

## Chapter 2: Basic Topology

*Author: Meng-Gen Tsai*

*Email: plover@gmail.com*

### Notation.

- (1)  $E^\circ$  or  $\text{int}(E)$  is the interior of  $E$ .
- (2)  $\overline{E}$  is the closure of  $E$ .
- (3)  $\tilde{E}$  is the complement of  $E$ .
- (4)  $B(p; r)$  or  $B(p)$  is the set of all points  $q$  in a metric space  $(M, d)$  such that  $d_M(p, q) < r$ .

**Exercise 2.1.** *Prove that the empty set is a subset of every set.*

*Proof.* By Definitions 1.3,

- (1) The set which contains no element will be called the **empty set**,
- (2) If  $A$  and  $B$  are sets, and if every element of  $A$  is an element of  $B$ , we say that  $A$  is a **subset** of  $B$ ,

every element of the empty set (there are none) belongs to every set. That is, the empty set is a subset of every set.  $\square$

**Exercise 2.2.** *A complex number  $z$  is said to be algebraic if there are integers  $a_0, \dots, a_n$ , not all zero, such that*

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

*Prove that the set of all algebraic numbers is countable. (Hint: For every positive integer  $N$  there are only finitely many equations with*

$$n + |a_0| + |a_1| + \cdots + |a_n| = N.$$

Might assume  $a_0 \neq 0$ .

For example, all rational numbers are algebraic since  $p = \frac{\alpha}{\beta}$  (where  $\alpha, \beta \in \mathbb{Z}$ ) is a root of  $\beta z - \alpha = 0$ .

Besides,  $z = \sqrt{2} + \sqrt{3}$  is algebraic since  $z^4 - 10z^2 + 1 = 0$ . In fact,  $z = \pm\sqrt{2} \pm \sqrt{3}$  are also algebraic since  $z^4 - 10z^2 + 1 = (z - \sqrt{2} - \sqrt{3})(z + \sqrt{2} - \sqrt{3})(z - \sqrt{2} + \sqrt{3})(z + \sqrt{2} + \sqrt{3})$ .

**Lemma.** *The set of all polynomials over  $\mathbb{Z}$  is countable implies that the set of algebraic numbers is countable.*

*Proof of Lemma.* By definition, we write the set of algebraic numbers as

$$S = \bigcup_{f(x) \in \mathbb{Z}[x]} \{z \in \mathbb{C} : f(z) = 0\}.$$

Since each polynomial of degree  $n$  has at most  $n$  roots,  $\{z \in \mathbb{C} : f(z) = 0\}$  is finite for each given  $f(x) \in \mathbb{Z}[x]$ . So  $S$  is a countable union (by assumption) of finite sets, and hence at most countable.  $S$  is infinite since every integer  $\alpha$  is a root of  $f(z) = z - \alpha$ . So  $S$  is countable.  $\square$

Thus, it suffices to show that *the set of all polynomials over  $\mathbb{Z}$  is countable*.

*Proof (Hint).* For every positive integer  $N$  there are only finitely many equations with  $n + |a_0| + |a_1| + \cdots + |a_n| = N$ . Write

$$P_N = \{f(x) \in \mathbb{Z}[x] : n + |a_0| + |a_1| + \cdots + |a_n| = N\}$$

where  $f(x) = a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n$  with  $a_0 \neq 0$ , and

$$P = \bigcup_{N=1}^{\infty} P_N.$$

$P$  is the set of all polynomials over  $\mathbb{Z}$ .

Each  $P_N$  is finite for given  $N$  (since the equation  $n + |a_0| + |a_1| + \cdots + |a_n| = N$  has finitely many solutions  $(n, a_0, a_1, \dots, a_n) \in \mathbb{Z}^{n+2}$ ). So  $P$  is a countable union of finite sets, and hence at most countable.  $P$  is infinite since  $\mathbb{Z}$  is a subring of  $\mathbb{Z}[x]$ . So  $P$  is countable.  $\square$

*Proof (Theorem 2.13).*

- (1)  $\mathbb{Z}^N$  is countable for any integer  $N > 0$ . Theorem 2.13.
- (2) The set of all polynomials over  $\mathbb{Z}$  is countable. Let

$$P_n = \{f \in \mathbb{Z}[x] : \deg f = n\},$$

and

$$P = \bigcup_{n=1}^{\infty} P_n = \mathbb{Z}[x].$$

*Claim:*  $P_n$  is countable. Define a 1-1 map  $\varphi_n : P_n \rightarrow \mathbb{Z}^{n+1}$  by

$$\varphi_n(a_0 z^n + a_1 z^{n-1} + \cdots + a_n) = (a_0, a_1, \dots, a_{n-1}, a_n).$$

By (1) and Theorem 2.8,  $P_n$  is countable. ( $P_n$  is infinite since  $a_n \in \mathbb{Z}$ .) Now  $P$  is a countable union of countable sets, and hence countable by Theorem 2.12.

□

*Proof (Unique factorization theorem).*

- (1) *The set of prime numbers is countable.* Write all primes in the ascending order as  $p_1, p_2, \dots, p_n, \dots$  where  $p_1 = 2, p_2 = 3, \dots, p_{10001} = 104743, \dots$  (See ProjectEuler 7: 10001st prime. Use sieve of Eratosthenes to get  $p_{10001}$ .)
- (2) *The set of all polynomials over  $\mathbb{Z}$  is countable.* Let

$$P_n = \{f \in \mathbb{Z}[x] : \deg f = n\},$$

and

$$P = \bigcup_{n=1}^{\infty} P_n = \mathbb{Z}[x].$$

*Claim:*  $P_n$  is countable. Define a map  $\varphi_n : P_n \rightarrow \mathbb{Z}^+$  by

$$\varphi_n(a_0 z^n + a_1 z^{n-1} + \cdots + a_n) = p_1^{\psi(a_0)} p_2^{\psi(a_1)} \cdots p_{n+1}^{\psi(a_n)},$$

where  $\psi$  is a 1-1 correspondence from  $\mathbb{Z}$  to  $\mathbb{Z}^+$  (Example 2.5). By the unique factorization theorem,  $\varphi_n$  is 1-1. So  $P_n$  is countable by Theorem 2.8. ( $P_n$  is infinite since  $a_n \in \mathbb{Z}$ .) Now  $P$  is a countable union of countable sets, and hence countable by Theorem 2.12.

□

**Exercise 2.3.** *Prove that there exist real numbers which are not algebraic.*

*Proof (Exercise 2.2).* If all real numbers were algebraic, then  $\mathbb{R}$  is countable by Exercise 2.2, contrary to the fact that  $\mathbb{R}$  is uncountable (Corollary to Theorem 2.43). □

*Proof (Liouville, 1844).*

- (1) **Lemma.** *If  $\xi$  is a real algebraic number of degree  $n > 1$ , then there is a constant  $A > 0$  (depending on  $\xi$ ) such that*

$$\left| \xi - \frac{h}{k} \right| \geq \frac{A}{k^n}$$

*for all rational numbers  $\frac{h}{k}$ .*

- (a) If  $\left|\xi - \frac{h}{k}\right| \geq 1$ , pick  $A = 1 > 0$ .
- (b) If  $\left|\xi - \frac{h}{k}\right| < 1$ , let  $f(x) = a_0 + a_1x + \cdots + a_nx^n$  be an irreducible polynomial of degree  $n > 1$  over  $\mathbb{Z}$  such that  $f(\xi) = 0$ . By the mean value theorem,

$$f(\xi) - f\left(\frac{h}{k}\right) = \left(\xi - \frac{h}{k}\right) f'(c)$$

for some  $c \in \left(\xi - \frac{h}{k}, \xi + \frac{h}{k}\right) \subseteq (\xi - 1, \xi + 1)$ . Notice that

- (i)  $f(\xi) = 0$  by definition.
- (ii)  $f\left(\frac{h}{k}\right) \neq 0$  since  $\frac{h}{k}$  cannot be a root of  $f(x)$ . Otherwise  $f$  is of degree 1, contrary to the assumption of  $f$ .
- (iii)  $\left|f\left(\frac{h}{k}\right)\right| \geq \frac{1}{k^n}$  since

$$\begin{aligned} f\left(\frac{h}{k}\right) &= a_0 + a_1\left(\frac{h}{k}\right) + \cdots + a_n\left(\frac{h}{k}\right)^n \neq 0, \\ k^n f\left(\frac{h}{k}\right) &= a_0k^n + hk^{n-1}a_1 + \cdots + h^na_n \neq 0, \\ k^n \left|f\left(\frac{h}{k}\right)\right| &\geq 1. \end{aligned}$$

- (iv)  $|f'(c)| \leq \sup_{x \in [\xi-1, \xi+1]} |f'(x)|$  since  $c \in [\xi - 1, \xi + 1]$  and  $f'(x)$  is continuous or bounded on a compact set  $[\xi - 1, \xi + 1]$ .

By (i)-(iv),

$$\begin{aligned} \left|f(\xi) - f\left(\frac{h}{k}\right)\right| &= \left|\left(\xi - \frac{h}{k}\right) f'(c)\right|, \\ \frac{1}{k^n} &\leq \left|f\left(\frac{h}{k}\right)\right| = \left|\xi - \frac{h}{k}\right| |f'(c)| \leq \left|\xi - \frac{h}{k}\right| \cdot \sup_{x \in [\xi-1, \xi+1]} |f'(x)|. \end{aligned}$$

Pick  $A = (1 + \sup_{x \in [\xi-1, \xi+1]} |f'(x)|)^{-1} > 0$ .

By (a)(b), we arrange  $A = \min(1, (1 + \sup_{x \in [\xi-1, \xi+1]} |f'(x)|)^{-1}) > 0$  to fit the inequality.

- (2)  $\xi = \sum_{n=0}^{\infty} 10^{-n!}$  is transcendental.

- (a) Let  $k_j = 10^{j!}$ ,  $h_j = 10^{j!} \sum_{n=0}^j 10^{-n!}$ . Then

$$\left|\xi - \frac{h_j}{k_j}\right| = \sum_{n=j+1}^{\infty} 10^{-n!} < \sum_{n=(j+1)!}^{\infty} 10^{-n} = \frac{A_j}{k_j^j}$$

where  $A_j = \frac{10}{9} \cdot 10^{-j!}$ .

- (b) If  $\xi$  were a real algebraic number of degree  $d > 1$ , then by Lemma and (a),

$$\frac{A}{k_j^d} < \left| \xi - \frac{h_j}{k_j} \right| < \frac{A_j}{k_j^j} < \frac{A_j}{k_j^d}$$

for some  $A > 0$  and  $j \geq d$ , or  $0 < A < A_j$ . Since  $j$  is arbitrary,  $A_j \rightarrow 0$  as  $j \rightarrow \infty$ , contrary to  $A > 0$ .

- (c) If  $\xi$  were a real algebraic number of degree  $d = 1$ ,  $\xi = \frac{h}{k}$  is a rational number. So

$$\left| \xi - \frac{h_j}{k_j} \right| = \left| \frac{h}{k} - \frac{h_j}{k_j} \right| = \left| \frac{hk_j - kh_j}{kk_j} \right| \geq \left| \frac{1}{kk_j} \right| = \frac{|k|^{-1}}{k_j}$$

for all  $j$ . (It is impossible that  $hk_j - kh_j = 0$  or  $\frac{h}{k} = \frac{h_j}{k_j}$  since  $|\frac{h}{k} - \frac{h_j}{k_j}| = \sum_{n=j+1}^{\infty} 10^{-n!} > 0$  for all  $j$ .) Again by (a),

$$\frac{|k|^{-1}}{k_j} \leq \left| \xi - \frac{h_j}{k_j} \right| < \frac{A_j}{k_j^j} < \frac{A_j}{k_j},$$

or  $0 < |k|^{-1} < A_j$ . (Similar to (b).) Since  $j$  is arbitrary,  $A_j \rightarrow 0$  as  $j \rightarrow \infty$ , contrary to  $|k|^{-1} > 0$ .

