Solutions to the book: *do Carmo, Differential Geometry of Curves and Surfaces*

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Contents

Chapter 1: Curves

1-1. Introduction

Classical differential geometry: the study of local properties of curves and surfaces.

Global differential geometry: the study of the influence of the local properties on the behavior of the entire curve and surface.

No exercises.

1-2. Parametrized Curves

Exercise 1-2.1.

Find a parametrized curve $\alpha(t)$ whose trace is the circle $x^2 + y^2 = 1$ such that $\alpha(t)$ runs clockwise around the circle with $\alpha(0) = (0, 1)$.

Proof. $\alpha(t) = (\sin t, \cos t)$, $t \in \mathbb{R}$. $\square$

Exercise 1-2.2.

Let $\alpha(t)$ be a parametrized curve which does not pass through the origin. If $\alpha(t_0)$ is the point of the trace of α closest to the origin and $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

Proof. Let $f(t) = |\alpha(t)|^2 = \alpha(t) \cdot \alpha(t)$. $f(t)$ is differentiable and $f(t)$ has a local minimum at a point $t = t_0 \in I$. So $f'(t_0) = 0$. [Theorem 5.8 in *W. Rudin, Principles of Mathematical Analysis, 3rd edition.*] Since

$$f'(t) = 2\alpha(t) \cdot \alpha'(t),$$

$f'(t_0) = 2\alpha(t_0) \cdot \alpha'(t_0) = 0$, or $\alpha(t_0) \cdot \alpha'(t_0) = 0$. Since $\alpha(t_0) \neq 0$ and $\alpha'(t_0) \neq 0$, $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$. $\square$

Exercise 1-2.3.

A parametrized curve $\alpha(t)$ has a property that its second derivative $\alpha''(t)$ is identically zero. What can be said about α?

Proof.

(1) $\alpha(t)$ is a straight line.

(2) Since $\alpha''(t)$ is identically zero, $\alpha'(t) = a$ is a constant. [Theorem 5.11 in *W. Rudin, Principles of Mathematical Analysis, 3rd edition.*] Define $f(t) = \alpha(t) - at$ (on I). Since $f'(t) = \alpha'(t) - a = 0$, $f(t) = \alpha(t) - at = b$ is a constant again. Therefore, $\alpha(t) = at + b$, which is a straight line (on I).

□

Exercise 1-2.4.

Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve and let $v \in \mathbb{R}^3$ be a fixed vector. Assume that $\alpha'(t)$ is orthogonal to v for all $t \in I$ and that $\alpha(0)$ is orthogonal to v. Prove that $\alpha(t)$ is orthogonal to v for all $t \in I$.

Need to assume that $\alpha(t) \neq 0$ for all $t \in I$.

Proof. Given any $t \neq 0 \in I$. (Nothing to do at $t = 0$.) Define $f : I \to \mathbb{R}$ by $f(t) = \alpha(t) \cdot v$. By the mean value theorem, there exists a point ξ between 0 and t such that

$$f(t) - f(0) = f'(\xi)(t - 0),$$

where $f'(t) = \alpha'(t) \cdot v + \alpha(t) \cdot v' = \alpha'(t) \cdot v$. Note that $f(0) = 0$ since $\alpha(0)$ is orthogonal to v, and $f'(\xi) = 0$ since $\alpha'(t)$ is orthogonal to v. So the identity is reduced to

$$f(t) = 0,$$

or $\alpha(t) \cdot v = 0$, or $\alpha(t)$ is orthogonal to v. □

Exercise 1-2.5.

Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve, with $\alpha'(t) \neq 0$ for all $t \in I$. Show that $|\alpha(t)|$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.

The same trick in Exercise 1-2.2.

Proof. It is equivalent to *show that* $|\alpha(t)|^2$ *is a nonzero constant if and only if* $\alpha(t)$ *is orthogonal to* $\alpha'(t)$ *for all* $t \in I$. Let

$$f(t) = |\alpha(t)|^2 = \alpha(t) \cdot \alpha(t).$$

Notice that $\alpha'(t) \neq 0$, and thus

$$\begin{aligned}
&|\alpha(t)| \text{ is a nonzero constant} \\
\iff & f(t) = |\alpha(t)|^2 \text{ is a nonzero constant} \\
\iff & f'(t) = 0 \text{ and } f(t) \text{ is a nonzero constant} \\
\iff & \alpha(t) \cdot \alpha'(t) = 0 \text{ and } \alpha(t) \text{ is a nonzero constant} \\
\iff & \alpha(t) \text{ is orthogonal to } \alpha'(t) \text{ for all } t \in I.
\end{aligned}$$

□

1-3. Regular Curves; Arc Length

Exercise 1-3.1.

Show that the tangent lines to the regular parametrized curve $\alpha(t) = (3t, 3t^2, 2t^3)$ *make a constant angle with the line* $y = 0$, $z = x$.

