Chapter 6: The Riemann-Stieltjes Integral

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Supplement. Another definition of Riemann-Stieltjes integral. *(Exercise 7.3, 7.4 of the book T. M. Apostol, Mathematical Analysis, Second Edition.) Let P be a partition of $[a, b]$. The norm of a partition P is the length of the largest subinterval $[x_{i-1}, x_i]$ of P and is denoted by $\|P\|$.*

We say $f \in \mathscr{R}(\alpha)$ if there exists $A \in \mathbb{R}$ having the property that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any partition P of $[a, b]$ with norm $\|P\| < \delta$ and for any choice of $t_i \in [x_{i-1}, x_i]$, we have $|\sum_{i=1}^n f(t_i)\Delta\alpha_i - A| < \varepsilon$.

Claim. *$f \in \mathscr{R}$ in the sense of Definition 6.2 implies that $f \in \mathscr{R}$ in the sense of this another definition.*

Proof of Claim. Let $A = \int f dx$, $M > 0$ be one upper bound of $|f|$ on $[a, b]$. Given $\varepsilon > 0$, there exists a partition $P_0 = \{a = x_0, x_1, \ldots, x_{N-1}, x_N = b\}$ such that $U(P_0, f) \leq A + \frac{\varepsilon}{2}$. Let $\delta = \frac{\varepsilon}{2MN} > 0$. Then for any partition P with norm $\|P\| < \delta$, write

$$U(P, f) = \sum_{i=1}^n M_i \Delta x_i = S_1 + S_2,$$

where S_1 is the sum of terms arising from those subintervals of P containing no point of P_0, S_2 is the sum of the remaining terms. Then

$$S_1 \leq U(P_0, f) < A + \frac{\varepsilon}{2},$$
$$S_2 \leq NM\|P\| < NM\delta < \frac{\varepsilon}{2}.$$

Therefore, $U(P, f) < A + \varepsilon$. Similarly, $L(P, f) > A - \varepsilon$ whenever $\|P\| < \delta'$. Hence, $|\sum_{i=1}^n f(t_i)\Delta x_i - A| < \varepsilon$ whenever $\|P\| < \min\{\delta, \delta'\}$. (Copy Apostol's hint and ensure $M > 0$. M in Apostol's hint might be zero if $f = 0$.) $\square$

This supplement will be used in computing $\int_0^\infty (\frac{\sin x}{x})^2 dx = \frac{\pi}{2}$ in Exercise 8.12.

Exercise 6.1. *Suppose α increases on $[a, b]$, $a \leq x_0 \leq b$, α is continuous at x_0, $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathscr{R}(\alpha)$ and that $\int f d\alpha = 0$.*

Given any partition $P = \{a = p_0, p_1, \ldots, p_{n-1}, p_n = b\}$, where $a = p_0 \leq p_1 \leq \cdots \leq p_{n-1} \leq p_n = b$. We might compute $L(P, f, \alpha)$ and $U(P, f, \alpha)$ by using ε-δ

argument since we are hinted by the condition that α is continuous. A function which is continuous at x_0 has a nice property near x_0 and this property would help us estimate $U(P, f, \alpha)$ near x_0. On the contrary, if both f and α are discontinuous at x_0, it might be $f \notin \mathscr{R}(\alpha)$. Besides, if f has too many points of discontinuity ($f(x) = 0$ if $x \in \mathbb{Q}$ and $f(x) = 1$ otherwise, for example), then f might not be Riemann-integrable on $[0, 1]$.

Claim 1. $L(P, f, \alpha) = 0$.

Proof of Claim 1. $m_i = 0$ since $\inf f(x) = 0$ on any subinterval of $[a, b]$. So $L(P, f, \alpha) = \sum m_i \Delta\alpha_i = 0$. Here we don't need the condition that α is continuous at x_0. $\square$

Claim 2. *For any $\varepsilon > 0$, there exists a partition P such that $U(P, f, \alpha) < \varepsilon$.*

Proof of Claim 2. Say $x_0 \in [p_{i_0-1}, p_{i_0}]$ for some i_0. Then

$$M_i = \sup_{p_{i-1} \leq x \leq p_i} f(x) = \begin{cases} 0 & \text{if } i \neq i_0, \\ 1 & \text{if } i = i_0. \end{cases}$$

So

$$U(P, f, \alpha) = \sum M_i \Delta\alpha_i = \Delta\alpha_{i_0}.$$

It is not true for any arbitrary α. (For example, α has a jump on $x = x_0$.) In fact, Exercise 6.3 shows this. Luckily, α is continuous at x_0. So for $\varepsilon > 0$, there exists $\delta > 0$ such that $|\alpha(x) - \alpha(x_0)| < \frac{\varepsilon}{2}$ whenever $|x - x_0| < \delta$ (and $x \in [a, b]$). Now we pick a nice partition

$$P = \{a, x_0 - \delta_1, x_0 + \delta_2, b\},$$

where $\delta_1 = \min\{\delta, x_0 - a\} \geq 0$ and $\delta_2 = \min\{\delta, b - x_0\} \geq 0$. (It is a trick about resizing "δ" to avoid considering the edge cases $x_0 = a$ or $x_0 = b$ or $a = b$.) Then $x_0 \in [x_0 - \delta_1, x_0 + \delta_2]$ and $\Delta\alpha$ on $[x_0 - \delta_1, x_0 + \delta_2]$ is

$$\begin{aligned} \alpha(x_0 + \delta_2) - \alpha(x_0 - \delta_1) &= (\alpha(x_0 + \delta_2) - \alpha(x_0)) + (\alpha(x_0) - \alpha(x_0 - \delta_1)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore, $U(P, f, \alpha) < \varepsilon$. $\square$

Proof (Definition 6.2). By Claim 1 and 2 and notice that $U(P, f, \alpha) \geq 0$ for any partition P,

$$\overline{\int_a^b} f d\alpha = \inf U(P, f, \alpha) = 0,$$
$$\underline{\int_a^b} f d\alpha = \sup L(P, f, \alpha) = 0,$$

the inf and sup again being taken over all partitions. Hence $f \in \mathscr{R}(\alpha)$ and that $\int f d\alpha = 0$ by Definition 6.2. $\square$

Proof (Theorem 6.6). By Claim 1 and 2,

$$0 \leq U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$$

Hence $f \in \mathscr{R}(\alpha)$ by Theorem 6.6. Furthermore,

$$\int f d\alpha = \underline{\int_a^b} f d\alpha = \sup L(P, f, \alpha) = 0.$$

$\square$

Proof (Theorem 6.10). $f \in \mathscr{R}(\alpha)$ by Theorem 6.10. Thus, by Claim 1

$$\int f d\alpha = \underline{\int_a^b} f d\alpha = \sup L(P, f, \alpha) = 0.$$

$\square$

Exercise 6.2. *Suppose $f \geq 0$, f is continuous on $[a, b]$, and $\int_a^b f(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$. (Compare with Exercise 6.1.)*

For one application, see Exercise 7.20.

Proof. (Reductio ad absurdum) If there were $p \in [a, b]$ such that $f(p) > 0$. Since f is continuous on $[a, b]$, given $\varepsilon = \frac{1}{64} f(p) > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(p)| \leq \frac{1}{64} f(p) \text{ whenever } |x - p| \leq \delta, x \in [a, b].$$

Hence

$$f(x) \geq \frac{63}{64} f(p)$$

whenever $x \in E = [\max\{a, p - \delta\}, \min\{b, p + \delta\}] \subseteq [a, b]$. Note that the length of E is $|E| > 0$. So

$$0 = \int_a^b f(x)dx \geq \int_E f(x)dx \geq \int_E \frac{63}{64} f(p)dx = \frac{63}{64} f(p)|E| > 0,$$

which is absurd. $\square$

Note. (Lebesgue integral) *Let f be a nonnegative measurable function. Then $\int f = 0$ implies $f = 0$ a.e.*

Exercise 6.3. *Define three functions β_1, β_2, β_3 as follows: $\beta_j(x) = 0$ if $x < 0$, $\beta_j(x) = 1$ if $x > 0$ for $j = 1, 2, 3$; and $\beta_1(0) = 0$, $\beta_2(0) = 1$, $\beta_3(0) = \frac{1}{2}$. Let f be a bounded functions on $[-1, 1]$.*

(a) *Prove that $f \in \mathscr{R}(\beta_1)$ if and only if $f(0+) = f(0)$ and that then*
$$\int f d\beta_1 = f(0).$$

(b) *State and prove a similar result for β_2.*

(c) *Prove that $f \in \mathscr{R}(\beta_3)$ if and only if f is continuous at 0.*

(d) *If f is continuous at 0 prove that*
$$\int f d\beta_1 = \int f d\beta_2 = \int f d\beta_3 = f(0).$$

Proof of (a).

