Chapter 5: Differentiation

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Exercise 5.1. *Let f be defined for all real x, and suppose that*

$$|f(x) - f(y)| \leq (x-y)^2$$

for all real x and y. Prove that f is a constant.

Proof.

(1) Write

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq |x - y|$$

if $x \neq y$.

(2) Given any $y \in \mathbb{R}$,

$$\left| \frac{f(x) - f(y)}{x - y} \right| \to 0 \quad \text{as} \quad x \to y,$$

or $|f'(y)| = 0$.

(3) Or using ε-δ argument. Fix $y \in \mathbb{R}$. Given any $\varepsilon > 0$, there exists $\delta = \varepsilon > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} - 0 \right| \leq |x - y| < \delta = \varepsilon$$

whenever $|x - y| < \delta$. That is, $|f'(y)| = 0$.

(4) So $f'(y) = 0$ for any $y \in \mathbb{R}$. By Theorem 5.11 (b), f is a constant.

□

Exercise 5.2. *Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b), and let g be its inverse function. Prove that g is differentiable, and that*

$$g'(f(x)) = \frac{1}{f'(x)} \qquad (a < x < b).$$

Proof. Let $E = (a, b)$.

(1) Theorem 5.10 implies that for any $a < p < q < b$ there exists $\xi \in (p, q)$ such that
$$f(p) - f(q) = (p - q)f'(\xi).$$
Since $\xi \in (p, q) \subseteq E$, by assumption $f'(\xi) > 0$. Hence $f(p) - f(q) = (p - q)f'(\xi) < 0$ (here $p - q < 0$), or
$$f(p) < f(q)$$
if $p < q$. Therefore, f is strictly increasing in (a, b).

(2) *Show that f is one-to-one in E if f is strictly increasing in E.* If $f(p) = f(q)$, then it cannot be $p > q$ or $p < q$ ((1)). So that $p = q$, or f is injective.

(3) *Show that g is well-defined.* Theorem 5.2 and Theorem 4.17.

(4) *Show that $g'(f(x)) = \frac{1}{f'(x)}$.* Given $y \in f(E)$, say $y = f(x)$ for some $x \in E$. Given any $s \in f(E)$ with $s \neq y$. Here $s = f(t)$ for some $t \in E$ and $t \neq x$.
$$\begin{aligned} \lim_{s \to y} \frac{g(s) - g(y)}{s - y} &= \lim_{f(t) \to f(x)} \frac{g(f(t)) - g(f(x))}{f(t) - f(x)} \\ &= \lim_{t \to x} \frac{t - x}{f(t) - f(x)} \\ &= \lim_{t \to x} \frac{1}{\frac{f(t) - f(x)}{t - x}} \\ &= \frac{1}{f'(x)}. && (f' > 0) \end{aligned}$$
Here $s \to y$ if and only if $t \to x$ since both f and g are continuous and one-to-one. Hence g is differentiable and $g'(f(x)) = \frac{1}{f'(x)}$.

□

Exercise 5.3. *Suppose g is a real function on $\mathbb{R}^1$, with bounded derivative (say $|g'| \leq M$). Fix $\varepsilon > 0$, and define $f(x) = x + \varepsilon g(x)$. Prove that f is one-to-one if ε is small enough. (A set of admissible values of ε can be determined which depends only on M.)*

Proof.

(1) Note that $f'(x) = 1 + \varepsilon g'(x)$ (Theorem 5.3). Since $|g'| \leq M$,
$$1 - \varepsilon M \leq f'(x) \leq 1 + \varepsilon M.$$

(2) Pick
$$\varepsilon = \frac{1}{M + 1} > 0.$$

Thus,

$$f'(x) \geq \frac{1}{M+1} > 0.$$

By Exercise 5.2, $f(x)$ is strictly increasing in $\mathbb{R}$ or one-to-one in $\mathbb{R}$.

$\square$

Exercise 5.4. *If*

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$

where $C_0, \ldots, C_n$ are real constants, prove that the equation

$$C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n = 0$$

has at least one real root between 0 *and* 1.