□

**Exercise 2.4.** *Is the set of all irrational real numbers countable?*

*Proof (Reductio ad absurdum).* If  $\mathbb{R} - \mathbb{Q}$  were countable, then  $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} - \mathbb{Q})$  is countable (Theorem 2.12), contrary to the fact that  $\mathbb{R}$  is uncountable (Corollary to Theorem 2.43). □

*Proof (Exercise 2.18).* Exercise 2.18 provides some examples of uncountable subset  $E$  of irrational real numbers.

- (1) Let  $A$  be the set of all  $y \in [0, 1]$  whose decimal expansion contains only the digits 4 and 7. Let  $\xi = \sum_{n=0}^{\infty} 10^{-n!}$  and

$$E = \{y + \xi : y \in A\}.$$

- (2) Let  $E$  be a subset of Liouville numbers as

$$E = \left\{ \sum_{n=0}^{\infty} \frac{a_n}{10^{n!}} : a_n \in \{4, 7\} \right\}.$$

- (3) Let

$$E = \left\{ \sum_{n=1989}^{\infty} \frac{a_n}{n!} : a_n \in \{6, 4\} \right\}.$$

We can apply the same argument of Theorem 2.14 to prove that each  $E$  is uncountable. Then use Theorem 2.8 to get all irrational real numbers cannot be countable.  $\square$

**Exercise 2.5.** Construct a bounded set of real numbers with exactly three limit points.

*Proof (Exercise 2.12).* Let

$$K_p = \{p\} \cup \left\{ p + \frac{1}{n} : n \in \mathbb{Z}^+ \right\} \subseteq \mathbb{R}^1$$

be a compact set of  $\mathbb{R}^1$  with exactly one limit point  $p \in \mathbb{R}^1$  (Exercise 2.12). Then

$$K_{1989} \cup K_6 \cup K_4$$

is a compact set of  $\mathbb{R}^1$  with exactly three limit points  $1989, 6, 4 \in \mathbb{R}^1$ .  $\square$

**Exercise 2.6.** Let  $E'$  be the set of all limit points of a set  $E$ . Prove that  $E'$  is closed. Prove that  $E$  and  $\bar{E}$  have the same limit points. (Recall that  $\bar{E} = E \cup E'$ .) Do  $E$  and  $E'$  always have the same limit points?

*Proof.*

(1) Show that  $E'$  is closed.

(a) Use Definition 2.18 (d).

- (i) It suffices to show every limit point of  $E'$  is a limit point of  $E$ . Given a limit point  $p$  of  $E'$ , so that every open neighborhood  $U$  of  $p$  contains a point  $q_0 \neq p$  such that  $q_0 \in E'$ .
- (ii) Since  $q_0$  is a limit point of  $E$ , there is an open neighborhood  $V$  of  $q_0$  contains a point  $q \neq q_0$  such that  $q \in E$ , where

$$V = U \cap B\left(q_0; \frac{1}{2}d(p, q_0)\right) \subseteq U$$

( $B(x; r)$  is the open ball with center at  $x$  and radius  $r$ ).

- (iii) By the construction of  $V$ , for such open neighborhood  $U$  of  $p$ , there is  $q \neq p$  and  $q \in V \subseteq U$  and  $q \in E$ . That is,  $p$  is a limit point of  $E$ .

(b) Use Definition 2.18 (e).

- (i) To show  $E'$  is closed or  $X - E'$  is open, it suffices to show every point of  $X - E'$  is an interior point of  $X - E'$ .
- (ii) Given a point  $p \in X - E'$ , or  $p$  is not a limit point of  $E$ . There is an open neighborhood  $U$  of  $p$  contains no point  $q \neq p$  such that  $q \in E$ .

- (iii) To show  $U$  is an open neighborhood of  $p$  such that  $U \subseteq X - E'$ , it suffices to no point  $q \neq p$  such that  $q \in E'$ . If there were a limit point  $q$  of  $E$  such that  $q \neq p$  and  $q \in U$ , then

$$V = U \cap B\left(q; \frac{1}{2}d(p, q)\right) \subseteq U$$

is an open neighborhood of  $q$  contains no point of  $E$ , contrary to the assumption  $q \in E'$ . So  $U \subseteq X - E'$  is an open neighborhood of  $p \in X - E'$ .

- (2) Show that  $E' = \overline{E}'$ . It suffices to show  $E' \supseteq \overline{E}'$ . ( $E' \subseteq \overline{E}'$  holds trivially since  $E \subseteq \overline{E}$ ). Given a limit point  $p$  of  $\overline{E} = E \cup E'$ .

- (a)  $p$  is a limit point of  $E$ . Nothing to do.  
(b)  $p$  is a limit point of  $E'$ . Since  $p$  is a limit point of  $E'$  and  $E'$  is a closed set,  $p \in E'$ , or  $p$  is a limit point of  $E$ .

In any case,  $E' \supseteq \overline{E}'$ .

- (3)  $E$  and  $E'$  might not have the same limit points. Let

$$E = \left\{ \frac{1}{n} : n \in \mathbb{Z}^+ \right\} \subseteq \mathbb{R}^1.$$

Then  $E' = \{0\}$  and thus  $(E')' = \emptyset$ .

□

**Exercise 2.7.** Let  $A_1, A_2, A_3, \dots$  be subsets of a metric space.

- (a) If  $B_n = \bigcup_{i=1}^n A_i$ , prove that  $\overline{B_n} = \bigcup_{i=1}^n \overline{A_i}$ , for  $n = 1, 2, 3, \dots$   
(b) If  $B = \bigcup_{i=1}^\infty A_i$ , prove that  $\overline{B} \supseteq \bigcup_{i=1}^\infty \overline{A_i}$ .

Show, by an example, that this inclusion can be proper.

*Proof of (a).*

- (1) Show that  $\overline{B_n} \subseteq \bigcup_{i=1}^n \overline{A_i}$ . Since  $A_i \subseteq \overline{A_i}$  for any  $i$ , we have

$$B_n = \bigcup_{i=1}^n A_i \subseteq \bigcup_{i=1}^n \overline{A_i}.$$

Since  $\bigcup_{i=1}^n \overline{A_i}$  is a union of finitely many closed set  $\overline{A_i}$ ,  $\bigcup_{i=1}^n \overline{A_i}$  is closed (Theorem 2.24(d)). By Theorem 2.27(c),  $\overline{B_n} \subseteq \bigcup_{i=1}^n \overline{A_i}$ .

- (2) Show that  $\overline{B_n} \supseteq \bigcup_{i=1}^n \overline{A_i}$ . Same argument in the proof of (b).

□

*Proof of (b).* Since  $\bigcup_{j=1}^{\infty} A_j \supseteq A_i$  for any  $i$ , by the monotonicity of closure, we have  $\overline{\bigcup_{j=1}^{\infty} A_j} \supseteq \overline{A_i}$  for any  $i$ , or  $\overline{B} \supseteq \bigcup_{i=1}^{\infty} \overline{A_i}$ . □

*Proof of proper inclusion in (b).* Let

$$A_n = \left( \frac{1}{n}, \infty \right) \subseteq \mathbb{R}^1$$

for any  $n \in \mathbb{Z}^+$ . Then

$$\begin{aligned} \bigcup_{n=1}^{\infty} A_n = (0, \infty) &\implies \overline{\bigcup_{n=1}^{\infty} A_n} = \overline{(0, \infty)} = [0, \infty), \\ \overline{A_n} = \left[ \frac{1}{n}, \infty \right) &\implies \bigcup_{n=1}^{\infty} \overline{A_n} = \bigcup_{n=1}^{\infty} \left[ \frac{1}{n}, \infty \right) = (0, \infty). \end{aligned}$$

□

**Exercise 2.8.** *Is every point of every open set  $E \subseteq \mathbb{R}^2$  a limit point of  $E$ ? Answer the same question for closed sets in  $\mathbb{R}^2$ .*

It is not true for all metric spaces  $X$ . The (discrete) metric in Exercise 2.10 implies no limit point exists in  $X$ .

*Proof.*

- (1) *Show that for every open set  $E \subseteq \mathbb{R}^k$ ,  $E \subseteq E'$ . Given any point  $\mathbf{p} \in E$ , we shall show  $\mathbf{p}$  is a limit point of  $E$ .*
  - (a) Since  $E$  is open, there is an open neighborhood  $B(\mathbf{p}; r_0) \subseteq E$  for some  $r_0 > 0$ .
  - (b) *In particular, given any  $s \in \mathbb{R}$  such that  $0 < s < r_0$ , we can find*

$$\mathbf{q} \in B(\mathbf{p}; s) \subseteq B(\mathbf{p}; r_0) \subseteq E$$

*such that  $\mathbf{q} \neq \mathbf{p}$ . Explicitly, write*

$$\mathbf{p} = (p_1, \dots, p_k)$$

*and choose*

$$\mathbf{q} = \left( p_1 + \frac{s}{89}, p_2, \dots, p_k \right) \neq \mathbf{p}$$

*(since  $s > 0$ ). Clearly,  $\mathbf{q}$  is well-defined in  $\mathbb{R}^k$  and  $|\mathbf{q} - \mathbf{p}| = \frac{s}{89} < s$  or  $\mathbf{q} \in B(\mathbf{p}; s)$ .*



- (c) Now given every open neighborhood  $B(\mathbf{p}, r)$  of  $\mathbf{p}$ . We can choose  $s \in \mathbb{R}$  such that  $0 < s < \min\{r_0, r\} \leq r_0$ . (might pick  $s = \frac{1}{64} \min\{r_0, r\}$ .)  
By (b), there exists  $\mathbf{q} \neq \mathbf{p}$  such that

$$\mathbf{q} \in B(\mathbf{p}; s) \subseteq B(\mathbf{p}; r) \subseteq E.$$

- (2) Give an example of a closed set  $E \subseteq \mathbb{R}^k$  such that  $E \not\subseteq E'$ . Pick  $E = \{\mathbf{0}\}$ .  
So  $E' = \emptyset$  and thus  $E \not\subseteq E'$ .

□

**Exercise 2.9.** Let  $E^\circ$  denote the set of all interior points of a set  $E$ . [See Definition 2.18(e);  $E^\circ$  is called the interior of  $E$ .]

- (a) Prove that  $E^\circ$  is always open.
- (b) Prove that  $E$  is open if and only if  $E^\circ = E$ .
- (c) If  $G$  is contained in  $E$  and  $G$  is open, prove that  $G$  is contained in  $E^\circ$ .
- (d) Prove that the complement of  $E^\circ$  is the closure of the complement of  $E$ .
- (e) Do  $E$  and  $\overline{E}$  always have the same interiors?
- (f) Do  $E$  and  $E^\circ$  always have the same closures?

Similar to Theorem 2.27.

*Proof of (a).* It is equivalent to show that  $E^\circ \subseteq (E^\circ)^\circ$ .

- (1) Given any point  $x \in E^\circ$ , there is  $r > 0$  such that  $B(x; r) \subseteq E$ .
- (2) It suffices to show that  $B(x; \frac{2}{r}) \subseteq E^\circ$ . Given any point  $y \in B(x; \frac{2}{r})$ , we will show that there is an open neighborhood  $B(y; \frac{2}{r})$  of  $y$  such that  $B(y; \frac{2}{r}) \subseteq E$ .
- (3) Given any point  $z \in B(y; \frac{2}{r})$ , we have

$$d(z, x) \leq d(z, y) + d(y, x) < \frac{2}{r} + \frac{2}{r} = r,$$

or  $z \in B(x; r) \subseteq E$ . Therefore,  $B(y; \frac{2}{r}) \subseteq E$ , or  $y \in E^\circ$ , or  $B(x; \frac{2}{r}) \subseteq E^\circ$ , or  $x \in (E^\circ)^\circ$ , or  $E^\circ \subseteq (E^\circ)^\circ$ .

□

*Proof of (b).*

- (1) ( $\implies$ )(Definition 2.18) Since  $E$  is open, every point of  $E$  is an interior point of  $E$ . Hence  $E \subseteq E^\circ$ . Note that  $E^\circ \subseteq E$  is trivial, and thus  $E^\circ = E$ .

- (2) ( $\Leftarrow$ )(a) By (a),  $E = E^\circ$  is always open.
- (3) ( $\Leftarrow$ )(Definition 2.18) Every point of  $E$  is an interior point of  $E$  since  $E = E^\circ$ . Hence  $E$  is open by Definition 2.18(f).

