Chapter 4: Continuity

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Exercise 4.1. *Suppose f is a real function define on $\mathbb{R}^1$ which satisfies*

$$\lim_{h \to 0} [f(x+h) - f(x-h)] = 0$$

for every $x \in \mathbb{R}^1$. Does this imply that f is continuous?

Proof. $\lim_{h \to 0}[f(x+h) - f(x-h)] = 0$ holds if f is continuous. But the converse of this statement and is not true. For example, define $f : \mathbb{R}^1 \to \mathbb{R}^1$ by

$$f(x) = \begin{cases} 1 & (x = 0), \\ 0 & (x \neq 0). \end{cases}$$

f is not continuous at $x = 0$ but

$$\lim_{h \to 0} [f(x+h) - f(x-h)] = 0$$

for any $x \in \mathbb{R}^1$. (The identity holds for $x \neq 0$ since f is continuous on $\mathbb{R}^1 - \{0\}$. Besides, $\lim_{h \to 0}[f(0+h) - f(0-h)] = \lim_{h \to 0}[0 - 0] = 0$.) $\square$

Exercise 4.2. *If f is a continuous mapping of a metric space X into a metric space Y, prove that $f(\overline{E}) \subseteq \overline{f(E)}$ for every set $E \subseteq X$. ($\overline{E}$ denotes the closure of E.) Show, by an example, that $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$.*

Proof.

(1) Since f is continuous and $\overline{f(E)}$ is closed, $f^{-1}(\overline{f(E)})$ is closed. Hence,

$$
\begin{aligned}
f^{-1}(\overline{f(E)}) &\supseteq f^{-1}(f(E)) & \text{(Monotonicity of } f^{-1}) \\
&\supseteq E, & \text{(Note in Theorem 4.14)} \\
\overline{E} &\subseteq f^{-1}(\overline{f(E)}), & \text{(Monotonicity of closure)} \\
f(\overline{E}) &\subseteq f(f^{-1}(\overline{f(E)})) & \text{(Monotonicity of } f) \\
&\subseteq \overline{f(E)}. & \text{(Note in Theorem 4.14)}
\end{aligned}
$$

(2) Let $f : (0, \infty) \to \mathbb{R}$ be a continuous function defined by

$$f(x) = \frac{1}{x}.$$

Consider $E = \mathbb{Z}^+ \subseteq (0, \infty)$. Then $f(E) = \left\{ \frac{1}{n} : n \in \mathbb{Z}^+ \right\}$, and thus

$$f(\overline{E}) = \left\{ \frac{1}{n} : n \in \mathbb{Z}^+ \right\}.$$

$$\overline{f(E)} = \left\{ \frac{1}{n} : n \in \mathbb{Z}^+ \right\} \bigcup \{0\}.$$

□

Supplement (Inverse image).

(1) $E \subseteq f^{-1}[f(E)]$ for $E \subseteq X$.

$$\forall\, x \in E \Longrightarrow f(x) \in f(E)$$
$$\Longleftrightarrow x \in f^{-1}[f(E)]. \qquad \text{(Definition of the inverse image)}$$

□

(2) $f[f^{-1}(E)] \subseteq E$ for $E \subseteq Y$.

$$\forall\, y \in f[f^{-1}(E)] \Longleftrightarrow \exists\, x \in f^{-1}(E) \text{ such that } y = f(x)$$
$$\Longleftrightarrow \exists\, x, f(x) \in E \text{ such that } y = f(x)$$
$$\Longrightarrow \exists\, x, y = f(x) \in E.$$

□

Supplement (Continuity). *Let f be a map from a topological space on X to a topological space on Y. Then, the following statements are equivalent:*

(1) *f is continuous: For each $x \in X$ and every neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$.*

(2) *For every open set O in Y, the inverse image $f^{-1}(O)$ is open in X.*

(3) *For every closed set C in Y, the inverse image $f^{-1}(C)$ is closed in X.*

(4) *$f(A)^\circ \subseteq f(A^\circ)$ for every subset A of X.*

(5) *$f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ$ for every subset B of Y.*

(6) *$f(\overline{A}) \subseteq \overline{f(A)}$ for every subset A of X.*

(7) *$\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for every subset B of Y.*

Exercise 4.3. *Let f be a continuous real function on a metric space X. Let $Z(f)$ (the zero set of f) be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.*

Proof (Corollary to Theorem 4.8). Since f is continuous, $f^{-1}(\{0\}) = Z(f)$ is closed in X for a closed subset $\{0\}$ in $\mathbb{R}^1$. $\square$

Denote the complement of any set E by $\widetilde{E}$.

Proof (Theorem 4.8). Consider the complement of $Z(f)$ in X,

$$\widetilde{Z(f)} = \{x \in X : f(x) \neq 0\}$$
$$= f^{-1}((-\infty, 0) \cup (0, \infty)).$$

Since f is continuous, $f^{-1}((-\infty, 0) \cup (0, \infty)) = \widetilde{Z(f)}$ is open in X for a open subset $(-\infty, 0) \cup (0, \infty)$ in $\mathbb{R}^1$. $\square$

Proof (Definition 2.18(d)). Given any limit point p of $Z(f)$. *Show that $f(p) = 0$ or $p \in Z(f)$.* Since f is continuous, given any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(p)| < \varepsilon$ for all $x \in X$ for which $d_X(x, p) < \delta$. Since p is a limit point of $Z(f)$, for such $\delta > 0$ we have a point $q \neq p$ such that $q \in Z(f)$, or $f(q) = 0$. So $|f(p)| < \varepsilon$ for any $\varepsilon > 0$. $f(p) = 0$. $\square$

Proof (Definition 2.18(f)). Consider the complement of $Z(f)$ in X,

$$\widetilde{Z(f)} = \{x \in X : f(x) \neq 0\} = \{f > 0\} \cup \{f < 0\}$$

where $\{f > 0\} = \{x \in X : f(x) > 0\}$ and $\{f < 0\} = \{x \in X : f(x) < 0\}$. It suffices to show $\{f > 0\}$ is open. ($\{f < 0\}$ is similar.) Given any point p of $\{f > 0\}$ or $f(p) > 0$. *Want to show p is an interior point of $\{f > 0\}$.* Since f is continuous, given any $\varepsilon = \frac{f(p)}{2} > 0$ there exists a $\delta > 0$ such that $|f(x) - f(p)| < \frac{f(p)}{2}$ for all $x \in X$ for which $d_X(x, p) < \delta$. For such x with $d_X(x, p) < \delta$ we have

$$\frac{1}{2} f(p) < f(x) < \frac{3}{2} f(p).$$

That is, $N = \{x : d_X(x, p) < \delta\}$ is a neighborhood p such that $N \subseteq \{f > 0\}$. $\square$

Exercise 4.4. *Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that $f(E)$ is dense in $f(X)$. If $g(p) = f(p)$ for all $p \in E$, prove that $g(p) = f(p)$ for all $p \in X$. (In other words, a continuous mapping is determined by its values on a dense subset of its domain.)*

Proof.

3

(1) *Show that $f(E)$ is dense in $f(X)$.* It suffices to show that every point $y \in f(X) - f(E)$ is a limit point of $f(E)$. Since $y \in f(X) - f(E)$, there exists a point $x \in X - E$ such that $y = f(x)$. Since E is dense in X, there exists a sequence $\{x_n\}$ in E such that $x_n \to x$ as $n \to \infty$. Let $y_n = f(x_n) \in f(E)$. Take limit and use the continuity of f, $y_n \to y$ as $n \to \infty$, or y is a limit point of $f(E)$.

(2) *Show that $g(p) = f(p)$ for all $p \in X$ if $g(p) = f(p)$ for all $p \in E$.* It suffices to show $g(p) = f(p)$ for all $p \in X - E$. Given any $p \in X - E$, there exists a sequence $\{p_n\}$ in E such that $p_n \to p$ as $n \to \infty$. Notice that $g(p_n) = f(p_n)$ by the assumption. Take limit and use the continuity of f and g, $g(p) = f(p)$ for $p \in X - E$.

□

Exercise 4.5. *If f is a real continuous function defined on a closed set $E \subseteq \mathbb{R}^1$, prove that there exist continuous real function g on $\mathbb{R}^1$ such that $g(x) = f(x)$ for all $x \in E$. (Such functions g are called **continuous extensions** of f from E to $\mathbb{R}^1$.) Show that the result becomes false if the word "closed" is omitted. Extend the result to vector valued functions. (Hint: Let the graph of g be a straight line on each of the segments which constitute the complement of E (compare Exercise 2.29). The result remains true if $\mathbb{R}^1$ is replaced by any metric space, but the proof is not so simple.)*

Proof.

(1) Every open set in $\mathbb{R}^1$ is the union of an at most countable collection of disjoint segments (Exercise 2.29).

(2) We need to construct a continuous real function on the complement of E. By (1), write $\widetilde{E} = \bigcup_{i \in \mathscr{C}}(a_i, b_i)$ where $\mathscr{C}$ is at most countable and $a_i < b_i$. (a_i, b_i could be $\pm\infty$.) Define $g(x)$ by

$$g(x) = \begin{cases} f(x) & (x \in E), \\ f(a_i) + \frac{f(b_i) - f(a_i)}{b_i - a_i}(x - a_i) & (x \in (a_i, b_i) : \text{finite interval}), \\ f(a_i) & (x \in (a_i, b_i) : a_i : \text{finite}, b_i = +\infty), \\ f(b_i) & (x \in (a_i, b_i) : a_i = -\infty, b_i : \text{finite}), \\ 0 & (x \in (a_i, b_i) : a_i = -\infty, b_i = +\infty). \end{cases}$$

Show that g is continuous in $\mathbb{R}^1$, or show that $g(x)$ is continuous at $x = p$ for any point $p \in \mathbb{R}^1$.

