

## Chapter 15: Bernoulli Numbers

Author: Meng-Gen Tsai

Email: plover@gmail.com

**Supplement.** Equation (4) on page 231. *Prove that*

$$x \cot x = 1 - 2 \sum_{n=1}^{\infty} \frac{x^2}{n^2 \pi^2 - x^2}.$$

*Proof (Exercise 6.73 in the book Graham, Knuth and Patashnik, Concrete Mathematics, Second Edition).*

(1) *Show that*

$$\cot x = \frac{1}{2^n} \sum_{k=0}^{2^n-1} \cot \frac{x + k\pi}{2^n}$$

for all integers  $n \geq 1$ . Notice that

$$\begin{aligned} \cot(x + \pi) &= \cot x, \\ \cot\left(x + \frac{\pi}{2}\right) &= -\tan x, \\ \cot x &= \frac{1}{2} \left( \cot \frac{x}{2} - \tan \frac{x}{2} \right). \end{aligned}$$

Use mathematical induction. The case  $n = 1$  is the same as the note.

Assume the case  $n = m$  holds. For  $n = m + 1$ ,

$$\begin{aligned} \sum_{k=0}^{2^{m+1}-1} \cot \frac{x + k\pi}{2^{m+1}} &= \sum_{k=0}^{2^m-1} \cot \frac{x + k\pi}{2^{m+1}} + \sum_{k=2^m}^{2^{m+1}-1} \cot \frac{x + k\pi}{2^{m+1}} \\ &= \sum_{k=0}^{2^m-1} \cot \frac{x + k\pi}{2^{m+1}} + \sum_{k=0}^{2^m-1} \cot \frac{x + (2^m + k)\pi}{2^{m+1}} \\ &= \sum_{k=0}^{2^m-1} \cot \frac{x + k\pi}{2^{m+1}} + \sum_{k=0}^{2^m-1} \cot \left( \frac{x + k\pi}{2^{m+1}} + \frac{\pi}{2} \right) \\ &= \sum_{k=0}^{2^m-1} \left( \cot \frac{x + k\pi}{2^{m+1}} - \tan \frac{x + k\pi}{2^{m+1}} \right) \\ &= \sum_{k=0}^{2^m-1} \left( \cot \frac{x + k\pi}{2^{m+1}} - \tan \frac{x + k\pi}{2^{m+1}} \right) \\ &= 2 \sum_{k=0}^{2^m-1} \cot \frac{x + k\pi}{2^m}. \end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{2^{m+1}} \sum_{k=0}^{2^{m+1}-1} \cot \frac{x+k\pi}{2^{m+1}} &= \frac{1}{2^{m+1}} \cdot 2 \sum_{k=0}^{2^m-1} \cot \frac{x+k\pi}{2^m} \\ &= \frac{1}{2^m} \sum_{k=0}^{2^m-1} \cot \frac{x+k\pi}{2^m} \\ &= \cot x.\end{aligned}$$

(2) By rearranging the index of summation of the identity in (1), we have

$$x \cot x = \frac{x}{2^n} \cot \frac{x}{2^n} - \frac{x}{2^n} \tan \frac{x}{2^n} + \sum_{k=1}^{2^{n-1}-1} \frac{x}{2^n} \left( \cot \frac{x+k\pi}{2^n} + \cot \frac{x-k\pi}{2^n} \right)$$

for all integers  $n \geq 1$ .

(3) Notice that  $\lim_{x \rightarrow 0} x \cot x = 1$ . Let  $n \rightarrow \infty$ , the result is established.

□

**Exercise 15.1.** Using the definition of the Bernoulli number show  $B_{10} = \frac{5}{66}$  and  $B_{12} = -\frac{691}{2730}$ .

*Proof.*

- (1) It is known that  $B_1 = -\frac{1}{2}$ ,  $B_2 = \frac{1}{6}$ ,  $B_4 = -\frac{1}{30}$ ,  $B_6 = \frac{1}{42}$ , and  $B_m = 0$  for odd  $m > 1$ .
- (2) Recall the implicit recurrence relation,

$$\sum_{k=0}^m \binom{m+1}{k} B_k = [m=0],$$

where  $[m=0]$  is the Iverson brackets which is equal to the Kronecker delta  $\delta_{m0}$ .

