Notes on the book: *Ash, Probability and Measure Theory, 2nd edition*

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Contents

Chapter 1: Fundamentals of Measure and Integration Theory

1.1. Introduction

Problem 1.1.1.

Establish formulas (1)-(5).

Formulas.

(1) If $A_n \uparrow A$, then $A_n^c \downarrow A^c$; If $A_n \downarrow A$, then $A_n^c \uparrow A^c$.

(2)

$$\bigcup_{i=1}^{n} A_i = A_1 \cup (A_1^c \cap A_2) \cup (A_1^c \cap A_2^c \cap A_3) \\ \cup \cdots \cup (A_1^c \cap \cdots \cap A_{n-1}^c \cap A_n).$$

(3) Furthermore,

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \left(A_1^c \cap \cdots \cap A_{n-1}^c \cap A_n \right).$$

(4) If the A_n form an increasing sequence, then

$$\bigcup_{i=1}^{n} A_i = A_1 \cup (A_2 - A_1) \cup \cdots \cup (A_n - A_{n-1}).$$

(5) If the A_n form an increasing sequence, then

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} (A_n - A_{n-1})$$

(take A_0 as the empty set).

Proof of Formula (1).

(1) Suppose that $A_n \uparrow A$ is an increasing sequence of sets with limit A. Then $A_1 \subset A_2 \subset \cdots$ and $\bigcup_{n=1}^{\infty} A_n = A$. So $A_1^c \supset A_2^c \supset \cdots$ and

$$\bigcap_n A_n^c = \left(\bigcup_n A_n \right)^c = A^c$$

by the De Morgan laws. Hence $A_n \uparrow A$ implies that $A_n^c \downarrow A^c$.

(2) Conversely, suppose that $A_n \downarrow A$ is an decreasing sequence of sets with limit A. Then $A_1 \supset A_2 \supset \cdots$ and $\bigcap_{n=1}^{\infty} A_n = A$. So $A_1^c \subset A_2^c \subset \cdots$ and

$$\bigcup_n A_n^c = \left(\bigcap_n A_n \right)^c = A^c$$

by the De Morgan laws. Hence $A_n \downarrow A$ implies that $A_n^c \uparrow A^c$.

□

Proof of Formula (2).

(1) Set

$$B_i = A_1^c \cap \cdots \cap A_{i-1}^c \cap A_i$$

for $i = 1, \cdots, n$. Observe that $B_1 = A_1$. So it is equivalent to show that

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i.$$

(2) Since each B_i is a subset of A_i, $\bigcup_{i=1}^{n} A_i \supset \bigcup_{i=1}^{n} B_i$.

(3) Conversely, given any $x \in \bigcup_{i=1}^{n} A_i$. $x \in A_j$ for some j. Now take the minimal value of j such that $x \in A_j$. The minimality of j implies that $x \notin A_1, A_2, \cdots, A_{j-1}$. Hence

$$x \in A_1^c \cap \cdots \cap A_{j-1}^c \cap A_j = B_j \subset \bigcup_{i=1}^{n} B_i.$$

Therefore, $\bigcup_{i=1}^{n} A_i \subset \bigcup_{i=1}^{n} B_i$.

(4) By (2)(3), $\bigcup_{i=1}^{n} A_i$ and $\bigcup_{i=1}^{n} B_i$ are equal.

□

Proof of Formula (3). Same as the proof of formula (2) since the minimality of j described in part (3) exists. □

Proof of Formula (4).

(1) As A_n form an increasing sequence, $A_1 \subset A_2 \subset \cdots$ or $A_1^c \supset A_2^c \supset \cdots$. Hence

$$A_1^c \cap \cdots \cap A_{i-1}^c = A_{i-1}^c.$$

Therefore, B_i is reduced to

$$B_i = A_1^c \cap \cdots \cap A_{i-1}^c \cap A_i = A_{i-1}^c \cap A_i = A_i - A_{i-1}.$$

(2) Now formula (2) becomes

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} (A_i - A_{i-1}).$$

$\square$

Proof of Formula (5). Note that $B_n = A_n - A_{n-1}$ in the proof of formula (4). Formula (3) becomes $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} (A_n - A_{n-1})$. $\square$

Problem 1.1.2.