Proof. $\alpha'(t) = (3, 6t, 6t^2)$. The line $y = 0$, $z = x$ is $\beta(t) = (1, 0, 1)$. The cosine of the angle θ between these to curves is

$$\begin{aligned}
\cos\theta &= \frac{(3, 6t, 6t^2) \cdot (1, 0, 1)}{|(3, 6t, 6t^2)||(1, 0, 1)|} \\
&= \frac{3 + 6t^2}{\sqrt{3^2 + (6t)^2 + (6t^2)^2}\sqrt{2}} \\
&= \frac{3 + 6t^2}{\sqrt{9 + 36t^2 + 36t^4}\sqrt{2}} \\
&= \frac{3 + 6t^2}{\sqrt{(3 + 6t^2)^2}\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}.
\end{aligned}$$

(Notice $3 + 6t^2 > 0$ for all $t \in \mathbb{R}$.) That is, the angle between α' and β is a constant ($= \pi/4$). □

Exercise 1-3.2. (Cycloid)

A circular disk of radius 1 in the plane xy rolls without slipping along the x axis. The figure described by a point of the circumference of of the disk is called a ***cycloid*** *(Figure 1-7 in Mantredo P. do Carmo, Differential Geometry of Curves and Surfaces).*

(a) *Obtain a parametrized curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ the trace of which is the cycloid and determine its singular points.*

(b) *Compute the arc length of the cycloid corresponding to a complete rotation of the disk.*

Proof of (a).

(1) Since

$$\begin{cases} x = t - \sin t \\ y = 1 - \cos t, \end{cases}$$

we define $\alpha(t) = (t - \sin t, 1 - \cos t)$.

(2) $\alpha'(t) = (1 - \cos t, \sin t)$. $\alpha'(t) = 0$ if and only if $t = 2n\pi$ where $n \in \mathbb{Z}$. That is, all singular points are $\alpha(2n\pi) = (2n\pi, 0)$ where $n \in \mathbb{Z}$.

□

Proof of (b). The arc length of the cycloid corresponding to a complete rotation of the disk is

$$\begin{aligned} \int_0^{2\pi} |\alpha'(t)| dt &= \int_0^{2\pi} \sqrt{(1-\cos t)^2 + (\sin t)^2} dt \\ &= \int_0^{2\pi} \sqrt{2}\sqrt{1 - \cos t} dt \\ &= \int_0^{2\pi} 2 \sin \frac{t}{2} dt \\ &= \left[-4 \cos \frac{t}{2} \right]_{t=0}^{t=2\pi} \\ &= 8. \end{aligned}$$

□

Supplement. The cycloid is not an algebraic curve.

Exercise 1-3.3. (Cissoid of Diocles)

Let $0A = 2a$ be the diameter of a circle $\mathbb{S}^1$ and $0Y$ and AV be the tangents to $\mathbb{S}^1$ at 0 and A, respectively. A half-line r is drawn from 0 which meets the circle $\mathbb{S}^1$ at C and the line AV at B. On $0B$ mark off the segment $0p = CB$. If we rotate r about 0, the point p will describe a curve called the ***cissoid of Diocles****. By taking $0A$ as the x axis and $0Y$ as the y axis, prove that*

(a) *The tract of*

$$\alpha(t) = \left(\frac{2at^2}{1+t^2}, \frac{2at^3}{1+t^2}\right), \qquad t \in \mathbb{R},$$

is the cissoid of Diocles ($t = \tan\theta$; see Figure 1-8 in Mantredo P. do Carmo, Differential Geometry of Curves and Surfaces).

(b) *The origin $(0,0)$ is a singular point of the cissoid.*

(c) *As $t \to \infty$, $\alpha(t)$ approaches the line $x = 2a$, and $\alpha'(t) \to (0, 2a)$. Thus, as $t \to \infty$, the curve and its tangent approach the line $x = 2a$; we say that $x = 2a$ is an* ***asymptote*** *to the cissoid.*

Proof of (a).

(1) The polar equations of the circle $\mathbb{S}^1$ and the half-line r is

$$r = 2a\cos\theta,$$
$$r = 2a\sec\theta,$$

respectively.

(2) By construction, the polar equation of the cissoid is

$$r = 2a\sec\theta - 2a\cos\theta = 2a\frac{\sin^2\theta}{\cos\theta} = 2a\sin\theta\tan\theta.$$

(3) Put $t = \tan\theta$, we have

$$x = r\cos\theta = 2a\sin^2\theta = \frac{2at^2}{1+t^2},$$
$$y = r\sin\theta = tx = \frac{2at^3}{1+t^2}.$$

So

$$\alpha(t) = (x, y) = \left(\frac{2at^2}{1+t^2}, \frac{2at^3}{1+t^2}\right).$$

□

Supplement. The cissoid is an algebraic curve $= V((x^2+y^2)x = 2ay^2)$.

Proof of (b). Note that $\alpha(0) = (0, 0)$ and

$$\alpha'(t) = \left(\frac{4at}{(t^2+1)^2}, \frac{2at^2(t^2+3)}{(t^2+1)^2} \right).$$

Hence $\alpha'(0) = (0, 0)$. That is, $(0, 0)$ is a singular point of the cissoid. (In fact, the origin is the unique singular point of the cissoid.) □

Proof of (c).

(1) Note that

$$\lim_{t \to \pm\infty} x(t) = \lim_{t \to \pm\infty} \frac{2at^2}{1+t^2} = 2a,$$
$$\lim_{t \to \pm\infty} y(t) = \lim_{t \to \pm\infty} \frac{2at^3}{1+t^2} = \pm\infty.$$

Hence, $\alpha(t)$ approaches the line $x = 2a$ as $t \to \pm\infty$.

(2) Similarly,

$$\lim_{t \to \pm\infty} x'(t) = \lim_{t \to \pm\infty} \frac{4at}{(t^2+1)^2} = 0,$$
$$\lim_{t \to \pm\infty} y'(t) = \lim_{t \to \pm\infty} \frac{2at^2(t^2+3)}{(t^2+1)^2} = 2a.$$

Therefore, $\alpha'(t) \to (0, 2a)$ as $t \to \pm\infty$.

