Notes on the book:
Apostol, Modular Functions and Dirichlet Series in Number Theory, 2nd edition

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Chapter 1: Elliptic functions

Exercise 1.11.

If $k \geq 2$ and $\tau \in H$ prove that the Eisenstein series

$$G_{2k}(\tau) = \sum_{(m,n) \neq (0,0)} (m + n\tau)^{-2k}$$

has the Fourier expansion

$$G_{2k}(\tau) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) e^{2\pi i n \tau}.$$

Proof.

(1) Similar to Lemma 1.3 on page 19, we have

$$(2k-1)! \sum_{m=-\infty}^{+\infty} \frac{1}{(\tau+m)^{2k}} = (2\pi i)^{2k} \sum_{r=1}^{\infty} r^{2k-1} e^{2\pi i r \tau}.$$

(2) Similar to Theorem 1.18, we have

$$G_{2k}(\tau) = \sum_{(m,n) \neq (0,0)} (m + n\tau)^{-2k}$$

$$= \sum_{\substack{m=-\infty \\ m \neq 0 (n=0)}}^{+\infty} m^{-2k} + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{+\infty} \left((m + n\tau)^{-2k} + (m - n\tau)^{-2k} \right)$$

$$= 2\zeta(2k) + 2 \sum_{n=1}^{\infty} \sum_{m=-\infty}^{+\infty} (m + n\tau)^{-2k}$$

$$= 2\zeta(2k) + 2 \sum_{n=1}^{\infty} \frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{r=1}^{\infty} r^{2k-1} e^{2\pi i n r \tau}$$

$$= 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \underbrace{\sum_{d|n} d^{2k-1}}_{=\sigma_{2k-1}(n)} e^{2\pi i n \tau}.$$

In the last double sum we collect together those terms for which nr is constant.

$\square$