

# Solutions to the book: *do Carmo, Differential Geometry of Curves and Surfaces*

Meng-Gen Tsai  
plover@gmail.com

May 3, 2021

## Contents

<b>Chapter 1: Curves</b>	<b>2</b>
1-1. Introduction . . . . .	2
1-2. Parametrized Curves . . . . .	2
Exercise 1-2.1. . . . .	2
Exercise 1-2.2. . . . .	2
Exercise 1-2.3. . . . .	3
Exercise 1-2.4. . . . .	3
Exercise 1-2.5. . . . .	3
1-3. Regular Curves; Arc Length . . . . .	4
Exercise 1-3.1. . . . .	4
Exercise 1-3.2. (Cycloid) . . . . .	5
Exercise 1-3.3. (Cisoid of Diocles) . . . . .	6
Exercise 1-3.4. (Tractrix) . . . . .	7
Exercise 1-3.5. (Folium of Descartes) . . . . .	8
Exercise 1-3.8. . . . .	10
Exercise 1-3.9. . . . .	11
Exercise 1-3.10. (Straight Lines as Shortest) . . . . .	13
1-4. The Vector Product in $\mathbb{R}^3$ . . . . .	14
Exercise 1-4.1. . . . .	14
Exercise 1-4.2. . . . .	14
Exercise 1-4.3. . . . .	15
Exercise 1-4.13. . . . .	16
1-5. The Local Theory of Curves Parametrized by Arc Length . . . . .	17
Exercise 1-5.2. . . . .	17
1-6. The Local Canonical Form . . . . .	17
1-7. Global Properties of Plane Curves . . . . .	17

# Chapter 1: Curves

## 1-1. Introduction

Classical differential geometry: the study of local properties of curves and surfaces.

Global differential geometry: the study of the influence of the local properties on the behavior of the entire curve and surface.

*No exercises.*

## 1-2. Parametrized Curves

### Exercise 1-2.1.

Find a parametrized curve  $\alpha(t)$  whose trace is the circle  $x^2 + y^2 = 1$  such that  $\alpha(t)$  runs clockwise around the circle with  $\alpha(0) = (0, 1)$ .

*Proof.*  $\alpha(t) = (\sin t, \cos t)$ ,  $t \in \mathbb{R}$ .  $\square$

### Exercise 1-2.2.

Let  $\alpha(t)$  be a parametrized curve which does not pass through the origin. If  $\alpha(t_0)$  is the point of the trace of  $\alpha$  closest to the origin and  $\alpha'(t_0) \neq 0$ , show that the position vector  $\alpha(t_0)$  is orthogonal to  $\alpha'(t_0)$ .

*Proof.* Let  $f(t) = |\alpha(t)|^2 = \alpha(t) \cdot \alpha(t)$ .  $f(t)$  is differentiable and  $f(t)$  has a local minimum at a point  $t = t_0 \in I$ . So  $f'(t_0) = 0$ . [Theorem 5.8 in *W. Rudin, Principles of Mathematical Analysis, 3rd edition.*] Since

$$f'(t) = 2\alpha(t) \cdot \alpha'(t),$$

$f'(t_0) = 2\alpha(t_0) \cdot \alpha'(t_0) = 0$ , or  $\alpha(t_0) \cdot \alpha'(t_0) = 0$ . Since  $\alpha(t_0) \neq 0$  and  $\alpha'(t_0) \neq 0$ ,  $\alpha(t_0)$  is orthogonal to  $\alpha'(t_0)$ .  $\square$

**Exercise 1-2.3.**

A parametrized curve  $\alpha(t)$  has a property that its second derivative  $\alpha''(t)$  is identically zero. What can be said about  $\alpha$ ?

*Proof.*

- (1)  $\alpha(t)$  is a straight line.
- (2) Since  $\alpha''(t)$  is identically zero,  $\alpha'(t) = a$  is a constant. [Theorem 5.11 in *W. Rudin, Principles of Mathematical Analysis, 3rd edition.*] Define  $f(t) = \alpha(t) - at$  (on  $I$ ). Since  $f'(t) = \alpha'(t) - a = 0$ ,  $f(t) = \alpha(t) - at = b$  is a constant again. Therefore,  $\alpha(t) = at + b$ , which is a straight line (on  $I$ ).

□

**Exercise 1-2.4.**

Let  $\alpha : I \rightarrow \mathbb{R}^3$  be a parametrized curve and let  $v \in \mathbb{R}^3$  be a fixed vector. Assume that  $\alpha'(t)$  is orthogonal to  $v$  for all  $t \in I$  and that  $\alpha(0)$  is orthogonal to  $v$ . Prove that  $\alpha(t)$  is orthogonal to  $v$  for all  $t \in I$ .

