Chapter 15: Bernoulli Numbers

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Supplement. Exercise 6.73 in the book Graham, Knuth and Patashnik, Concrete Mathematics, Second Edition.

Prove that

$$x \cot x = 1 - 2 \sum_{n=1}^{\infty} \frac{x^2}{n^2\pi^2 - x^2}.$$

Proof.

(1) *Show that*

$$\cot x = \frac{1}{2^n} \sum_{k=0}^{2^n-1} \cot \frac{x + k\pi}{2^n}$$

for all integers $n \geq 1$. Notice that

$$\cot(x + \pi) = \cot x,$$
$$\cot\left(x + \frac{\pi}{2}\right) = -\tan x,$$
$$\cot x = \frac{1}{2}\left(\cot \frac{x}{2} - \tan \frac{x}{2}\right).$$

Use mathematical induction. The case $n = 1$ is the same as the note.

Assume the case $n = m$ holds. For $n = m + 1$,

$$
\begin{aligned}
\sum_{k=0}^{2^{m+1}-1} \cot \frac{x + k\pi}{2^{m+1}} &= \sum_{k=0}^{2^m-1} \cot \frac{x + k\pi}{2^{m+1}} + \sum_{k=2^m}^{2^{m+1}-1} \cot \frac{x + k\pi}{2^{m+1}} \\
&= \sum_{k=0}^{2^m-1} \cot \frac{x + k\pi}{2^{m+1}} + \sum_{k=0}^{2^m-1} \cot \frac{x + (2^m + k)\pi}{2^{m+1}} \\
&= \sum_{k=0}^{2^m-1} \cot \frac{x + k\pi}{2^{m+1}} + \sum_{k=0}^{2^m-1} \cot \left(\frac{x + k\pi}{2^{m+1}} + \frac{\pi}{2} \right) \\
&= \sum_{k=0}^{2^m-1} \left(\cot \frac{x + k\pi}{2^{m+1}} - \tan \frac{x + k\pi}{2^{m+1}} \right) \\
&= \sum_{k=0}^{2^m-1} \left(\cot \frac{x + k\pi}{2^{m+1}} - \tan \frac{x + k\pi}{2^{m+1}} \right) \\
&= 2 \sum_{k=0}^{2^m-1} \cot \frac{x + k\pi}{2^m}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{1}{2^{m+1}} \sum_{k=0}^{2^{m+1}-1} \cot \frac{x + k\pi}{2^{m+1}} &= \frac{1}{2^{m+1}} \cdot 2 \sum_{k=0}^{2^m-1} \cot \frac{x + k\pi}{2^m} \\
&= \frac{1}{2^m} \sum_{k=0}^{2^m-1} \cot \frac{x + k\pi}{2^m} \\
&= \cot x.
\end{aligned}
$$

(2) By rearranging the index of summation of the identity in (1), we have

$$
x \cot x = \frac{x}{2^n} \cot \frac{x}{2^n} - \frac{x}{2^n} \tan \frac{x}{2^n} + \sum_{k=1}^{2^{n-1}-1} \frac{x}{2^n} \left(\cot \frac{x + k\pi}{2^n} + \cot \frac{x - k\pi}{2^n} \right)
$$

for all integers $n \geq 1$.

(3) Notice that $\lim_{x \to 0} x \cot x = 1$. Let $n \to \infty$, the result is established.

$\square$

Exercise 15.6. *For $m \geq 3$, show $|B_{2m+2}| > |B_{2m}|$. (Hint: Use Theorem 2.)*

Proof. By Theorem 2,

$$
2\zeta(2m) = (-1)^{m+1} \frac{(2\pi)^{2m}}{(2m)!} B_{2m}.
$$

Thus,

$$\frac{|B_{2m+2}|}{|B_{2m}|} = \frac{\zeta(2m+2)(2m+2)(2m+1)}{\zeta(2m)(2\pi)^2} > \frac{1 \cdot 8 \cdot 7}{\zeta(6) \cdot (2\pi)^2} = \frac{13230}{\pi^8} > 1,$$

or $|B_{2m+2}| > |B_{2m}|$. $\square$

Exercise 15.8. *Consider the power series expansion of* $\tan x$ *about the origin;*

$$\sum_{k=1}^{\infty} T_k \frac{x^{2k-1}}{(2k-1)!}.$$

Show

$$T_k = (-1)^{k-1} \frac{B_{2k}}{2k} (2^{2k} - 1) 2^{2k}.$$

Note that $T_k \in \mathbb{Z}$ *for all* k *by Exercise 3.*

Proof.

(1) By the equation (6) on page 232,

$$x \cot x = 1 + \sum_{k=2}^{\infty} B_k \frac{(2ix)^k}{k!}.$$

Since $B_k = 0$ for $k > 1$ and odd,

$$x \cot x = 1 + \sum_{k=1}^{\infty} B_{2k} \frac{(2ix)^{2k}}{(2k)!} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} x^{2k},$$

or

$$\cot x = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} x^{2k-1}.$$

Combine the first term $\frac{1}{x}$ into the summation,

$$\cot x = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} x^{2k-1}.$$

(2) Note that $\tan x = \cot x - 2\cot(2x)$. By (1),

$$\begin{aligned}
\tan x &= \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} x^{2k-1} - 2 \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} (2x)^{2k-1} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k (1 - 2^{2k}) 2^{2k} B_{2k}}{(2k)!} x^{2k-1} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k (1 - 2^{2k}) 2^{2k} B_{2k}}{(2k)!} x^{2k-1}.
\end{aligned}$$

Write $T_k = (-1)^{k-1}(2^{2k}-1)2^{2k}\frac{B_{2k}}{2k}$. Therefore, $\tan x = \sum_{k=1}^{\infty} T_k \frac{x^{2k-1}}{(2k-1)!}$.

By Exercise 3, $(2^{2k}-1)2^{2k}\frac{B_{2k}}{2k} \in \mathbb{Z}$, or $T_k \in \mathbb{Z}$ for all k. $\square$