  (a) Given a point $p \in \widetilde{E}$. There is an open interval $I = (a_i, b_i)$ such that $p \in I$. Since the graph of g in an open interval I is a straight line, g is continuous at $x = p$.

(b) Given an isolated point $p \in E$. There are two open intervals $I = (a_i, b_i)$ and $J = (a_j, b_j)$ such that $b_i = p = a_j$. So $\lim_{x \to p^-} g(x) = \lim_{x \to p^+} g(x) = f(p)$ by the construction of g, which says g is continuous at $x = p$.

(c) Given a limit point $p \in E$. So that $g(p) = f(p)$. Given $\varepsilon > 0$. Consider $\lim_{x \to p^+} g(x)$ first. (The case $\lim_{x \to p^-} g(x)$ is similar.)

(i) For such $\varepsilon > 0$, there is a $\delta' > 0$ such that

$$f(p) - \varepsilon < f(x) < f(p) + \varepsilon$$

whenever
$$x \in E \text{ and } p < x < \delta'.$$

Since p is a limit point of E, there is a point $q \neq p$ such that $|q - p| < \delta'$. Might assume that $q > p$, and then retake $\delta = \min\{\delta', q - p\} > 0$. (If no such q, $\lim_{x \to p^+} g(x) = f(p)$ trivially.)

(ii) For any x such that $p < x < q$, consider $x \in E$ or else $x \in \widetilde{E}$. As $x \in E$, nothing to do by (i).

(iii) As $x \in \widetilde{E}$, there exists an open interval $I = (a_i, b_i)$ such that $x \in I \subseteq (p, q)$. Therefore,

$$f(a_i) \leq g(x) \leq f(b_i) \text{ or } f(a_i) \geq g(x) \geq f(b_i).$$

By (i),

$$f(p) - \varepsilon < f(a_i) < f(p) + \varepsilon \text{ and}$$
$$f(p) - \varepsilon < f(b_i) < f(p) + \varepsilon,$$
$$f(p) - \varepsilon < f(a_i) \leq g(x) \leq f(b_i) < f(p) + \varepsilon \text{ or}$$
$$f(p) - \varepsilon < f(b_i) \leq g(x) \leq f(a_i) < f(p) + \varepsilon.$$

Hence, given $\varepsilon > 0$ there is a $\delta > 0$ such that $|g(x) - g(p)| < \varepsilon$ whenever $p < x < \delta$ (and $x \in \mathbb{R}^1$), or $\lim_{x \to p^+} g(x) = g(p)$.

(3) Consider $f(x) = \log(x)$ in $(0, \infty)$. Since $\lim_{x \to 0} f(x) = -\infty$, we cannot find any real continuous function g defined on $x = 0$.

(4) For a vector-valued function $\mathbf{f} = (f_1, \ldots, f_k)$, with each f_i is continuous on a closed set $E \subseteq \mathbb{R}^1$, extend f_i to a continuous function g_i on $\mathbb{R}^1$ as (2). Put $\mathbf{g} = (g_1, \ldots, g_k)$. Clearly $\mathbf{g}$ is an extension of $\mathbf{f}$. Besides, $\mathbf{g}$ is continuous in $\mathbb{R}^1$ by Theorem 4.10.

$\square$

Supplement (Tietze's Extension Theorem). *If X is a normal topological space and $f : A \to \mathbb{R}$ is a continuous map from a closed subset A of X into the real numbers carrying the standard topology, then there exists a continuous map $g : X \to \mathbb{R}$ with $g(a) = f(a)$ for all $a \in A$.*

Exercise 4.6. *If f is defined on E, the graph of f is the set of points $(x, f(x))$, for $x \in E$. In particular, if E is a set of real numbers, and f is real-valued, the graph of f is a subset of the plain. Suppose E is compact, and prove that that f is continuous on E if and only if its graph is compact.*

Proof. Let $G = \{(x, f(x)) : x \in E\}$ be the graph of f.

(1) ($\Longrightarrow$) Let $\mathbf{f} : E \to G$ defined by

$$\mathbf{f}(x) = (x, f(x)).$$

$\mathbf{f}(E) = G$ exactly. Since f and x are continuous in E, $\mathbf{f}$ is continuous (Theorem 4.10). As E is compact, $\mathbf{f}(E)$ is compact (Theorem 4.14).

(2) ($\Longleftarrow$) Let $\pi : G \to E$ be a projection map defined by

$$\pi(x, f(x)) = x.$$

Notice that $\pi \circ \mathbf{f} = \mathrm{id}_E$ and $\mathbf{f} \circ \pi = \mathrm{id}_G$. Besides, π is a continuous one-to-one mapping of a compact set G onto E. Then the inverse mapping $\pi^{-1} = \mathbf{f}$ is a continuous mapping of E onto G (Theorem 4.17). So f is continuous (Theorem 4.10).

$\square$

Exercise 4.7. *If $E \subseteq X$ and if f is a function defined on X, the **restriction** of f to E is the function g whose domain of definition is E, such that $g(p) = f(p)$ for $p \in E$. Define f and g on $\mathbb{R}^2$ by:*

$$f(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0) \\ \frac{xy^2}{x^2 + y^4} & \text{if } (x, y) \neq (0, 0), \end{cases}$$

$$g(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0) \\ \frac{xy^2}{x^2 + y^6} & \text{if } (x, y) \neq (0, 0), \end{cases}$$

Prove that f is bounded on $\mathbb{R}^2$, that g is unbounded in every neighborhood of $(0, 0)$, and that f is not continuous at $(0, 0)$; nevertheless, the restrictions of both f and g to every straight line in $\mathbb{R}^2$ are continuous!

Proof.

(1) *Show that f is bounded on $\mathbb{R}^2$.*

$$(|x| - |y^2|)^2 \geq 0 \iff |x|^2 - 2|x||y^2| + |y^2|^2 \geq 0$$
$$\iff |x|^2 + |y^2|^2 \geq 2|x||y^2|$$
$$\iff |x^2 + y^4| \geq 2|xy^2|$$
$$\implies \frac{1}{2} \geq \left| \frac{xy^2}{x^2 + y^2} \right| \text{ whenever } (x, y) \neq (0, 0)$$
$$\implies |f(x, y)| \leq \frac{1}{2} \text{ whenever } (x, y) \neq (0, 0).$$

Note that $f(0, 0) = 0 \leq \frac{1}{2}$. Hence f is bounded by $\frac{1}{2}$ on $\mathbb{R}^2$.

(2) *Show that g is unbounded in every neighborhood of $\mathbb{R}^2$.* Consider a sequence $\{\mathbf{p}_n\}_{n \geq 1} \subseteq \mathbb{R}^2$

$$\mathbf{p}_n = (x_n, y_n) = \left(\frac{1}{n^3}, \frac{1}{n} \right)$$

such that $\mathbf{p}_n \neq \mathbf{0}$ and $\lim \mathbf{p}_n = \mathbf{0}$. Thus,

$$\lim_{n \to \infty} g(\mathbf{p}_n) = \lim_{n \to \infty} \frac{x_n y_n^2}{x_n^2 + y_n^6} = \lim_{n \to \infty} \frac{(\frac{1}{n^3})(\frac{1}{n})^2}{(\frac{1}{n^3})^2 + (\frac{1}{n})^6} = \lim_{n \to \infty} \frac{n}{2} = \infty.$$

Hence g is unbounded in every neighborhood of $\mathbb{R}^2$.

(3) *Show that f is not continuous at $(0, 0)$.* Consider a sequence $\{\mathbf{p}_n\}_{n \geq 1} \subseteq \mathbb{R}^2$

$$\mathbf{p}_n = (x_n, y_n) = \left(\frac{1}{n^2}, \frac{1}{n} \right)$$

such that $\mathbf{p}_n \neq \mathbf{0}$ and $\lim \mathbf{p}_n = \mathbf{0}$. Thus,

$$\lim_{n \to \infty} f(\mathbf{p}_n) = \lim_{n \to \infty} \frac{x_n y_n^2}{x_n^2 + y_n^4} = \lim_{n \to \infty} \frac{(\frac{1}{n^2})(\frac{1}{n})^2}{(\frac{1}{n^2})^2 + (\frac{1}{n})^4} = \frac{1}{2}.$$

So, $\lim f(\mathbf{p}_n) = \frac{1}{2} \neq 0$. By Theorem 4.6, f is not continuous at $(0, 0)$.

