# Chapter 8: Some Special Functions

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**Supplement.** Fourier coefficients in Definition 8.9.

(1) Write
$$f(x) = a_0 + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}$$
(as the textbook Rudin, Principles of Mathematical Analysis, Third Edition). Then
$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx.$$
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)dx, n \in \mathbb{Z}^+.$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)dx, n \in \mathbb{Z}^+.$$

(2) One might write in one different form,
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$
The only difference between the new one and the old one is $a_0$, so $a_0$ should be
$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx.$$

(3) Again, one might write in one different form,
$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$
Similarly, $a_0$ should be
$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x)}{\sqrt{2}} dx.$$

(4) Recall $f(x) = \sum_{-N}^{N} c_n e^{inx}$ $(x \in \mathbb{R})$ where
$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

The relations among $a_n$, $b_n$ of this textbook and $c_n$ are

$$\begin{aligned} c_0 &= a_0 \\ c_n &= \frac{1}{2}\left(a_n + ib_n\right), n \in \mathbb{Z}^+. \end{aligned}$$

(5) In some textbooks (Henryk Iwaniec, Topics in Classical Automorphic Forms), it is convenient to consider periodic functions $f$ of period 1. Define

$$e(n) = e^{2\pi ix} = \cos(2\pi x) + i\sin(2\pi x).$$

Any periodic and piecewise continuous function $f$ has the Fourier series representation

$$f(x) = \sum_{-\infty}^{\infty} a_n e(nx)$$

with coefficients given by

$$a_n = \int_0^1 f(x)e(-nx)dx.$$

Here is one exercise for this representation. *Show that the fractional part of $x$, $\{x\} = x - [x]$, is given by*

$$\{x\} = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{\pi n}.$$

**Supplement.** Parseval's theorem 8.16.

(1) Given

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx)), x \in \mathbb{R}.$$

Then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

(2) Given

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx)), x \in \mathbb{R}.$$

Then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

(3) Given

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

Then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

**Exercise 8.1.** *Define*

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

*Prove that $f$ has derivatives of all orders at $x = 0$, and that $f^{(n)}(0) = 0$ for $n = 1, 2, 3, \ldots$*

$f(x)$ is an example of non-analytic smooth function, that is, infinitely differentiable functions are not necessarily analytic. In this exercise, we will show that Taylor series of $f$ at the origin converges everywhere to the zero function. So the Taylor series does not equal $f(x)$ for $x \neq 0$. Consequently, $f$ is not analytic at $x = 0$.

*Proof.*

(1) *Show that*

$$\lim_{x \to 0} g(x)e^{-\frac{1}{x^2}} = 0$$

*for any rational function $g(x) \in \mathbb{R}(x)$.*

(a) Write $g(x) = \frac{p(x)}{q(x)}$ for some $p(x), q(x) \in \mathbb{R}[x]$, $g(x) \neq 0$.

(b) Write $q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0$. $q(x)$ is not identically zero, that is, there exists the unique coefficient of the least power of $x$ in $q(x)$ which is non-zero, say $b_M \neq 0$.

(c) Thus,

$$g(x) = \frac{p(x)/x^M}{q(x)/x^M}.$$

The denominator of $g(x)$ tends to $b_M \neq 0$ as $x \to 0$. By the similar argument in Theorem 8.6(f), we have

$$\frac{p(x)}{x^M}e^{-\frac{1}{x^2}} \to 0 \text{ as } x \to 0.$$

Hence, $\lim_{x \to 0} g(x)e^{-\frac{1}{x^2}} = 0$ for any $g(x) \in \mathbb{R}(x)$.

(2) *Given any real $x \neq 0$, show that*

$$f^{(n)}(x) = g_n(x)e^{-\frac{1}{x^2}}$$

*for some rational function $g(x) \in \mathbb{R}(x)$.*

(a) Say $g_0(x) = 1 \in \mathbb{R}(x)$.

(b) $\mathbb{R}(x)$ is a field. *Show that $g'(x) \in \mathbb{R}(x)$ for any $g(x) \in \mathbb{R}(x)$.* Write $g(x) = \frac{p(x)}{q(x)}$ for some $p(x), q(x) \in \mathbb{R}[x]$, $q(x) \neq 0$. Thus

$$g'(x) = \frac{p'(x)q(x) - p(x)q'(x)}{q(x)^2}.$$

The numerator of $g'(x)$ is in $\mathbb{R}[x]$ since the differentiation operator on $\mathbb{R}[x]$ is closed in $\mathbb{R}[x]$. Also, the denominator of $g'(x) = q(x)^2 \neq 0$ since $\mathbb{R}[x]$ is an integral domain. Therefore, $g'(x) \in \mathbb{R}(x)$.

(c) Induction on $n$. For $n = 1$, we have

$$\begin{aligned}
f'(x) &= g_0'(x)e^{-\frac{1}{x^2}} + g_0(x) \cdot \left(-\frac{1}{x^2}\right)' e^{-\frac{1}{x^2}} \\
&= \left(g_0'(x) + g_0(x) \cdot \left(-\frac{1}{x^2}\right)'\right) e^{-\frac{1}{x^2}} \\
&= g_1(x)e^{-\frac{1}{x^2}}
\end{aligned}$$

where

$$g_1(x) = g_0'(x) + g_0(x) \cdot \left(-\frac{1}{x^2}\right)' \in \mathbb{R}(x).$$

Now assume that the conclusion holds for $n = k$. As $n = k + 1$, similar to the case $n = 1$,

$$f^{(k+1)}(x) = g_{k+1}(x)e^{-\frac{1}{x^2}}$$

where

$$g_{k+1}(x) = g_k'(x) + g_k(x) \cdot \left(-\frac{1}{x^2}\right)' \in \mathbb{R}(x).$$

By induction, the conclusion is true.

(3) Induction on $n$. For $n = 1$, by (1) we have

$$f'(0) = \lim_{t \to 0} \frac{e^{-\frac{1}{t^2}} - 0}{t} = 0.$$

Now assume that the statement holds for $n = k$. As $n = k + 1$, by (1)(2) we have

$$f^{(k+1)}(0) = \lim_{t \to 0} \frac{f^{(k)}(t) - f^{(k)}(0)}{t} = \lim_{t \to 0} \frac{g_k(t)e^{-\frac{1}{t^2}} - 0}{t} = 0.$$

Thus, $f^{(n)}(0) = 0$ for $n \in \mathbb{Z}^+$.

$\square$

**Exercise 8.2.** *Let $a_{ij}$ be the number in the $i$th row and $j$th column of the array*

$$\begin{array}{ccccc} -1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & -1 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{2} & -1 & 0 & \cdots \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

*so that*

$$a_{ij} = \begin{cases} 0 & (i < j), \\ -1 & (i = j), \\ 2^{j-i} & (i > j). \end{cases}$$

*Prove that*

$$\sum_i \sum_j a_{ij} = -2, \qquad \sum_j \sum_i a_{ij} = 0.$$

Also see Theorem 8.3.

*Proof (Brute-force).*

$$\begin{aligned} \sum_i \sum_j a_{ij} &= \sum_{i=1}^{\infty} \left( \sum_{j=i} a_{ij} + \sum_{j<i} a_{ij} \right) \\ &= \sum_{i=1}^{\infty} \left( -1 + \sum_{j=1}^{i-1} 2^{j-i} \right) \\ &= \sum_{i=1}^{\infty} (-1 + (1 - 2^{1-i})) \\ &= \sum_{i=1}^{\infty} -2^{1-i} \\ &= -2. \end{aligned}$$

$$
\begin{aligned}
\sum_j \sum_i a_{ij} &= \sum_{j=1}^{\infty} \left( \sum_{i=j} a_{ij} + \sum_{i>j} a_{ij} \right) \\
&= \sum_{j=1}^{\infty} \left( -1 + \sum_{i=j+1}^{\infty} 2^{j-i} \right) \\
&= \sum_{j=1}^{\infty} (-1+1) \\
&= \sum_{j=1}^{\infty} 0 \\
&= 0.
\end{aligned}
$$

□

**Exercise 8.3.** *Prove that*

$$\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$$

*if $a_{ij} \geq 0$ for all $i$ and $j$ (the case $+\infty = +\infty$ may occur).*

*Note.* It can be proved by Theorem 8.3 if both summations are finite.

*Proof.*

(1) Let $\mathscr{F}(I)$ be the collection of all finite subsets of $I$.

(2) Let

$$s = \sup \left\{ \sum_{(i,j) \in E} a_{ij} : E \in \mathscr{F}(\mathbb{N}^2) \right\}$$

(the case $s = +\infty$ may occur). *It suffices to show that $\sum_i \sum_j a_{ij} = s$.* The case $\sum_j \sum_i a_{ij} = s$ is similar, and thus $\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$.

(3) *Show that $\sum_i \sum_j a_{ij} \geq s$.* Given any $E \in \mathscr{F}(\mathbb{N}^2)$. It is clear that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \geq \sum_{(i,j) \in E} a_{ij}$$

(since $a_{ij} \geq 0$). Thus,

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \geq \sup \left\{ \sum_{(i,j) \in E} a_{ij} : E \in \mathscr{F}(\mathbb{N}^2) \right\} = s.$$

(4) *Show that* $\sum_i \sum_j a_{ij} \le s$. (Reductio ad absurdum) If $\sum_i \sum_j a_{ij} > s$, especially $s < \infty$, then there exists $\varepsilon > 0$ such that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} > s + \varepsilon,$$

or

$$\sum_{i=1}^{n} \sum_{j=1}^{\infty} a_{ij} > s + \varepsilon$$

for some integer $n$. Consider two possible cases.

(a) If there is some $1 \le i_0 \le n$ such that

$$\sum_{j=1}^{\infty} a_{i_0 j} = \infty,$$

then there is some $m$ such that

$$\sum_{j=1}^{m} a_{i_0 j} > s.$$

For $E = \{(i_0, 1), \dots, (i_0, m)\} \in \mathscr{F}(\mathbb{N}^2)$,

$$\sum_{(i,j)\in E} a_{ij} = \sum_{j=1}^{m} a_{i_0 j} > s,$$

contrary to the supremum of $s$.

(b) Otherwise, for each $1 \le i \le n$ we have

$$\sum_{j=1}^{\infty} a_{ij} < \infty,$$

or there exists some $m_i$ such that

$$\sum_{j=1}^{m_i} a_{ij} > \sum_{j=1}^{\infty} a_{ij} - \frac{\varepsilon}{n}.$$

For $E = \bigcup_{1 \leq i \leq n} \{(i, 1), \ldots, (i, m_i)\} \in \mathscr{F}(\mathbb{N}^2)$,

$$\begin{aligned}
\sum_{(i,j) \in E} a_{ij} &= \sum_{i=1}^{n} \sum_{j=1}^{m_i} a_{ij} \\
&> \sum_{i=1}^{n} \left( \sum_{j=1}^{\infty} a_{ij} - \frac{\varepsilon}{n} \right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{\infty} a_{ij} - \sum_{i=1}^{n} \frac{\varepsilon}{n} \\
&> s + \varepsilon - \varepsilon \\
&= s,
\end{aligned}$$

contrary to the supremum of $s$.

Therefore, $\sum_i \sum_j a_{ij} \leq s$.

(5) By (3)(4), $\sum_i \sum_j a_{ij} = s$. Similarly, $\sum_j \sum_i a_{ij} = s$. Hence, $\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$ (including the case $+\infty = +\infty$).

$\square$

**Exercise 8.4.** *Prove the following limit relations:*

(a) $\lim_{x \to 0} \frac{b^x - 1}{x} = \log b \qquad (b > 0)$.

(b) $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$.

(c) $\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$.

(d) $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$.

*Proof of (a).*

$$\begin{aligned}
\lim_{x \to 0} \frac{b^x - 1}{x} &= \lim_{x \to 0} \frac{\exp(x \log b) - 1}{x} \\
&= \frac{d}{dx} \exp(x \log b) \bigg|_{x=0} \\
&= \exp(x \log b) \cdot \log b \big|_{x=0} \\
&= \log b.
\end{aligned}$$

$\square$

*Proof of (b).*

$$\begin{aligned}\lim_{x\to 0}\frac{\log(1+x)}{x} &= \frac{d}{dx}\log(1+x)\bigg|_{x=0}\\ &= \frac{1}{x+1}\bigg|_{x=0}\\ &= 1.\end{aligned}$$

□

*Proof of (c).*

$$\begin{aligned}\lim_{x\to 0}(1+x)^{\frac{1}{x}} &= \lim_{x\to 0}\exp\left(\frac{\log(1+x)}{x}\right)\\ &= \exp\left(\lim_{x\to 0}\frac{\log(1+x)}{x}\right)\\ &= \exp(1)\\ &= e.\end{aligned}$$

□

*Proof of (d).*

$$\begin{aligned}\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n &= \lim_{n\to\infty}\left(\left(1+\frac{x}{n}\right)^{\frac{n}{x}}\right)^x\\ &= \left(\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{\frac{n}{x}}\right)^x\\ &= \left(\lim_{y\to 0}(1+y)^{\frac{1}{y}}\right)^x\\ &= \exp(x).\end{aligned}$$

□

**Exercise 8.5.** *Find the following limits*

(a) $\lim_{x\to 0}\frac{e-(1+x)^{\frac{1}{x}}}{x}$.

