# Chapter 2: Modules

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**Exercise 2.1.** *Show that* $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$ *if* $m$, $n$ *are coprime.*

It suffices to show that

$$(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$$

where $d$ is the greatest common divisor of $m$ and $n$.

*Outlines.*

(1) Define $\widetilde{\varphi}$ by

$$\begin{array}{rccc} \widetilde{\varphi}: & (\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z}) & \longrightarrow & \mathbb{Z}/d\mathbb{Z} \\ & \cup & & \cup \\ & (x + m\mathbb{Z}, y + n\mathbb{Z}) & \longmapsto & xy + d\mathbb{Z}. \end{array}$$

$\widetilde{\varphi}$ is well-defined and $\mathbb{Z}$-bilinear.

(2) By the universal property, $\widetilde{\varphi}$ factors through a $\mathbb{Z}$-bilinear map

$$\varphi : (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/d\mathbb{Z}$$

(such that $\varphi(x \otimes y) = \widetilde{\varphi}(x, y)$).

(3) To show that $\varphi$ is isomorphic, might find the inverse map $\psi : \mathbb{Z}/d\mathbb{Z} \to (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$ of $\varphi$. Define $\psi$ by

$$\begin{array}{rccc} \psi: & \mathbb{Z}/d\mathbb{Z} & \longrightarrow & (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \\ & \cup & & \cup \\ & z + d\mathbb{Z} & \longmapsto & (z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}). \end{array}$$

$\psi$ is well-defined and $\mathbb{Z}$-linear.

(4) $\psi \circ \varphi = \mathrm{id}$.

(5) $\varphi \circ \psi = \mathrm{id}$.

*Proof of (1).*

(a) $\widetilde{\varphi}$ *is well-defined.* Say $x' = x + am$ for some $a \in \mathbb{Z}$ and $y' = y + bn$ for some $b \in \mathbb{Z}$. Then $x'y' - xy = yam + xbn + abmn \in \mathbb{Z}/d\mathbb{Z}$. That is, $\widetilde{\varphi}$ is independent of coset representative.

(b) $\widetilde{\varphi}$ *is $\mathbb{Z}$-bilinear.*

(i) *For any $\lambda \in \mathbb{Z}$, $\widetilde{\varphi}(\lambda x, y) = \widetilde{\varphi}(x, \lambda y) = \lambda \widetilde{\varphi}(x, y)$.* In fact,

$$\begin{aligned}
\widetilde{\varphi}(\lambda(x + m\mathbb{Z}), y + n\mathbb{Z}) &= \widetilde{\varphi}(\lambda x + m\mathbb{Z}, y + n\mathbb{Z}) = \lambda xy + d\mathbb{Z}, \\
\widetilde{\varphi}(x + m\mathbb{Z}, \lambda(y + n\mathbb{Z})) &= \widetilde{\varphi}(x + m\mathbb{Z}, \lambda y + n\mathbb{Z}) = \lambda xy + d\mathbb{Z}, \\
\widetilde{\varphi}(x_1 + m\mathbb{Z}, y + n\mathbb{Z}) &= \lambda(xy + d\mathbb{Z}) = \lambda xy + d\mathbb{Z}.
\end{aligned}$$

(ii) $\widetilde{\varphi}(x_1 + x_2, y) = \widetilde{\varphi}(x_1, y) + \widetilde{\varphi}(x_2, y)$. In fact,

$$\begin{aligned}
\widetilde{\varphi}((x_1 + x_2) + m\mathbb{Z}, y + n\mathbb{Z}) &= (x_1 + x_2)y + d\mathbb{Z}, \\
\widetilde{\varphi}(x_1 + m\mathbb{Z}, y + n\mathbb{Z}) + \widetilde{\varphi}(x_2 + m\mathbb{Z}, y + n\mathbb{Z}) &= (x_1 y + d\mathbb{Z}) + (x_2 y + d\mathbb{Z}) \\
&= (x_1 + x_2)y + d\mathbb{Z}.
\end{aligned}$$

(iii) $\widetilde{\varphi}(x, y_1 + y_2) = \widetilde{\varphi}(x, y_1) + \widetilde{\varphi}(x, y_2)$. Similar to (ii).

