Chapter 6: The Riemann-Stieltjes Integral

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Supplement. Another definition of Riemann-Stieltjes integral. *(Exercise 7.3, 7.4 of the book T. M. Apostol, Mathematical Analysis, Second Edition.) Let P be a partition of $[a, b]$. The norm of a partition P is the length of the largest subinterval $[x_{i-1}, x_i]$ of P and is denoted by $\|P\|$.*

We say $f \in \mathscr{R}(\alpha)$ if there exists $A \in \mathbb{R}$ having the property that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any partition P of $[a, b]$ with norm $\|P\| < \delta$ and for any choice of $t_i \in [x_{i-1}, x_i]$, we have $|\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - A| < \varepsilon$.

Claim. *$f \in \mathscr{R}$ in the sense of Definition 6.2 implies that $f \in \mathscr{R}$ in the sense of this another definition.*

Proof of Claim. Let $A = \int f dx$, $M > 0$ be one upper bound of $|f|$ on $[a, b]$. Given $\varepsilon > 0$, there exists a partition $P_0 = \{a = x_0, x_1, \dots, x_{N-1}, x_N = b\}$ such that $U(P_0, f) \leq A + \frac{\varepsilon}{2}$. Let $\delta = \frac{\varepsilon}{2MN} > 0$. Then for any partition P with norm $\|P\| < \delta$, write

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i = S_1 + S_2,$$

where S_1 is the sum of terms arising from those subintervals of P containing no point of P_0, S_2 is the sum of the remaining terms. Then

$$S_1 \leq U(P_0, f) < A + \frac{\varepsilon}{2},$$
$$S_2 \leq NM\|P\| < NM\delta < \frac{\varepsilon}{2}.$$

Therefore, $U(P, f) < A + \varepsilon$. Similarly, $L(P, f) > A - \varepsilon$ whenever $\|P\| < \delta'$. Hence, $|\sum_{i=1}^{n} f(t_i)\Delta x_i - A| < \varepsilon$ whenever $\|P\| < \min\{\delta, \delta'\}$. (Copy Apostol's hint and ensure $M > 0$. M in Apostol's hint might be zero if $f = 0$.) $\square$

This supplement will be used in computing $\int_0^\infty (\frac{\sin x}{x})^2 dx = \frac{\pi}{2}$ in Exercise 8.12.

Exercise 6.1. *Suppose α increases on $[a, b]$, $a \leq x_0 \leq b$, α is continuous at x_0, $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathscr{R}(\alpha)$ and that $\int f d\alpha = 0$.*

Given any partition $P = \{a = p_0, p_1, \dots, p_{n-1}, p_n = b\}$, where $a = p_0 \leq p_1 \leq \cdots \leq p_{n-1} \leq p_n = b$. We might compute $L(P, f, \alpha)$ and $U(P, f, \alpha)$ by using ε-δ

argument since we are hinted by the condition that α is continuous. A function which is continuous at x_0 has a nice property near x_0 and this property would help us estimate $U(P, f, \alpha)$ near x_0. On the contrary, if both f and α are discontinuous at x_0, it might be $f \notin \mathscr{R}(\alpha)$. Besides, if f has too many points of discontinuity ($f(x) = 0$ if $x \in \mathbb{Q}$ and $f(x) = 1$ otherwise, for example), then f might not be Riemann-integrable on $[0, 1]$.

Claim 1. $L(P, f, \alpha) = 0$.

Proof of Claim 1. $m_i = 0$ since $\inf f(x) = 0$ on any subinterval of $[a, b]$. So $L(P, f, \alpha) = \sum m_i \Delta\alpha_i = 0$. Here we don't need the condition that α is continuous at x_0. □

Claim 2. *For any $\varepsilon > 0$, there exists a partition P such that $U(P, f, \alpha) < \varepsilon$.*

Proof of Claim 2. Say $x_0 \in [p_{i_0-1}, p_{i_0}]$ for some i_0. Then

$$M_i = \sup_{p_{i-1} \leq x \leq p_i} f(x) = \begin{cases} 0 & \text{if } i \neq i_0, \\ 1 & \text{if } i = i_0. \end{cases}$$

So

$$U(P, f, \alpha) = \sum M_i \Delta\alpha_i = \Delta\alpha_{i_0}.$$

It is not true for any arbitrary α. (For example, α has a jump on $x = x_0$.) In fact, Exercise 6.3 shows this. Luckily, α is continuous at x_0. So for $\varepsilon > 0$, there exists $\delta > 0$ such that $|\alpha(x) - \alpha(x_0)| < \frac{\varepsilon}{2}$ whenever $|x - x_0| < \delta$ (and $x \in [a, b]$). Now we pick a nice partition

$$P = \{a, x_0 - \delta_1, x_0 + \delta_2, b\},$$

where $\delta_1 = \min\{\delta, x_0 - a\} \geq 0$ and $\delta_2 = \min\{\delta, b - x_0\} \geq 0$. (It is a trick about resizing "δ" to avoid considering the edge cases $x_0 = a$ or $x_0 = b$ or $a = b$.) Then $x_0 \in [x_0 - \delta_1, x_0 + \delta_2]$ and $\Delta\alpha$ on $[x_0 - \delta_1, x_0 + \delta_2]$ is

$$\begin{aligned} \alpha(x_0 + \delta_2) - \alpha(x_0 - \delta_1) &= (\alpha(x_0 + \delta_2) - \alpha(x_0)) + (\alpha(x_0) - \alpha(x_0 - \delta_1)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore, $U(P, f, \alpha) < \varepsilon$. □

