

Notes on the book: *Patrick Morandi, Field and Galois Theory*

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I. Galois Theory

§1. Field Extensions

Problem 1.1.

Let K be a field extension of F . By defining scalar multiplication for $\alpha \in F$ and $a \in K$ by $\alpha \cdot a = \alpha a$, the multiplication in K , show that K is an F -vector space.

Proof.

(1) K is an additive group.

(2) Show that $(\alpha\beta) \cdot a = \alpha \cdot (\beta \cdot a)$ for $\alpha, \beta \in F$ and $a \in K$. In fact,

$$\begin{aligned}(\alpha\beta) \cdot a &= \alpha\beta a \in K, \\ \alpha \cdot (\beta \cdot a) &= \alpha\beta a \in K.\end{aligned}$$

(3) Show that $(\alpha + \beta) \cdot a = \alpha \cdot a + \beta \cdot a$ for $\alpha, \beta \in F$ and $a \in K$.

$$\begin{aligned}(\alpha + \beta) \cdot a &= (\alpha + \beta)a \\ &= \alpha a + \beta a \in K, \\ \alpha \cdot a + \beta \cdot a &= \alpha a + \beta a \in K.\end{aligned}$$

(4) Show that $\alpha \cdot (a + b) = \alpha \cdot a + \alpha \cdot b$ for $\alpha \in F$ and $a, b \in K$.

$$\begin{aligned}\alpha \cdot (a + b) &= \alpha(a + b) \\ &= \alpha a + \alpha b \in K, \\ \alpha \cdot a + \alpha \cdot b &= \alpha a + \alpha b \in K.\end{aligned}$$

(5) Show that $1 \cdot a = a$ for $a \in K$. $1 \cdot a = 1a = a \in K$.

By (1) to (5), K is an F -vector space. \square

Problem 1.2.

If K is a field extension of F , prove that $[K : F] = 1$ if and only if $K = F$.

Proof.

(1) $[K : F] = 1 \iff K = F$. Take a basis $\{1\}$ for K as an F -vector space.

- (2) $[K : F] = 1 \implies K = F$. Take a basis $\{a\}$ for K as an F -vector space where $a \in K$. Since $1 \in K$ as an F -vector space, there exists $\alpha \in F$ such that $1 = \alpha a$. $a = \alpha^{-1} \in F$, or $K \subseteq F$, or $K = F$.

□

Problem 1.3.

Let K be a field extension of F , and let $a \in K$. Show that the evaluation map $ev_a : F[x] \rightarrow K$ given by $ev_a(f(x)) = f(a)$ is a ring and F -vector space homomorphism. (Such a map is called an F -algebra homomorphism.)

Proof.

- (1) ev_a is a ring homomorphism.

$$(a) \quad ev_a(f(x) + g(x)) = f(a) + g(a) = ev_a(f(x)) + ev_a(g(x)).$$

$$(b) \quad ev_a(f(x)g(x)) = g(a)f(a) = ev_a(g(x))ev_a(f(x)).$$

$$(c) \quad ev_a(1) = 1.$$

- (2) ev_a is an F -vector space homomorphism.

$$(a) \quad ev_a(f(x) + g(x)) = f(a) + g(a) = ev_a(f(x)) + ev_a(g(x)).$$

$$(b) \quad \text{Given } c \in F, ev_a(cf(x)) = cf(a) = c ev_a(f(x)).$$

□

Problem 1.4.

Prove Proposition 1.9: Let K be a field extension of F and let $a_1, \dots, a_n \in K$. Then

$$F[a_1, \dots, a_n] = \{f(a_1, \dots, a_n) : f \in F[x_1, \dots, x_n]\}$$

and

$$F(a_1, \dots, a_n) = \left\{ \frac{f(a_1, \dots, a_n)}{g(a_1, \dots, a_n)} : f, g \in F[x_1, \dots, x_n], g(a_1, \dots, a_n) \neq 0 \right\},$$

so $F(a_1, \dots, a_n)$ is the quotient field of $F[x_1, \dots, x_n]$.

Proof (Proposition 1.8).

- (1) The evaluation map $ev_{(a_1, \dots, a_n)} : F[x_1, \dots, x_n] \rightarrow K$ has image

$$\{f(a_1, \dots, a_n) : f \in F[x_1, \dots, x_n]\},$$

so this set is a subring of K .

(2) If R is a subring of K that contains F and a_1, \dots, a_n , then

$$f(a_1, \dots, a_n) \in R$$

for any $f(x_1, \dots, x_n) \in F[x_1, \dots, x_n]$ by closure of addition and multiplication.

(3) So $\{f(a_1, \dots, a_n) : f \in F[x_1, \dots, x_n]\}$ is contained in all subrings of K that contains F and a_1, \dots, a_n . Hence

$$F[a_1, \dots, a_n] = \{f(a_1, \dots, a_n) : f \in F[x_1, \dots, x_n]\}.$$

(4) The quotient field of $F[a_1, \dots, a_n]$ is then the set

$$\left\{ \frac{f(a_1, \dots, a_n)}{g(a_1, \dots, a_n)} : f, g \in F[x_1, \dots, x_n], g(a_1, \dots, a_n) \neq 0 \right\}.$$

It is clearly is contained in any subfield of K that contains $F[a_1, \dots, a_n]$; hence, it is equal to $F(a_1, \dots, a_n)$.

□

Problem 1.5.

Show that $\mathbb{Q}(\sqrt{5}, \sqrt{7}) = \mathbb{Q}(\sqrt{5} + \sqrt{7})$.

Proof.

(1) $\mathbb{Q}(\sqrt{5}, \sqrt{7}) \supseteq \mathbb{Q}(\sqrt{5} + \sqrt{7})$ since $\sqrt{5} + \sqrt{7} \in \mathbb{Q}(\sqrt{5}, \sqrt{7})$.