Need to assume that  $\alpha(t) \neq 0$  for all  $t \in I$ .

*Proof.* Given any  $t \neq 0 \in I$ . (Nothing to do at  $t = 0$ .) Define  $f : I \rightarrow \mathbb{R}$  by  $f(t) = \alpha(t) \cdot v$ . By the mean value theorem, there exists a point  $\xi$  between 0 and  $t$  such that

$$f(t) - f(0) = f'(\xi)(t - 0),$$

where  $f'(t) = \alpha'(t) \cdot v + \alpha(t) \cdot v' = \alpha'(t) \cdot v$ . Note that  $f(0) = 0$  since  $\alpha(0)$  is orthogonal to  $v$ , and  $f'(\xi) = 0$  since  $\alpha'(\xi)$  is orthogonal to  $v$ . So the identity is reduced to

$$f(t) = 0,$$

or  $\alpha(t) \cdot v = 0$ , or  $\alpha(t)$  is orthogonal to  $v$ . □

**Exercise 1-2.5.**

Let  $\alpha : I \rightarrow \mathbb{R}^3$  be a parametrized curve, with  $\alpha'(t) \neq 0$  for all  $t \in I$ . Show that  $|\alpha(t)|$  is a nonzero constant if and only if  $\alpha(t)$  is orthogonal to  $\alpha'(t)$  for all  $t \in I$ .

The same trick in Exercise 1-2.2.

*Proof.* It is equivalent to show that  $|\alpha(t)|^2$  is a nonzero constant if and only if  $\alpha(t)$  is orthogonal to  $\alpha'(t)$  for all  $t \in I$ . Let

$$f(t) = |\alpha(t)|^2 = \alpha(t) \cdot \alpha(t).$$

Notice that  $\alpha'(t) \neq 0$ , and thus

$$\begin{aligned} & |\alpha(t)| \text{ is a nonzero constant} \\ \iff & f(t) = |\alpha(t)|^2 \text{ is a nonzero constant} \\ \iff & f'(t) = 0 \text{ and } f(t) \text{ is a nonzero constant} \\ \iff & \alpha(t) \cdot \alpha'(t) = 0 \text{ and } \alpha(t) \text{ is a nonzero constant} \\ \iff & \alpha(t) \text{ is orthogonal to } \alpha'(t) \text{ for all } t \in I. \end{aligned}$$

□

### 1-3. Regular Curves; Arc Length

#### Exercise 1-3.1.

Show that the tangent lines to the regular parametrized curve  $\alpha(t) = (3t, 3t^2, 2t^3)$  make a constant angle with the line  $y = 0, z = x$ .

*Proof.*  $\alpha'(t) = (3, 6t, 6t^2)$ . The line  $y = 0, z = x$  is  $\beta(t) = (1, 0, 1)$ . The cosine of the angle  $\theta$  between these two curves is

$$\begin{aligned} \cos \theta &= \frac{(3, 6t, 6t^2) \cdot (1, 0, 1)}{|(3, 6t, 6t^2)| |(1, 0, 1)|} \\ &= \frac{3 + 6t^2}{\sqrt{3^2 + (6t)^2 + (6t^2)^2} \sqrt{2}} \\ &= \frac{3 + 6t^2}{\sqrt{9 + 36t^2 + 36t^4} \sqrt{2}} \\ &= \frac{3 + 6t^2}{\sqrt{(3 + 6t^2)^2} \sqrt{2}} \\ &= \frac{1}{\sqrt{2}}. \end{aligned}$$

(Notice  $3 + 6t^2 > 0$  for all  $t \in \mathbb{R}$ .) That is, the angle between  $\alpha'$  and  $\beta$  is a constant ( $= \pi/4$ ). □

**Exercise 1-3.2. (Cycloid)**

A circular disk of radius 1 in the plane  $xy$  rolls without slipping along the  $x$  axis. The figure described by a point of the circumference of the disk is called a **cycloid** (Figure 1-7 in Mantredo P. do Carmo, *Differential Geometry of Curves and Surfaces*).

- (a) Obtain a parametrized curve  $\alpha : \mathbb{R} \rightarrow \mathbb{R}^2$  the trace of which is the cycloid and determine its singular points.
- (b) Compute the arc length of the cycloid corresponding to a complete rotation of the disk.

*Proof of (a).*

- (1) Since

$$\begin{cases} x = t - \sin t \\ y = 1 - \cos t, \end{cases}$$

we define  $\alpha(t) = (t - \sin t, 1 - \cos t)$ .

- (2)  $\alpha'(t) = (1 - \cos t, \sin t)$ .  $\alpha'(t) = 0$  if and only if  $t = 2n\pi$  where  $n \in \mathbb{Z}$ . That is, all singular points are  $\alpha(2n\pi) = (2n\pi, 0)$  where  $n \in \mathbb{Z}$ .

