

Chapter 2: Basic Topology

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Exercise 2.1. *Prove that the empty set is a subset of every set.*

Proof. By Definitions 1.3,

- (1) The set which contains no element will be called the **empty set**,
- (2) If A and B are sets, and if every element of A is an element of B , we say that A is a **subset** of B ,

every element of the empty set (there are none) belongs to every set. That is, the empty set is a subset of every set. \square

Exercise 2.2. *A complex number z is said to be algebraic if there are integers a_0, \dots, a_n , not all zero, such that*

$$a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic numbers is countable. (Hint: For every positive integer N there are only finitely many equations with

$$n + |a_0| + |a_1| + \dots + |a_n| = N.$$

Might assume $a_0 \neq 0$.

For example, all rational numbers are algebraic since $p = \frac{\alpha}{\beta}$ (where $\alpha, \beta \in \mathbb{Z}$) is a root of $\beta z - \alpha = 0$.

Besides, $z = \sqrt{2} + \sqrt{3}$ is algebraic since $z^4 - 10z^2 + 1 = 0$. In fact, $z = \pm\sqrt{2} + \pm\sqrt{3}$ are also algebraic since $z^4 - 10z^2 + 1 = (z - \sqrt{2} - \sqrt{3})(z + \sqrt{2} - \sqrt{3})(z - \sqrt{2} + \sqrt{3})(z + \sqrt{2} + \sqrt{3})$.

Lemma. *The set of all polynomials over \mathbb{Z} is countable implies that the set of algebraic numbers is countable.*

Proof of Lemma. By definition, we write the set of algebraic numbers as

$$S = \bigcup_{f(x) \in \mathbb{Z}[x]} \{z \in \mathbb{C} : f(z) = 0\}.$$

Since each polynomial of degree n has at most n roots, $\{z \in \mathbb{C} : f(z) = 0\}$ is finite for each given $f(x) \in \mathbb{Z}[x]$. So S is a countable union (by assumption) of finite sets, and hence at most countable. S is infinite since every integer α is a root of $f(z) = z - \alpha$. So S is countable. \square

Thus, it suffices to show that *the set of all polynomials over \mathbb{Z} is countable*.

Proof (Hint). For every positive integer N there are only finitely many equations with $n + |a_0| + |a_1| + \cdots + |a_n| = N$. Write

$$P_N = \{f(x) \in \mathbb{Z}[x] : n + |a_0| + |a_1| + \cdots + |a_n| = N\}$$

where $f(x) = a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n$ with $a_0 \neq 0$, and

$$P = \bigcup_{N=1}^{\infty} P_N.$$

P is the set of all polynomials over \mathbb{Z} .

Each P_N is finite for given N (since the equation $n + |a_0| + |a_1| + \cdots + |a_n| = N$ has finitely many solutions $(n, a_0, a_1, \dots, a_n) \in \mathbb{Z}^{n+2}$). So P is a countable union of finite sets, and hence at most countable. P is infinite since \mathbb{Z} is a subring of $\mathbb{Z}[x]$. So P is countable. \square

Proof (Theorem 2.13).

- (1) \mathbb{Z}^N is countable for any integer $N > 0$. Theorem 2.13.
- (2) The set of all polynomials over \mathbb{Z} is countable. Let

$$P_n = \{f \in \mathbb{Z}[x] : \deg f = n\},$$

and

$$P = \bigcup_{n=1}^{\infty} P_n = \mathbb{Z}[x].$$

Claim: P_n is countable. Define a 1-1 map $\varphi_n : P_n \rightarrow \mathbb{Z}^{n+1}$ by

$$\varphi_n(a_0z^n + a_1z^{n-1} + \cdots + a_n) = (a_0, a_1, \dots, a_{n-1}, a_n).$$

By (1) and Theorem 2.8, P_n is countable. (P_n is infinite since $a_n \in \mathbb{Z}$.) Now P is a countable union of countable sets, and hence countable by Theorem 2.12.

