

Solutions to the book: *Fulton, Algebraic Curves*

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July 6, 2021

Contents

Chapter 1: Affine Algebraic Sets	8
1.1. Algebraic Preliminaries	8
Problem 1.1.*	8
Problem 1.2.*	9
Problem 1.3.*	10
Problem 1.4.*	11
Problem 1.5.*	12
Problem 1.6.*	12
Problem 1.7.*	13
1.2. Affine Space and Algebraic Sets	15
Problem 1.8.*	15
Problem 1.9.	16
Problem 1.10.	16
Problem 1.11.	16
Problem 1.12.	17
Problem 1.13.	18
Problem 1.14.*	20
Problem 1.15.*	22
1.3. The Ideal of a Set of Points	22
Problem 1.16.*	22
Problem 1.17.*	23
Problem 1.18.*	24
Problem 1.19.	25
Problem 1.20.*	26
Problem 1.21.*	26
1.4. The Hilbert Basis Theorem	27
Problem 1.22.* (Correspondence theorem for rings)	27
1.5. Irreducible Components of an Algebraic Set	30
Problem 1.23.	30

Problem 1.24.	31
Problem 1.25.	31
Problem 1.26.	32
Problem 1.27.	33
Problem 1.28.	34
Problem 1.29.*	34
1.6. Algebraic Subsets of the Plane	35
Problem 1.30.	35
Problem 1.31.	35
1.7. Hilbert's Nullstellensatz	37
Problem 1.32.	37
Problem 1.33.	38
Problem 1.34.	40
Problem 1.35.	40
Problem 1.36.	41
Problem 1.37.*	42
Problem 1.38.*	43
Problem 1.39.	43
Problem 1.40.	44
1.8. Modules; Finiteness Conditions	46
Problem 1.41.*	46
Problem 1.42.	46
Problem 1.43.*	47
Problem 1.44.*	47
Problem 1.45.*	48
1.9. Integral Elements	49
Problem 1.46.* (Transitivity of integral extensions)	49
Problem 1.47.*	50
Problem 1.48.*	51
Problem 1.49.*	51
Problem 1.50.*	52
1.10. Field Extensions	53
Problem 1.51.*	53
Problem 1.52.* (Splitting fields)	55
Problem 1.53.*	55
Problem 1.54.*	56
Chapter 2: Affine Varieties	58
2.1. Coordinate Rings	58
Problem 2.1.*	58
Problem 2.2.*	58
Problem 2.3.*	59
Problem 2.4.*	60
Problem 2.5.	60
2.2. Polynomial Maps	61
Problem 2.6.*	61

Problem 2.7.*	62
Problem 2.8.	63
Problem 2.9.*	64
Problem 2.10.*	65
Problem 2.11.	65
Problem 2.12.	67
Problem 2.13.	68
2.3. Coordinate Changes	69
Problem 2.14.* (Linear subvariety)	69
Problem 2.15.* (Line)	72
Problem 2.16.	75
2.4. Rational Functions and Local Rings	77
Problem 2.17.	77
Problem 2.18.	77
Problem 2.19.	78
Problem 2.20. (Quadric surface)	79
Problem 2.21.*	80
Problem 2.22.*	80
2.5. Discrete Valuation Rings	81
Problem 2.23.*	81
Problem 2.24.*	81
Problem 2.25. (p -adic integers)	83
Problem 2.26.*	84
Problem 2.27.	85
Problem 2.28.*	86
Problem 2.29.*	88
Problem 2.30.*	89
Problem 2.31. (Formal power series)	90
Problem 2.32. (Power series expansion)	92
2.6. Forms	95
Problem 2.33.	95
Problem 2.34.	96
Problem 2.35.*	97
Problem 2.36.	98
2.7. Direct Products of Rings	99
Problem 2.37.	99
Problem 2.38.*	99
2.8. Operations with Ideals	99
Problem 2.39.*	99
Problem 2.40.* (Chinese remainder theorem)	101
Problem 2.41.*	103
Problem 2.42.* (Isomorphism theorems for rings)	105
Problem 2.43.*	106
Problem 2.44.*	106
Problem 2.45.*	107
Problem 2.46.*	107

2.9. Ideals with a Finite Number of Zeros	108
Problem 2.47.	108
2.10. Quotient Modules and Exact Sequences	109
Problem 2.48.*	109
Problem 2.49.*	109
Problem 2.50.*	112
Problem 2.51.	113
Problem 2.52.* (Isomorphism theorems for modules)	114
Problem 2.53.*	115
2.11. Free Modules	116
Problem 2.54.	116
Problem 2.55.	117
Problem 2.56.	117
Chapter 3: Local Properties of Plane Curves	119
3.1. Multiple Points and Tangent Lines	119
Problem 3.1.	119
Problem 3.2.	120
Problem 3.3.	123
Problem 3.4.	123
Problem 3.5.	124
Problem 3.6.	125
Problem 3.7.	125
Problem 3.8.	129
Problem 3.9.	131
Problem 3.10.	132
Problem 3.11. (Tangent space)	133
3.2. Multiplicities and Local Rings	134
Problem 3.12. (Flex)	134
Problem 3.13.*	135
Problem 3.14.	136
Problem 3.15.	136
Problem 3.16.	138
3.3. Intersection Numbers	139
Problem 3.17.	139
Problem 3.18.	142
Problem 3.19.*	143
Problem 3.20.	144
Problem 3.21.	144
Problem 3.22. (Cusp)	145
Problem 3.23. (Hypercusp)	147
Problem 3.24.*	147

Chapter 4: Projective Varieties	150
4.1. Projective Space	150
Problem 4.1.	150
Problem 4.2.*	150
Problem 4.3.	151
4.2. Projective Algebraic Sets	151
Problem 4.4.*	151
Problem 4.5.	152
Problem 4.6.	152
Problem 4.7.	153
Problem 4.8.	153
Problem 4.9.*	154
Problem 4.10.	155
Problem 4.11.* (Linear subvariety)	156
Problem 4.12.*	158
Problem 4.13.* (Line)	158
Problem 4.14.*	161
Problem 4.15.*	162
Problem 4.16.*	163
Problem 4.17.*	164
Problem 4.18. (Duality)	165
4.3. Affine and Projective Varieties	166
Problem 4.19.*	166
Problem 4.20.	166
Problem 4.21.	167
Problem 4.22.*	167
Problem 4.23.*	168
Problem 4.24.*	168
Problem 4.25.*	169
4.4. Multiprojective Space	170
Problem 4.26.*	170
Problem 4.27.*	172
Problem 4.28.* (Segre embedding)	173
Chapter 5: Projective Plane Curves	176
5.1. Definitions	176
Problem 5.1.*	176
Problem 5.2.	176
Problem 5.3.	179
Problem 5.4.*	184
Problem 5.5.*	185
Problem 5.6.*	185
Problem 5.7.*	186
Problem 5.8.*	187
Problem 5.9. (Conic)	188
Problem 5.10. (Cubic with a cusp)	189

Problem 5.11. (Cubic with a node)	191
Problem 5.12.*	192
Problem 5.13.	193
Problem 5.14.*	194
Problem 5.15.*	195
Problem 5.16.*	195
5.2. Linear Systems of Curves	196
Problem 5.17.	196
Problem 5.18.	197
Problem 5.19.	197
5.3. Bézout's Theorem	198
Problem 5.20.	198
Problem 5.21.*	200
Problem 5.22.*	200
Problem 5.23. (Hessian matrix)	201
Problem 5.24.	207
5.4. Multiple Points	209
Problem 5.25.	209
Problem 5.26.*	210
Problem 5.27.	211
Problem 5.28. (Terrible point)	211
5.5. Max Noether's Fundamental Theorem	212
Problem 5.29.	212
5.6. Applications of Noether's Theorem	213
Problem 5.33.	213
Problem 5.34.	214
Problem 5.35.	215
Problem 5.36.	215
Problem 5.37.	216
Problem 5.38.	219
Problem 5.39.	220
Problem 5.42.	221
Chapter 6: Varieties, Morphisms, and Rational Maps	223
6.1. The Zariski Topology	223
Problem 6.1.*	223
Problem 6.2.*	223
Problem 6.3.*	225
Problem 6.4.*	225
Problem 6.5.	226
Problem 6.6.*	227
Problem 6.7.*	228
Problem 6.8.*	229
6.2. Varieties	229
Problem 6.9.	229
Problem 6.10.*	230

Problem 6.11. (Noetherian topological space)	230
Problem 6.12.*	231
6.3. Morphisms of Varieties	232
Problem 6.13.*	232
Problem 6.14.*	233
Problem 6.15.*	234
Problem 6.17.	235
Problem 6.18.	236
Problem 6.21.	237
6.4. Products and Graphs	237
Problem 6.29. (Algebraic group)	237
6.5. Algebraic Function Fields and Dimension of Varieties	239
Problem 6.31.* (Theorem of the primitive element)	239
Problem 6.32.*	240
Problem 6.33. (Transcendence degree)	240
Problem 6.34.	243
6.6. Rational Maps	243
Chapter 7: Resolution of Singularities	244
7.1. Rational Maps of Curves	244
7.2. Blowing up a Point in \mathbf{A}^2	244
7.3. Blowing up a Point in \mathbf{P}^2	244
7.4. Quadratic Transformations	244
7.5. Nonsingular Models of Curves	244
Chapter 8: Riemann-Roch Theorem	245
8.1. Divisors	245
8.2. The Vector Spaces $L(D)$	245
8.3. Riemann's Theorem	245
8.4. Derivations and Differentials	245
8.5. Canonical Divisors	245
8.6. Riemann-Roch Theorem	245

Chapter 1: Affine Algebraic Sets

1.1. Algebraic Preliminaries

Problem 1.1.*

Let R be a domain.

- (a) If f, g are forms of degree r, s respectively in $R[x_1, \dots, x_n]$, show that fg is a form of degree $r + s$.
- (b) Show that any factor of a form in $R[x_1, \dots, x_n]$ is also a form.

Proof of (a).

- (1) Write

$$f = \sum_{(i)} a_{(i)} x^{(i)},$$
$$g = \sum_{(j)} b_{(j)} x^{(j)},$$

where $\sum_{(i)}$ is the summation over $(i) = (i_1, \dots, i_n)$ with $i_1 + \dots + i_n = r$ and $\sum_{(j)}$ is the summation over $(j) = (j_1, \dots, j_n)$ with $j_1 + \dots + j_n = s$.

- (2) Hence,

$$fg = \sum_{(i)} \sum_{(j)} a_{(i)} b_{(j)} x^{(i)} x^{(j)}$$
$$= \sum_{(i),(j)} a_{(i)} b_{(j)} x^{(k)}$$

where $(k) = (i_1 + j_1, \dots, i_n + j_n)$ with $(i_1 + j_1) + \dots + (i_n + j_n) = r + s$. Each $x^{(k)}$ is the form of degree $r + s$ and $a_{(i)} b_{(j)} \in R$. Hence fg is a form of degree $r + s$.

□

Proof of (b).

- (1) Given any form $f \in R[x_1, \dots, x_n]$, and write $f = gh$. It suffices to show that g is a form as well. (So does h .)
- (2) Write

$$g = g_0 + \dots + g_r, \quad h = h_0 + \dots + h_s$$

where $g_r \neq 0$ and $h_s \neq 0$. So

$$f = gh = g_0h_0 + \cdots + g_rh_s.$$

Since R is a domain, $R[x_1, \dots, x_n]$ is a domain and thus $g_rh_s \neq 0$. The maximality of r and s implies that $\deg f = r + s$. Therefore, by the maximality of $r + s$, $f = g_rh_s$, or $g = g_r$, or g is a form.

□

Problem 1.2.*

Let R be a UFD, K the quotient field of R . Show that every element z of K may be written $z = a/b$, where $a, b \in R$ have no common factors; this representative is unique up to units of R .

Proof.

- (1) Show that every element z of K may be written $z = a/b$, where $a, b \in R$ have no common factors. Given any $z = a/b \in K$ where $a, b \in R$. Write

$$\begin{aligned} a &= p_1 \cdots p_n, \\ b &= q_1 \cdots q_m \end{aligned}$$

where all $p_1, \dots, p_n, q_1, \dots, q_m$ are irreducible in R . (It is possible since R is a UFD.) For each i , suppose $p_i \mid q_j$ for some i, j . Write $q_j = p_i u$ for some $u \in R$. By the irreducibility of p_i and q_j , u is a unit. So

$$z = \frac{a}{b} = \frac{p_1 \cdots \widehat{p_i} \cdots p_n}{q_1 \cdots \widehat{q_j} \cdots q_m} = \frac{p_1 \cdots \widehat{p_i} \cdots p_n}{u q_1 \cdots \widehat{q_j} \cdots q_m}.$$

Continue this method we can write $z = \frac{a'}{b'}$ where a' and b' have no common factors.

- (2) Write $z = a/b = a'/b'$ where

- (a) $a, b, a', b' \in R$,
- (b) a and b have no common factors,
- (c) a' and b' have no common factors.

Write

$$\begin{aligned} a &= p_1 \cdots p_n, \\ b &= q_1 \cdots q_m, \\ a' &= p'_1 \cdots p'_{n'}, \\ b' &= q'_1 \cdots q'_{m'} \end{aligned}$$

where all $p_i, q_j, p'_{i'}, q'_{j'}$ are irreducible in R . As $z = a/b = a'/b'$, $ab' = a'b$ or

$$p_1 \cdots p_n q'_1 \cdots q'_{m'} = p'_1 \cdots p'_{n'} q_1 \cdots q_m.$$

- (3) For $i = 1$, $p_1 = u_1 p'_{i'}$ for some unit $u_1 \in R$ since a and b have no common factors and all $p_1, q_j, p'_{i'}$ are irreducible. Hence

$$u_1 \widehat{p_1} p_2 \cdots p_n q'_1 \cdots q'_{m'} = p'_1 \cdots \widehat{p'_{i'}} \cdots p'_{n'} q_1 \cdots q_m.$$

Continue this method, we have $n \leq n'$ and all p_1, \dots, p_n are canceled.

- (4) Conversely, we can apply the argument in (3) to $i' = 1, \dots, n'$ to conclude that $n' \leq n$. Therefore, $n = n'$ and

$$\underbrace{u_1 \cdots u_n}_{\text{a unit in } R} q'_1 \cdots q'_{m'} = q_1 \cdots q_m.$$

Hence, $b = ub'$ where $u = u_1 \cdots u_n$ is a unit in R . Similarly, $a = va'$ where v is a unit in R . So the representative of $z \in K$ is unique up to units of R .

□

Problem 1.3.*

Let R be a PID. Let \mathfrak{p} be a nonzero, proper, prime ideal in R .

- (a) Show that \mathfrak{p} is generated by an irreducible element.
- (b) Show that \mathfrak{p} is maximal.

Proof of (a).

- (1) Let $\mathfrak{p} = (a)$ be a nonzero, proper, prime ideal in R . It suffices to show that a is irreducible.
- (2) Suppose $a = bc$. By the primality of \mathfrak{p} , $b \in \mathfrak{p}$ or $c \in \mathfrak{p}$. Suppose $b \in \mathfrak{p} = (a)$. (The case $c \in \mathfrak{p}$ is similar.) Then there is a $d \in R$ such that $b = ad$. Hence, $a = bc = adc$ or $(1 - dc)a = 0$.
- (3) Since R is a domain, $1 = dc$ or $a = 0$. $a = 0$ implies that $\mathfrak{p} = (0)$ is a zero ideal, contrary to the assumption. Therefore, $1 = dc$, or c is a unit, or a is irreducible.

□

Proof of (b).

- (1) Given any ideal $I = (b)$ of R containing $\mathfrak{p} = (a)$. As the generator a of \mathfrak{p} is in $\mathfrak{p} \subseteq I$, there is some $c \in R$ such that $a = bc$. By the irreducibility of a (in (a)), b is a unit or c is a unit.
- (2) b is a unit implies that $I = R$. c is a unit implies that $I = \mathfrak{p}$. In any case, we conclude that \mathfrak{p} is maximal.

□

Problem 1.4.*

Let k be an infinite field, $f \in k[x_1, \dots, x_n]$. Suppose $f(a_1, \dots, a_n) = 0$ for all $a_1, \dots, a_n \in k$. Show that $f = 0$. (Hint: Write

$$f = \sum f_i x_n^i, \quad f_i \in k[x_1, \dots, x_{n-1}].$$

Use induction on n , and the fact that $f(a_1, \dots, a_{n-1}, x_n)$ has only a finite number of roots if any $f_i(a_1, \dots, a_{n-1}) \neq 0$.)

Proof.

- (1) Induction on n . The case $n = 1$. (Reductio ad absurdum) If there were a nonzero $f \in k[x_1]$ such that $f(a) = 0$ for all $a \in k$. Note that f has at most $\deg f < \infty$ roots, contrary to the infinity of k .
- (2) Assume that the conclusion holds for $n - 1$, then for any $f \in k[x_1, \dots, x_n]$ we can write

$$f = \sum f_i x_n^i, \quad f_i \in k[x_1, \dots, x_{n-1}]$$

as $f \in (k[x_1, \dots, x_{n-1}])[x_n]$. Suppose $f(a_1, \dots, a_n) = 0$ for all $a_1, \dots, a_n \in k$. For fixed a_1, \dots, a_{n-1} , the polynomial $f(a_1, \dots, a_{n-1}, x_n) \in k[x_n]$ has all distinct roots in an infinite field k . By (1), $f(a_1, \dots, a_{n-1}, x_n) = 0 \in k[x_n]$, or each $f_i(a_1, \dots, a_{n-1}) = 0$. As all a_1, \dots, a_{n-1} run over k , we can apply the induction hypothesis each $f_i(x_1, \dots, x_{n-1}) = 0 \in k[x_1, \dots, x_{n-1}]$. Hence, $f = 0 \in k[x_1, \dots, x_n]$.

□

Note. If k is a finite field of order $q = p^k$, then the polynomial $f(x) = x^q - x$ has q distinct roots in k .

Problem 1.5.*

Let k be any field. Show that there are an infinitely number of irreducible monic polynomials in $k[x]$. (Hint: Suppose f_1, \dots, f_n were all of them, and factor $f_1 \cdots f_n + 1$ into irreducible factors.)

Proof (Due to Euclid).

- (1) If f_1, \dots, f_n were all irreducible monic polynomials, then we consider

$$g = f_1 \cdots f_n + 1 \in k[x].$$

So there is an irreducible monic polynomial $f = f_i$ dividing g for some i since

$$\deg g = \deg f_1 + \cdots + \deg f_n \geq 1$$

and $k[x]$ is a UFD.

- (2) However, f would divide the difference

$$g - f_1 \cdots f_{i-1} f_{i+1} \cdots f_n = 1,$$

contrary to $\deg f_i \geq 1$.

□

Problem 1.6.*

Show that any algebraically closed field is infinite. (Hint: The irreducible monic polynomials are $x - a$, $a \in k$.)

Proof (Due to Euclid).

- (1) Let k be an algebraically closed field. If a_1, \dots, a_n were all elements in k , then we consider a monic polynomials

$$f(x) = (x - a_1) \cdots (x - a_n) + 1 \in k[x].$$

- (2) Since k is algebraically closed, there is an element $a \in k$ such that $f(a) = 0$. By assumption, $a = a_i$ for some $1 \leq i \leq n$, and thus $f(a) = f(a_i) = 1$, contrary to the fact that a field is a commutative ring where $0 \neq 1$ and all nonzero elements are invertible.

□

Problem 1.7.*

Let k be a field, $f \in k[x_1, \dots, x_n]$, $a_1, \dots, a_n \in k$.

(a) Show that

$$f = \sum \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}, \quad \lambda_{(i)} \in k.$$

(b) If $f(a_1, \dots, a_n) = 0$, show that $f = \sum_{i=1}^n (x_i - a_i)g_i$ for some (not unique) g_i in $k[x_1, \dots, x_n]$.

Proof of (a).

(1) Regard $k[x_1, \dots, x_n]$ as $(k[x_1, \dots, x_{n-1}])[x_n]$. Since $(k[x_1, \dots, x_{n-1}])[x_n]$ is a Euclidean domain with a function

$$f \in (k[x_1, \dots, x_{n-1}])[x_n] \mapsto \deg_{x_n} f \in \mathbb{Z}_{\geq 0}$$

satisfying the division-with-remainder property.

(2) Apply the division algorithm for f and nonzero $x_n - a_n$ to produce a quotient q and remainder r with $f = (x_n - a_n)q + r$ and either $r = 0$ or $\deg_{x_n}(r) < \deg_{x_n}(x_n - a_n) = 1$. That is, $r \in k[x_1, \dots, x_{n-1}]$ is a constant in $(k[x_1, \dots, x_{n-1}])[x_n]$. Continue this process to get that f is of the form

$$f = \sum_{i_n} f_{i_n} (x_n - a_n)^{i_n}$$

where $f_{i_n} \in k[x_1, \dots, x_{n-1}]$.

(3) Use the same argument in (2) for each $f_{i_n} \in k[x_1, \dots, x_{n-1}]$, we have

$$\begin{aligned} f_{i_n} &= \sum_{i_{n-1} \in k[x_1, \dots, x_{n-2}]} \underbrace{f_{i_n, i_{n-1}}}_{\in k[x_1, \dots, x_{n-2}]} (x_{n-1} - a_{n-1})^{i_{n-1}} \\ f_{i_n, i_{n-1}} &= \sum_{i_{n-2} \in k[x_1, \dots, x_{n-3}]} \underbrace{f_{i_n, i_{n-1}, i_{n-2}}}_{\in k[x_1, \dots, x_{n-3}]} (x_{n-2} - a_{n-2})^{i_{n-2}}, \\ &\dots \\ f_{i_n, \dots, i_2} &= \sum_{i_1 \in k} \underbrace{f_{i_n, \dots, i_1}}_{\in k} (x_1 - a_1)^{i_1}. \end{aligned}$$

Note that $f_{i_n, \dots, i_1} \in k$, we can write

$$f = \sum \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}, \quad \lambda_{(i)} \in k.$$

by replacing all f_{i_n, \dots, i_k} by $f_{i_n, \dots, i_{k-1}}$ for $k = n, n-1, \dots, 2$.

(4) Or use the induction on n .

□

Proof of (b).

(1) Write

$$f = \sum \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}, \quad \lambda_{(i)} \in k$$

by (a).

(2) As $f(a_1, \dots, a_n) = 0$, $\lambda_{(i)} = 0$ if all i_1, \dots, i_n are zero, that is, there is no nonzero constant term in the representation of f . Hence, for each term

$$f_{(i)} := \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}$$

with $\lambda_{(i)} \neq 0$, there exists one $i_k > 0$ for some $1 \leq k \leq n$. So we can write

$$f_{(i)} = (x_k - a_k) \underbrace{(\lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_k - a_k)^{i_k-1} \cdots (x_n - a_n)^{i_n})}_{:= g_{(i)} \in k[x_1, \dots, x_n]}.$$

Note that the expression of $f_{(i)}$ is not unique since there may exist more than one $i_k > 0$ as $1 \leq k \leq n$.

(3) Now we iterate each nonzero term in f , apply the factorization in (2), and then group by each $x_k - a_k$. Therefore, we can write

$$f = \sum_{i=1}^n (x_i - a_i) g_i$$

for some $g_1 \in k[x_1, \dots, x_n]$.

(4) The expression of f is not unique. For example, take $f(x, y) = x^2 + 2xy + y^2 \in k[x, y]$. As $f(0, 0) = 0$, we can write

$$\begin{aligned} f(x, y) &= x \cdot \underbrace{(x + 2y)}_{g_1} + y \cdot \underbrace{y}_{g_2}, \text{ or} \\ &= x \cdot \underbrace{(x + y)}_{g_1} + y \cdot \underbrace{(x + y)}_{g_2}, \text{ or} \\ &= x \cdot \underbrace{x}_{g_1} + y \cdot \underbrace{(2x + y)}_{g_2}. \end{aligned}$$

□

1.2. Affine Space and Algebraic Sets

Problem 1.8.*

Show that the algebraic subsets of $\mathbf{A}^1(k)$ are just the finite subsets, together with $\mathbf{A}^1(k)$ itself.

Proof.

(1) Show that $k[x]$ is a PID if k is a field.

- (a) Let I be an ideal of $k[x]$.
- (b) If $I = \{0\}$ then $I = (0)$ and I is principal.
- (c) If $I \neq \{0\}$, then take f to be a polynomial of minimal degree in I . It suffices to show that $I = (f)$. Clearly, $(f) \subseteq I$ since I is an ideal. Conversely, for any $g \in I$,

$$g(x) = f(x)h(x) + r(x)$$

for some $h, r \in k[x]$ with $r = 0$ or $\deg r < \deg f$ (as $k[x]$ is a Euclidean domain). Now as

$$r = g - fh \in I,$$

$r = 0$ (otherwise contrary to the minimality of f), we have $g = fh \in (f)$ for all $g \in I$.

(2) Let Y be an algebraic subset of $\mathbf{A}^1(k)$, say $Y = V(I)$ for some ideal I of $k[x]$. Since $k[x]$ is a PID, $I = (f)$ for some $f \in k[x]$.

- (a) If $f = 0$, then $I = (0)$ and $Y = V(0) = \mathbf{A}^1(k)$.
- (b) If $f \neq 0$, then $f(x) = 0$ has finitely many roots in k , say $a_1, \dots, a_m \in k$. Hence,

$$Y = V(I) = V(f) = \{f(a) = 0 : a \in k\} = \{a_1, \dots, a_m\}$$

is a finite subsets of $\mathbf{A}^1(k)$.

By (a)(b), the result is established.

□

Notes.

- (1) By the Hilbert basis theorem, $k[x]$ is Noetherian as k is Noetherian. Hence, for any algebraic subset $Y = V(I)$ of $\mathbf{A}^1(k)$, we can write $I = (f_1, \dots, f_m)$. Note that

$$Y = V(I) = V(f_1) \cap \dots \cap V(f_m).$$

Now apply the same argument to get the same conclusion.

- (2) Suppose $k = \bar{k}$. $\mathbf{A}^1(k)$ is irreducible, because its only proper closed subsets are finite, yet it is infinite (because k is algebraically closed, hence infinite).

Problem 1.9.

If k is a finite field, show that every subset of $\mathbf{A}^n(k)$ is algebraic.

Proof.

- (1) Every subset of $\mathbf{A}^n(k)$ is finite since $|\mathbf{A}^n(k)| = |k|^n$ is finite.
- (2) Note that $V(x_1 - a_1, \dots, x_n - a_n) = \{(a_1, \dots, a_n)\} \subseteq \mathbf{A}^n(k)$ (Property (5) in §1.2) and any finite union of algebraic sets is algebraic (Property (4) in §1.2). Thus, every subset of $\mathbf{A}^n(k)$ is algebraic (by (1)).

□

Problem 1.10.

Give an example of a countable collection of algebraic sets whose union is not algebraic.

Proof.

- (1) Let $k = \mathbb{Q}$ be an infinite field. $V(x - a) = \{a\}$ is an algebraic sets for all $a \in \mathbb{Q}$. In particular, $V(x - a) = \{a\}$ is algebraic for all $a \in \mathbb{Z}$.
- (2) Note that

$$Y := \bigcup_{a \in \mathbb{Z}} V(x - a) = \mathbb{Z}$$

is a countable union of algebraic sets. Since Y is a proper subset of $k = \mathbb{Q}$, it cannot be algebraic by Problem 1.8.

□

Problem 1.11.

Show that the following are algebraic sets:

- (a) $\{(t, t^2, t^3) \in \mathbf{A}^3(k) : t \in k\}$;
- (b) $\{(\cos(t), \sin(t)) \in \mathbf{A}^2(\mathbb{R}) : t \in \mathbb{R}\}$;
- (c) the set of points in $\mathbf{A}^2(\mathbb{R})$ whose polar coordinates (r, θ) satisfy the equation $r = \sin(\theta)$.

Proof of (a).

- (1) The twisted cubic curve

$$Y = \{(t, t^2, t^3) \in \mathbf{A}^3(k) : t \in k\} = V(x^2 - y) \cap V(x^3 - z)$$

is algebraic. We say that Y is given by the parametric representation $x = t, y = t^2, z = t^3$.

- (2) The generators for the ideal $I(Y)$ are $x^2 - y$ and $x^3 - z$.
 (3) Y is an affine variety of dimension 1.
 (4) The affine coordinate ring $A(Y)$ is isomorphic to a polynomial ring in one variable over k .

□

Proof of (b). The circle

$$\{(\cos(t), \sin(t)) \in \mathbf{A}^2(\mathbb{R}) : t \in \mathbb{R}\} = V(x^2 + y^2 - 1)$$

is algebraic. □

Proof of (c). The circle

$$\{(r, \theta) : r = \sin(\theta)\} = V(x^2 + y^2 - y)$$

is algebraic again. □

Problem 1.12.

Suppose C is an affine plane curve, and L is a line in $\mathbf{A}^2(k)$, $L \not\subseteq C$. Suppose $C = V(f)$, $f \in k[x, y]$ a polynomial of degree n . Show that $L \cap C$ is a finite set of no more than n points. (Hint: Suppose $L = V(y - (ax + b))$, and consider $f(x, ax + b) \in k[x]$.)

Proof.

- (1) Say $L = V(y - (ax + b))$ be a line in $\mathbf{A}^2(k)$. (The case $L = V(x - (ay + b))$ is similar.)
 (2) Note that $L \not\subseteq C$ implies that $(y - (ax + b)) \nmid f$. Hence, the polynomial

$$g : x \mapsto f(x, ax + b) \in k[x]$$

is nonzero and $\deg g \leq n$. Therefore, the number of roots of g in k is no more than n .

(3) Hence,

$$\begin{aligned}
L \cap C &= V(y - (ax + b)) \cap V(f) \\
&= \{(x, y) \in \mathbf{A}^2(k) : y = ax + b \text{ and } f(x, y) = 0\} \\
&= \{(x, y) \in \mathbf{A}^2(k) : f(x, ax + b) = 0\}
\end{aligned}$$

is finite of no more than n points.

□

Problem 1.13.

Show that each of the following sets is not algebraic:

- (a) $\{(x, y) \in \mathbf{A}^2(\mathbb{R}) : y = \sin(x)\}$.
- (b) $\{(z, w) \in \mathbf{A}^2(\mathbb{C}) : |z|^2 + |w|^2 = 1\}$, where $|x + iy|^2 = x^2 + y^2$ for $x, y \in \mathbb{R}$.
- (c) $\{(\cos(t), \sin(t), t) \in \mathbf{A}^3(\mathbb{R}) : t \in \mathbb{R}\}$.

Proof of (a).

- (1) (Reductio ad absurdum) If

$$Y := \{(x, y) \in \mathbf{A}^2(\mathbb{R}) : y = \sin(x)\}$$

were algebraic, then there is a subset S of $\mathbb{R}[x, y]$ such that

$$Y = V(S) = \bigcap_{f \in S} V(f).$$

- (2) $S \neq \emptyset$ since $Y \neq \mathbf{A}^2(\mathbb{R})$. ($(89, 64) \in \mathbf{A}^2(\mathbb{R}) - Y$.)
- (3) Take a fixed line $L = V(y)$ in $\mathbf{A}^2(\mathbb{R})$. For each affine curve $f \in S$, we have

$$V(f) \cap L \supseteq \bigcap_{f \in S} V(f) \cap L = Y \cap L = \{(n\pi, 0) \in \mathbf{A}^2(\mathbb{R}) : n \in \mathbb{Z}\},$$

which is infinite. By problem 1.12, $y \mid f$. As f runs over S , $Y \subseteq V(y) = L$, contradicts that $(0, \frac{\pi}{2}) \in L - Y$.

□

Proof of (b).

- (1) Similar to (a). (Reductio ad absurdum) If

$$Y := \{(x, y) \in \mathbf{A}^2(\mathbb{C}) : |x|^2 + |y|^2 = 1\}$$

were algebraic, then there is a subset S of $\mathbb{C}[x, y]$ such that

$$Y = V(S) = \bigcap_{f \in S} V(f).$$

- (2) $S \neq \emptyset$ since $Y \neq \mathbf{A}^2(\mathbb{C})$. $((89, 64) \in \mathbf{A}^2(\mathbb{C}) - Y)$
(3) Take a fixed line $L = V(x)$ in $\mathbf{A}^2(\mathbb{C})$. For each affine curve $f \in S$, we have

$$V(f) \cap L \supseteq \bigcap_{f \in S} V(f) \cap L = Y \cap L = \{(0, y) \in \mathbf{A}^2(\mathbb{C}) : |y| = 1\},$$

which is infinite (since Y contains a unit circle in the complex plane). By problem 1.12, $x \mid f$. As f runs over S , $Y \subseteq V(x) = L$, contradicts that the origin $(0, 0) \in L - Y$.

□

Proof of (c).

- (1) Similar to (a) and (b).
(2) Suppose C is an affine plane curve, and L is a line in $\mathbf{A}^3(k)$, $L \not\subseteq C$. Suppose $C = V(f)$, $f \in k[x, y, z]$ a polynomial of degree n . Show that $L \cap C$ is a finite set of no more than n points. The proof is similar to Problem 1.12.
(a) Say $L = V(y - (ax + b), z - (cx + d))$ be a line in $\mathbf{A}^3(k)$.
(b) Note that $L \not\subseteq C$ implies that $(y - (ax + b)) \nmid f$ and $(z - (cx + d)) \nmid f$. Hence, the polynomial

$$g : x \mapsto f(x, ax + b, cx + d) \in k[x]$$

is nonzero and $\deg g \leq n$. Therefore, the number of roots of g in k is no more than n .

- (c) Hence,

$$\begin{aligned} L \cap C &= V(y - (ax + b), z - (cx + d)) \cap V(f) \\ &= \{(x, y) \in \mathbf{A}^2(k) : y = ax + b, z = cx + d \text{ and } f(x, y) = 0\} \\ &= \{(x, y) \in \mathbf{A}^2(k) : f(x, ax + b, cx + d) = 0\} \end{aligned}$$

is finite of no more than n points.

(3) (Reductio ad absurdum) If

$$Y := \{(\cos(t), \sin(t), t) \in \mathbf{A}^3(\mathbb{R}) : t \in \mathbb{R}\}$$

were algebraic, then there is a subset S of $\mathbb{R}[x, y, z]$ such that

$$Y = V(S) = \bigcap_{f \in S} V(f).$$

(4) $S \neq \emptyset$ since $Y \neq \mathbf{A}^3(\mathbb{R})$. ((1989, 6, 4) $\in \mathbf{A}^3(\mathbb{R}) - Y$.)

(5) Take a fixed line $L = V(x - 1, y)$ in $\mathbf{A}^3(\mathbb{R})$. For each affine curve $f \in S$, we have

$$V(f) \cap L \supseteq \bigcap_{f \in S} V(f) \cap L = Y \cap L = \{(1, 0, 2n\pi) \in \mathbf{A}^3(\mathbb{R}) : n \in \mathbb{Z}\},$$

which is infinite. By (2), $(x - 1) \mid f$ and $y \mid f$. As f runs over S , $Y \subseteq V(x - 1, y) = L$, contradicts that $(1, 0, \pi) \in L - Y$.

□

Supplement. A circular disk of radius 1 in the plane xy rolls without slipping along the x axis. The figure described by a point of the circumference of the disk is called a **cycloid**. The parametrized curve $\alpha : \mathbb{R} \rightarrow \mathbb{R}^2$ is

$$\begin{cases} x = t - \sin t \\ y = 1 - \cos t. \end{cases}$$

The cycloid is not algebraic (as (a)).

Problem 1.14.*

Let f be a nonconstant polynomial in $k[x_1, \dots, x_n]$, k algebraically closed. Show that $\mathbf{A}^n(k) - V(f)$ is infinite if $n \geq 1$, and $V(f)$ is infinite if $n \geq 2$. Conclude that the complement of any proper algebraic set is infinite. (Hint: See Problem 1.4.)

Proof.

(1) Show that $\mathbf{A}^n(k) - V(f)$ is infinite if $n \geq 1$. Since f is a nonconstant polynomial in $k[x_1, \dots, x_n]$, we may assume that $\deg_{x_n}(f) > 0$. Hence

$$x_n \mapsto f(1, \dots, 1, x_n)$$

is a nonconstant polynomial of degree $\deg_{x_n}(f) > 0$ in $k[x_n]$. So f has finitely many roots in k , say ξ_1, \dots, ξ_m ($m \geq 0$). Hence,

$$(1, \dots, 1, x_n) \neq 0$$

whenever $x_n \neq \xi_m$. Such subset in $\mathbf{A}^1(k)$ is infinite since $k = \bar{k}$ (Problem 1.6). Therefore,

$$\begin{aligned}\mathbf{A}^n(k) - V(f) &= \{(a_1, \dots, a_n) \in \mathbf{A}^n(k) : f(a_1, \dots, a_n) \neq 0\} \\ &\supseteq \{a_n \in \mathbf{A}^1(k) : f(1, \dots, 1, x_n) \neq 0\}\end{aligned}$$

is infinite.

(2) Show that $V(f)$ is infinite if $n \geq 2$.

(a) Similar to (1). Since f is a nonconstant polynomial in $k[x_1, \dots, x_n]$, we may assume that $m := \deg_{x_n}(f) > 0$. Write

$$f = \sum_{i=0}^m f_i(x_1, \dots, x_{n-1})x_n^i.$$

Note that each f_i is well-defined since $n \geq 2$.

(b) If f_n is constant in $k[x_1, \dots, x_{n-1}]$, then f_n is nonzero (since $m > 0$) or $V(f_n) = \emptyset$. If f_n is nonconstant in $k[x_1, \dots, x_{n-1}]$, then the set $\mathbf{A}^{n-1}(k) - V(f_n)$ is infinite by (1). In any case,

$$\mathbf{A}^{n-1}(k) - V(f_n)$$

is infinite.

(c) For each $P = (a_1, \dots, a_{n-1}) \in \mathbf{A}^{n-1}(k) - V(f_n)$,

$$g_P : x_n \mapsto f(P, x_n) = f(a_1, \dots, a_{n-1}, x_n)$$

defines a polynomial in $k[x_n]$ of degree $m > 0$. Since $k = \bar{k}$, g_P has at least one root $Q \in k$. Hence

$$V(f) \supseteq \{(P, Q) \in \mathbf{A}^n(k) : P \in \mathbf{A}^{n-1}(k) - V(f_n), g_P(Q) = 0\}$$

is infinite since the set $\mathbf{A}^{n-1}(k) - V(f_n)$ is infinite.

Note. It is not true if $k \neq \bar{k}$. For example, $V(x^2 + y^2 + 1) = \emptyset$ in $\mathbf{A}^2(\mathbb{R})$.

(3) Note that

$$\mathbf{A}^n(k) - V(S) = \mathbf{A}^n(k) - \bigcap_{f \in S} V(f) = \bigcup_{f \in S} (\mathbf{A}^n(k) - V(f)).$$

Thus the complement of any proper algebraic set is infinite by (1).

□

Problem 1.15.*

Let $V \subseteq \mathbf{A}^n(k)$, $W \subseteq \mathbf{A}^m(k)$ be algebraic sets. Show that

$$V \times W = \{(a_1, \dots, a_n, b_1, \dots, b_m) : (a_1, \dots, a_n) \in V, (b_1, \dots, b_m) \in W\}$$

is an algebraic set in $\mathbf{A}^{n+m}(k)$. It is called the **product** of V and W .

Proof.

(1) Write

$$\begin{aligned} V &= V(S_V) = \{P \in \mathbf{A}^n(k) : f(P) = 0 \forall f \in S_V\} \\ W &= V(S_W) = \{Q \in \mathbf{A}^m(k) : g(Q) = 0 \forall g \in S_W\}, \end{aligned}$$

where $S_V \subseteq k[x_1, \dots, x_n]$ and $S_W \subseteq k[y_1, \dots, y_m]$. It suffices to show that

$$V \times W = V(S),$$

where $S \subseteq k[x_1, \dots, x_n, y_1, \dots, y_m]$ is the union of S_V and S_W .

(2) Here we can identify S_V with the subset of $k[x_1, \dots, x_n, y_1, \dots, y_m]$ by noting that

$$k[x_1, \dots, x_n] \hookrightarrow (k[y_1, \dots, y_m])[x_1, \dots, x_n] = k[x_1, \dots, x_n, y_1, \dots, y_m].$$

Here we regard k as a subring of $k[y_1, \dots, y_m]$. Similar treatment to S_W .

(3) By construction, $V \times W \subseteq V(S)$. Conversely, given any $(P, Q) \in V(S) \subseteq \mathbf{A}^{n+m}(k)$, we have $h(P, Q) = 0$ for all $h \in S = S_V \cup S_W$ (by (2)). By construction, $f(P) = 0$ for all $f \in S_V$ since f only involve x_1, \dots, x_n . Hence, $P \in V$. Similarly, $Q \in W$. Therefore, $(P, Q) \in V \times W$.

□

1.3. The Ideal of a Set of Points

Problem 1.16.*

Let V, W be algebraic sets in $\mathbf{A}^n(k)$. Show that $V = W$ if and only if $I(V) = I(W)$.

Proof.

(1) (Proof of Property (6) in §1.3.) Show that if $X \subseteq Y$, then $I(X) \supseteq I(Y)$. If $f \in I(Y)$ then $f(P) = 0$ for all $P \in Y$. So $f(P) = 0$ for all $P \in X \subseteq Y$ or $f \in I(X)$.

- (2) (Proof of Property (8) in §1.3.) $I(V(S)) \supseteq S$ for any set S of polynomials; $V(I(X)) \supseteq X$ for any set X of points.
- (a) If $f \in S$ then f vanishes on $V(S)$, hence $f \in IV(S)$.
 - (b) If $P \in X$ then every polynomial in $I(X)$ vanishes at P , so P belongs to the zero set of $I(X)$.
- (3) (Proof of Property (9) in §1.3.) $V(I(V(S))) = V(S)$ for any set S of polynomials, and $I(V(I(X))) = I(X)$ for any set X of points. So if V is an algebraic set, $V = V(I(V))$, and if I is the ideal of an algebraic set, $I = I(V(I))$.
- (a) In each case, it suffices to show that the left side is a subset of the right side. (by Properties (6)(8) in §1.3).
 - (b) If $P \in V(S)$ then $f(P) = 0$ for all $f \in I(V(S))$, so $P \in V(I(V(S)))$.
 - (c) If $f \in I(X)$ then $f(P) = 0$ for all $P \in V(I(X))$. Thus f vanishes on $V(I(X))$, so $f \in I(V(I(X)))$.
- (4) Show that $V = W$ if and only if $I(V) = I(W)$.
- (a) By Property (6) in §1.3, $I(V) \supseteq I(W)$ if $V \subseteq W$ and $I(V) \subseteq I(W)$ if $V \supseteq W$. Thus, $I(V) = I(W)$ if $V = W$.
 - (b) Conversely, $I(V) = I(W)$ implies that $V(I(V)) = V(I(W))$ by Property (3) in §1.2 and similar argument in (a). By Property (9) in §1.3, $V(I(V)) = V$ and $V(I(W)) = W$. Thus, $V = W$.

□

Problem 1.17.*

- (a) Let V be an algebraic set in $\mathbf{A}^n(k)$, $P \in \mathbf{A}^n(k)$ a point not in V . Show that there is a polynomial $f \in k[x_1, \dots, x_n]$ such that $f(Q) = 0$ for all $Q \in V$, but $f(P) = 1$. (Hint: $I(V) \neq I(V \cup \{P\})$.)
- (b) Let P_1, \dots, P_r be distinct points in $\mathbf{A}^n(k)$, not in an algebraic set V . Show that there are polynomials $f_1, \dots, f_r \in I(V)$ such that $f_i(P_j) = 0$ if $i \neq j$, and $f_i(P_i) = 1$. (Hint: Apply (a) to the union of V and all but one point.)
- (c) With P_1, \dots, P_r and V as in (b), and $a_{ij} \in k$ for $1 \leq i, j \leq r$, show that there are $g_i \in I(V)$ with $g_i(P_j) = a_{ij}$ for all i and j . (Hint: Consider $\sum_j a_{ij} f_j$.)

Proof of (a).

- (1) Since $I(V) \subsetneq I(V \cup \{P\})$ (by Problem 1.16), there is a polynomial $f \in k[x_1, \dots, x_n]$ such that $f(Q) = 0$ for all $Q \in V$, but $f(P) \neq 0$.

- (2) Since k is a field, $(f(P))^{-1} \in k$. Consider the polynomial $(f(P))^{-1}f \in k[x_1, \dots, x_n]$. It is well-defined. Also, $((f(P))^{-1}f)(Q) = (f(P))^{-1}f(Q) = 0$ for all $Q \in V$, but $(f(P))^{-1}f(P) = (f(P))^{-1}f(P) = 1$.

□

Proof of (b).

- (1) For $1 \leq i \leq$, define

$$W = V \cup \{P_1, \dots, P_r\}$$

$$W_i = V \cup \{P_1, \dots, \widehat{P_i}, \dots, P_r\}.$$

Here $W = W_i \cup \{P_i\} \neq W_i$.

- (2) By (a), there is a polynomial $f_i \in k[x_1, \dots, x_n]$ such that $f_i(Q) = 0$ for all $Q \in W_i$, but $f_i(P_i) = 1$. Here $f_i \in I(V)$ and $f_i(P_j) = \delta_{ij}$ where δ_{ij} is the Kronecker delta.

□

Proof of (c).

- (1) For each $1 \leq i \leq r$, define

$$g_i = \sum_j a_{ij} f_j \in k[x_1, \dots, x_n].$$

- (2) $g_i \in I(V)$ since g_i is a linear combination of f_j and $I(V)$ is an ideal.

- (3) Also,

$$g_i(P_j) = \sum_{j'} a_{ij'} f_{j'}(P_j) = \sum_{j'} a_{ij'} \delta_{j'j} = a_{ij}.$$

□

Problem 1.18.*

Let I be an ideal in a ring R . If $a^n \in I$, $b^m \in I$, show that $(a + b)^{n+m} \in I$. Show that $\text{rad}(I)$ is an ideal, in fact a radical ideal. Show that any prime ideal is radical.

Proof.

- (1) Show that $(a + b)^{n+m} \in I$ if $a^n \in I$, $b^m \in I$. By the binomial theorem,

$$(a + b)^{n+m} = \sum_{i=0}^{n+m} a^i b^{n+m-i}.$$

For each term $a^i b^{n+m-i}$, either $i \geq n$ holds or $n + m - i \geq m$ holds, and thus $a^i b^{n+m-i} \in I$ (since $a^n \in I$, $b^m \in I$ and I is an ideal). Hence, the result is established.

- (2) Show that $\text{rad}(I)$ is an ideal.

- (a) $0 \in \text{rad}(I)$ since $0 = 0^1 \in I$ for any ideal in R .
- (b) $(a + b)^{n+m} \in I$ if $a^n \in I$, $b^m \in I$ by (1).
- (c) $(-a)^{2n} = (a^n)^2 \in I$ if $a^n \in I$ (since I is an ideal).
- (d) $(ra)^n = r^n a^n \in I$ if $a^n \in I$ and $r \in R$ (since I is an ideal and R is commutative).

- (3) Show that $\text{rad}(\text{rad}(I)) = \text{rad}(I)$. It suffices to show $\text{rad}(\text{rad}(I)) \subseteq \text{rad}(I)$. Given any $a \in \text{rad}(\text{rad}(I))$. By definition $a^n \in \text{rad}(I)$ for some positive integer n . Again by definition $(a^n)^m = a^{nm} \in I$ for some positive integer m . As nm is a positive integer, $a \in \text{rad}(I)$.

- (4) Show that every prime ideal \mathfrak{p} is radical. Given any $a \in \text{rad}(\mathfrak{p})$, that is, $a^n \in \mathfrak{p}$ for some positive integer. Write $a^n = aa^{n-1}$ if $n > 1$. By the primality of \mathfrak{p} , $a \in \mathfrak{p}$ or $a^{n-1} \in \mathfrak{p}$. If $a \in \mathfrak{p}$, we are done. If $a^{n-1} \in \mathfrak{p}$, we continue this descending argument (or the mathematical induction) until the power of a is equal to 1. Hence \mathfrak{p} is radical.

□

Problem 1.19.

Show that $I = (x^2 + 1) \subseteq \mathbb{R}[x]$ is a radical (even a prime) ideal, but I is not the ideal of any set in $\mathbf{A}^1(\mathbb{R})$.

Proof.

- (1) Show that $I = (x^2 + 1)$ is a prime ideal in $\mathbb{R}[x]$. Given any $fg \in I$. It suffices to show that $f \in I$ or $g \in I$. By definition of I , there is a polynomial $h \in \mathbb{R}[x]$ such that $fg = (x^2 + 1)h$. So $(x^2 + 1) \mid f$ or $(x^2 + 1) \mid g$ since $x^2 + 1$ is irreducible in a unique factorization domain $\mathbb{R}[x]$. Therefore, $f \in I$ or $g \in I$.
- (2) Show that I is not the ideal of any set in $\mathbf{A}^1(\mathbb{R})$. Since $x^2 + 1$ has no roots in \mathbb{R} , I cannot be the ideal of any nonempty set in $\mathbf{A}^1(\mathbb{R})$. Besides, $I(\emptyset) = (1) \neq (x^2 + 1)$.

□

Problem 1.20.*

Show that for any ideal I in $k[x_1, \dots, x_n]$, $V(I) = V(\text{rad}(I))$, and $\text{rad}(I) \subseteq I(V(I))$.

Proof.

- (1) Show that $V(I) = V(\text{rad}(I))$. Since $I \subseteq \text{rad}(I)$, it suffices to show that $V(I) \subseteq V(\text{rad}(I))$. Given any $P \in V(I)$. For any $f \in \text{rad}(I)$, $f^n \in I$ for some positive integer $n > 0$. Note that

$$0 = (f^n)(P) = f(P)^n$$

since $f^n \in I$ and $P \in V(I)$. As k is a domain, $f(P)^n = 0$ implies $f(P) = 0$. So $P \in V(\text{rad}(I))$.

- (2) By Properties (6)(8) in §1.3,

$$I(V(I)) = I(V(\text{rad}(I))) \supseteq \text{rad}(I).$$

□

Note.

- (1) By the Hilbert's Nullstellensatz, $I(V(I)) = \text{rad}(I)$ if $k = \bar{k}$.
 (2) Take $I = (x^2 + 1)$ as an ideal in $\mathbb{R}[x]$. Note that $I(V(I)) = I(\emptyset) = (1)$ and $\text{rad}(I) = I = (x^2 + 1)$. So the equality in $\text{rad}(I) \subsetneq I(V(I))$ might not hold if $k \neq \bar{k}$. (See Problem 1.19.)

Problem 1.21.*

Show that $I = (x_1 - a_1, \dots, x_n - a_n) \subseteq k[x_1, \dots, x_n]$ is a maximal ideal, and that the natural homomorphism from k to $k[x_1, \dots, x_n]/I$ is an isomorphism.

Proof.

- (1) Show that I is a maximal ideal. Suppose that J is an ideal such that $J \supsetneq I$. Take any $f \in J - I$. By Problem 1.7(a),

$$f = \sum \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}.$$

As $f \notin I$, there is a nonzero constant term in f , say $\lambda \in k - \{0\}$. Note that $f - \lambda \in I \subsetneq J$. Hence,

$$\lambda = f - (f - \lambda) \in J$$

since J is an ideal. As $\lambda \neq 0$, $J = k[x_1, \dots, x_n]$ is not a proper ideal containing I .

- (2) Let $\varphi : k \rightarrow k[x_1, \dots, x_n]/I$ be the natural homomorphism. (That is, $\varphi : \lambda \rightarrow \lambda + I \in k[x_1, \dots, x_n]/I$.)
- (3) Show that φ is surjective. Given any $f + I \in k[x_1, \dots, x_n]/I$. By Problem 1.7(a),

$$f = \sum \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}.$$

So

$$\begin{aligned} f + I &= \sum \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n} + I \\ &= \left(f(a_1, \dots, a_n) + \sum_{\text{nonconstant}} \lambda_{(i)} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n} \right) + I \\ &= f(a_1, \dots, a_n) + I. \end{aligned}$$

(Here the summation over all nonconstant terms is in I .) Hence

$$\varphi : f(a_1, \dots, a_n) \in k \mapsto f + I.$$

- (4) Show that φ is injective. $\ker(\varphi) = \{\lambda \in k : \lambda \in I\} = k \cap I = \{0\}$ since I is a proper ideal.
- (5) By (2)(3)(4), $\varphi : k \rightarrow k[x_1, \dots, x_n]/(x_1 - a_1, \dots, x_n - a_n)$ is an isomorphism.

□

1.4. The Hilbert Basis Theorem

Problem 1.22.* (Correspondence theorem for rings)

Let I be an ideal in a ring R , $\pi : R \rightarrow R/I$ the natural homomorphism.

- (a) Show that for every ideal J' of R/I , $\pi^{-1}(J') = J$ is an ideal of R containing I , and for every ideal J of R containing I , $\pi(J) = J'$ is an ideal of R/I . This sets up a natural one-to-one correspondence between $\{\text{ideals of } R/I\}$ and $\{\text{ideals of } R \text{ that contain } I\}$.
- (b) Show that J' is a radical ideal if and only if J is radical. Similarly for prime and maximal ideals.

- (c) Show that J' is finitely generated if J is. Conclude that R/I is Noetherian if R is Noetherian. Any ring of the form $k[x_1, \dots, x_n]/I$ is Noetherian.

Proof of (a).

- (1) Show that for every ideal J' of R/I , $\pi^{-1}(J') = J$ is an ideal of R containing I .

- (a) Show that J contains I . Note that $\pi^{-1}(0) = I \subseteq \pi^{-1}(J') = J$. So J contains I . In particular, $J \neq \emptyset$ since $I \neq \emptyset$.
- (b) Show that J is a additive subgroup of R . It suffices to show that $a - b \in J$ for any $a \in J$ and $b \in J$. Actually,

$$\pi(a - b) = \pi(a) - \pi(b) \in J'$$

implies $a - b \in \pi^{-1}(J') = J$.

- (c) Show that for every $r \in R$ and every $a \in J$, the product $ra \in J$. In fact,

$$\pi(ra) = \pi(r)\pi(a) \in J'$$

implies $ra \in \pi^{-1}(J') = J$.

- (2) Show that for every ideal J of R containing I , $\pi(J) = J'$ is an ideal of R/I .

- (a) Show that J' is nonempty. Note that $\pi(a) = 0 \in \pi(I) \subseteq \pi(J) = J'$ for any $a \in I$. So J' is nonempty since J is nonempty.
- (b) Show that J' is a additive subgroup of R/I . It suffices to show that $\pi(a) - \pi(b) \in J'$ for any $\pi(a) \in J'$, $\pi(b) \in J'$, $a \in J$ and $b \in J$. It is trivial since

$$\pi(a) - \pi(b) = \pi(a - b) \in \pi(J) = J',$$

π is a ring homomorphism and J is an ideal.

- (c) Show that for every $\pi(r) \in R/I$ ($r \in R$) and every $\pi(a) \in J'$ ($a \in J$), the product $\pi(r)\pi(a) \in J'$. It is trivial since

$$\pi(r)\pi(a) = \pi(ra) \in \pi(J) = J',$$

π is a ring homomorphism and J is an ideal.

- (3) By (1)(2), we setup the correspondence between

$$\{\text{ideals of } R/I\} \longleftrightarrow \{\text{ideals of } R \text{ that contain } I\}.$$

Note that this correspondence preserves the subset relation, and thus this correspondence is one-to-one.

□

Proof of (b).

- (1) *Show that J' is radical if J is radical.* It suffices to show that $(a + I)^n = a^n + I \in J'$ implies that $a + I \in J'$. Note that

$$(a + I)^n = a^n + I \in J'$$

implies that $a^n \in J$ or $a \in J$ since J is radical. Hence $a + I \in J/I = J'$.

- (2) *Show that J is radical if J' is radical.* It suffices to show that $a^n \in J$ implies that $a \in J$. Note that

$$\pi(a^n) = \pi(a)^n \in J'$$

implies that $\pi(a) \in J'$ since J' is radical. $a \in \pi^{-1}(J') = J$.

- (3) *Show that J' is prime if J is prime.* It suffices to show that $(a + I)(b + I) = ab + I \in J'$ implies that $a + I \in J'$ or $b + I \in J'$. Note that

$$(a + I)(b + I) = ab + I \in J'$$

implies that $ab \in J$. So $a \in J$ or $b \in J$ by the primality of J . Hence $a + I \in J'$ or $b + I \in J'$.

- (4) *Show that J is prime if J' is prime.* It suffices to show that $ab \in J$ implies that $a \in J$ or $b \in J$. Note that

$$\pi(ab) = \pi(a)\pi(b) \in J'$$

implies that $\pi(a) \in J'$ or $\pi(b) \in J'$ by the primality of J' . So $a \in \pi^{-1}(J') = J$ or $b \in \pi^{-1}(J') = J$.

- (5) *Show that J' is maximal if J is maximal.* Suppose \mathfrak{m} is an ideal containing J' . By (a), $\pi^{-1}(\mathfrak{m})$ is an ideal containing J . So $\pi^{-1}(\mathfrak{m}) = J$ or $\pi^{-1}(\mathfrak{m}) = R$ by the maximality of J . Hence, $\mathfrak{m} = \pi(J) = J'$ or $\mathfrak{m} = \pi(R) = R/I$.

- (6) *Show that J is maximal if J' is maximal.* Suppose \mathfrak{m} is an ideal containing J . By (a), $\pi(\mathfrak{m})$ is an ideal containing J' . So $\pi(\mathfrak{m}) = J'$ or $\pi(\mathfrak{m}) = R/I$ by the maximality of J' . Hence, $\mathfrak{m} = \pi^{-1}(J') = J$ or $\mathfrak{m} = \pi^{-1}(R/I) = R$.

□

Note.

- (1) Note that

$$R/J \cong (R/I)/(J/I)$$

if J is an ideal of R such that $I \subseteq J$.

- (2) Hence, J is prime iff $R/J \cong (R/I)/(J/I)$ is a domain iff J/I is prime.
(3) Also, J is maximal iff $R/J \cong (R/I)/(J/I)$ is a field iff J/I is maximal.

Proof of (c).

- (1) *Show that J' is finitely generated if J is.* Suppose J is generated by a_1, \dots, a_m . It suffices to show that J' is generated by

$$a_1 + I, \dots, a_m + I \in J/I.$$

Given any $a + I \in J'$ where $a \in J$. Write $a = \sum_{1 \leq i \leq m} r_i a_i$ for some $r_i \in R$. Then

$$a + I = \sum r_i a_i + I = \sum (r_i + I)(a_i + I)$$

is generated by $a_1 + I, \dots, a_m + I$.

- (2) *Show that R/I is Noetherian if R is Noetherian.* Note that R is an ideal of itself.
(3) *Show that any ring of the form $k[x_1, \dots, x_n]/I$ is Noetherian.* By the corollary to the Hilbert basis theorem, $k[x_1, \dots, x_n]$ is Noetherian. By (2), the ring $k[x_1, \dots, x_n]/I$ is Noetherian.

□

1.5. Irreducible Components of an Algebraic Set

Problem 1.23.

Give an example of a collection of ideals \mathcal{S} ideals in a Noetherian ring such that no maximal member of \mathcal{S} is a maximal ideal.

Proof.

- (1) Let R be any Noetherian ring. Let \mathcal{S} be any collection of ideals containing R itself. Then the only maximal member of \mathcal{S} is R , which is not a maximal ideal.
(2) Or let R be any Noetherian ring and R is not a field. ($R = k[x_1, \dots, x_n]$ where k is a field for example.) Let $\mathcal{S} = \{(0)\}$. Then the only maximal member of \mathcal{S} is (0) , which is not maximal since R is not a field.

□

Problem 1.24.

Show that every proper ideal in a Noetherian ring is contained in a maximal ideal. (Hint: If I is the ideal, apply the lemma to $\{\text{proper ideals that contain } I\}$.)

Proof.

- (1) Say I be any proper ideal in a Noetherian ring. Let

$$\mathcal{S} = \{\text{proper ideals that contain } I\}.$$

Apply the lemma to \mathcal{S} to get that \mathcal{S} has a maximal member $\mathfrak{m} \in \mathcal{S}$.

- (2) Show that \mathfrak{m} is maximal. Since $\mathfrak{m} \in \mathcal{S}$, \mathfrak{m} is a proper ideal in R . Suppose $\mathfrak{m}' \supsetneq \mathfrak{m}$ is a proper ideal containing \mathfrak{m} . As \mathfrak{m} contains I , \mathfrak{m}' also contains I or $\mathfrak{m}' \in \mathcal{S}$. By the maximality of \mathfrak{m} , $\mathfrak{m}' \subseteq \mathfrak{m}$. So $\mathfrak{m}' = \mathfrak{m}$.

□

Problem 1.25.

- (a) Show that $V(y - x^2) \subseteq \mathbf{A}^2(\mathbb{C})$ is irreducible, in fact, $I(V(y - x^2)) = (y - x^2)$.
- (b) Decompose $V(y^4 - x^2, y^4 - x^2y^2 + xy^2 - x^3) \subseteq \mathbf{A}^2(\mathbb{C})$ into irreducible components.

Proof of (a).

- (1) Let $I = (y - x^2)$ be an ideal of $\mathbb{C}[x, y]$. Since \mathbb{C} is algebraically closed,

$$I(V(I)) = \text{rad}(I)$$

by the Hilbert's Nullstellensatz. It suffices to show that I is prime, or to show that $y - x^2$ is prime. Since $\mathbb{C}[x, y]$ is a UFD, it suffices to show that $y - x^2$ is irreducible.

- (2) Show that $y - x^2$ is irreducible in $\mathbb{C}[x, y]$. Write

$$y - x^2 \in (\mathbb{C}[y])[x].$$

Note that $\mathbb{C}[y]$ is a UFD and y is the constant term. If we can show that y is prime in $\mathbb{C}[y]$, then by the Eisenstein's criterion we can say $y - x^2$ is irreducible in $(\mathbb{C}[y])[x]$.

- (3) As $\mathbb{C}[y]/(y) \cong \mathbb{C}$ is a field or a domain, (y) is maximal or prime. Hence, $y - x^2$ is irreducible.

(4) Or apply Corollary 1 to Proposition 2 in the next section to (2)(3).

□

Proof of (b).

(1) Write

$$\begin{aligned}
 Y &:= V(y^4 - x^2, y^4 - x^2y^2 + xy^2 - x^3) \\
 &= V((y^2 - x)(y^2 + x), (y^2 - x^2)(y^2 + x)) \\
 &= V(y^2 + x) \cup V(y^2 - x, y^2 - x^2) \\
 &= V(y^2 + x) \cup V(y^2 - x, x(x - 1)) \\
 &= V(y^2 + x) \cup V(x, y) \cup V(y + 1, x - 1) \cup V(y - 1, x - 1).
 \end{aligned}$$

(2) Here $V(y^2 + x)$ is irreducible as (a). Besides, $V(x, y)$, $V(y + 1, x - 1)$ and $V(y - 1, x - 1)$ are irreducible since all corresponding ideals are maximal (by the Hilbert's Nullstellensatz and Problem 1.21).

□

Problem 1.26.

Show that $f = y^2 + x^2(x - 1)^2 \in \mathbb{R}[x, y]$ is an irreducible polynomial, but $V(f)$ is reducible.

Proof.

(1) Show that f is an irreducible polynomial.

(a) Suppose

$$f = (f_2(x)y^2 + f_1(x)y + f_0(x)) \cdot g(x)$$

for some $f_i(x), g(x) \in \mathbb{R}[x]$. So

$$f_2(x)g(x) = 1, \quad f_1(x)g(x) = 0, \quad f_0(x)g(x) = x^2(x - 1)^2.$$

Hence,

$$f_2(x)y^2 + f_1(x)y + f_0(x) = uf, \quad g(x) = u^{-1},$$

where u is a unit in \mathbb{R} .

(b) Suppose

$$f = (f_1(x)y + f_0(x)) \cdot (g_1(x)y + g_0(x))$$

for some $f_i(x), g_j(x) \in \mathbb{R}[x]$. So

$$\begin{aligned} f_1(x)g_1(x) &= 1, \\ f_1(x)g_0(x) + f_0(x)g_1(x) &= 0, \\ f_0(x)g_0(x) &= x^2(x-1)^2. \end{aligned}$$

So $f_1(x) = u, g_1(x) = u^{-1}$ for some unit $u \in \mathbb{R}$. Hence,

$$u^2 g_0(x)^2 = -x^2(x-1)^2,$$

which is absurd since \mathbb{R} is not algebraically closed.

(c) By (a)(b), f is irreducible in $\mathbb{R}[x, y]$.

- (2) Show that $V(f)$ is reducible. $V(f) = \{(0, 0), (1, 0)\} = V(x, y) \cup V(x-1, y)$.
Here $V(x, y)$ and $V(x-1, y)$ are all proper algebraic sets in $V(f)$.

□

Problem 1.27.

Let V, W be algebraic sets in $\mathbf{A}^n(k)$ with $V \subseteq W$. Show that each irreducible component of V is contained in some irreducible component of W .

Proof.

- (1) Write two decompositions of V, W into irreducible components as

$$\begin{aligned} V &= V_1 \cup \cdots \cup V_r, \\ W &= W_1 \cup \cdots \cup W_s, \end{aligned}$$

- (2) For each irreducible component V_i of V , consider $V_i \cap W$:

$$V_i \cap W = (V_i \cap W_1) \cup \cdots \cup (V_i \cap W_s).$$

By the irreducibility of V_i , there is only one j such that $V_i \cap W_j = V_i$ and other intersections are empty. Therefore, each irreducible component V_i is contained in some irreducible component W_j of W .

□

Problem 1.28.

If $V = V_1 \cup \cdots \cup V_r$ is the decomposition of an algebraic set into irreducible components, show that $V_i \not\subseteq \bigcup_{j \neq i} V_j$.

Proof.

- (1) (Reductio ad absurdum) If

$$V_i \subseteq \bigcup_{j \neq i} V_j$$

for some i , then

$$V = V_1 \cup \cdots \cup \widehat{V_i} \cup \cdots \cup V_r$$

is another decomposition of an algebraic set into irreducible components.

- (2) By Theorem 2 in §1.5, the number of irreducible components is unique determined, contrary to the assumption and (1).

□

Problem 1.29.*

Show that $\mathbf{A}^n(k)$ is irreducible if k is infinite.

Proof.

- (1) (Reductio ad absurdum) If $\mathbf{A}^n(k)$ were reducible, then $\mathbf{A}^n(k) = V_1 \cup V_2$ where V_1, V_2 are algebraic sets in $\mathbf{A}^n(k)$, V_1 and V_2 are nonempty and proper in $\mathbf{A}^n(k)$.
- (2) Take $P_i \in V_i$ for $i = 1, 2$. By Problem 1.17, there are two polynomials $f_1, f_2 \in k[x_1, \dots, x_n]$ such that $f_i(Q) = 0$ for all $Q \in V_i$ and $f_1(P_2) = f_2(P_1) = 1$.
- (3) By construction, $(f_1 f_2)(a_1, \dots, a_n) = 0$ for any $a_1, \dots, a_n \in k$. As k is infinite, $f_1 f_2 = 0$ by Problem 1.4. Since $k[x_1, \dots, x_n]$ is a domain, $f_1 = 0$ or $f_2 = 0$, contrary to $f_1(P_2) = f_2(P_1) \neq 0$.

□

Note. $\mathbf{A}^n(k)$ is reducible if k is finite.

1.6. Algebraic Subsets of the Plane

Problem 1.30.

Let $k = \mathbb{R}$.

