Notes on the book: *Apostol, Modular Functions and Dirichlet Series in Number Theory, 2nd edition*

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Chapter 1: Elliptic functions

Exercise 1.11.

If $k \geq 2$ and $\tau \in H$ prove that the Eisenstein series

$$G_{2k}(\tau) = \sum_{(m,n)\neq(0,0)} (m+n\tau)^{-2k}$$

has the Fourier expansion

$$G_{2k}(\tau) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)e^{2\pi i n\tau}.$$

Proof.

(1) Let $q = e^{2\pi i\tau}$. Similar to Lemma 1.3 on page 19, we have

$$(2k-1)!\sum_{m=-\infty}^{+\infty}\frac{1}{(\tau+m)^{2k}} = (2\pi i)^{2k}\sum_{r=1}^{\infty} r^{2k-1}q^r.$$

(2) Similar to Theorem 1.18, we have

$$\begin{aligned}
G_{2k}(\tau) &= \sum_{(m,n)\neq(0,0)} (m+n\tau)^{-2k} \\
&= \sum_{\substack{m=-\infty \\ m\neq 0 (n=0)}}^{+\infty} m^{-2k} + \sum_{n=1}^{\infty}\sum_{m=-\infty}^{+\infty}\left((m+n\tau)^{-2k} + (m-n\tau)^{-2k}\right) \\
&= 2\zeta(2k) + 2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{+\infty}(m+n\tau)^{-2k} \\
&= 2\zeta(2k) + 2\sum_{n=1}^{\infty}\frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{r=1}^{\infty} r^{2k-1}q^{nr} \\
&= 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\underbrace{\sum_{d|n} d^{2k-1}}_{=\sigma_{2k-1}(n)} q^n.
\end{aligned}$$

In the last double sum we collect together those terms for which nr is constant.

□

Exercise 1.12.

Refer to Exercise 1.11. If $\tau \in H$ prove that

$$G_{2k}\left(-\frac{1}{\tau}\right) = \tau^{2k} G_{2k}(\tau)$$

and deduce that

$$\begin{aligned} G_{2k}\left(\frac{i}{2}\right) &= (-4)^k G_{2k}(2i) && \textit{for all } k \geq 2, \\ G_{2k}(i) &= 0 && \textit{if } k \textit{ is odd}, \\ G_{2k}(e^{\frac{2\pi i}{3}}) &= 0 && \textit{if } k \not\equiv 0 \pmod 3. \end{aligned}$$

Proof.

(1)

$$\begin{aligned} G_{2k}\left(-\frac{1}{\tau}\right) &= \sum_{(m,n)\neq(0,0)} \left(m - \frac{n}{\tau}\right)^{-2k} \\ &= \tau^{2k} \sum_{(m,n)\neq(0,0)} (\tau m - n)^{-2k} \\ &= \tau^{2k} G_{2k}(\tau). \end{aligned}$$

(2) Let $\tau = 2i$. We have $G_{2k}\left(\frac{i}{2}\right) = (-4)^k G_{2k}(2i)$.

(3) Let $\tau = i$. We have $G_{2k}(i) = (-1)^k G_{2k}(i)$. Hence $G_{2k}(i) = 0$ if k is odd.

(4) Let $\tau = e^{\frac{\pi i}{3}}$. We have $G_{2k}(e^{\frac{2\pi i}{3}}) = e^{\frac{2k\pi i}{3}} G_{2k}(e^{\frac{\pi i}{3}})$. Since

$$e^{\frac{2\pi i}{3}} = -1 + e^{\frac{\pi i}{3}}$$

and each Eisenstein series is a periodic function of τ of period 1, we have $G_{2k}(e^{\frac{2\pi i}{3}}) = G_{2k}(e^{\frac{\pi i}{3}})$. So $G_{2k}(e^{\frac{2\pi i}{3}}) = e^{\frac{2k\pi i}{3}} G_{2k}(e^{\frac{2\pi i}{3}})$. Therefore $G_{2k}(e^{\frac{2\pi i}{3}}) = 0$ if $k \not\equiv 0 \pmod 3$.

□

Exercise 1.13.