(4) *The restrictions of f to every straight line in $\mathbb{R}^2$ is continuous.*

   (a) *The line $L_\infty = \{(0, y) : y \in \mathbb{R}\}$. Hence $f|_{L_\infty}(x, y) = 0$ for all $(x, y) \in L_\infty$ (including $(0, 0) \in L_\infty$). Therefore $f|_{L_\infty}$ is continuous.*

   (b) *The line $L_\alpha = \{(x, \alpha x) : x \in \mathbb{R}\}$ for some $\alpha \in \mathbb{R}$. $f|_{L_\alpha}(x, y)$ is continuous on $L_\alpha - \{(0, 0)\}$.*

$$f|_{L_\alpha}(x, y) = f|_{L_\alpha}(x, \alpha x) = \begin{cases} 0 & \text{if } (x, y) = (0, 0) \\ \frac{\alpha^2 x}{1 + \alpha^4 x^2} & \text{if } (x, y) \neq (0, 0). \end{cases}$$

   So

$$\lim_{(x,y) \to (0,0)} f|_{L_\alpha}(x, y) = \lim_{x \to 0} \frac{\alpha^2 x}{1 + \alpha^4 x^2} = 0 = f(0, 0),$$

or $f|_{L_\alpha}(x, y)$ is continuous at $(0, 0)$. Therefore, $f|_{L_\alpha}(x, y)$ is continuous on L_α.

(c) *The line L not passing $(0, 0)$.* It is clear since $f(x, y)$ is continuous on $\mathbb{R}^2 - \{(0, 0)\}$.

(5) *The restrictions of g to every straight line in $\mathbb{R}^2$ is continuous.* Similar to (4).

(a) *The line $L_\infty = \{(0, y) : y \in \mathbb{R}\}$.* Hence $g|_{L_\infty}(x, y) = 0$ for all $(x, y) \in L_\infty$ (including $(0, 0) \in L_\infty$). Therefore $g|_{L_\infty}$ is continuous.

(b) *The line $L_\alpha = \{(x, \alpha x) : x \in \mathbb{R}\}$ for some $\alpha \in \mathbb{R}$.* $g|_{L_\alpha}(x, y)$ is continuous on $L_\alpha - \{(0, 0)\}$.

$$g|_{L_\alpha}(x, y) = g|_{L_\alpha}(x, \alpha x) = \begin{cases} 0 & \text{if } (x, y) = (0, 0) \\ \frac{\alpha^2 x}{1 + \alpha^6 x^4} & \text{if } (x, y) \neq (0, 0). \end{cases}$$

So

$$\lim_{(x,y)\to(0,0)} g|_{L_\alpha}(x, y) = \lim_{x\to 0} \frac{\alpha^2 x}{1 + \alpha^6 x^4} = 0 = g(0, 0),$$

or $g|_{L_\alpha}(x, y)$ is continuous at $(0, 0)$. Therefore, $g|_{L_\alpha}(x, y)$ is continuous on L_α.

(c) *The line L not passing $(0, 0)$.* It is clear since $g(x, y)$ is continuous on $\mathbb{R}^2 - \{(0, 0)\}$.

$\square$

Exercise 4.8. *Let f be a real uniformly continuous function on the bounded set E in $\mathbb{R}$. Prove that f is bounded on E. Show that the conclusion is false if boundedness of E is omitted from the hypothesis.*

The conclusion is false if boundedness of E is omitted from the hypothesis. For example, $f(x) = x$ on $\mathbb{R}$ is uniformly continuous on $\mathbb{R}$ but $f(\mathbb{R}) = \mathbb{R}$ is unbounded.

Proof (Brute-force).

(1) Since $f : E \to \mathbb{R}$ is uniformly continuous, given any $\varepsilon > 0$, there is $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$. In particular, pick $\varepsilon = 1$.

(2) By the boundedness of E, there is $M > 0$ such that $|x| < M$ for all $x \in E$.

(3) For such $\delta > 0$, we construct a covering of $E \subseteq \mathbb{R}$. Construct a special collection $\mathscr{C}$ of intervals

$$I_a = \left[\frac{\delta}{2} a, \frac{\delta}{2}(a + 1) \right]$$

8

where $a \in \mathbb{Z}$ satisfying

$$|a| < \frac{2M}{\delta} + 1.$$

By construction, $\mathscr{C}$ is a finite covering of E.

(4) For every interval I_a of the collection $\mathscr{C}$, pick a point $x_a \in E \bigcap I_a$ if possible. This process will terminate eventually since $\mathscr{C}$ is a finite. Collect these representative points as $\mathscr{D} = \{x_a\}$. Notice that $\mathscr{D}$ is finite again.

(5) Now for any point $x \in E$, x lies in some I_a containing x_a. Both x and x_a are in the same interval and their distance satisfies

$$|x - x_a| \le \frac{\delta}{2} < \delta$$

and thus by (1)

$$\left|f(x) - f(x_a)\right| < 1, \text{ or } \left|f(x)\right| < 1 + \left|f(x_a)\right|.$$

(6) Let

$$M = 1 + \max_{x_a \in \mathscr{D}} \left|f(x_a)\right|.$$

So given any $x \in E$, $\left|f(x)\right| < M$.

□

Proof (Heine-Borel Theorem). Heine-Borel theorem provides the finiteness property to construct the boundedness property of f.

(1) *Let E be a bounded subset of a metric space X. Show that the closure of E in X is also bounded in X.* E is bounded if $E \subseteq B_X(a; r)$ for some $r > 0$ and some $a \in X$. (The ball $B_X(a; r)$ is defined to the set of all $x \in X$ such that $d_X(x, a) < r$.) Take the closure on the both sides,

$$\overline{E} \subseteq \overline{B_X(a; r)} = \{x \in X : d_X(x, a) \le r\} \subseteq B_X(a; 2r),$$

or $\overline{E}$ is bounded.

(2) Since $f : E \to \mathbb{R}$ is uniformly continuous, given any $\varepsilon > 0$, there is $\delta > 0$ such that $\left|f(x) - f(y)\right| < \varepsilon$ whenever $|x - y| < \delta$. In particular, pick $\varepsilon = 1$.

(3) For such $\delta > 0$, we construct an open covering of $\overline{E} \subseteq \mathbb{R}$. Pick a collection $\mathscr{C}$ of open balls $B(a; \delta) \subseteq \mathbb{R}$ where a runs over all elements of E. $\mathscr{C}$ covers $\overline{E}$ (by the definition of accumulation points). Since $\overline{E}$ is closed and bounded (by applying (1) on the boundedness of E), $\overline{E}$ is compact (Heine-Borel theorem). That is, there is a finite subcollection $\mathscr{C}'$ of $\mathscr{C}$ also covers $\overline{E}$, say

$$\mathscr{C}' = \left\{ B(a_1; \delta)), B(a_2; \delta), \dots, B(a_m; \delta) \right\}.$$

(4) Given any $x \in E \subseteq \overline{E}$, there is some $a_i \in E$ $(1 \le i \le m)$ such that $x \in B(a_i; \delta)$. In such ball, $|x - a_i| < \delta$. By (2), $|f(x) - f(a_i)| < 1$, or $|f(x)| < 1 + |f(a_i)|$. Almost done. Notice that a_i depends on x, and thus we might use finiteness of $\{a_1, a_2, \ldots, a_m\}$ to remove dependence of a_i.

(5) Let
$$M = 1 + \max_{1 \le i \le m} |f(a_i)|.$$
So given any $x \in E$, $|f(x)| < M$.

□

Supplement. Exercise about considering the closure. (Problem 3.5 in H. L. Royden, Real Analysis, 3rd Edition.) *Let $A = \mathbb{Q} \cap [0, 1]$, and let $\{I_n\}$ be a finite collection of open intervals covering A. Then $\sum l(I_n) \ge 1$.*

Proof.

$$1 = m^*[0, 1] = m^*\overline{A} \le m^* \left(\overline{\bigcup I_n} \right) = m^* \left(\bigcup \overline{I_n} \right)$$
$$\le \sum m^*(\overline{I_n}) = \sum l(\overline{I_n}) = \sum l(I_n).$$

□

Exercise 4.9. *Show that the requirement in the definition of uniformly continuity can be rephrased as follows, in terms of diameters of sets: To every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mathrm{diam} f(E) < \varepsilon$ for all $E \subseteq X$ with $\mathrm{diam} E < \delta$.*

Proof.

(1) ($\Longrightarrow$) Given $\varepsilon > 0$. By Definition 4.18, there exists a $\delta > 0$ such that
$$d(f(p), f(q)) < \frac{\varepsilon}{64}$$
for all p and q in X for which $d(p, q) < \delta$. Let E be any subset of X satisfying $\mathrm{diam} E < \delta$. Then for any $p, q \in E$,
$$d(p, q) \le \mathrm{diam} E < \delta.$$
So that
$$d(f(p), f(q)) < \frac{\varepsilon}{64},$$
or $\frac{\varepsilon}{64}$ is an upper bound of $S = \{d(f(p), f(q)) : p, q \in E\}$. Hence
$$\mathrm{diam} f(E) = \sup S \le \frac{\varepsilon}{64} < \varepsilon.$$
(Here we pick "$\frac{\varepsilon}{64}$" instead of ε since we want to get "$\mathrm{diam} f(E) < \varepsilon$" instead of $\mathrm{diam} f(E) \le \varepsilon$.)

(2) ($\Longleftarrow$) Easy. Given $\varepsilon > 0$ there exists a $\delta > 0$ such that $\text{diam} f(E) < \varepsilon$ for all $E \subseteq X$ with $\text{diam} E < \delta$. In particular, for any $p, q \in X$ with $d(p, q) < \delta$, we can take $E = \{p, q\} \subseteq X$ and its diameter

$$\text{diam} E = d(p, q) < \delta.$$

So that

$$d(f(p), f(q)) = \text{diam} f(E) < \varepsilon,$$

or Definition 4.18 holds.

