

Notes on the book:  
*Apostol, Modular Functions and  
Dirichlet Series in Number Theory,  
2nd edition*

Meng-Gen Tsai  
plover@gmail.com

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## Chapter 1: Elliptic functions

### Exercise 1.11.

If  $k \geq 2$  and  $\tau \in H$  prove that the Eisenstein series

$$G_{2k}(\tau) = \sum_{(m,n) \neq (0,0)} (m + n\tau)^{-2k}$$

has the Fourier expansion

$$G_{2k}(\tau) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) e^{2\pi i n \tau}.$$

*Proof.*

(1) Let  $q = e^{2\pi i \tau}$ . Similar to Lemma 1.3 on page 19, we have

$$(2k-1)! \sum_{m=-\infty}^{+\infty} \frac{1}{(\tau + m)^{2k}} = (2\pi i)^{2k} \sum_{r=1}^{\infty} r^{2k-1} q^r.$$

(2) Similar to Theorem 1.18, we have

$$\begin{aligned} G_{2k}(\tau) &= \sum_{(m,n) \neq (0,0)} (m + n\tau)^{-2k} \\ &= \sum_{\substack{m=-\infty \\ m \neq 0(n=0)}}^{+\infty} m^{-2k} + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{+\infty} ((m + n\tau)^{-2k} + (m - n\tau)^{-2k}) \\ &= 2\zeta(2k) + 2 \sum_{n=1}^{\infty} \sum_{m=-\infty}^{+\infty} (m + n\tau)^{-2k} \\ &= 2\zeta(2k) + 2 \sum_{n=1}^{\infty} \frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{r=1}^{\infty} r^{2k-1} q^{nr} \\ &= 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \underbrace{\sum_{d|n} d^{2k-1}}_{=\sigma_{2k-1}(n)} q^n. \end{aligned}$$

In the last double sum we collect together those terms for which  $nr$  is constant.

□

**Exercise 1.12.**

Refer to Exercise 1.11. If  $\tau \in H$  prove that

$$G_{2k}\left(-\frac{1}{\tau}\right) = \tau^{2k} G_{2k}(\tau)$$

and deduce that

$$\begin{aligned} G_{2k}\left(\frac{i}{2}\right) &= (-4)^k G_{2k}(2i) && \text{for all } k \geq 2, \\ G_{2k}(i) &= 0 && \text{if } k \text{ is odd,} \\ G_{2k}(e^{\frac{2\pi i}{3}}) &= 0 && \text{if } k \not\equiv 0 \pmod{3}. \end{aligned}$$

*Proof.*

(1)

$$\begin{aligned} G_{2k}\left(-\frac{1}{\tau}\right) &= \sum_{(m,n) \neq (0,0)} \left(m - \frac{n}{\tau}\right)^{-2k} \\ &= \tau^{2k} \sum_{(m,n) \neq (0,0)} (\tau m - n)^{-2k} \\ &= \tau^{2k} G_{2k}(\tau). \end{aligned}$$

(2) Let  $\tau = 2i$ . We have  $G_{2k}\left(\frac{i}{2}\right) = (-4)^k G_{2k}(2i)$ .

(3) Let  $\tau = i$ . We have  $G_{2k}(i) = (-1)^k G_{2k}(i)$ . Hence  $G_{2k}(i) = 0$  if  $k$  is odd.

(4) Let  $\tau = e^{\frac{\pi i}{3}}$ . We have  $G_{2k}(e^{\frac{2\pi i}{3}}) = e^{\frac{2k\pi i}{3}} G_{2k}(e^{\frac{\pi i}{3}})$ . Since

$$e^{\frac{2\pi i}{3}} = -1 + e^{\frac{\pi i}{3}}$$

and each Eisenstein series is a periodic function of  $\tau$  of period 1, we have  $G_{2k}(e^{\frac{2\pi i}{3}}) = G_{2k}(e^{\frac{\pi i}{3}})$ . So  $G_{2k}(e^{\frac{2\pi i}{3}}) = e^{\frac{2k\pi i}{3}} G_{2k}(e^{\frac{\pi i}{3}})$ . Therefore  $G_{2k}(e^{\frac{2\pi i}{3}}) = 0$  if  $k \not\equiv 0 \pmod{3}$ .