Proof. Let

$$g(x) = C_0 x + \frac{C_1}{2} x^2 + \cdots + \frac{C_{n-1}}{n} x^n + \frac{C_n}{n+1} x^{n+1} \in \mathbb{R}[x].$$

Then $g(0) = g(1) = 0$, and $g'(x) = C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n$. By the mean value theorem (Theorem 5.10), there exists a point $\xi \in (0, 1)$ at which

$$g(1) - g(0) = g'(\xi)(1 - 0),$$

or $g'(\xi) = 0$. That is, there exists a real root $x = \xi$ between 0 and 1 at which $C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n = 0$. $\square$

Exercise 5.5. *Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.*

Proof. Given any $x > 0$. Since f is differentiable for every $x > 0$, f is differentiable on $[x, x+1]$. By Theorem 5.2 and Theorem 5.10 (the mean value theorem), there is a point $\xi \in (x, x+1)$ at which

$$f(x+1) - f(x) = [(x+1) - x] f'(\xi)$$

or

$$g(x) = f'(\xi).$$

As $x \to +\infty$, $\xi \to +\infty$. Hence

$$\lim_{x \to +\infty} g(x) = \lim_{\xi \to +\infty} f'(\xi) = 0.$$

$\square$

Exercise 5.6. *Suppose*

(a) f *is continuous for* $x \geq 0$,

(b) $f'(x)$ *exists for* $x > 0$,

(c) $f(0) = 0$,

(d) f' *is monotonically increasing.*

Put

$$g(x) = \frac{f(x)}{x} \qquad (x > 0)$$

and prove that g *is monotonically increasing.*

Proof.

(1) It suffices to show that $g'(x) \geq 0$ for $x > 0$ (Theorem 5.11(a)), that is, to show that

$$g'(x) = \frac{xf'(x) - f(x)}{x^2} \geq 0 \qquad (x > 0),$$

or

$$xf'(x) - f(x) \geq 0 \qquad (x > 0)$$

since $x^2 > 0$ for all nonzero x.

(2) Given $x > 0$. By (a)(b), we apply the mean value theorem (Theorem 5.10) on f to get

$$f(x) - f(0) = (x - 0)f'(\xi)$$

for some $\xi \in (0, x)$. By (c),

$$f(x) = xf'(\xi).$$

By (d),

$$f(x) = xf'(\xi) \leq xf'(x).$$

Hence $xf'(x) - f(x) \geq 0$, or g is monotonically increasing.

$\square$

Note. g is increasing strictly if f is increasing strictly.

Exercise 5.7. *Suppose* $f'(x)$, $g'(x)$ *exist,* $g'(x) \neq 0$, *and* $f(x) = g(x) = 0$. *Prove that*

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}.$$

(This holds also for complex functions.)

Proof.

$$\begin{aligned}
\frac{f'(t)}{g'(t)} &= \frac{\lim_{t \to x} \frac{f(t)-f(x)}{t-x}}{\lim_{t \to x} \frac{g(t)-g(x)}{t-x}} \\
&= \lim_{t \to x} \frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}} && \text{(Both limits exist and } g' \neq 0\text{)} \\
&= \lim_{t \to x} \frac{f(t)}{g(t)}. && (f(x) = g(x) = 0)
\end{aligned}$$

This proof is also true for complex functions. $\square$

Exercise 5.8. *Suppose $f'(x)$ is continuous on $[a, b]$ and $\varepsilon > 0$. Prove that there exists $\delta > 0$ such that*

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \varepsilon$$

*whenever $0 < |t - x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$. (This could be expressed by saying f is **uniformly differentiable** on $[a, b]$ if f' is continuous on $[a, b]$.) Does this hold for vector-valued functions too?*

Proof.

(1) Since $f'(x)$ is continuous on a compact set $[a, b]$, $f'(x)$ is uniformly continuous on $[a, b]$. So given any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f'(t) - f'(x)| < \varepsilon$$

whenever $0 < |t - x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$.

(2) For such $t < x$ in (1), by the mean value theorem (Theorem 5.10), there exists a point $\xi \in (t, x)$ at which

$$f'(\xi) = \frac{f(t) - f(x)}{t - x}.$$

Note that ξ is also satisfying $0 < |t - \xi| < |t - x| < \delta$ and $a \leq \xi \leq b$. Hence by (1) we also have

$$|f'(\xi) - f'(x)| < \varepsilon,$$

or

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \varepsilon.$$

(3) *Suppose* $\mathbf{f}'(x)$ *is continuous on* $[a, b]$ *and* $\varepsilon > 0$. *Prove that there exists* $\delta > 0$ *such that*

$$\left|\frac{\mathbf{f}(t) - \mathbf{f}(x)}{t - x} - \mathbf{f}'(x)\right| < \varepsilon$$

whenever $0 < |t - x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$.

