Chapter 11: The Lebesuge Theory

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Exercise 11.1. *If $f \geq 0$ and $\int_E f d\mu = 0$, prove that $f(x) = 0$ almost everywhere on E. (Hint: Let E_n be the subset of E on which $f(x) > \frac{1}{n}$. Write $A = \bigcup E_n$. Then $\mu(A) = 0$ if and only if $\mu(E_n) = 0$ for every n.)*

Might assume that f is measurable on E.

Proof (Hint).

(1) Define $A = \{x \in E : f(x) > 0\}$. So $f(x) = 0$ almost everywhere on E if and only if $\mu(A) = 0$.

(2) Define
$$E_n = \left\{ x \in E : f(x) > \frac{1}{n} \right\}$$
for $n = 1, 2, 3, \ldots$. Note that $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$ and
$$A = \bigcup_{n=1}^{\infty} E_n.$$
Since μ is a measure,
$$\lim_{n \to \infty} \mu(E_n) = \mu(A)$$
(Theorem 11.3).

(3) (Reductio ad absurdum) If $\mu(A) > 0$, there is an integer N such that $\mu(E_n) \geq \frac{\mu(A)}{2}$ whenever $n \geq N$ (by (2)). In particular, take $n = N$ to get
$$\begin{aligned}
\int_E f d\mu &\geq \int_{E_N} f d\mu && (\mu \text{ is a measure and } E_N \subseteq E) \\
&\geq \frac{1}{N} \cdot \mu(E_N) && \text{(Remarks 11.23(b))} \\
&\geq \frac{1}{N} \cdot \frac{\mu(A)}{2} \\
&> 0,
\end{aligned}$$
contrary to the assumption that $\int_E f d\mu = 0$.

□

Note. Compare to Exercise 6.2.

Exercise 11.2. *If $\int_A f d\mu = 0$ for every measurable subset A of a measurable set E, then $f(x) = 0$ almost everywhere on E.*

Might assume that f is measurable on E.

Proof.

(1) Define

$$A = \{x \in E : f(x) \geq 0\} \qquad \text{and} \qquad B = \{x \in E : f(x) \leq 0\}.$$

A and B are measurable subsets of a measurable set E since f is measurable.

(2) Apply Exercise 11.1 to the fact that $f \geq 0$ on A (by construction) and $\int_A f d\mu = 0$ (by assumption), we have $f(x) = 0$ almost everywhere on A.

(3) Similarly, apply Exercise 11.1 to the fact that $-f \geq 0$ on B and $\int_B (-f) d\mu = -\int_B f d\mu = 0$, we have $f(x) = 0$ almost everywhere on B.

(4) As $E = A \cup B$, $f(x) = 0$ almost everywhere on E by (2)(3).

$\square$

Exercise 11.3. *If $\{f_n\}$ is a sequence of measurable functions, prove that the set of points x at which $\{f_n(x)\}$ converges is measurable.*

Proof.

(1) It suffices to show that

$$E = \{x : \{f_n(x)\} \text{ is convergent } \} = \{x : \{f_n(x)\} \text{ is Cauchy } \}$$

is measurable (since $\mathbb{R}^1$ is complete).

(2) Write

$$E = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n,m \geq N} \left\{ x : |f_n(x) - f_m(x)| \leq \frac{1}{k} \right\}$$

Since $\{f_n\}$ is a sequence of measurable functions, $x \mapsto |f_n(x) - f_m(x)|$ is measurable (Theorem 11.16 and Theorem 11.18). Hence

$$\left\{ x : |f_n(x) - f_m(x)| \leq \frac{1}{k} \right\}$$

is measurable (Theorem 11.15). Therefore E is measurable.

$\square$

Exercise 11.4. *If $f \in \mathscr{L}(\mu)$ on E and g is bounded and measurable on E, then $fg \in \mathscr{L}(\mu)$ on E.*

Proof (Theorem 11.27).

(1) fg is measurable since both f and g are measurable (Theorem 11.18).

