

## Chapter 4: Limits and Continuity

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### Continuity of real-valued functions

**Exercise 4.19.** Let  $f$  be continuous on  $[a, b]$  and define  $g$  as follows:  $g(a) = f(a)$  and, for  $a < x \leq b$ , let  $g(x)$  be the maximum value of  $f$  in the subinterval  $[a, x]$ . Show that  $g$  is continuous on  $[a, b]$ .

Indeed,  $g(x) = \max_{a \leq t \leq x} f(t)$  for  $x \in [a, b]$ .

*Proof.*

- (1)  $f$  is continuous on  $[a, b]$  at a point  $p \iff$  Given any  $\epsilon' > 0$ , there exists  $\delta' > 0$  such that  $|f(x) - f(p)| < \epsilon'$  whenever  $|x - p| < \delta'$  (and  $x \in [a, b]$ ). We left  $\epsilon'$  and  $\delta'$  undecided temporarily.

- (2) To estimate  $g$  on

$$[p - \delta', p + \delta'] \cap [a, b],$$

we need to study the behavior of  $f$  on  $[a, p + \delta'] \cap [a, b]$  (by the definition of  $g(x)$ ), and then use the continuity of  $f$  to establish the desired result.

- (3) Look at where  $f$  takes the maximum value over on  $[a, p + \delta'] \cap [a, b]$  at. There are two possible cases (might overlapped):

- (a) At a point in  $[a, p - \delta'] \cap [a, b]$ . In this case  $g$  is constant on  $[p - \delta', p + \delta'] \cap [a, b]$ , or  $|g(x) - g(p)| = 0$ .
- (b) At a point  $q \in (p - \delta', p + \delta'] \cap [a, b]$ . For any  $x \in [p - \delta', p + \delta'] \cap [a, b]$ ,
- (i)  $f(p) - \epsilon' < g(x)$  by the maximality of  $g$  on  $[a, x]$ .
  - (ii)  $g(x) \leq f(q) < f(p) + \epsilon'$  since  $g$  is an increasing function and  $f$  takes the maximum value over on  $[a, p + \delta'] \cap [a, b]$  at  $q \in (p - \delta', p + \delta'] \cap [a, b]$ .

By (i)(i),

$$f(p) - \epsilon' < g(x) < f(p) + \epsilon'$$

for any  $x \in [p - \delta', p + \delta'] \cap [a, b]$  (especially  $x = p$ ). Therefore,

$$|g(x) - g(p)| < 2\epsilon' \text{ whenever } |x - p| < \delta' \text{ (and } x \in [a, b]).$$

By (a)(b), we have  $|g(x) - g(p)| < 2\epsilon'$  whenever  $|x - p| < \delta'$  (and  $x \in [a, b]$ ) in any cases.

(4) Retake  $\epsilon' = \frac{\epsilon}{2} > 0$  and  $\delta = \delta' > 0$ .

□

## Continuity in metric spaces

In Exercise 4.29 through 4.33, we assume that  $f : S \rightarrow T$  is a function from one metric space  $(S, d_S)$  to another  $(T, d_T)$ .

**Exercise 4.29.** *Prove that  $f$  is continuous on  $S$  if and only if*

$$f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ \quad \text{for every subset } B \text{ of } T.$$

Denote the interior of any set  $S$  by  $S^\circ$ .

*Proof (On topological spaces).*

(1) ( $\implies$ )

$$\begin{aligned} \forall x \in f^{-1}(B^\circ) &\implies f(x) \in B^\circ \\ &\implies \exists \text{ open neighborhood } V \subseteq B^\circ \subseteq B \text{ containing } f(x) \\ &\implies x \in f^{-1}(V) \subseteq f^{-1}(B) \\ &\implies f^{-1}(V) \text{ is open in } S \text{ since } f \text{ is continuous} \\ &\implies f^{-1}(V) \text{ is open neighborhood } \subseteq f^{-1}(B) \text{ containing } x \\ &\implies x \in (f^{-1}(B))^\circ. \end{aligned}$$

(2) ( $\impliedby$ ) *Given any open subset  $V$  of  $T$ , need to show  $U = f^{-1}(V)$  is open in  $S$ .*

$$\begin{aligned} f^{-1}(V) &= f^{-1}(V^\circ) && (V \text{ is open}) \\ &\subseteq (f^{-1}(V))^\circ && (\text{Assumption}) \end{aligned}$$

So  $U \subseteq U^\circ$  or  $U = U^\circ$  is open.

□

**Exercise 4.30.** *Prove that  $f$  is continuous on  $S$  if and only if*

$$f(\overline{A}) \subseteq \overline{f(A)} \quad \text{for every subset } A \text{ of } S.$$

Denote the closure of any set  $S$  by  $\overline{S}$ .

