Chapter 2: Four Important Linear PDE

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Notes.

(1) (Equation (7) in Section 2.2)

$$|D\Phi(x)| \leq \frac{C}{|x|^{n-1}}, \qquad |D^2\Phi(x)| \leq \frac{C}{|x|^n} \qquad (x \neq 0)$$

for some constant $C > 0$. In fact,

$$\begin{aligned}\frac{\partial}{\partial x_i}\Phi(x) &= -\frac{1}{n\alpha(n)} x_i |x|^{-n}, \\ \frac{\partial^2}{\partial x_i \partial x_j}\Phi(x) &= \frac{1}{n\alpha(n)}(n x_i x_j - |x|^2 \delta_{ij})|x|^{-n-2}.\end{aligned}$$

(2) (Equation (12) in Section 2.2) The constant C is rescaled. It is just a constant.

(3) (Equation (13) in Section 2.2) Take $U \mapsto B(0, \varepsilon)$, $u(y) \mapsto \Phi(y)$ and $v(y) \mapsto f(x-y)$ in the integration by parts (Green's first identity):

$$\int_U Dv \cdot Du \, dx = -\int_U u \Delta v \, dx + \int_{\partial U} \frac{\partial v}{\partial \nu} u \, dS.$$

Problem 2.1. *Write down an explicit formula for a function u solving the initial-value problem*

$$\begin{cases} u_t + b \cdot Du + cu = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Here $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ are constants.

Proof (Transport equation). Define

$$z(s) = u(x + sb, t + s) \qquad (s \in \mathbb{R}).$$

So

$$\begin{aligned}\dot{z}(s) &= Du(x + sb, t + s) \cdot b + u_t(x + sb, t + s) \\ &= -cu(x + sb, t + s) \\ &= -cz(s).\end{aligned}$$

Solve this ODE to get

$$
\begin{aligned}
z(s) = z(0)e^{-cs} &\Longrightarrow u(x+sb, t+s) = u(x,t)e^{-cs} \\
&\Longrightarrow u(x-tb, 0) = u(x,t)e^{ct} && \text{(Let } s = -t\text{)} \\
&\Longrightarrow g(x-tb) = u(x,t)e^{ct} \\
&\Longrightarrow u(x,t) = g(x-tb)e^{-ct}.
\end{aligned}
$$

□

Problem 2.2. *Prove that Laplace's equation $\Delta u = 0$ is rotation invariant; that is, if O is an orthogonal $n \times n$ matrix and we define*

$$v(x) := u(Ox) \qquad (x \in \mathbb{R}^n),$$

then $\Delta v = 0$.

Proof.

(1) Let $O = [O_{ij}]$. O is orthogonal if $OO^T = O^T O = I$, or

$$\sum_{i=1}^{n} O_{pi} O_{qi} = \delta_{pq}$$

where δ_{pq} is the Kronecker delta.

(2) Let $y = Ox$. So that

$$
\begin{aligned}
D_i v(x) &= \sum_{p=1}^{n} D_p u(y) O_{pi}, \\
D_{ij} v(x) &= \sum_{q=1}^{n} \sum_{p=1}^{n} D_{pq} u(y) O_{pi} O_{qj}, \\
\Delta v(x) &= \sum_{i=1}^{n} D_{ii} v(x) \\
&= \sum_{i=1}^{n} \sum_{q=1}^{n} \sum_{p=1}^{n} D_{pq} u(y) O_{pi} O_{qi} \\
&= \sum_{q=1}^{n} \sum_{p=1}^{n} D_{pq} u(y) \left(\sum_{i=1}^{n} O_{pi} O_{qi} \right) \\
&= \sum_{q=1}^{n} \sum_{p=1}^{n} D_{pq} \delta_{pq} \\
&= \sum_{q=1}^{n} D_{qq} u(y) \\
&= \Delta u(y).
\end{aligned}
$$

(3) As $\Delta u(y) = 0$, $\Delta v(x) = 0$.

$\square$

Problem 2.3. *Modify the proof of the mean value formulas to show for $n \geq 3$ that*

$$u(0) = \fint_{\partial B(0,r)} g dS + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f dx,$$

provided

$$\begin{cases} -\Delta u = f & \text{in } B^0(0,r) = \mathrm{int}(B(0,r)) \\ u = g & \text{on } \partial B(0,r). \end{cases}$$

Proof.

(1) ...

(2) ...

$\square$

Problem 2.4. *We say $v \in C^2(\overline{U})$ is **subharmonic** if*

$$-\Delta v \leq 0 \qquad \text{in } U.$$

(a) *Prove for subharmonic v that*

$$v(x) \leq \fint_{B(x,r)} v dy \qquad \text{for all } B(x,r) \subseteq U.$$

(b) *Prove that therefore $\max_{\overline{U}} v = \max_{\partial U} v$.*

(c) *Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v := \phi(u)$. Prove that v is subharmonic.*

(d) *Prove $v := |Du|^2$ is subharmonic, whenever u is harmonic.*

Proof of (a). It is exactly the same as the proof of Theorem 2 (Mean-value theorem for Laplace's equation).

