Chapter 15: Bernoulli Numbers

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Supplement. Equation (4) on page 231. *Prove that*

$$x \cot x = 1 - 2 \sum_{n=1}^{\infty} \frac{x^2}{n^2\pi^2 - x^2}.$$

Proof (Exercise 6.73 in the book Graham, Knuth and Patashnik, Concrete Mathematics, Second Edition).

(1) *Show that*

$$\cot x = \frac{1}{2^n} \sum_{k=0}^{2^n-1} \cot \frac{x + k\pi}{2^n}$$

for all integers $n \geq 1$. Notice that

$$\begin{aligned} \cot(x+\pi) &= \cot x, \\ \cot\left(x + \frac{\pi}{2}\right) &= -\tan x, \\ \cot x &= \frac{1}{2}\left(\cot\frac{x}{2} - \tan\frac{x}{2}\right). \end{aligned}$$

Use mathematical induction. The case $n = 1$ is the same as the note. Assume the case $n = m$ holds. For $n = m + 1$,

$$\begin{aligned} \sum_{k=0}^{2^{m+1}-1} \cot \frac{x+k\pi}{2^{m+1}} &= \sum_{k=0}^{2^m-1} \cot \frac{x+k\pi}{2^{m+1}} + \sum_{k=2^m}^{2^{m+1}-1} \cot \frac{x+k\pi}{2^{m+1}} \\ &= \sum_{k=0}^{2^m-1} \cot \frac{x+k\pi}{2^{m+1}} + \sum_{k=0}^{2^m-1} \cot \frac{x+(2^m+k)\pi}{2^{m+1}} \\ &= \sum_{k=0}^{2^m-1} \cot \frac{x+k\pi}{2^{m+1}} + \sum_{k=0}^{2^m-1} \cot \left(\frac{x+k\pi}{2^{m+1}} + \frac{\pi}{2}\right) \\ &= \sum_{k=0}^{2^m-1} \left(\cot \frac{x+k\pi}{2^{m+1}} - \tan \frac{x+k\pi}{2^{m+1}}\right) \\ &= \sum_{k=0}^{2^m-1} \left(\cot \frac{x+k\pi}{2^{m+1}} - \tan \frac{x+k\pi}{2^{m+1}}\right) \\ &= 2 \sum_{k=0}^{2^m-1} \cot \frac{x+k\pi}{2^m}. \end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{2^{m+1}} \sum_{k=0}^{2^{m+1}-1} \cot \frac{x+k\pi}{2^{m+1}} &= \frac{1}{2^{m+1}} \cdot 2 \sum_{k=0}^{2^m-1} \cot \frac{x+k\pi}{2^m} \\
&= \frac{1}{2^m} \sum_{k=0}^{2^m-1} \cot \frac{x+k\pi}{2^m} \\
&= \cot x.
\end{aligned}$$

(2) By rearranging the index of summation of the identity in (1), we have

$$x \cot x = \frac{x}{2^n} \cot \frac{x}{2^n} - \frac{x}{2^n} \tan \frac{x}{2^n} + \sum_{k=1}^{2^{n-1}-1} \frac{x}{2^n} \left(\cot \frac{x+k\pi}{2^n} + \cot \frac{x-k\pi}{2^n} \right)$$

for all integers $n \geq 1$.

(3) Notice that $\lim_{x \to 0} x \cot x = 1$. Let $n \to \infty$, the result is established.

□

Exercise 15.1. *Using the definition of the Bernoulli number show* $B_{10} = \frac{5}{66}$ *and* $B_{12} = -\frac{691}{2730}$.

Proof.

(1) It is known that $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, and $B_m = 0$ for odd $m > 1$.

(2) Recall the implicit recurrence relation,

$$\sum_{k=0}^{m} \binom{m+1}{k} B_k = [m = 0],$$

where $[m = 0]$ is the Iverson brackets which is equal to the Kronecker delta δ_{m0}.

(3) So

$$\begin{aligned}
0 &= 1 + 9B_1 + 36B_2 + 84B_3 + 126B_4 + 126B_5 + 84B_6 + 36B_7 + 9B_8, \\
0 &= 1 + 9B_1 + 36B_2 + 126B_4 + 84B_6 + 9B_8, \\
0 &= 1 + 9\left(-\frac{1}{2}\right) + 36\left(\frac{1}{6}\right) + 126\left(-\frac{1}{30}\right) + 84\left(\frac{1}{42}\right) + 9B_8, \\
0 &= \frac{3}{10} + 9B_8,
\end{aligned}$$

Thus $B_8 = -\frac{1}{30}$.

