

Notes on the book:  
*Apostol, Modular Functions and  
Dirichlet Series in Number Theory,  
2nd edition*

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## Chapter 1: Elliptic functions

### Exercise 1.11.

If  $k \geq 2$  and  $\tau \in H$  prove that the Eisenstein series

$$G_{2k}(\tau) = \sum_{(m,n) \neq (0,0)} (m + n\tau)^{-2k}$$

has the Fourier expansion

$$G_{2k}(\tau) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) e^{2\pi i n \tau}.$$

*Proof.*

(1) Let  $q = e^{2\pi i \tau}$ . Similar to Lemma 1.3 on page 19, we have

$$(2k-1)! \sum_{m=-\infty}^{+\infty} \frac{1}{(\tau + m)^{2k}} = (2\pi i)^{2k} \sum_{r=1}^{\infty} r^{2k-1} q^r.$$

(2) Similar to Theorem 1.18, we have

$$\begin{aligned} G_{2k}(\tau) &= \sum_{(m,n) \neq (0,0)} (m + n\tau)^{-2k} \\ &= \sum_{\substack{m=-\infty \\ m \neq 0(n=0)}}^{+\infty} m^{-2k} + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{+\infty} ((m + n\tau)^{-2k} + (m - n\tau)^{-2k}) \\ &= 2\zeta(2k) + 2 \sum_{n=1}^{\infty} \sum_{m=-\infty}^{+\infty} (m + n\tau)^{-2k} \\ &= 2\zeta(2k) + 2 \sum_{n=1}^{\infty} \frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{r=1}^{\infty} r^{2k-1} q^{nr} \\ &= 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \underbrace{\sum_{d|n} d^{2k-1}}_{=\sigma_{2k-1}(n)} q^n. \end{aligned}$$

In the last double sum we collect together those terms for which  $nr$  is constant.

□

**Exercise 1.12.**

Refer to Exercise 1.11. If  $\tau \in H$  prove that

$$G_{2k}\left(-\frac{1}{\tau}\right) = \tau^{2k} G_{2k}(\tau)$$

and deduce that

$$\begin{aligned} G_{2k}\left(\frac{i}{2}\right) &= (-4)^k G_{2k}(2i) && \text{for all } k \geq 2, \\ G_{2k}(i) &= 0 && \text{if } k \text{ is odd,} \\ G_{2k}(e^{\frac{2\pi i}{3}}) &= 0 && \text{if } k \not\equiv 0 \pmod{3}. \end{aligned}$$

*Proof.*

(1)

$$\begin{aligned} G_{2k}\left(-\frac{1}{\tau}\right) &= \sum_{(m,n) \neq (0,0)} \left(m - \frac{n}{\tau}\right)^{-2k} \\ &= \tau^{2k} \sum_{(m,n) \neq (0,0)} (\tau m - n)^{-2k} \\ &= \tau^{2k} G_{2k}(\tau). \end{aligned}$$

(2) Let  $\tau = 2i$ . We have  $G_{2k}\left(\frac{i}{2}\right) = (-4)^k G_{2k}(2i)$ .

(3) Let  $\tau = i$ . We have  $G_{2k}(i) = (-1)^k G_{2k}(i)$ . Hence  $G_{2k}(i) = 0$  if  $k$  is odd.

(4) Let  $\tau = e^{\frac{\pi i}{3}}$ . We have  $G_{2k}(e^{\frac{2\pi i}{3}}) = e^{\frac{2k\pi i}{3}} G_{2k}(e^{\frac{\pi i}{3}})$ . Since

$$e^{\frac{2\pi i}{3}} = -1 + e^{\frac{\pi i}{3}}$$

and each Eisenstein series is a periodic function of  $\tau$  of period 1, we have  $G_{2k}(e^{\frac{2\pi i}{3}}) = G_{2k}(e^{\frac{\pi i}{3}})$ . So  $G_{2k}(e^{\frac{2\pi i}{3}}) = e^{\frac{2k\pi i}{3}} G_{2k}(e^{\frac{\pi i}{3}})$ . Therefore  $G_{2k}(e^{\frac{2\pi i}{3}}) = 0$  if  $k \not\equiv 0 \pmod{3}$ .