□

*Proof of (c).*  $G \subseteq E$  implies  $G^\circ \subseteq E^\circ$ .  $G = G^\circ$  since  $G$  is open ((b)). Hence  $G = G^\circ \subseteq E^\circ$ , that is,  $E^\circ$  is the largest open set contained in  $E$ . (Similarly,  $\overline{E}$  is the smallest closed set containing  $E$ .) □

*Proof of (d).* Show that  $X - E^\circ = \overline{X - E}$  and  $(X - E)^\circ = X - \overline{E}$ .

- (1) (Theorem 2.27 and (c))

$$\begin{aligned}
X - E^\circ &= X - \bigcup_{\text{Open } V \subseteq E} V \\
&= \bigcap_{\text{Open } V \subseteq E} (X - V) \\
&= \bigcap_{\text{Closed } W \supseteq X - E} W \\
&= \overline{X - E}. \\
X - \overline{E} &= X - \bigcap_{\text{Closed } W \supseteq E} W \\
&= \bigcup_{\text{Closed } W \supseteq E} (X - W) \\
&= \bigcup_{\text{Open } V \subseteq X - E} V \\
&= (X - E)^\circ.
\end{aligned}$$

- (2) (Brute-force)

$$\begin{aligned}
x \in E^\circ &\iff \exists r > 0 \text{ such that } B(x; r) \subseteq E \\
&\iff \exists r > 0 \text{ such that } B(x; r) \cap (X - E) = \emptyset \\
&\iff x \notin \overline{X - E} \\
&\iff x \in X - \overline{X - E}. \\
x \in (X - E)^\circ &\iff \exists r > 0 \text{ such that } B(x; r) \subseteq (X - E) \\
&\iff \exists r > 0 \text{ such that } B(x; r) \cap E = \emptyset \\
&\iff x \notin \overline{E} \\
&\iff x \in X - \overline{E}.
\end{aligned}$$

Note that  $X - E^\circ = \overline{X - E}$  is equivalent to  $(X - E)^\circ = X - \overline{E}$  by mapping  $E \mapsto X - E$ . □

*Proof of (e).* No.

- (1) Let  $X = \mathbb{R}$  equipped with the Euclidean metric, and  $E = \mathbb{Q} \subseteq X$ .
- (2)  $E^\circ = \emptyset$  since  $\widetilde{\mathbb{Q}}$  is dense in  $\mathbb{R}$ .
- (3)  $(\overline{E})^\circ = (\mathbb{R})^\circ = \mathbb{R}$  since  $\mathbb{Q}$  is dense in  $\mathbb{R}$  and  $\mathbb{R}$  is open.

□

*Proof of (f).* No.

- (1) Let  $X = \mathbb{R}$  equipped with the Euclidean metric, and  $E = \mathbb{Q} \subseteq X$ .
- (2)  $\overline{E} = \mathbb{R}$  since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ .
- (3)  $\overline{E}^\circ = \overline{\emptyset} = \emptyset$  since  $\widetilde{\mathbb{Q}}$  is dense in  $\mathbb{R}$ .

□

**Exercise 2.10.** Let  $X$  be an infinite set. For  $p \in X$  and  $q \in X$ , define

$$d(p, q) = \begin{cases} 1 & (\text{if } p \neq q) \\ 0 & (\text{if } p = q). \end{cases}$$

*Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?*

(The statement holds even if  $X$  is finite.) We called  $d$  the discrete metric, and the corresponding topology on  $X$  induces the discrete topology. Conversely, if  $X$  has the discrete topology,  $X$  is always metrizable by the discrete metric.

*Proof.*

- (1)  $d(p, q)$  is a metric.
  - (a)  $d(p, q) > 0$  if  $p \neq q$ ;  $d(p, p) = 0$ . Trivial.
  - (b)  $d(p, q) = d(q, p)$ . Trivial.
  - (c)  $d(p, q) \leq d(p, r) + d(r, q)$  for any  $r \in X$ . If  $p = q$ , nothing to do. If  $p \neq q$ ,  $r \neq p$  or  $r \neq q$  for any  $r \in X$ . (Assume not true,  $r = p$  and  $r = q$  implies that  $p = q$  which is a contradiction.) In any case  $d(p, r) + d(r, q) \geq 1 = d(p, q)$ .
- (2) Every subset is open. Let  $E$  be any subset of  $X$ . Then every point  $p \in E$  is an interior point of  $E$ . In fact, we can pick one open neighborhood  $U = B(p; \frac{1}{2})$  of  $p$  containing only one point  $p \in E$  or  $U = \{p\}$ , and such open neighborhood  $U$  is a subset of  $E$ . So every subset of  $X$  is open.

- (3) *Every subset is closed.* Since every subset is open, every subset is closed by Theorem 2.23.

**Supplement.** Might use Definition 2.18 (d) to prove directly since there are no limit points in  $X$  if we consider one open neighborhood  $U = B(p; \frac{1}{2})$  of  $p$ . Therefore, every subset is closed. Again we apply Theorem 2.23 to get that every subset is open without using Definition 2.18 (f).

- (4) *A subset is compact iff it is finite.*

- (a) *Any finite subset is compact.* Say  $E = \{p_1, p_2, \dots, p_k\}$ , and  $\{G_\alpha\}$  be an open covering of  $E$ . From  $\{G_\alpha\}$  we pick  $G_{\alpha_1}$  containing  $p_1$ ,  $G_{\alpha_2}$  containing  $p_2$ , ..., and  $G_{\alpha_k}$  containing  $p_k$ . This process can be done in the finitely many steps. Therefore,

$$\{G_{\alpha_1}, G_{\alpha_2}, \dots, G_{\alpha_k}\}$$

is a finite subcovering of  $\{G_\alpha\}$  covering  $E$ .

- (b) *Any infinite subset is not compact.* Take a collection

$$\mathcal{G} = \{G_p = \{p\}\}$$

of open subsets where  $p$  runs all points in  $E$ . Clearly,  $\{G_p\}$  is an open covering. Assume

$$\mathcal{G}' = \{G_{p_1}, G_{p_2}, \dots, G_{p_k}\}$$

is any finite subcovering of  $\mathcal{G}$ . Since  $E$  is infinite, there exist a point  $p \in E$  such that  $p \neq p_1, p \neq p_2, \dots, p \neq p_k$ . Therefore,  $\mathcal{G}'$  does not cover  $p$ , or  $\mathcal{G}$  does not contain any finite subcovering  $\mathcal{G}'$ .

□

Notice that every subset is bounded. Therefore, every subset is closed and bounded, but only finite subset is compact, i.e., Heine-Borel theorem is not true in the infinite discrete topology.

**Exercise 2.11.** For  $x \in \mathbb{R}^1$  and  $y \in \mathbb{R}^1$ , define

$$\begin{aligned} d_1(x, y) &= (x - y)^2, \\ d_2(x, y) &= \sqrt{|x - y|}, \\ d_3(x, y) &= |x^2 - y^2|, \\ d_4(x, y) &= |x - 2y|, \\ d_5(x, y) &= \frac{|x - y|}{1 + |x - y|}. \end{aligned}$$

Determine, for each of these, whether it is a metric or not.

*Proof.*

- (1)  $d = d_1$  is not a metric. (Reductio ad absurdum) If  $d$  were a metric, then

$$d(0, 2) > d(0, 1) + d(1, 2),$$

contrary to Definition 2.15(c) that  $d(p, q) \leq d(p, r) + d(r, q)$ .

- (2)  $d = d_2$  is a metric. It suffices to show that  $d'(x, y) = \sqrt{d(x, y)}$  is a metric if  $d(x, y)$  is a metric. For any  $p, q, r \in \mathbb{R}^1$ ,

(a)  $d'(p, q) = \sqrt{d(p, q)} > 0$  if  $p \neq q$ ;  $d'(p, p) = \sqrt{d(p, p)} = 0$ .

(b)  $d'(p, q) = \sqrt{d(p, q)} = \sqrt{d(q, p)} = d'(q, p)$ .

(c)

$$\begin{aligned} \sqrt{d(p, r) + d(r, q)} &\leq \sqrt{d(p, r)} + \sqrt{d(r, q)} \\ \iff (\sqrt{d(p, r) + d(r, q)})^2 &\leq (\sqrt{d(p, r)} + \sqrt{d(r, q)})^2 \\ \iff d(p, r) + d(r, q) &\leq d(p, r) + d(r, q) + 2\sqrt{d(p, r)}\sqrt{d(r, q)} \\ \iff 0 &\leq 2\sqrt{d(p, r)}\sqrt{d(r, q)}. \end{aligned}$$

(d)

$$\begin{aligned} d'(p, q) &= \sqrt{d(p, q)} \\ &\leq \sqrt{d(p, r) + d(r, q)} && \text{(Triangle inequality)} \\ &\leq \sqrt{d(p, r)} + \sqrt{d(r, q)} && ((c)) \\ &= d'(p, r) + d'(r, q). \end{aligned}$$

By Definition 2.15,  $d'$  is a metric.

- (3)  $d = d_3$  is not a metric. (Reductio ad absurdum) If  $d$  were a metric, then

$$d(1, -1) = 0,$$

contrary to Definition 2.15(a):  $d(p, q) > 0$  if  $p \neq q$ ;  $d(p, p) = 0$ .

- (4)  $d = d_4$  is not a metric. (Reductio ad absurdum) If  $d$  were a metric, then

$$d(1, 1) = 1,$$

contrary to Definition 2.15(a):  $d(p, q) > 0$  if  $p \neq q$ ;  $d(p, p) = 0$ .

- (5)  $d = d_5$  is a metric. It suffices to show that  $d'(x, y) = \frac{d(x, y)}{1+d(x, y)}$  is a metric if  $d(x, y)$  is a metric. For any  $p, q, r \in \mathbb{R}^1$ ,

- (a)  $d'(p, q) = \frac{d(p, q)}{1+d(p, q)} > 0$  if  $p \neq q$ ;  $d'(p, p) = \frac{d(p, p)}{1+d(p, p)} = 0$ .  
(b)  $d'(p, q) = \frac{d(p, q)}{1+d(p, q)} = \frac{d(q, p)}{1+d(q, p)} = d'(q, p)$ .  
(c) Write  $x = d(p, q)$ ,  $y = d(p, r)$  and  $z = d(r, q)$ . So  $x, y, z \geq 0$  and

$$\begin{aligned} x &\leq y + z \\ \iff x + x(y + z) &\leq y + z + x(y + z) \\ \iff x(1 + y + z) &\leq (1 + x)(y + z) \\ \iff \frac{x}{1 + x} &\leq \frac{y + z}{1 + y + z}. \end{aligned}$$

(d)

$$\begin{aligned} d'(p, q) &= \frac{d(p, q)}{1 + d(p, q)} \\ &\leq \frac{d(p, r) + d(r, q)}{1 + d(p, r) + d(r, q)} && ((c)) \\ &= \frac{d(p, r)}{1 + d(p, r) + d(r, q)} + \frac{d(r, q)}{1 + d(p, r) + d(r, q)} \\ &= \frac{d(p, r)}{1 + d(p, r)} + \frac{d(r, q)}{1 + d(r, q)} \\ &= d'(p, r) + d'(r, q). \end{aligned}$$

(e) Or we can show  $d'(p, q) \leq d'(p, r) + d'(r, q)$  by

$$\begin{aligned} \frac{x}{1 + x} &\leq \frac{y}{1 + y} + \frac{z}{1 + z} \\ \iff x(1 + y)(1 + z) &\leq y(1 + z)(1 + x) + z(1 + x)(1 + y) \\ \iff x + xy + xz + xyz & \\ &\leq (y + xy + yz + xyz) + (z + xz + yz + xyz) \\ \iff x &\leq y + z + 2yz + xyz \\ \iff x &\leq y + z && (d \text{ is nonnegative}) \end{aligned}$$

By Definition 2.15,  $d'$  is a metric.

□

**Exercise 2.12.** Let  $K \subseteq \mathbb{R}^1$  consist of 0 and the numbers  $\frac{1}{n}$ , for  $n = 1, 2, 3, \dots$ . Prove that  $K$  is compact directly from the definition (without using the Heine-Borel theorem).