(4) Again,

$$\begin{aligned}
0 =& 1 + 11B_1 + 55B_2 + 165B_3 + 330B_4 + 462B_5 + 462B_6 + \\
& 330B_7 + 165B_8 + 55B_9 + 11B_{10}, \\
0 =& 1 + 11B_1 + 55B_2 + 330B_4 + 462B_6 + 165B_8 + 11B_{10}, \\
0 =& 1 + 11\left(-\frac{1}{2}\right) + 55\left(\frac{1}{6}\right) + 330\left(-\frac{1}{30}\right) + 462\left(\frac{1}{42}\right) + \\
& 165\left(-\frac{1}{30}\right) + 11B_{10}, \\
0 =& -\frac{5}{6} + 11B_{10},
\end{aligned}$$

Thus $B_{10} = \frac{5}{66}$.

(4) Finally,

$$\begin{aligned}
0 =& 1 + 13B_1 + 78B_2 + 286B_3 + 715B_4 + 1287B_5 + 1716B_6 + \\
& 1716B_7 + 1287B_8 + 715B_9 + 286B_{10} + 78B_{11} + 13B_{12}, \\
0 =& 1 + 13B_1 + 78B_2 + 715B_4 + 1716B_6 + 1287B_8 + 286B_{10} + 13B_{12}, \\
0 =& 1 + 13\left(-\frac{1}{2}\right) + 78\left(\frac{1}{6}\right) + 715\left(-\frac{1}{30}\right) + 1716\left(\frac{1}{42}\right) + \\
& 1287\left(-\frac{1}{30}\right) + 286\left(\frac{5}{66}\right) + 13B_{12}, \\
0 =& \frac{691}{210} + 13B_{12},
\end{aligned}$$

Thus $B_{12} = -\frac{691}{2730}$.

□

Exercise 15.2. *If $a \in \mathbb{Z}$, show $a(a^m - 1)B_m \in \mathbb{Z}$ for all $m > 0$.*

Proof.

(1) *Trivial cases.* If $m = 1$, $a(a-1)B_1 = -\frac{1}{2}a(a-1) \in \mathbb{Z}$ for any $a \in \mathbb{Z}$. For odd $m > 1$, $B_m = 0$ or $a(a^m - 1)B_m = 0 \in \mathbb{Z}$ (Proposition 15.1.1).

(2) *Consider that $m > 1$ and even.* By Theorem 3,

$$B_{2m} + \sum_{p-1 \mid 2m} \frac{1}{p} \in \mathbb{Z}$$

where the sum is over all primes p such that $p-1 \mid 2m$. So it suffices to show

$$a(a^{2m}-1)\sum_{p-1|2m}\frac{1}{p} \in \mathbb{Z},$$

or

$$a(a^{2m}-1)\frac{1}{p} \in \mathbb{Z}$$

for any $a \in \mathbb{Z}$ and any prime p such that $p-1 \mid 2m$.

(3) *Consider all possible a.* If $p \mid a$, it is trivial. If $p \nmid a$, $a^{p-1} \equiv 1 \ (p)$ by Fermat's Little Theorem, or $a^{2m} \equiv 1 \ (p)$ by $p-1 \mid 2m$. In any cases, $a(a^{2m}-1)\frac{1}{p} \in \mathbb{Z}$.

□

Exercise 15.6. *For $m \geq 3$, show $|B_{2m+2}| > |B_{2m}|$. (Hint: Use Theorem 2.)*

Proof. By Theorem 2,

$$2\zeta(2m) = (-1)^{m+1}\frac{(2\pi)^{2m}}{(2m)!}B_{2m}.$$

Thus,

$$\frac{|B_{2m+2}|}{|B_{2m}|} = \frac{\zeta(2m+2)(2m+2)(2m+1)}{\zeta(2m)(2\pi)^2} > \frac{1\cdot 8\cdot 7}{\zeta(6)\cdot(2\pi)^2} = \frac{13230}{\pi^8} > 1,$$

or $|B_{2m+2}| > |B_{2m}|$. □

Exercise 15.8. *Consider the power series expansion of* $\tan x$ *about the origin;*

$$\sum_{k=1}^{\infty} T_k \frac{x^{2k-1}}{(2k-1)!}.$$

Show

$$T_k = (-1)^{k-1}\frac{B_{2k}}{2k}(2^{2k}-1)2^{2k}.$$

Note that $T_k \in \mathbb{Z}$ for all k by Exercise 3.