(2)

$$\begin{aligned} (\sqrt{7} + \sqrt{5})^{-1} &= \frac{1}{\sqrt{7} + \sqrt{5}} \\ &= \frac{\sqrt{7} - \sqrt{5}}{(\sqrt{7} + \sqrt{5})(\sqrt{7} - \sqrt{5})} \\ &= \frac{\sqrt{7} - \sqrt{5}}{2} \in \mathbb{Q}(\sqrt{5} + \sqrt{7}), \end{aligned}$$

Or $\sqrt{7} - \sqrt{5} \in \mathbb{Q}(\sqrt{5} + \sqrt{7})$. Thus

$$\begin{aligned} \sqrt{7} &= \frac{1}{2} \cdot ((\sqrt{7} + \sqrt{5}) + (\sqrt{7} - \sqrt{5})) \in \mathbb{Q}(\sqrt{5} + \sqrt{7}), \\ \sqrt{5} &= \frac{1}{2} \cdot ((\sqrt{7} + \sqrt{5}) - (\sqrt{7} - \sqrt{5})) \in \mathbb{Q}(\sqrt{5} + \sqrt{7}). \end{aligned}$$

Thus, $\mathbb{Q}(\sqrt{5}, \sqrt{7}) \subseteq \mathbb{Q}(\sqrt{5} + \sqrt{7})$.

By (1)(2), $\mathbb{Q}(\sqrt{5}, \sqrt{7}) = \mathbb{Q}(\sqrt{5} + \sqrt{7})$. □

Problem 1.9.

If K is an extension of F such that $[K : F]$ is prime, show that there are no intermediate fields between K and F .

Proof. Let L be any field such that $F \subseteq L \subseteq K$. By Proposition 1.20,

$$[K : F] = [K : L][L : F].$$

Since $[K : F]$ is prime, $[K : L] = 1$ or $[L : F] = 1$. By Problem 1.2, $L = K$ or $L = F$, or there are no intermediate fields between K and F . \square

Problem 1.11.

If K is an algebraic extension of F and if R is a subring of K with $F \subseteq R \subseteq K$, show that R is a field.

Proof.

- (1) R is a domain since R is contained in a field K . To show R is a field, it suffices to show that every nonzero element $\alpha \in R$ has an inverse in R .
- (2) Since $\alpha \in R \subseteq K$ is algebraic over F , there is a minimal polynomial

$$f(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0$$

such that $f(\alpha) = 0$, where each $b_i \in F$ and $b_0 \neq 0$ by the minimality of f .

- (3) Note that

$$\begin{aligned} f(\alpha) &= 0 \\ \iff b_n \alpha^n + b_{n-1} \alpha^{n-1} + \cdots + b_0 &= 0 \\ \iff b_n \alpha^n + b_{n-1} \alpha^{n-1} + \cdots + b_1 \alpha &= -b_0 \\ \iff \alpha(b_n \alpha^{n-1} + b_{n-1} \alpha^{n-2} + \cdots + b_1) &= -b_0 \\ \iff \alpha \underbrace{((-b_0)^{-1} b_n \alpha^{n-1} + (-b_0)^{-1} b_{n-1} \alpha^{n-2} + \cdots + (-b_0)^{-1} b_1)}_{:=\alpha'} &= 1. \end{aligned}$$

Hence $\alpha' \in F[\alpha] \subseteq R$. Therefore α' is the inverse of α in R .

\square

Problem 1.12.

Show that $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ are not isomorphic as fields but are isomorphic as vector spaces over \mathbb{Q} .

Proof.

- (1) Show that $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ are not isomorphic as fields. (Reductio ad absurdum) If $\varphi : \mathbb{Q}(\sqrt{2}) \rightarrow \mathbb{Q}(\sqrt{3})$ were an isomorphism as fields, then φ is an identity map on \mathbb{Q} , and

$$\begin{aligned}\varphi(\sqrt{2}) &= a + b\sqrt{3} \text{ for some } a, b \in \mathbb{Q} \\ \implies \varphi(\sqrt{2})\varphi(\sqrt{2}) &= (a + b\sqrt{3})^2 \\ \implies \varphi(\sqrt{2}\sqrt{2}) &= (a + b\sqrt{3})^2 \\ \implies \varphi(2) &= a^2 + 3b^2 + 2ab\sqrt{3} \\ \implies 2 &= a^2 + 3b^2 + 2ab\sqrt{3}.\end{aligned}$$

If $2ab \neq 0$, then $\sqrt{3} = \frac{2-a^2-3b^2}{2ab} \in \mathbb{Q}$, which is absurd. Hence $2ab = 0$.

- (a) $a = 0$. Write $b = \frac{m}{n} \in \mathbb{Q}$ where $m, n \in \mathbb{Z}$ and $(m, n) = 1$. Hence

$$2n^2 = 3m^2.$$

So $2 \mid 3m^2$, $2 \mid m^2$, $2 \mid m$. So $4 \mid 2n^2$, $2 \mid n^2$, $2 \mid n$. Hence $2 \mid (m, n)$, contrary to the assumption that $(m, n) = 1$.

- (b) $b = 0$. $2 = a^2$. Write $a = \frac{m}{n} \in \mathbb{Q}$ where $m, n \in \mathbb{Z}$ and $(m, n) = 1$. Similar to the argument in (a), we will reach a contradiction.

By (a)(b), no such isomorphism φ , that is, $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ are not isomorphic as fields.

- (2) Show that $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ are isomorphic as \mathbb{Q} -vector spaces. $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] = 2$. There is a natural map $\varphi : \mathbb{Q}(\sqrt{2}) \rightarrow \mathbb{Q}(\sqrt{3})$ defined by $\varphi(a + b\sqrt{2}) = a + b\sqrt{3}$. Clearly φ is well-defined, linear, injective and surjective.

□

Problem 1.16.

Let \mathbb{A} be the algebraic closure of \mathbb{Q} in \mathbb{C} . Prove that $[\mathbb{A} : \mathbb{Q}] = \infty$.

