

## Chapter 9: Functions of Several Variables

Author: Meng-Gen Tsai

Email: plover@gmail.com

**Exercise 9.1.** If  $S$  is a nonempty subset of a vector space  $X$ , prove (as asserted in Section 9.1) that the span of  $S$  is a vector space.

Denote the span of  $S$  by  $\text{span}(S)$ .

*Proof.*

- (1) Since  $S \neq \emptyset$ , there is  $\mathbf{z} \in S$ . So  $1\mathbf{z} = \mathbf{z} \in \text{span}(S) \neq \emptyset$ . (In fact,  $\text{span}(S) \supseteq S$ .)
- (2) If  $\mathbf{x}, \mathbf{y} \in \text{span}(S)$ , then there exist elements  $\mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{y}_1, \dots, \mathbf{y}_n \in S$  and scalars  $a_1, \dots, a_m, b_1, \dots, b_n$  such that

$$\mathbf{x} = a_1\mathbf{x}_1 + \dots + a_m\mathbf{x}_m,$$

$$\mathbf{y} = b_1\mathbf{y}_1 + \dots + b_n\mathbf{y}_n.$$

Then

$$\mathbf{x} + \mathbf{y} = a_1\mathbf{x}_1 + \dots + a_m\mathbf{x}_m + b_1\mathbf{y}_1 + \dots + b_n\mathbf{y}_n$$

is a linear combination of the elements of  $S$ . For any scalar  $c$ ,

$$c\mathbf{x} = (ca_1)\mathbf{x}_1 + \dots + (ca_m)\mathbf{x}_m$$

is again linear combination of the elements of  $S$ .

- (3) By (1)(2),  $\text{span}(S)$  is a vector space.

□

*Note.* Any subspace of  $X$  that contains  $S$  must also contain  $\text{span}(S)$ .

**Exercise 9.2.** Prove (as asserted in Section 9.6) that  $BA$  is linear if  $A$  and  $B$  are linear transformations. Prove also that  $A^{-1}$  is linear and invertible if  $A$  is invertible.

*Proof.* Use the notation in Definitions 9.6.

- (1) Show that  $BA$  is linear if  $A$  and  $B$  are linear transformations. Let  $X, Y, Z$  be vector spaces,  $A \in L(X, Y)$  and  $B \in L(Y, Z)$ .

(a) Given any  $\mathbf{x}_1, \mathbf{x}_2 \in X$ .

$$\begin{aligned}
(BA)(\mathbf{x}_1 + \mathbf{x}_2) &= B(A(\mathbf{x}_1 + \mathbf{x}_2)) \\
&= B(A\mathbf{x}_1 + A\mathbf{x}_2) && (A \text{ is a linear transformation}) \\
&= B(A\mathbf{x}_1) + B(A\mathbf{x}_2) && (B \text{ is a linear transformation}) \\
&= (BA)\mathbf{x}_1 + (BA)\mathbf{x}_2.
\end{aligned}$$

(b) For any  $\mathbf{x} \in X$  and scalar  $c$ ,

$$\begin{aligned}
(BA)(c\mathbf{x}) &= B(A(c\mathbf{x})) \\
&= B(cA\mathbf{x}) && (A \text{ is a linear transformation}) \\
&= cB(A\mathbf{x}) && (B \text{ is a linear transformation}) \\
&= c(BA)\mathbf{x}.
\end{aligned}$$

By (a)(b),  $BA \in L(X, Z)$ .

(2) Show that  $A^{-1}$  is linear if  $A$  is invertible.

(a) Given any  $\mathbf{y}_1, \mathbf{y}_2 \in X$ . Since  $A$  is surjective, there exist  $\mathbf{x}_1, \mathbf{x}_2 \in X$  such that

$$\begin{aligned}
\mathbf{y}_1 &= A\mathbf{x}_1 \\
\mathbf{y}_2 &= A\mathbf{x}_2.
\end{aligned}$$

So

$$\begin{aligned}
A^{-1}\mathbf{y}_1 &= A^{-1}(A\mathbf{x}_1) = \mathbf{x}_1 \\
A^{-1}\mathbf{y}_2 &= A^{-1}(A\mathbf{x}_2) = \mathbf{x}_2
\end{aligned}$$

