

Solutions to the book:
*Jürgen Neukirch, Algebraic Number
Theory*

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Chapter I: Algebraic Integers

I.1. The Gaussian Integers

Exercise I.1.1.

$\alpha \in \mathbb{Z}[i]$ is a unit if and only if $N(\alpha) = 1$.

Proof.

- (1) (\implies) Since α is a unit, there is $\beta \in \mathbb{Z}[i]$ such that $\alpha\beta = 1$. So $N(\alpha\beta) = N(1)$, or $N(\alpha)N(\beta) = 1$. Since the image of N is nonnegative integers, $N(\alpha) = 1$.
- (2) (\impliedby) $N(\alpha) = \alpha\bar{\alpha}$, or $1 = \alpha\bar{\alpha}$ since $N(\alpha) = 1$. That is, $\bar{\alpha} \in \mathbb{Z}[i]$ is the inverse of $\alpha \in \mathbb{Z}[i]$. (Or we solve the equation $N(\alpha) = a^2 + b^2 = 1$, and show that all four solutions (± 1 and $\pm i$) are units.)
- (3) Conclusion: a unit $\alpha = a + bi$ of $\mathbb{Z}[i]$ is satisfying the equation $N(\alpha) = a^2 + b^2 = 1$ by (1)(2). That is, the only unit of $\mathbb{Z}[i]$ are ± 1 and $\pm i$.

□

Exercise I.1.4.

Show that the ring $\mathbb{Z}[i]$ cannot be ordered.

Proof. Similar to the fact that i cannot be ordered in \mathbb{C} , i cannot be ordered in $\mathbb{Z}[i]$ either. □

Exercise I.1.5.

Show that the only units of the ring $\mathbb{Z}[\sqrt{-d}] = \mathbb{Z} + \mathbb{Z}\sqrt{-d}$, for any rational integer $d > 1$, are ± 1 .

Proof.

- (1) Define the norm N on $\mathbb{Z}[\sqrt{-d}]$ by

$$N(x + y\sqrt{-d}) = (x + y\sqrt{-d})(x - y\sqrt{-d}) = x^2 + y^2d,$$

i.e., by $N(z) = |z|^2$. It is multiplicative.

(2) Similar to Exercise I.1.1,

$$\begin{aligned}x + y\sqrt{-d} \in \mathbb{Z}[\sqrt{-d}] \text{ is a unit} &\iff N(x + y\sqrt{-d}) = x^2 + y^2d = 1 \\&\iff x^2 = 1 \text{ and } y = 0 \\&\iff x = \pm 1 \text{ and } y = 0.\end{aligned}$$

Hence the only units of the ring $\mathbb{Z}[\sqrt{-d}]$ are ± 1 ($d > 1$).

□

I.2. Integrality

Exercise I.2.1.

Is $\frac{3+2\sqrt{6}}{1-\sqrt{6}}$ an algebraic integer?

Proof.

- (1) $\alpha := \frac{3+2\sqrt{6}}{1-\sqrt{6}} = -3 - \sqrt{6}$. Since the set of all algebraic integers is a ring, α is an algebraic integer.
- (2) Or show that α satisfies a monic equation $x^2 + 6x + 3 = 0 \in \mathbb{Z}[x]$.

□