Chapter 4: Determinants

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Section 4.1: Determinants of Order 2

Exercise 4.1.1. *Label the following statements as being true or false.*

  (a) *The function* $\det : \mathsf{M}_{2\times 2}(F) \to F$ *is a linear transformation.*

  (b) *The determinant of a 2×2 matrix is a linear function of each row of the matrix when the other row is held fixed.*

  (c) *If $A \in \mathsf{M}_{2\times 2}(F)$ and $\det(A) = 0$, then A is invertible.*

  (d) *If u and v are vectors in $\mathbb{R}^2$ emanating from the origin, then the area of the parallelogram having u and v as adjacent side is*

$$\det \begin{pmatrix} u \\ v \end{pmatrix}.$$

  (e) *A coordinate system is right-handed if and only if its orientation equals 1.*

Proof of (a). False. Example 4.1.1, or take

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathsf{M}_{2\times 2}(F) \text{ and } B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \mathsf{M}_{2\times 2}(F).$$

Then $\det(A + B) = \det(I_2) = 1 \neq 0 = 0 + 0 = \det(A) + \det(B)$. $\square$

Proof of (b). True. Proposition 4.1. $\square$

Proof of (c). False. Proposition 4.2. $\square$

Proof of (d). False. The area should be

$$O \begin{pmatrix} u \\ v \end{pmatrix} \cdot \det \begin{pmatrix} u \\ v \end{pmatrix} = \left| \det \begin{pmatrix} u \\ v \end{pmatrix} \right|.$$

$\square$

Proof of (e). True. See Exercise 4.1.12. $\square$

Exercise 4.1.2. *Compute the determinants of the following elements of* $\mathsf{M}_{2\times 2}(\mathbb{R})$.

(a) $\begin{pmatrix} 6 & -3 \\ 2 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} -5 & 2 \\ 6 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 8 & 0 \\ 3 & -1 \end{pmatrix}$

Proof of (a).

$$\det \begin{pmatrix} 6 & -3 \\ 2 & 4 \end{pmatrix} = 6 \cdot 4 - (-3) \cdot 2 = 24 + 6 = 30.$$

□

Proof of (b).

$$\det \begin{pmatrix} -5 & 2 \\ 6 & 1 \end{pmatrix} = (-5) \cdot 1 - 2 \cdot 6 = -5 - 12 = -17.$$

□

Proof of (c).

$$\det \begin{pmatrix} 8 & 0 \\ 3 & -1 \end{pmatrix} = 8 \cdot (-1) - 0 \cdot 3 = -8.$$

□

Exercise 4.1.3. *Compute the determinants of the following elements of* $\mathsf{M}_{2\times 2}(\mathbb{C})$.

(a) $\begin{pmatrix} -1 + i & 1 - 4i \\ 3 + 2i & 2 - 3i \end{pmatrix}$

(b) $\begin{pmatrix} 5 - 2i & 6 + 4i \\ -3 + i & 7i \end{pmatrix}$

(c) $\begin{pmatrix} 2i & 3 \\ 4 & 6i \end{pmatrix}$

Proof of (a).

$$\det \begin{pmatrix} -1 + i & 1 - 4i \\ 3 + 2i & 2 - 3i \end{pmatrix} = (-1 + i) \cdot (2 - 3i) - (1 - 4i) \cdot (3 + 2i)$$
$$= (1 + 5i) - (11 - 10i)$$
$$= -10 + 15i.$$

□

Proof of (b).

$$\det \begin{pmatrix} 5-2i & 6+4i \\ -3+i & 7i \end{pmatrix} = (5-2i)\cdot(7i) - (6+4i)\cdot(-3+i)$$
$$= (14+35i) - (-22-6i)$$
$$= 36+41i.$$

□

Proof of (c).

$$\det \begin{pmatrix} 2i & 3 \\ 4 & 6i \end{pmatrix} = (2i)\cdot(6i) - 3\cdot4 = -12 - 12 = -24.$$

