Chapter 5: Differentiation

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Exercise 5.1. *Let f be defined for all real x, and suppose that*

$$|f(x) - f(y)| \leq (x - y)^2$$

for all real x and y. Prove that f is a constant.

Proof.

(1) Write
$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq |x - y|$$
if $x \neq y$.

(2) Given any $y \in \mathbb{R}$,
$$\left| \frac{f(x) - f(y)}{x - y} \right| \to 0 \quad \text{as} \quad x \to y,$$
or $|f'(y)| = 0$.

(3) Or using ε-δ argument. Fix $y \in \mathbb{R}$. Given any $\varepsilon > 0$, there exists $\delta = \varepsilon > 0$ such that
$$\left| \frac{f(x) - f(y)}{x - y} - 0 \right| \leq |x - y| < \delta = \varepsilon$$
whenever $|x - y| < \delta$. That is, $|f'(y)| = 0$.

(4) So $f'(y) = 0$ for any $y \in \mathbb{R}$. By Theorem 5.11 (b), f is a constant.

□

Exercise 5.2. *Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b), and let g be its inverse function. Prove that g is differentiable, and that*

$$g'(f(x)) = \frac{1}{f'(x)} \qquad (a < x < b).$$

Proof. Let $E = (a, b)$.

(1) Theorem 5.10 implies that for any $a < p < q < b$ there exists $\xi \in (p, q)$ such that

$$f(p) - f(q) = (p - q)f'(\xi).$$

Since $\xi \in (p, q) \subseteq E$, by assumption $f'(\xi) > 0$. Hence $f(p) - f(q) = (p - q)f'(\xi) < 0$ (here $p - q < 0$), or

$$f(p) < f(q)$$

if $p < q$. Therefore, f is strictly increasing in (a, b).

(2) *Show that f is one-to-one in E if f is strictly increasing in E.* If $f(p) = f(q)$, then it cannot be $p > q$ or $p < q$ ((1)). So that $p = q$, or f is injective.

(3) *Show that g is well-defined.* Theorem 5.2 and Theorem 4.17.

(4) *Show that $g'(f(x)) = \frac{1}{f'(x)}$.* Given $y \in f(E)$, say $y = f(x)$ for some $x \in E$. Given any $s \in f(E)$ with $s \neq y$. Here $s = f(t)$ for some $t \in E$ and $t \neq x$.

$$\begin{aligned}
\lim_{s \to y} \frac{g(s) - g(y)}{s - y} &= \lim_{f(t) \to f(x)} \frac{g(f(t)) - g(f(x))}{f(t) - f(x)} \\
&= \lim_{t \to x} \frac{t - x}{f(t) - f(x)} \\
&= \lim_{t \to x} \frac{1}{\frac{f(t) - f(x)}{t - x}} \\
&= \frac{1}{f'(x)}. && (f' > 0)
\end{aligned}$$

Here $s \to y$ if and only if $t \to x$ since both f and g are continuous and one-to-one. Hence g is differentiable and $g'(f(x)) = \frac{1}{f'(x)}$.

□

Exercise 5.3. *Suppose g is a real function on $\mathbb{R}^1$, with bounded derivative (say $|g'| \leq M$). Fix $\varepsilon > 0$, and define $f(x) = x + \varepsilon g(x)$. Prove that f is one-to-one if ε is small enough. (A set of admissible values of ε can be determined which depends only on M.)*

Proof.

(1) Note that $f'(x) = 1 + \varepsilon g'(x)$ (Theorem 5.3). Since $|g'| \leq M$,

$$1 - \varepsilon M \leq f'(x) \leq 1 + \varepsilon M.$$

(2) Pick

$$\varepsilon = \frac{1}{M + 1} > 0.$$

Thus,
$$f'(x) \geq \frac{1}{M+1} > 0.$$
By Exercise 5.2, $f(x)$ is strictly increasing in $\mathbb{R}$ or one-to-one in $\mathbb{R}$.