- (a) Show that $I(V(x^2 + y^2 + 1)) = (1)$.
- (b) Show that every algebraic subset of $\mathbf{A}^2(\mathbb{R})$ is equal to $V(f)$ for some $f \in \mathbb{R}[x, y]$.

This indicates why we usually require that k be algebraically closed.

Proof of (a). $I(V(x^2 + y^2 + 1)) = I(\emptyset) = (1)$ since $x^2 + y^2 + 1 \geq 1$ is never zero for any $x, y \in \mathbb{R}$. \square

Proof of (b).

- (1) Given any algebraic subset V of $\mathbf{A}^2(\mathbb{R})$. $V = V(1)$ if $V = \emptyset$. $V = V(0)$ if $V = \mathbf{A}^2(\mathbb{R})$. Now suppose V is a nonempty proper algebraic subset of $\mathbf{A}^2(\mathbb{R})$. Write $V = V_1 \cup \cdots \cup V_m$, where each V_i is irreducible. Here $V_i \neq \emptyset$ and $V_i \neq \mathbf{A}^2(\mathbb{R})$ for all i .
- (2) As $k = \mathbb{R}$ is infinite, Corollary 2 to Proposition 2 implies that each V_i is either a point or an irreducible plane curve $V(f_i)$, where f_i is an irreducible polynomial and $V(f_i)$ is infinite.
- (3) If $V_i = \{(a_i, b_i)\}$ is a point, then define

$$f_i(x, y) = (x - a_i)^2 + (y - b_i)^2.$$

By the property of \mathbb{R} , $V_i = V(f_i)$.

- (4) Define $f = f_1 \cdots f_m \in \mathbb{R}[x, y]$. Hence,

$$\begin{aligned} V &= V_1 \cup \cdots \cup V_m \\ &= V(f_1) \cup \cdots \cup V(f_m) \\ &= V(f_1 \cdots f_m) \\ &= V(f). \end{aligned}$$

\square

Problem 1.31.

- (a) Find the irreducible components of $V(y^2 - xy - x^2y + x^3)$ in $\mathbf{A}^2(\mathbb{R})$, and also in $\mathbf{A}^2(\mathbb{C})$.

(b) Do the same for $V(y^2 - x(x^2 - 1))$, and for $V(x^3 + x - x^2y - y)$.

Proof of (a).

(1) Note that

$$\begin{aligned} V(y^2 - xy - x^2y + x^3) &= V((y - x^2)(y - x)) \\ &= V(y - x^2) \cup V(y - x). \end{aligned}$$

(2) Note that $y - x^2$ and $y - x$ are irreducible in $\mathbb{C}[x, y]$ and thus also in $\mathbb{R}[x, y]$ by the similar argument in Problem 1.25(a). Also, $V(y - x^2)$ and $V(y - x)$ are infinite in $\mathbf{A}^2(\mathbb{R})$ and thus also in $\mathbf{A}^2(\mathbb{C})$.

(3) Therefore, $V(y - x^2)$ and $V(y - x)$ are the irreducible components of $V(y^2 - xy - x^2y + x^3)$ in $\mathbf{A}^2(\mathbb{R})$ and also in $\mathbf{A}^2(\mathbb{C})$.

□

Outline of (b).

- (1) The elliptic curve $V(y^2 - x(x + 1)(x - 1))$ is irreducible over $\mathbf{A}^2(\mathbb{R})$.
- (2) The elliptic curve $V(y^2 - x(x + 1)(x - 1))$ is irreducible over $\mathbf{A}^2(\mathbb{C})$.
- (3) The irreducible component of $V(x^3 + x - x^2y - y)$ over $\mathbf{A}^2(\mathbb{R})$ is $V(x - y)$.
- (4) The irreducible components of $V(x^3 + x - x^2y - y)$ over $\mathbf{A}^2(\mathbb{C})$ are $V(x + i)$, $V(x - i)$ and $V(x - y)$.

Proof of (b).

(1) Similar to Problem 1.25. To show $y^2 - x(x + 1)(x - 1)$ is irreducible in $\mathbb{C}[x, y]$, we write

$$y^2 - x(x + 1)(x - 1) \in (\mathbb{C}[x])[y].$$

Note that $\mathbb{C}[x]$ is a UFD and $-x(x + 1)(x - 1)$ is the constant term. As $\mathbb{C}[x]/(x) \cong \mathbb{C}$ is a domain, (x) is prime. Clearly, $x \mid x(x + 1)(x - 1)$ but $x^2 \nmid x(x + 1)(x - 1)$. By the Eisenstein's criterion, we can say $y^2 - x(x + 1)(x - 1)$ is irreducible over $(\mathbb{C}[x])[y]$.

- (2) Moreover, $V(y^2 - x(x + 1)(x - 1))$ is infinite over $\mathbf{A}^2(\mathbb{R})$ and thus also over $\mathbf{A}^2(\mathbb{C})$. ($y = f(x) = \sqrt{x(x + 1)(x - 1)}$ is continuous and strictly increasing on $[1, \infty)$ in the sense of calculus. As the measure of $[1, \infty)$ is ∞ , the set $V(y^2 - x(x + 1)(x - 1))$ is infinite over $\mathbf{A}^2(\mathbb{R})$.)
- (3) By Corollary 1 to Proposition 2, $V(y^2 - x(x^2 - 1))$ itself is irreducible over $\mathbf{A}^2(\mathbb{R})$ or $\mathbf{A}^2(\mathbb{C})$.

- (4) Consider $V(x^3 + x - x^2y - y) \subseteq \mathbf{A}^2(\mathbb{R})$.

$$\begin{aligned} V(x^3 + x - x^2y - y) &= V((x^2 + 1)(x - y)) \\ &= V(x^2 + 1) \cup V(x - y) \\ &= \emptyset \cup V(x - y) \\ &= V(x - y). \end{aligned}$$

Here we use that fact that $x^2 + 1 = 0$ has no real solution $x \in \mathbb{R}$. Similar to (a), $V(x - y)$ is the only irreducible component of $V(x^3 + x - x^2y - y)$ in $\mathbf{A}^2(\mathbb{R})$.

- (5) Consider $V(x^3 + x - x^2y - y) \subseteq \mathbf{A}^2(\mathbb{C})$.

$$\begin{aligned} V(x^3 + x - x^2y - y) &= V((x + i)(x - i)(x - y)) \\ &= V(x + i) \cup V(x - i) \cup V(x - y). \end{aligned}$$

Similar to (a), $V(x \pm i)$ and $V(x - y)$ are the irreducible components of $V(x^3 + x - x^2y - y)$ in $\mathbf{A}^2(\mathbb{C})$.

□

1.7. Hilbert's Nullstellensatz

Problem 1.32.

Show that both theorems and all of the corollaries are false if k is not algebraically closed.

Proof.

- (1) Weak Nullstellensatz: $I = (x^2 + 1)$ is a proper ideal in $\mathbb{R}[x]$ but $V(I) = \emptyset$.
(2) Hilbert's Nullstellensatz: Let $I = (y^2 + x^2(x - 1)^2)$ be an ideal in $\mathbb{R}[x, y]$. Hence,

$$\begin{aligned} I(V(I)) &= I(\{(0, 0), (1, 0)\}) && \text{(Problem 1.26.)} \\ &= (x(x - 1), y) \\ &\neq I \\ &= \text{rad}(I). \end{aligned}$$

The last equality holds since f is irreducible in a UFD $\mathbb{R}[x, y]$ and thus I is a prime ideal.

- (3) Corollary 1: Same example in the case Hilbert's Nullstellensatz. If $I = (y^2 + x^2(x - 1)^2)$ is a radical ideal in $\mathbb{R}[x, y]$. Then $I(V(I)) \neq I$.

- (4) Corollary 2: Same example in the case Hilbert's Nullstellensatz. If $I = (y^2 + x^2(x-1)^2)$ is a prime ideal in $\mathbb{R}[x, y]$, then

$$V(I) = \{(0, 0), (1, 0)\} = V(x, y) \cup V(x-1, y)$$

is reducible. Next, consider a prime ideal $J = (x^2 + y^2)$ in $\mathbb{R}[x, y]$. (Use the same argument in Problem 1.26 to get the irreducibility of $x^2 + y^2$.) $V(J) = \{(0, 0)\}$ is a point but J is not a maximal ideal (since $J \subsetneq (x^2 + y^2, x) \subsetneq (1)$).

- (5) Corollary 3: Same example in Corollary 2.
- (6) Corollary 4: Let $I = (x^2 + y^2)$ be an ideal in $\mathbb{R}[x, y]$. Then $V(I) = \{(0, 0)\}$ is a finite set. But $\mathbb{R}[x, y]/(x^2 + y^2)$ is an infinite dimensional vector space over \mathbb{R} . In fact, the monomials

$$\{\overline{x^m}, \overline{x^m y} : m = 0, 1, 2, \dots\}$$

is a basis for $\mathbb{R}[x, y]/(x^2 + y^2)$.

□

Problem 1.33.

- (a) Decompose $V(x^2 + y^2 - 1, x^2 - z^2 - 1) \subseteq \mathbf{A}^3(\mathbb{C})$ into irreducible components.
- (b) Let $V = \{(t, t^2, t^3) \in \mathbf{A}^3(\mathbb{C}) : t \in \mathbb{C}\}$. Find $I(V)$, and show that V is irreducible.

Proof of (a).

- (1) Write

$$\begin{aligned} & V(x^2 + y^2 - 1, x^2 - z^2 - 1) \\ &= V(x^2 + y^2 - 1, y^2 + z^2) \\ &= V(x^2 + y^2 - 1, (y + iz)(y - iz)) \\ &= V(x^2 + y^2 - 1, y + iz) \cup V(x^2 + y^2 - 1, y - iz). \end{aligned}$$

By the Hilbert's Nullstellensatz, it suffices to show that $(x^2 + y^2 - 1, y + iz)$ and $(x^2 + y^2 - 1, y - iz)$ are prime.

- (2) Show that $I = (x^2 + y^2 - 1, y + iz)$ is prime in $\mathbb{C}[x, y, z]$. Note that

$$\mathbb{C}[x, y, z]/I \cong \mathbb{C}[x, y]/(x^2 + y^2 - 1)$$

is a ring isomorphism defined by

$$f(x, y, z) + I \mapsto f(x, y, -iy) + (x^2 + y^2 - 1).$$

(Use the similar argument in (b) to prove it is indeed an isomorphism.)
So it suffices to show that

$$x^2 + y^2 - 1 \in \mathbb{C}[x, y]$$

is irreducible. (Thus, $\mathbb{C}[x, y]/(x^2 + y^2 - 1) \cong \mathbb{C}[x, y, z]/I$ is a domain, or I is prime.) We can use the similar argument in Problem 1.31 (b) to show $x^2 + y^2 - 1 = y^2 + (x+1)(x-1)$ is irreducible as showing the irreducibility of $y^2 - x(x+1)(x-1)$.

- (3) Similarly, $I = (x^2 + y^2 - 1, y - iz)$ is prime. Therefore, the irreducible components of $V(x^2 + y^2 - 1, x^2 - z^2 - 1)$ are $V(x^2 + y^2 - 1, y + iz)$ and $V(x^2 + y^2 - 1, y - iz)$.

□

Proof of (b).

- (1) Write

$$V = \{(t, t^2, t^3) \in \mathbf{A}^3(\mathbb{C}) : t \in \mathbb{C}\} = V(x^2 - y, x^3 - z).$$

Let $I = (x^2 - y, x^3 - z)$ in $\mathbb{C}[x, y, z]$. By the Hilbert's Nullstellensatz, $I(V) = \text{rad}(I)$. So it suffices to show that $I = (x^2 - y, x^3 - z)$ is prime (and thus V is irreducible).

- (2) *Show that*

$$\mathbb{C}[x, y, z]/I \cong \mathbb{C}[t]$$

is a domain, and thus $I = (x^2 - y, x^3 - z)$ is a prime ideal.

- (a) Define a ring homomorphism $\alpha : \mathbb{C}[x, y, z]/I \rightarrow \mathbb{C}[t]$ by

$$\alpha : f(x, y, z) + I \mapsto f(t, t^2, t^3).$$

α is well-defined since $\alpha((x^2 - y) + I) = 0$ and $\alpha((x^3 - z) + I) = 0$.

- (b) *Show that α is surjective.*

$$\alpha : g(x) + I \in \mathbb{C}[x, y, z]/I \mapsto g(t) \in \mathbb{C}[t]$$

for any $g(t)$.

- (c) *Show that α is injective.* Suppose $\alpha(f(x, y, z) + I) = 0$. Write

$$\begin{aligned} f(x, y, z) + I &= \sum_{(i)} \lambda_{(i)} x^{i_1} (y - x^2)^{i_2} (z - x^3)^{i_3} + I \\ &= \sum_i \lambda_i x^i + I. \end{aligned}$$

So

$$0 = \alpha(f(x, y, z) + I) = \alpha\left(\sum_i \lambda_i x^i + I\right) = \sum_i \lambda_i t^i.$$

Hence, $\ker(\alpha) = I$.

□

Problem 1.34.

Let R be a UFD.

- (a) Show that a monic polynomial of degree two or three in $R[x]$ is irreducible if and only if it has no root in R .
- (b) $x^2 - a \in R[x]$ is irreducible if and only if a is not a square in R .

Proof of (a).

- (1) It is equivalent to show that a monic polynomial of degree two or three in $R[x]$ is reducible if and only if it has one root in R .
- (2) Suppose f is reducible of degree 2 or 3. Then there exist nonconstant monic polynomials $g, h \in R[x]$ such that $f = gh$. By

$$\deg(g) + \deg(h) = \deg(f) = 2 \text{ or } 3,$$

we may assume that $\deg(g) = 1$. (Otherwise g or h will be a constant polynomial.) Say $g(x) = x - a$ where $a \in R$. Now

$$f(a) = g(a)h(a) = 0$$

implies that $a \in R$ is a root of f .

- (3) Conversely, if $a \in R$ is a root of f , then apply the same argument in Problem 1.7 we can write

$$f = (x - a)g$$

for some $g \in R[x]$. Here $\deg(g) \geq 1$ since $\deg(f) = 1 + \deg(g) \geq 2$. Therefore, f is reducible.

□

Proof of (b). By (a), $x^2 - a \in R[x]$ is reducible $\iff x^2 - a$ has one root $\alpha \in R$ $\iff a = \alpha^2$ is a square in R for some $\alpha \in R$. □

Problem 1.35.

Show that $V(y^2 - x(x - 1)(x - \lambda)) \subseteq \mathbf{A}^2(k)$ is an irreducible curve for any algebraically closed field k , and any $\lambda \in k$.

Proof.

- (1) By the Hilbert's Nullstellensatz, it suffices to show that

$$I = (y^2 - x(x-1)(x-\lambda))$$

is a prime ideal in $k[x, y]$, or show that

$$y^2 - x(x-1)(x-\lambda)$$

is irreducible (since $k[x, y]$ is a UFD).

- (2) By Problem 1.34(b), $y^2 - x(x-1)(x-\lambda) \in (\mathbb{C}[x])[y]$ is irreducible if $x(x-1)(x-\lambda)$ is not a square in $\mathbb{C}[x]$. Note that every square in $\mathbb{C}[x]$ is of even degree. So $x(x-1)(x-\lambda)$ cannot be a square in $\mathbb{C}[x]$ since $\deg(x(x-1)(x-\lambda)) = 3$ is odd.

□

Note. $V(y^2 - x(x-1)(x-\lambda))$ is the elliptic curve as Problem 1.31.

Problem 1.36.

Let $I = (y^2 - x^2, y^2 + x^2) \subseteq \mathbb{C}[x, y]$. Find $V(I)$ and $\dim_{\mathbb{C}}(\mathbb{C}[x, y]/I)$.

Proof.

- (1) Clearly, $V(I) = \{(0, 0)\}$ is a finite set. By Corollary 4 to the Hilbert's Nullstellensatz,

$$\dim_{\mathbb{C}}(\mathbb{C}[x, y]/I) < \infty.$$

In fact, $\dim_{\mathbb{C}}(\mathbb{C}[x, y]/I) = 4$.

- (2) Given any $f + I \in \mathbb{C}[x, y]/I$ where $f \in \mathbb{C}[x, y]$. Write

$$f(x, y) = \sum_i f_i(x) y^i$$

where $f_i(x) = \sum_j a_{ij} x^j \in \mathbb{C}[x]$. Note that

$$\begin{aligned} x^2 &= \frac{1}{2}(y^2 + x^2) - \frac{1}{2}(y^2 - x^2) \in I, \\ y^2 &= \frac{1}{2}(y^2 + x^2) + \frac{1}{2}(y^2 - x^2) \in I. \end{aligned}$$

So

$$\begin{aligned}
f(x, y) + I &= \sum_i f_i(x) y^i + I \\
&= f_0(x) + f_1(x) y + I \\
&= \sum_j a_{0j} x^j + \left(\sum_j a_{1j} x^j \right) y + I \\
&= a_{00} + a_{01} x + a_{10} y + a_{11} xy + I
\end{aligned}$$

is generated by $\mathcal{B} = \{\bar{1}, \bar{x}, \bar{y}, \bar{xy}\}$.

- (3) Note that \mathcal{B} is a basis since any linear combination of elements in \mathcal{B} is not in I . Therefore,

$$\dim_{\mathbb{C}}(\mathbb{C}[x, y]/I) = |\mathcal{B}| = 4.$$

□

Problem 1.37.*

Let K be any field, $f \in K[x]$ a polynomial of degree $n > 0$. Show that the residues $\bar{1}, \bar{x}, \dots, \bar{x}^{n-1}$ form a basis for $K[x]/(f)$ over K .

Proof.

- (1) Show that every element in $K[x]/(f)$ is generated by $\mathcal{B} = \{\bar{1}, \bar{x}, \dots, \bar{x}^{n-1}\}$. Given any $\bar{g} \in K[x]/(f)$ with $g \in K[x]$. By the division-with-remainder property of $K[x]$, there are some polynomials $q, r \in K[x]$ such that

$$g = fq + r$$

where $r = 0$ or $\deg(r) < n$ if $r \neq 0$. Therefore,

$$g + (f) = fq + r + (f) = r + (f).$$

Note that $r + (f)$ is generated by \mathcal{B} .

- (2) Show that \mathcal{B} is a basis for $K[x]/(f)$ over K . Suppose

$$a_0 + a_1 x + \dots + a_{n-1} x^{n-1} \in (f)$$

for $a_1, \dots, a_{n-1} \in K$. We can regard any linear combination of $\{1, x, \dots, x^{n-1}\}$ as a polynomial $r(x)$ in $K[x]$. $r \in (f)$ implies that there exists a polynomial $g \in K[x]$ such that $r = fg$. If $g \neq 0$, then $\deg(r) = \deg(f) + \deg(g) \geq n$, which is impossible. So $g = 0$ and thus $r = fg = 0 \in K[x]$. Therefore, $a_0 = a_1 = \dots = a_{n-1} = 0 \in K$ and

$$\dim_K(K[x]/(f)) = \deg(f).$$

□

Problem 1.38.*

Let $R = k[x_1, \dots, x_n]$, k algebraically closed, $V = V(I)$. Show that there is a natural one-to-one correspondence between algebraic subsets of V and radical ideals in $k[x_1, \dots, x_n]/I$, and that irreducible algebraic sets (resp. points) correspond to prime ideals (resp. maximal ideals). (See Problem 1.22.)

Proof.

- (1) Given any algebraic subset W of V . By the Hilbert's Nullstellensatz,

$$I(W) \supseteq I(V) = \text{rad}(I) \supseteq I.$$

- (2) By Corollary 1 to the Hilbert's Nullstellensatz and Problem 1.22(b), we have a one-to-one correspondence such that

$$\begin{aligned} & \{\text{algebraic subsets of } V\} \\ & \longleftrightarrow \{\text{radical ideals containing } I\} \\ & \longleftrightarrow \{\text{radical ideals of } k[x_1, \dots, x_n]/I\}. \end{aligned}$$

- (3) Again by Corollary 2 to the Hilbert's Nullstellensatz and Problem 1.22(b), we have a one-to-one correspondence such that

$$\begin{aligned} & \{\text{irreducible algebraic subsets (resp. points) of } V\} \\ & \longleftrightarrow \{\text{prime (resp. maximal) ideals containing } I\} \\ & \longleftrightarrow \{\text{prime (resp. maximal) ideals of } k[x_1, \dots, x_n]/I\}. \end{aligned}$$

□

Problem 1.39.

- (a) Let R be a UFD, and let $\mathfrak{p} = (t)$ be a principal proper prime ideal. Show that there is no prime ideal \mathfrak{q} such that $0 \subsetneq \mathfrak{q} \subsetneq \mathfrak{p}$.
- (b) Let $V = V(f)$ be irreducible hypersurface in \mathbf{A}^n . Show that there is no irreducible algebraic set W such that $V \subsetneq W \subsetneq \mathbf{A}^n$.

Proof of (a).

- (1) (Reductio ad absurdum) Suppose that \mathfrak{q} were a prime ideal in R such that $0 \subsetneq \mathfrak{q} \subsetneq \mathfrak{p}$.

- (2) Show that there is an irreducible element in \mathfrak{q} . Given any $q \in \mathfrak{q}$. Since \mathfrak{q} is proper, we can write

$$q = q_1 \cdots q_n$$

as a product of irreducible elements in a UFD. Since \mathfrak{q} is prime, there is one irreducible element $q_i \in \mathfrak{q}$.

- (3) Now $q_i \in \mathfrak{q} \subseteq \mathfrak{p} = (t)$. So $q_i = ut$ for some $u \in R$. By the irreducibility of q_i , u is a unit or t is a unit. If u is a unit, then

$$(t) = (q_i) \subseteq \mathfrak{q} \subseteq \mathfrak{p} = (t).$$

So $\mathfrak{q} = \mathfrak{p}$, which is absurd. If t is a unit, then $\mathfrak{p} = (1)$, contrary to the primality of \mathfrak{p} .

□

Proof of (b).

- (1) We might assume that $k = \bar{k}$. By Corollary 3 to the Hilbert's Nullstellensatz and the irreducibility of $V(f)$, there are an irreducible polynomial $g \in k[x_1, \dots, x_n]$ and an integer $m > 0$ such that

$$f = g^m,$$

and

$$I(V(f)) = (g).$$

- (2) (Reductio ad absurdum) Suppose that there were an irreducible algebraic set W such that $V \subsetneq W \subsetneq \mathbf{A}^n$. Then by Corollary 3 to the Hilbert's Nullstellensatz again,

$$(g) = I(V(f)) \supsetneq I(W) \supsetneq (1) \in k[x_1, \dots, x_n].$$

Here $(g) = I(V(f))$ and $I(W)$ are all prime.

- (3) Note that (g) is a principal proper prime ideal in a UFD $k[x_1, \dots, x_n]$. By (a), such ideal $I(W)$ cannot be prime, which is absurd.

□

Problem 1.40.

Let $I = (x^2 - y^3, y^2 - z^3) \subseteq k[x, y, z]$. Define $\alpha : k[x, y, z] \rightarrow k[t]$ by $\alpha(x) = t^9$, $\alpha(y) = t^6$, $\alpha(z) = t^4$.

- (a) Show that every element of $k[x, y, z]/I$ is the residue of an element $a + xb + yc + xzd$, for some $a, b, c, d \in k[z]$.

- (b) If $f = a + xb + yc + xyd$, $a, b, c, d \in k[z]$ and $\alpha(f) = 0$, compare like powers of t to conclude that $f = 0$.
- (c) Show that $\ker(\alpha) = I$, so I is prime, $V(I)$ is irreducible, and $I(V(I)) = I$.

Proof of (a).

- (1) Take any element $\bar{f} \in k[x, y, z]/I$ where $f \in k[x, y, z]$. Regard $f \in (k[y, z])[x]$, By the division-with-remainder property of $(k[y, z])[x]$,

$$f = (x^2 - y^3)q + r$$

where $q, r \in (k[y, z])[x]$ and $r = 0$ or $\deg_x(r) < 2$. In any case, $r = xr_1 + r_0$ for some $r_1, r_0 \in k[y, z]$.

- (2) Apply the same argument to (1), we have

$$\begin{aligned} r_0 &= (y^2 - z^3)q_0 + yc + a \\ r_1 &= (y^2 - z^3)q_1 + yd + b \end{aligned}$$

where $q_0, q_1 \in k[y, z]$ and $a, b, c, d \in k[z]$.

- (3) By $\bar{r}_0 = \overline{yc + a}$ and $\bar{r}_1 = \overline{yd + b}$,

$$\begin{aligned} \bar{f} &= \bar{r} \\ &= \overline{xr_1} + \bar{r}_0 \\ &= \overline{x(yd + b)} + (\overline{yc + a}) \\ &= \bar{a} + \bar{b} \cdot \bar{x} + \bar{c} \cdot \bar{y} + \bar{d} \cdot \overline{xy}. \end{aligned}$$

□

Proof of (b). As $0 = \alpha(f) = a + ct^6 + bt^9 + dt^{15} \in k[t]$, $a = b = c = d = 0 \in k$.

□

Proof of (c).

- (1) $I \subseteq \ker(\alpha)$ is trivial.
- (2) Show that $\ker(\alpha) \subseteq I$. Take any $f \in \ker(\alpha)$, or $\alpha(f) = 0$. By (a), $f = r + f_1$ where $f_1 \in I$ and $r = a + bx + cy + dxy \in k[x, y, z]$ for some $a, b, c, d \in k[z]$. Note that α is a ring homomorphism. Therefore,

$$0 = \alpha(f) = \alpha(r + f_1) = \alpha(r) + \alpha(f_1) = \alpha(r).$$

By (b), $r = 0 \in k[x, y, z]$ and thus $f = f_1 \in I$.

- (3) Therefore,

$$\alpha : k[x, y, z]/(x^2 - y^3, y^2 - z^3) \hookrightarrow k[t]$$

is injective.

□

1.8. Modules; Finiteness Conditions

Problem 1.41.*

If S is module-finite over R , then S is ring-finite over R .

Proof.

- (1) Write $S = \sum Rs_i$ for some $s_1, \dots, s_n \in S$ since S is module-finite over R .
- (2) Show that $\sum Rs_i = R[s_1, \dots, s_n]$. $\sum Rs_i \subseteq R[s_1, \dots, s_n]$ is trivial. Conversely, take any $v \in R[s_1, \dots, s_n]$. Write

$$v = \sum_{(j)} \overbrace{a_{(j)} s_1^{j_1} \cdots s_n^{j_n}}^{\in \sum Rs_i}$$

$\in R \quad \in S = \sum Rs_i$

Here each term $a_{(i)} s_1^{i_1} \cdots s_n^{i_n}$ is in $\sum Rs_i$. As $\sum Rs_i$ is an R -module,

$$v = \sum_{(i)} a_{(i)} s_1^{i_1} \cdots s_n^{i_n} \in \sum Rs_i.$$

□

Note. The converse is not true (by Problem 1.42).

Problem 1.42.

Show that $S = R[x]$ (the ring of polynomials in one variable) is ring-finite over R , but not module-finite.

Proof.

- (1) $S = R[x]$ is ring-finite over R by definition (as $x \in S$).
- (2) (Reductio ad absurdum) If $S = \sum Rs_i$ for some $s_1, \dots, s_n \in S$ were module-finite over R . Any element $s \in \sum Rs_i$ is of degree

$$\deg s \leq \max_{1 \leq i \leq n} \deg s_i := m.$$

So that $x^{m+1} \in S = R[x]$ but not in $\sum Rs_i$, which is absurd.

□

Problem 1.43.*

If L is ring-finite over K (K, L fields) then L is a finitely generated field extension of K .

Proof.

- (1) $L = K[v_1, \dots, v_n]$ for some $v_i \in L$ since L is ring-finite over K .
- (2) Apply Proposition 4 in §1.10, L is module-finite (and hence algebraic) over K , that is, $L = K[v_1, \dots, v_n] = K(v_1, \dots, v_n)$ is a finitely generated field extension of K .

□

Problem 1.44.*

Show that $L = K(x)$ (the field of rational functions in one variable) is a finitely generated field extension of K , but L is not ring-finite over K . (Hint: If L were ring-finite over K , a common denominator of ring generators would be an element $b \in K[x]$ such that for all $z \in L$, $b^n z \in K[x]$ for some n ; but let $z = 1/c$, where c doesn't divide b (Problem 1.5).)

Proof.

- (1) (Reductio ad absurdum) Suppose that L were ring-finite over K . Write $L = K[v_1, \dots, v_m]$ where $v_1, \dots, v_m \in L = K(x)$. Let $b \in K[x]$ be a common denominator of ring generators v_1, \dots, v_m . (So that all $bv_i \in K[x]$.) Therefore, for any $z \in L = K[v_1, \dots, v_m]$, there is an integer $n > 0$ such that $b^n z \in K[x]$.
- (2) Consider $z = 1/c \in K(x)$, where $c \in K[x]$ doesn't divide b . The existence of c is guaranteed by Problem 1.5. Hence, for any integer $n > 0$

$$b^n z = b^n / c$$

is never in $K[x]$ by the construction of c , which is absurd.

□

Problem 1.45.*

Let R be a subring of S , S a subring of T .

- (a) If $S = \sum Rv_i$, $T = \sum Sw_j$, show that $T = \sum Rv_iw_j$.
- (b) If $S = R[v_1, \dots, v_n]$, $T = S[w_1, \dots, w_m]$, show that $T = R[v_1, \dots, v_n, w_1, \dots, w_m]$.
- (c) If R, S, T are fields, and $S = R(v_1, \dots, v_n)$, $T = S(w_1, \dots, w_m)$, show that $T = R(v_1, \dots, v_n, w_1, \dots, w_m)$.

So each of the three finiteness conditions is a transitive relation.

Proof of (a).

- (1) Show that $T \subseteq \sum Rv_iw_j$. Given any $t \in T = \sum Sw_j$. There are some $s_j \in S$ such that $t = \sum_j s_j w_j$. As $s_j \in S = \sum Rv_i$, there are some $r_{ij} \in R$ such that $s_j = \sum_i r_{ij} v_i$. Hence,

$$t = \sum_j s_j w_j = \sum_j \left(\sum_i r_{ij} v_i \right) w_j = \sum_{i,j} r_{ij} v_i w_j \in \sum Rv_iw_j.$$

- (2) Show that $T \supseteq \sum Rv_iw_j$. Take any $\sum r_{ij} v_i w_j \in \sum Rv_iw_j$.

$$\sum r_{ij} v_i w_j = \sum_j \left(\sum_i r_{ij} v_i \right) w_j \in \sum_j Sw_j = T.$$

□

Proof of (b).

- (1) Note that $R[x_1, \dots, x_m]$ is canonically isomorphic to $R[x_1, \dots, x_{m-1}][x_m]$. Hence $R[x_1, \dots, x_m]$ is isomorphic to $R[x_1][x_2] \cdots [x_m]$.
- (2) Hence,

$$\begin{aligned} T &= S[w_1, \dots, w_m] \\ &= R[v_1, \dots, v_n][w_1, \dots, w_m] \\ &= R[v_1, \dots, v_n][w_1] \cdots [w_m] \\ &= R[v_1] \cdots [v_n][w_1] \cdots [w_m] \\ &= R[v_1, \dots, v_n, w_1, \dots, w_m]. \end{aligned}$$

□

Proof of (c).

- (1) By (b), $R(v_1, \dots, v_n)$ is canonically isomorphic to $R(v_1, \dots, v_{n-1})(v_n)$. Hence $R(v_1, \dots, v_n)$ is isomorphic to $R(v_1) \cdots (v_n)$. To see this, note that $R[x_1, \dots, x_m] \cong R[x_1, \dots, x_{m-1}][x_m]$ implies that

$$R(x_1, \dots, x_m) \cong R[x_1, \dots, x_{m-1}](x_m) \hookrightarrow R(x_1, \dots, x_{m-1})(x_m).$$

Conversely, for any $a/b \in R(x_1, \dots, x_{m-1})(x_m)$ where

$$\begin{aligned} a &= \sum_i a_i x_m^i \in R(x_1, \dots, x_{m-1})[x_m], \\ b &= \sum_j b_j x_m^j \in R(x_1, \dots, x_{m-1})[x_m] \end{aligned}$$

and $b \neq 0$, there is a nonzero polynomial $c \in R[x_1, \dots, x_{m-1}]$ such that all ca_i and cb_j are in $R[x_1, \dots, x_{m-1}]$. Hence,

$$\begin{aligned} \frac{a}{b} &= \frac{\sum_i a_i x_m^i}{\sum_j b_j x_m^j} \\ &= \frac{c \sum_i a_i x_m^i}{c \sum_j b_j x_m^j} \\ &= \frac{\sum_i ca_i x_m^i}{\sum_j cb_j x_m^j} \\ &\in R[x_1, \dots, x_{m-1}](x_m). \end{aligned}$$

(2) Hence,

$$\begin{aligned} T &= S(w_1, \dots, w_m) \\ &= R(v_1, \dots, v_n)(w_1, \dots, w_m) \\ &= R(v_1, \dots, v_n)(w_1) \cdots (w_m) \\ &= R(v_1) \cdots (v_n)(w_1) \cdots (w_m) \\ &= R(v_1, \dots, v_n, w_1, \dots, w_m). \end{aligned}$$

□

1.9. Integral Elements

Problem 1.46.* (Transitivity of integral extensions)

Let R be a subring of S , S a subring of (a domain) T . If S is integral over R , and T is integral over S , show that T is integral over R . (Hint: Let $z \in T$, so we have $z^n + a_1 z^{n-1} + \cdots + a_n = 0$, $a_i \in S$. Then $R[a_1, \dots, a_n, z]$ is module-finite

over R .)

Proof (Hint).

- (1) Let $z \in T$, so we have $z^n + a_1 z^{n-1} + \cdots + a_n = 0$, $a_i \in S$. Therefore, z is integral over $R[a_1, \dots, a_n]$, or $R[a_1, \dots, a_n, z]$ is module-finite over $R[a_1, \dots, a_n]$.
- (2) Show that $R[a_1, \dots, a_n]$ is module-finite over R if all $a_i \in S$. Note that

$$\begin{aligned} a_1 &\text{ is integral over } R, \\ a_2 &\text{ is integral over } R[a_1] \supseteq R, \\ &\dots \\ a_n &\text{ is integral over } R[a_1, \dots, a_{n-1}]. \end{aligned}$$

By Proposition 3,

$$\begin{aligned} R[a_1] &\text{ is module-finite over } R, \\ R[a_1][a_2] &\text{ is module-finite over } R[a_1], \\ &\dots \\ R[a_1, \dots, a_{n-1}][a_n] &\text{ is module-finite over } R[a_1, \dots, a_{n-1}]. \end{aligned}$$

Also note that $R[a_1, \dots, a_i] = R[a_1, \dots, a_{i-1}][a_i]$ if $i > 1$. By the transitive relation of the module-finiteness (Problem 1.45), $R[a_1, \dots, a_n]$ is module-finite over R .

- (3) Again by the transitive relation of the module-finiteness (Problem 1.45), $R[a_1, \dots, a_n, z]$ is module-finite over R . Hence, $R[a_1, \dots, a_n, z]$ is a subring of T containing $R[z]$ which is module-finite over R . By Proposition 3, z is integral over R .

□

Problem 1.47.*

Suppose (a domain) S is ring-finite over R . Show that S is module-finite over R if and only if S is integral over R .

Proof.

- (1) Write $S = R[v_1, \dots, v_m]$ for some $v_i \in S$.
- (2) Suppose that S is integral over R . Then all v_i are integral over R . Use the same argument in Problem 1.46, we have

$$S = R[v_1, \dots, v_n]$$

is module-finite over R .

- (3) Conversely, suppose that S is module-finite over R . Take any $v \in S$. Write $v = \sum_i r_i v_i \in S$ since S is module-finite over R . Note that $S = R[v_1, \dots, v_m]$ is a subring of S itself containing $R[v]$ which is module-finite over R . By Proposition 3, v is integral over R .

□

Problem 1.48.*

Let L be a field, k an algebraically closed subfield of L .

- (a) Show that any element of L that is algebraic over k is already in k .
 (b) An algebraically closed field has no module-finite field extensions except itself.

Proof of (a).

- (1) Let $\alpha \in L$ be algebraic over k . Then there is a nonzero polynomial $f(x) \in k[x]$ with $f(\alpha) = 0$. Note that $\deg f \geq 1$.
 (2) Since k is algebraically closed, every polynomial is a product of first degree polynomials, say

$$f(x) = c(x - \alpha_1) \cdots (x - \alpha_m)$$

where $c \in k - \{0\}$ and $\alpha_1, \dots, \alpha_m \in k$. As $f(\alpha) = 0$, $\alpha = \alpha_i \in k$ for some $1 \leq i \leq m$. Hence, $\alpha \in L$ is algebraic over k implies that $\alpha \in k$.

□

Proof of (b).

- (1) Suppose that L is module-finite field extensions of an algebraically closed field k .
 (2) By Problem 1.41, L is ring-finite over k . By Problem 1.47, L is integral or algebraic over k (since k is a field). By (a), $L = k$.

□

Problem 1.49.*

Let K be a field, $L = K(x)$ the field of rational functions in one variable over K .

- (a) Show that any element of L that is integral over $K[x]$ is already in $K[x]$.
(Hint: If $z^n + a_1z^{n-1} + \cdots + a_n = 0$, write $z = f/g$, f, g relatively prime.
Then $f^n + a_1f^{n-1}g + \cdots + a_ng^n = 0$, So g divides f .)
- (b) Show that there is no nonzero element $f \in K[x]$ such that for every $z \in L$, $f^n z$ is integral over $K[x]$ for some $n > 0$. (Hint: See Problem 1.44.)

Proof of (a).

- (1) Note that 0 is integral over $K[x]$ and $0 \in K[x]$ trivially.
- (2) Now we take any nonzero element $z \in L = K(x)$ which is integral over $K[x]$. So $z^n + a_1z^{n-1} + \cdots + a_n = 0$ for some $a_1, \dots, a_n \in K[x]$ and $a_n \neq 0$ (since $z \neq 0$).
- (3) Write $z = f/g$, f, g relatively prime in $K[x]$. Then

$$f^n + a_1f^{n-1}g + \cdots + a_ng^n = 0 \in K[x].$$

Since $a_n \neq 0$, $g \mid f^n$ or $g \mid f$ or $g = 1 \in K$. Therefore, $z = f \in K[x]$.

□

Proof of (b).

- (1) (Reductio ad absurdum) Suppose there were a nonzero element $f \in K[x]$ such that for every $z \in L$, $f^n z$ is integral over $K[x]$ for some $n > 0$.
- (2) Let $z = 1/g \in K(x)$, where g is an irreducible polynomial not dividing f . The existence of g is guaranteed by Problem 1.5.
- (3) By the hypothesis in (1), there is an integer $n > 0$ such that $f^n z$ is integral over $K[x]$. By (a), $f^n z = f^n/g$ is also in $K[x]$. So $g \mid f^n$ or $g \mid f$, which is absurd.

□

Problem 1.50.*

Let K be a subfield of a field L .

- (a) Show that the set of elements of L that are algebraic over K is a subfield of L containing K . (Hint: If $v^n + a_1v^{n-1} + \cdots + a_n = 0$, and $a_n \neq 0$, then $v(v^{n-1} + \cdots + a_{n-1}) = -a_n$.)
- (b) Suppose L is module-finite over K , and $K \subseteq R \subseteq L$, R a subring of L . Show that R is a field.

Proof of (a).

- (1) Let R be the set of elements of L that are algebraic over K . By Corollary to Proposition 3, R is a subring of L containing K . (Note that K is a field.) So it suffices to show that $v^{-1} \in R$ if $v \in R - \{0\}$.
- (2) Since v is algebraic over K , we can write

$$v^n + a_1 v^{n-1} + \cdots + a_n = 0$$

for some $a_1, \dots, a_n \in K$ and $a_n \neq 0$. So

$$(v^{-1})^n + \underbrace{\frac{a_{n-1}}{a_n}}_{\in K} (v^{-1})^{n-1} + \cdots + \underbrace{\frac{a_1}{a_n}}_{\in K} (v^{-1}) + \underbrace{\frac{1}{a_n}}_{\in K} = 0,$$

or v^{-1} is integral over K . Hence, $v^{-1} \in R$.

□

Proof of (b).

- (1) By Problem 1.47, L is algebraic over K . Hence, R is algebraic over K .
- (2) To show that R is a field, it suffices to show that $v^{-1} \in R$ if $v \in R - \{0\}$. Since v is algebraic over K , we can write

$$v^n + a_1 v^{n-1} + \cdots + a_n = 0$$

for some $a_1, \dots, a_n \in K$ and $a_n \neq 0$. So

$$v \left(- \underbrace{\frac{1}{a_n}}_{\in K \subseteq R} \underbrace{v^{n-1}}_{\in R} - \cdots - \underbrace{\frac{a_{n-1}}{a_n}}_{\in K \subseteq R} \right) = 1.$$

Here $v^{-1} = \left(-\frac{1}{a_n} v^{n-1} - \cdots - \frac{a_{n-1}}{a_n} \right)$ is the inverse of v in R (since R is a ring containing K).

□

1.10. Field Extensions

Problem 1.51.*

Let K be a field, $f \in K[x]$ an irreducible monic polynomial of degree $n > 0$.

- (a) Show that $L = K[x]/(f)$ is a field, and if \bar{x} is the residue of x in L , then $f(\bar{x}) = 0$.
- (b) Suppose L' is a field extension of K , $y \in L'$ such that $f(y) = 0$. Show that the homomorphism from $K[x]$ to L' that takes x to y induces an isomorphism of L with $K(y)$.
- (c) With L' , y as in (b), suppose $g \in K[x]$ and $g(y) = 0$. Show that f divides g .
- (d) Show that $f = (x - \bar{x})f_1$, $f_1 \in L[x]$.

Proof of (a).

- (1) (f) is a prime ideal in a UFD $K[x]$ since f is irreducible. Note that $K[x]$ is also a PID, (f) is maximal (Problem 1.3). Hence $L = K[x]/(f)$ is a field.
- (2) $f(\bar{x}) = f(x) + (f(x)) = (f(x)) = \bar{0}$.

□

Proof of (b).

- (1) Let $\alpha : K[x] \rightarrow L'$ be a homomorphism defined by

$$\alpha\left(\sum a_i x^i\right) = \sum a_i y^i$$

where $a_i \in K$. $\text{im}(\alpha) = K(y)$ clearly.

- (2) Note that $\ker(\alpha)$ is an ideal containing (f) since $\alpha(f) = 0$. $\ker(\alpha)$ is proper since $\alpha(1) = 1 \neq 0$. By the maximality of (f) , $\ker(\alpha) = (f)$.
- (3) Hence, α induces an isomorphism of L with $K(y)$:

$$L = K[x]/(f) \cong K(y) \hookrightarrow L'.$$

□

Proof of (c). By (b), $g \in \ker(\alpha) = (f)$. So $f \mid g$. □

Proof of (d).

- (1) By (a), $\bar{x} \in L$ is a root of $f \in L[x]$ (by embedding $K[x]$ in $L[x]$).
- (2) Since L is a field, by Problem 1.7(b) we have

$$f = (x - \bar{x})f_1$$

for some $f_1 \in L[x]$.

□

Problem 1.52.* (Splitting fields)

Let K be a field, $f \in K[x]$. Show that there is a field L containing K such that $f = \prod_{i=1}^n (x - x_i) \in L[x]$. (Hint: Use Problem 1.51(d) and induction on the degree.) L is called a **splitting field** of F .

Proof.

- (1) Let $p(x) \in K[x]$ be an irreducible factor of $f(x) \in K[x]$, and let L' be the field $K[x]/(p(x))$ (by Problem 1.51(a)).
- (2) Then we might regard K as a subfield of L' by sending $a \in K$ to $\bar{a} = a + (p(x)) \in L'$.
- (3) By Problem 1.51(a), \bar{x} is a root of $p \in L'$; therefore is a root of f .
- (4) Induction on n . By (1)(2)(3), there is a field $L' \supseteq K$ such that L' contains a root \bar{x} of $f(x)$, say $f(x) = (x - \bar{x})f_1(x)$ over $L'[x]$ (by Problem 1.51(d)). By induction, there is a field $L \supseteq L'$ such that f_1 splits over L . Hence, f splits over L .

□

Problem 1.53.*

Suppose K is a field of characteristic zero, f an irreducible monic polynomial in $K[x]$ of degree $n > 0$. Let L be a splitting field of f , so $f = \prod_{i=1}^n (x - x_i)$, $x_i \in L$. Show that the x_i are distinct. (Hint: Apply Problem 1.51(c) to $g = f_x$; if $(x - \bar{x})^2$ divides f , then $g(\bar{x}) = 0$.)

Proof.

- (1) Since $f \in K[x]$ is irreducible over K , $\gcd(f, f_x)$ is 1 or f . As $\text{char}(K) = 0$, $\deg(f_x) = \deg(f) - 1$. So f does not divide f_x or $\gcd(f, f_x) = 1$. Hence, there are polynomials $g, h \in K[x]$ such that

$$1 = fg + f_x h.$$

This equation is also true in $L[x]$.

- (2) Note that

$$f = \prod_{i=1}^n (x - x_i) \in L[x],$$

$$f_x = \sum_{i=1}^n (x - x_1) \cdots \widehat{(x - x_i)} \cdots (x - x_n) \in L[x].$$

If \bar{x} were a multiple root of f , then $f(\bar{x}) = f_x(\bar{x}) = 0$. By (1),

$$1 = f(\bar{x})g(\bar{x}) + f_x(\bar{x})h(\bar{x}) = 0,$$

which is absurd.

□

Problem 1.54.*

Let R be a domain with quotient field K , and let L be a finite algebraic extension of K .

- (a) For any $v \in L$, show that there is a nonzero $a \in R$ such that av is integral over R .
- (b) Show that there is a basis v_1, \dots, v_n for L over K (as a vector space) such that each v_i is integral over R .

Proof of (a).

- (1) Take any $v \in L$, which is algebraic over K . Write

$$v^n + a_1v^{n-1} + \dots + a_n = 0$$

for some $a_1, \dots, a_n \in K$ and $a_n \neq 0$. Since K is the quotient field of R , there is a common denominator $a \in R$ of a_1, \dots, a_n . Here $a \neq 0$ and $aa_i \in R$ for all $1 \leq i \leq n$.

- (2) Hence,

$$\begin{aligned} & a^n v^n + a^n a_1 v^{n-1} + \dots + a^n a_n = 0 \\ \iff & (av)^n + \underbrace{(aa_1)}_{\in R} (av)^{n-1} + \underbrace{a(aa_2)}_{\in R} (av)^{n-2} + \dots + \underbrace{a^{n-1}(aa_n)}_{\in R} = 0. \end{aligned}$$

av is integral over R .

□

Proof of (b).

- (1) Since L be a finite algebraic extension of K , there exists a basis

$$\{w_1, \dots, w_n\}$$

for L over K (as a vector space).

- (2) For each $w_i \in L$, there is a nonzero $a_i \in R$ such that $a_i w_i$ is integral over R (by (a)). So it suffices to show that

$$\{a_1 w_1, \dots, a_n w_n\}$$

is also a basis for L over K .

- (3) Suppose

$$0 = \sum_i \alpha_i (a_i w_i) = \sum_i (\alpha_i a_i) w_i$$

for some $\alpha_1, \dots, \alpha_n \in K$. Since $\{w_1, \dots, w_n\}$ is a basis, $\alpha_i a_i = 0$ for all i , or $\alpha_i = 0$ for all i (since all $a_i \neq 0$). Hence $\{a_1 w_1, \dots, a_n w_n\}$ is linearly independent.

- (4) Also, for any $w \in L$, we can write

$$\begin{aligned} w &= \underbrace{\beta_1}_{\in K} w_1 + \dots + \underbrace{\beta_n}_{\in K} w_n \\ &= \underbrace{\frac{\beta_1}{a_1}}_{\in K} (a_1 w_1) + \dots + \underbrace{\frac{\beta_n}{a_n}}_{\in K} (a_n w_n) \end{aligned}$$

as a linear combination of $\{a_1 w_1, \dots, a_n w_n\}$ over K .

□

Chapter 2: Affine Varieties

2.1. Coordinate Rings

Problem 2.1.*

Show that the map which associates to each $f \in k[x_1, \dots, x_n]$ a polynomial function in $\mathcal{F}(V, k)$ is a ring homomorphism whose kernel is $I(V)$.

Proof.

- (1) Define a map $\alpha : k[x_1, \dots, x_n] \rightarrow \mathcal{F}(V, k)$. Every polynomial $f \in k[x_1, \dots, x_n]$ defines a function from V to k by

$$\alpha(f)(a_1, \dots, a_n) = f(a_1, \dots, a_n)$$

for all $(a_1, \dots, a_n) \in V$.

- (2) α is a ring homomorphism by construction in (1).
(3) Show that $\ker(\alpha) = I(V)$. In fact, given any $f \in k[x_1, \dots, x_n]$, we have $\alpha(f) = 0$ (sending all $a \in V$ to $0 \in k$) if and only if $f(a) = 0$ for all $a \in V$ if and only if $f \in I(V)$.
(4) Hence,

$$k[x_1, \dots, x_n]/I(V) = \Gamma(V) \cong \{\text{polynomial functions in } \mathcal{F}(V, k)\}$$

as a ring isomorphism.

□

Problem 2.2.*

Let $V \subseteq \mathbf{A}^n$ be a variety. A **subvariety** of V is a variety $W \subseteq \mathbf{A}^n$ that is contained in V . Show that there is a natural one-to-one correspondence between algebraic subsets (resp. subvarieties, resp. points) of V and radical ideals (resp. prime ideals, resp. maximal ideals) of $\Gamma(V)$. (See Problems 1.22, 1.38.)

Proof. Repeat Problem 1.38 by replacing $k[x_1, \dots, x_n]/I$ by $\Gamma(V)$. □

Problem 2.3.*

Let W be a subvariety of a variety V , and let $I_V(W)$ be the ideal of $\Gamma(V)$ corresponding to W .

- (a) Show that every polynomial function on V restricts to a polynomial function on W .
- (b) Show that the map from $\Gamma(V)$ to $\Gamma(W)$ defined in part (a) is a surjective homomorphism with kernel $I_V(W)$, so that $\Gamma(W)$ is isomorphic to $\Gamma(V)/I_V(W)$.

Proof of (a).

- (1) Given any polynomial function $f \in \mathcal{F}(V, k)$ on V . There is a polynomial $g \in k[x_1, \dots, x_n]$ such that $f(P) = g(P)$ for all $P \in V \supseteq W$; thus $f(P) = g(P)$ for all $P \in W$, or $f|_W$ is a polynomial function on W .
- (2) The map $\alpha : \{\text{polynomial functions in } \mathcal{F}(V, k)\} \rightarrow \{\text{polynomial functions in } \mathcal{F}(W, k)\}$ in (1) is defined by

$$\alpha(f) = f|_W.$$

It is a ring homomorphism.

□

Proof of (b).

- (1) Identify $\Gamma(V)$ (resp. $\Gamma(W)$) with the set of all polynomial functions in $\mathcal{F}(V, k)$ (resp. in $\mathcal{F}(W, k)$) by Problem 2.1. The map

$$\alpha : \Gamma(V) \rightarrow \Gamma(W)$$

is defined by

$$\alpha(f + I(V)) = f + I(W).$$

It is well-defined by (a).

- (2) Show that α is surjective. For any $f + I(W) \in \Gamma(W)$, take $f + I(V) \in \Gamma(V)$ and then $\alpha(f + I(V)) = f + I(W)$. (The choice of $f + I(V)$ depends on the representation of $f + I(W)$ and thus might not be unique.)
- (3) Show that $\ker(\alpha) = I_V(W)$, and thus $\Gamma(W) \cong \Gamma(V)/I_V(W)$. Since α is a surjective homomorphism,

$$\begin{aligned} \ker(\alpha) &= \Gamma(V)/\Gamma(W) \\ &= (k[x_1, \dots, x_n]/I(V))/(k[x_1, \dots, x_n]/I(W)) \\ &= I(W)/I(V) \\ &= I_V(W). \end{aligned}$$

□

Problem 2.4.*

Let $V \subseteq \mathbf{A}^n$ be a nonempty variety. Show that the following are equivalent:

- (i) V is a point.
- (ii) $\Gamma(V) = k$.
- (iii) $\dim_k \Gamma(V) < \infty$.

Proof.

- (1) (i) \implies (ii). By Corollary 2 to the Hilbert's Nullstellensatz in §1.7, $V = \{(a_1, \dots, a_n)\}$ corresponds to the maximal ideal

$$I(V) = (x_1 - a_1, \dots, x_n - a_n)$$

in $k[x_1, \dots, x_n]$. Hence,

$$\Gamma(V) = k[x_1, \dots, x_n]/(x_1 - a_1, \dots, x_n - a_n) \cong k$$

(by Problem 1.24).

- (2) (ii) \implies (iii). $\dim_k(\Gamma(V)) = \dim_k(k) = 1 < \infty$.
- (3) (iii) \implies (i). By Corollary 4 to the Hilbert's Nullstellensatz in §1.7, V is a finite set of points in \mathbf{A}^n . Since V is a nonempty variety, V is exactly a point.

□

Problem 2.5.

Let f be an irreducible polynomial in $k[x, y]$, and suppose f is monic in y : $f = y^n + a_1(x)y^{n-1} + \dots + a_n(x)$, with $n > 0$. Let $V = V(f) \subseteq \mathbf{A}^2$. Show that the natural homomorphism from $k[x]$ to $\Gamma(V) = k[x, y]/(f)$ is one-to-one, so that $k[x]$ may be regarded as a subring of $\Gamma(V)$; show that the residues $\bar{1}, \bar{y}, \dots, \bar{y}^{n-1}$ generate $\Gamma(V)$ over $k[x]$ as a module.

Proof.

- (1) $\Gamma(V) = k[x, y]/(f)$ is well-defined since f is irreducible. Define a ring homomorphism $\alpha : k[x] \rightarrow \Gamma(V) = k[x, y]/(f)$ by

$$\alpha : g(x) \mapsto g(x) + (f(x, y)).$$

- (2) *Show that α is one-to-one.* If there were a nonzero polynomial $g \in k[x]$ such that $\alpha(g) = 0$, then $g = fh$ for some nonzero polynomial $h \in k[x, y]$. Hence

$$0 = \deg_y(g) = \deg_y(f) + \deg_y(h) \geq n > 0,$$

which is absurd. Therefore, α is one-to-one. Hence $k[x]$ may be regarded as a subring of $\Gamma(V)$, and thus the multiplication in $\Gamma(V)$ makes $\Gamma(V)$ a $k[x]$ -module.

- (3) Given any $g(x, y) + (f(x, y)) \in k[x, y]/(f)$ where $g \in k[x, y] = (k[x])[y]$. By the division-with-remainder property of $(k[x])[y]$,

$$g = fq + r$$

for some $q, r \in (k[x])[y]$ and

$$r = r_1(x)y^{n-1} + \cdots + r_n(x)$$

where $r_1, \dots, r_n \in k[x]$. Hence

$$\begin{aligned} g + (f) &= fq + r + (f) \\ &= r + (f) \\ &= r_1(x)y^{n-1} + \cdots + r_n(x) + (f) \\ &= \underbrace{r_1(x)}_{\in k[x]} \bar{y}^{n-1} + \cdots + \underbrace{r_n(x)}_{\in k[x]} \bar{1}, \end{aligned}$$

which means that the residues $\bar{1}, \bar{y}, \dots, \bar{y}^{n-1}$ generate $\Gamma(V)$ over $k[x]$ as a module.

□

2.2. Polynomial Maps

Problem 2.6.*

Let $\varphi : V \rightarrow W$, $\psi : W \rightarrow Z$. Show that $\widetilde{\psi \circ \varphi} = \widetilde{\varphi} \circ \widetilde{\psi}$. Show that the composition of polynomial maps is a polynomial map.

Proof.

- (1) *Show that $\widetilde{\psi \circ \varphi} = \widetilde{\varphi} \circ \widetilde{\psi}$.* It is equivalent to show that

$$(\widetilde{\psi \circ \varphi})(f) = (\widetilde{\varphi} \circ \widetilde{\psi})(f)$$

for all $f \in \mathcal{F}(Z, k)$. In fact,

$$\begin{aligned} (\widetilde{\psi \circ \varphi})(f) &= f \circ \psi \circ \varphi, \\ (\widetilde{\varphi} \circ \widetilde{\psi})(f) &= \widetilde{\varphi}(\widetilde{\psi}(f)) = \widetilde{\varphi}(f \circ \psi) = f \circ \psi \circ \varphi. \end{aligned}$$

- (2) Show that the composition of polynomial maps is a polynomial map. Say $V \subseteq \mathbf{A}^n, W \subseteq \mathbf{A}^m, Z \subseteq \mathbf{A}^r$. Since φ (resp. ψ) is a polynomial map, there are polynomials $t_1, \dots, t_m \in k[x_1, \dots, x_n]$ (resp. $s_1, \dots, s_r \in k[x_1, \dots, x_m]$) such that

$$\begin{aligned}\varphi(P) &= (t_1(P), \dots, t_m(P)) \\ \psi(Q) &= (s_1(Q), \dots, s_r(Q))\end{aligned}$$

for all $P \in V$ (resp. $Q \in W$). Hence the composition $\psi \circ \varphi$ is

$$\begin{aligned}(\psi \circ \varphi)(P) &= \psi(\varphi(P)) \\ &= \psi(t_1(P), \dots, t_m(P)) \\ &= (s_1(t_1(P), \dots, t_m(P)), \dots, s_r(t_1(P), \dots, t_m(P))).\end{aligned}$$

So there are polynomials $y_1, \dots, y_r \in k[x_1, \dots, x_n]$ defined by

$$y_i(P) = s_i(t_1(P), \dots, t_m(P))$$

for all $(a_1, \dots, a_n) \in \mathbf{A}^n$ such that

$$(\psi \circ \varphi)(P) = (y_1(P), \dots, y_r(P)).$$

(Note that the composition of polynomials is a polynomials.) Hence $\psi \circ \varphi$ is a polynomial map.

□

Problem 2.7.*

If $\varphi : V \rightarrow W$ is a polynomial map, and X is an algebraic subset of W , show that $\varphi^{-1}(X)$ is an algebraic subset of V . If $\varphi^{-1}(X)$ is irreducible, and X is contained in the image of φ , show that X is irreducible. This gives a useful test for irreducibility.

Proof.

- (1) Show that $\varphi^{-1}(X) = V(\tilde{\varphi}(I(X)))$ is algebraic.

$$\begin{aligned}P \in \varphi^{-1}(X) &\iff \varphi(P) \in X \\ &\iff f(\varphi(P)) = 0 \forall f \in I(X) \\ &\iff \tilde{\varphi}(f)(P) = 0 \forall f \in I(X) \\ &\iff g(P) = 0 \forall g \in \tilde{\varphi}(I(X)) \\ &\iff P \in V(\tilde{\varphi}(I(X))).\end{aligned}$$

Also note that $\tilde{\varphi}(I(X))$ is an ideal in $k[x_1, \dots, x_n]$ since φ is a polynomial map.

- (2) If $\varphi^{-1}(X)$ is irreducible, and X is contained in the image of φ , show that X is irreducible. (Reductio ad absurdum) Suppose that X were reducible or $I(X)$ were not prime. So that there exist two polynomials $f_1, f_2 \notin I(X)$ but $f_1 f_2 \in I(X)$. By definition of $I(X)$, there exist two points $P_1, P_2 \in X$ such that $f_i(P_i) \neq 0$ for $i = 1, 2$.
- (3) Since X is contained in the image of φ , there are two corresponding points $Q_1, Q_2 \in \varphi^{-1}(X)$ such that $\varphi(Q_i) = P_i$. So $\tilde{\varphi}(f_i)(Q_i) = f_i(P_i) \neq 0$, or $\tilde{\varphi}(f_i) \notin I(\varphi^{-1}(X))$. However

$$\tilde{\varphi}(f_1)\tilde{\varphi}(f_2) = \tilde{\varphi}(f_1 f_2) \in I(\varphi^{-1}(X))$$

since $f_1 f_2 \in I(X)$, contrary to the primality of $I(\varphi^{-1}(X))$.

□

Problem 2.8.

- (a) Show that $\{(t, t^2, t^3) \in \mathbf{A}^3(k) : t \in k\}$ is an affine variety.
- (b) Show that $V(xz - y^2, yz - x^3, z^2 - x^2y) \subseteq \mathbf{A}^3(\mathbb{C})$ is a variety. (Hint: $y^3 - x^4, z^3 - x^5, z^4 - y^5 \in I(V)$. Find a polynomial map from $\mathbf{A}^1(\mathbb{C})$ onto V .)

Proof of (a).

- (1) Let $Y := \{(t, t^2, t^3) \in \mathbf{A}^3(k) : t \in k\}$ be the twisted cubic curve. By Problem 2.7, it suffices to show that there is a polynomial map from $\mathbf{A}^1(k)$ onto Y . Here we use the fact that $\mathbf{A}^1(k)$ is irreducible as $k = \bar{k}$ is infinite (by Problem 1.29).
- (2) Define a mapping φ from $\mathbf{A}^1(k)$ to Y by $\varphi(t) = (t, t^2, t^3) \in Y$. φ is a polynomial map. Also, φ is surjective.

□

Note. Also see Problems 1.11 and 1.33 (for the case $k = \mathbb{C}$).

Proof of (b).

- (1) We prove for any algebraically closed field k .
- (2) Write

$$\begin{aligned} V &= V(xz - y^2, yz - x^3, z^2 - x^2y), \\ Y &= \{(t^3, t^4, t^5) \in \mathbf{A}^3(k) : t \in k\}. \end{aligned}$$

We want to show that $Y = V$. $Y \subseteq V$ is trivial. Now given any $(x, y, z) \in V$. If $x = 0$, then $y = z = 0$. So $(x, y, z) = (0, 0, 0) \in Y$. If $x \neq 0$, define

$$t = \frac{y}{x} \in k.$$

Hence,

$$\begin{aligned} t^3 &= \frac{y^3}{x^3} = \frac{y(xz)}{x^3} = \frac{yz}{x^2} = \frac{x^3}{x^2} = x, \\ t^4 &= tx = y, \\ t^5 &= ty = \frac{y^2}{x} = \frac{xz}{x} = z. \end{aligned}$$

- (3) Same as (a). Define a mapping φ from $\mathbf{A}^1(k)$ to $Y = V$ by $\varphi(t) = (t^3, t^4, t^5) \in Y = V$.

□

Note.

- (1) We don't use the hint.
- (2) In fact, it is easy to show that

$$Y = V(y^3 - x^4, z^3 - x^5, z^4 - y^5).$$

- (3) $I(V)$ is a prime ideal of height 2 in $k[x, y, z]$ which cannot be generated by 2 elements. We say V is **not a local complete intersection**.

Problem 2.9.*

Let $\varphi : V \rightarrow W$ be a polynomial map of affine varieties, $V' \subseteq V$, $W' \subseteq W$ subvarieties. Suppose $\varphi(V') \subseteq W'$.

- (a) Show that $\tilde{\varphi}(I_W(W')) \subseteq I_V(V')$ (see Problems 2.3).
- (b) Show that the restriction of φ gives a polynomial map from V' to W' .

Proof of (a).

- (1) It suffices to show that $f \in I_V(V')$ for any $f = \tilde{\varphi}(g) \in \tilde{\varphi}(I_W(W'))$ for some $g \in I_W(W')$.
- (2) To show $f \in I_V(V')$, it suffices to show that $f(P) = 0$ for all $P \in \varphi(V')$. In fact,

$$f(P) = \tilde{\varphi}(g)(P) = g(\varphi(P)) = 0$$

since $\varphi(V') \subseteq W'$ and $g \in I_W(W')$.

□

Proof of (b).

(1) Similar to Problem 2.3.

(2) Since φ is a polynomial map, there are polynomials $t_1, \dots, t_m \in k[x_1, \dots, x_n]$ such that

$$\varphi(P) = (t_1(P), \dots, t_m(P)) \in W$$

for all $P \in V$. So that $\varphi|_{V'} : V' \rightarrow \varphi(V') \subseteq W'$ is also a polynomial map which is equipped with the same polynomials t_1, \dots, t_m such that

$$\varphi(P) = (t_1(P), \dots, t_m(P)) \in W' \subseteq W$$

for all $P \in V' \subseteq V$. (Note that both V' and W' are affine varieties.)

□

Problem 2.10.*

Show that the **projection map** $\text{pr} : \mathbf{A}^n \rightarrow \mathbf{A}^r$, $n \geq r$, defined by $\text{pr}(a_1, \dots, a_n) = (a_1, \dots, a_r)$ is a polynomial map.

Proof.

(1) Define $t_i \in k[x_1, \dots, x_n]$ by $t_i(x_1, \dots, x_n) = x_i$ for $i = 1, \dots, r$.

(2) Clearly,

$$\text{pr}(P) = (t_1(P), \dots, t_r(P))$$

for $P = (a_1, \dots, a_n) \in \mathbf{A}^n$, and thus pr is a polynomial map.

□

Problem 2.11.

Let $f \in \Gamma(V)$, V a variety $\subseteq \mathbf{A}^n$. Define

$$G(f) = \{(a_1, \dots, a_n, a_{n+1}) \in \mathbf{A}^{n+1} : (a_1, \dots, a_n) \in V \text{ and } a_{n+1} = f(a_1, \dots, a_n)\},$$

the **graph** of f . Show that $G(f)$ is an affine variety, and that the map $(a_1, \dots, a_n) \mapsto (a_1, \dots, a_n, f(a_1, \dots, a_n))$ defines an isomorphism of V with $G(f)$. (Projection gives the inverse.)

Proof.

- (1) Define $I = I(V)$ as an ideal in $k[x_1, \dots, x_n]$. Note that

$$G(f) = V(\underbrace{(I, x_{n+1} - f)}_{:=J}).$$

Here we can view I as an ideal of $k[x_1, \dots, x_n, x_{n+1}]$.

- (2) To show that $G(f)$ is an affine variety, it suffices to show that

$$I(G(f)) = I(V(J)) = \text{rad}(J)$$

is prime (by Proposition 1 in §1.5 and the Hilbert's Nullstellensatz in §1.7). Suppose $gh \in I(G(f)) = \text{rad}(J)$. Write

$$\begin{aligned} g &= \sum_i g_i x_{n+1}^i = \sum_i g_i (\underbrace{(x_{n+1} - f) + f}_{\in J})^i, \\ h &= \sum_j h_j x_{n+1}^j = \sum_j h_j (\underbrace{(x_{n+1} - f) + f}_{\in J})^j \end{aligned}$$

where $g_i, h_j \in k[x_1, \dots, x_n]$.

- (3) Hence

$$\begin{aligned} \text{rad}(J) &= gh + \text{rad}(J) && (gh \in \text{rad}(J)) \\ &= (g + \text{rad}(J))(h + \text{rad}(J)) \\ &= \left(\sum_i g_i f^i + \text{rad}(J) \right) \left(\sum_j h_j f^j + \text{rad}(J) \right) && (x_{n+1} - f \in J) \\ &= \left(\sum_i g_i f^i \right) \left(\sum_j h_j f^j \right) + \text{rad}(J) \end{aligned}$$

or

$$\underbrace{\left(\sum_i g_i f^i \right)^N \left(\sum_j h_j f^j \right)^N}_{\in k[x_1, \dots, x_n]} \in J = (I, x_{n+1} - f)$$

for some positive integer N . So that $(\sum_i g_i f^i)^N (\sum_j h_j f^j)^N \in I$.