□

Exercise 4.1.4. *For each of the following pairs of vectors u and v in $\mathbb{R}^2$, compute the area of the parallelogram determined by u and v.*

(a) $u = (3, -2)$ *and* $v = (2, 5)$

(b) $u = (1, 3)$ *and* $v = (-3, 1)$

(c) $u = (4, -1)$ *and* $v = (-6, -2)$

(d) $u = (3, 4)$ *and* $v = (2, -6)$

Proof of (a).
$$\left| \det \begin{pmatrix} 3 & -2 \\ 2 & 5 \end{pmatrix} \right| = |19| = 19.$$

□

Proof of (b).
$$\left| \det \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix} \right| = |10| = 10.$$

□

Proof of (c).
$$\left| \det \begin{pmatrix} 4 & -1 \\ -6 & -2 \end{pmatrix} \right| = |-14| = 14.$$

□

Proof of (d).

$$\left| \det \begin{pmatrix} 3 & 4 \\ 2 & -6 \end{pmatrix} \right| = |-26| = 26.$$

□

Exercise 4.1.5. *Prove that if B is the matrix obtained by interchanging the rows of a 2×2 matrix A, then $\det(B) = -\det(A)$.*

Proof. Write

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{M}_{2 \times 2}(F).$$

Then

$$B = \begin{pmatrix} c & d \\ a & b \end{pmatrix} \in \mathsf{M}_{2 \times 2}(F).$$

Then $\det(B) = cb - ad = -(ad - bc) = -\det(A)$. □

Exercise 4.1.6. *Prove that if the two columns of $A \in \mathsf{M}_{2 \times 2}(F)$ are identical, then $\det(A) = 0$.*

Proof. By assumption, write

$$A = \begin{pmatrix} a & a \\ c & c \end{pmatrix} \in \mathsf{M}_{2 \times 2}(F).$$

Then $\det(A) = ac - ac = 0$. □

Exercise 4.1.7. *Prove that for any $A \in \mathsf{M}_{2 \times 2}(F)$, $\det(A^t) = \det(A)$.*

Proof. Write

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{M}_{2 \times 2}(F),$$

then

$$A^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathsf{M}_{2 \times 2}(F).$$

So $\det(A) = ad - bc = ad - cb = \det(A^t)$. □

Exercise 4.1.8. *Prove that if $A \in \mathsf{M}_{2 \times 2}(F)$ is upper triangular, then $\det(A)$ equals the product of the diagonal entries of A.*

Proof. Write

$$A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathsf{M}_{2\times 2}(F)$$

since A is upper triangular. Then $\det(A) = ad$, which is equal to the product of the diagonal entries, a and d, of A. $\square$

Exercise 4.1.9. *Prove that for any* $A, B \in \mathsf{M}_{2\times 2}(F)$ *we have* $\det(AB) = \det(A) \cdot \det(B)$.

Proof. Write

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{M}_{2\times 2}(F),$$

$$B = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in \mathsf{M}_{2\times 2}(F).$$

Then

$$AB = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} \in \mathsf{M}_{2\times 2}(F).$$

A direct calculation shows

$$
\begin{aligned}
\det(AB) &= (ae + bg)(cf + dh) - (af + bh)(ce + dg) \\
&= (acef + adeh + bcfg + bdgh) - (acef + adfg + bceh + bdgh) \\
&= adeh + bcfg - adfg - bceh \\
&= (ad - bc)(eh - fg) \\
&= \det(A)\det(B).
\end{aligned}
$$

$\square$

Exercise 4.1.10. *The **classical adjoint** of a* 2×2 *matrix* $A \in \mathsf{M}_{2\times 2}(F)$ *is the matrix*

$$C = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}.$$

Prove

(a) $CA = AC = [\det(A)]I$.

(b) $\det(C) = \det(A)$.

(c) *The classical adjoint of* A^t *is* C^t.

(d) *If* A *is invertible, then* $A^{-1} = [\det(A)]^{-1}C$.