\square

Proof (Unique factorization theorem).

- (1) *The set of prime numbers is countable.* Write all primes in the ascending order as $p_1, p_2, \dots, p_n, \dots$ where $p_1 = 2, p_2 = 3, \dots, p_{10001} = 104743, \dots$ (See ProjectEuler 7: 10001st prime. Use sieve of Eratosthenes to get p_{10001} .)
- (2) *The set of all polynomials over \mathbb{Z} is countable.* Let

$$P_n = \{f \in \mathbb{Z}[x] : \deg f = n\},$$

and

$$P = \bigcup_{n=1}^{\infty} P_n = \mathbb{Z}[x].$$

Claim: P_n is countable. Define a map $\varphi_n : P_n \rightarrow \mathbb{Z}^+$ by

$$\varphi_n(a_0 z^n + a_1 z^{n-1} + \dots + a_n) = p_1^{\psi(a_0)} p_2^{\psi(a_1)} \dots p_{n+1}^{\psi(a_n)},$$

where ψ is a 1-1 correspondence from \mathbb{Z} to \mathbb{Z}^+ (Example 2.5). By the unique factorization theorem, φ_n is 1-1. So P_n is countable by Theorem 2.8. (P_n is infinite since $a_n \in \mathbb{Z}$.) Now P is a countable union of countable sets, and hence countable by Theorem 2.12.

□

Exercise 2.3. *Prove that there exist real numbers which are not algebraic.*

Proof (Exercise 2.2). If all real numbers were algebraic, then \mathbb{R} is countable by Exercise 2.2, contrary to the fact that \mathbb{R} is uncountable (Corollary to Theorem 2.43). □

Proof (Liouville, 1844).

- (1) **Lemma.** *If ξ is a real algebraic number of degree $n > 1$, then there is a constant $A > 0$ (depending on ξ) such that*

$$\left| \xi - \frac{h}{k} \right| \geq \frac{A}{k^n}$$

for all rational numbers $\frac{h}{k}$.

- (a) If $\left| \xi - \frac{h}{k} \right| \geq 1$, pick $A = 1 > 0$.
- (b) If $\left| \xi - \frac{h}{k} \right| < 1$, let $f(x) = a_0 + a_1 x + \dots + a_n x^n$ be an irreducible polynomial of degree $n > 1$ over \mathbb{Z} such that $f(\xi) = 0$. By the mean value theorem,

$$f(\xi) - f\left(\frac{h}{k}\right) = \left(\xi - \frac{h}{k}\right) f'(c)$$

for some $c \in \left(\xi - \frac{h}{k}, \xi + \frac{h}{k}\right) \subseteq (\xi - 1, \xi + 1)$. Notice that

- (i) $f(\xi) = 0$ by definition.
- (ii) $f\left(\frac{h}{k}\right) \neq 0$ since $\frac{h}{k}$ cannot be a root of $f(x)$. Otherwise f is of degree 1, contrary to the assumption of f .
- (iii) $\left|f\left(\frac{h}{k}\right)\right| \geq \frac{1}{k^n}$ since

$$\begin{aligned} f\left(\frac{h}{k}\right) &= a_0 + a_1\left(\frac{h}{k}\right) + \cdots + a_n\left(\frac{h}{k}\right)^n \neq 0, \\ k^n f\left(\frac{h}{k}\right) &= a_0 k^n + h k^{n-1} a_1 + \cdots + h^n a_n \neq 0, \\ k^n \left|f\left(\frac{h}{k}\right)\right| &\geq 1. \end{aligned}$$

- (iv) $|f'(c)| \leq \sup_{x \in [\xi-1, \xi+1]} |f'(x)|$ since $c \in [\xi-1, \xi+1]$ and $f'(x)$ is continuous or bounded on a compact set $[\xi-1, \xi+1]$.

