Notes on the book: *P.J. Hilton and U. Stammbach, A Course in Homological Algebra*

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Contents

Chapter I: Modules

§1. Modules

Exercise 1.1. (Diagram chasing)

Complete the proof of Lemma 1.1. Show moreover that α is surjective (resp. injective) if α', α'' are surjective (resp. injective).

Lemma 1.1. Let $0 \to A' \to A \to A'' \to 0$ and $0 \to B' \to B \to B'' \to 0$ be two short exact sequences. Suppose that in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r, "\mu"] \arrow[d, "\alpha'"] & A \arrow[r, "\varepsilon"] \arrow[d, "\alpha"] & A'' \arrow[r] \arrow[d, "\alpha''"] & 0 \\
0 \arrow[r] & B' \arrow[r, "\mu'"] & B \arrow[r, "\varepsilon'"] & B'' \arrow[r] & 0
\end{tikzcd}$$

any two of the three homomorphisms α', α, α'' are isomorphisms. Then the third is an isomorphism, too.

Proof (Diagram chasing).

(1) *Show that α is surjective if α', α'' are surjective.*

   (a) Take any $b \in B$, it suffices to find $a \in A$ such that $\alpha a = b$.

   (b) Consider the commutative diagram

$$\begin{tikzcd}
A \arrow[r, two heads, "\varepsilon"] \arrow[d, "\alpha"] & A'' \arrow[d, "\alpha''"] \\
B \arrow[r, "\varepsilon'"] & B''
\end{tikzcd}$$

   $\varepsilon' b \in B'$. By the surjectivity of α'', $\exists\, a'' \in A''$ such that $\alpha'' a'' = \varepsilon' b$. By the surjectivity of ε, $\exists\, \overline{a} \in A$ such that $\varepsilon \overline{a} = a''$. Hence

$$\begin{aligned}
\varepsilon'(b - \alpha\overline{a}) &= \varepsilon' b - \varepsilon' \alpha \overline{a} \\
&= \varepsilon' b - \alpha'' \varepsilon \overline{a} && \text{(The diagram commutes)} \\
&= \varepsilon' b - \alpha'' a'' \\
&= \varepsilon' b - \varepsilon' b \\
&= 0.
\end{aligned}$$

   (c) Consider the short exact sequence

$$0 \longrightarrow B' \xrightarrow{\mu'} B \xrightarrow{\varepsilon'} B'' \longrightarrow 0$$

   As $\varepsilon'(b - \alpha\overline{a}) = 0$, $\exists\, b' \in B'$ such that $\mu' b' = b - \alpha\overline{a}$.

(d) Consider the commutative diagram

$$\begin{tikzcd} A' \arrow[r, "\mu"] \arrow[d, "\alpha'"] & A \arrow[d, "\alpha"] \\ B' \arrow[r, "\mu'"] & B \end{tikzcd}$$

By the surjectivity of α', $\exists\, a' \in A'$ such that $\alpha' a' = b'$. Hence

$$\begin{aligned} \alpha(\mu a' + \overline{a}) &= \alpha\mu a' + \alpha\overline{a} \\ &= \mu'\alpha' a' + \alpha\overline{a} && \text{(The diagram commutes)} \\ &= \mu' b' + \alpha\overline{a} \\ &= (b - \alpha\overline{a}) + \alpha\overline{a} \\ &= b. \end{aligned}$$

Therefore, there exists $a := \mu a' + \overline{a}$ such that $\alpha a = b$.

(2) *Show that α is injective if α', α'' are injective.*

(a) It suffices to show that $\ker\alpha = 0$. Take $a \in \ker\alpha$. ($\alpha(a) = \alpha a = 0$.)

(b) Consider the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\varepsilon"] \arrow[d, "\alpha"] & A'' \arrow[d, "\alpha''"] \\ B \arrow[r, "\varepsilon'"] & B'' \end{tikzcd}$$

we have $0 = \varepsilon'\alpha a = \alpha''\varepsilon a$. By the injectivity of α'', $\varepsilon a = 0$.