(b) $\lim_{n\to\infty}\frac{n}{\log n}\left[n^{\frac{1}{n}}-1\right]$.

(c) $\lim_{x\to 0}\frac{\tan x - x}{x(1-\cos x)}$.

(d) $\lim_{x\to 0}\frac{x-\sin x}{\tan x - x}$.

*Proof of (a).* By L'Hospital's rule (Theorem 5.13),

$$\begin{aligned}
\lim_{x \to 0} \frac{e - (1+x)^{\frac{1}{x}}}{x} &= \lim_{x \to 0} \frac{-(1+x)^{\frac{1}{x}} \cdot \frac{\frac{x}{x+1} - \log(x+1)}{x^2}}{1} \\
&= \lim_{x \to 0} \left( -(1+x)^{\frac{1}{x}} \cdot \frac{\frac{x}{x+1} - \log(x+1)}{x^2} \right) \\
&= -\lim_{x \to 0} (1+x)^{\frac{1}{x}} \cdot \lim_{x \to 0} \frac{\frac{x}{x+1} - \log(x+1)}{x^2} \\
&= -e \cdot \lim_{x \to 0} \frac{\frac{x}{x+1} - \log(x+1)}{x^2} && \text{(Exercise 8.4(c))} \\
&= -e \cdot \lim_{x \to 0} \frac{-\frac{x}{(x+1)^2}}{2x} \\
&= e \cdot \lim_{x \to 0} \frac{1}{2(x+1)^2} \\
&= e \cdot \frac{1}{2} \\
&= \frac{e}{2}.
\end{aligned}$$

Here

$$\begin{aligned}
\frac{d}{dx} \left( e - (1+x)^{\frac{1}{x}} \right) &= \frac{d}{dx} \left( e - \exp\left( \frac{\log(x+1)}{x} \right) \right) \\
&= -\exp\left( \frac{1}{x} \log(x+1) \right) \cdot \frac{\frac{1}{x+1} \cdot x - \log(x+1) \cdot 1}{x^2} \\
&= -(1+x)^{\frac{1}{x}} \cdot \frac{\frac{x}{x+1} - \log(x+1)}{x^2},
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dx} \left( \frac{x}{x+1} - \log(x+1) \right) &= \frac{(x+1) - x}{(x+1)^2} - \frac{1}{x+1} \\
&= -\frac{x}{(x+1)^2}.
\end{aligned}$$

□

*Proof of (b).*

(1) Let $x = \frac{\log n}{n}$. Note that $\lim_{n \to \infty} \frac{\log n}{n} = 0$.

(2)

$$
\begin{aligned}
\lim_{n\to\infty} \frac{n}{\log n}\left[n^{\frac{1}{n}}-1\right] &= \lim_{n\to\infty} \frac{n}{\log n}\left[\exp\left(\frac{\log n}{n}\right)-1\right] \\
&= \lim_{x\to 0} \frac{\exp(x)-1}{x} && ((1)) \\
&= \frac{d}{dx}\exp(x)\bigg|_{x=0} \\
&= \exp(x)\big|_{x=0} \\
&= 1.
\end{aligned}
$$

□

*Proof of (c) (L'Hospital's rule).* By L'Hospital's rule (Theorem 5.13) three times,

$$
\begin{aligned}
\lim_{x\to 0} \frac{\tan x - x}{x(1-\cos x)} &= \lim_{x\to 0} \frac{\sec^2 x - 1}{1-\cos x + x\sin x} \\
&= \lim_{x\to 0} \frac{2\sec x(\tan x \sec x)}{\sin x + \sin x + x\cos x} \\
&= \lim_{x\to 0} \frac{2\tan x \sec^2 x}{2\sin x + x\cos x} \\
&= \lim_{x\to 0} \frac{2[\sec^2 x \sec^2 x + \tan x \cdot 2\sec x(\tan x \sec x)]}{2\cos x + \cos x - x\sin x} \\
&= \lim_{x\to 0} \frac{2\sec^4 x + 2\sec^2 x \tan^2 x}{3\cos x - x\sin x} \\
&= \frac{2}{3}.
\end{aligned}
$$

□

*Proof of (c) (Taylor series).* Since

$$
\begin{aligned}
\cos x &= 1 - \frac{x^2}{2} + O(x^4) \\
\tan x &= x + \frac{x^3}{3} + O(x^5),
\end{aligned}
$$

we have

$$
\lim_{x\to 0} \frac{\tan x - x}{x(1-\cos x)} = \lim_{x\to 0} \frac{\frac{x^3}{3} + O(x^5)}{\frac{x^3}{2} + O(x^5)} = \frac{2}{3}.
$$

□

*Proof of (d) (L'Hospital's rule).* By L'Hospital's rule (Theorem 5.13) three times,

$$\begin{aligned}
\lim_{x\to 0}\frac{x-\sin x}{\tan x - x} &= \lim_{x\to 0}\frac{1-\cos x}{\sec^2 x - 1} \\
&= \lim_{x\to 0}\frac{\sin x}{2\sec x(\tan x\sec x)} \\
&= \lim_{x\to 0}\frac{\sin x}{2\tan x\sec^2 x} \\
&= \lim_{x\to 0}\frac{\cos x}{2\tan x\sec^2 x} \\
&= \lim_{x\to 0}\frac{\cos x}{2[\sec^2 x\sec^2 x + \tan x\cdot 2\sec x(\tan x\sec x)]} \\
&= \lim_{x\to 0}\frac{\cos x}{2\sec^4 x + 2\sec^2 x\tan^2 x} \\
&= \frac{1}{2}.
\end{aligned}$$

□

*Proof of (d) (Taylor series).* Since

$$\begin{aligned}
\sin x &= x - \frac{x^3}{6} + O(x^5) \\
\tan x &= x + \frac{x^3}{3} + O(x^5),
\end{aligned}$$

we have

$$\lim_{x\to 0}\frac{x-\sin x}{\tan x - x} = \lim_{x\to 0}\frac{\frac{x^3}{6}+O(x^5)}{\frac{x^3}{2}+O(x^5)} = \frac{1}{2}.$$

□

**Exercise 8.6.** *Suppose $f(x)f(y) = f(x+y)$ for all real $x$ and $y$.*

(a) *Assuming that $f$ is differentiable and not zero, prove that*

$$f(x) = e^{cx}$$

*where $c$ is a constant.*

(b) *Prove the same thing, assuming only that $f$ is continuous.*

Part (b) implies part (a). We prove part (b) directly.

*Proof of (b).*

(1) Since $f(x)$ is not zero, there exists $x_0 \in \mathbb{R}$ such that $f(x_0) \neq 0$. So $f(0)f(x_0) = f(x_0)$, or $f(0) = 1$ by cancelling $f(x_0) \neq 0$.

(2) Next, $f(\frac{n}{m}) = f(\frac{1}{m})^n$ for $m \in \mathbb{Z}$, $n \in \mathbb{Z}^+$. Since $f$ is continuous at $x = 0$, $f$ is positive in the neighborhood of $x = 0$. That is, there exists $N \in \mathbb{Z}^+$ such that $f(\frac{1}{m}) > 0$ whenever $|m| \geq N$. So, $f(\frac{n}{m}) = f(\frac{1}{m})^n > 0$. (Since $f(\frac{n}{m}) = f(\frac{kn}{km})$ for any $k \in \mathbb{Z}^+$, we can rescale $m$ to $km$ such that $|km| \geq N$.) That is, $f$ is positive on $\mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and $f$ is continuous on $\mathbb{R}$, $f$ is positive on $\mathbb{R}$.

(3) Now let $c = \log f(1)$ (which is well-defined since $f > 0$). We write $f(1)$ in the two ways. Firstly, $f(1) = f(\frac{n}{n}) = f(\frac{1}{n})^n$ where $n \in \mathbb{Z}^+$. Secondly, $f(1) = e^c = (e^{\frac{c}{n}})^n$. Since the positive $n$-th root is unique (Theorem 1.21), $f(\frac{1}{n}) = e^{\frac{c}{n}}$ for $n \in \mathbb{Z}^+$. By $f(x)f(-x) = f(0) = 1$ or $f(-x) = \frac{1}{f(x)}$, $f(-\frac{1}{n}) = \frac{1}{e^{\frac{c}{n}}} = e^{-\frac{c}{n}}$ for $n \in \mathbb{Z}^+$. Therefore,

$$f\left(\frac{1}{m}\right) = e^{\frac{c}{m}} \text{ where } m \in \mathbb{Z}.$$

(4) By using $f(\frac{n}{m}) = f(\frac{1}{m})^n$ for $m \in \mathbb{Z}$, $n \in \mathbb{Z}^+$ again, $f(\frac{n}{m}) = e^{c\frac{n}{m}}$ where $m \in \mathbb{Z}, n \in \mathbb{Z}^+$, or

$$f(x) = e^{cx} \text{ where } x \in \mathbb{Q}.$$

Since $g(x) = f(x) - e^{cx}$ vanishes on a dense set of $\mathbb{Q}$ and $g$ is continuous on $\mathbb{R}$, $g$ vanishes on $\mathbb{R}$. Therefore, $f(x) = e^{cx}$ for $x \in \mathbb{R}$.

$\square$

**Supplement.** *Proof of (a).*

(1) Since $f(x)$ is not zero, there exists $x_0 \in \mathbb{R}$ such that $f(x_0) \neq 0$. So $f(0)f(x_0) = f(x_0)$, or $f(0) = 1$ by cancelling $f(x_0) \neq 0$.

(2) Since $f$ is differentiable, for any $x \in \mathbb{R}$,

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x)f(h) - f(x)}{h} \\ &= f(x) \lim_{h \to 0} \frac{f(h) - 1}{h} \\ &= f(x) \lim_{h \to 0} \frac{f(h) - f(0)}{h} \\ &= f(x)f'(0). \end{aligned}$$

Let $c = f'(0)$ be a constant. Then $f'(x) = cf(x)$. So $f(x) = e^{cx}$ for $x \in \mathbb{R}$. (To see this, let $g(x) = \frac{f(x)}{e^{cx}}$ be well-defined on $\mathbb{R}$. $g(0) = 1$. $g'(x) = 0$ since $f'(x) = cf(x)$. So $g(x)$ is a constant, or $g(x) = 1$ since $g(0) = 1$. Therefore, $f(x) = e^{cx}$ on $\mathbb{R}$.)

$\square$

**Supplement.** Cauchy's functional equation.

(1) *(Cauchy's functional equation.) Suppose $f(x) + f(y) = f(x + y)$ for all real $x$ and $y$. Assuming that $f$ is continuous, prove that $f(x) = cx$ where $c$ is a constant.*

Notice that we cannot let $g(x) = \log f(x)$ and apply Cauchy's functional equation on $g(x)$ to prove Exercise 8.6 since $f(x)$ is not necessary positive and thus $g(x) = \log f(x)$ might be meaningless. However, this wrong approach gives you some useful ideas such as you need to prove that $f(x)$ is positive first, and $f(x)$ should be equal to $e^{cx}$ where $c = g(1) = \log f(1)$.

(2) *Suppose $f(xy) = f(x) + f(y)$ for all positive real $x$ and $y$. Assuming that $f$ is continuous, prove that $f(x) = c \log x$ where $c$ is a constant.*

(3) *Suppose $f(xy) = f(x)f(y)$ for all positive real $x$ and $y$. Assuming that $f$ is continuous and positive, prove that $f(x) = x^c$ where $c$ is a constant.*

(4) *Suppose $f(x + y) = f(x) + f(y) + xy$ for all real $x$ and $y$. Assuming that $f$ is continuous, prove that $f(x) = \frac{1}{2}x^2 + cx$ where $c$ is a constant.*

(5) *(USA 2002.) Suppose $f(x^2 - y^2) = xf(x) - yf(y)$ for all real $x$ and $y$. Assuming that $f$ is continuous, prove that $f(x) = cx$ where $c$ is a constant.*

**Supplement.** *Show that the only automorphism of $\mathbb{Q}$ is the identity.*

*Proof.* Given any $\sigma \in \mathrm{Aut}(\mathbb{Q})$.