Proof (Example 1.16). By Example 1.16, $[\mathbb{Q}(\sqrt[n]{2}) : \mathbb{Q}] = n$. Therefore,

$$[\mathbb{A} : \mathbb{Q}] = [\mathbb{A} : \mathbb{Q}(\sqrt[n]{2})][\mathbb{Q}(\sqrt[n]{2}) : \mathbb{Q}] = [\mathbb{A} : \mathbb{Q}(\sqrt[n]{2})]n$$

for arbitrary $n \in \mathbb{Z}^+$. Hence $[\mathbb{A} : \mathbb{Q}] = \infty$. \square

Proof (Example 1.16). Given a prime number p . By Example 1.16, $[\mathbb{Q}(\rho) : \mathbb{Q}] = p - 1$ where $\rho = \exp(2\pi i/p)$. Therefore,

$$[\mathbb{A} : \mathbb{Q}] = [\mathbb{A} : \mathbb{Q}(\rho)][\mathbb{Q}(\rho) : \mathbb{Q}] = [\mathbb{A} : \mathbb{Q}(\rho)](p - 1)$$

for arbitrary prime p . Hence $[\mathbb{A} : \mathbb{Q}] = \infty$. \square

Problem 1.23.

Recall that the characteristic of a ring R with identity is the smallest positive integer n for which $n \cdot 1 = 0$, if such an n exists, or else the characteristic is 0. Let R be a ring with identity. Define $\varphi : \mathbb{Z} \rightarrow R$ by $\varphi(n) = n \cdot 1$, where 1 is the identity of R . Show that φ is a ring homomorphism and that $\ker(\varphi) = m\mathbb{Z}$ for a unique nonnegative integer m , and show that m is the characteristic of R .

Proof.

(1) φ is a ring homomorphism.

$$(a) \quad \varphi(a+b) = \varphi(a) + \varphi(b). \quad \varphi(a+b) = (a+b) \cdot 1 = a \cdot 1 + b \cdot 1 = \varphi(a) + \varphi(b).$$

$$(b) \quad \varphi(ab) = \varphi(a)\varphi(b). \quad \varphi(ab) = (ab) \cdot 1 = (a \cdot 1)(b \cdot 1) = \varphi(a)\varphi(b) \text{ since } 1 \times 1 = 1. \text{ (Here } \times \text{ is the multiplication operator of } R.)$$

(2) $\ker(\varphi) = m\mathbb{Z}$ for a unique nonnegative integer m . Since $\ker(\varphi)$ is an ideal of a PID \mathbb{Z} , there is a unique nonnegative integer m such that $\ker(\varphi) = m\mathbb{Z}$.

(3) m is the characteristic of R . There are only two possible cases, $\text{char}(R) = 0$ or else $\text{char}(R) > 0$.

$$(a) \quad \text{char}(R) = 0. \quad \ker(\varphi) = 0. \quad \text{Thus } m = 0 = \text{char}(R).$$

$$(b) \quad \text{char}(R) = n > 0. \quad n \in \ker(\varphi), \text{ so } m > 0 \text{ and } m \mid n. \text{ By the minimality of } n, \quad m = n = \text{char}(R).$$

\square

Problem 1.24.

For any positive integer n , give an example of a ring of characteristic n .

Proof. The ring $\mathbb{Z}/n\mathbb{Z}$. \square

Problem 1.25.

If R is an integral domain, show that either $\text{char}(R) = 0$ or $\text{char}(R)$ is prime.

Proof.

- (1) 1 has infinite order. $\text{char}(R) = 0$. (Nothing to do.)
- (2) 1 has finite order n . Want to show n is prime. If $n = ab$ where $a, b \in \mathbb{Z}^+$, then

$$0 = n \cdot 1 = (a \cdot 1)(b \cdot 1).$$

Since R is an integral domain, $a \cdot 1 = 0$ or $b \cdot 1 = 0$. By the minimality of n , $a \geq n$ or $b \geq n$. $a = n$ or $b = n$. That is, n is prime.

□

§2. Automorphisms**Problem 2.1.**

Show that the only automorphism of \mathbb{Q} is the identity.

Proof. Given any $\sigma \in \text{Aut}(\mathbb{Q})$.

- (1) Show that $\sigma(1) = 1$. Since $1^2 = 1$, $\sigma(1)\sigma(1) = \sigma(1)$. $\sigma(1) = 0$ or 1 . There are only two possible cases.

- (a) Assume that $\sigma(1) = 0$. So

$$\sigma(a) = \sigma(a \cdot 1) = \sigma(a) \cdot \sigma(1) = \sigma(a) \cdot 0 = 0$$

for any $a \in \mathbb{Q}$. That is, $\sigma = 0 \in \text{Aut}(\mathbb{Q})$, which is absurd.

- (b) Therefore, $\sigma(1) = 1$.

- (2) Show that $\sigma(n) = n$ for all $n \in \mathbb{Z}^+$. Write $n = 1 + 1 + \cdots + 1$ (n times 1). Applying the additivity of σ , we have

$$\sigma(n) = \sigma(1) + \sigma(1) + \cdots + \sigma(1) = 1 + 1 + \cdots + 1 = n.$$

(Might use induction on n to eliminate \cdots symbols.)

- (3) Show that $\sigma(n) = n$ for all $n \in \mathbb{Z}$. By the additivity of σ , $\sigma(-n) = -\sigma(n) = -n$ for $n \geq 0$. The result is established.

For any $a = \frac{n}{m} \in \mathbb{Q}$ ($m, n \in \mathbb{Z}$, $n \neq 0$), applying the multiplication of σ on $am = n$, that is, $\sigma(a)\sigma(m) = \sigma(n)$. By (3), we have $\sigma(a)m = n$, or

$$\sigma(a) = \frac{m}{n} = a$$

provided $n \neq 0$, or σ is the identity. \square

Problem 2.2.

Show that the only automorphism of \mathbb{R} is the identity. (Hint: If σ is an automorphism, show that $\sigma|_{\mathbb{Q}} = \text{id}$, and if $a > 0$, then $\sigma(a) > 0$. It is an interesting fact that there are infinitely many automorphisms of \mathbb{C} , even though $[\mathbb{C} : \mathbb{R}] = 2$. Why is this fact not a contradiction to this problem?)

Proof (Hint). Given any $\sigma \in \text{Aut}(\mathbb{R})$.

