

## Chapter 7: Sequences and Series of Functions

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**Exercise 7.1.** Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

*Proof (Cauchy criterion).* Let  $\{f_n\}$  be a uniformly convergent sequence of bounded functions.

- (1) Since  $f_n$  is bounded, there exists  $M_n$  such that  $|f_n(x)| \leq M_n$ .
- (2) Since  $\{f_n\}$  converges uniformly, given  $1 > 0$  there exists an integer  $N$  such that

$$|f_n(x) - f_m(x)| \leq 1 \text{ whenever } n, m \geq N$$

(Theorem 7.8 (Cauchy criterion for uniform convergence)). Especially,

$$|f_n(x)| \leq |f_n(x) - f_N(x)| + |f_N(x)| \leq 1 + M_N \text{ whenever } n \geq N.$$

- (3) Thus,  $\{f_n\}$  is uniformly bounded by  $M = \max\{M_1, \dots, M_{N-1}, M_N + 1\}$ .

□

**Exercise 7.2.** If  $\{f_n\}$  and  $\{g_n\}$  converge uniformly on a set  $E$ , prove that  $\{f_n + g_n\}$  converge uniformly on  $E$ . If, in addition,  $\{f_n\}$  and  $\{g_n\}$  are sequences of bounded functions, prove that  $\{f_n g_n\}$  converges uniformly on  $E$ .

*Proof.* Let  $\{f_n\} \rightarrow f$  uniformly and  $\{g_n\} \rightarrow g$  uniformly.

- (1) Show that  $\{f_n + g_n\}$  converges uniformly. Given  $\varepsilon > 0$ . Since  $\{f_n\} \rightarrow f$  uniformly and  $\{g_n\} \rightarrow g$  uniformly, there exist two integers  $N_1$  and  $N_2$  such that

$$|f_n(x) - f(x)| \leq \frac{\varepsilon}{2} \text{ whenever } n \geq N_1, x \in E$$

$$|g_n(x) - g(x)| \leq \frac{\varepsilon}{2} \text{ whenever } n \geq N_2, x \in E.$$

Take  $N = \max\{N_1, N_2\}$ , we have

$$\begin{aligned} & |(f_n(x) + g_n(x)) - (f(x) + g(x))| \\ &= |(f_n(x) - f(x)) + (g_n(x) - g(x))| \\ &\leq |f_n(x) - f(x)| + |g_n(x) - g(x)| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

whenever  $n \geq N$ ,  $x \in E$ . Hence  $\{f_n + g_n\}$  converges to  $f + g$  uniformly on  $E$ .

(2) Show that  $\{f_n g_n\}$  converges uniformly if, in addition,  $\{f_n\}$  and  $\{g_n\}$  are sequences of bounded functions. Given  $\varepsilon > 0$ .

(a) By Exercise 7.1, both  $\{f_n\}$  and  $\{g_n\}$  are uniformly bounded. So there exist  $M_1$  and  $M_2$  such that

$$|f_n(x)| \leq M_1 \text{ and } |g_n(x)| \leq M_2$$

for all  $n$  and  $x \in E$ . Also,  $|f(x)| \leq M_1 + 1$  and  $|g(x)| \leq M_2 + 1$ .

(b) Since  $\{f_n\} \rightarrow f$  uniformly and  $\{g_n\} \rightarrow g$  uniformly, there exist two integers  $N_1$  and  $N_2$  such that

$$|f_n(x) - f(x)| \leq \frac{\varepsilon}{2(M_2 + 1)} \text{ whenever } n \geq N_1, x \in E$$

$$|g_n(x) - g(x)| \leq \frac{\varepsilon}{2(M_1 + 1)} \text{ whenever } n \geq N_2, x \in E.$$

(Note that each denominator of  $\frac{\varepsilon}{2(M_j + 1)}$  ( $j = 1, 2$ ) is well-defined and positive!) Take  $N = \max\{N_1, N_2\}$ , we have

$$\begin{aligned} & |f_n(x)g_n(x) - f(x)g(x)| \\ &= |[f_n(x) - f(x)]g_n(x) + f(x)[g_n(x) - g(x)]| \\ &\leq |f_n(x) - f(x)||g_n(x)| + |f(x)||g_n(x) - g(x)| \\ &\leq \frac{\varepsilon}{2(M_2 + 1)} \cdot M_2 + (M_1 + 1) \cdot \frac{\varepsilon}{2(M_1 + 1)} \\ &\leq \varepsilon \end{aligned}$$

whenever  $n \geq N$ ,  $x \in E$ . Hence  $\{f_n g_n\}$  converges to  $fg$  uniformly on  $E$ .