(1) *Show that $\sigma(1) = 1$.* Since $1^2 = 1$, $\sigma(1)\sigma(1) = \sigma(1)$. $\sigma(1) = 0$ or $1$. There are only two possible cases.

  (a) Assume that $\sigma(1) = 0$. So

$$\sigma(a) = \sigma(a \cdot 1) = \sigma(a) \cdot \sigma(1) = \sigma(a) \cdot 0 = 0$$

  for any $a \in \mathbb{Q}$. That is, $\sigma = 0 \in \mathrm{Aut}(\mathbb{Q})$, which is absurd.

  (b) Therefore, $\sigma(1) = 1$.

(2) *Show that $\sigma(n) = n$ for all $n \in \mathbb{Z}^+$.* Write $n = 1 + 1 + \cdots + 1$ ($n$ times 1). Applying the additivity of $\sigma$, we have

$$\sigma(n) = \sigma(1) + \sigma(1) + \cdots + \sigma(1) = 1 + 1 + \cdots + 1 = n.$$

(Might use induction on $n$ to eliminate $\cdots$ symbols.)

(3) *Show that $\sigma(n) = n$ for all $n \in \mathbb{Z}$.* By the additivity of $\sigma$, $\sigma(-n) = -\sigma(n) = -n$ for $n \geq 0$. The result is established.

For any $a = \frac{n}{m} \in \mathbb{Q}$ ($m, n \in \mathbb{Z}$, $n \neq 0$), applying the multiplication of $\sigma$ on $am = n$, that is, $\sigma(a)\sigma(m) = \sigma(n)$. By (3), we have $\sigma(a)m = n$, or

$$\sigma(a) = \frac{m}{n} = a$$

provided $n \neq 0$, or $\sigma$ is the identity. $\Box$

**Exercise 8.7.** *If* $0 < x < \frac{\pi}{2}$, *prove that*

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1.$$

*Proof.*

(1) Let

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

be a continuous function on $\left[0, \frac{\pi}{2}\right]$ (since $\lim_{x \to 0+} f(x) = 1$). So

$$f'(x) = \frac{x \cos x - \sin x}{x^2} < 0$$

on $\left(0, \frac{\pi}{2}\right)$ since $\tan x > x$ on $\left(0, \frac{\pi}{2}\right)$.

(2) *Show that* $\frac{\sin x}{x} < 1$ *on* $\left(0, \frac{\pi}{2}\right)$. Given any $x \in \left(0, \frac{\pi}{2}\right)$, there exists $\xi_1 \in (0, x)$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(\xi_1) < 0$$

by the mean value theorem (Theorem 5.10). So $f(x) < f(0) = 1$, or $\frac{\sin x}{x} < 1$.

(3) *Show that* $\frac{\sin x}{x} > \frac{2}{\pi}$ *on* $\left(0, \frac{\pi}{2}\right)$. Given any $x \in \left(0, \frac{\pi}{2}\right)$, there exists $\xi_2 \in (0, x)$ such that

$$\frac{f(\frac{\pi}{2}) - f(x)}{\frac{\pi}{2} - x} = f'(\xi_2) < 0$$

by the mean value theorem (Theorem 5.10). So $f(x) > f(\frac{\pi}{2}) = \frac{2}{\pi}$, or $\frac{\sin x}{x} > \frac{2}{\pi}$.

$\Box$

**Exercise 8.8.** *For* $n = 0, 1, 2, \ldots$, *and* $x$ *real, prove that*

$$|\sin(nx)| \leq n |\sin x|.$$

*Note that this inequality may be false for other values of $n$. For instance,*

$$\left|\sin\left(\frac{1}{2}\pi\right)\right| > \frac{1}{2}|\sin\pi|.$$

*Proof.* Induction on $n$.

(1) Note that
$$\sin(a+b) = \sin a\cos b + \cos a \sin b$$
for any $a, b \in \mathbb{R}$.

(2) $n = 0, 1$ are clearly true.

(3) Assume the induction hypothesis that for the single case $n = k$ holds, meaning
$$|\sin(kx)| \leq k|\sin x|$$
is true. It follows that
$$\begin{aligned}
|\sin((k+1)x)| &= |\sin(kx)\cos x + \cos(kx)\sin x| && ((1)) \\
&\leq |\sin(kx)||\cos x| + |\cos(kx)||\sin x| && \text{(Triangle inequality)} \\
&\leq |\sin(kx)| + |\sin x| && (|\cos(\cdot)| \leq 1) \\
&\leq k|\sin x| + |\sin x| && \text{(Induction hypothesis)} \\
&\leq (k+1)|\sin x|.
\end{aligned}$$

□

**Exercise 8.9 (The Euler-Mascheroni constant).**

(a) *Put $s_N = 1 + \frac{1}{2} + \cdots + \frac{1}{N}$. Prove that*
$$\lim_{N\to\infty}(s_N - \log N)$$
*exists. (The limit, often denoted by $\gamma$, is called Euler's constant. Its numerical value is $0.5772\ldots$. It is not known whether $\gamma$ is rational or not.)*

(b) *Roughly how large must $m$ be so that $N = 10^m$ satisfies $s_N > 100$?*

*Proof of (a) (Theorem 3.14).*

(1) Note that

$$\begin{aligned}
&\frac{1}{1+\frac{1}{n}} \leq \frac{1}{x} \leq 1 \text{ for } x \in \left[1, 1+\frac{1}{n}\right] \\
\Longrightarrow &\int_1^{1+\frac{1}{n}} \frac{dx}{1+\frac{1}{n}} \leq \int_1^{1+\frac{1}{n}} \frac{dx}{x} \leq \int_1^{1+\frac{1}{n}} dx && \text{(Theorem 6.12(b))} \\
\Longrightarrow &\frac{1}{n+1} \leq \int_1^{1+\frac{1}{n}} \frac{dx}{x} \leq \frac{1}{n} \\
\Longrightarrow &\frac{1}{n+1} \leq \log\left(1+\frac{1}{n}\right) \leq \frac{1}{n}. && \text{(Equation (39) on page 180)}
\end{aligned}$$

(2) Define

$$\gamma_n = s_n - \log n.$$

It suffices to show that $\{\gamma_n\}$ is monotonic and bounded (Theorem 3.14).

(3) *Show that* $\{\gamma_n\}$ *is decreasing.*

$$\begin{aligned}
\gamma_{n+1} - \gamma_n &= (s_{n+1} - \log(n+1)) - (s_n - \log n) \\
&= (s_{n+1} - s_n) - (\log(n+1) - \log n) \\
&= \frac{1}{n+1} - \log\left(\frac{n+1}{n}\right) \\
&= \frac{1}{n+1} - \log\left(1+\frac{1}{n}\right) \\
&\leq 0. && ((1))
\end{aligned}$$

*Note.* $\gamma_n \leq \cdots \leq \gamma_1 = 1$ for all $n = 1, 2, 3, \ldots$.

(4) *Show that* $\gamma_n \geq 0$ *for all* $n = 1, 2, 3, \ldots$. Since

$$\begin{aligned}
\log n &= \sum_{k=1}^{n-1} (\log(k+1) - \log k) \\
&= \sum_{k=1}^{n-1} \log \frac{k+1}{k} \\
&= \sum_{k=1}^{n-1} \log\left(1+\frac{1}{k}\right) \\
&\leq \sum_{k=1}^{n-1} \frac{1}{k} && ((1)) \\
&= s_{n-1},
\end{aligned}$$

we have

$$\gamma_n = s_n - \log n \geq s_n - s_{n-1} = \frac{1}{n} > 0.$$

By (3)(4), $\{\gamma_n\}$ converges to $\lim_{N\to\infty}(s_N - \log N) = \gamma$. $\square$

**Supplement.** *Show that if $f \geq 0$ on $[0, \infty)$ and $f$ is monotonically decreasing, and if*

$$c_n = \sum_{k=1}^{n} f(k) - \int_1^n f(x)dx,$$

*then $\lim_{n\to\infty} c_n$ exists.* (Exercise 10 of Section 5.2 in the textbook: *R Creighton Buck, Advanced Calculus, 3rd edition*. See page 235.) If this exercise is true, we can get the existence of $\gamma$ by taking $f(x) = \frac{1}{x}$.

(1) Note that

$$f(n+1) \leq \int_n^{n+1} f(x)dx \leq f(n).$$

(2) *Show that $\{c_n\}$ is decreasing.*

$$c_{n+1} - c_n = f(n+1) - \int_n^{n+1} f(x)dx \leq 0.$$

(3) *Show that $c_n \geq 0$.* Since $f(k) \geq \int_k^{k+1} f(x)dx$,

$$\begin{aligned}
\sum_{k=1}^{n} f(k) &\geq \sum_{k=1}^{n} \int_k^{k+1} f(x)dx \\
&= \int_1^{n+1} f(x)dx \\
&\geq \int_1^n f(x)dx. && (f \geq 0)
\end{aligned}$$

So that $c_n = \sum_{k=1}^n f(k) - \int_1^n f(x)dx \geq 0$.

(4) By (2)(3), $\{c_n\}$ converges (Theorem 3.14).

$\square$

*Proof of (a) (Limit comparison test).* Inspired by this paper: *Philippe Flajolet and Ilan Vardi, Zeta Function Expansions of Classical Constants.*

(1) Rewrite

$$\gamma_n + \log n - \log(n+1) = \sum_{k=1}^{n} \left( \frac{1}{k} - \log\left(1 + \frac{1}{k}\right)\right)$$

(similar to the argument in (a)(4)(Theorem 3.14)). Let

$$c_k = \frac{1}{k} - \log\left(1 + \frac{1}{k}\right).$$

(2) *Show that*

$$\lim_{k\to\infty} \frac{c_k}{\frac{1}{k^2}} = \frac{1}{2}.$$

In fact,

$$\begin{aligned}
&\lim_{k\to\infty} \frac{c_k}{\frac{1}{k^2}} \\
=&\lim_{x\to 0} \frac{x - \log(1+x)}{x^2} && (\text{Put } x = \frac{1}{k}) \\
=&\lim_{x\to 0} \frac{1 - \frac{1}{1+x}}{2x} && (\text{L'Hospital's rule}) \\
=&\lim_{x\to 0} \frac{1}{2(x+1)} \\
=&\frac{1}{2}.
\end{aligned}$$

(3) By limit comparison test or comparison test, $\sum c_k$ converges since $\sum \frac{1}{k^2}$ converges. Also,

$$\lim_{n\to\infty} \log n - \log(n+1) = 0.$$

Therefore, $\lim_{n\to\infty} \gamma_n$ exists.

□

*Note.* This proof is based on **limit comparison test** (Theorem 8.21) in this textbook: *Tom. M. Apostol, Mathematical Analysis, 2nd edition.* It is easy to prove by the original comparison test.

*Proof of (a) (Comparison test).*

(1) Note that

$$0 \leq x - \log(x+1) \leq \frac{x^2}{2}$$

for all $x \geq 0$.

(2) Write

$$c_n = \frac{1}{n} - \log\left(1 + \frac{1}{n}\right).$$

as in the the proof of (a) (Limit comparison test). By (1),

$$|c_n| \leq \frac{1}{2n^2}$$

for all $n = 1, 2, \ldots$. Hence, by the comparison test (Theorem 3.25(a), $\sum c_n$ converges since $\sum \frac{1}{n^2}$ converges (to $\frac{\pi^2}{6}$). Use the same argument in the proof of (a) (Limit comparison test), since

$$\gamma_n + \log n - \log(n+1) = \sum c_n \text{ and } \lim_{n\to\infty} \log n - \log(n+1) = 0,$$

we have the existence of $\lim \gamma_n = \gamma$.

$\square$

*Proof of (a) (Uniformly convergence of $\sum \frac{x}{n(x+n)}$).* (One example to Exercise 7 of Section 6.2 in the textbook: *R Creighton Buck, Advanced Calculus, 3rd edition*. See pages 270 to 271.)

(1) Let
$$f_n(x) = \frac{x}{n(x+n)} = \frac{1}{n} - \frac{1}{x+n}$$
defined on $E = [0, 1]$.