*Proof.* Let  $\{G_\alpha\}$  be an open covering of  $K$ . There is an open set  $G_0 \in \{G_\alpha\}$  containing 0. So there exists an open neighborhood  $U = B(0; r)$  of 0 such that

$U \subseteq G_0$ . So  $U$  contains all points  $q = \frac{1}{n}$  of  $K$  whenever  $n > \frac{1}{r}$ . To construct a finite subcovering of  $\{G_\alpha\}$ , we need to pick finitely many open sets from  $\{G_\alpha\}$  to cover the remaining points  $q = \frac{1}{n}$  where  $n = 1, 2, \dots, [\frac{1}{r}]$ , say  $G_1$  contains  $q = \frac{1}{1}$ ,  $G_2$  contains  $q = \frac{1}{2}$ , ...,  $G_{[\frac{1}{r}]}$  contains  $q = \frac{1}{[\frac{1}{r}]}$ . (Might be duplicated.) Hence,

$$\left\{G_0, G_1, G_2, \dots, G_{[\frac{1}{r}]}\right\}$$

is a finite subcovering of  $\{G_\alpha\}$  covering  $K$ .  $\square$

*Proof (Heine-Borel theorem).*

- (1)  $K$  is closed. In fact, the only limit point of  $K$  is 0, which is in  $K$ .
  - (a)  $p = 0$  is a limit point. Given  $r > 0$ . There always exists  $n \in \mathbb{Z}^+$  such that  $r > \frac{1}{n}$ . So any open neighborhood  $B(0; r)$  of  $p = 0$  contains at least one point  $q = \frac{1}{n} \neq 0$  in  $K$ .
  - (b)  $p < 0$  is not a limit point. Pick an open neighborhood  $B(p; r)$  of  $p$  where  $r = |p| > 0$ . Then  $B(p; r) \cap K = \emptyset$ .
  - (c)  $p > 0$  is not a limit point. There always exists  $m \in \mathbb{Z}^+$  such that  $p > \frac{1}{n}$  whenever  $n \geq m$ . Pick an open neighborhood  $B(p; r)$  of  $p$  where  $r = p - \frac{1}{m} > 0$ . Then  $B(p; r)$  does not have all points  $q = \frac{1}{n} \in K$  whenever  $n \geq m$ . By Theorem 2.20,  $p$  cannot be a limit point of  $K$ .
- (2)  $K$  is bounded. There is a real number  $M = 2$  and a point  $q = 0 \in \mathbb{R}^1$  such that  $|p - q| = |p| < 2$  for all  $p \in K$ .

By Heine-Borel theorem,  $K$  is compact in  $\mathbb{R}^1$ .  $\square$

**Exercise 2.13.** Construct a compact set of real numbers whose limit points form a countable set.

*Proof (Exercise 2.12).* Let  $K(p; r) \subseteq \mathbb{R}^1$  be

$$K(p; r) = \left\{p + \frac{r}{n} : n = 2, 3, \dots\right\} \cup \{p\}$$

and

$$K = \left(\bigcup_{i=0}^{\infty} K(2^{-i}; 2^{-i})\right) \cup \{0\}.$$

- (1) The set of limit points of  $K$  is  $K' = \{2^{-i} : i = 0, 1, 2, \dots\} \cup \{0\}$ , which is (infinitely) countable.
  - (a) The unique limit point of  $K(2^{-i}; 2^{-i})$  is  $2^{-i}$  for each  $i = 0, 1, 2, \dots$  (Exercise 2.12).

- (b) 0 is a limit point of  $K$ .
  - (c) No other limit points of  $K$ . Similar to the argument of the proof of Exercise 2.12.
- (2)  $K$  is closed. All limit points are in  $K$ .
- (3)  $K$  is bounded. There is a real number  $M = 2$  and a point  $q = 0 \in \mathbb{R}^1$  such that  $|p - q| = |p| < 2$  for all  $p \in K$ .

By Heine-Borel theorem,  $K$  is compact in  $\mathbb{R}^1$ , and has infinitely countable limit points.  $\square$

**Exercise 2.14.** Give an example of an open cover of the segment  $(0, 1)$  which has no finite subcover.

*Proof.* In  $\mathbb{R}^1$ , take a collection

$$\mathcal{G} = \left\{ G_n = \left( \frac{1}{n}, 1 \right) \right\}$$

of open subsets where  $n \in \mathbb{Z}^+$ .

- (1)  $\mathcal{G}$  is an open covering of  $(0, 1) \subseteq \mathbb{R}^1$ . Actually, given  $x \in (0, 1)$ , there exists a positive integer  $n$  such that  $x > \frac{1}{n}$ . That is,  $x \in (\frac{1}{n}, 1) = G_n$ .
- (2) There is no finite subcovering of  $\mathcal{G}$ . Assume

$$\mathcal{G}' = \{G_{n_1}, G_{n_2}, \dots, G_{n_k}\}$$

is any finite subcovering of  $\mathcal{G}$  where  $n_1 < n_2 < \dots < n_k$ . Take  $x \in \left(0, \frac{1}{n_k}\right) \neq \emptyset$ ,  $x = \frac{1}{2n_k}$  for example. Then  $x \notin G_{n_1}$ ,  $x \notin G_{n_2}, \dots, x \notin G_{n_k}$ , which contradicts that  $\mathcal{G}'$  is a finite subcovering of  $\mathcal{G}$  covering  $(0, 1)$ .

$\square$

**Exercise 2.15.** Show that Theorem 2.36 and its Corollary become false (in  $\mathbb{R}^1$ , for example) if the word “compact” is replaced by “closed” or by “bounded.”

*Recall:*

- (1) Theorem 2.36: If  $\{K_\alpha\}$  is a collection of compact subsets of a metric space  $X$  such that the intersection of every finite subcollection of  $\{K_\alpha\}$  is nonempty, then  $\bigcap K_\alpha$  is nonempty.
- (2) Corollary: If  $\{K_n\}$  is a sequence of nonempty compact sets such that  $K_n$  contains  $K_{n+1}$  ( $n = 1, 2, 3, \dots$ ), then  $\bigcap K_n$  is not empty.



*Proof.* Let  $X = \mathbb{R}^1$  with the usual Euclidean metric.

(1) For the closeness, let  $K_n = [n, \infty) \subseteq X$ .

(2) For the boundedness, let  $K_n = (0, \frac{1}{n}) \subseteq X$ .

In any case,  $K_1 \supseteq K_2 \supseteq \cdots$  and  $\bigcap K_n = \emptyset$ .  $\square$

**Exercise 2.16.** Regard  $\mathbb{Q}$ , the set of all rational numbers, as a metric space, with  $d(p, q) = |p - q|$ . Let  $E$  be the set of all  $p \in \mathbb{Q}$  such that  $2 < p^2 < 3$ . Show that  $E$  is closed and bounded in  $\mathbb{Q}$ , but that  $E$  is not compact. Is  $E$  open in  $\mathbb{Q}$ ?

**Lemma.** Assume  $S \subseteq T \subseteq M$ . Then  $S$  is compact in  $(M, d)$  if, and only if,  $S$  is compact in the metric subspace  $(T, d)$ .

*Proof of Lemma.*

(1) ( $\implies$ ) Let  $\mathcal{F}$  be an open covering of  $S$  in  $(T, d)$ , say  $S \subseteq \bigcup_{A \in \mathcal{F}} A$  where  $A$  is open in  $T$ . Then  $A = B \cap T$  for some open set  $B$  in  $M$  (Theorem 3.33). Let  $\mathcal{G}$  be the collection of  $B$ . Then

$$S \subseteq \bigcup_{A \in \mathcal{F}} A = \bigcup_{B \in \mathcal{G}} (B \cap T) \subseteq \bigcup_{B \in \mathcal{G}} B,$$

or  $\mathcal{G}$  be an open covering of  $S$  in  $(M, d)$ . Since  $S$  is compact in  $(M, d)$ ,  $\mathcal{G}$  contains a finite subcovering, say

$$S \subseteq B_1 \cap \cdots \cap B_p.$$

So

$$S \cap T \subseteq (B_1 \cap T) \cap \cdots \cap (B_p \cap T),$$

or

$$S \subseteq A_1 \cap \cdots \cap A_p$$

(since  $S \subseteq T$  or  $S \cap T = S$ ). So there is a finite subcovering of  $\mathcal{F}$  covering  $S$ , or  $S$  is compact in  $(T, d)$ .

(2) ( $\impliedby$ ) Let  $\mathcal{G}$  be an open covering of  $S$  in  $(M, d)$ , say  $S \subseteq \bigcup_{B \in \mathcal{G}} B$  where  $B$  is open in  $M$ . Then  $A = B \cap T$  is open in  $T$ . Let  $\mathcal{F}$  be the collection of  $A$ . Then

$$S \cap T \subseteq \bigcup_{B \in \mathcal{G}} (B \cap T) = \bigcup_{A \in \mathcal{F}} A,$$

or  $\mathcal{F}$  be an open covering of  $S \cap T = S$  in  $(T, d)$ . Since  $S$  is compact in  $(T, d)$ ,  $\mathcal{F}$  contains a finite subcovering, say

$$S \subseteq A_1 \cap \cdots \cap A_p.$$

Clearly,  $S \subseteq B_1 \cap \cdots \cap B_p$  since  $A = B \cap T \subseteq B$ . So there is a finite subcovering of  $\mathcal{G}$  covering  $S$ , or  $S$  is compact in  $(M, d)$ .

□

*Proof.* Write  $E_0 = (\sqrt{2}, \sqrt{3}) \cup (-\sqrt{3}, -\sqrt{2})$ , and  $E = E_0 \cap \mathbb{Q}$ .

- (1)  $E$  is a subset of  $\mathbb{Q}$ .
- (2) *Show that  $E$  is bounded in  $\mathbb{Q}$ .* Since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , there is  $p \in \mathbb{Q}$  such that  $\sqrt{2} < p < \sqrt{3}$ , or  $p \in E$ . Let  $r = p + \sqrt{3} > 0$ . Therefore,  $E \subseteq B(p; r)$  for some  $r > 0$  and  $p \in E$ , or  $E$  is bounded.
- (3) *Show that  $E$  is closed in  $\mathbb{Q}$ .* It suffices to show its complement is open in  $\mathbb{Q}$ . Given any

$$p \in \tilde{E} = ((-\infty, -\sqrt{3}] \cup [-\sqrt{2}, \sqrt{2}] \cup [\sqrt{3}, \infty)) \cap \mathbb{Q}.$$

$$p \leq -\sqrt{3} \text{ or } -\sqrt{2} \leq p \leq \sqrt{2} \text{ or } p \geq \sqrt{3}.$$

- (a)  $p \leq -\sqrt{3}$ .  $p \neq -\sqrt{3}$  since  $p \in \mathbb{Q}$  and  $-\sqrt{3}$  is irrational. So  $p < -\sqrt{3}$  and thus there exists  $q \in \mathbb{Q}$  such that  $p < q < -\sqrt{3}$  since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ . Let  $r = \max\{-\sqrt{3} - q, q - p\} > 0$ . The ball  $B(q; r)$  is contained in  $\tilde{E}$ .
- (b)  $-\sqrt{2} \leq p \leq \sqrt{2}$ . Similar to (a).
- (c)  $p \geq \sqrt{3}$ . Similar to (a).

By (a)(b),  $\tilde{E}$  is open in  $\mathbb{Q}$ , or  $E$  is closed in  $\mathbb{Q}$ .

- (4) *Show that  $E$  is not compact in  $\mathbb{Q}$ .* (Reductio ad absurdum) If  $E_0$  were compact in the metric space  $\mathbb{Q}$ ,  $E_0$  is compact in the metric space  $\mathbb{R}$  (Lemma), which is absurd.
- (5) *Show that  $E$  is open.* Similar to (3).

□

**Exercise 2.17.** Let  $E$  be the set of all  $x \in [0, 1]$  whose decimal expansion contains only the digits 4 and 7. Is  $E$  countable? Is  $E$  dense in  $[0, 1]$ ? Is  $E$  compact? Is  $E$  perfect?

