Chapter 1: The Real And Complex Number Systems

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Integers

Exercise 1.1 *Prove that there is no largest prime. (A proof was known to Euclid.)*

There are many proofs of this result. We provide some of them.

Proof (Due to Euclid). If $p_1, p_2, ..., p_t$ were all primes, then write

$$n = p_1 p_2 \cdots p_t + 1$$

and there were a prime number p dividing n.

(1) p can not be any of $p_i (1 \leq i \leq t)$, otherwise p would divide the difference $n - p_1 p_2 \cdots p_t = 1$.

(2) This prime p is another prime $\neq p_i$ for $1 \leq i \leq t$, which is absurd.

□

Proof (Unique factorization theorem). Given N.

(1) *Show that* $\sum_{n \leq N} \frac{1}{n} \leq \prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1}$.
By the unique factorization theorem on $n \leq N$,

$$\sum_{n \leq N} \frac{1}{n} \leq \prod_{p \leq N} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1}.$$

(2) By (1) and the fact that $\sum \frac{1}{n}$ diverges, there are infinitely many primes.

□

Proof (Due to Eckford Cohen).

(1) $ord_p n! = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots$. For any $k = 1, 2, ..., n$, we can express k as $k = p^s t$ where $s = \mathrm{ord}_p k$ is a non-negative integer and $(t, p) = 1$. There are $[\frac{n}{p^a}]$ numbers such that $p^a \mid k$ for $a = 1, 2, ...$ Therefore, there are

$$\left[\frac{n}{p^a}\right] - \left[\frac{n}{p^{a+1}}\right]$$

numbers such that $\text{ord}_p k = a$ for $a = 1, 2, \ldots$. Hence,

$$\begin{aligned}\text{ord}_p n! &= \left(\left[\frac{n}{p}\right] - \left[\frac{n}{p^2}\right]\right) + 2\left(\left[\frac{n}{p^2}\right] - \left[\frac{n}{p^3}\right]\right) + 3\left(\left[\frac{n}{p^3}\right] - \left[\frac{n}{p^4}\right]\right) + \cdots \\ &= \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots .\end{aligned}$$

(2) *$\text{ord}_p n! \leq \frac{n}{p-1}$ and that $n!^{\frac{1}{n}} \leq \prod_{p|n!} p^{\frac{1}{p-1}}$.*

$$\begin{aligned}\text{ord}_p n! &= \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots \\ &\leq \frac{n}{p} + \frac{n}{p^2} + \frac{n}{p^3} + \cdots \\ &= \frac{\frac{n}{p}}{1 - \frac{1}{p}} \\ &= \frac{n}{p-1}.\end{aligned}$$

Thus,

$$n! = \prod_{p|n!} p^{\text{ord}_p n!} \leq \prod_{p|n!} p^{\frac{n}{p-1}} = \left(\prod_{p|n!} p^{\frac{1}{p-1}}\right)^n,$$

or

$$n!^{\frac{1}{n}} \leq \prod_{p|n!} p^{\frac{1}{p-1}}.$$

(3) $(n!)^2 \geq n^n$. Write $(n!)^2 = \prod_{k=1}^n k \prod_{k=1}^n (n+1-k) = \prod_{k=1}^n k(n+1-k)$, and $n^n = \prod_{k=1}^n n$. It suffices to show that $k(n+1-k) \geq n$ for each $1 \leq k \leq n$. Notice that $k(n+1-k) - n = (n-k)(k-1) \geq 0$ for $1 \leq k \leq n$. The inequality holds.

(4) By (3)(4), $\prod_{p|n!} p^{\frac{1}{p-1}} \geq \sqrt{n}$. Assume that there are finitely many primes, the value $\prod_{p|n!} p^{\frac{1}{p-1}}$ is a finite number whenever the value of n. However, $\sqrt{n} \to \infty$ as $n \to \infty$, which leads to a contradiction. Hence there are infinitely many primes.

$\square$

Proof (Formula for $\phi(n)$). If $p_1, p_2, \ldots, p_t$ were all primes, then let $n = p_1 p_2 \cdots p_t$ and all numbers between 2 and n are NOT relatively prime to n. Thus, $\phi(n) = 1$ by the definition of ϕ. By the formula for ϕ,

$$\begin{aligned}\phi(n) &= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_t}\right) \\ 1 &= (p_1 p_2 \cdots p_t)\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_t}\right) \\ &= (p_1 - 1)(p_2 - 1)\cdots(p_t - 1) > 1,\end{aligned}$$

which is a contradiction (since 3 is a prime). Hence there are infinitely many primes. □

Exercise 1.2 *If n is a positive integer, prove the algebraic identity*

$$a^n - b^n = (a-b)\sum_{k=0}^{n-1} a^k b^{n-1-k}.$$

Proof.

