

Chapter 1: The Real And Complex Number Systems

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Integers

Exercise 1.1. *Prove that there is no largest prime. (A proof was known to Euclid.)*

There are many proofs of this result. We provide some of them.

Proof (Due to Euclid). If p_1, p_2, \dots, p_t were all primes, then we consider

$$n = p_1 p_2 \cdots p_t + 1.$$

Thus there is a prime number p dividing n . p can not be any of p_i for $1 \leq i \leq t$; otherwise p would divide the difference $n - p_1 p_2 \cdots p_t = 1$. That is, $p \neq p_i$ for $1 \leq i \leq t$, contrary to the assumption. \square

Supplement (Due to Euclid).

- (1) *Show that $k[x]$, with k a field, has infinitely many irreducible polynomials.*
If f_1, f_2, \dots, f_t were all irreducible polynomials, then we consider

$$g = f_1 f_2 \cdots f_t + 1 \in k[x].$$

So there is an irreducible polynomial f dividing g (since $\deg g = \deg f_1 + \deg f_2 + \cdots + \deg f_t \geq 1$). f can not be any of $c_i f_i$ for $1 \leq i \leq t$ and $c_i \in k - \{0\}$; otherwise f would divide the difference $g - f_1 f_2 \cdots f_t = 1$. That is, $f \neq c_i f_i$ for $1 \leq i \leq t$ and $c_i \in k - \{0\}$, contrary to the assumption.

- (2) *Show that any algebraically closed field is infinite.* Let k be an algebraically closed field. If a_1, \dots, a_n were all elements in k , then we consider a monic polynomials

$$F(X) = (X - a_1) \cdots (X - a_n) + 1 \in k[X].$$

Since k is algebraically closed, there is an element $a \in k$ such that $F(a) = 0$. By assumption, $a = a_i$ for some $1 \leq i \leq n$, and thus $F(a) = F(a_i) = 1$, contrary to the fact that a field is a commutative ring where $0 \neq 1$ and all nonzero elements are invertible.

\square

Proof (Unique factorization theorem). Given N .

- (1) Show that $\sum_{n \leq N} \frac{1}{n} \leq \prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1}$.

By the unique factorization theorem on $n \leq N$,

$$\sum_{n \leq N} \frac{1}{n} \leq \prod_{p \leq N} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1}.$$

- (2) By (1) and the fact that $\sum \frac{1}{n}$ diverges, there are infinitely many primes.

□

Proof (Due to Eckford Cohen).

- (1) $\text{ord}_p n! = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots$. For any $k = 1, 2, \dots, n$, we can express k as $k = p^s t$ where $s = \text{ord}_p k$ is a non-negative integer and $(t, p) = 1$. There are $\left[\frac{n}{p^a}\right]$ numbers such that $p^a \mid k$ for $a = 1, 2, \dots$. Therefore, there are

$$\left[\frac{n}{p^a}\right] - \left[\frac{n}{p^{a+1}}\right]$$

numbers such that $\text{ord}_p k = a$ for $a = 1, 2, \dots$. Hence,

$$\begin{aligned} \text{ord}_p n! &= \left(\left[\frac{n}{p}\right] - \left[\frac{n}{p^2}\right]\right) + 2\left(\left[\frac{n}{p^2}\right] - \left[\frac{n}{p^3}\right]\right) + 3\left(\left[\frac{n}{p^3}\right] - \left[\frac{n}{p^4}\right]\right) + \cdots \\ &= \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots. \end{aligned}$$

- (2) $\text{ord}_p n! \leq \frac{n}{p-1}$ and that $n!^{\frac{1}{n}} \leq \prod_{p|n!} p^{\frac{1}{p-1}}$.

$$\begin{aligned} \text{ord}_p n! &= \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots \\ &\leq \frac{n}{p} + \frac{n}{p^2} + \frac{n}{p^3} + \cdots \\ &= \frac{\frac{n}{p}}{1 - \frac{1}{p}} \\ &= \frac{n}{p-1}. \end{aligned}$$

Thus,

$$n! = \prod_{p|n!} p^{\text{ord}_p n!} \leq \prod_{p|n!} p^{\frac{n}{p-1}} = \left(\prod_{p|n!} p^{\frac{1}{p-1}}\right)^n,$$

or

$$n!^{\frac{1}{n}} \leq \prod_{p|n!} p^{\frac{1}{p-1}}.$$

(3) $(n!)^2 \geq n^n$. Write $(n!)^2 = \prod_{k=1}^n k \prod_{k=1}^n (n+1-k) = \prod_{k=1}^n k(n+1-k)$, and $n^n = \prod_{k=1}^n n$. It suffices to show that $k(n+1-k) \geq n$ for each $1 \leq k \leq n$. Notice that $k(n+1-k) - n = (n-k)(k-1) \geq 0$ for $1 \leq k \leq n$. The inequality holds.