□

*Proof of (b).* The arc length of the cycloid corresponding to a complete rotation of the disk is

$$\begin{aligned} \int_0^{2\pi} |\alpha'(t)| dt &= \int_0^{2\pi} \sqrt{(1 - \cos t)^2 + (\sin t)^2} dt \\ &= \int_0^{2\pi} \sqrt{2} \sqrt{1 - \cos t} dt \\ &= \int_0^{2\pi} 2 \sin \frac{t}{2} dt \\ &= \left[ -4 \cos \frac{t}{2} \right]_{t=0}^{t=2\pi} \\ &= 8. \end{aligned}$$

□

**Supplement.** The cycloid is not an algebraic curve.

**Exercise 1-3.3. (Cisoid of Diocles)**

Let  $0A = 2a$  be the diameter of a circle  $\mathbb{S}^1$  and  $0Y$  and  $AV$  be the tangents to  $\mathbb{S}^1$  at  $0$  and  $A$ , respectively. A half-line  $r$  is drawn from  $0$  which meets the circle  $\mathbb{S}^1$  at  $C$  and the line  $AV$  at  $B$ . On  $0B$  mark off the segment  $0p = CB$ . If we rotate  $r$  about  $0$ , the point  $p$  will describe a curve called the **cisoid of Diocles**. By taking  $0A$  as the  $x$  axis and  $0Y$  as the  $y$  axis, prove that

(a) The tract of

$$\alpha(t) = \left( \frac{2at^2}{1+t^2}, \frac{2at^3}{1+t^2} \right), \quad t \in \mathbb{R},$$

is the cisoid of Diocles ( $t = \tan \theta$ ; see Figure 1-8 in Mantredo P. do Carmo, *Differential Geometry of Curves and Surfaces*).

(b) The origin  $(0, 0)$  is a singular point of the cisoid.

(c) As  $t \rightarrow \infty$ ,  $\alpha(t)$  approaches the line  $x = 2a$ , and  $\alpha'(t) \rightarrow (0, 2a)$ . Thus, as  $t \rightarrow \infty$ , the curve and its tangent approach the line  $x = 2a$ ; we say that  $x = 2a$  is an **asymptote** to the cisoid.

*Proof of (a).*

(1) The polar equations of the circle  $\mathbb{S}^1$  and the half-line  $r$  is

$$\begin{aligned} r &= 2a \cos \theta, \\ r &= 2a \sec \theta, \end{aligned}$$

respectively.

(2) By construction, the polar equation of the cisoid is

$$r = 2a \sec \theta - 2a \cos \theta = 2a \frac{\sin^2 \theta}{\cos \theta} = 2a \sin \theta \tan \theta.$$

(3) Put  $t = \tan \theta$ , we have

$$\begin{aligned} x &= r \cos \theta = 2a \sin^2 \theta = \frac{2at^2}{1+t^2}, \\ y &= r \sin \theta = tx = \frac{2at^3}{1+t^2}. \end{aligned}$$

So

$$\alpha(t) = (x, y) = \left( \frac{2at^2}{1+t^2}, \frac{2at^3}{1+t^2} \right).$$

□

**Supplement.** The cisoid is an algebraic curve  $= V((x^2 + y^2)x = 2ay^2)$ .

*Proof of (b).* Note that  $\alpha(0) = (0, 0)$  and

$$\alpha'(t) = \left( \frac{4at}{(t^2 + 1)^2}, \frac{2at^2(t^2 + 3)}{(t^2 + 1)^2} \right).$$

Hence  $\alpha'(0) = (0, 0)$ . That is,  $(0, 0)$  is a singular point of the cissoid. (In fact, the origin is the unique singular point of the cissoid.)  $\square$

*Proof of (c).*

(1) Note that

$$\begin{aligned} \lim_{t \rightarrow \pm\infty} x(t) &= \lim_{t \rightarrow \pm\infty} \frac{2at^2}{1+t^2} = 2a, \\ \lim_{t \rightarrow \pm\infty} y(t) &= \lim_{t \rightarrow \pm\infty} \frac{2at^3}{1+t^2} = \pm\infty. \end{aligned}$$

Hence,  $\alpha(t)$  approaches the line  $x = 2a$  as  $t \rightarrow \pm\infty$ .

(2) Similarly,

$$\begin{aligned} \lim_{t \rightarrow \pm\infty} x'(t) &= \lim_{t \rightarrow \pm\infty} \frac{4at}{(t^2 + 1)^2} = 0, \\ \lim_{t \rightarrow \pm\infty} y'(t) &= \lim_{t \rightarrow \pm\infty} \frac{2at^2(t^2 + 3)}{(t^2 + 1)^2} = 2a. \end{aligned}$$

Therefore,  $\alpha'(t) \rightarrow (0, 2a)$  as  $t \rightarrow \pm\infty$ .