(1) Given any $\delta > 0$, we have
$$|f(x) - f(0)| \leq \sup_{x \in [0,\delta]} f(x) - \inf_{x \in [0,\delta]} f(x)$$
if $x \in [0, \delta]$.

(2) Given any $\varepsilon > 0$ and $\delta > 0$. *Show that if f is bounded and $|f(x) - f(0)| < \varepsilon$ on $[0, \delta]$ then*
$$\sup_{x \in [0,\delta]} f(x) - \inf_{x \in [0,\delta]} f(x) < 2\varepsilon.$$

Since f is bounded, there exists $x_1, x_2 \in [0, \delta]$ such that
$$f(x_1) = \sup_{x \in [0,\delta]} f(x) \quad \text{and} \quad f(x_2) = \inf_{x \in [0,\delta]} f(x).$$

By assumption,
$$f(x_1) - f(x_2) \leq |f(x_1) - f(0)| + |f(0) - f(x_2)| < 2\varepsilon.$$

(3) *Show that* $f \in \mathscr{R}(\beta_1)$ *iff* $f(0+) = f(0)$.

$$
\begin{aligned}
& f \in \mathscr{R}(\beta_1) \\
\Longleftrightarrow & \forall \varepsilon > 0 \text{ there is } P \text{ such that } U(P, f, \beta_1) - L(P, f, \beta_1) < \varepsilon && \text{(Theorem 6.6)} \\
\Longleftrightarrow & \forall \varepsilon > 0 \text{ there is } P \text{ containing } 0 \text{ such that } U(P, f, \beta_1) - L(P, f, \beta_1) < \varepsilon && \text{(Theorem 6.4)} \\
& \text{where } P = \{-1 = x_0 < x_1 < \ldots < x_k = 0 < \ldots < x_n = 1\} \\
\Longleftrightarrow & \forall \varepsilon > 0 \text{ there is } P \text{ containing } 0 \text{ such that } M_{k+1} - m_{k+1} < \varepsilon \\
\Longleftrightarrow & \forall \varepsilon > 0 \text{ there is } P \text{ containing } 0 \text{ such that } \sup_{x \in [0,\delta]} f(x) - \inf_{x \in [0,\delta]} f(x) < \varepsilon \\
& \text{where } [x_k, x_{k+1}] = [0, \delta],\ \delta > 0 \\
& \text{(Take } P = \{-1, 0, \delta, 1\} \text{ in ``}\Longleftarrow\text{'' direction)} \\
\Longleftrightarrow & \forall \varepsilon > 0 \text{ there is } \delta > 0 \text{ such that } |f(x) - f(0)| < \varepsilon \text{ whenever } x \in [0, \delta] && ((1)(2)) \\
& \text{(Replace } \varepsilon \text{ by } \frac{\varepsilon}{2} \text{ in ``}\Longleftarrow\text{'' direction)} \\
\Longleftrightarrow & \lim_{x \to 0+} f(x) = f(0).
\end{aligned}
$$

(4) *Show that* $\int f d\beta_1 = f(0)$ *if* $f \in \mathscr{R}(\beta_1)$. By (3) and Theorem 6.7,

$$\left| f(0) - \int_a^b f d\beta_1 \right| < \varepsilon.$$

Since ε is arbitrary, $\int f d\beta_1 = f(0)$.

$\square$

Proof of (b). Show that $f \in \mathscr{R}(\beta_2)$ *if and only if* $f(0-) = f(0)$ *and that then*

$$\int f d\beta_2 = f(0).$$

Similar to (a).

(1) Given any $\delta > 0$, we have

$$|f(x) - f(0)| \leq \sup_{x \in [-\delta,0]} f(x) - \inf_{x \in [-\delta,0]} f(x)$$

if $x \in [-\delta, 0]$.

(2) Given any $\varepsilon > 0$ and $\delta > 0$. *Show that if* f *is bounded and* $|f(x) - f(0)| < \varepsilon$ *on* $[-\delta, 0]$ *then*

$$\sup_{x \in [-\delta,0]} f(x) - \inf_{x \in [-\delta,0]} f(x) < 2\varepsilon.$$

Since f is bounded, there exists $x_1, x_2 \in [-\delta, 0]$ such that

$$f(x_1) = \sup_{x \in [-\delta,0]} f(x) \quad \text{and} \quad f(x_2) = \inf_{x \in [-\delta,0]} f(x).$$

By assumption,

$$f(x_1) - f(x_2) \leq |f(x_1) - f(0)| + |f(0) - f(x_2)| < 2\varepsilon.$$

(3) *Show that $f \in \mathscr{R}(\beta_1)$ iff $f(0-) = f(0)$.*

$$\begin{aligned}
&f \in \mathscr{R}(\beta_2) \\
\Longleftrightarrow &\forall \varepsilon > 0 \text{ there is } P \text{ such that } U(P, f, \beta_2) - L(P, f, \beta_2) < \varepsilon && \text{(Theorem 6.6)} \\
\Longleftrightarrow &\forall \varepsilon > 0 \text{ there is } P \text{ containing } 0 \text{ such that } U(P, f, \beta_2) - L(P, f, \beta_2) < \varepsilon && \text{(Theorem 6.4)} \\
&\text{where } P = \{-1 = x_0 < x_1 < \ldots < x_k = 0 < \ldots < x_n = 1\} \\
\Longleftrightarrow &\forall \varepsilon > 0 \text{ there is } P \text{ containing } 0 \text{ such that } M_k - m_k < \varepsilon \\
\Longleftrightarrow &\forall \varepsilon > 0 \text{ there is } P \text{ containing } 0 \text{ such that } \sup_{x \in [-\delta, 0]} f(x) - \inf_{x \in [-\delta, 0]} f(x) < \varepsilon \\
&\text{where } [x_{k-1}, x_k] = [-\delta, 0],\ \delta > 0 \\
&\text{(Take } P = \{-1, -\delta, 0, 1\} \text{ in ``}\Longleftarrow\text{'' direction)} \\
\Longleftrightarrow &\forall \varepsilon > 0 \text{ there is } \delta > 0 \text{ such that } |f(x) - f(0)| < \varepsilon \text{ whenever } x \in [-\delta, 0] && ((1)(2)) \\
&\text{(Replace } \varepsilon \text{ by } \frac{\varepsilon}{2} \text{ in ``}\Longleftarrow\text{'' direction)} \\
\Longleftrightarrow &\lim_{x \to 0-} f(x) = f(0).
\end{aligned}$$

(4) *Show that $\int f d\beta_2 = f(0)$ if $f \in \mathscr{R}(\beta_2)$.* By (3) and Theorem 6.7,

$$\left| f(0) - \int_a^b f d\beta_2 \right| < \varepsilon.$$

Since ε is arbitrary, $\int f d\beta_2 = f(0)$.

□

Proof of (c). Note that f is continuous at 0 iff $f(0+) = f(0-) = f(0)$. Apply the same argument in (a) and (b), we have $f \in \mathscr{R}(\beta_3)$ if and only if $f(0+) = f(0-) = f(0)$. □

Proof of (d). It suffices to show that

$$\int_a^b f d\beta_3 = f(0).$$

We can apply Theorem 6.12(d)(e) to $\beta_3 = \frac{1}{2}(\beta_1 + \beta_2)$. That is,

$$\int_a^b f d\beta_3 = \frac{1}{2} \left[\int_a^b f d\beta_1 + \int_a^b f d\beta_2 \right] = \frac{1}{2}[f(0) + f(0)] = f(0).$$

Or apply the same argument in (a) and (b) to get

$$\left| f(0) - \int_a^b f d\beta_3 \right| < \varepsilon$$

for any $\varepsilon > 0$, or $\int_a^b f d\beta_3 = f(0)$. $\square$

Exercise 6.4. *If*

$$f(x) = \begin{cases} 0 & \text{for all irrational } x, \\ 1 & \text{for all rational } x, \end{cases}$$

prove that $f \notin \mathscr{R}$ on $[a,b]$ for any $a < b$.

Proof. Given any partition

$$P = \{a = p_0, p_1, \ldots, p_{n-1}, p_n = b\}$$

of $[a,b]$ where $a = p_0 \leq p_1 \leq \cdots \leq p_{n-1} \leq p_n = b$. Since $a < b$, we might assume that $a = p_0 < p_1 < \cdots < p_{n-1} < p_n = b$ by removing duplicated points. Since $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$ are dense in $\mathbb{R}$, we have

$$\begin{aligned} M_i &= \sup_{p_{i-1} \leq x \leq p_i} f(x) = 1, \\ m_i &= \inf_{p_{i-1} \leq x \leq p_i} f(x) = 0, \\ U(P,f) &= \sum_{i=1}^n M_i \Delta x_i = \sum_{i=1}^n \Delta x_i = b - a, \\ L(P,f) &= \sum_{i=1}^n m_i \Delta x_i = \sum_{i=1}^n 0 = 0. \end{aligned}$$

Since P is arbitrary,

$$\begin{aligned} \overline{\int_a^b} f dx &= \inf U(P,f) = b - a > 0, \\ \underline{\int_a^b} f dx &= \sup L(P,f) = 0. \end{aligned}$$

Hence $f \notin \mathscr{R}$ on $[a,b]$ for any $a < b$. $\square$

Note.