(c) Consider the short exact sequence

$$0 \longrightarrow A' \xrightarrow{\mu} A \xrightarrow{\varepsilon} A'' \longrightarrow 0$$

As $\varepsilon a = 0$, $\exists\, a' \in A'$ such that $\mu a' = a$.

(d) Consider the commutative diagram

$$\begin{tikzcd} A' \arrow[r, "\mu"] \arrow[d, "\alpha'"] & A \arrow[d, "\alpha"] \\ B' \arrow[r, tail, "\mu'"] & B \end{tikzcd}$$

$0 = \alpha a = \alpha\mu a' = \mu'\alpha' a'$. By the injectivity of $\mu'\alpha'$, $a' = 0$. Therefore, $a = \mu a' = 0$.

(3) *Suppose α is surjective. Show that α'' is surjective.*

(a) Take any $b'' \in B''$, it suffices to find $a'' \in A''$ such that $\alpha'' a'' = b''$.

(b) Consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varepsilon} & A'' \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha''} \\ B & \xrightarrow{\varepsilon'} & B'' \end{array}$$

By the surjectivity of ε', $\exists\, b \in B$ such that $\varepsilon' b = b''$. By the surjectivity of α, $\exists\, a \in A$ such that $\alpha a = b$. Take $a'' := \varepsilon a \in A''$. Hence

$$\begin{aligned} \alpha'' a'' &= \alpha'' \varepsilon a \\ &= \varepsilon' \alpha a && \text{(The diagram commutes)} \\ &= \varepsilon' b \\ &= b''. \end{aligned}$$

(4) *Suppose α' is surjective and α is injective. Show that α'' is injective.*

(a) It suffices to show that $\ker \alpha'' = 0$. Take $a'' \in \ker \alpha''$. ($\alpha''(a'') = \alpha'' a'' = 0$.)

(b) Consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varepsilon} & A'' \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha''} \\ B & \xrightarrow{\varepsilon'} & B'' \end{array}$$

By the surjectivity of ε, $\exists\, a \in A$ such that $\varepsilon a = a''$. So

$$\begin{aligned} 0 &= \alpha'' a'' \\ &= \alpha'' \varepsilon a \\ &= \varepsilon' \alpha a. && \text{(The diagram commutes)} \end{aligned}$$

(c) Consider the short exact sequence

$$0 \longrightarrow B' \xrightarrow{\mu'} B \xrightarrow{\varepsilon'} B'' \longrightarrow 0$$

As $\varepsilon'(\alpha a) = 0$, $\exists\, b' \in B'$ such that $\mu' b' = \alpha a$.

(d) Consider the commutative diagram

$$\begin{array}{ccc} A' & \xrightarrow{\mu} & A \\ \downarrow{\scriptstyle\alpha'} & & \downarrow{\scriptstyle\alpha} \\ B' & \xrightarrow{\mu'} & B \end{array}$$

By surjectivity of α', $\exists\, a' \in A'$ such that $\alpha' a' = b'$. So

$$\begin{aligned} \alpha a &= \mu' b' \\ &= \mu' \alpha' a' \\ &= \alpha \mu a'. && \text{(The diagram commutes)} \end{aligned}$$

By the injectivity of α, $a = \mu a'$. Hence

$$a'' = \varepsilon a = \varepsilon \mu a' = 0.$$

Therefore $\ker \alpha'' = 0$.

(5) By (3)(4), α'' is an isomorphism if both α' and α are isomorphisms.

(6) *Suppose α is surjective and α'' is injective. Show that α' is surjective.*

(a) Take any $b' \in B'$, it suffices to find $a' \in A'$ such that $\alpha' a' = b'$. Let $b := \mu' b' \in B$ and note that $\varepsilon' b = 0$ by the exactness of

$$0 \to B' \to B \to B'' \to 0.$$

(b) Consider the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\varepsilon"] \arrow[d, "\alpha"] & A'' \arrow[d, "\alpha''"] \\ B \arrow[r, "\varepsilon'"] & B'' \end{tikzcd}$$

By the surjectivity of α, $\exists\, a \in A$ such that $\alpha a = b$. So

$$\begin{aligned} 0 &= \varepsilon' b \\ &= \varepsilon' \alpha a \\ &= \alpha'' \varepsilon a. && \text{(The diagram commutes)} \end{aligned}$$

By the injectivity of α'', $\varepsilon a = 0$.