(a) Write

$$\mathbf{f}(x) = (f_1(x), \ldots, f_k(x)) \in \mathbb{R}^k.$$

By Remarks 5.16, $\mathbf{f}(x)$ is differentiable at a point x if and only if each $f_1, \ldots, f_k$ is differentiable at x. So that

$$\mathbf{f}'(x) = (f_1'(x), \ldots, f_k'(x)) \in \mathbb{R}^k.$$

By Theorem 4.10, $\mathbf{f}'(x)$ is continuous if and only if each $f_1, \ldots, f_k$ is continuous.

(b) Similar to (1)(2), Since $f_i'(x)$ is continuous on a compact set $[a, b]$ where $1 \leq i \leq k$, $f_i'(x)$ is uniformly continuous on $[a, b]$. So given any $\varepsilon > 0$ there exists $\delta_i > 0$ such that

$$|f_i'(t) - f_i'(x)| < \frac{\varepsilon}{\sqrt{k}}$$

whenever $0 < |t - x| < \delta_i$, $a \leq x \leq b$, $a \leq t \leq b$. Take $\delta = \min_{1 \leq i \leq k} \delta_i > 0$.

(c) For such $t < x$ in (1), by the mean value theorem (Theorem 5.10), there exists a point $\xi_i \in (t, x)$ at which

$$f_i'(\xi_i) = \frac{f_i(t) - f_i(x)}{t - x}.$$

Note that ξ_i is also satisfying $0 < |t - \xi_i| < |t - x| < \delta$ and $a \leq \xi_i \leq b$. Hence by (1) we also have

$$|f_i'(\xi_i) - f_i'(x)| < \frac{\varepsilon}{\sqrt{k}},$$

or

$$\left|\frac{f_i(t) - f_i(x)}{t - x} - f_i'(x)\right| < \frac{\varepsilon}{\sqrt{k}}.$$

(d) Hence

$$\left|\frac{\mathbf{f}(t) - \mathbf{f}(x)}{t - x} - \mathbf{f}'(x)\right| = \left(\sum_{i=1}^{k} \left|\frac{f_i(t) - f_i(x)}{t - x} - f_i'(x)\right|^2\right)^{\frac{1}{2}} < \varepsilon.$$

$\square$

Exercise 5.9. *Let f be a continuous real function on $\mathbb{R}^1$, of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$. Dose it follow that $f'(0)$ exists?*

Proof.

(1) *Show that $f'(0) = 3$.* It is equivalent to show that

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 3.$$

Write $F(x) = f(x) - f(0)$ and $G(x) = x - 0$ on $\mathbb{R}^1$. So that

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{F(x)}{G(x)} = 0.$$

(2) Note that

$$\lim_{x \to 0} \frac{F'(x)}{G'(x)} = \lim_{x \to 0} \frac{f'(x)}{1} = 3.$$

(3) Since f is continuous on $\mathbb{R}^1$, F is continuous on $\mathbb{R}^1$. Hence

$$\lim_{x \to 0} F(x) = F(\lim_{x \to 0} x) = F(0) = 0.$$

Also, G is continuous on $\mathbb{R}^1$ implies that

$$\lim_{x \to 0} G(x) = G(\lim_{x \to 0} x) = G(0) = 0.$$

(4) Apply L'Hospital's rule (Theorem 5.13) to (2)(3), we have

$$\lim_{x \to 0} \frac{F(x)}{G(x)} = 3,$$

or $f'(0) = 3$.

□

Exercise 5.10. *Suppose f and g are complex differentiable functions on $(0, 1)$, $f(x) \to 0$, $g(x) \to 0$, $f'(x) \to A$, $g'(x) \to B$ as $x \to 0$, where A and B are complex numbers, $B \neq 0$. Prove that*

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{A}{B}.$$

Compare with Example 5.18. (Hint:

$$\frac{f(x)}{g(x)} = \left(\frac{f(x)}{x} - A \right) \frac{x}{g(x)} + A \frac{x}{g(x)}.$$

Apply Theorem 5.13 to the real and imaginary parts of $\frac{f(x)}{x}$ and $\frac{g(x)}{x}$.)