(3) By (1)(2), the curve and its tangent approach the line  $x = 2a$  as  $t \rightarrow \pm\infty$ , or  $x = 2a$  is an asymptote to the cissoid.

$\square$

### Exercise 1-3.4. (Tractrix)

Let  $\alpha : (0, \pi) \rightarrow \mathbb{R}^2$  be given by

$$\alpha(t) = \left( \sin t, \cos t + \log \tan \frac{t}{2} \right),$$

where  $t$  is the angle that the  $y$  axis makes with the vector  $\alpha(t)$ . The trace of  $\alpha$  is called the **tractrix**. (Figure 1-9 in Mantredo P. do Carmo, *Differential Geometry of Curves and Surfaces*). Show that

- (a)  $\alpha$  is a differentiable parametrized curve, regular except at  $t = \frac{\pi}{2}$ .
- (b) The length of the segment of the tangent of the tractrix between the point of tangency and the  $y$  axis is constantly equal to 1.

*Proof of (a).*

$$\begin{aligned}\alpha'(t) &= \left( \cos t, -\sin t + \frac{1}{\tan \frac{t}{2}} \frac{1}{\cos^2 \frac{t}{2}} \frac{1}{2} \right) \\ &= \left( \cos t, -\sin t + \frac{1}{2 \sin \frac{t}{2} \cos \frac{t}{2}} \right) \\ &= \left( \cos t, \frac{\cos^2 t}{\sin t} \right)\end{aligned}$$

exists. And  $\alpha'(t) = 0$  if and only if  $t = \frac{\pi}{2}$ . That is, there is a unique singular point at  $t = \frac{\pi}{2}$ .  $\square$

*Proof of (b).* The tangent line of the tractrix through the regular point  $t$  is parametrized by  $\beta : \mathbb{R} \rightarrow \mathbb{R}^2$  which is defined by

$$\begin{aligned}\beta(u) &= u\alpha'(t) + \alpha(t) \\ &= \left( u \cos t + \sin t, u \frac{\cos^2 t}{\sin t} + \cos t + \log \tan \frac{t}{2} \right).\end{aligned}$$

By construction, this tangent line  $\beta(u)$  meets the tractrix at  $u = 0$ , and meets the  $y$ -axis when  $u \cos t + \sin t = 0$  or  $u = -\tan t$ . So the length of the segment is

$$\begin{aligned}|\beta(0) - \beta(-\tan t)| &= \sqrt{(-\tan t \cos t)^2 + \left(-\tan t \frac{\cos^2 t}{\sin t}\right)^2} \\ &= \sqrt{(\sin t)^2 + (\cos t)^2} \\ &= 1.\end{aligned}$$

$\square$

### Exercise 1-3.5. (Folium of Descartes)

Let  $\alpha : (-1, +\infty) \rightarrow \mathbb{R}^2$  be given by

$$\alpha(t) = \left( \frac{3at}{1+t^3}, \frac{3at^2}{1+t^3} \right).$$

*Prove that:*

- (a) For  $t = 0$ ,  $\alpha$  is tangent to the  $x$  axis.
- (b) As  $t \rightarrow +\infty$ ,  $\alpha(t) \rightarrow (0, 0)$  and  $\alpha'(t) = (0, 0)$ .
- (c) Take the curve the the opposite orientation. Now, as  $t \rightarrow -1$ , the curve and its tangent approach the line  $x + y + a = 0$ .



The figure obtained by completing the trace of  $\alpha$  in such a way that it becomes symmetric relative the the line  $y = x$  is called the **folium of Descartes** (See Figure 1-10 in Mantredo P. do Carmo, *Differential Geometry of Curves and Surfaces*).

*Proof of (a).* Note that

$$\alpha'(t) = \left( \frac{3a(1-2t^3)}{(1+t^3)^2}, \frac{3at(2-t^3)}{(1+t^3)^2} \right).$$

Hence,  $\alpha'(0) = (3a, 0)$ , or  $\alpha$  is tangent to the  $x$  axis when  $t = 0$ .  $\square$

*Proof of (b).*

(1)

$$\begin{aligned} \lim_{t \rightarrow +\infty} \alpha(t) &= \lim_{t \rightarrow +\infty} \left( \frac{3at}{1+t^3}, \frac{3at^2}{1+t^3} \right) \\ &= \left( \lim_{t \rightarrow +\infty} \frac{3at}{1+t^3}, \lim_{t \rightarrow +\infty} \frac{3at^2}{1+t^3} \right) \\ &= (0, 0). \end{aligned}$$