$\square$

Exercise 4.10. *Complete the details of the following alternative proof of Theorem 4.19 (Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X): If f is not uniformly continuous, then for some $\varepsilon > 0$ there are sequences $\{p_n\}$, $\{q_n\}$ in X such that $d_X(p_n, q_n) \to 0$ but $d_Y(f(p_n), f(q_n)) > \varepsilon$. Use Theorem 2.37 to obtain a contradiction.*

Proof.

(1) (Reductio ad absurdum) If f were not uniformly continuous, then for some $\varepsilon > 0$ there are sequences $\{p_n\}$, $\{q_n\}$ in X such that $d_X(p_n, q_n) \to 0$ but $d_Y(f(p_n), f(q_n)) > \varepsilon$.

(2) By Theorem 2.37, there is a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ such that $\{p_{n_k}\}$ converges to $p \in X$. Similar argument to $\{q_n\}$, we have a subsequence $\{q_{n_k'}\}$ of $\{q_n\}$ converging to $q \in X$.

(3) Since

$$d_X(p, q) \le d_X(p, p_{n_k}) + d_X(p_{n_k}, q_{n_k'}) + d_X(q_{n_k'}, q) \to 0$$

(by assumption and (2)) and $d_X(p, q)$ is a constant, $d_X(p, q) = 0$ or $p = q$.

(4) Since f is continuous,

$$\lim_{k \to \infty} f(p_{n_k}) = f(p) = f(q) = \lim_{k \to \infty} f(q_{n_k'})$$

or $d_Y(f(p_{n_k}), f(q_{n_k'})) \to 0$, contrary to the assumption.

$\square$

Exercise 4.11. *Suppose f is a uniformly continuous mapping of a metric space X into a metric space Y and prove that $\{f(x_n)\}$ is a Cauchy sequence in Y for every Cauchy sequence $\{x_n\}$ in X. Use this result to give an alternative proof*

of the theorem stated in Exercise 4.13.

An alternative proof of Exercise 4.13 will be in Exercise 4.13 itself.

Proof (Definition 4.18). Given any Cauchy sequence $\{x_n\}$ in X.

(1) Given any $\varepsilon > 0$. Since f is uniformly continuous, there exists a $\delta > 0$ such that
$$d_Y(f(p), f(q)) < \varepsilon$$
for all p and q in X for which $d_X(p, q) < \delta$.

(2) Since $\{x_n\}$ is Cauchy in X, for such $\delta > 0$ there is an integer N such that
$$d_X(x_n, x_m) < \delta$$
whenever $n, m \geq N$.

(3) By (1)(2),
$$d_Y(f(x_n), f(x_m)) < \varepsilon$$
whenever $n, m \geq N$. Hence $\{f(x_n)\}$ is Cauchy in Y.

□

Proof (Exercise 4.9). Given any Cauchy sequence $\{x_n\}$ in X.

(1) Given any $\varepsilon > 0$. Since f is uniformly continuous, there exists a $\delta > 0$ such that $\mathrm{diam} f(E) < \varepsilon$ for all $E \subseteq X$ with $\mathrm{diam} E < \delta$.

(2) Since $\{x_n\}$ is Cauchy in X, for such $\delta > 0$ there is an integer N such that
$$d_X(x_n, x_m) < \frac{\delta}{64}$$
whenever $n, m \geq N$.

(3) Consider $E = \{x_N, x_{N+1}, \ldots\}$. By (2), $\mathrm{diam} E \leq \frac{\delta}{64} < \delta$. By (1),
$$d_Y(f(x_n), f(x_m)) \leq \mathrm{diam} f(E) < \varepsilon$$
whenever $n, m \geq N$. Hence $\{f(x_n)\}$ is Cauchy in Y.

□

Exercise 4.12. *A uniformly continuous function of a uniformly continuous function is uniformly continuous. State this more precisely and prove it.*

Statement (similar to Theorem 4.7): *suppose X, Y, Z are metric space, $E \subseteq X$, f maps E into Y, g maps the range of f, $f(E)$, into Z, and h is the mapping of E into Z defined by*

$$h(x) = g(f(x)) \qquad (x \in E).$$

If f is uniformly continuous on E and g is uniformly continuous on $f(E)$, then h is uniformly continuous on E.

Proof.

(1) Given $\varepsilon > 0$. Since g is uniformly continuous on $f(E)$, there exists $\eta > 0$ such that

$$d_Z(g(f(p)), g(f(q))) < \varepsilon \quad \text{if} \quad d_Y(f(p), f(q)) < \eta \quad \text{and} \quad f(p), f(q) \in f(E).$$

(2) Since f is uniformly continuous on E, there exists $\delta > 0$ such that

$$d_Y(f(p), f(q)) < \eta \quad \text{if} \quad d_X(p, q) < \delta \quad \text{and} \quad p, q \in E.$$

(3) By (1)(2),
$$d_Z(h(p), h(q)) = d_Z(g(f(p)), g(f(q))) < \varepsilon$$

if $d_X(p, q) < \delta$ and $p, q \in E$. Hence h is uniformly continuous on E.

$\square$

Exercise 4.13. *Let E be a dense subset of a metric space X, and let f be a uniformly continuous real function defined on E. Prove that f has a continuous extension from E to X (see Exercise 4.5 for terminology). (Uniqueness follows from Exercise 4.4.) (Hint: For each $p \in X$ and each positive integer n, let $V_n(p)$ be the set of all $q \in E$ with $d(p, q) < \frac{1}{n}$. Use Exercise 4.9 to show that the intersection of the closures of the sets $f(V_1(p)), f(V_2(p)), \ldots$, consists of a single point, say $g(p)$, of $\mathbb{R}^1$. Prove that the function g so define on X is the desired extension of f.) Could the range space $\mathbb{R}^1$ be replaced by $\mathbb{R}^k$? By any compact metric space? By any complete metric space? By any metric space?*

Proof (Hint). We prove the case that the range metric space is complete.

(1) Given any $p \in X$. We will extend f on $x = p$. For any positive integer n, let $V_n(p)$ be the set of all $q \in E$ with $d(p, q) < \frac{1}{n}$.

(2) *Show that* $\overline{f(V_n(p))} \supseteq \overline{f(V_{n+1}(p))}$. By construction, $V_n(p) \supseteq V_{n+1}(p)$. Thus $f(V_n(p)) \supseteq f(V_{n+1}(p))$ and

$$\overline{f(V_n(p))} \supseteq \overline{f(V_{n+1}(p))}.$$

(3) *Show that* $\lim_{n \to \infty} \mathrm{diam}\overline{f(V_n(p))} = 0$.

(a) Since E is dense in X, $V_n(p) \neq \varnothing$ and thus

$$f(V_n(p)) \neq \varnothing.$$

Especially,
$$\overline{f(V_n(p))} \supseteq f(V_n(p)) \neq \varnothing.$$

Hence $\mathrm{diam}V_n(p)$ and $\mathrm{diam}f(V_n(p))$ are well-defined.

(b) By the definition of $V_n(p)$ or $0 \leq \mathrm{diam}V_n(p) \leq \frac{2}{n}$,

$$\lim_{n \to \infty} \mathrm{diam}V_n(p) = 0.$$

(c) By the uniformly continuity of f (Exercise 4.9),

$$\lim_{n \to \infty} \mathrm{diam}f(V_n(p)) = 0.$$

(d) Since $\mathrm{diam}\overline{f(V_n(p))} = \mathrm{diam}f(V_n(p))$ (Theorem 3.10(a)),

$$\lim_{n \to \infty} \mathrm{diam}\overline{f(V_n(p))} = 0.$$

(4) *Show that there is an integer N such that $\overline{f(V_n(p))}$ is closed and bounded whenever $n \geq N$.*

   (a) (Closeness.) Each $\overline{f(V_n(p))}$ is closed.

   (b) (Boundedness.) Since $\lim_{n\to\infty} \mathrm{diam}\overline{f(V_n(p))} = 0$ by (3), there is an integer N such that

$$\mathrm{diam}\overline{f(V_n(p))} \leq \frac{1}{89}$$

   whenever $n \geq N$. By the definition of diameters of $\overline{f(V_n(p))}$, each $\overline{f(V_n(p))}$ is bounded by $\frac{1}{64}$ whenever $n \geq N$.

   (c) *Note.* If we apply Exercise 4.8 instead, we need extra efforts to generalize Exercise 4.8 to different range spaces for answering the following questions.

(5) By (2)(3)(4) and Exercise 3.21,

$$\bigcap_{n=N}^{\infty} \overline{f(V_n(p))}$$

or

$$\bigcap_{n=1}^{\infty} \overline{f(V_n(p))}$$

consists of exactly one point, say $g(p)$. This point $g(p)$ is an extension of f at $x = p$. Clearly, $g(p) = f(p)$ if $p \in E$.

(6) *Define*

$$g(p) = \begin{cases} \bigcap_{n=1}^{\infty} \overline{f(V_n(p))} = f(p) & (p \in E), \\ \bigcap_{n=1}^{\infty} \overline{f(V_n(p))} & (p \notin E). \end{cases}$$

Show that g is uniformly continuous.