- (4) Since $I = I(V)$ is a prime ideal, we might get $\sum_i g_i f^i \in I \subseteq \text{rad}(J)$. (The case $\sum_j h_j f^j$ is similar.) Hence $\text{rad}(J) = I(G(f))$ is a prime ideal, or $G(f)$ is irreducible.

- (5) As $G(f)$ is an affine variety, the map $\alpha : V \rightarrow G(f)$ defined by

$$\alpha : (a_1, \dots, a_n) \mapsto (a_1, \dots, a_n, f(a_1, \dots, a_n))$$

is a polynomial map. (Here $t_1 = x_1, \dots, t_n = x_n$ and $t_{n+1} = f$.)

- (6) By Problem 2.10, the projection map pr is a polynomial map. Also note that $\text{pr} \circ \alpha = 1_V$ and $\alpha \circ \text{pr} = 1_{G(f)}$. Therefore, $V \cong G(f)$ as an affine variety isomorphism.

□

Problem 2.12.

- (a) Let $\varphi : \mathbf{A}^1 \rightarrow V = V(y^2 - x^3) \subseteq \mathbf{A}^2$ be defined by $\varphi(t) = (t^2, t^3)$. Show that although φ is a one-to-one, onto polynomial map, φ is not an isomorphism. (Hint: $\tilde{\varphi}(\Gamma(V)) = k[t^2, t^3] \subsetneq k[t] = \Gamma(\mathbf{A}^1)$.)
- (b) Let $\varphi : \mathbf{A}^1 \rightarrow V = V(y^2 - x^2(x+1))$ be defined by $\varphi(t) = (t^2 - 1, t(t^2 - 1))$. Show that φ is one-to-one and onto, except that $\varphi(\pm 1) = (0, 0)$.

Proof of (a).

- (1) Similar to Problem 2.8(a), φ is a polynomial map.
- (2) Similar to Problem 2.8(a) again,

$$V = V(y^2 - x^3) = \{(t^2, t^3) \in \mathbf{A}^2(k) : t \in k\}.$$

Hence the map $\varphi : t \mapsto (t^2, t^3)$ is surjective.

- (3) Show that φ is injective. Suppose $(t^2, t^3) = (s^2, s^3)$ for some $t, s \in k$. If $t = 0$, then $s = 0$. If $t \neq 0$, then $t = \frac{t^3}{t^2} = \frac{s^3}{s^2} = s$. In any case, $t = s$ whenever $(t^2, t^3) = (s^2, s^3)$.
- (4) Show that φ is not an isomorphism. It suffices to show that $\tilde{\varphi}(\Gamma(V)) \subsetneq \Gamma(\mathbf{A}^1)$ by Proposition 1. For any $f \in \Gamma(V)$,

$$\tilde{\varphi}(f)(t) = (f \circ \varphi)(t) = f(t^2, t^3) \in k[t^2, t^3].$$

Hence,

$$\tilde{\varphi}(\Gamma(V)) \subseteq k[t^2, t^3] \subsetneq k[t] = \Gamma(\mathbf{A}^1).$$

(Here note that $t \notin k[t^2, t^3]$ but $t \in k[t]$.)

□

Proof of (b).

- (1) Write

$$Y = \{(t^2 - 1, t(t^2 - 1)) \in \mathbf{A}^2(k) : t \in k\}.$$

Show that $Y = V$. Similar to Problem 2.8(a). It suffices to show that $(x, y) \in Y$ for any $(x, y) \in V$. If $x = 0$, then $y = 0$ or $(x, y) = (0, 0) \in Y$

whenever $t = \pm 1$. (In fact, $(0, 0) = (t^2 - 1, t(t^2 - 1))$ iff $t^2 - 1 = 0$ iff $t = \pm 1$ in any field.) If $x \neq 0$, define

$$t = \frac{y}{x} \in k.$$

So $y = tx$ and thus

$$0 = y^2 - x^2(x + 1) = t^2x^2 - x^2(x + 1) = x^2(t^2 - (x + 1)).$$

Since $x \neq 0$ and k is a field, we have

$$t^2 - (x + 1) = 0 \iff x = t^2 - 1.$$

Hence, $y = tx = t(t^2 - 1)$ and therefore $(x, y) \in Y$.

- (2) By (1), φ is surjective and $\varphi(\pm 1) = (0, 0)$.
- (3) Show that φ is injective except that $\varphi(\pm 1) = (0, 0)$. Given $t, s \in k$. It suffices to show that $t = s$ whenever $(t^2 - 1, t(t^2 - 1)) = (s^2 - 1, s(s^2 - 1)) \neq (0, 0)$. In fact, by assumption we have $t^2 - 1 = s^2 - 1 \neq 0$ by assumption. Therefore,

$$t = \frac{t(t^2 - 1)}{t^2 - 1} = \frac{s(s^2 - 1)}{s^2 - 1} = s.$$

□

Problem 2.13.

Let $V = V(x^2 - y^3, y^2 - z^3) \subseteq \mathbf{A}^3$ as in Problem 1.40, $\bar{\alpha} : \Gamma(V) \rightarrow k[t]$ induced by the homomorphism α of that problem.

- (a) What is the polynomial map f from \mathbf{A}^1 to V such that $\tilde{f} = \bar{\alpha}$?
- (b) Show that f is one-to-one and onto, but not an isomorphism.

Proof of (a).

- (1) Write

$$Y = \{(t^9, t^6, t^4) \in \mathbf{A}^3(k) : t \in k\}.$$

Show that $Y = V$. Similar to Problem 2.8(a). It suffices to show that $(x, y, z) \in Y$ for any $(x, y, z) \in V$. If $x = 0$, then $y = z = 0$ or $(x, y, z) = (0, 0, 0) \in Y$ by taking $t = 0$. If $x \neq 0$, define

$$t = \frac{yz}{x} \in k.$$

Hence,

$$\begin{aligned} t^9 &= \frac{y^9 z^9}{x^9} = \frac{y^{15}}{x^9} = \frac{x^{10}}{x^9} = x, \\ t^6 &= \frac{y^6 z^6}{x^6} = \frac{y^5 z^6}{x^6} y = \frac{y^9}{x^6} y = \frac{x^6}{x^6} y = y, \\ t^4 &= \frac{y^4 z^4}{x^4} = \frac{y^4 z^3}{x^4} z = \frac{y^6}{x^4} z = \frac{x^4}{x^4} z = z. \end{aligned}$$

(2) Define a mapping $f : \mathbf{A}^1 \rightarrow \mathbf{A}^3$ by

$$f : t \mapsto (t^9, t^6, t^4).$$

f is a polynomial map by construction. By (1), $f : \mathbf{A}^1 \rightarrow f(\mathbf{A}^1) = V$ and thus $\tilde{f} = \bar{\alpha}$ by the definition of α .

□

Proof of (b).

(1) Similar to Problem 2.12(a).

(2) f is surjective by the proof of (a).

(3) *Show that f is injective.* Suppose $(t^9, t^6, t^4) = (s^9, s^6, s^4)$ for some $t, s \in k$. If $t = 0$, then $s = 0$. If $t \neq 0$, then $t = \frac{t^6 t^4}{t^9} = \frac{s^6 s^4}{s^9} = s$. In any case, $t = s$ whenever $(t^9, t^6, t^4) = (s^9, s^6, s^4)$.

(4) *Show that f is not an isomorphism.* It suffices to show that $\tilde{f}(\Gamma(V)) \subsetneq \Gamma(\mathbf{A}^1)$ by Proposition 1. For any $g \in \Gamma(V)$,

$$\tilde{f}(g)(t) = (g \circ f)(t) = g(t^9, t^6, t^4) \in k[t^4, t^6, t^9].$$

Hence,

$$\tilde{\varphi}(\Gamma(V)) \subseteq k[t^4, t^6, t^9] \subsetneq k[t] = \Gamma(\mathbf{A}^1).$$

(Here note that $t \notin k[t^4, t^6, t^9]$ but $t \in k[t]$.)

□

2.3. Coordinate Changes

Problem 2.14.* (Linear subvariety)

A set $V \subseteq \mathbf{A}^n(k)$ is called a **linear subvariety** of $\mathbf{A}^n(k)$ if $V = V(f_1, \dots, f_r)$ for some polynomials f_i of degree 1.

- (a) Show that if t is an affine change of coordinates on $\mathbf{A}^n(k)$, then V^t is also a linear subvariety of $\mathbf{A}^n(k)$.
- (b) If $V \neq \emptyset$, show that there is an affine change of coordinates t of \mathbf{A}^n such that $V^t = V(x_{m+1}, \dots, x_n)$. (Hint: use induction on r .) So V is a variety.
- (c) Show that the m that appears in part (b) is independent of the choice of t . It is called the **dimension** of V . Then V is then isomorphic (as a variety) to $\mathbf{A}^m(k)$. (Hint: Suppose there were an affine change of coordinates t such that $V(x_{m+1}, \dots, x_n)^t = V(x_{s+1}, \dots, x_n)$, $m < s$; show that t_{m+1}, \dots, t_n would be dependent.)

Proof of (a).

- (1) Say $t = (t_1, \dots, t_n)$ is an affine change of coordinates, and $V = V(f_1, \dots, f_r)$ for some polynomials f_i of degree 1.
- (2) Show that V is a variety and thus $I(V) = (f_1, \dots, f_r)$ by the Hilbert's Nullstellensatz. $V(f_1, \dots, f_r)$ is the set of all solutions of the system of linear equations:

$$\begin{aligned} f_1 &= a_{11}x_1 + \dots + a_{1n}x_n - b_1 = 0, \\ &\dots \\ f_r &= a_{r1}x_1 + \dots + a_{rn}x_n - b_r = 0. \end{aligned}$$

Write $Ax = b$ and $V = V(Ax = b)$ where

$$A = \underbrace{\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{r1} & \dots & a_{rn} \end{pmatrix}}_{\in \mathbf{M}_{r \times n}(k)}, \quad x = \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{\in \mathbf{M}_{n \times 1}(k)}, \quad b = \underbrace{\begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix}}_{\in \mathbf{M}_{r \times 1}(k)}.$$

- (3) The Gaussian elimination in linear algebra says that $(A|b)$ has the same solutions as its reduced row echelon form $(A'|b')$, that is, $V(Ax = b) = V(A'x = b')$.
- (4) If $V(f_1, \dots, f_r) = \emptyset$, nothing to do. If $V(f_1, \dots, f_r) \neq \emptyset$, then

$$V(f_1, \dots, f_r) = V(g_1, \dots, g_m)$$

where $m = \text{rank}(A)$ is the number of nonzero rows in A' ($m \leq r, n$) and $g_i = a'_{i1}x_1 + \dots + a'_{in}x_n - b'_i$ for $1 \leq i \leq m$. (a'_{ij} is the entry of the matrix A' .)

- (5) Now given any $f + I(V) \in k[x_1, \dots, x_n]/I(V)$, we replace the leading term x_{i_1} of g_1 by $x_{i_1} - g_1$ to get

$$f + I(V) = f(x_1, \dots, \underbrace{x_{i_1} - g_1}_{i_1 \text{th position}}, \dots, x_n) + I(V) := f_1 + I(V)$$

where $f_1 \in k[x_1, \dots, \widehat{x_{i_1}} \dots, x_n]$. Continue this process to replace each leading term x_{i_j} of g_j by $x_{i_j} - g_j$ to get one by one to get

$$f + I(V) = f_1 + I(V) \text{ where } f_1 \in k[x_1, \dots, \widehat{x_{i_1}} \dots, x_n].$$

...

$$f_{m-1} + I(V) = f_m + I(V) \text{ where } f_m \in k[x_1, \dots, \widehat{x_{i_1}} \dots, \widehat{x_{i_m}} \dots, x_n].$$

Hence, a routine shows that there is a ring isomorphism

$$\alpha : k[x_1, \dots, x_n]/I(V) \rightarrow \underbrace{k[x_1, \dots, \widehat{x_{i_1}} \dots, \widehat{x_{i_m}} \dots, x_n]}_{\text{a domain}}$$

sending f to f_m . Therefore, V is a variety.

- (6) As $I(V) = (f_1, \dots, f_r)$, $I(V)^t = (f_1^t, \dots, f_r^t)$ where each f_i^t is linear. Thus $V^t = V(I(V)^t) = V(f_1^t, \dots, f_r^t)$ is also a linear subvariety of $\mathbf{A}^n(k)$.

□

Proof of (b).

- (1) Suppose $A \in M_{r \times n}(k)$ is of rank $n - m$. Linear algebra says that there exist invertible matrices $B \in M_{r \times r}(k)$ and $C \in M_{n \times n}(k)$ such that $D = BAC$, where

$$D = BAC = \underbrace{\begin{pmatrix} O_1 & O_2 \\ O_3 & I_{n-m} \end{pmatrix}}_{\in M_{r \times n}(k)}$$

in which $I_{n-m} \in M_{(n-m) \times (n-m)}(k)$ is the identity matrix and O_1, O_2 , and O_3 are zero matrices.

- (2) Let t' be the linear map corresponding to the matrix C . So

$$\begin{aligned} V^{t'} &= V(Ax = b)^{t'} \\ &= V(ACx = b) \\ &= V(BACx = Bb) && (B: \text{invertible}) \\ &= V(Dx = Bb) \\ &= V(-\beta_1, \dots, -\beta_m, x_{m+1} - \beta_{m+1}, \dots, x_n - \beta_n) && (V \neq \emptyset) \\ &= V(x_{m+1} - \beta_{m+1}, \dots, x_n - \beta_n) \end{aligned}$$

$$\text{where } Bb = \underbrace{\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}}_{\in M_{n \times 1}(k)}.$$

- (3) Let t'' be the translation corresponding to the matrix Bb . Let $t = t'' \circ t'$ be the desired affine change of coordinates. Therefore,

$$\begin{aligned} V^t &= (V^{t'})^{t''} \\ &= V(x_{m+1} - \beta_{m+1}, \dots, x_n - \beta_n)^{t''} \\ &= V(x_{m+1}, \dots, x_n). \end{aligned}$$

□

Proof of (c).

- (1) Linear algebra says that the rank of any matrix is uniquely determined. Therefore, $\dim(V) = n - \text{rank}(A|b) = n - \text{rank}(A'|b')$ is uniquely determined.
- (2) V is then isomorphic to $\mathbf{A}^m(k)$ as a variety.

□

Problem 2.15.* (Line)

Let $P = (a_1, \dots, a_n)$, $Q = (b_1, \dots, b_n)$ be distinct points of \mathbf{A}^n . The **line** through P and Q is defined to be $\{(a_1 + s(b_1 - a_1), \dots, a_n + s(b_n - a_n)) : s \in k\}$.

- (a) Show that if L is the line through P and Q , and t is an affine change of coordinates, then $t(L)$ is the line through $t(P)$ and $t(Q)$.
- (b) Show that a line is a linear subvariety of dimension 1, and that a linear subvariety of dimension 1 is the line through any two of its points.
- (c) Show that, in \mathbf{A}^2 , a line is the same thing as a hyperplane.
- (d) Let $P, P' \in \mathbf{A}^2$, L_1, L_2 two distinct lines through P , L'_1, L'_2 distinct lines through P' . Show that there is an affine change of coordinates t of \mathbf{A}^2 such that $t(P) = P'$ and $t(L_i) = L'_i$, $i = 1, 2$.

Proof of (a).

- (1) Write $t = (t_1, \dots, t_n)$ as

$$t_i = \sum_j c_{ij}x_j + c_{i0}.$$

Take any point $P_s = (a_1 + s(b_1 - a_1), \dots, a_n + s(b_n - a_n)) \in L$ for some $s \in k$. (In particular, $P_0 = P$ and $P_1 = Q$.)

(2) As

$$\begin{aligned}
t_i(P_s) &= \sum_j c_{ij}(a_j + s(b_j - a_j)) + c_{i0} \\
&= \left(\sum_j c_{ij}a_j + c_{i0} \right) \\
&\quad + s \left[\left(\sum_j c_{ij}b_j + c_{i0} \right) - \left(\sum_j c_{ij}a_j + c_{i0} \right) \right] \\
&= t_i(P) + s(t_i(Q) - t_i(P)),
\end{aligned}$$

we have

$$\begin{aligned}
t(L) &= \{(t_1(P) + s(t_1(Q) - t_1(P)), \dots, t_n(P) + s(t_n(Q) - t_n(P))) \\
&\quad : s \in k\}.
\end{aligned}$$

Moreover, $t(P) \in t(L)$ as $s = 0$ and $t(Q) \in t(L)$ as $s = 1$. Therefore, $t(L)$ is the line through $t(P)$ and $t(Q)$.

□

Proof of (b).

(1) Note that $a_\alpha \neq b_\alpha$ for some $1 \leq \alpha \leq n$ since $P \neq Q$. Write

$$L = V \left(x_i = a_i + \frac{x_\alpha - a_\alpha}{b_\alpha - a_\alpha} (b_i - a_i) : 1 \leq i \leq n \right).$$

(Here we solve $s = \frac{x_\alpha - a_\alpha}{b_\alpha - a_\alpha}$ and then replace s in the equation $x_i = a_i + t(b_i - a_i)$.) By Problem 2.14, L is a linear subvariety.

(2) Note that

$$\begin{aligned}
n - \dim(L) &= \text{the rank of the corresponding augmented matrix } (A'|b') \\
&= \text{the maximal number of the linearly independent rows of } (A'|b') \\
&= n - 1,
\end{aligned}$$

which is uniquely determined. Therefore, $\dim(V) = 1$.

(3) Conversely, $\dim(V) = 1$ implies that $\text{rank}(A'|b') = n - 1$. So all leading terms are all x_i except only one x_j for some j . Hence V is of the form

$$V = (x_i + a_{ij}x_j = b_i)$$

for $1 \leq i \leq n$ and $i \neq j$. So

$$\begin{aligned} V &= \{(b_1 - a_{1j}s, \dots, \underbrace{s}_{j\text{th position}}, \dots, b_n - a_{nj}s) : s \in k\} \\ &= \{(b_1 + s((b_1 - a_{1j}) - b_1), \dots, \underbrace{0 + s(1 - 0)}_{j\text{th position}}, \dots, \\ &\quad (b_n + s((b_n - a_{nj}) - b_n)) : s \in k\} \end{aligned}$$

is a line passing

$$\begin{aligned} P &= (b_1, \dots, 0, \dots, b_n) \\ Q &= (b_1 - a_{1j}, \dots, 1, \dots, b_n - a_{nj}) \end{aligned}$$

with $P \neq Q$ (since they are different in the j th position). (Here we can change P and Q to any two different points on V .)

□

Proof of (c).

- (1) A line $L \subseteq \mathbf{A}^2$ is $V(x + ay = b)$ or $V(x + ay = b)$ by (b). In any case, L is a hyperplane in \mathbf{A}^2 .
- (2) Conversely, given any hyperplane $V = V(ax + by + c = 0) \subseteq \mathbf{A}^2$ where a and b are not all zero. Might assume that $a \neq 0$. (The case $b \neq 0$ is similar.) So

$$V = \left\{ \left(-\frac{c}{a} - \frac{b}{a}s, s \right) : s \in k \right\}$$

is a line passing $(-\frac{c}{a}, 0)$ and $(-\frac{c+b}{a}, 1)$.

□

Proof of (d).

- (1) It suffices to show that there is a bijective affine change of coordinates t of \mathbf{A}^2 such that $t(P) = (0, 0)$, $t(L_1) = V(x = 0)$ and $t(L_2) = V(y = 0)$. Write $P = (p_1, p_2)$ and $L_i = a_ix + b_iy + c_i$ for $i = 1, 2$.
- (2) Let $t'' = (t''_1, t''_2)$ be a translation defined by

$$\begin{pmatrix} t''_1 \\ t''_2 \end{pmatrix} = \begin{pmatrix} x - p_1 \\ y - p_2 \end{pmatrix}.$$

So $L_1^{t''} = a_1x + b_1y$ and $L_2^{t''} = a_2x + b_2y$. Let

$$A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

and $t' = (t'_1, t'_2)$ be a linear map defined by

$$\begin{pmatrix} t'_1 \\ t'_2 \end{pmatrix} = \frac{1}{\det(A)} \begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

(t' is well-defined since L_1 and L_2 are distinct lines and thus $\det(A) \neq 0$.)
Write $t = (t_1, t_2) = t' \circ t''$. So

$$\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \frac{1}{\det(A)} \begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix} \begin{pmatrix} x - p_1 \\ y - p_2 \end{pmatrix}$$

and

$$\begin{aligned} L_1^t &= (L_1^{t''})^{t'} = x \\ L_2^t &= (L_2^{t''})^{t'} = y. \end{aligned}$$

(3) Conversely, define an affine change of coordinates $s = (s_1, s_2)$ of \mathbf{A}^2 by

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} a_1x + b_1y + c_1 \\ a_2x + b_2y + c_2 \end{pmatrix}$$

so that $x^s = L_1$ and $y^s = L_2$.

(4) By (2)(3), the statement in (1) is established.

□

Problem 2.16.

Let $k = \mathbb{C}$. Give $\mathbf{A}^n(\mathbb{C}) = \mathbb{C}^n$ the usual topology (obtained by identifying \mathbb{C} with \mathbb{R}^2 , and hence \mathbb{C}^n with \mathbb{R}^{2n}). Recall that a topological space X is path-connected if for any $P, Q \in X$, there is a continuous mapping $\gamma : [0, 1] \rightarrow X$ such that $\gamma(0) = P$, $\gamma(1) = Q$.

- (a) Show that $\mathbb{C} - S$ is path-connected for any finite set S .
- (b) Let V be an algebraic set in $\mathbf{A}^n(\mathbb{C})$. Show that $\mathbf{A}^n(\mathbb{C}) - V$ is path-connected. (Hint: If $P, Q \in \mathbf{A}^n(\mathbb{C}) - V$, let L be the line through P and Q . Then $L \cap V$ is finite, and L is isomorphic to $\mathbf{A}^1(\mathbb{C})$.)

Proof of (a).

- (1) Regard \mathbb{C}^n as \mathbb{R}^{2n} . Given any $P, Q \in \mathbb{C}^n - S$. Write $S := \{P_1, \dots, P_m\} \subseteq \mathbb{C}^n$.
- (2) Let L_{P_i} (resp. L'_{P_i}) be a line passing P (resp. Q) and P_i for every $P_i \in S$. (It is well-defined since P, Q are not in S .) So $\mathcal{C} := \{L_{P_i} : P_i \in S\}$ (resp. $\mathcal{C}' := \{L'_{P_i} : P_i \in S\}$) is a collection of finitely many lines.

- (3) Consider a unit sphere $\mathbb{S}^{2n-1}(P)$ centered at P .

$$\left(\bigcup_{L_i \in \mathcal{C}} L_i \right) \cap \mathbb{S}^{2n-1}(P)$$

is again a finite set (of order $\leq 2|S| = 2m$). Since \mathbb{S}^{2n-1} is infinite, we can always take a line L passing P and some point in $\mathbb{S}^{2n-1}(P)$ where $L \cap S = \emptyset$.

- (4) Similarly, we take a line L' passing Q and some point in $\mathbb{S}^{2n-1}(Q)$ where $L' \cap S = \emptyset$ and $L' \cap L \neq \emptyset$. (There are only two points in $\mathbb{S}^{2n-1}(Q)$ such that $L' \cap L = \emptyset$. Note that \mathbb{S}^{2n-1} is infinite.)
- (5) Take any point $A \in L' \cap L$. So there is a path from P to A (on a segment contained in L) and then to Q (on a segment contained in L'). Therefore, $\mathbb{C}^n - S$ is path-connected.

□

Proof of (b).

- (1) Given any $P, Q \in \mathbf{A}^n(\mathbb{C}) - V$. Let L be the line through P and Q . To show $\mathbf{A}^n(\mathbb{C}) - V$ is path-connected, it suffices to show that

$$L - V = L - (V \cap L)$$

is path-connected.

- (2) Similar to Problem 1.12, we have $V \cap L$ is finite. In fact, write $V = (f_1, \dots, f_r)$ and $L = \{(a_1 + t(b_1 - a_1), \dots, a_n + t(b_n - a_n)) : t \in k\}$ where $P = (a_1, \dots, a_n)$ and $Q = (b_1, \dots, b_n)$. Then

$$\begin{aligned} V \cap L &= \bigcap_i (V(f_i) \cap L) \\ &= \bigcap_i \{f_i(a_1 + t(b_1 - a_1), \dots, a_n + t(b_n - a_n)) = 0 : t \in k\} \\ &= \bigcap_i \text{finite set} \\ &= \text{finite set.} \end{aligned}$$

Here $f_i(a_1 + t(b_1 - a_1), \dots, a_n + t(b_n - a_n))$ is a nonzero polynomial since $P, Q \notin V(f_i)$.

- (3) Note that the path-connectedness is a topological invariant under homeomorphisms. Since any line is homeomorphic to \mathbb{C}^1 , $L - V$ is homeomorphic to $\mathbb{C}^1 - S$ for some finite set S (by (2)). By (a), $L - V$ is path-connected and so is $\mathbf{A}^n(\mathbb{C}) - V$.

□

2.4. Rational Functions and Local Rings

Problem 2.17.

Let $V = V(y^2 - x^2(x + 1)) \subseteq \mathbf{A}^2$, and \bar{x}, \bar{y} the residues of x, y in $\Gamma(V)$; let $z = \bar{y}/\bar{x} \in k(V)$. Find the pole sets of z and of z^2 .

Proof.

(1) Show that the pole set of z is $\{(0, 0)\}$.

(a) Since V is irreducible by Problem 2.12(b), V is a variety. Note that the pole set of z is

$$\bigcap_{z=\bar{f}/\bar{g}} V(\bar{g}).$$

(b) By (a), $\{(0, 0)\}$ contains the pole set of z . (As the denominator $x = 0$, we solve the equation $y^2 - x^2(x + 1) = 0$ to get $y = 0$.)

(c) (Reductio ad absurdum) If $(0, 0)$ were not a pole, then there were $\bar{f}, \bar{g} \in \Gamma(V)$ such that $z = \bar{y}/\bar{x} = \bar{f}/\bar{g}$ where $\bar{g}(0, 0) \neq 0$. So

$$\begin{aligned} z &= \bar{y}/\bar{x} = \bar{f}/\bar{g} \\ \implies \overline{xf} &= \overline{yg} \\ \implies xf - yg &\in (y^2 - x^2(x + 1)) \\ \implies xf - yg &= h(y^2 - x^2(x + 1)) \text{ for some } h \\ \implies y(g + hy) &= x(f + hx(x + 1)) \in (x) \\ \implies g + hy &\in (x) \\ \implies g(0, 0) &= 0, \end{aligned}$$

which is absurd.

(2) Show that the pole set of z^2 is empty. Note that $z^2 = \overline{y^2}/\overline{x^2} = \overline{x+1}$. So the pole set of z^2 is

$$\bigcap_{z^2=\bar{f}/\bar{g}} V(\bar{g}) = \emptyset.$$

□

Problem 2.18.

Let $\mathcal{O}_P(V)$ be the local ring of a variety V at a point P . Show that there is a natural one-to-one correspondence between the prime ideals in $\mathcal{O}_P(V)$ and the subvarieties of V that pass through P . (Hint: If I is prime in $\mathcal{O}_P(V)$, $I \cap \Gamma(V)$

is prime in $\Gamma(V)$, and I is generated by $I \cap \Gamma(V)$; use Problem 2.2.)

Proof.

- (1) Write $P = (a_1, \dots, a_n)$ and $\mathfrak{m} := (x_1 - a_1, \dots, x_n - a_n)$. It suffices to show that there is a natural one-to-one correspondence between the prime ideals in $\mathcal{O}_P(V)$ and prime ideals in $\Gamma(V)$ which is contained in $I(V(P)) = \mathfrak{m}$ by Problem 2.2.
- (2) If \mathfrak{p} is prime in $\mathcal{O}_P(V)$, $\mathfrak{p} \cap \Gamma(V)$ is prime in $\Gamma(V)$ since $\Gamma(V)$ is a subring of $\mathcal{O}_P(V)$. Note that $\mathfrak{p} \subseteq \mathfrak{m}_P(V)$ and thus

$$\mathfrak{p} \cap \Gamma(V) \subseteq \mathfrak{m}_P(V) \cap \Gamma(V) = (x_1 - a_1, \dots, x_n - a_n).$$

- (3) Conversely, if \mathfrak{q} is prime in $\Gamma(V)$ which is contained in \mathfrak{m} then we need to show that $\mathfrak{p} := \mathfrak{q}\mathcal{O}_P(V)$ is prime in $\mathcal{O}_P(V)$.
- (4) Note that \mathfrak{p} is proper (since $\mathfrak{q} \subseteq \mathfrak{m}$). Suppose $\frac{a}{b} \frac{c}{d} \in \mathfrak{p}$ with $b(P) \neq 0$ and $d(P) \neq 0$. Hence

$$ac = \frac{a}{b} \frac{c}{d} \cdot (bd) \in \mathfrak{p} \cap \Gamma(V) = \mathfrak{q}.$$

By the primality of \mathfrak{q} , might assume that $a \in \mathfrak{q}$. (The case $c \in \mathfrak{q}$ is the same.) So that $\frac{a}{b} = a \cdot \frac{1}{b} \in \mathfrak{q}\mathcal{O}_P(V) = \mathfrak{p}$. Therefore, \mathfrak{p} is prime.

□

Problem 2.19.

Let f be a rational function on a variety V . Let

$$U = \{P \in V : f \text{ is defined at } P\}.$$

Then f defines a function from U to k . Show that this function determines f uniquely. So a rational function may be considered as a type of function, but only on the complement of an algebraic subset of V , not on V itself.

Proof.

- (1) Write $f = a/b \in k(V)$ with $b(P) \neq 0$. Define $f : U \rightarrow k$ by $f : P \mapsto f(P) = a(P)/b(P)$.
- (2) Show that this function is well-defined. Given any $P \in U$. Suppose that $f = a/b = c/d$ with $b(P) \neq 0$ and $d(P) \neq 0$. So, $ad = bc \in \Gamma(V)$ implies that $a(P)d(P) = b(P)c(P)$. So, $a(P)/b(P) = c(P)/d(P)$ (since $b(P) \neq 0$ and $d(P) \neq 0$). Therefore, $f : U \rightarrow k$ is well-defined.

□

Problem 2.20. (Quadric surface)

Let

$$V = V(xw - yz) \subseteq \mathbf{A}^4(k),$$

and

$$\Gamma(V) = k[x, y, z, w]/(xw - yz).$$

Let $\bar{x}, \bar{y}, \bar{z}, \bar{w}$ be the residues of x, y, z, w in $\Gamma(V)$. Then

$$\bar{x}/\bar{y} = \bar{z}/\bar{w} = f \in k(V)$$

is defined at $P = (x, y, z, w) \in V$ if $y \neq 0$ or $w \neq 0$. Show that it is impossible to write $f = a/b$, where $a, b \in \Gamma(V)$, and $b(P) \neq 0$ for every P where f is defined. Show that the pole set of f is exactly $\{(x, y, z, w) : y = 0 \text{ and } w = 0\}$.

Proof.

- (1) Note that the pole set of f is

$$\bigcap_{f=\bar{a}/\bar{b}} V(\bar{b}) \subseteq \{(x, y, z, w) \in \mathbf{A}^4(k) : y = w = 0\}.$$

- (2) Show that f is not defined at the origin $O = (0, 0, 0, 0)$. (Reductio ad absurdum) Suppose $f = \bar{x}/\bar{y} = \bar{a}/\bar{b}$ with $\bar{b}(O) \neq 0$. So $\bar{b}x - \bar{a}y = 0$, or

$$bx - ay = g(xw - yz)$$

for some $g \in k[x, y, z, w]$. Take 1-forms on the both sides to get

$$b(O)x - a(O)y = 0 \in k[x, y, z, w],$$

or $a(O) = b(O) = 0$, which is absurd.

- (3) Show that it is impossible to write $f = \bar{a}/\bar{b}$, where $\bar{a}, \bar{b} \in \Gamma(V)$, and $\bar{b}(P) \neq 0$ for every P where f is defined. (Reductio ad absurdum) Consider the polynomial

$$\beta(y, w) := b(0, y, 0, w) \in k[y, w].$$

β is not a constant polynomial since $V(\beta) = \{(0, 0)\}$ by (1)(2). By Problem 1.14, $V(\beta) = \{(0, 0)\}$ is infinite, which is absurd.

- (4) Show that the pole set of f is exactly $\{(x, y, z, w) : y = 0 \text{ and } w = 0\}$. (Reductio ad absurdum) Given any $P = (x_0, 0, z_0, 0) \in \{(x, y, z, w) : y = 0 \text{ and } w = 0\}$. Suppose $f = \bar{x}/\bar{y} = \bar{a}/\bar{b}$ where $\bar{b}(P) \neq 0$. Similar to (2),

$$bx - ay = g(xw - yz)$$

for some $g \in k[x, y, z, w]$. So

$$b(P)x_0 = b(P)x_0 - a(P) \cdot 0 = g(P)(x_0 \cdot 0 - 0 \cdot z_0) = 0.$$

As $b(P) \neq 0$, $x_0 = 0$. Similarly, $z_0 = 0$ by noting that $bz - aw = h(xw - yz)$ for some $h \in k[x, y, z, w]$. Hence $P = (0, 0, 0, 0)$, contrary to (2).

□

Note. It is equal to the Segre embedding of $\mathbf{P}^1 \times \mathbf{P}^1$ in \mathbf{P}^3 , for suitable choice of coordinates.

Problem 2.21.*

Let $\varphi : V \rightarrow W$ be a polynomial map of affine varieties, $\tilde{\varphi} : \Gamma(W) \rightarrow \Gamma(V)$ the induced map on coordinate rings. Suppose $P \in V$, $\varphi(P) = Q$. Show that $\tilde{\varphi}$ extends uniquely to a ring homomorphism (also written $\tilde{\varphi}$) from $\mathcal{O}_Q(W)$ to $\mathcal{O}_P(V)$. (Note that $\tilde{\varphi}$ may not extend to all of $k(W)$.) Show that $\tilde{\varphi}(\mathfrak{m}_Q(W)) \subseteq \mathfrak{m}_P(V)$.

Proof.

- (1) Define $\tilde{\varphi} : \mathcal{O}_Q(W) \rightarrow \mathcal{O}_P(V)$ by

$$\tilde{\varphi} : a/b \mapsto \tilde{\varphi}(a)/\tilde{\varphi}(b).$$

It is well-defined since $b(Q) \neq 0$ implies that

$$\tilde{\varphi}(b)(P) = b(\varphi(P)) = b(Q) \neq 0.$$

- (2) Note that $\tilde{\varphi}$ may not extend to all of $k(W)$ since $\tilde{\varphi} : k(W) \rightarrow k(V)$ might not be well-defined if $\tilde{\varphi}(b) = 0$ for all $b \in \Gamma(W)$.
- (3) Show that $\tilde{\varphi}(\mathfrak{m}_Q(W)) \subseteq \mathfrak{m}_P(V)$. Take any $a/b \in \mathfrak{m}_Q(W)$ with $a(Q) = 0$ and $b(Q) \neq 0$. As

$$\tilde{\varphi}(a)(P) = a(\varphi(P)) = a(Q) = 0,$$

we have $\tilde{\varphi}(a/b) \in \mathfrak{m}_P(V)$.

□

Problem 2.22.*

Let $t : \mathbf{A}^n \rightarrow \mathbf{A}^n$ be an affine change of coordinates, $t(P) = Q$. Show that $\tilde{t} : \mathcal{O}_Q(\mathbf{A}^n) \rightarrow \mathcal{O}_P(\mathbf{A}^n)$ is an isomorphism. Show that \tilde{t} induces an isomorphism from $\mathcal{O}_Q(V)$ to $\mathcal{O}_P(V^t)$ if $P \in V^t$, for V a subvariety of \mathbf{A}^n .

Proof.

- (1) Since $\tilde{t} : \Gamma(\mathbf{A}^n) \rightarrow \Gamma(\mathbf{A}^n)$ is a ring isomorphism, it extends uniquely to a ring isomorphism (also written \tilde{t}) from $\mathcal{O}_Q(\mathbf{A}^n)$ to $\mathcal{O}_P(\mathbf{A}^n)$ by Problem 2.21.

- (2) Note that $\mathcal{O}_Q(V) \hookrightarrow \mathcal{O}_Q(\mathbf{A}^n)$, $\mathcal{O}_P(V^t) \hookrightarrow \mathcal{O}_P(\mathbf{A}^n)$, and $\tilde{t}(\mathcal{O}_Q(V)) = \mathcal{O}_P(V^t)$, $\tilde{t}: \mathcal{O}_Q(V) \rightarrow \mathcal{O}_P(V^t)$ is an isomorphism.

□

2.5. Discrete Valuation Rings

Problem 2.23.*

Show that the order function on K is independent of the choice of uniformizing parameter.

Proof.

- (1) Show that a uniformizing parameter is unique up to a unit. Suppose t and t' are two uniformizing parameters for a discrete valuation ring R with the quotient field K . Since R is a DVR, the maximal ideal is

$$\mathfrak{m} = (t) = (s).$$

As $s \in (t)$, there is an element $a \in R$ such that $s = at$. As s is irreducible (by the maximality of \mathfrak{m}), a is a unit or t is a unit (which is impossible). Hence $s = at$ for some unit $a \in R$.

- (2) For any $z \in K$, write

$$z = ut^n = vs^m$$

for some units u, v and integers $n \geq m$. (The case $n \leq m$ is similar.) Replace $s = at$ to get $ut^n = va^mt^m$. So $t^{n-m} = u^{-1}va^m$ is a unit. Hence, $m = n$, or the order function on K is independent of the choice of uniformizing parameter.

□

Problem 2.24.*

Let $V = \mathbf{A}^1$, $\Gamma(V) = k[x]$, $K = k(V) = k(x)$.

- (a) For each $a \in k = V$, show that $\mathcal{O}_a(V)$ is a DVR with uniformizing parameter $t = x - a$.
- (b) Show that $\mathcal{O}_\infty = \{f/g \in k(x) : \deg(g) \geq \deg(f)\}$ is also a DVR, with uniformizing parameter $t = 1/x$.

Proof of (a).

- (1) By Proposition 7 in §2.4, $\mathcal{O}_a(V)$ is a (Noetherian) local domain. It suffices to show that $t = x - a$ is an irreducible element in $\mathcal{O}_a(V)$ such that every nonzero $z \in \mathcal{O}_a(V)$ might be written uniquely in the form $z = ut^n$, u a unit in $\mathcal{O}_a(V)$, n a nonnegative integer (by Proposition 4).
- (2) Write $z = f/g \in \mathcal{O}_a(V)$ where $g(a) \neq 0$. By Problem 1.7,

$$f = \sum_{i=0}^{\deg(f)} \lambda_i (x - a)^i.$$

Let n be the smallest integer such that $\lambda_n \neq 0$. (Such n is existed since z or f is nonzero.) Hence, $f = f_1(x - a)^n$ where $f_1 = \sum_{i=n}^{\deg(f)} \lambda_i (x - a)^{i-n} \neq 0$ and $f_1(a) = \lambda_n \neq 0$. So

$$z = f/g = (f_1/g)(x - a)^n.$$

Here f_1/g is a unit in $\mathcal{O}_a(V)$. Besides, it is easy to show that n is unique by the similar argument in Problem 2.23. Hence, $\mathcal{O}_a(V)$ is a DVR with uniformizing parameter $t = x - a$.

□

Proof of (b).

- (1) Show that \mathcal{O}_∞ is a subring of $k(x)$. Clearly, $1 = 1/1 \in \mathcal{O}_\infty$. Also, given any $f = a/b, g = c/d \in \mathcal{O}_\infty$. So

$$\begin{aligned} f - g &= a/b - c/d = \frac{ad - bc}{bd} \in \mathcal{O}_\infty \\ fg &= a/b \cdot c/d = \frac{ac}{bd} \in \mathcal{O}_\infty \end{aligned}$$

since

$$\begin{aligned} \deg(ad - bc) &\leq \max(\deg(ad), \deg(bc)) \\ &\leq \max(\deg(a) + \deg(d), \deg(b) + \deg(c)) \\ &\leq \max(\deg(b) + \deg(d), \deg(b) + \deg(d)) \\ &\leq \deg(b) + \deg(d) \\ &\leq \deg(bd) \end{aligned}$$

and

$$\deg(ac) = \deg(a) + \deg(c) \leq \deg(b) + \deg(d) = \deg(bd).$$

(Here we define $\deg(0) = -\infty$ by convention.) By the subring test, \mathcal{O}_∞ is a subring of $k(x)$.

- (2) Show that \mathcal{O}_∞ is a DVR. Clearly \mathcal{O}_∞ is not a field since $1/x \in \mathcal{O}_\infty$ but $x = x/1 \notin \mathcal{O}_\infty$. Let $t = 1/x$ be an irreducible element of \mathcal{O}_∞ . ($\deg(x) = 1$ implies the irreducibility of t .) Now for any nonzero $f/g \in \mathcal{O}_\infty$, write

$$f/g = ((fx^n)/g)(1/x^n) = ((fx^n)/g)t^n$$

where $n := \deg(g) - \deg(f) \geq 0$. Note that $\deg(fx^n) = \deg(f) + n = \deg(g)$. So $(fx^n)/g$ is a unit since the inverse $g/(fx^n)$ is also in \mathcal{O}_∞ . Besides, it is easy to show that n is unique by the similar argument in Problem 2.23. Hence, \mathcal{O}_∞ is a DVR.

□

Note.

- (1) The quotient field of \mathcal{O}_∞ is $K = k(V) = k(x)$.
- (2) The set of units in $\mathcal{O}_\infty(V)$ is $\{f/g \in k(x) : \deg(g) = \deg(f)\}$.
- (3) The maximal ideal of $\mathcal{O}_\infty(V)$ is $\{f/g \in k(x) : \deg(g) > \deg(f)\}$.

Problem 2.25. (p -adic integers)

Let $p \in \mathbb{Z}$ be a prime number. Show that

$$\{r \in \mathbb{Q} : r = a/b, a, b \in \mathbb{Z}, p \text{ doesn't divide } b\}$$

is a DVR with quotient field \mathbb{Q} .

Proof.

- (1) Let

$$\mathbb{Z}_p = \{r \in \mathbb{Q} : r = a/b, a, b \in \mathbb{Z}, p \nmid b\}$$

be the set of all p -adic integers.

- (2) Show that \mathbb{Z}_p is a subring of \mathbb{Q} . Clearly, $1 = 1/1 \in \mathbb{Z}_p$ (since $p \nmid 1$). Also, given any $r = a/b, s = c/d \in \mathbb{Z}_p$. So

$$\begin{aligned} r - s &= a/b - c/d = \frac{ad - bc}{bd} \in \mathbb{Z}_p \\ rs &= a/b \cdot c/d = \frac{ac}{bd} \in \mathbb{Z}_p \end{aligned}$$

since $p \nmid b, p \nmid d$ and p is a prime number. By the subring test, \mathbb{Z}_p is a subring of \mathbb{Q} .

- (3) Note that $\mathbb{Z}_p \subseteq \mathbb{Q}$ is a domain and \mathbb{Z}_p is not a field (since $p = p/1 \in \mathbb{Z}_p$ but $p^{-1} = 1/p \notin \mathbb{Z}_p$).

- (4) Let $t = p$ be an irreducible element in \mathbb{Z}_p . For the irreducibility of $t = p$, we write $p = a/b \cdot c/d = \frac{ac}{bd}$ where $p \nmid b, p \nmid d$. So $pbd = ac$ or

$$1 = \text{ord}_p(ac) = \text{ord}_p(a) + \text{ord}_p(c).$$

Here $\text{ord}_p : \mathbb{Z} \rightarrow \mathbb{Z}_{\geq 0}$ is defined by $\text{ord}_p(a) = n$ where n is the largest number such that p^n divides a , that is, $p^n \mid a$ and $p^{n+1} \nmid a$. So $(\text{ord}_p(a), \text{ord}_p(c)) = (0, 1)$ or $(1, 0)$. Hence, a/b or c/d is a unit in \mathbb{Z}_p , or p is irreducible in \mathbb{Z}_p .

- (5) For any nonzero $r = a/b \in \mathbb{Z}_p$, $a \neq 0$ can be written as $a = p^n c$ for some nonnegative integer n and $c \in \mathbb{Z}^+$ uniquely. Hence

$$r = a/b = (c/b)p^n = (c/b)t^n,$$

where c/b is a unit and n is a nonnegative integer. Besides, it is easy to show that n is unique by the similar argument in Problem 2.23. By Proposition 4, \mathbb{Z}_p is a DVR.

- (6) Show that the quotient field of \mathbb{Z}_p is \mathbb{Q} . It suffices to show that r is in the quotient field of \mathbb{Z}_p if $r \in \mathbb{Q} - \mathbb{Z}_p$. Note that $r \neq 0$. Write $r = a/b$ with $\gcd(a, b) = 1$. As $r \notin \mathbb{Z}_p$, $p \mid b$ and $p \nmid a$. Therefore, $1/r = b/a \in \mathbb{Z}_p$, or r is in the quotient field of \mathbb{Z}_p .

□

Note.

- (1) $p\mathbb{Z}_p$ is the maximal ideal of \mathbb{Z}_p .
- (2) The residue field $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$.

Problem 2.26.*

Let R be a DVR with quotient field K ; let \mathfrak{m} be the maximal ideal of R .

- (a) Show that if $z \in K$, $z \notin R$, then $z^{-1} \in \mathfrak{m}$.
- (b) Suppose $R \subseteq S \subseteq K$, and S is also a DVR. Suppose the maximal ideal of S contains \mathfrak{m} . Show that $S = R$.

Proof of (a).

- (1) Suppose t is one uniformizing parameter for R . If $z \in K - R$, then we can write $z = ut^{-n}$ for some unit $u \in R$ and $n \in \mathbb{Z}^+$.
- (2) Hence,

$$z^{-1} = u^{-1}t^n.$$

Since u^{-1} is a unit in R and $n > 0$, $z^{-1} \in \mathfrak{m}$.

□

Proof of (b).

- (1) (Reductio ad absurdum) Suppose $z \in S - R \subseteq K - R$. By (a), $z^{-1} \in \mathfrak{m}$. So z^{-1} is in the maximal ideal \mathfrak{m}' of S containing \mathfrak{m} .
- (2) As \mathfrak{m}' is an ideal, $1 = z \cdot z^{-1} \in \mathfrak{m}'$, which is absurd. Therefore, $S = R$.

□

Problem 2.27.

Show that the DVR's of Problem 2.24 are the only DVR's with quotient field $k(x)$ that contain k . Show that those of Problem 2.25 are the only DVR's with quotient field \mathbb{Q} .

Proof (Problem 2.26).

- (1) *Show that $\mathcal{O}_a(V)$ and \mathcal{O}_∞ are the only DVR's with quotient field $k(x)$ that contain k .*
 - (a) Let $k \subseteq R \subsetneq k(x)$ be a DVR with quotient field $k(x)$, \mathfrak{m} be the unique maximal ideal of R . $\mathfrak{m} \neq (0)$ and the set of units in R is $R - \mathfrak{m}$.
 - (b) There are two possible cases: $x \in R$ or $x \notin R$.
 - (c) Suppose $x \in R$. So R contains $k[x]$ as a subring. Consider the subset

$$S := \{x - a \in k[x] : a \in k\} \cap \mathfrak{m} \subseteq \mathfrak{m}.$$

Suppose there were two distinct elements $x - a, x - b \in S$. Then $1 \in \mathfrak{m}$, contrary to the maximality of \mathfrak{m} . Suppose $S = \emptyset$, then every $x - a$ is a unit in R . Since $k = \bar{k}$, $R = k(x)$ is a field, which is absurd. Hence, there is only one $x - a \in \mathfrak{m}$ for one unique $a \in k$ and other $x - b$ with $b \neq a$ is a unit in R . Thus, $R \supseteq \mathcal{O}_a(V)$ and \mathfrak{m} contains $(x - a)\mathcal{O}_a(V)$, which is the maximal ideal of $\mathcal{O}_a(V)$. By Problem 2.26, $R = \mathcal{O}_a(V)$.

- (d) If $x \notin R$, then $x - a \notin R$ whenever $a \in k \subseteq R$. Hence $(x - a)^{-1} \in \mathfrak{m}$ whenever $a \in k$ by Problem 2.26(a). Next, given any $f/g \in \mathcal{O}_\infty$, by $k = \bar{k}$ we have

$$f/g = \underbrace{u}_{\in k} \underbrace{\frac{x - \alpha_1}{x - \beta_1}}_{\in R} \cdots \underbrace{\frac{x - \alpha_n}{x - \beta_n}}_{\in R} \underbrace{\frac{1}{x - \beta_{n+1}}}_{\in \mathfrak{m}} \cdots \underbrace{\frac{1}{x - \beta_m}}_{\in \mathfrak{m}},$$

where $n := \deg(f)$, $m := \deg(g)$ and $n \leq m$. Here

$$\frac{x - \alpha_i}{x - \beta_i} = \underbrace{1}_{\in k} + \underbrace{\frac{\beta_i - \alpha_i}{x - \beta_i}}_{\in \mathfrak{m} \subseteq R} \in R.$$

Therefore, $R \supseteq \mathcal{O}_\infty$ and \mathfrak{m} contains the maximal ideal $x^{-1}\mathcal{O}_\infty$ of \mathcal{O}_∞ . By Problem 2.26, $R = \mathcal{O}_\infty$.

(2) Show that \mathbb{Z}_p are the only DVR's with quotient field \mathbb{Q} .

- (a) Let $R \subsetneq \mathbb{Q}$ be a DVR with quotient field \mathbb{Q} , \mathfrak{m} be the unique maximal ideal of R . $\mathfrak{m} \neq (0)$ and the set of units in R is $R - \mathfrak{m}$.
- (b) Note that $R \subseteq \mathbb{Q}$ contains \mathbb{Z} as a subring. Consider the subset

$$S := \{p \in \mathbb{Z} : p \text{ is a prime number}\} \cap \mathfrak{m} \subseteq \mathfrak{m}.$$

- (c) Suppose there were two distinct prime integers $p, q \in S$. By the Bézout's identity, there exist integers a and b such that $pa + qb = 1$. $1 \in \mathfrak{m}$, contrary to the maximality of \mathfrak{m} .
- (d) Suppose no prime integer were in S , then every prime integer is a unit in R . By the fundamental theorem of arithmetic, $R = \mathbb{Q}$ is a field, which is absurd.
- (e) By (c)(d), $p \in \mathfrak{m}$ for one unique prime $p \in \mathbb{Z}$. Thus, $R \supseteq \mathbb{Z}_p$ by the definition of \mathbb{Z}_p and \mathfrak{m} contains $p\mathbb{Z}_p$, which is the maximal ideal of \mathbb{Z}_p . By Problem 2.26, $R = \mathbb{Z}_p$.

□

Problem 2.28.*

An order function on a field K is a function φ from K onto $\mathbb{Z} \cup \{\infty\}$, satisfying:

- (i) $\varphi(a) = \infty$ if and only if $a = 0$.
- (ii) $\varphi(ab) = \varphi(a) + \varphi(b)$.
- (iii) $\varphi(a + b) \geq \min(\varphi(a), \varphi(b))$.

Show that $R = \{z \in K : \varphi(z) \geq 0\}$ is a DVR with maximal ideal $\mathfrak{m} = \{z \in K : \varphi(z) > 0\}$, and quotient field K . Conversely, show that if R is a DVR with quotient field K , then the function $\text{ord} : K \rightarrow \mathbb{Z} \cup \{\infty\}$ is an order function on K . Giving a DVR with quotient field K is equivalent to defining an order function on K .

Proof.

- (1) Show that $\varphi(1) = 0$. Note that $\varphi(1) = \varphi(1 \cdot 1) = \varphi(1) + \varphi(1)$ by (ii). By Property (i) of φ , we cancel $\varphi(1) \in \mathbb{Z}$ on the both side to get $\varphi(1) = 0$.

- (2) Show that $\varphi(-z) = \varphi(z)$ for all $z \in K$, and $\varphi(z^{-1}) = -\varphi(z)$ for all $z \in K - \{0\}$. Note that $\varphi(-1) = 0$ since $0 = \varphi(1) = \varphi((-1) \cdot (-1)) = \varphi(-1) + \varphi(-1)$ (by (1)). Therefore,

$$\varphi(-z) = \varphi((-1) \cdot z) = \varphi(-1) + \varphi(z) = \varphi(z).$$

Besides,

$$0 = \varphi(1) = \varphi(zz^{-1}) = \varphi(z) + \varphi(z^{-1})$$

if $z \neq 0$. So $\varphi(z^{-1}) = -\varphi(z)$ if $z \neq 0$.

- (3) Show that $R = \{z \in K : \varphi(z) \geq 0\}$ is a ring.

(a) $R \neq \emptyset$ since $1 \in R$.

(b) If $a, b \in R$, then

$$\varphi(a - b) \geq \min(\varphi(a), \varphi(-b)) = \min(\varphi(a), \varphi(b)) \geq 0$$

(by (2)), or $a - b \in R$.

(c) If $a, b \in R$, then $\varphi(ab) = \varphi(a) + \varphi(b) \geq 0$.

By the subring test, R is a subring of K .

- (4) Show that $\{z \in K - \{0\} : \varphi(z) = 0\}$ is the set of all units in R . Given any $z \in K - \{0\}$, we have

$$0 = \varphi(z) + \varphi(z^{-1})$$

(by (2)). Hence z is a unit in R iff $z, z^{-1} \in R$ iff $\varphi(z) = \varphi(z^{-1}) = 0$.

- (5) Show that $\mathfrak{m} = \{z \in K : \varphi(z) > 0\}$ is a maximal ideal of R .

(a) If $a, b \in \mathfrak{m}$, then $\varphi(a + b) \geq \min(\varphi(a), \varphi(b)) > 0$.

(b) If $a \in \mathfrak{m}$ and $r \in R$, then $\varphi(ra) = \varphi(r) + \varphi(a) \geq \varphi(a) > 0$.

(c) By (a)(b), \mathfrak{m} is an ideal of R .

(d) Note that each proper ideal in R does not have any unit, that is, such proper ideal is contained in $\{z \in K : \varphi(z) > 0\} = \mathfrak{m}$ exactly (by (4)). Therefore, \mathfrak{m} is maximal. (Such maximal ideal \mathfrak{m} is unique and thus R is a local ring.)

- (6) Show that R is a DVR. It suffices to show that there is an irreducible element $t \in R$ such that every nonzero $z \in R$ may be written uniquely in the form $z = ut^n$, u a unit in R , n a nonnegative integer. Since φ is surjective, there is an element $t \in R$ such that $\varphi(t) = 1$. Note that $t \neq 0$ and irreducible (by using Property (ii) of φ). Hence for any nonzero $z \in R$ with $n := \varphi(z) \in \mathbb{Z}$ and $n \geq 0$, the order of $zt^{-n} \in K$ is

$$\varphi(zt^{-n}) = \varphi(z) - n\varphi(t) = n - n \cdot 1 = 0$$

(by (2)). That is, $zt^{-n} = u$ is a unit in R (by (4)). Hence $z = ut^n$ for some unit $u \in R$ and nonnegative integer n . Note that n is uniquely determined by $\varphi(z)$. By Proposition 4, R is a DVR.

- (7) Show that the quotient field of R is K . Since R is a DVR, the quotient field of R is contained in K . Conversely, given any $z \in K$. If $\varphi(z) \geq 0$, then $z \in R \subseteq K$. If $\varphi(z) < 0$, then $\varphi(z^{-1}) = -\varphi(z) > 0$ or $z^{-1} \in R$. Hence $z = 1/z^{-1} \in K$ is in the quotient field of R .
- (8) Show that giving a DVR with quotient field K is equivalent to defining an order function on K . It suffices to show that $\text{ord}(\cdot)$ on K defines an order function φ on K . By Problem 2.29, it suffices to show that

$$\text{ord}(a + b) \geq \min(\text{ord}(a), \text{ord}(b))$$

if $\text{ord}(a) = \text{ord}(b) := n$. Write $a = ut^n, b = vt^n$ where u, v are units in R . Hence,

$$\begin{aligned} \text{ord}(a + b) &= \text{ord}(ut^n + vt^n) \\ &= \text{ord}((u + v)t^n) \\ &= \text{ord}(u + v) + n \\ &\geq n && (u + v \in R) \\ &= \min(\text{ord}(a), \text{ord}(b)). \end{aligned}$$

□

Problem 2.29.*

Let R be a DVR with quotient field K , ord the order function on K .

- (a) If $\text{ord}(a) < \text{ord}(b)$, show that $\text{ord}(a + b) = \text{ord}(a)$.
- (b) If $a_1, \dots, a_n \in K$, and for some i , $\text{ord}(a_i) < \text{ord}(a_j)$ (all $j \neq i$), then $a_1 + \dots + a_n \neq 0$.

Proof of (a).

- (1) Let t be a uniformizing parameter for R . Given any $a, b \in K$. Write $a = ut^n, b = vt^m$ where u, v are units in R and n, m are integers.
- (2) Since $\text{ord}(a) < \text{ord}(b)$, $n < m$. Hence,

$$a + b = (u + vt^{m-n})t^n.$$

To show that $\text{ord}(a + b) = \text{ord}(a) = n$, it suffices to show that $u + vt^{m-n}$ is a unit in R .

- (3) (Reductio ad absurdum) Suppose that $u + vt^{m-n}$ were not a unit. Since R is local, the maximal ideal (t) contains all nonunit elements in R . Hence, $u + vt^{m-n} \in (t)$. As $m - n > 0$, $vt^{m-n} \in (t)$ and thus a unit $u \in (t)$, contrary to the maximality of (t) .

□

Proof of (b).

- (1) Might assume that $\text{ord}(a_1) < \text{ord}(a_j)$ (all $j \neq 1$). In particular, $\text{ord}(a_1) < \infty$.
- (2) Similar to (a). Let t be a uniformizing parameter for R . Write $a_i = u_i t^{m_i}$ where u_i are units in R and m_i are integers. ($i = 1, \dots, n$.) Since $\text{ord}(a_1) < \text{ord}(a_j)$ (all $j \neq 1$), $m_1 < m_j$. Hence,

$$a_1 + \cdots + a_n = (u_1 + \underbrace{u_2 t^{m_2 - m_1} + \cdots + u_n t^{m_n - m_1}}_{\in (t)}) t^{m_1}.$$

So $u_1 + u_2 t^{m_2 - m_1} + \cdots + u_n t^{m_n - m_1}$ is a unit in R .

- (3) By (1)(2),

$$\text{ord}(a_1 + \cdots + a_n) = \text{ord}(a_1) < \infty,$$

or $a_1 + \cdots + a_n \neq 0$ (since ord is an order function on K).

□

Problem 2.30.*

Let R be a DVR with maximal ideal \mathfrak{m} , and quotient field K , and suppose a field k is a subring of R , and that the composition $k \rightarrow R \rightarrow R/\mathfrak{m}$ is an isomorphism of k with R/\mathfrak{m} (as for example in Problem 2.24). Verify the following assertions:

- (a) For any $z \in R$, there is a unique $\lambda \in k$ such that $z - \lambda \in \mathfrak{m}$.
- (b) Let t be a uniformizing parameter for R , $z \in R$. Then for any $n \geq 0$ there are unique $\lambda_0, \lambda_1, \dots, \lambda_n \in k$ and $z_n \in R$ such that

$$z = \lambda_0 + \lambda_1 t + \lambda_2 t^2 + \cdots + \lambda_n t^n + z_n t^{n+1}.$$

(Hint: For uniqueness use Problem 2.29; for existence use (a) and induction.)

Proof of (a).

- (1) Note that

$$k \xrightarrow{i} R \xrightarrow{\pi} R/\mathfrak{m}$$

is an isomorphism.

- (2) For $z + \mathfrak{m} \in R/\mathfrak{m}$, there exists the unique $\lambda \in k$ such that

$$z + \mathfrak{m} = \pi(i(\lambda)) = \pi(\lambda) = \lambda + \mathfrak{m}.$$

So $z - \lambda \in \mathfrak{m}$ for one unique $\lambda \in k$.

□

Proof of (b).

(1) Note that

$$\mathfrak{m} = \{z \in K : \text{ord}(z) > 0\}.$$

By (a),

$$z = \lambda_0 + \underbrace{tz_0}_{\in \mathfrak{m}}$$

for one unique $\lambda_0 \in k$ and $z_0 \in R$. Continue this process or by induction, we have the expression

$$z = \lambda_0 + \lambda_1 t + \lambda_2 t^2 + \cdots + \lambda_n t^n + z_n t^{n+1}.$$

(2) For the uniqueness, suppose

$$0 = \lambda_0 + \lambda_1 t + \lambda_2 t^2 + \cdots + \lambda_n t^n + z_n t^{n+1}.$$

Note that

$$\text{ord}(\lambda_i t^i) = \begin{cases} \infty & (\lambda_i = 0) \\ i & (\lambda_i \neq 0) \end{cases}$$

since every nonzero element in k is a unit in $k \subseteq R$. Also, $\text{ord}(z_n t^{n+1}) = \infty$ if $z_n = 0$; $\text{ord}(z_n t^{n+1}) \geq n+1$ if $z_n \neq 0$.

(3) Suppose i_0 is the smallest integer such that $\lambda_{i_0} \neq 0$, then $\text{ord}(\lambda_{i_0} t^{i_0}) = i_0 < \text{ord}(\lambda_j t^j)$ if $i_0 \neq j$ and $\text{ord}(\lambda_{i_0} t^{i_0}) = i_0 < n+1 \leq \text{ord}(z_n t^{n+1})$. By Problem 2.29(b), such i_0 does not exist. Hence all $\lambda_i = 0$. So as R is a domain, z_n is also equal to 0. Therefore, the uniqueness is established.

□

Problem 2.31. (Formal power series)

Let k be a field. The ring of **formal power series** over k , written $k[[x]]$, is defined to be

$$\left\{ \sum_{i=0}^{\infty} a_i x^i : a_i \in k \right\}.$$

(As with polynomials, a rigorous definition is best given in terms of sequences (a_0, a_1, \dots) of elements in k ; here we allow an infinite number of nonzero terms.) Define the sum by

$$\sum a_i x^i + \sum b_i x^i = \sum (a_i + b_i) x^i,$$

and the (Cauchy) product by

$$\left(\sum a_i x^i\right) \left(\sum b_i x^i\right) = \sum c_i x^i,$$

where $c_i = \sum_{j+k=i} a_j b_k$. Show that $k[[x]]$ is a ring containing $k[x]$ as a subring. Show that $k[[x]]$ is a DVR with uniformizing parameter x . Its quotient field is denoted $k((x))$.

Proof.

- (1) Two formal power series $\sum a_i x^i$ and $\sum b_i x^i$ in $k[[x]]$ are considered equal if $a_i = b_i$ for all integers $i \geq 0$.
- (2) The zero element in $k[[x]]$ is $0 = \sum_{i=0}^{\infty} 0x^i$, and the multiplicative identity is

$$1 = 1 + 0x + \cdots + 0x^n + \cdots.$$

Hence, $k[[x]]$ is a ring (by a tedious argument). Moreover, $k[[x]]$ is a domain (again by a tedious argument).

- (3) Show that $k[[x]] \supseteq k[x]$. In fact, for any $f = \sum_{i=0}^n a_i x^i \in k[x]$, we can write

$$f = a_0 + a_1 x + \cdots + a_n x^n + 0x^{n+1} + \cdots \in k[[x]].$$

- (4) Show that $f = \sum_{i=0}^{\infty} a_i x^i$ is a unit in $k[[x]]$ if and only if $a_0 \neq 0$. Suppose $g = \sum_{i=0}^{\infty} b_i x^i \in k[[x]]$ such that $fg = 1$. Then

$$\begin{aligned} 1 &= a_0 b_0, \\ 0 &= \sum_{j=0}^k a_j b_{k-j}. \end{aligned}$$

So f is not a unit in $k[[x]]$ if $a_0 = 0$. Now if $a_0 \neq 0$ then $b_0 := a_0^{-1} \in k$. Then by observing that

$$\begin{aligned} 0 &= \sum_{j=0}^k a_j b_{k-j} \iff a_0 b_k = - \sum_{j=1}^k a_j b_{k-j} \\ &\iff b_k = -b_0 \sum_{j=1}^k a_j b_{k-j}, \end{aligned}$$

we can solve b_1, b_2, \dots by induction, and (b_0, b_1, \dots) gives the existence of $g \in k[[x]]$.

- (5) By (4), $k[[x]]$ is not a field since $x \in k[[x]]$ but $x^{-1} \notin k[[x]]$. Let $t = x$ be an irreducible element in $k[[x]]$. ($\deg(x) = 1$ implies the irreducibility of t .) Hence every nonzero $f \in k[[x]]$ can be written uniquely in the form

$$f = ux^n$$

where n is the smallest integer such that $a_n \neq 0$. By (4),

$$u = a_n + a_{n+1}x + \cdots$$

is a unit in $k[[x]]$ as $a_n \neq 0$. Besides, it is easy to show that n is unique by the similar argument in Problem 2.23. Therefore, $k[[x]]$ is a DVR with uniformizing parameter x .

□

Problem 2.32. (Power series expansion)

Let R be a DVR satisfying the conditions of Problem 2.30. Any $z \in R$ then determines a power series $\sum \lambda_i x^i$, if $\lambda_0, \lambda_1, \dots$ are determined as in Problem 2.30(b).

- (a) Show that the map $z \rightarrow \sum \lambda_i x^i$ is a one-to-one ring homomorphism of R into $k[[x]]$. We often write $z = \sum \lambda_i t^i$, and call this the **power series expansion** of z in terms of t .
- (b) Show that the homomorphism extends to a homomorphism of K into $k((x))$, and that the order function on $k((x))$ restricts to that on K .
- (c) Let $a = 0$ in Problem 2.24, $t = x$. Find the power series expansion of $z = (1 - x)^{-1}$ and of $(1 - x)(1 + x^2)^{-1}$ in terms of t .

Proof of (a).

- (1) Define the map $\alpha : R \rightarrow k[[x]]$ by

$$\alpha : z \mapsto \sum_{i=0}^{\infty} \lambda_i x^i$$

where λ_i are determined as in Problem 2.30(b).

- (2) Show that α is well-defined and one-to-one. Write

$$\alpha(z) = \sum_{i=0}^{\infty} \lambda_i x^i = \sum_{i=0}^{\infty} \lambda'_i x^i.$$

If there were $\lambda_n \neq \lambda'_n$ for some n , then Problem 2.30(b) implies that two expressions of z

$$\begin{aligned} z &= \lambda_0 + \lambda_1 t + \cdots + \lambda_n t^n + z_n t^{n+1} \\ &= \lambda'_0 + \lambda'_1 t + \cdots + \lambda'_n t^n + z'_n t^{n+1} \end{aligned}$$

are the same. That is, $\lambda_n = \lambda'_n$, which is absurd. Hence, α is well-defined. Also, $0 = 0 + 0t + 0t^2 + \cdots + 0t^n + 0t^{n+1}$ implies that α is one-to-one.

(3) *Show that α is addition preserving.* Given $a, b \in R$. By Problem 2.30(b),

$$a + b = \lambda_0 + \lambda_1 t + \cdots + \lambda_n t^n + c_n t^{n+1}$$

and

$$\begin{aligned} a &= \mu_0 + \mu_1 t + \cdots + \mu_n t^n + a_n t^{n+1} \\ b &= \nu_0 + \nu_1 t + \cdots + \nu_n t^n + b_n t^{n+1} \end{aligned}$$

for any integer $n \geq 0$. So

$$a + b = \underbrace{(\mu_0 + \nu_0)}_{\in k} + \underbrace{(\mu_1 + \nu_1)}_{\in k} t + \cdots + \underbrace{(\mu_n + \nu_n)}_{\in k} t^n + \underbrace{(a_n + b_n)}_{\in R} t^{n+1}.$$

Since the expression of $a + b$ is unique (by Problem 2.30(b)),

$$\lambda_i = \mu_i + \nu_i$$

for all $i = 0, 1, \dots, n$. Since n is arbitrary, $\lambda_i = \mu_i + \nu_i$ is true for all nonnegative integers. Hence, $\alpha(a + b) = \alpha(a) + \alpha(b)$.