(3) So

$$0 = 1 + 9B_1 + 36B_2 + 84B_3 + 126B_4 + 126B_5 + 84B_6 + 36B_7 + 9B_8,$$

$$0 = 1 + 9B_1 + 36B_2 + 126B_4 + 84B_6 + 9B_8,$$

$$0 = 1 + 9 \left( -\frac{1}{2} \right) + 36 \left( \frac{1}{6} \right) + 126 \left( -\frac{1}{30} \right) + 84 \left( \frac{1}{42} \right) + 9B_8,$$

$$0 = \frac{3}{10} + 9B_8,$$

$$\text{Thus } B_8 = -\frac{1}{30}.$$

(4) Again,

$$\begin{aligned}
0 &= 1 + 11B_1 + 55B_2 + 165B_3 + 330B_4 + 462B_5 + 462B_6 + \\
&\quad 330B_7 + 165B_8 + 55B_9 + 11B_{10}, \\
0 &= 1 + 11B_1 + 55B_2 + 330B_4 + 462B_6 + 165B_8 + 11B_{10}, \\
0 &= 1 + 11 \left( -\frac{1}{2} \right) + 55 \left( \frac{1}{6} \right) + 330 \left( -\frac{1}{30} \right) + 462 \left( \frac{1}{42} \right) + \\
&\quad 165 \left( -\frac{1}{30} \right) + 11B_{10}, \\
0 &= -\frac{5}{6} + 11B_{10},
\end{aligned}$$

$$\text{Thus } B_{10} = \frac{5}{66}.$$

(4) Finally,

$$\begin{aligned}
0 &= 1 + 13B_1 + 78B_2 + 286B_3 + 715B_4 + 1287B_5 + 1716B_6 + \\
&\quad 1716B_7 + 1287B_8 + 715B_9 + 286B_{10} + 78B_{11} + 13B_{12}, \\
0 &= 1 + 13B_1 + 78B_2 + 715B_4 + 1716B_6 + 1287B_8 + 286B_{10} + 13B_{12}, \\
0 &= 1 + 13 \left( -\frac{1}{2} \right) + 78 \left( \frac{1}{6} \right) + 715 \left( -\frac{1}{30} \right) + 1716 \left( \frac{1}{42} \right) + \\
&\quad 1287 \left( -\frac{1}{30} \right) + 286 \left( \frac{5}{66} \right) + 13B_{12}, \\
0 &= \frac{691}{210} + 13B_{12},
\end{aligned}$$

$$\text{Thus } B_{12} = -\frac{691}{2730}.$$

□

**Exercise 15.2.** If  $a \in \mathbb{Z}$ , show  $a(a^m - 1)B_m \in \mathbb{Z}$  for all  $m > 0$ .

*Proof.*

(1) *Trivial cases.* If  $m = 1$ ,  $a(a - 1)B_1 = -\frac{1}{2}a(a - 1) \in \mathbb{Z}$  for any  $a \in \mathbb{Z}$ . For odd  $m > 1$ ,  $B_m = 0$  or  $a(a^m - 1)B_m = 0 \in \mathbb{Z}$  (Proposition 15.1.1).

(2) *Consider that  $m > 1$  and even.* By Theorem 3,

$$B_{2m} + \sum_{p-1 \mid 2m} \frac{1}{p} \in \mathbb{Z}$$

where the sum is over all primes  $p$  such that  $p - 1 \mid 2m$ . So it suffices to show

$$a(a^{2m} - 1) \sum_{p-1 \mid 2m} \frac{1}{p} \in \mathbb{Z},$$

or

$$a(a^{2m} - 1) \frac{1}{p} \in \mathbb{Z}$$

for any  $a \in \mathbb{Z}$  and any prime  $p$  such that  $p - 1 \mid 2m$ .

- (3) Consider all possible  $a$ . If  $p \mid a$ , it is trivial. If  $p \nmid a$ ,  $a^{p-1} \equiv 1 \pmod{p}$  by Fermat's Little Theorem, or  $a^{2m} \equiv 1 \pmod{p}$  by  $p - 1 \mid 2m$ . In any cases,  $a(a^{2m} - 1) \frac{1}{p} \in \mathbb{Z}$ .

