Chapter 4: Limits and Continuity

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Continuity of real-valued functions

Exercise 4.19. *Let f be continuous on $[a, b]$ and define g as follows: $g(a) = f(a)$ and, for $a < x \leq b$, let $g(x)$ be the maximum value of f in the subinterval $[a, x]$. Show that g is continuous on $[a, b]$.*

Indeed, $g(x) = \max_{a \leq t \leq x} f(t)$ for $x \in [a, b]$.

Proof.

(1) f is continuous on $[a, b]$ at a point $p \Longleftrightarrow$ Given any $\epsilon' > 0$, there exists $\delta' > 0$ such that $|f(x) - f(p)| < \epsilon'$ whenever $|x - p| < \delta'$ (and $x \in [a, b]$). We left ϵ' and δ' undecided temporarily.

(2) To estimate g on
$$[p - \delta', p + \delta'] \cap [a, b],$$
we need to study the behavior of f on $[a, p + \delta'] \cap [a, b]$ (by the definition of $g(x)$), and then use the continuity of f to establish the desired result.

(3) Look at where f takes the maximum value over on $[a, p + \delta'] \cap [a, b]$ at. There are two possible cases (might overlapped):

   (a) *At a point in* $[a, p - \delta'] \cap [a, b]$. In this case g is constant on $[p - \delta', p + \delta'] \cap [a, b]$, or $|g(x) - g(p)| = 0$.

   (b) *At a point* $q \in (p - \delta', p + \delta'] \cap [a, b]$. For any $x \in [p - \delta', p + \delta'] \cap [a, b]$,

      (i) $f(p) - \epsilon' < g(x)$ by the maximality of g on $[a, x]$.

      (ii) $g(x) \leq f(q) < f(p) + \epsilon'$ since g is an increasing function and f takes the maximum value over on $[a, p + \delta'] \cap [a, b]$ at $q \in (p - \delta', p + \delta'] \cap [a, b]$.

   By (i)(ii),
$$f(p) - \epsilon' < g(x) < f(p) + \epsilon'$$
   for any $x \in [p - \delta', p + \delta'] \cap [a, b]$ (especially $x = p$). Therefore,
$$|g(x) - g(p)| < 2\epsilon' \text{ whenever } |x - p| < \delta' (\text{and } x \in [a, b]).$$

By (a)(b), we have $|g(x) - g(p)| < 2\epsilon'$ whenever $|x - p| < \delta'$(and $x \in [a, b]$) in any cases.

(4) Retake $\epsilon' = \frac{\epsilon}{2} > 0$ and $\delta = \delta' > 0$.

$\square$

Continuity in metric spaces

In Exercise 4.29 through 4.33, we assume that $f : S \to T$ is a function from one metric space (S, d_S) to another (T, d_T).

Exercise 4.29. *Prove that f is continuous on S if and only if*

$$f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ \quad \textit{for every subset } B \textit{ of } T.$$

Denote the interior of any set S by S°.

Proof (On topological spaces).

(1) $(\Longrightarrow)$

$$\begin{aligned}
\forall x \in f^{-1}(B^\circ) &\Longrightarrow f(x) \in B^\circ \\
&\Longrightarrow \exists \text{ open neighborhood } V \subseteq B^\circ \subseteq B \text{ containing } f(x) \\
&\Longrightarrow x \in f^{-1}(V) \subseteq f^{-1}(B) \\
&\Longrightarrow f^{-1}(V) \text{ is open in } S \text{ since } f \text{ is continuous} \\
&\Longrightarrow f^{-1}(V) \text{ is open neighborhood} \subseteq f^{-1}(B) \text{ containing } x \\
&\Longrightarrow x \in (f^{-1}(B))^\circ.
\end{aligned}$$

(2) $(\Longleftarrow)$ *Given any open subset V of T, need to show $U = f^{-1}(V)$ is open in S.*

$$\begin{aligned}
f^{-1}(V) &= f^{-1}(V^\circ) && (V \text{ is open}) \\
&\subseteq (f^{-1}(V))^\circ && (\text{Assumption})
\end{aligned}$$

So $U \subseteq U^\circ$ or $U = U^\circ$ is open.