(4) *Show that α is multiplication preserving.* Given $a, b \in R$. By Problem 2.30(b),

$$ab = \lambda_0 + \lambda_1 t + \cdots + \lambda_n t^n + c_n t^{n+1}$$

and

$$\begin{aligned} a &= \mu_0 + \mu_1 t + \cdots + \mu_n t^n + a_n t^{n+1} \\ b &= \nu_0 + \nu_1 t + \cdots + \nu_n t^n + b_n t^{n+1} \end{aligned}$$

for any integer $n \geq 0$. So

$$\begin{aligned} ab &= \underbrace{(\mu_0 \nu_0)}_{\in k} + \underbrace{(\mu_1 \nu_0 + \mu_0 \nu_1)}_{\in k} t + \cdots \\ &\quad + \underbrace{(\mu_n \nu_0 + \mu_{n-1} \nu_1 + \cdots + \mu_1 \nu_{n-1} + \mu_0 \nu_n)}_{\in k} t^n \\ &\quad + \underbrace{(\text{other terms})}_{\in R} t^{n+1}. \end{aligned}$$

Since the expression of $a + b$ is unique (by Problem 2.30(b)),

$$\lambda_i = \sum_{j+k=i} \mu_j \nu_k$$

for all $i = 0, 1, \dots, n$. Since n is arbitrary, $\lambda_i = \sum_{j+k=i} \mu_j \nu_k$ is true for all nonnegative integers. Hence, $\alpha(ab) = \alpha(a)\alpha(b)$.

(5) Show that α is multiplicative identity preserving. Note that

$$1 = \underbrace{1}_{\in k} + \underbrace{0}_{\in k}t + \cdots + \underbrace{0}_{\in k}t^n + \underbrace{0}_{\in k}t^{n+1}$$

for every nonnegative integer n . Hence $\alpha : 1 \mapsto 1 \in k[[x]]$.

(6) By (3)(4)(5), α is a ring homomorphism.

□

Proof of (b).

(1) Define the mapping β from K to $k((x))$ by

$$\beta : a/b \mapsto \alpha(a)/\alpha(b)$$

where $a, b \in R$ and $b \neq 0$.

(2) β is well-defined since:

(a) $\alpha(b) \neq 0$ if $b \neq 0$ by the injectivity of α .

(b) The value of $\beta(a/b)$ is independent of the choice of $a/b \in K$ since α is a ring homomorphism.

(3) Also, β is a ring homomorphism since α is a ring homomorphism.

(4) To show that the order function on $k((x))$ restricts to that on K , it suffices to show that

$$\text{ord}_R(z) = \text{ord}_{k[[x]]}(\alpha(z)).$$

In fact,

$$\begin{aligned} m := \text{ord}_R(z) &\iff z = \lambda_m t^m + \cdots + \lambda_n t^n + z_n t^{n+1} \text{ with } \lambda_m \neq 0 \\ &\iff \alpha(z) = \lambda_m x^m + \cdots \text{ with } \lambda_m \neq 0 \\ &\iff \text{ord}_{k[[x]]}(\alpha(z)) = m. \end{aligned}$$

□

Proof of (c).

(1) In calculus we have

$$(1-x)^{-1} = 1 + x + x^2 + \cdots = \sum_{i=0}^{\infty} x^i$$

for $|x| < 1$. In the ring of formal power series $k[[x]]$, $1-x$ is a unit (by (4) in the proof of Problem 2.31) and satisfies

$$(1-x) \left(\sum_{i=0}^{\infty} x^i \right) = 1 \in k[[x]].$$

Hence, the power expansion of $(1 - x)^{-1}$ is

$$(1 - x)^{-1} = \sum_{i=0}^{\infty} x^i \in k((x)).$$

(2) Note that $1 + x^2$ is a unit in $k[[x]]$ and satisfies

$$(1 + x^2) \left(\sum_{i=0}^{\infty} (-1)^i x^{2i} \right) = 1 \in k[[x]].$$

Hence, the power expansion of $(1 - x)(1 + x^2)^{-1}$ is

$$\begin{aligned} (1 - x) \left(\sum_{i=0}^{\infty} (-1)^i x^{2i} \right) &= \left(\sum_{i=0}^{\infty} (-1)^i x^{2i} \right) - x \left(\sum_{i=0}^{\infty} (-1)^i x^{2i} \right) \\ &= \sum_{i=0}^{\infty} (-1)^i x^{2i} + \sum_{i=0}^{\infty} (-1)^{i+1} x^{2i+1} \\ &= \sum_{i=0}^{\infty} (-1)^i x^i \in k[[x]]. \end{aligned}$$

□

2.6. Forms

Problem 2.33.

Factor $y^3 - 2xy^2 + 2x^2y + x^3$ into linear factors in $\mathbb{C}[x, y]$.

Proof.

- (1) Let $f(x, y) = y^3 - 2xy^2 + 2x^2y + x^3$. Then $f_*(x) = 1 - 2x + 2x^3 + x^3$.
- (2) Solve $f_*(x) = 0$ over \mathbb{C} by WolframAlpha (a computational knowledge engine) to get

$$\begin{aligned} \alpha_1 &= -\frac{2}{3} - \frac{10}{3} \sqrt[3]{\frac{2}{79 - 3\sqrt{249}}} - \frac{1}{3} \sqrt[3]{\frac{79 - 3\sqrt{249}}{2}} \\ \alpha_2 &= -\frac{2}{3} + \frac{5}{3}(1 - \sqrt{3}i) \sqrt[3]{\frac{2}{79 - 3\sqrt{249}}} + \frac{1}{6}(1 + \sqrt{3}i) \sqrt[3]{\frac{79 - 3\sqrt{249}}{2}} \\ \alpha_3 &= -\frac{2}{3} + \frac{5}{3}(1 + \sqrt{3}i) \sqrt[3]{\frac{2}{79 - 3\sqrt{249}}} + \frac{1}{6}(1 - \sqrt{3}i) \sqrt[3]{\frac{79 - 3\sqrt{249}}{2}}. \end{aligned}$$

So $f_*(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$.

(3) Hence,

$$\begin{aligned} f(x, y) &= (f_*)^* \\ &= ((x - \alpha_1)(x - \alpha_2)(x - \alpha_3))^* \\ &= (x - \alpha_1 y)(x - \alpha_2 y)(x - \alpha_3 y). \end{aligned}$$

□

Note. If $f(x, y) = y^3 - 2xy^2 + 2x^2y + 4x^3$, then

$$f(x, y) = (x - \alpha_1 y)(x - \alpha_2 y)(x - \alpha_3 y)$$

where

$$\begin{aligned} \alpha_1 &= -\frac{1}{6} - \frac{7}{6} \sqrt[3]{\frac{1}{37 - 3\sqrt{114}}} - \frac{1}{6} \sqrt[3]{37 - 3\sqrt{114}} \\ \alpha_2 &= -\frac{1}{6} + \frac{7}{12}(1 - \sqrt{3}i) \sqrt[3]{\frac{1}{37 - 3\sqrt{114}}} + \frac{1}{12}(1 + \sqrt{3}i) \sqrt[3]{37 - 3\sqrt{114}} \\ \alpha_3 &= -\frac{1}{6} + \frac{7}{12}(1 + \sqrt{3}i) \sqrt[3]{\frac{1}{37 - 3\sqrt{114}}} + \frac{1}{12}(1 - \sqrt{3}i) \sqrt[3]{37 - 3\sqrt{114}}. \end{aligned}$$

Problem 2.34.

Suppose $f, g \in k[x_1, \dots, x_n]$ are forms of degree $r, r + 1$ respectively, with no common factors (k a field). Show that $f + g$ is irreducible.

Proof.

(1) Suppose $f + g = rs \in k[x_1, \dots, x_n]$. Proposition 5 implies that

$$(f + g)^* = (rs)^* \implies x_{n+1}f + g = r^*s^*.$$

Note that $\deg_{x_{n+1}}(x_{n+1}f + g) = 1$. So $\deg_{x_{n+1}}(r^*) = 0$ or $\deg_{x_{n+1}}(s^*) = 0$. Might assume $\deg_{x_{n+1}}(r^*) = 0$. (The case $\deg_{x_{n+1}}(s^*) = 0$ is similar.)

(2) Since $\deg_{x_{n+1}}(r^*) = 0$, $r^* \mid f$ and $r^* \mid g$. Note that $\deg_{x_{n+1}}(r^*) = 0$ implies that $r^* = r$ is a form in $k[x_1, \dots, x_n]$. Hence r is a common factor of f and g , or r is a constant in $k[x_1, \dots, x_n]$. So $f + g$ is irreducible.

□

Problem 2.35.*

- (a) Show that there are $d + 1$ monomials of degree d in $R[x, y]$, and $1 + 2 + \cdots + (d + 1) = \frac{(d+1)(d+2)}{2}$ monomials of degree d in $R[x, y, z]$.
- (b) Let $V(d, n) = \{\text{forms of degree } d \text{ in } k[x_1, \dots, x_n]\}$, k a field. Show that $V(d, n)$ is a vector space over k , and that the monomials of degree d form a basis. So $\dim V(d, 1) = 1$; $\dim V(d, 2) = d + 1$; $\dim V(d, 3) = \frac{(d+1)(d+2)}{2}$.
- (c) Let ℓ_1, ℓ_2, \dots and m_1, m_2, \dots be sequences of nonzero linear forms in $k[x, y]$, and assume no $\ell_i = \lambda m_j$, $\lambda \in k$. Let $A_{ij} = \ell_1 \ell_2 \cdots \ell_i m_1 m_2 \cdots m_j$, $i, j \geq 0$ ($A_{00} = 1$). Show that $\{A_{ij} : i + j = d\}$ forms a basis for $V(d, 2)$.

Proof of (a).

- (1) All monomials of degree d in $R[x, y]$ are

$$x^d, x^{d-1}y, \dots, xy^{d-1}, y^d,$$

or of the form $x^i y^j$ with $i, j \geq 0$ and $i + j = d$. So there are $d + 1$ monomials of degree d in $R[x, y]$.

- (2) Similar to (1), all monomials of degree d in $R[x, y]$ are of the form $x^i y^j z^k$ with $i, j, k \geq 0$ and $i + j + k = d$. By the stars and bars (combinatorics) method, there are

$$\binom{d + 3 - 1}{3 - 1} = \frac{(d + 2)(d + 1)}{2}$$

monomials of degree d in $R[x, y, z]$.

□

Proof of (b).

- (1) To show $V(d, n)$ is a vector space, it suffices to show that $V(d, n)$ is a subspace of $k[x_1, \dots, x_n]$ since $k[x_1, \dots, x_n]$ is a vector space over k .
- (2) Note that $0 \in V(d, n)$ is nonempty. For any $f, g \in V(d, n)$ and $a, b \in k$, we have $af + bg \in V(d, n)$. Hence $V(d, n)$ is subspace.
- (3) Let

$$\mathcal{B} = \{x_1^{i_1} \cdots x_n^{i_n} : i_1, \dots, i_n \geq 0, i_1 + \cdots + i_n = d\}.$$

\mathcal{B} is an independent set, and \mathcal{B} generates $V(d, n)$. So \mathcal{B} is a basis for $V(d, n)$.

- (4) Similar to (a),

$$\dim_k V(d, n) = |\mathcal{B}| = \binom{d + n - 1}{n - 1}$$

by the stars and bars (combinatorics) method. In particular, $\dim V(d, 1) = 1$; $\dim V(d, 2) = d + 1$; $\dim V(d, 3) = \frac{(d+1)(d+2)}{2}$.

□

Proof of (c).

- (1) Show that $\mathcal{B}' := \{A_{ij} : i + j = d\}$ is an independent set. (Reductio ad absurdum) Suppose that there were a nontrivial linear combination of A_{ij} such that

$$\sum_{i+j=d} c_{ij} A_{ij} = 0.$$

- (2) Let p be the smallest index i such that $c_{ij} \neq 0$. Write $q := d - p$. So

$$\begin{aligned} c_{pq} A_{pq} &= - \sum_{\substack{i+j=d \\ i \neq p, j \neq q}} c_{ij} A_{ij} = - \sum_{\substack{i+j=d \\ i > p, j < q}} c_{ij} A_{ij} \\ \iff A_{pq} &= - \sum_{\substack{i+j=d \\ i > p, j < q}} \frac{c_{ij}}{c_{pq}} A_{ij} \\ \iff \ell_1 \cdots \ell_p m_1 \cdots m_q &= - \sum_{\substack{i+j=d \\ i > p, j < q}} \frac{c_{ij}}{c_{pq}} \ell_1 \cdots \ell_p \ell_{p+1} \cdots \ell_i m_1 \cdots m_j \\ \iff m_1 \cdots m_q &= -\ell_{p+1} \sum_{\substack{i+j=d \\ i > p, j < q}} \frac{c_{ij}}{c_{pq}} \underbrace{\ell_{p+2} \cdots \ell_i}_{:=1 \text{ if } i=p+1} m_1 \cdots m_j \\ \iff \ell_{p+1} &| m_1 \cdots m_q. \end{aligned}$$

Since all ℓ_i, m_j are linear forms, $\ell_{p+1} | m_j$ for some $1 \leq j \leq q$, which is absurd since no $\ell_i = \lambda m_j$, $\lambda \in k$. Therefore, \mathcal{B}' is an independent set.

- (3) Since

$$|\mathcal{B}'| = d + 1 = \dim_k V(d, 2),$$

\mathcal{B}' is also a basis for $V(d, 2)$.

□

Problem 2.36.

With the above notation, show that

$$\dim V(d, n) = \binom{d+n-1}{n-1},$$

the binomial coefficient.

Proof. See the proof of Problem 2.35(b). □

2.7. Direct Products of Rings

Problem 2.37.

What are the additive and multiplicative identities in $\times R_i$? Is the map from R_i to $\times R_i$ taking a_i to $(0, \dots, a_i, \dots, 0)$ a ring homomorphism?

Proof.

- (1) $(0, \dots, 0)$ is the additive identity in $\times R_i$.
- (2) $(1, \dots, 1)$ is the multiplicative identity in $\times R_i$.
- (3) The map $\alpha : R_i \rightarrow \times R_i$ taking a_i to $(0, \dots, a_i, \dots, 0)$ is not a ring homomorphism since

$$\alpha(1) = (0, \dots, 1, \dots, 0) \neq (1, \dots, 1),$$

or α is not multiplicative identity preserving (if R_j is not the zero ring for some $j \neq i$).

□

Problem 2.38.*

Show that if $k \subseteq R_i$, and each R_i is finite-dimensional over k , then $\dim(\times R_i) = \sum \dim(R_i)$.

Proof.

- (1) In the terminology of linear algebra, $\times R_i$ is the direct sum $\bigoplus R_i$ of R_i .
- (2) Hence,

$$\dim_k \left(\bigoplus R_i \right) = \sum \dim_k(R_i).$$

□

2.8. Operations with Ideals

Problem 2.39.*

Prove the following relations among ideals I_i, J in a ring R :

- (a) $(I_1 + I_2)J = I_1J + I_2J$.
(b) $(I_1 \cdots I_N)^n = I_1^n \cdots I_N^n$.

Proof of (a).

- (1) Note that $(I_1 + I_2)J$ and $I_1J + I_2J$ are ideals.
(2) Show that $(I_1 + I_2)J \subseteq I_1J + I_2J$. Given any

$$(x_1 + x_2)y \in (I_1 + I_2)J$$

where $x_i \in I_i$ and $y \in J$. It suffices to show that $(x_1 + x_2)y \in I_1J + I_2J$ (by (1)). In fact,

$$(x_1 + x_2)y = x_1y + x_2y \in I_1J + I_2J.$$

- (3) Show that $(I_1 + I_2)J \supseteq I_1J + I_2J$. Given any

$$x_1y_1 + x_2y_2 \in I_1J + I_2J$$

where $x_i \in I_i$ and $y_i \in J$. It suffices to show that $x_1y_1 + x_2y_2 \in (I_1 + I_2)J$ (by (1)). In fact,

$$x_1y_1 + x_2y_2 = (x_1 + \underbrace{0}_{\in I_2})y_1 + (\underbrace{0}_{\in I_1} + x_2)y_2 \in (I_1 + I_2)J$$

since $(I_1 + I_2)J$ is an ideal.

□

Proof of (b).

- (1) Note that $(I_1 \cdots I_N)^n$ and $I_1^n \cdots I_N^n$ are ideals.
(2) Show that $(I_1 \cdots I_N)^n \subseteq I_1^n \cdots I_N^n$. Given any

$$x = x_1 \cdots x_n$$

where $x_i \in I_1 \cdots I_N$. It suffices to show that $x \in I_1^n \cdots I_N^n$ (by (1)). For each $x_i \in I_1 \cdots I_N$, write

$$x_i = \sum_{j(i)} x_{j(i),1} \cdots x_{j(i),N}$$

where $x_{j(i),k} \in I_k$ for $1 \leq k \leq N$. Hence

$$\begin{aligned}
x &= x_1 \cdots x_n \\
&= \left(\sum_{j(1)} x_{j(1),1} \cdots x_{j(1),N} \right) \cdots \left(\sum_{j(n)} x_{j(n),1} \cdots x_{j(n),N} \right) \\
&= \sum_{j(1), \dots, j(n)} (x_{j(1),1} \cdots x_{j(1),N}) \cdots (x_{j(n),1} \cdots x_{j(n),N}) \\
&= \sum_{j(1), \dots, j(n)} \underbrace{(x_{j(1),1} \cdots x_{j(n),1})}_{\in I_1^n} \cdots \underbrace{(x_{j(1),N} \cdots x_{j(n),N})}_{\in I_N^n} \\
&\in I_1^n \cdots I_N^n.
\end{aligned}$$

(3) Show that $(I_1 \cdots I_N)^n \supseteq I_1^n \cdots I_N^n$. Given any

$$x = x_1 \cdots x_N \in I_1^n \cdots I_N^n$$

where $x_i \in I_i^n$ ($1 \leq i \leq N$). It suffices to show that $x \in (I_1 \cdots I_N)^n$ (by (1)). For each $x_i \in I_i^n$, write

$$x_i = \sum_{j(i)} x_{j(i),1} \cdots x_{j(i),n}$$

where $x_{j(i),k} \in I_k$ for $1 \leq k \leq n$. Hence

$$\begin{aligned}
x &= x_1 \cdots x_N \\
&= \left(\sum_{j(1)} x_{j(1),1} \cdots x_{j(1),n} \right) \cdots \left(\sum_{j(N)} x_{j(N),1} \cdots x_{j(N),n} \right) \\
&= \sum_{j(1), \dots, j(N)} (x_{j(1),1} \cdots x_{j(1),n}) \cdots (x_{j(N),1} \cdots x_{j(N),n}) \\
&= \sum_{j(1), \dots, j(N)} \underbrace{(x_{j(1),1} \cdots x_{j(N),1})}_{\in I_1 \cdots I_N} \cdots \underbrace{(x_{j(1),n} \cdots x_{j(N),n})}_{\in I_1 \cdots I_N} \\
&\in (I_1 \cdots I_N)^n.
\end{aligned}$$

□

Problem 2.40.* (Chinese remainder theorem)

- (a) Suppose I, J are comaximal ideals in R . Show that $I + J^2 = R$. Show that I^m and J^n are comaximal for all m, n .

- (b) Suppose I_1, \dots, I_N are ideals in R , and I_i and $J_i = \bigcap_{j \neq i} I_j$ are comaximal for all i . Show that

$$I_1^n \cap \dots \cap I_N^n = (I_1 \cdots I_N)^n = (I_1 \cap \dots \cap I_N)^n$$

for all n .

Proof of (a).

- (1) It suffices to show that $I^m + J^n = R$.
(2) Since $I^m + J^n \subseteq R$ is always true, it suffices to show that $I^m + J^n \supseteq R$.
In fact,

$$\begin{aligned} R &= R^{m+n-1} && (1 \in R) \\ &= (I + J)^{m+n-1} && (I, J \text{ are comaximal}) \\ &= \sum_{i=0}^{m+n-1} I^i J^{m+n-1-i} && (\text{Problem 2.39}) \\ &\subseteq I^m + J^n \end{aligned}$$

for all positive integers m, n . (If $m = 0$ or $n = 0$, then nothing to prove.)

□

Proof of (b).

- (1) Show that I_i and I_j are comaximal if $i \neq j$. Note that

$$R = I_i + J_i \subseteq I_i + I_j \subseteq R$$

if $i \neq j$.

- (2) If I_i is comaximal to I_j and $I_{j'}$. Show that I_i is also comaximal to $I_j I_{j'}$.

$$\begin{aligned} R &= (I_i + I_j)(I_i + I_{j'}) \\ &= I_i(I_i + I_j + I_{j'}) + I_j I_{j'} && (\text{Problem 2.39(a)}) \\ &\subseteq I_i + I_j I_{j'} \subseteq R. \end{aligned}$$

- (3) By (2), it is easy to get that I_i and $\prod_{j \neq i} I_j$ are comaximal by induction on the number of I_j for $j \neq i$.
(4) Show that $I_1 \cdots I_N = I_1 \cap \dots \cap I_N$. Induction on N .

$$\begin{aligned} I_1 \cap \dots \cap I_N &= I_1 \cap (I_2 \cap \dots \cap I_N) \\ &= I_1 \cap (I_2 \cdots I_N) && (\text{Induction hypothesis}) \\ &= I_1 \cdot (I_2 \cdots I_N) && ((3)) \\ &= I_1 \cdots I_N. \end{aligned}$$

- (5) Note that I_i^n and I_j^n are comaximal if $i \neq j$ by (a). We can apply the same argument in (2)(3)(4) to show that

$$I_1^n \cdots I_N^n = I_1^n \cap \cdots \cap I_N^n.$$

- (6) Therefore,

$$\begin{aligned} (I_1 \cap \cdots \cap I_N)^n &= (I_1 \cdots I_N)^n && ((4)) \\ &= I_1^n \cdots I_N^n && (\text{Problem 2.39(b)}) \\ &= I_1^n \cap \cdots \cap I_N^n && ((5)). \end{aligned}$$

□

Problem 2.41.*

Let I, J be ideals in R . Suppose I is finitely generated and $I \subseteq \text{rad}(J)$. Show that $I^n \subseteq J$ for some n .

Proof.

- (1) Let I be generated by $x_1, \dots, x_m \in I$. As $I \subseteq \text{rad}(J)$, there are integers $n_i > 0$ such that $x_i^{n_i} \in J$.
- (2) Let $N = n_1 + \cdots + n_m$. Given any $x = \sum_{i=1}^m r_i x_i \in I$, so

$$\begin{aligned} x^N &= \left(\sum_{i=1}^m r_i x_i \right)^N \\ &= \sum_{k_1 + \cdots + k_m = N} \binom{N}{k_1, \dots, k_m} r_1^{k_1} x_1^{k_1} \cdots r_m^{k_m} x_m^{k_m}. \end{aligned}$$

- (3) Note that for each term there is some j such that $k_j \geq n_j$. Hence,

$$\begin{aligned} x_j^{k_j} &= x_j^{k_j - n_j} x_j^{n_j} \in J && (J \text{ is an ideal}) \\ \implies r_1^{k_1} x_1^{k_1} \cdots r_m^{k_m} x_m^{k_m} &\in J \text{ for each term} && (J \text{ is an ideal}) \\ \implies x^N &\in J. && (J \text{ is an ideal}) \\ \implies I^N &\subseteq J. \end{aligned}$$

□

Supplement. (Exercise 1.13 in the textbook: Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*.) Suppose that I is an ideal in a

commutative ring. Show that if $\text{rad}(I)$ is finitely generated, then for some integer N we have $(\text{rad}(I))^N \subseteq I$. Conclude that in a Noetherian ring the ideals I and J have the same radical iff there is some integer N such that $I^N \subseteq J$ and $J^N \subseteq I$. Use the Nullstellensatz to deduce that if $I, J \subseteq S = k[x_1, \dots, x_n]$ are ideals and k is algebraically closed, then $Z(I) = Z(J)$ iff $I^N \subseteq J$ and $J^N \subseteq I$ for some N .

Proof.

- (1) Show that if $\text{rad}(I)$ is finitely generated, then for some integer N we have $(\text{rad}(I))^N \subseteq I$. Say $x_1, \dots, x_m \in \text{rad}(I)$ generate $\text{rad}(I)$.

- (a) For each i , there exists an integer $n_i > 0$ such that $x_i^{n_i} \in I$ (since $\text{rad}(I)$ is radical).
(b) Let $N = n_1 + \dots + n_m$. Given any $x = \sum_{i=1}^m r_i x_i \in \text{rad}(I)$, so

$$\begin{aligned} x^N &= \left(\sum_{i=1}^m r_i x_i \right)^N \\ &= \sum_{k_1 + \dots + k_m = N} \binom{N}{k_1, \dots, k_m} r_1^{k_1} x_1^{k_1} \dots r_m^{k_m} x_m^{k_m}. \end{aligned}$$

- (c) Note that for each term there is some j such that $k_j \geq n_j$. Hence,

$$\begin{aligned} x_j^{k_j} &= x_j^{k_j - n_j} x_j^{n_j} \in I && (I \text{ is an ideal}) \\ \implies r_1^{k_1} x_1^{k_1} \dots r_m^{k_m} x_m^{k_m} &\in I \text{ for each term} && (I \text{ is an ideal}) \\ \implies x^N &\in I. && (I \text{ is an ideal}) \\ \implies (\text{rad}(I))^N &\subseteq I. \end{aligned}$$

- (2) Show that in a Noetherian ring the ideals I and J have the same radical iff there is some integer N such that $I^N \subseteq J$ and $J^N \subseteq I$.

- (a) (\implies) Since in a Noetherian ring every ideal is finitely generated, $\text{rad}(I)$ and $\text{rad}(J)$ are finitely generated. By (1), there is a common integer N such that

$$(\text{rad}(I))^N \subseteq I \quad \text{and} \quad (\text{rad}(J))^N \subseteq J.$$

Note that $I^N \subseteq (\text{rad}(I))^N$ and $J^N \subseteq (\text{rad}(J))^N$. Since $\text{rad}(I) = \text{rad}(J)$ by assumption,

$$\begin{aligned} I^N &\subseteq (\text{rad}(I))^N = (\text{rad}(J))^N \subseteq J, \\ J^N &\subseteq (\text{rad}(J))^N = (\text{rad}(I))^N \subseteq I. \end{aligned}$$

- (b) (\impliedby) It suffices to show that $\text{rad}(I) \subseteq \text{rad}(J)$. $\text{rad}(J) \subseteq \text{rad}(I)$ is similar. Given any $x \in \text{rad}(I)$, there is an integer $M > 0$ such that $x^M \in I$. Hence $x^{MN} \in I^N \subseteq J$, or $x \in \text{rad}(J)$.

- (3) Show that if $I, J \subseteq S = k[x_1, \dots, x_n]$ are ideals and k is algebraically closed, then $Z(I) = Z(J)$ iff $I^N \subseteq J$ and $J^N \subseteq I$ for some N . Note that S is Noetherian and we can apply step (2). By the Nullstellensatz, $Z(I) = Z(J)$ iff $\text{rad}(I) = \text{rad}(J)$ iff $I^N \subseteq J$ and $J^N \subseteq I$ for some N .

□

Problem 2.42.* (Isomorphism theorems for rings)

- (a) Let $I \subseteq J$ be ideals in a ring R . Show that there is a natural ring homomorphism from R/I onto R/J .
- (b) Let I be an ideal in a ring R , R a subring of a ring S . Show that there is a natural ring homomorphism from R/I to S/IS .

Proof of (a).

- (1) Define a map $\alpha : R/I \rightarrow R/J$ by $\alpha(r + I) = r + J$.
- (2) Show that α is well-defined. If $a + I = b + I$, then $a - b \in I \subseteq J$ or $a + J = b + J$. Hence, $\alpha(a + I) = a + J = b + J = \alpha(b + I)$.
- (3) Show that α is a surjective homomorphism.
- (a) α is addition preserving.

$$\begin{aligned}\alpha((a + I) + (b + I)) &= \alpha(a + b + I) \\ &= a + b + J \\ &= (a + J) + (b + J) \\ &= \alpha(a + I) + \alpha(b + I).\end{aligned}$$

- (b) α is multiplication preserving.

$$\begin{aligned}\alpha((a + I)(b + I)) &= \alpha(ab + I) \\ &= ab + J \\ &= (a + J)(b + J) \\ &= \alpha(a + I)\alpha(b + I).\end{aligned}$$

- (c) α is multiplicative identity preserving. $\alpha(1 + I) = 1 + J$.

- (d) α is surjective since for any $a + J \in R/J$ there is an element $a + I \in R/I$ such that $\alpha(a + I) = a + J$.

- (4) Note that $\ker(\alpha) = J/I$. So $(R/I)/(J/I) \cong R/J$.

□

Proof of (b).

- (1) I is not necessary an ideal of S ; IS an ideal of S (and thus S/IS is well-defined).
- (2) Define a map $\alpha : R/I \rightarrow S/IS$ by $\alpha(r + I) = r + IS$. Note that $I \subseteq IS$ as a subset in S . Apply the same argument in (a), α is well-defined and α is a surjective homomorphism.
- (3) Note that $\ker(\alpha) = (R \cap SI)/I$. So $(R/I)/((R \cap SI)/I) \cong S/IS$.

□

Problem 2.43.*

Let $P = (0, \dots, 0) \in \mathbf{A}^n$, $\mathcal{O} = \mathcal{O}_P(\mathbf{A}^n)$, $\mathfrak{m} = \mathfrak{m}_P(\mathbf{A}^n)$. Let $I = (x_1, \dots, x_n) \subseteq k[x_1, \dots, x_n]$ be the ideal generated by x_1, \dots, x_n . Show that $I\mathcal{O} = \mathfrak{m}$, so $I^r\mathcal{O} = \mathfrak{m}^r$ for all r .

Proof.

- (1) By the definition

$$\mathfrak{m} = \{f \in \mathcal{O} : f(P) = 0\},$$

$I\mathcal{O} \subseteq \mathfrak{m}$. Conversely, by Problem 1.7(b) we have $I\mathcal{O} \supseteq \mathfrak{m}$.

- (2) By Problem 2.39(b),

$$\mathfrak{m}^r = (I\mathcal{O})^r = I^r\mathcal{O}^r = I^r\mathcal{O}.$$

Here $\mathcal{O}^r = \mathcal{O}$ since $1 \in \mathcal{O}$.

□

Problem 2.44.*

Let V be a variety in \mathbf{A}^n , $I = I(V) \subseteq k[x_1, \dots, x_n]$, $P \in V$, and let J be an ideal of $k[x_1, \dots, x_n]$ that contains I . Let J' be the image of J in $\Gamma(V)$. Show that there is a natural homomorphism φ from $\mathcal{O}_P(\mathbf{A}^n)/J\mathcal{O}_P(\mathbf{A}^n)$ to $\mathcal{O}_P(V)/J'\mathcal{O}_P(V)$, and that φ is an isomorphism. In particular, $\mathcal{O}_P(\mathbf{A}^n)/I\mathcal{O}_P(\mathbf{A}^n)$ is isomorphic to $\mathcal{O}_P(V)$.

Proof.

- (1) Define φ from $\mathcal{O}_P(\mathbf{A}^n)/J\mathcal{O}_P(\mathbf{A}^n)$ to $\mathcal{O}_P(V)/J'\mathcal{O}_P(V)$ by

$$\varphi : a/b + J\mathcal{O}_P(\mathbf{A}^n) \mapsto \bar{a}/\bar{b} + J'\mathcal{O}_P(V).$$

It is well-defined since $\varphi(J\mathcal{O}_P(\mathbf{A}^n)) = J'\mathcal{O}_P(V)$ and $b(P) \neq 0$ implies that $\bar{b}(P) \neq 0$.

- (2) Note that V is a subvariety of \mathbf{A}^n . So $\varphi : \Gamma(\mathbf{A}^n) \rightarrow \Gamma(V)$ is a ring homomorphism by Problem 2.3 and then φ extends uniquely to a ring homomorphism by using the similar argument in Problem 2.21.
- (3) φ is surjective since $\mathcal{O}_P(\mathbf{A}^n) \hookrightarrow \mathcal{O}_P(V)$ and $\varphi(J\mathcal{O}_P(\mathbf{A}^n)) = J'\mathcal{O}_P(V)$. φ is injective since $\varphi(J\mathcal{O}_P(\mathbf{A}^n)) = J'\mathcal{O}_P(V)$. Hence $\varphi : \mathcal{O}_P(\mathbf{A}^n)/J \rightarrow \mathcal{O}_P(V)/J'$ is isomorphic. In particular, $\mathcal{O}_P(\mathbf{A}^n)/I\mathcal{O}_P(\mathbf{A}^n) \cong \mathcal{O}_P(V)$ (by taking $J = I$ and noting that $J' = I' = 0$).

□

Problem 2.45.*

Show that ideals $I, J \subseteq k[x_1, \dots, x_n]$ (k algebraically closed) are comaximal if and only if $V(I) \cap V(J) = \emptyset$.

Proof.

- (1) Show that $V(I) \cap V(J) = V(I + J)$.

$$\begin{aligned} P \in V(I) \cap V(J) &\iff f(P) = 0 \forall f \in I \text{ and } g(P) = 0 \forall g \in J \\ &\iff f(P) = 0 \forall f \in I + J \\ &\iff P \in V(I + J). \end{aligned}$$

- (2) Hence,

$$\begin{aligned} \emptyset = V(I) \cap V(J) &\iff \emptyset = V(I + J) && ((1)) \\ &\iff I + J = k[x_1, \dots, x_n] && (\text{Weak Nullstellensatz}) \\ &\iff I \text{ and } J \text{ are comaximal.} \end{aligned}$$

□

Problem 2.46.*

Let $I = (x, y) \subseteq k[x, y]$. Show that

$$\dim_k(k[x, y]/I^n) = 1 + 2 + \dots + n = \frac{n(n+1)}{2}.$$

Proof.

(1) The set

$$\mathcal{B} = \{x^i y^j + I^n : i, j \in \mathbb{Z}, i, j \geq 0, i + j < n\}$$

generates $k[x, y]/I^n$ as a k -vector space. Besides, each nonzero element in I^n has the degree $\geq n$, and thus \mathcal{B} is an independent set. Therefore, \mathcal{B} is a basis for $k[x, y]/I^n$.

(2) Hence,

$$\dim_k(k[x, y]/I^n) = |\mathcal{B}| = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

□

2.9. Ideals with a Finite Number of Zeros

Problem 2.47.

Suppose R is a ring containing k , and R is finite dimensional over k . Show that R is isomorphic to a direct product of local rings.

Proof.

- (1) Let $\{v_1, \dots, v_n\}$ be a basis for R over k (as a vector space). Define a k -module homomorphism $\alpha : k[x_1, \dots, x_n] \rightarrow R$ by $\alpha(x_i) = v_i$. Clearly, α is surjective and thus

$$R \cong k[x_1, \dots, x_n] / \ker(\alpha)$$

as a k -module isomorphism. Note that $\ker(\alpha)$ is an ideal of $k[x_1, \dots, x_n]$.

- (2) Write $I := \ker(\alpha)$. Hence,

$$\dim_k(k[x_1, \dots, x_n]/I) = \dim_k(R) < \infty.$$

By Corollary 4 to the Hilbert's Nullstellensatz in §1.7, $V(I)$ is finite.

- (3) Write $V(I) = \{P_1, \dots, P_N\}$ and $\mathcal{O}_i = \mathcal{O}_{P_i}(\mathbf{A}^n)$. By Proposition 6,

$$R \cong k[x_1, \dots, x_n]/I \cong \prod_{i=1}^N \mathcal{O}_i / I \mathcal{O}_i,$$

which is isomorphic to a direct product of local rings.

□

2.10. Quotient Modules and Exact Sequences

Problem 2.48.*

Verify that for any R -module homomorphism $\varphi : M \rightarrow M'$, $\ker(\varphi)$ and $\text{im}(\varphi)$ are submodules of M and M' respectively. Show that

$$0 \rightarrow \ker(\varphi) \rightarrow M \xrightarrow{\varphi} \text{im}(\varphi) \rightarrow 0$$

is exact.

Proof.

- (1) Show that $\ker(\varphi)$ is a subgroup of M . It suffices to show that $a - b \in \ker(\varphi)$ for all $a, b \in \ker(\varphi)$. In fact, $\varphi(a - b) = \varphi(a) - \varphi(b) = 0 - 0 = 0$, or $a - b \in \ker(\varphi)$.
- (2) Show that $\ker(\varphi)$ is a submodule of M . By (1), it suffices to show that $ra \in \ker(\varphi)$ for all $r \in R$ and $a \in \ker(\varphi)$. In fact, $\varphi(ra) = r \cdot \varphi(a) = r \cdot 0 = 0$, or $ra \in \ker(\varphi)$.
- (3) Show that $\text{im}(\varphi)$ is a subgroup of M' . It suffices to show that $a - b \in \text{im}(\varphi)$ for all $a, b \in \text{im}(\varphi)$. As $a, b \in \text{im}(\varphi)$, there are two elements $a', b' \in M$ such that $\varphi(a') = a$ and $\varphi(b') = b$. So $\varphi(a' - b') = \varphi(a') - \varphi(b') = a - b$, or $a - b \in \text{im}(\varphi)$.
- (4) Show that $\text{im}(\varphi)$ is a submodule of M' . By (3), it suffices to show that $ra \in \text{im}(\varphi)$ for all $r \in R$ and $a \in \text{im}(\varphi)$. As $a \in \text{im}(\varphi)$, there is one element $a' \in M$ such that $\varphi(a') = a$. So $\varphi(ra') = r\varphi(a') = ra$, or $ra \in \text{im}(\varphi)$.
- (5) Show that

$$0 \rightarrow \ker(\varphi) \xrightarrow{i} M \xrightarrow{\varphi} \text{im}(\varphi) \rightarrow 0$$

is exact. Note that $\ker(\varphi) \xrightarrow{i} M$ is the natural inclusion and $M \xrightarrow{\varphi} \text{im}(\varphi)$ is surjective. Also, it is trivial that $\text{im}(i) = \ker(\varphi)$.

□

Problem 2.49.*

- (a) (Factor theorem for modules) Let N be a submodule of M , $\pi : M \rightarrow M/N$ the natural homomorphism. Suppose $\varphi : M \rightarrow M'$ is a homomorphism of R -modules, and $\varphi(N) = 0$. Show that there is a unique homomorphism $\bar{\varphi} : M/N \rightarrow M'$ such that $\bar{\varphi} \circ \pi = \varphi$.

- (b) (Isomorphism theorems for modules) If N and P are submodules of a module M , with $P \subseteq N$, then there are natural homomorphisms from M/P onto M/N and from N/P into M/P . Show that the resulting sequence

$$0 \rightarrow N/P \rightarrow M/P \rightarrow M/N \rightarrow 0$$

is exact.

- (c) Let $U \subseteq W \subseteq V$ be vector spaces, with V/U finite-dimensional. Then $\dim V/U = \dim V/W + \dim W/U$.
- (d) If $J \subseteq I$ are ideals in a ring R , there is a natural exact sequence of R -modules:

$$0 \rightarrow I/J \rightarrow R/J \rightarrow R/I \rightarrow 0.$$

- (e) If \mathcal{O} is a local ring with maximal ideal \mathfrak{m} , there is a natural exact sequence of \mathcal{O} -modules

$$0 \rightarrow \mathfrak{m}^n/\mathfrak{m}^{n+1} \rightarrow \mathcal{O}/\mathfrak{m}^{n+1} \rightarrow \mathcal{O}/\mathfrak{m}^n \rightarrow 0.$$

Proof of (a).

- (1) Define $\bar{\varphi} : M/N \rightarrow M'$ by

$$\bar{\varphi}(m + N) = \varphi(m).$$

$\bar{\varphi}$ is well-defined since $m + N = n + N$ implies that $m - n \in N \subseteq \ker(\varphi)$.

- (2) $\bar{\varphi}$ is a homomorphism of R -modules since φ is a homomorphism of R -modules.
- (3) $\bar{\varphi} \circ \pi = \varphi$ by construction.
- (4) Suppose there is a homomorphism $\psi : M/N \rightarrow M'$ such that $\psi \circ \pi = \varphi$. For any $m + N \in M/N$, we have

$$\bar{\varphi}(m + N) = \varphi(m) = (\psi \circ \pi)(m) = \psi(\pi(m)) = \psi(m + N).$$

That is, $\psi = \bar{\varphi}$.

□

Proof of (b).

- (1) Define $\pi : M/P \rightarrow M/N$ by

$$\pi : \underbrace{m + P}_{\in M/P} \mapsto \underbrace{m + N}_{\in M/N}.$$

- (a) *Show that π is well-defined.* If $m + P = n + P \in M/P$, then $m - n \in P \subseteq N$ or $m + N = n + N \in M/N$.
- (b) π is a module homomorphism since M/N is a module.
- (c) π is surjective by construction.
- (2) Define $i : N/P \hookrightarrow M/P$ by

$$i : \underbrace{m + P}_{\in N/P} \mapsto \underbrace{m + P}_{\in M/P}.$$

- (a) *Show that i is well-defined.* If $m + P = n + P \in N/P$, then $m, n \in N \subseteq M$ and $m - n \in P$. So $m + P = n + P \in M/P$.
- (b) i is a module homomorphism since M/P is a module.
- (c) i is injective by construction.
- (3) To show that $0 \rightarrow N/P \rightarrow M/P \rightarrow M/N \rightarrow 0$ is exact, it suffices to show that $\ker(\pi) = \text{im}(i) = N/P$ (by the injectivity of i). It is trivial since

$$m + P \in \ker(\pi) \iff m \in N \iff m + P \in N/P.$$

□

Proof of (c).

- (1) By (b),

$$0 \rightarrow W/U \rightarrow V/U \xrightarrow{\varphi} V/W \rightarrow 0$$

is exact.

- (2) By the rank-nullity theorem for a linear transformation,

$$\dim V/U = \dim \text{im}(\varphi) + \dim \ker(\varphi) = \dim V/W + \dim W/U.$$

□

Proof of (d).

- (1) Regard R as a R -module and I, J as submodules of a R -module R .
- (2) As $J \subseteq I$, by (b) we have

$$0 \rightarrow I/J \rightarrow R/J \rightarrow R/I \rightarrow 0.$$

□

Proof of (e).

- (1) Note that $\mathfrak{m}^{n+1} \subseteq \mathfrak{m}^n$ are ideals in a local ring \mathcal{O} .

(2) By (d), there is a natural exact sequence of \mathcal{O} -modules:

$$0 \rightarrow \mathfrak{m}^n/\mathfrak{m}^{n+1} \rightarrow \mathcal{O}/\mathfrak{m}^{n+1} \rightarrow \mathcal{O}/\mathfrak{m}^n \rightarrow 0.$$

□

Problem 2.50.*

Let R be a DVR satisfying the conditions of Problem 2.30. Then $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is an R -module, and so also a k -module, since $k \subseteq R$.

- (a) Show that $\dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = 1$ for all $n \geq 0$.
- (b) Show that $\dim_k(R/\mathfrak{m}^n) = n$ for all $n > 0$.
- (c) Let $z \in R$. Show that $\text{ord}(z) = n$ if $(z) = \mathfrak{m}^n$, and hence that $\text{ord}(z) = \dim_k(R/(z))$.

Proof of (a).

- (1) By Problem 2.49(e),

$$0 \rightarrow \mathfrak{m}^n/\mathfrak{m}^{n+1} \rightarrow R/\mathfrak{m}^{n+1} \rightarrow R/\mathfrak{m}^n \rightarrow 0$$

is exact.

- (2) By the rank-nullity theorem (Proposition 3),

$$\begin{aligned} \dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1}) &= \dim_k(R/\mathfrak{m}^{n+1}) - \dim_k(R/\mathfrak{m}^n) \\ &= (n+1) - n \\ &= 1. \end{aligned} \tag{((b))}$$

□

Proof of (b).

- (1) Let t be a uniformizing parameter for R , $z \in R$. By Problem 2.30(b), there are unique $\lambda_0, \dots, \lambda_{n-1} \in k$ and $z_{n-1} \in R$ such that

$$z = \lambda_0 + \lambda_1 t + \dots + \lambda_{n-1} t^{n-1} + z_{n-1} t^n.$$

Hence we can define a map $\varphi : R/\mathfrak{m}^n \rightarrow k^n$ by

$$\varphi : z + \mathfrak{m}^n \mapsto (\lambda_0, \dots, \lambda_{n-1}).$$

- (2) φ is well-defined by the uniqueness of the expression of z in Problem 2.30(b). φ is a k -module homomorphism and φ is surjective (since $k \subseteq R$). φ is injective by the uniqueness of the expression of z in Problem 2.30(b).

- (3) Hence, $R/\mathfrak{m}^n \cong k^n$ or $\dim_k(R/\mathfrak{m}^n) = n$ for $n > 0$. (It is also true for $n = 0$ since $\dim_k(\{0\}) = 0$.)

□

Proof of (c).

- (1) Note that $\mathfrak{m}^n = (t^n)$ as $\mathfrak{m} = (t)$ where t is a uniformizing parameter for R .
- (2) Since $(z) = (t^n) = \mathfrak{m}^n$, $\text{ord}(z) = n$ by Problem 2.28. (Here $\text{ord}(z) \geq n$ by $z \in (t^n)$ and $n \geq \text{ord}(z)$ by $t^n \in (z)$.)

- (3) Hence,

$$\text{ord}(z) = n = \dim_k(R/\mathfrak{m}^n) = \dim_k(R/(z))$$

by (b).

□

Problem 2.51.

Let

$$0 \longrightarrow V_1 \longrightarrow \cdots \longrightarrow V_n \longrightarrow 0$$

be an exact sequence of finite-dimensional vector spaces. Show that $\sum (-1)^i \dim(V_i) = 0$.

Proof (Proposition 7 in §2.10).

- (1) For $i = 0, \dots, n$, by the rank-nullity theorem for a linear transformation $\varphi_i : V_i \rightarrow V_{i+1}$, we have

$$\dim V_i = \dim \text{im}(\varphi_i) + \dim \ker(\varphi_i).$$

(Here $V_0 = V_{n+1} := 0$ by convention.)

- (2) By the exactness of the sequence, we have

$$(a) \quad \text{im}(\varphi_i) = \ker(\varphi_{i+1}) \text{ for } i = 0, \dots, n-1. \text{ In particular, } \ker(\varphi_1) = \text{im}(\varphi_0) = 0.$$

$$(b) \quad \ker(\varphi_n) = V_n.$$

Hence,

$$\begin{aligned}
\sum_{i=1}^{n-1} (-1)^i \dim(V_i) &= \sum_{i=1}^{n-1} (-1)^i \dim \operatorname{im}(\varphi_i) + \sum_{i=1}^{n-1} (-1)^i \dim \ker(\varphi_i) \\
&= \sum_{i=1}^{n-1} (-1)^i \dim \ker(\varphi_{i+1}) + \sum_{i=1}^{n-1} (-1)^i \dim \ker(\varphi_i) \\
&= (-1)^{n-1} \underbrace{\dim \ker(\varphi_n)}_{=V_n} + (-1)^1 \underbrace{\dim \ker(\varphi_1)}_{=0} \\
&= -(-1)^n \dim V_n,
\end{aligned}$$

$$\text{or } \sum (-1)^i \dim(V_i) = 0.$$

□

Problem 2.52.* (Isomorphism theorems for modules)

Let N, P be submodules of a module M . Show that the subgroup

$$N + P = \{n + p : n \in N, p \in P\}$$

is a submodule of M . Show that there is a natural R -module isomorphism of $N/(N \cap P)$ onto $(N + P)/P$.

Proof.

- (1) Show that $N + P$ is a submodule of M . Given any $n_1 + p_1, n_2 + p_2 \in N + P$,

$$(n_1 + p_1) + (n_2 + p_2) = \underbrace{n_1 + n_2}_{\in N} + \underbrace{p_1 + p_2}_{\in P} \in N + P.$$

Given any $n + p \in N + P$ and $r \in R$,

$$r(n + p) = \underbrace{rn}_{\in N} + \underbrace{rp}_{\in P} \in N + P.$$

Here we use the fact that N and P are modules.

- (2) Define a module homomorphism $\varphi : N \rightarrow M/P$ by

$$\varphi : m \mapsto m + P.$$

$\ker(\varphi) = N \cap P$ and $\operatorname{im}(\varphi) = \{m + P : m \in N\} = (N + P)/P$. By Problem 2.48, φ induces a natural R -module isomorphism of $N/\ker(\varphi) = N/(N \cap P)$ onto $\operatorname{im}(\varphi) = (N + P)/P$ (which is sending $m + (N \cap P)$ to $m + P$).

□

Problem 2.53.*

Let V be a vector space, W a subspace, $T : V \rightarrow V$ a one-to-one linear map such that $T(W) \subseteq W$, and assume V/W and $W/T(W)$ are finite-dimensional.

- (a) Show that T induces an isomorphism of V/W with $T(V)/T(W)$.
- (b) Construct an isomorphism between $T(V)/(W \cap T(V))$ and $(W + T(V))/W$, and an isomorphism between $W/(W \cap T(V))$ and $(W + T(V))/T(V)$.
- (c) Use Problem 2.49(c) to show that $\dim V/(W + T(V)) = \dim(W \cap T(V))/T(W)$.
- (d) Conclude finally that $\dim V/T(V) = \dim W/T(W)$.

Proof of (a).

- (1) Define a map $\bar{T} : V/W \rightarrow T(V)/T(W)$ by

$$\bar{T} : v + W \mapsto T(v) + T(W).$$

- (2) Show that \bar{T} is well-defined. Suppose $u + W = v + W \in V/W$. So $u - v \in W$. So $T(u - v) = T(u) - T(v) \in T(W)$ (since T is a linear map). Hence, $T(u) + T(W) = T(v) + T(W)$.
- (3) \bar{T} is a linear map since T is a linear map.
- (4) \bar{T} is surjective by construction. Also, \bar{T} is injective since T is injective. Therefore, $\bar{T} : V/W \xrightarrow{\sim} T(V)/T(W)$ is isomorphic.

□

Proof of (b).

- (1) Put $N = T(V)$ and $P = W$ in Problem 2.52 to get

$$T(V)/(W \cap T(V)) \cong (W + T(V))/W.$$

- (2) Put $N = W$ and $P = T(V)$ in Problem 2.52 to get

$$W/(W \cap T(V)) \cong (W + T(V))/T(V).$$

□

Proof of (c).

- (1) Note that $W \subseteq W + T(V) \subseteq V$ as vector spaces and V/W is finite-dimensional. By Problem 2.49(c),

$$\dim V/W = \dim V/(W + T(V)) + \dim(W + T(V))/W.$$

- (2) Similarly, $T(V) \subseteq W \cap T(V) \subseteq T(W)$ as vector spaces and $T(V)/T(W)$ is finite-dimensional (since $V/T(W)$ is finite-dimensional and $T(V)/T(W)$ is a subspace of $V/T(W)$). Again by Problem 2.49(c),

$$\dim T(V)/T(W) = \dim T(V)/(W \cap T(V)) + \dim(W \cap T(V))/T(W).$$

- (3) By (a),

$$\dim V/W = \dim T(V)/T(W).$$

By (b),

$$\dim(W + T(V))/W = \dim T(V)/(W \cap T(V)).$$

Hence, the result is established by (1)(2).

□

Proof of (d).

- (1) Note that $V/T(V)$ is finite-dimensional. By Problem 2.49(c),

$$\dim V/T(V) = \dim V/(W + T(V)) + \dim(W + T(V))/T(V).$$

- (2) Similarly,

$$\dim W/T(W) = \dim W/(W \cap T(V)) + \dim(W \cap T(V))/T(W).$$

- (3) By (b),

$$\dim(W + T(V))/T(V) = \dim W/(W \cap T(V)).$$

By (c),

$$\dim V/(W + T(V)) = \dim(W \cap T(V))/T(W).$$

Hence, the result is established by (1)(2).

□

2.11. Free Modules

Problem 2.54.

What does M being free on m_1, \dots, m_n say in terms of the elements of M ?

Proof.

- (1) Any element $m \in M$ can be written uniquely as

$$m = \sum_{i=1}^n r_i m_i$$

for some $r_i \in R$ (which is analogous to the vector space).

- (2) The number of members in a basis for M is called the **rank** of M . That is, $n = \text{rank}(M)$.

□

Problem 2.55.

Let $f = x^n + a_1x^{n-1} + \cdots + a_n$ be a monic polynomial in $R[x]$. Show that $R[x]/(f)$ is a free R -module with basis $\bar{1}, \bar{x}, \dots, \bar{x}^{n-1}$, where \bar{x} is the residue of x .

Proof.

- (1) Given any $\bar{g} \in R[x]/(f)$ where

$$g = b_0x^m + b_1x^{m-1} + \cdots + b_mx \in R[x],$$

it suffices to show that \bar{g} is a linear combination of

$$\mathcal{B} := \{\bar{1}, \bar{x}, \dots, \bar{x}^{n-1}\}.$$

- (2) By the division-with-remainder property of $R[x]$,

$$g = fq + r$$

where $q, r \in R[x]$ with $r = c_0x^{n-1} + \cdots + c_{n-1}$. Hence,

$$\bar{g} = \bar{f}\bar{q} + \bar{r} = \bar{r} = c_0\bar{x}^{n-1} + \cdots + c_{n-1}\bar{1}$$

is a linear combination of \mathcal{B} .

□

Problem 2.56.

Show that a subset X of a module M generates M if and only if the homomorphism $M_X \rightarrow M$ is onto. Every module is isomorphic to a quotient of a free module.

Proof.

- (1) If X generates M , then for any $m \in M$ we can write

$$m = \sum_{x \in X} a_x x$$

as a finite sum where $a_x \in R$ and $x \in X \subseteq M$. Define $\varphi_x \in M_X$ by $\varphi_x(y) = \delta_{xy}$ where δ_{xy} is the Kronecker delta. Hence, the homomorphism $\alpha : M_X \rightarrow M$ maps the finite sum $\varphi := \sum_{x \in X} a_x \varphi_x$ to $\sum_{x \in X} a_x x = m$.

- (2) Conversely, if the homomorphism $\alpha : M_X \rightarrow M$ is onto, then for any $m \in M$ there is a finite sum $\varphi := \sum_{x \in X} a_x \varphi_x$ such that $\alpha(\varphi) = m$. Hence,

$$m = \alpha(\varphi) = \alpha \left(\sum_{x \in X} a_x \varphi_x \right) = \sum_{x \in X} a_x x$$

is generated by X .

- (3) Let

$$F = \bigoplus_{m \in M} R$$

be a free module. Define a map $\varphi : F \rightarrow M$ by

$$\varphi : (0, \dots, 0, \underbrace{1}_{m\text{th position}}, 0, \dots, 0) \mapsto m.$$

φ is well-defined. φ is a module homomorphism. φ is surjective. Hence

$$M \cong F/\ker(\varphi)$$

is isomorphic to a quotient of a free module.

□

Chapter 3: Local Properties of Plane Curves

3.1. Multiple Points and Tangent Lines

Problem 3.1.

Prove that in the above examples $P = (0, 0)$ is the only multiple point on the curves $c = y^2 - x^3$, $d = y^2 - x^3 - x^2$, $e = (x^2 + y^2)^2 + 3x^2y - y^3$, and $f = (x^2 + y^2)^3 - 4x^2y^2$.

Proof.

(1)

$$\begin{aligned}\frac{\partial c}{\partial x} &= -3x^2 = 0 \\ \frac{\partial c}{\partial y} &= 2y = 0\end{aligned}$$

implies that $(x, y) = (0, 0)$. Note that $c(0, 0) = \frac{\partial f}{\partial c}(0, 0) = \frac{\partial c}{\partial y}(0, 0) = 0$. So $(x, y) = (0, 0)$ is the only multiple point on c .

(2)

$$\begin{aligned}\frac{\partial d}{\partial x} &= -3x^2 - 2x = 0 \\ \frac{\partial d}{\partial y} &= 2y = 0\end{aligned}$$

implies that $(x, y) = (0, 0) \in d$ is the only multiple point on d . (Note that $(x, y) = (-\frac{2}{3}, 0) \notin d$.)

(3)

$$\begin{aligned}\frac{\partial e}{\partial x} &= 4x(x^2 + y^2) + 6xy = 0 \\ \frac{\partial e}{\partial y} &= 4y(x^2 + y^2) + 3x^2 - 3y^2 = 0\end{aligned}$$

implies that $x = 0$ or $4(x^2 + y^2) + 6y = 0$.

(a) $x = 0$ implies that $(x, y) = (0, 0)$ or $(0, 1)$. Note that $(0, 1)$ is a simple point (since $\frac{\partial e}{\partial y}(0, 1) = 1$).

(b) $4(x^2 + y^2) + 6y = 0$ implies that $x^2 + y^2 = -\frac{3y}{2}$ and thus

$$\begin{aligned} 0 &= 4y(x^2 + y^2) + 3x^2 - 3y^2 \\ &= 4y\left(-\frac{3y}{2}\right) + 3x^2 - 3y^2 \\ &= 3(x^2 - 3y^2). \end{aligned}$$

$(x, y) \in e$ implies that

$$\begin{aligned} 0 &= (x^2 + y^2)^2 + 3x^2y - y^3 \\ &= (3y^2 + y^2)^2 + 3(3y^2)y - y^3 \\ &= 8y^3(2y + 1). \end{aligned}$$

So $(x, y) = (0, 0)$ or $\left(\pm\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$. Note that $\frac{\partial e}{\partial x}\left(\pm\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) \neq 0$.

Therefore, $(x, y) = (0, 0)$ is the only multiple point on e .

(4)

$$\begin{aligned} \frac{\partial f}{\partial x} &= 6x(x^2 + y^2)^2 - 8xy^2 = x(6(x^2 + y^2)^2 - 8y^2) = 0 \\ \frac{\partial f}{\partial y} &= 6y(x^2 + y^2)^2 - 8x^2y = y(6(x^2 + y^2)^2 - 8x^2) = 0 \end{aligned}$$

implies that $(x, y) = (0, 0)$ or $6(x^2 + y^2)^2 = 8x^2 = 8y^2$. $6(x^2 + y^2)^2 = 8x^2 = 8y^2$ implies that $x^2 = y^2$. So $(x, y) \in f$ implies that $x^2 = y^2 = \frac{1}{2}$, contrary that $6 = 6(x^2 + y^2)^2 \neq 8x^2 = 4$. Therefore, $(x, y) = (0, 0)$ is the only multiple point on f .

□

Problem 3.2.

Find the multiple points, and the tangent lines at the multiple points, for each of the following curves:

(a) $y^3 - y^2 + x^3 - x^2 + 3xy^2 + 3x^2y + 2xy$.

(b) $x^4 + y^4 - x^2y^2$.

(c) $x^3 + y^3 - 3x^2 - 3y^2 + 3xy + 1$.

(d) $y^2 + (x^2 - 5)(4x^4 - 20x^2 + 25)$.

Sketch the part of the curve in (d) that is contained in $\mathbf{A}^2(\mathbb{R}) \subseteq \mathbf{A}^2(\mathbb{C})$.

Proof of (a).

- (1) Let $f = y^3 - y^2 + x^3 - x^2 + 3xy^2 + 3x^2y + 2xy \in k[x, y]$. So

$$\begin{aligned}\frac{\partial f}{\partial x} &= 3x^2 + 6xy + 3y^2 - 2x + 2y = 0 \\ \frac{\partial f}{\partial y} &= 3x^2 + 6xy + 3y^2 + 2x - 2y = 0\end{aligned}$$

implies that

$$\begin{aligned}6(x + y)^2 &= 0 \\ -4(x - y) &= 0\end{aligned}$$

Note that $f(0, 0) = \frac{\partial f}{\partial x}(0, 0) = \frac{\partial f}{\partial y}(0, 0) = 0$. Hence, $(x, y) = (0, 0)$ is the only multiple point on f .

- (2) Write $f = (y^3 + 3xy^2 + 3x^2y + x^3) + (-x^2 + 2xy - y^2)$. The tangent lines at $(x, y) = (0, 0)$ is the linear factors of $-x^2 + 2xy - y^2 = -(x - y)^2$. Hence, the line $x - y = 0$ is the only tangent line at $(x, y) = (0, 0)$ of the multiplicity = 2.

□

Proof of (b).

- (1) Let $f = x^4 + y^4 - x^2y^2 \in k[x, y]$. So

$$\begin{aligned}\frac{\partial f}{\partial x} &= 4x^3 - 2xy^2 = 0 \\ \frac{\partial f}{\partial y} &= 4y^3 - 2x^2y = 0\end{aligned}$$

implies that $(x, y) = (0, 0)$. Note that $f(0, 0) = \frac{\partial f}{\partial x}(0, 0) = \frac{\partial f}{\partial y}(0, 0) = 0$. Hence, $(x, y) = (0, 0)$ is the only multiple point on f .

- (2) The tangent lines at $(x, y) = (0, 0)$ is the linear factors of $x^4 + y^4 - x^2y^2$. Hence, there are four distinct tangent lines

$$x \pm \sqrt{\frac{1 \pm \sqrt{-3}}{2}}y$$

at $(x, y) = (0, 0)$. Each tangent line is simple.

□

Proof of (c).

- (1) Let $f = x^3 + y^3 - 3x^2 - 3y^2 + 3xy + 1 \in k[x, y]$. So

$$\begin{aligned}\frac{\partial f}{\partial x} &= 3(x^2 - 2x + y) = 0 \\ \frac{\partial f}{\partial y} &= 3(y^2 - 2y + x) = 0\end{aligned}$$

implies that $(x - y)(x + y - 3) = 0$.

- (2) The case $x - y = 0$. Take $x = y$ in $3x^2 - 6x + 3y = 0$ to get $(x, y) = (1, 1), (0, 0)$. $(x, y) = (1, 1)$ is a multiple point on f since $f(1, 1) = \frac{\partial f}{\partial x}(1, 1) = \frac{\partial f}{\partial y}(1, 1) = 0$. $(x, y) = (0, 0)$ is impossible since $f(0, 0) = 1 \neq 0$.
- (3) The case $x + y - 3 = 0$. Take $x = -y + 3$ in f to get $1 = 0$, which is absurd.
- (4) By (2)(3), the only multiple point on f is $(x, y) = (1, 1)$.
- (5) Let $t(x, y) = (x + 1, y + 1)$. Then

$$f^t = f(x + 1, y + 1) = x^3 + y^3 + 3xy.$$

The tangent lines at $(x, y) = (1, 1)$ is the linear factors of $x^3 + y^3 + 3xy$. Hence, there are two distinct simple tangent lines x and y at $(x, y) = (1, 1)$.

□

Proof of (d).

- (1) Let $f = y^2 + (x^2 - 5)(4x^4 - 20x^2 + 25) \in k[x, y]$. So

$$\begin{aligned}\frac{\partial f}{\partial x} &= 2x(2x^2 - 5)(6x^2 - 25) = 0 \\ \frac{\partial f}{\partial y} &= 2y = 0\end{aligned}$$

implies that there are only two multiple points

$$(x, y) = \left(\pm \sqrt{\frac{5}{2}}, 0 \right)$$

on f .

- (2) Let $t(x, y) = \left(x + \sqrt{\frac{5}{2}}, y \right)$. Then

$$\begin{aligned}f^t &= f\left(x + \sqrt{\frac{5}{2}}, y\right) \\ &= 4x^6 + 12\sqrt{10}x^5 + 110x^4 + 20\sqrt{10}x^3 - 100x^2 + y^2.\end{aligned}$$

The tangent lines at $(x, y) = \left(\sqrt{\frac{5}{2}}, 0 \right)$ is the linear factors of $-100x^2 + y^2 = -(10x + y)(10x - y)$. Hence, there are two distinct simple tangent lines

$$10x \pm y$$

at $(x, y) = \left(\sqrt{\frac{5}{2}}, 0 \right)$.

(3) Similarly, there are also two distinct simple tangent lines

$$10x \pm y$$

$$\text{at } (x, y) = \left(-\sqrt{\frac{5}{2}}, 0\right).$$

□

Problem 3.3.

If a curve f of degree n has a point P of multiplicity n , show that f consists of n lines through P (not necessarily distinct).

Proof.

(1) Might assume that $P = (0, 0)$. (Note that any translation of f preserves the degree of f .)

(2) Write

$$f = f_m + f_{m+1} + \cdots + f_n$$

where f_i is a form in $k[x, y]$. Since m is the multiplicity of f at P , $m = n$. Hence, f is a form in two variables of degree n , or f consists of n lines through P .

□

Problem 3.4.

Let P be a double point on a curve f . Show that P is a node if and only if

$$\frac{\partial^2 f}{\partial x \partial y}(P)^2 \neq \frac{\partial^2 f}{\partial x^2}(P) \cdot \frac{\partial^2 f}{\partial y^2}(P).$$

Proof.

(1) Might assume that $P = (0, 0)$ is a double point on f . Write

$$f = f_2 + f_3 + \cdots + f_n \in k[x, y]$$

(where f_i is a form in $k[x, y]$), and

$$f_2 = ax^2 + bxy + cy^2 \in k[x, y].$$

(2) P is a node if and only if the discriminant

$$b^2 - 4ac \neq 0.$$

Note that

$$\begin{aligned}\frac{\partial^2 f}{\partial x \partial y}(P) &= b, \\ \frac{\partial^2 f}{\partial x^2}(P) &= 2a, \\ \frac{\partial^2 f}{\partial y^2}(P) &= 2c.\end{aligned}$$

Hence, P is a node if and only if

$$b^2 - 4ac = b^2 - (2a)(2c) = \frac{\partial^2 f}{\partial x \partial y}(P)^2 - \frac{\partial^2 f}{\partial x^2}(P) \cdot \frac{\partial^2 f}{\partial y^2}(P) \neq 0.$$

□

Problem 3.5.

($\text{char}(k) = 0$) Show that $m_P(f)$ is the smallest integer m such that for some $i + j = m$,

$$\frac{\partial^m f}{\partial x^i \partial y^j}(P) \neq 0.$$

Find an explicit description for the leading form for f at P in terms of these derivatives.

Proof.

(1) Might assume that $P = (0, 0)$. Write $f = f_0 + f_1 + \cdots + f_n$ where f_i is a form in $k[x, y]$. Consider any form f_m of f . Write

$$f_m = c_m x^m + c_{m-1} x^{m-1} y + \cdots + c_0 y^m$$

where $c_i \in k$ and not all c_i are zero.

(2) Similar to Problem 3.4, $\frac{\partial^m f}{\partial x^i \partial y^j}(P) = c_i i! j!$. Here $i + j = m$. Hence,

$$c_i = \frac{1}{i! j!} \frac{\partial^m f}{\partial x^i \partial y^j}(P) = \frac{1}{m!} \binom{m}{i} \frac{\partial^m f}{\partial x^i \partial y^j}(P)$$

and thus

$$f_m = \frac{1}{m!} \sum_{i=0}^m \binom{m}{i} \frac{\partial^m f}{\partial x^i \partial y^j}(P) x^i y^j.$$

- (3) Suppose m is the smallest integer such that for some $i + j = m$,

$$\frac{\partial^m f}{\partial x^i \partial y^j}(P) \neq 0.$$

Then $f_0 = f_1 = \cdots = f_{m-1} = 0$ and $f_m \neq 0$ in $k[x, y]$ by (2). Therefore, $m = m_P(f)$. The explicit description for the leading form f_m for f at P is already stated in (2).