(1) (Lebesgue integral) f is Lebesgue integrable.

(2) $f \in \mathscr{R}$ on $[a,b]$ iff $a = b$.

(3) (Problem 4.1 in *H. L. Royden, Real Analysis, 3rd edition.*) *Construct a sequence $\{f_n\}$ of nonnegative, Riemann integrable functions such that f_n increases monotonically to f. What does this imply about changing the order of integration and the limiting process?* (Since $\mathbb{Q}$ is countable, write

$$\mathbb{Q} = \{r_1, r_2, \ldots\}.$$

Define

$$f_n(x) = \begin{cases} 0 & \text{if } x \not\in \{r_1, \ldots, r_n\}, \\ 1 & \text{if } x \in \{r_1, \ldots, r_n\}. \end{cases}$$

By construction, f_n increases monotonically to f pointwise. Note that $f_n \to f$ not uniformly. Also, $\int_a^b f_n(x)dx = 0$ by using the same argument in Theorem 6.10. Therefore, $\lim_{n\to\infty} \int_a^b f_n(x)dx = 0$ but $\int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx$ does not exist.)

Exercise 6.5. *Suppose f is a bounded real function on $[a,b]$, and $f^2 \in \mathscr{R}$ on $[a,b]$. Does it follow that $f \in \mathscr{R}$? Does the answer change if we assume that $f^3 \in \mathscr{R}$?*

Actually we can omit the boundedness assumption of f since $f^2 \in \mathscr{R}$ or $f^3 \in \mathscr{R}$.

Proof.

(1) *Show that $f^2 \in \mathscr{R}$ on $[a,b]$ does not imply that $f \in \mathscr{R}$ (unless $f \geq 0$ on $[a,b]$).* Similar to Exercise 6.4, define

$$f(x) = \begin{cases} -1 & \text{for all irrational } x, \\ 1 & \text{for all rational } x. \end{cases}$$

$f^2 = 1 \in \mathscr{R}$ on $[a,b]$ but $f \not\in \mathscr{R}$ on $[a,b]$ for any $a < b$. (The proof for the "unless" part is similar to (2).)

(2) *Show that $f^3 \in \mathscr{R}$ on $[a,b]$ implies that $f \in \mathscr{R}$.* Let $\phi(x) = x^{\frac{1}{3}}$ on $\mathbb{R}$. By Theorem 6.11, $f(x) = \phi(f(x)^3) \in \mathscr{R}$. (The boundedness condition in Theorem 6.11 is unnecessary.)

□

Note. (Lebesgue integral) *Suppose that f^2 is Lebesgue integrable. Does it follow that f is Lebesgue integrable? Does the answer change if we assume that f^3 is Lebesgue integrable?* Both answers are no.

Exercise 6.6. *Let P be the Cantor set constructed in Sec. 2.44. Let f be a bounded real function on $[0,1]$ which is continuous at every point outside P.*

Prove that $f \in \mathscr{R}$ on $[0,1]$. (Hint: P can be covered by finitely many segments whose total length can be made as small as desired. Proceed as in Theorem 6.10.)

Proof (Theorem 6.10). Given any $\varepsilon > 0$.

(1) Note that in Section 2.44, we have

$$P = \bigcap_{n=1}^{\infty} E_n$$

and each E_n is the union of 2^n intervals, each of length $\frac{1}{3^n}$. For each interval $[u_j, v_j] \subseteq E_n$ of E_n ($1 \leq j \leq 2^n$), we construct a slightly larger open set

$$(u_j - \lambda, v_j + \lambda) \supsetneq [u, v]$$

where $\lambda = \frac{1}{2}\left(\frac{1}{2.28^n} - \frac{1}{3^n}\right) > 0$. Each length of $(u_j - \lambda, v_j + \lambda)$ is $\frac{1}{2.28^n}$. Write

$$G_n = \bigcup_{1 \leq j \leq 2^n} (u_j - \lambda, v_j + \lambda).$$

Hence

$$G_n \supsetneq \bigcup_{1 \leq j \leq 2^n} [u_j, v_j] = E_n \supseteq P,$$

and the total length $|G_n|$ of G_n satisfies

$$|G_n| \leq \sum_{1 \leq j \leq 2^n} |(u_j - \lambda, v_j + \lambda)| = \left(\frac{2}{2.28}\right)^n.$$

(Two different subintervals might be overlapped.) As $n \to \infty$, P can be covered by finitely many open segments whose total length can be made as small as desired. Now we take an integer N such that $\left(\frac{2}{2.28}\right)^N < \frac{\varepsilon}{64(M+1)}$.

(2) Let $K = [0,1] - G_N$ be a compact set (Theorem 2.35). By construction, f is continuous on K and thus f is uniformly continuous. So there is $\delta > 0$ such that $|f(s) - f(t)| < \frac{\varepsilon}{89}$ if $s, t \in K$ and $|s - t| < \delta$.

(3) Now we construct a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$, as the following steps:

   (a) Put $\frac{0}{m}, \frac{1}{m}, \ldots, \frac{m}{m}$ in P for some integer $m \geq \frac{1}{\delta}$.
   (b) Put $u_j - \lambda$ and $v_j + \lambda$ in P.
   (c) Remove any points in the segment $(u_j - \lambda, v_j + \lambda)$ except 0 and 1.

(4) Note that $M_i - m_i \leq 2M$ $(1 \leq i \leq n)$ where $M = \sup |f(x)|$ is defined. Hence,

$$U(P, f) - L(P, f) \leq \frac{\varepsilon}{89} + 2M \cdot \frac{\varepsilon}{64(M+1)} \leq \varepsilon.$$

Since ε is arbitrary, Theorem 6.6 shows that $f \in \mathscr{R}$.

□

Supplement (Lebesgue's criterion for Riemann-integrability). *Let f be a bounded real function on $[a,b]$ and let D be the set of discontinuities of f in $[a,b]$. Then $f \in \mathscr{R}$ on $[a,b]$ if and only if D has measure zero.*

For a proof, see Theorem 7.48 in *Tom M. Apostol, Mathematical Analysis, 2nd edition.*

Exercise 6.7. *Suppose f is a real function on $(0,1]$ and $f \in \mathscr{R}$ on $[c,1]$ for every $c > 0$. Define*

$$\int_0^1 f(x)dx = \lim_{c \to 0} \int_c^1 f(x)dx$$

if this limit exists (and is finite).

(a) *If $f \in \mathscr{R}$ on $[0,1]$, show that this definition of the integral agrees with the old one.*

(b) *Construct a function such that the above limit exists, although it fails to exist with $|f|$ in place of f.*

Proof of (a).

(1) Since $f \in \mathscr{R}$ on $[0,1]$, f is bounded or $|f| \leq M$ for some real M.

(2) For any $0 < c < 1$, we have

$$\begin{aligned} \left| \int_0^1 f(x)dx - \int_c^1 f(x)dx \right| &= \left| \int_0^c f(x)dx \right| && \text{(Theorem 6.12(c))} \\ &\leq Mc. && \text{(Theorem 6.12(d))} \end{aligned}$$

(3) Given any $\varepsilon > 0$, there exists $\delta = \frac{\varepsilon}{M+1} > 0$ such that

$$\left| \int_0^c f(x)dx - \int_0^1 f(x)dx \right| \leq Mc < M\delta = M \cdot \frac{\varepsilon}{M+1} < \varepsilon$$

whenever $0 < c < \delta$. Hence $\lim_{c \to 0} \int_0^c f(x)dx = \int_0^1 f(x)dx$.

□

Proof of (b)(Construct by nonabsolutely convergent series).

(1) Given any nonabsolutely (conditionally) convergent series $\sum_{k=1}^n a_k$ (take $\sum \frac{(-1)^n}{n}$ for example and then see Remark 3.46), we define f on $(0,1]$ by

$$f(x) = 2^n a_n$$

if $\frac{1}{2^n} < x \leq \frac{1}{2^{n-1}}$ as $n = 1, 2, \ldots$.