(4) By (3)(4), $\prod_{p|n!} p^{\frac{1}{p-1}} \geq \sqrt{n}$. Assume that there are finitely many primes, the value $\prod_{p|n!} p^{\frac{1}{p-1}}$ is a finite number whenever the value of n . However, $\sqrt{n} \rightarrow \infty$ as $n \rightarrow \infty$, which leads to a contradiction. Hence there are infinitely many primes.

□

Proof (Formula for $\phi(n)$). If p_1, p_2, \dots, p_t were all primes, then let $n = p_1 p_2 \cdots p_t$ and all numbers between 2 and n are NOT relatively prime to n . Thus, $\phi(n) = 1$ by the definition of ϕ . By the formula for ϕ ,

$$\begin{aligned} \phi(n) &= n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_t}\right) \\ 1 &= (p_1 p_2 \cdots p_t) \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_t}\right) \\ &= (p_1 - 1)(p_2 - 1) \cdots (p_t - 1) > 1, \end{aligned}$$

which is a contradiction (since 3 is a prime). Hence there are infinitely many primes. □

Exercise 1.2. If n is a positive integer, prove the algebraic identity

$$a^n - b^n = (a - b) \sum_{k=0}^{n-1} a^k b^{n-1-k}.$$

Proof.

(1)

$$\begin{aligned} (a - b) \sum_{k=0}^{n-1} a^k b^{n-1-k} &= a \sum_{k=0}^{n-1} a^k b^{n-1-k} - b \sum_{k=0}^{n-1} a^k b^{n-1-k} \\ &= \sum_{k=0}^{n-1} a^{k+1} b^{n-1-k} - \sum_{k=0}^{n-1} a^k b^{n-k}. \end{aligned}$$

(2) Arrange summation index:

$$\begin{aligned}\sum_{k=0}^{n-1} a^{k+1} b^{n-1-k} &= \sum_{k=1}^n a^k b^{n-k} = a^n + \sum_{k=1}^{n-1} a^k b^{n-k}, \\ \sum_{k=0}^{n-1} a^k b^{n-k} &= b^n + \sum_{k=1}^{n-1} a^k b^{n-k}.\end{aligned}$$

(3) By (1)(2),

$$\begin{aligned}(a-b) \sum_{k=0}^{n-1} a^k b^{n-1-k} &= \left(a^n + \sum_{k=1}^{n-1} a^k b^{n-k} \right) - \left(b^n + \sum_{k=1}^{n-1} a^k b^{n-k} \right) \\ &= a^n - b^n.\end{aligned}$$

□

Supplement. Some exercises without proof.

- (1) Let x be a nilpotent element of A . Show that $1+x$ is a unit of A . Deduce that the sum of a nilpotent element and a unit is a unit. (Exercise 1.1 in the textbook: *Atiyah and Macdonald, Introduction to Commutative Algebra*)
- (2) Prove that $1^k + 2^k + \cdots + (p-1)^k \equiv 0(p)$ if $p-1 \nmid k$ and $-1(p)$ if $p-1 \mid k$. (Exercise 4.11 in the textbook: *Kenneth Ireland and Michael Rosen, A Classical Introduction to Modern Number Theory, 2nd edition*)
- (3) Use the existence of a primitive root to give another proof of Wilson's theorem $(p-1)! \equiv -1(p)$. (Exercise 4.12 in the textbook: *Kenneth Ireland and Michael Rosen, A Classical Introduction to Modern Number Theory, 2nd edition*)
- (4) Suppose n and F are integers and $n, F > 0$. Show that

$$B_n(Fx) = F^{n-1} \sum_{a=0}^{F-1} B_n\left(x + \frac{a}{F}\right).$$

where $B_n(x)$ are Bernoulli polynomials. (Exercise 15.19 in the textbook: *Kenneth Ireland and Michael Rosen, A Classical Introduction to Modern Number Theory, 2nd edition*)

- (5) Exercise 1.3.
- (6) Exercise 1.4.

□

Exercise 1.3. *If $2^n - 1$ is a prime, prove that n is prime. A prime of the form $2^p - 1$, where p is prime, is called a Mersenne prime.*

It suffices to prove that: *If $a^n - 1$ is a prime, show that $a = 2$ and that n is a prime.* Primes of the form $2^p - 1$ are called Mersenne primes. For example, $2^3 - 1 = 7$ and $2^5 - 1 = 31$. It is not known if there are infinitely many Mersenne primes.

Proof.