*Proof (On topological spaces).*

(1) ( $\implies$ ) Since  $f$  is continuous and  $\overline{f(A)}$  is closed,  $f^{-1}(\overline{f(A)})$  is closed. Hence,

$$\begin{aligned}
f^{-1}(\overline{f(A)}) &\supseteq f^{-1}(f(A)) && \text{(Monotonicity of } f^{-1}) \\
&\supseteq A, && \text{(Exercise 2.7(a))} \\
\overline{A} &\subseteq f^{-1}(\overline{f(A)}), && \text{(Monotonicity of closure)} \\
f(\overline{A}) &\subseteq f(f^{-1}(\overline{f(A)})) && \text{(Monotonicity of } f) \\
&\subseteq \overline{f(A)}. && \text{(Exercise 2.7(b))}
\end{aligned}$$

(2) ( $\impliedby$ ) Given any closed subset  $D$  of  $T$ , need to show  $C = f^{-1}(D)$  is closed in  $S$ .

$$\begin{aligned}
f(\overline{C}) &\subseteq \overline{f(C)} && \text{(Assumption)} \\
&= \overline{f(f^{-1}(D))} && (C = f^{-1}(D)) \\
&\subseteq \overline{D} && \text{(Exercise 2.7(b))} \\
&= D, && (D \text{ is closed}) \\
f^{-1}(f(\overline{C})) &\subseteq f^{-1}(D), && \text{(Monotonicity of } f^{-1}) \\
\overline{C} &\subseteq f^{-1}(f(\overline{C})) \subseteq f^{-1}(D) = C. && \text{(Exercise 2.7(a))}
\end{aligned}$$

So  $C \supseteq \overline{C}$  or  $C = \overline{C}$  is closed.

□

**Supplement.** Let  $f$  be a map from a topological space on  $X$  to a topological space on  $Y$ . Then, the following statements are equivalent:

- (1)  $f$  is continuous: For each  $x \in X$  and every neighborhood  $V$  of  $f(x)$ , there is a neighborhood  $U$  of  $x$  such that  $f(U) \subseteq V$ .
- (2) For every open set  $O$  in  $Y$ , the inverse image  $f^{-1}(O)$  is open in  $X$ .
- (3) For every closed set  $C$  in  $Y$ , the inverse image  $f^{-1}(C)$  is closed in  $X$ .
- (4)  $f(A)^\circ \subseteq f(A^\circ)$  for every subset  $A$  of  $X$ .
- (5)  $f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ$  for every subset  $B$  of  $Y$ .
- (6)  $f(\overline{A}) \subseteq \overline{f(A)}$  for every subset  $A$  of  $X$ .
- (7)  $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$  for every subset  $B$  of  $Y$ .

**Exercise 4.33.** Give an example of a continuous  $f$  and a Cauchy sequence  $\{x_n\}$  in some metric space  $S$  for which  $\{f(x_n)\}$  is not a Cauchy sequence in  $T$ .

Compare with Exercise 4.54 to get some hints.

*Proof.* Let

$$S = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots \right\} = \left\{ \frac{1}{n} : n \in \mathbb{Z}^+ \right\}.$$

Define  $f : S \rightarrow \mathbb{R}$  by  $f\left(\frac{1}{n}\right) = (-1)^n$ . Then  $f$  is continuous (but not uniformly continuous). The sequence  $\{x_n\} = \left\{\frac{1}{n}\right\}$  in  $S$  is a Cauchy sequence, but the sequence  $\{f(x_n)\} = \{(-1)^n\}$  is not a Cauchy sequence in  $\mathbb{R}$ .  $\square$

## Uniform continuity

**Exercise 4.50.** *Prove that a function which is uniformly continuous on  $S$  is also continuous on  $S$ .*

*Proof.* The proof is straightforward.

- (1) Suppose  $f : S \rightarrow T$  is uniformly continuous on  $S$ . Given any  $\epsilon > 0$ , there is  $\delta > 0$  such that  $d_T(f(x), f(y)) < \epsilon$  whenever  $d_S(x, y) < \delta$ .
- (2) Show that  $f$  is continuous at any point  $p$  in  $S$ . Set  $y = p$  in (1).

$\square$

**Exercise 4.51.** *If  $f(x) = x^2$  for  $x \in \mathbb{R}$ , prove that  $f$  is not uniformly continuous on  $\mathbb{R}$ .*

*Proof.* Prove by contradiction.

- (1) If  $f$  were uniformly continuous on  $\mathbb{R}$ , then for any  $\epsilon > 0$ , there is  $\delta > 0$  such that  $|f(x) - f(y)| < \epsilon$  whenever  $|x - y| < \delta$ . Here we pick  $\epsilon = 1 > 0$ .
- (2) So

$$|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y| < 1$$

for any  $|x - y| < \delta$ . In particular, we pick  $x = \frac{1}{\delta}$  and  $y = \frac{1}{\delta} + \frac{\delta}{2}$ . Now  $|x - y| = \frac{\delta}{2} < \delta$ , and thus  $|f(x) - f(y)| = |x + y||x - y| < 1$  would be true. However,

$$|f(x) - f(y)| = |x + y||x - y| = \left(\frac{2}{\delta} + \frac{\delta}{2}\right) \left(\frac{\delta}{2}\right) > \frac{2}{\delta} \cdot \frac{\delta}{2} = 1,$$

contrary to  $|f(x) - f(y)| = |x + y||x - y| < 1$ .