By (i)-(iv),

$$\begin{aligned} \left|f(\xi) - f\left(\frac{h}{k}\right)\right| &= \left|\left(\xi - \frac{h}{k}\right) f'(c)\right|, \\ \frac{1}{k^n} &\leq \left|f\left(\frac{h}{k}\right)\right| = \left|\xi - \frac{h}{k}\right| |f'(c)| \leq \left|\xi - \frac{h}{k}\right| \cdot \sup_{x \in [\xi-1, \xi+1]} |f'(x)|. \end{aligned}$$

Pick $A = (1 + \sup_{x \in [\xi-1, \xi+1]} |f'(x)|)^{-1} > 0$.

By (a)(b), we arrange $A = \min(1, (1 + \sup_{x \in [\xi-1, \xi+1]} |f'(x)|)^{-1}) > 0$ to fit the inequality.

- (2) $\xi = \sum_{n=0}^{\infty} 10^{-n!}$ is transcendental.

- (a) Let $k_j = 10^{j!}$, $h_j = 10^{j!} \sum_{n=0}^j 10^{-n!}$. Then

$$\left|\xi - \frac{h_j}{k_j}\right| = \sum_{n=j+1}^{\infty} 10^{-n!} < \sum_{n=(j+1)!}^{\infty} 10^{-n} = \frac{A_j}{k_j^j}$$

where $A_j = \frac{10}{9} \cdot 10^{-j!}$.

- (b) If ξ were a real algebraic number of degree $d > 1$, then by Lemma and (a),

$$\frac{A}{k_j^d} < \left|\xi - \frac{h_j}{k_j}\right| < \frac{A_j}{k_j^j} < \frac{A_j}{k_j^d}$$

for some $A > 0$ and $j \geq d$, or $0 < A < A_j$. Since j is arbitrary, $A_j \rightarrow 0$ as $j \rightarrow \infty$, contrary to $A > 0$.

- (c) If ξ were a real algebraic number of degree $d = 1$, $\xi = \frac{h}{k}$ is a rational number. So

$$\left|\xi - \frac{h_j}{k_j}\right| = \left|\frac{h}{k} - \frac{h_j}{k_j}\right| = \left|\frac{hk_j - kh_j}{kk_j}\right| \geq \left|\frac{1}{kk_j}\right| = \frac{|k|^{-1}}{k_j}$$

for all j . (It is impossible that $hk_j - kh_j = 0$ or $\frac{h}{k} = \frac{h_j}{k_j}$ since $|\frac{h}{k} - \frac{h_j}{k_j}| = \sum_{n=j+1}^{\infty} 10^{-n!} > 0$ for all j .) Again by (a),

$$\frac{|k|^{-1}}{k_j} \leq \left| \xi - \frac{h_j}{k_j} \right| < \frac{A_j}{k_j^j} < \frac{A_j}{k_j},$$

or $0 < |k|^{-1} < A_j$. (Similar to (b).) Since j is arbitrary, $A_j \rightarrow 0$ as $j \rightarrow \infty$, contrary to $|k|^{-1} > 0$.

□

Exercise 2.4. *Is the set of all irrational real numbers countable?*

Proof (Reductio ad absurdum). If $\mathbb{R} - \mathbb{Q}$ were countable, then $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} - \mathbb{Q})$ is countable (Theorem 2.12), contrary to the fact that \mathbb{R} is uncountable (Corollary to Theorem 2.43). □

Exercise 2.5. *Construct a bounded set of real numbers with exactly three limit points.*

Proof (Exercise 2.12). Let

$$K_p = \{p\} \cup \left\{ p + \frac{1}{n} : n \in \mathbb{Z}^+ \right\} \subseteq \mathbb{R}^1$$

be a compact set of \mathbb{R}^1 with exactly one limit point $p \in \mathbb{R}^1$ (Exercise 2.12). Then

$$K_{1989} \cup K_6 \cup K_4$$

is a compact set of \mathbb{R}^1 with exactly three limit points $1989, 6, 4 \in \mathbb{R}^1$. □

Exercise 2.6. *Let E' be the set of all limit points of a set E . Prove that E' is closed. Prove that E and \overline{E} have the same limit points. (Recall that $\overline{E} = E \cup E'$.) Do E and E' always have the same limit points?*

Proof.