(2)

$$\begin{aligned} \lim_{t \rightarrow +\infty} \alpha'(t) &= \lim_{t \rightarrow +\infty} \left( \frac{3a(1-2t^3)}{(1+t^3)^2}, \frac{3at(2-t^3)}{(1+t^3)^2} \right) \\ &= \left( \lim_{t \rightarrow +\infty} \frac{3a(1-2t^3)}{(1+t^3)^2}, \lim_{t \rightarrow +\infty} \frac{3at(2-t^3)}{(1+t^3)^2} \right) \\ &= (0, 0). \end{aligned}$$

$\square$

*Proof of (c).*

(1) Note that

$$\begin{aligned} \lim_{t \rightarrow -1^+} \alpha(t) &= \lim_{t \rightarrow -1^+} \left( \frac{3at}{1+t^3}, \frac{3at^2}{1+t^3} \right) \\ &= \left( \lim_{t \rightarrow -1^+} \frac{3at}{1+t^3}, \lim_{t \rightarrow -1^+} \frac{3at^2}{1+t^3} \right) \\ &= (-\infty, +\infty) \end{aligned}$$

and

$$\begin{aligned} \lim_{t \rightarrow -1^+} (x(t) + y(t)) &= \lim_{t \rightarrow -1^+} \left( \frac{3at}{1+t^3} + \frac{3at^2}{1+t^3} \right) \\ &= \lim_{t \rightarrow -1^+} \frac{3at}{1-t+t^2} \\ &= -a. \end{aligned}$$

Therefore, as  $t \rightarrow -1$ , the curve approaches the line  $x + y + a = 0$ .

(2) Note that

$$\begin{aligned}\lim_{t \rightarrow -1^+} \frac{y'(t)}{x'(t)} &= \lim_{t \rightarrow -1^+} \frac{\frac{3a(1-2t^3)}{(1+t^3)^2}}{\frac{3at(2-t^3)}{(1+t^3)^2}} \\ &= \lim_{t \rightarrow -1^+} \frac{1-2t^3}{t(2-t^3)} \\ &= 1.\end{aligned}$$

□

### Exercise 1-3.8.

Let  $\alpha : I \rightarrow \mathbb{R}^3$  be a differentiable curve and let  $[a, b] \subseteq I$  be a closed interval. For every partition

$$a = t_0 < t_1 < \cdots < t_n = b$$

of  $[a, b]$ , consider the sum

$$\sum_{i=1}^n |\alpha(t_i) - \alpha(t_{i-1})| = l(\alpha, P),$$

where  $P$  stands for the given partition. The norm  $|P|$  of a partition  $P$  is defined as

$$|P| = \max(t_i - t_{i-1}), i = 1, \dots, n.$$

Geometrically,  $l(\alpha, P)$  is the length of a polygon inscribed in  $\alpha([a, b])$  with vertices in  $\alpha(t_i)$  (see Figure 1-3 in Mantred P. do Carmo, *Differential Geometry of Curves and Surfaces*). The point of the exercise is to show that the arc length of  $\alpha([a, b])$  is, in some sense, a limit of lengths of inscribed polygons. Prove that given  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $|P| < \delta$  then

$$\left| \int_a^b |\alpha'(t)| dt - l(\alpha, P) \right| < \varepsilon.$$

Assume that  $\alpha'(t)$  is continuous.

*Proof.* Given  $\varepsilon > 0$ .

- (1) Since  $\alpha'(t)$  is continuous on a compact set  $[a, b]$ ,  $\alpha'(t)$  is uniformly continuous, that is, there exists  $\delta > 0$  such that

$$|\alpha'(s) - \alpha'(t)| < \frac{\varepsilon}{2(b-a)} \text{ whenever } |s - t| < \delta.$$

- (2) Let  $P = \{a = t_0, t_1, \dots, t_n = b\}$  be a partition of  $[a, b]$ , with  $\Delta t_i = t_i - t_{i-1} < \delta$  for all  $i = 1, \dots, n$ . If  $t_{i-1} \leq t \leq t_i$ , it follows that

$$|\alpha'(t_i)| - \frac{\varepsilon}{2(b-a)} \leq |\alpha'(t)| \leq |\alpha'(t_i)| + \frac{\varepsilon}{2(b-a)}.$$