(3) By (1)(2), the curve and its tangent approach the line $x = 2a$ as $t \to \pm\infty$, or $x = 2a$ is an asymptote to the cissoid.

□

Exercise 1-3.4. (Tractrix)

Let $\alpha : (0, \pi) \to \mathbb{R}^2$ be given by

$$\alpha(t) = \left(\sin t, \cos t + \log \tan \frac{t}{2} \right),$$

where t is the angle that the y axis makes with the vector $\alpha(t)$. The trace of α is called the ***tractrix****. (Figure 1-9 in Mantredo P. do Carmo, Differential Geometry of Curves and Surfaces). Show that*

(a) *α is a differentiable parametrized curve, regular except at $t = \frac{\pi}{2}$.*

(b) *The length of the segment of the tangent of the tractrix between the point of tangency and the y axis is constantly equal to 1.*

Proof of (a).

$$\begin{aligned}
\alpha'(t) &= \left(\cos t, -\sin t + \frac{1}{\tan\frac{t}{2}}\frac{1}{\cos^2\frac{t}{2}}\frac{1}{2}\right)\\
&= \left(\cos t, -\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right)\\
&= \left(\cos t, \frac{\cos^2 t}{\sin t}\right)
\end{aligned}$$

exists. And $\alpha'(t) = 0$ if and only if $t = \frac{\pi}{2}$. That is, there is an unique singular point at $t = \frac{\pi}{2}$. □

Proof of (b). The the tangent line of the tractrix through the regular point t is parametrized by $\beta : \mathbb{R} \to \mathbb{R}^2$ which is defined by

$$\begin{aligned}
\beta(u) &= u\alpha'(t) + \alpha(t)\\
&= \left(u\cos t + \sin t, u\frac{\cos^2 t}{\sin t} + \cos t + \log\tan\frac{t}{2}\right).
\end{aligned}$$

By construction, this tangent line $\beta(u)$ meets the tractrix at $u = 0$, and meets the y-axis when $u\cos t + \sin t = 0$ or $u = -\tan t$. So the length of the segment is

$$\begin{aligned}
|\beta(0) - \beta(-\tan t)| &= \sqrt{(-\tan t\cos t)^2 + \left(-\tan t\frac{\cos^2 t}{\sin t}\right)^2}\\
&= \sqrt{(\sin t)^2 + (\cos t)^2}\\
&= 1.
\end{aligned}$$

□

Exercise 1-3.5. (Folium of Descartes)

Let $\alpha : (-1, +\infty) \to \mathbb{R}^2$ be given by

$$\alpha(t) = \left(\frac{3at}{1+t^3}, \frac{3at^2}{1+t^3}\right).$$

Prove that:

(a) *For $t = 0$, α is tangent to the x axis.*

(b) *As $t \to +\infty$, $\alpha(t) \to (0,0)$ and $\alpha'(t) = (0,0)$.*

(c) *Take the curve the the opposite orientation. Now, as $t \to -1$, the curve and its tangent approach the line $x + y + a = 0$.*

The figure obtained by completing the trace of α in such a way that it becomes symmetric relative the the line $y = x$ is called the ***folium of Descartes*** *(See Figure 1-10 in Mantredo P. do Carmo, Differential Geometry of Curves and Surfaces).*

Proof of (a). Note that

$$\alpha'(t) = \left(\frac{3a(1-2t^3)}{(1+t^3)^2}, \frac{3at(2-t^3)}{(1+t^3)^2} \right).$$

Hence, $\alpha'(0) = (3a, 0)$, or α is tangent to the x axis when $t = 0$. $\square$

Proof of (b).

(1)

$$\begin{aligned}
\lim_{t\to+\infty} \alpha(t) &= \lim_{t\to+\infty} \left(\frac{3at}{1+t^3}, \frac{3at^2}{1+t^3} \right) \\
&= \left(\lim_{t\to+\infty} \frac{3at}{1+t^3}, \lim_{t\to+\infty} \frac{3at^2}{1+t^3} \right) \\
&= (0, 0).
\end{aligned}$$

(2)

$$\begin{aligned}
\lim_{t\to+\infty} \alpha'(t) &= \lim_{t\to+\infty} \left(\frac{3a(1-2t^3)}{(1+t^3)^2}, \frac{3at(2-t^3)}{(1+t^3)^2} \right) \\
&= \left(\lim_{t\to+\infty} \frac{3a(1-2t^3)}{(1+t^3)^2}, \lim_{t\to+\infty} \frac{3at(2-t^3)}{(1+t^3)^2} \right) \\
&= (0, 0).
\end{aligned}$$

$\square$

Proof of (c).

(1) Note that

$$\begin{aligned}
\lim_{t\to-1^+} \alpha(t) &= \lim_{t\to-1^+} \left(\frac{3at}{1+t^3}, \frac{3at^2}{1+t^3} \right) \\
&= \left(\lim_{t\to-1^+} \frac{3at}{1+t^3}, \lim_{t\to-1^+} \frac{3at^2}{1+t^3} \right) \\
&= (-\infty, +\infty)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{t\to-1^+} (x(t) + y(t)) &= \lim_{t\to-1^+} \left(\frac{3at}{1+t^3} + \frac{3at^2}{1+t^3} \right) \\
&= \lim_{t\to-1^+} \frac{3at}{1-t+t^2} \\
&= -a.
\end{aligned}$$

Therefore, as $t \to -1$, the curve approaches the line $x + y + a = 0$.