□

*Proof (Cauchy criterion).*

(1) Show that  $\{f_n + g_n\}$  converges uniformly. Given  $\varepsilon > 0$ . Since  $\{f_n\}$  and  $\{g_n\}$  converge uniformly, there exist two integers  $N_1$  and  $N_2$  such that

$$|f_n(x) - f_m(x)| \leq \frac{\varepsilon}{2} \text{ whenever } n, m \geq N_1, x \in E$$

$$|g_n(x) - g_m(x)| \leq \frac{\varepsilon}{2} \text{ whenever } n, m \geq N_2, x \in E.$$

Take  $N = \max\{N_1, N_2\}$ , we have

$$\begin{aligned}
& |(f_n(x) + g_n(x)) - (f_m(x) + g_m(x))| \\
&= |(f_n(x) - f_m(x)) + (g_n(x) - g_m(x))| \\
&\leq |f_n(x) - f_m(x)| + |g_n(x) - g_m(x)| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

whenever  $n, m \geq N$ ,  $x \in E$ . Hence  $\{f_n + g_n\}$  converges uniformly on  $E$ .

(2) Show that  $\{f_n g_n\}$  converges uniformly if, in addition,  $\{f_n\}$  and  $\{g_n\}$  are sequences of bounded functions. Given  $\varepsilon > 0$ .

(a) By Exercise 7.1, both  $\{f_n\}$  and  $\{g_n\}$  are uniformly bounded. So there exist  $M_1$  and  $M_2$  such that

$$|f_n(x)| \leq M_1 \text{ and } |g_n(x)| \leq M_2$$

for all  $n$  and  $x \in E$ . Also,  $|f(x)| \leq M_1 + 1$  and  $|g(x)| \leq M_2 + 1$ .

(b) Since  $\{f_n\} \rightarrow f$  uniformly and  $\{g_n\} \rightarrow g$  uniformly, there exist two integers  $N_1$  and  $N_2$  such that

$$\begin{aligned}
|f_n(x) - f_m(x)| &\leq \frac{\varepsilon}{2(M_2 + 1)} \text{ whenever } n, m \geq N_1, x \in E \\
|g_n(x) - g_m(x)| &\leq \frac{\varepsilon}{2(M_1 + 1)} \text{ whenever } n, m \geq N_2, x \in E.
\end{aligned}$$

Take  $N = \max\{N_1, N_2\}$ , we have

$$\begin{aligned}
& |f_n(x)g_n(x) - f_m(x)g_m(x)| \\
&= |[f_n(x) - f_m(x)]g_n(x) + f_m(x)[g_n(x) - g_m(x)]| \\
&\leq |f_n(x) - f_m(x)||g_n(x)| + |f_m(x)||g_n(x) - g_m(x)| \\
&\leq \frac{\varepsilon}{2(M_2 + 1)} \cdot M_2 + M_1 \cdot \frac{\varepsilon}{2(M_1 + 1)} \\
&\leq \varepsilon
\end{aligned}$$

whenever  $n \geq N$ ,  $x \in E$ . Hence  $\{f_n g_n\}$  converges uniformly on  $E$ .

□

**Exercise 7.3.** Construct sequences  $\{f_n\}$ ,  $\{g_n\}$  which converge uniformly on some set  $E$ , but such that  $\{f_n g_n\}$  does not converge uniformly on  $E$  (of course,  $\{f_n g_n\}$  must converge on  $E$ ).

We provide some examples here.

*Proof* ( $f_n(x) = x + \frac{1}{n}$ ).

- (1) Define  $\{f_n(x)\}$  on  $E = \mathbb{R}$  by  $f_n(x) = x + \frac{1}{n}$  and  $f(x) = x$ . Clearly,  $\{f_n(x)\}$  converges to  $f(x)$  pointwise.
- (2) Show that  $\{f_n\}$  converges uniformly. Given  $\varepsilon > 0$ . There exists an integer  $N \geq \frac{1}{\varepsilon}$  such that

$$|f_n(x) - f(x)| = \frac{1}{n} \leq \frac{1}{N} \leq \varepsilon$$

whenever  $n \geq N$  and  $x \in E$ . Hence  $\{f_n\} \rightarrow f$  uniformly.

- (3) Show that  $\{f_n^2\}$  does not converge uniformly. Clearly,  $\{f_n(x)^2\}$  converges to  $f(x)^2$  pointwise. Hence

$$\sup_{x \in E} |f_n(x)^2 - f(x)^2| = \sup_{x \in E} \left| \frac{2x}{n} + \frac{1}{n^2} \right| \rightarrow \infty$$

as  $n \rightarrow \infty$  (by considering  $x = n^2 \in E$ ). Hence  $\{f_n^2\}$  does not converge uniformly (Theorem 7.9).