□

**Exercise 15.6.** For  $m \geq 3$ , show  $|B_{2m+2}| > |B_{2m}|$ . (Hint: Use Theorem 2.)

*Proof.* By Theorem 2,

$$2\zeta(2m) = (-1)^{m+1} \frac{(2\pi)^{2m}}{(2m)!} B_{2m}.$$

Thus,

$$\frac{|B_{2m+2}|}{|B_{2m}|} = \frac{\zeta(2m+2)(2m+2)(2m+1)}{\zeta(2m)(2\pi)^2} > \frac{1 \cdot 8 \cdot 7}{\zeta(6) \cdot (2\pi)^2} = \frac{13230}{\pi^8} > 1,$$

or  $|B_{2m+2}| > |B_{2m}|$ . □

**Exercise 15.8.** Consider the power series expansion of  $\tan x$  about the origin;

$$\sum_{k=1}^{\infty} T_k \frac{x^{2k-1}}{(2k-1)!}.$$

Show

$$T_k = (-1)^{k-1} \frac{B_{2k}}{2k} (2^{2k} - 1) 2^{2k}.$$

Note that  $T_k \in \mathbb{Z}$  for all  $k$  by Exercise 3.

*Proof.*

- (1) By the equation (6) on page 232,

$$x \cot x = 1 + \sum_{k=2}^{\infty} B_k \frac{(2ix)^k}{k!}.$$

Since  $B_k = 0$  for odd  $k > 1$ ,

$$x \cot x = 1 + \sum_{k=1}^{\infty} B_{2k} \frac{(2ix)^{2k}}{(2k)!} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} x^{2k},$$

or

$$\cot x = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} x^{2k-1}.$$

Combine the first term  $\frac{1}{x}$  into the summation,

$$\cot x = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} x^{2k-1}.$$

(2) Note that  $\tan x = \cot x - 2 \cot(2x)$ . By (1),

$$\begin{aligned} \tan x &= \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} x^{2k-1} - 2 \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} (2x)^{2k-1} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (1 - 2^{2k}) 2^{2k} B_{2k}}{(2k)!} x^{2k-1} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k (1 - 2^{2k}) 2^{2k} B_{2k}}{(2k)!} x^{2k-1}. \end{aligned}$$

Write  $T_k = (-1)^{k-1} (2^{2k} - 1) 2^{2k} \frac{B_{2k}}{2k}$ . Therefore,  $\tan x = \sum_{k=1}^{\infty} T_k \frac{x^{2k-1}}{(2k-1)!}$ .

By Exercise 15.3,  $(2^{2k} - 1) 2^{2k} \frac{B_{2k}}{2k} \in \mathbb{Z}$ , or  $T_k \in \mathbb{Z}$  for all  $k$ .  $\square$

**Exercise 15.12.** Recall the definition of the Bernoulli polynomials;

$$B_m(x) = \sum_{k=0}^m \binom{m}{k} B_k x^{m-k}.$$

Show that

$$\frac{te^{tx}}{e^t - 1} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!}.$$

*Proof.* By Lemma 1,

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}.$$

So

$$\frac{te^{tx}}{e^t - 1} = \left( \sum_{m=0}^{\infty} B_m \frac{t^m}{m!} \right) \left( \sum_{m=0}^{\infty} x^m \frac{t^m}{m!} \right).$$

Write  $\frac{te^{tx}}{e^t-1} = \sum_{m=0}^{\infty} b_m(x) \frac{t^m}{m!}$  and we want to check if  $b_m(x) = B_m(x)$  or not. The result is established if  $b_m(x) = B_m(x)$  holds. Equating coefficients of  $t^m$  gives

$$\begin{aligned} \frac{b_m(x)}{m!} &= \sum_{k=0}^m \frac{B_k x^{m-k}}{k!(m-k)!}, \\ b_m(x) &= \sum_{k=0}^m \frac{m!}{k!(m-k)!} B_k x^{m-k} \\ &= \sum_{k=0}^m \binom{m}{k} B_k x^{m-k} \\ &= B_m(x). \end{aligned}$$

□

**Exercise 15.13.** Show  $B_m(x+1) - B_m(x) = mx^{m-1}$ .

*Proof.* Let  $f(t, x) = \frac{te^{tx}}{e^t-1}$ .