(2) Note that

$$\begin{aligned}
\lim_{t \to -1^+} \frac{y'(t)}{x'(t)} &= \lim_{t \to -1^+} \frac{\frac{3a(1-2t^3)}{(1+t^3)^2}}{\frac{3at(2-t^3)}{(1+t^3)^2}} \\
&= \lim_{t \to -1^+} \frac{1-2t^3}{t(2-t^3)} \\
&= -1.
\end{aligned}$$

Hence, as $t \to -1$, its tangent also approaches the line $x + y + a = 0$.

□

Exercise 1-3.6. (Logarithmic spiral)

Let $\alpha(t) = (ae^{bt}\cos t, ae^{bt}\sin t)$, $t \in \mathbb{R}$, a and b constants, $a > 0$, $b < 0$, be a parametrized curve.

(a) *Show that as $t \to +\infty$, $\alpha(t)$ approaches the origin 0, spiraling around it (because of this, the trace of α is called the **logarithmic spiral**; See Figure 1-11 in Mantredo P. do Carmo, Differential Geometry of Curves and Surfaces).*

(b) *Show that $\alpha'(t) \to (0,0)$ as $t \to +\infty$ and that*

$$\lim_{t \to +\infty} \int_{t_0}^{t} |\alpha'(t)| dt$$

is finite; that is, α has finite arc length in $[t_0, \infty)$.

Proof of (a).

(1) Note that

$$\lim_{t \to +\infty} x(t) = \lim_{t \to +\infty} \frac{\overbrace{a\cos t}^{\text{bounded}}}{\underbrace{e^{-bt}}_{\to +\infty}} = 0$$

and $\lim_{t \to +\infty} y(t) = 0$ (by the similar argument). Hence $\alpha(t)$ approaches the origin 0 as $t \to +\infty$.

(2) $\alpha(t) = (ae^{bt}\cos t, ae^{bt}\sin t)$ is moving in counter-clockwise on a circle path and sweeping out a length ae^{bt} as t is moving from t_0 to $+\infty$. Note that $t \mapsto ae^{bt}$ is decreasing strictly (as t is moving from t_0 to $+\infty$). Hence α spiraling around the origin.

$\square$

Proof of (b).

(1) Note that

$$\alpha'(t) = (ae^{bt}(\underbrace{b\cos t - \sin t}_{\text{bounded}}), ae^{bt}(\underbrace{b\sin t + \cos t}_{\text{bounded}})).$$

As $t \to +\infty$, $\alpha'(t) \to (0,0)$.

(2) As

$$\begin{aligned}\int_{t_0}^{+\infty} |\alpha'(t)|dt &= \int_{t_0}^{+\infty} ae^{bt}\sqrt{b^2+1}dt \\ &= \left[\frac{a}{b}e^{bt}\sqrt{b^2+1}\right]_{t=t_0}^{t=+\infty} \\ &= -\frac{a}{b}e^{bt_0}\sqrt{b^2+1} \\ &< +\infty,\end{aligned}$$

α has finite arc length in $[t_0, \infty)$.

$\square$

Exercise 1-3.7.

A map $\alpha : I \to \mathbb{R}^3$ is called ***a curve of class*** $\mathcal{C}^k$ *if each of the coordinate functions in the expression $\alpha(t) = (x(t), y(t), z(t))$ has continuous derivatives up to order k. If α is merely continuous, we say that α is of class $\mathcal{C}^0$. A curve α is called* ***simple*** *is the map α is one-to-one. Thus, the curve $\alpha(t) = (t^3-4t, t^2-4)$ ($t \in \mathbb{R}$) is not simple.*

Let $\alpha : I \to \mathbb{R}^3$ be a simple curve of class $\mathcal{C}^0$. We say that α has a ***weak tangent*** *at $t = t_0 \in I$ if the line determined by $\alpha(t_0+h)$ and $\alpha(t_0)$ has a limit position when $h \to 0$. We say that α has a* ***strong tangent*** *at $t = t_0 \in I$ if the line determined by $\alpha(t_0+h)$ and $\alpha(t_0+k)$ has a limit position when $h, k \to 0$. Show that*

(a) *$\alpha(t) = (t^3, t^2)$, $t \in \mathbb{R}$, has a weak tangent but not a strong tangent at $t = 0$.*

(b) *If $\alpha : I \to \mathbb{R}^3$ is of class $\mathcal{C}^1$ and regular at $t = t_0$, then it has a strong tangent at $t = t_0$.*

(c) *The curve given by*

$$\alpha(t) = \begin{cases} (t^2, t^2), & t \geq 0, \\ (t^2, -t^2), & t \leq 0, \end{cases}$$

is of class $\mathcal{C}^1$ but not of class $\mathcal{C}^2$. Draw a sketch of the curve and its tangent vectors.

Proof of (a).

(1) Note that $\alpha(0) = (0, 0)$ and $\alpha(h) = (h^3, h^2)$. The line passing $\alpha(0)$ and $\alpha(h)$ is

$$\begin{aligned} &(x - 0)(h^2 - 0) - (y - 0)(h^3 - 0) = 0 \\ \Longleftrightarrow\ & x - hy = 0. \end{aligned}$$

As $h \to 0$, the line has a limit position $x = 0$. Therefore, $\alpha(t)$ has a weak tangent.

(2) The line passing $\alpha(h)$ and $\alpha(k)$ is

$$\begin{aligned} &(x - k^2)(h^2 - k^2) - (y - k^3)(h^3 - k^3) = 0 \\ \Longleftrightarrow\ & (x - k^2)(h + k) - (y - k^3)(h^2 + hk + k^2) = 0. \end{aligned}$$

As $h \to 0$, the line has a limit position

$$\begin{aligned} &(x - k^2) - (y - k^3)k = 0 \\ \Longleftrightarrow\ & x - ky + k^4 - k^2 = 0. \end{aligned}$$

As $k \to 0$, the line has a limit position $x = 0$.