□

Problem 3.6.

Irreducible curves with given tangent lines ℓ_i of multiplicity r_i may be constructed as follows: if $\sum r_i = m$, let $f = \prod \ell_i^{r_i} + f_{m+1}$, where f_{m+1} is chosen to make f irreducible (see Problem 2.34).

Proof.

- (1) Let $f_m = \prod \ell_i^{r_i} \in k[x, y]$. Problem 1.4 implies that there exists a point $P = (a, b)$ such that $f_m(P) \neq 0$ since $f_m \neq 0 \in k[x, y]$.
- (2) Let $\ell : bx - ay = 0$ and $f_{m+1} = \ell^{m+1}$. Since $(a, b) \neq (0, 0)$, $\deg(\ell) = 1$. Also, $\ell_i \nmid \ell$ by (1). Hence, f_m and f_{m+1} have no common factors. By Problem 2.34, $f = f_m + f_{m+1}$ is irreducible.

□

Problem 3.7.

- (a) Show that the real part of the curve e of the examples is the set of points in $\mathbf{A}^2(\mathbb{R})$ whose polar coordinates (r, θ) satisfy the equation $r = -\sin(3\theta)$. Find the polar equation for the curve f .
- (b) If n is an odd integer ≥ 1 , show that the equation $r = \sin(n\theta)$ defines the real part of a curve of degree $n + 1$ with an ordinary n -tuple point at $(0, 0)$. (Use the fact that $\sin(n\theta) = \operatorname{im}(e^{in\theta})$ to get the equation; note that rotation by $\frac{\pi}{n}$ is a linear transformation that takes the curve into itself.)
- (c) Analyze the singularities that arise by looking at $r^2 = \sin^2(n\theta)$, n even.
- (d) Show that the curves constructed in (b) and (c) are all irreducible in $\mathbf{A}^2(\mathbb{C})$. (Hint: Make the polynomials homogeneous with respect to a variable z , and use §2.6.)

Proof of (a).

(1) De Moivre's theorem implies that

$$\begin{aligned}
 \cos(n\theta) + i \sin(n\theta) &= (\cos \theta + i \sin \theta)^n \\
 &= \sum_{k=0}^n \binom{n}{k} (\cos \theta)^{n-k} i^k (\sin \theta)^k \\
 &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{2k} (\cos \theta)^{n-2k} (\sin \theta)^{2k} \\
 &\quad + i \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{2k+1} (\cos \theta)^{n-2k-1} (\sin \theta)^{2k+1}.
 \end{aligned}$$

Hence,

$$\sin(n\theta) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{2k+1} (\cos \theta)^{n-2k-1} (\sin \theta)^{2k+1}.$$

In particular,

$$\sin(3\theta) = 3 \cos^2 \theta \sin \theta - \sin^3 \theta.$$

(2) $r = -\sin(3\theta)$ implies that

$$\begin{aligned}
 r^4 &= r^3(-\sin(3\theta)) \\
 &= r^3(-3 \cos^2 \theta \sin \theta + \sin^3 \theta) \\
 &= -3(r \cos \theta)^2(r \sin \theta) + (r \sin \theta)^3.
 \end{aligned}$$

Hence, $r = -\sin(3\theta)$ implies that

$$(x^2 + y^2)^2 = -3x^2y + y^3$$

in $\mathbf{A}^2(\mathbb{R})$.

(3) As $f(x, y) = (x^2 + y^2)^3 - 4x^2y^2$, $f(r \cos \theta, r \sin \theta) = 0$ implies that

$$r^6 = 4r^4 \cos^2 \theta \sin^2 \theta = r^4 \sin^2 2\theta$$

or

$$r^2 = \sin^2 2\theta.$$

□

Proof of (b).

(1) By (a), $r = \sin(n\theta)$ with odd $n \geq 1$ implies that

$$\begin{aligned} r &= \sin(n\theta) = \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{2k+1} (\cos \theta)^{n-2k-1} (\sin \theta)^{2k+1} \\ \implies r^{n+1} &= \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{2k+1} (r \cos \theta)^{n-2k-1} (r \sin \theta)^{2k+1} \\ \implies (x^2 + y^2)^{\frac{n+1}{2}} &= \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1}. \end{aligned}$$

Hence, $r = \sin(n\theta)$ defines the real part of a curve

$$\alpha : (x^2 + y^2)^{\frac{n+1}{2}} - \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1}$$

of degree $n+1$.

(2) Note that $(0,0) \in \alpha$. Since

$$\begin{aligned} \frac{\partial \alpha}{\partial x} &= (n+1)(x^2 + y^2)^{\frac{n-1}{2}} x \\ &\quad - \sum_{k=0}^{\frac{n-3}{2}} (-1)^k (n-2k-1) \binom{n}{2k+1} x^{n-2k-2} y^{2k+1} \\ \frac{\partial \alpha}{\partial y} &= (n+1)(x^2 + y^2)^{\frac{n-1}{2}} y \\ &\quad - \sum_{k=0}^{\frac{n-1}{2}} (-1)^k (2k+1) \binom{n}{2k+1} x^{n-2k-1} y^{2k}, \end{aligned}$$

$\frac{\partial \alpha}{\partial x}(0,0) = \frac{\partial \alpha}{\partial y}(0,0) = 0$. $(0,0)$ is a multiple point.

(3) The tangents at $(0,0)$ are the linear factors of

$$\sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1}.$$

Clearly, y is a tangent line at $(0,0)$. Note that rotation by $\frac{\pi}{n}$ is a linear transformation that takes the curve into itself. Hence, all tangents at $(0,0)$ are

$$\ell_k : \sin \frac{k\pi}{n} x - \cos \frac{k\pi}{n} y$$

for $k = 0, 1, \dots, n-1$. All ℓ_k are pairwise distinct, and thus $(0,0)$ is an ordinary n -tuple point.

□

Proof of (c).

- (1) Similar to (b), $r^2 = \sin^2(n\theta)$ defines the real part of a curve

$$\beta : (x^2 + y^2)^{n+1} - \left(\sum_{k=0}^{\frac{n-2}{2}} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1} \right)^2$$

of degree $2n + 2$.

- (2) Note that

$$\beta(0, 0) = \frac{\partial \beta}{\partial x}(0, 0) = \frac{\partial \beta}{\partial y}(0, 0) = 0.$$

Hence, $(0, 0)$ is a multiple point.

- (3) Similar to (b), all tangents at $(0, 0)$ are

$$\ell_k : \sin \frac{k\pi}{n} x - \cos \frac{k\pi}{n} y$$

of multiplicity $= 2$ for $k = 0, 1, \dots, n-1$.

□

Proof of (d).

- (1) The case n is odd.

(a) Consider

$$\alpha : (x^2 + y^2)^{\frac{n+1}{2}} - \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1}.$$

(b) Since

$$\begin{aligned} \alpha_{n+1} &:= (x^2 + y^2)^{\frac{n+1}{2}} \\ &= (x + iy)^{\frac{n+1}{2}} (x - iy)^{\frac{n+1}{2}} \\ \alpha_n &= \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1} \\ &= c \prod_{k=0}^{n-1} \left(\sin \frac{k\pi}{n} x - \cos \frac{k\pi}{n} y \right) \end{aligned}$$

(with some $c \in \mathbb{C}$), α_{n+1} and α_n have no common factors in $\mathbb{C}[x, y]$.
By Problem 2.34, α is irreducible.

(2) The case n is even.

(a) Consider

$$\beta : \underbrace{(x^2 + y^2)^{n+1}}_{:=\beta_{2n+2}} - \underbrace{\left(\sum_{k=0}^{\frac{n-2}{2}} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1} \right)^2}_{:=\beta_{2n}}.$$

(b) Similar to the proof of Problem 2.34, suppose $\beta = \beta_{2n} + \beta_{2n+2} = rs \in \mathbb{C}[x, y]$. So

$$(\beta_{2n} + \beta_{2n+2})^* = (rs)^* \implies z^2 \beta_{2n} + \beta_{2n+2} = r^* s^*.$$

Note that $\deg_z(z^2 \beta_{2n} + \beta_{2n+2}) = 2$. So $\deg_z(r^*) = 0, 1, 2$.

(c) The case $\deg_z(r^*) = 0, 2$ is similar to the proof of Problem 2.34 because β_{2n} and β_{2n+2} have no common factors in $\mathbb{C}[x, y]$.

(d) The case $\deg_z(r^*) = 1$. (So $\deg_z(s^*) = 1$.) Write $r = r_p + r_{p+1}$ and $s = s_q + s_{q+1}$. Hence, $\beta = rs$ implies that

$$\begin{aligned} r_p s_q &= - \left(\sum_{k=0}^{\frac{n-2}{2}} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1} \right)^2 \\ r_p s_{q+1} + r_{p+1} s_q &= 0 \\ r_{p+1} s_{q+1} &= (x^2 + y^2)^{n+1} = (x + iy)^{n+1} (x - iy)^{n+1}. \end{aligned}$$

Since $n+1$ is odd and $x \pm iy \nmid \left(\sum_{k=0}^{\frac{n-2}{2}} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1} \right)^2$, $r_p s_{q+1} + r_{p+1} s_q = 0$ implies that $r_p = s_q = 0$, which is absurd.

(e) By (b)(c)(d), β is irreducible over \mathbb{C} .

□

Problem 3.8.

Let $t : \mathbf{A}^2 \rightarrow \mathbf{A}^2$ be a polynomial map, $t(Q) = P$.

(a) Show that $m_Q(f^t) \geq m_P(f)$.

(b) Let $t = (t_1, t_2)$, and define

$$J_Q t = \left(\frac{\partial t_i}{\partial x_j}(Q) \right)$$

to be the **Jacobian matrix** of t at Q . Show that $m_Q(f^t) = m_P(f)$ if $J_Q t$ is invertible.

- (c) Show that the converse of (b) is false: let $t = (x^2, y)$, $f = y - x^2$, $P = Q = (0, 0)$.

Proof of (a).

- (1) Might assume that $P = Q = (0, 0)$. Write $t = (t_1, t_2)$ and thus $(0, 0) = t(0, 0) = (t_1(0, 0), t_2(0, 0))$. So there is no nonzero constant term in t_i ($i = 1, 2$).
- (2) Might assume that $f \neq 0$ (since there is nothing to prove when $f = 0$). Write $f = f_m + f_{m+1} + \cdots + f_n$ where f_i is a form in $k[x, y]$ and $f_m \neq 0$. So,

$$f(t_1(x, y), t_2(x, y)) = f_m(t_1(x, y), t_2(x, y)) + \cdots + f_n(t_1(x, y), t_2(x, y))$$

has the multiplicity $= \infty$ or $\geq m = m_P(f)$. In any case, $m_Q(f^t) \geq m_P(f)$.

□

Proof of (b).

- (1) Might assume that $P = Q = (0, 0)$. Since $J_Q t$ is invertible,

$$\begin{pmatrix} \frac{\partial t_1}{\partial x}(Q) & \frac{\partial t_1}{\partial y}(Q) \\ \frac{\partial t_2}{\partial x}(Q) & \frac{\partial t_2}{\partial y}(Q) \end{pmatrix} \neq 0$$

(as vectors). Hence t_1 (resp. t_2) has the multiplicity $= 1$.

- (2) Define

$$s = (s_1, s_2) : \mathbf{A}^2 \rightarrow \mathbf{A}^2$$

be a polynomial map such that s_i is the linear term of t_i . Note that $J_Q s = J_Q t$ is invertible and $m_Q(f^s) = m_Q(f^t)$ for any f .

- (3) Show that $m_Q(f^s) = m_Q(f^t)$ for any $f \in k[x, y]$. Might assume that $f \neq 0$ (since there is nothing to prove when $f = 0$). Write $f = f_m + f_{m+1} + \cdots + f_n$ where f_i is a form in $k[x, y]$ and $f_m \neq 0$. Since

$$m_Q(f_i^t) = m_Q(f_i^s) = i \text{ or } \infty,$$

$$m_Q(f^s) = m_Q(f^t).$$

- (4) Since $J_Q s$ is invertible, s^{-1} is also a polynomial map with an invertible Jacobian matrix $J_Q s^{-1}$. By (a),

$$m_Q(f^s) \geq m_P(f) = m_P((f^s)^{s^{-1}}) = m_Q(f^s)$$

or $m_Q(f^s) = m_P(f)$. Therefore, $m_Q(f^t) = m_Q(f^s) = m_P(f)$.

□

Proof of (c). $m_P(f) = 1$ and $m_Q(f^t) = 1$ since $f^t = y - x^4$. However,

$$J_Q t = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

is not invertible. □

Problem 3.9.

Let $f \in k[x_1, \dots, x_n]$ define a hypersurface $V(f) \subseteq \mathbf{A}^n$. Let $P \in \mathbf{A}^n$.

- (a) Define the multiplicity $m_P(f)$ of f at P .
- (b) If $m_P(f) = 1$, define the tangent hyperplane to f at P .
- (c) Examine $f = x^2 + y^2 - z^2$, $P = (0, 0, 0)$. Is it possible to define tangent hyperplanes at multiple points?

Proof of (a).

- (1) Let $P = (0, \dots, 0)$. Write $f = f_m + f_{m+1} + \dots + f_n$, where f_i is a form in $k[x_1, \dots, x_n]$ of degree i , $f_m \neq 0$. We define m to be the multiplicity of f at $P = (0, \dots, 0)$, write $m = m_P(f)$.
- (2) To extend these definitions to a point $P = (a_1, \dots, a_n) \neq (0, \dots, 0)$, let t be the translation that takes $(0, \dots, 0)$ to P , i.e.,

$$t(x_1, \dots, x_n) = (x_1 + a_1, \dots, x_n + a_n).$$

Then

$$f^t = f(x_1 + a_1, \dots, x_n + a_n).$$

Define $m_P(f)$ to be $m_{(0, \dots, 0)}(f^t)$, i.e., write $f^t = g_m + g_{m+1} + \dots$, g_i forms, $g_m \neq 0$, and let $m = m_P(f)$.

□

Proof of (b).

- (1) Let $P = (0, \dots, 0)$. Write $f = f_m + f_{m+1} + \dots + f_n$, where f_i is a form in $k[x_1, \dots, x_n]$ of degree i , $f_m \neq 0$. If $m = 1$, then $f_m = f_1$ is a hyperplane. Hence, we can define the tangent hyperplane to f at P to be f_1 .
- (2) Similar to (a), the tangent hyperplane to f at $P = (a_1, \dots, a_n) \neq (0, \dots, 0)$ is g_1 where

$$f^t = g_1 + g_2 + \dots$$

and $t(x_1, \dots, x_n) = (x_1 + a_1, \dots, x_n + a_n)$.

□

Proof of (c).

- (1) No.
- (2) *Show that $x^2 + y^2 - z^2$ is irreducible over k .* (Reductio ad absurdum)
 Suppose $x^2 + y^2 - z^2$ were reducible. By Problem 1.1, we can write

$$x^2 + y^2 - z^2 = (a_1x + a_2y + a_3z)(b_1x + b_2y + b_3z)$$

for some $a_i, b_i \in k$ ($i = 1, 2, 3$). Expanding out the right hand side and comparing coefficients to get

$$\begin{aligned} a_1b_1 &= a_2b_2 = -a_3b_3 = 1 \\ a_1b_2 + a_2b_1 &= a_2b_3 + a_3b_2 = a_3b_1 + a_1b_3 = 0. \end{aligned}$$

So $a_i, b_i \neq 0$ for all i and

$$b_1 = \frac{-a_1}{a_3}b_3 = \frac{-a_1}{a_3} \cdot \frac{-a_3}{a_2}b_2 = \frac{-a_1}{a_3} \cdot \frac{-a_3}{a_2} \cdot \frac{-a_2}{a_1}b_1 = -b_1.$$

Hence, $b_1 = 0$, which is absurd.

- (3) Since $x^2 + y^2 - z^2$ is irreducible over any field k with $\text{char}(k) = 0$, it is impossible to define tangent hyperplanes at $(0, 0, 0)$.

□

Problem 3.10.

Show that an irreducible plane curve has only a finite number of multiple points. Is this true for hypersurfaces?

Proof.

- (1) Let $f \in k[x, y]$ be an irreducible plane curve, and

$$V = V\left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

be the set of the multiple (singular) points of f . Moreover, V is an algebraic set.

- (2) Suppose $\frac{\partial f}{\partial x} \neq 0$ (since not all $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are zero). It is nothing to do if $\frac{\partial f}{\partial x}$ is a nonzero constant. Suppose $\deg(\frac{\partial f}{\partial x}) \geq 1$. Since f is irreducible and $\deg(\frac{\partial f}{\partial x}) = \deg f - 1$, f and $\frac{\partial f}{\partial x}$ have no common factors. By Proposition 2 in §1.6, $V\left(f, \frac{\partial f}{\partial x}\right)$ is a finite set. Hence, $V \subseteq V\left(f, \frac{\partial f}{\partial x}\right)$ is finite as a subset of a finite set.

- (3) The conclusion is not true for hypersurfaces when $n \geq 3$. Consider $f = x_2^2 - x_1^3 \in k[x_1, \dots, x_n]$. The set of the multiple (singular) points of f is

$$\{(0, 0, a_3, \dots, a_n) : a_3, \dots, a_n \in k\},$$

which is infinite as k is infinite.

□

Problem 3.11. (Tangent space)

Let $V \subseteq \mathbf{A}^n$ be an affine variety, $P \in V$. The **tangent space** $T_P(V)$ is defined to be

$$\left\{ (v_1, \dots, v_n) \in \mathbf{A}^n : \sum_i \frac{\partial g}{\partial x_i}(P) v_i = 0 \ \forall g \in I(V) \right\}.$$

If $V = V(f)$ is a hypersurface, f irreducible, show that

$$T_P(V) = \left\{ (v_1, \dots, v_n) \in \mathbf{A}^n : \sum_i \frac{\partial f}{\partial x_i}(P) v_i = 0 \right\}.$$

How does the dimension of $T_P(V)$ relate to the multiplicity of f at P ?

Proof.

- (1) By the Hilbert's Nullstellensatz, the irreducibility of f implies that $I(V) = I(V(f)) = (f)$.

- (2) Let

$$W = \left\{ (v_1, \dots, v_n) \in \mathbf{A}^n : \sum_i \frac{\partial f}{\partial x_i}(P) v_i = 0 \right\}.$$

$W \supseteq T_P(V)$ is true since $f \in I(V) = (f)$.

- (3) Show that $W \subseteq T_P(V)$. Given any $(v_1, \dots, v_n) \in W$. Now for any $g \in I(V) = (f)$, there exists a $h \in k[x_1, \dots, x_n]$ such that $g = fh$. Hence,

$$\begin{aligned} \sum_i \frac{\partial g}{\partial x_i}(P) v_i &= \sum_i \frac{\partial(fh)}{\partial x_i}(P) v_i \\ &= \sum_i \left(\underbrace{\frac{\partial(f)}{\partial x_i}(P)}_{=0} h(P) + f(P) \frac{\partial h}{\partial x_i}(P) \right) v_i \\ &= h(P) \underbrace{\sum_i \frac{\partial f}{\partial x_i}(P) v_i}_{=0} \\ &= 0 \end{aligned}$$

implies that $(v_1, \dots, v_n) \in T_P(V)$.

(4) By definition of $T_P(V)$,

$$\dim_k(T_P(V)) = \begin{cases} n-1 & \text{if } m_P(f) = 1 \\ n & \text{if } m_P(f) > 1. \end{cases}$$

□

3.2. Multiplicities and Local Rings

Problem 3.12. (Flex)

A simple point P on a curve f is called a **flex** if $\text{ord}_P^f(L) \geq 3$, where L is the tangent to f at P . The flex is called **ordinary** if $\text{ord}_P(L) = 3$, a **higher flex** otherwise.

- (a) Let $f = y - x^n$. For which n does f have a flex at $P = (0, 0)$, and what kind of flex?
- (b) Suppose $P = (0, 0)$, $L = y$ is the tangent line, $f = y + ax^2 + \cdots$. Show that P is a flex on f if and only if $a = 0$. Give a simple criterion for calculating $\text{ord}_P^f(y)$, and therefore for determining if P is a higher flex.

Proof of (a).

- (1) When $n = 0$ or 1 , the tangent line L to f at any point is $L = f$ itself. So

$$\text{ord}_P^f(L) = \text{ord}_P^f(f) = \text{ord}_P^f(0) = \infty.$$

P is a higher flex.

- (2) When $n > 1$, the tangent line L to f at $P = (0, 0)$ is $L = y$. So

$$\text{ord}_P^f(L) = \text{ord}_P^f(y) = \text{ord}_P^f(x^n) = n.$$

Here x is a uniformizing parameter for $\mathcal{O}_P(f)$ since the line x is not tangent to f (Theorem 1). Hence, P is a flex if $n \geq 3$, P is an ordinary flex if $n = 3$, and P is a higher flex if $n > 3$.

□

Proof of (b).

- (1) Since y is the tangent line, $\text{ord}_P^f(y) \geq 2$. By Problem 2.29(a),

$$\text{ord}_P^f(y) = \text{ord}_P^f(ax^2 + \cdots) = 2$$

if and only if $a \neq 0$. Hence, P is flex iff $\text{ord}_P^f(y) \geq 3$ iff $a = 0$.

(2) In general,

$$\text{ord}_P^f(y) = \text{ord}_P^f(ax^2 + \cdots) = m_P(ax^2 + \cdots) = m_P(f - y).$$

Hence, P is a higher flex if $f - y$ has no nonzero form of degree 3.

□

Problem 3.13.*

With the notation of Theorem 2, and $\mathfrak{m} = \mathfrak{m}_P(f)$, show that $\dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = n + 1$ for $0 \leq n < m_P(f)$. In particular, P is a simple point if and only if $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 1$; otherwise $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 2$.

Proof.

(1) From the exact sequence

$$0 \rightarrow \mathfrak{m}^n/\mathfrak{m}^{n+1} \rightarrow \mathcal{O}/\mathfrak{m}^{n+1} \rightarrow \mathcal{O}/\mathfrak{m}^n \rightarrow 0,$$

it suffices to show that

$$\dim_k(\mathcal{O}/\mathfrak{m}^n) = \frac{n(n+1)}{2}$$

as $0 \leq n < m_P(f)$. (Problem 2.49 and Proposition 7 in §2.10.)

(2) We may assume that $P = (0, 0)$. Similar to the proof of Theorem 2, we are reduced to calculating the dimension of $k[x, y]/(I^n, f)$. Let $m = m_P(f)$. By the definition of m ,

$$f \in \underbrace{(x, y)}_{=I}^m = I^m.$$

So if $0 \leq n < m_P(f)$, then $f \in I^m \subseteq I^n$ and thus $(I^n, f) = I^n$. Therefore,

$$\begin{aligned} \dim_k(\mathcal{O}/\mathfrak{m}^n) &= \dim_k(k[x, y]/(I^n, f)) \\ &= \dim_k(k[x, y]/I^n) \\ &= \frac{n(n+1)}{2}. \end{aligned} \quad (\text{Problem 2.46})$$

So

$$\dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = \dim_k(\mathcal{O}/\mathfrak{m}^{n+1}) - \dim_k(\mathcal{O}/\mathfrak{m}^n) = n + 1.$$

(3) P is a simple point if $m = m_P(f) = 1$ by definition. Note that

$$\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \begin{cases} 1 & \text{if } m = 1 \text{ (Theorem 1)} \\ 2 & \text{if } m > 1 \text{ ((2))}. \end{cases}$$

Therefore, P is a simple iff $m = 1$ iff $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 1$.

□

Problem 3.14.

Let $V = V(x^2 - y^3, y^2 - z^3) \subseteq \mathbf{A}^3$, $P = (0, 0, 0)$, $\mathfrak{m} = \mathfrak{m}_P(V)$. Find $\dim_k(\mathfrak{m}/\mathfrak{m}^2)$. (See Problem 1.40.)

Proof.

(1) $\mathfrak{m} = (x, y, z)$.

(2) Write $\mathcal{O} = \mathcal{O}_P(V)$. By Problem 1.40(a), every element of \mathcal{O} is of the form

$$\bar{a} + \bar{x}\bar{b} + \bar{y}\bar{c} + \bar{x}\bar{y}\bar{d}$$

for some $a, b, c, d \in k[z]$.

(3) By (1)(2), $\{\bar{1}\}$ (resp. $\{\bar{1}, \bar{z}, \bar{y}, \bar{z}\}$) is a basis for \mathcal{O}/\mathfrak{m} (resp. $\mathcal{O}/\mathfrak{m}^2$). Hence, $\dim_k(\mathcal{O}/\mathfrak{m}) = 1$ (resp. $\dim_k(\mathcal{O}/\mathfrak{m}^2) = 4$). Therefore,

$$\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \dim_k(\mathcal{O}/\mathfrak{m}^2) - \dim_k(\mathcal{O}/\mathfrak{m}) = 3$$

by Proposition 7 in §2.10.

(4) By Theorem I.5.1 in *Robin Hartshorne, Algebraic Geometry*,

$$3 = \dim_k(\mathfrak{m}/\mathfrak{m}^2) + \text{rank} J = \dim_k(\mathfrak{m}/\mathfrak{m}^2).$$

Here the Jacobian matrix of V at P

$$J = \left[\frac{\partial f_i}{\partial x_j}(P) \right] = \begin{pmatrix} 2x & -3y^2 & 0 \\ 0 & 2y & -3z^2 \end{pmatrix}_{P=(0,0,0)} = 0$$

has rank zero.

□

Problem 3.15.

(a) Let $\mathcal{O} = \mathcal{O}_P(\mathbf{A}^2)$ for some $P \in \mathbf{A}^2$, $\mathfrak{m} = \mathfrak{m}_P(\mathbf{A}^2)$. Calculate $\chi(n) = \dim_k(\mathcal{O}/\mathfrak{m}^n)$.

(b) Let $\mathcal{O} = \mathcal{O}_P(\mathbf{A}^r(k))$. Show that $\chi(n)$ is a polynomial of degree r in n , with leading coefficient $\frac{1}{r!}$ (see Problem 2.36).

Proof of (a). Might assume that $P = (0, 0)$. By Problem 2.46, the Hilbert-Samuel polynomial is

$$\chi(n) = \dim_k(\mathcal{O}/\mathfrak{m}^n) = \dim_k(k[x, y]/(x, y)^n) = \frac{n(n+1)}{2}.$$

□

Proof of (b).

(1) Might assume that $P = (0, \dots, 0)$. Similar to (a),

$$\chi(n) = \dim_k(\mathcal{O}/\mathfrak{m}^n) = \dim_k(k[x_1, \dots, x_r]/(x_1, \dots, x_r)^n).$$

(2) Since

$$\mathcal{B} = \{x_1^{i_1} x_2^{i_2} \cdots x_r^{i_r} : i_1 + \cdots + i_r < n\}$$

is a basis for $k[x_1, \dots, x_r]/(x_1, \dots, x_r)^n$,

$$\dim_k(k[x_1, \dots, x_r]/(x_1, \dots, x_r)^n) = |\mathcal{B}|.$$

(3) By the stars and bars (combinatorics) method (as Problem 2.35(b)),

$$\begin{aligned} |\mathcal{B}| &= |\{(i_1, \dots, i_r) : i_1 + \cdots + i_r \leq n\}| \\ &\quad - |\{(i_1, \dots, i_r) : i_1 + \cdots + i_r = n\}| \\ &= |\{(i_1, \dots, i_r, j) : i_1 + \cdots + i_r + j = n\}| \\ &\quad - |\{(i_1, \dots, i_r) : i_1 + \cdots + i_r = n\}| \\ &= \binom{n+r}{r} - \binom{n+r-1}{r-1} \\ &= \binom{n+r-1}{r} \\ &= \frac{1}{r!} (n+r-1)(n+r-2) \cdots (n+1)(n). \end{aligned}$$

So

$$\chi(n) = \frac{1}{r!} (n+r-1)(n+r-2) \cdots (n+1)(n)$$

is a polynomial of degree r in n , with leading coefficient $\frac{1}{r!}$.

(4) By Problem 2.36, we can also deduce that

$$\begin{aligned} \dim_k(k[x_1, \dots, x_r]/(x_1, \dots, x_r)^n) &= \sum_{i=0}^{n-1} \dim_k V(i, r) \\ &= \sum_{i=0}^{n-1} \binom{i+r-1}{r-1} \\ &= \binom{n+r-1}{r} \end{aligned}$$

by the Pascal's identity.

□

Problem 3.16.

Let $f \in k[x_1, \dots, x_r]$ define a hypersurface in \mathbf{A}^r . Write $f = f_m + f_{m+1} + \dots$, and let $m = m_P(f)$ where $P = (0, \dots, 0)$. Suppose f is irreducible, and let $\mathcal{O} = \mathcal{O}_P(V(f))$, \mathfrak{m} its maximal ideal. Show that $\chi(n) = \dim_k(\mathcal{O}/\mathfrak{m}^n)$ is a polynomial of degree $r-1$ for sufficiently large n , and that the leading coefficient of χ is $\frac{m_P(f)}{(r-1)!}$. Can you find a definition for the multiplicity of a local ring that makes sense in all the cases you know?

Proof.

- (1) Similar to the proof of Theorem 2. By Problem 2.43,

$$\mathfrak{m}^n = I^n \mathcal{O}$$

where $I = (x_1, \dots, x_r) \subseteq k[x_1, \dots, x_r]$. Since $V(I^n) = \{P\}$,

$$k[x_1, \dots, x_r]/(I^n, f) \cong \mathcal{O}_P(\mathbf{A}^r)/(I^n, f) \mathcal{O}_P(\mathbf{A}^r) \cong \mathcal{O}/I^n \mathcal{O} \cong \mathcal{O}/\mathfrak{m}^n$$

(Corollary 2 to Proposition 6 in §2.9 and Problem 2.44).

- (2) So we are reduced to calculating the dimension of $k[x_1, \dots, x_r]/(I^n, f)$. As $n \geq m = m_P(f)$, there is a natural ring homomorphism

$$\varphi : k[x_1, \dots, x_r]/I^n \rightarrow k[x_1, \dots, x_r]/(I^n, f)$$

and a k -linear map

$$\psi : k[x_1, \dots, x_r]/I^{n-m} \rightarrow k[x_1, \dots, x_r]/I^n$$

defined by $\bar{g} \mapsto \overline{fg}$. It is easy to verify that the sequence

$$0 \rightarrow k[x_1, \dots, x_r]/I^{n-m} \xrightarrow{\psi} k[x_1, \dots, x_r]/I^n \xrightarrow{\varphi} k[x_1, \dots, x_r]/(I^n, f) \rightarrow 0$$

is exact.

- (3) By Problem 3.15,

$$\begin{aligned} & \dim_k(k[x_1, \dots, x_r]/(I^n, f)) \\ &= \binom{n+r-1}{r} - \binom{n-m+r-1}{r} \\ &= \frac{1}{r!} ((n+r-1) \cdots n - (n-m+r-1) \cdots (n-m)) \\ &= \frac{1}{r!} (n^{r-1}(rm) + \dots) \\ &= \frac{m}{(r-1)!} n^{r-1} + \dots \end{aligned}$$

Therefore, $\chi(n) = \dim_k(\mathcal{O}/\mathfrak{m}^n)$ is a polynomial of degree $r-1$ for $n \geq m$, and that the leading coefficient of χ is $\frac{m_P(f)}{(r-1)!}$.

(4) It is reasonable to define the multiplicity of a Noetherian local ring \mathcal{O} by

$$(d!) \cdot (\text{leading coefficient of } \chi(n))$$

for sufficiently large n , where d is the dimension (or Krull dimension) of \mathcal{O} . (Note that the dimension of a hypersurface in \mathbf{A}^r is $r - 1$.)

□

3.3. Intersection Numbers

Problem 3.17.

Find the intersection numbers of various pairs of curves

(a) $a = y - x^2$

(b) $b = y^2 - x^3 + x$

(c) $c = y^2 - x^3$

(d) $d = y^2 - x^3 - x^2$

(e) $e = (x^2 + y^2)^2 + 3x^2y - y^3$

(f) $f = (x^2 + y^2)^3 - 4x^2y^2$

at the point $P = (0, 0)$.

Proof.

(1) Note that Example in §3.3 shows that $I(P, e \cap f) = 14$. Also,

$$I(P, a \cap b) = m_P(a)m_P(b) = 1 \cdot 1 = 1$$

$$I(P, a \cap d) = m_P(a)m_P(d) = 1 \cdot 2 = 2$$

$$I(P, b \cap c) = m_P(b)m_P(c) = 1 \cdot 2 = 2$$

$$I(P, b \cap d) = m_P(b)m_P(d) = 1 \cdot 2 = 2$$

$$I(P, b \cap e) = m_P(b)m_P(e) = 1 \cdot 3 = 3$$

$$I(P, c \cap d) = m_P(c)m_P(d) = 2 \cdot 2 = 4$$

$$I(P, d \cap e) = m_P(d)m_P(e) = 2 \cdot 3 = 6$$

$$I(P, d \cap f) = m_P(d)m_P(f) = 2 \cdot 4 = 8$$

by Property (5).

(2) Show that $I(P, a \cap c) = 3$.

$$\begin{aligned}
I(P, a \cap c) &= I(P, a \cap (c + (-x^2 - y)a)) && \text{(Property (7))} \\
&= I(P, a \cap x^3(x - 1)) \\
&= 3I(P, a \cap x) + I(P, a \cap (x - 1)) && \text{(Property (6))} \\
&= 3I(P, a \cap x) && \text{(Property (2))} \\
&= 3. && \text{(Property (5))}
\end{aligned}$$

(3) Show that $I(P, a \cap e) = 4$.

$$\begin{aligned}
I(P, a \cap e) &= I(P, a \cap (e + (x^2 + 2y^2 + 4y)a)) && \text{(Property (7))} \\
&= I(P, a \cap y^2(y^2 + y + 4)) \\
&= 2I(P, a \cap y) + I(P, a \cap (y^2 + y + 4)) && \text{(Property (6))} \\
&= 2I(P, a \cap y) && \text{(Property (2))} \\
&= 2I(P, (a - y) \cap y) && \text{(Property (7))} \\
&= 2I(P, (-x^2) \cap y) \\
&= 4. && \text{(Property (5))}
\end{aligned}$$

(4) Show that $I(P, a \cap f) = 6$. Similar to (3). Let

$$q_4 = x^4 + 3x^2y^2 + 3y^4 + x^2y + 3y^3 - 3y^2.$$

So

$$\begin{aligned}
I(P, a \cap f) &= I(P, a \cap \underbrace{(f + q_4a)}_{\text{(Property (7))}}) \\
&= I(P, a \cap y^3(y^3 + 3y^2 + 3y - 3)) \\
&\quad \text{(Property (6))} \\
&= \underbrace{3I(P, a \cap y)}_{=2 \text{ (by (3))}} + \underbrace{I(P, a \cap (y^3 + 3y^2 + 3y - 3))}_{=0 \text{ (Property (2))}} \\
&= 6.
\end{aligned}$$

(5) Show that $I(P, b \cap f) = 6$. Similar to (3). Let

$$q_5 = -x^6 - 3x^5 - x^3y^2 - x^4 - 3x^2y^2 - y^4 + 3x^3 + xy^2 + 3x^2.$$

So

$$\begin{aligned}
& I(P, b \cap f) \\
&= I(P, b \cap \underbrace{(f + q_5 b)}_{\text{(Property (7))}}) \\
&= I(P, b \cap x^3(x^6 + 3x^5 - 5x^3 - 4x^2 + 3x + 3)) \\
&\quad \text{(Property (6))} \\
&= \underbrace{3I(P, b \cap x)}_{=2} + \underbrace{I(P, b \cap (x^6 + 3x^5 - 5x^3 - 4x^2 + 3x + 3))}_{=0 \text{ (Property (2))}} \\
&= 6.
\end{aligned}$$

Here

$$I(P, b \cap x) = I(P, (b + (x^2 - 1)x) \cap x) = I(P, y^2 \cap x) = 2.$$

(6) Show that $I(P, c \cap e) = 7$. Similar to (3).

$$\begin{aligned}
I(P, c \cap e) &= I(P, c \cap \underbrace{(e + (-x^3 - 2x^2 - y^2 + y)c)}_{\text{(Property (7))}}) \\
&= I(P, c \cap x^2(x^4 + 2x^3 + x^2 - xy + 3y)) \\
&\quad \text{(Property (6))} \\
&= 2 \underbrace{I(P, c \cap x)}_{=2 \text{ (Property (5))}} + I(P, c \cap \underbrace{(x^4 + 2x^3 + x^2 - xy + 3y)}_{:=h_6}) \\
&= 4 + I(P, c \cap h_6).
\end{aligned}$$

Here

$$\begin{aligned}
& I(P, c \cap h_6) \\
&= I(P, (3c) \cap h_6) \\
&= I(P, (3c - yh_6) \cap h_6) \\
&= I(P, (-x^4y - 2x^3y - \underbrace{3x^3 - x^2y + xy^2}_{\text{}}) \cap (x^4 + 2x^3 + x^2 - xy + \underbrace{3y}_{\text{}})) \\
&= 3 \cdot 1 \\
&= 3
\end{aligned}$$

by Properties (5) and (7). Therefore, $I(P, c \cap e) = 4 + 3 = 7$.

(7) Show that $I(P, c \cap f) = 10$. Similar to (5). Let

$$q_7 = -x^6 - 3x^5 - x^3y^2 - 3x^4 - 3x^2y^2 - y^4 + 4x^2.$$

So

$$\begin{aligned}
& I(P, c \cap f) \\
&= I(P, c \cap \underbrace{(f + q_7 c)}_{\text{(Property (7))}}) \\
&= I(P, c \cap x^5(x^4 + 3x^3 + 3x^2 + x - 4)) \\
&\quad \text{(Property (6))} \\
&= \underbrace{5 I(P, c \cap x)}_{=2} + \underbrace{I(P, c \cap (x^4 + 3x^3 + 3x^2 + x - 4))}_{=0 \text{ (Property (2))}} \\
&= 10.
\end{aligned}$$

□

Problem 3.18.

Give a proof of Property (8) that uses only Properties (1)-(7).

Recall Properties (1)-(8):

- (1) $I(P, f \cap g)$ is a nonnegative integer for any f, g , and P such that f and g intersect properly at P . $I(P, f \cap g) = \infty$ if f and g do not intersect properly at P .
- (2) $I(P, f \cap g) = 0$ if and only if $P \notin f \cap g$. $I(P, f \cap g)$ depends only on the components of f and g that pass through P .
- (3) If t is an affine change of coordinates on \mathbf{A}^2 , and $t(Q) = P$, then

$$I(P, f \cap g) = I(Q, f^t \cap g^t).$$

- (4) $I(P, f \cap g) = I(P, g \cap f)$.
- (5) $I(P, f \cap g) \geq m_P(f)m_P(g)$, with equality occurring if and only if f and g have not tangent lines in common at P .
- (6) If $f = \prod f_i^{r_i}$, and $g = \prod g_j^{s_j}$, then

$$I(P, f \cap g) = \sum_{i,j} r_i s_j I(P, f_i \cap g_j).$$

- (7) $I(P, f \cap g) = I(P, f \cap (g + af))$ for any $a \in k[x, y]$.
- (8) If P is a simple point on f , then $I(P, f \cap g) = \text{ord}_P^f(g)$.

Proof.

- (1) Might assume that f is irreducible. There is nothing to prove if $\text{ord}_P^f(g) = \infty$. Might assume that $n = \text{ord}_P^f(g) < \infty$.
- (2) Similar to the proof of Theorem 1 in §3.2, Property (3) implies that we might assume that $P = (0, 0)$, that y is the tangent line, and that x is one line through P which is not tangent to f at P . Here x is a uniformizing parameter for $\mathcal{O}_P(f)$ (Theorem 1 in §3.2).
- (3) By definition,

$$g = ux^n$$

for some unit in $\mathcal{O}_P(f)$. Write $u = \frac{a}{b}$ where $a, b \in k[x, y]$, $a(P) \neq 0$ and $b(P) \neq 0$. Hence,

$$bg = ax^n + cf$$

in $k[x, y]$ for some $c \in k[x, y]$.

- (4) Therefore,

$$\begin{aligned}
 I(P, f \cap g) &= I(P, f \cap bg) - I(P, f \cap b) && \text{(Property (6))} \\
 &= I(P, f \cap bg) - 0 && \text{(Property (2))} \\
 &= I(P, f \cap (ax^n + cf)) && \text{(Step (3))} \\
 &= I(P, f \cap ax^n) && \text{(Property (7))} \\
 &= I(P, f \cap a) + nI(P, f \cap x) && \text{(Property (6))} \\
 &= 0 + nI(P, f \cap x) && \text{(Property (2))} \\
 &= n. && \text{(Property (5))}
 \end{aligned}$$

□

Problem 3.19.*

A line L is tangent to a curve f at a point P if and only if $I(P, f \cap L) > m_P(f)$.

Proof.

- (1) Note that $m_P(L) = 1$ and the only tangent line of L is itself.
- (2) By Property (5),

$$\begin{aligned}
 I(P, f \cap L) &> m_P(f) = m_P(f)m_P(L) \\
 \iff f \text{ and } L &\text{ have one common tangent line at } P \\
 \iff L \text{ is tangent} &\text{ to a curve } f \text{ at } P. && \text{(By (1))}
 \end{aligned}$$

□

Problem 3.20.

If P is a simple point on f , then $I(P, f \cap (g+h)) \geq \min\{I(P, f \cap g), I(P, f \cap h)\}$. Give an example to show that this may be false if P is not simple on f .

Proof.

(1)

$$\begin{aligned} I(P, f \cap (g+h)) &= \text{ord}_P^f(g+h) && \text{(Property (8))} \\ &\geq \min\{\text{ord}_P^f(g), \text{ord}_P^f(h)\} && \text{(Problem 2.28)} \\ &= \min\{I(P, f \cap g), I(P, f \cap h)\}. && \text{(Property (8))} \end{aligned}$$

(2) Pick $P = (0, 0)$, $f = (x^2 + y^2)^3 - 4x^2y^2$, $g = x$ and $h = y$. By Property (5), $I(P, f \cap (g+h)) = 4$, $I(P, f \cap g) > 4$ and $I(P, f \cap h) > 4$. So

$$I(P, f \cap (g+h)) < \min\{I(P, f \cap g), I(P, f \cap h)\}$$

for such example.

□

Problem 3.21.

Let f be an affine plane curve. Let L be a line that is not a component of f . Suppose $L = \{(a+tb, c+td) : t \in k\}$. Define $g(t) = f(a+tb, c+td)$. Factor $g(t) = \prod (t - \lambda_i)^{e_i}$, λ_i distinct. Show that there is a natural one-to-one correspondence between the λ_i and the points $P_i \in L \cap f$. Show that under this correspondence, $I(P_i, L \cap f) = e_i$. In particular, $\sum I(P, L \cap f) \leq \deg(f)$.

Proof.

(1) Show that there is a natural one-to-one correspondence between the λ_i and the points $P_i \in L \cap f$.

$$\begin{aligned} P_i \in L \cap f &\iff P_i \in L \text{ and } P \in f \\ &\iff \exists \lambda \in k \text{ such that } 0 = f(P_i) = f(a + \lambda b, c + \lambda d) = g(\lambda) \\ &\iff \lambda \in k \text{ is a root of } g(t) = \prod (t - \lambda_i)^{e_i} \\ &\iff \lambda = \lambda_i \in k \text{ for some } i. \end{aligned}$$

(2) Show that $I(P_i, L \cap f) = e_i$. By Property (3), we may suppose $P_i = (0, 0)$ and $L = y$. Write

$$f(x, y) = f_0(x) + f_1(x)y + f_2(x)y^2 + \cdots \in (k[x])[y]$$

where $f_j \in k[x]$. Note that $f_0(x) = f(x, 0) = g(x)$. So

$$\begin{aligned}
I(P_i, L \cap f) &= I(P_i, y \cap (f_0(x) + f_1(x)y + \cdots)) \\
&= I(P_i, y \cap f_0(x)) && \text{(Property (7))} \\
&= I(P_i, y \cap g(x)) \\
&= I\left(P_i, y \cap \prod (x - \lambda_i)^{e_i}\right) \\
&= \sum_j e_j I(P_j, y \cap (x - \lambda_j)) && \text{(Property (6))} \\
&= e_i I(P_i, y \cap x) && \text{(Property (2))} \\
&= e_i. && \text{(Property (5))}
\end{aligned}$$

Here $\lambda_i = 0$ by the correspondence of (1).

(3) In particular,

$$\sum_i I(P_i, L \cap f) = \sum_i e_i = \deg(g(x)) = \deg(f(x, 0)) \leq \deg(f(x, y)).$$

□

Problem 3.22. (Cusp)

Suppose P is a double point on a curve f , and suppose f has only one tangent L at P .

- (a) Show that $I(P, f \cap L) \geq 3$. The curve f is said to have an (ordinary) **cusp** at P if $I(P, f \cap L) = 3$.
- (b) Suppose $P = (0, 0)$, and $L = y$. Show that P is a cusp if and only if $\frac{\partial^3 f}{\partial x^3}(P) \neq 0$. Give some examples.
- (c) Show that if P is a cusp on f , then f has only one component passing through P .

Might assume that $\text{char}(k) = 0$.

Proof of (a). Since $I(P, f \cap L) > m_P(f) = 2$ (Problem 3.19), $I(P, f \cap L) \geq 3$ (Property (1)). □

Proof of (b).

(1) By assumption,

$$f = y^2 + f_3 + f_4 + \cdots$$

where f_i is a form in $k[x, y]$.

(2) Hence, P is a cusp of f if and only if

$$\begin{aligned}
3 &= I(P, f \cap y) \\
&= I(P, (y^2 + f_3 + f_4 + \cdots) \cap y) \\
&= I(P, (f_3 + f_4 + \cdots) \cap y) \\
&\geq m_P(f_3 + f_4 + \cdots)m_P(y) \\
&\geq 3.
\end{aligned}$$

Here the equality is occurring if and only if y is not a tangent line of $f_3 + f_4 + \cdots$ (Property (5)).

(3) Note that

$$\begin{aligned}
&y \text{ is not a tangent line of } f_3 + f_4 + \cdots \\
&\iff y \nmid f_3 \\
&\iff \frac{\partial^3 f}{\partial x^3}(P) \neq 0.
\end{aligned} \tag{Problem 3.5}$$

(4) Examples: $y^2 = x^3$, $y^2 = -x^2y^2 + x^3$ and so on.

□

Proof of (c).

(1) Might assume $P = (0, 0)$ and $L = y$ by Property (3).

(2) Given $f = gh$. Write

$$\begin{aligned}
f &= y^2 + \text{higher terms} \\
g &= g_r + \text{higher terms} \\
h &= h_s + \text{higher terms}
\end{aligned}$$

where g_r (resp. h_s) is a form of degree r (resp. s) in $k[x, y]$. So $f = gh$ implies that

$$\begin{aligned}
y^2 + \text{higher terms} &= (g_r + \text{higher terms})(h_s + \text{higher terms}) \\
&= (g_r h_s) + \text{higher terms}.
\end{aligned}$$

Hence $y^2 = g_r h_s$. In particular, $2 = r + s$.

(3) If $y \mid g_r$ and $y \mid h_s$, then

$$\begin{aligned}
I(P, g \cap L) &> m_P(g)m_P(L) = r \implies I(P, g \cap L) \geq r + 1 \\
I(P, h \cap L) &> m_P(h)m_P(L) = s \implies I(P, h \cap L) \geq s + 1.
\end{aligned}$$

So,

$$\begin{aligned}
3 &= I(P, f \cap L) \\
&= I(P, g \cap L) + I(P, h \cap L) \\
&\geq (r+1) + (s+1) \\
&= 4,
\end{aligned}$$

which is absurd. So we might assume that $g_r = 1$ and $h_s = y^2$. $f = gh$ implies that $g = 1 + g_1 + \cdots$ is not passing through P . Hence the conclusion is established.

□

Problem 3.23. (Hypercusp)

A point P on a curve f is called a **hypercusp** if $m_P(f) > 1$, f has only one tangent line L at P , and $I(P, L \cap f) = m_P(f) + 1$. Generalize the results of the preceding problem to this case.

Generalization.

- (a) $I(P, f \cap L) \geq m_P(f) + 1$.
- (b) Suppose $P = (0, 0)$, $L = y$ and $m = m_P(f)$. Then P is a hypercusp if and only if $\frac{\partial^{m+1} f}{\partial x^{m+1}}(P) \neq 0$. Give some examples.
- (c) If P is a hypercusp on f , then f has only one component passing through P .

The proof is almost the same as Problem 3.22 by replacing 2 by $m_P(f)$. □

Problem 3.24.*

The object of this problem is to find a property of the local ring $\mathcal{O}_P(f)$ that determines whether or not P is an ordinary multiple point on f . Let f be an irreducible plane curve, $P = (0, 0)$, $\mathfrak{m} = \mathfrak{m}_P(f) > 1$. Let $m = m_P(f)$. For $g \in k[x, y]$ or $\in \Gamma(f)$, denote its residue in $\mathfrak{m}/\mathfrak{m}^2$ by \bar{g} .

- (a) Show that the map from $V = \{\text{forms of degree 1 in } k[x, y]\}$ to $\mathfrak{m}/\mathfrak{m}^2$ taking $ax + by$ to $\bar{ax + by}$ is an isomorphism of vector spaces (see Problem 3.13).
- (b) Suppose P is an ordinary multiple point, with tangents L_1, \dots, L_m . Show that $I(P, f \cap L_i) > m$ and $\bar{L}_i \neq \lambda \bar{L}_j$ for all $i \neq j$, and all $\lambda \in k$.

- (c) Suppose there are $g_1, \dots, g_m \in k[x, y]$ such that $I(P, f \cap g_i) > m$ and $\overline{g_i} \neq \lambda \overline{g_j}$ for all $i \neq j$, and all $\lambda \in k$. Show that P is an ordinary multiple point on f . (Hint: Write $g_i = L_i + \text{higher terms} \in k[x, y]$. $\overline{L_i} = \overline{g_i} \neq 0$, and L_i is the tangent to g_i , so L_i is tangent to f by Property (5) of intersection numbers. Thus f has m tangents at P .)
- (d) Show that P is an ordinary multiple point on f if and only if there are $g_1, \dots, g_m \in \mathfrak{m}$ such that $\overline{g_i} \neq \lambda \overline{g_j}$ for all $i \neq j$, $\lambda \in k$, and

$$\dim_k \mathcal{O}_P(f)/(g_i) > m.$$

Proof of (a).

- (1) $\mathcal{B} = \{x, y\}$ is a basis for V as a k -vector space.
- (2) $\mathcal{B}' = \{\overline{x}, \overline{y}\}$ is a basis for $\mathfrak{m}/\mathfrak{m}^2$ as a k -vector space.
- (3) By (1)(2), we can define a canonical isomorphism

$$\alpha : V \rightarrow \mathfrak{m}/\mathfrak{m}^2$$

by sending \mathcal{B} to \mathcal{B}' , that is, $\alpha(x) = \overline{x}$ and $\alpha(y) = \overline{y}$.

□

Proof of (b).

- (1) Write

$$f = \prod_{i=1}^m L_i + \text{higher terms}.$$

Problem 3.19 says that $I(P, f \cap L_i) > m_P(f) = m$.

- (2) Since P is an ordinary multiple point on f , L_i and L_j are linearly independent in V in the sense of (a). Hence, $\alpha(L_i) = \overline{L_i}$ and $\alpha(L_j) = \overline{L_j}$ are linearly independent in $\mathfrak{m}/\mathfrak{m}^2$ (since α is an isomorphism). The conclusion holds.

□

Proof of (c).

- (1) Write $g_i = L_i + \text{higher terms} \in k[x, y]$. Here g_i has no constant term since $P \in g_i$ by the definition of intersection numbers.
- (2) Pick $\lambda = 0 \in k$ and $j \neq i$. ($m > 1$ implies the existence of j .) So

$$\alpha(L_i) = \overline{L_i} = \overline{g_i} \neq \lambda \overline{g_j} = 0$$

or $L_i \neq 0$ (since α is an isomorphism).

- (3) Hence, L_i is the tangent to g_i . So Property (5) implies that

$$I(P, f \cap g_i) \geq m_P(f)m_P(g_i) = m.$$

Note that L_i is the only tangent line of g_i . By Property (5), the assumption $I(P, f \cap g_i) > m$ implies that L_i is tangent to f .

- (4) Note that $\overline{g_i} \neq \lambda \overline{g_j}$ for all $i \neq j$, and all $\lambda \in k$. So $\overline{L_i} \neq \lambda \overline{L_j}$ for all $i \neq j$, and all $\lambda \in k$. Since α is an isomorphism, all L_i are linearly independent and thus f has m tangents L_1, \dots, L_m at P . Therefore, P is an ordinary multiple point.

□

Proof of (d).

- (1) Note that

$$\dim_k \mathcal{O}_P(f)/(g_i) = \dim_k \mathcal{O}_P(\mathbf{A}^2)/(f, g_i) = I(P, f \cap g_i)$$

(by Problem 2.44).

- (2) (\implies) Suppose that P is an ordinary multiple point, with tangents L_1, \dots, L_m . By (b),

$$\dim_k \mathcal{O}_P(f)/(L_i) = I(P, f \cap L_i) > m$$

and $\overline{L_i} \neq \lambda \overline{L_j}$ for all $i \neq j$, and all $\lambda \in k$. Take

$$g_i = L_i + I(f) \in \Gamma(f) \subseteq \mathcal{O}_P(f).$$

Since $g_i \in \mathfrak{m}$ (by $L_i(P) = 0$) and $\overline{g_i} = \overline{L_i}$, the conclusion is proved.

- (3) (\impliedby) Suppose that there are $g_1, \dots, g_m \in \mathfrak{m}$ such that $\overline{g_i} \neq \lambda \overline{g_j}$, and $\dim_k \mathcal{O}_P(f)/(g_i) > m$. For each $i = 1, \dots, m$, we take $g'_i + I(f) = \underline{g_i} \in \mathfrak{m}$ for some $g'_i \in k[x, y]$. Although g'_i is not uniquely determined by g_i , $\overline{g'_i} = \overline{g_i}$ and thus $\overline{g'_i} \neq \lambda \overline{g'_j}$. By (1),

$$\dim_k \mathcal{O}_P(f)/(g_i) = \dim_k \mathcal{O}_P(f)/(g'_i) = I(P, f \cap g'_i) > m$$

Hence, by (c) f has m tangents at P .

□

Chapter 4: Projective Varieties

4.1. Projective Space

Problem 4.1.

What points in \mathbf{P}^2 do not belong to two of the three sets U_1, U_2, U_3 ?

Proof.

- (1) The point $[1 : 0 : 0]$ does not belong to U_2 and U_3 .
- (2) The point $[0 : 1 : 0]$ does not belong to U_3 and U_1 .
- (3) The point $[0 : 0 : 1]$ does not belong to U_1 and U_2 .

□

Problem 4.2.*

Let $f \in k[x_1, \dots, x_{n+1}]$ (k infinite). Write $f = \sum f_i$, f_i a form of degree i . Let $P \in \mathbf{P}^n(k)$, and suppose $f(x_1, \dots, x_{n+1}) = 0$ for every choice of homogeneous coordinates (x_1, \dots, x_{n+1}) for P . Show that each $f_i(x_1, \dots, x_{n+1}) = 0$ for all homogeneous coordinates for P . (Hint: consider

$$g(\lambda) = f(\lambda x_1, \dots, \lambda x_{n+1}) = \sum \lambda^i f_i(x_1, \dots, x_{n+1})$$

for fixed (x_1, \dots, x_{n+1}) .)

Proof.

- (1) Consider

$$g(\lambda) = f(\lambda x_1, \dots, \lambda x_{n+1}) = \sum \lambda^i f_i(x_1, \dots, x_{n+1})$$

for fixed (x_1, \dots, x_{n+1}) . $g(\lambda)$ is a polynomial in $k[\lambda]$.

- (2) Since $g(\lambda) = f(\lambda x_1, \dots, \lambda x_{n+1}) = 0$ for all $\lambda \in k - \{0\}$, $g(\lambda) = 0$ has infinitely many solutions in k . Similar to Problem 1.4, $g = 0 \in k[\lambda]$, that is, each $f_i(x_1, \dots, x_{n+1}) = 0$ for all homogeneous coordinates for P .

□

Problem 4.3.

- (a) Show that the definitions of this section carry over without change to the case where k is an arbitrary field.
- (b) If k_0 is a subfield of k , show that $\mathbf{P}^n(k_0)$ may be identified with a subset of $\mathbf{P}^n(k)$.

Proof of (a). Note that a field is a commutative ring where $0 \neq 1$ and all nonzero elements are invertible. Hence the definitions in this section are well-defined for any field k . \square

Proof of (b). Note that $0 \in k_0$ and $0 \in k$. So any point $P \in \mathbf{P}^n(k_0)$ is also in $\mathbf{P}^n(k)$ since $P \neq (0, \dots, 0)$ and

$$\{(\lambda x_1, \dots, \lambda x_{n+1}) : \lambda \in k_0\} \subseteq \{(\lambda x_1, \dots, \lambda x_{n+1}) : \lambda \in k\}$$

as a subset. \square

4.2. Projective Algebraic Sets**Problem 4.4.***

Let I be a homogeneous ideal in $k[x_1, \dots, x_{n+1}]$. Show that I is prime if and only if the following condition is satisfied: for any forms $f, g \in k[x_1, \dots, x_{n+1}]$, if $fg \in I$, then $f \in I$ or $g \in I$.

Proof.

- (1) (\implies) Trivial.
- (2) (\impliedby) Suppose that $f, g \in k[x_1, \dots, x_{n+1}]$ and $fg \in I$. Write $f = \sum_{i=0}^r f_i$ (resp. $g = \sum_{j=0}^s g_j$), f_i a form of degree i (resp. g_j a form of degree j). Induction on the $\deg(fg) = r + s$.
- (3) When $r + s = 0$, nothing to do.
- (4) Assume that the result is true for smaller values of $r + s$. Then the highest homogeneous component $f_r g_s$ of fg is also in I since I is homogeneous. By assumption, $f_r \in I$ or $g_s \in I$, and might say that $f_r \in I$. Therefore,

$$(f - f_r)g \in I.$$

By the induction hypothesis, $f - f_r \in I$ or $g \in I$. Hence, $f = (f - f_r) + f_r \in I$ or $g \in I$.

(5) Therefore, (3)(4) implies that I is prime.

□

Problem 4.5.

If I is a homogeneous ideal, show that $\text{rad}(I)$ is also homogeneous.

Proof.

- (1) Given any $f = \sum_{i=0}^r f_i \in \text{rad}(I)$, f_i a form of degree i . It suffices to show that each $f_i \in \text{rad}(I)$. Note that $f^m \in I$ for some $m > 0$.
- (2) The highest homogeneous component f_r^m of f^m is also in I since I is homogeneous. Hence, $f_r \in \text{rad}(I)$. Again note that $f - f_r \in \text{rad}(I)$ and $\deg(f - f_r) < r$. Continue this process (or by induction), and we have $f_{r-1}, \dots, f_0 \in \text{rad}(I)$.

□

Problem 4.6.

State and prove the projective analogues of properties (1)-(10) of Chapter 1, Sections 2 and 3.

Statements.

- (1) *If I is the homogeneous ideal in $k[x_1, \dots, x_{n+1}]$ generated by a set of forms S , then $V(S) = V(I)$; so every algebraic set is equal to $V(I)$ for some homogeneous ideal I .*
- (2) *If $\{I_\alpha\}$ is any collection of homogeneous ideals, then $V(\cup_\alpha I_\alpha) = \cap_\alpha V(I_\alpha)$; so the intersection of any collection of algebraic sets is an algebraic set.*
- (3) *If $I \subseteq J$, then $V(I) \supseteq V(J)$.*
- (4) *$V(fg) = V(f) \cup V(g)$ for any forms f, g ; $V(I) \cup V(J) = V(\{fg : f \in I, g \in J\})$; so any finite union of algebraic sets is an algebraic set.*
- (5) *$V(0) = \mathbf{P}^n(k)$; $V(I) = \emptyset$ if I contains all forms of degree $\geq N$ for some N ; $V(a_i x_j - a_j x_i) = \{[a_1 : \dots : a_{n+1}]\}$ for $[a_1 : \dots : a_{n+1}] \in \mathbf{P}^n(k)$. So any finite subset of $\mathbf{P}^n(k)$ is an algebraic set.*
- (6) *If $X \subseteq Y$ are nonempty, then $I(X) \supseteq I(Y)$.*
- (7) *$I(\emptyset) = k[x_1, \dots, x_{n+1}]$; $I(\mathbf{P}^n(k)) = (0)$ if k is an infinite field;*

- (8) $I(V(S)) \supseteq S$ for any set S of forms; $V(I(X)) \supseteq X$ for any set X of points.
- (9) $V(I(V(S))) = V(S)$ for any set S of forms, and $I(V(I(X))) = I(X)$ for any set X of points. So if V is an algebraic set, $V = V(I(V))$, and if I is the homogeneous ideal of an algebraic set, $I = I(V(I))$.
- (10) $I(X)$ is a radical ideal for any nonempty $X \subseteq \mathbf{P}^n(k)$.

Proof. Proposition 1 and the projective Nullstellensatz give all. \square

Problem 4.7.

Show that each irreducible component of a cone is also a cone.

Proof.

- (1) Let V is an algebraic set in \mathbf{P}^n , and $C(V)$ be the cone over V . Might assume that $V \neq \emptyset$.
- (2) *Show that V is irreducible if and only if $C(V)$ is irreducible.* V is irreducible if and only if $I_p(V) = I_a(C(V))$ is prime if and only if $C(V)$ is irreducible.
- (3) Let $V = V_1 \cup \cdots \cup V_r$ be the decomposition of an algebraic set into irreducible components. Note that

$$C(V_1 \cup \cdots \cup V_r) = C(V_1) \cup \cdots \cup C(V_r)$$

(by the definition of the cone). Here each $C(V_i)$ is irreducible (by (2)). By Theorem 2 in §1.5, each irreducible component of $C(V)$ must be one of $C(V_i)$, which is also a cone.

\square

Problem 4.8.

Let $V = \mathbf{P}^1$, $\Gamma_h(V) = k[x, y]$. Let $t = x/y \in k(V)$, and show that $k(V) = k(t)$. Show that there is a natural one-to-one correspondence between the points of \mathbf{P}^1 and the DVRs with quotient field $k(V)$ that contain k (see Problem 2.27); which DVR corresponds to the point at infinity?

Proof.

- (1) *Show that $k(V) = k(t)$. Given any $f/g \in k(V)$ where $f, g \in \Gamma_h(V)$ are of the same degree. Then*

$$f(x, y)/g(x, y) = f(t, 1)/g(t, 1) \in k(t).$$

Conversely, given any $f/g \in k(t)$,

$$\frac{f(t)}{g(t)} = \frac{f(x/y)}{g(x/y)} = \frac{y^d f(x/y)}{y^d g(x/y)} \in k(V)$$

where $d = \max\{\deg(f), \deg(g)\}$.

- (2) Note that $k = \bar{k}$. By Problem 2.27, the DVR's with quotient field $k(V) = k(t)$ are

$$\mathcal{O}_a(\mathbf{A}^1) \text{ where } a \in \mathbf{A}^1 = k \text{ and } \mathcal{O}_\infty(\mathbf{A}^1),$$

which correspond to

$$\mathbf{P}^1(k) = \{[a : 1] : a \in k\} \cup \{[1 : 0]\}.$$

In particular, the DVR $\mathcal{O}_\infty(\mathbf{A}^1)$ corresponds to the point at infinity ($= [1 : 0]$).

□

Problem 4.9.*

Let I be a homogeneous ideal in $k[x_1, \dots, x_{n+1}]$, and

$$\Gamma = k[x_1, \dots, x_{n+1}]/I.$$

Show that the forms of degree d in Γ form a finite-dimensional vector space over k .

Proof.

- (1) Write $R = k[x_1, \dots, x_{n+1}]$. For R (resp. Γ), define $R_{(d)}$ (resp. $\Gamma_{(d)}$) to be the corresponding homogeneous component of degree d . Consider a natural homomorphism

$$\alpha : R_{(d)} \rightarrow R_{(d)}/I \cong \Gamma_{(d)}$$

by $\alpha(h) = \bar{h}$ for any form h of degree d .

- (2) $\Gamma_{(d)}$ can be regarded as a subspace of $R_{(d)}$ since α is surjective. Since $R_{(d)}$ is finite-dimensional with $\dim R_{(d)} = \binom{d+n-1}{n-1}$, $\Gamma_{(d)}$ is finite-dimensional by linear algebra.

□

Problem 4.10.

Let $R = k[x, y, z]$, $f \in R$ an irreducible form of degree n , $V = V(f) \subseteq \mathbf{P}^2$, and $\Gamma = \Gamma_h(V)$.

(a) Construct an exact sequence

$$0 \rightarrow R \xrightarrow{\psi} R \xrightarrow{\varphi} \Gamma \rightarrow 0$$

where ψ is multiplication by f .

(b) Show that

$$\dim_k \{\text{forms of degree } d \text{ in } \Gamma\} = dn - \frac{n(n-3)}{2}$$

if $d > n$.

Proof of (a).

- (1) ψ is defined by $\psi(g) = fg$ and φ is naturally defined by $\varphi(h) = \bar{h}$. ψ and φ are well-defined and homomorphisms of vector space.
- (2) $\text{im}(\psi) = \ker(\varphi) = (f) = I(f)$ (since f is irreducible). ψ is injective since $R = k[x, y, z]$ is a domain. φ is surjective trivially. Hence, the sequence of vector spaces over k is exact.

□

Proof of (b).

(1) The exact sequence

$$0 \rightarrow R \xrightarrow{\psi} R \xrightarrow{\varphi} \Gamma \rightarrow 0$$

induces the exact sequence

$$0 \rightarrow R_{(d-n)} \xrightarrow{\psi} R_{(d)} \xrightarrow{\varphi} \Gamma_{(d)} \rightarrow 0,$$

where $*_{(d)}$ (resp. $*_{(d-n)}$) denotes the corresponding homogeneous component of degree d (resp. $d-n$).

(2) By Problem 2.36,

$$\dim_k R_{(d-n)} = \binom{d-n+2}{2}, \quad \dim_k R_{(d)} = \binom{d+2}{2}.$$

(3) Since $R_{(d)}$ is finite-dimensional, $\Gamma_{(d)}$ is also finite-dimensional (by regarding $\Gamma_{(d)}$ as a subspace of $R_{(d)}$). Proposition 7 in §2.9 shows that

$$\begin{aligned} \dim_k \Gamma_{(d)} &= \dim_k R_{(d)} - \dim_k R_{(d-n)} \\ &= \binom{d+2}{2} - \binom{d-n+2}{2} \\ &= dn - \frac{n^2}{2} + \frac{3n}{2}. \end{aligned}$$

□

Problem 4.11.* (Linear subvariety)

A set $V \subseteq \mathbf{P}^n(k)$ is called a **linear subvariety** of $\mathbf{P}^n(k)$ if $V = V(h_1, \dots, h_r)$, where each h_i is a form of degree 1.

- (a) Show that if t is a projective change of coordinates, then $V^t = t^{-1}(V)$ is also a linear subvariety.
- (b) Show that there is a projective change of coordinates t of \mathbf{P}^n such that $V^t = V(x_{m+2}, \dots, x_{n+1})$, so V is a variety.
- (c) Show that the m that appears in part (b) is independent of the choice of t . It is called the **dimension** of V ($m = -1$ if $V = \emptyset$).