- (1) Apply the same argument in Problem 2.1, we have $\sigma|_{\mathbb{Q}} = \text{id}$. Notice that $\sigma(a) \neq 0$ for any $a \neq 0$.
- (2) Show that $\sigma(a) > 0$ if $a > 0$. Given any $a > 0$. Write $a = \sqrt{a}\sqrt{a}$ (well-defined) and then apply σ on the both sides,

$$\sigma(a) = \sigma(\sqrt{a})\sigma(\sqrt{a}) = \sigma(\sqrt{a})^2 > 0$$

(since $\sqrt{a} \neq 0$ and thus $\sigma(\sqrt{a})$ cannot be zero).

- (3) Show that $\sigma(a) > \sigma(b)$ if $a > b$. It is a corollary to (2) by applying σ on $a - b > 0$. ($\sigma(a - b) > 0$, or $\sigma(a) - \sigma(b) > 0$, or $\sigma(a) > \sigma(b)$.)
- (4) For any real number $x \in \mathbb{R}$, choose two sequences $\{p_n\}, \{q_n\}$ of rational numbers such that $p_n < x < q_n$ and $p_n, q_n \rightarrow x$ as $n \rightarrow \infty$. Take σ on the inequality, $\sigma(p_n) < \sigma(x) < \sigma(q_n)$. So $p_n < \sigma(x) < q_n$ since $\sigma|_{\mathbb{Q}} = \text{id}$. Let $n \rightarrow \infty$, we get $x \leq \sigma(x) \leq x$, or $\sigma(x) = x$.

\square

Supplement. Automorphisms of the Complex Numbers. by Paul B. Yale (Pomona College) [Link].

Problem 2.4.

Let B be an integral domain with quotient field F . If $\sigma : B \rightarrow B$ is a ring automorphism, show that σ induces a ring automorphism $\sigma' : F \rightarrow F$ defined by $\sigma'(a/b) = \sigma(a)/\sigma(b)$ if $a, b \in B$ with $b \neq 0$.

Proof.

(1) Show that σ' is well-defined.

- (a) $\sigma' : F \rightarrow F$ is defined. $\sigma(a), \sigma(b) \in B$ since σ is a homomorphism.
 $\sigma(b) \neq 0$ since $b \neq 0$ and σ is a one-on-one homomorphism.
- (b) σ' is independent of the representation of $a/b \in F$. Suppose $a/b = c/d$ where $a, b, c, d \in B$ and $b, d \neq 0$. Hence,

$$\begin{aligned}
 a/b = c/d &\iff ad = bc \\
 &\iff \sigma(ad) = \sigma(bc) \\
 &\iff \sigma(a)\sigma(d) = \sigma(b)\sigma(c) \quad (\sigma: \text{homomorphism}) \\
 &\iff \sigma(a)/\sigma(d) = \sigma(c)/\sigma(b) \quad (\sigma(b), \sigma(d) \neq 0) \\
 &\iff \sigma'(a/b) = \sigma'(c/d).
 \end{aligned}$$

(2) Show that σ' is a ring homomorphism.

- (a) Show that $\sigma'(a/b + c/d) = \sigma'(a/b) + \sigma'(c/d)$.

$$\begin{aligned}
 \sigma'(a/b + c/d) &= \sigma'((ad + bc)/(bd)) \\
 &= \sigma(ad + bc)/\sigma(bd) \\
 &= (\sigma(a)\sigma(d) + \sigma(b)\sigma(c))/(\sigma(b)\sigma(d)) \quad (\sigma: \text{homomorphism}) \\
 &= \sigma(a)/\sigma(b) + \sigma(c)/\sigma(d) \\
 &= \sigma'(a/b) + \sigma'(c/d).
 \end{aligned}$$

- (b) Show that $\sigma'(a/b \cdot c/d) = \sigma'(a/b) \cdot \sigma'(c/d)$.

$$\begin{aligned}
 \sigma'(a/b \cdot c/d) &= \sigma'((ac)/(bd)) \\
 &= \sigma(ac)/\sigma(bd) \\
 &= (\sigma(a)\sigma(c))/(\sigma(b)\sigma(d)) \quad (\sigma: \text{homomorphism}) \\
 &= \sigma(a)/\sigma(b) \cdot \sigma(c)/\sigma(d) \\
 &= \sigma'(a/b) \cdot \sigma'(c/d).
 \end{aligned}$$

(3) Show that σ' is injective.

$$\begin{aligned}
 \sigma'(a/b) = 0 &\iff \sigma(a)/\sigma(b) = 0 \\
 &\iff \sigma(a) = 0 \\
 &\iff a = 0 \quad (\sigma: \text{injective}) \\
 &\iff a/b = 0/b = 0 \in F
 \end{aligned}$$

(4) Show that σ' is a surjective. Given any $c/d \in F$, want to show there is $a/b \in F$ such that $\sigma'(a/b) = c/d$.

$$\begin{aligned}
 c/d \in F &\implies c, d \in B \\
 &\implies \exists a, b \in B \text{ such that } \sigma(a) = c, \sigma(b) = d \quad (\sigma: \text{surjective}) \\
 &\implies \exists a, b \in B \text{ such that } \sigma(a)/\sigma(b) = c/d \\
 &\implies \exists a, b \in B \text{ such that } \sigma'(a/b) = c/d.
 \end{aligned}$$

II. Some Galois Extensions

§10. Hilbert Theorem 90 and Group Cohomology

Supplement.

- (1) Corollary 10.4 (Cohomological Hilbert Theorem 90). Let K be a cyclic Galois extension of F . Then $H^1(\text{Gal}(K/F), K^\times) = 0$.
- (2) (*Exercise 10.24 in the textbook: Rudin, Principles of Mathematical Analysis, 3rd edition.*) Let $\omega = \sum a_i(\mathbf{x})dx_i$ be a 1-form of class C'' in a convex open set $E \subseteq \mathbb{R}^n$. Assume $d\omega = 0$ and prove that ω is exact in E . Hence the first de Rham cohomology $H_{\text{dR}}^1(E) = 0$.
- (3) $H_{\text{dR}}^1(E) = 0$ if E is simply connected. (The converse is not true.)
- (4) (*Exercise 10.21 in the textbook: Rudin, Principles of Mathematical Analysis, 3rd edition.*) Consider the 1-form

$$\eta = \frac{xdy - ydx}{x^2 + y^2}$$

in $\mathbb{R}^2 - \{\mathbf{0}\}$.