Ramanujan's tau function $\tau(n)$ is defined by the Fourier expansion

$$\Delta(\tau) = (2\pi)^{12} \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n \tau},$$

derived in Theorem 1.19. Prove that

$$\tau(n) = 8000\{(\sigma_3 \circ \sigma_3) \circ \sigma_3\}(n) - 147(\sigma_5 \circ \sigma_5)(n),$$

where $f \circ g$ denotes the Cauchy product of two sequences,

$$(f \circ g)(n) = \sum_{k=0}^{n} f(k)g(n-k),$$

and $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$ for $n \geq 1$, with $\sigma_3(0) = \frac{1}{240}$, $\sigma_5(0) = -\frac{1}{504}$. (Hint: Theorem 1.18.)

Proof.

(1) Let $q = e^{2\pi i\tau}$. Write

$$g_2(\tau) = \frac{4\pi^4}{3}\left\{1 + 240\sum_{k=1}^{\infty}\sigma_3(k)q^k\right\} = \frac{4\pi^4}{3}\left\{240\sum_{k=0}^{\infty}\sigma_3(k)q^k\right\},$$

$$g_3(\tau) = \frac{8\pi^6}{27}\left\{1 - 504\sum_{k=1}^{\infty}\sigma_5(k)q^k\right\} = \frac{8\pi^6}{27}\left\{-504\sum_{k=0}^{\infty}\sigma_5(k)q^k\right\}$$

(Theorem 1.18).

(2) Similar to the proof of Theorem 1.19,

$$\begin{aligned}
\Delta(\tau) &= g_2(\tau)^3 - 27g_3(\tau)^2 \\
&= \frac{64\pi^{12}}{27}\left\{\left(240\sum_{k=0}^{\infty}\sigma_3(k)q^k\right)^3 - \left(-504\sum_{k=0}^{\infty}\sigma_5(k)q^k\right)^2\right\} \\
&= (2\pi)^{12}\left\{8000\left(\sum_{k=0}^{\infty}\sigma_3(k)q^k\right)^3 - 147\left(\sum_{k=0}^{\infty}\sigma_5(k)q^k\right)^2\right\} \\
&= (2\pi)^{12}\sum_{n=0}^{\infty}\{8000\{(\sigma_3 \circ \sigma_3) \circ \sigma_3\}(n) - 147(\sigma_5 \circ \sigma_5)(n)\}\, q^n \\
&= (2\pi)^{12}\sum_{n=1}^{\infty}\{8000\{(\sigma_3 \circ \sigma_3) \circ \sigma_3\}(n) - 147(\sigma_5 \circ \sigma_5)(n)\}\, q^n.
\end{aligned}$$

(Here $8000\{(\sigma_3 \circ \sigma_3) \circ \sigma_3\}(0) - 147(\sigma_5 \circ \sigma_5)(0) = 0$.)

(3) Therefore

$$\tau(n) = 8000\{(\sigma_3 \circ \sigma_3) \circ \sigma_3\}(n) - 147(\sigma_5 \circ \sigma_5)(n)$$

for $n \geq 1$.

□

Exercise 1.14. (Lambert series)

A series of the form $\sum_{n=1}^{\infty} f(n)\frac{x^n}{1-x^n}$ *is called a **Lambert series**. Assuming absolute convergence, prove that*

$$\sum_{n=1}^{\infty} f(n)\frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} F(n)x^n,$$

where

$$F(n) = \sum_{d|n} f(d).$$

Apply this result to obtain the following formulas, valid for $|x| < 1$.

(a)

$$\sum_{n=1}^{\infty} \frac{\mu(n)x^n}{1-x^n} = x.$$

(b)

$$\sum_{n=1}^{\infty} \frac{\varphi(n)x^n}{1-x^n} = \frac{x}{(1-x)^2}.$$

(c)

$$\sum_{n=1}^{\infty} \frac{n^\alpha x^n}{1-x^n} = \sum_{n=1}^{\infty} \sigma_\alpha(n)x^n.$$

(d)

$$\sum_{n=1}^{\infty} \frac{\lambda(n)x^n}{1-x^n} = \sum_{n=1}^{\infty} x^{n^2}.$$

(e) *Use the result in (c) to express* $g_2(\tau)$ *and* $g_3(\tau)$ *in terms of Lambert series in* $x = e^{2\pi i\tau}$.

