

# Chapter 1: Curves

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## Section 1-1: Introduction

Classical differential geometry: the study of local properties of curves and surfaces.

Global differential geometry: the study of the influence of the local properties on the behavior of the entire curve and surface.

*No exercises.*

## Section 1-2: Parametrized Curves

**Exercise 1-2.1.** Find a parametrized curve  $\alpha(t)$  whose trace is the circle  $x^2 + y^2 = 1$  such that  $\alpha(t)$  runs clockwise around the circle with  $\alpha(0) = (0, 1)$ .

*Proof.*  $\alpha(t) = (\sin t, \cos t)$ ,  $t \in \mathbb{R}$ .  $\square$

**Exercise 1-2.2.** Let  $\alpha(t)$  be a parametrized curve which does not pass through the origin. If  $\alpha(t_0)$  is the point of the trace of  $\alpha$  closest to the origin and  $\alpha'(t_0) \neq 0$ , show that the position vector  $\alpha(t_0)$  is orthogonal to  $\alpha'(t_0)$ .

*Proof.* Let  $f(t) = |\alpha(t)|^2 = \alpha(t) \cdot \alpha(t)$ .  $f(t)$  is differentiable and  $f(t)$  has a local minimum at a point  $t = t_0 \in I$ . So  $f'(t_0) = 0$ . [Theorem 5.8 in W. Rudin, Principles of Mathematical Analysis, 3rd edition.] Since

$$f'(t) = 2\alpha(t) \cdot \alpha'(t),$$

$f'(t_0) = 2\alpha(t_0) \cdot \alpha'(t_0) = 0$ , or  $\alpha(t_0) \cdot \alpha'(t_0) = 0$ . Since  $\alpha(t_0) \neq 0$  and  $\alpha'(t_0) \neq 0$ ,  $\alpha(t_0)$  is orthogonal to  $\alpha'(t_0)$ .  $\square$

**Exercise 1-2.3.** A parametrized curve  $\alpha(t)$  has a property that its second derivative  $\alpha''(t)$  is identically zero. What can be said about  $\alpha$ ?

$\alpha(t)$  is a straight line.

*Proof.* Since  $\alpha''(t)$  is identically zero,  $\alpha'(t) = a$  is a constant. [Theorem 5.11 in W. Rudin, Principles of Mathematical Analysis, 3rd edition.] Define

$f(t) = \alpha(t) - at$  (on  $I$ ). Since  $f'(t) = \alpha'(t) - a = 0$ ,  $f(t) = \alpha(t) - at = b$  is a constant again. Therefore,  $\alpha(t) = at + b$ , which is a straight line (on  $I$ ).  $\square$

**Exercise 1-2.4.** Let  $\alpha : I \rightarrow \mathbb{R}^3$  be a parametrized curve and let  $v \in \mathbb{R}^3$  be a fixed vector. Assume that  $\alpha'(t)$  is orthogonal to  $v$  for all  $t \in I$  and that  $\alpha(0)$  is orthogonal to  $v$ . Prove that  $\alpha(t)$  is orthogonal to  $v$  for all  $t \in I$ .

Need to assume that  $\alpha(t) \neq 0$  for all  $t \in I$ .

*Proof.* Given any  $t \neq 0 \in I$ . (Nothing to do at  $t = 0$ .) Define  $f : I \rightarrow \mathbb{R}$  by  $f(t) = \alpha(t) \cdot v$ . By the mean value theorem, there exists a point  $\xi$  between 0 and  $t$  such that

$$f(t) - f(0) = f'(\xi)(t - 0),$$

where  $f'(t) = \alpha'(t) \cdot v + \alpha(t) \cdot v' = \alpha'(t) \cdot v$ . Note that  $f(0) = 0$  since  $\alpha(0)$  is orthogonal to  $v$ , and  $f'(\xi) = 0$  since  $\alpha'(\xi)$  is orthogonal to  $v$ . So the identity is reduced to

$$f(t) = 0,$$

or  $\alpha(t) \cdot v = 0$ , or  $\alpha(t)$  is orthogonal to  $v$ .  $\square$

**Exercise 1-2.5.** Let  $\alpha : I \rightarrow \mathbb{R}^3$  be a parametrized curve, with  $\alpha'(t) \neq 0$  for all  $t \in I$ . Show that  $|\alpha(t)|$  is a nonzero constant if and only if  $\alpha(t)$  is orthogonal to  $\alpha'(t)$  for all  $t \in I$ .