- (1) *n is a prime.* Assume n were not prime, say $n = rs$ for some $r, s > 1$. By Exercise 1.2, $a^{rs} - 1 = (a^s - 1)(\sum_{k=0}^{r-1} a^{sk})$. $a^s - 1 = 1$ since $a^s - 1 < a^{rs} - 1$ and $a^{rs} - 1$ is a prime. Hence $s = 1$ and $(a = 2)$, which is absurd.
- (2) *$a = 2$.* If a is odd, then $a^p - 1 > 2$ is even, which is not a prime. If $a > 2$ is even, $a^p - 1 = (a - 1)(\sum_{k=0}^{p-1} a^k)$. Both $a - 1 > 1$ and $\sum_{k=0}^{p-1} a^k > 1$, which is absurd.

By (1)(2), $a = 2$ and that n is a prime if $a^n - 1$ is a prime. □

Exercise 1.6. *Prove that every nonempty set of positive integers contains a smallest member. This is called the well-ordering principle.*

Proof. Use mathematical induction to establish that the well-ordering principle.

- (1) Given a set S of positive integers, let $P(n)$ be the proposition ‘If $m \in S$ for some $m \leq n$, then S has a least element’. Want to show $P(n)$ is true for all $n \in \mathbb{N}$.
 - (a) $P(1)$ is true. For $m \in S$ with $m \leq n = 1$, or $m = 1$ by the minimality of $1 \in \mathbb{N}$, S has a least element 1 (m itself) in \mathbb{N} .
 - (b) Suppose $P(n)$ is true. If $n + 1 \in S$, then there are only two possible cases.
 - (i) There is a positive integer $m \in S$ less than $n + 1$. So $n \geq m \in S$. Since $P(n)$ is true, S has a least element.
 - (ii) There is no positive integer $m \in S$ less than $n + 1$. In this case $n + 1$ is the least element in S .

In any cases (i)(ii), S has a least element, or $P(n + 1)$ is true.

By mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

- (2) *Show that the well-ordering principle holds.* Let T be a nonempty subset of \mathbb{N} , so there exists a positive integer $k \in T$. Notice that $P(k)$ is true by (1), thus T has a least element since $k \leq k$.

□

Supplement. Show that the well-ordering principle implies the principle of mathematical induction.

Proof. Suppose that

- (1) $P(n)$ be a proposition defined for each $n \in \mathbb{N}$,
- (2) $P(1)$ is true,
- (3) $[P(n) \Rightarrow P(n + 1)]$ is true.

Consider the set

$$S = \{n \in \mathbb{N} : P(n) \text{ is false}\} \subseteq \mathbb{N}.$$

Want to show S is empty, or the principle of mathematical induction holds. If S were nonempty, by the well-ordering principle S has a smallest element m . m cannot be 1 by (2). Say $m > 1$. Therefore, $m - 1 \in \mathbb{N}$ and $P(m - 1)$ is true by the minimality of m . By (3), $P((m - 1) + 1) = P(m)$ is true, which is absurd.

□

Rational and irrational numbers

Exercise 1.11. Given any real $x > 0$, prove that there is an irrational number between 0 and x .

Proof. There are only two possible cases: x is rational, or x is irrational.

- (1) x is rational. Pick $y = \frac{x}{\sqrt{89}} \in (0, x) \subseteq \mathbb{R}$. y is irrational.
- (2) x is irrational. Pick $y = \frac{x}{\sqrt{64}} \in (0, x) \subseteq \mathbb{R}$. y is irrational.

□

Proof (Exercise 4.12). Pick

$$y = \lim_{m \rightarrow \infty} [\lim_{n \rightarrow \infty} \cos^{2n}(m!\pi x)] \cdot \frac{x}{\sqrt{89}} + (1 - \lim_{m \rightarrow \infty} [\lim_{n \rightarrow \infty} \cos^{2n}(m!\pi x)]) \cdot \frac{x}{\sqrt{64}}.$$

- (1) x is rational. $y = \frac{x}{\sqrt{89}} \in (0, x) \subseteq \mathbb{R}$ is irrational.
- (2) x is irrational. $y = \frac{x}{\sqrt{64}} \in (0, x) \subseteq \mathbb{R}$ is irrational.

□

Upper bounds

Inequalities

Exercise 1.23. Prove Lagrange's identity for real numbers:

$$\left(\sum_{k=1}^n a_k b_k\right)^2 = \left(\sum_{k=1}^n a_k\right)^2 \left(\sum_{k=1}^n b_k\right)^2 - \sum_{1 \leq k < j \leq n} (a_k b_j - a_j b_k)^2.$$

Note that this identity implies the Cauchy-Schwarz inequality.