(1) *Show that E' is closed.*

(a) *Use Definition 2.18 (d).*

(i) It suffices to show every limit point of E' is a limit point of E . Given a limit point p of E' , so that every open neighborhood U of p contains a point $q_0 \neq p$ such that $q_0 \in E'$.

- (ii) Since q_0 is a limit point of E , there is an open neighborhood V of q_0 contains a point $q \neq q_0$ such that $q \in E$, where

$$V = U \cap B\left(q_0; \frac{1}{2}d_E(p, q_0)\right) \subseteq U$$

($B(x; r)$ is the open ball with center at x and radius r).

- (iii) By the construction of V , for such open neighborhood U of p , there is $q \neq p$ and $q \in V \subseteq U$ and $q \in E$. That is, p is a limit point of E .

(b) Use Definition 2.18 (e).

- (i) To show E' is closed or $X - E'$ is open, it suffices to show every point of $X - E'$ is an interior point of $X - E'$.
- (ii) Given a point $p \in X - E'$, or p is not a limit point of E . There is an open neighborhood U of p contains no point $q \neq p$ such that $q \in E$.
- (iii) To show U is an open neighborhood of p such that $U \subseteq X - E'$, it suffices to no point $q \neq p$ such that $q \in E'$. If there were a limit point q of E such that $q \neq p$ and $q \in U$, then

$$V = U \cap B\left(q; \frac{1}{2}d_E(p, q)\right) \subseteq U$$

is an open neighborhood of q contains no point of E , contrary to the assumption $q \in E'$. So $U \subseteq X - E'$ is an open neighborhood of $p \in X - E'$.

- (2) Show that $E' = \overline{E}'$. It suffices to show $E' \supseteq \overline{E}'$. ($E' \subseteq \overline{E}'$ holds trivially since $E \subseteq \overline{E}$). Given a limit point p of $\overline{E} = E \cup E'$.

- (a) p is a limit point of E . Nothing to do.
- (b) p is a limit point of E' . Since p is a limit point of E' and E' is a closed set, $p \in E'$, or p is a limit point of E .

In any case, $E' \supseteq \overline{E}'$.

- (3) E and E' might not have the same limit points. Let

$$E = \left\{ \frac{1}{n} : n \in \mathbb{Z}^+ \right\} \subseteq \mathbb{R}^1.$$

Then $E' = \{0\}$ and thus $(E')' = \emptyset$.

□

Exercise 2.7. Let A_1, A_2, A_3, \dots be subsets of a metric space.

(a) If $B_n = \bigcup_{i=1}^n A_i$, prove that $\overline{B_n} = \bigcup_{i=1}^n \overline{A_i}$, for $n = 1, 2, 3, \dots$

(b) If $B = \bigcup_{i=1}^{\infty} A_i$, prove that $\overline{B} \supseteq \bigcup_{i=1}^{\infty} \overline{A_i}$.

Show, by an example, that this inclusion can be proper.

Proof of (a).

(1) Show that $\overline{B_n} \subseteq \bigcup_{i=1}^n \overline{A_i}$. Since $A_i \subseteq \overline{A_i}$ for any i , we have

$$B_n = \bigcup_{i=1}^n A_i \subseteq \bigcup_{i=1}^n \overline{A_i}.$$

Since $\bigcup_{i=1}^n \overline{A_i}$ is a union of finitely many closed set $\overline{A_i}$, $\bigcup_{i=1}^n \overline{A_i}$ is closed (Theorem 2.24(d)). By Theorem 2.27(c), $\overline{B_n} \subseteq \bigcup_{i=1}^n \overline{A_i}$.

(2) Show that $\overline{B_n} \supseteq \bigcup_{i=1}^n \overline{A_i}$. Same argument in the proof of (b).