*Proof.*

- (1) *Show that  $E$  is uncountable.* Same as Theorem 2.14. Or show that  $E$  is perfect and then apply Theorem 2.43.
- (2) *Show that  $E$  is not dense in  $[0, 1]$ .* Note that  $E \subseteq [\frac{4}{9}, \frac{7}{9}]$ . So

$$B\left(0; \frac{1}{64}\right) \cap E \subseteq B\left(0; \frac{1}{64}\right) \cap \left[\frac{4}{9}, \frac{7}{9}\right] = \emptyset$$

or 0 is not a limit point of  $E$ . Hence  $E$  is not dense in  $[0, 1]$ .

(3) Show that  $E$  is compact. It is equivalent to show that  $E$  is closed and bounded (Theorem 2.41). Let a decimal expansion of  $x \in (0, 1)$  be  $0.x_1x_2\cdots$ .

- (a) Show that  $\tilde{E}$  is open. Since  $E \subseteq [\frac{4}{9}, \frac{7}{9}]$ , it suffices to show that every point  $x \in (0, 1) \cap \tilde{E}$  is an interior point of  $\tilde{E}$ . Say a decimal expansion of  $x$  containing at least one digit  $x_n \neq 4, 7$ . Note that

$$|x - y| \geq 10^{-n} > 0$$

for any  $y = 0.y_1y_2\cdots \in E$ . Hence there is an open neighborhood  $B(x; 10^{-n})$  of  $x$  such that  $B(x; 10^{-n}) \cap E = \emptyset$ , or  $B(x; 10^{-n}) \subseteq \tilde{E}$ , or  $x$  is an interior point of  $\tilde{E}$ .

- (b) Show that  $E$  is closed. Given any limit point  $x \in \mathbb{R}^1$  of  $E$ , we want to show that  $x \in E$ . (Reductio ad absurdum) Similar to (a).

- (c) Show that  $E$  is bounded.  $E \subseteq B(0; 1)$ .

(4) Show that  $E$  is perfect.

- (a)  $E$  is closed (by (3)).

- (b) Show that every point of  $E$  is a limit point of  $E$ . Given any  $x \in E$ . Given any open neighborhood  $B(x; r)$  of  $x$ , there is a positive integer  $n$  such that

$$\frac{3}{10^n} < r.$$

For such  $n$ , pick  $y = 0.x_1x_2\cdots x_{n-1}y_n\cdots x_{n+1}\cdots \in E$  where

$$y_n = \begin{cases} 4 & (x_n = 7), \\ 7 & (x_n = 4). \end{cases}$$

$y \neq x$ , and  $|y - x| = \frac{3}{10^n} < r$ . So that there is  $y \neq x$  such that  $y \in B(x; r)$ , or  $x$  is a limit point of  $E$ .

□

**Exercise 2.18.** Is there a nonempty perfect set in  $\mathbb{R}^1$  which contains no rational number?

Yes.

**Lemma.**  $x \in \mathbb{Q}$  if and only if has repeating decimal expansion.

*Proof of Lemma.*

- (1) ( $\Leftarrow$ ) Given any repeating decimal

$$x = x_0.x_1x_2\cdots x_n\overline{x_{n+1}\cdots x_{n+m}}$$

where  $x_0 \in \mathbb{Z}$  and  $x_1, \dots, x_{n+m} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ . Thus  $x = p/q$  where

$$p = (10^m - 1) \sum_{i=0}^n 10^{n-i} x_i + \sum_{j=1}^m 10^{m-j} x_{n+j} \in \mathbb{Z}$$

and

$$q = 10^n(10^m - 1) \in \mathbb{Z}.$$

(2) ( $\implies$ ) (Euler's totient function) Given any  $x = p/q$  where  $p, q \in \mathbb{Z}$ ,  $q > 0$ .

- (a) Write  $q = 2^a 5^b q_1$  where  $a, b$  are nonnegative integers and  $(q_1, 10) = 1$  (Unique factorization theorem).
- (b) Let  $n = \max\{a, b\}$ . Then  $2^{n-a} 5^{n-b} q = 10^n q_1$ .
- (c) Since  $(q_1, 10) = 1$ ,  $10^m \equiv 1 \pmod{q_1}$  where  $m = \varphi(q_1)$  is Euler's totient function of  $q_1$ . Hence  $10^m - 1 = q_1 q_2$  for some  $q_2 \in \mathbb{Z}$ , or

$$2^{n-a} 5^{n-b} q_2 q = 10^n (10^m - 1).$$

Here  $2^{n-a} 5^{n-b} q_2$ ,  $n, m$  are nonnegative integers.

(d) Now write

$$x = \frac{p}{q} = \frac{2^{n-a} 5^{n-b} q_2 p}{10^n (10^m - 1)} = \frac{(10^m - 1) q_3 + r}{10^n (10^m - 1)} = \frac{q_3}{10^n} + \frac{r}{10^n (10^m - 1)}$$

where  $q_3, r \in \mathbb{Z}$  with  $0 \leq r < 10^m - 1$ . Might assume  $q_3 \geq 0$ . (If  $q_3 < 0$ , apply the same argument to  $-q_3$  and then add the minus symbol “−” in the front of a decimal expansion.) Hence

$$x = x_0.x_1x_2\cdots x_n\overline{x_{n+1}\cdots x_{n+m}}$$

where

$$\begin{aligned} x_0 &= \left\lfloor \frac{q_3}{10^n} \right\rfloor \\ x_i &= \text{last digit of } \left\lfloor \frac{q_3}{10^{n-i}} \right\rfloor & (1 \leq i \leq n) \\ x_{n+j} &= \text{last digit of } \left\lfloor \frac{r}{10^{m-j}} \right\rfloor & (1 \leq j \leq m) \end{aligned}$$

(3) ( $\implies$ ) (Pigeonhole principle) Given any  $x = p/q$  where  $p, q \in \mathbb{Z}$ ,  $q > 0$ .

- (a) Might assume  $p \geq 0$ . (If  $p < 0$ , apply the same argument to  $-p$  and then add the minus symbol “−” in the front of the decimal expansion.) Write

$$x = x_0.x_1x_2\cdots$$

(b) Apply Euclidean algorithm to get

$$p = x_0q + r_0 \quad \text{with} \quad 0 \leq r_0 < q.$$

$x_0$  is the integer part of  $x = p/q$ . Continue Euclidean algorithm to get  $x_1$  by

$$10r_0 = x_1q + r_1 \quad \text{with} \quad 0 \leq r_1 < q.$$

In general, for  $n \geq 1$ ,  $x_n$  is given by

$$10r_{i-1} = x_iq + r_i \quad \text{with} \quad 0 \leq r_i < q.$$

(c) The pigeonhole principle shows that there must be two equal remainders, that is,

$$r_n = r_{n+m} \quad \text{with} \quad m > 0.$$

By induction,  $r_{n+k} = r_{n+m+k}$  for any  $k \geq 0$ . Thus  $x_{n+k} = x_{n+m+k}$  holds for any  $k > 0$ , that is,  $x$  has a decimal expansion

$$x = x_0.x_1x_2 \cdots x_n \overline{x_{n+1} \cdots x_{n+m}}.$$

□

*Proof (Exercise 2.17).* Let  $A$  be the set of all  $y \in [0, 1]$  whose decimal expansion contains only the digits 4 and 7. Though  $A \cap \mathbb{Q} \neq \emptyset$  since  $\frac{4}{9} \in A$ , we can shift  $A$  by a number  $\xi = \sum_{n=0}^{\infty} 10^{-n!}$  (Exercise 2.3), that is, we construct

$$E = \{y + \xi : y \in A\}$$

and show that  $E$  is our desired nonempty perfect set in  $\mathbb{R} - \mathbb{Q}$ .

- (1) Any number  $x \in E$  has decimal expansion  $x = 0.x_1x_2 \cdots$  with  $x_n \in \{5, 8\}$  if  $n$  is a factorial number; otherwise  $x_n \in \{4, 7\}$ .
- (2)  $E$  is a perfect set (Exercise 2.17).
- (3)  $E \subseteq \mathbb{R} - \mathbb{Q}$ . It suffices to show that each  $x \in E$  has no repeating decimal expansions (Lemma). It is clear by the construction of  $\xi = \sum_{n=0}^{\infty} 10^{-n!}$ .

□

*Proof (Exercise 2.3).* Let  $E$  be a subset of Liouville numbers as

$$E = \left\{ \sum_{n=0}^{\infty} \frac{a_n}{10^{n!}} : a_n \in \{4, 7\} \right\}.$$

$E$  is perfect. (The same argument of Exercise 2.17.) Besides, all numbers of  $E$  are transcendental. (Set  $k_j = 10^{j!}$  and  $h_j = 10^{j!} \sum_{n=0}^j \frac{a_n}{10^{n!}}$  and apply the same argument of Exercise 2.3.) □

*Note.* Or using Lemma to prove all numbers of  $E$  are irrational.

*Proof (Theorem 3.32).* Let

$$E = \left\{ \sum_{n=1989}^{\infty} \frac{a_n}{n!} : a_n \in \{6, 4\} \right\}.$$

$E$  is perfect. (The same argument of Exercise 2.17.) Besides, all numbers of  $E$  are irrational (The same argument of Theorem 3.32.)  $\square$

*Proof (Non constructive existence proof).* By Cantor-Bendixson theorem (Exercise 2.28), it suffices to find a uncountable closed set in  $\mathbb{R} - \mathbb{Q}$ .

(1) Write  $\mathbb{Q} = \{r_1, r_2, \dots\}$  since  $\mathbb{Q}$  is countable. Let

$$I_n = B\left(r_n; \frac{1}{2^{n+1}}\right) \supseteq \{r_n\}$$

and

$$A = \bigcup_{n=1}^{\infty} I_n \supseteq \mathbb{Q}.$$

Hence  $A$  is an open subset in  $\mathbb{R}$ .

(2) Let  $E = \mathbb{R} - A$ . By construction,  $E$  is closed and  $E \cap \mathbb{Q} = \emptyset$ .

(3) *Show that  $E$  is uncountable. It suffices to show that  $m^*(E) > 0$ .* In fact, the outer measure of  $U$  is

$$m^*(A) \leq \sum_{n=1}^{\infty} m^*(I_n) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Thus,

$$m^*(E) \geq m^*(\mathbb{R}) - m^*(A) = \infty - 1 = \infty.$$

Hence, the set of all condensation points of  $E$  is our desired nonempty perfect set in  $\mathbb{R} - \mathbb{Q}$ .  $\square$

*Note.* In fact, we can replace  $\mathbb{Q}$  by the set of all real algebraic numbers (Exercise 2.2).

### Exercise 2.19.

- (a) *If  $A$  and  $B$  are disjoint closed sets in some metric space  $X$ , prove that they are separated.*
- (b) *Prove the same for disjoint open sets.*

- (c) Fix  $p \in X$ ,  $\delta > 0$ , define  $A$  to be the set of all  $q \in X$  for which  $d(p, q) < \delta$ , define  $B$  similarly, with  $>$  in place of  $<$ . Prove that  $A$  and  $B$  are separated.
- (d) Prove that every connected metric space with at least two points is uncountable. Hint: Use (c).

*Proof of (a).* Since

$$\begin{aligned} A \cap \overline{B} &= A \cap B && (B \text{ is closed}) \\ &= \emptyset, && (A \text{ and } B \text{ are disjoint}) \\ \overline{A} \cap B &= A \cap B && (A \text{ is closed}) \\ &= \emptyset. && (A \text{ and } B \text{ are disjoint}) \end{aligned}$$

$A$  and  $B$  are separated.  $\square$

*Proof of (b)(Theorem 2.27(c)).* Note that  $\tilde{A}$  is a closed set containing  $B$ . Since  $\overline{B}$  is the smallest closed set containing  $B$ ,  $\tilde{A} \supseteq \overline{B}$  (Theorem 2.27(c)). Hence

$$A \cap \overline{B} \subseteq A \cap \tilde{A} = \emptyset.$$

Similarly,  $\overline{A} \cap B = \emptyset$ . Hence  $A$  and  $B$  are separated.  $\square$

*Proof of (c).* Since both

$$A = \{q \in X : d(p, q) < \delta\} \text{ and } B = \{q \in X : d(p, q) > \delta\}$$

are open in  $X$ , they are separated by (b).  $\square$

*Proof of (d).* Let  $X$  be a connected metric space.