(3) On the other hand, as $h = -k$ we have $y - k^3 = 0$. As $k \to 0$, the line has a limit position $y = 0$, contrary to (2). Therefore, $\alpha(t)$ has a strong tangent.

□

Proof of (b).

(1) The line L passing $\alpha(t_0 + h)$ and $\alpha(t_0 + k)$ is

$$\begin{aligned} x(s) &= x(t_0) + \frac{x(t_0 + h) - x(t_0 + k)}{h - k} s, \\ y(s) &= y(t_0) + \frac{y(t_0 + h) - y(t_0 + k)}{h - k} s, \\ z(s) &= z(t_0) + \frac{z(t_0 + h) - z(t_0 + k)}{h - k} s. \end{aligned}$$

(2) By the mean value theorem,

$$\frac{x(t_0 + h) - x(t_0 + k)}{h - k} = x'(t_0 + \xi)$$

for some ξ between h and k. Since $\alpha \in \mathcal{C}^1$, $x(t) \in \mathcal{C}^1$. Hence

$$\begin{aligned}\lim_{h,k\to 0} \frac{x(t_0+h) - x(t_0+k)}{h-k} &= \lim_{h,k\to 0} x'(t_0+\xi) \\ &= \lim_{\xi\to 0} x'(t_0+\xi) \\ &= x'(t_0).\end{aligned}$$

Similarly, we have $\lim_{h,k\to 0} \frac{y(t_0+h)-y(t_0+k)}{h-k} = y'(t_0)$ and $\lim_{h,k\to 0} \frac{z(t_0+h)-z(t_0+k)}{h-k} = z'(t_0)$. Since α is regular, $\lim_{h,k\to 0} L$ is a non degenerate line

$$\begin{aligned}x(s) &= x(t_0) + x'(t_0)s, \\ y(s) &= y(t_0) + y'(t_0)s, \\ z(s) &= z(t_0) + z'(t_0)s\end{aligned}$$

and thus $\lim_{h,k\to 0} L$ is a strong tangent at $t = t_0$.

□

Proof of (c).

(1) Since

$$\alpha'(t) = \begin{cases} (2t, 2t), & t \geq 0, \\ (2t, -2t), & t \leq 0, \end{cases}$$

α is of class $\mathcal{C}^1$.

(2) Since

$$\alpha''(t) = \begin{cases} (2, 2), & t > 0, \\ \text{undefined}, & t = 0 \\ (2, -2), & t < 0, \end{cases}$$

α is not of class $\mathcal{C}^2$.

(Skip drawing a sketch of the curve and its tangent vectors.) □

Exercise 1-3.8.

Let $\alpha : I \to \mathbb{R}^3$ be a differentiable curve and let $[a, b] \subseteq I$ be a closed interval. For every partition

$$a = t_0 < t_1 < \cdots < t_n = b$$

of $[a, b]$, consider the sum

$$\sum_{i=1}^{n} |\alpha(t_i) - \alpha(t_{i-1})| = l(\alpha, P),$$

where P stands for the given partition. The norm $|P|$ of a partition P is defined as

$$|P| = \max(t_i - t_{i-1}), i = 1, \cdots, n.$$

Geometrically, $l(\alpha, P)$ is the length of a polygon inscribed in $\alpha([a, b])$ with vertices in $\alpha(t_i)$ (see Figure 1-3 in Mantredo P. do Carmo, Differential Geometry of Curves and Surfaces). The point of the exercise is to show that the arc length of $\alpha([a, b])$ is, in some sense, a limit of lengths of inscribed polygons. Prove that given $\varepsilon > 0$ there exists $\delta > 0$ such that if $|P| < \delta$ then

$$\left| \int_a^b |\alpha'(t)| dt - l(\alpha, P) \right| < \varepsilon.$$

Assume that $\alpha'(t)$ is continuous.

Proof. Given $\varepsilon > 0$.

(1) Since $\alpha'(t)$ is continuous on a compact set $[a, b]$, $\alpha'(t)$ is uniformly continuous, that is, there there exists $\delta > 0$ such that

$$|\alpha'(s) - \alpha'(t)| < \frac{\varepsilon}{2(b-a)} \text{ whenever } |s - t| < \delta.$$

(2) Let $P = \{a = t_0, t_1, \ldots, t_n = b\}$ be a partition of $[a, b]$, with $\Delta t_i = t_i - t_{i-1} < \delta$ for all $i = 1, \ldots, n$. If $t_{i-1} \leq t \leq t_i$, it follows that

$$|\alpha'(t_i)| - \frac{\varepsilon}{2(b-a)} \leq |\alpha'(t)| \leq |\alpha'(t_i)| + \frac{\varepsilon}{2(b-a)}.$$