(2) Note that
$$|f_n(x)| \leq \frac{1}{n^2}$$
for all $x \in [0, 1]$. Since $\sum \frac{1}{n^2}$ converges, $\sum f_n$ converges uniformly on $[0, 1]$ (Theorem 7.10).

(3) Corollary to Theorem 7.16 implies that
$$\begin{aligned}
\int_0^1 \sum_{n=1}^{\infty} \frac{x}{n(x+n)} dx &= \sum_{n=1}^{\infty} \int_0^1 \frac{x}{n(x+n)} dx \\
&= \sum_{n=1}^{\infty} \int_0^1 \left( \frac{1}{n} - \frac{1}{x+n} \right) dx \\
&= \sum_{n=1}^{\infty} \left( \frac{1}{n} - \log \frac{n+1}{n} \right) \\
&= \lim_{N \to \infty} \left( \sum_{n=1}^{N} \frac{1}{n} - \log(N+1) \right) \\
&= \lim_{N \to \infty} (s_N - \log(N+1))
\end{aligned}$$
exists. Since $\lim_{N \to \infty}(\log(N+1) - \log N) = 0$,
$$\begin{aligned}
\gamma &= \lim_{N \to \infty} (s_N - \log N) \\
&= \lim_{N \to \infty} (s_N - \log(N+1)) + \lim_{N \to \infty} (\log(N+1) - \log N)
\end{aligned}$$
exists.

$\square$

*Proof of (a) (Existence of $\int_1^{\infty} \frac{\{x\}}{x^2} dx$).*

(1) Define $\{x\} = x - [x]$ where $[x]$ is the greatest integer $\leq x$ (Exercise 6.16). *Show that*

$$\int_1^\infty \frac{\{x\}}{x^2} dx < \infty.$$

Use the similar argument in Exercise 6.16(b). Since $\frac{\{x\}}{x^2} \leq \frac{1}{x^2}$ on $[1, \infty)$ and $\int_1^\infty \frac{1}{x^2} dx = 1$ exists, the result is established (Theorem 6.12(b)).

(2) *Show that*

$$\int_1^N \frac{[x]}{x^2} dx = s_N - 1.$$

Use the similar argument in Exercise 6.16(a),

$$\begin{aligned}
\int_1^N \frac{[x]}{x^2} dx &= \sum_{k=1}^{N-1} \int_k^{k+1} \frac{[x]}{x^2} dx \\
&= \sum_{k=1}^{N-1} \int_k^{k+1} \frac{k}{x^2} dx \\
&= \sum_{k=1}^{N-1} \int_k^{k+1} \frac{k}{x^2} dx \\
&= \sum_{k=1}^{N-1} \frac{1}{k+1} \\
&= \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} \\
&= s_N - 1.
\end{aligned}$$

**Supplement (Euler's summation formula).** (Theorem 7.13 in the textbook: *Tom. M. Apostol, Mathematical Analysis, 2nd edition.*) *If $f$ has a continuous derivative $f'$ on $[a, b]$, then we have*

$$\sum_{a<n\leq b} f(n) = \int_a^b f(x)dx + \int_a^b f'(x)\{x\}dx + f(a)\{a\} - f(b)\{b\},$$

*where $\sum_{a<n\leq b}$ means the sum from $n = [a] + 1$ to $n = [b]$. When $a$ and $b$ are integers, this becomes*

$$\sum_{n=a}^{b} f(n) = \int_a^b f(x)dx + \int_a^b f'(x)\left(\{x\} - \frac{1}{2}\right) dx + \frac{f(a) + f(b)}{2}.$$

By taking $f(x) = \frac{1}{x}$ we can get the same result.

(3) *Show that*

$$\int_1^N \frac{\{x\}}{x^2} dx = \log N - s_N + 1 = 1 - \gamma_N.$$

In fact,

$$\begin{aligned}
\int_1^N \frac{\{x\}}{x^2} dx &= \int_1^N \frac{x-[x]}{x^2} dx \\
&= \int_1^N \frac{1}{x} dx - \int_1^N \frac{[x]}{x^2} dx \\
&= \log N - (s_N - 1) \\
&= \log N - s_N + 1 \\
&= 1 - \gamma_N.
\end{aligned}$$

(4) Since

$$\lim_{N\to\infty} \int_1^N \frac{\{x\}}{x^2} dx = \int_1^\infty \frac{\{x\}}{x^2} dx$$

exists (by (1)), $\gamma = \lim \gamma_N$ exists.

$\square$

*Proof of (b).* By $s_n - \log n > 0$ in (a)(4)(Theorem 3.14), it suffices to choose $N = 10^m$ such that $s_N \geq \log(N+1) > 100$, or

$$m > \frac{\log(\exp(100)-1)}{\log 10},$$

or choose $m$ satisfying

$$m > \frac{100}{\log 10} > \frac{\log(\exp(100)-1)}{\log 10},$$

or $m = 44$. $\square$

*Note.* The exact value of $N$ is

$$15092688622113788323693563264538101449859497 \approx 1.509 \times 10^{43}.$$

**Exercise 8.10.** *Prove that* $\sum \frac{1}{p}$ *diverges; the sum extends over all primes.*

There are many proofs of this result. We provide some of them.

*Proof (Due to hint).* Given $N$.

(1) *Show that*

$$\sum_{n \leq N} \frac{1}{n} \leq \prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1}.$$

By the unique factorization theorem on $n \leq N$,

$$\sum_{n \leq N} \frac{1}{n} \leq \prod_{p \leq N} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1}.$$

(2) By (1) and the fact that $\sum \frac{1}{n}$ diverges, there are infinitely many primes.

(3) *Show that*

$$\prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1} \leq \exp\left(\sum_{p \leq N} \frac{2}{p}\right).$$

By applying the inequality $(1-x)^{-1} < e^{2x}$ where $x \in (0, \frac{1}{2}]$ on any prime $p$,

$$\left(1 - \frac{1}{p}\right)^{-1} < \exp\left(\frac{2}{p}\right).$$

Now multiplying the inequality over all primes $p \leq N$ and noticing that $\exp(x) \cdot \exp(y) = \exp(x+y)$, we have

$$\prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1} \leq \exp\left(\sum_{p \leq N} \frac{2}{p}\right).$$

(4) By (1)(3),

$$\sum_{n \leq N} \frac{1}{n} \leq \exp\left(\sum_{p \leq N} \frac{2}{p}\right).$$

Since $\sum_{n \leq N} \frac{1}{n}$ diverges, the result holds.

□

*Proof (Due to Kenneth Ireland and Michael Rosen).* The proof in Kenneth Ireland and Michael Rosen, A Classical Introduction to Modern Number Theory, Second Edition (Theorem 3 in Chapter 2) does not use the inequality $(1-x)^{-1} < e^{2x}$ ($x \in (0, \frac{1}{2}]$) directly. Instead, the authors take the logarithm on $(1-p^{-1})^{-1}$ and estimate it. (So the length of proof is longer than the proof due to hint.)

That is,

$$
\begin{aligned}
-\log(1-p^{-1}) &= \sum_{n=1}^{\infty} \frac{p^{-n}}{n} \\
&= \frac{1}{p} + \sum_{n=2}^{\infty} \frac{p^{-n}}{n} \\
&< \frac{1}{p} + \sum_{n=2}^{\infty} p^{-n} \\
&= \frac{1}{p} + \frac{p^{-2}}{1-p^{-1}} \\
&< \frac{1}{p} + 2 \cdot \frac{1}{p^2}.
\end{aligned}
$$

Now we sum over all primes $p \le N$,

$$
\log\left(\prod_{p\le N}\left(1-\frac{1}{p}\right)^{-1}\right) < \sum_{p\le N}\frac{1}{p} + 2\sum_{p\le N}\frac{1}{p^2}.
$$

So

$$
\log \sum_{n\le N}\frac{1}{n} < \sum_{p\le N}\frac{1}{p} + 2\sum_{p\le N}\frac{1}{p^2}.
$$

Notice that $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{p^2}$ converges (since $\sum \frac{1}{n^2}$ converges). Therefore, $\sum \frac{1}{p}$ diverges. $\square$

*Proof (Due to I. Niven).* It is an exercise in Kenneth Ireland and Michael Rosen, A Classical Introduction to Modern Number Theory, Second Edition. See Exercise 27 in Chapter 2.

(1) *Show that $\sum' \frac{1}{n}$, the sum being over square free integers, diverges.* For any positive integers $n$, we can write $n = a^2 b$ where $a \in \mathbb{Z}^+$ and $b$ is a square free integer. Given $N$,

$$
\sum_{n\le N}\frac{1}{n} \le \left(\sum_{a=1}^{\infty}\frac{1}{a^2}\right)\left(\sum_{b\le N}{}'\frac{1}{b}\right).
$$

Notice that $\sum_{a=1}^{\infty}\frac{1}{a^2}$ converges. Since $\sum_{n\le N}\frac{1}{n} \to \infty$ as $N \to \infty$, $\sum'_{b\le N}\frac{1}{b} \to \infty$ as $N \to \infty$.

(2) *Show that*

$$
\prod_{p\le N}(1+\frac{1}{p}) \to \infty \textit{ as } N \to \infty.
$$

By the unique factorization theorem on $n \leq N$,

$$\prod_{p \leq N} \left(1 + \frac{1}{p}\right) \geq {\sum_{n \leq N}}' \frac{1}{n}.$$

Since ${\sum_{n \leq N}}' \frac{1}{n} \to \infty$ as $N \to \infty$ by (1), the conclusion is established.

(3) By applying the inequality $e^x > 1 + x$ on any prime $p$,

$$\exp\left(\frac{1}{p}\right) > 1 + \frac{1}{p}.$$

Now multiplying the inequality over all primes $p \leq N$ and noticing that $\exp(x) \cdot \exp(y) = \exp(x + y)$, we have

$$\exp\left(\sum_{p \leq N} \frac{1}{p}\right) > \prod_{p \leq N} \left(1 + \frac{1}{p}\right).$$

By (2), $\exp\left(\sum_{p \leq N} \frac{1}{p}\right) \to \infty$ as $N \to \infty$, or $\sum_{p \leq N} \frac{1}{p} \to \infty$ as $N \to \infty$.

$\square$

**Exercise 8.11.** *Suppose $f \in \mathscr{R}$ on $[0, A]$ for all $A < \infty$, and $f(x) \to 1$ as $x \to +\infty$. Prove that*

$$\lim_{t \to 0} t \int_0^\infty e^{-tx} f(x) dx = 1 \qquad (t > 0),$$

It is similar to Exercise 3.14(a).

*Proof.* Given any $\varepsilon > 0$.

(1) The integral $\int_0^\infty e^{-tx} f(x) dx$ is well-defined. (It suffices to show that $\int_0^\infty e^{-tx} f(x) dx$ converges absolutely in the sense of Exercise 6.8. It is quite easy since $f(x) \to 1$ as $x \to +\infty$ and well-behavior of $\int_{A_0}^\infty e^{-tx} f(x) dx$ for any $A_0 > 0$.)

(2) Note that

$$t \int_0^\infty e^{-tx} dx = 1$$

for any $t > 0$.

(3) Since $f(x) \to 1$ as $x \to +\infty$, there is $A_0 > 0$ such that

$$|f(x) - 1| < \frac{\varepsilon}{64} \text{ whenever } x \geq A_0.$$

(4) Since $f \in \mathscr{R}$ on $[0, A_0]$, $f$ is bounded on $[0, A_0]$, or $|f| \leq M$ on $[0, A_0]$ for some $M$ (Theorem 6.7(c)).

(5) As $t > 0$,

$$\begin{aligned}
&\left|\left(t\int_0^\infty e^{-tx}f(x)dx\right) - 1\right| \\
=&\left|t\int_0^\infty e^{-tx}(f(x)-1)dx\right| && ((2)) \\
\leq& t\int_0^\infty e^{-tx}|f(x)-1|dx && ((1) \text{ with Theorem 6.13}) \\
=& t\int_0^{A_0} e^{-tx}|f(x)-1|dx + t\int_{A_0}^\infty e^{-tx}|f(x)-1|dx \\
\leq& t\int_0^{A_0} (M+1)dx + t\int_{A_0}^\infty e^{-tx}|f(x)-1|dx && ((3) \text{ and } e^{-tx} \leq 1) \\
\leq& t\int_0^{A_0} (M+1)dx + t\int_{A_0}^\infty e^{-tx}\frac{\varepsilon}{64}dx && ((4)) \\
=& tA_0(M+1) + \exp(-A_0 t)\frac{\varepsilon}{64} \\
\leq& tA_0(M+1) + \frac{\varepsilon}{64}. && (e^{-tx} \leq 1)
\end{aligned}$$

Since $t$ is arbitrary, take $t = \frac{\varepsilon}{89A_0(M+1)} > 0$ to get

$$\left|\left(t\int_0^\infty e^{-tx}f(x)dx\right) - 1\right| < \frac{\varepsilon}{89} + \frac{\varepsilon}{64} < \varepsilon,$$

or

$$\lim_{t\to 0^+} t\int_0^\infty e^{-tx}f(x)dx = 1.$$

□

**Exercise 8.12.** *Suppose* $0 < \delta < \pi$,

$$f(x) = \begin{cases} 1 & \textit{if } |x| \leq \delta, \\ 0 & \textit{if } \delta < |x| \leq \pi, \end{cases}$$

*and* $f(x + 2\pi) = f(x)$ *for all* $x$.