□

*Proof of (3).*

(a) $\psi$ *is well-defined.* Say $z' = z + cd$ for some $c \in \mathbb{Z}$. Note that $d = \alpha m + \beta n$ for some $\alpha, \beta \in \mathbb{Z}$. Thus

$$\begin{aligned}
\psi(z' + d\mathbb{Z}) &= \psi(z + cd + d\mathbb{Z}) \\
&= \psi(z + c(\alpha m + \beta n) + d\mathbb{Z}) \\
&= (z + c(\alpha m + \beta n) + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\
&= (z + c\beta n + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\
&= (z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) + (c\beta n + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\
&= \psi(z + d\mathbb{Z}) + (1 + m\mathbb{Z}) \otimes (c\beta n + n\mathbb{Z}) \\
&= \psi(z + d\mathbb{Z}).
\end{aligned}$$

(b) $\psi$ *is $\mathbb{Z}$-linear.*

(i) *For any $\lambda \in \mathbb{Z}$, $\psi(\lambda z) = \lambda \psi(z)$.* In fact,

$$\begin{aligned}
\psi(\lambda(z + d\mathbb{Z})) &= \psi(\lambda z + d\mathbb{Z}) = (\lambda z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}), \\
\lambda \psi(z + d\mathbb{Z}) &= \lambda((z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z})) = (\lambda z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}).
\end{aligned}$$

(ii) $\psi(z_1 + z_2) = \psi(z_1) + \psi(z_2)$.

$$\begin{aligned}
\psi((z_1 + z_2) + d\mathbb{Z}) &= (z_1 + z_2 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}), \\
\psi(z_1 + d\mathbb{Z}) + \psi(z_2 + d\mathbb{Z}) &= (z_1 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) + (z_2 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\
&= (z_1 + z_2 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}).
\end{aligned}$$

$\square$

*Proof of (4).* For any $(x+m\mathbb{Z}) \otimes (y+n\mathbb{Z}) \in (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$,

$$\begin{aligned}\psi(\varphi((x+m\mathbb{Z}) \otimes (y+n\mathbb{Z}))) &= \psi(xy+d\mathbb{Z}) \\ &= (xy+m\mathbb{Z}) \otimes (1+n\mathbb{Z}) \\ &= (x+m\mathbb{Z}) \otimes (y+n\mathbb{Z}).\end{aligned}$$

$\square$

*Proof of (5).* For any $z+d\mathbb{Z} \in \mathbb{Z}/d\mathbb{Z}$,

$$\begin{aligned}\varphi(\psi(z+d\mathbb{Z}) &= \varphi((z+m\mathbb{Z}) \otimes (1+n\mathbb{Z})) \\ &= z+d\mathbb{Z}.\end{aligned}$$

$\square$

**Exercise 2.2.** *Let $A$ be a ring, $\mathfrak{a}$ an ideal, $M$ an $A$-module. Show that $(A/\mathfrak{a}) \otimes_A M$ is isomorphic to $M/\mathfrak{a}M$. (Hint: Tensor the exact sequence $0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$ with $M$.*

*Proof (Hint).* There is a natural exact sequence $E$:

$$E : 0 \to \mathfrak{a} \xrightarrow{i} A \xrightarrow{\pi} A/\mathfrak{a} \to 0$$

where $i$ is the inclusion map (and $\pi$ is the projection map). Tensor $E$ with $M$:

$$E' : \mathfrak{a} \otimes_A M \xrightarrow{i\otimes 1} A \otimes_A M \xrightarrow{\pi\otimes 1} (A/\mathfrak{a}) \otimes_A M \to 0$$

is exact, or

$$(A/\mathfrak{a}) \otimes_A M \cong A \otimes_A M/\mathrm{im}(i \otimes 1).$$

By Proposition 2.14, There is an unique isomorphism $A \otimes_A M \to M$ defined by $a \otimes x \mapsto ax$. This isomorphism sends $\mathrm{im}(i \otimes 1)$ to $\mathfrak{a}M$. Therefore,

$$(A/\mathfrak{a}) \otimes_A M \cong M/\mathfrak{a}M.$$

$\square$

*Proof (Brute-force).*

(1) Define $\widetilde{\varphi}$ by

$$\begin{array}{rccc} \widetilde{\varphi}: & A/\mathfrak{a} \times M & \longrightarrow & M/\mathfrak{a}M \\ & \cup & & \cup \\ & (a+\mathfrak{a}, x) & \longmapsto & ax+\mathfrak{a}M. \end{array}$$

$\widetilde{\varphi}$ is well-defined and $A$-bilinear.