(2) By construction,

$$\int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} f(x)dx = \left(\frac{1}{2^{n-1}} - \frac{1}{2^n}\right) 2^n a_n = a_n.$$

and thus

$$\int_{\frac{1}{2^n}}^{1} f(x)dx = \int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} f(x)dx + \cdots + \int_{\frac{1}{2}}^{1} f(x)dx = \sum_{k=1}^{n} a_k.$$

(3) Given any $\varepsilon > 0$. Since $\sum a_n$ is convergent, there exists a common integer N such that

$$|a_n| \leq \frac{\varepsilon}{89}$$

and

$$\left|\sum_{k=1}^{n} a_k - A\right| \leq \frac{\varepsilon}{64}$$

for some real A whenever $n \geq N$ (Definition 3.21 and Theorem 3.23). Therefore, for any $0 < c \leq \frac{1}{2^N}$, say $\frac{1}{2^{n+1}} < c \leq \frac{1}{2^n} \leq \frac{1}{2^N}$ for some $n \geq N$, we have

$$\begin{aligned}
\left|\int_c^1 f(x)dx - A\right| &= \left|\int_c^{\frac{1}{2^n}} f(x)dx + \int_{\frac{1}{2^n}}^{1} f(x)dx - A\right| \\
&\leq \left|\left(\frac{1}{2^n} - c\right) 2^{n+1} a_{n+1}\right| + \left|\sum_{k=1}^{n} a_k - A\right| \\
&\leq |a_{n+1}| + \left|\sum_{k=1}^{n} a_k - A\right| \\
&\leq \frac{\varepsilon}{89} + \frac{\varepsilon}{64} \\
&\leq \varepsilon.
\end{aligned}$$

Hence, $\lim_{c\to 0} \int_c^1 f(x)dx = A$ exists.

(4) Since

$$\int_{\frac{1}{2^n}}^{1} |f(x)|dx = \int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} |f(x)|dx + \cdots + \int_{\frac{1}{2}}^{1} |f(x)|dx = \sum_{k=1}^{n} |a_k| \to \infty$$

as $n \to \infty$, $\lim_{c\to 0} \int_c^1 f(x)dx$ does not exist. (Or show that $\lim_{c\to 0} \int_c^1 f(x)dx = \infty$ by definition directly.)

$\square$

Exercise 6.8.
PLACEHOLDER

Exercise 6.9.
PLACEHOLDER

Exercise 6.10. *Let p and q be positive real integers such that*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Prove the following statements.

(a) *If $u \geq 0$ and $v \geq 0$, then*

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

Equality holds if and only if $u^p = v^q$.

(b) *If $f \in \mathscr{R}(\alpha)$, $g \in \mathscr{R}(\alpha)$, $f \geq 0$, $g \geq 0$, and*

$$\int_a^b f^p d\alpha = \int_a^b g^q d\alpha = 1,$$

then

$$\int_a^b fg d\alpha \leq 1.$$

(c) *If f and g are complex functions in $\mathscr{R}(\alpha)$, then*

$$\left| \int_a^b fg d\alpha \right| \leq \left\{ \int_a^b |f|^p d\alpha \right\}^{\frac{1}{p}} \left\{ \int_a^b |g|^q d\alpha \right\}^{\frac{1}{q}}.$$

*This is **Hölder's inequality**. When $p = q = 2$ it is usually called the Schwarz inequality. (Note that Theorem 1.35 is a very special case of this.)*

(d) *Show that Hölder's inequality is also true for the "improper" integrals described in Exercise 6.7 and 6.8.*

Proof of (a)(Young's inequality).

(1) $u = 0$ or $v = 0$ is nothing to do. For $u > 0$ and $v > 0$, we give some different proofs.

(2) First proof.

$$\begin{aligned}
uv &= \exp(\log(uv)) \\
&= \exp\left(\frac{1}{p}\log(u^p) + \frac{1}{q}\log(v^q)\right) \\
&\leq \frac{1}{p}\exp(\log(u^p)) + \frac{1}{q}\exp(\log(v^q)) \qquad \text{(Convexity of } \exp(x)\text{)} \\
&= \frac{u^p}{p} + \frac{v^q}{q}.
\end{aligned}$$

Here the convexity of $\exp(x)$ can be derived by the fact that $(\exp(x))'' > 0$ and Exercise 5.14. The fact that the equality holds if and only if $u^p = v^q$ is derived from the strictly convexity of $\exp(x)$ additionally. (For the details about the exponential and logarithmic functions, might see Chapter 8.)

(3) Second proof.

$$\begin{aligned}
\log\left(\frac{u^p}{p} + \frac{v^q}{q}\right) &\geq \frac{1}{p}\log(u^p) + \frac{1}{q}\log(v^q) \qquad \text{(Concavity of } \log(x)\text{)} \\
&= \log(u) + \log(v) \\
&= \log(uv).
\end{aligned}$$

Since $\log(x)$ increases monotonically ($(\log(x))' = \frac{1}{x} > 0$ if $x > 0$), $\frac{u^p}{p} + \frac{v^q}{q} \geq uv$ (or take the exponential function to get the same conclusion). Here the concavity of $\log(x)$ can be derived by the fact that $(\log(x))'' < 0$ and a statement that $f''(x) \leq 0$ if and only if f is concave. The fact that the equality holds if and only if $u^p = v^q$ is derived from the strictly concavity of $\log(x)$ additionally. (The proof is analogous to Exercise 5.14.)

(4) Third proof. *Suppose that $f : [0, \infty) \to [0, \infty)$ is a strictly increasing continuous function such that $f(0) = 0$ and $\lim_{x\to\infty} f(x) = \infty$. Then*

$$uv \leq \int_0^u f(x)dx + \int_0^v f^{-1}(x)dx$$

for every $u, v \geq 0$, and equality occurs if and only if $v = f(u)$. Define

$$F(x) = -xf(x) + \int_0^x f(t)dt + \int_0^{f(x)} f^{-1}(t)dt.$$

By Theorem 6.20 (the fundamental theorem of calculus) and Theorem 5.5 (chain rule),

$$F'(x) = -(f(x) + xf'(x)) + f(x) + f'(x)f^{-1}(f(x)) = 0.$$

Hence $F(x)$ is a constant on $(0, u)$ (Theorem 5.11(b)). Note that $F(x)$ is continuous on $[0, u]$ and $F(0) = 0$, so $F(x) = 0$ on $[0, u]$ or

$$\int_0^x f(t)dt + \int_0^{f(x)} f^{-1}(t)dt = xf(x).$$

Take $x = u$ to get

$$\int_0^u f(x)dx + \int_0^{f(u)} f^{-1}(x)dx = uf(u).$$

Hence

$$\begin{aligned}
&\int_0^u f(x)dx + \int_0^v f^{-1}(x)dx - uv \\
=&\int_0^u f(x)dx + \int_0^{f(u)} f^{-1}(x)dx + \int_{f(u)}^v f^{-1}(x)dx - uv \\
=&uf(u) + \int_{f(u)}^v f^{-1}(x)dx - uv \\
=&\int_{f(u)}^v [f^{-1}(x) - f^{-1}(f(u))]dx \\
\geq&0.
\end{aligned}$$

The last inequality holds since f is strictly increasing and thus f^{-1} is strictly increasing too. Besides, the equality holds if and only if $f(u) = v$. Now the conclusion holds by taking $f(x) = x^{p-1}$ in

$$uv \leq \int_0^u f(x)dx + \int_0^v f^{-1}(x)dx$$

and the equality holds if and only if $u^p = v^q$.

□

Proof of (b). Every integral is well-defined (Theorem 6.11 and Theorem 6.13(a)). Let $u = f \geq 0$ and $v = g \geq 0$ in (a). Integrate both sides of the inequality

$$fg \leq \frac{f^p}{p} + \frac{g^q}{q}$$

to get

$$\begin{aligned}
\int_a^b fg d\alpha &\leq \int_a^b \left(\frac{f^p}{p} + \frac{g^q}{q} \right) d\alpha && \text{(Theorem 6.12(b))} \\
&= \int_a^b \frac{f^p}{p} d\alpha + \int_a^b \frac{g^q}{q} d\alpha && \text{(Theorem 6.12(a))} \\
&= \frac{1}{p} \int_a^b f^p d\alpha + \frac{1}{q} \int_a^b g^q d\alpha && \text{(Theorem 6.12(a))} \\
&= \frac{1}{p} + \frac{1}{q} && \text{(Assumption)} \\
&= 1.
\end{aligned}$$

The equality holds if $f^p = g^q$. Note that the equality does not hold only if $f^p = g^q$. (Consider α is constant on some subinterval $[c, d] \subsetneq [a, b]$.) Luckily, it is true for the additional assumption that $\alpha(x) = x$ and f, g are continuous on $[a, b]$. $\square$

Proof of (c). There are three possible cases.

(1) The case $\left\{ \int_a^b |f|^p d\alpha \right\}^{\frac{1}{p}} = 0$. So $\int_a^b |f|^p d\alpha = 0$.