Proof of (a).

- (1) Say $t = (t_1, \dots, t_{n+1})$ is a projective change of coordinates, and $V = V(h_1, \dots, h_r)$, where each h_i is a form of degree 1.
- (2) Show that V is a variety and thus $I(V) = (h_1, \dots, h_r)$ by the projective Nullstellensatz. V is the set of all non-trivial solutions of the system of linear equations:

$$\begin{aligned} h_1 &= a_{1,1}x_1 + \dots + a_{1,n+1}x_{n+1} = 0, \\ &\dots \\ h_r &= a_{r,1}x_1 + \dots + a_{r,n+1}x_{n+1} = 0. \end{aligned}$$

(Here we identify $[x_1 : \dots : x_{n+1}] \in \mathbf{P}^n$.) Write $Ax = 0$ and thus $V = V(Ax = 0)$, where

$$A = \underbrace{\begin{pmatrix} a_{1,1} & \dots & a_{1,n+1} \\ \vdots & \ddots & \vdots \\ a_{r,1} & \dots & a_{r,n+1} \end{pmatrix}}_{\in \mathbf{M}_{r \times (n+1)}(k)}, \quad x = \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix}}_{\in \mathbf{M}_{(n+1) \times 1}(k)}.$$

- (3) The Gaussian elimination in linear algebra says that $(A|0)$ has the same solutions as its reduced row echelon form $(A'|0)$, that is, $V(Ax = 0) = V(A'x = 0)$.
- (4) If $V(h_1, \dots, h_r) = \emptyset$, nothing to do. If $V(h_1, \dots, h_r) \neq \emptyset$, then

$$V(h_1, \dots, h_r) = V(g_1, \dots, g_{m+1})$$

where $m+1 = \text{rank}(A)$ is the number of nonzero rows in A' ($m+1 \leq r, n+1$) and $g_i = a'_{i,1}x_1 + \dots + a'_{i,n+1}x_{n+1}$ for $1 \leq i \leq m+1$. ($a'_{i,j}$ is the entry of the matrix A' .)

- (5) Now given any $f + I(V) \in k[x_1, \dots, x_{n+1}]/I(V)$, we replace the leading term x_{i_1} of g_1 by $x_{i_1} - g_1$ to get

$$f + I(V) = f(x_1, \dots, \underbrace{x_{i_1} - g_1}_{i_1 \text{th position}}, \dots, x_{n+1}) + I(V) := f_1 + I(V)$$

where $f_1 \in k[x_1, \dots, \widehat{x_{i_1}} \dots, x_{n+1}]$. Continue this process to replace each leading term x_{i_j} of g_j by $x_{i_j} - g_j$ to get one by one to get

$$f + I(V) = f_1 + I(V), f_1 \in k[x_1, \dots, \widehat{x_{i_1}} \dots, x_{n+1}].$$

...

$$f_m + I(V) = f_{m+1} + I(V), f_{m+1} \in k[x_1, \dots, \widehat{x_{i_1}}, \dots, \widehat{x_{i_{m+1}}} \dots, x_{n+1}].$$

Hence, a routine shows that there is a ring isomorphism

$$\alpha : k[x_1, \dots, x_{n+1}]/I(V) \rightarrow \underbrace{k[x_1, \dots, \widehat{x_{i_1}}, \dots, \widehat{x_{i_{m+1}}} \dots, x_{n+1}]}_{\text{a domain}}$$

sending f to f_{m+1} . Therefore, V is a variety.

- (6) As $I(V) = (h_1, \dots, h_r)$, $I(V)^t = (h_1^t, \dots, h_r^t)$ where each h_i^t is a form of degree 1. Thus $V^t = V(I(V)^t) = V(h_1^t, \dots, h_r^t)$ is also a linear subvariety of $\mathbf{P}^n(k)$.

□

Proof of (b).

- (1) Suppose $A \in \mathbf{M}_{r \times (n+1)}(k)$ is of rank $(n+1) - (m+1) = n - m$. Linear algebra says that there exist invertible matrices $B \in \mathbf{M}_{r \times r}(k)$ and $C \in \mathbf{M}_{(n+1) \times (n+1)}(k)$ such that $D = BAC$, where

$$D = BAC = \underbrace{\begin{pmatrix} O_1 & O_2 \\ O_3 & I_{n-m} \end{pmatrix}}_{\in \mathbf{M}_{r \times (n+1)}(k)}$$

in which $I_{n-m} \in \mathbf{M}_{(n-m) \times (n-m)}(k)$ is the identity matrix and O_1, O_2 , and O_3 are zero matrices.

- (2) Let t' be the linear map corresponding to the matrix C . So

$$\begin{aligned} V^{t'} &= V(Ax = 0)^{t'} \\ &= V(ACx = 0) \\ &= V(BACx = 0) && (B: \text{invertible}) \\ &= V(Dx = 0) \\ &= V(0, \dots, 0, x_{m+2}, \dots, x_{n+1}) && (V \neq \emptyset) \\ &= V(x_{m+2}, \dots, x_{n+1}). \end{aligned}$$

□

Proof of (c). Linear algebra says that the rank of any matrix is uniquely determined. Therefore, $m = n - \text{rank}(A)$ is uniquely determined. □

Problem 4.12.*

Let H_1, \dots, H_m be hyperplanes in \mathbf{P}^n , $m \leq n$. Show that

$$H_1 \cap H_2 \cap \cdots \cap H_m \neq \emptyset.$$

Proof.

(1) Let

$$H_i : a_{i,1}x_1 + \cdots + a_{i,n+1}x_{n+1} = 0$$

for $i = 1, 2, \dots, m$.

(2) View (1) as the system of linear equations. Let the coefficient matrix A be

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n+1} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n+1} \end{pmatrix}.$$

Note that $\text{rank}(A) \leq \min\{m, n+1\} = m$. The rank-nullity theorem shows that

$$\dim_k \ker(A) = (n+1) - \text{rank}(A) \geq (n+1) - m \geq 1.$$

Hence, there is a nonzero solution of $\bigcap_{i=1}^m H_i$, or $\bigcap_{i=1}^m H_i \neq \emptyset \in \mathbf{P}^n$.

□

Problem 4.13.* (Line)

Let $P = [a_1 : \cdots : a_{n+1}]$, $Q = [b_1 : \cdots : b_{n+1}]$ be distinct points of \mathbf{P}^n . The **line** L through P and Q is defined by

$$L = \{[\lambda a_1 + \mu b_1 : \cdots : \lambda a_{n+1} + \mu b_{n+1}] : \lambda, \mu \in k, \lambda \neq 0 \text{ or } \mu \neq 0\}.$$

Prove the projective analogue of Problem 2.15.

- (a) Show that if L is the line through P and Q , and t is a projective change of coordinates, then $t(L)$ is the line through $t(P)$ and $t(Q)$.
- (b) Show that a line is a linear subvariety of dimension 1, and that a linear subvariety of dimension 1 is the line through any two of its points.

- (c) Show that, in \mathbf{P}^2 , a line is the same thing as a hyperplane.
- (d) Let $P, P' \in \mathbf{P}^2$, L_1, L_2 two distinct lines through P , L'_1, L'_2 distinct lines through P' . Show that there is a projective change of coordinates t of \mathbf{P}^2 such that $t(P) = P'$ and $t(L_i) = L'_i$, $i = 1, 2$.

Proof of (a).

- (1) Write $t = (t_1, \dots, t_{n+1})$ as

$$t_i = \sum_j c_{ij} x_j.$$

Given any point $P_{\lambda, \mu} = [\lambda a_1 + \mu b_1 : \dots : \lambda a_{n+1} + \mu b_{n+1}] \in L$ for some not all zeros $\lambda, \mu \in k$. (In particular, $P_{1,0} = P$ and $P_{0,1} = Q$.)

- (2) As

$$\begin{aligned} t_i(P_{\lambda, \mu}) &= \sum_j c_{ij} (\lambda a_j + \mu b_j) \\ &= \lambda \sum_j c_{ij} a_j + \mu \sum_j c_{ij} b_j \\ &= \lambda t_i(P) + \mu t_i(Q), \end{aligned}$$

we have

$$\begin{aligned} t(L) &= \{[\lambda t_1(P) + \mu t_1(Q) : \dots : \lambda t_{n+1}(P) + \mu t_{n+1}(Q)] \\ &\quad : \lambda, \mu \in k, \lambda \neq 0 \text{ or } \mu \neq 0\}. \end{aligned}$$

Moreover, $t(P) \in t(L)$ as $(\lambda, \mu) = (1, 0)$, $t(Q) \in t(L)$ as $(\lambda, \mu) = (0, 1)$, and $t(P) \neq t(Q)$ (since $P \neq Q$ and t is a projective change of coordinates.) Therefore, $t(L)$ is the line through $t(P)$ and $t(Q)$.

□

Proof of (b).

- (1) First, write L as the system of equations

$$x_i = \lambda a_i + \mu b_i$$

($i = 1, \dots, n+1$) where $\lambda, \mu \in k, \lambda \neq 0$ or $\mu \neq 0$. Since $P \neq Q \in \mathbf{P}^n$, there exist $1 \leq \alpha, \beta \leq n$ such that $a_\alpha b_\beta - a_\beta b_\alpha \neq 0$. So we can solve λ and μ in terms of x_α and x_β by Cramer's rule, say

$$\lambda = \frac{x_\alpha b_\beta - x_\beta b_\alpha}{a_\alpha b_\beta - a_\beta b_\alpha}, \quad \mu = \frac{a_\alpha x_\beta - a_\beta x_\alpha}{a_\alpha b_\beta - a_\beta b_\alpha}.$$

(2) Define

$$\begin{aligned} V &= V \left(x_i = \frac{x_\alpha b_\beta - x_\beta b_\alpha}{a_\alpha b_\beta - a_\beta b_\alpha} a_i + \frac{a_\alpha x_\beta - a_\beta x_\alpha}{a_\alpha b_\beta - a_\beta b_\alpha} b_i : 1 \leq i \leq n+1 \right) \\ &= V \left(\begin{vmatrix} x_i & a_i & b_i \\ x_\alpha & a_\alpha & b_\alpha \\ x_\beta & a_\beta & b_\beta \end{vmatrix} = 0 : 1 \leq i \leq n+1 \right). \end{aligned}$$

By construction, $L = V$ is a linear subvariety in \mathbf{P}^n . (See Problem 4.11.)

(3) Might assume that $\alpha = n$ and $\beta = n+1$. View

$$\begin{vmatrix} x_i & a_i & b_i \\ x_\alpha & a_\alpha & b_\alpha \\ x_\beta & a_\beta & b_\beta \end{vmatrix} = 0, i = 1, \dots, n+1$$

as the system of linear equations. Write A as the corresponding coefficient matrix. A is a reduced row echelon form of rank $(n+1) - 2 = n-1$. So $\dim(V) = n - \text{rank}(A) = n - (n-1) = 1$.

(4) Conversely, $\dim(V) = 1$ implies that $\text{rank}(A'|0) = n-1$. So all leading terms are all x_i except two x_α, x_β for some $\alpha \neq \beta$, might say $\alpha = n$ and $\beta = n+1$. Hence V is of the form

$$V = (x_i + a_i x_n + b_i x_{n+1} = 0)$$

for $1 \leq i \leq n-1$. So

$$\begin{aligned} V &= \{[-a_1 \lambda - b_1 \mu : \dots : -a_{n-1} \lambda - b_{n-1} \mu : \lambda : \mu] : \\ &\quad \lambda, \mu \in k, \lambda \neq 0 \text{ or } \mu \neq 0\} \end{aligned}$$

is a line passing two different points

$$\begin{aligned} P &= [-a_1 : \dots : -a_{n-1} : 1 : 0] \\ Q &= [-b_1 : \dots : -b_{n-1} : 0 : 1]. \end{aligned}$$

□

Proof of (c).

(1) By part (b), a line $L \subseteq \mathbf{P}^2$ is

$$\begin{aligned} &V((a_2 b_3 - b_2 a_3)x + (a_3 b_1 - a_1 b_3)y + (a_1 b_2 - a_2 b_1)z = 0) \\ &= V \left(\begin{vmatrix} x & a_1 & b_1 \\ y & a_2 & b_2 \\ z & a_3 & b_3 \end{vmatrix} = 0 \right), \end{aligned}$$

which is also a plane in \mathbf{P}^2 .

(2) Conversely, given any plane

$$V = V(ax + by + cz = 0) \subseteq \mathbf{P}^2$$

where a, b, c are not all zero. Might assume that $a \neq 0$. (Other cases are similar.) So

$$V = \left\{ \left[-\frac{b}{a}\lambda - \frac{c}{a}\mu : \lambda : \mu \right] \in \mathbf{P}^2 : \lambda, \mu \in k, \lambda \neq 0 \text{ or } \mu \neq 0 \right\}$$

is a line passing $P = \left[-\frac{b}{a} : 1 : 0 \right] \in \mathbf{P}^2$ and $Q = \left[-\frac{c}{a} : 0 : 1 \right] \in \mathbf{P}^2$.

□

Proof of (d).

- (1) Take one point $P_i \in L_i$ (resp. $P'_i \in L'_i$) other than P (resp. P') for $i = 1, 2$. It is possible since every line is passing two distinct points.
- (2) By Problem 4.15, $P_1 \notin L_2$ (resp. $P'_1 \notin L'_2$) and $P_2 \notin L_1$ (resp. $P'_2 \notin L'_1$).
- (3) By Problem 4.14, there is a unique projective change of coordinates $t : \mathbf{P}^2 \rightarrow \mathbf{P}^2$ such that $t(P) = P'$, $t(P_1) = P'_1$ and $t(P_2) = P'_2$.
- (4) Hence, the line $t(L_i)$ (by part (a)) and the line L'_i are both passing P' and P'_i for $i = 1, 2$. Since $P' \neq P'_i$ by construction, Problem 4.15 implies that $t(L_i) = L'_i$.

□

Problem 4.14.*

Let P_1, P_2, P_3 (resp. Q_1, Q_2, Q_3) be three points in \mathbf{P}^2 not lying on a line. Show that there is a projective change of coordinates $t : \mathbf{P}^2 \rightarrow \mathbf{P}^2$ such that $t(P_i) = Q_i$, $i = 1, 2, 3$. Extend this to $n + 1$ points in \mathbf{P}^n , not lying on a hyperplane.

Proof.

- (1) Write

$$P_i = [a_{i1} : a_{i2} : a_{i3}] \in \mathbf{P}^2(k)$$

$$Q_i = [b_{i1} : b_{i2} : b_{i3}] \in \mathbf{P}^2(k)$$

for $i = 1, 2, 3$.

(2) Define

$$A = \begin{pmatrix} P_1 & P_2 & P_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

$$B = \begin{pmatrix} Q_1 & Q_2 & Q_3 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \end{pmatrix}.$$

Note that A (resp. B) is depending on the representations of P_i (resp. Q_i) up to a nonzero constant in k .

(3) Here A (resp. B) is invertible since P_1, P_2, P_3 (resp. Q_1, Q_2, Q_3) are not lying on a line. Define $t : \mathbf{P}^2 \rightarrow \mathbf{P}^2$ by sending $P = [x : y : z] \in \mathbf{P}^2$ to

$$t(P) = BA^{-1}P = \begin{pmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Note that the matrix $BA^{-1} \in \mathrm{GL}_3(k)$ depends on the representations of P_i (resp. Q_i) up to a nonzero constant in k . Hence, t is a well-defined map from \mathbf{P}^2 to \mathbf{P}^2 . Besides, t is a projective change of coordinates mapping P_i to Q_i ($i = 1, 2, 3$).

(4) *Generalization.* Let P_i (resp. Q_i) be $n+1$ points in \mathbf{P}^n ($i = 1, \dots, n+1$) not lying on a hyperplane. Then there is a projective change of coordinates $t : \mathbf{P}^n \rightarrow \mathbf{P}^n$ such that $t(P_i) = Q_i$ ($i = 1, \dots, n+1$). The proof is the same except replacing 2 by n .

□

Problem 4.15.*

Show that any two distinct lines in \mathbf{P}^2 intersect in one point.

Proof.

(1) Let

$$L_1 : a_1x + b_1y + c_1z = 0$$

$$L_2 : a_2x + b_2y + c_2z = 0$$

be two distinct lines in \mathbf{P}^2 .

(2) View (1) as the system of linear equations. Let the coefficient matrix A be

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}.$$

Since L_1 and L_2 are distinct, $\text{rank}(A) = 2$. The rank-nullity theorem shows that

$$\dim_k \ker(A) = 3 - \text{rank}(A) = 1.$$

- (3) Might take a basis $\{(x_0, y_0, z_0)\}$ for $\ker(A)$. Here $(x_0, y_0, z_0) \neq 0$ and any other nonzero solutions of $L_1 \cap L_2$ is of the form $(\lambda x_0, \lambda y_0, \lambda z_0)$ ($\lambda \neq 0$). Therefore,

$$L_1 \cap L_2 = \{[x_0 : y_0 : z_0] \in \mathbf{P}^2\}.$$

□

Problem 4.16.*

Let L_1, L_2, L_3 (resp. M_1, M_2, M_3) be lines in $\mathbf{P}^2(k)$ that do not all pass through a point. Show that there is a projective change of coordinates: $t : \mathbf{P}^2 \rightarrow \mathbf{P}^2$ such that $t(L_i) = M_i$. (Hint: Let $P_{ij} = L_i \cap L_j$, $Q_{ij} = M_i \cap M_j$, $i \neq j$, and apply Problem 4.14.) Extend this to $n + 1$ hyperplanes in \mathbf{P}^n , not passing through a point.

Proof.

- (1) Let $P_{ij} = L_i \cap L_j$ (resp. $Q_{ij} = M_i \cap M_j$), $i \neq j$. P_{ij} (resp. Q_{ij}) is uniquely determined by L_i and L_j (resp. M_i and M_j) (Problem 4.15). Also, $P_{ij} \neq P_{i'j'}$ (resp. $Q_{ij} \neq Q_{i'j'}$) if $\{i, j\} \neq \{i', j'\}$ (as sets) by assumption.
- (2) Problem 4.14 shows that there is a projective change of coordinates $t : \mathbf{P}^2 \rightarrow \mathbf{P}^2$ such that $t(P_{ij}) = Q_{ij}$, $i \neq j$. Similar to the argument in Problem 4.13(d), we conclude that $t(L_i) = M_i$.
- (3) Show that we can extend this to $n + 1$ hyperplanes in \mathbf{P}^n , not passing through a point. We cannot apply steps (1)(2) to the generalized case \mathbf{P}^n . Instead, we can apply the proof of Problem 4.14.
- (4) Let E_i (resp. F_i) ($i = 1, \dots, n + 1$) be $(n + 1)$ hyperplanes in \mathbf{P}^n that do not passing through a point. Write

$$E_i : a_{i,1}x_1 + \dots + a_{i,n+1}x_{n+1} = 0 \in \mathbf{P}^n(k)$$

$$F_i : b_{i,1}x_1 + \dots + b_{i,n+1}x_{n+1} = 0 \in \mathbf{P}^n(k)$$

for $i = 1, \dots, n + 1$.

- (5) View E_i (resp. F_i) as the system of linear equations. Let the coefficient

matrix A (resp. B) be

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{n+1,1} \\ \vdots & \ddots & \vdots \\ a_{1,n+1} & \cdots & a_{n+1,n+1} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{1,1} & \cdots & b_{n+1,1} \\ \vdots & \ddots & \vdots \\ b_{1,n+1} & \cdots & b_{n+1,n+1} \end{pmatrix}.$$

Note that A (resp. B) is depending on the representations of E_i (resp. F_i) up to a nonzero constant in k .

- (6) Here A (resp. B) is invertible since E_i (resp. F_i) are not passing through a point. Define $t : \mathbf{P}^n \rightarrow \mathbf{P}^n$ by sending $P = [x_1 : \cdots : x_{n+1}] \in \mathbf{P}^n$ to

$$t(P) = BA^{-1}P$$

$$= \begin{pmatrix} b_{1,1} & \cdots & b_{n+1,1} \\ \vdots & \ddots & \vdots \\ b_{1,n+1} & \cdots & b_{n+1,n+1} \end{pmatrix} \begin{pmatrix} a_{1,1} & \cdots & a_{n+1,1} \\ \vdots & \ddots & \vdots \\ a_{1,n+1} & \cdots & a_{n+1,n+1} \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix}.$$

Note that the matrix $BA^{-1} \in \mathrm{GL}_{n+1}(k)$ depends on the representations of E_i (resp. F_i) up to a nonzero constant in k . Hence, t is a well-defined map from \mathbf{P}^n to \mathbf{P}^n . Besides, t is a projective change of coordinates mapping E_i to F_i ($i = 1, \dots, n+1$).

□

Note. It is the duality of Problem 4.14. See Problem 4.18 for more details.

Problem 4.17.*

Let z be a rational function on a projective variety V . Show that the pole set of z is an algebraic subset of V .

Proof.

- (1) Similar to the proof of Proposition 2 in §2.4. For $g \in k[x_1, \dots, x_{n+1}]$, denote the residue of g in $\Gamma_h(V)$ by \bar{g} .

- (2) Let

$$J_z = \{g \in k[x_1, \dots, x_{n+1}] : \bar{g}z \in \Gamma_h(V)\}.$$

J_z is an ideal in $k[x_1, \dots, x_{n+1}]$ containing $I(V)$, and the points of $V(J_z)$ are exactly those points where z is not defined if J_z is homogeneous.

- (3) Show that J_z is homogeneous by the homogeneous property of $I(V)$. Given any $g = g_r + g_{r+1} + \cdots \in J_z$, where g_i is a form of $k[x_1, \dots, x_{n+1}]$. Write $z = a/b$ for some form $a, b \in \Gamma_h(V)$ of the same degree. So $\bar{g}z = \bar{g}a/b \in \Gamma_h(V)$. So there is $f = f_s + f_{s+1} + \cdots \in k[x_1, \dots, x_{n+1}]$ (where f_i is a form of $k[x_1, \dots, x_{n+1}]$) such that $ga - fb \in I(V)$. Since $I(V)$ is homogeneous, each form $g_i a - f_i b$ is in $I(V)$. So $\bar{g}_i z = \bar{f}_i \in \Gamma_h(V)$ for each i (since $\Gamma_h(V)$ is homogeneous), that is, $g_i \in J_z$ for each i .

□

Problem 4.18. (Duality)

Let $H = V(\sum a_i x_i)$ be a hyperplane in \mathbf{P}^n . Note that (a_1, \dots, a_{n+1}) is determined by H up to a constant.

- (a) Show that assigning $[a_1 : \cdots : a_{n+1}] \in \mathbf{P}^n$ to H sets up a natural one-to-one correspondence between $\{\text{hyperplanes in } \mathbf{P}^n\}$ and \mathbf{P}^n . If $P \in \mathbf{P}^n$, let P^* be the corresponding hyperplane; if H is a hyperplane, H^* denotes the corresponding point.
- (b) Show that $P^{**} = P$, $H^{**} = H$. Show that $P \in H$ if and only if $H^* \in P^*$.

This is the well-known **duality** of the projective space.

Proof of (a).

- (1) Define $\alpha : \{\text{hyperplanes}\} \rightarrow \mathbf{P}^n$ (resp. $\beta : \mathbf{P}^n \rightarrow \{\text{hyperplanes}\}$) by

$$\begin{aligned}\alpha : V\left(\sum a_i x_i\right) &\mapsto [a_1 : \cdots : a_{n+1}], \\ \beta : [a_1 : \cdots : a_{n+1}] &\mapsto V\left(\sum a_i x_i\right).\end{aligned}$$

- (2) As $H = V(\sum a_i x_i)$ is a hyperplane, the corresponding

$$\alpha(H) = [a_1 : \cdots : a_{n+1}] \in \mathbf{P}^n$$

and thus α is well-defined. Similarly, β is well-defined.

- (3) Note that both $\alpha \circ \beta$ and $\beta \circ \alpha$ are identity maps. α (resp. β) is an isomorphism, that is, there is a natural one-to-one correspondence between

$$\{\text{hyperplanes in } \mathbf{P}^n\} \longleftrightarrow \mathbf{P}^n.$$

□

Proof of (b).

- (1) We've showed that $P^{**} = P$, $H^{**} = H$ in (a). It suffices to show that $P \in H$ iff $H^* \in P^*$.
- (2) Write $H = V(\sum a_i x_i) \subseteq \mathbf{P}^n$ and $P = [b_1 : \cdots : b_{n+1}] \in \mathbf{P}^n$. Hence,

$$\begin{aligned} P \in H &\iff a_1 b_1 + \cdots + a_{n+1} b_{n+1} = 0 \\ &\iff b_1 a_1 + \cdots + b_{n+1} a_{n+1} = 0 \\ &\iff H^* \in P^*. \end{aligned}$$

□

4.3. Affine and Projective Varieties

Problem 4.19.*

If $I = (f)$ is the ideal of an affine hypersurface, show that $I^* = (f^*)$.

Proof.

- (1) Note that I^* is the ideal in $k[x_1, \dots, x_{n+1}]$ generated by $\{g^* : g \in I = (f)\}$. In particular, $f^* \in I^*$. Thus $(f^*) \subseteq I^*$.
- (2) Conversely, given any $g = \sum a_i (h_i f)^* \in I^*$, where $a_i, h_i \in k[x_1, \dots, x_{n+1}]$. Thus,

$$g = \sum_i a_i (h_i f)^* = \sum_i a_i h_i^* f^* = \left(\sum_i a_i h_i^* \right) f^* \in (f^*)$$

(by Proposition 5 in §2.6).

□

Problem 4.20.

Let $V = V(y - x^2, z - x^3) \subseteq \mathbf{A}^3$. Prove:

- (a) $I(V) = (y - x^2, z - x^3)$.
- (b) $zw - xy \in I(V)^* \subseteq k[x, y, z, w]$, but $zw - xy \notin I((y - x^2)^*, (z - x^3)^*)$. So if $I(V) = (f_1, \dots, f_r)$, it does not follow that $I(V)^* = (f_1^*, \dots, f_r^*)$.

Proof of (a). By Problems 1.11 and 2.8, V is an affine variety. Thus $I(V) = (y - x^2, z - x^3)$ is a prime ideal. \square

Proof of (b).

- (1) Since $z - xy = (z - x^3) - x(y - x^2) \in I(V)$, $zw - xy = (z - xy)^* \in I(V)^*$.
- (2) Suppose $zw - xy \in I((y - x^2)^*, (z - x^3)^*) = I(yw - x^2, zw^2 - x^3)$. Write $zw - xy = f \cdot (yw - x^2) + g \cdot (zw^2 - x^3)$ for some $f, g \in k[x, y, z, w]$. Then $\deg_x(zw - xy) = 1$ but $\deg_x(f \cdot (yw - x^2) + g \cdot (zw^2 - x^3))$ cannot be 1, which is absurd.

\square

Problem 4.21.

Show that if $V \subseteq W \subseteq \mathbf{P}^n$ are varieties, and V is a hypersurface, then $W = V$ or $W = \mathbf{P}^n$ (see Problem 1.39).

Proof.

- (1) Write $V = V(f)$ for some irreducible form $f \in k[x_1, \dots, x_{n+1}]$. So $I(V) = (f)$ by the projective Nullstellensatz.
- (2) Note that $I(W)$ is a prime ideal such that

$$(f) \supseteq I(W) \supseteq (0).$$

By Problem 1.39, $I(W) = (f)$ or (0) . Thus, $W = V$ or $W = \mathbf{P}^n$.

\square

Problem 4.22.*

Suppose V is a variety in \mathbf{P}^n and $V \supseteq H_\infty$. Show that $V = \mathbf{P}^n$ or $V = H_\infty$. If $V = \mathbf{P}^n$, $V_* = \mathbf{A}^n$, while if $V = H_\infty$, $V_* = \emptyset$.

Proof.

- (1) Note that $H_\infty = V(x_{n+1})$ is a hypersurface. By Problem 4.21, $V = \mathbf{P}^n$ or $V = H_\infty$.
- (2) If $V = \mathbf{P}^n$, then $I = I(V) = (0)$. So $V_* = V(I_*) = V(0) = \mathbf{A}^n$.
- (3) If $V = H_\infty$, then $I = I(V) = (x_{n+1})$. So $V_* = V(I_*) = V(1) = \emptyset$.

\square

Problem 4.23.*

Describe all subvarieties in \mathbf{P}^1 and in \mathbf{P}^2 .

Proof.

- (1) Show that all subvarieties in \mathbf{P}^1 are \emptyset , \mathbf{P}^1 , and single points. (Also compare to Problem 1.8.)
 - (a) \emptyset , single points and \mathbf{P}^1 are all varieties.
 - (b) Let V be a nonempty subvariety in \mathbf{P}^1 . Write $\{[a : b]\} = V(bx - ay) \subseteq V$ for some point $[a : b] \in \mathbf{P}^1$. So $V(bx - ay) \subseteq V \subseteq \mathbf{P}^1$.
 - (c) Since $V(bx - ay)$ is a hypersurface and $bx - ay$ is irreducible, Problem 4.21 implies that $V = V(bx - ay)$ (a single point) or $V = \mathbf{P}^1$ itself.
- (2) Show that all subvarieties in \mathbf{P}^2 are \emptyset , \mathbf{P}^2 , single points, and hypersurfaces $V(f)$, where f is an irreducible form. Similar to Corollary 2 of Proposition 2 in §1.6.
 - (a) Let V be a subvariety in \mathbf{P}^2 . If V is finite or $I(V) = (0)$, V is of one required type.
 - (b) Otherwise $I(V)$ contains a non-constant form f ; since $I(V)$ is prime, we may assume f is irreducible.
 - (c) Hence, $I(V) = (f)$; for if $g \in I(V)$, $g \notin (f)$, then $V \subseteq V(f, g)$ is finite by Problem 5.7, which is a projective analogue of Proposition 2 in §1.6.

□

Problem 4.24.*

Let $P = [0 : 1 : 0] \in \mathbf{P}^2(k)$. Show that the lines through P consist of the following:

- (a) The vertical lines $L_\lambda = V(x - \lambda z) = \{[\lambda : t : 1] : t \in k\} \cup \{P\}$.
- (b) The line at infinity $L_\infty = V(z) = \{[x : y : 0] : x, y \in k, x \neq 0 \text{ or } y \neq 0\}$.

Proof.

- (1) Let L be one line passing through P and $Q = [b_1 : b_2 : b_3] \neq P$. Thus,

$$\begin{aligned}
 L &= \{[\mu b_1 : \lambda + \mu b_2 : \mu b_3] : \lambda, \mu \in k, \lambda \neq 0 \text{ or } \mu \neq 0\} \\
 &= \{[\mu b_1 : \lambda + \mu b_2 : \mu b_3] : \lambda, \mu \in k, \mu \neq 0\} \cup \{P\} \\
 &= \{[b_1 : t + b_2 : b_3] : t \in k\} \cup \{P\} \qquad (t := \lambda/\mu)
 \end{aligned}$$

Consider two cases: $b_3 \neq 0$ and $b_3 = 0$.

(2) $b_3 \neq 0$. So

$$\begin{aligned}
L &= \{[b_1 : t + b_2 : b_3] : t \in k\} \cup \{P\} \\
&= \left\{ \left[\frac{b_1}{b_3} : \frac{t}{b_3} + \frac{b_2}{b_3} : 1 \right] : t \in k \right\} \cup \{P\} \\
&= \{[\lambda : t : 1] : t \in k\} \cup \{P\} & \left(\frac{b_1}{b_3} \mapsto \lambda, \frac{t + b_2}{b_3} \mapsto t \right) \\
&= L_\lambda.
\end{aligned}$$

(3) $b_3 = 0$. So

$$\begin{aligned}
L &= \{[b_1 : t + b_2 : 0] : t \in k\} \cup \{P\} \\
&= \{[x : y : 0] : y \in k\} \cup \{P\} & (b_1 \mapsto x, t + b_2 \mapsto y) \\
&= \{[x : y : 0] : y \in k, x \neq 0\} \cup \{P\} \\
&= \{[x : y : 0] : y \in k, x \neq 0 \text{ or } y \neq 0\} \\
&= L_\infty.
\end{aligned}$$

□

Problem 4.25.*

Let $P = [x : y : z] \in \mathbf{P}^2$.

- (a) Show that $\{(a, b, c) \in \mathbf{A}^3 : ax + by + cz = 0\}$ is a hyperplane in \mathbf{A}^3 .
- (b) Show that for any finite set of points in \mathbf{P}^2 , there is a line not passing through any of them.

Proof of (a).

- (1) Let $V = \{(a, b, c) \in \mathbf{A}^3 : ax + by + cz = 0\}$.
- (2) V is well-defined and the form $ax + by + cz$ is of degree one since not all a, b, c are zero. Hence V is a hyperplane (plane) in \mathbf{A}^3 .

□

Proof of (b).

- (1) Let S be any finite set of points in $\mathbf{P}^2(k)$. Write $S = (S \cap H_\infty) \cup (S - H_\infty)$.
- (2) $S \cap H_\infty$ is a finite subset of $H_\infty = \mathbf{P}^1(k) \supseteq \mathbf{A}^1(k) = k$. Since $k = \bar{k}$ is infinite (Problem 1.6), there is one point $[\beta : 1 : 0] \in H_\infty - S$. Here $\beta \in k$ is uniquely determined.

- (3) $S - H_\infty$ is finite too. Write $S - H_\infty = \{[\lambda_1 : \mu_1 : 1], \dots, [\lambda_r : \mu_r : 1]\}$. Here λ_i and μ_i are all uniquely determined ($i = 1, \dots, r$). Take any $\gamma \in k$ such that $\gamma \notin \{\lambda_1 - \beta\mu_1, \dots, \lambda_r - \beta\mu_r\}$. (It is possible since k is infinite.)

- (4) Let

$$L = V(x - \beta y - \gamma z)$$

be a line in \mathbf{P}^2 . By construction, $L \cap S = \emptyset$.

□

4.4. Multiprojective Space

Problem 4.26.*

- (a) Define maps $\varphi_{i,j} : \mathbf{A}^{n+m} \rightarrow U_i \times U_j \subseteq \mathbf{P}^n \times \mathbf{P}^m$. Using $\varphi_{n+1,m+1}$, define the “biprojective closure” of an algebraic set in \mathbf{A}^{n+m} . Prove an analogue of Proposition 3 of §4.3.
- (b) Generalize part (a) to maps

$$\varphi : \mathbf{A}^{n_1} \times \dots \times \mathbf{A}^{n_r} \times \mathbf{A}^m \rightarrow \mathbf{P}^{n_1} \times \dots \times \mathbf{P}^{n_r} \times \mathbf{A}^m.$$

Show that this sets up a correspondence between

$$\{\text{nonempty affine varieties in } \mathbf{A}^{n_1+\dots+n_r+m}\}$$

and

$$\{\text{varieties in } \mathbf{P}^{n_1} \times \dots \times \mathbf{P}^{n_r} \times \mathbf{A}^m \\ \text{that intersect } U_{n_1+1} \times \dots \times U_{n_r+1} \times \mathbf{A}^m\}.$$

Show that this correspondence preserves function fields and local rings.

Proof of (a).

- (1) Define maps $\varphi_{i,j} : \mathbf{A}^{n+m} \rightarrow U_i \times U_j \subseteq \mathbf{P}^n \times \mathbf{P}^m$ by

$$\varphi_{i,j} : (x_1, \dots, x_n, y_1, \dots, y_m) \mapsto \\ [x_1 : \dots : x_{i-1} : 1 : x_i : \dots : x_n] \times [y_1 : \dots : y_{j-1} : 1 : y_j : \dots : y_m].$$

Actually, we can write $\varphi_{i,j}(x, y) = (\varphi_i(x), \varphi_j(y))$ for $x = (x_1, \dots, x_n) \in \mathbf{A}^n$ and $y = (y_1, \dots, y_m) \in \mathbf{A}^m$.

- (2) Given any algebraic set V in \mathbf{A}^{n+m} . Let I^* be the bihomogeneous ideal in $k[x, y] = k[x_1, \dots, x_{n+1}, y_1, \dots, y_{m+1}]$ generated by $\{f^* : f \in I\}$. Here

$$f^* = x_{n+1}^r y_{m+1}^s f\left(\frac{x_1}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}}, \frac{y_1}{y_{m+1}}, \dots, \frac{y_m}{y_{m+1}}\right)$$

where r (resp. s) is the degree of f in x (resp. y). Define $V^* = V(I^*)$ as the “biprojective closure.”

- (3) Proposition 3 for multiprojective space.

- (i) If $V \subseteq \mathbf{A}^{n+m}$, $\varphi_{n+1, m+1}(V) = V^* \cap (U_{n+1} \times U_{m+1})$ and $(V^*)_* = V$.
- (ii) If $V \subseteq W \subseteq \mathbf{A}^{n+m}$, then $V^* \subseteq W^* \subseteq \mathbf{P}^n \times \mathbf{P}^m$. If $V \subseteq W \subseteq \mathbf{P}^n \times \mathbf{P}^m$, then $V_* \subseteq W_* \subseteq \mathbf{A}^{n+m}$.
- (iii) If V is irreducible in \mathbf{A}^{n+m} , then V^* is irreducible in $\mathbf{P}^n \times \mathbf{P}^m$.
- (iv) If $V = \bigcup_i V_i$ is the irreducible decomposition of V in \mathbf{A}^{n+m} , then $V^* = \bigcup_i V_i^*$ is the irreducible decomposition of V^* in $\mathbf{P}^n \times \mathbf{P}^m$.
- (v) If $V \subseteq \mathbf{A}^{n+m}$, then V^* is the smallest algebraic set in $\mathbf{P}^n \times \mathbf{P}^m$ that contains $\varphi_{n+1, m+1}(V)$.
- (vi) If $V \subsetneq \mathbf{A}^{n+m}$, then no component of V^* lies in or contains $H_\infty \times H_\infty$.
- (vii) If $V \subseteq \mathbf{P}^n \times \mathbf{P}^m$, and no component of V lies in or contains $(H_\infty \times \mathbf{P}^m) \cup (\mathbf{P}^n \times H_\infty)$, then $V_* \subsetneq \mathbf{A}^{n+m}$ and $(V_*)^* = V$.

- (4) Proof of Proposition 3 for multiprojective space.

- (i) It follows from Proposition 5 in §2.6.
- (ii) It is obvious.
- (iii) If $V \subseteq \mathbf{A}^{n+m}$, $I = I(V)$, then a biform f belongs to I^* if and only if $f_* \in I$. If I is prime, it follows readily that I^* is also prime.
- (iv) It follows from (ii)(iii)(v).
- (v) Suppose W is an algebraic set in $\mathbf{P}^n \times \mathbf{P}^m$ which contains $\varphi_{n+1, m+1}(V)$. If $f \in I(W)$, then $f_* \in I(V)$, so $f = x_{n+1}^r y_{m+1}^s (f_*)^* \in I(V)^*$. Therefore $I(W) \subseteq I(V)^*$, so $W \supseteq V^*$, as desired.
- (vi) We may assume V is irreducible. $V^* \not\subseteq H_\infty \times H_\infty$ by (i). If $V \subseteq H_\infty \times H_\infty$, then $I(V)^* \subseteq I(V^*) \subseteq I(H_\infty \times H_\infty) = (x_{n+1}, y_{m+1})$. But if $0 \neq f \in I(V)$, then $f^* \in I(V)^*$, but $f^* \notin (x_{n+1}, y_{m+1})$. So $V^* \not\subseteq H_\infty \times H_\infty$.
- (vii) We may assume V is irreducible. Since $\varphi_{n+1, m+1}(V_*) \subseteq V$, it suffices to show that $V \subseteq (V_*)^*$, or that $I(V_*)^* \subseteq I(V)$. Let $f \in I(V_*)$. Then $f^N \in I(V)_*$ for some N (Nullstellensatz), so $x_{n+1}^t y_{m+1}^s (f^N)^* \in I(V)$ for some t, s (Proposition 5 in §2.6). But $I(V)$ is prime, and $x_{n+1}, y_{m+1} \notin I(V)$ since $V \not\subseteq (H_\infty \times \mathbf{P}^m) \cup (\mathbf{P}^n \times H_\infty)$, so $f^* \in I(V)$, as desired.

□

Proof of (b).

- (1) Define maps $\varphi_{i_1, \dots, i_r} : \mathbf{A}^{n_1} \times \dots \times \mathbf{A}^{n_r} \times \mathbf{A}^m \rightarrow U_{i_1} \times \dots \times U_{i_r} \times \mathbf{A}^m$ by

$$\varphi_{i_1, \dots, i_r} : (\mathbf{x}_1, \dots, \mathbf{x}_r, \mathbf{y}) \mapsto (\varphi_{i_1}(\mathbf{x}_1), \dots, \varphi_{i_r}(\mathbf{x}_r), \mathbf{y}),$$

where $\mathbf{x}_i = (x_{i,1}, \dots, x_{i,n_i}) \in \mathbf{A}^{n_i}$ and $\mathbf{y} = (y_1, \dots, y_m) \in \mathbf{A}^m$.

- (2) Applying the same argument in (a), we have that there is a correspondence between

$$\{\text{nonempty affine varieties in } \mathbf{A}^{n_1+\dots+n_r+m}\}$$

and

$$\{\text{varieties in } \mathbf{P}^{n_1} \times \dots \times \mathbf{P}^{n_r} \times \mathbf{A}^m \\ \text{that intersect } U_{n_1+1} \times \dots \times U_{n_r+1} \times \mathbf{A}^m\}.$$

- (3) Let V be a nonempty affine varieties in $\mathbf{A}^{n_1+\dots+n_r+m}$. Let $V^* \subseteq \mathbf{P}^{n_1} \times \dots \times \mathbf{P}^{n_r} \times \mathbf{A}^m$ be the multiprojective closure of V intersecting $U_{n_1+1} \times \dots \times U_{n_r+1} \times \mathbf{A}^m$.
- (4) If $\bar{f} \in \Gamma(V^*)$ is a multi-form, we may define \bar{f}_* as follows: take a multi-form $f \in k[\mathbf{x}_1, \dots, \mathbf{x}_r, \mathbf{y}]$ such whose $I(V^*)$ -residue in \bar{f} , and let \bar{f}_* to be $I(V)$ -residue of f_* (one checks that this is independent of the choice of f).
- (5) Define a natural isomorphism $\alpha : k(V^*) \rightarrow k(V)$ by $\alpha(f/g) = f_*/g_*$, where f, g are multi-forms of the same multi-degree on V^* .
- (6) If $P \in V$, we may consider $P \in V^*$ (by means of $\varphi_{n_1+1, \dots, n_r+1}$) and then α induces an isomorphism of $\mathcal{O}_P(V^*)$ with $\mathcal{O}_P(V)$.

□

Problem 4.27.*

Show that the pole set of a rational function on a variety in any multispace is an algebraic subset.

Proof.

- (1) Similar to Problem 4.17. Let V be a variety in one multispace $\mathbf{P}^{n_1} \times \dots \times \mathbf{P}^{n_r} \times \mathbf{A}^m$. Let z be a rational function on V .

(2) Let

$$J_z = \{g \in k[\mathbf{x}_1, \dots, \mathbf{x}_r, \mathbf{y}] : \bar{g}z \in \Gamma(V)\},$$

where $\mathbf{x}_i = [x_{i,1} : \dots : x_{i,n_i+1}] \in \mathbf{P}^{n_i}$ for $1 \leq i \leq r$, $\mathbf{y} = (y_1, \dots, y_m) \in \mathbf{A}^m$, and \bar{g} is the residue of g in the multi-homogeneous coordinate ring $\Gamma(V)$.

- (3) J_z is an ideal in $k[\mathbf{x}_1, \dots, \mathbf{x}_r, \mathbf{y}]$ containing the multi-homogeneous ideal $I(V)$, and the points of $V(J_z)$ are exactly those points where z is not defined if J_z is multi-homogeneous.
- (4) Show that J_z is multi-homogeneous by the multi-homogeneous property of $I(V)$. Induction on r and then apply the proof of Problem 4.17.

□

Problem 4.28.* (Segre embedding)

For simplicity of notation, in this problem we let x_0, \dots, x_n be coordinates for \mathbf{P}^n , y_0, \dots, y_m coordinates for \mathbf{P}^m , and $t_{00}, t_{01}, \dots, t_{0m}, t_{10}, \dots, t_{nm}$ coordinates for \mathbf{P}^N , where $N = (n+1)(m+1) - 1 = n + m + nm$. Define $S : \mathbf{P}^n \times \mathbf{P}^m \rightarrow \mathbf{P}^N$ by the formula: $S([x_0 : \dots : x_n], [y_0 : \dots : y_m]) = [x_0y_0 : x_0y_1 : \dots : x_ny_m]$. S is called the **Segre embedding** of $\mathbf{P}^n \times \mathbf{P}^m$ in \mathbf{P}^{n+m+nm} .

- (a) Show that S is a well-defined, one-to-one mapping.
- (b) Show that if W is an algebraic subset of \mathbf{P}^N , then $S^{-1}(W)$ is an algebraic subset of $\mathbf{P}^n \times \mathbf{P}^m$.
- (c) Let $V = V(\{t_{ij}t_{k\ell} - t_{i\ell}t_{kj} : i, k = 0, \dots, n; j, \ell = 0, \dots, m\}) \subseteq \mathbf{P}^N$. Show that $S(\mathbf{P}^n \times \mathbf{P}^m) = V$. In fact, $S(U_i \times U_j) = V \cap U_{ij}$, where $U_{ij} = \{[t] : t_{ij} \neq 0\}$.
- (d) Show that V is a variety.

Proof of (a).

- (1) Show that S is well-defined. Given any non-zero $\lambda, \mu \in k$.

$$\begin{aligned} & S([\lambda x_0 : \dots : \lambda x_n], [\mu y_0 : \dots : \mu y_m]) \\ &= [(\lambda x_0)(\mu y_0) : (\lambda x_0)(\mu y_1) : \dots : (\lambda x_n)(\mu y_m)] \\ &= [(\lambda \mu)x_0y_0 : (\lambda \mu)x_0y_1 : \dots : (\lambda \mu)x_ny_m] \\ &= [x_0y_0 : x_0y_1 : \dots : x_ny_m] \quad (\lambda \mu \neq 0) \\ &= S([x_0 : \dots : x_n], [y_0 : \dots : y_m]). \end{aligned}$$

So S is independent of the choices of $[x_0 : \dots : x_n]$ and $[y_0 : \dots : y_m]$. Since there is an index i (resp. j) such that $x_i \neq 0$ (resp. $y_j \neq 0$), $x_iy_j \neq 0$ and thus $\text{im}(S) \subseteq \mathbf{P}^N$.

- (2) *Show that S is one-to-one.* Given any $P = [x_0y_0 : x_0y_1 : \dots : x_ny_m] \in \text{im}(S) \subseteq \mathbf{P}^N$. Might assume that $x_0 = y_0 = 1$. So

$$\begin{aligned} P &= [x_0y_0 : x_0y_1 : \dots : x_ny_m] \\ &= [1 : y_1 : y_2 : \dots : y_m : x_1 : \dots : x_2 : \dots : x_n : \dots : x_ny_m]. \end{aligned}$$

By the expression of P , x_1, \dots, x_n and y_1, \dots, y_m are uniquely determined. Hence $[1 : x_1 : \dots : x_n]$ and $[1 : y_1 : \dots : y_m]$ are uniquely determined by P . S is injective.

□

Proof of (b).

- (1) $S^{-1}(W) = \{P \in \mathbf{P}^n \times \mathbf{P}^m : S(P) \in W\}$ is the inverse image of W under S . Write

$$W = V(h_1, \dots, h_r)$$

where each $h_i \in k[t_{00}, \dots, t_{nm}]$ is a form of degree d_i .

- (2) Define the pullback $\tilde{h}_i \in k[x_0, \dots, x_n, y_0, \dots, y_m]$ as

$$\tilde{h}_i(x_0, \dots, x_n, y_0, \dots, y_m) = h_i(x_0y_0, \dots, x_ny_m)$$

for each i . Clearly, \tilde{h}_i is a biform of bidegree (d_i, d_i) .

- (3) As

$$\begin{aligned} P \in S^{-1}(W) &\iff S(P) \in W \\ &\iff h_i(S(P)) = 0 \forall h_i \\ &\iff \tilde{h}_i(P) = 0 \forall \tilde{h}_i \\ &\iff P \in V(\tilde{h}_1, \dots, \tilde{h}_r), \end{aligned}$$

$S^{-1}(W) = V(\tilde{h}_1, \dots, \tilde{h}_r)$ is algebraic.

□

Proof of (c).

- (1) *Show that $\text{im}(S) = V$ is algebraic.* $\text{im}(S) \subseteq V$ by the construction of S . Conversely, take $P = [a_{ij}]_{\substack{i=0, \dots, n \\ j=0, \dots, m}} \in V \subseteq \mathbf{P}^N$. So there is one entry $a_{k\ell} \neq 0$. So

$$\begin{aligned} P &= [a_{ij}]_{\substack{i=0, \dots, n \\ j=0, \dots, m}} \\ &= [a_{ij}a_{k\ell}]_{\substack{i=0, \dots, n \\ j=0, \dots, m}} \\ &= [a_{i\ell}a_{kj}]_{\substack{i=0, \dots, n \\ j=0, \dots, m}} \\ &= S([a_{i\ell}]_{i=0, \dots, n}, [a_{kj}]_{j=0, \dots, m}) \end{aligned}$$

is in $\text{im}(S)$.

- (2) In the proof of (1), we have $S(U_i \times U_j) = V \cap U_{ij}$ where $U_{ij} = \{[t] : t_{ij} \neq 0\}$ (by setting $(k, \ell) \rightarrow (i, j)$).

□

Proof of (d).

- (1) Write $V = V_1 \cup V_2$ where V_1, V_2 are algebraic sets in \mathbf{P}^N . By (b)(c),

$$\mathbf{P}^n \times \mathbf{P}^m = S^{-1}(V) = S^{-1}(V_1) \cup S^{-1}(V_2).$$

- (2) Since $\mathbf{P}^n \times \mathbf{P}^m$ is irreducible (Problem 4.26), we might assume that $\mathbf{P}^n \times \mathbf{P}^m = S^{-1}(V_1)$. Note that $S(S^{-1}(W)) = W$ holds for any set $W \subseteq V$. Hence

$$V_1 = S(S^{-1}(V_1)) = S(\mathbf{P}^n \times \mathbf{P}^m) = V,$$

that is, V is irreducible.

□

Chapter 5: Projective Plane Curves

5.1. Definitions

Problem 5.1.*

Let f be a projective plane curve. Show that a point P is a multiple point of f if and only if $f(P) = \frac{\partial f}{\partial x}(P) = \frac{\partial f}{\partial y}(P) = \frac{\partial f}{\partial z}(P) = 0$.

Proof.

- (1) Let P be a point in f . Might assume $P \in U_3$. By dehomogenizing f with respect to z , $m_P(f) = m_P(f_*)$. Thus,

$$\begin{aligned} m_P(f) &= m_P(f_*) > 1 \\ \iff f_*(P) &= \frac{\partial f_*}{\partial x}(P) = \frac{\partial f_*}{\partial y}(P) = 0 \\ \iff f(P) &= \frac{\partial f}{\partial x}(P) = \frac{\partial f}{\partial y}(P) = 0. \quad (f \text{ is a form}) \end{aligned}$$

- (2) By Euler's Theorem in §1.1, $f(P) = \frac{\partial f}{\partial x}(P) = \frac{\partial f}{\partial y}(P) = 0$ implies that $\frac{\partial f}{\partial z}(P) = 0$.

□

Problem 5.2.

Show that the following curves are irreducible; find their multiple points, and the multiplicities and tangents at the multiple points.

- (a) $xy^4 + yz^4 + xz^4$.
- (b) $x^2y^3 + x^2z^3 + y^2z^3$.
- (c) $y^2z - x(x - z)(x - \lambda z)$, $\lambda \in k$.
- (d) $x^n + y^n + z^n$, $n > 0$.

Proof of (a).

- (1) Let $f = xy^4 + yz^4 + xz^4$. It suffices to show that $f_* = x(y^4 + 1) + y$ is irreducible. View $f_* \in (k[y])[x]$ as a linear function in x . Since $y^4 + 1$ and y have no common factors in $k[x]$, f_* is irreducible in $(k[y])[x] = k[x, y]$.

(2) By solving the system of equations $f(P) = \frac{\partial f}{\partial x}(P) = \frac{\partial f}{\partial y}(P) = \frac{\partial f}{\partial z}(P) = 0$, there is only one multiple point $[1 : 0 : 0]$ of f .

(3) Since $P = [1 : 0 : 0]$, we dehomogenize f w.r.t. x to get

$$m_P(f) = m_P(f_*) = m_{(0,0)}(yz^4 + y^4 + z^4) = 4.$$

P is an ordinary multiple point of 4 distinct tangents $y \pm (-1)^{\frac{1}{4}}z$, $y \pm (-1)^{\frac{3}{4}}z$.

□

Proof of (b).

(1) Let $f = x^2y^3 + x^2z^3 + y^2z^3$. It suffices to show that $f_* = x^2(y^3 + 1) + y^2$ is irreducible. View $f_* \in (k[y])[x]$ as a quadratic function in x . Note that $y^3 + 1$ and y^2 have no common factors in $k[x]$. We may write

$$f_*(x, y) = (a(y)x + b(y))(c(y)x + d(y))$$

if f_* were reducible. So

$$\begin{aligned} ac &= y^3 + 1 \\ ad + bc &= 0 \\ bd &= y^2. \end{aligned}$$

There is no solutions for $a, b, c, d \in k[y]$. Hence, f_* is irreducible in $(k[y])[x] = k[x, y]$.

(2) By solving the system of equations $f(P) = \frac{\partial f}{\partial x}(P) = \frac{\partial f}{\partial y}(P) = \frac{\partial f}{\partial z}(P) = 0$, we have $P = [1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$.

(3) Suppose $P = [1 : 0 : 0]$. We dehomogenize f w.r.t. x to get

$$m_P(f) = m_P(f_*) = m_{(0,0)}(y^2z^3 + y^3 + z^3) = 3.$$

P is an ordinary multiple point of 3 distinct tangents $y + z$, $y + (-1)^{\frac{1}{3}}z$, $y - (-1)^{\frac{2}{3}}z$.

(4) Suppose $P = [0 : 1 : 0]$. We dehomogenize f w.r.t. y to get

$$m_P(f) = m_P(f_*) = m_{(0,0)}(x^2z^3 + z^3 + x^2) = 2.$$

x is the tangent to f at P of the multiplicity = 2.

(5) Suppose $P = [0 : 0 : 1]$. We dehomogenize f w.r.t. z to get

$$m_P(f) = m_P(f_*) = m_{(0,0)}(x^2y^3 + x^2 + y^2) = 2.$$

P is an ordinary multiple point of 2 distinct tangents $x \pm (-1)^{\frac{1}{2}}y$.

□

Proof of (c).

- (1) Note that $y^2 - x(x-1)(x-\lambda)$ is irreducible (Problem 1.35). Hence, the homogenizing form $f = y^2z - x(x-z)(x-\lambda z)$ of $y^2 - x(x-1)(x-\lambda)$ w.r.t. z is also irreducible.
- (2) By solving the system of equations $f(P) = \frac{\partial f}{\partial x}(P) = \frac{\partial f}{\partial y}(P) = \frac{\partial f}{\partial z}(P) = 0$, we have

$$\begin{aligned} P &= [0 : 0 : 1] \text{ if } \lambda = 0 \\ P &= [1 : 0 : 1] \text{ if } \lambda = 1 \end{aligned}$$

are all multiple points of f .

- (3) Suppose $P = [0 : 0 : 1]$ with $\lambda = 0$. Then

$$m_P(f) = m_P(f_*) = m_{(0,0)}(y^2 - x^2(x-1)) = 2.$$

y is the tangent to f at P of the multiplicity = 2. (The projective closure of y is y .)

- (4) Similarly, suppose $P = [1 : 0 : 1]$ with $\lambda = 1$. Then

$$\begin{aligned} m_P(f) &= m_P(f_*) \\ &= m_{(1,0)}(y^2 - x(x-1)^2) \\ &= m_{(0,0)}(y^2 - x^2(x+1)) \\ &= 2. \end{aligned}$$

y is the tangent to f at P of the multiplicity = 2.

□

Proof of (d).

- (1) Let $f = x^n + y^n + z^n$. It suffices to show that $f_* = x^n + y^n + 1$ is irreducible. Write

$$f_* = x^n + (y^n + 1) \in (k[y])[x].$$

Note that $k[y]$ is a UFD and $y^n + 1$ is separable. Thus by the Eisenstein's criterion, f_* irreducible in $(k[y])[x] = k[x, y]$.

- (2) By solving the system of equations $f(P) = \frac{\partial f}{\partial x}(P) = \frac{\partial f}{\partial y}(P) = \frac{\partial f}{\partial z}(P) = 0$, there are no multiple points on f for $n \geq 1$.

□

Problem 5.3.

Find all points of intersection of the following pairs of curves, and the intersection numbers at these points:

- (a) $y^2z - x(x - 2z)(x + z)$ and $y^2 + x^2 - 2xz$.
- (b) $(x^2 + y^2)z + x^3 + y^3$ and $x^3 + y^3 - 2xyz$.
- (c) $y^5 - x(y^2 - xz)^2$ and $y^4 + y^3z - x^2z^2$.
- (d) $(x^2 + y^2)^2 + 3x^2yz - y^3z$ and $(x^2 + y^2)^3 - 4x^2y^2z^2$.

Proof of (a).

- (1) Say $f = y^2z - x(x - 2z)(x + z)$ and $g = y^2 + x^2 - 2xz$. By $f - zg = 0$, $x(x - 2z)(x + 2z) = 0$. So the intersection points in \mathbf{P}^2 are

$$\underbrace{[0 : 0 : 1]}_{x=0}, \underbrace{[2 : 0 : 1]}_{x-2z=0}, \underbrace{[-2, \pm\sqrt{-8} : 1]}_{x+2z=0}.$$

- (2) Suppose $P = [0 : 0 : 1]$. We dehomogenize f and g w.r.t. z to get

$$\begin{aligned} I(P, f \cap g) &= I((0, 0), f_* \cap g_*) \\ &= I((0, 0), (-x(x - 2)(x + 1) + y^2) \cap (y^2 + x^2 - 2x)) \\ &= 2 \end{aligned}$$

(since they have no common tangents).

- (3) Suppose $P = [2 : 0 : 1]$. We dehomogenize f and g w.r.t. z to get

$$\begin{aligned} I(P, f \cap g) &= I(P, f_* \cap g_*) \\ &= I(P, (-x(x - 2)(x + 1) + y^2) \cap (y^2 + x^2 - 2x)) \\ &= I((0, 0), (-x(x + 2)(x + 3) + y^2) \cap (y^2 + x^2 + 2x)) \\ &= 2. \end{aligned}$$

- (4) Suppose $P = [-2, \pm\sqrt{-8} : 1]$. We dehomogenize f and g w.r.t. z to get

$$\begin{aligned} I(P, f \cap g) &= I(P, f_* \cap g_*) \\ &= I(P, (-x(x - 2)(x + 1) + y^2) \cap (y^2 + x^2 - 2x)) \\ &= I((0, 0), (-x^3 + 7x^2 + y^2 - 14x \pm 2\sqrt{-8}y) \cap \\ &\quad (y^2 + x^2 - 6x \pm 2\sqrt{-8}y)) \\ &= 1. \end{aligned}$$

(Or by Bézout's theorem, each intersection point has the intersection number 1 in this case.)

□

Proof of (b).

- (1) Say $f = (x^2 + y^2)z + x^3 + y^3$ and $g = x^3 + y^3 - 2xyz$. By $f - g = 0$, $(x + y)^2 z = 0$. So the intersection points in \mathbf{P}^2 are

$$\underbrace{[0 : 0 : 1]}_{x+y=0}, \underbrace{[1 : -1 : 0], [1 : \omega : 0], [1 : 1 - \omega : 0]}_{z=0}.$$

where $\omega = (-1)^{\frac{1}{3}} \in k$.

- (2) Suppose $P = [0 : 0 : 1]$. We dehomogenize f and g w.r.t. z to get

$$\begin{aligned} I(P, f \cap g) &= I((0, 0), f_* \cap g_*) \\ &= I((0, 0), (x^3 + y^3 + x^2 + y^2) \cap (x^3 + y^3 - 2xy)) \\ &= 4. \end{aligned}$$

- (3) Suppose $P = [1 : -1 : 0]$. We dehomogenize f and g w.r.t. x to get

$$\begin{aligned} I(P, f \cap g) &= I(P, f_* \cap g_*) \\ &= I(P, (f_* - g_*) \cap g_*) \\ &= I(P, (y + 1)^2 z \cap (1 + y^3 - 2yz)) \\ &= I((0, 0), y^2 z \cap (y^3 - 3y^2 - 2yz + 3y + 2z)) \\ &= 3. \end{aligned}$$

- (4) Suppose $P = [1 : \omega : 0]$. Similar to (3), we dehomogenize f and g w.r.t. x to get

$$\begin{aligned} I(P, f \cap g) &= I(P, (y + 1)^2 z \cap (1 + y^3 - 2yz)) \\ &= I((0, 0), (y^2 z + 2(1 + \omega)yz + (1 + \omega)^2 z) \cap \\ &\quad (y^3 + 3\omega y^2 - 2yz + 3\omega^2 y - 2\omega z)) \\ &= I((0, 0), (y^2 z + 2(1 + \omega)yz + 3\omega z) \cap \\ &\quad (y^3 + 3\omega y^2 - 2yz + 3\omega^2 y - 2\omega z)) \\ &= 1. \end{aligned}$$

- (5) Suppose $P = [1 : 1 - \omega : 0]$. We dehomogenize f and g w.r.t. x to get

$$\begin{aligned} I(P, f \cap g) &= I(P, (y + 1)^2 z \cap (1 + y^3 - 2yz)) \\ &= I((0, 0), (y^2 z + 2(2 - \omega)yz + (2 - \omega)^2 z) \cap \\ &\quad (y^3 + 3(1 - \omega)y^2 - 2yz + 3(1 - \omega)^2 y - 2(1 - \omega)z)) \\ &= I((0, 0), (y^2 z + 2(2 - \omega)yz + 3(1 - \omega)z) \cap \\ &\quad (y^3 + 3(1 - \omega)y^2 - 2yz - 3\omega y - 2(1 - \omega)z)) \\ &= 1. \end{aligned}$$

□

Proof of (c).

- (1) Say $f = y^5 - x(y^2 - xz)^2$ and $g = y^4 + y^3z - x^2z^2$. If $y = 0$, the intersection points in $\mathbf{P}^2 - U_2$ are

$$[0 : 0 : 1], [1 : 0 : 0].$$

If $y \neq 0$, the intersection points in $2_3 \cong \mathbf{A}^2$ are the solutions of $f_* = 1 - x(1 - xz)^2$ and $g_* = 1 + z - x^2z^2$:

$$\left[\frac{1}{2} : 1 : 2 \pm 2\sqrt{2} \right].$$

Note that there are only 4 different intersection points (Bézout's theorem).

- (2) Suppose $P = [0 : 0 : 1]$. We dehomogenize f and g w.r.t. z to get

$$\begin{aligned} & I(P, f \cap g) \\ &= I((0, 0), f_* \cap g_*) \\ &= I((0, 0), (y^5 - x(y^2 - x)^2) \cap (y^4 + y^3 - x^2)) \\ &= I((0, 0), ((y^5 - x(y^2 - x)^2) - x(y^4 + y^3 - x^2)) \cap (y^4 + y^3 - x^2)) \\ &= I((0, 0), y^2(y^2 - x)(2x - y) \cap (y^4 + y^3 - x^2)) \\ &= I((0, 0), y^2 \cap (y^4 + y^3 - x^2)) \\ &\quad + I((0, 0), (y^2 - x) \cap (y^4 + y^3 - x^2)) \\ &\quad + I((0, 0), (2x - y) \cap (y^4 + y^3 - x^2)) \\ &= 4 + I((0, 0), (y^2 - x) \cap (y^4 + y^3 - x^2)) + 2 \\ &= 9. \end{aligned}$$

Here

$$\begin{aligned} & I((0, 0), (y^2 - x) \cap (y^4 + y^3 - x^2)) \\ &= I((0, 0), (y^2 - x) \cap ((y^4 + y^3 - x^2) - x(y^2 - x))) \\ &= I((0, 0), (y^2 - x) \cap (y^4 - xy^2 + y^3)) \\ &= 3. \end{aligned}$$

(3) Suppose $P = [1 : 0 : 0]$. We dehomogenize f and g w.r.t. x to get

$$\begin{aligned}
& I(P, f \cap g) \\
&= I((0, 0), f_* \cap g_*) \\
&= I((0, 0), (y^5 - (y^2 - z)^2) \cap (y^4 + y^3z - z^2)) \\
&= I((0, 0), ((y^5 - (y^2 - z)^2) - (y^4 + y^3z - z^2)) \cap (y^4 + y^3z - z^2)) \\
&= I((0, 0), y^2(y^2 - z)(y - 2) \cap (y^4 + y^3z - z^2)) \\
&= I((0, 0), y^2 \cap (y^4 + y^3z - z^2)) \\
&\quad + I((0, 0), (y^2 - z) \cap (y^4 + y^3z - z^2)) \\
&\quad + I((0, 0), (y - 2) \cap (y^4 + y^3z - z^2)) \\
&= 4 + I((0, 0), (y^2 - z) \cap (y^4 + y^3z - z^2)) + 0 \\
&= 4 + 5 + 0 \\
&= 9.
\end{aligned}$$

Here

$$\begin{aligned}
& I((0, 0), (y^2 - z) \cap (y^4 + y^3z - z^2)) \\
&= I((0, 0), (y^2 - z) \cap ((y^4 + y^3z - z^2) - z(y^2 - z))) \\
&= I((0, 0), (y^2 - z) \cap y^2(y^2 + yz - z)) \\
&= I((0, 0), (y^2 - z) \cap y^2) + I((0, 0), (y^2 - z) \cap (y^2 + yz - z)) \\
&= 2 + I((0, 0), (y^2 - z) \cap yz) \\
&= 2 + I((0, 0), (y^2 - z) \cap y) + I((0, 0), (y^2 - z) \cap z) \\
&= 2 + 1 + I((0, 0), y^2 \cap z) \\
&= 2 + 1 + 2 \\
&= 5.
\end{aligned}$$

(4) Suppose $P = [\frac{1}{2} : 1 : 2 + 2\sqrt{2}]$. We dehomogenize f and g w.r.t. y to get

$$\begin{aligned}
& I(P, f \cap g) \\
&= I(P, f_* \cap g_*) \\
&= I(P, (1 - x(1 - xz)^2) \cap (1 + z - x^2z^2)) \\
&= I\left((0, 0), \left(-(6 + 2\sqrt{2})x - \sqrt{2}/2 \cdot z + \text{higher terms}\right) \cap \right. \\
&\quad \left.(-(12 + 8\sqrt{2})x - \sqrt{2}z + \text{higher terms})\right) \\
&= 1.
\end{aligned}$$

Similarly, $I(P, f \cap g) = 1$ as $P = [\frac{1}{2} : 1 : 2 - 2\sqrt{2}]$.

□

Proof of (d).

