

## Chapter 15: Bernoulli Numbers

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**Supplement.** Equation (4) on page 231. *Prove that*

$$x \cot x = 1 - 2 \sum_{n=1}^{\infty} \frac{x^2}{n^2 \pi^2 - x^2}.$$

*Proof (Exercise 6.73 in the book Graham, Knuth and Patashnik, Concrete Mathematics, Second Edition).*

(1) *Show that*

$$\cot x = \frac{1}{2^n} \sum_{k=0}^{2^n-1} \cot \frac{x + k\pi}{2^n}$$

for all integers  $n \geq 1$ . Notice that

$$\begin{aligned} \cot(x + \pi) &= \cot x, \\ \cot\left(x + \frac{\pi}{2}\right) &= -\tan x, \\ \cot x &= \frac{1}{2} \left( \cot \frac{x}{2} - \tan \frac{x}{2} \right). \end{aligned}$$

Use mathematical induction. The case  $n = 1$  is the same as the note. Assume the case  $n = m$  holds. For  $n = m + 1$ ,

$$\begin{aligned} \sum_{k=0}^{2^{m+1}-1} \cot \frac{x + k\pi}{2^{m+1}} &= \sum_{k=0}^{2^m-1} \cot \frac{x + k\pi}{2^{m+1}} + \sum_{k=2^m}^{2^{m+1}-1} \cot \frac{x + k\pi}{2^{m+1}} \\ &= \sum_{k=0}^{2^m-1} \cot \frac{x + k\pi}{2^{m+1}} + \sum_{k=0}^{2^m-1} \cot \frac{x + (2^m + k)\pi}{2^{m+1}} \\ &= \sum_{k=0}^{2^m-1} \cot \frac{x + k\pi}{2^{m+1}} + \sum_{k=0}^{2^m-1} \cot \left( \frac{x + k\pi}{2^{m+1}} + \frac{\pi}{2} \right) \\ &= \sum_{k=0}^{2^m-1} \left( \cot \frac{x + k\pi}{2^{m+1}} - \tan \frac{x + k\pi}{2^{m+1}} \right) \\ &= \sum_{k=0}^{2^m-1} \left( \cot \frac{x + k\pi}{2^{m+1}} - \tan \frac{x + k\pi}{2^{m+1}} \right) \\ &= 2 \sum_{k=0}^{2^m-1} \cot \frac{x + k\pi}{2^m}. \end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{2^{m+1}} \sum_{k=0}^{2^{m+1}-1} \cot \frac{x+k\pi}{2^{m+1}} &= \frac{1}{2^{m+1}} \cdot 2 \sum_{k=0}^{2^m-1} \cot \frac{x+k\pi}{2^m} \\ &= \frac{1}{2^m} \sum_{k=0}^{2^m-1} \cot \frac{x+k\pi}{2^m} \\ &= \cot x.\end{aligned}$$

(2) By rearranging the index of summation of the identity in (1), we have

$$x \cot x = \frac{x}{2^n} \cot \frac{x}{2^n} - \frac{x}{2^n} \tan \frac{x}{2^n} + \sum_{k=1}^{2^{n-1}-1} \frac{x}{2^n} \left( \cot \frac{x+k\pi}{2^n} + \cot \frac{x-k\pi}{2^n} \right)$$

for all integers  $n \geq 1$ .

(3) Notice that  $\lim_{x \rightarrow 0} x \cot x = 1$ . Let  $n \rightarrow \infty$ , the result is established.

□

**Exercise 15.1.** Using the definition of the Bernoulli number show  $B_{10} = \frac{5}{66}$  and  $B_{12} = -\frac{691}{2730}$ .

*Proof.*

- (1) It is known that  $B_1 = -\frac{1}{2}$ ,  $B_2 = \frac{1}{6}$ ,  $B_4 = -\frac{1}{30}$ ,  $B_6 = \frac{1}{42}$ , and  $B_m = 0$  for  $m > 1$  and odd.
- (2) Recall the implicit recurrence relation,

$$\sum_{k=0}^m \binom{m+1}{k} B_k = [m=0],$$

where  $[m=0]$  is the Iverson brackets which is equal to the Kronecker delta  $\delta_{m0}$ .