(by Definitions 9.4). Hence

$$\begin{aligned}
A^{-1}(\mathbf{y}_1 + \mathbf{y}_2) &= A^{-1}(A\mathbf{x}_1 + A\mathbf{x}_2) \\
&= A^{-1}(A(\mathbf{x}_1 + \mathbf{x}_2)) && (A \text{ is a linear transformation}) \\
&= \mathbf{x}_1 + \mathbf{x}_2 && (\text{Definitions 9.4}) \\
&= A^{-1}\mathbf{y}_1 + A^{-1}\mathbf{y}_2.
\end{aligned}$$

(b) For any  $\mathbf{y} \in X$  and scalar  $c$ , there is a corresponding  $\mathbf{x} \in X$  such that  $\mathbf{y} = A\mathbf{x}$  since  $A$  is surjective. So  $A^{-1}\mathbf{y} = \mathbf{x}$  by Definition 9.4. Hence

$$\begin{aligned}
A^{-1}(c\mathbf{y}) &= A^{-1}(cA\mathbf{x}) \\
&= A^{-1}(A(c\mathbf{x})) && (A \text{ is a linear transformation}) \\
&= c\mathbf{x} && (\text{Definitions 9.4}) \\
&= cA^{-1}\mathbf{y}.
\end{aligned}$$

By (a)(b),  $A^{-1} \in L(X)$ .

(3) Show that  $A^{-1}$  is invertible if  $A$  is invertible. It suffices to show that  $A^{-1}$  is injective and surjective.

(a) Show that  $A^{-1}$  is injective. Given any  $\mathbf{y}_1, \mathbf{y}_2 \in X$ . Since  $A$  is surjective, there exist  $\mathbf{x}_1, \mathbf{x}_2 \in X$  such that

$$\begin{aligned}\mathbf{y}_1 &= A\mathbf{x}_1 \\ \mathbf{y}_2 &= A\mathbf{x}_2.\end{aligned}$$

Suppose  $A^{-1}\mathbf{y}_1 = A^{-1}\mathbf{y}_2$ . So  $A^{-1}(A\mathbf{x}_1) = A^{-1}(A\mathbf{x}_2)$ , or  $\mathbf{x}_1 = \mathbf{x}_2$ , or  $\mathbf{y}_1 = A\mathbf{x}_1 = A\mathbf{x}_2 = \mathbf{y}_2$ .

(b) Show that  $A^{-1}$  is surjective. For any  $\mathbf{x} \in X$ , there exists  $A\mathbf{x} \in X$  such that  $A^{-1}(A\mathbf{x}) = \mathbf{x}$  by Definitions 9.4.

□

**Exercise 9.3.** Assume  $A \in L(X, Y)$  and  $A\mathbf{x} = \mathbf{0}$  only when  $\mathbf{x} = \mathbf{0}$ . Prove that  $A$  is then 1-1.

*Proof.* Suppose  $A\mathbf{x} = A\mathbf{y}$ . Since  $A$  is a linear transformation,  $A(\mathbf{x} - \mathbf{y}) = A\mathbf{x} - A\mathbf{y} = \mathbf{0}$ . By assumption,  $\mathbf{x} - \mathbf{y} = \mathbf{0}$  or  $\mathbf{x} = \mathbf{y}$ . □

**Exercise 9.4.** Prove (as asserted in Section 9.30) that null spaces and ranges of linear transformations are vector spaces.

*Proof.* Use the notation in Definitions 9.30. Suppose  $X, Y$  are vector spaces, and  $A \in L(X, Y)$ , as in Definition 9.6.

(1) Show that  $\mathcal{N}(A)$  is a vector space in  $X$ .

(a) Note that  $\mathbf{0} \in X$ . Since  $A\mathbf{0} = \mathbf{0}$ ,  $\mathbf{0} \in \mathcal{N}(A) \neq \emptyset$ .

(b) Suppose  $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{N}(A)$ . Then

$$\begin{aligned}A(\mathbf{x}_1 + \mathbf{x}_2) &= A\mathbf{x}_1 + A\mathbf{x}_2 && (A \text{ is a linear transformation}) \\ &= \mathbf{0} + \mathbf{0} && (\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{N}(A)) \\ &= \mathbf{0}.\end{aligned}$$

So  $\mathbf{x}_1 + \mathbf{x}_2 \in \mathcal{N}(A)$ .