- (a) Carry out the computation that leads to

$$\int_{\gamma} \eta = 2\pi \neq 0,$$

and prove that $d\eta = 0$.

- (b) Let $\gamma(t) = (r \cos t, r \sin t)$, for some $r > 0$, and let Γ be a C'' -curve in $\mathbb{R}^2 - \{\mathbf{0}\}$, with parameter interval $[0, 2\pi]$, with $\Gamma(0) = \Gamma(2\pi)$, such that the intervals $[\gamma(t), \Gamma(t)]$ do not contain $\mathbf{0}$ for any $t \in [0, 2\pi]$. Prove that

$$\int_{\Gamma} \eta = 2\pi.$$

- (c) Take $\Gamma(t) = (a \cos t, b \sin t)$ where $a > 0$, $b > 0$ are fixed. Show that

$$\int_0^{2\pi} \frac{ab}{a^2 \cos^2 t + b^2 \sin^2 t} dt = 2\pi.$$

- (d) Show that

$$\eta = d\left(\arctan \frac{y}{x}\right)$$

in any convex open set in which $x \neq 0$, and that

$$\eta = d\left(-\arctan \frac{x}{y}\right)$$

in any convex open set in which $y \neq 0$. Explain why this justifies the notation $\eta = d\theta$, in spite of the fact that η is not exact in $\mathbb{R}^2 - \{0\}$.

- (5) (Exercise 10.22 in the textbook: Rudin, *Principles of Mathematical Analysis*, 3rd edition.) Define ζ in $\mathbb{R}^3 - \{0\}$ by

$$\zeta = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{r^3}$$

where $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$, let D be the rectangle given by $0 \leq u \leq \pi$, $0 \leq v \leq 2\pi$, and let Σ be the 2-surface in \mathbb{R}^3 , with parameter domain D , given by

$$x = \sin u \cos v, \quad y = \sin u \sin v, \quad z = \cos u.$$

- (a) Prove that $d\zeta = 0$ in $\mathbb{R}^3 - \{0\}$.
 (b) Let S denote the restriction of Σ to a parameter domain $E \subseteq D$. Prove that

$$\int_S \zeta = \int_E \sin u \, du \, dv = A(S),$$

where A denotes area, as in Section 10.46. Note that this contains

$$\int_{\Sigma} \zeta = \int_D \sin u \, du \, dv = 4\pi \neq 0$$

as a special case.

- (c) Suppose g, h_1, h_2, h_3 , are C'' -functions on $[0, 1]$, $g > 0$. Let $(x, y, z) = \Phi(s, t)$ define a 2-surface Φ , with parameter domain I^2 , by

$$x = g(t)h_1(s), \quad y = g(t)h_2(s), \quad z = g(t)h_3(s).$$

Prove that

$$\int_{\Phi} \zeta = 0.$$

Note the shape of the range of Φ : For fixed s , $\Phi(s, t)$ runs over an interval on a line through 0 . The range of Φ thus lies in a “cone” with vertex at the origin.

- (d) Let E be a closed rectangle in D , with edges parallel to those of D . Suppose $f \in C''(D)$, $f > 0$. Let Ω be the 2-surface with parameter domain E , defined by

$$\Omega(u, v) = f(u, v)\Sigma(u, v).$$

Define S as in (b) and prove that

$$\int_{\Omega} \zeta = \int_S \zeta = A(S).$$

(e) Put $\lambda = -\frac{z}{r}\eta$, where

$$\eta = \frac{xdy - ydx}{x^2 + y^2}.$$

Then λ is a 1-form in the open set $V \subseteq \mathbb{R}^3$ in which $x^2 + y^2 > 0$. Show that ζ is exact in V by showing that

$$\zeta = d\lambda.$$

(f) Is ζ exact in the complement of every line through the origin?

- (6) (Exercise 10.23 in the textbook: Rudin, Principles of Mathematical Analysis, 3rd edition.) Fix n . Define $r_k = (x_1^2 + \cdots + x_k^2)^{\frac{1}{2}}$ for $1 \leq k \leq n$, let E_k be the set of all $\mathbf{x} \in \mathbb{R}^n$ at which $r_k > 0$, and let ω_k be the $(k-1)$ -form defined in E_k by

$$\omega_k = (r_k)^{-k} \sum_{i=1}^k (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k$$

Note that $\omega_2 = \eta$, $\omega_3 = \zeta$ in the terminology of Exercise 10.21 and Exercise 10.22. Note also that

$$E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n = \mathbb{R}^n.$$

(a) Prove that $d\omega_k = 0$ in E_k .

(b) For $k = 2, \dots, n$, prove that ω_k is exact in E_{k-1} , by showing that

$$\omega_k = d(f_k \omega_{k-1}) = df_k \wedge \omega_{k-1}$$

where $f_k(\mathbf{x}) = (-1)^k g_k \left(\frac{x_k}{r_k} \right)$ where

$$g_k(t) = \int_{-1}^t (1 - s^2)^{\frac{k-3}{2}} ds \quad (-1 < t < 1).$$

(c) Is ω_n exact in E_n ?

- (7) $H_{\text{dR}}^{n-1}(\mathbb{R}^n - \{\mathbf{0}\}) = \mathbb{R}^1$. (Compare to (5)(6)(7).)

Problem 10.1.

Let M be a G -module. Show that the boundary map $\delta_n : C^n(G, M) \rightarrow C^{n+1}(G, M)$ defined in this section is a homomorphism.

Proof.

(1) δ_n is defined by

$$\begin{aligned}\delta_n(f)(\sigma_1, \dots, \sigma_{n+1}) &= \sigma_1 f(\sigma_2, \dots, \sigma_{n+1}) \\ &\quad + \sum_{i=1}^n (-1)^i f(\sigma_1, \dots, \sigma_i \sigma_{i+1}, \dots, \sigma_{n+1}) \\ &\quad + (-1)^{n+1} f(\sigma_1, \dots, \sigma_n)\end{aligned}$$

if $n > 0$. If $n = 0$, then the map $\delta_0 : M = C^0(G, M) \rightarrow C^1(G, M)$ is defined by $\delta_0(m)(\sigma) = \sigma m - m$.

