# Chapter 3: Elements of Point Set Topology

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## Compact subsets of a metric space

*Prove each of the following statements concerning an arbitrary metric space $(M, d)$ and subsets $S$, $T$ of $M$.*

**Exercise 3.39.** *If $S$ is closed and $T$ is compact, then $S \cap T$ is compact.*

*Proof (On topological spaces).* Let $\mathscr{F}$ be an open covering of $S \cap T$, say $S \cap T \subseteq \bigcup_{A \in \mathscr{F}} A$. We will show that a finite number of the sets $A$ cover $S \cap T$. Since $S$ is closed its complement $\widetilde{S}$ in $M$ is open, so $\mathscr{F} \cup \{\widetilde{S}\}$ is an open covering of $T$. Since $T$ is compact, so this covering contains a finite subcovering which we can assume includes $\widetilde{S}$. Therefore,

$$T \subseteq A_1 \cup \cdots \cup A_p \cup \widetilde{S}.$$

This subcovering also covers $S \cap T$ and, since $\widetilde{S}$ contains no points of $S$, we can delete the set $\widetilde{S}$ for the subcovering and still covers $S \cap T$. Thus

$$S \cap T \subseteq A_1 \cup \cdots \cup A_p$$

so $S \cap T$ is compact. $\square$

*Proof (Theorem 3.39).*

$$\begin{aligned}
& T \text{ is compact in } (M, d) && \\
\Longrightarrow & T \text{ is compact in } (T, d) && \text{(Exercise 3.38)} \\
\Longrightarrow & S \cap T \text{ is compact in } (T, d) && (S \cap T\text{: closed in } (T, d)\text{, Theorem 3.38)} \\
\Longrightarrow & S \cap T \text{ is compact in } (M, d). && \text{(Exercise 3.38)}
\end{aligned}$$

$\square$

**Exercise 3.41.** *The union of a finite number of compact subsets of $M$ is compact.*

*Proof (On topological spaces).* Let $K_1, \ldots, K_n$ be compact subsets of $M$. Let $\mathscr{F}$ be an open covering of $K_1 \cup \cdots \cup K_n$, say

$$K_1 \cup \cdots \cup K_n \subseteq \bigcup_{A \in \mathscr{F}} A.$$

We will show that a finite number of the sets $A$ cover $K_1 \cup \cdots \cup K_n$. Clearly $\mathscr{F}$ is an open covering of every $K_i$. Since $K_i$ is compact, this covering contains a finite subcovering $\mathscr{F}_i$, say

$$K_i \subseteq A_{1(i)} \cup \cdots \cup A_{p(i)}.$$

Union all finite subcovering $\mathscr{F}_i$ to get a finite subcovering of $K_1 \cup \cdots \cup K_n$, say

$$K_1 \cup \cdots \cup K_n \subseteq \bigcup_{A \in \bigcup_{1 \leq i \leq n} \mathscr{F}_i} A.$$

□

**Supplement (Zariski topology).** *Let $A$ be a ring and let $X$ be the set of all prime ideals of $A$. For each subset $E$ of $A$, let $V(E)$ denote the set of all prime ideals of $A$ which contain $E$. The sets $V(E)$ satisfy the axioms for closed sets in a topological space. The resulting topology is called the Zariski topology. The topological space $X$ is called the prime spectrum of $A$, and is written Spec$(A)$.*

*For each $f \in A$, let $X_f$ denote the complement of $V(f)$ in $X =$ Spec$(A)$. The sets $X_f$ are open. Show that they form a basis of open sets for the Zariski topology, and that*

1. *Each $X_f$ is quasi-compact (compact), that is, every open covering of $X$ has a finite subcovering.*
2. *An open subset of $X$ is quasi-compact if and only if it is a finite union of sets $X_f$.*

By Exercise 3.41, we know that $X$ is quasi-compact if $X$ is a finite union of quasi-compact sets $X_f$.