(1)

$$f(t, x+1) - f(t, x) = \frac{te^{t(x+1)}}{e^t-1} - \frac{te^{tx}}{e^t-1} = te^{tx}.$$

Expand  $te^{tx}$  in a power series about the origin as follows

$$\begin{aligned} te^{tx} &= t \sum_{m=0}^{\infty} x^m \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} x^m \frac{t^{m+1}}{m!} \\ &= \sum_{m=1}^{\infty} x^{m-1} \frac{t^m}{(m-1)!} \\ &= \sum_{m=1}^{\infty} mx^{m-1} \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} mx^{m-1} \frac{t^m}{m!}. \end{aligned}$$

So,

$$f(t, x+1) - f(t, x) = \sum_{m=0}^{\infty} mx^{m-1} \frac{t^m}{m!}.$$

(2) By Exercise 15.12,

$$\begin{aligned} f(t, x+1) - f(t, x) &= \sum_{m=0}^{\infty} B_m(x+1) \frac{t^m}{m!} - \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} (B_m(x+1) - B_m(x)) \frac{t^m}{m!}. \end{aligned}$$

By (1)(2), comparing coefficients of  $t^m$  yields

$$mx^{m-1} = B_m(x+1) - B_m(x).$$

□

**Exercise 15.14.** Use Exercise 13 to give a new proof of Theorem 1:

$$S_m(n) = \frac{1}{m+1} (B_{m+1}(n) - B_{m+1}).$$

*Proof.* By Exercise 13,

$$B_{m+1}(k) - B_{m+1}(k-1) = (m+1)(k-1)^m$$

for any  $k$ . So,

$$\begin{aligned} \sum_{k=1}^n (B_{m+1}(k) - B_{m+1}(k-1)) &= \sum_{k=1}^n (m+1)(k-1)^m, \\ B_{m+1}(n) - B_{m+1}(0) &= (m+1)S_m(n). \end{aligned}$$

Note that  $B_{m+1}(0) = B_{m+1}$  for any  $m$ . So Theorem 1 is established by a new proof. □

**Exercise 15.15.** Suppose  $f(x) = \sum_{k=0}^n a_k x^k$  be a polynomial with complex coefficients. Use Exercise 13 to find a polynomial  $F(x)$  such that  $F(x+1) - F(x) = f(x)$ .

*Proof.* By Exercise 15.13,

$$x^k = \frac{1}{k+1} (B_{k+1}(x+1) - B_{k+1}(x))$$

for  $k \geq 0$ . Thus,

$$\begin{aligned} f(x) &= \sum_{k=0}^n a_k x^k \\ &= \sum_{k=0}^n a_k \cdot \frac{1}{k+1} (B_{k+1}(x+1) - B_{k+1}(x)) \\ &= \sum_{k=0}^n \frac{a_k}{k+1} B_{k+1}(x+1) - \sum_{k=0}^n \frac{a_k}{k+1} B_{k+1}(x). \end{aligned}$$

Let

$$F(x) = \sum_{k=0}^n \frac{a_k}{k+1} B_{k+1}(x),$$

and we get  $f(x) = F(x+1) - F(x)$ .  $\square$

**Exercise 15.16.** For  $n \geq 1$ , show  $\frac{d}{dx} B_n(x) = n B_{n-1}(x)$ .

*Proof.* For  $n \geq 1$ ,

$$\frac{d}{dx} B_n(x) = \sum_{k=0}^n (n-k) \binom{n}{k} B_k x^{n-k-1} = \sum_{k=0}^{n-1} (n-k) \binom{n}{k} B_k x^{n-k-1}.$$

Note that

$$(n-k) \binom{n}{k} = n \binom{n-1}{k}.$$

So

$$\begin{aligned} \frac{d}{dx} B_n(x) &= \sum_{k=0}^{n-1} n \binom{n-1}{k} B_k x^{n-k-1} \\ &= n \sum_{k=0}^{n-1} \binom{n-1}{k} B_k x^{n-k-1} \\ &= n B_{n-1}(x). \end{aligned}$$