(2) It suffices to show that $\delta_n(f+g) = \delta_n(f) + \delta_n(g)$ for all n and all n -cochains f and g .

(3) If $n = 0$, then

$$\begin{aligned}\delta_0(f+g)(\sigma) &= \sigma(f+g) - (f+g) \\ &= \sigma f + \sigma g - f - g && (M: G\text{-module}) \\ &= (\sigma f - f) + (\sigma g - g) && (M: \text{abelian group}) \\ &= \delta_0(f) + \delta_0(g).\end{aligned}$$

(4) If $n \geq 1$, then

$$\begin{aligned}&\delta_n(f+g)(\sigma) \\ &= \sigma_1(f+g)(\sigma_2, \dots, \sigma_{n+1}) + \sum_{i=1}^n (-1)^i (f+g)(\sigma_1, \dots, \sigma_i \sigma_{i+1}, \dots, \sigma_{n+1}) \\ &\quad + (-1)^{n+1} (f+g)(\sigma_1, \dots, \sigma_n) \\ &= \sigma_1 f(\sigma_2, \dots, \sigma_{n+1}) + \sigma_1 g(\sigma_2, \dots, \sigma_{n+1}) \\ &\quad + \sum_{i=1}^n (-1)^i f(\sigma_1, \dots, \sigma_i \sigma_{i+1}, \dots, \sigma_{n+1}) \\ &\quad + \sum_{i=1}^n (-1)^i g(\sigma_1, \dots, \sigma_i \sigma_{i+1}, \dots, \sigma_{n+1}) \\ &\quad + (-1)^{n+1} f(\sigma_1, \dots, \sigma_n) + (-1)^{n+1} g(\sigma_1, \dots, \sigma_n) \\ &= \left\{ \sigma_1 f(\sigma_2, \dots, \sigma_{n+1}) + \sum_{i=1}^n (-1)^i f(\sigma_1, \dots, \sigma_i \sigma_{i+1}, \dots, \sigma_{n+1}) \right. \\ &\quad \left. + (-1)^{n+1} f(\sigma_1, \dots, \sigma_n) \right\} + \left\{ \sigma_1 g(\sigma_2, \dots, \sigma_{n+1}) \right. \\ &\quad \left. + \sum_{i=1}^n (-1)^i g(\sigma_1, \dots, \sigma_i \sigma_{i+1}, \dots, \sigma_{n+1}) + (-1)^{n+1} g(\sigma_1, \dots, \sigma_n) \right\} \\ &= \delta_n(f)(\sigma) + \delta_n(g)(\sigma).\end{aligned}$$

(Here note that $C^n(G, M)$ is an abelian group).

□

Problem 10.2.

With notation as in the previous problem, show that $\delta_{n+1} \circ \delta_n$ is the zero map.

Proof.

(1) If $n = 0$, then

$$\begin{aligned} (\delta_1 \circ \delta_0)(f)(\sigma_1, \sigma_2) &= \delta_1(\delta_0(f))(\sigma_1, \sigma_2) \\ &= \sigma_1 \delta_0(f)(\sigma_2) - \delta_0(f)(\sigma_1 \sigma_2) + \delta_0(f)(\sigma_1) \\ &= \sigma_1(\sigma_2 f - f) - (\sigma_1 \sigma_2 f - f) + (\sigma_1 f - f) \\ &= 0. \end{aligned}$$

(2) If $n \geq 1$, then we write

$$\begin{aligned} &(\delta_{n+1} \circ \delta_n)(f)(\sigma_1, \dots, \sigma_{n+2}) \\ &= \delta_{n+1}(\delta_n(f))(\sigma_1, \dots, \sigma_{n+2}) \\ &= \underbrace{\sigma_1 \delta_n(f)(\sigma_2, \dots, \sigma_{n+2})}_{\text{Part (3)}} \\ &\quad + \underbrace{\sum_{j=1}^{n+1} (-1)^j \delta_n(f)(\sigma_1, \dots, \sigma_j \sigma_{j+1}, \dots, \sigma_{n+2})}_{\text{Parts (4)(5)(6)}} \\ &\quad + \underbrace{(-1)^{n+2} \delta_n(f)(\sigma_1, \dots, \sigma_{n+1})}_{\text{Part (7)}}. \end{aligned}$$

(3) The first term is

$$\begin{aligned} &\sigma_1 \delta_n(f)(\sigma_2, \dots, \sigma_{n+2}) \\ &= \sigma_1 \sigma_2 f(\sigma_3, \dots, \sigma_{n+2}) \\ &\quad + \sum_{i=1}^n (-1)^i \sigma_1 f(\sigma_2, \dots, \sigma_{i+1} \sigma_{i+2}, \dots, \sigma_{n+2}) \\ &\quad + (-1)^{n+1} \sigma_1 f(\sigma_2, \dots, \sigma_{n+1}). \end{aligned}$$

(4) The first term ($j = 1$) in the summation is

$$\begin{aligned} &(-1)^1 \delta_n(f)(\sigma_1 \sigma_2, \dots, \sigma_{n+2}) \\ &= -\sigma_1 \sigma_2 f(\sigma_3, \dots, \sigma_{n+2}) \\ &\quad + f(\sigma_1 \sigma_2 \sigma_3, \dots, \sigma_{n+2}) - \sum_{i=2}^n (-1)^i f(\sigma_1 \sigma_2, \dots, \sigma_{i+1} \sigma_{i+2}, \dots, \sigma_{n+2}) \\ &\quad - (-1)^{n+1} f(\sigma_1 \sigma_2, \dots, \sigma_{n+1}) \end{aligned}$$

