Chapter 7: Sequences and Series of Functions

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Exercise 7.1. *Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.*

Proof (Cauchy criterion). Let $\{f_n\}$ be a uniformly convergent sequence of bounded functions.

(1) Since f_n is bounded, there exists M_n such that $|f_n(x)| \leq M_n$.

(2) Since $\{f_n\}$ converges uniformly, given $1 > 0$ there exists an integer N such that
$$|f_n(x) - f_m(x)| \leq 1 \text{ whenever } n, m \geq N$$
(Theorem 7.8 (Cauchy criterion for uniformly convergence)). Especially,
$$|f_n(x)| \leq |f_n(x) - f_N(x)| + |f_N(x)| \leq 1 + M_N \text{ whenever } n \geq N.$$

(3) Thus, $\{f_n\}$ is uniformly bounded by $M = \max\{M_1, \ldots, M_{N-1}, M_N + 1\}$.

□

Exercise 7.2. *If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n + g_n\}$ converge uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E.*

Proof. Let $\{f_n\} \to f$ uniformly and $\{g_n\} \to g$ uniformly.

(1) *Show that $\{f_n + g_n\}$ converges uniformly.* Given $\varepsilon > 0$. Since $\{f_n\} \to f$ uniformly and $\{g_n\} \to g$ uniformly, there exist two integers N_1 and N_2 such that
$$|f_n(x) - f(x)| \leq \frac{\varepsilon}{2} \text{ whenever } n \geq N_1, x \in E$$
$$|g_n(x) - g(x)| \leq \frac{\varepsilon}{2} \text{ whenever } n \geq N_2, x \in E.$$
Take $N = \max\{N_1, N_2\}$, we have
$$\begin{aligned}
&|(f_n(x) + g_n(x)) - (f(x) + g(x))| \\
=&|(f_n(x) - f(x)) + (g_n(x) - g(x))| \\
\leq&|f_n(x) - f(x)| + |g_n(x) - g(x)| \\
\leq&\frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
=&\varepsilon
\end{aligned}$$

whenever $n \geq N$, $x \in E$. Hence $\{f_n + g_n\}$ converges to $f + g$ uniformly on E.

(2) *Show that $\{f_n g_n\}$ converges uniformly if, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions.* Given $\varepsilon > 0$.

(a) By Exercise 7.1, both $\{f_n\}$ and $\{g_n\}$ are uniformly bounded. So there exist M_1 and M_2 such that

$$|f_n(x)| \leq M_1 \text{ and } |g_n(x)| \leq M_2$$

for all n and $x \in E$. Also, $|f(x)| \leq M_1 + 1$ and $|g(x)| \leq M_2 + 1$.

(b) Since $\{f_n\} \to f$ uniformly and $\{g_n\} \to g$ uniformly, there exist two integers N_1 and N_2 such that

$$|f_n(x) - f(x)| \leq \frac{\varepsilon}{2(M_2+1)} \text{ whenever } n \geq N_1, x \in E$$
$$|g_n(x) - g(x)| \leq \frac{\varepsilon}{2(M_1+1)} \text{ whenever } n \geq N_2, x \in E.$$

(Note that each denominator of $\frac{\varepsilon}{2(M_j+1)}$ $(j = 1, 2)$ is well-defined and positive!) Take $N = \max\{N_1, N_2\}$, we have

$$\begin{aligned}
&|f_n(x)g_n(x) - f(x)g(x)| \\
=&|[f_n(x) - f(x)]g_n(x) + f(x)[g_n(x) - g(x)]| \\
\leq&|f_n(x) - f(x)||g_n(x)| + |f(x)||g_n(x) - g(x)| \\
\leq&\frac{\varepsilon}{2(M_2+1)} \cdot M_2 + (M_1+1) \cdot \frac{\varepsilon}{2(M_1+1)} \\
\leq&\varepsilon
\end{aligned}$$

whenever $n \geq N$, $x \in E$. Hence $\{f_n g_n\}$ converges to fg uniformly on E.

□

Proof (Cauchy criterion).

(1) *Show that $\{f_n + g_n\}$ converges uniformly.* Given $\varepsilon > 0$. Since $\{f_n\}$ and $\{g_n\}$ converge uniformly, there exist two integers N_1 and N_2 such that

$$|f_n(x) - f_m(x)| \leq \frac{\varepsilon}{2} \text{ whenever } n, m \geq N_1, x \in E$$
$$|g_n(x) - g_m(x)| \leq \frac{\varepsilon}{2} \text{ whenever } n, m \geq N_2, x \in E.$$

Take $N = \max\{N_1, N_2\}$, we have

$$\begin{aligned}
&|(f_n(x) + g_n(x)) - (f_m(x) + g_m(x))| \\
=&|(f_n(x) - f_n(x)) + (g_n(x) - g_m(x))| \\
\leq&|f_n(x) - f_n(x)| + |g_n(x) - g_m(x)| \\
\leq&\frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
=&\varepsilon
\end{aligned}$$

whenever $n, m \geq N$, $x \in E$. Hence $\{f_n + g_n\}$ converges uniformly on E.

(2) *Show that $\{f_n g_n\}$ converges uniformly if, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions.* Given $\varepsilon > 0$.

(a) By Exercise 7.1, both $\{f_n\}$ and $\{g_n\}$ are uniformly bounded. So there exist M_1 and M_2 such that

$$|f_n(x)| \leq M_1 \text{ and } |g_n(x)| \leq M_2$$

for all n and $x \in E$. Also, $|f(x)| \leq M_1 + 1$ and $|g(x)| \leq M_2 + 1$.

(b) Since $\{f_n\} \to f$ uniformly and $\{g_n\} \to g$ uniformly, there exist two integers N_1 and N_2 such that

$$\begin{aligned}
|f_n(x) - f_m(x)| &\leq \frac{\varepsilon}{2(M_2+1)} \text{ whenever } n, m \geq N_1, x \in E \\
|g_n(x) - g_m(x)| &\leq \frac{\varepsilon}{2(M_1+1)} \text{ whenever } n, m \geq N_2, x \in E.
\end{aligned}$$

Take $N = \max\{N_1, N_2\}$, we have

$$\begin{aligned}
&|f_n(x)g_n(x) - f_m(x)g_m(x)| \\
=&|[f_n(x) - f_m(x)]g_n(x) + f_m(x)[g_n(x) - g_m(x)]| \\
\leq&|f_n(x) - f_m(x)||g_n(x)| + |f_m(x)||g_n(x) - g_m(x)| \\
\leq&\frac{\varepsilon}{2(M_2+1)} \cdot M_2 + M_1 \cdot \frac{\varepsilon}{2(M_1+1)} \\
\leq&\varepsilon
\end{aligned}$$

whenever $n \geq N$, $x \in E$. Hence $\{f_n g_n\}$ converges uniformly on E.

□