Define sets of real numbers as follows. Let $A_n = (-\frac{1}{n}, 1]$ if n is odd, and $A_n = (-1, \frac{1}{n}]$ if n is even. Find $\limsup_n A_n$ and $\liminf_n A_n$.

Proof.

(1) Write

$$\begin{aligned}
\bigcup_{k=n}^{\infty} A_k &= \left(\bigcup_{k=\lfloor \frac{n}{2} \rfloor}^{\infty} A_{2k+1} \right) \cup \left(\bigcup_{k=\lfloor \frac{n+1}{2} \rfloor}^{\infty} A_{2k} \right) \\
&= \left(\bigcup_{k=\lfloor \frac{n}{2} \rfloor}^{\infty} \left(-\frac{1}{2k+1}, 1 \right] \right) \cup \left(\bigcup_{k=\lfloor \frac{n+1}{2} \rfloor}^{\infty} \left(-1, \frac{1}{2k} \right] \right) \\
&= \left(-\frac{1}{2\lfloor \frac{n}{2} \rfloor + 1}, 1 \right] \cup \left(-1, \frac{1}{2\lfloor \frac{n+1}{2} \rfloor} \right] \\
&= (-1, 1]
\end{aligned}$$

for each k. Hence

$$\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \bigcap_{n=1}^{\infty} (-1, 1] = (-1, 1].$$

(2) Similarly, for each k we have

$$\begin{aligned}
\bigcap_{k=n}^{\infty} A_k &= \left(\bigcap_{k=\lfloor \frac{n}{2} \rfloor}^{\infty} A_{2k+1} \right) \cap \left(\bigcap_{k=\lfloor \frac{n+1}{2} \rfloor}^{\infty} A_{2k} \right) \\
&= \left(\bigcap_{k=\lfloor \frac{n}{2} \rfloor}^{\infty} \left(-\frac{1}{2k+1}, 1 \right] \right) \cap \left(\bigcap_{k=\lfloor \frac{n+1}{2} \rfloor}^{\infty} \left(-1, \frac{1}{2k} \right] \right) \\
&= [0, 1] \cup (-1, 0] \\
&= \{0\}.
\end{aligned}$$

Hence

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} \{0\} = \{0\}.$$

□

Problem 1.1.5.

Establish formulas (10)-(13).

Formulas.

(10)

$$\left(\limsup_n A_n\right)^c = \liminf_n A_n^c.$$

(11)

$$\left(\liminf_n A_n\right)^c = \limsup_n A_n^c.$$

(12)

$$\liminf_n A_n \subset \limsup_n A_n.$$

(13) If $A_n \uparrow A$ or $A_n \downarrow A$, then $\liminf_n A_n = \limsup_n A_n = A$.

Proof of Formula (10). The De Morgan laws shows that

$$\begin{aligned}
\left(\limsup_n A_n\right)^c &= \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right)^c \\
&= \bigcup_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k\right)^c \\
&= \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k^c \\
&= \limsup_n A_n^c.
\end{aligned}$$

□

Proof of Formula (11). Similar to the proof of formula (10).

$$
\begin{aligned}
\left(\liminf_n A_n\right)^c &= \left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k\right)^c \\
&= \bigcap_{n=1}^{\infty} \left(\bigcap_{k=n}^{\infty} A_k\right)^c \\
&= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k^c \\
&= \limsup_n A_n^c.
\end{aligned}
$$

□

Proof of Formula (12). Formulas (7) and (9) give all. □

Proof of Formula (13).

(1) If $A_n \uparrow A$, then

$$\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \bigcap_{n=1}^{\infty} A = A$$

and

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} A_n = A.$$

(2) If $A_n \downarrow A$, then

$$\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \bigcap_{n=1}^{\infty} A_n = A$$

and

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} A = A.$$

□

Problem 1.1.6.

Let $A = (a, b)$ and $B = (c, d)$ be disjoint open intervals of $\mathbb{R}$, and let $C_n = A$ if n is odd, $C_n = B$ if n is even. Find $\limsup_n C_n$ and $\liminf_n C_n$.

Proof.

(1)
$$\limsup_n C_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} C_k = \bigcap_{n=1}^{\infty} (A \cup B) = A \cup B.$$
(2)
$$\liminf_n C_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} C_k = \bigcup_{n=1}^{\infty} \varnothing = \varnothing.$$
□

1.2. Fields, σ-Fields, and Measures

Problem 1.2.1.