Note. In (a), $\mu(n)$ *is the Möbius function; In (b),* $\varphi(n)$ *is Euler's totient; and in (d),* $\lambda(n)$ *is Liouville's function.*

Proof. Similar to the proof of Exercise 1.11.

$$\begin{aligned}
\sum_{n=1}^{\infty} f(n)\frac{x^n}{1-x^n} &= \sum_{n=1}^{\infty} f(n)\sum_{r=1}^{\infty} x^{rn} \\
&= \sum_{n=1}^{\infty}\sum_{r=1}^{\infty} f(n)x^{rn} \\
&= \sum_{n=1}^{\infty} \underbrace{\left(\sum_{d|n} f(d)\right)}_{=F(n)} x^n.
\end{aligned}$$

$\square$

Proof of (a). Theorem 2.1 in the textbook: *T. M. Apostol, Introduction to Analytic Number Theory* shows that

$$F(n) := \sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

Hence

$$\sum_{n=1}^{\infty} \mu(n) \frac{x^n}{1 - x^n} = \sum_{n=1}^{\infty} F(n) x^n = x.$$

$\square$

Proof of (b). Theorem 2.2 in the textbook: *T. M. Apostol, Introduction to Analytic Number Theory* shows that $F(n) := \sum_{d|n} \varphi(d) = n$. Hence

$$\sum_{n=1}^{\infty} \varphi(n) \frac{x^n}{1 - x^n} = \sum_{n=1}^{\infty} F(n) x^n = \sum_{n=1}^{\infty} n x^n = \frac{x}{(1 - x)^2}.$$

$\square$

Proof of (c). Since

$$F(n) := \sum_{d|n} d^{\alpha} = \sigma_{\alpha}(n),$$

we have

$$\sum_{n=1}^{\infty} n^{\alpha} \frac{x^n}{1 - x^n} = \sum_{n=1}^{\infty} F(n) x^n = \sum_{n=1}^{\infty} \sigma_{\alpha}(n) x^n.$$

$\square$

Proof of (d). Theorem 2.19 in the textbook: *T. M. Apostol, Introduction to Analytic Number Theory* shows that

$$F(n) := \sum_{d|n} \lambda(d) = \begin{cases} 1 & \text{if } n \text{ is a square}, \\ 0 & \text{otherwise}. \end{cases}$$

Hence

$$\sum_{n=1}^{\infty} \lambda(n) \frac{x^n}{1 - x^n} = \sum_{n=1}^{\infty} F(n) x^n = \sum_{n=1}^{\infty} x^{n^2}.$$

$\square$

Proof of (e).

(1) Let $q = x = e^{2\pi i\tau}$.

$$\begin{aligned} g_2(\tau) &= \frac{4\pi^4}{3}\left\{1 + 240\sum_{k=1}^{\infty}\sigma_3(k)q^k\right\} && \text{(Theorem 1.18)} \\ &= \frac{4\pi^4}{3}\left\{1 + 240\sum_{k=1}^{\infty}\frac{k^3q^k}{1-q^k}\right\} && \text{((c)).} \end{aligned}$$

(2) Similarly,

$$\begin{aligned} g_3(\tau) &= \frac{8\pi^6}{27}\left\{1 - 504\sum_{k=1}^{\infty}\sigma_5(k)q^k\right\} && \text{(Theorem 1.18)} \\ &= \frac{8\pi^6}{27}\left\{1 - 504\sum_{k=1}^{\infty}\frac{k^5q^k}{1-q^k}\right\} && \text{((c)).} \end{aligned}$$

□

Exercise 1.15.

Let

$$G(x) = \sum_{n=1}^{\infty}\frac{n^5x^n}{1-x^n},$$

and let

$$F(x) = \sum_{\substack{n=1 \\ (n\ odd)}}^{\infty}\frac{n^5x^n}{1+x^n}.$$

(a) *Prove that* $F(x) = G(x) - 34G(x^2) + 64(x^4)$.

(b) *Prove that*

$$\sum_{\substack{n=1 \\ (n\ odd)}}^{\infty}\frac{n^5}{1+e^{n\pi}} = \frac{31}{504}.$$

(c) *Use Theorem 12.17 in the textbook: T. M. Apostol, Introduction to Analytic Number Theory to prove the more general result*

$$\sum_{\substack{n=1 \\ (n\text{ odd})}}^{\infty}\frac{n^{4k+1}}{1+e^{n\pi}} = \frac{2^{4k+1}-1}{8k+4}B_{4k+2}.$$

Proof of (a).

(1) Consider the general case. *Let*

$$G(x) = \sum_{n=1}^{\infty} \frac{n^{4k+1}x^n}{1-x^n},$$

and let

$$F(x) = \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^{4k+1}x^n}{1+x^n}.$$

Show that $F(x) = G(x) - (2^{4k+1}+2)G(x^2) + 2^{4k+2}G(x^4)$.