Proof (Definition 6.2). By Claim 1 and 2 and notice that $U(P, f, \alpha) \geq 0$ for any partition P,

$$\overline{\int_a^b} f d\alpha = \inf U(P, f, \alpha) = 0,$$
$$\underline{\int_a^b} f d\alpha = \sup L(P, f, \alpha) = 0,$$

the inf and sup again being taken over all partitions. Hence $f \in \mathscr{R}(\alpha)$ and that $\int f d\alpha = 0$ by Definition 6.2. $\square$

Proof (Theorem 6.5). By Claim 1 and 2,

$$0 \leq U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$$

Hence $f \in \mathscr{R}(\alpha)$ by Theorem 6.5. Furthermore,

$$\int f d\alpha = \underline{\int_a^b} f d\alpha = \sup L(P, f, \alpha) = 0.$$

$\square$

Proof (Theorem 6.10). $f \in \mathscr{R}(\alpha)$ by Theorem 6.10. Thus, by Claim 1

$$\int f d\alpha = \underline{\int_a^b} f d\alpha = \sup L(P, f, \alpha) = 0.$$

$\square$

Exercise 6.2. *Suppose $f \geq 0$, f is continuous on $[a, b]$, and $\int_a^b f(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$. (Compare with Exercise 6.1.)*

For one application, see Exercise 7.20.

Proof. (Reductio ad absurdum) If there were $p \in [a, b]$ such that $f(p) > 0$. Since f is continuous on $[a, b]$, given $\varepsilon = \frac{1}{64} f(p) > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(p)| \leq \frac{1}{64} f(p) \text{ whenever } |x - p| \leq \delta, x \in [a, b].$$

Hence

$$f(x) \geq \frac{63}{64} f(p)$$

whenever $x \in E = [\max\{a, p - \delta\}, \min\{b, p + \delta\}] \subseteq [a, b]$. Note that the length of E is $|E| > 0$. So

$$0 = \int_a^b f(x)dx \geq \int_E f(x)dx \geq \int_E \frac{63}{64} f(p)dx = \frac{63}{64} f(p)|E| > 0,$$

which is absurd. $\square$

Exercise 6.3.
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Exercise 6.4. *If*

$$f(x) = \begin{cases} 0 & \textit{for all irrational } x, \\ 1 & \textit{for all rational } x, \end{cases}$$

prove that $f \notin \mathscr{R}$ on $[a,b]$ for any $a < b$.

Proof. Given any partition

$$P = \{a = p_0, p_1, \ldots, p_{n-1}, p_n = b\}$$

of $[a,b]$ where $a = p_0 \leq p_1 \leq \cdots \leq p_{n-1} \leq p_n = b$. Since $a < b$, we might assume that $a = p_0 < p_1 < \cdots < p_{n-1} < p_n = b$ by removing duplicated points. Since $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$ are dense in $\mathbb{R}$, we have

$$\begin{aligned} M_i &= \sup_{p_{i-1} \leq x \leq p_i} f(x) = 1, \\ m_i &= \inf_{p_{i-1} \leq x \leq p_i} f(x) = 0, \\ U(P,f) &= \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} \Delta x_i = b - a, \\ L(P,f) &= \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} 0 = 0. \end{aligned}$$

Since P is arbitrary,

$$\begin{aligned} \overline{\int_a^b} f dx &= \inf U(P,f) = b - a > 0, \\ \underline{\int_a^b} f dx &= \sup L(P,f) = 0. \end{aligned}$$

Hence $f \notin \mathscr{R}$ on $[a,b]$ for any $a < b$. $\square$

Note.

(1) $f \in \mathscr{R}$ on $[a,b]$ iff $a = b$.

(2) (Problem 4.1 in *H. L. Royden, Real Analysis, 3rd edition.*) *Construct a sequence $\{f_n\}$ of nonnegative, Riemann integrable functions such that f_n*

increases monotonically to f. What does this imply about changing the order of integration and the limiting process? (Since $\mathbb{Q}$ is countable, write

$$\mathbb{Q} = \{r_1, r_2, \ldots\}.$$

Define

$$f_n(x) = \begin{cases} 0 & \text{if } x \notin \{r_1, \ldots, r_n\} , \\ 1 & \text{if } x \in \{r_1, \ldots, r_n\} . \end{cases}$$

By construction, f_n increases monotonically to f pointwise. Note that $f_n \to f$ not uniformly. Also, $\int_a^b f_n(x)dx = 0$ by using the same argument in Theorem 6.10. Therefore, $\lim_{n\to\infty} \int_a^b f_n(x)dx = 0$ but $\int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx$ does not exist.)

