Chapter 4: Determinants

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Section 4.3: Properties of Determinants

Exercise 4.3.11. *A matrix $Q \in \mathsf{M}_{n\times n}(\mathbb{R})$ is called orthogonal if $QQ^t = I$. Prove that if Q is orthogonal, then $\det(Q) = \pm 1$.*

Proof. By the orthogonality of Q, $QQ^t = I$. So

$$\begin{aligned}
QQ^t = I &\Longrightarrow \det(QQ^t) = \det(I) \\
&\Longleftrightarrow \det(Q)\det(Q^t) = \det(I) && \text{(Theorem 4.7)} \\
&\Longleftrightarrow \det(Q)\det(Q) = \det(I) && \text{(Theorem 4.8)} \\
&\Longleftrightarrow \det(Q)^2 = 1 && \text{(Example 4.2.4)} \\
&\Longleftrightarrow \det(Q) = \pm 1.
\end{aligned}$$

□

Exercise 4.3.14. *Prove that if $A, B \in \mathsf{M}_{n\times n}(F)$ are similar, then $\det(A) = \det(B)$.*

Proof. Since A, B are similar, there exists an invertible matrix Q such that $B = Q^{-1}AQ$. So

$$\begin{aligned}
\det(B) &= \det(Q^{-1}AQ) \\
&= \det(Q^{-1})\det(A)\det(Q) && \text{(Theorem 4.7)} \\
&= \det(Q)\det(Q^{-1})\det(A) && \text{(F is field)} \\
&= \det(QQ^{-1})\det(A) && \text{(Theorem 4.7)} \\
&= \det(I)\det(A) \\
&= 1 \cdot \det(A) && \text{(Example 4.2.4)} \\
&= \det(A).
\end{aligned}$$

□