

## Chapter 3: Numerical Sequences and Series

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**Exercise 3.1.** Prove that the convergence of  $\{s_n\}$  implies convergence of  $\{|s_n|\}$ . Is the converse true?

*Proof.*

- (1) Since  $\{s_n\}$  is convergent, there is  $s \in \mathbb{R}^1$  with the following property: given any  $\epsilon > 0$ , there is  $N$  such that  $|s_n - s| < \epsilon$  whenever  $n \geq N$ . So

$$||s_n| - |s|| \leq |s_n - s| < \epsilon$$

(Exercise 1.13). That is,  $\{|s_n|\}$  converges to  $|s|$ .

- (2) The converse is not true by considering  $s_n = (-1)^{n+1}$ .

□

**Exercise 3.2** Calculate  $\lim_{n \rightarrow \infty} (\sqrt{n^2 + n} - n)$ .

*Proof.*

$$\sqrt{n^2 + n} - n = \frac{n}{\sqrt{n^2 + n} + n} = \frac{1}{\sqrt{1 + \frac{1}{n}} + 1} \rightarrow \frac{1}{1 + 1} = \frac{1}{2}$$

as  $n \rightarrow \infty$ . □

*Proof ( $\epsilon - N$  argument).* Let  $s_n = \sqrt{n^2 + n} - n$ . Show that the sequence  $\{s_n\}$  converges to  $s = \frac{1}{2}$ . Given any  $\epsilon > 0$ , there is  $N > \frac{1}{\epsilon}$  such that

$$\begin{aligned} |s_n - s| &= \left| (\sqrt{n^2 + n} - n) - \frac{1}{2} \right| = \left| \frac{1}{\sqrt{1 + \frac{1}{n}} + 1} - \frac{1}{2} \right| \\ &= \left| \frac{2 - \left( \sqrt{1 + \frac{1}{n}} + 1 \right)}{2 \left( \sqrt{1 + \frac{1}{n}} + 1 \right)} \right| = \left| \frac{1 - \sqrt{1 + \frac{1}{n}}}{2 \left( \sqrt{1 + \frac{1}{n}} + 1 \right)} \right| \\ &= \left| \frac{1 - \left( 1 - \frac{1}{n} \right)}{2 \left( \sqrt{1 + \frac{1}{n}} + 1 \right)^2} \right| = \left| \frac{-\frac{1}{n}}{2 \left( \sqrt{1 + \frac{1}{n}} + 1 \right)^2} \right| < \frac{1}{n} \leq \frac{1}{N} < \epsilon \end{aligned}$$

whenever  $n \geq N$ .  $\square$

**Exercise 3.3** If  $s_1 = \sqrt{2}$  and

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} \quad (n = 1, 2, 3, \dots),$$

prove that  $\{s_n\}$  converges, and that  $s_n < 2$  for  $n = 1, 2, 3, \dots$ .

The convergence of  $\{s_n\}$  implies there is  $s \in \mathbb{R}$  such that  $s_n \rightarrow s$  where  $s = \sqrt{2 + \sqrt{s}}$  and  $\sqrt{2} < s \leq 2$ . WolframAlpha shows that

$$s = \frac{1}{3} \left( -1 + \sqrt[3]{\frac{1}{2}(79 - 3\sqrt{249})} + \sqrt[3]{\frac{1}{2}(79 + 3\sqrt{249})} \right).$$

*Proof (Theorem 3.14).*

(1) Show that  $\{s_n\}$  is increasing (by mathematical induction).

(a) Show that  $s_2 > s_1$ . In fact,

$$s_2 = \sqrt{2 + \sqrt{s_1}} = \sqrt{2 + \sqrt{\sqrt{2}}} < \sqrt{2} = s_1.$$

(a) Show that  $s_{n+1} > s_n$  if  $s_n > s_{n-1}$ .

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} > \sqrt{2 + \sqrt{s_{n-1}}} = s_n.$$

By mathematical induction,  $\{s_n\}$  is (strictly) increasing.

(2) Show that  $\{s_n\}$  is bounded (by mathematical induction).