- (1) Let  $p, q \in X$  with  $p \neq q$ . Hence  $d_X(p, q) = r > 0$  (Definition 2.15(a)).
- (2) Given any  $\delta \in (0, r)$ . Define

$$A = \{x \in X : d(p, x) < \delta\} \text{ and } B = \{x \in X : d(p, x) > \delta\}.$$

$$p \in A \neq \emptyset \text{ and } q \in B \neq \emptyset.$$

- (3) If there were no  $y_\delta \in X$  such that  $d(p, y_\delta) = \delta$ , we can write  $X = A \cup B$  as a union of two nonempty separated sets ((c)), contrary to the connectedness of  $X$ .
- (4) Collect these  $y$  as  $E$ . Since  $d$  is a function, there is a one-to-one map from  $(0, r)$  to  $E$  defined by  $\delta \mapsto y_\delta$  in (3). Since  $(0, r)$  is uncountable,  $X \supseteq E$  is uncountable.

□

**Exercise 2.20.** *Are closures and interiors of connected sets always connected? (Look at subsets of  $\mathbb{R}^2$ .)*

*Proof.*

- (1) *Interiors of connected sets are not always connected.* Let  $X = \mathbb{R}^2$  with the usual Euclidean metric be a metric space. Take

$$E = B(89; 1) \bigcup B(64; 1) \bigcup \{(x, 0) \in \mathbb{R}^2 : 64 \leq x \leq 89\}.$$

$E$  is connected and

$$E^\circ = B(89; 1) \bigcup B(64; 1)$$

is disconnected.

- (2) *Closures of connected sets are always connected. It suffices to show that  $E$  is disconnected if  $\overline{E}$  is disconnected.*

- (a) Write  $\overline{E} = A \cup B$  as a union of two nonempty separated sets. Here  $A \neq \emptyset$ ,  $B \neq \emptyset$ ,  $A \cap \overline{B} = \emptyset$  and  $\overline{A} \cap B = \emptyset$ .

- (b) Write

$$E = (A \cap E) \bigcup (B \cap E)$$

and we will show that  $E$  is disconnected.

- (c) *Show that  $A \cap E$  and  $B \cap E$  are separated.* In fact,

$$(A \cap E) \cap \overline{B \cap E} \subseteq A \cap \overline{B} = \emptyset,$$

$$\overline{A \cap E} \cap (B \cap E) \subseteq \overline{A} \cap B = \emptyset.$$

- (d) *Show that  $A \cap E$  and  $B \cap E$  are nonempty.* (Reductio ad absurdum)  
If  $A \cap E = \emptyset$ , then

$$E = (A \cap E) \bigcup (B \cap E) = B \cap E \implies E \subseteq B.$$

So

$$\begin{aligned} A &= (A \cup B) \bigcap A && (A \subseteq A \cup B) \\ &= \overline{E} \bigcap A \\ &\subseteq \overline{B} \bigcap A && (E \subseteq B) \\ &= \emptyset \end{aligned}$$

which contradicts  $A \neq \emptyset$  in (a). Therefore,  $A \cap E \neq \emptyset$ . Similarly,  $B \cap E \neq \emptyset$ .



Hence,  $E$  is disconnected if  $\overline{E}$  is disconnected, or closures of connected sets are always connected.

□

**Exercise 2.21.** Let  $A$  and  $B$  be separated subsets of some  $\mathbb{R}^k$ , suppose  $\mathbf{a} \in A$ ,  $\mathbf{b} \in B$ , and define

$$\mathbf{p}(t) = (1-t)\mathbf{a} + t\mathbf{b}$$

for  $t \in \mathbb{R}^1$ . Put  $A_0 = \mathbf{p}^{-1}(A)$ ,  $B_0 = \mathbf{p}^{-1}(B)$ . [Thus  $t \in A_0$  if and only if  $\mathbf{p}(t) \in A$ .]

- (a) Prove that  $A_0$  and  $B_0$  are separated subsets of  $\mathbb{R}^1$ .
- (b) Prove that there exists  $t_0 \in (0, 1)$  such that  $\mathbf{p}(t_0) \notin A \cup B$ .
- (c) Prove that every convex subset of  $\mathbb{R}^k$  is connected.

*Proof of (a).*

(1) Note that

- (a)  $\mathbf{a} \neq \mathbf{b}$  or  $|\mathbf{a} - \mathbf{b}| > 0$  since  $A \cap B = \emptyset$ .
  - (b)  $|\mathbf{p}(t) - \mathbf{p}(s)| = |t - s||\mathbf{a} - \mathbf{b}|$  by a direct calculation.
  - (c)  $\mathbf{p}(t) = \mathbf{p}(s)$  if and only if  $t = s$  by (a)(b).
- (2) Show that  $A_0 \cap \overline{B_0} = \emptyset$ . (Reductio ad absurdum) If there were  $t \in A_0 \cap \overline{B_0}$ , then  $t \in A_0$  and  $t$  is a limit point of  $B_0$ .
- (a)  $t \in A_0$  implies that  $\mathbf{p}(t) \in A$ .
  - (b) Show that  $t$  is a limit point of  $B_0 \implies \mathbf{p}(t)$  is a limit point of  $B$ .  
Given any  $\varepsilon > 0$ , there is  $s \in B_0$  such that

$$|t - s| < \frac{\varepsilon}{|\mathbf{a} - \mathbf{b}|} \quad \text{with} \quad s \neq t$$

since  $t$  is a limit point of  $B_0$ . So by (1),

$$|\mathbf{p}(t) - \mathbf{p}(s)| = |t - s||\mathbf{a} - \mathbf{b}| < \varepsilon.$$

Here  $\mathbf{p}(s) \in B$  and  $\mathbf{p}(s) \neq \mathbf{p}(t)$ . So  $\mathbf{p}(t)$  is a limit point of  $B$ .

By (a)(b),  $\mathbf{p}(t) \in A \cap \overline{B} = \emptyset$ , contrary to the assumption that  $A$  and  $B$  are separated.

(3) Show that  $\overline{A_0} \cap B_0 = \emptyset$ . Similar to (2).

By (2)(3),  $A_0$  and  $B_0$  are separated.  $\square$

*Proof of (b).* (Reductio ad absurdum) If  $\mathbf{p}(t)$  were in  $A \cup B$  for all  $t \in (0, 1)$ , we will show that  $[0, 1]$  is separated by  $A_0 \cap [0, 1]$  and  $B_0 \cap [0, 1]$  to get a contradiction.

- (1)  $\mathbf{p}(t)$  were in  $A \cup B$  for all  $t \in [0, 1]$  since  $\mathbf{p}(0) = \mathbf{a} \in A \cup B$  and  $\mathbf{p}(1) = \mathbf{b} \in A \cup B$ . Therefore,

$$[0, 1] \subseteq \mathbf{p}^{-1}(A \cup B) = \mathbf{p}^{-1}(A) \cup \mathbf{p}^{-1}(B) = A_0 \cup B_0.$$

- (2) Let  $A_1 = A_0 \cap [0, 1]$  and  $B_1 = B_0 \cap [0, 1]$ . So  $[0, 1] = A_1 \cup B_1$ .

- (3) Show that  $A_1 \neq \emptyset$  and  $B_1 \neq \emptyset$ .

$$\begin{aligned} \mathbf{p}(0) \in A &\iff 0 \in \mathbf{p}^{-1}(A) = A_0 \\ &\iff 0 \in A_0 \text{ and } 0 \in [0, 1] \\ &\iff 0 \in A_0 \cap [0, 1] = A_1. \end{aligned}$$

Similarly,  $1 \in B_1$ .

*Note.* That's why we consider  $[0, 1]$  instead of  $(0, 1)$ .

- (4) Show that  $A_1 \cap \overline{B_1} = \emptyset$  and  $\overline{A_1} \cap B_1 = \emptyset$ . Since  $A_1 \subseteq A_0$  and  $B_1 \subseteq B_0$ ,  $A_1 \cap \overline{B_1} \subseteq A_0 \cap \overline{B_0} = \emptyset$  or  $A_1 \cap \overline{B_1} = \emptyset$ . Similarly,  $\overline{A_1} \cap B_1 = \emptyset$ .

By (2)(3)(4),  $[0, 1]$  is separated, contrary to the connectedness of  $[0, 1]$  (Theorem 2.47).  $\square$

*Proof of (c).*

- (1) Let  $E$  be a convex subset of  $\mathbb{R}^k$ . Recall

$$\mathbf{p}(t) = (1 - t)\mathbf{a} + t\mathbf{b} \in E$$

whenever  $\mathbf{a}, \mathbf{b} \in E$  and  $t \in (0, 1)$ .

- (2) (Reductio ad absurdum) If  $E$  were separated by  $A$  and  $B$ , pick  $\mathbf{a} \in A \subseteq E$  and  $\mathbf{b} \in B \subseteq E$ .
- (3) By (b), there exists  $t_0 \in (0, 1)$  such that  $\mathbf{p}(t_0) \notin A \cup B = E$ , contrary to the convexity of  $E$ .

$\square$

**Exercise 2.22.** A metric space is called separable if it contains a countable dense subset. Show that  $\mathbb{R}^k$  is separable. (Hint: Consider the set of points which have only rational coordinates.)

*Proof.* Let  $E$  be the set of points which have only rational coordinates.

- (1) *Show that  $E$  is countable.*  $\mathbb{Q}$  is countable and thus  $E = \mathbb{Q}^k$  is countable (Theorem 2.13).
- (2) *Show that  $E$  is dense.* Given any  $\mathbf{p} = (p_1, \dots, p_k) \in \mathbb{R}^k$ . We want to show that  $\mathbf{p}$  is a limit point of  $E$ .
  - (a) Given any open neighborhood  $B(\mathbf{p}; r)$  of  $\mathbf{p}$ ,  $r > 0$ .
  - (b) Since  $\mathbb{Q}$  is dense in  $\mathbb{R}$  (Theorem 1.20), every coordinate of  $\mathbf{p}$  is a limit point of  $\mathbb{Q}$ . In particular, for every  $i = 1, 2, \dots, k$ , the open neighborhood  $B\left(p_i, \frac{r}{\sqrt{k}}\right)$  of  $p_i$  contains a point  $q_i \neq p_i$  and  $q_i \in \mathbb{Q}$ .
  - (c) Collect all  $q_i$  in (b) and define  $\mathbf{q} = (q_1, \dots, q_k) \in \mathbb{Q}^k = E$ . By construction  $\mathbf{q} \neq \mathbf{p}$  and

$$\begin{aligned}
 |\mathbf{p} - \mathbf{q}| &= \sqrt{(p_1 - q_1)^2 + \dots + (p_k - q_k)^2} \\
 &< \sqrt{\left(\frac{r}{\sqrt{k}}\right)^2 + \dots + \left(\frac{r}{\sqrt{k}}\right)^2} \\
 &= \sqrt{k \cdot \frac{r^2}{k}} \\
 &= r
 \end{aligned}$$

or  $\mathbf{q} \in B(\mathbf{p}; r)$ .

By (a)(b)(c),  $E$  is dense in  $\mathbb{R}^k$ .

By (1)(2),  $\mathbb{R}^k$  is separable.  $\square$

**Exercise 2.23.** A collection  $\{V_\alpha\}$  of open subsets of  $X$  is said to be a base for  $X$  if the following is true: For every  $x \in X$  and every open set  $G \subseteq X$  such that  $x \in G$ , we have  $x \in V_\alpha \subseteq G$  for some  $\alpha$ . In other words, every open set in  $X$  is the union of a subcollection of  $\{V_\alpha\}$ .

*Prove that every separable metric space has a countable base. (Hint: Take all neighborhoods with rational radius and center in some countable dense subset of  $X$ .)*

*Note.*  $\mathbb{R}^k$  has a countable base (Exercise 2.22).

*Proof (Hint).* Let  $X$  be a separable metric space, and  $E$  be a countable dense subset of  $X$ . Let  $\mathcal{B}$  be a collection of all neighborhoods with rational radius and center in  $E$ .

- (1)  $\mathcal{B}$  is countable (Theorem 2.12).
- (2)  $\mathcal{B}$  is a base for  $X$ . Similar to Exercise 2.9(a). Given any  $p \in X$  and every open set  $G \subseteq X$  such that  $p \in G$ . Since  $p$  is in an open set  $G$ , there exists an open neighborhood  $B(p; r)$  of  $p$  such that  $B(p; r) \subseteq G$ .