$\square$

Exercise 4.30. *Prove that f is continuous on S if and only if*

$$f(\overline{A}) \subseteq \overline{f(A)} \quad \textit{for every subset } A \textit{ of } S.$$

Denote the closure of any set S by $\overline{S}$.

Proof (On topological spaces).

(1) ($\Longrightarrow$) Since f is continuous and $\overline{f(A)}$ is closed, $f^{-1}(\overline{f(A)})$ is closed. Hence,

$$\begin{aligned}
f^{-1}(\overline{f(A)}) &\supseteq f^{-1}(f(A)) && \text{(Monotonicity of } f^{-1}\text{)} \\
&\supseteq A, && \text{(Exercise 2.7(a))} \\
\overline{A} &\subseteq f^{-1}(\overline{f(A)}), && \text{(Monotonicity of closure)} \\
f(\overline{A}) &\subseteq f(f^{-1}(\overline{f(A)})) && \text{(Monotonicity of } f\text{)} \\
&\subseteq \overline{f(A)}. && \text{(Exercise 2.7(b))}
\end{aligned}$$

(2) ($\Longleftarrow$) *Given any closed subset D of T, need to show $C = f^{-1}(D)$ is closed in S.*

$$\begin{aligned}
f(\overline{C}) &\subseteq \overline{f(C)} && \text{(Assumption)} \\
&= \overline{f(f^{-1}(D))} && (C = f^{-1}(D)) \\
&\subseteq \overline{D} && \text{(Exercise 2.7(b))} \\
&= D, && (D \text{ is closed}) \\
f^{-1}(f(\overline{C})) &\subseteq f^{-1}(D), && \text{(Monotonicity of } f^{-1}\text{)} \\
\overline{C} \subseteq f^{-1}(f(\overline{C})) &\subseteq f^{-1}(D) = C. && \text{(Exercise 2.7(a))}
\end{aligned}$$

So $C \supseteq \overline{C}$ or $C = \overline{C}$ is closed.

□

Supplement (Continuity). *Let f be a map from a topological space on X to a topological space on Y. Then, the following statements are equivalent:*

(1) *f is continuous: For each $x \in X$ and every neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$.*

(2) *For every open set O in Y, the inverse image $f^{-1}(O)$ is open in X.*

(3) *For every closed set C in Y, the inverse image $f^{-1}(C)$ is closed in X.*

(4) *$f(A)^\circ \subseteq f(A^\circ)$ for every subset A of X.*

(5) *$f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ$ for every subset B of Y.*

(6) *$f(\overline{A}) \subseteq \overline{f(A)}$ for every subset A of X.*

(7) *$\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for every subset B of Y.*

Exercise 4.33. *Give an example of a continuous f and a Cauchy sequence $\{x_n\}$ in some metric space S for which $\{f(x_n)\}$ is not a Cauchy sequence in T.*

Compare with Exercise 4.54 to get some hints.

Proof. Let
$$S = \left\{1, \frac{1}{2}, \frac{1}{3}, ...\right\} = \left\{\frac{1}{n} : n \in \mathbb{Z}^+\right\}.$$
Define $f : S \to \mathbb{R}$ by $f\left(\frac{1}{n}\right) = (-1)^n$. Then f is continuous (but not uniformly continuous). The sequence $\{x_n\} = \left\{\frac{1}{n}\right\}$ in S is a Cauchy sequence, but the sequence $\{f(x_n)\} = \{(-1)^n\}$ is not a Cauchy sequence in $\mathbb{R}$. □

Uniform continuity

Exercise 4.50. *Prove that a function which is uniformly continuous on S is also continuous on S.*

Proof. The proof is straightforward.

(1) Suppose $f : S \to T$ is uniformly continuous on S. Given any $\epsilon > 0$, there is $\delta > 0$ such that $d_T(f(x), f(y)) < \epsilon$ whenever $d_S(x, y) < \delta$.

(2) *Show that f is continuous at any point p in S.* Set $y = p$ in (1).

□

Exercise 4.51. *If $f(x) = x^2$ for $x \in \mathbb{R}$, prove that f is not uniformly continuous on $\mathbb{R}$.*

Proof. Prove by contradiction.