Proof.

(1) By the equation (6) on page 232,

$$x\cot x = 1 + \sum_{k=2}^{\infty} B_k \frac{(2ix)^k}{k!}.$$

Since $B_k = 0$ for odd $k > 1$,

$$x \cot x = 1 + \sum_{k=1}^{\infty} B_{2k} \frac{(2ix)^{2k}}{(2k)!} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} x^{2k},$$

or

$$\cot x = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} x^{2k-1}.$$

Combine the first term $\frac{1}{x}$ into the summation,

$$\cot x = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} x^{2k-1}.$$

(2) Note that $\tan x = \cot x - 2\cot(2x)$. By (1),

$$\begin{aligned}
\tan x &= \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} x^{2k-1} - 2 \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} (2x)^{2k-1} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k (1 - 2^{2k}) 2^{2k} B_{2k}}{(2k)!} x^{2k-1} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k (1 - 2^{2k}) 2^{2k} B_{2k}}{(2k)!} x^{2k-1}.
\end{aligned}$$

Write $T_k = (-1)^{k-1}(2^{2k}-1)2^{2k}\frac{B_{2k}}{2k}$. Therefore, $\tan x = \sum_{k=1}^{\infty} T_k \frac{x^{2k-1}}{(2k-1)!}$.
By Exercise 15.3, $(2^{2k}-1)2^{2k}\frac{B_{2k}}{2k} \in \mathbb{Z}$, or $T_k \in \mathbb{Z}$ for all k. $\square$

Exercise 15.12. *Recall the definition of the Bernoulli polynomials;*

$$B_m(x) = \sum_{k=0}^{m} \binom{m}{k} B_k x^{m-k}.$$

Show that

$$\frac{te^{tx}}{e^t - 1} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!}.$$

Proof. By Lemma 1,

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}.$$

So

$$\frac{te^{tx}}{e^t - 1} = \left(\sum_{m=0}^{\infty} B_m \frac{t^m}{m!} \right) \left(\sum_{m=0}^{\infty} x^m \frac{t^m}{m!} \right).$$

Write $\frac{te^{tx}}{e^t-1} = \sum_{m=0}^{\infty} b_m(x)\frac{t^m}{m!}$ and we want to check if $b_m(x) = B_m(x)$ or not. The result is established if $b_m(x) = B_m(x)$ holds. Equating coefficients of t^m gives

$$\begin{aligned}\frac{b_m(x)}{m!} &= \sum_{k=0}^{m} \frac{B_k x^{m-k}}{k!(m-k)!},\\ b_m(x) &= \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} B_k x^{m-k}\\ &= \sum_{k=0}^{m} \binom{m}{k} B_k x^{m-k}\\ &= B_m(x).\end{aligned}$$

□

Exercise 15.13. *Show $B_m(x+1) - B_m(x) = mx^{m-1}$.*

Proof. Let $f(t,x) = \frac{te^{tx}}{e^t-1}$.

(1)

$$f(t,x+1) - f(t,x) = \frac{te^{t(x+1)}}{e^t-1} - \frac{te^{tx}}{e^t-1} = te^{tx}.$$

Expand te^{tx} in a power series about the origin as follows

$$\begin{aligned}te^{tx} &= t\sum_{m=0}^{\infty} x^m \frac{t^m}{m!}\\ &= \sum_{m=0}^{\infty} x^m \frac{t^{m+1}}{m!}\\ &= \sum_{m=1}^{\infty} x^{m-1} \frac{t^m}{(m-1)!}\\ &= \sum_{m=1}^{\infty} mx^{m-1} \frac{t^m}{m!}\\ &= \sum_{m=0}^{\infty} mx^{m-1} \frac{t^m}{m!}.\end{aligned}$$

So,

$$f(t,x+1) - f(t,x) = \sum_{m=0}^{\infty} mx^{m-1} \frac{t^m}{m!}.$$

(2) By Exercise 15.12,

$$\begin{aligned} f(t,x+1)-f(t,x) &= \sum_{m=0}^{\infty} B_m(x+1)\frac{t^m}{m!} - \sum_{m=0}^{\infty} B_m(x)\frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}(B_m(x+1)-B_m(x))\frac{t^m}{m!}. \end{aligned}$$

