Chapter 1: Vector Spaces

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Section 1.2: Vector Spaces

Exercise 1.2.2. *Write the zero vector of* $\mathsf{M}_{3\times 4}(F)$*.*

Proof. $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathsf{M}_{3\times 4}(F)$. $\square$

Exercise 1.2.3. *If* $M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ *what are* M_{13}, M_{21}, M_{22}*?*

Proof. Since $M_{ij} = 3(i-1)+j$, $M_{13} = 3$, $M_{21} = 4$ and $M_{22} = 5$. $\square$

Exercise 1.2.22. *How many elements are there in the vector space* $\mathsf{M}_{m\times n}(\mathbb{Z}/2\mathbb{Z})$*?*

Proof. 2^{mn}. $\square$

Section 1.6: Bases and Dimension

Exercise 1.6.19. *Let* V *be a vector space having dimension* n*, and let* S *be a subset of* V *that generates* V*.*

(a) *Prove that there is a subset of* S *that is a basis for* V*. (Be careful not to assume that* S *is finite.)*

(b) *Prove that* S *contains at least* n *elements.*

Proof of (a). Similar to the argument in Theorem 1.9.

(1) If $S = \varnothing$ or $S = \{0\}$, then $\mathsf{V} = \{0\}$ and $\varnothing$ is a subset of S that is a basis for V.

(2) Otherwise S contains a nonzero element u_1. $\{u_1\}$ is a linearly independent set. Continue, if possible, choosing elements $u_2, ..., u_k$ in S such that $\{u_1, u_2, ..., u_k\}$ is linearly independent. By the Replacement Theorem (Theorem 1.10), we must eventually reach a stage at which $\beta = \{u_1, u_2, ..., u_k\}$ is a linearly independent subset of S with $k \leq n$.

(3) β generates S by the construction of β, and S generates V. Therefore, β generates V (and thus $k = n$ by the definition of dimension).

Therefore, there is a subset of S that is a basis for V. $\square$

Proof of (b). By (a), there is a subset $\beta \subseteq S$ of size n that is a basis for V. So S contains at least n elements of β. $\square$