(a) *Compute the Fourier coefficients of* $f$.

(b) *Compute that*

$$\sum_{n=1}^\infty \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2} \qquad (0 < \delta < \pi).$$

(c) *Deduce from Parseval's theorem that*

$$\sum_{n=1}^{\infty} \frac{(\sin(n\delta))^2}{n^2\delta} = \frac{\pi - \delta}{2}.$$

(d) *Let* $\delta \to 0$ *and prove that*

$$\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}.$$

(e) *Put* $\delta = \frac{\pi}{2}$ *in (c). What do you get?*

It is a centered square pulse around $x = 0$ with shift $\delta$. Besides, $f(x)$ is an even function.

*Proof of (a).*

$$\begin{aligned} c_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx \\ &= \frac{1}{2\pi} \int_{-\delta}^{\delta} dx \\ &= \frac{\delta}{\pi}. \end{aligned}$$

For $0 \neq n \in \mathbb{Z}$,

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx}dx \\ &= \frac{1}{2\pi} \int_{-\delta}^{\delta} e^{-inx}dx \\ &= \frac{1}{2\pi} \cdot \frac{2\sin(n\delta)}{n} \\ &= \frac{\sin(n\delta)}{n\pi}. \end{aligned}$$

□

**Supplement.** Find $a_n$ and $b_n$ of this textbook.
By (a), $a_0 = \frac{\delta}{\pi}$, $a_n = \frac{2\sin(n\delta)}{n\pi}$, $b_n = 0$ for $n \in \mathbb{Z}^+$. Surely, we can compute $a_n$

and $b_n$ $(n > 0)$ directly. Since $f(x)$ is an even function, $b_n = 0$. And

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \\ &= \frac{2}{\pi}\int_{0}^{\delta} \cos(nx)dx \\ &= \frac{2\sin(n\delta)}{n\pi}. \end{aligned}$$

*Proof of (b).* Given $x = 0$, there are constants $\delta' = \delta > 0$ and $M = 1 < \infty$ such that

$$|f(0+t) - f(0)| \leq M|t|$$

for all $t \in (-\delta', \delta')$. By Theorem 8.14,

$$\sum_{-\infty}^{\infty} c_n = f(0).$$

Notice that $c_{-n} = c_n$ for $n \in \mathbb{Z}^+$, so

$$\begin{aligned} \frac{\delta}{\pi} + 2\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n\pi} &= 1 \\ \sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} &= \frac{\pi - \delta}{2}. \end{aligned}$$

□

We can also use the expression $a_n$ and $b_n$ to prove the same thing. Besides, taking $\delta = 1$ yields

$$\sum_{n=1}^{\infty} \frac{\sin n}{n} = \frac{\pi - 1}{2}.$$

*Proof of (c).* Since $f(x)$ is a Riemann-integrable function with period $2\pi$, by Parseval's theorem

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{-\infty}^{\infty} |c_n|^2.$$

So

$$\frac{\delta}{\pi} = \frac{\delta^2}{\pi^2} + 2\sum_{n=1}^{\infty} \frac{(\sin(n\delta))^2}{n^2\pi^2},$$

or

$$\sum_{n=1}^{\infty} \frac{(\sin(n\delta))^2}{n^2\delta} = \frac{\pi - \delta}{2}.$$

□

Notices that
$$\sum_{n=1}^{\infty} \frac{(\sin n)^2}{n^2} = \frac{\pi - 1}{2}$$
as $\delta = 1$.

*Proof of (d).* Given $\varepsilon > 0$. By Exercise 6.8,
$$\int_0^{\infty} \left( \frac{\sin x}{x} \right)^2 dx$$
exists. So there exists $b > 0$ such that
$$\left| \int_0^{b} \left( \frac{\sin x}{x} \right)^2 dx - \int_0^{\infty} \left( \frac{\sin x}{x} \right)^2 dx \right| < \frac{\varepsilon}{4}$$
By Supplement in Chapter 6, there exists $\delta > 0$ such that for any partition $P_m = \{0, \frac{b}{m}, \frac{2b}{m}, \dots, \frac{(m-1)b}{m}, b\}$ of $[0, b]$ with $\|P\| = \frac{b}{m} < \delta$, or $m > \frac{b}{\delta}$, we have
$$\left| \sum_{n=1}^{m} \frac{(\sin(n\frac{b}{m}))^2}{(n\frac{b}{m})^2} \cdot \frac{b}{m} - \int_0^{b} \left( \frac{\sin x}{x} \right)^2 dx \right| < \frac{\varepsilon}{4},$$
$$\left| \sum_{n=1}^{m} \frac{(\sin(n\frac{b}{m}))^2}{n^2\frac{b}{m}} - \int_0^{b} \left( \frac{\sin x}{x} \right)^2 dx \right| < \frac{\varepsilon}{4}.$$
For simplicity we resize $\delta$ to $\delta < \pi$ to make $0 < \frac{b}{m} < \delta < \pi$. Besides, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, there exists $N > 0$ such that
$$\left| \sum_{n=1}^{\infty} \frac{(\sin(n\frac{b}{m}))^2}{n^2\frac{b}{m}} - \sum_{n=1}^{m} \frac{(\sin(n\frac{b}{m}))^2}{n^2\frac{b}{m}} \right| < \frac{\varepsilon}{4}$$
whenever $m \geq N$. By (c),
$$\left| \frac{\pi - \frac{b}{m}}{2} - \sum_{n=1}^{m} \frac{(\sin(n\frac{b}{m}))^2}{n^2\frac{b}{m}} \right| < \frac{\varepsilon}{4}$$
whenever $m \geq N$. Last, it is easy to get
$$\left| \frac{\pi}{2} - \frac{\pi - \frac{b}{m}}{2} \right| < \frac{\varepsilon}{4}$$
whenever $m > \frac{2b}{\varepsilon}$. Now we have
$$\left| \frac{\pi}{2} - \int_0^{\infty} \left( \frac{\sin x}{x} \right)^2 dx \right| < \varepsilon$$

whenever $m > \max(\frac{b}{\delta}, N, \frac{2b}{\varepsilon})$. Since $\varepsilon$ is arbitrary, $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$. □

*Proof of (e).*

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

Write

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^2} &= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \\ &= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2},\end{aligned}$$

so

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{6}.$$

□

**Exercise 8.13.** *Put $f(x) = x$ if $0 \le x < 2\pi$, and apply Parseval's theorem to conclude that*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi}{6}.$$

*Proof.*

$$\begin{aligned}c_0 &= \frac{1}{2\pi}\int_0^{2\pi} x dx \\ &= \pi,\end{aligned}$$

For $n \neq 0$,

$$\begin{aligned}c_n &= \frac{1}{2\pi}\int_0^{2\pi} x e^{-inx} dx \\ &= \frac{1}{2\pi}\left(\left[-\frac{1}{in} x e^{-inx}\right]_{x=0}^{x=2\pi} - \int_0^{2\pi} -\frac{1}{in} e^{-inx} dx\right) \\ &= \frac{i}{n}.\end{aligned}$$

Since $f(x)$ is a Riemann-integrable function with period $2\pi$, by Parseval's theorem

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{-\infty}^{\infty} |c_n|^2.$$

So

$$\frac{1}{2\pi} \cdot \frac{(2\pi)^3}{3} = \pi^2 + 2\sum_{n=1}^{\infty} \frac{1}{n^2},$$

or

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

$\square$

**Supplement.** *Put $f(x) = x^n$ if $n \in \mathbb{Z}^+$ and $0 \le x < 2\pi$. Might get*

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!}.$$

**Exercise 8.14.** PLACEHOLDER.

**Exercise 8.15.** *With the Dirichlet kernel $D_n$ as defined by*

$$D_n(x) = \sum_{k=-n}^{n} \exp(ikx) = \frac{\sin\left(n + \frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)},$$

*put the* ***Fejér kernel***

$$K_N(x) = \frac{1}{N+1} \sum_{n=0}^{N} D_n(x).$$

*Prove that*

$$K_N(x) = \frac{1}{N+1} \cdot \frac{1 - \cos(N+1)x}{1 - \cos x}$$

*and that*

(a) $K_N \ge 0$,

(b) $\frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(x) dx = 1$,

(c) $K_N(x) \le \frac{1}{N+1} \cdot \frac{2}{1 - \cos\delta}$ *if* $0 < \delta \le |x| \le \pi$.

*If $s_N = s_N(f; x)$ is the $N$th partial sum of the Fourier series of $f$, consider the arithmetic means*

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_N}{N+1}.$$

*Prove that*

$$s_N(f; x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x - t) K_N(t) dt,$$

*and hence prove* ***Fejér's theorem:***

*If $f$ is continuous, with period $2\pi$, then $\sigma_N(f;x) \to f(x)$ uniformly on $[-\pi,\pi]$.*

*(Hint: Use properties (a),(b),(c) to proceed as in Theorem 7.26.)*

*Proof of $K_N(x) = \frac{1}{N+1} \cdot \frac{1-\cos(N+1)x}{1-\cos x}$.* Since

$$\begin{aligned}(1-\cos x)K_N(x) =& 2\left(\sin\frac{x}{2}\right)^2 \frac{1}{N+1}\sum_{n=0}^{N}\frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)} \\ =& \frac{1}{N+1}\sum_{n=0}^{N} 2\sin\frac{x}{2}\sin\left(n+\frac{1}{2}\right)x \\ =& \frac{1}{N+1}\sum_{n=0}^{N}(\cos(nx)-\cos(n+1)x) \\ =& \frac{1-\cos(N+1)x}{N+1},\end{aligned}$$

$$K_N(x) = \frac{1}{N+1}\cdot\frac{1-\cos(N+1)x}{1-\cos x}$$

if $x \neq 2k\pi$ for $k \in \mathbb{Z}$. □

*Proof of (a).* It is clear since $\cos x \leq 1$ for all $x \in \mathbb{R}$. Or we may write

$$K_N(x) = \frac{1}{N+1}\left(\frac{\sin\frac{(N+1)x}{2}}{\sin\frac{x}{2}}\right)^2 \geq 0.$$

□

*Proof of (b).* By the definition of $D_n(x)$,

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x)dx =& \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{N+1}\sum_{n=0}^{N} D_n(x)dx \\ =& \frac{1}{N+1}\sum_{n=0}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(x)dx \\ =& \frac{1}{N+1}\sum_{n=0}^{N} 1 \\ =& 1.\end{aligned}$$

□

*Proof of (c).* Since $\cos x$ is bounded by 1 and monotonically decreasing on $(0, \pi]$,

$$\begin{aligned}K_N(x) =& \frac{1}{N+1} \cdot \frac{1-\cos(N+1)x}{1-\cos x} \\ \leq& \frac{1}{N+1} \cdot \frac{2}{1-\cos\delta}.\end{aligned}$$

□

*Proof of $s_N(f; x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)dt$.*

$$\begin{aligned}\sigma_N(f;x) &= \frac{1}{N+1}\sum_{n=0}^{N} s_N(f;x) \\ &= \frac{1}{N+1}\sum_{n=0}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_N(t)dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\left(\frac{1}{N+1}\sum_{n=0}^{N} D_N(t)\right)dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)dt.\end{aligned}$$

□

*Proof of Fejér's theorem.* Given any $\varepsilon > 0$.

(1)

$$\begin{aligned}|\sigma_N(f;x) - f(x)| =& \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)K_N(t)dt\right| \\ =& \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x-t)-f(x))K_N(t)dt\right| \\ \leq& \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x-t)-f(x)|K_N(t)dt.\end{aligned}$$

(2) Since $f$ is continuous on a compact set $[-\pi, \pi]$, $f$ is continuous uniformly. For such $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(y) - f(x)| < \frac{\varepsilon}{2}$$

whenever $x, y \in [-\pi, \pi]$ and $|y - x| < \delta$.