(5) The j th term for $2 \leq j \leq n$ in the summation is

$$\begin{aligned}
& (-1)^j \delta_n(f)(\sigma_1, \dots, \sigma_j \sigma_{j+1}, \dots, \sigma_{n+2}) \\
&= (-1)^j \sigma_1 f(\sigma_2, \dots, \sigma_j \sigma_{j+1}, \dots, \sigma_{n+2}) \\
&\quad + (-1)^j \sum_{i=1}^{j-2} (-1)^i f(\sigma_1, \dots, \sigma_i \sigma_{i+1}, \dots, \sigma_j \sigma_{j+1}, \dots, \sigma_{n+2}) \\
&\quad + (-1)^j (-1)^{j-1} f(\sigma_1, \dots, \sigma_{j-1} \sigma_j \sigma_{j+1}, \dots, \sigma_{n+2}) \\
&\quad + (-1)^j (-1)^j f(\sigma_1, \dots, \sigma_j \sigma_{j+1} \sigma_{j+2}, \dots, \sigma_{n+2}) \\
&\quad + (-1)^j \sum_{i=j+1}^n (-1)^i f(\sigma_1, \dots, \sigma_j \sigma_{j+1}, \dots, \sigma_{i+1} \sigma_{i+2}, \dots, \sigma_{n+2}) \\
&\quad + (-1)^j (-1)^{n+1} f(\sigma_1, \dots, \sigma_j \sigma_{j+1}, \dots, \sigma_{n+1}).
\end{aligned}$$

(6) The last term ($j = n + 1$) in the summation is

$$\begin{aligned}
& (-1)^{n+1} \delta_n(f)(\sigma_1, \dots, \sigma_n, \sigma_{n+1} \sigma_{n+2}) \\
&= (-1)^{n+1} \sigma_1 f(\sigma_2, \dots, \sigma_{n+1} \sigma_{n+2}) \\
&\quad + (-1)^{n+1} \sum_{i=1}^{n-1} (-1)^i f(\sigma_1, \dots, \sigma_i \sigma_{i+1}, \dots, \sigma_{n+1} \sigma_{n+2}) \\
&\quad + (-1)^{n+1} (-1)^n f(\sigma_1, \dots, \sigma_n \sigma_{n+1} \sigma_{n+2}) \\
&\quad + (-1)^{n+1} (-1)^{n+1} f(\sigma_1, \dots, \sigma_n).
\end{aligned}$$

(7) The last term is

$$\begin{aligned}
& (-1)^{n+2} \delta_n(f)(\sigma_1, \dots, \sigma_{n+1}) \\
&= (-1)^{n+2} \sigma_1 f(\sigma_2, \dots, \sigma_{n+1}) \\
&\quad + (-1)^{n+2} \sum_{i=1}^n (-1)^i f(\sigma_1, \dots, \sigma_i \sigma_{i+1}, \dots, \sigma_{n+1}) \\
&\quad + (-1)^{n+2} (-1)^{n+1} f(\sigma_1, \dots, \sigma_n).
\end{aligned}$$

(8) Hence we have $(\delta_{n+1} \circ \delta_n)(f)(\sigma_1, \dots, \sigma_{n+2}) = 0$.

□

Supplement.

- (1) (Theorem 10.20 in the textbook: *Rudin, Principles of Mathematical Analysis*, 3rd edition.) If ω is a k -form of class \mathcal{C}'' in some open set $E \subseteq \mathbb{R}^n$, then $d^2\omega = 0$.

- (2) (Exercise 10.16 in the textbook: Rudin, *Principles of Mathematical Analysis*, 3rd edition.) If $k \geq 2$ and $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_k]$ is an oriented affine k -simplex, prove that $\partial^2 \sigma = 0$, directly from the definition of the boundary operator ∂ . Deduce from this that $\partial^2 \Psi = 0$ for every chain Ψ .

Problem 10.3.

Let M be a G -module, and let $f \in Z^2(G, M)$. Show that $f(1, 1) = f(1, \sigma) = \sigma^{-1}f(\sigma, 1)$ for all $\sigma \in G$.

Proof.

- (1) $f \in Z^2(G, M)$ if and only if $\delta_2(f) = 0$. So

$$\begin{aligned} \delta_2(f)(\sigma_1, \sigma_2, \sigma_3) &= \sigma_1 f(\sigma_2, \sigma_3) - f(\sigma_1 \sigma_2, \sigma_3) + f(\sigma_1, \sigma_2 \sigma_3) - f(\sigma_1, \sigma_2) \\ &= 0. \end{aligned}$$

for any $\sigma_1 \sigma_2, \sigma_3 \in G$.

- (2) Take $\sigma_1 = \sigma_2 = 1$ and $\sigma_3 = \sigma$ to get

$$f(1, \sigma) - f(1, \sigma) + f(1, \sigma) - f(1, 1) = 0.$$

So $f(1, 1) = f(1, \sigma)$.

- (3) Take $\sigma_1 = \sigma$ and $\sigma_2 = \sigma_3 = 1$ to get

$$\sigma f(1, 1) - f(\sigma, 1) + f(\sigma, 1) - f(\sigma, 1) = 0.$$

So $\sigma f(1, 1) = f(\sigma, 1)$ or $f(1, 1) = \sigma^{-1}f(\sigma, 1)$.

□

Problem 10.4.

If E is a group with an abelian normal subgroup M , and if $G = E/M$, show that the action of G on M given by $\sigma m = eme^{-1}$ if $eM = \sigma$ is well-defined and makes M into a G -module.

Proof.

- (1) Show that $G \times M \rightarrow M$ defined by $\sigma m = eme^{-1}$ is independent of the choice of the coset representation of $\sigma = eM$. Suppose $\sigma = e_1 M = e_2 M$. $e_2 = e_1 m_1$ for some $m_1 \in M$.

(2) Therefore

$$e_2 m e_2^{-1} = (e_1 m_1) m (e_1 m_1)^{-1} = e_1 m_1 m m_1^{-1} e_1^{-1} = e_1 m e_1^{-1}.$$

Here $(e_1 m_1)^{-1} = m_1^{-1} e_1^{-1}$ holds in a group E and $m_1 m m_1^{-1} = m$ since M is an abelian group.

(3) Show that M is a G -module where $G \times M \rightarrow M$ is defined by $\sigma m = e m e^{-1}$.

(a) Show that $1m = m$. $1m = 1m1^{-1} = m$ where $1 = 1M \in G = E/M$.