By (1)(2), comparing coefficients of t^m yields

$$mx^{m-1} = B_m(x+1) - B_m(x).$$

□

Exercise 15.14. *Use Exercise 13 to give a new proof of Theorem 1:*

$$S_m(n) = \frac{1}{m+1}(B_{m+1}(n) - B_{m+1}).$$

Proof. By Exercise 13,

$$B_{m+1}(k) - B_{m+1}(k-1) = (m+1)(k-1)^m$$

for any k. So,

$$\begin{aligned} \sum_{k=1}^{n}(B_{m+1}(k) - B_{m+1}(k-1)) &= \sum_{k=1}^{n}(m+1)(k-1)^m, \\ B_{m+1}(n) - B_{m+1}(0) &= (m+1)S_m(n). \end{aligned}$$

Note that $B_{m+1}(0) = B_{m+1}$ for any m. So Theorem 1 is established by a new proof. □

Exercise 15.15. *Suppose* $f(x) = \sum_{k=0}^{n} a_k x^k$ *be a polynomial with complex coefficients. Use Exercise 13 to find a polynomial* $F(x)$ *such that* $F(x+1) - F(x) = f(x)$*.*

Proof. By Exercise 15.13,

$$x^k = \frac{1}{k+1}(B_{k+1}(x+1) - B_{k+1}(x))$$

for $k \geq 0$. Thus,

$$\begin{aligned}
f(x) &= \sum_{k=0}^{n} a_k x^k \\
&= \sum_{k=0}^{n} a_k \cdot \frac{1}{k+1}(B_{k+1}(x+1) - B_{k+1}(x)) \\
&= \sum_{k=0}^{n} \frac{a_k}{k+1} B_{k+1}(x+1) - \sum_{k=0}^{n} \frac{a_k}{k+1} B_{k+1}(x).
\end{aligned}$$

Let

$$F(x) = \sum_{k=0}^{n} \frac{a_k}{k+1} B_{k+1}(x),$$

and we get $f(x) = F(x+1) - F(x)$. □

Exercise 15.16. *For $n \geq 1$, show $\frac{d}{dx} B_n(x) = nB_{n-1}(x)$.*

Proof. For $n \geq 1$,

$$\frac{d}{dx} B_n(x) = \sum_{k=0}^{n} (n-k)\binom{n}{k} B_k x^{n-k-1} = \sum_{k=0}^{n-1} (n-k)\binom{n}{k} B_k x^{n-k-1}.$$

Note that

$$(n-k)\binom{n}{k} = n\binom{n-1}{k}.$$

So

$$\begin{aligned}
\frac{d}{dx} B_n(x) &= \sum_{k=0}^{n-1} n\binom{n-1}{k} B_k x^{n-k-1} \\
&= n\sum_{k=0}^{n-1} \binom{n-1}{k} B_k x^{n-k-1} \\
&= nB_{n-1}(x).
\end{aligned}$$

□

Exercise 15.17. *Show $B_n(1-x) = (-1)^n B_n(x)$.*

Proof. Let $f(t,x) = \frac{te^{tx}}{e^t - 1}$.

(1) $f(t, 1-x) = f(-t, x)$.

$$f(t, 1-x) = \frac{te^{t(1-x)}}{e^t - 1} = e^t \cdot \frac{te^{-tx}}{e^t - 1} = \frac{-te^{-tx}}{e^{-t} - 1} = f(-t, x).$$

(2) By Exercise 15.12,

$$f(t, 1-x) = \sum_{n=0}^{\infty} B_n(1-x)\frac{t^n}{n!}$$
$$f(-t, x) = \sum_{n=0}^{\infty} (-1)^n B_n(x)\frac{t^n}{n!}.$$

By (1), comparing coefficients of t^n yields $B_n(1-x) = (-1)^n B_n(x)$.

□

Exercise 15.18. *Use Exercise 13 and 17 to give a new proof that $B_n = 0$ for n odd and $n > 1$.*

Proof.

(1) *$B_m(1) - B_m(0) = 0$ for any $m > 1$.* Taking $x = 0$ in Exercise 15.13 and keeping $m - 1 > 0$ or $m > 1$.