(2) $|g| \leq M$ for some real $M \in \mathbb{R}^1$ by the boundedness of g. Hence

$$|fg| \leq M|f|$$

on E.

(3) To apply Theorem 11.27, it suffices to show that $M|f| \in \mathscr{L}(\mu)$ on E. Theorem 11.26 implies that $|f| \in \mathscr{L}(\mu)$ if $f \in \mathscr{L}(\mu)$. And Remarks 11.23(d) implies that $M|f| \in \mathscr{L}(\mu)$ if $|f| \in \mathscr{L}(\mu)$.

$\square$

Note (Riemann integral). *If $f \in \mathscr{R}$ on $[a, b]$ and g is bounded and measurable on $[a, b]$, then fg might be not Riemann integrable.*

Exercise 11.5. *Put*

$$g(x) = \begin{cases} 0 & (0 \leq x \leq \frac{1}{2}), \\ 1 & (\frac{1}{2} < x \leq 1), \end{cases}$$

and

$$\begin{aligned} f_{2k}(x) &= g(x) & (0 \leq x \leq 1), \\ f_{2k+1}(x) &= g(1-x) & (0 \leq x \leq 1). \end{aligned}$$

Show that

$$\liminf_{n \to \infty} f_n(x) = 0 \qquad (0 \leq x \leq 1),$$

but

$$\int_0^1 f_n(x) dx = \frac{1}{2}.$$

(Compare with the Fatou's theorem.)

Proof.

(1) *Show that* $\liminf_{n \to \infty} f_n(x) = 0$. Note that

$$g(1-x) = \begin{cases} 1 & (0 \leq x < \frac{1}{2}), \\ 0 & (\frac{1}{2} < x \leq 1). \end{cases}$$

Since $f_n(x) \geq 0$ by definition, $\liminf_{n\to\infty} f_n(x) \geq 0$. Since $f_{2k}(0) = f_{2k+1}(1) = 0$ for all positive integers k, $\liminf_{n\to\infty} f_n(x) \leq 0$. Therefore the result is established.

(2) *Show that* $\int_0^1 f_n(x)dx = \frac{1}{2}$. Since

$$\int_0^1 f_{2k}(x)dx = \int_0^1 g(x)dx = \frac{1}{2},$$
$$\int_0^1 f_{2k+1}(x)dx = \int_0^1 g(1-x)dx = \frac{1}{2},$$

in any case $\int_0^1 f_n(x)dx = \frac{1}{2}$ for all positive integers n.

(3) This example shows that we may have the strict inequality in the Fatou's theorem.

□

Supplement (Similar exercise). *Consider the sequence* $\{f_n\}$ *defined by* $f_n(x) = 1$ *if* $n \leq x < n+1$, *with* $f_n(x) = 0$ *otherwise. Show that we may have the strict inequality in the Fatou's theorem.*

Exercise 11.6. *Let*

$$f_n(x) = \begin{cases} \frac{1}{n} & (|x| \leq n), \\ 0 & (|x| > n). \end{cases}$$

Then $f_n(x) \to 0$ *uniformly on* $\mathbb{R}^1$, *but*

$$\int_{-\infty}^{\infty} f_n(x)dx = 2 \qquad (n = 1, 2, 3, \ldots).$$

(We write $\int_{-\infty}^{\infty}$ *in place of* $\int_{\mathbb{R}^1}$*.) Thus uniform convergence does not imply dominated convergence in the sense of Theorem 11.32. However, on sets of finite measure, uniformly convergent sequences of bounded functions do satisfy Theorem 11.32.*

Proof.

(1) *Show that* $f_n(x) \to 0$ *uniformly on* $\mathbb{R}^1$. Given any $\varepsilon > 0$, there is an integer $N > \frac{1}{\varepsilon}$ such that

$$|f_n(x) - 0| \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

whenever $n \geq N$ and $x \in \mathbb{R}^1$. Hence $f_n(x) \to 0$ uniformly.