(a) Given any $\varepsilon > 0$. Since f is uniformly continuous on E, there exists $\delta > 0$ such that

$$d(f(p), f(q)) < \frac{\varepsilon}{3} < \varepsilon$$

whenever $d(p, q) < \delta$ and $p, q \in E$. We will show that such δ also holds for g. Now given any $p, q \in X$ with $d(p, q) < \delta$.

(b) Since $\operatorname{diam} f(V_n(p)) = \operatorname{diam} \overline{f(V_n(p))}$ and $\lim_{n \to \infty} \operatorname{diam} \overline{f(V_n(p))} = 0$ (whether $p \in E$ or not), there is an integer N_1 such that

$$\operatorname{diam} f(V_n(p)) < \frac{\varepsilon}{3}$$

whenever $n \geq N_1$. Similarly, there is an integer N_2 such that

$$\operatorname{diam} f(V_n(q)) < \frac{\varepsilon}{3}$$

whenever $n \geq N_2$.

(c) Take an integer N_3 satisfying

$$N_3 > \frac{4}{\delta - d(p, q)} > 0.$$

For any $p' \in V_n(p) \neq \varnothing$ and $q' \in V_n(q) \neq \varnothing$ as $n \geq N_3$, we have

$$d(p', q') \leq d(p', p) + d(p, q) + d(q, q')$$
$$\leq \frac{2}{n} + d(p, q) + \frac{2}{n}$$
$$\leq \frac{2}{N_3} + d(p, q) + \frac{2}{N_3}$$
$$< \frac{2(\delta - d(p, q))}{4} + d(p, q) + \frac{2(\delta - d(p, q))}{4}$$
$$= \delta.$$

(d) Take $N = \max\{N_1, N_2, N_3\}$. For any $p' \in V_N(p)$ and $q' \in V_N(p)$, we have

$$d(g(p), f(p')) \leq \operatorname{diam} f(V_N(p)) < \frac{\varepsilon}{3},$$
$$d(f(p'), f(q')) < \frac{\varepsilon}{3},$$
$$d(f(q'), g(q)) \leq \operatorname{diam} f(V_N(q)) < \frac{\varepsilon}{3}.$$

15

Hence

$$d(g(p), g(q)) \leq d(g(p), f(p')) + d(f(p'), f(q')) + d(f(q'), g(q))$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$
$$= \varepsilon.$$

(7) *Show that the range space $\mathbb{R}^1$ cannot be replaced by any metric space.*

    (a) Take $X = \mathbb{R}$ and $Y = \mathbb{Q}$ with the Euclidean metric. Let $E = \mathbb{Q}$ be a dense subset of $X = \mathbb{R}$. Define $f : E \to Y$ by

$$f(x) = x.$$

    (b) f is uniformly continuous on E.

    (c) (Reductio ad absurdum) If f were having a continuous extension g on X, then

$$\lim_{n \to \infty} g(p_n) = g(p)$$

    for any sequence $\{p_n\}$ in X such that $p_n \neq p$ and $\lim_{n \to \infty} p_n = p$.

    (d) In particular, for some rational sequence $\{p_n\}$ in $E = \mathbb{Q}$ converging to $\sqrt{2} \in X$, we have

$$\lim_{n \to \infty} g(p_n) = \lim_{n \to \infty} f(p_n) = \lim_{n \to \infty} p_n = \sqrt{2} = g(p) \in \mathbb{Q},$$

    which is absurd.

$\square$

Proof (Exercise 4.11). We prove the case that the range metric space is complete.

(1) Given any $p \in X$. We will extend f on $x = p$. Since E is dense in X, there exists a sequence $\{p_n\}$ in E converging to p (whether $p \in E$ or not).

(2) Since E is dense in X, there exists a sequence $\{p_n\}$ in E converging to p (whether $p \in E$ or not). Hence $\{p_n\}$ is Cauchy in E (Theorem 3.11(a)). Since f is uniformly continuous, $\{f(p_n)\}$ is Cauchy. Since the range space is complete, $\{f(p_n)\}$ converges to a point, say $g(p)$. This point $g(p)$ is an extension of f at $x = p$. Clearly, $g(p) = f(p)$ if $p \in E$.

(3) *Show that $g(p)$ is well-defined.* If $\{p'_n\}$ is another sequence in E converging to p, we construct a new sequence $\{p''_n\}$ based on $\{p_n\}$ and $\{p'_n\}$ by

$$p''_n = \begin{cases} p_{\frac{n+1}{2}} & (n \equiv 1 \pmod 2), \\ p'_{\frac{n}{2}} & (n \equiv 0 \pmod 2). \end{cases}$$

Clearly $\{p''_n\}$ also converges to p. So $\{f(p''_n)\}$ converges to a single point. Note that $\{f(p_n)\}$ and $\{f(p'_n)\}$ are two subsequences of $\{f(p''_n)\}$, and thus both subsequences converge to the same point.

(4) *Define*

$$g(p) = \begin{cases} f(p) & (p \in E), \\ \lim_{n \to \infty} f(p_n) & (p \notin E) \end{cases}$$

where $\{p_n\}$ is any sequence in E converging to p. Show that g is uniformly continuous.

(a) Given any $\varepsilon > 0$. Since f is uniformly continuous on E, there exists $\delta > 0$ such that

$$d(f(p), f(q)) < \frac{\varepsilon}{3} < \varepsilon$$

whenever $d(p, q) < \delta$ and $p, q \in E$. We will show that such δ also holds for g. Now given any $p, q \in X$ with $d(p, q) < \delta$.

(b) By (2), there exists a sequence $\{p_n\}$ in E such that $\lim p_n = p$. Take an integer N_1 such that

$$d(p_n, p) < \frac{\delta - d(p, q)}{2}$$

whenever $n \geq N_1$. Similarly, there exists a sequence $\{q_n\}$ in E such that $\lim q_n = q$. Take an integer N_2 such that

$$d(q_n, q) < \frac{\delta - d(p, q)}{2}$$

whenever $n \geq N_2$. Therefore,

$$\begin{aligned} d(p_n, q_n) &\leq d(p_n, p) + d(p, q) + d(q, q_n) \\ &< \frac{\delta - d(p, q)}{2} + d(p, q) + \frac{\delta - d(p, q)}{2} \\ &= \delta. \end{aligned}$$

whenever $n \geq N_1$ and $n \geq N_2$.

(c) Since $\lim f(p_n) = g(p)$, there is an integer N_3 such that

$$d(f(p_n), g(p)) < \frac{\varepsilon}{3}$$

whenever $n \geq N_3$. Similarly, since $\lim f(q_n) = g(q)$, there is an integer N_4 such that

$$d(f(q_n), g(q)) < \frac{\varepsilon}{3}$$

whenever $n \geq N_4$.

(d) Take $N = \max\{N_1, N_2, N_3, N_4\}$, we have

$$\begin{aligned} d(g(p), f(p_N)) &< \frac{\varepsilon}{3}, \\ d(f(p_N), f(q_N)) &< \frac{\varepsilon}{3}, \\ d(f(q_N), g(q)) &< \frac{\varepsilon}{3}. \end{aligned}$$

Hence

$$d(g(p), g(q)) \leq d(g(p), f(p_N)) + d(f(p_N), f(q_N)) + d(f(q_N), g(q))$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$
$$= \varepsilon.$$

$\square$

Exercise 4.14 (Brouwer's fixed-point theorem). *Let $I = [0, 1]$ be the closed unit interval. Suppose f is continuous mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$.*

Proof (Theorem 4.23). Let $g(x) = f(x) - x$ in I.

(1) $g(0) = 0$. Take $x = 0$.

(2) $g(1) = 0$. Take $x = 1$.

(3) Suppose $g(0) \neq 0$ ($f(0) \neq 0$) and $g(1) \neq 0$ ($f(1) \neq 1$). Since $f : I \to I$, $f(0) > 0$ and $f(1) < 1$. That is, $g(0) > 0$ and $g(1) < 0$. Applying the intermediate value theorem (Theorem 4.23), there is a point in $\xi \in (0, 1)$ such that $g(\xi) = 0$. That is, $f(\xi) = \xi$ for some $\xi \in (0, 1)$.

In any case, the conclusion holds. $\square$

Supplement. Brouwer's fixed-point theorem.

(1) In the $\mathbb{R}^1$, see Exercise 4.14 itself.

(2) In the $\mathbb{R}^2$, see Exercise 8.29.

(3) In the $\mathbb{R}^n$, every continuous function from a closed ball of a Euclidean space $\mathbb{R}^n$ into itself has a fixed point (without proof).

(4) In a Banach space, Schauder fixed-point theorem.

Exercise 4.15. *Call a mapping of X into Y **open** if $f(V)$ is an open set in Y whenever V is an open set in X. Prove that every continuous open mapping of $\mathbb{R}^1$ into $\mathbb{R}^1$ is monotonic.*

In fact, f is strictly monotonic.

Proof.

(1) (Reductio ad absurdum) If f were not strictly monotonic, then there exist $a < c < b \in \mathbb{R}^1$ such that

$$f(a) \leq f(c) \geq f(b)$$

or

$$f(a) \geq f(c) \leq f(b).$$

(2) In any case, f is a real continuous function on a compact set $[a, b]$. By Theorem 4.16, there exists $p, q \in [a, b]$ such that

$$M = \sup_{x \in [a,b]} f(x) = f(p),$$
$$m = \inf_{x \in [a,b]} f(x) = f(q).$$

(3) As $f(a) \leq f(c) \geq f(b)$, we consider where f reaches its maximum value M (by (2)).