(2) *$B_m(1) = -B_m(0)$ for any odd m.* Taking $x = 0$ in Exercise 15.17 and keeping m is odd.

$$f(t, 1-x) = \sum_{n=0}^{\infty} B_n(1-x)\frac{t^n}{n!}$$
$$f(-t, x) = \sum_{n=0}^{\infty} (-1)^n B_n(x)\frac{t^n}{n!}.$$

By (1)(2), for m odd and $m > 1$, $B_m(0) = 0$ or $B_m = 0$. □

Exercise 15.19 (Multiplication Theorem). *Suppose n and F are integers and $n, F > 0$. Show that*

$$B_n(Fx) = F^{n-1} \sum_{a=0}^{F-1} B_n\left(x + \frac{a}{F}\right).$$

(Hint: Use Exercise 12.)

Proof. By $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + y^{n-1}$ (Exercise 1.24),

$$e^{Ft} - 1 = (e^t - 1)(1 + e^t + e^{2t} + \cdots + e^{(F-1)t}) = (e^t - 1)\sum_{a=0}^{F-1} e^{at}.$$

So,

$$\begin{aligned}
\frac{1}{e^t-1} &= \frac{1}{e^{Ft}-1}\sum_{a=0}^{F-1} e^{at},\\
\frac{te^{tFx}}{e^t-1} &= \frac{te^{tFx}}{e^{Ft}-1}\sum_{a=0}^{F-1} e^{at}\\
&= \sum_{a=0}^{F-1} \frac{te^{(Fx+a)t}}{e^{Ft}-1}\\
&= \sum_{a=0}^{F-1} \frac{te^{(Fx+a)t}}{e^{Ft}-1}\\
&= \sum_{a=0}^{F-1} F^{-1}\frac{(Ft)e^{(x+\frac{a}{F})(Ft)}}{e^{Ft}-1}.
\end{aligned}$$

By Exercise 15.12,

$$\begin{aligned}
\sum_{n=0}^{\infty} B_n(Fx)\frac{t^n}{n!} &= \sum_{a=0}^{F-1} F^{-1}\sum_{n=0}^{\infty} B_n\left(x+\frac{a}{F}\right)\frac{(Ft)^n}{n!}\\
&= \sum_{n=0}^{\infty}\sum_{a=0}^{F-1} F^{-1}B_n\left(x+\frac{a}{F}\right)\frac{(Ft)^n}{n!}\\
&= \sum_{n=0}^{\infty}\sum_{a=0}^{F-1} F^{n-1}B_n\left(x+\frac{a}{F}\right)\frac{t^n}{n!}.
\end{aligned}$$

Comparing coefficients of t^n on the both sides of the above equation and yields $B_n(Fx) = F^{n-1}\sum_{a=0}^{F-1} B_n\left(x+\frac{a}{F}\right)$. □

Exercise 15.21. *Show* $2^{n-1}B_n(\frac{1}{2}) = (1-2^{n-1})B_n$.

The original identity $B_n(\frac{1}{2}) = (1-2^{n-1})B_n$ is wrong. For $n=2$, $B_2(x) = x^2 - x + \frac{1}{6}$ and thus $-\frac{1}{12} = B_2(\frac{1}{2}) \neq (1-2^{2-1})B_2 = -\frac{1}{6}$.

Proof. Taking $F=2$ in Exercise 15.19,

$$\begin{aligned}
B_n(2x) &= 2^{n-1}\sum_{a=0}^{1} B_n\left(x+\frac{a}{2}\right)\\
&= 2^{n-1}B_n(x) + 2^{n-1}B_n\left(x+\frac{1}{2}\right).
\end{aligned}$$

Let $x = 0$,

$$B_n(0) = 2^{n-1}B_n(0) + 2^{n-1}B_n\left(\frac{1}{2}\right),$$

So

$$2^{n-1}B_n\left(\frac{1}{2}\right) = (1-2^{n-1})B_n(0) = (1-2^{n-1})B_n.$$

□

Exercise 15.22. *More generally, show that* $(1-F^{n-1})B_n = F^{n-1}\sum_{a=1}^{F-1} B_n(\frac{a}{F})$.

The original identity $(1-F^{n-1})B_n = \sum_{a=1}^{F-1} B_n(\frac{a}{F})$ is wrong again.

Proof. Let $x = 0$ in Exercise 15.19,

$$B_n(0) = F^{n-1}\sum_{a=0}^{F-1} B_n\left(\frac{a}{F}\right) = F^{n-1}B_n(0) + F^{n-1}\sum_{a=1}^{F-1} B_n\left(\frac{a}{F}\right),$$

So

$$F^{n-1}\sum_{a=1}^{F-1} B_n\left(\frac{a}{F}\right) = (1-F^{n-1})B_n(0) = (1-F^{n-1})B_n.$$

□