  (a) *Show that* $\int_a^b |f| d\alpha = 0$ *if* $\int_a^b |f|^p d\alpha = 0$. (Reductio ad absurdum) If $\int_a^b |f| d\alpha = A > 0$, then given $\varepsilon = \frac{A}{2} > 0$, there exists a partition $P_0 = \{a = x_0 \leq \cdots \leq x_n = b\}$ such that

$$\sum_{i=0}^{n} m_i \Delta\alpha_i > \frac{A}{2},$$

  where $m_i = \inf_{x \in [x_{i-1}, x_i]} |f|$ and $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$. By the pigeonhole principle, there exists $1 \leq i_0 \leq n$ such that

$$L(P_0, |f|, \alpha) = m_{i_0} \Delta\alpha_{i_0} > \frac{A}{2n} > 0.$$

  Especially, $m_{i_0} > 0$ and $\Delta\alpha_{i_0} > 0$. Now we consider $L(P, |f|^p, \alpha)$. Hence

$$L(P_0, |f|^p, \alpha) = \sum_{i=0}^{n} m_i^p \Delta\alpha_i \geq m_{i_0}^p \Delta\alpha_{i_0} > 0,$$

  or

$$\underline{\int_a^b} |f| d\alpha = \sup L(P, f, \alpha) \geq m_{i_0}^p \Delta\alpha_{i_0} > 0,$$

  which is absurd.

  (b) *Show that* $\int_a^b |fg| d\alpha = 0$ *if* $\int_a^b |f| d\alpha = 0$. Since $g \in \mathscr{R}(\alpha)$, $|g|$ is bounded by some real M on $[a, b]$, that is, $|g(x)| \leq M$. Hence

$$0 \leq \int_a^b |fg| d\alpha \leq \int_a^b M|f| d\alpha = M \int_a^b |f| d\alpha = 0.$$

  Therefore $\int_a^b |fg| d\alpha = 0$.

By (a)(b), $\int_a^b |fg| d\alpha = 0$ and thus Hölder's inequality holds for this case.

(2) The case $\left\{ \int_a^b |g|^q d\alpha \right\}^{\frac{1}{q}} = 0$. Similar to (1).

(3) If both $\left\{\int_a^b |f|^p d\alpha\right\}^{\frac{1}{p}} > 0$ and $\left\{\int_a^b |g|^q d\alpha\right\}^{\frac{1}{q}} > 0$, then we apply (b) to

$$F(x) = \frac{|f(x)|}{\left\{\int_a^b |f(x)|^p d\alpha\right\}^{\frac{1}{p}}} \qquad \text{and} \qquad G(x) = \frac{|g(x)|}{\left\{\int_a^b |g(x)|^q d\alpha\right\}^{\frac{1}{q}}}.$$

Here $F(x) \geq 0$ and $G(x) \geq 0$ are well-defined and Riemann integrable. Thus the conclusion holds. The equality holds if $F(x)^p = G(x)^q$ or

$$\frac{|f|^p}{\int_a^b |f|^p d\alpha} = \frac{|g|^q}{\int_a^b |g|^q d\alpha}.$$

Note that the equality does not hold only if $\frac{|f|^p}{\int_a^b |f|^p d\alpha} = \frac{|g|^q}{\int_a^b |g|^q d\alpha}$. Luckily, it is true for the additional assumption that $\alpha(x) = x$ and f, g are continuous on $[a, b]$.

By (1)(2)(3), in any case the equality holds if

$$|f|^p \int_a^b |g|^q d\alpha = |g|^q \int_a^b |f|^p d\alpha.$$

In addition, if $\alpha(x) = x$ and f, g are continuous on $[a, b]$, then the equality holds if and only if

$$|f|^p \int_a^b |g|^q d\alpha = |g|^q \int_a^b |f|^p d\alpha.$$

□

Proof of (d).

(1) *Suppose f and g are real functions on $(0, 1]$ and $f, g \in \mathscr{R}$ on $[c, 1]$ for every $c > 0$. Show that*

$$\left|\int_0^1 fg dx\right| \leq \left\{\int_0^1 |f|^p dx\right\}^{\frac{1}{p}} \left\{\int_0^1 |g|^q dx\right\}^{\frac{1}{q}}.$$

Here $\int_0^1$ is one improper integral defined in Exercise 6.7.

(a) By (c), we have

$$\left|\int_c^1 fg dx\right| \leq \left\{\int_c^1 |f|^p dx\right\}^{\frac{1}{p}} \left\{\int_c^1 |g|^q dx\right\}^{\frac{1}{q}}$$

for any $c \in (0, 1]$. Here every integral is well-defined (Theorem 6.11 and Theorem 6.13).

(b) Since every integral is ≥ 0, by taking the limit in the right hand side we have

$$\begin{aligned}\left|\int_c^1 fgdx\right| &\leq \left\{\int_c^1 |f|^p dx\right\}^{\frac{1}{p}} \left\{\int_c^1 |g|^q dx\right\}^{\frac{1}{q}} \\ &\leq \left\{\int_0^1 |f|^p dx\right\}^{\frac{1}{p}} \left\{\int_0^1 |g|^q dx\right\}^{\frac{1}{q}}.\end{aligned}$$

It is possible that $\left\{\int_0^1 |f|^p dx\right\}^{\frac{1}{p}} = \infty$ or $\left\{\int_0^1 |g|^q dx\right\}^{\frac{1}{q}} = \infty$.

(c) Now $\left|\int_c^1 fgdx\right|$ is bounded by $\left\{\int_0^1 |f|^p dx\right\}^{\frac{1}{p}} \left\{\int_0^1 |g|^q dx\right\}^{\frac{1}{q}}$. Take limit to get

$$\left|\int_0^1 fgdx\right| \leq \left\{\int_0^1 |f|^p dx\right\}^{\frac{1}{p}} \left\{\int_0^1 |g|^q dx\right\}^{\frac{1}{q}}$$

even if some limit is divergent.

(2) *Suppose f and g are real functions on $[a, b]$ and $f, g \in \mathscr{R}$ on $[a, b]$ for every $b > a$ where a is fixed. Show that*

$$\left|\int_a^\infty fgdx\right| \leq \left\{\int_a^\infty |f|^p dx\right\}^{\frac{1}{p}} \left\{\int_a^\infty |g|^q dx\right\}^{\frac{1}{q}}.$$

Here $\int_a^\infty$ is one improper integral defined in Exercise 6.8. Same as (1).

□

Exercise 6.11. *Let α be a fixed increasing function on $[a, b]$. For $u \in \mathscr{R}(\alpha)$, define*

$$\|u\|_2 = \left\{\int_a^b |u|^2 d\alpha\right\}^{\frac{1}{2}}.$$

Suppose $f, g, h \in \mathscr{R}(\alpha)$, and prove the triangle inequality

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2$$

as a consequence of the Schwarz inequality, as in the proof of Theorem 1.37.

Proof.

(1) By Exercise 6.10(c) with $p = q = 2$, we have

$$\begin{aligned}\int_a^b |f-g||g-h|d\alpha &= \left|\int_a^b |f-g||g-h|d\alpha\right| \\ &\leq \left\{\int_a^b |f-g|^2 dx\right\}^{\frac{1}{2}} \left\{\int_a^b |g-h|^2 dx\right\}^{\frac{1}{2}} \\ &= \|f-g\|_2 \|g-h\|_2.\end{aligned}$$

Every integral is well-defined (Theorem 6.12 and Theorem 6.13 (or Theorem 6.11)).

(2) Since

$$\begin{aligned}
\|f-h\|_2^2 &= \int_a^b |f-h|^2 d\alpha \\
&\leq \int_a^b (|f-g|+|g-h|)^2 d\alpha && \text{(Triangle inequality)} \\
&= \int_a^b (|f-g|^2 + 2|f-g||g-h| + |g-h|^2) d\alpha \\
&= \int_a^b |f-g|^2 d\alpha + 2\int_a^b |f-g||g-h| d\alpha + \int_a^b |g-h|^2 d\alpha \\
&\leq \|f-g\|_2^2 + 2\|f-g\|_2\|g-h\|_2 + \|g-h\|_2^2 && ((1)) \\
&= (\|f-g\|_2 + \|g-h\|_2)^2,
\end{aligned}$$

we have

$$\|f-h\|_2 \leq \|f-g\|_2 + \|g-h\|_2.$$

Here every integral is well-defined (Theorem 6.12 and Theorem 6.13 (or Theorem 6.11)).

□

Exercise 6.12. *With the notations of Exercise 6.11, suppose $f \in \mathscr{R}(\alpha)$ and $\varepsilon > 0$. Prove that there exists a continuous function g on $[a,b]$ such that $\|f-g\|_2 < \varepsilon$. (Hint: Let $P = \{a = x_0 \leq \cdots \leq x_n = b\}$ be a suitable partition of $[a,b]$, define*

$$g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i)$$

if $x_{i-1} \leq t \leq x_i$.)

Proof. Given $\varepsilon > 0$.