□

*Proof* ( $f_n(x) = \frac{1}{x}$ ,  $g_n(x) = \frac{1}{n}$ ).

- (1) Let  $E = (0, 1)$ . Let  $\{f_n(x)\}$  on  $E$  be  $f_n(x) = \frac{1}{x}$  and  $\{g_n(x)\}$  on  $E$  be  $g_n(x) = \frac{1}{n}$ . Clearly,  $\{f_n(x)\}$  converges to  $f(x) = \frac{1}{x}$  pointwise and  $\{g_n(x)\}$  converges to  $g(x) = 0$  pointwise.
- (2) Show that  $\{f_n\}$  converges uniformly. Given  $\varepsilon > 0$ . There exists an integer  $N = 1$  such that

$$|f_n(x) - f(x)| = 0 \leq \varepsilon$$

whenever  $n \geq N$  and  $x \in E$ . Hence  $\{f_n\} \rightarrow f$  uniformly.

- (3) Show that  $\{g_n\}$  converges uniformly. Given  $\varepsilon > 0$ . There exists an integer  $N \geq \frac{1}{\varepsilon}$  such that

$$|g_n(x) - g(x)| = \frac{1}{n} \leq \frac{1}{N} \leq \varepsilon$$

whenever  $n \geq N$  and  $x \in E$ . Hence  $\{g_n\} \rightarrow g$  uniformly.

- (4) Show that  $\{f_n g_n\}$  does not converge uniformly. Clearly,  $\{f_n(x)g_n(x)\}$  converges to  $f(x)g(x) = 0$  pointwise. Hence

$$\sup_{x \in E} |f_n(x)g_n(x) - 0| = \sup_{x \in E} \left| \frac{1}{nx} \right| \rightarrow \infty$$

as  $n \rightarrow \infty$  (by considering  $x = \frac{1}{n^2} \in E$ ). Hence  $\{f_n g_n\}$  does not converge uniformly (Theorem 7.9).

□

*Proof* (Exercise 9.2 in Tom M. Apostol, *Mathematical Analysis*, 2nd edition).

- (1) Let  $E = [\alpha, \beta] \subseteq \mathbb{R}$  be a bounded interval. Define two sequences  $\{f_n\}$  and  $\{g_n\}$  on  $E$  as follows:

$$f_n(x) = x \left(1 + \frac{1}{n}\right) \text{ if } x \in \mathbb{R}, n = 1, 2, \dots,$$

$$g_n(x) = \begin{cases} \frac{1}{n} & \text{if } x = 0 \text{ or if } x \text{ is irrational,} \\ b + \frac{1}{n} & \text{if } x \text{ is rational } \neq 0, \text{ say } x = \frac{a}{b}, b > 0. \end{cases}$$

Here we assume that  $\gcd(a, b) = 1$ . Clearly,  $f(x) = x$  and

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or if } x \text{ is irrational,} \\ b & \text{if } x \text{ is rational } \neq 0, \text{ say } x = \frac{a}{b}, b > 0. \end{cases}$$

Let  $M = \max\{|\alpha|, |\beta|\} \geq 0$ .

- (2) *Show that  $\{f_n\}$  converges uniformly.* Given  $\varepsilon > 0$ . There exists an integer  $N \geq \frac{M}{\varepsilon}$  such that

$$|f_n(x) - f(x)| = \frac{|x|}{n} \leq \frac{M}{N} \leq \varepsilon$$

whenever  $n \geq N$  and  $x \in E$ . Hence  $\{f_n\} \rightarrow f$  uniformly.

- (3) *Show that  $\{g_n\}$  converges uniformly.* Given  $\varepsilon > 0$ . There exists an integer  $N \geq \frac{1}{\varepsilon}$  such that

$$|g_n(x) - g(x)| = \frac{1}{n} \leq \frac{1}{N} \leq \varepsilon$$

whenever  $n \geq N$  and  $x \in E$ . Hence  $\{g_n\} \rightarrow g$  uniformly.

- (4) *Show that  $\{f_n g_n\}$  does not converge uniformly.*

(a) Clearly,  $\{f_n(x)g_n(x)\}$  converges to  $f(x)g(x)$  pointwise where

$$f(x)g(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or if } x \text{ is irrational,} \\ a & \text{if } x = \frac{a}{b} \text{ is rational } \neq 0, b > 0. \end{cases}$$

(b) Note that

$$f_n(x)g_n(x) = \begin{cases} \frac{x}{n} \left(1 + \frac{1}{n}\right) & \text{if } x = 0 \text{ or if } x \text{ is irrational,} \\ \left(a + \frac{x}{n}\right) \left(1 + \frac{1}{n}\right) & \text{if } x = \frac{a}{b} \text{ is rational } \neq 0, b > 0. \end{cases}$$