    (a) $f(a) = M$ or $f(b) = M$. Since $f(a) \leq f(c) \geq f(b)$, by the maximality of M, $f(c) = M$ or $M \in f((a, b))$.

    (b) $f(a) < M$ and $f(b) < M$. Hence $M \in f((a, b))$ clearly.

In any case, $M \in f((a, b))$. Note that $f((a, b))$ is open since f is an open mapping and (a, b) is open.

Since M is in an open set $f((a, b))$, there exists an open neighborhood $B(M; r) \subseteq f((a, b))$ where $r > 0$. Hence

$$M + \frac{r}{64} \in B(M; r) \subseteq f((a, b)),$$

contrary to the maximality of M.

(4) As $f(a) \geq f(c) \leq f(b)$, we consider where f reaches its minimum value m (by (2)). Similar to (3), we can reach a contradiction again.

(5) By (3)(4), (1) is absurd, and thus f is strictly monotonic.

$\square$

Exercise 4.16. *Let $[x]$ denote the largest integer contained in x, this is, $[x]$ is a integer such that $x - 1 < [x] \leq x$; and let $(x) = x - [x]$ denote the fractional part of x. What discontinuities do the function $[x]$ and (x) have?*

Proof.

(1) *The function $[x]$ only has discontinuities at $x \in \mathbb{Z}$.*

19

(a) For any $p \notin \mathbb{Z}$, there is an integer n such that $n < p < n+1$. Given any $\varepsilon > 0$, there is a $\delta = \min\{p - n, (n+1) - p\} > 0$ such that $\big|[x] - [p]\big| < \varepsilon$ whenever $|x - p| < \delta$. In fact, $|x - p| < \delta$ is equivalent to $n < x < n+1$ and therefore $\big|[x] - [p]\big| = |n - n| = 0 < \varepsilon$.

(b) For any $p \in \mathbb{Z}$, $\lim_{x \to p^+} [x] = p$ and $\lim_{x \to p^-} [x] = p - 1$.

(2) *The function (x) only has discontinuities at $x \in \mathbb{Z}$.*

(a) Since $[x]$ is continuous on $\mathbb{R} - \mathbb{Z}$ and x is continuous on $\mathbb{R}$, especially on $\mathbb{R} - \mathbb{Z}$, $(x) = x - [x]$ is continuous on $\mathbb{R} - \mathbb{Z}$.

(b) For any $p \in \mathbb{Z}$, $\lim_{x \to p^+} (x) = 0$ and $\lim_{x \to p^-} (x) = 1$.

□

Exercise 4.17. *Let f be a real function defined on (a, b). Prove that the set of points at which f has a simple discontinuity is at most countable. (Hint: Let E be the set on which $f(x-) < f(x+)$. With each point x of E, associate a triple (p, q, r) of rational numbers such that*

(a) $f(x-) < p < f(x+)$,

(b) $a < q < t < x$ implies $f(t) < p$,

(c) $x < t < r < b$ implies $f(t) > p$.

The set of all such triples is countable. Show that each triple is associated with at most one point of E. Deal similarly with the other possible types of simple discontinuities.)

Proof (Hint).

(1) By Definition 4.26, *it suffices to show that each following set is at most countable and thus the union of four sets are at most countable (Corollary to Theorem 2.12).*

(a) $E_1 = \{x : f(x+) > f(x-)\}$,
(b) $E_2 = \{x : f(x+) < f(x-)\}$,
(c) $E_3 = \{x : f(x) > f(x+) = f(x-)\}$,
(d) $E_4 = \{x : f(x) < f(x+) = f(x-)\}$,

(2) *Show that with each point x of E_1, we can associate a triple (p, q, r) of rational numbers such that*

(a) $f(x-) < p < f(x+)$,
(b) $a < q < t < x$ implies $f(t) < p$,
(c) $x < t < r < b$ implies $f(t) > p$.

(i) For (a), as $(f(x-), f(x+))$ is a nonempty open set in $\mathbb{R}$ and $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a rational number $p \in (f(x-), f(x+))$ or $f(x-) < p < f(x+)$.

(ii) For (b), since $f(x-)$ exists, for $\varepsilon = p - f(x-) > 0$ there exists $\delta > 0$ such that

$$f(x-) - \varepsilon < f(t) < f(x-) + \varepsilon \quad \text{whenever} \quad a < x - \delta < t < x.$$

So
$$f(t) < p \quad \text{whenever} \quad a < x - \delta < t < x.$$

Since $(x - \delta, x)$ is a nonempty open set in $\mathbb{R}$ and $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a rational number $q \in (x - \delta, x)$ or $a < x - \delta < q < x$. Hence if $a < q < t < x$ we have $f(t) < p$.

(iii) For (c), similar to (ii). Since $f(x+)$ exists, for $\varepsilon = f(x+) - p > 0$ there exists $\delta > 0$ such that

$$f(x+) - \varepsilon < f(t) < f(x+) + \varepsilon \quad \text{whenever} \quad x < t < x + \delta < b.$$

So
$$f(t) > p \quad \text{whenever} \quad x < t < x + \delta < b.$$

Since $(x, x + \delta)$ is a nonempty open set in $\mathbb{R}$ and $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a rational number $r \in (x, x + \delta)$ or $x < r < x + \delta < b$. Hence if $x < t < r < b$ we have $f(t) > p$.

(3) *Show that each triple in (2) is associated with at most one point of E_1.* (Reductio ad absurdum) If there were another $x' > x$ such that

(a) $f(x'-) < p < f(x'+)$,

(b) $a < q < t < x'$ implies $f(t) < p$,

(c) $x' < t < r < b$ implies $f(t) > p$,

then there exists $t_0 \in \mathbb{R}$ satisfying $x < t_0 < x'$ such that $f(t_0) > p$ (property (c) of x) and $f(t_0) < p$ (property (b) of x'), which is absurd. The case $x' < x$ is similar.

(4) By (2)(3), the set E_1 is a subset of a countable set

$$\{(p, q, r) : (p, q, r) \in \mathbb{Q}^3\},$$

which is at most countable (Theorem 2.8 and Corollary to Theorem 2.12). Similarly, E_2 is at most countable.

(5) Similar to (2), for each point x of E_3, we can associate a triple (p, q, r) of rational numbers such that

(a) $f(x) > p > f(x+) = f(x-)$,

(b) $a < q < t < x$ implies $f(t) < p$,

21

(c) $x < t < r < b$ implies $f(t) < p$.

(6) *Show that each triple in (5) is associated with at most one point of E_3.*
(Reductio ad absurdum) If there were another $x' > x$ such that

(a) $f(x') > p > f(x'+) = f(x'-)$,
(b) $a < q < t < x'$ implies $f(t) < p$,
(c) $x' < t < r < b$ implies $f(t) < p$,

then $f(x) < p$ (property (b) of x'), which is absurd. The case $x' < x$ is
similar.

(7) By (5)(6), the set E_3 is a subset of a countable set

$$\{(p, q, r) : (p, q, r) \in \mathbb{Q}^3\},$$

which is at most countable (Theorem 2.8 and Corollary to Theorem 2.12).
Similarly, E_4 is at most countable.

(8) By (4)(7), all E_j $(j = 1, 2, 3, 4)$ are at most countable. Hence the set of
points at which f has a simple discontinuity is at most countable.

$\square$

Exercise 4.18 (Thomae's function). *Every rational x can be written in the
form $x = \frac{m}{n}$, where $n > 0$, and m and n are integers without any common
divisors. When $x = 0$, we take $n = 1$. Consider the function f defined on $\mathbb{R}^1$
by*

$$f(x) = \begin{cases} 0 & (x \text{ irrational}), \\ \frac{1}{n} & (x = \frac{m}{n}). \end{cases}$$

*Prove that f is continuous at every irrational point, and that f has a simple
discontinuity at every rational point.*

Proof.

(1) *Show that f has period 1.*

(a) As x is irrational, $x + 1$ is irrational too. Hence $f(x + 1) = 0 = f(x)$.
(b) As $x = \frac{m}{n} \in \mathbb{Q}$, where $n > 0$, and m and n are integers without any
common divisors.
$$x + 1 = \frac{m + n}{n} \in \mathbb{Q},$$
where $n > 0$, and $\gcd(m + n, n) = \gcd(m, n) = 1$. Hence $f(x + 1) = \frac{1}{n} = f(x)$.

In any case, $f(x + 1) = f(x)$ for any $x \in \mathbb{R}$.

(2) *Show that $f(p+) = f(p-) = 0$ for any $p \in \mathbb{R}$.*

    (a) By (1), we might assume $p \in [0, 1]$. For the edge cases $f(0-)$ and $f(1+)$, note that $f(0-) = f(1-)$ and $f(1+) = f(0+)$.

    (b) Now given any point $p \in [0, 1]$. Given $\varepsilon > 0$, it suffices to find $\delta > 0$ such that

$$\left|f(x) - 0\right| < \varepsilon \quad \text{whenever} \quad p < x < p + \delta.$$

    (The case $f(p-) = 0$ is similar.)