(a) Show that  $s_1 \leq 2$ .  $\sqrt{2} \leq 2$ .

(a) Show that  $s_{n+1} \leq 2$  if  $s_n \leq 2$ .

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} \leq \sqrt{2 + \sqrt{2}} < 2.$$

By mathematical induction,  $\{s_n\}$  is bounded by 2.

Hence,  $\{s_n\}$  converges since  $\{s_n\}$  is increasing and bounded (Theorem 3.14).  $\square$

**Exercise 3.4** Find the upper and lower limits of the sequences  $\{s_n\}$  defined by

$$s_1 = 0; s_{2m} = \frac{s_{2m-1}}{2}; s_{2m+1} = \frac{1}{2} + s_{2m}.$$

Write out the first few terms of  $\{s_n\}$ :

$$0, 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{3}{8}, \frac{7}{8}, \frac{7}{16}, \frac{15}{16}, \dots$$

It suggests us

$$\begin{aligned} s_{2m+1} &= 1 - \frac{1}{2^m} \quad (m = 0, 1, 2, \dots), \\ s_{2m} &= \frac{1}{2} - \frac{1}{2^m} \quad (m = 1, 2, 3, \dots). \end{aligned}$$

*Proof.*

(1) *Show that*

$$\begin{aligned} s_{2m+1} &= 1 - \frac{1}{2^m} \quad (m = 0, 1, 2, \dots), \\ s_{2m} &= \frac{1}{2} - \frac{1}{2^m}. \quad (m = 1, 2, 3, \dots) \end{aligned}$$

Apply mathematical induction.

(2) The upper limit is 1.

(3) The lower limit is  $\frac{1}{2}$ .

□

**Exercise 3.7** *Prove that the convergence of  $\sum a_n$  implies the convergence of*

$$\sum \frac{\sqrt{a_n}}{n},$$

*if  $a_n \geq 0$ .*

*Proof (Cauchy's inequality).*

(1) *Show that  $\sum \frac{\sqrt{a_n}}{n}$  is bounded.* For any  $k \in \mathbb{Z}^+$ ,

$$\begin{aligned} \left( \sum_{n=1}^k \frac{\sqrt{a_n}}{n} \right)^2 &\leq \left( \sum_{n=1}^k a_n \right) \left( \sum_{n=1}^k \frac{1}{n^2} \right) && \text{(Cauchy's inequality)} \\ &\leq \left( \sum_{n=1}^{\infty} a_n \right) \left( \sum_{n=1}^{\infty} \frac{1}{n^2} \right). && \left( \sum a_n, \sum \frac{1}{n^2} : \text{convergent} \right) \end{aligned}$$

Thus,  $\left( \sum_{n=1}^k \frac{\sqrt{a_n}}{n} \right)^2$  is bounded, or  $\sum_{n=1}^k \frac{\sqrt{a_n}}{n}$  is bounded.

(2) Show that  $\sum_{n=1}^k \frac{\sqrt{a_n}}{n}$  is increasing. It is clear due to  $\frac{\sqrt{a_n}}{n} \geq 0$ .

By Theorem 3.14,  $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$  converges.  $\square$

*Proof (AM-GM inequality). Show that  $\sum \frac{\sqrt{a_n}}{n}$  is bounded.*

$$\begin{aligned} \frac{\sqrt{a_n}}{n} &\leq \frac{1}{2} \left( a_n + \frac{1}{n^2} \right) && \text{(AM-GM inequality)} \\ \sum_{n=1}^k \frac{\sqrt{a_n}}{n} &\leq \frac{1}{2} \left( \sum_{n=1}^k a_n + \sum_{n=1}^k \frac{1}{n^2} \right) \\ &\leq \frac{1}{2} \left( \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} \frac{1}{n^2} \right). && \left( \sum a_n, \sum \frac{1}{n^2} : \text{convergent} \right) \end{aligned}$$

Thus,  $\sum_{n=1}^k \frac{\sqrt{a_n}}{n}$  is bounded. The rest proof is the same as previous.  $\square$