(c) Consider the short exact sequence

$$0 \longrightarrow A' \xrightarrow{\mu} A \xrightarrow{\varepsilon} A'' \longrightarrow 0$$

As $\varepsilon a = 0$, $\exists\, a' \in A'$ such that $\mu a' = a$.

(d) Consider the commutative diagram

$$\begin{tikzcd} A' \arrow[r, "\mu"] \arrow[d, "\alpha'"] & A \arrow[d, "\alpha"] \\ B' \arrow[r, tail, "\mu'"] & B \end{tikzcd}$$

Note that

$$\begin{aligned} \mu'(\alpha' a') &= \mu' \alpha' a' \\ &= \alpha \mu a' && \text{(The diagram commutes)} = \alpha a \\ &= b \\ &= \mu' b'. \end{aligned}$$

By the injectivity of μ', $b' = \alpha' a'$ for some $a' \in A'$.

(7) *Suppose α is injective. Show that α' is injective.*

  (a) It suffices to show that $\ker \alpha' = 0$. Take $a' \in \ker \alpha'$. ($\alpha'(a') = \alpha' a' = 0$.)

  (b) Consider the commutative diagram

$$\begin{tikzcd} A' \arrow[r, tail, "\mu"] \arrow[d, "\alpha'"] & A \arrow[d, "\alpha"] \\ B' \arrow[r, "\mu'"] & B \end{tikzcd}$$

  Note that

$$\begin{aligned} 0 &= \mu' 0 \\ &= \mu' \alpha' a' \\ &= \alpha \mu a'. && \text{(The diagram commutes)} \end{aligned}$$

  The injectivity of $\alpha\mu$ shows that $a' = 0$.

(8) By (6)(7), α' is an isomorphism if both α and α'' are isomorphisms.

□

Exercise 1.2. (Five lemma)

Show that, given a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & A_1 \arrow[r] \arrow[d, "\varphi_1"] & A_2 \arrow[r] \arrow[d, "\varphi_2"] & A_3 \arrow[r] \arrow[d, "\varphi_3"] & A_4 \arrow[r] \arrow[d, "\varphi_4"] & A_5 \arrow[r] \arrow[d, "\varphi_5"] & \cdots \\ \cdots \arrow[r] & B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5 \arrow[r] & \cdots \end{tikzcd}$$

with exact rows, in which φ_1, φ_2, φ_4, φ_5 are isomorphisms, then φ_3 is also an isomorphism. Can we weaken the hypotheses in a reasonable way?

One reasonable hypotheses:

(a) *If φ_1 is surjective and φ_2, φ_4 is injective, then φ_3 is injective.*

(b) *If φ_5 is injective and φ_2, φ_4 is surjective, then φ_3 is surjective.*

Proof of (a).

(1) Write

$$\begin{tikzcd} \cdots \arrow[r] & A_1 \arrow[r, "\alpha_1"] \arrow[d, "\varphi_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "\varphi_2"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "\varphi_3"] & A_4 \arrow[r, "\alpha_4"] \arrow[d, "\varphi_4"] & A_5 \arrow[r] \arrow[d, "\varphi_5"] & \cdots \\ \cdots \arrow[r] & B_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & B_4 \arrow[r, "\beta_4"] & B_5 \arrow[r] & \cdots \end{tikzcd}$$

Take $a \in \ker(\varphi_3)$ and then we need to show $a = 0$.

(2) The commutative diagram

$$\begin{tikzcd} A_3 \arrow[r, "\alpha_3"] \arrow[d, "\varphi_3"] & A_4 \arrow[d, tail, "\varphi_4"] \\ B_3 \arrow[r, "\beta_3"] & B_4 \end{tikzcd}$$

implies that $0 = \beta_3 0 = \beta_3\varphi_3 a = \varphi_4\alpha_3 a$. The injectivity of φ_4 implies that $\alpha_3 a = 0$.