(3) So

$$0 = 1 + 9B_1 + 36B_2 + 84B_3 + 126B_4 + 126B_5 + 84B_6 + 36B_7 + 9B_8,$$

$$0 = 1 + 9B_1 + 36B_2 + 126B_4 + 84B_6 + 9B_8,$$

$$0 = 1 + 9 \left( -\frac{1}{2} \right) + 36 \left( \frac{1}{6} \right) + 126 \left( -\frac{1}{30} \right) + 84 \left( \frac{1}{42} \right) + 9B_8,$$

$$0 = \frac{3}{10} + 9B_8,$$

$$\text{Thus } B_8 = -\frac{1}{30}.$$

(4) Again,

$$\begin{aligned}
0 &= 1 + 11B_1 + 55B_2 + 165B_3 + 330B_4 + 462B_5 + 462B_6 + \\
&\quad 330B_7 + 165B_8 + 55B_9 + 11B_{10}, \\
0 &= 1 + 11B_1 + 55B_2 + 330B_4 + 462B_6 + 165B_8 + 11B_{10}, \\
0 &= 1 + 11 \left( -\frac{1}{2} \right) + 55 \left( \frac{1}{6} \right) + 330 \left( -\frac{1}{30} \right) + 462 \left( \frac{1}{42} \right) + \\
&\quad 165 \left( -\frac{1}{30} \right) + 11B_{10}, \\
0 &= -\frac{5}{6} + 11B_{10},
\end{aligned}$$

$$\text{Thus } B_{10} = \frac{5}{66}.$$

(4) Finally,

$$\begin{aligned}
0 &= 1 + 13B_1 + 78B_2 + 286B_3 + 715B_4 + 1287B_5 + 1716B_6 + \\
&\quad 1716B_7 + 1287B_8 + 715B_9 + 286B_{10} + 78B_{11} + 13B_{12}, \\
0 &= 1 + 13B_1 + 78B_2 + 715B_4 + 1716B_6 + 1287B_8 + 286B_{10} + 13B_{12}, \\
0 &= 1 + 13 \left( -\frac{1}{2} \right) + 78 \left( \frac{1}{6} \right) + 715 \left( -\frac{1}{30} \right) + 1716 \left( \frac{1}{42} \right) + \\
&\quad 1287 \left( -\frac{1}{30} \right) + 286 \left( \frac{5}{66} \right) + 13B_{12}, \\
0 &= \frac{691}{210} + 13B_{12},
\end{aligned}$$

$$\text{Thus } B_{12} = -\frac{691}{2730}.$$

□

**Exercise 15.2.** If  $a \in \mathbb{Z}$ , show  $a(a^m - 1)B_m \in \mathbb{Z}$  for all  $m > 0$ .

*Proof.*

- (1) *Trivial cases.* If  $m = 1$ ,  $a(a - 1)B_1 = -\frac{1}{2}a(a - 1) \in \mathbb{Z}$  for any  $a \in \mathbb{Z}$ .  
For  $m > 1$  and odd,  $B_m = 0$  or  $a(a^m - 1)B_m = 0 \in \mathbb{Z}$  (Proposition 15.1.1).

- (2) *Consider that  $m > 1$  and even.* By Theorem 3,

$$B_{2m} + \sum_{p-1|2m} \frac{1}{p} \in \mathbb{Z}$$

where the sum is over all primes  $p$  such that  $p - 1 \mid 2m$ . So it suffices to show

$$a(a^{2m} - 1) \sum_{p-1 \mid 2m} \frac{1}{p} \in \mathbb{Z},$$

or

$$a(a^{2m} - 1) \frac{1}{p} \in \mathbb{Z}$$

for any  $a \in \mathbb{Z}$  and any prime  $p$  such that  $p - 1 \mid 2m$ .

- (3) Consider all possible  $a$ . If  $p \mid a$ , it is trivial. If  $p \nmid a$ ,  $a^{p-1} \equiv 1 \pmod{p}$  by Fermat's Little Theorem, or  $a^{2m} \equiv 1 \pmod{p}$  by  $p - 1 \mid 2m$ . In any cases,  $a(a^{2m} - 1) \frac{1}{p} \in \mathbb{Z}$ .

□

**Exercise 15.6.** For  $m \geq 3$ , show  $|B_{2m+2}| > |B_{2m}|$ . (Hint: Use Theorem 2.)

*Proof.* By Theorem 2,

$$2\zeta(2m) = (-1)^{m+1} \frac{(2\pi)^{2m}}{(2m)!} B_{2m}.$$

Thus,

$$\frac{|B_{2m+2}|}{|B_{2m}|} = \frac{\zeta(2m+2)(2m+2)(2m+1)}{\zeta(2m)(2\pi)^2} > \frac{1 \cdot 8 \cdot 7}{\zeta(6) \cdot (2\pi)^2} = \frac{13230}{\pi^8} > 1,$$

or  $|B_{2m+2}| > |B_{2m}|$ . □

**Exercise 15.8.** Consider the power series expansion of  $\tan x$  about the origin;

$$\sum_{k=1}^{\infty} T_k \frac{x^{2k-1}}{(2k-1)!}.$$

Show

$$T_k = (-1)^{k-1} \frac{B_{2k}}{2k} (2^{2k} - 1) 2^{2k}.$$

Note that  $T_k \in \mathbb{Z}$  for all  $k$  by Exercise 3.