Proof (Hint).

(1) Write
$$f(x) = f_1(x) + if_2(x)$$
for $x \in (0,1)$, where both f_1 and f_2 are real functions. By Remarks 5.16, it is clear that
$$f'(x) = f_1'(x) + if_2'(x).$$

(2) Write
$$A = A_1 + iA_2$$
where both A_1 and A_2 are real numbers. Then as $x \to 0$, we have

(a) $f(x) \to 0$ if and only if $f_1(x) \to 0$ and $f_2(x) \to 0$.

(b) $f'(x) \to A$ if and only if $f_1'(x) \to A_1$ and $f_2'(x) \to A_2$.

Hence by L'Hospital's rule (Theorem 5.13),
$$\lim_{x\to 0} \frac{f_i(x)}{x} = \lim_{x\to 0} \frac{f_i'(x)}{1} = A_i$$
$(i = 1, 2)$ or
$$\begin{aligned}
\lim_{x\to 0} \frac{f(x)}{x} &= \lim_{x\to 0} \frac{f_1(x) + if_2(x)}{x} \\
&= \lim_{x\to 0} \frac{f_1(x)}{x} + i \lim_{x\to 0} \frac{f_2(x)}{x} \\
&= A_1 + iA_2 \\
&= A.
\end{aligned}$$
Similarly,
$$\lim_{x\to 0} \frac{g(x)}{x} = B.$$
Note that $B \neq 0$, and thus
$$\lim_{x\to 0} \frac{x}{g(x)} = \frac{1}{B}.$$

(3) Hence
$$\begin{aligned}
\lim_{x\to 0} \frac{f(x)}{g(x)} &= \lim_{x\to 0} \left[\left(\frac{f(x)}{x} - A \right) \frac{x}{g(x)} + A \frac{x}{g(x)} \right] \\
&= \lim_{x\to 0} \left(\frac{f(x)}{x} - A \right) \cdot \lim_{x\to 0} \frac{x}{g(x)} + \lim_{x\to 0} A \frac{x}{g(x)} \\
&= 0 \cdot \frac{1}{B} + \frac{A}{B} \\
&= \frac{A}{B}.
\end{aligned}$$

(4) *Compare with Example 5.18.* Define $f(x) = x$ and $g(x) = x + x^2 \exp\left(\frac{i}{x^2}\right)$ as in Example 5.18. Note that $f(x) \to 0$, $g(x) \to 0$, $f'(x) \to 1$ and $g'(x) \to \infty$ as $x \to 0$. By Example 5.18

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = 1 \neq 0 = \frac{1}{\infty} = \lim_{x \to 0} \frac{A}{B}.$$

$\square$

Exercise 5.11. *Suppose f is defined in a neighborhood of x, and suppose $f''(x)$ exists. Show that*

$$\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

Show by an example that the limit may exist even if $f''(x)$ dose not. (Hint: Use Theorem 5.13.)

Proof (Theorem 5.13).

(1) Write $F(h) = f(x+h) + f(x-h) - 2f(x)$ and $G(h) = h^2$. It is equivalent to show that

$$\lim_{h \to 0} \frac{F(h)}{G(h)} = f''(x).$$

We might apply Theorem 5.13 (L'Hospital rule) to get it.

(2) *Show that* $\lim_{h \to 0} F(h) = 0$ *and* $\lim_{h \to 0} G(h) = 0$. It is clear that $\lim_{h \to 0} G(h) = \lim_{h \to 0} h^2 = 0$ since $x \mapsto x^2$ is continuous on $\mathbb{R}^1$. Besides, since f is continuous at x (by applying Theorem 5.2 twice),

$$\lim_{h \to 0} F(h) = f(x) + f(x) - 2f(x) = 0.$$

(3) *Show that*

$$\lim_{h \to 0} \frac{F'(h)}{G'(h)} = \lim_{h \to 0} \frac{f'(x+h) - f'(x-h)}{2h}$$

is well-defined. Since $f''(x)$ exists in a neighborhood $B(x; r)$ of x (where $r > 0$), $f'(x)$ exists and is continuous in $B(x; r)$ (Theorem 5.2). As $0 < |h| < \frac{r}{2}$,

$$x + h \in B\left(x + h; \frac{r}{2}\right) \subseteq B(x; r)$$

and

$$x - h \in B\left(x - h; \frac{r}{2}\right) \subseteq B(x; r).$$

So $f'(x+h)$ and $f'(x-h)$ exist in $B(x; r)$ as $0 < |h| < \frac{r}{2}$. Hence

$$\lim_{h \to 0} \frac{F'(h)}{G'(h)} = \lim_{h \to 0} \frac{f'(x+h) - f'(x-h)}{2h}$$

is well-defined (Theorem 5.3 and Theorem 5.5 (the chain rule)).

