

Chapter 1: Set Theory

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Problem 1.1. Show that $\{x : x \neq x\} = \emptyset$.

Proof. Every element x of $\{x : x \neq x\}$ satisfying $x \neq x$, contrary to $x = x$. That is, there are no elements in $\{x : x \neq x\}$, or $\{x : x \neq x\} = \emptyset$. \square

Problem 1.2. Show that if $x \in \emptyset$, then x is a green-eyed lion.

Proof. $\emptyset \subseteq \{\text{a green-eyed lion}\}$. \square

Problem 1.4. Show that the well-ordering principle implies the principle of mathematical induction. (Hint: Consider the set $\{n \in \mathbb{N} : P(n) \text{ is false}\}$.)

Proof (Hint). Suppose that

- (1) $P(n)$ be a proposition defined for each $n \in \mathbb{N}$,
- (2) $P(1)$ is true,
- (3) $[P(n) \Rightarrow P(n+1)]$ is true.

Consider the set

$$S = \{n \in \mathbb{N} : P(n) \text{ false}\} \subseteq \mathbb{N}.$$

Want to show S is empty, or the principle of mathematical induction holds. If S were nonempty, by the well-ordering principle S has a smallest element m . m cannot be 1 by (2). Say $m > 1$. Therefore, $m-1 \in \mathbb{N}$ and $P(m-1)$ is true by the minimality of m . By (3), $P((m-1)+1) = P(m)$ is true, which is absurd. \square

Problem 1.5. Use mathematical induction to establish that the well-ordering principle. (Hint: Given a set S of positive integers, let $P(n)$ be the proposition ‘If $n \in S$, then S has a least element’.)

Proof (Modified hint).

- (1) Given a set S of positive integers, let $P(n)$ be the proposition ‘If $m \in S$ for some $m \leq n$, then S has a least element’. Want to show $P(n)$ is true for all $n \in \mathbb{N}$.

- (a) $P(1)$ is true. For $m \in S$ with $m \leq n = 1$, or $m = 1$ by the minimality of $1 \in \mathbb{N}$, S has a least element 1 (m itself) in \mathbb{N} .
- (b) Suppose $P(n)$ is true. If $n + 1 \in S$, then there are only two possible cases.
 - (i) There is a positive integer $m \in S$ less than $n + 1$. So $n \geq m \in S$. Since $P(n)$ is true, S has a least element.
 - (ii) There is no positive integer $m \in S$ less than $n + 1$. In this case $n + 1$ is the least element in S .

In any cases (i)(ii), S has a least element, or $P(n + 1)$ is true.

By mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

- (2) *Show that the well-ordering principle holds.* Let T be a nonempty subset of \mathbb{N} , so there exists a positive integer $k \in T$. Notice that $P(k)$ is true by (1), thus T has a least element since $k \leq k$.

□

Problem 1.9. *Show that $A \subseteq B \Leftrightarrow A \cap B = A \Leftrightarrow A \cup B = B$.*

Proof.

- (1) $A \subseteq B \Leftrightarrow A \cap B = A$.
 - (a) (\Rightarrow) It suffices to show $A \cap B \supseteq A$. For any $x \in A$, $x \in B$ by $A \subseteq B$, so $x \in A \cap B$, so $A \cap B \supseteq A$.
 - (b) (\Leftarrow) $A = A \cap B \subseteq B$.
- (2) $A \subseteq B \Leftrightarrow A \cup B = B$.
 - (a) (\Rightarrow) It suffices to show $A \cup B \subseteq B$. For any $x \in A \cup B$, $x \in A$ or $x \in B$. By $A \subseteq B$, $x \in B$ or $x \in B$. $x \in B$, so $A \cup B \subseteq B$.
 - (b) (\Leftarrow) $A \subseteq A \cup B = B$.

□

Problem 1.11. *Show that $A \subseteq B \Leftrightarrow \tilde{B} \subseteq \tilde{A}$.*

Proof.

$$\begin{aligned}
 A \subseteq B &\Leftrightarrow x \in A \Rightarrow x \in B \\
 &\Leftrightarrow x \notin B \Rightarrow x \notin A \\
 &\Leftrightarrow \tilde{B} \subseteq \tilde{A}.
 \end{aligned}$$

□

Problem 1.14. *Show that*

$$B \cap \left[\bigcup_{A \in \mathcal{C}} A \right] = \bigcup_{A \in \mathcal{C}} (B \cap A).$$

Proof.

$$\begin{aligned} x \in B \cap \left[\bigcup_{A \in \mathcal{C}} A \right] &\iff x \in B \text{ and } x \in \bigcup_{A \in \mathcal{C}} A \\ &\iff x \in B \text{ and } x \in A \text{ for some } A \in \mathcal{C} \\ &\iff x \in B \cap A \text{ for some } A \in \mathcal{C} \\ &\iff x \in \bigcup_{A \in \mathcal{C}} (B \cap A). \end{aligned}$$

□