(1) There are some real numbers m and M such that $m \leq f(x) \leq M$ if $x \in [a,b]$ since $f \in \mathscr{R}(\alpha)$ or f is bounded on $[a,b]$. By Theorem 6.6, there exists a partition $P = \{a = x_0 \leq \cdots \leq x_n = b\}$ such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \frac{\varepsilon^2}{M-m+1}.$$

Here

$$\begin{aligned}
U(P,f,\alpha) &= \sum_{i=1}^n M_i \Delta\alpha_i \text{ where } M_i = \sup_{x_{i-1}\leq x\leq x_i} f(x) \\
L(P,f,\alpha) &= \sum_{i=1}^n m_i \Delta\alpha_i \text{ where } m_i = \inf_{x_{i-1}\leq x\leq x_i} f(x).
\end{aligned}$$

(2) For such partition P, define g on $[a,b]$ by

$$g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i)$$

if $x_{i-1} \leq t \leq x_i$. So that

$$\begin{aligned}
|f(t) - g(t)| &= \left| \left(\frac{x_i - t}{\Delta x_i} + \frac{t - x_{i-1}}{\Delta x_i} \right) f(t) - \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i) \right| \\
&= \left| \frac{x_i - t}{\Delta x_i} (f(t) - f(x_{i-1})) + \frac{t - x_{i-1}}{\Delta x_i} (f(t) - f(x_i)) \right| \\
&\leq \frac{x_i - t}{\Delta x_i} |f(t) - f(x_{i-1})| + \frac{t - x_{i-1}}{\Delta x_i} |f(t) - f(x_i)| \\
&\leq \frac{x_i - t}{\Delta x_i} (M_i - m_i) + \frac{t - x_{i-1}}{\Delta x_i} (M_i - m_i) \\
&= M_i - m_i
\end{aligned}$$

if $x_{i-1} \leq t \leq x_i$. Especially,

$$|f(t) - g(t)| \leq M - m$$

if $a \leq t \leq b$.

(3) Note that the integral $\int_a^b |f-g|^2 d\alpha$ is well-defined (Theorem 6.8, Theorem 6.11 and Theorem 6.12). So that

$$\begin{aligned}
\int_a^b |f - g|^2 d\alpha =& \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f - g|^2 d\alpha \\
\leq& \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (M - m)(M_i - m_i) d\alpha \\
=& (M - m) \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (M_i - m_i) \Delta\alpha_i \\
=& (M - m)[U(P, f, \alpha) - L(P, f, \alpha)] \\
\leq& (M - m) \cdot \frac{\varepsilon^2}{M - m + 1} \\
<& \varepsilon^2.
\end{aligned}$$

Hence,

$$\|f - g\|_2 = \left\{ \int_a^b |f - g|^2 d\alpha \right\}^{\frac{1}{2}} < \varepsilon.$$

□

Note.

(1) Apply the same argument we can prove the following statement:

> *Suppose $f \in \mathscr{R}(\alpha)$ and $\varepsilon > 0$. Prove that there exists a continuous function g on $[a,b]$ such that $\int_a^b |f-g|d\alpha < \varepsilon$.*

(2) (Lebesgue integral)

(a) *Let f be Lebesgue integrable over E. Then, given $\varepsilon > 0$, there is a simple function φ such that*

$$\int_E |f-\varphi| < \varepsilon.$$

(b) *Under the same hypothesis there is a step function ψ such that*

$$\int_E |f-\psi| < \varepsilon.$$

(c) *Under the same hypothesis there is a continuous function g vanishing outside a finite interval such that*

$$\int_E |f-g| < \varepsilon.$$

Exercise 6.13. *Define*

$$f(x) = \int_x^{x+1} \sin\left(t^2\right) dt.$$

(a) *Prove that $|f(x)| < \frac{1}{x}$ if $x > 0$. (Hint: Put $t^2 = u$ and integrate by parts, to show that $f(x)$ is equal to*

$$\frac{\cos\left(x^2\right)}{2x} - \frac{\cos[(x+1)^2]}{2(x+1)} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{\frac{3}{2}}} du.$$

Replace $\cos u$ by -1.)

(b) *Prove that*

$$2xf(x) = \cos\left(x^2\right) - \cos[(x+1)^2] + r(x)$$

where $|r(x)| < \frac{c}{x}$ and c is a constant.

(c) *Find the upper and lower limits of $xf(x)$, as $x \to \infty$.*

(d) *Does $\int_0^\infty \sin\left(t^2\right) dt$ converges?*

Proof of (a).

(1) Put $t^2 = u$ and integrate by parts to get

$$\begin{aligned} f(x) &= \int_x^{x+1} \sin\left(t^2\right) dt \\ &= \int_{x^2}^{(x+1)^2} \frac{\sin u}{2u^{\frac{1}{2}}} du \\ &= -\frac{\cos[(x+1)^2]}{2(x+1)} + \frac{\cos\left(x^2\right)}{2x} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{\frac{3}{2}}} du. \end{aligned}$$

(2)

$$\begin{aligned} |f(x)| &\leq \left| \frac{\cos[(x+1)^2]}{2(x+1)} \right| + \left| \frac{\cos\left(x^2\right)}{2x} \right| + \left| \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{\frac{3}{2}}} du \right| \\ &\leq \left| \frac{\cos[(x+1)^2]}{2(x+1)} \right| + \left| \frac{\cos\left(x^2\right)}{2x} \right| + \int_{x^2}^{(x+1)^2} \frac{|\cos u|}{4u^{\frac{3}{2}}} du \\ &\leq \frac{1}{2(x+1)} + \frac{1}{2x} + \int_{x^2}^{(x+1)^2} \frac{1}{4u^{\frac{3}{2}}} du \\ &= \frac{1}{2(x+1)} + \frac{1}{2x} + \left[\frac{1}{2x} - \frac{1}{2(x+1)} \right] \\ &= \frac{1}{x}. \end{aligned}$$

(3) The equality in (2) holds only if $\left|\cos[(x+1)^2]\right| = 1$, $\left|\cos\left(x^2\right)\right| = 1$, and

$$\left| \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{\frac{3}{2}}} du \right| = \int_{x^2}^{(x+1)^2} \frac{|\cos u|}{4u^{\frac{3}{2}}} du = \int_{x^2}^{(x+1)^2} \frac{1}{4u^{\frac{3}{2}}} du.$$

Since $\cos u$ has two absolute minimums or maximums at two different points $u = x^2$ and $u = (x+1)^2$, by the property of $\cos(u)$ there is some $u_0 \in [x^2, (x+1)^2]$ such that $\cos(u_0) = 0$. Hence given $\varepsilon = \frac{1}{2} > 0$ there exists $\delta > 0$ such that

$$|\cos(u)| \leq \frac{1}{2}$$

whenever

$$u \in E = [\max\{u_0 - \delta, x^2\}, \min\{u_0 + \delta, (x+1)^2\}] \subseteq [x^2, (x+1)^2].$$

Here $|E| > 0$. So that

$$\int_{x^2}^{(x+1)^2} \frac{|\cos u|}{4u^{\frac{3}{2}}} du = \int_{x^2}^{(x+1)^2} \frac{1}{4u^{\frac{3}{2}}} du - \frac{1}{2} \int_E \frac{1}{4u^{\frac{3}{2}}} du < \int_{x^2}^{(x+1)^2} \frac{1}{4u^{\frac{3}{2}}} du,$$

which is absurd. Hence the equality in (2) does not hold.

□

Proof of (b).

(1) By (a),

$$2xf(x) = \cos\bigl(x^2\bigr) - \cos[(x+1)^2] + r(x)$$

where

$$r(x) = \frac{\cos[(x+1)^2]}{x+1} - 2x \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{\frac{3}{2}}} du.$$

(2) Similar to (a),

$$\begin{aligned} |2xf(x)| &\leq \frac{1}{x+1} + 2x \int_{x^2}^{(x+1)^2} \frac{1}{4u^{\frac{3}{2}}} du \\ &= \frac{1}{x+1} + 2x \left[\frac{1}{2x} - \frac{1}{2(x+1)} \right] \\ &= \frac{2}{x+1} \\ &< \frac{2}{x}. \end{aligned}$$

□

Proof of (c).
□

Note.

$$\int_0^\infty \sin\bigl(t^2\bigr) dt = \int_0^\infty \cos\bigl(t^2\bigr) dt = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

Exercise 6.14. *Deal similarly with*

$$f(x) = \int_x^{x+1} \sin\bigl(e^t\bigr) dt.$$

Show that

$$e^x |f(x)| < 2$$

and that

$$e^x f(x) = \cos(e^x) - e^{-1} \cos\bigl(e^{x+1}\bigr) + r(x)$$

where $|r(x)| < Ce^{-x}$ *for some constant* C.

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Exercise 6.15. *Suppose f is a real, continuously differentiable function on $[a,b]$, $f(a) = f(b) = 0$, and*

$$\int_a^b f(x)^2 dx = 1.$$

Prove that

$$\int_a^b x f(x) f'(x) dx = -\frac{1}{2}$$

and that

$$\int_a^b [f'(x)]^2 dx \int_a^b x^2 f(x)^2 dx > \frac{1}{4}.$$

Proof. Every integral is well-defined (Theorem 4.9 and Theorem 6.8).