Hence,

$$\begin{aligned}
&\int_{t_{i-1}}^{t_i} |\alpha'(t)| dt \\
&\geq |\alpha'(t_i)| \Delta t_i - \frac{\varepsilon}{2(b-a)} \Delta t_i \\
&= \left| \int_{t_{i-1}}^{t_i} [\alpha'(t) + \alpha'(t_i) - \alpha'(t)] dt \right| - \frac{\varepsilon}{2(b-a)} \Delta t_i \\
&\geq \left| \int_{t_{i-1}}^{t_i} \alpha'(t) dt \right| - \left| \int_{t_{i-1}}^{t_i} [\alpha'(t_i) - \alpha'(t)] dt \right| - \frac{\varepsilon}{2(b-a)} \Delta t_i \\
&\geq |\alpha(t_i) - \alpha(t_{i-1})| - \frac{\varepsilon}{b-a} \Delta t_i
\end{aligned}$$

and

$$
\begin{aligned}
&\int_{t_{i-1}}^{t_i} |\alpha'(t)|dt \\
&\leq |\alpha'(t_i)|\Delta t_i + \frac{\varepsilon}{2(b-a)}\Delta t_i \\
&= \left| \int_{t_{i-1}}^{t_i} [\alpha'(t) + \alpha'(t_i) - \alpha'(t)]dt \right| + \frac{\varepsilon}{2(b-a)}\Delta t_i \\
&\leq \left| \int_{t_{i-1}}^{t_i} \alpha'(t)dt \right| + \left| \int_{t_{i-1}}^{t_i} [\alpha'(t_i) - \alpha'(t)]dt \right| + \frac{\varepsilon}{2(b-a)}\Delta t_i \\
&\leq |\alpha(t_i) - \alpha(t_{i-1})| + \frac{\varepsilon}{b-a}\Delta t_i.
\end{aligned}
$$

(3) If we add these inequalities, we obtain

$$
l(\alpha, P) - \varepsilon \leq \int_a^b |\alpha'(t)|dt \leq l(\alpha, P) + \varepsilon.
$$

□

Exercise 1-3.9.

(a) *Let $\alpha : I \to \mathbb{R}^3$ be a curve of class $\mathcal{C}^0$ (compare Exercise 1-3.7). Use the approximation by polygons described in Exercise 1-3.8 to give a reasonable defintion of arc length of α.*

(b) *(A Nonrectifiable Curve.) The following example shows that, with any reasonable definition, the arc length of a $\mathcal{C}^0$ curve in a closed interval may be unbounded. Let $\alpha : [0,1] \to \mathbb{R}^2$ be given as $\alpha(t) = (t, t\sin(\frac{\pi}{t}))$ if $t \neq 0$, and $\alpha(0) = (0,0)$. Show, geometrically, that the arc length of the portion of the curve corresponding to $\frac{1}{n+1} \leq t \leq \frac{1}{n}$ is at least $\frac{2}{n+\frac{1}{2}}$. Use this to show that the length of curve in the interval $\frac{1}{N} \leq t \leq 1$ is greater than $2\sum_{n=1}^{N-1} \frac{1}{n+1}$, and thus it tends to infinity as $N \to \infty$.*

Proof of (a). Define

$$
l(\alpha) = \sup\{l(\alpha, P) : P \text{ is a partition of } [a,b]\}.
$$

□

Note. (Theorem 6.17 in *Tom. M. Apostol, Mathematical Analysis, 2nd edition.*). α is rectifiable if and only α is of bounded variation on $[a,b]$.

Proof of (b).

(1) Consider a partition $P = \left\{\frac{1}{n+1}, \frac{1}{n+\frac{1}{2}}, \frac{1}{n}\right\}$ of $\left[\frac{1}{n+1}, \frac{1}{n}\right]$. So that $\alpha(\frac{1}{n+1}) = \alpha(\frac{1}{n}) = 0$ and $\alpha(\frac{1}{n+\frac{1}{2}}) = \pm 1$.

(2) Thus,

$$\begin{aligned}
&\text{The arc length of the portion of } \alpha \text{ over } \left[\frac{1}{n+1}, \frac{1}{n}\right] \\
\geq &\text{ The sum of each length of the individual chords} \\
= &\sqrt{\left(\frac{1}{n+\frac{1}{2}} - \frac{1}{n+1}\right)^2 + \left(\frac{1}{n+\frac{1}{2}}\right)^2} \\
&+ \sqrt{\left(\frac{1}{n} - \frac{1}{n+\frac{1}{2}}\right)^2 + \left(\frac{1}{n+\frac{1}{2}}\right)^2} \\
\geq &\frac{2}{n+\frac{1}{2}}.
\end{aligned}$$

(3) So

$$\begin{aligned}
&\text{The arc length of } \alpha \text{ over } \left[\frac{1}{N}, 1\right] \\
= &\sum_{n=1}^{N-1} \left\{\text{The arc length of } \alpha \text{ over } \left[\frac{1}{n+1}, \frac{1}{n}\right]\right\} \\
\geq &\sum_{n=1}^{N-1} \frac{2}{n+\frac{1}{2}} \\
> &2\sum_{n=1}^{N-1} \frac{1}{n+1}.
\end{aligned}$$

It tends to infinity as $N \to \infty$, or α is nonrectifiable.

□

Exercise 1-3.10. (Straight Lines as Shortest)

Let $\alpha : I \to \mathbb{R}^3$ *be a parametrized curve. Let* $[a, b] \subseteq I$ *and set* $\alpha(a) = p$, $\alpha(b) = q$.

(a) *Show that, for any constant vector* v, $|v| = 1$,

$$(q - p) \cdot v = \int_a^b \alpha'(t) \cdot v dt \leq \int_a^b |\alpha'(t)| dt.$$

(b) *Set*

$$v = \frac{q-p}{|q-p|}$$

and show that

$$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)| dt;$$

that is, the curve of shortest length from $\alpha(a)$ to $\alpha(b)$ is the straight line joining these points.

Assume $p \neq q$ (otherwise $v = \frac{q-p}{|q-p|}$ is meaningless).