The same trick in Exercise 1-2.2.

*Proof.* It is equivalent to show that  $|\alpha(t)|^2$  is a nonzero constant if and only if  $\alpha(t)$  is orthogonal to  $\alpha'(t)$  for all  $t \in I$ . Let

$$f(t) = |\alpha(t)|^2 = \alpha(t) \cdot \alpha(t).$$

Notice that  $\alpha'(t) \neq 0$ , and thus

$$\begin{aligned} & |\alpha(t)| \text{ is a nonzero constant} \\ \iff & f(t) = |\alpha(t)|^2 \text{ is a nonzero constant} \\ \iff & f'(t) = 0 \text{ and } f(t) \text{ is a nonzero constant} \\ \iff & \alpha(t) \cdot \alpha'(t) = 0 \text{ and } \alpha(t) \text{ is a nonzero constant} \\ \iff & \alpha(t) \text{ is orthogonal to } \alpha'(t) \text{ for all } t \in I. \end{aligned}$$

$\square$

## Section 1-3: Regular Curves; Arc Length

**Exercise 1-3.1.** Show that the tangent lines to the regular parametrized curve  $\alpha(t) = (3t, 3t^2, 2t^3)$  make a constant angle with the line  $y = 0, z = x$ .

*Proof.*  $\alpha'(t) = (3, 6t, 6t^2)$ . The line  $y = 0, z = x$  is  $\beta(t) = (1, 0, 1)$ . The cosine of the angle  $\theta$  between these two curves is

$$\begin{aligned}\cos \theta &= \frac{(3, 6t, 6t^2) \cdot (1, 0, 1)}{|(3, 6t, 6t^2)| |(1, 0, 1)|} \\ &= \frac{3 + 6t^2}{\sqrt{3^2 + (6t)^2 + (6t^2)^2} \sqrt{2}} \\ &= \frac{3 + 6t^2}{\sqrt{9 + 36t^2 + 36t^4} \sqrt{2}} \\ &= \frac{3 + 6t^2}{\sqrt{(3 + 6t^2)^2} \sqrt{2}} \\ &= \frac{1}{\sqrt{2}}.\end{aligned}$$

(Notice  $3 + 6t^2 > 0$  for all  $t \in \mathbb{R}$ .) That is, the angle between  $\alpha'$  and  $\beta$  is a constant ( $= \pi/4$ ).  $\square$

**Exercise 1-3.2.** A circular disk of radius 1 in the plane  $xy$  rolls without slipping along the  $x$  axis. The figure described by a point of the circumference of the disk is called a **cycloid** (Figure 1-7 in Mantredio P. do Carmo, *Differential Geometry of Curves and Surfaces*).

- (a) Obtain a parametrized curve  $\alpha : \mathbb{R} \rightarrow \mathbb{R}^2$  the trace of which is the cycloid and determine its singular points.
- (b) Compute the arc length of the cycloid corresponding to a complete rotation of the disk.

*Proof of (a).*

- (1) Since

$$\begin{cases} x = t - \sin t \\ y = 1 - \cos t, \end{cases}$$

we define  $\alpha(t) = (t - \sin t, 1 - \cos t)$ .

- (2)  $\alpha'(t) = (1 - \cos t, \sin t)$ .  $\alpha'(t) = 0$  if and only if  $t = 2n\pi$  where  $n \in \mathbb{Z}$ . That is, all singular points are  $\alpha(2n\pi) = (2n\pi, 0)$  where  $n \in \mathbb{Z}$ .