(3) The exact sequence

$$\cdots \longrightarrow A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} A_4 \longrightarrow \cdots$$

shows that $a \in \ker(\alpha_3) = \operatorname{im}(\alpha_2)$. So there exists $a_2 \in A_2$ such that $\alpha_2 a_2 = a$.

(4) The commutative diagram

$$\begin{tikzcd} A_2 \arrow[r, "\alpha_2"] \arrow[d, "\varphi_2"] & A_3 \arrow[d, "\varphi_3"] \\ B_2 \arrow[r, "\beta_2"] & B_3 \end{tikzcd}$$

implies that $0 = \varphi_3 a = \varphi_3\alpha_2 a_2 = \beta_2\varphi_2 a_2$.

(5) The exact sequence

$$\cdots \longrightarrow B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \longrightarrow \cdots$$

shows that $\varphi_2 a_2 \in \ker(\beta_2) = \operatorname{im}(\beta_1)$. So there exists $b_1 \in B_1$ such that $\varphi_2 a_2 = \beta_1 b_1$.

(6) Consider the commutative diagram

$$\begin{tikzcd} A_1 \arrow[r, "\alpha_1"] \arrow[d, two heads, "\varphi_1"] & A_2 \arrow[d, tail, "\varphi_2"] \\ B_1 \arrow[r, "\beta_1"] & B_2 \end{tikzcd}$$

The surjectivity of φ_i implies that $\exists\, a_1 \in A_1$ such that $\varphi_1 a_1 = b_1$. Hence the commutative diagram implies that $\varphi_2(\alpha_1 a_1) = \varphi_2\alpha_1 a_1 = \beta_1\varphi_1 a_1 = \beta_1 b_1 = \varphi_2 a_2$. The injectivity of φ_2 implies that $\alpha_1 a_1 = a_2$.

(7) The exact sequence

$$\cdots \longrightarrow A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \longrightarrow \cdots$$

shows that $a = \alpha_2 a_2 = \alpha_2\alpha_1 a_1 = 0$. Therefore φ_3 is injective.

□

Proof of (b).

(1) Take any $b \in B_3$, it suffices to find $a \in A$ such that $\varphi_3 a = b$.

(2) Let $b_4 := \beta_3 b \in B_4$. The exact sequence

$$\cdots \longrightarrow B_3 \xrightarrow{\beta_3} B_4 \xrightarrow{\beta_4} B_5 \longrightarrow \cdots$$

shows that $\beta_4 b_4 = \beta_4(\beta_3 b) = 0$.

(3) Look at the commutative diagram

$$\begin{tikzcd} A_4 \arrow[r, "\alpha_4"] \arrow[d, two heads, "\varphi_4"] & A_5 \arrow[d, tail, "\varphi_5"] \\ B_4 \arrow[r, "\beta_4"] & B_5 \end{tikzcd}$$

By the surjectivity of φ_4, $\exists\ a_4 \in A_4$ such that $\varphi_4 a_4 = b_4$. So the commutative diagram says that $0 = \beta_4 b_4 = \beta_4\varphi_4 a_4 = \varphi_5\alpha_4 a_4$. By the injectivity of φ_5, $\alpha_4 a_4 = 0$.

(4) The exact sequence

$$\cdots \longrightarrow A_3 \xrightarrow{\alpha_3} A_4 \xrightarrow{\alpha_4} A_5 \longrightarrow \cdots$$

shows that $a_4 \in \ker(\alpha_4) = \mathrm{im}(\alpha_3)$. So there exists $a_3 \in A_3$ such that $\alpha_3 a_3 = a_4$.