(3) Since $f$ is continuous on a compact set $[-\pi, \pi]$, $f$ is bounded on $[-\pi, \pi]$, say $M = \sup |f(x)|$.

(4) Therefore,

$$\begin{aligned}
&|\sigma_N(f;x) - f(x)| \\
\leq &\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x-t) - f(x)| K_N(t) dt \\
= &\frac{1}{2\pi}\int_{-\pi}^{-\delta} |f(x-t) - f(x)| K_N(t) dt \\
&+ \frac{1}{2\pi}\int_{-\delta}^{\delta} |f(x-t) - f(x)| K_N(t) dt \\
&+ \frac{1}{2\pi}\int_{\delta}^{\pi} |f(x-t) - f(x)| K_N(t) dt \\
\leq &\frac{1}{2\pi}\int_{-\pi}^{-\delta} 2M \cdot \frac{1}{N+1} \cdot \frac{2}{1-\cos\delta} dt \\
&+ \frac{1}{2\pi}\int_{-\delta}^{\delta} \frac{\varepsilon}{2} K_N(t) dt \\
&+ \frac{1}{2\pi}\int_{\delta}^{\pi} 2M \cdot \frac{1}{N+1} \cdot \frac{2}{1-\cos\delta} dt \\
= &\frac{4M(\pi-\delta)}{(N+1)(1-\cos\delta)\pi} + \frac{\varepsilon}{2} \cdot \frac{1}{2\pi}\int_{-\delta}^{\delta} K_N(t) dt \\
\leq &\frac{4M(\pi-\delta)}{(N+1)(1-\cos\delta)\pi} + \frac{\varepsilon}{2}.
\end{aligned}$$

(5) Since $N$ is arbitrary, we can take an integer $N > \frac{4M(\pi-\delta)}{(1-\cos\delta)\pi\varepsilon} - 1$ so that

$$\begin{aligned}
|\sigma_N(f;x) - f(x)| \leq &\frac{4M(\pi-\delta)}{(N+1)(1-\cos\delta)\pi} + \frac{\varepsilon}{2} \\
< &\frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
= &\varepsilon.
\end{aligned}$$

Therefore, the conclusion holds.

□

**Exercise 8.16.** *Prove a pointwise version of Fejér's theorem: If $f \in \mathscr{R}$ and $f(x+)$, $f(x-)$ exist for some $x$, then*

$$\lim_{N\to\infty} \sigma_N(f;x) = \frac{1}{2}[f(x+) + f(x-)].$$

*Proof.* Given any $\varepsilon > 0$.

(1) Since $K_N(-t) = K_N(t)$, we have

$$\sigma_N(f;x) = \frac{1}{2\pi}\int_0^{\pi} f(x-t)K_N(t)dt + \frac{1}{2\pi}\int_0^{\pi} f(x+t)K_N(t)dt$$

and

$$\frac{1}{2\pi}\int_0^{\pi} K_N(t)dt = \frac{1}{2}.$$

(2) Since $f \in \mathscr{R}$, $f$ is bounded on $[-\pi,\pi]$, say $M = \sup|f(x)|$.

(3) Therefore,

$$\begin{aligned}
&\left|\frac{1}{2\pi}\int_0^{\pi} f(x-t)K_N(t)dt - \frac{1}{2}f(x-)\right| \\
=&\left|\frac{1}{2\pi}\int_0^{\pi} (f(x-t)-f(x-))K_N(t)dt\right| \\
\leq&\frac{1}{2\pi}\int_0^{\pi} |f(x-t)-f(x-)|K_N(t)dt.
\end{aligned}$$

Since $f(x-)$ exists, for fixed $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(y) - f(x-)| < \frac{\varepsilon}{2}$$

whenever $y \in (x-\delta, x) \cap [-\pi,\pi]$. Hence,

$$\begin{aligned}
&\left|\frac{1}{2\pi}\int_0^{\pi} f(x-t)K_N(t)dt - \frac{1}{2}f(x-)\right| \\
\leq&\frac{1}{2\pi}\int_0^{\pi} |f(x-t)-f(x-)|K_N(t)dt \\
=&\frac{1}{2\pi}\int_0^{\delta} |f(x-t)-f(x-)|K_N(t)dt \\
&+\frac{1}{2\pi}\int_{\delta}^{\pi} |f(x-t)-f(x-)|K_N(t)dt \\
\leq&\frac{1}{2\pi}\int_0^{\delta} \frac{\varepsilon}{2}K_N(t)dt + \frac{1}{2\pi}\int_{\delta}^{\pi} 2M \cdot \frac{1}{N+1}\cdot\frac{2}{1-\cos\delta}dt \\
=&\frac{\varepsilon}{2}\cdot\frac{1}{2\pi}\int_0^{\delta} K_N(t)dt + \frac{2M(\pi-\delta)}{(N+1)(1-\cos\delta)\pi} \\
\leq&\frac{\varepsilon}{4} + \frac{2M(\pi-\delta)}{(N+1)(1-\cos\delta)\pi}.
\end{aligned}$$

(4) Since $N$ is arbitrary, we can take an integer $N_1 > \frac{8M(\pi-\delta)}{(1-\cos\delta)\pi\varepsilon} - 1$ such that

$$\begin{aligned}\left|\frac{1}{2\pi}\int_0^\pi f(x-t)K_n(t)dt - \frac{1}{2}f(x-)\right| \leq& \frac{\varepsilon}{4} + \frac{2M(\pi-\delta)}{(n+1)(1-\cos\delta)\pi} \\ <&\frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ =&\frac{\varepsilon}{2}\end{aligned}$$

whenever $n \geq N_1$. Similarly, we can take an integer $N_2$ such that

$$\begin{aligned}\left|\frac{1}{2\pi}\int_0^\pi f(x+t)K_n(t)dt - \frac{1}{2}f(x+)\right| \leq& \frac{\varepsilon}{4} + \frac{2M(\pi-\delta)}{(n+1)(1-\cos\delta)\pi} \\ <&\frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ =&\frac{\varepsilon}{2}.\end{aligned}$$

whenever $n \geq N_2$.

(5) Hence,

$$\begin{aligned}&\left|\sigma_n(f;x) - \frac{1}{2}[f(x+)+f(x-)]\right| \\ \leq&\left|\frac{1}{2\pi}\int_0^\pi f(x-t)K_n(t)dt - \frac{1}{2}f(x-)\right| \\ &+\left|\frac{1}{2\pi}\int_0^\pi f(x+t)K_n(t)dt - \frac{1}{2}f(x+)\right| \\ <&\frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ =&\varepsilon.\end{aligned}$$

whenever $n \geq \max\{N_1, N_2\}$. Hence, $\lim \sigma_n(f;x) = \frac{1}{2}[f(x+)+f(x-)]$.

□

**Supplement.** Poisson's equation. (Theorem 1 of Section 2.2 in the textbook: *Lawrence C. Evans, Partial Differential Equations.*) *Let the fundamental solution of Laplace's equation be*

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log|x| & (n=2) \\ \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}} & (n\geq 3), \end{cases}$$

*where* $x \in \mathbb{R}^n$, $x \neq 0$. *Let*

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)dy.$$

*Then* $-\Delta u = f$ *in* $\mathbb{R}^n$. Note that $\Phi(x)$ blows up at 0. To calculate $\Delta u(x)$, we need to isolate this singularity inside a small ball, say $B(0;\varepsilon)$. Therefore,

$$\Delta u(x) = \int_{B(0;\varepsilon)} \Phi(y)\Delta_x f(x-y)dy + \int_{\mathbb{R}^n - B(0;\varepsilon)} \Phi(y)\Delta_x f(x-y)dy,$$

and we can continue estimating two integrals individually as the textbook did.

**Exercise 8.17.** PLACEHOLDER.

**Exercise 8.18.** PLACEHOLDER.

**Exercise 8.19.** *Suppose $f$ is a continuous function on $\mathbb{R}$, $f(x+2\pi) = f(x)$, and $\frac{\alpha}{\pi}$ is irrational. Prove that*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x+n\alpha) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt$$

*for every $x$. (Hint: Do it first for $f(x) = \exp(ikx)$.)*

*Proof (Hint).* Given any $\varepsilon > 0$.

(1) Do it first for $f(x) = \exp(ikx)$. Note that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(ikx)dt = \begin{cases} 1 & (k=0), \\ 0 & (k \neq 0). \end{cases}$$

   (a) $k = 0$ is nothing to do.
   (b) Suppose $k \neq 0$.

$$\begin{aligned} \frac{1}{N}\sum_{n=1}^{N} f(x+n\alpha) &= \frac{1}{N}\sum_{n=1}^{N} \exp(ik(x+n\alpha)) \\ &= \frac{1}{N}\sum_{n=1}^{N} \exp(ikx)\exp(ik\alpha n) \\ &= \frac{1}{N}\exp(ikx) \cdot \frac{\exp(ik\alpha) - \exp(ik\alpha(N+1))}{1-\exp(ik\alpha)} \\ &= \exp(ik(x+\alpha))\left[\frac{1}{N} \cdot \frac{1-\exp(ik\alpha N)}{1-\exp(ik\alpha)}\right] \\ &= f(x+\alpha)\frac{1}{N}\frac{1-\exp(ik\alpha N)}{1-\exp(ik\alpha)} \to 0 \end{aligned}$$

   as $N \to \infty$ since $\exp(iy)$ is bounded ($y \in \mathbb{R}$). (Note that the denominator $1 - \exp(ik\alpha) \neq 0$ since $k \neq 0$ and $\frac{\alpha}{\pi}$ is irrational.)

By (a)(b),

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x+n\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)dt.$$

for $f(x) = \exp(ikx)$ and any $x \in \mathbb{R}$.

(2) Therefore,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x+n\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)dt$$

is also true for trigonometric polynomials $f(x)$.

(3) By Theorem 8.15, there is a trigonometric polynomial

$$P(x) = \sum_{n=-N_1}^{N_1} c_n \exp(inx)$$

such that

$$|P(x) - f(x)| < \frac{\varepsilon}{89}.$$

By (2), there is an integer $N_2$ such that

$$\left| \frac{1}{N} \sum_{n=1}^{N} P(x+n\alpha) - \frac{1}{2\pi} \int_{-\pi}^{\pi} P(t)dt \right| < \frac{\varepsilon}{64}$$

whenever $N \geq N_2$. Therefore,

$$
\begin{aligned}
&\left|\frac{1}{N}\sum_{n=1}^{N} f(x+n\alpha) - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt\right| \\
\leq &\left|\frac{1}{N}\sum_{n=1}^{N} f(x+n\alpha) - \frac{1}{N}\sum_{n=1}^{N} P(x+n\alpha)\right| \\
&+ \left|\frac{1}{N}\sum_{n=1}^{N} P(x+n\alpha) - \frac{1}{2\pi}\int_{-\pi}^{\pi} P(t)dt\right| \\
&+ \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} P(t)dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt\right| \\
\leq &\frac{1}{N}\sum_{n=1}^{N} |f(x+n\alpha) - P(x+n\alpha)| \\
&+ \left|\frac{1}{N}\sum_{n=1}^{N} P(x+n\alpha) - \frac{1}{2\pi}\int_{-\pi}^{\pi} P(t)dt\right| \\
&+ \frac{1}{2\pi}\int_{-\pi}^{\pi} |P(t) - f(t)|dt \\
< &\frac{1}{N}\sum_{n=1}^{N} \frac{\varepsilon}{89} + \frac{\varepsilon}{64} + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\varepsilon}{89}dt \\
= &\frac{\varepsilon}{89} + \frac{\varepsilon}{64} + \frac{\varepsilon}{89} \\
< &\varepsilon
\end{aligned}
$$

whenever $N \geq N_2$. Hence

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x+n\alpha) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt$$

is also true for continuous function $f(x)$ (with period $2\pi$).