$\square$

*Proof of (b).* The arc length of the cycloid corresponding to a complete rotation of the disk is

$$\begin{aligned}
 \int_0^{2\pi} |\alpha'(t)| dt &= \int_0^{2\pi} \sqrt{(1 - \cos t)^2 + (\sin t)^2} dt \\
 &= \int_0^{2\pi} \sqrt{2} \sqrt{1 - \cos t} dt \\
 &= \int_0^{2\pi} 2 \sin \frac{t}{2} dt \\
 &= \left[ -4 \cos \frac{t}{2} \right]_{t=0}^{t=2\pi} \\
 &= 8.
 \end{aligned}$$

□

**Supplement.** The cycloid is not an algebraic curve.

**Exercise 1-3.4.** Let  $\alpha : (0, \pi) \rightarrow \mathbb{R}^2$  be given by

$$\alpha(t) = \left( \sin t, \cos t + \log \tan \frac{t}{2} \right),$$

where  $t$  is the angle that the  $y$  axis makes with the vector  $\alpha(t)$ . The trace of  $\alpha$  is called the **tractrix**. (Figure 1-9 in Mantredo P. do Carmo, *Differential Geometry of Curves and Surfaces*). Show that

- (a)  $\alpha$  is a differentiable parametrized curve, regular except at  $t = \frac{\pi}{2}$ .
- (b) The length of the segment of the tangent of the tractrix between the point of tangency and the  $y$  axis is constantly equal to 1.

*Proof of (a).*

$$\begin{aligned}
 \alpha'(t) &= \left( \cos t, -\sin t + \frac{1}{\tan \frac{t}{2}} \frac{1}{\cos^2 \frac{t}{2}} \frac{1}{2} \right) \\
 &= \left( \cos t, -\sin t + \frac{1}{2 \sin \frac{t}{2} \cos \frac{t}{2}} \right) \\
 &= \left( \cos t, \frac{\cos^2 t}{\sin t} \right)
 \end{aligned}$$

exists. And  $\alpha'(t) = 0$  if and only if  $t = \frac{\pi}{2}$ . That is, there is an unique singular point at  $t = \frac{\pi}{2}$ . □

*Proof of (b).* The the tangent line of the tractrix through the regular point  $t$  is parametrized by  $\beta : \mathbb{R} \rightarrow \mathbb{R}^2$  which is defined by

$$\begin{aligned}\beta(u) &= u\alpha'(t) + \alpha(t) \\ &= \left( u \cos t + \sin t, u \frac{\cos^2 t}{\sin t} + \cos t + \log \tan \frac{t}{2} \right).\end{aligned}$$

By construction, this tangent line  $\beta(u)$  meets the tractrix at  $u = 0$ , and meets the  $y$ -axis when  $u \cos t + \sin t = 0$  or  $u = -\tan t$ . So the length of the segment is

$$\begin{aligned}|\beta(0) - \beta(-\tan t)| &= \sqrt{(-\tan t \cos t)^2 + \left(-\tan t \frac{\cos^2 t}{\sin t}\right)^2} \\ &= \sqrt{(\sin t)^2 + (\cos t)^2} \\ &= 1.\end{aligned}$$

□

**Exercise 1-3.10.** (*Straight Lines as Shortest.*) Let  $\alpha : I \rightarrow \mathbb{R}^3$  be a parametrized curve. Let  $[a, b] \subseteq I$  and set  $\alpha(a) = p$ ,  $\alpha(b) = q$ .

(a) Show that, for any constant vector  $v$ ,  $|v| = 1$ ,

$$(q - p) \cdot v = \int_a^b \alpha'(t) \cdot v dt \leq \int_a^b |\alpha'(t)| dt.$$

(b) Set

$$v = \frac{q - p}{|q - p|}$$

and show that

$$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)| dt;$$

that is, the curve of shortest length from  $\alpha(a)$  to  $\alpha(b)$  is the straight line joining these points.

Assume  $p \neq q$  (otherwise  $v = \frac{q-p}{|q-p|}$  is meaningless).