□

Exercise 5.4. *If*
$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$
where $C_0, ..., C_n$ are real constants, prove that the equation
$$C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n = 0$$
has at least one real root between 0 and 1.

Proof. Let
$$g(x) = C_0 x + \frac{C_1}{2}x^2 + \cdots + \frac{C_{n-1}}{n}x^n + \frac{C_n}{n+1}x^{n+1} \in \mathbb{R}[x].$$
Then $g(0) = g(1) = 0$, and $g'(x) = C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n$. By the mean value theorem (Theorem 5.10), there exists a point $\xi \in (0,1)$ at which
$$g(1) - g(0) = g'(\xi)(1-0),$$
or $g'(\xi) = 0$. That is, there exists a real root $x = \xi$ between 0 and 1 at which $C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n = 0$. □

Exercise 5.14. *Let f be a differentiable real function defined in (a,b). Prove that f is convex if and only if f' is monotonically increasing. Assume next $f''(x)$ exists for every $x \in (a,b)$, and prove that f is convex if and only if $f''(x) \geq 0$ for all $x \in (a,b)$.*

Proof.

(1) *Show that f' is monotonically increasing if f is convex.*

   (a) Since f is convex, by definition (Exercise 4.23)
   $$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$
   whenever $a < x < b$, $a < y < b$, $0 < \lambda < 1$.

(b) As $x \neq y$, we have

$$\begin{aligned} f(y) - f(x) &\geq \frac{f(x + \lambda(y-x)) - f(x)}{\lambda} \\ &= \frac{f(x + \lambda(y-x)) - f(x)}{\lambda(y-x)} \cdot (y - x) \end{aligned}$$

and let $\lambda \to 0$ to get

$$f(y) - f(x) \geq f'(x)(y - x)$$

(since $f'(x)$ exists). Similarly, we have

$$f(x) - f(y) \geq f'(y)(x - y).$$

(c) Given any $y > x$, we have

$$f'(y)(y - x) \geq f(y) - f(x) \geq f'(x)(y - x).$$

Hence $f'(y) \geq f'(x)$ whenever $y > x$, or f' is monotonically increasing.

(2) *Show that f is convex if f' is monotonically increasing.* Given any $y > x$ and any $0 < \lambda < 1$.

(a) By Theorem 5.10 (the mean value theorem), there is a point $x < \xi < y$ such that

$$f(y) - f(x) = f'(\xi)(y - x).$$

Since f' is monotonically increasing,

$$f'(y)(y - x) \geq f(y) - f(x) \geq f'(x)(y - x).$$

(b) Write $z = \lambda x + (1 - \lambda)y$. Hence

$$\begin{aligned} f(y) - f(z) &\geq f'(z)(y - z), \\ f(z) - f(x) &\leq f'(z)(z - x), \end{aligned}$$

or

$$\begin{aligned} f(y) &\geq f(z) + f'(z)(y - z), \\ f(x) &\geq f(z) + f'(z)(x - z), \end{aligned}$$

or

$$\begin{aligned} \lambda f(x) + (1 - \lambda) f(y) \geq & \lambda[f(z) + f'(z)(x - z)] \\ & + (1 - \lambda)[f(z) + f'(z)(y - z)] \\ = & f(z) \\ = & f(\lambda x + (1 - \lambda)y). \end{aligned}$$

Hence f is convex.

(3) *Show that* $f''(x) \geq 0$ *if* f *is convex and* f'' *exists.* By (1), f' is monotonically increasing since f is convex. Given any $x \neq y$, we have

$$\frac{f'(y) - f'(x)}{y - x} \geq 0.$$

Let $y \to x$, we have $f''(x) \geq 0$ if f'' exists.

(4) *Show that* f *is convex if* f'' *exists and* $f''(x) \geq 0$. By Theorem 5.11(a), f' is monotonically increasing. By (2), f is convex.

□