(2) By the universal property, $\widetilde{\varphi}$ factors through a $A$-bilinear map

$$\varphi : A/\mathfrak{a} \otimes_A M \to M/\mathfrak{a}M$$

(such that $\varphi(a \otimes x) = \widetilde{\varphi}(a, x)$).

(3) To show that $\varphi$ is isomorphic, might find the inverse map $\psi : M/\mathfrak{a}M \to A/\mathfrak{a} \otimes_A M$ of $\varphi$. Define $\psi$ by

$$\begin{array}{rccc} \psi : & M/\mathfrak{a}M & \longrightarrow & A/\mathfrak{a} \otimes_A M \\ & \cup & & \cup \\ & x + \mathfrak{a}M & \longmapsto & (1 + \mathfrak{a}) \otimes x. \end{array}$$

$\psi$ is well-defined and $A$-linear.

(4) $\psi \circ \varphi = \mathrm{id}$.

(5) $\varphi \circ \psi = \mathrm{id}$.

$\square$

**Exercise 2.3.** *Let $A$ be a local ring, $M$ and $N$ finitely generated $A$-modules. Prove that if $M \otimes_A N = 0$, then $M = 0$ or $N = 0$. (Hint: Let $\mathfrak{m}$ be the maximal ideal, $k = A/\mathfrak{m}$ the residue field. Let $M_k = k \otimes_A M \cong M/\mathfrak{m}M$ by Exercise 2.2. By Nakayama's lemma, $M_k = 0 \Longrightarrow M = 0$. But $M \otimes_A N = 0 \Longrightarrow (M \otimes_A N)_k = 0 \Longrightarrow M_k \otimes_k N_k = 0 \Longrightarrow M_k = 0$ or $N_k = 0$ since $M_k$, $N_k$ are vector spaces over a field.)*

The conclusion might be false if $A$ is not local. For example, Exercise 2.1.

*Proof (Hint).* Let $\mathfrak{m}$ be the maximal ideal, $k = A/\mathfrak{m}$ the residue field. Let $M_k = k \otimes_A M$.

(1) *(Base extension) Show that $(M \otimes_A N)_k = M_k \otimes_k N_k$.* In fact, by Proposition 2.14

$$\begin{aligned} (M \otimes_A N)_k &= k \otimes_A (M \otimes_A N) \\ &= (k \otimes_A M) \otimes_A N \\ &= M_k \otimes_A N \\ &= (M_k \otimes_k k) \otimes_A N \\ &= M_k \otimes_k (k \otimes_A N) \\ &= M_k \otimes_k N_k. \end{aligned}$$

(2)

$$\begin{aligned}
M \otimes_A N = 0 &\Longrightarrow (M \otimes_A N)_k = 0 \\
&\Longrightarrow M_k \otimes_k N_k = 0 && ((1)) \\
&\Longrightarrow M_k = 0 \text{ or } N_k = 0 && (M_k, N_k\text{: vector spaces}) \\
&\Longrightarrow M/\mathfrak{m}M = 0 \text{ or } M/\mathfrak{m}M = 0 && (\text{Exercise 2.2}) \\
&\Longrightarrow M = 0 \text{ or } N = 0. && (\text{Nakayama's lemma})
\end{aligned}$$

□

**Exercise 2.4.** *Let $M_i$ $(i \in I)$ be any family of $A$-modules, and let $M$ be their direct sum. Prove that $M$ is flat $\Leftrightarrow$ each $M_i$ is flat.*

*Proof.* Given any $A$-module homomorphism $f : N' \to N$.

(1) Similar to Proposition 2.14(iii), we have two isomorphisms

(a)

$$\varphi : \bigoplus_{i \in I} (N' \otimes M_i) \cong N' \otimes_A \bigoplus_{i \in I} M_i$$

defined by

$$\varphi((x \otimes m_i)_{i \in I}) = x \otimes (m_i)_{i \in I}$$

where $x \in N'$, $m_i \in M_i$ $(i \in I)$.

(b)

$$\psi : N \otimes_A \bigoplus_{i \in I} M_i \cong \bigoplus_{i \in I} (N \otimes M_i)$$

defined by

$$\psi(y \otimes (m_i)_{i \in I}) = (y \otimes m_i)_{i \in I}$$

where $y \in N$, $m_i \in M_i$ $(i \in I)$.