$\square$

**Exercise 4.52.** Assume that  $f$  is uniformly continuous on a bounded set  $S$  in  $\mathbb{R}^n$ . Prove that  $f$  must be bounded on  $S$ .

The conclusion is false if boundedness of  $S$  is omitted from the hypothesis. For example,  $f(x) = x$  on  $\mathbb{R}$  is uniformly continuous on  $\mathbb{R}$  but  $f(\mathbb{R}) = \mathbb{R}$  is unbounded.

*Proof.* Heine-Borel theorem provides the finiteness property to construct the boundedness property of  $f$ .

- (1) Let  $S$  be a bounded subset of a metric space  $X$ . Show that the closure of  $S$  in  $X$  is also bounded in  $X$ .  $S$  is bounded if  $S \subseteq B_X(a; r)$  for some  $r > 0$  and some  $a \in X$ . (The ball  $B_X(a; r)$  is defined to be the set of all  $x \in X$  such that  $d_X(x, a) < r$ .) Take the closure on the both sides,

$$\overline{S} \subseteq \overline{B_X(a; r)} = \{x \in X : d_X(x, a) \leq r\} \subseteq B_X(a; 2r),$$

or  $\overline{S}$  is bounded.

- (2) Since  $f : S \rightarrow T$  is uniformly continuous, given any  $\epsilon > 0$ , there is  $\delta > 0$  such that  $d_T(f(x), f(y)) < \epsilon$  whenever  $d_S(x, y) < \delta$ . In particular, pick  $\epsilon = 1$ .
- (3) For such  $\delta > 0$ , we construct an open covering of  $\overline{S} \subseteq \mathbb{R}^n$ . Pick a collection  $\mathcal{F}$  of open balls  $B(a; \delta) \subseteq \mathbb{R}^n$  where  $a$  runs over all elements of  $S$ .  $\mathcal{F}$  covers  $\overline{S}$  (by the definition of accumulation points). Since  $\overline{S}$  is closed and bounded (by applying (1) on the boundedness of  $S$ ),  $\overline{S}$  is compact (Heine-Borel theorem on  $\mathbb{R}^n$ ). That is, there is a finite subcollection  $\mathcal{F}'$  of  $\mathcal{F}$  also covers  $\overline{S}$ , say

$$\mathcal{F}' = \{B(a_1; \delta), B(a_2; \delta), \dots, B(a_m; \delta)\}.$$

- (4) Given any  $x \in S \subseteq \overline{S}$ , there is some  $a_i \in S$  ( $1 \leq i \leq m$ ) such that  $x \in B(a_i; \delta)$ . In such ball,  $d_S(x, a_i) < \delta$ . By (2),  $\|f(x) - f(a_i)\| < 1$ , or  $\|f(x)\| < 1 + \|f(a_i)\|$ . Almost done. Notice that  $a_i$  depends on  $x$ , and thus we might use finiteness of  $\{a_1, a_2, \dots, a_m\}$  to remove dependence of  $a_i$ .
- (5) Take the maximum value over all  $a_i$  ( $1 \leq i \leq m$ ) and then assign

$$M = 1 + \max_{1 \leq i \leq m} \|f(a_i)\|.$$

So given any  $x \in S$ ,  $\|f(x)\| < M$ .

□

**Exercise 4.54.** Assume  $f : S \rightarrow T$  is uniformly continuous on  $S$ , where  $S$  and  $T$  are metric spaces. If  $\{x_n\}$  is any Cauchy sequence in  $S$ , prove that  $\{f(x_n)\}$

is a Cauchy sequence in  $T$ . (Compare with Exercise 4.33.)

Therefore, we need to find a continuous but not uniformly continuous function to solve Exercise 4.33: *Give an example of a continuous  $f$  and a Cauchy sequence  $\{x_n\}$  in some metric space  $S$  for which  $\{f(x_n)\}$  is not a Cauchy sequence in  $T$ .*

*Proof.* The proof is straightforward.

- (1) Since  $f : S \rightarrow T$  is uniformly continuous on  $S$ , given any  $\epsilon > 0$ , there is  $\delta > 0$  such that  $d_T(f(x), f(y)) < \epsilon$  whenever  $d_S(x, y) < \delta$ .
- (2) Since  $\{x_n\}$  is any Cauchy sequence in  $S$ , especially for such  $\delta > 0$  in (1), there is an integer  $N$  such that  $d_S(x_m, x_n) < \delta$  whenever  $m \geq N$  and  $n \geq N$ . So as  $m \geq N$  and  $n \geq N$ , we have  $d_T(f(x_m), f(x_n)) < \epsilon$  by (1), or  $\{f(x_n)\}$  itself is a Cauchy sequence in  $T$ .

□