(1) By Theorem 6.22 (integration by parts),

$$\int_a^b x \left(\frac{f(x)^2}{2} \right)' dx = \left[x \cdot \frac{f(x)^2}{2} \right]_{x=a}^{x=b} - \int_a^b \frac{f(x)^2}{2} dx,$$

or

$$\int_a^b x f(x) f'(x) dx = \left[b \cdot \frac{f(b)^2}{2} - a \cdot \frac{f(a)^2}{2} \right] - \frac{1}{2} \int_a^b f(x)^2 dx = -\frac{1}{2}.$$

(2) By Exercise 6.10(c),

$$\int_a^b [f'(x)]^2 dx \int_a^b x^2 f(x)^2 dx \geq \left(\int_a^b x f(x) f'(x) dx \right)^2 = \frac{1}{4}.$$

(3) (Reductio ad absurdum) If the equality were holding, then by Exercise 6.10(c)

$$(f'(x))^2 \int_a^b x^2 f(x)^2 dx = x^2 f(x)^2 \int_a^b [f'(x)]^2 dx$$

on $[a,b]$ (since x, $f(x)$ and $f'(x)$ are continuous on $[a,b]$).

(a) *Show that both integrals are nonzero.* (Reductio ad absurdum) If $\int_a^b x^2 f(x)^2 dx = 0$, then $x^2 f(x)^2 = 0$ or $x f(x) = 0$ on $[a,b]$ (Exercise 6.2). So that

$$\int_a^b x f(x) f'(x) dx = 0 \neq -\frac{1}{2},$$

which is absure. Similarly, $\int_a^b [f'(x)]^2 dx \neq 0$.

(b) By (a), we write

$$C = \left\{ \frac{\int_a^b [f'(x)]^2 dx}{\int_a^b x^2 f(x)^2 dx} \right\}^{\frac{1}{2}} > 0$$

be a positive constant. Hence

$$f'(x) = \pm Cxf(x).$$

Here the sign "$\pm$" is not necessary unchanged on $[a, b]$. Luckily, we can show that the sign "$\pm$" is unchanged on some subinterval of $[a, b]$.

(c) To find such subinterval of $[a, b]$, we consider the zero set $Z(f')$ and $Z(xf)$ on $[a, b]$. Since $f'(x) = \pm Cxf(x)$ with $C > 0$, we have

$$Z(f') = Z(xf).$$

Note that $Z(f') = Z(xf)$ is closed (Exercise 4.3) and not equal to $[a, b]$ (by applying the same argument in (a)). Hence the complement of $Z(f') = Z(xf)$ is open and nonempty, which can be written as the union of an at most countable collection of disjoint segments (Exercise 2.29).

(d) Consider any nonempty open interval in (c), say

$$(c, d) \subseteq [a, b].$$

By construction, $f'(x) \neq 0$ for all $x \in (c, d)$. Since $f'(x)$ is continuous, by Theorem 4.23 there are only two mutually exclusive possible cases:

(i) $f'(x) > 0$ for all $x \in (c, d)$,
(ii) $f'(x) < 0$ for all $x \in (c, d)$.

Similar result for $xf(x)$. Therefore, the sign "$\pm$" of $f'(x) = \pm Cxf(x)$ are unchanged on (c, d), that is,

(i) $f'(x) = Cxf(x)$ for all $x \in (c, d)$,
(ii) $f'(x) = -Cxf(x)$ for all $x \in (c, d)$,

(e) Suppose $f'(x) = Cxf(x)$ on (c, d). Since $f'(x)$ and $xf(x)$ are both vanishing at $x = c$ and $x = d$, $f'(x) = Cxf(x)$ at $x = c$ and $x = d$. So

$$f'(x) = Cxf(x) \text{ if } x \in [c, d].$$

Define

$$\phi(x, y) = Cxy$$

be a real function on $R = [c, d] \times \mathbb{R}$. And consider the initial-value problem

$$y' = \phi(x, y) \qquad \text{with} \qquad y(c) = 0.$$

Then

$$|\phi(x, y_2) - \phi(x, y_1)| = Cx|y_2 - y_1| \leq A|y_2 - y_1|$$

where $A = C \cdot \max\{|c|, |d|\}$ is a constant. By Exercise 5.27, this initial-value problem has at most one solution. Clearly, $y = f(x) = 0$ on $[c, d]$ is one solution of this initial-value problem, contrary to the construction of $[c, d]$. Similar result for the case $f'(x) = -Cxf(x)$.

Therefore, the equality does not hold.

□

Exercise 6.16. *For* $1 < s < \infty$*, define*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

(This is Riemann's zeta function, of great importance in the study of the distribution of prime numbers.) Prove that

(a)

$$\zeta(s) = s \int_1^{\infty} \frac{[x]}{x^{s+1}} dx$$

and that

(b)

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{\infty} \frac{x - [x]}{x^{s+1}} dx,$$

where $[x]$ *denotes the greatest integer* $\leq x$*. Prove that the integral in (b) converges for all* $s > 0$*. (Hint: To prove (a), compute the difference between the integral over* $[1, N]$ *and the* N*th partial sum of the series that defines* $\zeta(s)$*.)*

Proof of (a) (Hint).

(a) Define

$$a_N = s \int_1^N \frac{[x]}{x^{s+1}} dx - \sum_{n=1}^{N} \frac{1}{n^s}.$$

Hence

$$\begin{aligned}
s \int_1^N \frac{[x]}{x^{s+1}} dx &= \sum_{n=1}^{N-1} s \int_n^{n+1} \frac{[x]}{x^{s+1}} dx \\
&= \sum_{n=1}^{N-1} s \int_n^{n+1} \frac{n}{x^{s+1}} dx \\
&= \sum_{n=1}^{N-1} n \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) \\
&= \left(\sum_{n=1}^{N} \frac{1}{n^s} \right) - \frac{1}{N^{s-1}},
\end{aligned}$$

or

$$a_N = -\frac{1}{N^{s-1}}.$$

So

$$\lim_{N \to \infty} a_N = 0$$

(since $s - 1 > 0$). By Theorem 3.28, $\zeta(s)$ converges if $s > 1$. Hence

$$\lim_{N \to \infty} s \int_1^N \frac{[x]}{x^{s+1}} dx = \zeta(s)$$

converges.

(b) Hence given any real $b > 1$, there exists an integer N such that $N \leq b < N + 1$. Since $x \mapsto \frac{[x]}{x^{s+1}} \geq 0$ on $[1, \infty)$,

$$s \int_1^N \frac{[x]}{x^{s+1}} dx \leq s \int_1^b \frac{[x]}{x^{s+1}} dx \leq s \int_1^{N+1} \frac{[x]}{x^{s+1}} dx.$$

Since $b \to \infty$ if and only if $N \to \infty$,

$$\begin{aligned} &\lim_{N \to \infty} s \int_1^N \frac{[x]}{x^{s+1}} dx \leq \lim_{b \to \infty} s \int_1^b \frac{[x]}{x^{s+1}} dx \leq \lim_{N \to \infty} s \int_1^{N+1} \frac{[x]}{x^{s+1}} dx \\ \Longrightarrow &\zeta(s) \leq \lim_{b \to \infty} s \int_1^b \frac{[x]}{x^{s+1}} dx \leq \zeta(s). \end{aligned}$$

Hence

$$\lim_{b \to \infty} s \int_1^b \frac{[x]}{x^{s+1}} dx = s \int_1^\infty \frac{[x]}{x^{s+1}} dx = \zeta(s)$$

(in the sense of Exercise 6.8).

$\square$

Proof of (b).

(a) *Show that*

$$s \int_1^\infty \frac{1}{x^s} dx = \frac{s}{s-1}.$$

Given any real $b > 1$. By the fundamental theorem of calculus (Theorem 6.21),

$$s \int_1^b \frac{1}{x^s} dx = \frac{s}{s-1} - \frac{s}{(s-1)b^{s-1}}.$$

Hence

$$\lim_{b \to \infty} s \int_1^b \frac{1}{x^s} dx = \frac{s}{s-1}$$

since $\frac{1}{b^{s-1}} \to 0$ as $b \to \infty$ (in the sense of Exercise 6.8).

(b) By $\zeta(s) = s\int_1^\infty \frac{[x]}{x^{s+1}}dx$ and $s\int_1^\infty \frac{1}{x^s}dx = \frac{s}{s-1}$, $s\int_1^\infty \frac{x-[x]}{x^{s+1}}dx$ exists and equal to

$$s\int_1^\infty \frac{x-[x]}{x^{s+1}}dx = s\int_1^\infty \frac{1}{x^s}dx - s\int_1^\infty \frac{[x]}{x^{s+1}}dx = \frac{s}{s-1} - \zeta(s).$$

The result is established.