    (c) For such $\varepsilon > 0$, there is an integer $N \geq s$ such that

$$\frac{1}{N} < \varepsilon.$$

    Hence

$$E = \left\{ \frac{m}{n} \in [0, 1] : n \leq N \text{ where } \frac{m}{n} \text{ is defined as this exercise} \right\} - \{p\}$$

    is finite since m and n are satisfying the relation $0 \leq m \leq n \leq N$ and there are only finitely many pairs of (m, n).

    (d) Now we can take

$$\delta = \min\{|q - p| : q \in E\} > 0.$$

    As $p < x < p + \delta$, $\left|f(x) - 0\right| = 0 < \varepsilon$ if x is irrational;

$$\left|f(x) - 0\right| < \frac{1}{N} < \varepsilon$$

    if x is rational. (Note that $x \notin E$ by the construction of δ and thus the denominator of x is $> N$.) In any case, $\left|f(x) - 0\right| < \varepsilon$.

(3) Since $f(x) = 0$ if $x \in \mathbb{R} - \mathbb{Q}$ and $f(x) \neq 0$ if $x \in \mathbb{Q}$, f is continuous at every irrational point and f has a simple discontinuity at every rational point (by (2)).

$\square$

Note.

(1) *f is nowhere differentiable.*

    (a) For any rational number, this follows from discontinuity.

    (b) For any irrational number x, given $\varepsilon = \frac{1}{64} > 0$, *it suffices to show that for any $\delta > 0$, there exists $x < t < x + \delta$ such that*

$$\frac{f(t) - f(x)}{t - x} > \frac{1}{64}.$$

(c) It is clear to find t in $\mathbb{Q}$. For such $\delta > 0$, there is an integer n such that

$$n > \frac{1}{\delta} > 0.$$

For such $n > 0$, there is an integer m such that

$$m - 1 < nx < m.$$

($m - 1 \leq nx < m$ is true trivially and notice that the equality does not hold since x is irrational.) Take

$$t = \frac{m}{n}.$$

So that

$$x < t < x + \frac{1}{n} < x + \delta.$$

(d) Let $\gcd(m, n) = g \geq 1$. Then

$$f(t) = \frac{1}{\frac{n}{g}} = \frac{g}{n} \geq \frac{1}{n}$$

and thus

$$\frac{f(t) - f(x)}{t - x} > \frac{\frac{1}{n}}{\frac{1}{n}} = 1 > \frac{1}{64}.$$

□

(2) $f \in \mathscr{R}$ on any bounded interval $[a, b]$ and $\int_a^b f \, dx = 0$. To prove this, use the similar argument in Exercise 6.6.

Exercise 4.19. *Suppose f is a real function with domain $\mathbb{R}^1$ which has the intermediate value property: If $f(a) < c < f(b)$, then $f(x) = c$ for some x between a and b. Suppose also, for every rational r, that the set of all x with $f(x) = r$ is closed. Prove that f is continuous. (Hint: If $x_n \to x_0$ but $f(x_n) > r > f(x_0)$ for some r and all n, then $f(t_n) = r$ for some t_n between x_0 and x_n; thus $t_n \to x_0$. Find a contradiction. (N. J. Fine, Amer. Math. Monthly, vol. 73, 1966, p.782.))*

Proof.

(1) (Reductio ad absurdum) If f were not continuous at $x = x_0$, then there exists a sequence $\{x_n\}$ in $\mathbb{R}^1$ such that $\{f(x_n)\}$ in $\mathbb{R}^1$ not converging to $f(x_0)$. So there exists $\varepsilon_0 > 0$ and a subsequence $\{x_{n'}\}$ of $\{x_n\}$ such that

$$\left| f\left(x_{n'}\right) - f(x_0) \right| \geq \varepsilon_0$$

for all n'. Hence

$$f\left(x_{n'}\right) \geq f(x_0) + \varepsilon_0 \quad \text{or} \quad f\left(x_{n'}\right) \leq f(x_0) - \varepsilon_0$$

for all n'.

(2) Since there are infinitely many $\{x_{n'}\}$, there are only two possible cases:

(a) There is a subsequence $\{x_{n''}\}$ of $\{x_{n'}\}$ such that $f\left(x_{n''}\right) \geq f(x_0) + \varepsilon_0$ for all n''.

(b) There is a subsequence $\{x_{n''}\}$ of $\{x_{n'}\}$ such that $f\left(x_{n''}\right) \leq f(x_0) - \varepsilon_0$ for all n''.

In any case, we may assume that there exists $\varepsilon_0 > 0$ and a sequence $\{x_n\}$ such that
$$f(x_n) \geq f(x_0) + \varepsilon_0$$
for all n. (The case that $f(x_n) \leq f(x_0) - \varepsilon_0$ is similar.)

(3) Pick any rational number $r \in (f(x_0), f(x_0) + \varepsilon_0)$ since $\mathbb{Q}$ is dense in $\mathbb{R}$. (In fact, $\mathbb{Q}$ can be replaced by any dense subset of $\mathbb{R}$ in this exercise.) Since

$$f(x_n) \geq f(x_0) + \varepsilon_0 > r > f(x_0),$$

by the intermediate value property, there exists some t_n between x_0 and x_n such that $f(t_n) = r$.

(4) Let $E = \{t : f(t) = r\}$ be a closed set (since $r \in \mathbb{Q}$ and the assumption). Hence $t_n \in E$ for all n (by (3)).

(5) $\lim t_n = x_0$ since $\lim x_n = x_0$. Hence x_0 is a limit point of E. (Note that $x_0 \notin E$ since $f(x_0) < r$ (by (3)).) Since E is closed, $x_0 \in E$ or $f(x_0) = r$, which is absurd.

□

Supplement. *Suppose f is a differentiable real function with domain $\mathbb{R}^1$. Suppose for every rational r, that the set of all x with $f'(x) = r$ is closed. Prove that f' is continuous.* Theorem 5.12 (Darboux's theorem) and Exercise 4.19 itself implies all.

Exercise 4.20. *If E is a nonempty subset of a metric space X, define the distance from $x \in X$ to E by*

$$\rho_E(x) = \inf_{z \in E} d(x, z).$$

(a) *Prove that $\rho_E(x) = 0$ if and only if $x \in \overline{E}$.*

25

(b) *Prove that $\rho_E(x)$ is a uniformly continuous function on X, by showing that*

$$\big|\rho_E(x) - \rho_E(y)\big| \le d(x, y)$$

for all $x \in X$, $y \in X$. (Hint: $\rho_E(x) \le d(x, z) \le d(x, y) + d(y, z)$, so that

$$\rho_E(x) \le d(x, y) + \rho_E(y).)$$

Proof of (a).

$$\rho_E(x) > 0 \iff \inf_{z \in E} d(x, z) > 0$$
$$\iff \exists\, r > 0 \text{ such that } B(x; r) \cap E = \varnothing$$
$$\iff x \notin \overline{E}.$$

□

Proof of (b) (Hint). Given any $x \in X$, $y \in X$. For any $z \in E$, we have

$$\rho_E(x) \le d(x, z) \le d(x, y) + d(y, z).$$

$\rho_E(x) - d(x, y)$ is a lower bound of the set $\{d(y, z) : z \in E\}$. So that

$$\rho_E(x) \le d(x, y) + \rho_E(y).$$

Similarly,

$$\rho_E(y) \le d(x, y) + \rho_E(x).$$

Hence

$$\big|\rho_E(x) - \rho_E(y)\big| \le d(x, y),$$

or $\rho_E(x)$ is uniformly continuous on X (by taking $\delta = \varepsilon > 0$ in Definition 4.18).
□

Exercise 4.21. *Suppose K and F are disjoint sets in a metric space X. K is compact. F is closed. Prove that there exists $\delta > 0$ such that $d(p, q) > \delta$ if $p \in K$, $q \in F$. (Hint: ρ_F is a continuous positive function on K.) Show that the conclusion may fail for two disjoint closed sets if neither is compact.*

Proof.

(1)

(2)

□

Exercise 4.22. *Let A and B be disjoint nonempty closed sets in a metric space X, and define*

$$f(p) = \frac{\rho_A(p)}{\rho_A(p) + \rho_B(p)}, \qquad (p \in X).$$

Show that f is a continuous function on X whose range lies in $[0,1]$, that $f(p) = 0$ precisely on A and $f(p) = 1$ precisely on B. This establishes a converse of Exercise 4.3: Every closed set A contained in X is $Z(f)$ for some continuous real f on X. Setting

$$V = f^{-1}\left(\left[0, \frac{1}{2}\right)\right), \qquad W = f^{-1}\left(\left(\frac{1}{2}, 1\right]\right),$$

show that V and W are open and disjoint, and that A is contained in V, B is contained in W. (Thus pairs of disjoint closed set in a metric space can be covered by pairs of disjoint open sets. This property of metric spaces is called normality.)

Proof.

(1)

(2)

□

Exercise 4.23. *A real-valued function f defined in (a,b) is said to be **convex** if*

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

whenever $a < x < b$, $a < y < b$, $0 < \lambda < 1$. Prove that every convex function is continuous. Prove that every increasing convex function of a convex function is convex. (For example, if f is convex, so is e^f.)

If f is convex in (a,b) and if $a < s < t < u < b$, show that

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

Proof.