Hence,

$$\begin{aligned} & \int_{t_{i-1}}^{t_i} |\alpha'(t)| dt \\ & \geq |\alpha'(t_i)| \Delta t_i - \frac{\varepsilon}{2(b-a)} \Delta t_i \\ & = \left| \int_{t_{i-1}}^{t_i} [\alpha'(t) + \alpha'(t_i) - \alpha'(t)] dt \right| - \frac{\varepsilon}{2(b-a)} \Delta t_i \\ & \geq \left| \int_{t_{i-1}}^{t_i} \alpha'(t) dt \right| - \left| \int_{t_{i-1}}^{t_i} [\alpha'(t_i) - \alpha'(t)] dt \right| - \frac{\varepsilon}{2(b-a)} \Delta t_i \\ & \geq |\alpha(t_i) - \alpha(t_{i-1})| - \frac{\varepsilon}{b-a} \Delta t_i \end{aligned}$$

and

$$\begin{aligned} & \int_{t_{i-1}}^{t_i} |\alpha'(t)| dt \\ & \leq |\alpha'(t_i)| \Delta t_i + \frac{\varepsilon}{2(b-a)} \Delta t_i \\ & = \left| \int_{t_{i-1}}^{t_i} [\alpha'(t) + \alpha'(t_i) - \alpha'(t)] dt \right| + \frac{\varepsilon}{2(b-a)} \Delta t_i \\ & \leq \left| \int_{t_{i-1}}^{t_i} \alpha'(t) dt \right| + \left| \int_{t_{i-1}}^{t_i} [\alpha'(t_i) - \alpha'(t)] dt \right| + \frac{\varepsilon}{2(b-a)} \Delta t_i \\ & \leq |\alpha(t_i) - \alpha(t_{i-1})| + \frac{\varepsilon}{b-a} \Delta t_i. \end{aligned}$$

- (3) If we add these inequalities, we obtain

$$l(\alpha, P) - \varepsilon \leq \int_a^b |\alpha'(t)| dt \leq l(\alpha, P) + \varepsilon.$$

□

### Exercise 1-3.9.

- (a) Let  $\alpha : I \rightarrow \mathbb{R}^3$  be a curve of class  $C^0$  (compare Exercise 1-3.7). Use the approximation by polygons described in Exercise 1-3.8 to give a reasonable definition of arc length of  $\alpha$ .

- (b) (A Nonrectifiable Curve.) The following example shows that, with any reasonable definition, the arc length of a  $C^0$  curve in a closed interval may be unbounded. Let  $\alpha : [0, 1] \rightarrow \mathbb{R}^2$  be given as  $\alpha(t) = (t, t \sin(\frac{\pi}{t}))$  if  $t \neq 0$ , and  $\alpha(0) = (0, 0)$ . Show, geometrically, that the arc length of the portion of the curve corresponding to  $\frac{1}{n+1} \leq t \leq \frac{1}{n}$  is at least  $\frac{2}{n+\frac{1}{2}}$ . Use this to show that the length of curve in the interval  $\frac{1}{N} \leq t \leq 1$  is greater than  $2 \sum_{n=1}^{N-1} \frac{1}{n+1}$ , and thus it tends to infinity as  $N \rightarrow \infty$ .

*Proof of (a).* Define

$$l(\alpha) = \sup\{l(\alpha, P) : P \text{ is a partition of } [a, b]\}.$$

□

*Note.* (Theorem 6.17 in Tom. M. Apostol, *Mathematical Analysis*, 2nd edition.).  $\alpha$  is rectifiable if and only if  $\alpha$  is of bounded variation on  $[a, b]$ .

*Proof of (b).*

- (1) Consider a partition  $P = \left\{ \frac{1}{n+1}, \frac{1}{n+\frac{1}{2}}, \frac{1}{n} \right\}$  of  $\left[ \frac{1}{n+1}, \frac{1}{n} \right]$ . So that  $\alpha(\frac{1}{n+1}) = \alpha(\frac{1}{n}) = 0$  and  $\alpha(\frac{1}{n+\frac{1}{2}}) = \pm 1$ .
- (2) Thus,

$$\begin{aligned} & \text{The arc length of the portion of } \alpha \text{ over } \left[ \frac{1}{n+1}, \frac{1}{n} \right] \\ & \geq \text{The sum of each length of the individual chords} \\ & = \sqrt{\left( \frac{1}{n+\frac{1}{2}} - \frac{1}{n+1} \right)^2 + \left( \frac{1}{n+\frac{1}{2}} \right)^2} \\ & \quad + \sqrt{\left( \frac{1}{n} - \frac{1}{n+\frac{1}{2}} \right)^2 + \left( \frac{1}{n+\frac{1}{2}} \right)^2} \\ & \geq \frac{2}{n+\frac{1}{2}}. \end{aligned}$$

(3) So

$$\begin{aligned}
& \text{The arc length of } \alpha \text{ over } \left[ \frac{1}{N}, 1 \right] \\
&= \sum_{n=1}^{N-1} \left\{ \text{The arc length of } \alpha \text{ over } \left[ \frac{1}{n+1}, \frac{1}{n} \right] \right\} \\
&\geq \sum_{n=1}^{N-1} \frac{2}{n + \frac{1}{2}} \\
&> 2 \sum_{n=1}^{N-1} \frac{1}{n+1}.
\end{aligned}$$

It tends to infinity as  $N \rightarrow \infty$ , or  $\alpha$  is nonrectifiable.

