfrak{N}$ is a subset of the Jacobson radical $\mathfrak{R}$. It suffices to show that $\mathfrak{R} \subseteq \mathfrak{N}$.

(2) Given any $f \in \mathfrak{R}$. By Proposition 1.9, $f \in \mathfrak{R}$ if and only if $1 - fy$ is a unit in $A[x]$ for all $y \in A[x]$. Especially, pick $y = x \in A[x]$ and then $1 - xf$ is a unit in $A[x]$.

(3) By Exercise 1.2 (i), all coefficients of f are nilpotent. By Exercise 1.2 (ii), f is nilpotent, or $f \in \mathfrak{N}$.

$\square$