□

**Exercise 1.13.**

Ramanujan's tau function  $\tau(n)$  is defined by the Fourier expansion

$$\Delta(\tau) = (2\pi)^{12} \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n \tau},$$

derived in Theorem 1.19. Prove that

$$\tau(n) = 8000\{(\sigma_3 \circ \sigma_3) \circ \sigma_3\}(n) - 147(\sigma_5 \circ \sigma_5)(n),$$

where  $f \circ g$  denotes the Cauchy product of two sequences,

$$(f \circ g)(n) = \sum_{k=0}^n f(k)g(n-k),$$

and  $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$  for  $n \geq 1$ , with  $\sigma_3(0) = \frac{1}{240}$ ,  $\sigma_5(0) = -\frac{1}{504}$ . (Hint: Theorem 1.18.)

*Proof.*

(1) Let  $q = e^{2ki\tau}$ . Write

$$\begin{aligned} g_2(\tau) &= \frac{4\pi^4}{3} \left\{ 1 + 240 \sum_{k=1}^{\infty} \sigma_3(k)q^k \right\} = \frac{4\pi^4}{3} \cdot \left( 240 \sum_{k=0}^{\infty} \sigma_3(k)q^k \right), \\ g_3(\tau) &= \frac{8\pi^6}{27} \left\{ 1 - 504 \sum_{k=1}^{\infty} \sigma_5(k)q^k \right\} = \frac{8\pi^6}{27} \cdot \left( -504 \sum_{k=0}^{\infty} \sigma_5(k)q^k \right) \end{aligned}$$

(Theorem 1.18).

(2) Similar to the proof of Theorem 1.19,

$$\begin{aligned} \Delta(\tau) &= g_2(\tau)^3 - 27g_3(\tau)^2 \\ &= \frac{64\pi^{12}}{27} \left\{ \left( 240 \sum_{k=0}^{\infty} \sigma_3(k)q^k \right)^3 - \left( -504 \sum_{k=0}^{\infty} \sigma_5(k)q^k \right)^2 \right\} \\ &= (2\pi)^{12} \left\{ 8000 \left( \sum_{k=0}^{\infty} \sigma_3(k)q^k \right)^3 - 147 \left( \sum_{k=0}^{\infty} \sigma_5(k)q^k \right)^2 \right\} \\ &= (2\pi)^{12} \sum_{n=0}^{\infty} \{8000\{(\sigma_3 \circ \sigma_3) \circ \sigma_3\}(n) - 147(\sigma_5 \circ \sigma_5)(n)\} q^n \\ &= (2\pi)^{12} \sum_{n=1}^{\infty} \{8000\{(\sigma_3 \circ \sigma_3) \circ \sigma_3\}(n) - 147(\sigma_5 \circ \sigma_5)(n)\} q^n. \end{aligned}$$

(Here  $8000\{(\sigma_3 \circ \sigma_3) \circ \sigma_3\}(0) - 147(\sigma_5 \circ \sigma_5)(0) = 0$ .)

(3) Therefore

$$\tau(n) = 8000\{(\sigma_3 \circ \sigma_3) \circ \sigma_3\}(n) - 147(\sigma_5 \circ \sigma_5)(n)$$

for  $n \geq 1$ .

□

**Exercise 1.14. (Lambert series)**

A series of the form  $\sum_{n=1}^{\infty} f(n) \frac{x^n}{1-x^n}$  is called a **Lambert series**. Assuming absolute convergence, prove that

$$\sum_{n=1}^{\infty} f(n) \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} F(n) x^n,$$

where

$$F(n) = \sum_{d|n} f(d).$$

Apply this result to obtain the following formulas, valid for  $|x| < 1$ .

(a)

$$\sum_{n=1}^{\infty} \frac{\mu(n) x^n}{1-x^n} = x.$$

(b)

$$\sum_{n=1}^{\infty} \frac{\varphi(n) x^n}{1-x^n} = \frac{x}{(1-x)^2}.$$

(c)

$$\sum_{n=1}^{\infty} \frac{n^{\alpha} x^n}{1-x^n} = \sum_{n=1}^{\infty} \sigma_{\alpha}(n) x^n.$$

(d)

$$\sum_{n=1}^{\infty} \frac{\lambda(n) x^n}{1-x^n} = \sum_{n=1}^{\infty} x^{n^2}.$$

(e) Use the result in (c) to express  $g_2(\tau)$  and  $g_3(\tau)$  in terms of Lambert series in  $x = e^{2\pi i \tau}$ .