□

**Exercise 1-3.10. (Straight Lines as Shortest)**

Let  $\alpha : I \rightarrow \mathbb{R}^3$  be a parametrized curve. Let  $[a, b] \subseteq I$  and set  $\alpha(a) = p$ ,  $\alpha(b) = q$ .

(a) Show that, for any constant vector  $v$ ,  $|v| = 1$ ,

$$(q - p) \cdot v = \int_a^b \alpha'(t) \cdot v dt \leq \int_a^b |\alpha'(t)| dt.$$

(b) Set

$$v = \frac{q - p}{|q - p|}$$

and show that

$$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)| dt;$$

that is, the curve of shortest length from  $\alpha(a)$  to  $\alpha(b)$  is the straight line joining these points.

Assume  $p \neq q$  (otherwise  $v = \frac{q-p}{|q-p|}$  is meaningless).

*Proof of (a).* Let  $f(t) = \alpha(t) \cdot v$  defined on  $I$ . By the fundamental theorem of calculus,

$$\int_a^b f'(t) dt = f(b) - f(a).$$

Since  $f'(t) = \alpha'(t) \cdot v$ ,

$$(\alpha(b) - \alpha(a)) \cdot v = \int_a^b \alpha'(t) \cdot v dt.$$

Therefore,

$$\begin{aligned} (q - p) \cdot v &= \int_a^b \alpha'(t) \cdot v dt \\ &\leq \int_a^b |\alpha'(t) \cdot v| dt \\ &\leq \int_a^b |\alpha'(t)| |v| dt \\ &= \int_a^b |\alpha'(t)| dt. \end{aligned}$$

□

*Proof of (b).*  $|v| = \frac{|q-p|}{|q-p|} = 1$ . So,

$$\begin{aligned} (q - p) \cdot \frac{q - p}{|q - p|} &\leq \int_a^b |\alpha'(t)| dt, \\ |q - p| &\leq \int_a^b |\alpha'(t)| dt. \end{aligned}$$

□

## 1-4. The Vector Product in $\mathbb{R}^3$

### Exercise 1-4.1.

*Check whether the following bases are positive:*

- (a) *The basis  $\{(1, 3), (4, 2)\}$  in  $\mathbb{R}^2$ .*
- (b) *The basis  $\{(1, 3, 5), (2, 3, 7), (4, 8, 3)\}$  in  $\mathbb{R}^3$ .*

*Proof of (a).* Write  $u = (1, 3)$  and  $v = (4, 2)$ . Then

$$\det(u, v) = \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = -10 < 0.$$

Thus  $\{u, v\}$  is negative w.r.t. the natural order basis  $\{e_1 = (1, 0), e_2 = (0, 1)\}$ . □

*Proof of (b).* Write  $u = (1, 3, 5)$ ,  $v = (2, 3, 7)$ ,  $w = (4, 8, 3)$ . Then

$$\det(u, v, w) = \begin{vmatrix} 1 & 3 & 5 \\ 2 & 3 & 7 \\ 4 & 8 & 3 \end{vmatrix} = 39 > 0.$$

Thus  $\{u, v, w\}$  is positive w.r.t. the natural order basis  $\{e_1, e_2, e_3\}$ .  $\square$

### Exercise 1-4.2.

A plane  $P$  contained in  $\mathbb{R}^3$  is given by the equation  $ax+by+cz+d=0$ . Show that the vector  $v = (a, b, c)$  is perpendicular to the plane and that  $|d|/\sqrt{a^2+b^2+c^2}$  measures the distance from the plane to the origin  $(0, 0, 0)$ .

Say  $v$  is a normal vector of  $E$ .

In general, the distance from the plane  $E$  to any point  $(x_0, y_0, z_0) \in \mathbb{R}^3$  is

$$\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

*Proof.*

- (1) To show  $v = (a, b, c)$  is perpendicular to the plane, it suffices to show that  $v \cdot u = 0$  for any vector  $u$  lying on the plane  $E$ . Write  $u = \overrightarrow{PQ}$  where  $P = (x_1, y_1, z_1) \in E$  and  $Q = (x_2, y_2, z_2) \in E$ . Hence  $u = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$ .

$$\begin{aligned} v \cdot u &= (a, b, c) \cdot (x_2 - x_1, y_2 - y_1, z_2 - z_1) \\ &= a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) \\ &= (ax_2 + by_2 + cz_2) - (ax_1 + by_1 + cz_1) \\ &= (-d) - (-d) \\ &= 0. \end{aligned}$$