*Proof.*

- (1) By the equation (6) on page 232,

$$x \cot x = 1 + \sum_{k=2}^{\infty} B_k \frac{(2ix)^k}{k!}.$$

Since  $B_k = 0$  for  $k > 1$  and odd,

$$x \cot x = 1 + \sum_{k=1}^{\infty} B_{2k} \frac{(2ix)^{2k}}{(2k)!} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} x^{2k},$$

or

$$\cot x = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} x^{2k-1}.$$

Combine the first term  $\frac{1}{x}$  into the summation,

$$\cot x = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} x^{2k-1}.$$

(2) Note that  $\tan x = \cot x - 2 \cot(2x)$ . By (1),

$$\begin{aligned} \tan x &= \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} x^{2k-1} - 2 \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} (2x)^{2k-1} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (1 - 2^{2k}) 2^{2k} B_{2k}}{(2k)!} x^{2k-1} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k (1 - 2^{2k}) 2^{2k} B_{2k}}{(2k)!} x^{2k-1}. \end{aligned}$$

Write  $T_k = (-1)^{k-1} (2^{2k} - 1) 2^{2k} \frac{B_{2k}}{2k}$ . Therefore,  $\tan x = \sum_{k=1}^{\infty} T_k \frac{x^{2k-1}}{(2k-1)!}$ .

By Exercise 3,  $(2^{2k} - 1) 2^{2k} \frac{B_{2k}}{2k} \in \mathbb{Z}$ , or  $T_k \in \mathbb{Z}$  for all  $k$ .  $\square$

**Exercise 15.13.** Show  $B_m(x+1) - B_m(x) = mx^{m-1}$ .

*Proof.* Let  $f(t, x) = \frac{te^{tx}}{e^t - 1}$ .

(1)

$$f(t, x+1) - f(t, x) = \frac{te^{t(x+1)}}{e^t - 1} - \frac{te^{tx}}{e^t - 1} = te^{tx}.$$

Expand  $te^{tx}$  in a power series about the origin as follows

$$\begin{aligned}
te^{tx} &= t \sum_{m=0}^{\infty} x^m \frac{t^m}{m!} \\
&= \sum_{m=0}^{\infty} x^m \frac{t^{m+1}}{m!} \\
&= \sum_{m=1}^{\infty} x^{m-1} \frac{t^m}{(m-1)!} \\
&= \sum_{m=1}^{\infty} mx^{m-1} \frac{t^m}{m!} \\
&= \sum_{m=0}^{\infty} mx^{m-1} \frac{t^m}{m!}.
\end{aligned}$$

So,

$$f(t, x+1) - f(t, x) = \sum_{m=0}^{\infty} mx^{m-1} \frac{t^m}{m!}.$$

(2) By Exercise 15.12,

$$\begin{aligned}
f(t, x+1) - f(t, x) &= \sum_{m=0}^{\infty} B_m(x+1) \frac{t^m}{m!} - \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!} \\
&= \sum_{m=0}^{\infty} (B_m(x+1) - B_m(x)) \frac{t^m}{m!}.
\end{aligned}$$

By (1)(2), comparing coefficients of  $t^m$  yields

$$mx^{m-1} = B_m(x+1) - B_m(x).$$

□

**Exercise 15.14.** Use Exercise 13 to give a new proof of Theorem 1:

$$S_m(n) = \frac{1}{m+1} (B_{m+1}(n) - B_{m+1}).$$

*Proof.* By Exercise 13,

$$B_{m+1}(k) - B_{m+1}(k-1) = (m+1)(k-1)^m$$

for any  $k$ . So,

$$\begin{aligned}
\sum_{k=1}^n (B_{m+1}(k) - B_{m+1}(k-1)) &= \sum_{k=1}^n (m+1)(k-1)^m, \\
B_{m+1}(n) - B_{m+1}(0) &= (m+1)S_m(n).
\end{aligned}$$

Note that  $B_{m+1}(0) = B_{m+1}$  for any  $m$ . So Theorem 1 is established by a new proof.  $\square$

**Exercise 15.16.** For  $n \geq 1$ , show  $\frac{d}{dx}B_n(x) = nB_{n-1}(x)$ .

*Proof.* For  $n \geq 1$ ,

$$\frac{d}{dx}B_n(x) = \sum_{k=0}^n (n-k) \binom{n}{k} B_k x^{n-k-1} = \sum_{k=0}^{n-1} (n-k) \binom{n}{k} B_k x^{n-k-1}.$$

Note that

$$(n-k) \binom{n}{k} = n \binom{n-1}{k}.$$

So

$$\begin{aligned} \frac{d}{dx}B_n(x) &= \sum_{k=0}^{n-1} n \binom{n-1}{k} B_k x^{n-k-1} \\ &= n \sum_{k=0}^{n-1} \binom{n-1}{k} B_k x^{n-k-1} \\ &= nB_{n-1}(x). \end{aligned}$$

$\square$