Let Ω be a countably infinite set, and let $\mathscr{F}$ consist of all subsets of Ω. Define $\mu(A) = 0$ if A is finite, $\mu(A) = \infty$ if A is infinite.

(a) *Show that μ is finitely additive but not countably additive.*

(b) *Show that Ω is the limit of an increasing sequence of sets A_n with $\mu(A_n) = 0$ for all n, but $\mu(\Omega) = \infty$.*

Proof of (a).

(1) *Show that μ is finitely additive.* Given a finitely collection of disjoint sets $A_1, A_2, \ldots, A_n$ in $\mathscr{F}$. If each set A_k $(k = 1, 2, \ldots, n)$ is finite, then $\bigcup A_k$ is also finite and thus we have
$$\mu\left(\bigcup_{k=1}^{n} A_k\right) = 0 = \sum_{k=1}^{n} \mu(A_k).$$
If there is some $A_{k'}$ is infinite, then $\bigcup A_k \supset A_{k'}$ is also infinite and thus
$$\mu\left(\bigcup_{k=1}^{n} A_k\right) = \infty = \sum_{k=1}^{n} \mu(A_k).$$

(2) *Show that μ is not countably additive.* Write
$$\Omega = \{\omega_1, \omega_2, \ldots\}$$
(since Ω is countably infinite) and $A_n = \{\omega_n\}$ for all $n = 1, 2, \ldots$. Hence $A_1, A_2, \ldots$ is a countably infinitely collection of disjoint sets and $\bigcup_{n=1}^{\infty} A_n = \Omega$. Therefore,
$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu(\Omega) = \infty$$

but

$$\sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} 0 = 0.$$

□

Proof of (b).

(1) Similar to the proof of (a). Write $\Omega = \{\omega_1, \omega_2, \dots\}$ and

$$A_n = \{\omega_1, \omega_2, \dots, \omega_n\}.$$

for all $n = 1, 2, \dots$.

(2) Therefore, $A_n \uparrow \Omega$, $\mu(A_n) = 0$ for all n but $\mu(\Omega) = \infty$. (Theorem 1.2.7 implies that μ cannot be a countably additive.)

□

Problem 1.2.2.

Let μ be counting measure on Ω, where Ω is an infinite set. Show that there is a sequence of sets $A_n \downarrow \varnothing$ with $\lim_{n \to \infty} \mu(A_n) \neq 0$.

Proof.

(1) Take a sequence of elements

$$\omega_1, \omega_2, \dots$$

from Ω. It is possible since Ω is an infinite set.

(2) Define

$$A_n = \{\omega_n, \omega_{n+1}, \dots\} \subset \Omega$$

for all $n = 1, 2, \dots$. So $A_n \downarrow \varnothing$ and each $\mu(A_n) = \infty$ (since each A_n is infinite). Hence

$$\lim_{n \to \infty} \mu(A_n) = \infty.$$

□

Problem 1.2.3.

Let Ω be a countably infinite set, and let $\mathscr{F}$ be the field consisting of all finite subsets of Ω and their complements. If A is finite, set $\mu(A) = 0$, and if A^c is finite, set $\mu(A) = 1$.

(a) *Show that μ is finitely additive but not countably additive on $\mathscr{F}$.*

(b) *Show that Ω is the limit of an increasing sequence of sets $A_n \in \mathscr{F}$ with $\mu(A_n) = 0$ for all n, but $\mu(\Omega) = 1$.*

Proof of (a).

(1) *Show that μ is finitely additive.* Given a finitely collection of disjoint sets $A_1, A_2, \ldots, A_n$ in $\mathscr{F}$. If each set A_k $(k = 1, 2, \ldots, n)$ is finite, then $\bigcup A_k$ is also finite and thus we have

$$\mu\left(\bigcup_{k=1}^{n} A_k\right) = 0 = \sum_{k=1}^{n} \mu(A_k).$$

(2) If there is some $A_{k'}$ is infinite, then there is only one such k'. (Assume that there were another k'' such that $A_{k''}$ is infinite. Since $A_{k'} \cap A_{k''} = \varnothing$, the De Morgan laws shows that

$$A_{k'}^c \cup A_{k''}^c = \Omega.$$

That is, a countably infinite set is a union of two finite subsets, which is absurd.) Hence

$$\mu\left(\bigcup_{k=1}^{n} A_k\right) = 1 = 0 + \cdots + 0 + \underbrace{1}_{k'\text{-th}} + 0 + \cdots + 0 = \sum_{k=1}^{n} \mu(A_k).$$