(2) The identity

$$\sum_{n=1}^{\infty} \frac{x^n}{1+x^n} = \sum_{n=1}^{\infty} \frac{x^n}{1-x^n} - 2\sum_{n=1}^{\infty} \frac{x^{2n}}{1-x^{2n}}$$

is always true. Hence $H(x) := \sum_{n=1}^{\infty} \frac{n^{4k+1}x^n}{1+x^n} = G(x) - 2G(x^2)$.

(3) Note that

$$\begin{aligned} H(x) &= \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^{4k+1}x^n}{1+x^n} + \sum_{\substack{n=1 \\ (n \text{ even})}}^{\infty} \frac{n^{4k+1}x^n}{1+x^n} \\ &= F(x) + \sum_{n=1}^{\infty} \frac{(2n)^{4k+1}x^{2n}}{1+x^{2n}} \\ &= F(x) + 2^{4k+1}\sum_{n=1}^{\infty} \frac{n^{4k+1}x^{2n}}{1+x^{2n}} \\ &= F(x) + 2^{4k+1}H(x^2). \end{aligned}$$

Hence

$$\begin{aligned} F(x) &= H(x) - 2^{4k+1}H(x^2) \\ &= [G(x) - 2G(x^2)] - 2^{4k+1}[G(x^2) - 2G(x^4)] \\ &= G(x) - (2^{4k+1}+2)G(x^2) + 2^{4k+2}G(x^4). \end{aligned}$$

□

Proof of (b). Take $k = 1$ in part (c), we have

$$\sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^5}{1+e^{n\pi}} = \frac{31}{12} \cdot \frac{1}{42} = \frac{31}{504}.$$

□

Proof of (c).

(1) Let $q = e^{2\pi i\tau}$. So

$$\begin{aligned}
G_{4k+2}(\tau) &= 2\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!}\sum_{n=1}^{\infty}\sigma_{4k+1}(n)q^n && \text{(Exercise 1.11)} \\
&= 2\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!}G(q) && \text{(Exercise 1.14(c))}
\end{aligned}$$

Hence

$$\begin{aligned}
&G_{4k+2}(\tau) - (2^{4k+1}+2)G_{4k+2}(2\tau) + 2^{4k+2}G_{4k+2}(4\tau) \\
={}& \left[2\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!}G(q)\right] \\
&- (2^{4k+1}+2)\left[2\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!}G(q^2)\right] \\
&+ 2^{4k+2}\left[2\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!}G(q^4)\right] \\
={}& (1 - (2^{4k+1}+2) + 2^{4k+2}) \cdot 2\zeta(4k+2) \\
&+ \frac{2(2\pi i)^{4k+2}}{(4k+1)!}[G(q) - (2^{4k+1}+2)G(q^2) + 2^{4k+2}G(q^4)] \\
={}& (2^{4k+2}-2)\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!}F(q).
\end{aligned}$$

(2) By taking $\tau = \frac{i}{2}$, we have

$$F(q) = F(e^{-\pi}) = \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^{4k+1}}{1+e^{n\pi}}$$

and

$$\begin{aligned}
&G_{4k+2}(\tau) - (2^{4k+1}+2)G_{4k+2}(2\tau) + 2^{4k+2}G_{4k+2}(4\tau) \\
={}& G_{4k+2}\left(\frac{i}{2}\right) - (2^{4k+1}+2)G_{4k+2}(i) + 2^{4k+2}G_{4k+2}(2i) \\
={}& (-4)^{2k+1}G_{4k+2}(2i) - (2^{4k+1}+2)\cdot 0 + 2^{4k+2}G_{4k+2}(2i) \\
={}& 0.
\end{aligned}$$

(Exercise 1.12). Hence

$$0 = (2^{4k+2}-2)\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!}\sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^{4k+1}}{1+e^{n\pi}}.$$

(3) Theorem 12.17 in the textbook: *T. M. Apostol, Introduction to Analytic Number Theory* shows that

$$\zeta(4k+2) = (-1)^{2k+1+1}\frac{(2\pi)^{4k+2}B_{4k+2}}{2(4k+2)!} = \frac{(2\pi)^{4k+2}B_{4k+2}}{2(4k+2)!}.$$

Hence

$$\sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^{4k+1}}{1+e^{n\pi}} = \frac{2^{4k+1}-1}{8k+4} B_{4k+2}.$$

$\square$