Note that

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Proof of (a).

$$\begin{aligned} CA &= \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \\ &= \begin{pmatrix} A_{22}A_{11} - A_{12}A_{21} & A_{22}A_{12} - A_{12}A_{22} \\ -A_{21}A_{11} + A_{11}A_{21} & -A_{21}A_{12} + A_{11}A_{22} \end{pmatrix} \\ &= \begin{pmatrix} \det(A) & 0 \\ 0 & \det(A) \end{pmatrix} \\ &= [\det(A)]I. \end{aligned}$$

$$\begin{aligned} AC &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \\ &= \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & -A_{11}A_{12} + A_{12}A_{11} \\ A_{21}A_{22} - A_{22}A_{21} & -A_{21}A_{12} + A_{22}A_{11} \end{pmatrix} \\ &= \begin{pmatrix} \det(A) & 0 \\ 0 & \det(A) \end{pmatrix} \\ &= [\det(A)]I. \end{aligned}$$

□

Proof of (b).

$$\begin{aligned} \det(C) &= A_{22}A_{11} - (-A_{12})(-A_{21}) \\ &= A_{11}A_{22} - A_{12}A_{21} \\ &= \det(A). \end{aligned}$$

□

Proof of (c).

$$A^t = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix}.$$

The classical adjoint of A^t is

$$\begin{pmatrix} A_{22} & -A_{21} \\ -A_{12} & A_{11} \end{pmatrix} = C^t.$$

□

Proof of (d). Proposition 4.2. □

Exercise 4.1.11. *Let $\delta : \mathsf{M}_{2\times 2}(F) \to F$ be a function with the following three properties.*

   (i) *δ is a linear function of each row of the matrix when the other row is held fixed.*

   (ii) *If the two rows of $A \in \mathsf{M}_{2\times 2}(F)$ are identical, then $\delta(A) = 0$.*

   (iii) *If I is the 2×2 identity matrix, then $\delta(I) = 1$.*

Prove that $\delta(A) = \det(A)$ for all $A \in \mathsf{M}_{2\times 2}(F)$. (This result is generalized in Section 4.5.)

Proof. Write

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

   (1) *If u, v are elements of F^2 and k is a scalar, then*

$$\delta \begin{pmatrix} u \\ v + ku \end{pmatrix} = \delta \begin{pmatrix} u + kv \\ v \end{pmatrix} = \delta \begin{pmatrix} u \\ v \end{pmatrix}.$$

   In fact,

$$
\begin{aligned}
\delta \begin{pmatrix} u \\ v + ku \end{pmatrix} &= \delta \begin{pmatrix} u \\ v \end{pmatrix} + \delta \begin{pmatrix} u \\ ku \end{pmatrix} && \text{(Property (i))} \\
&= \delta \begin{pmatrix} u \\ v \end{pmatrix} + k\delta \begin{pmatrix} u \\ u \end{pmatrix} && \text{(Property (i))} \\
&= \delta \begin{pmatrix} u \\ v \end{pmatrix}. && \text{(Property (ii))}
\end{aligned}
$$

   Similarly, $\delta \begin{pmatrix} u + kv \\ v \end{pmatrix} = \delta \begin{pmatrix} u \\ v \end{pmatrix}.$

   (2) *If u, v are elements of F^2, then*

$$\delta \begin{pmatrix} u \\ v \end{pmatrix} = -\delta \begin{pmatrix} v \\ u \end{pmatrix}.$$

   In fact,

$$
\begin{aligned}
0 &= \delta \begin{pmatrix} u + v \\ u + v \end{pmatrix} && \text{(Property (ii))} \\
&= \delta \begin{pmatrix} u + v \\ u \end{pmatrix} + \delta \begin{pmatrix} u + v \\ v \end{pmatrix} && \text{(Property (i))} \\
&= \delta \begin{pmatrix} v \\ u \end{pmatrix} + \delta \begin{pmatrix} u \\ v \end{pmatrix}. && \text{((1))}
\end{aligned}
$$

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(3) *If v is an element of F^2, then*

$$\delta\begin{pmatrix} 0 \\ v \end{pmatrix} = 0.$$

In fact,

$$\delta\begin{pmatrix} 0 \\ v \end{pmatrix} = \delta\begin{pmatrix} 0+0 \\ v \end{pmatrix}$$

$$= \delta\begin{pmatrix} 0 \\ v \end{pmatrix} + \delta\begin{pmatrix} 0 \\ v \end{pmatrix}. \qquad\qquad \text{(Property (i))}$$

In particular, $\delta\begin{pmatrix} 0 \\ v \end{pmatrix} = 0 = \det\begin{pmatrix} 0 \\ v \end{pmatrix}$.