*Proof of (a).* Let  $f(t) = \alpha(t) \cdot v$  defined on  $I$ . By the fundamental theorem of calculus,

$$\int_a^b f'(t) dt = f(b) - f(a).$$

Since  $f'(t) = \alpha'(t) \cdot v$ ,

$$(\alpha(b) - \alpha(a)) \cdot v = \int_a^b \alpha'(t) \cdot v dt.$$

Therefore,

$$\begin{aligned}
(q-p) \cdot v &= \int_a^b \alpha'(t) \cdot v dt \\
&\leq \int_a^b |\alpha'(t) \cdot v| dt \\
&\leq \int_a^b |\alpha'(t)| |v| dt \\
&= \int_a^b |\alpha'(t)| dt.
\end{aligned}$$

□

*Proof of (b).*  $|v| = \frac{|q-p|}{|q-p|} = 1$ . So,

$$\begin{aligned}
(q-p) \cdot \frac{q-p}{|q-p|} &\leq \int_a^b |\alpha'(t)| dt, \\
|q-p| &\leq \int_a^b |\alpha'(t)| dt.
\end{aligned}$$

□

## Section 1-4: The Vector Product in $\mathbb{R}^3$

**Exercise 1-4.1.** Check whether the following bases are positive:

- (a) The basis  $\{(1, 3), (4, 2)\}$  in  $\mathbb{R}^2$ .
- (b) The basis  $\{(1, 3, 5), (2, 3, 7), (4, 8, 3)\}$  in  $\mathbb{R}^3$ .

*Proof of (a).* Write  $u = (1, 3)$  and  $v = (4, 2)$ . Then

$$\det(u, v) = \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = -10 < 0.$$

Thus  $\{u, v\}$  is negative w.r.t. the natural order basis  $\{e_1 = (1, 0), e_2 = (0, 1)\}$ . □

*Proof of (b).* Write  $u = (1, 3, 5)$ ,  $v = (2, 3, 7)$ ,  $w = (4, 8, 3)$ . Then

$$\det(u, v, w) = \begin{vmatrix} 1 & 3 & 5 \\ 2 & 3 & 7 \\ 4 & 8 & 3 \end{vmatrix} = 39 > 0.$$

Thus  $\{u, v, w\}$  is positive w.r.t. the natural order basis  $\{e_1, e_2, e_3\}$ . □

**Exercise 1-4.2.** A plane  $P$  contained in  $\mathbb{R}^3$  is given by the equation  $ax + by + cz + d = 0$ . Show that the vector  $v = (a, b, c)$  is perpendicular to the plane and that  $|d|/\sqrt{a^2 + b^2 + c^2}$  measures the distance from the plane to the origin  $(0, 0, 0)$ .

Say  $v$  is a normal vector of  $E$ .

In general, the distance from the plane  $E$  to any point  $(x_0, y_0, z_0) \in \mathbb{R}^3$  is

$$\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

*Proof.*

- (1) To show  $v = (a, b, c)$  is perpendicular to the plane, it suffices to show that  $v \cdot u = 0$  for any vector  $u$  lying on the plane  $E$ . Write  $u = \overrightarrow{PQ}$  where  $P = (x_1, y_1, z_1) \in E$  and  $Q = (x_2, y_2, z_2) \in E$ . Hence  $u = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$ .

$$\begin{aligned} v \cdot u &= (a, b, c) \cdot (x_2 - x_1, y_2 - y_1, z_2 - z_1) \\ &= a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) \\ &= (ax_2 + by_2 + cz_2) - (ax_1 + by_1 + cz_1) \\ &= (-d) - (-d) \\ &= 0. \end{aligned}$$

- (2) Pick any point  $(x_1, y_1, z_1) \in E$ . The distance from the plane  $E$  to the point  $(x_0, y_0, z_0)$  is

$$\begin{aligned} &\left| (x_1 - x_0, y_1 - y_0, z_1 - z_0) \cdot \frac{v}{|v|} \right| \\ &= \left| (x_1 - x_0, y_1 - y_0, z_1 - z_0) \cdot \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}} \right| \\ &= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|-d - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}. \end{aligned}$$

□

**Exercise 1-4.3.** Determine the angle of intersection of the two planes  $5x + 3y + 2z - 4 = 0$  and  $3x + 4y - 7z = 0$ .