(1) Set

$$\phi(r) := \fint_{\partial B(x,r)} v(y) dS(y) = \fint_{\partial B(0,1)} v(x+rz) dS(z)$$

$(r > 0)$. Then

$$
\begin{aligned}
\phi'(r) &= ⨍_{\partial B(0,1)} Dv(\underbrace{x + rz}_{=y}) \cdot z dS(z) \\
&= ⨍_{\partial B(x,y)} Dv(y) \cdot \underbrace{\frac{y - x}{r}}_{=\nu} dS(y) \\
&= ⨍_{\partial B(x,y)} \frac{\partial v}{\partial \nu} dS(y) \\
&= \frac{r}{n} ⨍_{B(x,y)} \Delta u(y) dy && \text{(Green's first identity)} \\
&\geq 0 && \text{(By assumption)}
\end{aligned}
$$

or $\phi(r)$ is increasing.

(2) Note that

$$\lim_{t \to 0} \phi(t) = \lim_{t \to 0} ⨍_{\partial B(x,t)} v(y) dS(y) = v(x).$$

So that

$$v(x) = \lim_{t \to 0} \phi(t) \leq \phi(r) = ⨍_{\partial B(x,r)} v(y) dS(y).$$

(3) Hence, for all $B(x,r) \subseteq U$ we have

$$
\begin{aligned}
⨍_{B(x,r)} v dy &= \frac{1}{\alpha(n) r^n} \int_{B(x,r)} v dy \\
&= \frac{1}{\alpha(n) r^n} \int_0^r \int_{\partial B(x,\rho)} v(y) dS(y) d\rho && \text{(Polar coordinates)} \\
&\geq \frac{1}{\alpha(n) r^n} \int_0^r n \alpha(n) \rho^{n-1} v(x) d\rho && \text{((2))} \\
&= v(x) \frac{1}{r^n} \underbrace{\int_0^r n \rho^{n-1} d\rho}_{=r^n} \\
&= v(x).
\end{aligned}
$$

□

Proof of (b). Similar to the proof of Theorem 4 (Strong maximum principle).

(1) Suppose there exists a point $x_0 \in U$ with $v(x_0) = M := \max_{\overline{U}} v$. Then for $0 < r < \operatorname{dist}(x_0, \partial U)$, the mean-value property (in (a)) asserts

$$M = v(x_0) \leq ⨍_{B(x_0,r)} v dy \leq M.$$

As equality holds only if $v \equiv M$ within $B(x_0, r)$, we see $v = M$ for all $y \in B(x, r)$. Hence the set $\{x \in U : v(x) = M\}$ is both open and closed in U (since $v \in C(\overline{U})$), and thus eqauls to one connected component U_α of U. By the definition of $\partial U_\alpha \subseteq \overline{U_\alpha}$ and continuity of v, $v|_{\partial U_\alpha} \equiv M$. As $\partial U_\alpha \subseteq \partial U$, the result is established.

(2) If no such point $x_0 \in U$ with $v(x_0) = \max_{\overline{U}} v$, then $\max_{\overline{U}} v = \max_{\partial U} v$ is trivial.

□

Proof of (c).

(1)

$$\begin{aligned}
\Delta v &= \sum_{i=1}^{n} v_{x_i x_i} \\
&= \sum_{i=1}^{n} (\phi'(u) u_{x_i})_{x_i} \\
&= \sum_{i=1}^{n} \phi''(u) u_{x_i}^2 + \phi'(u) u_{x_i x_i} \\
&= \phi''(u) |Du|^2 + \phi'(u) \Delta u.
\end{aligned}$$

(2) As u is harmonic ($\Delta u = 0$) and ϕ is convex ($\phi''(u) \geq 0$ by Exercise 5.14 in the textbook: *Rudin, Principles of Mathematical Analysis, 3rd edition*), $\Delta v \geq 0$ (by (1)).

□

Proof of (d).

(1) Since u is smooth, u is harmonic implies that u_{x_j} is harmonic for all x_j. In fact,

$$\begin{aligned}
\Delta(u_{x_j}) &= \sum_{i=1}^{n} (u_{x_j})_{x_i x_i} \\
&= \sum_{i=1}^{n} u_{x_i x_i x_j} && \text{(Smoothness of } u\text{)} \\
&= \left(\sum_{i=1}^{n} u_{x_i x_i} \right)_{x_j} \\
&= (\Delta u)_{x_j} \\
&= 0.
\end{aligned}$$

(2) Since $x \mapsto x^2$ is convex and u_{x_i} is harmonic (by (1)),

$$v := |Du|^2 = \sum_{i=1}^{n} (u_{x_i})^2$$

is a finite sum of subharmonic functions by (3), which is also subharmonic.

□

Problem 2.5. ...

Proof.

(1) ...

(2) ...

□

Problem 2.6. ...

Proof.

(1) ...

(2) ...

□

Problem 2.7. ...

Proof.

(1) ...

(2) ...

□

Problem 2.8. ...

Proof.

(1) ...

(2) ...

□

Problem 2.9. *...*

Proof.

(1) ...

(2) ...

□

Problem 2.10. *...*

Proof.

(1) ...

(2) ...

□

Problem 2.11. *...*

Proof.

(1) ...

(2) ...

□

Problem 2.12. *...*

Proof.

(1) ...

(2) ...

□

Problem 2.13. *...*

Proof.

(1) ...

(2) ...

□

Problem 2.14. *...*

Proof.

(1) ...

(2) ...

□

Problem 2.15. *...*

Proof.

(1) ...

(2) ...

□

Problem 2.16. *...*

Proof.

(1) ...

(2) ...

□

Problem 2.17. *...*

Proof.

(1) ...

(2) ...

□

Problem 2.18. *...*

Proof.

(1) ...

(2) ...

□