Chapter 3: L^p-Spaces

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Exercise 3.3. *Assume that φ is a continuous real function on (a, b) such that*

$$\varphi\left(\frac{x+y}{2}\right) \leq \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(y)$$

for all x and $y \in (a, b)$. Prove that φ is convex. (The conclusion does not follow if continuity is omitted from the hypotheses.)

Proof.

(1) *Show that*

$$\varphi\left(\frac{x_1 + \cdots + x_n}{n}\right) \leq \frac{\varphi(x_1) + \cdots + \varphi(x_n)}{n}$$

whenever $a < x_i < b$ $(1 \leq i \leq n)$. Apply Cauchy induction and use the same argument in proving the AM-GM inequality. As $n = 1, 2$, the inequality holds by assumption. Suppose $n = 2^k$ $(k \geq 1)$ the inequality holds. As $n = 2^{k+1}$,

$$\begin{aligned}
&\varphi\left(\frac{x_1 + \cdots + x_{2^{k+1}}}{2^{k+1}}\right) \\
=&\varphi\left(\frac{1}{2}\left(\frac{x_1 + \cdots + x_{2^k}}{2^k} + \frac{x_{2^k+1} + \cdots + x_{2^{k+1}}}{2^k}\right)\right) \\
\leq&\frac{1}{2}\left(\varphi\left(\frac{x_1 + \cdots + x_{2^k}}{2^k}\right) + \varphi\left(\frac{x_{2^k+1} + \cdots + x_{2^{k+1}}}{2^k}\right)\right) \\
\leq&\frac{1}{2}\left(\frac{\varphi(x_1) + \cdots + \varphi(x_{2^k})}{2^k} + \frac{\varphi(x_{2^k+1}) + \cdots + \varphi(x_{2^{k+1}})}{2^k}\right) \\
=&\frac{\varphi(x_1) + \cdots + \varphi(x_{2^k}) + \varphi(x_{2^k+1}) + \cdots + \varphi(x_{2^{k+1}})}{2^{k+1}} \\
=&\frac{\varphi(x_1) + \cdots + \varphi(x_{2^{k+1}})}{2^{k+1}}.
\end{aligned}$$

As n is not a power of 2, then it is certainly less than some natural power of 2, say $n < 2^m$ for some m. Let

$$x_{n+1} = \cdots = x_{2^m} = \frac{x_1 + \cdots + x_n}{n} = \alpha.$$

Then by the induction hypothesis,

$$\begin{aligned}
\varphi(\alpha) &= \varphi\left(\frac{x_1 + \cdots + x_n + \alpha + \cdots + \alpha}{2^m}\right) \\
&\leq \frac{\varphi(x_1) + \cdots + \varphi(x_n) + \varphi(\alpha) + \cdots + \varphi(\alpha)}{2^m} \\
&\leq \frac{\varphi(x_1) + \cdots + \varphi(x_n) + (2^m - n)\varphi(\alpha)}{2^m}, \\
2^m\varphi(\alpha) &\leq \varphi(x_1) + \cdots + \varphi(x_n) + (2^m - n)\varphi(\alpha), \\
n\varphi(\alpha) &\leq \varphi(x_1) + \cdots + \varphi(x_n),
\end{aligned}$$

or $\varphi\left(\frac{1}{n}(x_1 + \cdots + x_n)\right) \leq \frac{1}{n}(\varphi(x_1) + \cdots \varphi(x_n))$.

(2) Hence,

$$\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y)$$

for any rational λ in $(0,1)$. (Given any positive integers $p < q$, put $n = q$, $x_1 = \cdots = x_p = x$ and $x_{p+1} = \cdots = x_n = y$ in (1).)

(3) Given any real $\lambda \in (0,1)$, there is a sequence of rational numbers $\{r_n\} \subseteq (0,1)$ such that $r_n \to \lambda$. By (2),

$$\varphi(r_n x + (1-r_n)y) \leq r_n\varphi(x) + (1-r_n)\varphi(y)$$

for any rational r_n in $(0,1)$. Taking limit on the both sides and using the continuity of f, we have

$$\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y).$$

$\square$

Proof (Reductio ad absurdum). If φ were not convex, then there is a subinterval $[c,d] \subseteq (a,b)$ such that

$$\frac{\varphi(d) - \varphi(c)}{d - c} < \frac{\varphi(x_0) - \varphi(c)}{x_0 - c}$$

for some $x_0 \in [c,d]$. Let

$$\psi(x) = \varphi(x) - \varphi(c) - \frac{\varphi(d) - \varphi(c)}{d - c}(x - c)$$

for $x \in [c,d]$. Therefore,

(1) $\psi(x)$ is continuous and midpoint convex.

(2) $\psi(c) = \psi(d) = 0$.

(3) Let $M = \sup\{\psi(x) : x \in [c,d]\}$. $\infty > M > 0$ due to the continuity of ψ and the existence of x_0. And let $\xi = \inf\{x \in [c,d] : \psi(x) = M\}$. By the continuity of g, $\psi(\xi) = M$. $\xi \in (c,d)$ by (2).

(4) Since (c,d) is open, there is $h > 0$ such that $(\xi - h, \xi + h) \subseteq (c,d)$. By the minimality of ξ and M, $\psi(\xi - h) < \psi(\xi)$ and $\psi(\xi + h) \leq \psi(\xi)$.

Therefore,

$$\begin{aligned}
\psi(\xi - h) + \psi(\xi + h) &< 2\psi(\xi), \\
\frac{\psi(\xi - h) + \psi(\xi + h)}{2} &< \psi(h) \\
&= \psi\left(\frac{(\xi - h) + (\xi + h)}{2}\right),
\end{aligned}$$

contrary to the midpoint convexity of ψ. □