(c) Suppose  $\mathbf{x} \in \mathcal{N}(A)$  and  $c$  is a scalar. Then

$$\begin{aligned}A(c\mathbf{x}) &= cA\mathbf{x} && (A \text{ is a linear transformation}) \\ &= c\mathbf{0} && (\mathbf{x} \in \mathcal{N}(A)) \\ &= \mathbf{0}.\end{aligned}$$

So  $c\mathbf{x} \in \mathcal{N}(A)$ .

By (a)(b)(c),  $\mathcal{N}(A)$  is a vector space.

(2) Show that  $\mathcal{R}(A)$  is a vector space in  $Y$ .

(a) Note that  $\mathbf{0} \in X$ . So  $A\mathbf{0} = \mathbf{0} \in \mathcal{R}(A) \neq \emptyset$ .

(b) Suppose  $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{R}(A)$ . Then there exist  $\mathbf{x}_1, \mathbf{x}_2 \in X$  such that  $A\mathbf{x}_1 = \mathbf{y}_1$  and  $A\mathbf{x}_2 = \mathbf{y}_2$ . Hence

$$\begin{aligned}\mathbf{y}_1 + \mathbf{y}_2 &= A\mathbf{x}_1 + A\mathbf{x}_2 \\ &= A(\mathbf{x}_1 + \mathbf{x}_2) \quad (A \text{ is a linear transformation}).\end{aligned}$$

So  $\mathbf{y}_1 + \mathbf{y}_2 \in \mathcal{R}(A)$ .

(c) Suppose  $\mathbf{y} \in \mathcal{R}(A)$  and  $c$  is a scalar. Then there exists  $\mathbf{x} \in X$  such that  $A\mathbf{x} = \mathbf{y}$ . Hence

$$\begin{aligned}c\mathbf{y} &= cA\mathbf{x} \\ &= A(c\mathbf{x}) \quad (A \text{ is a linear transformation}).\end{aligned}$$

So  $c\mathbf{y} \in \mathcal{R}(A)$ .

By (a)(b)(c),  $\mathcal{R}(A)$  is a vector space.

□

**Exercise 9.5.** Prove that to every  $A \in L(\mathbb{R}^n, \mathbb{R}^1)$  corresponds a unique  $\mathbf{y} \in \mathbb{R}^n$  such that  $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$ . Prove also that  $\|A\| = \|\mathbf{y}\|$ . (Hint: Under certain conditions, equality holds in the Schwarz inequality.)

*Proof.*

(1) Recall that  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  is the standard basis of  $\mathbb{R}^n$  (Definitions 9.1).

Given any  $\mathbf{x} \in \mathbb{R}^n$ , write  $\mathbf{x} = (x_1, \dots, x_n)$  as  $\mathbf{x} = \sum x_j \mathbf{e}_j$ .

(2) Show that  $\mathbf{y}$  exists. Since  $A$  is a linear transformation,

$$\begin{aligned}A\mathbf{x} &= A\left(\sum x_j \mathbf{e}_j\right) \\ &= \sum x_j A\mathbf{e}_j \\ &= (x_1, \dots, x_n) \cdot (A\mathbf{e}_1, \dots, A\mathbf{e}_n) \\ &= \mathbf{x} \cdot \sum (A\mathbf{e}_j) \mathbf{e}_j.\end{aligned}$$

Define  $\mathbf{y} = \sum (A\mathbf{e}_j) \mathbf{e}_j \in \mathbb{R}^n$  so that  $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$ .

(3) Show that  $\mathbf{y}$  is unique. Suppose there exists some  $\mathbf{z} \in \mathbb{R}^n$  such that  $A\mathbf{x} = \mathbf{x} \cdot \mathbf{z}$ . So

$$\begin{aligned}0 &= A\mathbf{x} - A\mathbf{x} \\ &= \mathbf{x} \cdot \mathbf{y} - \mathbf{x} \cdot \mathbf{z} \\ &= \mathbf{x} \cdot (\mathbf{y} - \mathbf{z})\end{aligned}$$

for any  $\mathbf{x} \in \mathbb{R}^n$ . In particular, take  $\mathbf{x} = \mathbf{y} - \mathbf{z} \in \mathbb{R}^n$  to get

$$0 = (\mathbf{y} - \mathbf{z}) \cdot (\mathbf{y} - \mathbf{z}) = |\mathbf{y} - \mathbf{z}|^2$$

or  $\mathbf{y} - \mathbf{z} = \mathbf{0}$  or  $\mathbf{y} = \mathbf{z}$ .