(5) Let $\bar{b} = b - \varphi_3 a_3 \in B_3$. The commutative diagram

$$\begin{tikzcd} A_3 \arrow[r, "\alpha_3"] \arrow[d, "\varphi_3"] & A_4 \arrow[d, "\varphi_4"] \\ B_3 \arrow[r, "\beta_3"] & B_4 \end{tikzcd}$$

implies that $\beta_3\bar{b} = \beta_3 b - \beta_3\varphi_3 a_3 = \beta_3 b - \varphi_4\alpha_3 a_3 = \beta_3 b - \varphi_4 a_4 = \beta_3 b - b_4 = \beta_3 b - \beta_3 b = 0$. So $\bar{b} \in \ker(\beta_3)$.

(6) The exact sequence

$$\cdots \longrightarrow B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} B_4 \longrightarrow \cdots$$

shows that $\overline{b} \in \ker(\beta_3) = \operatorname{im}(\beta_2)$. Hence $\exists\, b_2 \in B_2$ such that $\overline{b} = \beta_2 b_2$.

(7) Look at the commutative diagram

$$\begin{array}{ccc} A_2 & \xrightarrow{\alpha_2} & A_3 \\ \Big\downarrow{\scriptstyle \varphi_2} & & \Big\downarrow{\scriptstyle \varphi_3} \\ B_2 & \xrightarrow{\beta_2} & B_3 \end{array}$$

The surjectivity of φ_2 implies that $\exists\, a_2 \in A_2$ such that $b_2 = \varphi_2 a_2$. Let $a := \alpha_2 a_2 + a_3$. Hence

$$\begin{aligned} \varphi_3(a) &= \varphi_3 \alpha_2 a_2 + \varphi_3 a_3 \\ &= \beta_2 \varphi_2 a_2 + \varphi_3 a_3 && \text{(The diagram commutes)} \\ &= \beta_2 b_2 + \varphi_3 a_3 \\ &= \overline{b} + \varphi_3 a_3 \\ &= (b - \varphi_3 a_3) + \varphi_3 a_3 \\ &= b. \end{aligned}$$

□

Exercise 1.4.

Show that the abelian group A admits the structure of a $\mathbb{Z}/(m)$-module if and only if $mA = 0$.

Proof.

(1) ($\Longrightarrow$) It suffices to show that $ma = 0$ for all $a \in A$. Let $\Lambda = \mathbb{Z}/(m)$.

$$\begin{aligned} ma &= \underbrace{a + \cdots + a}_{m \text{ times}} \\ &= \underbrace{1_\Lambda a + \cdots + 1_\Lambda a}_{m \text{ times}} && \text{(Axiom M3)} \\ &= (\underbrace{1_\Lambda + \cdots + 1_\Lambda}_{m \text{ times}})a && \text{(Axiom M1)} \\ &= 0_\Lambda a && (\operatorname{char}(\Lambda) = m) \\ &= 0. && \text{(Axiom M1)} \end{aligned}$$

(2) ($\Longleftarrow$) Write $\overline{\lambda} \in \Lambda := \mathbb{Z}/(m)$ where $\lambda \in \mathbb{Z}$ and $\overline{\lambda}$ is the residue class of λ in Λ. Define $\omega : \Lambda \to \operatorname{End}(A, A)$ by

$$\omega(\overline{\lambda})(a) = \lambda a$$

for all $a \in A$ and $\overline{\lambda} \in \Lambda$. ω is well-defined since $mA = 0$. Note that all four module axioms hold for A (as a Λ-module).

□

§2. The Group of Homomorphisms

Exercise 2.1.

Show that in the setting of Theorem 2.1 $\varepsilon_* = \mathrm{Hom}(A, \varepsilon)$ *is not, in general, surjective even if* ε *is. (Hint: Take* $\Lambda = \mathbb{Z}$, $A = \mathbb{Z}/(n)$, *the integers mod* n, *and the short exact sequence* $\mathbb{Z} \overset{\mu}{\rightarrowtail} \mathbb{Z} \overset{\varepsilon}{\twoheadrightarrow} \mathbb{Z}/(n)$ *where* μ *is multiplication by* n.*)*

Theorem 2.1. Let $B' \overset{\mu}{\rightarrowtail} B \xrightarrow{\varepsilon} B''$ be an exact sequence of Λ-modules. For every Λ-module A the induced sequence

$$0 \longrightarrow \mathrm{Hom}_\Lambda(A, B') \xrightarrow{\mu_*} \mathrm{Hom}_\Lambda(A, B) \xrightarrow{\varepsilon_*} \mathrm{Hom}_\Lambda(A, B'')$$

is exact.