(1) If f were uniformly continuous on $\mathbb{R}$, then for any $\epsilon > 0$, there is $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$. Here we pick $\epsilon = 1 > 0$.

(2) So
$$|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y| < 1$$
for any $|x - y| < \delta$. In particular, we pick $x = \frac{1}{\delta}$ and $y = \frac{1}{\delta} + \frac{\delta}{2}$. Now $|x - y| = \frac{\delta}{2} < \delta$, and thus $|f(x) - f(y)| = |x + y||x - y| < 1$ would be true. However,
$$|f(x) - f(y)| = |x + y||x - y| = \left(\frac{2}{\delta} + \frac{\delta}{2}\right)\left(\frac{\delta}{2}\right) > \frac{2}{\delta} \cdot \frac{\delta}{2} = 1,$$
contrary to $|f(x) - f(y)| = |x + y||x - y| < 1$.

□

Exercise 4.52. *Assume that f is uniformly continuous on a bounded set S in $\mathbb{R}^n$. Prove that f must be bounded on S.*

The conclusion is false if boundedness of S is omitted from the hypothesis. For example, $f(x) = x$ on $\mathbb{R}$ is uniformly continuous on $\mathbb{R}$ but $f(\mathbb{R}) = \mathbb{R}$ is unbounded.

Proof (Brute-force).

(1) Since $f : S \to T$ is uniformly continuous, given any $\epsilon > 0$, there is $\delta > 0$ such that $d_T(f(x), f(y)) < \epsilon$ whenever $d_S(x, y) < \delta$. In particular, pick $\epsilon = 1$.

(2) By the boundedness of S, there is $M > 0$ such that $\|x\| < M$ for all $x \in S$. In particular, each coordinate of $x \in \mathbb{R}^n$ is less than M.

(3) For such $\delta > 0$, we construct a covering of $S \subseteq \mathbb{R}^n$. Construct a special collection $\mathscr{C}$ of n-cells

$$I_{\mathbf{a}} = \left[\frac{\delta}{2\sqrt{n}}a_1, \frac{\delta}{2\sqrt{n}}(a_1+1)\right] \times \cdots \times \left[\frac{\delta}{2\sqrt{n}}a_n, \frac{\delta}{2\sqrt{n}}(a_n+1)\right]$$

where $\mathbf{a} = (a_1, ..., a_n) \in \mathbb{Z}^n$ satisfying

$$|a_i| < \frac{2\sqrt{n}M}{\delta} + 1 \ (1 \leq i \leq n).$$

By construction, $\mathscr{C}$ is a finite covering of S.

(4) For every n-cell $I_{\mathbf{a}}$ of the collection $\mathscr{C}$, pick a point $x_{\mathbf{a}} \in S \bigcap I_{\mathbf{a}}$ if possible. This process will terminate eventually since $\mathscr{C}$ is a finite. Collect these representative points as $\mathscr{D} = \{x_{\mathbf{a}}\}$. Notice that $\mathscr{D}$ is finite again.

(5) Now for any point $x \in S$, x lies in some $I_{\mathbf{a}}$ containing $x_{\mathbf{a}}$. Both x and $x_{\mathbf{a}}$ are in the same cell and their distance satisfies

$$\|x - x_{\mathbf{a}}\| \leq \sqrt{\left(\frac{\delta}{2\sqrt{n}}\right)^2 + \cdots + \left(\frac{\delta}{2\sqrt{n}}\right)^2} = \frac{\delta}{2} < \delta$$

and thus by (1)

$$\|f(x) - f(x_{\mathbf{a}})\| < 1, \text{ or } \|f(x)\| < 1 + \|f(x_{\mathbf{a}})\|.$$

(6) Let

$$M = 1 + \max_{x_{\mathbf{a}} \in \mathscr{D}} \|f(x_{\mathbf{a}})\|.$$

So given any $x \in S$, $\|f(x)\| < M$.

$\square$

Proof (Heine-Borel Theorem). Heine-Borel theorem provides the finiteness property to construct the boundedness property of f.