(1)

$$\begin{aligned}(a-b)\sum_{k=0}^{n-1} a^k b^{n-1-k} &= a\sum_{k=0}^{n-1} a^k b^{n-1-k} - b\sum_{k=0}^{n-1} a^k b^{n-1-k} \\ &= \sum_{k=0}^{n-1} a^{k+1} b^{n-1-k} - \sum_{k=0}^{n-1} a^k b^{n-k}.\end{aligned}$$

(2) Arrange index in summation symbols.

$$\sum_{k=0}^{n-1} a^{k+1} b^{n-1-k} = \sum_{k=1}^{n} a^k b^{n-k} = a^n + \sum_{k=1}^{n-1} a^k b^{n-k},$$

$$\sum_{k=0}^{n-1} a^k b^{n-k} = b^n + \sum_{k=1}^{n-1} a^k b^{n-k}.$$

(3) By (1)(2),

$$\begin{aligned}(a-b)\sum_{k=0}^{n-1} a^k b^{n-1-k} &= \left(a^n + \sum_{k=1}^{n-1} a^k b^{n-k}\right) - \left(b^n + \sum_{k=1}^{n-1} a^k b^{n-k}\right) \\ &= a^n - b^n.\end{aligned}$$

□

Supplement (Exercise 1.1 in Atiyah and Macdonald, Introduction to Commutative Algebra). *Let x be a nilpotent element of A. Show that $1+x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit.*

Proof.

(1) Suppose $x^m = 0$ for some odd integer $m \geq 0$. Then

$$1 = 1 + x^m = (1+x)(1 - x + x^2 - \cdots + (-1)^{m-1}x^{m-1}),$$

or $1+x$ is a unit.

(2) If u is any unit and x is any nilpotent, $u + x = u \cdot (1 + u^{-1}x)$ is a product of two units (using that $u^{-1}x$ is nilpotent and applying (1)) and hence a unit again.

□

Exercise 1.3 *If $2^n - 1$ is a prime, prove that n is prime. A prime of the form $2^p - 1$, where p is prime, is called a Mersenne prime.*

It suffices to prove that: *If $a^n - 1$ is a prime, show that $a = 2$ and that n is a prime.* Primes of the form $2^p - 1$ are called Mersenne primes. For example, $2^3 - 1 = 7$ and $2^5 - 1 = 31$. It is not known if there are infinitely many Mersenne primes.

Proof.

(1) *n is a prime.* Assume n were not prime, say $n = rs$ for some $r, s > 1$. By Exercise 1.2, $a^{rs} - 1 = (a^s - 1)(\sum_{k=0}^{r-1} a^{sk})$. $a^s - 1 = 1$ since $a^s - 1 < a^{rs} - 1$ and $a^{rs} - 1$ is a prime. Hence $s = 1$ and $(a = 2)$, which is absurd.

(2) $a = 2$. If a is odd, then $a^p - 1 > 2$ is even, which is not a prime. If $a > 2$ is even, $a^p - 1 = (a - 1)(\sum_{k=0}^{p-1} a^k)$. Both $a - 1 > 1$ and $\sum_{k=0}^{p-1} a^k > 1$, which is absurd.

By (1)(2), $a = 2$ and that n is a prime if $a^n - 1$ is a prime. □

Rational and irrational numbers

Exercise 1.11 *Given any real $x > 0$, prove that there is an irrational number between 0 and x.*

Proof. There are only two possible cases: x is rational, or x is irrational.

(1) *x is rational.* Pick $y = \frac{x}{\sqrt{89}} \in (0, x) \subseteq \mathbb{R}$. y is irrational.

(2) *x is irrational.* Pick $y = \frac{x}{\sqrt{64}} \in (0, x) \subseteq \mathbb{R}$. y is irrational.

□

Proof (Exercise 4.12). Pick

$$y = \lim_{m \to \infty} [\lim_{n \to \infty} \cos^{2n}(m!\pi x)] \cdot \frac{x}{\sqrt{89}} + (1 - \lim_{m \to \infty} [\lim_{n \to \infty} \cos^{2n}(m!\pi x)]) \cdot \frac{x}{\sqrt{64}}.$$

(1) *x is rational.* $y = \frac{x}{\sqrt{89}} \in (0, x) \subseteq \mathbb{R}$ is irrational.

(2) *x is irrational.* $y = \frac{x}{\sqrt{64}} \in (0, x) \subseteq \mathbb{R}$ is irrational.

$\square$