(2) *Show that* $\int_{-\infty}^{\infty} f_n(x)dx = 2$.

$$\int_{-\infty}^{\infty} f_n(x)dx = \int_{-n}^{n} \frac{1}{n} dx = 2.$$

(3) By (1)(2),

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx \neq \int_{-\infty}^{\infty} \lim_{n\to\infty} f_n(x)dx$$

suggests that the Lebesgue's dominated convergence theorem (Theorem 11.32) does not hold in this case. In fact, if there were $g \in \mathscr{L}$ such that $|f_n(x)| \leq g(x)$, then

$$\begin{aligned}
\int_{-\infty}^{\infty} g(x)dx &\geq \int_{0}^{\infty} g(x)dx && \text{(Theorem 11.24)} \\
&= \sum_{n=1}^{\infty} \int_{n-1}^{n} g(x)dx && \text{(Theorem 11.24)} \\
&\geq \sum_{n=1}^{\infty} \int_{n-1}^{n} |f_n(x)|dx \\
&= \sum_{n=1}^{\infty} \int_{n-1}^{n} \frac{1}{n} dx \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \\
&= \infty,
\end{aligned}$$

which is absurd.

(4) *Show that on sets of finite measure, uniformly convergent sequences of bounded functions* $\{f_n\}$ *do satisfy Theorem 11.32.*

(a) Since $\{f_n\}$ is uniformly convergent, $\{f_n\}$ is uniformly bounded (Exercise 7.1), or there exists a real number M such that

$$|f_n(x)| \leq M$$

for all positive integer n and $x \in E$.

(b) Define $g(x) = M$ on E. It is clear that

$$\int_E g(x)dx = M\mu(E) < +\infty.$$

Now we can apply the Lebesgue's dominated convergence theorem (Theorem 11.32) to get

$$\lim_{n\to\infty} \int_E f_n d\mu = \int_E \lim_{n\to\infty} f_n d\mu.$$

$\square$

Exercise 11.7. *...*

Proof.

(1)

(2)

$\square$

Exercise 11.8. *If $f \in \mathscr{R}$ on $[a, b]$ and if $F(x) = \int_a^x f(t)dt$, prove that $F'(x) = f(x)$ almost everywhere on $[a, b]$.*

Proof.

(1) Theorem 6.20 implies that $F'(x_0) = f(x_0)$ if f is continuous at $x_0 \in [a, b]$.

(2) Since $f \in \mathscr{R}$ on $[a, b]$, f is bounded on $[a, b]$. Theorem 11.33 implies that f is continuous almost everywhere on $[a, b]$.

By (1)(2), $F'(x) = f(x)$ almost everywhere on $[a, b]$. $\square$

Exercise 11.9. *Prove that the function F given by*

$$F(x) = \int_a^x f dt \qquad (a \leq x \leq b)$$

(where $f \in \mathscr{L}$ on $[a, b]$) is continuous on $[a, b]$.

Proof.

(1) *Let $f \in \mathscr{L}$ on E. Show that given any $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$\int_A f d\mu < \varepsilon$$

whenever $A \subseteq E$ with $\mu(A) < \delta$.

(a) Define $f_n(x) = \min\{f(x), n\}$ on E for $n = 1, 2, 3, \ldots$. Then $\{f_n\}$ is a sequence of measurable functions such that

$$0 \leq f_1(x) \leq f_2(x) \leq \cdots .$$

Also, $f_n \to f$. Then by the Lebesuge's monotone convergenece theorem (Theorem 11.28),

$$\lim_{n \to \infty} \int_E f_n d\mu = \int_E f d\mu.$$

(b) For such $\varepsilon > 0$, there is an integer $N \geq 1$ such that

$$\int_E (f - f_N) d\mu < \frac{\varepsilon}{2}.$$

Choose $\delta > 0$ so that $\delta < \frac{\varepsilon}{2N}$. If $\mu(A) < \delta$, we have

$$\begin{aligned}
\int_A f d\mu &= \int_A (f - f_N) d\mu + \int_A f_N d\mu \\
&\leq \int_E (f - f_N) d\mu + N\mu(A) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