Proof.

(1) Consider

$$\mathrm{Hom}_\mathbb{Z}(\mathbb{Z}/(n), \mathbb{Z}) \xrightarrow{\varepsilon_*} \mathrm{Hom}_\mathbb{Z}(\mathbb{Z}/(n), \mathbb{Z}/(n)).$$

Note that $\mathrm{Hom}_\mathbb{Z}(\mathbb{Z}/(n), \mathbb{Z}/(n)) \cong \mathbb{Z}/(n)$ is not trivial. So to prove that ε_* is not surjective, it suffices to show that $\mathrm{Hom}_\mathbb{Z}(\mathbb{Z}/(n), \mathbb{Z}) = 0$.

(2) *Show that* $\mathrm{Hom}_\mathbb{Z}(\mathbb{Z}/(n), \mathbb{Z}) = 0$. Suppose $\alpha \in \mathrm{Hom}_\mathbb{Z}(\mathbb{Z}/(n), \mathbb{Z})$. Given any $a \in \mathbb{Z}/(n)$. So $na = 0$ by the Lagrange's theorem in group theory. So

$$0 = \alpha(0) = \alpha(na) = n\alpha(a) \in \mathbb{Z}.$$

So $\alpha(a) = 0 \in \mathbb{Z}$. Hence α is a zero map.

□

Exercise 2.2.

Prove Theorem 2.2. Show that $\mu^* = \mathrm{Hom}_\Lambda(\mu, B)$ *is not, in general, surjective even if* μ *is injective. (Hint: Take* $\Lambda = \mathbb{Z}$, $B = \mathbb{Z}/(n)$, *the integers mod* n, *and* *the short exact sequence* $\mathbb{Z} \overset{\mu}{\rightarrowtail} \mathbb{Z} \overset{\varepsilon}{\twoheadrightarrow} \mathbb{Z}/(n)$ *where* μ *is multiplication by* n.*)*

Theorem 2.2. Let $A' \xrightarrow{\mu} A \overset{\varepsilon}{\twoheadrightarrow} A''$ be an exact sequence of Λ-modules. For every Λ-module B the induced sequence

$$0 \longrightarrow \mathrm{Hom}_\Lambda(A'', B) \xrightarrow{\varepsilon^*} \mathrm{Hom}_\Lambda(A, B) \xrightarrow{\mu^*} \mathrm{Hom}_\Lambda(A', B)$$

is exact.

Proof of Theorem 2.2.

(1) *Show that* ε^* *is injective.* Take $\alpha \in \ker(\varepsilon^*) \subseteq \mathrm{Hom}_\Lambda(A'', B)$. It suffices to show that $\alpha a'' = 0$ for all $a'' \in A''$. By the surjectivity of ε, there exists $a \in A$ such that $\varepsilon a = a''$. Hence

$$\alpha a'' = \alpha\varepsilon a = (\varepsilon^*(\alpha))(a) = (0)(a) = 0.$$

(2) *Show that* $\mathrm{im}(\varepsilon^*) \subseteq \ker(\mu^*)$. A map in $\mathrm{im}(\varepsilon^*)$ is of the form $\alpha\varepsilon$. Plainly, $\varepsilon\mu\alpha$ is a zero map, since $\varepsilon\mu$ already is.

(3) *Show that* $\ker(\mu^*) \subseteq \mathrm{im}(\varepsilon^*)$. Consider the diagram

$$\begin{tikzcd} A' \arrow[r, "\mu"] & A \arrow[r, two heads, "\varepsilon"] \arrow[d, "\alpha"] & A'' \arrow[dl, dashed, "\exists\beta"] \\ & B & \end{tikzcd}$$

We have to show that if $\mu^*\alpha = \alpha\mu$ is the zero map, then α is of the form $\varepsilon^*\beta = \beta\varepsilon$ for some $\beta : A'' \to B$. But if $\alpha\mu = 0$, $\ker(\alpha) \supseteq \mathrm{im}(\mu) = \ker(\varepsilon)$. Since ε is surjective, α gives rise to a (unique) map $\beta : A'' \to B$ such that $\alpha = \beta\varepsilon$. In brief,

(a) Define β by $a'' \mapsto \alpha(a)$ where $a \in A$ satisfying $\varepsilon(a) = a''$. The existence of a is guaranteed by the surjectivity of ε.