Exercise 6.5.
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Exercise 6.6.
PLACEHOLDER

Exercise 6.7.
PLACEHOLDER

Exercise 6.8.
PLACEHOLDER

Exercise 6.9.
PLACEHOLDER

Exercise 6.10. *Let p and q be positive real integers such that*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Prove the following statements.

(a) *If $u \geq 0$ and $v \geq 0$, then*

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

Equality holds if and only if $u^p = v^q$.

(b)

(c)

Proof of (a)(Young's inequality). $u = 0$ or $v = 0$ is nothing to do. For $u > 0$ and $v > 0$,

$$\begin{aligned}
uv &= \exp(\log(uv)) \\
&= \exp\left(\frac{1}{p}\log(u^p) + \frac{1}{q}\log(v^q)\right) \\
&\leq \frac{1}{p}\exp(\log(u^p)) + \frac{1}{q}\exp(\log(v^q)) && \text{(exp is convex)} \\
&= \frac{u^p}{p} + \frac{v^q}{q}.
\end{aligned}$$

Or

$$\begin{aligned}
\log\left(\frac{u^p}{p} + \frac{v^q}{q}\right) &\geq \frac{1}{p}\log(u^p) + \frac{1}{q}\log(v^q) && \text{(log is concave)} \\
&= \log(u) + \log(v) \\
&= \log(uv)
\end{aligned}$$

and notice that log increases monotonically.
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Exercise 6.11.
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Exercise 6.12.
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Exercise 6.13.
PLACEHOLDER

Exercise 6.14.
PLACEHOLDER

Exercise 6.15. *Suppose f is a real, continuously differentiable function on $[a, b]$, $f(a) = f(b) = 0$, and*

$$\int_a^b f(x)^2 dx = 1.$$

Prove that

$$\int_a^b x f(x) f'(x) dx = -\frac{1}{2}$$

and that

$$\int_a^b [f'(x)]^2 dx \int_a^b x^2 f(x)^2 dx > \frac{1}{4}.$$

Proof.

(1) By Theorem 6.22 (integration by parts),

$$\int_a^b x \left(\frac{f(x)^2}{2} \right)' dx = \left[x \cdot \frac{f(x)^2}{2} \right]_{x=a}^{x=b} - \int_a^b \frac{f(x)^2}{2} dx,$$

or

$$\int_a^b x f(x) f'(x) dx = \left[b \cdot \frac{f(b)^2}{2} - a \cdot \frac{f(a)^2}{2} \right] - \frac{1}{2} \int_a^b f(x)^2 dx = \frac{1}{2}.$$

(2) By Exercise 6.10(c),

$$\int_a^b [f'(x)]^2 dx \int_a^b x^2 f(x)^2 dx \geq \left(\int_a^b x f(x) f'(x) dx \right)^2 = \frac{1}{4}.$$

The equality holds iff

$$f'(x) = \lambda x f(x) \qquad \text{or} \qquad x f(x) = \mu f'(x)$$

on $[a, b]$ for some constant $\lambda, \mu \in \mathbb{R}$.

(a) If $\lambda = 0$, then $f'(x) = 0$ or $f(x)$ is a constant. Since f is continuous and $f(a) = f(b) = 0$, $f(x) = 0$ on $[a, b]$, contrary to $\int_a^b f(x)^2 dx = 1$.

(b) If $\mu = 0$, then $x f(x) = 0$, contrary to $\int_a^b x f(x) f'(x) dx = -\frac{1}{2}$.

By (a)(b), The equality holds iff

$$f'(x) = c_1 x f(x)$$

on $[a, b]$ for some constant $c_1 \in \mathbb{R}$.

(3) Let $g(x) = f(x) \cdot \exp\left(-\frac{c_1}{2} x^2\right)$. Since

$$\begin{aligned} g'(x) &= f'(x) \cdot \exp\left(-\frac{c_1}{2} x^2\right) + f(x) \cdot (-c_1 x) \exp\left(-\frac{c_1}{2} x^2\right) \\ &= c_1 x f(x) \cdot \exp\left(-\frac{c_1}{2} x^2\right) + f(x) \cdot (-c_1 x) \exp\left(-\frac{c_1}{2} x^2\right) \\ &= 0 \end{aligned}$$

for all $x \in (a, b)$, $g(x) = c_2$ is a constant. Hence $f(x) = c_2 \exp\left(\frac{c_1}{2} x^2\right)$ on (a, b). Since f is continuous on $[a, b]$, $\lim_{x \to a} f(x) = f(a)$, or $c_2 \exp\left(\frac{c_1}{2} a^2\right) = 0$, or $c_2 = 0$, or $f(x) = 0$ on $[a, b]$, contrary to $\int_a^b f(x)^2 dx = 1$.

(4) Therefore, the equality does not hold, or

$$\int_a^b [f'(x)]^2 dx \int_a^b x^2 f(x)^2 dx > \frac{1}{4}.$$

□

PLACEHOLDER

Exercise 6.16.
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Exercise 6.17.
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Exercise 6.18.
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Exercise 6.19.
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