*Note.* In (a),  $\mu(n)$  is the Möbius function; In (b),  $\varphi(n)$  is Euler's totient; and in (d),  $\lambda(n)$  is Liouville's function.

*Proof.* Similar to the proof of Exercise 1.11.

$$\begin{aligned} \sum_{n=1}^{\infty} f(n) \frac{x^n}{1-x^n} &= \sum_{n=1}^{\infty} f(n) \sum_{r=1}^{\infty} x^{rn} \\ &= \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} f(n) x^{rn} \\ &= \sum_{n=1}^{\infty} \underbrace{\left( \sum_{d|n} f(d) \right)}_{=F(n)} x^n. \end{aligned}$$

□

*Proof of (a).* Theorem 2.1 in the textbook: *T. M. Apostol, Introduction to Analytic Number Theory* shows that

$$F(n) := \sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

Hence

$$\sum_{n=1}^{\infty} \mu(n) \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} F(n) x^n = x.$$

□

*Proof of (b).* Theorem 2.2 in the textbook: *T. M. Apostol, Introduction to Analytic Number Theory* shows that  $F(n) := \sum_{d|n} \varphi(d) = n$ . Hence

$$\sum_{n=1}^{\infty} \varphi(n) \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} F(n) x^n = \sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}.$$

□

*Proof of (c).* Since

$$F(n) := \sum_{d|n} d^\alpha = \sigma_\alpha(n),$$

we have

$$\sum_{n=1}^{\infty} n^\alpha \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} F(n) x^n = \sum_{n=1}^{\infty} \sigma_\alpha(n) x^n.$$

□

*Proof of (d).* Theorem 2.19 in the textbook: *T. M. Apostol, Introduction to Analytic Number Theory* shows that

$$F(n) := \sum_{d|n} \lambda(d) = \begin{cases} 1 & \text{if } n \text{ is a square,} \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\sum_{n=1}^{\infty} \lambda(n) \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} F(n) x^n = \sum_{n=1}^{\infty} x^{n^2}.$$

□

*Proof of (e).*

(1) Let  $q = x = e^{2\pi i\tau}$ .

$$\begin{aligned} g_2(\tau) &= \frac{4\pi^4}{3} \left\{ 1 + 240 \sum_{k=1}^{\infty} \sigma_3(k) q^k \right\} && \text{(Theorem 1.18)} \\ &= \frac{4\pi^4}{3} \left\{ 1 + 240 \sum_{k=1}^{\infty} \frac{k^3 q^k}{1 - q^k} \right\} && ((c)). \end{aligned}$$

(2) Similarly,

$$\begin{aligned} g_3(\tau) &= \frac{8\pi^6}{27} \left\{ 1 - 504 \sum_{k=1}^{\infty} \sigma_5(k) q^k \right\} && \text{(Theorem 1.18)} \\ &= \frac{8\pi^6}{27} \left\{ 1 - 504 \sum_{k=1}^{\infty} \frac{k^5 q^k}{1 - q^k} \right\} && ((c)). \end{aligned}$$

□

**Exercise 1.15.**

Let

$$G(x) = \sum_{n=1}^{\infty} \frac{n^5 x^n}{1 - x^n},$$

and let

$$F(x) = \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^5 x^n}{1 + x^n}.$$

(a) Prove that  $F(x) = G(x) - 34G(x^2) + 64(x^4)$ .

(b) Prove that

$$\sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^5}{1 + e^{n\pi}} = \frac{31}{504}.$$

(c) Use Theorem 12.17 in the textbook: *T. M. Apostol, Introduction to Analytic Number Theory* to prove the more general result

$$\sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^{4k+1}}{1 + e^{n\pi}} = \frac{2^{4k+1} - 1}{8k + 4} B_{4k+2}.$$

*Proof of (a).*

(1) Consider the general case. *Let*

$$G(x) = \sum_{n=1}^{\infty} \frac{n^{4k+1}x^n}{1-x^n},$$

*and let*

$$F(x) = \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^{4k+1}x^n}{1+x^n}.$$

*Show that*  $F(x) = G(x) - (2^{4k+1} + 2)G(x^2) + 2^{4k+2}G(x^4)$ .

