Chapter 1: Rings and Ideals

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Exercise 1.1 *Let x be a nilpotent element of A. Show that $1 + x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit.*

Proof.

(1) Suppose $x^m = 0$ for some odd integer $m \geq 0$. Then

$$1 = 1 + x^m = (1 + x)(1 - x + x^2 - \cdots + (-1)^{m-1}x^{m-1}),$$

or $1 + x$ is a unit.

(2) If u is any unit and x is any nilpotent, $u + x = u \cdot (1 + u^{-1}x)$ is a product of two units (using that $u^{-1}x$ is nilpotent and applying (1)) and hence a unit again.

□