- (1) Say $f = (x^2 + y^2)^2 + 3x^2yz - y^3z$ and $g = (x^2 + y^2)^3 - 4x^2y^2z^2$. If $z = 0$, the intersection points in $\mathbf{P}^2 - U_3$ are

$$[1 : \pm\sqrt{-1} : 0].$$

If $z \neq 0$, the intersection points in $U_3 \cong \mathbf{A}^2$ are the solutions of $f_* = (x^2 + y^2)^2 + 3x^2y - y^3$ and $g_* = (x^2 + y^2)^3 - 4x^2y^2$:

$$[0 : 0 : 1], \left[\pm \frac{\sqrt{5+2\sqrt{5}}}{4} : \frac{-\sqrt{5}}{4} : 1 \right], \left[\pm \frac{\sqrt{5-2\sqrt{5}}}{4} : \frac{\sqrt{5}}{4} : 1 \right].$$

(To find these solutions, we might solve $f_*(r \cos \theta, r \sin \theta) = g_*(r \cos \theta, r \sin \theta) = 0$ over $k = \mathbb{C}$. As $r = 0$, nothing to do. As $r > 0$, we get $\theta = \frac{m\pi}{5}$ for all integers m .) Note that there are only 7 different intersection points (Bézout's theorem).

- (2) Suppose $P = [1 : \sqrt{-1} : 0]$. We dehomogenize f and g w.r.t. x to get

$$\begin{aligned} & I(P, f \cap g) \\ &= I(P, f_* \cap g_*) \\ &= I(P, ((1 + y^2)^2 + 3yz - y^3z) \cap ((1 + y^2)^3 - 4y^2z^2)) \\ &= I(P, ((1 + y^2)^2 + 3yz - y^3z) \cap (yz(y^4 - 2y^2 - 4yz - 3))) \\ &= I(P, ((1 + y^2)^2 + 3yz - y^3z) \cap y) \\ &\quad + I(P, ((1 + y^2)^2 + 3yz - y^3z) \cap z) \\ &\quad + I(P, ((1 + y^2)^2 + 3yz - y^3z) \cap (y^4 - 2y^2 - 4yz - 3)) \\ &= 2I(P, (1 + y^2) \cap z) \\ &\quad + I(P, ((1 + y^2)^2 + 3yz - y^3z) \cap (y^4 - 2y^2 - 4yz - 3)) \\ &= 3. \end{aligned}$$

Here

$$I(P, (1 + y^2) \cap z) = I((0, 0), (y^2 + 2\sqrt{-1}y) \cap z) = 1$$

and

$$\begin{aligned} & I(P, ((1 + y^2)^2 + 3yz - y^3z) \cap (y^4 - 2y^2 - 4yz - 3)) \\ &= I((0, 0), (y^4 - y^3z + 4\sqrt{-1}y^3 - 3\sqrt{-1}y^2z - 4y^2 + 6yz + 4\sqrt{-1}z) \cap \\ &\quad (y^4 + 4\sqrt{-1}y^3 - 8y^2 - 4yz - 8\sqrt{-1}y - 4\sqrt{-1}z)) \\ &= 1. \end{aligned}$$

Similarly, $I(P, f \cap g) = 3$ if $P = [1 : -\sqrt{-1} : 0]$.

- (3) Suppose $P = [0 : 0 : 1]$. We dehomogenize f and g w.r.t. z to get

$$\begin{aligned} I(P, f \cap g) &= I((0, 0), f_* \cap g_*) \\ &= I((0, 0), ((x^2 + y^2)^2 + 3x^2y - y^3) \cap ((x^2 + y^2)^3 - 4x^2y^2)) \\ &= 14 \end{aligned}$$

(Example in §3.3).

- (4) Suppose $P = \left[\frac{\sqrt{5+2\sqrt{5}}}{4} : \frac{-\sqrt{5}}{4} : 1 \right]$. We dehomogenize f and g w.r.t. z to get

$$\begin{aligned}
& I(P, f \cap g) \\
&= I(P, f_* \cap g_*) \\
&= I(P, ((x^2 + y^2)^2 + 3x^2y - y^3) \cap ((x^2 + y^2)^3 - 4x^2y^2)) \\
&= I(P, ((x^2 + y^2)^2 + 3x^2y - y^3) \cap y(-3x^4 - 2x^2y^2 + y^4 - 4x^2y)) \\
&= 4I(P, x \cap y) \\
&\quad + I(P, ((x^2 + y^2)^2 + 3x^2y - y^3) \cap (-3x^4 - 2x^2y^2 + y^4 - 4x^2y)) \\
&= 1.
\end{aligned}$$

Here

$$I(P, x \cap y) = I\left((0, 0), \left(x + \frac{\sqrt{5+2\sqrt{5}}}{4}\right) \cap \left(y + \frac{-\sqrt{5}}{4}\right)\right) = 0$$

and

$$\begin{aligned}
& I(P, ((x^2 + y^2)^2 + 3x^2y - y^3) \cap (-3x^4 - 2x^2y^2 + y^4 - 4x^2y)) \\
&= I\left((0, 0), \left(\frac{(5+2\sqrt{5})\sqrt{5+2\sqrt{5}}}{8}x - \frac{5+2\sqrt{5}}{8}y + \text{higher terms}\right) \cap \right. \\
&\quad \left.\left(\frac{(-10+\sqrt{5})\sqrt{5+2\sqrt{5}}}{8}x - \frac{5+4\sqrt{5}}{8}y + \text{higher terms}\right)\right) \\
&= 1.
\end{aligned}$$

Similarly, $I(P, f \cap g) = 1$ for rest three cases.

□

Problem 5.4.*

Let P be a simple point on f . Show that the tangent line to f at P has the equation $\frac{\partial f}{\partial x}(P)x + \frac{\partial f}{\partial y}(P)y + \frac{\partial f}{\partial z}(P)z = 0$.

Proof.

- (1) Similar to Problem 5.1. Might assume $P = [x_0 : y_0 : 1] \in U_3$. By

dehomogenizing f with respect to z , the tangent line to f_* at P is

$$\begin{aligned} L : 0 &= \frac{\partial f}{\partial x}(P)(x - x_0) + \frac{\partial f}{\partial y}(P)(y - y_0) \\ &= \frac{\partial f}{\partial x}(P)x + \frac{\partial f}{\partial y}(P)y - \left(\frac{\partial f}{\partial x}(P)x_0 + \frac{\partial f}{\partial y}(P)y_0 \right) \\ &= \frac{\partial f}{\partial x}(P)x + \frac{\partial f}{\partial y}(P)y + \frac{\partial f}{\partial z}(P). \end{aligned}$$

(The last equation is due to Euler's Theorem in §1.1.)

(2) Hence, the projective closure of L , say

$$L^* : 0 = \frac{\partial f}{\partial x}(P)x + \frac{\partial f}{\partial y}(P)y + \frac{\partial f}{\partial z}(P)z,$$

is the tangent line to f at P .

□

Problem 5.5.*

Let $P = [0 : 1 : 0]$, f a curve of degree n , $f = \sum f_i(x, z)y^{n-i}$, f_i a form of degree i . Show that $m_P(f)$ is the smallest m such that $f_m \neq 0$, and the factors of f_m determine the tangents to f at P .

Proof.

(1) By dehomogenizing f with respect to y , we have

$$m_P(f) = m_P(f_*) = m_{(0,0)}\left(\sum f_i(x, z)\right) = m$$

where m is the smallest integer such that $f_m \neq 0$.

(2) The factors of f_m determine the tangents to f_* at P . Note that each factor of f_m is a form. Therefore, these factors of f_m also determine the tangents to f at P .

□

Problem 5.6.*

For any f , $P \in f$, show that $m_P\left(\frac{\partial f}{\partial x}\right) \geq m_P(f) - 1$.

Proof.

- (1) If $P = [1 : 0 : 0]$, by dehomogenizing f with respect to x we get

$$m_P\left(\frac{\partial f}{\partial x}\right) = m_P\left(\frac{\partial f_*}{\partial x}\right) = m_P(0) = \infty.$$

The result is established.

- (2) Might assume $P \in U_3$. By dehomogenizing f with respect to z , it suffices to show that

$$m_P\left(\frac{\partial f_*}{\partial x}\right) \geq m_P(f_*) - 1.$$

- (3) Write $m = m_P(f_*)$ and $f_* = g_m + g_{m+1} + \cdots \in k[x, y]$ where $g_i \neq 0$ is a form of degree m . Note that

$$\frac{\partial f_*}{\partial x} = \frac{\partial g_m}{\partial x} + \frac{\partial g_{m+1}}{\partial x} + \cdots.$$

So that $m_P\left(\frac{\partial f_*}{\partial x}\right) \geq m - 1 = m_P(f_*) - 1.$

□

Problem 5.7.*

Show that two plane curves with no common components intersect in a finite number of points.

Proof.

- (1) Similar to Proposition 2 in §1.6. Let f and g be forms in $k[x, y, z]$ with no common components. In particular, $f \neq 0$ and $g \neq 0$.
- (2) Show that the conclusion is true for a line g . Similar to Problem 1.12. Let $\alpha : \mathbf{P}^1 \rightarrow \mathbf{P}^2$ be the map

$$\alpha([\lambda : \mu]) = [\lambda a_1 + \mu b_1 : \lambda a_2 + \mu b_2 : \lambda a_3 + \mu b_3]$$

and $g = \text{im}(\alpha)$ be a line. Note that the points of $V(f, g)$ correspond to the zero of

$$\tilde{f}(\lambda, \mu) = f(\lambda a_1 + \mu b_1, \lambda a_2 + \mu b_2, \lambda a_3 + \mu b_3) = 0.$$

in \mathbf{P}^1 . \tilde{f} is a form in $k[\lambda, \mu]$ of degree $m := \deg(f)$ since f and g have no common components. Hence, \tilde{f} has exactly m zeros in \mathbf{P}^1 (with multiplicity). $V(f, g)$ has exactly m zeros (with multiplicity).

Note. It is the same as Problem 5.12(b).

- (3) Show that the conclusion is true for any f, g . Note that f and g have no common factors in $k(x, y)[z]$. Since $k(x, y)[z]$ is a PID, $(f, g) = (1) \in k(x, y)[z]$. So $\alpha f + \beta g = 1$ for some $\alpha, \beta \in k(x, y)[z]$. Clearing denominators from α, β gives us

$$af + bg = d$$

where d is a non-zero polynomial in $k[x, y]$, $a = d\alpha \in k[x, y, z]$ and $b = d\beta \in k[x, y, z]$. Taking suitable homogeneous parts of a, b and d , we might assume a, b and d are all homogeneous.

- (4) By Corollary to Proposition 5 in §2.6, d is a product of linear forms, say $d = L_1 \cdots L_r$. A common zero of f and g is also a zero of d , and therefore it is a zero of L_j for some j . So it suffices to show that $V = (f, g) = V(f, g, L_j)$ is finite for each j .
- (5) (Reductio ad absurdum) If not, there were one j such that both $V(f, L_j)$ and $V(g, L_j)$ were infinite. By (2), $L_j \mid f$ and $L_j \mid g$. f and g have one common factor, which is absurd.

□

Problem 5.8.*

Let f be an irreducible curve.

- (a) Show that $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, or $\frac{\partial f}{\partial z} \neq 0$.
- (b) Show that f has only a finite number of multiple points.

Proof of (a). If $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$ are all zero, f is a constant, which is not a projective plane curve. □

Proof of (b).

- (1) Let

$$V = V\left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

be the set of the multiple points of f (Problem 5.1). Moreover, V is an algebraic set.

- (2) Suppose $\frac{\partial f}{\partial x} \neq 0$ (since not all $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ are zero). It is nothing to do if $\frac{\partial f}{\partial x}$ is a nonzero constant. Suppose $\deg\left(\frac{\partial f}{\partial x}\right) \geq 1$. Since f is irreducible and $\deg\left(\frac{\partial f}{\partial x}\right) = \deg f - 1$, f and $\frac{\partial f}{\partial x}$ have no common factors. By Problem 5.7, $V\left(f, \frac{\partial f}{\partial x}\right)$ is a finite set. Hence, V is finite as a subset of a finite set $V\left(f, \frac{\partial f}{\partial x}\right)$.

□

Problem 5.9. (Conic)

- (a) Let f be an irreducible conic, $P = [0 : 1 : 0]$ a simple point on f , and $z = 0$ the tangent line to f at P . Show that $f = ayz - bx^2 - cxz - dz^2$, $a, b \neq 0$. Find a projective change of coordinates t so that $f^t = yz - x^2 - c'xz - d'z^2$. Find t' so that $(f^t)^{t'} = yz - x^2$. ($t' = (x, y + c'x + d'z, z)$.)
- (b) Show that, up to projective equivalence, there is only one irreducible conic: $yz = x^2$. Any irreducible conic is nonsingular.

Proof of (a).

- (1) Given any conic

$$f = ayz - bx^2 - cxz - dz^2 + \alpha y^2 + \beta xy.$$

- (2) $\alpha = 0$ since $P \in f$.

- (3) Show that $a \neq 0$ and $\beta = 0$. We dehomogenize f w.r.t. y to get $f_* = -bx^2 - cxz - dz^2 + az + \beta x$. Thus, the tangent line of f_* at $(0, 0)$ is $az + \beta x$. So $a \neq 0$ and $\beta = 0$ since z is the tangent line to f at P .

- (4) Show that $b \neq 0$. If $b = 0$, $f = ayz - cxz - dz^2 = z(ay - cx - dz)$ is reducible, which is absurd.

- (5) Write

$$\begin{aligned} f &= ayz - bx^2 - cxz - dz^2 \\ &= (\sqrt{ay})(\sqrt{az}) - (\sqrt{bx})^2 - \frac{c}{\sqrt{ab}}(\sqrt{bx})(\sqrt{az}) - \frac{d}{a}(\sqrt{az})^2. \end{aligned}$$

Apply a projective change of coordinates $t = \left(\frac{x}{\sqrt{b}}, \frac{y}{\sqrt{a}}, \frac{z}{\sqrt{a}}\right)$ to f :

$$f^t = yz - x^2 - c'xz - d'z^2,$$

where $c' = \frac{c}{\sqrt{ab}}$ and $d' = \frac{d}{a}$.

- (6) Write

$$f^t = yz - x^2 - c'xz - d'z^2 = (y - c'x - d'z)z - x^2.$$

Apply a projective change of coordinates $t' = (x, y + c'x + d'z, z)$ to f^t :

$$(f^t)^{t'} = yz - x^2.$$

□

Proof of (b).

- (1) Take one simple point $P_1 \in f$ and the corresponding tangent line L . (It is possible since Problem 5.8 and $|f| = \infty$.) Take $P_2 \in L - f$ and $P_3 \in f - L$. (It is possible since Problem 5.7 or 5.12 and both $|f|$ and $|L|$ are ∞ .)
- (2) Let $Q_1 = [0 : 1 : 0]$, $Q_2 = [1 : 0 : 0]$, $Q_3 = [0 : 0 : 1]$. Since P_1, P_2, P_3 (resp. Q_1, Q_2, Q_3) are not lying on a common line, there is a unique projective change of coordinates t such that $t(P_i) = Q_i$ ($i = 1, 2, 3$) by Problem 4.14.
- (3) Hence, f is projectively equivalent to an irreducible conic g such that $P = [0 : 1 : 0]$ is a simple point on g and $z = 0$ is the tangent line to g at P . By (a),

$$f \sim g \sim (yz - x^2).$$

- (4) Nonsingularity is preserved under projectively equivalence. Might assume that $f = yz - x^2$. $\frac{\partial f}{\partial x}(P) = \frac{\partial f}{\partial y}(P) = \frac{\partial f}{\partial z}(P) = 0$ implies that $P = [0 : 0 : 0]$, which is not in \mathbf{P}^2 . Hence f is nonsingular (Problem 5.1).

□

Problem 5.10. (Cubic with a cusp)

Let f be an irreducible cubic, $P = [0 : 0 : 1]$ a cusp on f , $y = 0$ the tangent line to f at P . Show that $f = ay^2z - bx^3 - cx^2y - dxy^2 - ey^3$. Find projective changes of coordinates

- (i) to make $a = b = 1$;
- (ii) to make $c = 0$ (change x to $x - \frac{c}{3}y$);
- (iii) to make $d = e = 0$ (z to $z + dx + ey$).

Up to projective equivalence, there is only one irreducible cubic with a cusp: $y^2z = x^3$. It has no other singularities.

Proof.

- (1) Given any cubic

$$\begin{aligned} f = & ay^2z - bx^3 - cx^2y - dxy^2 - ey^3 \\ & + \alpha_1z^3 + \alpha_2x^2z + \alpha_3z^2x + \alpha_4z^2y + \alpha_5xyz. \end{aligned}$$

- (2) $\alpha_1 = 0$ since $P \in f$.

- (3) Show that $b \neq 0$ and $\alpha_2 = \alpha_3 = 0$. Since $P = [0 : 0 : 1]$ is a cusp on f and $y = 0$ is the tangent line to f at P , $I(f \cap y) = 3$. So

$$\begin{aligned} 3 &= I(P, f \cap y) \\ &= I(P, (-bx^3 + \alpha_2 x^2 z + \alpha_3 z^2 x) \cap y) \\ &= I(P, (-bx^3 + \alpha_2 x^2 + \alpha_3 x) \cap y) \quad (\text{Dehomogenize w.r.t. } z) \end{aligned}$$

if and only if $b \neq 0$ and $\alpha_2 = \alpha_3 = 0$.

- (4) Show that $a \neq 0$ and $\alpha_4 = \alpha_5 = 0$. We dehomogenize f w.r.t. z to get

$$f_* = -bx^3 - cx^2y - dxy^2 - ey^3 + ay^2 + \alpha_5xy + \alpha_4y.$$

Since $(0, 0)$ is a double point, $\alpha_4 = 0$. Since f_* has only one tangent y on $(0, 0)$,

$$ay^2 + \alpha_5xy = y(ay + \alpha_5x)$$

implies that $a \neq 0$ and $\alpha_5 = 0$.

- (5) Therefore, $f = ay^2z - bx^3 - cx^2y - dxy^2 - ey^3$ with $a \neq 0$ and $b \neq 0$. Similar to Problem 5.9,

(i) Write

$$\begin{aligned} f &= ay^2z - bx^3 - cx^2y - dxy^2 - ey^3 \\ &= (\sqrt[3]{ay})^2(\sqrt[3]{az}) - (\sqrt[3]{bx})^3 - \frac{c}{\sqrt[3]{ab^2}}(\sqrt[3]{bx})^2(\sqrt[3]{ay}) \\ &\quad - \frac{d}{\sqrt[3]{a^2b}}(\sqrt[3]{bx})(\sqrt[3]{ay})^2 - \frac{e}{a}(\sqrt[3]{ay})^3. \end{aligned}$$

Apply a projective change of coordinates $t_1 = \left(\frac{x}{\sqrt[3]{b}}, \frac{y}{\sqrt[3]{a}}, \frac{z}{\sqrt[3]{a}}\right)$ to f :

$$f^{t_1} = y^2z - x^3 - c_1x^2y - d_1xy^2 - e_1y^3,$$

where $c_1 = \frac{c}{\sqrt[3]{ab^2}}$ and $d_1 = \frac{d}{\sqrt[3]{a^2b}}$ and $e_1 = \frac{e}{a}$.

(ii) Write

$$\begin{aligned} f^{t_1} &= y^2z - x^3 - c_1x^2y - d_1xy^2 - e_1y^3 \\ &= y^2z - \left(x + \frac{c_1}{3}y\right)^3 - \left(d_1 - \frac{c_1^2}{3}\right)\left(x + \frac{c_1}{3}y\right)y^2 \\ &\quad - \left(e_1 - \frac{c_1d_1}{3} + \frac{2c_1^3}{27}\right)y^3. \end{aligned}$$

Apply a projective change of coordinates $t_2 = \left(x - \frac{c_1}{3}y, y, z\right)$ to f^{t_1} :

$$(f^{t_1})^{t_2} = y^2z - x^3 - d_2xy^2 - e_2y^3$$

where $d_2 = d_1 - \frac{c_1^2}{3}$ and $e_2 = e_1 - \frac{c_1d_1}{3} + \frac{2c_1^3}{27}$. (Compare to Cardano's formula.)

(iii) Write

$$\begin{aligned}(f^{t_1})^{t_2} &= y^2z - x^3 - d_2xy^2 - e_2y^3 \\ &= y^2(z - d_2x - e_2y) - x^3.\end{aligned}$$

Apply a projective change of coordinates $t_3 = (x, y, z + d_2x + e_2y)$ to $(f^{t_1})^{t_2}$:

$$((f^{t_1})^{t_2})^{t_3} = y^2z - x^3.$$

- (6) Given any irreducible cubic f with a cusp. Apply the proof of Problem 5.9(b), f is projectively equivalent to an irreducible cubic g such that $P = [0 : 0 : 1]$ is a cusp on g and $y = 0$ is the tangent line to g at P . By (5),

$$f \sim g \sim (y^2z - x^3).$$

Similar to Problem 5.9(b), $y^2z - x^3$ has no other singularities except one point $P = [0 : 0 : 1]$.

□

Problem 5.11. (Cubic with a node)

Up to projective equivalence, there is only one irreducible cubic with a node: $xyz = x^3 + y^3$. It has no other singularities.

Proof.

- (1) Let f be an irreducible cubic, $P = [0 : 0 : 1]$ a node on f , $x = 0$, $y = 0$ the tangent lines to f at P . Show that $f = ax^3 + by^3 - cxyz - dx^2y - exy^2$, $a \neq 0$, $b \neq 0$, $c \neq 0$.

(a) Given any cubic

$$\begin{aligned}f &= ax^3 + by^3 - cxyz - dx^2y - exy^2 \\ &\quad + \alpha_1z^3 + \alpha_2x^2z + \alpha_3y^2z + \alpha_4z^2x + \alpha_5z^2y.\end{aligned}$$

(b) $\alpha_1 = 0$ since $P \in f$.

(c) We dehomogenize f w.r.t. z to get

$$f_* = ax^3 + by^3 - cxy - dx^2y - exy^2 + \alpha_2x^2 + \alpha_3y^2 + \alpha_4x + \alpha_5y.$$

Since $(0, 0)$ is a double point, $\alpha_4 = \alpha_5 = 0$. Since f_* have only two tangents x, y on $(0, 0)$, $-cxy + \alpha_2x^2 + \alpha_3y^2$ implies that $c \neq 0$ and $\alpha_2 = \alpha_3 = 0$.

- (d) Now $f = ax^3 + by^3 - cxyz - dx^2y - exy^2$ with $c \neq 0$. If $a = 0$ or $b = 0$, f is reducible, which is absurd. So $a \neq 0$ and $b \neq 0$.

(2) Hence,

$$f \sim (x^3 + y^3 - xyz - dx^2y - exy^2) \sim (x^3 + y^3 - xyz)$$

by $(x, y, z) \mapsto \left(\frac{1}{\sqrt[3]{a}}x, \frac{1}{\sqrt[3]{b}}y, \frac{\sqrt[3]{ab}}{c}z\right)$ and $(x, y, z) \mapsto (x, y, z - dx - ey)$.

(3) Given any irreducible cubic f with a node. Apply the proof of Problem 5.9(b), f is projectively equivalent to an irreducible cubic g such that $P = [0 : 0 : 1]$ is a node on g and x, y are the tangent lines to g at P . By (2),

$$f \sim g \sim (x^3 + y^3 - xyz).$$

Similar to Problem 5.9(b), $x^3 + y^3 - xyz$ has no other singularities except one point $P = [0 : 0 : 1]$.

□

Problem 5.12.*

(a) Assume $P = [0 : 1 : 0] \notin f$, f a curve of degree n . Show that

$$\sum_P I(P, f \cap x) = n.$$

(b) Show that if f is a curve of degree n , L a line not contained in f , then

$$\sum_P I(P, f \cap L) = n.$$

Proof of (a).

(1) Similar to Problem 5.7. The intersection points of f and x are the solutions of

$$f(0, y, z) = 0.$$

Write

$$f(0, y, z) = cy^n + \cdots$$

as a form in $k[y, z]$ of degree n .

(2) Note that $c \neq 0$ since $P = [0 : 1 : 0] \notin f$. By Corollary to Proposition 5 in §2.6,

$$f(0, y, z) = c \prod_i (y - \alpha_i z)^{r_i}$$

where $\alpha_i \in k$ and $\sum r_i = n$. So all intersection points are $P_i = [0 : \alpha_i : 1]$ with multiplicity r_i .

(3) Hence,

$$\begin{aligned}
& \sum_P I(P, f \cap x) \\
&= \sum_i I(P_i, f \cap x) \\
&= \sum_i I(P_i, f(0, y, z) \cap x) && \text{(Property (7) in §3.3)} \\
&= \sum_i \sum_j r_j I(P_i, (y - \alpha_j z) \cap x) && \text{(Property (6) in §3.3)} \\
&= \sum_i \sum_j r_j I((0, \alpha_i), (y - \alpha_j) \cap x) && \text{(Dehomogenize w.r.t. } z) \\
&= \sum_i \sum_j r_j I((0, 0), (y + \alpha_i - \alpha_j) \cap x) \\
&= \sum_i \sum_j r_j \delta_{ij} \\
&= \sum_i r_i \\
&= n.
\end{aligned}$$

□

Proof of (b).

- (1) Take any point $Q_1 \notin f \cup L$ (since \mathbf{P}^2 is irreducible). Suppose L is a line passing Q_2 and Q_3 , $Q_2 \neq Q_3$.
- (2) By Problem 4.14, there is a projective change of coordinates $t : \mathbf{P}^2 \rightarrow \mathbf{P}^2$ such that $t(Q_1) = [0 : 1 : 0]$ and $t(L) = x$. In particular, $t(Q_1) = [0 : 1 : 0] \notin f^t$ (by the construction in (1)).
- (3) Hence,

$$\sum_P I(P, f \cap L) = \sum_P I(P, f^t \cap L^t) = \sum_P I(P, f^t \cap x) = n.$$

□

Problem 5.13.

Prove that an irreducible cubic is either nonsingular or has at most one double point (a node or a cusp). (Hint: Use Problem 5.12, where L is a line through

two multiple points; or use Problems 5.10 and 5.11.)

Proof (Problem 5.12).

- (1) (Reductio ad absurdum) Suppose there were a line L passing through two multiple points P, Q of an irreducible cubic f .
- (2) Note that

$$\begin{aligned}
 \sum I(P, f \cap L) &\geq I(P, f \cap L) + I(Q, f \cap L) \\
 &\geq m_P(f)m_P(L) + m_Q(f)m_Q(L) \\
 &\geq 2 \cdot 1 + 2 \cdot 1 \\
 &= 4.
 \end{aligned}$$

By Problem 5.12, $\sum I(P, f \cap L) = 3$, a contradiction. Hence f has at most one multiple point. In particular, f has at most one double point.

□

Proof (Problems 5.10 and 5.11).

- (1) Suppose one irreducible cubic f has one multiple point P . It suffices to show that P is a cusp or a node, or it suffices to show that P is a double point.
- (2) Up to projective equivalence, might assume that $P = [0 : 0 : 1]$. If $m_P(f) = m_P(f_*) = 3$, then f_* is a form in $k[x, y]$ of degree 3. Thus f is in $k[x, y] \hookrightarrow k[x, y, z]$, which is reducible. Hence P is a double point. By Problems 5.10 and 5.11, f has no other singularities.

□

Problem 5.14.*

Let $P_1, \dots, P_n \in \mathbf{P}^2$. Show that there are an infinite number of lines passing through P_1 , but not through P_2, \dots, P_n . If P_1 is a simple point on f , we may take these lines transversal to f at P_1 .

Proof.

- (1) Might assume that all P_1, \dots, P_n are distinct. Write a line L_{ij} passing through P_i and P_j .
- (2) (Reductio ad absurdum) Suppose there were finite number of lines passing through P_1 , but not through P_2, \dots, P_n , say M_1, \dots, M_r . Hence, we can write \mathbf{P}^2 as

$$\mathbf{P}^2 = L_{12} \cup \dots \cup L_{1n} \cup M_1 \cup \dots \cup M_r,$$

contrary to the fact that \mathbf{P}^2 is irreducible.

□

Problem 5.15.*

Let C be an irreducible projective plane curve, P_1, \dots, P_n simple points on C , m_1, \dots, m_n integers. Show that there is a $z \in k(C)$ with $\text{ord}_{P_i}(z) = m_i$ for $i = 1, \dots, n$. (Hint: Take lines L_i as in Problem 5.14 for P_i , and a line L_0 not through any P_j , and let $z = \prod L_i^{m_i} L_0^{-\sum m_i}$.)

Proof.

- (1) Take lines L_i not tangent to C at P_i as in Problem 5.14 for P_i . Since $\mathbf{P}^2 = V(0) \supsetneq V(C)$, we can pick a point P outside C . Take one line L_0 through P but not through any P_j as in Problem 5.14.
- (2) Define $z = \prod L_i^{m_i} / L_0^{\sum m_i} \in k(C)$ since $\prod L_i^{m_i}$ and $L_0^{\sum m_i}$ are in $\Gamma(C)$ of the same degree $\sum m_i$. Note that $L_0(P_j) \neq 0$ for any P_j . So $\text{ord}_{P_i}(z)$ is well-defined. Moreover,

$$\text{ord}_{P_i}(z) = \sum_j m_j \text{ord}_{P_i}(L_j) - \left(\sum m_j \right) \text{ord}_{P_i}(L_0) = m_i$$

since L_j ($j \neq i$) and L_0 are units in $k(C)$ and $L_i \ni P_i$ is not tangent to C .

□

Problem 5.16.*

Let f be an irreducible curve in \mathbf{P}^2 . Suppose $I(P, f \cap z) = 1$, and $P \neq [1 : 0 : 0]$. Show that $\frac{\partial f}{\partial x}(P) \neq 0$. (Hint: If not, use Euler's Theorem to show that $\frac{\partial f}{\partial y}(P) = 0$; but z is not tangent to f at P .)

Proof.

- (1) Might write $P = [\alpha : 1 : 0] \in \mathbf{P}^2$ since $z(P) = 0$ (by $I(P, f \cap z) > 0$) and $P \neq [1 : 0 : 0]$.
- (2) Since $I(P, f \cap z) = 1$, $m_P(f) = 1$ and z is not tangent to f at P . Thus, P is a simple point of f . By Problem 5.4, the tangent line L to f at P is

$$\frac{\partial f}{\partial x}(P)x + \frac{\partial f}{\partial y}(P)y + \frac{\partial f}{\partial z}(P)z = 0.$$

In particular, $P = [\alpha : 1 : 0] \in L$, that is,

$$\frac{\partial f}{\partial x}(P)\alpha + \frac{\partial f}{\partial y}(P) = 0.$$

- (3) (Reductio ad absurdum) If $\frac{\partial f}{\partial x}(P) = 0$, $\frac{\partial f}{\partial y}(P) = 0$ by (2). Hence, the tangent line $L = z$, which is absurd.

□

5.2. Linear Systems of Curves

Problem 5.17.

Let P_1, P_2, P_3, P_4 are distinct points in \mathbf{P}^2 . Let V be the linear system of conics passing through these points. Show that $\dim(V) = 2$ if P_1, P_2, P_3, P_4 lie on a line, and $\dim(V) = 1$ otherwise.

Proof.

- (1) Let $V_i = V(2; P_1, \dots, P_i)$. (So $V = V_4$.) By Theorem 1 in §5.2, we have

$$\dim V_4 \geq \frac{2(2+3)}{2} - \sum_{i=1}^4 \frac{1(1+1)}{2} = 1,$$

$$\dim V_3 = \frac{2(2+3)}{2} - \sum_{i=1}^3 \frac{1(1+1)}{2} = 2.$$

Also note that $V_4 \subseteq V_3$. Hence

$$\dim V_4 = 1 \text{ or } 2.$$

Therefore, it suffices to show that P_1, P_2, P_3, P_4 lie on a line if and only if

$$V_4 = V_3.$$

- (2) The case P_1, P_2, P_3, P_4 lie on a line L . Take any conic $C \in V_3$. By Bézout's theorem or Problem 5.12, $L \mid C$. Hence $P_4 \in L \subseteq C$ and thus $C \in V_4$.
- (3) The case P_1, P_2, P_3, P_4 are not collinear.
- (a) Suppose there are some three points, say P_1, P_2, P_3 , are in a line L . Since $P_4 \notin L$ by assumption, the conic L^2 is in V_3 but not in V_4 .
- (b) Suppose there are no three points lying on a line. Let L_{ij} be the line passing through P_i and P_j . Then the conic $L_{12}L_{13} \in V_3$ but not in V_4 .

By (a)(b), $V_4 \subsetneq V_3$ if P_1, P_2, P_3, P_4 are not collinear.

□

Problem 5.18.

Show that there is only one conic passing through the five points $[0 : 0 : 1]$, $[0 : 1 : 0]$, $[1 : 0 : 0]$, $[1 : 1 : 1]$, and $[1 : 2 : 3]$; show that it is nonsingular.

Proof.

- (1) Let $C : ax^2 + bxy + cxz + dy^2 + eyz + fz^2$ be the conic passing these five points. That is,

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 6 & 9 \end{pmatrix}}_{\text{say } A \in \mathbf{M}_{5 \times 6}(k)} \underbrace{\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}}_{\in \mathbf{P}^5(k)} = 0.$$

- (2) The row echelon form A' of A is

$$A' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \mathbf{M}_{5 \times 6}(k).$$

$\text{rank}(A') = 5$ and thus there is a unique solution in \mathbf{P}^5 , say $[0 : 3 : -4 : 0 : 1 : 0] \in \mathbf{P}^5$. Hence,

$$C : 3xy - 4xz + yz = 0$$

is only one conic passing through these five points.

- (3) By solving the system of equations $C(P) = \frac{\partial C}{\partial x}(P) = \frac{\partial C}{\partial y}(P) = \frac{\partial C}{\partial z}(P) = 0$, $P = [0 : 0 : 0]$, which is absurd. Hence C is nonsingular.

□

Problem 5.19.

Consider the nine points $[0 : 0 : 1]$, $[0 : 1 : 1]$, $[1 : 0 : 1]$, $[1 : 1 : 1]$, $[0 : 2 : 1]$, $[2 : 0 : 1]$, $[1 : 2 : 1]$, $[2 : 1 : 1]$, and $[2 : 2 : 1] \in \mathbf{P}^2$ (Sketch). Show that there are an infinite number of cubics passing through these points.

Proof.

- (1) Let $C : a_1x^3 + a_2y^3 + a_3z^3 + a_4x^2y + a_5x^2z + a_6y^2x + a_7y^2z + a_8z^2x + a_9z^2y + a_{10}xyz$ be the cubic passing these five points. That is,

$$\underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 8 & 1 & 0 & 0 & 0 & 4 & 0 & 2 & 0 \\ 8 & 0 & 1 & 0 & 4 & 0 & 0 & 2 & 0 & 0 \\ 1 & 8 & 1 & 2 & 1 & 4 & 4 & 1 & 2 & 2 \\ 8 & 1 & 1 & 4 & 4 & 2 & 1 & 2 & 1 & 2 \\ 8 & 8 & 1 & 8 & 4 & 8 & 4 & 2 & 2 & 4 \end{pmatrix}}_{\text{say } A \in M_{9 \times 10}(k)} \underbrace{\begin{pmatrix} a_1 \\ \vdots \\ a_{10} \end{pmatrix}}_{\in \mathbf{P}^9(k)} = 0.$$

- (2) The row echelon form A' of A is

$$A' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in M_{9 \times 10}(k).$$

$\text{rank}(A') = 8 < 9$ and thus there are an infinite number of cubics passing through these points, say

$$C : \lambda x(x - z)(x - 2z) + \mu y(y - z)(y - 2z),$$

where $[\lambda : \mu] \in \mathbf{P}^1(k)$.

□

5.3. Bézout's Theorem

Problem 5.20.

Check your answers of Problem 5.3 with Bézout's theorem

Proof.

$$(1) \quad f = y^2z - x(x - 2z)(x + z) \text{ and } g = y^2 + x^2 - 2xz.$$

$$I([0 : 0 : 1], f \cap g) = 2,$$

$$I([2 : 0 : 1], f \cap g) = 2,$$

$$I([-2, \sqrt{-8} : 1], f \cap g) = 1,$$

$$I([-2, -\sqrt{-8} : 1], f \cap g) = 1.$$

$$\text{Hence, } \sum_P I(P, f \cap g) = 2 + 2 + 1 + 1 = 6 = \deg(f) \deg(g).$$

$$(2) \quad f = (x^2 + y^2)z + x^3 + y^3 \text{ and } g = x^3 + y^3 - 2xyz.$$

$$I([0 : 0 : 1], f \cap g) = 4,$$

$$I([1 : -1 : 0], f \cap g) = 3,$$

$$I([1 : \sqrt[3]{-1} : 0], f \cap g) = 1,$$

$$I([1 : 1 - \sqrt[3]{-1} : 0], f \cap g) = 1.$$

$$\text{Hence, } \sum_P I(P, f \cap g) = 4 + 3 + 1 + 1 = 9 = \deg(f) \deg(g).$$

$$(3) \quad f = y^5 - x(y^2 - xz)^2 \text{ and } g = y^4 + y^3z - x^2z^2.$$

$$I([0 : 0 : 1], f \cap g) = 9,$$

$$I([1 : 0 : 0], f \cap g) = 9,$$

$$I\left(\left[\frac{1}{2} : 1 : 2 + 2\sqrt{2}\right], f \cap g\right) = 1,$$

$$I\left(\left[\frac{1}{2} : 1 : 2 - 2\sqrt{2}\right], f \cap g\right) = 1.$$

$$\text{Hence, } \sum_P I(P, f \cap g) = 9 + 9 + 1 + 1 = 20 = \deg(f) \deg(g).$$

$$(4) \quad f = (x^2 + y^2)^2 + 3x^2yz - y^3z \text{ and } g = (x^2 + y^2)^3 - 4x^2y^2z^2.$$

$$I([1 : \sqrt{-1} : 0], f \cap g) = 3,$$

$$I([1 : -\sqrt{-1} : 0], f \cap g) = 3,$$

$$I([0 : 0 : 1], f \cap g) = 14,$$

$$I\left(\left[\frac{\sqrt{5} + 2\sqrt{5}}{4} : \frac{-\sqrt{5}}{4} : 1\right], f \cap g\right) = 1,$$

$$I\left(\left[-\frac{\sqrt{5} + 2\sqrt{5}}{4} : \frac{-\sqrt{5}}{4} : 1\right], f \cap g\right) = 1,$$

$$I\left(\left[\frac{\sqrt{5} - 2\sqrt{5}}{4} : \frac{\sqrt{5}}{4} : 1\right], f \cap g\right) = 1,$$

$$I\left(\left[-\frac{\sqrt{5} - 2\sqrt{5}}{4} : \frac{\sqrt{5}}{4} : 1\right], f \cap g\right) = 1.$$

$$\text{Hence, } \sum_P I(P, f \cap g) = 3 + 3 + 14 + 1 + 1 + 1 + 1 = 24 = \deg(f) \deg(g).$$

□

Problem 5.21.*

Show that every nonsingular projective plane curve is irreducible. Is this true for affine curves?

Proof.

- (1) Let f be a nonsingular projective plane curve. (Reductio ad absurdum) Suppose $f = gh$ were reducible. Then g and h intersect in \mathbf{P}^2 (Bézout's theorem).
- (2) Show that each intersection point P is a singularity of f . Note that $g(P) = h(P) = 0$ and the chain rule for derivatives. Thus,

$$\frac{\partial f}{\partial x_i}(P) = g(P) \frac{\partial h}{\partial x_i}(P) + \frac{\partial g}{\partial x_i}(P) h(P) = 0$$

for $i = 1, 2, 3$. P is a singularity of f (Problem 5.1).

- (3) Show that it is not true for affine curves. $V(x^2 - x) \subseteq \mathbf{A}^2$ is nonsingular but not irreducible.

□

Problem 5.22.*

Let f be an irreducible curve of degree n . Assume $\frac{\partial f}{\partial x} \neq 0$. Apply Corollary 1 to f and $\frac{\partial f}{\partial x}$, and conclude that $\sum m_P(f)(m_P(f) - 1) \leq n(n - 1)$. In particular, f has at most $\frac{n(n-1)}{2}$ multiple points. (See Problems 5.6, 5.8.)

Proof.

- (1) There is nothing to do for $n = 1$. Might assume that $n > 1$.
- (2) Since f and $\frac{\partial f}{\partial x}$ have no common component (Problem 5.8), Corollary 1 to Bézout's theorem gives us

$$\sum m_P(f)m_P\left(\frac{\partial f}{\partial x}\right) \leq \deg(f) \deg\left(\frac{\partial f}{\partial x}\right) \leq n(n - 1).$$

By Problem 5.6, the conclusion is established.

(3) Then

$$\begin{aligned}
n(n-1) &\geq \sum_P m_P(f)(m_P(f)-1) \\
&\geq \sum_{m_P(f)>1} 2(2-1) \\
&\geq \sum_{m_P(f)>1} 2 \\
&= 2 \cdot (\text{number of multiple points}).
\end{aligned}$$

□

Problem 5.23. (Hessian matrix)

A problem about flexes (see Problem 3.12): Let f be a projective plane curve of degree n , and assume f contains no lines.

Let $D_i f = \frac{\partial f}{\partial x_i}$ and $D_{ij} f = \frac{\partial^2 f}{\partial x_i \partial x_j}$, forms of degree $n-1$ and $n-2$ respectively. Form a 3×3 matrix with the entry in the (i, j) th place being $D_{ij} f$ ($i, j = 1, 2, 3$). Let H be the determinant of this matrix, a form of degree $3(n-2)$. This H is called the **Hessian** of f . Problems 5.22 and 6.47 show that $H \neq 0$, for f irreducible. The following theorem shows the relationship between flexes and the Hessian.

Theorem. ($\text{char}(k) = 0$)

- (1) $P \in H \cap f$ if and only if P is either a flex or a multiple point of f .
- (2) $I(P, H \cap f) = 1$ if and only if P is an ordinary flex.

Outline of proof.

- (a) Let t be a projective change of coordinates. Then the Hessian of f^t is $\det(t)^2(H^t)$. So we can assume $P = [0 : 0 : 1]$; write $f_*(x, y) = f(x, y, 1)$ and $H_*(x, y) = H(x, y, 1)$.
- (b) $(n-1)D_j f = \sum_i x_i D_{ij} f$. (Use Euler's Theorem.)
- (c) $I(P, f_* \cap H_*) = I(P, f_* \cap g)$ where

$$g = (D_2 f_*)^2 (D_{11} f_*) + (D_1 f_*)^2 (D_{22} f_*) - 2(D_1 f_*)(D_2 f_*)(D_{12} f_*).$$

(Hint: Perform row and column operations on the matrix for H_* . Add x times the first row plus y times the second row to the third row, then apply part (b). Do the same with the columns. Then calculate the determinant.)

(d) If P is a multiple point on f , then $I(P, f_* \cap g) > 1$.

(e) Suppose P is a simple point, $y = 0$ is the tangent line to f at P , so

$$f_* = y + ax^2 + bxy + cy^2 + dx^3 + ex^2y + \cdots.$$

Then P is a flex if and only if $a = 0$, and P is an ordinary flex if and only if $a = 0$ and $d \neq 0$. A short calculation shows that

$$g = 2a + 6dx + (8ac - 2b^2 + 2e)y + \text{higher terms},$$

which concludes the proof.

Corollary.

- (1) A nonsingular curve of degree > 2 always has a flex.
- (2) A nonsingular cubic has nine flexes, all ordinary.

Proof of (a).

- (1) Write $t = (t_1, t_2, t_3)$ where $t_1, t_2, t_3 \in k[x, y, z]$.
- (2) By the chain rule for differentiation,

$$\frac{\partial f^t}{\partial x_j} = \sum_{\beta=1}^3 \frac{\partial f^t}{\partial t_\beta} \frac{\partial t_\beta}{\partial x_j}$$

and thus

$$\begin{aligned} \frac{\partial^2 f^t}{\partial x_i \partial x_j} &= \frac{\partial}{\partial x_i} \left(\frac{\partial f^t}{\partial x_j} \right) \\ &= \frac{\partial}{\partial x_i} \left(\sum_{\beta=1}^3 \frac{\partial f^t}{\partial t_\beta} \frac{\partial t_\beta}{\partial x_j} \right) \\ &= \sum_{\beta=1}^3 \frac{\partial}{\partial x_i} \left(\frac{\partial f^t}{\partial t_\beta} \right) \cdot \frac{\partial t_\beta}{\partial x_j} + \frac{\partial f^t}{\partial t_\beta} \cdot \underbrace{\frac{\partial^2 t_\beta}{\partial x_i \partial x_j}}_{=0} \\ &= \sum_{\beta=1}^3 \sum_{\alpha=1}^3 \frac{\partial t_\alpha}{\partial x_i} \left(\frac{\partial f^t}{\partial t_\alpha \partial t_\beta} \right) \cdot \frac{\partial t_\beta}{\partial x_j} \\ &= \sum_{\alpha=1}^3 \sum_{\beta=1}^3 \frac{\partial t_\alpha}{\partial x_i} \left(\frac{\partial f^t}{\partial t_\alpha \partial t_\beta} \right) \frac{\partial t_\beta}{\partial x_j}. \end{aligned}$$

(3) Hence, the Hessian of f^t is

$$\begin{aligned}
H(f^t) &= \det \left(\left(\frac{\partial^2 f^t}{\partial x_i \partial x_j} \right)_{i,j} \right) \\
&= \det \left(\left(\sum_{\alpha=1}^3 \sum_{\beta=1}^3 \frac{\partial t_\alpha}{\partial x_i} \left(\frac{\partial f^t}{\partial t_\alpha \partial t_\beta} \right) \frac{\partial t_\beta}{\partial x_j} \right)_{i,j} \right) \\
&= \det \left(\left(\frac{\partial t_\alpha}{\partial x_i} \right)_{\alpha,i}^T \left(\frac{\partial f^t}{\partial t_\alpha \partial t_\beta} \right)_{\alpha,\beta} \left(\frac{\partial t_\beta}{\partial x_j} \right)_{\beta,j} \right) \\
&= \det \left(\left(\frac{\partial t_\alpha}{\partial x_i} \right)_{\alpha,i}^T \right) \det \left(\left(\frac{\partial f^t}{\partial t_\alpha \partial t_\beta} \right)_{\alpha,\beta} \right) \det \left(\left(\frac{\partial t_\beta}{\partial x_j} \right)_{\beta,j} \right) \\
&= \det(t^T) \cdot H(f)^t \cdot \det(t) \\
&= \det(t)^2 H(f)^t.
\end{aligned}$$

(4) Note that t is invertible. So we can assume $P = [0 : 0 : 1]$.

□

Proof of (b). Since $D_j f$ is a form of degree $n - 1$, the result is established by Euler's theorem in §1.1. □

Proof of (c).

(1) Note that

$$\begin{aligned}
& z^2 H \\
&= z^2 \begin{vmatrix} D_{11}f & D_{12}f & D_{13}f \\ D_{21}f & D_{22}f & D_{23}f \\ D_{31}f & D_{32}f & D_{33}f \end{vmatrix} \\
&= z \begin{vmatrix} D_{11}f & D_{12}f & D_{13}f \\ D_{21}f & D_{22}f & D_{23}f \\ xD_{11}f + yD_{21}f + zD_{31}f & xD_{12}f + yD_{22}f + zD_{32}f & xD_{13}f + yD_{23}f + zD_{33}f \end{vmatrix} \\
&= z \begin{vmatrix} D_{11}f & D_{12}f & D_{13}f \\ D_{21}f & D_{22}f & D_{23}f \\ (n-1)D_1f & (n-1)D_2f & (n-1)D_3f \end{vmatrix} \\
&= \begin{vmatrix} D_{11}f & D_{12}f & xD_{11}f + yD_{12}f + zD_{13}f \\ D_{21}f & D_{22}f & xD_{21}f + yD_{22}f + zD_{23}f \\ (n-1)D_1f & (n-1)D_2f & (n-1)xD_1f + (n-1)yD_2f + (n-1)zD_3f \end{vmatrix} \\
&= \begin{vmatrix} D_{11}f & D_{12}f & (n-1)D_1f \\ D_{21}f & D_{22}f & (n-1)D_2f \\ (n-1)D_1f & (n-1)D_2f & n(n-1)f \end{vmatrix} \\
&= (n-1)^2(D_1f)(D_{21}fD_2f - D_{22}fD_1f) \\
&\quad - (n-1)^2(D_2f)(D_{11}fD_2f - D_{12}fD_1f) \\
&\quad + n(n-1)f(D_{11}fD_{22}f - D_{12}fD_{21}f) \\
&= (n-1)^2(-(D_1f)^2D_{22}f - (D_2f)^2D_{11}f + 2(D_1f)(D_2f)(D_{12}f)) \\
&\quad + n(n-1)f(D_{11}fD_{22}f - D_{12}fD_{21}f).
\end{aligned}$$

(2) Dehomogenize z^2H w.r.t. z :

$$H_* = -(n-1)^2g + n(n-1)f_*[(D_{11}f_*)(D_{22}f_*) - (D_{12}f_*)^2].$$

Hence if $n > 1$, then

$$I(P, f_* \cap H_*) = I(P, f_* \cap (-(n-1)^2g)) = I(P, f_* \cap g).$$

(If $n \leq 1$, then $I(P, f_* \cap H_*) = I(P, f_* \cap g) = \infty$.)

□

Proof of (d).

- (1) Since $I(P, f_* \cap g) \geq m_P(f_*)m_P(g) \geq 2m_P(g)$, it suffices to show that $m_P(g) > 0$ or $P \in g$.

(2) Problem 5.1 shows that

$$\begin{aligned}
g(P) &= (D_1 f_*(P))^2 D_{22} f_*(P) \\
&\quad + (D_2 f_*(P))^2 D_{11} f_*(P) \\
&\quad - 2(D_1 f_*(P))(D_2 f_*(P))(D_{12} f_*(P)) \\
&= 0^2 \cdot D_{22} f_*(P) + 0^2 \cdot D_{11} f_*(P) - 2 \cdot 0 \cdot 0 \cdot D_{12} f_*(P) \\
&= 0.
\end{aligned}$$

□

Proof of (e).

(1) Suppose P is a simple point of f , $y = 0$ is the tangent line to f at P . Write

$$f_* = y + ax^2 + bxy + cy^2 + dx^3 + ex^2y + \cdots.$$

Then P is a flex if and only if $a = 0$, and P is an ordinary flex if and only if $a = 0$ and $d \neq 0$ (Problem 3.12).

(2) A short calculation shows that

$$\begin{aligned}
g &= (D_2 f_*)^2 (D_{11} f_*) + (D_1 f_*)^2 (D_{22} f_*) - 2(D_1 f_*)(D_2 f_*)(D_{12} f_*) \\
&= [2a + (6d + 4ab)x + (8ac + 2e)y] + 0 - 2[2abx + b^2y] + \text{higher terms} \\
&= 2a + 6dx + (8ac - 2b^2 + 2e)y + \text{higher terms}.
\end{aligned}$$

(3) Hence,

$$\begin{aligned}
P &\in f_* \cap g \\
\iff I(P, f_* \cap g) &\geq 1 \\
\iff I(P, (y + \cdots) \cap (2a + 6dx + (8ac - 2b^2 + 2e)y + \cdots)) &\geq 1 \\
\iff a &= 0 \\
\iff P &\text{ is a flex}
\end{aligned}$$

if P is a simple point of f .

(4) Also note that

$$\begin{aligned}
I(P, f_* \cap g) &= 1 \\
\iff I(P, (y + \cdots) \cap (2a + 6dx + (8ac - 2b^2 + 2e)y + \cdots)) &= 1 \\
\iff a = 0 \text{ and } d \neq 0 \\
\iff P &\text{ is an ordinary flex}
\end{aligned}$$

if P is a simple point of f .

□

Proof of Theorem (1).

- (1) (\implies) Suppose $P \in H \cap f$ and P is not a multiple point of f . So P is a simple point of f . Note that

$$I(P, H \cap f) = I(P, H_* \cap f_*) = I(P, g \cap f_*) \geq 1.$$

So P is a flex by part (e).

- (2) (\impliedby) If P is a flex of f , then part (e) shows that

$$I(P, H \cap f) = I(P, H_* \cap f_*) = I(P, g \cap f_*) \geq 1.$$

If P is a multiple point of f , then part (d) shows that

$$I(P, H \cap f) = I(P, H_* \cap f_*) \geq 2.$$

In any case, $P \in H \cap f$.

□

Proof of Theorem (2).

- (1) (\implies) By the proof of Theorem (1), P is a simple point of f . Therefore, the conclusion is established by part (e).
 (2) (\impliedby) Part (e).

□

Proof of Corollary (1).

- (1) Let f be a nonsingular curve of degree > 2 . So $\deg(H) = 3(n-2) > 0$. $H \cap f \neq \emptyset$ (Bézout's theorem).
 (2) Since f is nonsingular, every point in $H \cap f$ is a simple point. So every point in $H \cap f$ is a flex (Theorem (1)).

□

Proof of Corollary (2).

- (1) *Show that every flex of a nonsingular cubic f is ordinary.* Given any flex $P \in f$ and the corresponding tangent line L to f at P . By definition,

$$\text{ord}_P^f(L) = I(P, f \cap L) \geq 3.$$

Bézout's theorem or Problem 5.12 shows that

$$\sum_P I(P, f \cap L) = I(P, f \cap L) = 3.$$

That is, $\text{ord}_P^f(L) = 3$ or P is an ordinary flex.

(2) Since f is nonsingular, we have

$$\begin{aligned} H \cap f &= \{\text{all flexes of } f\} && \text{(Theorem (1))} \\ &= \{\text{all ordinary flexes of } f\} && ((1)) \\ &= \{P \in f : I(P, H \cap f) = 1\}. && \text{(Theorem (2))} \end{aligned}$$

(3) Hence

$$\sum_P I(P, H \cap f) = \deg(H) \deg(f) = 3 \cdot 3 = 9$$

(Bézout's theorem) shows that there are exactly 9 (ordinary) flexes of f .
(Also see Problem 5.37.)

□

Problem 5.24.

(char(k) = 0)

- (a) Let $P := [0 : 1 : 0]$ be a flex on an irreducible cubic f , $z = 0$ the tangent line to f at $[0 : 1 : 0]$. Show that $f = zy^2 + byz^2 + cyxz + \text{terms in } x, z$. Find a projective change of coordinates (using $y \mapsto y - \frac{b}{2}z - \frac{c}{2}x$) to get f to the form $zy^2 = \text{cubic in } x, z$.
- (b) Show that any irreducible cubic is projectively equivalent to one of the following: $y^2z = x^3$, $y^2z = x^2(x + z)$, or $y^2z = x(x - z)(x - \lambda z)$, $\lambda \in k$, $\lambda \neq 0, 1$. (See Problems 5.10, 5.11.)

Proof of (a).

- (1) As $z = 0$ is the tangent line to f at $[0 : 1 : 0]$, we can write

$$f(x, 1, z) = z + ax^2 + bz^2 + cxz - \alpha_0x^3 - \alpha_1x^2z - \alpha_2xz^2 - \alpha_3z^3.$$

By assumption,

$$I(P, f \cap z) = I(P, (ax^2 - \alpha_0x^3) \cap z) \geq 3$$

implies that $a = 0$. Also, $\alpha_0 \neq 0$ since f is irreducible. Hence,

$$f(x, y, z) = zy^2 + byz^2 + cyxz - \underbrace{\alpha_0}_{\neq 0} x^3 - \alpha_1x^2z - \alpha_2xz^2 - \alpha_3z^3.$$

- (2) So

$$\begin{aligned} f &\sim y^2z - \underbrace{\alpha_0}_{:=\beta_0} x^3 - \underbrace{\left(\alpha_1 - \frac{c^2}{4}\right)}_{:=\beta_1} x^2z - \underbrace{\left(\alpha_2 - \frac{bc}{2}\right)}_{:=\beta_2} xz^2 - \underbrace{\left(\alpha_3 - \frac{b^2}{4}\right)}_{:=\beta_3} z^3 \\ &= y^2z - \beta_0x^3 - \beta_1x^2z - \beta_2xz^2 - \beta_3z^3 \end{aligned}$$

by $y \mapsto y - \frac{b}{2}z - \frac{c}{2}x$. Hence f is of the form $zy^2 = \text{cubic in } x, z$ (up to projective equivalence).

□

Proof of (b).

(1) Since $\beta_0 \neq 0$ and $k = \bar{k}$, by $x \mapsto \beta_0^{-\frac{1}{3}}x$ we might write

$$f \sim y^2z - x^3 - \beta_1x^2z - \beta_2xz^2 - \beta_3z^3.$$

(2) Since $k = \bar{k}$,

$$x^3 + \beta_1x^2z + \beta_2xz^2 + \beta_3z^3 = (x - \gamma_1z)(x - \gamma_2z)(x - \gamma_3z)$$

for some $\gamma_1, \gamma_2, \gamma_3 \in k$. Consider the cases about $\gamma_1, \gamma_2, \gamma_3$.

(3) The case $\gamma_1 = \gamma_2 = \gamma_3$.

$$\begin{aligned} f &\sim y^2z - (x - \gamma_1z)(x - \gamma_2z)(x - \gamma_3z) \\ &\sim y^2z - x^3. \end{aligned} \quad (x \mapsto x + \gamma_1z)$$

(4) The case $\gamma_1 = \gamma_2 \neq \gamma_3$.

$$\begin{aligned} f &\sim y^2z - (x - \gamma_1z)(x - \gamma_2z)(x - \gamma_3z) \\ &\sim y^2z - x^2(x + (\gamma_1 - \gamma_3)z) && (x \mapsto x + \gamma_1z) \\ &\sim (\gamma_1 - \gamma_3)^{-1}y^2z - x^2(x + z) && (z \mapsto (\gamma_1 - \gamma_3)^{-1}z) \\ &\sim y^2z - x^2(x + z). && (y \mapsto (\gamma_1 - \gamma_3)^{\frac{1}{2}}y) \end{aligned}$$

(5) The case all $\gamma_1, \gamma_2, \gamma_3$ are different.

$$\begin{aligned} f &\sim y^2z - (x - \gamma_1z)(x - \gamma_2z)(x - \gamma_3z) \\ &\sim y^2z - x(x - (\gamma_2 - \gamma_1)z)(x - (\gamma_3 - \gamma_1)z) && (x \mapsto x + \gamma_1z) \\ &\sim (\gamma_2 - \gamma_1)^{-1}y^2z - x^2(x - z) \left(x - \frac{\gamma_3 - \gamma_1}{\gamma_2 - \gamma_1}z \right) && (z \mapsto (\gamma_2 - \gamma_1)^{-1}z) \\ &\sim y^2z - x^2(x - z) \left(x - \frac{\gamma_3 - \gamma_1}{\gamma_2 - \gamma_1}z \right) && (y \mapsto (\gamma_2 - \gamma_1)^{\frac{1}{2}}y) \\ &\sim y^2z - x^2(x - z)(x - \lambda z). && \left(\lambda := \frac{\gamma_3 - \gamma_1}{\gamma_2 - \gamma_1}, \lambda \neq 0, 1 \right) \end{aligned}$$

□

5.4. Multiple Points

Problem 5.25.

Let f be a projective plane curve of degree n with no multiple components, and c simple components. Show that

$$\sum \frac{m_P(m_P - 1)}{2} \leq \frac{(n-1)(n-2)}{2} + c - 1 \leq \frac{n(n-1)}{2}.$$

(Hint: Let $f = f_1 f_2$; consider separately the points on one f_i or on both.)

Proof.

(1) Write

$$f = f_1 \cdots f_c,$$

where each f_i is irreducible. By assumption, f_i and f_j have no common factor whenever $i \neq j$. By Bézout's theorem, we might consider separately the points on one f_i or on many components.

(2) Hence

$$\begin{aligned} & \sum \frac{m_P(m_P - 1)}{2} \\ & \leq \sum_{P \in f_1} \frac{m_P(m_P - 1)}{2} + \sum_{P \notin f_1} \frac{m_P(m_P - 1)}{2} \\ & \leq \underbrace{\frac{(d-1)(d-2)}{2}}_{\text{Theorem 2 in §5.4}} + \underbrace{\frac{(n-d-1)(n-d-2)}{2}}_{\text{Mathematical induction}} + (c-1) - 1 \\ & = \frac{(n-1)(n-2)}{2} + c - 1 + \underbrace{1 - d(n-d)}_{\leq 0} \\ & \leq \frac{(n-1)(n-2)}{2} + c - 1 \end{aligned}$$

where $d = \deg(f_1) \geq 1$.

(3) Note that $c \leq n$. So

$$\frac{(n-1)(n-2)}{2} + c - 1 \leq \frac{(n-1)(n-2)}{2} + n - 1 = \frac{n(n-1)}{2}.$$

□

Problem 5.26.*

($\text{char}(k) = 0$) Let f be an irreducible curve of degree n in \mathbf{P}^2 . Suppose $P \in \mathbf{P}^2$, with $m_P(f) = r \geq 0$. Then for all but a finite number of lines L through P , L intersects f in $n - r$ distinct points other than P . We outline a proof:

- (a) We may assume $P = [0 : 1 : 0]$. If $L_\lambda = \{[\lambda : t : 1] : t \in k\} \cup \{P\}$, we need only consider the L_λ . Then $f = a_r(x, z)y^{n-r} + \cdots + a_n(x, z)$, $a_r \neq 0$. (See Problems 4.24, 5.5).
- (b) Let $g_\lambda(t) = f(\lambda, t, 1)$. It is enough to show that for all but a finite number of λ , g_λ has $n - r$ distinct points.
- (c) Show that g_λ has $n - r$ distinct roots if $a_r(\lambda, 1) \neq 0$, and $f \cap \frac{\partial f}{\partial y} \cap L_\lambda = \{P\}$ if $P \in f$ (see Problem 1.53).

Proof of (a).

- (1) Might assume $P = [0 : 1 : 0]$ (up to projective equivalence).
- (2) By Problem 4.24, we might consider the lines $L_\lambda = \{[\lambda : t : 1] : t \in k\} \cup \{P\}$ except finitely one line L_∞ .
- (3) Write $f = a_r(x, z)y^{n-r} + \cdots + a_n(x, z)$, $a_r \neq 0$ (Problem 5.5).

□

Proof of (b). Let $g_\lambda(t) = f(\lambda, t, 1) = a_r(\lambda, 1)t^{n-r} + \cdots + a_n(\lambda, 1)$. It suffices to show that all but a finite number of λ , $g_\lambda(t)$ has $n - r$ distinct roots in k . □

Proof of (c).

- (1) There are finitely number of $\lambda \in k$ such that $a_r(\lambda, 1) = 0$ since k is a field.
- (2) $g_\lambda(t)$ is irreducible over $(k[\lambda])[t]$ since f is irreducible. Hence $a_r(\lambda, 1) \neq 0$ and $\text{char}(k) = 0$ implies that $g_\lambda(t)$ has $n - r$ distinct roots in k (Problem 1.53).
- (3) By (1)(2), the result is established.
- (4) Suppose $P \in f$. Show that $f \cap \frac{\partial f}{\partial y} \cap L_\lambda = \{P\}$ (for all but finitely number of λ). Since $f \cap L_\lambda = \{P\}$, it suffices to show that $P \in \frac{\partial f}{\partial y}$.
 - (a) It is nothing to do if P is a multiple point of f (Problem 5.1).
 - (b) If P is a simple point of f , then the tangent line to f at P is $L = \frac{\partial f}{\partial x}(P)x + \frac{\partial f}{\partial y}(P)y + \frac{\partial f}{\partial z}(P)z$ (Problem 5.4). Since $P \in L$, $\frac{\partial f}{\partial y}(P) = 0$.

In any case, $\frac{\partial f}{\partial y}(P) = 0$.

□

Problem 5.27.

Show that Problem 5.26 remains true if f is reducible, provided it has no multiple components.

Proof.

- (1) Write $f = f_1 \cdots f_s$ where each f_i is irreducible and f_i, f_j have no common factor whenever $i \neq j$.
- (2) Apply Problem 5.26 to each f_i to get that for all but a finite number of lines L_i through P , L_i intersects f_i in $n_i - r_i$ distinct points other than P , where $n_i = \deg(f_i)$ and $r_i = m_P(f_i)$. Collect these lines as \mathcal{L}_i .
- (3) Since $f_i \cap f_j$ is finite whenever $i \neq j$ (Problem 5.7), there are finitely many lines through P also passing through $f_i \cap f_j$ for any $i \neq j$. Collect these lines as \mathcal{L}' .
- (4) Hence any line from

$$(\mathcal{L}_1 \cup \cdots \cup \mathcal{L}_s) - \mathcal{L}'$$

passes through P and intersects f in

$$\sum (n_i - r_i) = \sum n_i - \sum r_i = n - r$$

distinct points other than P (by part (2) and our construction). The result is established.

□

Problem 5.28. (Terrible point)

($\text{char}(k) = p > 0$) $f = x^{p+1} - y^p z$, $P = [0 : 1 : 0]$. Find $L \cap f$ for all lines L passing through P . Show that every line that is tangent to f at a simple point passes through P !

Proof.

- (1) The case $\text{char}(k) = p > 0$ for Problem 4.24 shows that $L = L_\lambda = V(x - \lambda z)$ for any $\lambda \in k$ or $L = L_\infty = V(z)$. Note that $L_0 \cap f = \{P\}$ and $L_\infty \cap f = \{P\}$. It suffices to consider the case $L_\lambda \cap f$ for $\lambda \neq 0$.

(2) So

$$\begin{aligned}
0 &= x^{p+1} - y^p z \\
&= x^{p+1} - y^p (\lambda^{-1} x) && (x - \lambda z = 0) \\
&= x(x^p - \lambda^{-1} y^p) \\
&= x \left(x^p - \left(\lambda^{-\frac{1}{p}} y \right)^p \right) && (k = \bar{k}) \\
&= x \left(x - \lambda^{-\frac{1}{p}} y \right)^p. && (\text{char}(k) = p > 0)
\end{aligned}$$

Here $\lambda^{-\frac{1}{p}}$ is the unique p^{th} root of λ^{-1} (since $\text{char}(k) = p$). Hence

$$L_\lambda \cap f = \left\{ P, [\lambda^{-\frac{1}{p}} : 1 : \lambda^{-\frac{p+1}{p}}] \right\}.$$

(3) Show that every line that is tangent to f at a simple point passes through P . Let L be the tangent line to f at a simple point $Q = [a : b : c]$ passing through P . Problem 5.4 shows that

$$\begin{aligned}
L &= \frac{\partial f}{\partial x}(Q)x + \frac{\partial f}{\partial y}(Q)y + \frac{\partial f}{\partial z}(Q)z \\
&= \frac{\partial f}{\partial x}(Q)x + \frac{\partial f}{\partial z}(Q)z.
\end{aligned}$$

Here $\frac{\partial f}{\partial y} = -py^{p-1} = 0 \in k[x, y, z]$ since $\text{char}(k) = p > 0$. So $P = [0 : 1 : 0] \in L$ for any tangent line to f at a simple point of f .

□

5.5. Max Noether's Fundamental Theorem

Problem 5.29.

Fix f , g , and P . Show that in cases (1) and (2) - but not (3) - of Proposition 1 the conditions on h are equivalent to Noether's conditions.

Recall Proposition 1 in §5.5:

- (1) f and g meet transversally at P , and $P \in h$.
- (2) P is a simple point on f , and $I(P, h \cap f) \geq I(P, g \cap f)$.
- (3) f and g have distinct tangents at P , and $m_P(h) \geq m_P(f) + m_P(g) - 1$.

Proof.