□

**Exercise 8.20.** *The following simple computation yields a good approximation to Stirling's formula. For* $m = 1, 2, 3, \ldots,$ *define*

$$f(x) = (m+1-x)\log m + (x-m)\log(m+1)$$

*if* $m \leq x \leq m+1$, *and define*

$$g(x) = \frac{x}{m} - 1 + \log m$$

*if* $m-\frac{1}{2} \leq x < m+\frac{1}{2}$. *Draw the graphs of* $f$ *and* $g$. *Note that* $f(x) \leq \log x \leq g(x)$ *if* $x \geq 1$ *and that*

$$\int_1^n f(x)dx = \log(n!) - \frac{1}{2}\log n > -\frac{1}{8} + \int_1^n g(x)dx.$$

*Integrate* $\log x$ *over* $[1, n]$. *Conclude that*

$$\frac{7}{8} < \log(n!) - \left(n + \frac{1}{2}\right)\log n + n < 1$$

*for* $n = 2, 3, 4, \ldots$. *(Note:* $\log\sqrt{2\pi} \approx 0.918\ldots$.*) Thus*

$$e^{\frac{7}{8}} < \frac{n!}{(\frac{n}{e})^n\sqrt{n}} < e.$$

*Proof.*

(1) Omit the graphs of $f$ and $g$. Note that the concavity of $\log(x)$ implies that $f(x) \leq \log(x)$. Here the equality holds if and only if $x \in \mathbb{Z}^+$. Besides, since $g(x)$ is the tangent line at $(x, \log x)$ whenever $x \in \mathbb{Z}^+$, $g(x) \geq \log(x)$ and the equality holds if and only if $x \in \mathbb{Z}^+$.

(2)

$$\begin{aligned}
\int_1^n f(x)dx &= \sum_{m=1}^{n-1}\int_m^{m+1} f(x)dx \\
&= \sum_{m=1}^{n-1}\int_m^{m+1} (m+1-x)\log m + (x-m)\log(m+1)dx \\
&= \sum_{m=1}^{n-1}\int_m^{m+1} (\log(m+1) - \log m)x + (m+1)\log m - m\log(m+1)dx \\
&= \sum_{m=1}^{n-1} (\log(m+1) - \log m)\left(\frac{(m+1)^2 - m^2}{2}\right) + (m+1)\log m - m\log(m+1) \\
&= \sum_{m=1}^{n-1} \log m + \frac{1}{2}\sum_{m=1}^{n-1}(\log(m+1) - \log m) \\
&= \log((n-1)!) + \frac{1}{2}\log n \\
&= \log(n!) - \frac{1}{2}\log n.
\end{aligned}$$

(3) Write

$$\int_1^n g(x)dx = \left(\sum_{m=1}^{n}\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} g(x)dx\right) - \int_{\frac{1}{2}}^{1} g(x)dx - \int_n^{n+\frac{1}{2}} g(x)dx.$$

(a)

$$\begin{aligned}\sum_{m=1}^{n}\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} g(x)dx &= \sum_{m=1}^{n}\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} \left(\frac{x}{m} - 1 + \log m\right) dx \\ &= \sum_{m=1}^{n} \log m \\ &= \log(n!).\end{aligned}$$

(b)

$$\int_{\frac{1}{2}}^{1} g(x)dx = \int_{\frac{1}{2}}^{1} (x - 1 + \log 1)dx = -\frac{1}{8}.$$

(c)

$$\int_{n}^{n+\frac{1}{2}} g(x)dx = \int_{\frac{1}{2}}^{1} \left(\frac{x}{n} - 1 + \log n\right) dx = \frac{1}{2}\log n - \frac{1}{8n}.$$

By (a)(b)(c),

$$\int_{1}^{n} g(x)dx = \log(n!) - \frac{1}{2}\log n + \frac{1}{8}(1 - \frac{1}{n}) < \log(n!) - \frac{1}{2}\log n + \frac{1}{8}.$$

(4) Since $f(x) \leq \log x \leq g(x)$ and the equality holds if and only if $x \in \mathbb{Z}^+$ (by (1)),

$$\int_{1}^{n} f(x)dx \leq \int_{1}^{n} \log x dx \leq \int_{1}^{n} g(x)dx$$

for all $n = 1, 2, 3, \ldots$. The equality holds if and only if $n = 1$. Hence by (2)(3)

$$\log(n!) - \frac{1}{2}\log n \leq n\log n - n + 1 \leq \log(n!) - \frac{1}{2}\log n + \frac{1}{8}.$$

Arrange the inequality to get

$$\frac{7}{8} < \log(n!) - \left(n + \frac{1}{2}\right)\log n + n \leq 1$$

for $n = 1, 2, 3, \ldots$. Note that the equality holds if and only if $n = 1$. Therefore

$$\frac{7}{8} < \log(n!) - \left(n + \frac{1}{2}\right)\log n + n < 1$$

for $n = 2, 3, \ldots$.

(5) Exponentiate to get

$$\exp\left(\frac{7}{8}\right) < \exp\left[\log(n!) - \left(n + \frac{1}{2}\right)\log n + n\right] < \exp(1),$$

or

$$e^{\frac{7}{8}} < \frac{\exp(\log(n!))\exp(n)}{\exp\left[\left(n+\frac{1}{2}\right)\log n\right]} < e,$$

or $e^{\frac{7}{8}} < \frac{n!}{(\frac{n}{e})^n\sqrt{n}} < e$ (since $\exp(x)$ is a strictly increasing function of $x$).

$\square$

**Exercise 8.21 (Norm of Dirichlet kernel).** *Let*

$$L_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}|D_n(t)|dt \qquad (n = 1, 2, 3, \ldots).$$

*Prove that there exists a constant $C > 0$ such that*

$$L_n > C\log n \qquad (n = 1, 2, 3, \ldots),$$

*or, more precisely, that the sequence*

$$\left\{L_n - \frac{4}{\pi^2}\log n\right\}$$

*is bounded.*

*Proof.*

(1) Write

$$\begin{aligned}
L_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|D_n(t)|dt \\
&= \frac{1}{\pi}\int_{0}^{\pi}|D_n(t)|dt && (D_n(-t) = D_n(t)) \\
&= \frac{1}{\pi}\int_{0}^{\pi}\frac{\left|\sin\left(n+\frac{1}{2}\right)t\right|}{\sin\left(\frac{t}{2}\right)}dt. && \left(\sin\left(\frac{t}{2}\right) \geq 0 \text{ on } [0,\pi]\right)
\end{aligned}$$

(2) So,

$$\begin{aligned}
L_n &= \frac{1}{\pi}\int_{0}^{\pi}\frac{\left|\sin\left(n+\frac{1}{2}\right)t\right|}{\sin\left(\frac{t}{2}\right)}dt \\
&= \frac{1}{\pi}\int_{0}^{\pi}\left|\sin\left(n+\frac{1}{2}\right)t\right|\left(\frac{1}{\sin\left(\frac{t}{2}\right)} - \frac{1}{\frac{t}{2}} + \frac{1}{\frac{t}{2}}\right)dt \\
&= \underbrace{\frac{1}{\pi}\int_{0}^{\pi}\left|\sin\left(n+\frac{1}{2}\right)t\right|\left(\frac{1}{\sin\left(\frac{t}{2}\right)} - \frac{1}{\frac{t}{2}}\right)dt}_{:=I_n} + \underbrace{\frac{2}{\pi}\int_{0}^{\pi}\frac{\left|\sin\left(n+\frac{1}{2}\right)t\right|}{t}dt}_{:=J_n}.
\end{aligned}$$

(3) *Show that $I_n$ is uniformly bounded.* Note that $f(x) = \frac{1}{\sin(x)} - \frac{1}{x}$ is bounded (since $\lim_{x\to 0} f(x) = 0$ by using L'Hospital's rule twice). Also, $\left|\sin\left(n+\frac{1}{2}\right)t\right| \le 1$ for any $n$. Hence

$$0 \le I_n < \sup(f(x)) = \frac{2}{\pi}.$$

(4) *Show that $J_n - \frac{4}{\pi^2}\log n$ is uniformly bounded.* Since

$$\begin{aligned} J_n &= \frac{2}{\pi}\int_0^{\pi} \frac{\left|\sin\left(n+\frac{1}{2}\right)t\right|}{t}dt \\ &= \frac{2}{\pi}\int_0^{\left(n+\frac{1}{2}\right)\pi} \frac{|\sin x|}{x}dx, \qquad\qquad (\text{Let } x = \left(n+\frac{1}{2}\right)t) \end{aligned}$$

we have

$$\underbrace{\frac{2}{\pi}\sum_{k=0}^{n-1}\int_{k\pi}^{(k+1)\pi}\frac{|\sin x|}{x}dx}_{:=J_n^{(1)}} \le J_n \le \underbrace{\frac{2}{\pi}\sum_{k=0}^{n}\int_{k\pi}^{(k+1)\pi}\frac{|\sin x|}{x}dx}_{:=J_n^{(2)}}.$$

So

$$\begin{aligned} J_n^{(1)} &\ge \frac{2}{\pi}\sum_{k=0}^{n-1}\int_{k\pi}^{(k+1)\pi}\frac{|\sin x|}{(k+1)\pi}dx \\ &= \frac{2}{\pi}\sum_{k=0}^{n-1}\frac{2}{(k+1)\pi} \qquad\qquad (\int_0^{\pi}|\sin x|dx = 0) \\ &\ge \frac{4}{\pi^2}\log n, \qquad\qquad (\text{Exercise 8.9}) \end{aligned}$$

and

$$\begin{aligned} J_n^{(2)} &= \frac{2}{\pi}\int_0^{\pi}\frac{|\sin x|}{x}dx + \frac{2}{\pi}\sum_{k=1}^{n}\int_{k\pi}^{(k+1)\pi}\frac{|\sin x|}{x}dx \\ &\le \frac{2}{\pi}\int_0^{\pi}\frac{|\sin x|}{x}dx + \frac{2}{\pi}\sum_{k=1}^{n}\int_{k\pi}^{(k+1)\pi}\frac{|\sin x|}{k\pi}dx \\ &= \frac{2}{\pi}\int_0^{\pi}\frac{|\sin x|}{x}dx + \frac{2}{\pi}\sum_{k=1}^{n}\frac{2}{k\pi} \\ &\le \frac{2}{\pi}\int_0^{\pi}\frac{|\sin x|}{x}dx + \frac{4}{\pi^2}(\log n + 1) \\ &= \frac{4}{\pi^2}\log n + \frac{4}{\pi^2} + \frac{2}{\pi}\int_0^{\pi}\frac{|\sin x|}{x}dx. \end{aligned}$$

Hence,

$$0 \le J_n - \frac{4}{\pi^2}\log n \le \frac{4}{\pi^2} + \frac{2}{\pi}\int_0^{\pi}\frac{|\sin x|}{x}dx.$$

(5) By (3)(4),

$$0 \leq L_n - \frac{4}{\pi^2}\log n \leq \frac{2}{\pi} + \frac{4}{\pi^2} + \frac{2}{\pi}\int_0^{\pi} \frac{|\sin x|}{x} dx.$$

$\square$

**Exercise 8.22 (Newton's generalized binomial theorem).** *If $\alpha$ is a real and $-1 < x < 1$, prove Newton's binomial theorem*

$$(1+x)^{\alpha} = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n.$$

*(Hint: Denote the right side by $f(x)$. Prove that the series converges. Prove that*

$$(1+x)f'(x) = \alpha f(x)$$

*and solve this differential equation.) Show also that*

$$(1-x)^{-\alpha} = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} x^n$$

*if $-1 < x < 1$ and $\alpha > 0$.*

*Proof.*

(1) Let

$$f(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$

where $\binom{\alpha}{n}$ is defined by

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

(2) *Show that* $\binom{\alpha-1}{n} + \binom{\alpha-1}{n-1} = \binom{\alpha}{n}$.

$$\begin{aligned}
\binom{\alpha-1}{n} + \binom{\alpha-1}{n-1} &= \frac{(\alpha-1)\cdots(\alpha-n+1)(\alpha-n)}{n!} + \frac{(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!} \\
&= \frac{(\alpha-1)\cdots(\alpha-n+1)}{n!}[(\alpha-n)+n] \\
&= \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} \\
&= \binom{\alpha}{n}.
\end{aligned}$$

(3) *Show that $f(x)$ converges.* Write $c_n = \binom{\alpha}{n}$. Since

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty}\left|\frac{\alpha-n}{n+1}\right| = 1,$$

we have

$$\lim_{n\to\infty}\sqrt[n]{|c_n|} = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = 1$$

(Theorem 3.37) and thus the radius of convergence is 1. $f(x)$ converges if $|x| < 1$.