(c) *Show that*

$$\int_1^\infty \frac{x-[x]}{x^{s+1}}dx$$

converges for all $s > 0$. Note that $0 \le x - [x] < 1$ on $[1, \infty)$. So

$$\int_1^b \frac{x-[x]}{x^{s+1}}dx \le \int_1^b \frac{1}{x^{s+1}}dx = \frac{1}{s} - \frac{1}{sb^s}.$$

Since $\frac{1}{sb^s} \to 0$ as $b \to \infty$,

$$\int_1^\infty \frac{x-[x]}{x^{s+1}}dx = \lim_{b\to\infty}\int_1^b \frac{x-[x]}{x^{s+1}}dx \le \lim_{b\to\infty} \frac{1}{s} - \frac{1}{sb^s} = \frac{1}{s}.$$

Note that $\frac{1}{s}$ is finite, and thus the integral $\int_1^\infty \frac{x-[x]}{x^{s+1}}dx$ converges.

$\square$

Note. The integral $\int_1^\infty \frac{[x]}{x^{s+1}}dx$ does not converge for all $1 \ge s > 0$.

Supplement (Euler's summation formula). (Theorem 7.13 in the textbook: *Tom. M. Apostol, Mathematical Analysis, 2nd edition.*) *If f has a continuous derivative f' on $[a,b]$, then we have*

$$\sum_{a<n\le b} f(n) = \int_a^b f(x)dx + \int_a^b f'(x)\{x\}dx + f(a)\{a\} - f(b)\{b\},$$

where $\sum_{a<n\le b}$ means the sum from $n = [a]+1$ to $n = [b]$. When a and b are integers, this becomes

$$\sum_{n=a}^b f(n) = \int_a^b f(x)dx + \int_a^b f'(x)\left(\{x\} - \frac{1}{2}\right)dx + \frac{f(a)+f(b)}{2}.$$

By taking $f(x) = \frac{1}{x^s}$ we can get (a) as well.

Exercise 6.17. *Suppose α increases monotonically on $[a,b]$, g is continuous, and $g(x) = G'(x)$ for $a \le x \le b$. Prove that*

$$\int_a^b \alpha(x)g(x)dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G d\alpha.$$

(Hint: Take g real, without loss of generality. Given $P = \{a = x_0, x_1, \ldots, x_n = b\}$, choose $t_i \in (x_{i-1}, x_i)$ so that $g(t_i)\Delta x_i = G(x_i) - G(x_{i-1})$. Show that

$$\sum_{i=1}^{n} \alpha(x_i) g(t_i) \Delta x_i = G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^{n} G(x_{i-1}) \Delta\alpha_i.)$$

Proof (Hint). Given $\varepsilon > 0$.

(1) Take g real, without loss of generality. Given any partition

$$P = \{a = x_0 < x_1 < \ldots < x_n = b\}$$

of $[a, b]$.

(2) By the mean value theorem (Theorem 5.10), there is $t_i \in (x_{i-1}, x_i)$ such that
$$G(x_i) - G(x_{i-1}) = (x_i - x_{i-1})G'(t_i) = g(t_i)\Delta x_i.$$

(3) Hence,

$$\begin{aligned}
\sum_{i=1}^{n} \alpha(x_i) g(t_i) \Delta x_i &= \sum_{i=1}^{n} \alpha(x_i)(G(x_i) - G(x_{i-1})) \\
&= \sum_{i=1}^{n} \alpha(x_i) G(x_i) - \sum_{i=1}^{n} \alpha(x_i) G(x_{i-1}) \\
&= \underbrace{G(b)\alpha(b) - G(a)\alpha(a) + \sum_{i=1}^{n} \alpha(x_{i-1}) G(x_{i-1})}_{\text{adjust the index of } \sum_{i=1}^{n} \alpha(x_i)G(x_i)} - \sum_{i=1}^{n} \alpha(x_i) G(x_{i-1}) \\
&= G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^{n} G(x_{i-1}) \Delta\alpha_i.
\end{aligned}$$

(4) Since $G(x)$ is differentiable on $[a, b]$, $G(x)$ is continuous (Theorem 5.2) and thus $G \in \mathscr{R}(\alpha)$ (Theorem 6.8). So there is a partition P_1 such that

$$\left| \sum_{j=1}^{n} G(t_j) \Delta\alpha_j - \int_a^b G d\alpha \right| < \varepsilon$$

whenever $t_j \in [x_{j-1}, x_j]$ (Theorem 6.7). In particular, we pick $t_j = x_{j-1} \in [x_{j-1}, x_j]$ for all j, that is,

$$\left| \sum_{j=1}^{n} G(x_{j-1}) \Delta\alpha_j - \int_a^b G d\alpha \right| < \varepsilon.$$

Note that if P^* is a refinement of P, the result is true too (Theorem 6.4).

(5) Since α increases monotonically, $\alpha \in \mathscr{R}$ (Theorem 6.9). Since g is continuous, $g \in \mathscr{R}$ (Theorem 6.8). Hence $\alpha g \in \mathscr{R}$ (Theorem 6.13). So there is a partition P_2 such that

$$\left|\sum_{k=1}^{m} \alpha(t_k) g(t_k) \Delta x_k - \int_a^b \alpha g dx\right| < \varepsilon$$

whenever $t_k \in [x_{k-1}, x_k]$ (Theorem 6.7). In particular, we pick $t_k = x_k \in [x_{k-1}, x_k]$ for all k, that is,

$$\left|\sum_{k=1}^{m} \alpha(x_k) g(x_k) \Delta x_k - \int_a^b \alpha g dx\right| < \varepsilon.$$

Note that if P^* is a refinement of P, the result is true too (Theorem 6.4).

(6) Since g is continuous on a compact set $[a, b]$, g is uniformly continuous. Hence there exists $\delta > 0$ such that

$$|g(y) - g(x)| < \varepsilon$$

whenever $|y - x| < \delta$ and $x, y \in [a, b]$. For such δ, we construct a partition P_3 such that

$$|g(t_l) - g(x_l)| < \varepsilon$$

whenever $t_l \in [x_{l-1}, x_l]$. (For example, we might take

$$P_3 = \left\{a, a + \frac{1}{N}(b-a), a + \frac{2}{N}(b-a), \ldots, a + \frac{N-1}{N}(b-a), b\right\}$$

where N is an integer $\geq \frac{b-a}{\delta}$.) Hence

$$\begin{aligned}
&\left|\sum_{l=1}^{N} \alpha(x_l) g(t_l) \Delta x_l - \sum_{l=1}^{N} \alpha(x_l) g(x_l) \Delta x_l\right| \\
=&\left|\sum_{l=1}^{N} \alpha(x_l)[g(t_l) - g(x_l)] \Delta x_l\right| \\
\leq&\sum_{l=1}^{N} |\alpha(x_l)| \cdot |g(t_l) - g(x_l)| \cdot \Delta x_l \\
\leq&M\varepsilon \sum_{l=1}^{N} \Delta x_l \\
=&M(b-a)\varepsilon.
\end{aligned}$$

Note that if P^* is a refinement of P, the result is true too (by the uniformly convergence of g).

(7) Let $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ be a common refinement of P_1, P_2 and P_3. By (3)(4)(5)(6) we have

$$\begin{aligned}
&\left|\int_a^b \alpha(x)g(x)dx - G(b)\alpha(b) + G(a)\alpha(a) + \int_a^b G d\alpha\right| \\
=&\left|\int_a^b \alpha(x)g(x)dx - \sum_{i=1}^n \alpha(x_i)g(t_i)\Delta x_i + \int_a^b G d\alpha - \sum_{i=1}^n G(x_{i-1})\Delta\alpha_i\right| \\
\leq&\left|\int_a^b \alpha(x)g(x)dx - \sum_{i=1}^n \alpha(x_i)g(t_i)\Delta x_i\right| + \left|\int_a^b G d\alpha - \sum_{i=1}^n G(x_{i-1})\Delta\alpha_i\right| \\
\leq&\left|\int_a^b \alpha(x)g(x)dx - \sum_{i=1}^n \alpha(x_i)g(x_i)\Delta x_i\right| + \left|\sum_{i=1}^n \alpha(x_i)g(x_i)\Delta x_i - \sum_{i=1}^n \alpha(x_i)g(t_i)\Delta x_i\right| \\
&+ \left|\int_a^b G d\alpha - \sum_{i=1}^n G(x_{i-1})\Delta\alpha_i\right| \\
\leq&\varepsilon + M(b-a)\varepsilon + \varepsilon \\
=&(M(b-a)+2)\varepsilon.
\end{aligned}$$

Since ε is arbitrary,

$$\left|\int_a^b \alpha(x)g(x)dx - G(b)\alpha(b) + G(a)\alpha(a) + \int_a^b G d\alpha\right| = 0,$$

or

$$\int_a^b \alpha(x)g(x)dx - G(b)\alpha(b) + G(a)\alpha(a) + \int_a^b G d\alpha = 0,$$

or

$$\int_a^b \alpha(x)g(x)dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G d\alpha.$$

□

Exercise 6.18.
PLACEHOLDER

Exercise 6.19.
PLACEHOLDER