(b) Show that $\sigma(\tau m) = (\sigma\tau)m$. Write $\sigma = e_\sigma M$ and $\tau = e_\tau M$. Hence $\sigma\tau = e_\sigma e_\tau M$ and

$$\begin{aligned} \sigma(\tau m) &= \sigma(e_\tau m e_\tau^{-1}) \\ &= e_\sigma (e_\tau m e_\tau^{-1}) e_\sigma^{-1} \\ &= (e_\sigma e_\tau) m (e_\sigma e_\tau)^{-1} \\ &= (\sigma\tau)m. \end{aligned}$$

(c) Show that $\sigma(m_1 + m_2) = \sigma m_1 + \sigma m_2$.

$$\begin{aligned} \sigma(m_1 + m_2) &= e(m_1 + m_2)e^{-1} \\ &= e m_1 e^{-1} + e m_2 e^{-1} \\ &= \sigma m_1 + \sigma m_2 \end{aligned}$$

where $\sigma = eM$ for some $e \in E$.

□

Problem 10.5.

With E , M , G as in the previous problem, if e_σ is a coset representative of σ , show that the function defined by $f(\sigma, \tau) = e_\sigma e_\tau e_{\sigma\tau}^{-1}$ is a 2-cocycle.

Proof. It suffices to show that $\delta_2(f)(\sigma, \tau, v) = 0$ for any $\sigma, \tau, v \in G$. That is,

$$\begin{aligned} &\delta_2(f)(\sigma, \tau, v) \\ &= \sigma f(\tau, v) f(\sigma\tau, v)^{-1} f(\sigma, \tau v) f(\sigma, \tau)^{-1} \\ &= \sigma f(\tau, v) f(\sigma, \tau v) f(\sigma\tau, v)^{-1} f(\sigma, \tau)^{-1} \quad (M: \text{abelian}) \\ &= \sigma (e_\tau e_v e_{\tau v}^{-1}) (e_\sigma e_{\tau v} e_{\sigma\tau v}^{-1}) (e_{\sigma\tau} e_v e_{\sigma\tau v}^{-1})^{-1} (e_\sigma e_\tau e_{\sigma\tau}^{-1})^{-1} \\ &= (e_\sigma e_\tau e_v e_{\tau v}^{-1} e_\sigma^{-1}) (e_\sigma e_{\tau v} e_{\sigma\tau v}^{-1}) (e_{\sigma\tau v} e_v^{-1} e_{\sigma\tau}^{-1}) (e_{\sigma\tau} e_\tau^{-1} e_\sigma^{-1}) \\ &= 1. \end{aligned}$$

□

Problem 10.6.

Suppose that M is a G -module. For each $\sigma \in G$, let $m_\sigma \in M$. Show that the cochain f defined by $f(\sigma, \tau) = m_\sigma + \sigma m_\tau - m_{\sigma\tau}$ is a coboundary.

Proof.

- (1) To show f is a 2-coboundary, it suffices to show that there is a $g \in C^1(G, M)$ such that $f = \delta_1(g)$.
- (2) Actually, we can define $g : G \rightarrow M$ by $\sigma \mapsto m_\sigma$. So

$$\delta_1(g)(\sigma, \tau) = \sigma g(\tau) - g(\sigma\tau) + g(\sigma) = \sigma m_\tau - m_{\sigma\tau} + m_\sigma = f(\sigma, \tau)$$

for all $\sigma, \tau \in G$. Hence $f \in B^2(G, M)$.

□

Problem 10.7.

If M is a G -module and $f \in Z^2(G, M)$, show that $E_f = M \times G$ with multiplication defined by

$$(m, \sigma)(n, \tau) = (m \cdot \sigma n \cdot f(\sigma, \tau), \sigma\tau)$$

makes E_f into a group.

Proof.

- (1) The multiplication is a binary operation on E_f .
- (2) (Associativity) Show that

$$((m, \sigma)(n, \tau))(k, v) = (m, \sigma)((n, \tau)(k, v)).$$

for all $(m, \sigma), (n, \tau), (k, v)$. Note that

$$\begin{aligned} ((m, \sigma)(n, \tau))(k, v) &= (m \cdot \sigma n \cdot f(\sigma, \tau), \sigma\tau)(k, v) \\ &= (m \cdot \sigma n \cdot f(\sigma, \tau) \cdot \sigma\tau k \cdot f(\sigma\tau, v), \sigma\tau v) \\ &= (m \cdot \sigma n \cdot \sigma\tau k \cdot f(\sigma, \tau) \cdot f(\sigma\tau, v), \sigma\tau v) \end{aligned}$$

and

$$\begin{aligned} (m, \sigma)((n, \tau)(k, v)) &= (m, \sigma)(n \cdot \tau k \cdot f(\tau, v), \tau v) \\ &= (m \cdot \sigma(n \cdot \tau k \cdot f(\tau, v)) \cdot f(\sigma, \tau v), \sigma\tau v) \\ &= (m \cdot \sigma n \cdot \sigma\tau k \cdot \underbrace{\sigma f(\tau, v) \cdot f(\sigma, \tau v)}_{=f(\sigma, \tau) \cdot f(\sigma\tau, v)}, \sigma\tau v) \end{aligned}$$

(since $f \in Z^2(G, M)$).

(3) (Identity element) *Show that there exists an element*

$$1 := (f(1, 1)^{-1}, 1) \in E_f$$

such that $1(m, \sigma) = (m, \sigma)1 = (m, \sigma)$ for every $(m, \sigma) \in E_f$. It is the same as Problem 10.3.

(4) (Inverse element) *Show that for each $(m, \sigma) \in E_f$, there exists an element*

$$(n, \tau) := (\sigma^{-1} \{f(\sigma, \sigma^{-1})^{-1} \cdot m^{-1} \cdot f(1, 1)^{-1}\}, \sigma^{-1}) \in E_f$$

such that $(m, \sigma)(n, \tau) = (n, \tau)(m, \sigma) = 1$, where 1 is the identity element in E_f . A direct calculation with the fact that $f \in Z^2(G, M)$ gives all.

□