(4) *Show that  $\|A\| = |\mathbf{y}|$ .* By the Schwarz inequality (Theorem 1.37(d)),

$$|A\mathbf{x}| = |\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}| \leq |\mathbf{y}|$$

as  $|\mathbf{x}| \leq 1$ . Take the sup over all  $|\mathbf{x}| \leq 1$  to get

$$\|A\| \leq |\mathbf{y}|.$$

If  $\mathbf{y} = \mathbf{0}$ , then  $\|A\| = |\mathbf{y}| = 0$ . If  $\mathbf{y} \neq \mathbf{0}$ , then the equality holds when  $\mathbf{x} = \frac{\mathbf{y}}{|\mathbf{y}|} \in \mathbb{R}^n$ . (Here  $|\mathbf{x}| = 1$ .)

□

**Exercise 9.6.** *If  $f(0,0) = 0$  and*

$$f(x,y) = \frac{xy}{x^2 + y^2} \quad \text{if } (x,y) \neq (0,0),$$

*prove that  $(D_1f)(x,y)$  and  $(D_2f)(x,y)$  exist at every point of  $\mathbb{R}^2$ , although  $f$  is not continuous at  $(0,0)$ .*

*Proof.*

(1) *Show that*

$$(D_1f)(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0), \\ \frac{y(y^2 - x^2)}{(x^2 + y^2)^2} & \text{if } (x,y) \neq (0,0). \end{cases}$$

Write

$$\begin{aligned} (D_1f)(x,y) &= \lim_{t \rightarrow 0} \frac{f((x,y) + t(1,0)) - f(x,y)}{t} \\ &= \lim_{t \rightarrow 0} \frac{f(x+t,y) - f(x,y)}{t}. \end{aligned}$$

If  $(x,y) = (0,0)$ ,

$$(D_1f)(0,0) = \lim_{t \rightarrow 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t \rightarrow 0} \frac{0 - 0}{t} = 0.$$

If  $(x, y) \neq (0, 0)$ ,

$$\begin{aligned}
 (D_1 f)(x, y) &= \lim_{t \rightarrow 0} \frac{f(x+t, y) - f(x, y)}{t} \\
 &= \lim_{t \rightarrow 0} \frac{\frac{(x+t)y}{(x+t)^2 + y^2} - \frac{xy}{x^2 + y^2}}{t} \\
 &= \lim_{t \rightarrow 0} \frac{y(y^2 - x^2) - txy}{((x+t)^2 + y^2)(x^2 + y^2)} \\
 &= \frac{y(y^2 - x^2)}{(x^2 + y^2)^2}.
 \end{aligned}$$

(2) Show that

$$(D_2 f)(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0), \\ \frac{x(x^2 - y^2)}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0). \end{cases}$$

Similar to (1).

(3) Show that  $f$  is not continuous at  $(0, 0)$ . Note that

$$\lim_{n \rightarrow \infty} f\left(\frac{1}{n}, \frac{1}{n}\right) = \lim_{n \rightarrow \infty} \frac{\frac{1}{n} \cdot \frac{1}{n}}{\frac{1}{n^2} + \frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{1}{2} = \frac{1}{2}$$

and

$$\lim_{n \rightarrow \infty} f\left(\frac{1}{n}, 0\right) = \lim_{n \rightarrow \infty} \frac{0}{\frac{1}{n^2} + 0} = \lim_{n \rightarrow \infty} 0 = 0.$$

Hence the limit  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  does not exist.

□

### Exercise 9.7. ...

*Proof.*

(1)

(2)

□

### Exercise 9.8. ...

*Proof.*

(1)

(2)

□

**Exercise 9.9. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.10. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.11. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.12. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.13. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.14. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.15. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.16. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.17. ...**

*Proof.*

(1)

(2)



□

**Exercise 9.18. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.19. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.20. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.21. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.22. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.23. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.24. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.25. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.26. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.27. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.28. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.29. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.30. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.31. ...**

*Proof.*

(1)

(2)

□