(2) $f : N' \to N$ induces an $A$-module homomorphism

$$f \otimes \mathrm{id}_M : N' \otimes_A M \to N \otimes_A M.$$

(3) $\psi \circ f \otimes \mathrm{id}_M \circ \varphi$ defines an $A$-module homomorphism

$$\psi \circ f \otimes \mathrm{id}_M \circ \varphi : \bigoplus_{i \in I} (N' \otimes M_i) \to \bigoplus_{i \in I} (N \otimes M_i)$$

which sends $(x \otimes m_i)_{i \in I}$ to $(f(x) \otimes m_i)_{i \in I}$. That is,

$$\psi \circ f \otimes \mathrm{id}_M \circ \varphi = \bigoplus_{i \in I} f \otimes \mathrm{id}_{M_i}$$

.

(4) *Show that $M$ is flat if and only if each $M_i$ is flat.* Suppose $f$ is injective.

$$\begin{aligned}
&M_i \text{ is flat } \forall\, i \in I \\
\Longleftrightarrow\, & f \otimes \mathrm{id}_{M_i} \text{ is injective } \forall\, i \in I \\
\Longleftrightarrow\, & \bigoplus_{i \in I} f \otimes \mathrm{id}_{M_i} \text{ is injective} && \text{(Injectivity)} \\
\Longleftrightarrow\, & \psi \circ f \otimes \mathrm{id}_M \circ \varphi \text{ is injective} && \text{((3))} \\
\Longleftrightarrow\, & f \otimes \mathrm{id}_M \text{ is injective} && (\varphi,\, \psi \text{ are isomorphic}) \\
\Longleftrightarrow\, & M \text{ is flat.}
\end{aligned}$$

□

**Exercise 2.5.** *Let $A[x]$ be the ring of polynomials in one indeterminate over a ring $A$. Prove that $A[x]$ is a flat $A$-algebra. (Hint: Use Exercise 2.4.)*

*Proof (Hint).*

(1) $A$ is a flat $A$-module by Proposition 2.14(iv).

(2) As an $A$-module,
$$A[x] \cong \bigoplus_{n \in \mathbb{Z}^+} Ax^n \cong \bigoplus_{n \in \mathbb{Z}^+} A$$
(since $Ax^n \cong A$).

(3) By Exercise 2.4, $A[x] \cong \bigoplus_{n \in \mathbb{Z}^+} A$ is flat.

□

**Exercise 2.8.**

(i) *If $M$ and $N$ are flat $A$-modules, then so is $M \otimes_A N$.*

(ii) *If $B$ is a flat $A$-algebra and $N$ is a flat $B$-module, then $N$ is flat as $A$-module.*

*Proof of (i).* Given any exact sequence of $A$-modules $0 \to N_1 \to N_2 \to N_3 \to 0$. Since $M$ is flat,
$$0 \to N_1 \otimes_A M \to N_2 \otimes_A M \to N_3 \otimes_A M \to 0$$
is exact. Since $N$ is flat,
$$0 \to (N_1 \otimes_A M) \otimes_A N \to (N_2 \otimes_A M) \otimes_A N \to (N_3 \otimes_A M) \otimes_A N \to 0$$

is exact. By Proposition 2.14 (ii),

$$0 \to N_1 \otimes_A (M \otimes_A N) \to N_2 \otimes_A (M \otimes_A N) \to N_3 \otimes_A (M \otimes_A N) \to 0$$

is exact, or $M \otimes_A N$ is flat. $\square$

*Proof of (ii).* Given any exact sequence of $A$-modules $0 \to N_1 \to N_2 \to N_3 \to 0$. Since $B$ is a flat $A$-algebra ($A$-module),

$$0 \to N_1 \otimes_A B \to N_2 \otimes_A B \to N_3 \otimes_A B \to 0$$

is exact. Since $N$ is a flat $B$-module,

$$0 \to (N_1 \otimes_A B) \otimes_B N \to (N_2 \otimes_A B) \otimes_B N \to (N_3 \otimes_A B) \otimes_B N \to 0$$

is exact. By "Exercise 2.15" on page 27,

$$0 \to N_1 \otimes_A (B \otimes_B N) \to N_2 \otimes_A (B \otimes_B N) \to N_3 \otimes_A (B \otimes_B N) \to 0$$

is exact. By Proposition 2.14 (iv),

$$0 \to N_1 \otimes_A N \to N_2 \otimes_A N \to N_3 \otimes_A N \to 0$$

is exact, or $N$ is flat. $\square$