□

Proof of (b). Since $\bigcup_{j=1}^{\infty} A_j \supseteq A_i$ for any i , by the monotonicity of closure, we have $\overline{\bigcup_{j=1}^{\infty} A_j} \supseteq \overline{A_i}$ for any i , or $\overline{B} \supseteq \bigcup_{i=1}^{\infty} \overline{A_i}$. □

Proof of proper inclusion in (b). Let

$$A_n = \left(\frac{1}{n}, \infty \right) \subseteq \mathbb{R}^1$$

for any $n \in \mathbb{Z}^+$. Then

$$\begin{aligned} \bigcup_{n=1}^{\infty} A_n = (0, \infty) &\implies \overline{\bigcup_{n=1}^{\infty} A_n} = \overline{(0, \infty)} = [0, \infty), \\ \overline{A_n} = \left[\frac{1}{n}, \infty \right) &\implies \bigcup_{n=1}^{\infty} \overline{A_n} = \bigcup_{n=1}^{\infty} \left[\frac{1}{n}, \infty \right) = (0, \infty). \end{aligned}$$

□

Exercise 2.10. Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p, q) = \begin{cases} 1 & (\text{if } p \neq q) \\ 0 & (\text{if } p = q). \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

(The statement holds even if X is finite.) We called d the discrete metric, and the corresponding topology on X induces the discrete topology. Conversely, if X has the discrete topology, X is always metrizable by the discrete metric.

Proof.

- (1) $d(p, q)$ is a metric.
 - (a) $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$. Trivial.
 - (b) $d(p, q) = d(q, p)$. Trivial.
 - (c) $d(p, q) \leq d(p, r) + d(r, q)$ for any $r \in X$. If $p = q$, nothing to do. If $p \neq q$, $r \neq p$ or $r \neq q$ for any $r \in X$. (Assume not true, $r = p$ and $r = q$ implies that $p = q$ which is a contradiction.) In any case $d(p, r) + d(r, q) \geq 1 = d(p, q)$.
- (2) Every subset is open. Let E be any subset of X . Then every point $p \in E$ is an interior point of E . In fact, we can pick one open neighborhood $U = B(p; \frac{1}{2})$ of p containing only one point $p \in E$ or $U = \{p\}$, and such open neighborhood U is a subset of E . So every subset of X is open.
- (3) Every subset is closed. Since every subset is open, every subset is closed by Theorem 2.23.

Supplement. Might use Definition 2.18 (d) to prove directly since there are no limit points in X if we consider one open neighborhood $U = B(p; \frac{1}{2})$ of p . Therefore, every subset is closed. Again we apply Theorem 2.23 to get that every subset is open without using Definition 2.18 (f).

- (4) A subset is compact iff it is finite.
 - (a) Any finite subset is compact. Say $E = \{p_1, p_2, \dots, p_k\}$, and $\{G_\alpha\}$ be an open covering of E . From $\{G_\alpha\}$ we pick G_{α_1} containing p_1 , G_{α_2} containing p_2 , ..., and G_{α_k} containing p_k . This process can be done in the finitely many steps. Therefore,

$$\{G_{\alpha_1}, G_{\alpha_2}, \dots, G_{\alpha_k}\}$$

is a finite subcovering of $\{G_\alpha\}$ covering E .

- (b) Any infinite subset is not compact. Take a collection

$$\mathcal{G} = \{G_p = \{p\}\}$$

of open subsets where p runs all points in E . Clearly, $\{G_p\}$ is an open covering. Assume

$$\mathcal{G}' = \{G_{p_1}, G_{p_2}, \dots, G_{p_k}\}$$

is any finite subcovering of \mathcal{G} . Since E is infinite, there exist a point $p \in E$ such that $p \neq p_1, p \neq p_2, \dots, p \neq p_k$. Therefore, \mathcal{G}' does not cover p , or \mathcal{G} does not contains any finite subcovering \mathcal{G}' .