- (2) Pick any point  $(x_1, y_1, z_1) \in E$ . The distance from the plane  $E$  to the

point  $(x_0, y_0, z_0)$  is

$$\begin{aligned}
& \left| (x_1 - x_0, y_1 - y_0, z_1 - z_0) \cdot \frac{v}{|v|} \right| \\
&= \left| (x_1 - x_0, y_1 - y_0, z_1 - z_0) \cdot \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}} \right| \\
&= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \\
&= \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} \\
&= \frac{|-d - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} \\
&= \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.
\end{aligned}$$

□

### Exercise 1-4.3.

Determine the angle of intersection of the two planes  $5x + 3y + 2z - 4 = 0$  and  $3x + 4y - 7z = 0$ .

*Proof.*

- (1) The angle of intersection of the two planes is equal to a angle between two normal vectors of planes.
- (2) Let
  - (a) the angle of intersection of the two planes be  $\theta$ .
  - (b) the normal vector of  $5x + 3y + 2z - 4 = 0$  be  $n_1 = (5, 3, 2)$ .
  - (c) the normal vector of  $3x + 4y - 7z = 0$  be  $n_2 = (3, 4, -7)$ .

- (3) Hence,

$$\cos \theta = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{13}{2\sqrt{703}}.$$

$$\theta = \cos^{-1} \left( \frac{13}{2\sqrt{703}} \right).$$

□



**Exercise 1-4.13.**

Let  $u(t) = (u_1(t), u_2(t), u_3(t))$  and  $v(t) = (v_1(t), v_2(t), v_3(t))$  be differentiable maps from the interval  $(a, b)$  into  $\mathbb{R}^3$ . If the derivatives  $u'(t)$  and  $v'(t)$  satisfy the conditions

$$u'(t) = au(t) + bv(t), \quad v'(t) = cu(t) - av(t),$$

where  $a$ ,  $b$ , and  $c$  are constants, show that  $u(t) \wedge v(t)$  is a constant vector.

*Proof.* Since

$$\begin{aligned} \frac{d}{dt}(u(t) \wedge v(t)) &= u'(t) \wedge v(t) + u(t) \wedge v'(t) \\ &= (au(t) + bv(t)) \wedge v(t) + u(t) \wedge (cu(t) - av(t)) \\ &= au(t) \wedge v(t) + u(t) \wedge (-av(t)) \\ &= a(u(t) \wedge v(t)) + (-a)(u(t) \wedge v(t)) \\ &= (0, 0, 0), \end{aligned}$$

$u(t) \wedge v(t)$  is a constant vector.  $\square$

## 1-5. The Local Theory of Curves Parametrized by Arc Length

**Exercise 1-5.2.**

Show that the torsion  $\tau$  of  $\alpha$  is given by

$$\tau(s) = -\frac{\alpha'(s) \wedge \alpha''(s) \cdot \alpha'''(s)}{|\kappa(s)|^2}.$$

*Proof.*

- (1) Take inner product  $n(s)$  to the definition of torsion  $\tau(s)n(s) = b'(s)$ , we have

$$\tau(s) = b'(s) \cdot n(s).$$

Since  $b'(s) = t(s) \wedge n'(s)$ , we have to compute  $n'(s)$  first.

- (2) Compute  $n'(s)$ .

$$n'(s) = \frac{d}{ds} \left( \frac{\alpha''(s)}{\kappa(s)} \right) = \frac{\alpha'''(s)}{\kappa(s)} - \frac{\alpha''(s)\kappa'(s)}{\kappa(s)^2}.$$

(3) By (1)(2),

$$\begin{aligned}
\tau(s) &= b'(s) \cdot n(s) \\
&= (t(s) \wedge n'(s)) \cdot n(s) \\
&= \left( \alpha'(s) \wedge \left( \frac{\alpha'''(s)}{\kappa(s)} - \frac{\alpha''(s)\kappa'(s)}{\kappa(s)^2} \right) \right) \cdot \frac{\alpha''(s)}{\kappa(s)} \\
&= \left( \alpha'(s) \wedge \frac{\alpha'''(s)}{\kappa(s)} \right) \cdot \frac{\alpha''(s)}{\kappa(s)} \\
&= \frac{\alpha'(s) \wedge \alpha'''(s) \cdot \alpha''(s)}{|\kappa(s)|^2},
\end{aligned}$$

or

$$\tau(s) = \frac{\alpha'(s) \wedge \alpha'''(s) \cdot \alpha''(s)}{\alpha''(s)^2}.$$

□

## 1-6. The Local Canonical Form

## 1-7. Global Properties of Plane Curves