(b) β is well-defined since $\ker(\alpha) \supseteq \ker(\varepsilon)$.

(c) β is a homomorphism since both α, ε are homomorphisms.

□

Proof.

(1) *Show that* $\mu^* = \mathrm{Hom}_\Lambda(\mu, B)$ *is not, in general, surjective even if* μ *is injective.* Consider

$$\mathrm{Hom}_\mathbb{Z}(\mathbb{Z}, \mathbb{Z}/(n)) \xrightarrow{\mu^*} \mathrm{Hom}_\mathbb{Z}(\mathbb{Z}, \mathbb{Z}/(n)).$$

It suffices to show that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/(n)) \cong \mathbb{Z}/(n)$ canonically. If so, the homomorphism μ^* maps each $\alpha \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/(n))$ to the zero map in $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/(n))$, which means μ^* is not surjective.

(2) *Show that* $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/(n)) \cong \mathbb{Z}/(n)$. Take $\alpha \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/(n))$. Note that $\mathbb{Z} = (1)$. So α is uniquely determined by $\alpha(1)$. Conversely, each element $a \in \mathbb{Z}/(n)$ determines a unique homomorphism $\alpha : \mathbb{Z} \to \mathbb{Z}/(n)$ by $\alpha(1) = a$. Hence there is a group isomorphism

$$\Phi : \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/(n)) \to \mathbb{Z}/(n)$$

such that $\Phi : \alpha \mapsto \alpha(1)$. (It is easy to verify that Φ is a group homomorphism.)

$\square$

Exercise 2.6.

Compute $\operatorname{Hom}(\mathbb{Z}, \mathbb{Z}/(n))$, $\operatorname{Hom}(\mathbb{Z}/(m), \mathbb{Z}/(n))$, $\operatorname{Hom}(\mathbb{Z}/(m), \mathbb{Z})$, $\operatorname{Hom}(\mathbb{Q}, \mathbb{Z})$, $\operatorname{Hom}(\mathbb{Q}, \mathbb{Q})$. *Here "*$\operatorname{Hom}$*" means "*$\operatorname{Hom}_{\mathbb{Z}}$*" and* $\mathbb{Q}$ *is the group of rationals.*

Proof.

(1) *Show that* $\operatorname{Hom}(\mathbb{Z}, \mathbb{Z}/(n)) \cong \mathbb{Z}/(n)$. Each $\alpha \in \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}/(n))$ is uniquely determined by $\alpha(1) \in \mathbb{Z}/(n)$. Conversely, each element $a \in \mathbb{Z}/(n)$ determines a unique homomorphism $\alpha : \mathbb{Z} \to \mathbb{Z}/(n)$ by $\alpha(1) = a$. Hence there is a group isomorphism

$$\Phi : \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}/(n)) \to \mathbb{Z}/(n).$$

(2) *Show that* $\operatorname{Hom}(\mathbb{Z}/(m), \mathbb{Z}/(n)) \cong \mathbb{Z}/(m, n)$. Define a map

$$\Phi : \operatorname{Hom}(\mathbb{Z}/(m), \mathbb{Z}/(n)) \to \mathbb{Z}/(m, n)$$

by mapping $\alpha \in \operatorname{Hom}(\mathbb{Z}/(m), \mathbb{Z}/(n))$ to $\overline{\alpha(1)}$ where $\overline{\alpha(1)}$ is the residue class of $\alpha(1) \in \mathbb{Z}/(n)$ in $\mathbb{Z}/(m, n)$. Φ is well-defined. Φ is a group homomorphism. Φ is surjective and injective.