$\square$

**Exercise 15.17.** Show  $B_n(1-x) = (-1)^n B_n(x)$ .

*Proof.* Let  $f(t, x) = \frac{te^{tx}}{e^t - 1}$ .

$$(1) \quad f(t, 1-x) = f(-t, x).$$

$$f(t, 1-x) = \frac{te^{t(1-x)}}{e^t - 1} = e^t \cdot \frac{te^{-tx}}{e^t - 1} = \frac{-te^{-tx}}{e^{-t} - 1} = f(-t, x).$$



(2) By Exercise 15.12,

$$f(t, 1-x) = \sum_{n=0}^{\infty} B_n(1-x) \frac{t^n}{n!}$$

$$f(-t, x) = \sum_{n=0}^{\infty} (-1)^n B_n(x) \frac{t^n}{n!}.$$

By (1), comparing coefficients of  $t^n$  yields  $B_n(1-x) = (-1)^n B_n(x)$ .

□

**Exercise 15.18.** Use Exercise 13 and 17 to give a new proof that  $B_n = 0$  for  $n$  odd and  $n > 1$ .

*Proof.*

- (1)  $B_m(1) - B_m(0) = 0$  for any  $m > 1$ . Taking  $x = 0$  in Exercise 15.13 and keeping  $m-1 > 0$  or  $m > 1$ .
- (2)  $B_m(1) = -B_m(0)$  for any odd  $m$ . Taking  $x = 0$  in Exercise 15.17 and keeping  $m$  is odd.

$$f(t, 1-x) = \sum_{n=0}^{\infty} B_n(1-x) \frac{t^n}{n!}$$

$$f(-t, x) = \sum_{n=0}^{\infty} (-1)^n B_n(x) \frac{t^n}{n!}.$$

By (1)(2), for  $m$  odd and  $m > 1$ ,  $B_m(0) = 0$  or  $B_m = 0$ . □

**Exercise 15.19 (Multiplication theorem for Bernoulli polynomial).** Suppose  $n$  and  $F$  are integers and  $n, F > 0$ . Show that

$$B_n(Fx) = F^{n-1} \sum_{a=0}^{F-1} B_n\left(x + \frac{a}{F}\right).$$

(Hint: Use Exercise 12.)

*Proof.* By  $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + y^{n-1})$  (Exercise 1.24),

$$e^{Ft} - 1 = (e^t - 1)(1 + e^t + e^{2t} + \cdots + e^{(F-1)t}) = (e^t - 1) \sum_{a=0}^{F-1} e^{at}.$$

So,

$$\begin{aligned}
\frac{1}{e^t - 1} &= \frac{1}{e^{Ft} - 1} \sum_{a=0}^{F-1} e^{at}, \\
\frac{te^{tFx}}{e^t - 1} &= \frac{te^{tFx}}{e^{Ft} - 1} \sum_{a=0}^{F-1} e^{at} \\
&= \sum_{a=0}^{F-1} \frac{te^{(Fx+a)t}}{e^{Ft} - 1} \\
&= \sum_{a=0}^{F-1} \frac{te^{(Fx+a)t}}{e^{Ft} - 1} \\
&= \sum_{a=0}^{F-1} F^{-1} \frac{(Ft)e^{(x+\frac{a}{F})(Ft)}}{e^{Ft} - 1}.
\end{aligned}$$

By Exercise 15.12,

$$\begin{aligned}
\sum_{n=0}^{\infty} B_n(Fx) \frac{t^n}{n!} &= \sum_{a=0}^{F-1} F^{-1} \sum_{n=0}^{\infty} B_n\left(x + \frac{a}{F}\right) \frac{(Ft)^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{a=0}^{F-1} F^{-1} B_n\left(x + \frac{a}{F}\right) \frac{(Ft)^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{a=0}^{F-1} F^{n-1} B_n\left(x + \frac{a}{F}\right) \frac{t^n}{n!}.
\end{aligned}$$

Comparing coefficients of  $t^n$  on the both sides of the above equation and yields  $B_n(Fx) = F^{n-1} \sum_{a=0}^{F-1} B_n\left(x + \frac{a}{F}\right)$ .  $\square$

**Supplement 15.12.1 (Multiplication Theorem for  $\frac{1}{\exp(z)-1}$ ).**

$$\frac{1}{\exp(nz) - 1} = \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{\exp\left(z + \frac{2k\pi i}{n}\right) - 1}.$$

*Proof.* Let  $\zeta$  be one  $n$ -th root of unity. Write  $f(x) = x^n - 1 = \prod_{k=0}^{n-1} (x - \zeta^k)$ .