(3) *Show that μ is not countably additive.* Write

$$\Omega = \{\omega_1, \omega_2, \ldots\}$$

(since Ω is countably infinite) and $A_n = \{\omega_n\}$ for all $n = 1, 2, \ldots$. Hence $A_1, A_2, \ldots$ is a countably infinitely collection of disjoint sets and $\bigcup_{n=1}^{\infty} A_n = \Omega$. Therefore,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu(\Omega) = 1$$

but

$$\sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} 0 = 0.$$

□

Proof of (b). Write $\Omega = \{\omega_1, \omega_2, \ldots\}$ and

$$A_n = \{\omega_1, \omega_2, \ldots, \omega_n\} \in \mathscr{F}.$$

for all $n = 1, 2, \ldots$. Therefore, $A_n \uparrow \Omega$, $\mu(A_n) = 0$ for all n but $\mu(\Omega) = 1$. (Theorem 1.2.7 implies that μ cannot be a countably additive.) □

Problem 1.2.5.

Let μ be a nonnegative, finitely additive set function on the field $\mathscr{F}$. If $A_1, A_2, \ldots$ are disjoint sets in $\mathscr{F}$ and $\bigcup_{n=1}^{\infty} A_n \in \mathscr{F}$, show that

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \geq \sum_{n=1}^{\infty} \mu(A_n).$$

Proof.

(1) Note that μ is a nonnegative, finitely additive set function on $\mathscr{F}$. Hence,

$$\begin{aligned} \mu\left(\bigcup_{n=1}^{\infty} A_n\right) &\geq \mu\left(\bigcup_{n=1}^{m} A_n\right) && \text{(Theorem 1.2.5)} \\ &= \sum_{n=1}^{m} \mu(A_n) \end{aligned}$$

for every m.

(2) Since $\sum_{n=1}^{m} \mu(A_n)$ is bounded by $\mu\big(\bigcup_{n=1}^{\infty} A_n\big)$ and μ is nonnegative, the result is established as letting $m \to \infty$.

□

Problem 1.2.6.

Let $f : \Omega \to \Omega'$, and let $\mathscr{C}$ be a class of subsets of Ω'. Show that

$$\sigma(f^{-1}(\mathscr{C})) = f^{-1}(\sigma(\mathscr{C})),$$

where $f^{-1}(\mathscr{C}) = \{f^{-1}(A) : A \in \mathscr{C}\}$. (Use the good sets principle.)

Proof.

(1) *Show that* $\sigma(f^{-1}(\mathscr{C})) \subset f^{-1}(\sigma(\mathscr{C}))$. Note that $f^{-1}(\sigma(\mathscr{C}))$ is a σ-field. Hence by $\mathscr{C} \subset \sigma(\mathscr{C})$ we have

$$\sigma(f^{-1}(\mathscr{C})) \subset \sigma(f^{-1}(\sigma(\mathscr{C}))) = f^{-1}(\sigma(\mathscr{C})).$$

(2) *Show that* $\sigma(f^{-1}(\mathscr{C})) \supset f^{-1}(\sigma(\mathscr{C}))$. Let

$$\mathscr{S} = \{A \subset \Omega' : f^{-1}(A) \in \sigma(f^{-1}(\mathscr{C}))\}.$$

So $\mathscr{S}$ is a σ-field containing $\mathscr{C}$ (by observing that $f^{-1}\big(\bigcup A_n\big) = \bigcup f^{-1}(A_n)$). Hence $\mathscr{S} \supset \sigma(\mathscr{C})$. Now given any $f^{-1}(A) \in f^{-1}(\sigma(\mathscr{C}))$ with $A \in \sigma(\mathscr{C})$. As $\sigma(\mathscr{C}) \subset \mathscr{S}$, $f^{-1}(A) \in \sigma(f^{-1}(\mathscr{C}))$ or $f^{-1}(\sigma(\mathscr{C})) \subset \sigma(f^{-1}(\mathscr{C}))$.

$\square$