Chapter 2: Four Important Linear PDE

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Notes.

(1) (Equation (7))

$$|D\Phi(x)| \leq \frac{C}{|x|^{n-1}}, \qquad |D^2\Phi(x)| \leq \frac{C}{|x|^n} \qquad (x \neq 0)$$

for some constant $C > 0$. In fact,

$$\begin{aligned}
\frac{\partial}{\partial x_i}\Phi(x) &= -\frac{1}{n\alpha(n)} x_i |x|^{-n}, \\
\frac{\partial^2}{\partial x_i \partial x_j}\Phi(x) &= \frac{1}{n\alpha(n)}(n x_i x_j - |x|^2 \delta_{ij})|x|^{-n-2}.
\end{aligned}$$

(2) (Equation (12)) The constant C is rescaled. It is just a constant.

(3) (Equation (13)) Take $U \mapsto B(0,\varepsilon)$, $u(y) \mapsto \Phi(y)$ and $v(y) \mapsto f(x-y)$ in the integration by parts (Green's first identity):

$$\int_U Dv \cdot Du \, dx = -\int_U u \Delta v \, dx + \int_{\partial U} \frac{\partial v}{\partial \nu} u \, dS.$$

Problem 2.1. *Write down an explicit formula for a function u solving the initial-value problem*

$$\begin{cases} u_t + b \cdot Du + cu = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Here $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ are constants.

Proof (Transport equation). Define

$$z(s) = u(x + sb, t + s) \qquad (s \in \mathbb{R}).$$

So

$$\begin{aligned}
\dot{z}(s) &= Du(x + sb, t + s) \cdot b + u_t(x + sb, t + s) \\
&= -cu(x + sb, t + s) \\
&= -cz(s).
\end{aligned}$$

Solve this ODE to get

$$\begin{aligned} z(s) = z(0)e^{-cs} &\Longrightarrow u(x+sb, t+s) = u(x,t)e^{-cs} \\ &\Longrightarrow u(x-tb, 0) = u(x,t)e^{ct} \qquad\qquad \text{(Let } s = -t\text{)} \\ &\Longrightarrow g(x-tb) = u(x,t)e^{ct} \\ &\Longrightarrow u(x,t) = g(x-tb)e^{-ct}. \end{aligned}$$

□

Problem 2.2. *Prove that Laplace's equation $\Delta u = 0$ is rotation invariant; that is, if O is an orthogonal $n \times n$ matrix and we define*

$$v(x) := u(Ox) \qquad (x \in \mathbb{R}^n),$$

then $\Delta v = 0$.

Proof.

(1) Let $O = [O_{ij}]$. O is orthogonal if $OO^t = O^t O = I$, or

$$\sum_{i=1}^n O_{pi} O_{qi} = \delta_{pq}$$

where δ_{pq} is the Kronecker delta.

(2) Let $y = Ox$. So that

$$\begin{aligned} D_i v(x) &= \sum_{p=1}^n D_p u(y) O_{pi}, \\ D_{ij} v(x) &= \sum_{q=1}^n \sum_{p=1}^n D_{pq} u(y) O_{pi} O_{qj}, \\ \Delta v(x) &= \sum_{i=1}^n D_{ii} v(x) \\ &= \sum_{i=1}^n \sum_{q=1}^n \sum_{p=1}^n D_{pq} u(y) O_{pi} O_{qi} \\ &= \sum_{q=1}^n \sum_{p=1}^n D_{pq} u(y) \left(\sum_{i=1}^n O_{pi} O_{qi} \right) \\ &= \sum_{q=1}^n \sum_{p=1}^n D_{pq} \delta_{pq} \\ &= \sum_{q=1}^n D_{qq} u(y) \\ &= \Delta u(y). \end{aligned}$$

(3) As $\Delta u(y) = 0$, $\Delta v(x) = 0$.

$\square$

Problem 2.3. *Modify the proof of the mean value formulas to show for $n \geq 3$ that*

$$u(0) = ⨍_{\partial B(0,r)} g dS + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f dx,$$

provided

$$\begin{cases} -\Delta u = f & \text{in } B^0(0,r) \\ u = g & \text{on } \partial B(0,r). \end{cases}$$

Proof.