□

**Exercise 1.14. (Lambert series)**

A series of the form  $\sum_{n=1}^{\infty} f(n) \frac{x^n}{1-x^n}$  is called a **Lambert series**. Assuming absolute convergence, prove that

$$\sum_{n=1}^{\infty} f(n) \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} F(n) x^n,$$

where

$$F(n) = \sum_{d|n} f(d).$$

Apply this result to obtain the following formulas, valid for  $|x| < 1$ .

(a)

$$\sum_{n=1}^{\infty} \frac{\mu(n)x^n}{1-x^n} = x.$$

(b)

$$\sum_{n=1}^{\infty} \frac{\varphi(n)x^n}{1-x^n} = \frac{x}{(1-x)^2}.$$

(c)

$$\sum_{n=1}^{\infty} \frac{n^{\alpha}x^n}{1-x^n} = \sum_{n=1}^{\infty} \sigma_{\alpha}(n)x^n.$$

(d)

$$\sum_{n=1}^{\infty} \frac{\lambda(n)x^n}{1-x^n} = \sum_{n=1}^{\infty} x^{n^2}.$$

(e) Use the result in (c) to express  $g_2(\tau)$  and  $g_3(\tau)$  in terms of Lambert series in  $x = e^{2\pi i\tau}$ .

*Note.* In (a),  $\mu(n)$  is the Möbius function; In (b),  $\varphi(n)$  is Euler's totient; and in (d),  $\lambda(n)$  is Liouville's function.

*Proof.* Similar to the proof of Exercise 1.11.

$$\begin{aligned} \sum_{n=1}^{\infty} f(n) \frac{x^n}{1-x^n} &= \sum_{n=1}^{\infty} f(n) \sum_{r=1}^{\infty} x^{rn} \\ &= \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} f(n) x^{rn} \\ &= \sum_{n=1}^{\infty} \underbrace{\left( \sum_{d|n} f(d) \right)}_{=F(n)} x^n. \end{aligned}$$

□

*Proof of (a).* Theorem 2.1 in the textbook: *T. M. Apostol, Introduction to Analytic Number Theory* shows that

$$F(n) := \sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

Hence

$$\sum_{n=1}^{\infty} \mu(n) \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} F(n) x^n = x.$$

□

*Proof of (b).* Theorem 2.2 in the textbook: *T. M. Apostol, Introduction to Analytic Number Theory* shows that  $F(n) := \sum_{d|n} \varphi(d) = n$ . Hence

$$\sum_{n=1}^{\infty} \varphi(n) \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} F(n) x^n = \sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}.$$

□

*Proof of (c).* Since

$$F(n) := \sum_{d|n} d^\alpha = \sigma_\alpha(n),$$

we have

$$\sum_{n=1}^{\infty} n^\alpha \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} F(n) x^n = \sum_{n=1}^{\infty} \sigma_\alpha(n) x^n.$$

□

*Proof of (d).* Theorem 2.19 in the textbook: *T. M. Apostol, Introduction to Analytic Number Theory* shows that

$$F(n) := \sum_{d|n} \lambda(d) = \begin{cases} 1 & \text{if } n \text{ is a square,} \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\sum_{n=1}^{\infty} \lambda(n) \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} F(n) x^n = \sum_{n=1}^{\infty} x^{n^2}.$$

□

*Proof of (e).*

(1) Let  $q = x = e^{2\pi i\tau}$ .

$$\begin{aligned} g_2(\tau) &= \frac{4\pi^4}{3} \left\{ 1 + 240 \sum_{k=1}^{\infty} \sigma_3(k) q^k \right\} && \text{(Theorem 1.18)} \\ &= \frac{4\pi^4}{3} \left\{ 1 + 240 \sum_{k=1}^{\infty} \frac{k^3 q^k}{1 - q^k} \right\} && ((c)). \end{aligned}$$

(2) Similarly,

$$\begin{aligned} g_3(\tau) &= \frac{8\pi^6}{27} \left\{ 1 - 504 \sum_{k=1}^{\infty} \sigma_5(k) q^k \right\} && \text{(Theorem 1.18)} \\ &= \frac{8\pi^6}{27} \left\{ 1 - 504 \sum_{k=1}^{\infty} \frac{k^5 q^k}{1 - q^k} \right\} && ((c)). \end{aligned}$$

□