(1) *Show that* $\frac{f(t)-f(s)}{t-s} \le \frac{f(u)-f(s)}{u-s} \le \frac{f(u)-f(t)}{u-t}$. Since

$$t = \frac{t-s}{u-s}u + \left(1 - \frac{t-s}{u-s}\right)s$$
$$= \left(1 - \frac{u-t}{u-s}\right)u + \frac{u-t}{u-s}s$$

and $0 < \frac{t-s}{u-s}, \frac{u-t}{u-s} < 1$, by the convexity of f we have

$$f(t) \le \frac{t-s}{u-s}f(u) + \left(1 - \frac{t-s}{u-s}\right)f(s),$$
$$f(t) \le \left(1 - \frac{u-t}{u-s}\right)f(u) + \frac{u-t}{u-s}f(s).$$

It is equivalent to

$$\frac{f(t)-f(s)}{t-s} \le \frac{f(u)-f(s)}{u-s} \le \frac{f(u)-f(t)}{u-t}.$$

□

(2) *If x, y, x', y' are points of (a,b) with $x \le x' < y'$ and $x < y \le y'$, then the chord over (x', y') has larger slope than the chord over (x, y); that is,*

$$\frac{f(y)-f(x)}{y-x} \le \frac{f(y')-f(x')}{y'-x'}.$$

It is a corollary to (1).

(3) *Show that f is continuous.* Let $[c,d] \subseteq (a,b)$. Then by (2),

$$\frac{f(c)-f(a)}{c-a} \le \frac{f(y)-f(x)}{y-x} \le \frac{f(b)-f(d)}{b-d}$$

for x, y in $[c,d]$. Thus $|f(y) - f(x)| \le M|y - x|$ in $[c,d]$ (where $M = \max\left(|\frac{f(c)-f(a)}{c-a}|, |\frac{f(b)-f(d)}{b-d}|\right)$), and so f is absolutely continuous on each closed subinterval of (a,b). Especially, f is continuous.

(4) *Let f be a convex function, g be an increasing convex function, and $h = g \circ f$. Show that h is convex.*

$$
\begin{aligned}
f(\lambda x + (1-\lambda)y) &\le \lambda f(x) + (1-\lambda)f(y), &\text{(Convexity of } f\text{)}\\
g(f(\lambda x + (1-\lambda)y)) &\le g(\lambda f(x) + (1-\lambda)f(y)) &\text{(Increasing of } g\text{)}\\
&\le \lambda g(f(x)) + (1-\lambda)g(f(y)), &\text{(Convexity of } g\text{)}\\
h(\lambda x + (1-\lambda)y) &\le \lambda h(x) + (1-\lambda)h(y).
\end{aligned}
$$

28

□

Exercise 4.24. *Assume that f is a continuous real function defined in (a, b) such that*
$$f\left(\frac{x+y}{2}\right) \le \frac{f(x) + f(y)}{2}$$
for all $x, y \in (a, b)$. Prove that f is convex.

Proof.

(1) *Show that*
$$f\left(\frac{x_1 + \cdots + x_n}{n}\right) \le \frac{f(x_1) + \cdots + f(x_n)}{n}$$

whenever $a < x_i < b$ ($1 \le i \le n$). Apply Cauchy induction and use the same argument in proving the AM-GM inequality. As $n = 1, 2$, the inequality holds by assumption. Suppose $n = 2^k$ ($k \ge 1$) the inequality holds. As $n = 2^{k+1}$,

$$f\left(\frac{x_1 + \cdots + x_{2^{k+1}}}{2^{k+1}}\right)$$
$$= f\left(\frac{1}{2}\left(\frac{x_1 + \cdots + x_{2^k}}{2^k} + \frac{x_{2^k+1} + \cdots + x_{2^{k+1}}}{2^k}\right)\right)$$
$$\le \frac{1}{2}\left(f\left(\frac{x_1 + \cdots + x_{2^k}}{2^k}\right) + f\left(\frac{x_{2^k+1} + \cdots + x_{2^{k+1}}}{2^k}\right)\right)$$
$$\le \frac{1}{2}\left(\frac{f(x_1) + \cdots + f(x_{2^k})}{2^k} + \frac{f(x_{2^k+1}) + \cdots + f(x_{2^{k+1}})}{2^k}\right)$$
$$= \frac{f(x_1) + \cdots + f(x_{2^k}) + f(x_{2^k+1}) + \cdots + f(x_{2^{k+1}})}{2^{k+1}}$$
$$= \frac{f(x_1) + \cdots + f(x_{2^{k+1}})}{2^{k+1}}.$$

As n is not a power of 2, then it is certainly less than some natural power of 2, say $n < 2^m$ for some m. Let

$$x_{n+1} = \cdots = x_{2^m} = \frac{x_1 + \cdots + x_n}{n} = \alpha.$$

Then by the induction hypothesis,

$$f(\alpha) = f\left(\frac{x_1 + \cdots + x_n + \alpha + \cdots + \alpha}{2^m}\right)$$
$$\leq \frac{f(x_1) + \cdots + f(x_n) + f(\alpha) + \cdots + f(\alpha)}{2^m}$$
$$\leq \frac{f(x_1) + \cdots + f(x_n) + (2^m - n)f(\alpha)}{2^m},$$
$$2^m f(\alpha) \leq f(x_1) + \cdots + f(x_n) + (2^m - n)f(\alpha),$$
$$nf(\alpha) \leq f(x_1) + \cdots + f(x_n),$$

or $f\left(\frac{1}{n}(x_1 + \cdots + x_n)\right) \leq \frac{1}{n}(f(x_1) + \cdots f(x_n))$.

(2) Hence,
$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for any rational λ in $(0, 1)$. (Given any positive integers $p < q$, put $n = q$, $x_1 = \cdots = x_p = x$ and $x_{p+1} = \cdots = x_n = y$ in (1).)

(3) Given any real $\lambda \in (0, 1)$, there is a sequence of rational numbers $\{r_n\} \subseteq (0, 1)$ such that $r_n \to \lambda$. By (2),

$$f(r_n x + (1 - r_n)y) \leq r_n f(x) + (1 - r_n)f(y)$$

for any rational r_n in $(0, 1)$. Taking limit on the both sides and using the continuity of f, we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

□

Proof (Reductio ad absurdum). If f were not convex, then there is a subinterval $[c, d] \subseteq (a, b)$ such that

$$\frac{f(d) - f(c)}{d - c} < \frac{f(x_0) - f(c)}{x_0 - c}$$

for some $x_0 \in [c, d]$. Let

$$g(x) = f(x) - f(c) - \frac{f(d) - f(c)}{d - c}(x - c)$$

for $x \in [c, d]$. Therefore,

(1) $g(x)$ is continuous and midpoint convex.

(2) $g(c) = g(d) = 0$.

(3) Let $M = \sup\{g(x) : x \in [c, d]\}$. $\infty > M > 0$ due to the continuity of g and the existence of x_0. And let $\xi = \inf\{x \in [c, d] : g(x) = M\}$. By the continuity of g, $g(\xi) = M$. $\xi \in (c, d)$ by (2).

(4) Since (c, d) is open, there is $h > 0$ such that $(\xi - h, \xi + h) \subseteq (c, d)$. By the minimality of ξ and M, $g(\xi - h) < g(\xi)$ and $g(\xi + h) \leq g(\xi)$.

Therefore,

$$g(\xi - h) + g(\xi + h) < 2g(\xi),$$
$$\frac{g(\xi - h) + g(\xi + h)}{2} < g(h)$$
$$= g\left(\frac{(\xi - h) + (\xi + h)}{2}\right),$$

contrary to the midpoint convexity of g. □

The result becomes false if "continuity of f" is omitted.

Exercise 4.25. *If $A \subset \mathbb{R}^k$ and $B \subset \mathbb{R}^k$, define $A + B$ to be the set of all sums $\mathbf{x} + \mathbf{y}$ with $\mathbf{x} \in A$, $\mathbf{y} \in B$.*

(a) *If K is compact and C is closed in $\mathbb{R}^k$, prove that $K + C$ is closed. (Hint: Take $\mathbf{z} \notin K + C$, put $F = \mathbf{z} - C$, the set of all $\mathbf{z} - \mathbf{y}$ with $\mathbf{y} \in C$. Then K and F are disjoint. Choose δ as in Exercise 4.21. Show that the open ball with center $\mathbf{z}$ and radius δ does not intersect $K + C$.)*

(b) *Let α be an irrational real number. Let C_1 be the set of all integers, let C_2 be the set of all $n\alpha$ with $n \in C_1$. Show that C_1 and C_2 are closed subsets of $\mathbb{R}^1$ whose sum $C_1 + C_2$ is not closed, by showing that $C_1 + C_2$ is a countable dense subset of $\mathbb{R}^1$.*

Proof.

(1)

(2)

□

Exercise 4.26. *Suppose X, Y, Z are metric spaces, and Y is compact. Let f map X into Y, let g be a continuous one-to-one mapping of Y into Z, and put $h(x) = g(f(x))$ for $x \in X$.*

Prove that f is uniformly continuous if h is uniformly continuous. (Hint: g^{-1} has compact domain $g(Y)$, and $f(x) = g^{-1}(h(x))$.)

Prove also that f is continuous if h is continuous.

Show (by modifying Example 4.21, or by finding a different example) that the compactness of Y cannot be omitted from the hypotheses, even when X and Z are compact.

Proof.

(1)

(2)

□