(2) Apply (1) to f^+ and f^- on $E = [a, b]$. Given any $\varepsilon > 0$, there is a common $\delta > 0$ such that

$$\left| \int_x^y f^+ dt \right| < \frac{\varepsilon}{2} \qquad \text{and} \qquad \left| \int_x^y f^- dt \right| < \frac{\varepsilon}{2}$$

whenever $|y - x| < \delta$. So

$$|F(y) - F(x)| \leq \left| \int_x^y f^+ dt \right| + \left| \int_x^y f^- dt \right| < \varepsilon$$

whenever $|y - x| < \delta$. Hence F is uniformly continuous. (In fact, F is absolutely continuous by the same argument.)

□

Note. Compare to Theorem 6.20.

Exercise 11.10. *If $\mu(X) < +\infty$ and $f \in \mathscr{L}^2(\mu)$ on X, prove that $f \in \mathscr{L}$ on X. If*

$$\mu(X) = +\infty,$$

this is false. For instance, if

$$f(x) = \frac{1}{1 + |x|},$$

then $f^2 \in \mathscr{L}$ on $\mathbb{R}^1$, but $f \notin \mathscr{L}$ on $\mathbb{R}^1$.

Proof.

(1) Since $\mu(X) < +\infty$, $1 \in \mathscr{L}^2(\mu)$ on X. By Theorem 11.35, $f \in \mathscr{L}(\mu)$, and

$$\int_X |f| d\mu \leq \|f\| \|1\|.$$

(2) *Show that* $f^2 \in \mathscr{L}$ *on* $\mathbb{R}^1$. To apply Theorem 11.33, we might restrict the measure space $X = \mathbb{R}^1$ to some interval $[a, b]$. Then apply the Lebesgue's monotone convegence theorem (Theorem 11.28) to get the conclusion.

(a) Write

$$f(x)^2 = \left(\frac{1}{1+|x|}\right)^2 = \frac{1}{1+2|x|+x^2} \leq \frac{1}{1+x^2}.$$

By Theorem 11.27, *it suffices to show that* $\frac{1}{1+x^2} \in \mathscr{L}$ *on* $\mathbb{R}^1$.

(b) Consider the sequence $\{f_n\}$ defined by

$$f_n(x) = \frac{1}{1+x^2}\chi_{[-n,n]}(x).$$

(Here $\chi_{[-n,n]} = K_{[-n,n]}$ is the characteristic function of $[-n, n]$ defined in Definition 11.19.) By construction,

$$0 \leq f_1(x) \leq f_2(x) \leq \cdots \qquad (x \in \mathbb{R}^1)$$

and

$$f_n(x) \to \frac{1}{1+x^2} \qquad (x \in \mathbb{R}^1).$$

(c) Hence

$$\begin{aligned}
\int_{\mathbb{R}^1} \frac{1}{1+x^2} dx &= \lim_{n\to\infty} \int_{\mathbb{R}^1} f_n(x) dx && \text{(Theorem 11.28)} \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^1} \frac{1}{1+x^2}\chi_{[-n,n]}(x) dx \\
&= \lim_{n\to\infty} \int_{-n}^{n} \frac{1}{1+x^2} dx \\
&= \lim_{n\to\infty} \mathscr{R}\int_{-n}^{n} \frac{1}{1+x^2} dx && \text{(Theorem 11.33)} \\
&= \lim_{n\to\infty} 2\arctan(n) \\
&= \pi < \infty.
\end{aligned}$$

(4) *Show that* $f \notin \mathscr{L}$ *on* $\mathbb{R}^1$.