(1) *Let S be a bounded subset of a metric space X. Show that the closure of S in X is also bounded in X.* S is bounded if $S \subseteq B_X(a;r)$ for some $r > 0$ and some $a \in X$. (The ball $B_X(a;r)$ is defined to the set of all $x \in X$ such that $d_X(x,a) < r$.) Take the closure on the both sides,

$$\overline{S} \subseteq \overline{B_X(a;r)} = \{x \in X : d_X(x,a) \leq r\} \subseteq B_X(a;2r),$$

or $\overline{S}$ is bounded.

(2) Since $f : S \to T$ is uniformly continuous, given any $\epsilon > 0$, there is $\delta > 0$ such that $d_T(f(x), f(y)) < \epsilon$ whenever $d_S(x,y) < \delta$. In particular, pick $\epsilon = 1$.

(3) For such $\delta > 0$, we construct an open covering of $\overline{S} \subseteq \mathbb{R}^n$. Pick a collection $\mathscr{C}$ of open balls $B(a;\delta) \subseteq \mathbb{R}^n$ where a runs over all elements of S. $\mathscr{C}$ covers $\overline{S}$ (by the definition of accumulation points). Since $\overline{S}$ is closed and bounded (by applying (1) on the boundedness of S), $\overline{S}$ is compact (Heine-Borel theorem on $\mathbb{R}^n$). That is, there is a finite subcollection $\mathscr{C}'$ of $\mathscr{C}$ also covers $\overline{S}$, say

$$\mathscr{C}' = \{B(a_1;\delta)), B(a_2;\delta), ..., B(a_m;\delta)\}.$$

(4) Given any $x \in S \subseteq \overline{S}$, there is some $a_i \in S$ $(1 \leq i \leq m)$ such that $x \in B(a_i;\delta)$. In such ball, $d_S(x,a_i) < \delta$. By (2), $\|f(x) - f(a_i)\| < 1$, or $\|f(x)\| < 1 + \|f(a_i)\|$. Almost done. Notice that a_i depends on x, and thus we might use finiteness of $\{a_1, a_2, ..., a_m\}$ to remove dependence of a_i.

(5) Let

$$M = 1 + \max_{1 \leq i \leq m} \|f(a_i)\|.$$

So given any $x \in S$, $\|f(x)\| < M$.

$\square$

Supplement. Exercise about considering the closure. (Problem 3.5 in H. L. Royden, Real Analysis, 3rd Edition.) *Let $A = \mathbb{Q} \cap [0,1]$, and let $\{I_n\}$ be a finite collection of open intervals covering A. Then $\sum l(I_n) \geq 1$.*

Proof.

$$\begin{aligned} 1 = m^*[0,1] = m^*\overline{A} \leq m^*\left(\overline{\bigcup I_n}\right) &= m^*\left(\bigcup \overline{I_n}\right) \\ &\leq \sum m^*(\overline{I_n}) = \sum l(\overline{I_n}) = \sum l(I_n). \end{aligned}$$

□

Exercise 4.54. *Assume $f : S \to T$ is uniformly continuous on S, where S and T are metric spaces. If $\{x_n\}$ is any Cauchy sequence in S, prove that $\{f(x_n)\}$ is a Cauchy sequence in T. (Compare with Exercise 4.33.)*

Therefore, we need to find a continuous but not uniformly continuous function to solve Exercise 4.33: *Give an example of a continuous f and a Cauchy sequence $\{x_n\}$ in some metric space S for which $\{f(x_n)\}$ is not a Cauchy sequence in T.*

Proof. The proof is straightforward.

(1) Since $f : S \to T$ is uniformly continuous on S, given any $\epsilon > 0$, there is $\delta > 0$ such that $d_T(f(x), f(y)) < \epsilon$ whenever $d_S(x, y) < \delta$.

(2) Since $\{x_n\}$ is any Cauchy sequence in S, especially for such $\delta > 0$ in (1), there is an integer N such that $d_S(x_m, x_n) < \delta$ whenever $m \geq N$ and $n \geq N$. So as $m \geq N$ and $n \geq N$, we have $d_T(f(x_m), f(x_n)) < \epsilon$ by (1), or $\{f(x_n)\}$ itself is a Cauchy sequence in T.

□