Proof. Put $(a_k, b_k, A_k, B_k) \mapsto (a_k, b_k, a_k, b_k)$ in the following generalization (Binet-Cauchy identity). \square

Generalization (Binet-Cauchy identity).

$$\begin{aligned} & \sum_{1 \leq k < j \leq n} (a_k b_j - a_j b_k)(A_k B_j - A_j B_k) \\ &= \left(\sum_{k=1}^n a_k A_k\right) \left(\sum_{k=1}^n b_k B_k\right) - \left(\sum_{k=1}^n a_k B_k\right) \left(\sum_{k=1}^n b_k A_k\right). \end{aligned}$$

Proof.

$$\begin{aligned} & \sum_{1 \leq k < j \leq n} (a_k b_j - a_j b_k)(A_k B_j - A_j B_k) \\ &= \sum_{1 \leq k < j \leq n} (a_k b_j A_k B_j + a_j b_k A_j B_k) - \sum_{1 \leq k < j \leq n} (a_k b_j A_j B_k - a_j b_k A_k B_j) \\ &= \sum_{1 \leq k < j \leq n} (a_k A_k b_j B_j + a_j A_j b_k B_k) - \sum_{1 \leq k < j \leq n} (a_k B_k b_j A_j + a_j B_j b_k A_k) \\ &= \sum_{1 \leq k \neq j \leq n} a_k A_k b_j B_j - \sum_{1 \leq k \neq j \leq n} a_k B_k b_j A_j \\ &= \sum_{1 \leq k, j \leq n} a_k A_k b_j B_j - \sum_{1 \leq k, j \leq n} a_k B_k b_j A_j \\ & \quad (\text{since } a_k A_k b_j B_j - a_k B_k b_j A_j = 0 \text{ as } k = j) \\ &= \left(\sum_{k=1}^n a_k A_k\right) \left(\sum_{j=1}^n b_j B_j\right) - \left(\sum_{k=1}^n a_k B_k\right) \left(\sum_{j=1}^n b_j A_j\right) \\ &= \left(\sum_{k=1}^n a_k A_k\right) \left(\sum_{k=1}^n b_k B_k\right) - \left(\sum_{k=1}^n a_k B_k\right) \left(\sum_{k=1}^n b_k A_k\right). \end{aligned}$$

□

Supplement ($\mathbb{Z}[i]$). As $n = 2$, $(a_1^2 + a_2^2)(b_1^2 + b_2^2) = (a_1b_1 + a_2b_2)^2 + (a_1b_2 - a_2b_1)^2$.

Define $N : \mathbb{Z}[i] \rightarrow \mathbb{Z}$ by $N(a + bi) = a^2 + b^2$.

- (1) Verify that for all $\alpha, \beta \in \mathbb{Z}[i]$, $N(\alpha\beta) = N(\alpha)N(\beta)$, either by direct computation or using the fact that $N(a + bi) = (a + bi)(a - bi)$. Conclude that if $\alpha \mid \gamma$ in $\mathbb{Z}[i]$, then $N(\alpha) \mid N(\gamma)$ in \mathbb{Z} .
- (2) Let $\alpha \in \mathbb{Z}[i]$. Show that α is a unit iff $N(\alpha) = 1$. Conclude that the only units are ± 1 and $\pm i$.
- (3) Let $\alpha \in \mathbb{Z}[i]$. Show that if $N(\alpha)$ is a prime in \mathbb{Z} then α is irreducible in $\mathbb{Z}[i]$. Show that the same conclusion holds if $N(\alpha) = p^2$, where p is a prime in \mathbb{Z} , $p \equiv 3 \pmod{4}$.
- (4) Show that $1 - i$ is irreducible in \mathbb{Z} and that $2 = u(1 - i)^2$ for some unit u .
- (5) Show that every nonzero, non-unit Gaussian integer α is a product of irreducible elements, by induction on $N(\alpha)$.
- (6) Use the unique factorization in $\mathbb{Z}[i]$ to prove that every prime $p \equiv 1 \pmod{4}$ is a sum of two squares.
- (7) Describe all irreducible elements in $\mathbb{Z}[i]$.

Complex numbers

Exercise 1.48. Prove Lagrange's identity for complex numbers:

$$\left| \sum_{k=1}^n a_k b_k \right|^2 = \sum_{k=1}^n |a_k|^2 \sum_{k=1}^n |b_k|^2 - \sum_{1 \leq k < j \leq n} |a_k \overline{b_j} - a_j \overline{b_k}|^2.$$

Proof. Put $(a_k, b_k, A_k, B_k) \mapsto (a_k, \overline{b_k}, \overline{a_k}, b_k)$ in the generalization to Exercise 1.23 (Binet-Cauchy identity) and use the identity $|z| = z\overline{z}$.