- (3) Let  $r_0$  be rational such that  $0 < r_0 < \frac{r}{2}$  (Theorem 1.20(b)). Since  $E$  is dense in  $X$ , there is  $q \in E$  such that  $d_X(p, q) < r_0$ . For such  $r_0 \in \mathbb{Q}$  we pick an open neighborhood  $B(q; r_0)$  of  $q$ . Clearly,  $B(q; r_0) \in \mathcal{B}$ .
- (4)  $p \in B(q; r_0)$  since  $d_X(p, q) < r_0$ .
- (5) Show that  $B(q; r_0) \subseteq B(p; r) \subseteq G$ . For any  $z \in B(q; r_0)$ ,  $d_X(z, p) \leq d_X(z, q) + d_X(q, p) < r_0 + r_0 < \frac{r}{2} + \frac{r}{2} = r$ . That is,  $z \in B(p; r)$ .

By (3)(4)(5), (2) is established. By (1)(2),  $\mathcal{B}$  is a countable base for  $X$ .  $\square$

### Supplement.

- (1) In topology, a second-countable space, also called a completely separable space, is a topological space whose topology has a countable base.
- (2) Every second-countable space is separable.
- (3) The reverse implication of (2) does not hold in general. However, for metric spaces the properties of being second-countable and separable are equivalent.
- (4) Show that every second-countable metric space  $X$  is separable.

(a) Let  $\mathcal{B} = \{B_n : n \in \mathbb{Z}^+\}$  be a countable base of  $X$ .

(b) For every  $B_n \in \mathcal{B}$ , pick any point  $p_n$  of  $B_n$  and collect them as

$$E = \{p_n : p_n \in B_n \text{ for } n \in \mathbb{Z}^+\}.$$

(c)  $E$  is countable.

(d) Show that  $E$  is dense. Given any  $x \in X$ . For any open neighborhood  $B(x)$  of  $x$ ,  $B(x)$  is a union of subcollection of  $\mathcal{B}$ . That is, there is always a point in  $E$  by the construction of  $E$ .

$\square$

**Exercise 2.24.** Let  $X$  be a metric space in which every infinite subset has a limit point. Prove that  $X$  is separable.

(Hint: Fix  $\delta > 0$ , and pick  $x_1 \in X$ . Having chosen  $x_1, \dots, x_j \in X$ , choose  $x_{j+1}$ , if possible, so that  $d(x_i, x_{j+1}) \geq \delta$  for  $i = 1, \dots, j$ . Show that this process must stop after finite number of steps, and that  $X$  can therefore be covered by finite many neighborhoods of radius  $\delta$ . Take  $\delta = \frac{1}{n}$  ( $n = 1, 2, 3, \dots$ ) and consider the centers of the corresponding neighborhoods.)

Note. The reverse implication does not hold (Exercise 2.10).

Proof (Hint).

(1) Fix  $\delta > 0$ , and pick  $x_1 \in X$ . Show that every limit point compact metric space  $X$  is totally bounded.

- (a) Having chosen  $x_1, \dots, x_j \in X$ , choose  $x_{j+1}$ , if possible, so that  $d(x_i, x_{j+1}) \geq \delta$  for  $i = 1, \dots, j$ . Let  $E_\delta$  be the set of these  $x_i$ .
- (b) Show that this process must stop after finite number of steps, and that  $X$  can therefore be covered by finite many neighborhoods of radius  $\delta$ . (Reductio ad absurdum)
  - (i) If not,  $E_\delta$  is an infinite subset of  $X$ . By assumption there is a limit point of  $E_\delta$ , say  $p \in X$ .
  - (ii) In particular, an open neighborhood  $B(p; \frac{\delta}{64})$  of  $p$  contains a point  $x_n \in E_\delta$  with  $p \neq x_n$ .
  - (iii) The neighborhood  $B(p; \frac{\delta}{64})$  contains no other point  $x_m \in E_\delta$  with  $m \neq n$ . If so,

$$d_X(x_n, x_m) \leq d_X(x_n, p) + d_X(p, x_m) < \frac{\delta}{64} + \frac{\delta}{64} < \delta,$$

contrary to the construction of  $E_\delta$ .

- (iv) Note that  $p \notin E_\delta$  as a corollary to (iii).
- (v) So another open neighborhood  $B(p; r)$  of  $p$  with  $r = d_X(p, x_n) > 0$  contains no points  $x_m \in E_\delta$  with  $p \neq x_m$ , contrary to the assumption that  $p$  is a limit point of  $E_\delta$ .

(2) Show that every totally bounded metric space  $X$  is separable. Take  $\delta = \frac{1}{n}$  ( $n = 1, 2, 3, \dots$ ) in (1), and union all  $E_{\frac{1}{n}}$  as

$$E = \bigcup_{n=1}^{\infty} E_{\frac{1}{n}} \subseteq X.$$

Show that  $E$  is a countable dense subset of  $X$ .

- (a) Show that  $E$  is countable. Since  $E$  is the countable union of finite set  $E_{\frac{1}{n}}$ ,  $E$  is countable (Theorem 2.12).
- (b) Show that  $E$  is dense in  $X$ . Given any  $p \in X$ . It suffices to show that given any open neighborhood  $B(p; r)$  of  $p \in X - E$ , there exists  $q \in E$  such that  $q \in B(p; r)$ . Pick any  $n \in \mathbb{Z}^+$  such that  $\frac{1}{n} < r$  (Theorem 1.20(a)). By the construction of  $E_{\frac{1}{n}}$ , there is  $q \in E_{\frac{1}{n}}$  such that  $p \in B(q; \frac{1}{n})$ , or  $d_X(p, q) < \frac{1}{n} < r$ , or  $q \in B(p; r)$ .

□

### Supplement.

- (1) A topological space  $X$  is said to be limit point compact or weakly countably compact if every infinite subset of  $X$  has a limit point in  $X$ .

- (2) In a metric space, limit point compactness, compactness, and sequential compactness are all equivalent. For general topological spaces, however, these three notions of compactness are not equivalent.
- (3) A metric space  $X$  is totally bounded if and only if for every real number  $\delta > 0$ , there exists a finite collection of open balls in  $X$  of radius  $\delta$  whose union contains  $X$ .

**Exercise 2.25.** *Prove that every compact metric space  $K$  has a countable base, and that  $K$  is therefore separable. (Hint: For every positive integer  $n$ , there are finitely many neighborhood of radius  $\frac{1}{n}$  whose union covers  $K$ .)*

*Proof (Exercise 2.24(a)).*

- (1) *Show that every compact metric space  $K$  is limit point compact.* Given any subset  $E \subseteq K$ . It suffices to show that if  $E$  has no limit point, then  $E$  must be finite.
  - (a) Since  $E$  has no limit point,  $E$  is closed.
  - (b) For any point  $p \in E$ . Since  $p$  is not a limit point, there is an open neighborhood  $B(p)$  such that  $B(p)$  contains no point other than  $p$ .
  - (c) Similar to the proof of Theorem 2.35, let

$$\mathcal{F} = \{B(p) : p \in E \text{ with } B(p) \cap E = \{p\}\} \bigcup \tilde{E}.$$

Hence  $\mathcal{F}$  is an open covering of  $K$ .

- (d) Since  $K$  is compact by assumption, there is a finitely subcovering  $\mathcal{F}'$  of  $K$ . Since  $\tilde{E}$  does not intersect  $E$ , each  $B(p) \in \mathcal{F}'$  contains only one point of  $E$  and so  $E$  is finite.
- (2) Since  $K$  is limit point compact,  $K$  is separable (Exercise 2.24).

□

*Proof (Exercise 2.24(b)).*

- (1) *Show that every compact metric space  $K$  is totally bounded.* Given any real number  $\delta > 0$ , define an open covering  $\mathcal{F}$  of  $K$  by

$$\mathcal{F} = \{B(p; \delta) : p \in K\}.$$

Since  $K$  is compact, there exists a finite subcovering  $\mathcal{F}'$  of  $K$ .  $\mathcal{F}'$  is our desired finite collection of open balls in  $X$  of radius  $\delta$  whose union contains  $X$ .

- (2) Since  $K$  is totally bounded,  $K$  is separable (Exercise 2.24).

□

*Proof (Hint).*

- (1) Given any positive integer  $n > 0$ , define an open covering  $\mathcal{F}_n$  of  $K$  by

$$\mathcal{F}_n = \left\{ B\left(p; \frac{1}{n}\right) : p \in K \right\}.$$

Since  $K$  is compact, there exists a finite subcovering  $\mathcal{G}_n$  of  $K$ .

- (2) *Show that every compact metric space  $K$  is second-countable.*

- (a) Define

$$\mathcal{B} = \bigcup_{n \geq 1} \mathcal{G}_n$$

be a collection. Since  $\mathcal{B}$  is a countable union of finite set  $\mathcal{G}_n$ ,  $\mathcal{B}$  is countable. Hence it suffices to show that for every  $p \in K$  and every open set  $G \subseteq K$  such that  $p \in G$ , there is  $B \in \mathcal{B}$  such that  $x \in B \subseteq G$ .

- (b) Since  $G$  is open, there is an open neighborhood  $B(p; r)$  of  $p$  such that  $B(p; r) \subseteq G$ .

- (c) For such  $r > 0$ , there is  $n \in \mathbb{Z}^+$  with  $0 < \frac{1}{n} < \frac{r}{2}$  (Theorem 1.20(a)). So  $p$  is in some  $B(q; \frac{1}{n}) \in \mathcal{G}_n \subseteq \mathcal{B}$  since  $\mathcal{G}_n$  is a subcovering of  $K$ .

- (d) *Show that  $B(q; \frac{1}{n}) \subseteq B(p; r) \subseteq G$ . For any  $z \in B(q; \frac{1}{n})$ ,*

$$d_K(z, p) \leq d_K(z, q) + d_K(q, p) < \frac{1}{n} + \frac{1}{n} < \frac{r}{2} + \frac{r}{2} = r.$$

That is,  $z \in B(p; r)$ , or  $B(q; \frac{1}{n}) \subseteq B(p; r) \subseteq G$ .

By (a)(b)(c)(d),  $K$  is second-countable.

- (3) *Show that every second-countable metric space is separable.* Supplement (4) to Exercise 2.23.

□

**Exercise 2.26.** *Let  $X$  be a metric space in which every infinite subsets has a limit point. Prove that  $X$  is compact.*

*By Exercises 2.23 and 2.24,  $X$  has a countable base. It follows that every open cover of  $X$  has a countable subcovering  $\{G_n\}$ ,  $n = 1, 2, 3, \dots$ . If no finite subcollection of  $\{G_n\}$  covers  $X$ , then the complement  $F_n$  of  $G_1 \cup \dots \cup G_n$  is nonempty for each  $n$ , but  $\bigcap F_n$  is empty. If  $E$  is a set contains a point from each  $F_n$ , consider a limit point of  $E$ , and obtain a contradiction.*

*Note.* In every metric space, we have

$$\begin{aligned}
\{\text{compact}\} &\iff \{\text{limit point compact}\} \\
&\iff \{\text{complete and totally bounded}\} \\
&\implies \{\text{totally bounded}\} \\
&\implies \{\text{separable}\} \\
&\iff \{\text{second-countable}\} \\
&\iff \{\text{Lindelof}\}.
\end{aligned}$$

*Proof (Hint).*

- (1) Since  $X$  is limit point compact,  $X$  is separable (Exercise 2.24). Since  $X$  is separable,  $X$  is second-countable (Exercise 2.23).
- (2) *Show that  $X$  is Lindelof if  $X$  is second-countable.* Let  $X$  be a second-countable metric space. Let  $\mathcal{B} = \{B_n\}$  be a countable base of  $X$ . Given any open covering  $\mathcal{F}$  of  $X$ .
  - (a) Iterate each  $B_n \in \mathcal{B}$ , pick one  $G_n \in \mathcal{F}$  containing  $B_n$ , and collect them as
 
$$\mathcal{G} = \{G_n : G_n \supseteq B_n \text{ for } n \in \mathbb{Z}^+\}.$$
 ( $G_n$  might be duplicated.)
  - (b)  $\mathcal{G}$  is a countable subset of  $\mathcal{F}$ .
  - (c)  $\mathcal{G}$  covers  $X$  since  $\mathcal{B}$  is a countable base of  $X$ .
- (3) Hence, given any open covering  $\mathcal{F}$  of  $X$ , there is a countable subcovering  $\mathcal{G} = \{G_n\}$  of  $X$ . (Reductio ad absurdum) If there were no finite subcovering of  $\mathcal{G}$ , then the complement  $F_n$  of  $G_1 \cup \cdots \cup G_n$  is nonempty for each  $n$ , but  $\cap F_n$  is empty.
- (4) Let  $E$  be a set contains a point from each  $F_n$ .  $E$  is infinite and thus  $E$  has a limit point, say  $p$ .  $p \in G_n$  for some  $n$  since  $\mathcal{G} = \{G_n\}$  is an open covering of  $X$ . Since  $G_n$  is open, there is an open neighborhood  $B(p)$  of  $p$  such that  $B(p) \subseteq G_n$ . By the construction of  $F_n$ ,

$$B(p) \cap F_m = \emptyset$$

whenever  $m \geq n$ , contrary to the assumption that  $p$  is a limit point of  $E$ .