- (1) (Noether's condition \implies (1) of Proposition 1) Since $h_* = af_* + bg_*$ for some $a, b \in \mathcal{O}_P(\mathbf{P}^2)$,

$$\begin{aligned}
 I(P, h \cap f) &= I(P, h_* \cap f_*) \\
 &= I(P, (af_* + bg_*) \cap f_*) \\
 &= I(P, (bg_*) \cap f_*) \\
 &= I(P, b \cap f_*) + I(P, g_* \cap f_*) \\
 &\geq I(P, g_* \cap f_*) \\
 &= 1.
 \end{aligned}$$

Hence $P \in h$.

- (2) (Noether's condition \implies (2) of Proposition 1) As $h_* \in (f_*, g_*) \subseteq \mathcal{O}_P(\mathbf{P}^2)$, $(f_*, h_*) \subseteq (f_*, g_*)$. So

$$\begin{aligned}
 I(P, h \cap f) &= I(P, h_* \cap f_*) \\
 &= \dim_k(\mathcal{O}_P(\mathbf{P}^2)/(f_*, h_*)) \\
 &\geq \dim_k(\mathcal{O}_P(\mathbf{P}^2)/(f_*, g_*)) \\
 &= I(P, g_* \cap f_*) \\
 &= I(P, g \cap f).
 \end{aligned}$$

- (3) (Noether's condition $\not\Rightarrow$ (3) of Proposition 1) Consider $h = y^2z - x^3$, $f = y^2$, $g = x^3$ and $P = [0 : 0 : 1]$. Noether's condition holds for h_* , f_* , g_* at the point $P = [0 : 0 : 1]$, and f, g have distinct tangents at P . However,

$$\underbrace{m_P(h)}_{=2} < \underbrace{m_P(f)}_{=2} + \underbrace{m_P(g)}_{=3} - 1.$$

□

5.6. Applications of Noether's Theorem

Problem 5.33.

Let C be an irreducible cubic, L a line such that $L \bullet C = P_1 + P_2 + P_3$, P_i distinct. Let L_i be the tangent line to C at P_i : $L_i \bullet C = 2P_i + Q_i$ for some Q_i . Show that Q_1, Q_2, Q_3 lie on a line. (L^2 is a conic!)

Proof.

- (1) Similar to the proof of Proposition 2 in §5.6. Corollary 2 to Bézout's theorem shows that each P_i is simple. So all the points of $L^2 \cap C$ are simple points of C .

(2) Note that

$$\begin{aligned}
(L_1 L_2 L_3) \bullet C &= L_1 \bullet C + L_2 \bullet C + L_3 \bullet C \\
&= (2P_1 + Q_1) + (2P_2 + Q_2) + (2P_3 + Q_3) \\
&= (2P_1 + 2P_2 + 2P_3) + (Q_1 + Q_2 + Q_3)
\end{aligned}$$

and

$$L^2 \bullet C = 2(L \bullet C) = 2(P_1 + P_2 + P_3) = 2P_1 + 2P_2 + 2P_3.$$

So $(L_1 L_2 L_3) \bullet C > L^2 \bullet C$.

(3) By Corollary (2) to max Noether's fundamental theorem, there is a curve M such that $M \bullet C = Q_1 + Q_2 + Q_3$. Bézout's theorem shows that $\deg(M) = 1$ or M is a line. Therefore, M is a line passing Q_1, Q_2, Q_3 .

□

Problem 5.34.

Show that a line through two flexes on a cubic passes through a third flex.

Proof.

- (1) Let C be a cubic, L a line such that $L \bullet C = P_1 + P_2 + P_3$.
(2) Let P_i be a simple point in C , L_i the tangent line to C at P_i . Show that P_i is a flex in C if and only if $L_i \bullet C = 3P_i$.

P_i is a flex in C

$$\iff I(P_i, L_i \cap C) = \text{ord}_{P_i}^C(L_i) \geq 3$$

$$\iff I(P_i, L_i \cap C) = 3 \text{ and } L_i \cap C = \{P_i\}$$

$$\iff L_i \bullet C = 3P_i. \quad (\text{Bézout's theorem})$$

(3) Suppose P_1 and P_2 are flexes in C . $I(P_3, L_3 \cap C) > 1$ since L_3 is one tangent line to C at P_3 . So $L_3 \bullet C = 2P_3 + P'_3$ for some $P'_3 \in C$. Now it suffices to show that $P'_3 = P_3$. (P_3 is simple since P_1 and P_2 are simple points by Corollary 2 to Bézout's theorem. So P_3 is a flex if $L_3 \bullet C = 3P_3$ by (2).)

(4) Consider $C' = L^3$ and $C'' = L_1 L_2 L_3$. So

$$C' \bullet C = 3(L \bullet C) = 3(P_1 + P_2 + P_3) = 3P_1 + 3P_2 + 3P_3$$

$$C'' \bullet C = L_1 \bullet C + L_2 \bullet C + L_3 \bullet C = 3P_1 + 3P_2 + 2P_3 + P'_3.$$

By Proposition 3 in §5.6, $P_3 = P'_3$.

□

Problem 5.35.

Let C be any irreducible cubic, or any cubic without multiple components, C° the set of simple points of C , $O \in C^\circ$. Show that the same definition as in the nonsingular case makes C° into an abelian group.

Proof.

- (1) It suffices to show that $\varphi(C^\circ \times C^\circ) \subseteq C^\circ$.
- (2) Let C be any irreducible cubic. Given any $P_1, P_2 \in C^\circ$. Then there is a unique line L such that $L \bullet C = P_1 + P_2 + P_3$ for some $P_3 \in C$. (If $P_1 = P_2$, L is the tangent to C at P .) So it suffices to show that $P_3 \in C^\circ$.
- (3) Might assume $P_3 \neq P_1, P_3 \neq P_2$. Bézout's theorem shows that

$$\sum_P I(P, L \cap C) = 3.$$

So $I(P_3, L \cap C) = 1$ whenever $P_1 = P_2$ or $P_1 \neq P_2$. Therefore, $P_3 \in C^\circ$ (Property (5) in §3.3).

□

Problem 5.36.

Let C be an irreducible cubic, O a simple point on C giving rise to the addition \oplus on the set C° of simple points. Suppose another O' gives rise to an addition \oplus' . Define $\alpha : (C, O, \oplus) \rightarrow (C, O', \oplus')$ by $\alpha(P) = \varphi(\varphi(O, O'), P)$. Show that α is a group isomorphism. So the structure of the group is independent of the choice of O .

Proof.

- (1) Show that α is a group homomorphism, that is, $\alpha(P \oplus Q) = \alpha(P) \oplus' \alpha(Q)$ for all $P, Q \in C^\circ$. The definition is well-defined (Problem 5.35).
- (2) Similar to Proposition 4 in §5.6. Let

$$\begin{aligned} L_1 \bullet C &= P + Q + R', \\ M_1 \bullet C &= R' + O + R, & (R = P \oplus Q) \\ L_2 \bullet C &= O + O' + O'' \\ L_3 \bullet C &= R + O'' + T. & (T = \alpha(P \oplus Q)) \end{aligned}$$

Let

$$\begin{aligned}
M_2 \bullet C &= P + O'' + P' & (P' = \alpha(P)) \\
M_3 \bullet C &= Q + O'' + Q' & (Q' = \alpha(Q)) \\
L_4 \bullet C &= P' + Q' + S \\
M_4 \bullet C &= S + O' + T'. & (T' = \alpha(P) \oplus' \alpha(Q))
\end{aligned}$$

Let $f = L_1 L_2 L_3 L_4$, $f' = M_1 M_2 M_3 M_4$. Apply Problem 5.42 on C° to get $T = T'$ or $\alpha(P \oplus Q) = \alpha(P) \oplus' \alpha(Q)$.

- (3) Define $\beta : (C, O', \oplus') \rightarrow (C, O, \oplus)$ by $\beta(P) = \varphi(\varphi(O', O), P)$. β is a group homomorphism too. By step (1) in the proof of Problem 5.37(b), $\alpha \circ \beta = 1_{(C, O', \oplus')}$ and $\beta \circ \alpha = 1_{(C, O, \oplus)}$. Hence α is a group isomorphism.

□

Problem 5.37.

In Proposition 4, suppose O is a flex on C .

- Show that the flexes form a subgroup of C ; as an abelian group, this subgroup is isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.
- Show that the flexes are exactly the elements of order three in the group. (i.e., exactly those elements P such that $P \oplus P \oplus P = O$).
- Show that a point P is of order two in the group if and only if the tangent to C at P passes through O .
- Let $C = y^2z - x(x - z)(x - \lambda z)$, $\lambda \neq 0, 1$, $O = [0 : 1 : 0]$. Find the points of order two.
- Show that the points of order two on a nonsingular cubic form a group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
- Let C be a nonsingular cubic, $P \in C$. How many lines through P are tangent to C at some point $Q \neq P$? (The answer depends on whether P is a flex.)

Proof of (a).

- (1) Let G be the set of all flexes of C . Show that G is a subgroup of C . $G \neq \emptyset$ since $O \in G$. Given any two flexes $P, Q \in C$, we have

$$\begin{aligned}
-P &= \varphi(O, P) \in G \\
P \oplus Q &= \varphi(O, \varphi(P, Q)) \in G
\end{aligned}$$

by using Problem 5.34 three times. Thus G is a subgroup of C .

- (2) Show that $P \oplus P \oplus P = O$ for all flexes of C . Let L be the tangent line to C at P . By step (2) in the proof of Problem 5.34, P is a flex in C if and only if $L \bullet C = 3P$. Hence $\varphi(P, P) = P$,

$$P \oplus P = \varphi(O, \varphi(P, P)) = \varphi(O, P) = -P.$$

- (3) Note that $|G| = 9$ since a nonsingular cubic C has 9 ordinary flexes (Corollary (2) to Problem 5.23). So G is an abelian group of order 9 ((1)). So $G \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ (the fundamental theorem of finite abelian groups and (2)).

□

Proof of (b).

- (1) Let L be one line such that $L \bullet C = P_1 + P_2 + P_3$. Then L is uniquely determined by P_1 and P_2 . (If $P_1 \neq P_2$, nothing to do. If $P_1 = P_2$, the tangent line is unique since $P_1 = P_2$ is simple.)
- (2) Might assume $P \neq O$. (O is of order 1.) So

$$\begin{aligned}
 & P \oplus P \oplus P = O \\
 \iff & P \oplus P = -P \\
 \iff & \varphi(O, \varphi(P, P)) = \varphi(O, P) \\
 \iff & \varphi(O, \varphi(O, P)) = \varphi(P, P) & ((1)) \\
 \iff & P = \varphi(P, P) \\
 \iff & L \bullet C = 3P \text{ where } L \text{ is the tangent line to } C \text{ at } P \\
 \iff & P \text{ is a flex in } C. & (\text{Problem 5.34})
 \end{aligned}$$

□

Proof of (c).

- (1) $P \neq O$ since O is of order 1.
- (2) So

$$\begin{aligned}
 & P \oplus P = O \\
 \iff & \varphi(O, \varphi(P, P)) = O \\
 \iff & \varphi(O, O) = \varphi(P, P) & ((b)(1)) \\
 \iff & O = \varphi(P, P) & (O: \text{flex}) \\
 \iff & \text{The tangent line to } C \text{ at } P \text{ passing through } O.
 \end{aligned}$$

□

Proof of (d).

- (1) Problem 5.4 shows that the tangent line L to C at $P := [a : b : c] \in C$ is

$$L = \frac{\partial C}{\partial x}(P)x + \frac{\partial C}{\partial y}(P)y + \frac{\partial C}{\partial z}(P)z.$$

Might assume $P \neq O$. By (c), $O \in L$ and thus $\frac{\partial C}{\partial y}(P) = 2bc = 0$. $b = 0$ or $c = 0$.

- (2) The case $b = 0$. $P \in C$ implies that $a(a - c)(a - \lambda c) = 0$. So $P = [0 : 0 : 1], [1 : 0 : 1], [\lambda : 0 : 1]$. It is easy to verify that $P \oplus P = O$ and $P \neq O$.
- (3) The case $c = 0$. $P = [0 : 1 : 0] = O$.
- (4) By (2)(3), the points of order two (including O) are

$$[0 : 0 : 1], [1 : 0 : 1], [\lambda : 0 : 1], [0 : 1 : 0].$$

□

Proof of (e).

- (1) Let H be the set of all points $P \in C$ with $P \oplus P = O$. Show that H is a subgroup of C . $H \neq \emptyset$ since $O \in H$. Given any two $P, Q \in H$, we have

$$\begin{aligned} (-P) \oplus (-P) &= P \oplus P = O, \\ (P \oplus Q) \oplus (P \oplus Q) &= (P \oplus P) \oplus (Q \oplus Q) = O \oplus O = O. \end{aligned}$$

So H is a subgroup of C .

- (2) Note that the nonsingularity is preserved under projectively equivalence. By Problems 5.21, 5.24, 5.10 and 5.11, it suffices to assume that $C = y^2z - x(x - z)(x - \lambda z)$.
- (3) By (d) and the fundamental theorem of finite abelian groups, $H \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

□

Proof of (f).

- (1) To find lines through P are tangent to C at some point $Q \in C$, it suffices to find $Q \in C$ satisfying $\varphi(Q, Q) = P$. Construct an abelian group (C, P, \oplus) on C .
- (2) The case $P \in C$ is a flex. (c)(e) implies that there are 4 lines through P are tangent to C at some point $Q \in C$. Since $\varphi(P, P) = P$, there are 3 lines through P are tangent to C at some point $Q \neq P$.

- (3) The case $P \in C$ is not a flex. Construct another abelian group (C, P', \oplus') on C where $P' \in C$ is any flex. (The existence of P' is guaranteed by (a).) By Problem 5.36, the map

$$\alpha : (C, P, \oplus) \rightarrow (C, P', \oplus')$$

defined by $\alpha(Q) = \varphi(\varphi(P, P'), Q)$ is an isomorphism. Here $\alpha^{-1}(Q') = \varphi(\varphi(P, P'), Q')$.

- (4) By (2), there are 4 points $Q'_i \in C$ ($i = 1, 2, 3, 4$) such that

$$Q'_i \oplus' Q'_i = P'.$$

Take α^{-1} on the both sides to get

$$Q_i \oplus Q_i = P,$$

where $Q_i := \alpha^{-1}(Q'_i)$. Note that each $Q_i \neq P$ since P is not a flex. Therefore, there are 4 lines through P are tangent to C at some point $Q \in C$.

□

Problem 5.38.

Let C be a nonsingular cubic given by the equation

$$y^2z = x^3 + ax^2z + bxz^2 + cz^3,$$

$O = [0 : 1 : 0]$. Let $P_i = [x_i : y_i : 1]$, $i = 1, 2, 3$, and suppose $P_1 \oplus P_2 = P_3$. Let

$$\lambda = \begin{cases} \frac{y_1 - y_2}{x_1 - x_2} & \text{if } x_1 \neq x_2 \\ \frac{3x_1^2 + 2ax_1 + b}{2y_1} & \text{if } P_1 = P_2 \text{ and } y_1 \neq 0. \end{cases}$$

Let $\mu = y_i - \lambda x_i$, $i = 1, 2$. Show that

$$\begin{cases} x_3 = \lambda^2 - a - x_1 - x_2 \\ y_3 = -\lambda x_3 - \mu. \end{cases}$$

This gives an explicit method for calculating in the group.

Proof.

- (1) Suppose $x_1 \neq x_2$. Let the line L_1 passing through P_1 and P_2 . Since $L_1 \subseteq U_3 = \{[x : y : z] : z \neq 0\}$, we dehomogenize C (resp. L_1) to get

$$\begin{aligned} C_* : y^2 &= x^3 + ax^2 + bx + c, \\ (L_1)_* : y - y_1 &= \lambda(x - x_1). \end{aligned}$$

Here $\lambda = \frac{y_2 - y_1}{x_2 - x_1}$. Write $y = \lambda(x - x_1) + y_1 = \lambda x + \mu$ where $\mu = y_1 - \lambda x_1$. (μ is also equal to $y_2 - \lambda x_2$.) So

$$\begin{aligned} (\lambda x + \mu)^2 &= x^3 + ax^2 + \text{lower terms} \\ \iff x^3 + (a - \lambda^2)x^2 + \text{lower terms} &= 0 \\ \iff x_1 + x_2 + x'_3 &= \lambda^2 - a & (\text{Vieta's formulas}) \\ \iff \alpha = \lambda^2 - a - x_1 - x_2, \beta &= \lambda\alpha + \mu. \end{aligned}$$

Here $P'_3 = [\alpha : \beta : 1]$ is the third point on $L_1 \cap C$. ($L_1 \bullet C = P_1 + P_2 + P'_3$.)

- (2) Suppose $P_1 = P_2$ and $y_1 \neq 0$. Write $C = y^2z - x^3 - ax^2z - bxz^2 - cz^3$. By Problem 5.4, the tangent line L_1 at $P_1 = P_2$ is

$$L_1 : y = \lambda x + \frac{bx_1 + 2c}{2y_1}z.$$

(Note that $\lambda = \frac{3x_1^2 + 2ax_1 + b}{2y_1}$ and $P_1 \in C$.) In particular, $O \notin L_1$. By dehomogenizing C and replacing y by $\lambda x + \frac{bx_1 + 2c}{2y_1}$, we have

$$\begin{aligned} \left(\lambda x + \frac{bx_1 + 2c}{2y_1} \right)^2 &= x^3 + ax^2 + \text{lower terms} \\ \iff x^3 + (a - \lambda^2)x^2 + \text{lower terms} &= 0 \\ \iff x_1 + x_2 + x'_3 &= \lambda^2 - a & (\text{Vieta's formulas}) \\ \iff \alpha = \lambda^2 - a - x_1 - x_2, \beta &= \lambda\alpha + \mu. \end{aligned}$$

Here $\mu = y_1 - \lambda x_1$ and $P'_3 = [\alpha : \beta : 1]$ is the third point on $L_1 \cap C$. ($L_1 \bullet C = P_1 + P_2 + P'_3$.)

- (3) In any case of (1) or (2), we let the line L_2 passing through P'_3 and O . $L_2 = x - \alpha z$. So

$$\begin{aligned} y^2z &= \alpha^3z^3 + a\alpha^2z^3 + b\alpha z^3 + cz^3 = \beta^2z \\ \iff (y + \beta)(y - \beta)z &= 0 \\ \iff P_3 = [\alpha : -\beta : 1] &= [\lambda^2 - a - x_1 - x_2 : -\lambda x_3 - \mu : 1]. \end{aligned}$$

Here P_3 is the third point on $L_2 \cap C$. ($L_2 \bullet C = P'_3 + O + P_3$.)

□

Problem 5.39.

- (a) Let $C = y^2z - x^3 - 4xz^2$, $O = P_0 = [0 : 1 : 0]$, $P_1 = [2 : 4 : 1]$, $P_2 = [0 : 0 : 1]$ and $P_3 = [2 : -4 : 1]$. Show that $G = \{P_0, P_1, P_2, P_3\}$ form a subgroup of C that is cyclic of order 4.

- (b) Let $C = y^2z - x^3 + 43xz^2 - 166z^3$. Let $O = [0 : 1 : 0]$, $P = [3 : 8 : 1]$. Show that P is an element of order 7 in C .

Proof of (a).

- (1) A direct calculation shows that

$$P_i \oplus P_j = P_{i+j}$$

for all $i, j \in \mathbb{Z}/4\mathbb{Z}$ (Problem 5.38). Hence G is a subgroup of C .

- (2) Note that $G \xrightarrow{\sim} \mathbb{Z}/4\mathbb{Z}$ (by $P_i \mapsto i$) is an isomorphism. So G is a cyclic group of order 4.

□

Proof of (b).

- (1) A direct calculation shows that $2P = P \oplus P = [-5 : -16 : 1]$, $4P = 2P \oplus 2P = [11 : 32 : 1]$, and $8P = 4P \oplus 4P = [3 : 8 : 1] = P$ (Problem 5.38).
- (2) Since $8P = P$, $7P = O$. The order of P is 1 or 7 (since 7 is a prime integer). Note that $P \neq O$. Therefore, P is of order 7.

□

Problem 5.42.

Let C be a nonsingular cubic, f, f' curves of degree m such that $f \bullet C = \sum_{i=1}^{3m} P_i$, $f' \bullet C = \sum_{i=1}^{3m-1} P_i + Q$. Show that $P_{3m} = Q$.

Proof.

- (1) Similar to Proposition 3 in §5.6.
- (2) Let L be a line through P_{3m} that doesn't pass through Q ; $L \bullet C = P_{3m} + R + S$. Then

$$\begin{aligned} (Lf') \bullet C &= L \bullet C + f' \bullet C \\ &= (P_{3m} + R + S) + \left(\sum_{i=1}^{3m-1} P_i + Q \right) \\ &= \sum_{i=1}^{3m} P_i + Q + R + S \\ &= f \bullet C + Q + R + S, \end{aligned}$$

so there is a line L' such that $L' \bullet C = Q + R + S$ (max Noether's fundamental theorem). But then $L' = L$ and so $P_{3m} = Q$.

□

Chapter 6: Varieties, Morphisms, and Rational Maps

6.1. The Zariski Topology

Problem 6.1.*

Let $Z \subseteq Y \subseteq X$, X a topological space. Give Y the induced topology. Show that the topology induced by Y on Z is the same as that induced by X on Z .

Proof.

- (1) Take any open set U in Z whose topology is induced by Y . $U = Z \cap B$ for some open subset B in Y . Since Y is induced by X , $B = Y \cap A$ for some open subset A in X . Hence,

$$U = Z \cap B = Z \cap (Y \cap A) = Z \cap A,$$

that is, U is also an open set in Z whose topology is induced by X .

- (2) Conversely, take any open set U in Z whose topology is induced by X . $U = Z \cap A$ for some open subset A in X . Hence

$$U = Z \cap A = Z \cap (Y \cap A),$$

where $Y \cap A$ is an open set in Y whose topology is induced by X . So U is also an open set in Z whose topology is induced by Y .

□

Problem 6.2.*

- (a) Let X be a topological space, $X = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$, U_α open in X . Show that a subset W of X is closed if and only if each $W \cap U_\alpha$ is closed (in the induced topology) in U_α .
- (b) Suppose similarly $Y = \bigcup_{\alpha \in \mathcal{A}} V_\alpha$, V_α open in Y , and suppose $f : X \rightarrow Y$ is a mapping such that $f(U_\alpha) \subseteq V_\alpha$. Show that f is continuous if and only if the restriction of f to each U_α is a continuous mapping from U_α to V_α .

Proof of (a).

- (1) Show that a subset E of X is open if and only if each $E \cap U_\alpha$ is open (in the induced topology) in U_α . One way is nothing to do. Conversely,

$$E = E \cap X = E \cap \left(\bigcup_{\alpha \in \mathcal{A}} U_\alpha \right) = \bigcup_{\alpha \in \mathcal{A}} (E \cap U_\alpha)$$

is a union of open sets, which is open again.

- (2) Let $E = X - W$ be an open set in X . Note that

$$E \cap U_\alpha = (X - W) \cap U_\alpha = X \cap U_\alpha - W \cap U_\alpha = U_\alpha - W \cap U_\alpha.$$

Hence the conclusion holds.

- (3) It motivates the local study of topologies.

□

Proof of (b).

- (1) (\implies) Suppose f is continuous. It suffices to show that $(f|_{U_\alpha})^{-1}(W)$ is closed in U_α for every closed set W in V_α . Write $W = W_\alpha \cap V_\alpha$ where W_α is closed in Y . Note that

$$\begin{aligned} (f|_{U_\alpha})^{-1}(W) &= f^{-1}(W) \cap U_\alpha \\ &= f^{-1}(W_\alpha \cap V_\alpha) \cap U_\alpha \\ &= f^{-1}(W_\alpha) \cap f^{-1}(V_\alpha) \cap U_\alpha \\ &= f^{-1}(W_\alpha) \cap U_\alpha. \end{aligned} \quad (f^{-1}(V_\alpha) \supseteq U_\alpha)$$

Since $f^{-1}(W_\alpha)$ is closed in X , $(f|_{U_\alpha})^{-1}(W)$ is closed in U_α .

- (2) (\impliedby) Conversely, suppose $f|_{U_\alpha}$ is a continuous mapping from U_α to V_α . It suffices to show that $f^{-1}(W)$ is closed for every closed set W in Y . Note that $W \cap V_\alpha$ is closed in each V_α . By the continuity of $f|_{U_\alpha}$,

$$\begin{aligned} (f|_{U_\alpha})^{-1}(W \cap V_\alpha) &= f^{-1}(W \cap V_\alpha) \cap U_\alpha \\ &= f^{-1}(W) \cap f^{-1}(V_\alpha) \cap U_\alpha \\ &= f^{-1}(W) \cap U_\alpha \end{aligned}$$

is closed in U_α . Since $\alpha \in \mathcal{A}$ is arbitrary and $X = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$, $f^{-1}(W)$ is closed in X by (a).

□

Problem 6.3.*

- (a) Let V be an affine variety, $f \in \Gamma(V)$. Considering f as a mapping from V to $k = \mathbf{A}^1$, show that f is continuous.
- (b) Show that any polynomial map of affine varieties is continuous.

Proof of (a).

- (1) It is a special case of (b) since f itself is a polynomial map from V to \mathbf{A}^1 .
- (2) By Problem 6.6, it suffices to show that

$$f^{-1}(\lambda) = \{P \in V : f(P) = \lambda\}$$

is closed for any $\lambda \in k$. Note that $f^{-1}(\lambda) = V \cap V(f - \lambda = 0)$ is closed.

□

Proof of (b).

- (1) Let $V \subseteq \mathbf{A}^n$, $W \subseteq \mathbf{A}^m$ be affine varieties. Let φ be a polynomial map from V to W .
- (2) Problem 2.7 shows that $\varphi^{-1}(X)$ is closed for every closed subset X of W . Hence, φ is continuous.

□

Problem 6.4.*

Let $U_i \subseteq \mathbf{P}^n$, $\varphi_i : \mathbf{A}^n \rightarrow U_i$ as in Chapter 4. Give U_i the topology induced from \mathbf{P}^n .

- (a) Show that φ_i is a homeomorphism.
- (b) Show that a set $W \subseteq \mathbf{P}^n$ is closed if and only if each $\varphi_i^{-1}(W)$ is closed in \mathbf{A}^n , $i = 1, \dots, n+1$.
- (c) Show that if $V \subseteq \mathbf{A}^n$ is an affine variety, then the projective closure V^* of V is the closure of $\varphi_{n+1}(V)$ in \mathbf{P}^n .

Proof of (a).

- (1) Note that φ_i is one-to-one and onto. It suffices to show that φ_i and φ_i^{-1} are continuous.

- (2) *Show that φ_i is continuous.* Suppose $W \subseteq \mathbf{P}^n$ is closed in U_i . Let \overline{W} be its closure in \mathbf{P}^n . $\overline{W} = V(I)$ for some homogeneous ideal I in $k[x_1, \dots, x_{n+1}]$. Thus $\varphi_i^{-1}(W) = V(I_*)$ (w.r.t. x_i) is closed.
- (3) *Show that φ_i^{-1} is continuous.* Suppose $V \subseteq \mathbf{A}^n$ is closed. Proposition 3 in §4.3 gives us $\varphi_i(V) = V^* \cap U_i$ is closed in U_i .

□

Proof of (b).

$$\begin{aligned}
 & W \subseteq \mathbf{P}^n \text{ is closed} \\
 \iff & W \text{ is closed in } U_i \text{ for each } i && \text{(Problem 6.2.)} \\
 \iff & \varphi_i^{-1}(W) \text{ is closed in } \mathbf{A}^n \text{ for each } i. && (\varphi_i \text{ is a homeomorphism})
 \end{aligned}$$

□

Proof of (c). By Proposition 3 in §4.3, the projective closure V^* of V is the smallest algebraic set in \mathbf{P}^n containing $V = \varphi_{n+1}(V)$. That is the same as the definition of the closure of V . □

Problem 6.5.

Any infinite subset of an irreducible plane curve V is dense in V . Any one-to-one mapping from one irreducible plane curve onto another is a homeomorphism.

Proof.

- (1) *Show that any infinite subset of an irreducible plane curve V is dense in V . It is not true if V is reducible.*
- (a) Let W be the closure of an infinite subset of a plane curve V . $W \subseteq V$.
 - (b) By Corollary 2 to Proposition 2 in §1.6 or Problem 4.23, W is a finite union of points and irreducible plane curves. By Proposition 2 in §1.6 or Problem 5.7, each of these plane curves intersects V in finite number of points, or is equal to V .
 - (c) Since W is infinite, $W = V$. That is, any infinite subset of an irreducible plane curve is dense in V .
- (2) *Show that any one-to-one mapping from one irreducible plane curve onto another is a homeomorphism.*
- (a) Let $\varphi : f \rightarrow g$ be a bijective mapping. It suffices to show that φ is continuous.

- (b) The closed sets in $V(g)$ are $V(g)$ itself and points (by the same argument in (1)). $\varphi^{-1}(V(g)) = V(f)$ is closed. Given any point $Q \in V(g)$, $\varphi^{-1}(Q) = P$ is closed for some point $P \in V(f)$ since φ is bijective. So, φ is continuous.
- (c) Similarly, φ^{-1} is continuous too. Hence, φ is homeomorphic.

□

Problem 6.6.*

Let X be a topological space, $f : X \rightarrow \mathbf{A}^n$ a mapping. Then f is continuous if and only if for each hypersurface $V = V(f)$ of \mathbf{A}^n , $f^{-1}(V)$ is closed in X . A mapping $f : X \rightarrow k = \mathbf{A}^1$ is continuous if and only if $f^{-1}(\lambda)$ is closed for any $\lambda \in k$.

Proof.

- (1) We give a stronger version of the statement. Show that f is continuous iff $f^{-1}(V)$ is closed for each irreducible hypersurface $V = V(f) \subseteq \mathbf{A}^n$. One way is nothing to do.
- (2) Conversely, note that $f^{-1}(\mathbf{A}^n) = X$ and $f^{-1}(\emptyset) = \emptyset$ are closed. Now given any proper closed set V in \mathbf{A}^n , V is of the form

$$V = V(f_1, \dots, f_r) = V(f_1) \cap \dots \cap V(f_r),$$

where each f_i is irreducible polynomial. Hence,

$$f^{-1}(V) = f^{-1}(V(f_1)) \cap \dots \cap f^{-1}(V(f_r))$$

is a union of closed sets in a topological space X , which is closed again.

- (3) f is continuous iff $f^{-1}(V)$ is closed for each irreducible hypersurface $V = V(f) \subseteq \mathbf{A}^1$ (by (1)). Note that the irreducible hypersurface f in \mathbf{A}^1 is $f = x - \lambda$ for some $\lambda \in k$. So f is continuous iff

$$f^{-1}(V) = f^{-1}(x - \lambda) = f^{-1}(\lambda)$$

is closed for each $\lambda \in k$.

□

Problem 6.7.*

Let V be an affine variety, $f \in \Gamma(V)$.

- (a) Show that $V(f) = \{P \in V : f(P) = 0\}$ is a closed subset of V , and $V(f) \neq V$ unless $f = 0$.
- (b) Suppose U is a dense subset of V and $f(P) = 0$ for all $P \in U$. Then $f = 0$.

Proof of (a).

- (1) Write $f = \tilde{f} + I(V) \in \Gamma(V)$ for some $\tilde{f} \in k[x_1, \dots, x_n]$. Note that $f(P) = \tilde{f}(P)$ is well-defined for any $P \in V$. Thus,

$$\begin{aligned} V(f) &= \{P \in V : f(P) = 0\} \\ &= \{P \in V : \tilde{f}(P) = 0\} \\ &= V \cap \{P \in \mathbf{A}^n : \tilde{f}(P) = 0\} \\ &= V \cap V(\tilde{f}) \end{aligned}$$

is closed in V .

- (2) Also,

$$\begin{aligned} V(f) = V &\iff V \cap V(\tilde{f}) = V \\ &\iff V(\tilde{f}) \supseteq V \\ &\iff \tilde{f} \in I(V) \\ &\iff f = 0 \in \Gamma(V). \end{aligned}$$

□

Proof of (b).

- (1) Since $V(f) \supseteq U$ (by assumption) and $V(f)$ is closed (by (a)), $V(f) \supseteq \overline{U}$ by the definition of the closure of U .
- (2) Since U is dense in V , $\overline{U} = V$. Thus $V \supseteq V(f) \supseteq \overline{U} = V$ or $V(f) = V$. By (a), $f = 0$.

□

Problem 6.8.*

Let U be an open subset of a variety V , $z \in k(V)$. Suppose $z \in \mathcal{O}_P(V)$ for all $P \in U$. Show that $U_z = \{P \in U : z(P) \neq 0\}$ is open, and that the mapping from U to $k = \mathbf{A}^1$ defined by $P \mapsto z(P)$ is continuous.

Proof.

- (1) Show that U_z is open. By Problem 6.1, it suffices to show that U_z is open in U , or show that

$$W_z := U - U_z = \{P \in U : z(P) = 0\}$$

is closed in U .

- (2) Write $z = f/g$ where $f, g \in \Gamma(V)$, $g(P) \neq 0$ for all $P \in V$. $z(P) = 0$ if and only if $f(P) = 0$ on V . Hence,

$$W_z = \{P \in U : f(P) = 0\} = U \cap \{P \in V : f(P) = 0\}$$

is closed in U (Problem 6.7(a)).

- (3) Show that the mapping $\varphi : U \rightarrow k = \mathbf{A}^1$ defined by $\varphi : P \mapsto z(P)$ is continuous. By Problem 6.6, it suffices to show that $\varphi^{-1}(\lambda)$ is closed in U for any $\lambda \in k$.

- (4) Note that $z - \lambda \in \mathcal{O}_P(V)$ for all $P \in V$. Hence,

$$\varphi^{-1}(\lambda) = \{P \in U : z(P) = \lambda\} = \{P \in U : (z - \lambda)(P) = 0\} = W_{z-\lambda}$$

is closed in U by (2).

□

6.2. Varieties**Problem 6.9.**

Let $X = \mathbf{A}^2 - \{(0, 0)\}$, an open subvariety of \mathbf{A}^2 . Show that $\Gamma(X) = \Gamma(\mathbf{A}^2) = k[x, y]$.

Proof.

- (1) Let $z = f/g \in \Gamma(X)$ be a regular function on X . So $f = 0$ (nothing to do) or $g(P) \neq 0$ for all $P \in X = \mathbf{A}^2 - \{(0, 0)\}$.
- (2) As $V(g)$ is contained in a finite set $\{(0, 0)\}$, g is a nonzero constant. Therefore, $z \in k[x, y] = \Gamma(\mathbf{A}^2)$ (Problem 1.4).

□

Problem 6.10.*

Let U be an open subvariety of a variety X , Y a closed subvariety of U . Let \overline{Y} be the closure of Y in X . Show that

- (a) \overline{Y} is a closed subvariety of X .
- (b) Y is an open subvariety of \overline{Y} .

Proof of (a).

- (1) It suffices to show that \overline{Y} is irreducible if Y is irreducible (since \overline{Y} is closed in X already).
- (2) (Reductio ad absurdum) If \overline{Y} were reducible, there are two closed sets Y_1 and Y_2 such that $\overline{Y} \subseteq Y_1 \cup Y_2$ and $\overline{Y} \not\subseteq Y_1, Y_2$.
- (3) Show that $Y \subseteq \overline{Y} \subseteq Y_1 \cup Y_2$ and $Y \not\subseteq Y_1, Y_2$ if \overline{Y} were reducible. If not, might assume that $Y \subseteq Y_1$. Take closure to get $\overline{Y} \subseteq \overline{Y_1} = Y_1$ (since Y_1 is closed), contrary to the assumption.
- (4) Therefore, Y is reducible, a contradiction.

□

Proof of (b).

- (1) It suffices to show that $Y = \overline{Y} \cap X$.
- (2) $Y \subseteq \overline{Y} \cap X$ is trivial.
- (3) Conversely, write $Y = \tilde{Y} \cap X$ for some closed set \tilde{Y} in X . $\tilde{Y} \supseteq \overline{Y}$ since \overline{Y} is the smallest closed set containing Y . So $Y = \tilde{Y} \cap X \supseteq \overline{Y} \cap X$.

□

Problem 6.11. (Noetherian topological space)

- (a) Show that every family of closed subsets of a variety has a minimal member.
- (b) Show that if a variety is a union of a collection of open subsets, it is a union of a finite number of these subsets. (All varieties are **quasi-compact**.)

Proof of (a).

- (1) Similar to the proof of Proposition 1 in §6.2, we may assume that X is an affine variety in \mathbf{A}^N .

- (2) By Problem 1.38, there is a natural one-to-one correspondence between closed subsets of X and radical ideals in $k[x_1, \dots, x_N]/I(X)$.
- (3) Note that $I(X)$ is finitely generated (by the Hilbert basis theorem in §1.4). By Problem 1.22, $k[x_1, \dots, x_N]/I(X)$ is Noetherian. Hence, Lemma in §1.5 shows that every family of closed subsets of X has a minimal member.

□

Proof of (b).

- (1) Let $\{U_\alpha\}$ be an open covering of X . Let Σ be the family of closed sets being finite intersections of complements of members of $\{U_\alpha\}$, say

$$\bigcap_{\text{finite } \alpha} (X - U_\alpha) = X - \bigcup_{\text{finite } \alpha} U_\alpha.$$

By (a), Σ has a minimal member S . Now it suffices to show that $S = \emptyset$.

- (2) By the minimality of S , $S \cap (X - U_\alpha) = S$ for all α . So $S \subseteq X - U_\alpha$, or $X - S \supseteq U_\alpha$ for all α . Hence,

$$X \supseteq X - S \supseteq \bigcup_{\text{all } \alpha} U_\alpha \supseteq X.$$

- (3) Besides, we can show that X is a Noetherian topological space if every open subset of X is quasi-compact.

□

Problem 6.12.*

Let X be a variety, $z \in k(X)$. Show that the pole set of z is closed. If $z \in \mathcal{O}_P(X)$, there is a neighborhood U of z such that $z \in \Gamma(U)$; so $\mathcal{O}_P(X)$ is the union of all $\Gamma(U)$, where U runs through all neighborhoods of P .

Proof.

- (1) Similar to the proof of Proposition 1 in §6.2, we may assume that X is an affine variety in \mathbf{A}^N .
- (2) Proposition 2 in §2.4 shows that the pole set $V(J_z) \subseteq X$ of z is algebraic or closed.
- (3) Given $z \in \mathcal{O}_P(X)$ ($P \notin V(J_z)$), there is an open neighborhood $U := X - V(J_z)$ of z such that $z \in \Gamma(U)$. So

$$\mathcal{O}_P(X) = \bigcup_U \Gamma(U)$$

where U runs through all neighborhoods of P .

□

6.3. Morphisms of Varieties

Problem 6.13.*

Let R be a domain with quotient field K , $f \neq 0$ in R . Let

$$R[1/f] = \{a/f^n : a \in R, n \in \mathbb{Z}\},$$

a subring of K .

- (a) Show that if $\varphi : R \rightarrow S$ is any ring homomorphism such that $\varphi(f)$ is a unit in S , then φ extends uniquely to a ring homomorphism from $R[1/f]$ to S .
- (b) Show that the ring homomorphism from $R[x]/(xf-1)$ to $R[1/f]$ that takes x to $1/f$ is an isomorphism.

Proof of (a).

- (1) Suppose $\psi : R[1/f] \rightarrow S$ is a ring homomorphism such that $\psi = \varphi$ on R .
- (2) To see ψ is uniquely determined, it suffices to show that $\psi(1/f)$ is uniquely determined (since ψ is a homomorphism).
- (3) By $1 = f/f$, we have $\psi(1) = \psi(f)\psi(1/f)$. $\varphi(1) = \varphi(f)\psi(1/f)$ since $\psi = \varphi$ on R . Since $\varphi(1) = 1$ and $\varphi(f)$ is a unit in S , $\psi(1/f) = \varphi(f)^{-1}$ is uniquely determined.
- (4) Lastly, it is easy to check $\psi(a/f^n) = \varphi(a)\varphi(f)^{-n}$ is a ring homomorphism.

□

Proof of (b).

- (1) Define a ring homomorphism $\varphi : R \hookrightarrow R[x] \rightarrow S := R[x]/(xf-1)$ by

$$\varphi : r \mapsto r \mapsto \bar{r} = r + (xf-1).$$

- (2) Note that $\varphi(f) = \bar{f}$ is a unit in S since $\bar{x}\bar{f} = \bar{1}$ in S . By (a), φ extends uniquely to a ring homomorphism from $R[1/f]$ to S . In particular, $\varphi(1/f) = \varphi(f)^{-1} = \bar{x}$.
- (3) Show that φ is surjective. Note that $\varphi(r) = \bar{r}$ for any $r \in R$ and $\varphi(1/f) = \bar{x}$. So, φ is surjective since φ is a ring homomorphism.

- (4) Show that φ is injective. Suppose $\varphi(r/f^n) = 0$ for some $r/f^n \in R[1/f]$. May assume that $n > 0$. Write

$$rx^n = (xf - 1)g(x)$$

for some $g(x) \in R[x]$. It suffices to show that $g(x) = 0$. (Reductio ad absurdum) If $g(x) \neq 0$, then $r \neq 0$ and $\deg(g) = n - 1$. Take $x = 0$ in $rx^n = (xf - 1)g(x)$ to get $g(0) = 0$. So $g = xg_1$ for some $g_1 \in R[x]$ if $n > 1$. Hence,

$$rx^{n-1} = (xf - 1)g_1(x)$$

where $\deg g_1 = n - 2$ if $n > 1$. Continue this process to get $g(x) = cx^{n-1} \in R[x]$ for some nonzero $c \in R$. So $rx = cfx - c$, which is absurd.

□

Problem 6.14.*

Let X, Y be varieties, $f : X \rightarrow Y$ a mapping. Let $X = \bigcup_{\alpha} U_{\alpha}$, $Y = \bigcup_{\alpha} V_{\alpha}$, with U_{α} , V_{α} open subvarieties, and suppose $f(U_{\alpha}) \subseteq V_{\alpha}$ for all α .

- (a) Show that f is a morphism if and only if each restriction $f_{\alpha} : U_{\alpha} \rightarrow V_{\alpha}$ of f is a morphism.
- (b) If each U_{α} , V_{α} is affine, f is a morphism if and only if each $\tilde{f}(\Gamma(V_{\alpha})) \subseteq \Gamma(U_{\alpha})$.

Proof of (a).

- (1) Problem 6.2 shows that f is continuous if and only if each restriction $f_{\alpha} : U_{\alpha} \rightarrow V_{\alpha}$ of f is continuous.
- (2) Suppose $f : X \rightarrow Y$ is a morphism. The diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i_X \uparrow & & \uparrow i_Y \\ U_{\alpha} & \xrightarrow{f_{\alpha}} & V_{\alpha} \end{array}$$

is commutative. By Problem 6.15(a), two inclusion maps i_X and i_Y are morphisms. By Problem 6.15(b), $f \circ i_X$ is a morphism. So it suffices to show that f_{α} is a morphism if $i_Y \circ f_{\alpha}$ is a morphism.

- (3) Given a rational function $g \in \Gamma(V, \mathcal{O}_{V_{\alpha}})$ on an open set V of V_{α} . So $g \circ i_Y$ is a rational function in $\Gamma(V, \mathcal{O}_Y)$ on an open set V of Y (Problem 6.1).

Therefore, given any $y \in V$ we have

$$\begin{aligned}(\widetilde{f_\alpha}(g))(y) &= (g \circ f_\alpha)(y) \\ &= g(f_\alpha(y)) \\ &= g(i_Y(f_\alpha(y))) \\ &= (\widetilde{i_Y \circ f_\alpha}(g))(y).\end{aligned}$$

Hence $\widetilde{f_\alpha}$ pulls back g to a rational function $\widetilde{f_\alpha}(g)$ on U_α . Therefore, f_α is a morphism if f is a morphism.

- (4) Suppose $f_\alpha : U_\alpha \rightarrow V_\alpha$ of f is a morphism for all α . Given a rational function $g \in \Gamma(V, \mathcal{O}_Y)$ on an open set V of Y . g is a rational function in $\Gamma(V \cap V_\alpha, \mathcal{O}_{V_\alpha})$ on an open set $V \cap V_\alpha$ of $V_\alpha \subseteq Y$ for all α . Therefore,

$$\widetilde{f_\alpha}(g) = g \circ f_\alpha \in \Gamma(f_\alpha^{-1}(V \cap V_\alpha), \mathcal{O}_{U_\alpha}) = \Gamma(f_\alpha^{-1}(V), \mathcal{O}_{U_\alpha})$$

for all α . So each restriction $g \circ f_\alpha = (g \circ f)|_{U_\alpha}$ is a rational function on U_α . Hence, $\widetilde{f}(g) = g \circ f$ is a rational function on $X = \bigcup_\alpha U_\alpha$.

□

Proof of (b).

- (1) Part (a) and Proposition 2 in §6.2 show that f is a morphism if and only if each $\widetilde{f_\alpha}(\Gamma(V_\alpha)) \subseteq \Gamma(U_\alpha)$.
- (2) Since f and f_α are the same on U_α , $\widetilde{f_\alpha} = \widetilde{f}$ on $\Gamma(V_\alpha)$.

□

Problem 6.15.*

- (a) If Y is an open or closed subvariety of X , the inclusion $i : Y \rightarrow X$ is a morphism.
- (b) The composition of morphisms is a morphism.

Proof of (a).

- (1) The inclusion map is continuous in any topological space. ($i^{-1}(U) = U \cap Y$ is open in Y for every open set U of X .)
- (2) For every open set U of X , if $f \in \Gamma(U, \mathcal{O}_X)$, then $\widetilde{i}(f)$ is a rational function defined on $U \cap Y = i^{-1}(U)$. Thus, $\widetilde{i}(f) = f \circ i \in \Gamma(i^{-1}(U), \mathcal{O}_Y)$. (The ring $\Gamma(i^{-1}(U), \mathcal{O}_Y)$ is well-defined since Y is a variety.)

□

Proof of (b).

- (1) Given two morphisms $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$. The composition continuous functions $\psi \circ \varphi$ is continuous in any topological space.
- (2) For every open set U of Z , if $f \in \Gamma(U, \mathcal{O}_Z)$, then $\tilde{\psi}(f) = f \circ \psi \in \Gamma(\psi^{-1}(U), \mathcal{O}_Y)$. Note that $\psi^{-1}(U)$ is open since ψ is continuous. So

$$\tilde{\varphi}(\tilde{\psi}(f)) = f \circ \psi \circ \varphi \in \Gamma(\varphi^{-1}(\psi^{-1}(U)), \mathcal{O}_X).$$

Problem 2.6 gives that $\tilde{\varphi} \circ \tilde{\psi} = \widetilde{\psi \circ \varphi}$. Besides, $\varphi^{-1} \circ \psi^{-1} = (\psi \circ \varphi)^{-1}$. Therefore, $\widetilde{\psi \circ \varphi}$ pulls back rational functions to rational functions on every open set.

□

Problem 6.17.

- (a) Show that $\mathbf{A}^2 - \{(0, 0)\}$ is not an affine variety (see Problem 6.9).
- (b) The union of two open affine subvarieties of a variety may not be affine.

Note that $\mathbf{A}^1 - \{0\}$ is an affine variety which is isomorphic to $V(1 - xy) \subseteq \mathbf{A}^2$ (Proposition 5 in §6.3).

Proof of (a).

- (1) (Reductio ad absurdum) Suppose $X := \mathbf{A}^2 - \{(0, 0)\}$ were an affine variety. By Problem 6.15(a), the inclusion map $i : X \hookrightarrow \mathbf{A}^2$ is a morphism. By Proposition 2 in §6.2, $\tilde{i} : \Gamma(\mathbf{A}^2) \hookrightarrow \Gamma(X)$ is a ring homomorphism.
- (2) Problem 6.9 shows that $\Gamma(\mathbf{A}^2) = \Gamma(X) = k[x, y]$. Hence \tilde{i} is an isomorphism, so i is an isomorphism. However, i is not surjective on the point $(0, 0)$, which is absurd.

□

Proof of (b).

- (1) Write $\mathbf{A}^2 - \{(0, 0)\} = (\mathbf{A}^2 - V(x)) \cup (\mathbf{A}^2 - V(y))$. It suffices to show that $\mathbf{A}^2 - V(x)$ is an affine variety.
- (2) Note that $\mathbf{A}^2 - V(x)$ is isomorphic to a closed subvariety $V(1 - xz)$ of \mathbf{A}^3 .

□

Problem 6.18.

Show that the natural map π from $\mathbf{A}^{n+1} - \{(0, \dots, 0)\}$ to \mathbf{P}^n is a morphism of varieties, and that a subset U of \mathbf{P}^n is open if and only if $\pi^{-1}(U)$ is open.

Proof.

- (1) Show that π is the morphism by Problem 6.14. Write

$$X := \mathbf{A}^{n+1} - \{(0, \dots, 0)\} = \bigcup_{i=1}^{n+1} U_i$$

where $U_i = \{(a_1, \dots, a_{n+1}) \in X : a_i \neq 0\}$ are open subvarieties in X . Similarly, write

$$Y := \mathbf{P}^n = \bigcup_{i=1}^{n+1} V_i$$

where $V_i = \{[a_1 : \dots : a_{n+1}] \in Y : a_i \neq 0\}$ are open subvarieties. Hence $\pi(U_i) = V_i$ for all i .

- (2) Now it suffices to show $\pi|_{U_i} : U_i \rightarrow V_i$ is a morphism for all i (Problem 6.14). Given any closed subset $V(f) \cap V_i$ of V_i , where $f \in k[x_1, \dots, x_{n+1}]$. $(\pi|_{U_i})^{-1}(V(f) \cap V_i) = V(f) \cap U_i$ is also closed in U_i . Hence $\pi|_{U_i}$ is continuous. Also, $\pi|_{U_i}$ pulls back rational functions to a rational functions.
- (3) Show that $U \subseteq \mathbf{P}^n$ is open if and only if $\pi^{-1}(U)$ is open. $\pi^{-1}(U)$ is open if U is open since π is continuous. It suffices to show that $\pi(E)$ is open if E is open in X .
- (4) Let \tilde{E} be the union of all the lines through the origin that meet E , that is, $\tilde{E} = \pi^{-1}(\pi(E))$. Note that

$$\tilde{E} = \bigcup_{a \in k - \{0\}} (aE)$$

is open in X since each aE is open in X . (Write $E = X - V(f_1, \dots, f_r)$. Then $aE = X - V(g_1, \dots, g_r)$, where

$$g_i(x_1, \dots, x_{n+1}) := f_i(a^{-1}x_1, \dots, a^{-1}x_{n+1}) \in k[x_1, \dots, x_{n+1}]$$

for each i .) By the continuity of π , $\pi(E)$ is open since $\tilde{E} = \pi^{-1}(\pi(E))$ is open.

□

Problem 6.21.

Any variety is the union of a finite number of open affine subvarieties.

Proof.

- (1) Propositions 4 and 3 in §6.2 give all.
- (2) *Proof of Proposition 3 in §6.2.* By Problem 6.4, $V_i = \varphi_i^{-1}(V)$ is a closed subvariety of \mathbf{A}^{n+1} . Since $\varphi : \varphi_i^{-1}(V) \rightarrow V \cap U_i$ is an isomorphism, $V \cap U_i$ is an open affine subvariety. Therefore, $V = \bigcup_{i=1}^{n+1} (V \cap U_i)$ is the union of a finite number of open affine subvarieties.

□

6.4. Products and Graphs**Problem 6.29. (Algebraic group)**

Let V be a variety, and suppose V is also a group, i.e., there are mappings $\varphi : V \times V \rightarrow V$ (multiplication or addition), and $\psi : V \rightarrow V$ (inverse) satisfying the group axioms. If φ and ψ are morphisms, V is said to be an **algebraic group**. Show that each of the following is an algebraic group:

- (a) $\mathbf{A}^1 = k$, with the usual addition on k ; this group is often denoted \mathbb{G}_a .
- (b) $\mathbf{A}^1 - \{(0)\} = k - \{0\}$, with the usual multiplication on k ; this is denoted \mathbb{G}_m .
- (c) $\mathbf{A}^n(k)$ with addition; likewise $M_n(k) = \{n \text{ by } n \text{ matrices}\}$ under addition may be identified with $A^{n^2}(k)$.
- (d) $GL_n(k) = \{\text{invertible } n \times n \text{ matrices}\}$ is an affine open subvariety of $M_n(k)$, and a group under multiplication.
- (e) C a nonsingular plane cubic, $O \in C$, \oplus the resulting addition (see Problem 5.38).

Proof of (a).

- (1) $\mathbf{A}^1 = k$ is an affine variety.
- (2) k is an abelian group under addition with 0 as the additive identity.
- (3) $\varphi : (a, b) \mapsto a + b$ is continuous (by identifying $\mathbf{A}^1 \times \mathbf{A}^1$ to \mathbf{A}^2) and pulls back rational functions to a rational functions.

- (4) $\psi : a \mapsto -a$ is continuous and pulls back rational functions to a rational functions.

□

Proof of (b).

- (1) $\mathbf{A}^1 - \{0\}$ is an affine variety which is isomorphic to $V(1 - xy) \subseteq \mathbf{A}^2$. It is the same as Proposition 5 in §6.3.
- (2) k is an abelian group under multiplication with 1 as the multiplicative identity.
- (3) $\varphi : (a, b) \mapsto ab$ is continuous (by identifying $\mathbf{A}^1 \times \mathbf{A}^1$ to \mathbf{A}^2) and pulls back rational functions to a rational functions.
- (4) $\psi : a \mapsto a^{-1}$ is continuous and pulls back rational functions to a rational functions.

□

Proof of (c).

- (1) $\mathbf{A}^n = k^n$ is an affine variety (or apply Proposition 6 in §6.4).
- (2) k^n is a direct sum of n copies of abelian groups k under addition.
- (3) By Proposition 7 in §6.4 and (a), both φ and ψ are morphisms.

□

Proof of (d).

- (1) Similar to (b). $\mathrm{GL}_n(k)$ is an affine open subvariety of $\mathbf{M}_n(k)$ which is isomorphic to $V(1 - \det(x_{ij})y) \subseteq \mathbf{A}^{n^2+1}$ (where $(x_{ij}) \in \mathbf{M}_n(k)$).
- (2) $\mathrm{GL}_n(k)$ is a group under multiplication with I_n as the multiplicative identity.
- (3) Similar to (b). Both φ and ψ are morphisms.

□

Proof of (e).

- (1) Similar to Problem 5.10, we might assume that $C = y^2z - x^3 - ax^2z - bxz^2 - cz^3$ and $O = [0 : 1 : 0] \in C$. C is an affine variety in \mathbf{P}^2 .
- (2) By Proposition 4 in §5.6, C is an abelian group with the addition \oplus . Now we can use Problem 5.38 to compute $P \oplus Q$ for any $P, Q \in C$.

- (3) Define the addition φ as in Problem 5.38. Let

$$\begin{aligned}U_1 &= \{x_1 \neq x_2\}, \\U_2 &= \{x_1 = x_2, y_1 \neq 0\}, \\U_3 &= \{x_1 = x_2, y_1 = 0, z_1 \neq 0\}\end{aligned}$$

be an open covering of $C \times C$ (Proposition 7 in §6.4). It suffices to show that φ is a morphism on each U_α ($\alpha = 1, 2, 3$) (Problem 6.14). Since φ is a polynomial map on each U_α , φ is continuous on each U_α . Also, φ pulls back rational functions to a rational functions on each U_α . Hence φ is a morphism.

- (4) Define the inverse ψ as $\psi(O) = O$ and $\psi([x : y : 1]) = [x : -y : 1]$ (Problem 5.38). ψ is continuous and pulls back rational functions to a rational functions. So ψ is a morphism.

□

6.5. Algebraic Function Fields and Dimension of Varieties

Problem 6.31.* (Theorem of the primitive element)

Let K be a field of a characteristic zero, L a finite (algebraic) extension of K . Then there is a $z \in L$ such that $L = K(z)$.

Outline of Proof.

- (1) Suppose $L = K(x, y)$. Let f and g be monic irreducible polynomials in $K[t]$ such that $f(x) = 0$, $g(y) = 0$. Let L' be a field in which $f = \prod_{i=1}^n (t - x_i)$, $g = \prod_{j=1}^m (t - y_j)$, $x = x_1$, $y = y_1$, $L' \supseteq L$ (see Problems 1.52, 1.53).
- (2) Choose $\lambda \neq 0$ in K so that $\lambda x + y \neq \lambda x_i + y_j$ for all $i \neq 1, j \neq 1$. Let $z = \lambda x + y$, $K' = K(z)$. Set $h(t) = g(z - \lambda t) \in K'[t]$. Then $h(x) = 0$, $h(x_i) \neq 0$ if $i \neq 1$. Therefore $(h, f) = (t - x) \in K'[t]$. Then $x \in K'$, so $y \in K'$, so $L = K'$.
- (3) If $L = K(x_1, \dots, x_n)$, use induction on n to find $\lambda_1, \dots, \lambda_n \in K$ such that $L = K(\sum \lambda_i x_i)$.

Proof.

- (1) Suppose $L = K(x, y)$. Let f and g be monic irreducible polynomials in $K[t]$ such that $f(x) = 0$, $g(y) = 0$. Let L' be a splitting field of L in which $f = \prod_{i=1}^n (t - x_i)$, $g = \prod_{j=1}^m (t - y_j)$, $x = x_1$, $y = y_1$, $L' \supseteq L$ (Problems 1.52). Note that the x_i (resp. y_j) are distinct by the irreducibility of f (resp. g) (Problem 1.53).

- (2) Choose $\lambda \neq 0$ in K so that $\lambda x + y \neq \lambda x_i + y_j$ for all $i \neq 1, j \neq 1$. (Note that $\mathbb{Q} \hookrightarrow K$ since $\text{char}(K) = 0$. Hence, such one λ exists since K contains an infinite set \mathbb{Q} .) Let $z = \lambda x + y$ and $K' = K(z)$.
- (3) Show that $L = K'$. $K' = K(z) \subseteq L$ since $z \in L$. Conversely, it suffices to show that $x \in K'$. Define $h(t) = g(z - \lambda t) \in K'[t]$. Then $h(x) = 0$ and $h(x_i) \neq 0$ if $i \neq 1$ by the construction of λ . Therefore $(h, f) = (t - x) \in K'[t]$. Then $x \in K'$, so $y \in K'$, so $L = K'$.
- (4) If $L = K(x_1, \dots, x_n)$ is a finite (algebraic) extension of K , use induction on n to find $\lambda_1, \dots, \lambda_n \in K$ such that

$$\begin{aligned} L &= K(x_1, \dots, x_{n-1})(x_n) \\ &= K(\lambda_1 x_1 + \dots + \lambda_{n-1} x_{n-1})(x_n) && \text{(Induction hypothesis)} \\ &= K(\lambda_1 x_1 + \dots + \lambda_n x_n) && \text{(Base step)} \end{aligned}$$

where $\lambda_1 := 1$.

□

Problem 6.32.*

Let $L = K(x_1, \dots, x_n)$ as in Problem 6.31. Suppose $k \subseteq K$ is an algebraically closed subfield, and $V \subsetneq \mathbf{A}^n(k)$ is an algebraic set. Show that $L = K(\sum \lambda_i x_i)$ for some $(\lambda_1, \dots, \lambda_n) \in \mathbf{A}^n - V$.

Proof.

- (1) Since $V \subsetneq \mathbf{A}^n$, there is one index $1 \leq i \leq n$ such that the projection map $\pi_i : V \rightarrow \mathbf{A}^1$ defined by $\pi_i(a_1, \dots, a_n) = a_i$ satisfies $\pi_i(V) \subsetneq \mathbf{A}^1$ (Problem 2.10). Here $\pi_i(V)$ is a finite subset of \mathbf{A}^1 (Problem 1.8).
- (2) In the proof of Problem 6.31, we also take $\lambda_i \notin \pi_i(V)$. It is possible since k is infinite. Therefore, $(\lambda_1, \dots, \lambda_n) \in \mathbf{A}^n - V$.

□

Problem 6.33. (Transcendence degree)

The notion of transcendence degree is analogous to the idea of the dimension of a vector space. If $k \subseteq K$, we say that $x_1, \dots, x_n \in K$ are **algebraically independent** if there is no nonzero polynomial $f \in k[x_1, \dots, x_n]$ such that $f(x_1, \dots, x_n) = 0$. By methods entirely analogous to those for bases of vector spaces, one can prove:

- (a) Let $x_1, \dots, x_n \in K$, K a finitely generated extension of k . Then x_1, \dots, x_n is a minimal set such that K is algebraic over $k(x_1, \dots, x_n)$ if and only if x_1, \dots, x_n is a maximal set of algebraically independent elements of K . Such $\{x_1, \dots, x_n\}$ is called a **transcendence basis** of K over k .
- (b) Any algebraically independent set may be completed to a transcendence basis. Any set $\{x_1, \dots, x_n\}$ such that K is algebraic over $k(x_1, \dots, x_n)$ contains a transcendence basis.
- (c) $\text{tr. deg}_k K$ is the number of elements in any transcendence basis of K over k .

Proof of (a).

- (1) Let $S = \{s_1, \dots, s_n\} \subseteq K$. Show that the following statements are equivalent:

- (i) S is algebraically independent over k .
- (ii) For each i , s_i is transcendental over $k(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$.
- (iii) For each i , s_i is transcendental over $k(s_1, \dots, s_{i-1})$.

(Lemma 19.7 in the textbook: *Patrick Morandi, Field and Galois Theory*.)

The result is established if we can show that (i) \iff (ii).

- (2) ((i) \implies (ii)) Suppose there are $a_j \in k(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$ such that

$$s_i^m + \dots + a_1 s_i + a_0 = 0.$$

Clearing denominators, might assume $a_j \in k[s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n]$. ($a_m = 1$.) Then

$$f = \sum_j a_j(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n) s_i^j \in k[s_1, \dots, s_n]$$

has a root $(s_1, \dots, s_n) \in \mathbf{A}^n(k)$. Since S is algebraically independent over k , $f = 0$. So $a_j = 0$. So s_i is transcendental over $k(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$.

- (3) ((ii) \implies (iii)) Clearly.

- (4) ((iii) \implies (i)) Suppose S were not algebraically independent over k . Choose m minimal such that there exists a nonzero $f \in k[x_1, \dots, x_m]$ having a root $(s_1, \dots, s_m) \in \mathbf{A}^n$. Write $f = \sum_j a_j x_m^j$ where $a_j \in k[x_1, \dots, x_{m-1}]$. So

$$\sum_j a_j(s_1, \dots, s_{m-1}) s_m^j = 0.$$

Since s_m is not algebraic over $k(s_1, \dots, s_{m-1})$ by assumption, all $a_j(s_1, \dots, s_{m-1}) = 0$ in k . By the minimality of m , all $a_j = 0$ in $k[x_1, \dots, x_{m-1}]$. So $f = 0$, which is absurd.

□

Proof of (b).

- (1) *Show that any algebraically independent set may be completed to a transcendence basis.* Let A be any algebraically independent subset of K . We claim we can find a transcendence basis \mathcal{B} such that $A \subseteq \mathcal{B} \subseteq K$.
- (2) (Zorn's lemma) Let

$$\Sigma = \{S \text{ is algebraically independent} : A \subseteq S \subseteq K\}.$$

Order Σ by inclusion. Σ is non empty since $A \in \Sigma$. To apply Zorn's lemma we must show that every chain in Σ has an upper bound in Σ .

- (3) Let (S_α) be a chain in Σ , so that for each pair of indices α, β we have either $S_\alpha \subseteq S_\beta$, or $S_\beta \subseteq S_\alpha$. Let $B = \bigcup_\alpha S_\alpha$. Clearly, $B \in \Sigma$ since $A \subseteq B \subseteq K$ and polynomials only involve finitely many variables. By Zorn's lemma, Σ has a maximal element \mathcal{B} .
- (4) *Show that K is algebraic over $k(\mathcal{B})$.* (Reductio ad absurdum) Suppose there were an element $s \in K$ transcendental over $k(\mathcal{B})$. Then $\mathcal{B} \cup \{s\}$ is algebraically independent, which contradicts the maximality of \mathcal{B} .
- (5) *Show that any set $A = \{x_1, \dots, x_n\}$ such that K is algebraic over $k(x_1, \dots, x_n)$ contains a transcendence basis.* If A is algebraically independent, nothing to do. If A is algebraically dependent, then there is some x_i such that x_i algebraic over $k(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. So K is algebraic over $k(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. Continue this process, we can find a minimal subset \mathcal{B} of A such that K is algebraic over $k(x_1, \dots, x_r)$ (with suitable relabeling). Here $\mathcal{B} \subseteq A$ is a transcendence basis.

□

Proof of (c).

- (1) It suffices to show that $\text{tr. deg}_k K$ is well-defined. *Let K be a field extension of k . If S and T are transcendence bases for K/k , then $|S| = |T|$.* (Theorem 19.15 in the textbook: *Patrick Morandi, Field and Galois Theory*.)
- (2) Suppose $S = \{s_1, \dots, s_n\}$ is finite. In the proof of (b), $K/k(S - \{s_1\})$ is not algebraic. Since $K/k(T)$ is algebraic, some $t \in T$, say $t_1 \in T$ (by relabeling), must be transcendental over $k(S - \{s_1\})$. By the proof of (a),

$$S_1 := \{t_1, s_2, \dots, s_n\}$$

is algebraically independent. Note that s_1 is algebraic over $k(S_1)$, or else $\{t_1, s_1, \dots, s_n\}$ is algebraically independent, contrary to the assumption of S . Therefore, $K/k(S_1)$ is algebraic.

- (3) Continue this process to replace each element in S to get $S_n = \{t_1, \dots, t_n\} \subseteq T$. So $|S| \leq |T|$. Similarly, $|T| \leq |S|$. Therefore, $|S| = |T|$ if S is finite.
- (4) Suppose S (and thus T) are infinite. Since t is algebraic over $k(S)$ for each $t \in T$, there is a corresponding finite subset $S_t \subseteq S$ such that t is algebraic over $k(S_t)$.
- (5) Let

$$S' = \bigcup_{t \in T} S_t.$$

Then each $t \in T$ is algebraic over $k(S')$. Since $K/k(T)$ is algebraic, $K/k(S')$ is algebraic. Thus $S' = S$ since $S' \subseteq S$ and S is a transcendental basis for K/k .

- (6) Hence,

$$|S| = |S'| = \left| \bigcup_{t \in T} S_t \right| \leq |T \times \mathbb{N}| = |T|,$$

where the last equality holds since $|T| = \infty$. Similarly, $|T| \leq |S|$. Therefore, $|S| = |T|$ if S is infinite.

□

Problem 6.34.

Show that $\dim \mathbf{A}^n = \dim \mathbf{P}^n = n$.

Proof.

- (1) By Proposition 10 in §6.5,

$$\dim \mathbf{P}^n = \dim \mathbf{A}^n$$

since \mathbf{P}^n is the projective closure of \mathbf{A}^n .

- (2) A direct calculation shows

$$\dim \mathbf{A}^n = \text{tr. deg}_k k(\mathbf{A}^n) = \text{tr. deg}_k k(x_1, \dots, x_n) = n.$$

□

6.6. Rational Maps

Chapter 7: Resolution of Singularities

7.1. Rational Maps of Curves

7.2. Blowing up a Point in A^2

7.3. Blowing up a Point in P^2

7.4. Quadratic Transformations

7.5. Nonsingular Models of Curves

Chapter 8: Riemann-Roch Theorem

8.1. Divisors

8.2. The Vector Spaces $L(D)$

8.3. Riemann's Theorem

8.4. Derivations and Differentials

8.5. Canonical Divisors

8.6. Riemann-Roch Theorem