(4) To show $\delta(A) = \det(A)$, we consider three possible cases about the first row: $A_{11} \neq 0$, $A_{12} \neq 0$, or $A_{11} = A_{12} = 0$. The case $A_{11} = A_{12} = 0$ is proved in (3). We prove the rest two cases in (5) and (6). Write

$$u = (A_{11}, A_{12}) \text{ and } v = (A_{21}, A_{22}).$$

(5) *Show that $\delta(A) = \det(A)$ if $A_{11} \neq 0$.* So

$$\delta(A) = \delta\begin{pmatrix} u \\ v \end{pmatrix}$$

$$= \delta\begin{pmatrix} u \\ v - \frac{A_{21}}{A_{11}}u \end{pmatrix} \qquad\qquad\qquad ((1))$$

$$= \delta\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} - \frac{A_{12}A_{21}}{A_{11}} \end{pmatrix}$$

$$= \left(A_{22} - \frac{A_{12}A_{21}}{A_{11}}\right)\delta\begin{pmatrix} A_{11} & A_{12} \\ 0 & 1 \end{pmatrix} \qquad \text{(Property (i))}$$

$$= \left(A_{22} - \frac{A_{12}A_{21}}{A_{11}}\right)\delta\begin{pmatrix} A_{11} & 0 \\ 0 & 1 \end{pmatrix} \qquad\qquad ((1))$$

$$= A_{11}\left(A_{22} - \frac{A_{12}A_{21}}{A_{11}}\right)\delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{(Property (i))}$$

$$= \det(A)\delta(I)$$

$$= \det(A). \qquad\qquad\qquad\qquad\qquad \text{(Property (iii))}$$

(6) *Show that $\delta(A) = \det(A)$ if $A_{12} \neq 0$. So*

$$\delta(A) = \delta\begin{pmatrix} u \\ v \end{pmatrix}$$

$$= \delta\begin{pmatrix} u \\ v - \frac{A_{22}}{A_{12}}u \end{pmatrix} \qquad\qquad ((1))$$

$$= \delta\begin{pmatrix} A_{11} & A_{12} \\ A_{21} - \frac{A_{22}A_{11}}{A_{12}} & 0 \end{pmatrix}$$

$$= \left(A_{21} - \frac{A_{22}A_{11}}{A_{12}}\right)\delta\begin{pmatrix} A_{11} & A_{12} \\ 1 & 0 \end{pmatrix} \qquad\qquad \text{(Property (i))}$$

$$= \left(A_{21} - \frac{A_{22}A_{11}}{A_{12}}\right)\delta\begin{pmatrix} 0 & A_{12} \\ 1 & 0 \end{pmatrix} \qquad\qquad ((1))$$

$$= A_{12}\left(A_{21} - \frac{A_{22}A_{11}}{A_{12}}\right)\delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad\qquad \text{(Property (i))}$$

$$= -A_{12}\left(A_{21} - \frac{A_{22}A_{11}}{A_{12}}\right)\delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad\qquad ((2))$$

$$= \det(A)\delta(I)$$

$$= \det(A). \qquad\qquad \text{(Property (iii))}$$

□

Exercise 4.1.12. *Let $\{u, v\}$ be an ordered basis for $\mathbb{R}^2$. Prove that*

$$O\begin{pmatrix} u \\ v \end{pmatrix} = 1$$

if and only if $\{u, v\}$ forms a right-handed coordinate system. (Hint: Recall the definition of a rotation given in Example 2.1.2.)