Therefore,

$$f_n(x)g_n(x) - f(x)g(x) = \begin{cases} \frac{x}{n} \left(1 + \frac{1}{n}\right) & \text{if } x = 0 \text{ or if } x \text{ is irrational,} \\ \frac{x}{n} \left(1 + b + \frac{1}{n}\right) & \text{if } x = \frac{a}{b} \text{ is rational } \neq 0, b > 0. \end{cases}$$

(c) Hence

$$\begin{aligned}
\sup_{x \in E} |f_n(x)g_n(x) - f(x)g(x)| &\geq \sup_{x \in E \cap \mathbb{Q}} |f_n(x)g_n(x) - f(x)g(x)| \\
&= \sup_{x \in E \cap \mathbb{Q}} |a| \left( \frac{1}{n} + \frac{1}{bn} + \frac{1}{bn^2} \right) \\
&\geq \sup_{x \in E \cap \mathbb{Q}} |a| \left( \frac{1}{n} \right) \\
&= \sup_{x \in E \cap \mathbb{Q}} \frac{|a|}{n}.
\end{aligned}$$

(d) Given any irrational number  $\gamma \in E$ , there exists a sequence

$$\left\{ r_m = \frac{a_m}{b_m} \right\}$$

of nonzero rational numbers in  $E$  such that  $\lim r_m = \gamma$ . Show that  $\{a_m\}$  is unbounded. If it is true, we can find  $x_n = r_{m_n} = \frac{a_{m_n}}{b_{m_n}}$  such that  $|a_{m_n}| \geq n^2$  and

$$\sup_{x \in E} |f_n(x)g_n(x) - f(x)g(x)| \geq \sup_{x \in E \cap \mathbb{Q}} \frac{|a|}{n} \geq \frac{n^2}{n} = n \rightarrow \infty$$

as  $n \rightarrow \infty$ .

(e) (Reductio ad absurdum) If  $\{a_m\}$  were bounded, then there exists a **constant** subsequence of  $\{a_{m_k}\}$  such that  $\lim a_{m_k} = a \in \mathbb{Z}$ . Since  $\lim_{m \rightarrow \infty} r_m = \gamma$ ,  $\lim_{k \rightarrow \infty} r_{m_k} = \gamma$  or

$$\lim_{k \rightarrow \infty} b_{m_k} = \lim_{k \rightarrow \infty} \frac{a_{m_k}}{r_{m_k}} = \frac{a}{\gamma}$$

(it is well-defined since  $r_{m_k}$  and  $\gamma$  cannot be zero). Since all  $b_{m_k}$  are positive integers, the limit  $\lim b_{m_k} = b$  is a positive integer too, or  $b = \frac{a}{\gamma} \in \mathbb{Z}^+$ , or  $\gamma = \frac{a}{b} \in \mathbb{Z}$ , which is absurd.

Therefore,  $\{f_n g_n\}$  does not converge uniformly.

□

**Exercise 7.8.** If

$$I(x) = \begin{cases} 0 & (x \leq 0), \\ 1 & (x > 0), \end{cases}$$

if  $\{x_n\}$  is a sequence of distinct points of  $(a, b)$ , and if  $\sum |c_n|$  converges, prove that the series

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n) \quad (a \leq x \leq b)$$

converges uniformly, and that  $f$  is continuous for every  $x \neq x_n$ .

*Proof.*

- (1) Define  $f_n(x) = c_n I(x - x_n)$  on  $(a, b)$ . So

$$|f_n(x)| = |c_n| |I(x - x_n)| \leq |c_n| \quad (x \in (a, b), n = 1, 2, 3, \dots).$$

Since  $\sum |c_n|$  converges,  $f = \sum f_n$  converges uniformly (Theorem 7.10).

- (2) Given any  $p \in (a, b)$  with  $p \neq x_n$  for all  $n = 1, 2, 3, \dots$ . So each  $I(x - x_n)$  is continuous at  $x = p$ , and thus each partial sum  $\sum_{n=1}^N f_n(x)$  is continuous.

- (3) By Theorem 7.11

$$\begin{aligned} \lim_{x \rightarrow p} f(x) &= \lim_{x \rightarrow p} \sum_{n=1}^{\infty} f_n(x) \\ &= \lim_{N \rightarrow \infty} \left( \lim_{x \rightarrow p} \sum_{n=1}^N f_n(x) \right) \\ &= \lim_{N \rightarrow \infty} \sum_{n=1}^N f_n(p) \\ &= \sum_{n=1}^{\infty} f_n(p) \\ &= f(p). \end{aligned}$$

$f(x)$  is continuous at  $x = p$  too.

□