*Proof.*

- (1) The angle of intersection of the two planes is equal to a angle between two normal vectors of planes.
- (2) Let
  - (a) the angle of intersection of the two planes be  $\theta$ .
  - (b) the normal vector of  $5x + 3y + 2z - 4 = 0$  be  $n_1 = (5, 3, 2)$ .
  - (c) the normal vector of  $3x + 4y - 7z = 0$  be  $n_2 = (3, 4, -7)$ .
- (3) Hence,

$$\cos \theta = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{13}{2\sqrt{703}}.$$

$$\theta = \cos^{-1} \left( \frac{13}{2\sqrt{703}} \right).$$

□

**Exercise 1-4.13.** Let  $u(t) = (u_1(t), u_2(t), u_3(t))$  and  $v(t) = (v_1(t), v_2(t), v_3(t))$  be differentiable maps from the interval  $(a, b)$  into  $\mathbb{R}^3$ . If the derivatives  $u'(t)$  and  $v'(t)$  satisfy the conditions

$$u'(t) = au(t) + bv(t), v'(t) = cu(t) - av(t),$$

where  $a$ ,  $b$ , and  $c$  are constants, show that  $u(t) \wedge v(t)$  is a constant vector.

*Proof.* Since

$$\begin{aligned} \frac{d}{dt}(u(t) \wedge v(t)) &= u'(t) \wedge v(t) + u(t) \wedge v'(t) \\ &= (au(t) + bv(t)) \wedge v(t) + u(t) \wedge (cu(t) - av(t)) \\ &= au(t) \wedge v(t) + u(t) \wedge (-av(t)) \\ &= a(u(t) \wedge v(t)) + (-a)(u(t) \wedge v(t)) \\ &= (0, 0, 0), \end{aligned}$$

$u(t) \wedge v(t)$  is a constant vector. □

## Section 1-5: The Local Theory of Curves Parametrized by Arc Length



**Exercise 1-5.2.** Show that the torsion  $\tau$  of  $\alpha$  is given by

$$\tau(s) = -\frac{\alpha'(s) \wedge \alpha''(s) \cdot \alpha'''(s)}{|\kappa(s)|^2}.$$

*Proof.*

- (1) Take inner product  $n(s)$  to the definition of torsion  $\tau(s)n(s) = b'(s)$ , we have

$$\tau(s) = b'(s) \cdot n(s).$$

Since  $b'(s) = t(s) \wedge n'(s)$ , we have to compute  $n'(s)$  first.

- (2) Compute  $n'(s)$ .

$$n'(s) = \frac{d}{ds} \left( \frac{\alpha''(s)}{\kappa(s)} \right) = \frac{\alpha'''(s)}{\kappa(s)} - \frac{\alpha''(s)\kappa'(s)}{\kappa(s)^2}.$$

- (3) By (1)(2),

$$\begin{aligned} \tau(s) &= b'(s) \cdot n(s) \\ &= (t(s) \wedge n'(s)) \cdot n(s) \\ &= \left( \alpha'(s) \wedge \left( \frac{\alpha'''(s)}{\kappa(s)} - \frac{\alpha''(s)\kappa'(s)}{\kappa(s)^2} \right) \right) \cdot \frac{\alpha''(s)}{\kappa(s)} \\ &= \left( \alpha'(s) \wedge \frac{\alpha'''(s)}{\kappa(s)} \right) \cdot \frac{\alpha''(s)}{\kappa(s)} \\ &= \frac{\alpha'(s) \wedge \alpha'''(s) \cdot \alpha''(s)}{|\kappa(s)|^2}, \end{aligned}$$

or

$$\tau(s) = \frac{\alpha'(s) \wedge \alpha'''(s) \cdot \alpha''(s)}{|\alpha''(s)|^2}.$$

□

## Section 1-6: The Local Canonical Form

## Section 1-7: Global Properties of Plane Curves