If $\beta = \{u, v\}$ is an ordered basis for $\mathbb{R}^2$, define the orientation of β as

$$O\begin{pmatrix} u \\ v \end{pmatrix} = \frac{\det\begin{pmatrix} u \\ v \end{pmatrix}}{\left|\det\begin{pmatrix} u \\ v \end{pmatrix}\right|}.$$

A coordinate system $\{u, v\}$ is called right-handed if u can be rotated in a counterclockwise direction through an angle θ $(0 < \theta < \pi)$ to coincide with v.

Example 2.1.2. For any angle θ, define $\mathsf{T}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\mathsf{T}_\theta(a_1, a_2) = (a_1\cos\theta - a_2\sin\theta, a_1\sin\theta + a_2\cos\theta).$$

T_θ is called the rotation by θ.

Proof.

(1) By Example 2.1.2, for any coordinate system $\{u, v\}$, there is $0 < \theta < 2\pi$ and $\alpha > 0$ such that $v = \alpha \mathsf{T}_\theta(u)$. Write $u = (u_1, u_2) \in \mathbb{R}^2, v = (v_1, v_2) \in \mathbb{R}^2$.

(2) Calculate $\det \begin{pmatrix} u \\ v \end{pmatrix}$.

$$\det \begin{pmatrix} u \\ v \end{pmatrix} = \det \begin{pmatrix} u \\ \alpha \mathsf{T}_\theta(u) \end{pmatrix}$$
$$= \alpha \det \begin{pmatrix} u \\ \mathsf{T}_\theta(u) \end{pmatrix}$$
$$= \alpha \det \begin{pmatrix} u_1 & u_2 \\ u_1 \cos\theta - u_2 \sin\theta & u_1 \sin\theta + u_2 \cos\theta \end{pmatrix}$$
$$= \alpha(u_1^2 + u_2^2) \sin\theta.$$

(3)

$$O \begin{pmatrix} u \\ v \end{pmatrix} = 1 \iff \det \begin{pmatrix} u \\ v \end{pmatrix} = \alpha(u_1^2 + u_2^2) \sin\theta > 0$$
$$\iff \sin\theta > 0$$
$$\iff 0 < \theta < \pi$$
$$\iff \{u, v\} \text{ is a right-handed coordinate system.}$$

$\square$

Section 4.3: Properties of Determinants

Exercise 4.3.11. *A matrix $Q \in \mathsf{M}_{n\times n}(\mathbb{R})$ is called orthogonal if $QQ^t = I$. Prove that if Q is orthogonal, then $\det(Q) = \pm 1$.*

Proof. By the orthogonality of Q, $QQ^t = I$. So

$$QQ^t = I \implies \det(QQ^t) = \det(I)$$
$$\iff \det(Q)\det(Q^t) = \det(I) \qquad \text{(Theorem 4.7)}$$
$$\iff \det(Q)\det(Q) = \det(I) \qquad \text{(Theorem 4.8)}$$
$$\iff \det(Q)^2 = 1 \qquad \text{(Example 4.2.4)}$$
$$\iff \det(Q) = \pm 1.$$

□

Exercise 4.3.14. *Prove that if $A, B \in \mathsf{M}_{n \times n}(F)$ are similar, then $\det(A) = \det(B)$.*

Proof. Since A, B are similar, there exists an invertible matrix Q such that $B = Q^{-1}AQ$. So

$$
\begin{aligned}
\det(B) &= \det\big(Q^{-1}AQ\big) \\
&= \det\big(Q^{-1}\big)\det(A)\det(Q) && \text{(Theorem 4.7)} \\
&= \det(Q)\det\big(Q^{-1}\big)\det(A) && \text{(F is field)} \\
&= \det\big(QQ^{-1}\big)\det(A) && \text{(Theorem 4.7)} \\
&= \det(I)\det(A) \\
&= 1 \cdot \det(A) && \text{(Example 4.2.4)} \\
&= \det(A).
\end{aligned}
$$

□