Proof of (a). Let $f(t) = \alpha(t) \cdot v$ defined on I. By the fundamental theorem of calculus,

$$\int_a^b f'(t) dt = f(b) - f(a).$$

Since $f'(t) = \alpha'(t) \cdot v$,

$$(\alpha(b) - \alpha(a)) \cdot v = \int_a^b \alpha'(t) \cdot v dt.$$

Therefore,

$$\begin{aligned}(q-p) \cdot v &= \int_a^b \alpha'(t) \cdot v dt \\ &\leq \int_a^b |\alpha'(t) \cdot v| dt \\ &\leq \int_a^b |\alpha'(t)||v| dt \\ &= \int_a^b |\alpha'(t)| dt.\end{aligned}$$

□

Proof of (b). $|v| = \frac{|q-p|}{|q-p|} = 1$. So,

$$\begin{aligned}(q-p) \cdot \frac{q-p}{|q-p|} &\leq \int_a^b |\alpha'(t)| dt, \\ |q-p| &\leq \int_a^b |\alpha'(t)| dt.\end{aligned}$$

□

1-4. The Vector Product in $\mathbb{R}^3$

Exercise 1-4.1.

Check whether the following bases are positive:

(a) *The basis* $\{(1,3),(4,2)\}$ *in* $\mathbb{R}^2$.

(b) *The basis* $\{(1,3,5),(2,3,7),(4,8,3)\}$ *in* $\mathbb{R}^3$.

Proof of (a). Write $u=(1,3)$ and $v=(4,2)$. Then

$$\det(u,v)=\begin{vmatrix}1 & 3\\ 4 & 2\end{vmatrix}=-10<0.$$

Thus $\{u,v\}$ is negative w.r.t. the natural order basis $\{e_1=(1,0),e_2=(0,1)\}$. $\square$

Proof of (b). Write $u=(1,3,5)$, $v=(2,3,7)$, $w=(4,8,3)$. Then

$$\det(u,v,w)=\begin{vmatrix}1 & 3 & 5\\ 2 & 3 & 7\\ 4 & 8 & 3\end{vmatrix}=39>0.$$

Thus $\{u,v,w\}$ is positive w.r.t. the natural order basis $\{e_1,e_2,e_3\}$. $\square$

Exercise 1-4.2.

A plane P contained in $\mathbb{R}^3$ is given by the equation $ax+by+cz+d=0$. Show that the vector $v=(a,b,c)$ is perpendicular to the plane and that $|d|/\sqrt{a^2+b^2+c^2}$ measures the distance from the plane to the origin $(0,0,0)$.

Say v is a normal vector of E.

In general, the distance from the plane E to any point $(x_0,y_0,z_0)\in\mathbb{R}^3$ is

$$\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}.$$

Proof.

(1) To show $v=(a,b,c)$ is perpendicular to the plane, it suffices to show that $v\cdot u=0$ for any vector u lying on the plane E. Write $u=\overrightarrow{PQ}$ where $P=(x_1,y_1,z_1)\in E$ and $Q=(x_2,y_2,z_2)\in E$. Hence $u=(x_2-x_1,y_2-$

$y_1, z_2 - z_1)$.

$$\begin{aligned} v \cdot u &= (a, b, c) \cdot (x_2 - x_1, y_2 - y_1, z_2 - z_1) \\ &= a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) \\ &= (ax_2 + by_2 + cz_2) - (ax_1 + by_1 + cz_1) \\ &= (-d) - (-d) \\ &= 0. \end{aligned}$$

(2) Pick any point $(x_1, y_1, z_1) \in E$. The distance from the plane E to the point (x_0, y_0, z_0) is

$$\begin{aligned} &\left| (x_1 - x_0, y_1 - y_0, z_1 - z_0) \cdot \frac{v}{|v|} \right| \\ =& \left| (x_1 - x_0, y_1 - y_0, z_1 - z_0) \cdot \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}} \right| \\ =& \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ =& \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ =& \frac{|-d - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ =& \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}. \end{aligned}$$

□

Exercise 1-4.3.

Determine the angle of intersection of the two planes $5x + 3y + 2z - 4 = 0$ *and* $3x + 4y - 7z = 0$.

Proof.

(1) The angle of intersection of the two planes is equal to a angle between two normal vectors of planes.

(2) Let

  (a) the angle of intersection of the two planes be θ.
  (b) the normal vector of $5x + 3y + 2z - 4 = 0$ be $n_1 = (5, 3, 2)$.
  (c) the normal vector of $3x + 4y - 7z = 0$ be $n_2 = (3, 4, -7)$.

(3) Hence,
$$\cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{13}{2\sqrt{703}}.$$
$\theta = \cos^{-1}\left(\frac{13}{2\sqrt{703}}\right)$.

□

Exercise 1-4.7.

Prove that a necessary and sufficient condition for the plane
$$ax + by + cz + d = 0$$
and the line
$$x - x_0 = u_1 t, \qquad y - y_0 = u_2 t, \qquad z - z_0 = u_3 t$$
to be parallel is
$$au_1 + bu_2 + cu_3 = 0.$$
Proof. Write
$$\begin{aligned} E &: ax + by + cz + d = 0 \\ L &: x - x_0 = u_1 t, \qquad y - y_0 = u_2 t, \qquad z - z_0 = u_3 t. \end{aligned}$$
By Exercise 1-4.2, the vector (a, b, c) is perpendicular to the plane E. Hence,
$$\begin{aligned} E \text{ is parallel to } L &\iff (a, b, c) \text{ is perpendicular to } L \\ &\iff (a, b, c) \text{ is perpendicular to } u = (u_1, u_2, u_3) \\ &\iff 0 = (a, b, c) \cdot (u_1, u_2, u_3) = au_1 + bu_2 + cu_3. \end{aligned}$$
□

Exercise 1-4.8.