(3) *Show that* $\operatorname{Hom}(\mathbb{Z}/(m), \mathbb{Z}) = 0$. See part (2) in the proof of Exercise 2.1.

(4) *Show that* $\operatorname{Hom}(\mathbb{Q}, \mathbb{Z}) = 0$. (Reductio ad absurdum) Suppose there were a non zero map $\alpha : \mathbb{Q} \to \mathbb{Z}$. So $\exists\, a \in \mathbb{Q}$ such that $\alpha(a) = N \neq 0$. Note that

$$\alpha(a) = \alpha\left(\underbrace{\frac{a}{n} + \cdots + \frac{a}{n}}_{n \text{ times}}\right) = \underbrace{\alpha\left(\frac{a}{n}\right) + \cdots + \alpha\left(\frac{a}{n}\right)}_{n \text{ times}} = n\alpha\left(\frac{a}{n}\right)$$

for all integers n. As $\alpha\left(\frac{a}{n}\right) \in \mathbb{Z}$, $n \mid \alpha(a)$ for all $n \in \mathbb{Z}$, which is absurd.

(5) *Show that* $\mathrm{Hom}(\mathbb{Q},\mathbb{Q}) \cong \mathbb{Q}$. Note that each $\alpha \in \mathrm{Hom}(\mathbb{Q},\mathbb{Q})$ is uniquely determined by $\alpha(1) \in \mathbb{Q}$. ($\alpha(r) = r\alpha(1)$ by the similar argument in (4) and part (2) in the proof of Exercise 2.1.) Conversely, each element $a \in \mathbb{Q}$ determines a unique homomorphism $\alpha : \mathbb{Q} \to \mathbb{Q}$ by $\alpha(1) = a$. Hence there is a group isomorphism

$$\Phi : \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\mathbb{Q}) \to \mathbb{Q}.$$

□

§3. Sums and Products

Exercise 3.1.

Show that there is a canonical map $\sigma : \oplus_j A_j \to \prod_j A_j$.

Proof.

(1) Define $\sigma : (a_j)_{j\in J} \mapsto (a_j)_{j\in J}$.

(2) σ is well-defined since there are no restrictions on $\sigma((a_j)_{j\in J})$ though $(a_j)_{j\in J} \in \oplus_j A_j$ has one restriction on $(a_j)_{j\in J}$ (say $a_j \neq 0$ for only a finite number of subscripts).

(3) σ is a Λ-module homomorphism and σ is injective.

□

Exercise 3.6.

Show that $\mathbb{Z}/(m) \oplus \mathbb{Z}/(n) = \mathbb{Z}/(mn)$ *if and only if* m *and* n *are mutually prime.*

Proof.

(1) ($\Longrightarrow$) Given any $g := (g_1, g_2) \in \mathbb{Z}/(m) \oplus \mathbb{Z}/(n) \cong \mathbb{Z}/(mn)$. As $mg_1 = 0 \in \mathbb{Z}/(m)$ and $ng_2 = 0 \in \mathbb{Z}/(n)$,

$$\mathrm{lcm}(m,n)g = (\mathrm{lcm}(m,n)g_1, \mathrm{lcm}(m,n)g_2) = (0,0).$$

So $\mathrm{lcm}(m,n)$ is divisible by the order of $\mathbb{Z}/(mn)$, that is, $mn|\mathrm{lcm}(m,n)$. Hence $\gcd(m,n) = 1$ since $mn = \mathrm{lcm}(m,n)\gcd(m,n)$.

(2) ($\Longleftarrow$) Define $\alpha : \mathbb{Z} \to \mathbb{Z}/(m) \oplus \mathbb{Z}/(n)$ by

$$\alpha : g \mapsto (g + (m), g + (n)).$$

α is a group homomorphism. The Chinese remainder theorem shows that α is surjective. (Note that $\gcd(m, n) = 1$.) The kernel of α is (mn). Hence α induces a group isomorphism

$$\overline{\alpha} : \mathbb{Z}/(mn) \to \mathbb{Z}/(m) \oplus \mathbb{Z}/(n).$$

$\square$