Hence,  $X$  is compact if  $X$  is limit point compact.  $\square$

### Supplement.

- (1) Lindelof space is a topological space in which every open covering has a countable subcovering.



- (2) Show that  $X$  is second-countable if  $X$  is Lindelof. Same as the Proof (Hint) of Exercise 2.25 except changing the word “compact” to “Lindelof” and “finite” to “countable.”  $\square$
- (3) In every metric space, we have
- $$\{\text{compact}\} \iff \{\text{limit point compact}\} \iff \{\text{sequentially compact}\}.$$

**Exercise 2.27.** Define a point  $p$  in a metric space  $X$  to be a condensation point of a set  $E \subseteq X$  if every neighborhood of  $p$  contains uncountably many points of  $E$ .

Suppose  $E \subseteq \mathbb{R}^k$ ,  $E$  is uncountable, and let  $P$  be the set of all condensation points of  $E$ . Prove that  $P$  is perfect and that at most countably many points of  $E$  are not in  $P$ . In other words, show that  $\tilde{P} \cap E$  is at most countable.

(Hint: Let  $\{V_n\}$  be a countable base of  $\mathbb{R}^k$ , let  $W$  be the union of those  $V_n$  for which  $E \cap V_n$  is at most countable, and show that  $P = \widetilde{W}$ .)

Note. The statement is also true for separable metric space.

*Proof.*

- (1) Let  $\{V_n\}$  be a countable base of  $\mathbb{R}^k$  (Exercise 2.22 and 2.23). Let  $W$  be the union of those  $V_n$  for which  $E \cap V_n$  is at most countable.
- (2) Show that  $P = \widetilde{W}$ .

- (a) ( $P \subseteq \widetilde{W}$ ) Given any  $x \in P$ .

$$\begin{aligned} x \in P &\implies x \text{ is a condensation point of } E \\ &\implies \forall V_n \ni x, \exists B(x) \subseteq V_n \text{ such that } E \cap B(x) \text{ is uncountable} \\ &\implies E \cap V_n \text{ is uncountable} \\ &\implies x \notin W. \end{aligned}$$

- (b) ( $P \supseteq \widetilde{W}$ ) Given any  $x \in \widetilde{W}$ . Let  $P(V_n)$  be the proposition that  $E \cap V_n$  is at most countable.

$$\begin{aligned} x \in \widetilde{W} &\implies x \notin W = \bigcup_{P(V_n)} V_n \\ &\implies x \notin V_n \text{ for which } E \cap V_n \text{ is at most countable} \\ &\implies \forall B(x) \text{ of } x, x \in V_m \subseteq B(x) \text{ for some } V_m \quad (\{V_n\}: \text{ base of } X) \\ &\implies E \cap V_m \text{ is uncountable} \\ &\implies E \cap B(x) \supseteq E \cap V_m \text{ is uncountable} \\ &\implies x \text{ is a condensation point of } E \\ &\implies x \in P. \end{aligned}$$

- (3) Show that  $P$  is closed.  $P$  is the complement of an open subset  $W$ .
- (4) Show that  $P \subseteq P'$ . (Reductio ad absurdum)
- (a) If there were an isolated point  $x \in P$ , then there exists an open neighborhood  $B(x)$  of  $x$  such that  $B(x) \cap P = \{x\}$ .
  - (b) Since  $x$  is a condensation point of  $E$ , there are uncountably many points of  $E$  in  $B(x)$ , and such points  $y$  are not a condensation points of  $E$  except  $y = x$ .
  - (c) Given any point  $y \in E \cap B(x)$  with  $y \neq x$ . Since  $y$  is not a condensation point, there exists a neighborhood  $B(y)$  of  $y$  such that  $B(y) \cap E$  is at most countable. Since  $\{V_n\}$  is a base, for each  $B(y)$  there exists  $V_{n(y)}$  such that  $y \in V_{n(y)} \subseteq B(y)$ . Hence

$$V_{n(y)} \cap E \subseteq B(y) \cap E$$

is at most countable.

- (d) Hence,

$$\begin{aligned} E \cap B(x) - \{x\} &\subseteq \bigcup_{y \in E \cap B(x) - \{x\}} V_{n(y)} \\ &= \bigcup_{n(y)} V_{n(y)} \end{aligned}$$

is a countable union of at most countable sets, which is countable. Hence  $E \cap B(x) - \{x\}$  or  $E \cap B(x)$  is countable, contrary to the assumption that  $E \cap B(x)$  is uncountable.

- (5) Show that  $E \cap \tilde{P}$  is at most countable.

$$E \cap \tilde{P} = E \cap \left( \bigcup_{P(V_n)} V_n \right) = \bigcup_{P(V_n)} (E \cap V_n)$$

is at most countable.

□

**Exercise 2.28.** Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is at most countable. (Corollary: Every countable closed set in  $\mathbb{R}^k$  has isolated points.) (Hint: Use Exercise 2.27.)

*Proof (Exercise 2.27).* Let  $E$  be a closed set in a separable metric space.

- (1)  $E$  contains all limit points of  $E$ , especially contains all condensation points of  $E$ . So we can write

$$E = P \cup (E - P)$$

where  $P$  is the set of all condensation points of  $E$ .

- (2) By Exercise 2.27,  $P$  is perfect and  $E - P = E \cap \tilde{P}$  is at most countable.

□

**Cantor-Bendixson theorem.**

- (1) Closed sets of a Polish space  $X$  have the perfect set property in a particularly strong form: any closed subset of  $X$  may be written uniquely as the disjoint union of a perfect set and a countable set.
- (2) A Polish space is a separable completely metrizable topological space; that is, a space homeomorphic to a complete metric space that has a countable dense subset.

**Exercise 2.29.** *Prove that every open set in  $\mathbb{R}^1$  is the union of an at most countable collection of disjoint segments. (Hint: Use Exercise 2.22.)*

*Proof.* Let  $E$  be an open subset of  $\mathbb{R}^1$ .

- (1) For each  $x \in E$ , let  $I_x$  denote the largest open interval containing  $x$  and contained in  $E$ . More precisely, since  $E$  is open,  $x$  is contained in some small (non-trivial) interval, and therefore if

$$a_x = \inf\{a < x : (a, x) \subseteq E\} \text{ and } b_x = \sup\{b > x : (x, b) \subseteq E\}$$

we must have  $a_x < x < b_x$  (with possibly infinite values for  $a_x$  and  $b_x$ ).

- (2) Let  $I_x = (a_x, b_x)$ , then by construction we have  $x \in I_x$  as well as  $I_x \subseteq E$ . Hence

$$E = \bigcup_{I_x \in \mathcal{F}} I_x,$$

where  $\mathcal{F} = \{I_x\}_{x \in E}$ .

- (3) Suppose that two intervals  $I_x$  and  $I_y$  intersect. Then their union (which is also an open interval) is contained in  $E$  and contains  $x$  (and  $y$ ). Since  $I_x$  is maximal,  $I_x \cup I_y \subseteq I_x$ , and similarly  $I_x \cup I_y \subseteq I_y$ . This can happen only if  $I_x = I_y$ .
- (4) Therefore, any two distinct intervals in  $\mathcal{F}$  must be disjoint. Hence  $\mathcal{F}$  is countable since each open interval  $I_x \in \mathcal{F}$  contains a rational number.

□

**Exercise 2.30.** Imitate the proof of Theorem 2.43 to obtain the following result:

If  $\mathbb{R}^k = \bigcup_{n=1}^{\infty} F_n$ , where each  $F_n$  is a closed subset of  $\mathbb{R}^k$ , then at least one  $F_n$  has a nonempty interior.

Equivalent statement: If  $G_n$  is a dense open subset of  $\mathbb{R}^k$ , for  $n = 1, 2, 3, \dots$ , then  $\bigcap_{n=1}^{\infty} G_n$  is not empty (in fact, it is dense in  $\mathbb{R}^k$ ).

(This is a special case of Baire's theorem; see Exercise 3.22 for the general case.)

**Baire category theorem.** If  $G_n$  is a dense open subset of  $\mathbb{R}^k$ , for  $n = 1, 2, \dots$ , then

$$\bigcap_{n=1}^{\infty} G_n$$

is dense in  $\mathbb{R}^k$ .

*Proof of Baire category theorem.* Given any open set  $G_0$  in  $\mathbb{R}^k$ , will show that

$$\bigcap_{n=0}^{\infty} G_n \neq \emptyset.$$

- (1) Since  $G_1$  is dense,  $G_0 \cap G_1$  is nonempty. Take any one point  $\mathbf{x}_1$  in the open set  $G_0 \cap G_1$ , then there exists an open neighborhood

$$V_1 = \{\mathbf{y} \in \mathbb{R}^k : |\mathbf{y} - \mathbf{x}_1| < r_1\}$$

of  $\mathbf{x}_1$  such that

$$\overline{V_1} = \{\mathbf{y} \in \mathbb{R}^k : |\mathbf{y} - \mathbf{x}_1| \leq r_1\} \subseteq G_0 \cap G_1.$$

- (2) Suppose  $V_n$  has been constructed, take any one point  $\mathbf{x}_{n+1}$  in the open set  $V_n \cap G_{n+1}$ , then there exists an open neighborhood

$$V_{n+1} = \{\mathbf{y} \in \mathbb{R}^k : |\mathbf{y} - \mathbf{x}_{n+1}| < r_{n+1}\}$$

of  $\mathbf{x}_{n+1}$  with  $r_{n+1}$  such that

$$\overline{V_{n+1}} = \{\mathbf{y} \in \mathbb{R}^k : |\mathbf{y} - \mathbf{x}_{n+1}| \leq r_{n+1}\} \subseteq V_n \cap G_{n+1}.$$

- (3) Note that

- (a) each  $\overline{V_n}$  is nonempty (containing  $\mathbf{x}_n$ ) and compact.
- (b)  $\overline{V_1} \supseteq \overline{V_2} \supseteq \dots$  (since  $\overline{V_{n+1}} \subseteq V_n \cap G_{n+1} \subseteq V_n \subseteq \overline{V_n}$ ).

By Corollary to Theorem 2.36,

$$\bigcap_{n=1}^{\infty} \overline{V_n} \neq \emptyset.$$

(4) Pick  $\mathbf{x} \in \bigcap_{n=1}^{\infty} \overline{V_n}$ . Hence

$$\begin{aligned} \mathbf{x} \in \bigcap_{n=1}^{\infty} \overline{V_n} &\iff \mathbf{x} \in \overline{V_n} \text{ for all } n = 1, 2, 3, \dots \\ &\implies \mathbf{x} \in \overline{V_1} \subseteq G_0 \cap G_1 \text{ and } \mathbf{x} \in \overline{V_{n+1}} \subseteq V_n \cap G_{n+1} \subseteq G_{n+1} \\ &\implies \mathbf{x} \in G_0 \cap G_1 \cap \dots = \bigcap_{n=0}^{\infty} G_n \\ &\implies \bigcap_{n=0}^{\infty} G_n \neq \emptyset. \end{aligned}$$

□