□

Notice that every subset is bounded. Therefore, every subset is closed and bounded, but only finite subset is compact, i.e., Heine-Borel theorem is not true in the infinite discrete topology.

Exercise 2.12. Let $K \subseteq \mathbb{R}^1$ consist of 0 and the numbers $\frac{1}{n}$, for $n = 1, 2, 3, \dots$. Prove that K is compact directly from the definition (without using the Heine-Borel theorem).

Proof. Let $\{G_\alpha\}$ be an open covering of K . There is an open set $G_0 \in \{G_\alpha\}$ containing 0. So there exists an open neighborhood $U = B(0; r)$ of 0 such that $U \subseteq G_0$. So U contains all points $q = \frac{1}{n}$ of K whenever $n > \frac{1}{r}$. To construct a finite subcovering of $\{G_\alpha\}$, we need to pick finitely many open sets from $\{G_\alpha\}$ to cover the remaining points $q = \frac{1}{n}$ where $n = 1, 2, \dots, [\frac{1}{r}]$, say G_1 contains $q = \frac{1}{1}$, G_2 contains $q = \frac{1}{2}$, ..., $G_{[\frac{1}{r}]}$ contains $q = \frac{1}{[\frac{1}{r}]}$. (Might be duplicated.) Hence,

$$\left\{ G_0, G_1, G_2, \dots, G_{[\frac{1}{r}]} \right\}$$

is a finite subcovering of $\{G_\alpha\}$ covering K . □

Proof (Heine-Borel theorem).

- (1) K is closed. In fact, the only limit point of K is 0, which is in K .
 - (a) $p = 0$ is a limit point. Given $r > 0$. There always exists $n \in \mathbb{Z}^+$ such that $r > \frac{1}{n}$. So any open neighborhood $B(0; r)$ of $p = 0$ contains at least one point $q = \frac{1}{n} \neq 0$ in K .
 - (b) $p < 0$ is not a limit point. Pick an open neighborhood $B(p; r)$ of p where $r = |p| > 0$. Then $B(p; r) \cap K = \emptyset$.
 - (c) $p > 0$ is not a limit point. There always exists $m \in \mathbb{Z}^+$ such that $p > \frac{1}{n}$ whenever $n \geq m$. Pick an open neighborhood $B(p; r)$ of p where $r = p - \frac{1}{m} > 0$. Then $B(p; r)$ does not have all points $q = \frac{1}{n} \in K$ whenever $n \geq m$. By Theorem 2.20, p cannot be a limit point of K .
- (2) K is bounded. There is a real number $M = 2$ and a point $q = 0 \in \mathbb{R}^1$ such that $|p - q| = |p| < 2$ for all $p \in K$.

By Heine-Borel theorem, K is compact in \mathbb{R}^1 . □

Exercise 2.14. Give an example of an open cover of the segment $(0, 1)$ which has no finite subcover.

Proof. In \mathbb{R}^1 , take a collection

$$\mathcal{G} = \left\{ G_n = \left(\frac{1}{n}, 1 \right) \right\}$$

of open subsets where $n \in \mathbb{Z}^+$.

- (1) \mathcal{G} is an open covering of $(0, 1) \subseteq \mathbb{R}^1$. Actually, given $x \in (0, 1)$, there exists a positive integer n such that $x > \frac{1}{n}$. That is, $x \in (\frac{1}{n}, 1) = G_n$.
- (2) There is no finite subcovering of \mathcal{G} . Assume

$$\mathcal{G}' = \{G_{n_1}, G_{n_2}, \dots, G_{n_k}\}$$

is any finite subcovering of \mathcal{G} where $n_1 < n_2 < \dots < n_k$. Take $x \in \left(0, \frac{1}{n_k}\right) \neq \emptyset$, $x = \frac{1}{2n_k}$ for example. Then $x \notin G_{n_1}$, $x \notin G_{n_2}$, ..., $x \notin G_{n_k}$, which contradicts that \mathcal{G}' is a finite subcovering of \mathcal{G} covering $(0, 1)$.

□