Prove that the distance ρ between the nonparallel lines
$$\begin{aligned} x - x_0 = u_1 t, \qquad y - y_0 = u_2 t, \qquad z - z_0 = u_3 t, \\ x - x_0 = v_1 t, \qquad y - y_0 = v_2 t, \qquad z - z_0 = v_3 t \end{aligned}$$
is given by
$$\rho = \frac{|(u \wedge v) \cdot r|}{|u \wedge v|}$$

where $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$, $r = (x_0 - x_1, y_0 - y_1, z_0 - z_1)$.

Proof.

$$\rho = |r||\cos\angle(u \wedge v, r)| = |r|\frac{|(u \wedge v) \cdot r|}{|u \wedge v||r|} = \frac{|(u \wedge v) \cdot r|}{|u \wedge v|}.$$

It is well-defined ($|u \wedge v| > 0$) since two lines are nonparallel. □

Exercise 1-4.11.

(a) *Show that the volume V of a parallelepiped generated by three linearly independent vectors $u, v, w \in \mathbb{R}^3$ is given by $V = |(u \wedge v) \cdot w|$, and introduce an* ***oriented volume*** *in $\mathbb{R}^3$.*

(b) *Prove that*

$$V^2 = \begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix}.$$

Proof of (a).

(1) We can calculate the volume V by multiplying the area of the base $|u \wedge v|$ and the height h.

(2) Note that

$$h = |w||\cos\angle(u \wedge v, w)| = |w|\frac{|(u \wedge v) \cdot w|}{|u \wedge v||w|} = \frac{|(u \wedge v) \cdot w|}{|u \wedge v|}.$$

It is well-defined since u and v are linearly independent. Therefore,

$$V = |u \wedge v|h = |(u \wedge v) \cdot w|.$$

(3) The oriented volume is defined by

$$(u \wedge v) \cdot w.$$

□

Proof of (b). Recall $(u \wedge v) \cdot w = \det(u, v, w)$. Also note that $\det\big((u, v, w)^T\big) = \det(u, v, w)$. So

$$\begin{aligned} \begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix} &= \det\big((u, v, w) \cdot (u, v, w)^T\big) \\ &= \det(u, v, w)\det\big((u, v, w)^T\big) \\ &= \det(u, v, w)^2 \\ &= ((u \wedge v) \cdot w)^2 \\ &= V^2. \end{aligned}$$

□

Exercise 1-4.13.

Let $u(t) = (u_1(t), u_2(t), u_3(t))$ and $v(t) = (v_1(t), v_2(t), v_3(t))$ be differentiable maps from the interval (a, b) into $\mathbb{R}^3$. If the derivatives $u'(t)$ and $v'(t)$ satisfy the conditions

$$u'(t) = au(t) + bv(t), \qquad v'(t) = cu(t) - av(t),$$

where a, b, and c are constants, show that $u(t) \wedge v(t)$ is a constant vector.

Proof. Since

$$\begin{aligned}
\frac{d}{dt}(u(t) \wedge v(t)) &= u'(t) \wedge v(t) + u(t) \wedge v'(t) \\
&= (au(t) + bv(t)) \wedge v(t) + u(t) \wedge (cu(t) - av(t)) \\
&= au(t) \wedge v(t) + u(t) \wedge (-av(t)) \\
&= a(u(t) \wedge v(t)) + (-a)(u(t) \wedge v(t)) \\
&= (0, 0, 0),
\end{aligned}$$

$u(t) \wedge v(t)$ is a constant vector. □

1-5. The Local Theory of Curves Parametrized by Arc Length

Exercise 1-5.2.

Show that the torsion τ of α is given by

$$\tau(s) = -\frac{\alpha'(s) \wedge \alpha''(s) \cdot \alpha'''(s)}{|\kappa(s)|^2}.$$

Proof.

(1) Take inner product $n(s)$ to the definition of torsion $\tau(s)n(s) = b'(s)$, we have

$$\tau(s) = b'(s) \cdot n(s).$$

Since $b'(s) = t(s) \wedge n'(s)$, we have to compute $n'(s)$ first.

(2) Compute $n'(s)$.

$$n'(s) = \frac{d}{ds}\left(\frac{\alpha''(s)}{\kappa(s)}\right) = \frac{\alpha'''(s)}{\kappa(s)} - \frac{\alpha''(s)\kappa'(s)}{\kappa(s)^2}.$$

(3) By (1)(2),

$$\begin{aligned}
\tau(s) &= b'(s) \cdot n(s) \\
&= (t(s) \wedge n'(s)) \cdot n(s) \\
&= \left(\alpha'(s) \wedge \left(\frac{\alpha'''(s)}{\kappa(s)} - \frac{\alpha''(s)\kappa'(s)}{\kappa(s)^2}\right)\right) \cdot \frac{\alpha''(s)}{\kappa(s)} \\
&= \left(\alpha'(s) \wedge \frac{\alpha'''(s)}{\kappa(s)}\right) \cdot \frac{\alpha''(s)}{\kappa(s)} \\
&= \frac{\alpha'(s) \wedge \alpha'''(s) \cdot \alpha''(s)}{|\kappa(s)|^2},
\end{aligned}$$

or

$$\tau(s) = \frac{\alpha'(s) \wedge \alpha'''(s) \cdot \alpha''(s)}{\alpha''(s)^2}.$$

□

1-6. The Local Canonical Form

1-7. Global Properties of Plane Curves