By Lagrange interpolation,

$$\begin{aligned}\frac{1}{f(x)} &= \sum_{k=0}^{n-1} \frac{1}{f'(\zeta^k)} \cdot \frac{1}{x - \zeta^k} \\ \frac{1}{x^n - 1} &= \sum_{k=0}^{n-1} \frac{1}{n\zeta^{-k}} \cdot \frac{1}{x - \zeta^k} \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \frac{\zeta^k}{x - \zeta^k}.\end{aligned}$$

Let  $x = \exp(z)$ ,  $\zeta = \exp(-\frac{2\pi i}{n})$ .  $\square$

**Supplement 15.12.2 (Multiplication theorem for  $\cot z$ .)**

$$\cot z = \frac{1}{n} \sum_{k=0}^{n-1} \cot \frac{z + k\pi}{n}.$$

This equation yields  $x \cot x = 1 - 2 \sum_{n=1}^{\infty} \frac{x^2}{n^2 \pi^2 - x^2}$  again.

*Proof.* By Supplement 15.12.1,

$$\begin{aligned}\frac{1}{\exp(z) - 1} &= \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{\exp(\frac{z+2k\pi i}{n}) - 1} \\ \frac{1}{\exp(2iz) - 1} &= \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{\exp(\frac{2i(z+k\pi)}{n}) - 1}.\end{aligned}$$

Notice that  $\cot z = i + \frac{2i}{e^{2iz} - 1}$ ,  $\cot z = \frac{1}{n} \sum_{k=0}^{n-1} \cot \frac{z+k\pi}{n}$ .  $\square$

**Supplement 15.12.3 (Multiplication theorem for Gamma function) (Gauss's multiplication formula).**

$$\Gamma(z) \Gamma\left(z + \frac{1}{k}\right) \Gamma\left(z + \frac{2}{k}\right) \cdots \Gamma\left(z + \frac{k-1}{k}\right) = (2\pi)^{\frac{k-1}{2}} k^{\frac{1-2kz}{2}} \Gamma(kz).$$

**Exercise 15.20.** Suppose  $H(x)$  is a polynomial of degree  $n$  with complex coefficients. Suppose that for all integers  $n$ ,  $F > 0$  we have  $H(Fx) = F^{n-1} \sum_{a=0}^{F-1} H(x + \frac{a}{F})$ . Show that  $H(x) = CB_n(x)$  for some constant  $C$ . (Hint: Use Exercise 16 and induction on  $n$ .)

Use induction on  $n$  to show that  $H(x) = CB_n(x)$  where  $C$  is the leading coefficient of  $H(x)$  (since the leading coefficient of every Bernoulli polynomial is 1).

- (1) As  $n = 1$ , write  $H(x) = C_1x + C_0 \in \mathbb{C}$ . Then

$$\begin{aligned} H(Fx) &= \sum_{a=0}^{F-1} H\left(x + \frac{a}{F}\right), \\ C_1Fx + C_0 &= \sum_{a=0}^{F-1} \left(C_1\left(x + \frac{a}{F}\right) + C_0\right) \\ &= C_1Fx + C_1 \cdot \frac{F-1}{2} + C_0F, \\ C_0 &= \frac{-1}{2}C_1 \end{aligned}$$

if  $F > 1$ . That is,  $H(x) = C_1B_1(x)$  where  $C = C_1$  is a constant. In fact,  $C$  is the leading coefficient of  $H(x)$ .