(a) Consider the sequence $\{f_n\}$ defined by

$$f_n(x) = f(x)\chi_{[-n,n]}(x) = \frac{1}{1+|x|}\chi_{[-n,n]}(x).$$

By construction,

$$0 \leq f_1(x) \leq f_2(x) \leq \cdots \qquad (x \in \mathbb{R}^1)$$

and

$$f_n(x) \to f(x) \qquad (x \in \mathbb{R}^1).$$

(b) Hence

$$\begin{aligned}
\int_{\mathbb{R}^1} f(x)dx &= \lim_{n\to\infty} \int_{\mathbb{R}^1} f_n(x)dx && \text{(Theorem 11.28)} \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^1} \frac{1}{1+|x|} \chi_{[-n,n]}(x)dx \\
&= \lim_{n\to\infty} \int_{-n}^{n} \frac{1}{1+|x|} dx \\
&= \lim_{n\to\infty} \mathscr{R} \int_{-n}^{n} \frac{1}{1+|x|} dx && \text{(Theorem 11.33)} \\
&= \lim_{n\to\infty} 2\log(n+1) \\
&= \infty,
\end{aligned}$$

or $f \not\in \mathscr{L}$ on $\mathbb{R}^1$.

□

Note. Compare to Exercise 6.5.

Exercise 11.11. *If $f, g \in \mathscr{L}(\mu)$ on X, defined the distance between f and g by*

$$\int_X |f-g|d\mu.$$

Prove that $\mathscr{L}(\mu)$ is a complete metric space.

Proof.

(1) Define

$$\|f-g\|_1 = \int_X |f-g|d\mu.$$

Thus $\|f-g\|_1 = 0$ if and only if $f = g$ almost everywhere on X (Exercise 11.1). As in Remark 11.37, we identify two functions to be equivalent if they are equal almost everywhere.

(2) *Show that $\mathscr{L}(\mu)$ is a metric space.*

(a) By definition, $\|f-g\|_1 \geq 0$. Besides, $\|f-g\|_1 = 0$ if and only if $f = g$ almost everywhere by (1).

(b) $\|f-g\|_1 = \|g-f\|_1$ since $|f(x)-g(x)| = |g(x)-f(x)|$ for all $x \in X$.

(c) Since $|f(x)-g(x)| \leq |f(x)-h(x)| + |h(x)-g(x)|$ for all $x \in X$, Remarks 11.23(c) and Theorem 11.29 imply that

$$\|f-g\|_1 \leq \|f-h\|_1 + \|h-g\|_1.$$

(3) *Show that $\mathscr{L}(\mu)$ is complete.* Similar to the proof of Theorem 11.42.

(a) *Let $\{f_n\}$ be a Cauchy sequence in $\mathscr{L}(\mu)$, show that there exists a function $f \in \mathscr{L}(\mu)$ such that $\{f_n\}$ converges to $f \in \mathscr{L}(\mu)$.*

(b) Since $\{f_n\}$ is a Cauchy sequence, we can find a sequence $\{n_k\}$, $k = 1, 2, 3, \ldots$, such that

$$\|f_{n_k} - f_{n_{k+1}}\|_1 = \int_X |f_{n_k} - f_{n_{k+1}}| d\mu < \frac{1}{2^k} \qquad (k = 1, 2, 3, \ldots).$$

Hence

$$\sum_{k=1}^{\infty} \int_X |f_{n_k} - f_{n_{k+1}}| d\mu \leq \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 < +\infty.$$

(c) By Theorem 11.30, we may interchange the summation and integration to get

$$\int_X \sum_{k=1}^{\infty} |f_{n_k} - f_{n_{k+1}}| d\mu < +\infty,$$

or

$$\sum_{k=1}^{\infty} |f_{n_k}(x) - f_{n_{k+1}}(x)| = \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)| < +\infty$$

almost everywhere on X.

(d) Since the kth partial sum of the series

$$\sum_{k=1}^{\infty} (f_{n_{k+1}}(x) - f_{n_k}(x))$$

which converges almost everywhere on X (Theorem 3.45), is

$$f_{n_{k+1}}(x) - f_{n_1}(x),$$

we see that the equation

$$f(x) = \lim_{k \to \infty} f_{n_k}(x)$$

defines $f(x)$ for almost all $x \in X$, and it does not matter how we define $f(x)$ at the remaining points of X.