(2) The identity

$$\sum_{n=1}^{\infty} \frac{x^n}{1+x^n} = \sum_{n=1}^{\infty} \frac{x^n}{1-x^n} - 2 \sum_{n=1}^{\infty} \frac{x^{2n}}{1-x^{2n}}$$

is always true. Hence  $H(x) := \sum_{n=1}^{\infty} \frac{n^{4k+1}x^n}{1+x^n} = G(x) - 2G(x^2)$ .

(3) Note that

$$\begin{aligned} H(x) &= \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^{4k+1}x^n}{1+x^n} + \sum_{\substack{n=1 \\ (n \text{ even})}}^{\infty} \frac{n^{4k+1}x^n}{1+x^n} \\ &= F(x) + \sum_{n=1}^{\infty} \frac{(2n)^{4k+1}x^{2n}}{1+x^{2n}} \\ &= F(x) + 2^{4k+1} \sum_{n=1}^{\infty} \frac{n^{4k+1}x^{2n}}{1+x^{2n}} \\ &= F(x) + 2^{4k+1}H(x^2). \end{aligned}$$

Hence

$$\begin{aligned} F(x) &= H(x) - 2^{4k+1}H(x^2) \\ &= [G(x) - 2G(x^2)] - 2^{4k+1}[G(x^2) - 2G(x^4)] \\ &= G(x) - (2^{4k+1} + 2)G(x^2) + 2^{4k+2}G(x^4). \end{aligned}$$

□

*Proof of (b).* Take  $k = 1$  in part (c), we have

$$\sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^5}{1+e^{n\pi}} = \frac{31}{12} \cdot \frac{1}{42} = \frac{31}{504}.$$

□

*Proof of (c).*



(1) Let  $q = e^{2\pi i\tau}$ . So

$$\begin{aligned} G_{4k+2}(\tau) &= 2\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!} \sum_{n=1}^{\infty} \sigma_{4k+1}(n) q^n && \text{(Exercise 1.11)} \\ &= 2\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!} G(q) && \text{(Exercise 1.14(c))} \end{aligned}$$

Hence

$$\begin{aligned} &G_{4k+2}(\tau) - (2^{4k+1} + 2)G_{4k+2}(2\tau) + 2^{4k+2}G_{4k+2}(4\tau) \\ &= \left[ 2\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!} G(q) \right] \\ &\quad - (2^{4k+1} + 2) \left[ 2\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!} G(q^2) \right] \\ &\quad + 2^{4k+2} \left[ 2\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!} G(q^4) \right] \\ &= (1 - (2^{4k+1} + 2) + 2^{4k+2}) \cdot 2\zeta(4k+2) \\ &\quad + \frac{2(2\pi i)^{4k+2}}{(4k+1)!} [G(q) - (2^{4k+1} + 2)G(q^2) + 2^{4k+2}G(q^4)] \\ &= (2^{4k+2} - 2)\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!} F(q). \end{aligned}$$

(2) By taking  $\tau = \frac{i}{2}$ , we have

$$F(q) = F(e^{-\pi}) = \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^{4k+1}}{1 + e^{n\pi}}$$

and

$$\begin{aligned} &G_{4k+2}(\tau) - (2^{4k+1} + 2)G_{4k+2}(2\tau) + 2^{4k+2}G_{4k+2}(4\tau) \\ &= G_{4k+2}\left(\frac{i}{2}\right) - (2^{4k+1} + 2)G_{4k+2}(i) + 2^{4k+2}G_{4k+2}(2i) \\ &= (-4)^{2k+1}G_{4k+2}(2i) - (2^{4k+1} + 2) \cdot 0 + 2^{4k+2}G_{4k+2}(2i) \\ &= 0. \end{aligned}$$

(Exercise 1.12). Hence

$$0 = (2^{4k+2} - 2)\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!} \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^{4k+1}}{1 + e^{n\pi}}.$$

(3) Theorem 12.17 in the textbook: *T. M. Apostol, Introduction to Analytic Number Theory* shows that

$$\zeta(4k+2) = (-1)^{2k+1+1} \frac{(2\pi)^{4k+2} B_{4k+2}}{2(4k+2)!} = \frac{(2\pi)^{4k+2} B_{4k+2}}{2(4k+2)!}.$$

Hence

$$\sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^{4k+1}}{1+e^{n\pi}} = \frac{2^{4k+1}-1}{8k+4} B_{4k+2}.$$

□