- (2) Assume that the conclusion holds for  $n = k$ . As  $n = k + 1$ , it suffices to show  $f(x) = H(x) - CB_{k+1}(x) = 0$ , where  $C$  is the leading coefficient of  $H(x)$ .
- (3) Differentiate  $f(x) = H(x) - CB_{k+1}(x)$  and use Exercise 15.16,

$$f'(x) = H'(x) - C \cdot (k+1) \cdot B_k(x).$$

Might show  $f'(x) = 0$  and then get that  $H(x) - CB_{k+1}(x)$  is a constant.

- (4) Notice that the leading coefficient of  $H'(x)$  is  $C \cdot (k+1)$ . Besides, by differentiating  $H(Fx) = F^k \sum_{a=0}^{F-1} H\left(x + \frac{a}{F}\right)$ ,

$$\begin{aligned} H'(Fx) \cdot F &= F^k \sum_{a=0}^{F-1} H'\left(x + \frac{a}{F}\right), \\ H'(Fx) &= F^{k-1} \sum_{a=0}^{F-1} H'\left(x + \frac{a}{F}\right). \end{aligned}$$

By the induction hypothesis,  $H'(x) = (C \cdot (k+1))B_k(x)$  since  $H'(x)$  has degree  $(k+1) - 1 = k$ . Therefore,  $f'(x) = 0$  or  $f(x) = H(x) - CB_{k+1}(x) = A$  is a constant.

(5) By  $f(Fx) = H(Fx) - CB_{k+1}(Fx) = A$ ,

$$\begin{aligned}
A &= F^k \sum_{a=0}^{F-1} H\left(x + \frac{a}{F}\right) - CF^k \sum_{a=0}^{F-1} B_{k+1}\left(x + \frac{a}{F}\right) \\
&= F^k \sum_{a=0}^{F-1} \left( H\left(x + \frac{a}{F}\right) - CB_{k+1}\left(x + \frac{a}{F}\right) \right) \\
&= F^k \sum_{a=0}^{F-1} A \\
&= F^{k+1} A,
\end{aligned}$$

or  $(F^{k+1} - 1)A = 0$ . For  $F > 1$ ,  $A = 0$ . That is,  $H(x) = CB_{k+1}(x)$ .

By mathematical induction the result is established.  $\square$

**Exercise 15.21.** Show  $2^{n-1}B_n(\frac{1}{2}) = (1 - 2^{n-1})B_n$ .

The original identity  $B_n(\frac{1}{2}) = (1 - 2^{n-1})B_n$  is wrong. For  $n = 2$ ,  $B_2(x) = x^2 - x + \frac{1}{6}$  and thus  $-\frac{1}{12} = B_2(\frac{1}{2}) \neq (1 - 2^{2-1})B_2 = -\frac{1}{6}$ .

*Proof.* Taking  $F = 2$  in Exercise 15.19,

$$\begin{aligned}
B_n(2x) &= 2^{n-1} \sum_{a=0}^1 B_n\left(x + \frac{a}{2}\right) \\
&= 2^{n-1}B_n(x) + 2^{n-1}B_n\left(x + \frac{1}{2}\right).
\end{aligned}$$

Let  $x = 0$ ,

$$B_n(0) = 2^{n-1}B_n(0) + 2^{n-1}B_n\left(\frac{1}{2}\right),$$

So

$$2^{n-1}B_n\left(\frac{1}{2}\right) = (1 - 2^{n-1})B_n(0) = (1 - 2^{n-1})B_n.$$

$\square$

**Exercise 15.22.** More generally, show that  $(1 - F^{n-1})B_n = F^{n-1} \sum_{a=1}^{F-1} B_n(\frac{a}{F})$ .

The original identity  $(1 - F^{n-1})B_n = \sum_{a=1}^{F-1} B_n(\frac{a}{F})$  is wrong again.

*Proof.* Let  $x = 0$  in Exercise 15.19,

$$B_n(0) = F^{n-1} \sum_{a=0}^{F-1} B_n\left(\frac{a}{F}\right) = F^{n-1}B_n(0) + F^{n-1} \sum_{a=1}^{F-1} B_n\left(\frac{a}{F}\right),$$

So

$$F^{n-1} \sum_{a=1}^{F-1} B_n \left( \frac{a}{F} \right) = (1 - F^{n-1}) B_n(0) = (1 - F^{n-1}) B_n.$$

□