(1) ...

(2) ...

$\square$

Problem 2.4. *We say $v \in C^2(\overline{U})$ is **subharmonic** if*

$$-\Delta v \leq 0 \qquad \text{in } U.$$

(a) *Prove for subharmonic v that*

$$v(x) \leq ⨍_{B(x,r)} v dy \qquad \text{for all } B(x,r) \subseteq U.$$

(b) ...

(c) ...

(d) *Prove $v := |Du|^2$ is subharmonic, whenever u is harmonic.*

Proof of (a). It is exactly the same as the proof of Theorem 2 (Mean-value theorem for Laplace's equation).

(1) Set

$$\phi(r) := ⨍_{\partial B(x,r)} v(y) dS(y) = ⨍_{\partial B(0,1)} v(x + rz) dS(z)$$

$(r > 0)$. Then

$$
\begin{aligned}
\phi'(r) &= \fint_{\partial B(0,1)} Dv(\underbrace{x + rz}_{=y}) \cdot z dS(z) \\
&= \fint_{\partial B(x,y)} Dv(y) \cdot \underbrace{\frac{y-x}{r}}_{=\nu} dS(y) \\
&= \fint_{\partial B(x,y)} \frac{\partial v}{\partial \nu} dS(y) \\
&= \frac{r}{n} \fint_{B(x,y)} \Delta u(y) dy && \text{(Green's first identity)} \\
&\geq 0 && \text{(By assumption)}
\end{aligned}
$$

or $\phi(r)$ is increasing.

(2) Note that

$$
\lim_{t \to 0} \phi(t) = \lim_{t \to 0} \fint_{\partial B(x,t)} v(y) dS(y) = v(x).
$$

So that

$$
v(x) = \lim_{t \to 0} \phi(t) \leq \phi(r) = \fint_{\partial B(x,r)} v(y) dS(y).
$$

(3) Hence, for all $B(x,r) \subseteq U$ we have

$$
\begin{aligned}
\fint_{B(x,r)} v dy &= \frac{1}{\alpha(n) r^n} \int_{B(x,r)} v dy \\
&= \frac{1}{\alpha(n) r^n} \int_0^r \int_{\partial B(x,\rho)} v(y) dS(y) d\rho && \text{(Polar coordinates)} \\
&\geq \frac{1}{\alpha(n) r^n} \int_0^r n \alpha(n) \rho^{n-1} v(x) d\rho && \text{((2))} \\
&= v(x) \frac{1}{r^n} \underbrace{\int_0^r n \rho^{n-1} d\rho}_{=r^n} \\
&= v(x).
\end{aligned}
$$

□

Proof of (b).

(1) ...

(2) ...

□

Proof of (c).

(1) ...

(2) ...

□

Proof of (d).

(1) ...

(2) ...

□

Problem 2.5. ...

Proof.

(1) ...

(2) ...

□

Problem 2.6. ...

Proof.

(1) ...

(2) ...

□

Problem 2.7. ...

Proof.

(1) ...

(2) ...

□

Problem 2.8. ...

Proof.

(1) ...

(2) ...

□

Problem 2.9. ...

Proof.

(1) ...

(2) ...

□

Problem 2.10. ...

Proof.

(1) ...

(2) ...

□

Problem 2.11. ...

Proof.

(1) ...

(2) ...

□

Problem 2.12. ...

Proof.

(1) ...

(2) ...

□

Problem 2.13. *...*

Proof.

(1) ...

(2) ...

□

Problem 2.14. *...*

Proof.

(1) ...

(2) ...

□

Problem 2.15. *...*

Proof.

(1) ...

(2) ...

□

Problem 2.16. *...*

Proof.

(1) ...

(2) ...

□

Problem 2.17. *...*

Proof.

(1) ...

(2) ...

□

Problem 2.18. *...*

Proof.

(1) ...

(2) ...

□