(e) We shall now show that this function f has the desired properties. Let $\varepsilon > 0$ be given, and choose N such that

$$\|f_n - f_m\|_1 \leq \varepsilon$$

whenever $n, m \geq N$. If $n_k > N$, Fatou's theorem shows that

$$\|f - f_{n_k}\|_1 \leq \liminf_{i \to \infty} \|f_{n_i} - f_{n_k}\|_1 \leq \varepsilon.$$

Thus $f - f_{n_k} \in \mathscr{L}(\mu)$, and since $f = (f - f_{n_k}) + f_{n_k} \in \mathscr{L}(\mu)$, we see that $f \in \mathscr{L}(\mu)$. Also, since ε is arbitrary,

$$\lim_{k \to \infty} \|f - f_{n_k}\|_1 = 0.$$

(f) Finally, the inequality

$$\|f - f_n\|_1 \leq \|f - f_{n_k}\|_1 + \|f_{n_k} - f_n\|_1$$

shows that $\{f_n\}$ converges to $f \in \mathscr{L}(\mu)$; for if we take n and n_k large enough, each of the two terms can be made arbitrary small.

□

Exercise 11.12. *Suppose*

(a) $|f(x, y)| \leq 1$ *if* $0 \leq x \leq 1$, $0 \leq y \leq 1$.

(b) *for fixed* x, $f(x, y)$ *is a continuous function of* y.

(c) *for fixed* y, $f(x, y)$ *is a continuous function of* x.

Put

$$g(x) = \int_0^1 f(x, y) dy \qquad (0 \leq x \leq 1).$$

Is g *continuous?*

Proof.

(1) *Show that* g *is continuous.*

(2) Let $\{x_n\}$ be a sequence in $[0, 1]$ such that $x_n \neq x$ and $\lim x_n = x$. It suffices to show that

$$\begin{aligned}
\lim_{n \to \infty} g(x_n) &= \lim_{n \to \infty} \int_0^1 f(x_n, y) dy \\
&= \int_0^1 \lim_{n \to \infty} f(x_n, y) dy \\
&= \int_0^1 f(x, y) dy \\
&= g(x)
\end{aligned}$$

(Theorem 4.2). Since $\lim_{n \to \infty} f(x_n, y) = f(x, y)$ for any fixed y (by (c)), it suffices to show that

$$\lim_{n \to \infty} \int_0^1 f(x_n, y) dy = \int_0^1 \lim_{n \to \infty} f(x_n, y) dy.$$

(3) Define $\{f_n\}$ by $f_n(y) = f(x_n, y)$. $f_n(y)$ is a continuous function of y for every fixed n (by (b)). Thus $f_n(y)$ is measurable (Example 11.14). Besides, $|f_n(y)| \leq 1$ and $1 \in \mathscr{L}$ on $[0,1]$ (by (a)). The Lebesgue's dominated convergence theorem (Theorem 11.32) implies that

$$\lim_{n\to\infty} \int_0^1 f(x_n, y)dy = \int_0^1 \lim_{n\to\infty} f(x_n, y)dy.$$

□

Supplement (Similar exercise). *Suppose*

(a) $|f(x,y)| \leq g(y)$ *if* $0 \leq x \leq 1$, $0 \leq y \leq 1$, *where* $g \in \mathscr{L}$ *on* $[0,1]$.

(b) *for fixed* x, $f(x,y)$ *is a measurable function of* y.

(c) *for fixed* y, $f(x,y)$ *is a continuous function of* x.

Show that

$$h(x) = \int_0^1 f(x,y)dy \qquad (0 \leq x \leq 1).$$

is continuous.

Exercise 11.13. ...

Proof.

(1)

(2)

□

Exercise 11.14. ...

Proof.

(1)

(2)

□

Exercise 11.15. ...

Proof.

(1)

(2)

□

Exercise 11.16. ...

Proof.

(1)

(2)

□

Exercise 11.17. ...

Proof.

(1)

(2)

□

Exercise 11.18. ...

Proof.

(1)

(2)

□