(4) *Show that $(1+x)f'(x) = \alpha f(x)$.* By Theorem 8.1,

$$\begin{aligned} f'(x) &= \sum_{n=0}^{\infty}\binom{\alpha}{n}nx^{n-1} \\ &= \sum_{n=1}^{\infty}\binom{\alpha}{n}nx^{n-1} \\ &= \sum_{n=1}^{\infty}\alpha\binom{\alpha-1}{n-1}x^{n-1} \\ &= \sum_{n=0}^{\infty}\alpha\binom{\alpha-1}{n}x^{n}. \end{aligned}$$

Besides,

$$xf'(x) = \sum_{n=0}^{\infty}\binom{\alpha}{n}nx^n = \sum_{n=0}^{\infty}\alpha\binom{\alpha-1}{n-1}x^n.$$

Hence,

$$\begin{aligned} (1+x)f'(x) &= \sum_{n=0}^{\infty}\alpha\binom{\alpha-1}{n}x^n + \sum_{n=0}^{\infty}\alpha\binom{\alpha-1}{n-1}x^n \\ &= \alpha\sum_{n=0}^{\infty}\left[\binom{\alpha-1}{n} + \binom{\alpha-1}{n-1}\right]x^n \\ &= \alpha\sum_{n=0}^{\infty}\binom{\alpha}{n}x^n && ((2)) \\ &= \alpha f(x). \end{aligned}$$

(5) *Solve the differential equation $(1+x)f'(x) = \alpha f(x)$.* Given any $1 > \varepsilon > 0$. Use the notations in Exercise 5.27. Let

$$\phi(x,y) = \frac{\alpha y}{1+x}$$

defined on $[-1+\varepsilon, 1-\varepsilon] \times \mathbb{R}$. Let

$$g(x) = (1+x)^{\alpha}$$

defined on $[-1+\varepsilon, 1-\varepsilon]$. Thus,

$$g'(x) = \alpha(1+x)^{\alpha-1} = \frac{\alpha(1+x)^\alpha}{1+x} = \frac{\alpha g(x)}{1+x} = \phi(x, g(x))$$

and $g(0) = 1$. (Clearly, $f'(x) = \phi(x, f(x))$ and $f(0) = 1$.) *To show $f(x) = g(x)$, it suffices to show that there is a constant $A$ such that*

$$|\phi(x, g(x)) - \phi(x, f(x))| \le A|g(x) - f(x)|$$

*whenever* $(x, f(x)) \in \mathbb{R}$ *and* $(x, g(x)) \in \mathbb{R}$. In fact,

$$\begin{aligned} |\phi(x, g(x)) - \phi(x, f(x))| &= \left|\frac{\alpha g(x)}{1+x} - \frac{\alpha f(x)}{1+x}\right| \\ &= \frac{\alpha}{1+x}|g(x) - f(x)| \\ &\le \frac{\alpha}{\varepsilon}|g(x) - f(x)|. \end{aligned}$$

(Here $A = \frac{\alpha}{\varepsilon}$ is a constant.) By Exercise 5.27, $f(x) = g(x)$ on $[-1+\varepsilon, 1-\varepsilon]$ for any $1 > \varepsilon > 0$. So $f(x) = g(x)$ on $(-1, 1)$, or

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = (1+x)^\alpha$$

if $x \in (-1, 1)$.

(6) *Show that*

$$(1-x)^{-\alpha} = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} x^n$$

*if* $-1 < x < 1$ *and* $\alpha > 0$. In fact,

$$\begin{aligned} (1-x)^{-\alpha} &= \sum_{n=0}^{\infty} \binom{-\alpha}{n} (-x)^n \\ &= \sum_{n=0}^{\infty} \frac{(-\alpha)(-\alpha-1)\cdots(-\alpha-n+1)}{n!}(-1)^n x^n \\ &= \sum_{n=0}^{\infty} \frac{(\alpha)(\alpha+1)\cdots(\alpha+n-1)}{n!} x^n \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} x^n. \end{aligned}$$

□

**Exercise 8.23.** *Let $\gamma$ be a continuously differentiable* ***closed*** *curve in the complex plain, with parameter interval $[a,b]$, and assume that $\gamma(t) \neq 0$ for every $t \in [a,b]$. Define the* ***index*** *of $\gamma$ to be*

$$\mathrm{Ind}(\gamma) = \frac{1}{2\pi i}\int_a^b \frac{\gamma'(t)}{\gamma(t)}dt.$$

*Prove that $\mathrm{Ind}(\gamma)$ is always an integer. (Hint: There exists $\varphi$ on $[a,b]$ with $\varphi' = \frac{\gamma'}{\gamma}$, $\varphi(a) = 0$. Hence $\gamma\exp(-\varphi)$ is constant. Since $\gamma(a) = \gamma(b)$ it follows that $\exp(\varphi(b)) = \exp(\varphi(b)) = 1$. Note that $\varphi(b) = 2\pi i \mathrm{Ind}(\gamma)$.) Compute $\mathrm{Ind}(\gamma)$ when $\gamma(t) = \exp(int)$, $a = 0$, $b = 2\pi$. Explain why $\mathrm{Ind}(\gamma)$ is often called the* ***winding number*** *of $\gamma$ around $0$.*

*Proof.*

(1) *Show that $\mathrm{Ind}(\gamma)$ is always an integer.* Define

$$\varphi(x) = \int_a^x \frac{\gamma'(t)}{\gamma(t)}dt$$

if $x \in [a,b]$.

(a) *Show that $\varphi(x)$ is well-defined.* Since $\gamma$ is continuously differentiable with $\gamma(t) \neq 0$ on $[a,b]$, $\frac{\gamma'(t)}{\gamma(t)}$ is continuous on $[a,b]$. Hence $\varphi(x)$ is well-defined.

(b) *Show that $\varphi' = \frac{\gamma'}{\gamma}$ and $\varphi(a) = 0$.* By Theorem 6.20, $\varphi(x)$ is continuous. Furthermore, $\varphi(x)$ is differentiable on $[a,b]$ and $\varphi'(x) = \frac{\gamma'(x)}{\gamma(x)}$. By the definition of $\varphi$, $\varphi(a) = 0$.

(c) *Show that $\gamma\exp(-\varphi)$ is constant.* Write $f(x) = \gamma(x)\exp(-\varphi(x))$.

$$\begin{aligned} f'(x) &= \gamma'(x)\exp(-\varphi(x)) + \gamma(x)(-\varphi'(x))\exp(-\varphi(x)) \\ &= (\gamma'(x) - \gamma(x)\varphi'(x))\exp(-\varphi(x)) \\ &= 0. \end{aligned}$$

Hence $f = \gamma\exp(-\varphi)$ is constant (Theorem 5.11(b)).

(d) *Show that $\mathrm{Ind}(\gamma) \in \mathbb{Z}$.* By (c),

$$\begin{aligned} &\gamma(b)\exp(-\varphi(b)) = \gamma(a)\exp(-\varphi(a)) \\ \Longrightarrow &\exp(-\varphi(b)) = \exp(-\varphi(a)) && (\gamma \text{ is closed}) \\ \Longrightarrow &\exp(\varphi(b)) = \exp(\varphi(a)) \\ \Longrightarrow &\exp(2\pi i \mathrm{Ind}(\gamma)) = \exp(0) = 1 && ((\text{b})) \\ \Longrightarrow &2\pi i \mathrm{Ind}(\gamma) = 2\pi i n \text{ for some } n \in \mathbb{Z} && (\text{Theorem 8.7}) \\ \Longrightarrow &\mathrm{Ind}(\gamma) = n \text{ for some } n \in \mathbb{Z}. \end{aligned}$$

(2) *Compute* $\mathrm{Ind}(\gamma)$ *when* $\gamma(t) = \exp(int)$*,* $a = 0$*,* $b = 2\pi$*.*

$$\begin{aligned}\mathrm{Ind}(\gamma) &= \frac{1}{2\pi i}\int_a^b \frac{\gamma'(t)}{\gamma(t)}dt \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{in\exp(int)}{\exp(int)}dt \\ &= \frac{1}{2\pi i}\int_0^{2\pi} indt \\ &= n.\end{aligned}$$

(3) *Explain why* $\mathrm{Ind}(\gamma)$ *is often called the* ***winding number*** *of* $\gamma$ *around* $0$*.* As (2) suggested, $\mathrm{Ind}(\gamma)$ is an integer representing the total number of times that curve travels counterclockwise around 0. That's why we might say $\mathrm{Ind}(\gamma)$ is the winding number.

□

**Exercise 8.24.** PLACEHOLDER.

**Exercise 8.25.** PLACEHOLDER.

**Exercise 8.26.** PLACEHOLDER.

**Exercise 8.27.** PLACEHOLDER.

**Exercise 8.28.** PLACEHOLDER.

**Exercise 8.29.** PLACEHOLDER.

**Exercise 8.30.** *Use Stirling's formula to prove that*

$$\lim_{x\to\infty} \frac{\Gamma(x+c)}{x^c\Gamma(x)} = 1$$

*for every real constant* $c$*.*

*Proof.* By Stirling's formula,

$$\lim_{x\to\infty} \frac{\Gamma(x+c)}{\left(\frac{x+c-1}{e}\right)^{x+c-1}\sqrt{2\pi(x+c-1)}} = 1$$
$$\lim_{x\to\infty} \frac{\Gamma(x)}{\left(\frac{x-1}{e}\right)^{x-1}\sqrt{2\pi(x-1)}} = 1,$$

we have

$$\begin{aligned}
\lim_{x\to\infty} \frac{\Gamma(x+c)}{x^c\Gamma(x)} &= \lim_{x\to\infty} \frac{\Gamma(x+c)}{x^c\Gamma(x)} \\
&\quad \times \lim_{x\to\infty} \frac{\left(\frac{x+c-1}{e}\right)^{x+c-1}\sqrt{2\pi(x+c-1)}}{\Gamma(x+c)} \\
&\quad \times \lim_{x\to\infty} \frac{\Gamma(x)}{\left(\frac{x-1}{e}\right)^{x-1}\sqrt{2\pi(x-1)}} \\
&= \lim_{x\to\infty} \frac{\left(\frac{x+c-1}{e}\right)^{x+c-1}\sqrt{2\pi(x+c-1)}}{x^c\left(\frac{x-1}{e}\right)^{x-1}\sqrt{2\pi(x-1)}} \\
&= \lim_{x\to\infty} \frac{\left(\frac{x+c-1}{e}\right)^{c}}{x^c}\frac{\left(\frac{x+c-1}{e}\right)^{x-1}}{\left(\frac{x-1}{e}\right)^{x-1}}\sqrt{\frac{x+c-1}{x-1}} \\
&= \frac{1}{e^c}\cdot e^c \cdot 1 \\
&= 1
\end{aligned}$$

since

(1)
$$\lim_{x\to\infty} \frac{\left(\frac{x+c-1}{e}\right)^{c}}{x^c} = \frac{1}{e^c}\lim_{x\to\infty}\left(\frac{x+c-1}{x}\right)^c = \frac{1}{e^c}.$$

(2)
$$\lim_{x\to\infty} \frac{\left(\frac{x+c-1}{e}\right)^{x-1}}{\left(\frac{x-1}{e}\right)^{x-1}} = \lim_{x\to\infty}\left(\frac{x+c-1}{x-1}\right)^{x-1} = \lim_{x\to\infty}\left(1+\frac{c}{x-1}\right)^{x-1} = e^c.$$

(3) and
$$\lim_{x\to\infty}\sqrt{\frac{x+c-1}{x-1}} = \lim_{x\to\infty}\sqrt{1+\frac{c}{x-1}} = 1.$$

□

**Exercise 8.31.** *In the proof of Theorem 7.26 it was shown that*

$$\int_{-1}^{1}(1-x^2)^n dx \geq \frac{4}{3\sqrt{n}}$$

*for* $n = 1, 2, 3, \ldots$. *Use Theorem 8.20 and Exercise 8.30 to show the more precise result*

$$\lim_{n\to\infty} \sqrt{n} \int_{-1}^{1} (1 - x^2)^n dx = \sqrt{\pi}.$$

*Proof.*

$$\begin{aligned}
&\lim_{n\to\infty} \sqrt{n} \int_{-1}^{1} (1 - x^2)^n dx \\
=& \lim_{n\to\infty} \sqrt{n} \int_{0}^{1} u^{-\frac{1}{2}} (1 - u)^n dx && (u = x^2) \\
=& \lim_{n\to\infty} \sqrt{n} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma(n+1)}{\Gamma\left(n + \frac{3}{2}\right)} && \text{(Theorem 8.20)} \\
=& \Gamma\left(\frac{1}{2}\right) \lim_{n\to\infty} \frac{n^{\frac{1}{2}} \Gamma(n+1)}{\Gamma\left(n + \frac{3}{2}\right)} \\
=& \Gamma\left(\frac{1}{2}\right) && \text{(Exercise 8.30)} \\
=& \sqrt{\pi}. && \text{(Some consequences 8.21)}
\end{aligned}$$

□