(4) *Show that*

$$\lim_{h \to 0} \frac{f'(x+h) - f'(x-h)}{2h} = f''(x).$$

Since $f''(x)$ exists, by definition

$$\lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = f''(x)$$

and

$$\lim_{h \to 0} \frac{f'(x-h) - f'(x)}{-h} = f''(x).$$

Sum up two expressions to get

$$2f''(x) = \lim_{h \to 0} \frac{f'(x-h) - f'(x-h)}{h}.$$

(5) By (2)(3)(4) and Theorem 5.13 (L'Hospital rule), the result is established.

(6) *Given* $f(x) = x|x|$ *on* $\mathbb{R}^1$. *Show that*

$$\lim_{h \to 0} \frac{f(h) + f(-h) - 2f(0)}{h^2} = 0$$

but $f''(x)$ *does not exist at* $x = 0$. Clearly,

$$\begin{aligned}
\lim_{h \to 0} \frac{f(h) + f(-h) - 2f(0)}{h^2} &= \lim_{h \to 0} \frac{h|h| + (-h)| - h| - 2 \cdot 0}{h^2} \\
&= \lim_{h \to 0} \frac{h|h| - h|h| - 0}{h^2} \\
&= \lim_{h \to 0} 0 \\
&= 0.
\end{aligned}$$

But $f''(x)$ does not exist by Exercise 5.12.

□

Exercise 5.12. *If* $f(x) = |x|^3$, *compute* $f'(x)$, $f''(x)$ *for all real* x, *and show that* $f^{(3)}(0)$ *does not exist.*

Proof.

(1) Write

$$f(x) = \begin{cases} x^3 & (x \geq 0), \\ -x^3 & (x \leq 0). \end{cases}$$

(2) *Show that* $f'(x) = 3x|x|$. It is trivial that

$$f'(x) = \begin{cases} 3x^2 & (x > 0), \\ -3x^2 & (x < 0). \end{cases}$$

Note that

$$\lim_{x \to 0} f'(x) = 0.$$

Apply the same argument in Exercise 5.9, we have

$$f'(0) = 0.$$

Hence f' exists and $f'(x) = 3x|x|$ for any $x \in \mathbb{R}$.

(3) *Show that* $f''(x) = 6|x|$. Similar to (2).

$$f''(x) = \begin{cases} 6x & (x > 0), \\ -6x & (x < 0). \end{cases}$$

Note that

$$\lim_{x \to 0} f''(x) = 0.$$

Apply the same argument in Exercise 5.9, we have

$$f''(0) = 0.$$

Hence f'' exists and $f''(x) = 6|x|$ for any $x \in \mathbb{R}$.

(4) *Show that* $f^{(3)}(0)$ *does not exist.*

$$f'''(x) = \begin{cases} 6 & (x > 0), \\ -6 & (x < 0). \end{cases}$$

There are some proofs for showing that $f^{(3)}(0)$ does not exist.

(a) Since

$$\lim_{t \to 0+} \frac{f''(t) - f''(0)}{t - 0} = \lim_{t \to 0+} \frac{6t}{t} = 6$$

and

$$\lim_{t \to 0-} \frac{f''(t) - f''(0)}{t - 0} = \lim_{t \to 0-} \frac{-6t}{t} = -6,$$

$f^{(3)}(0)$ does not exist.

(b) (Reductio ad absurdum) If f were differentiable on $\mathbb{R}^1$, then

$$\lim_{t \to 0+} f'''(t) = 6$$

and

$$\lim_{t \to 0-} f'''(t) = -6,$$

or f''' has a simple discontinuity at $x = 0$, contrary to Corollary to Theorem 5.12.

$\square$

Note. Given $k > 0$. We can construct one real function f on $\mathbb{R}^1$, say

$$f(x) = \begin{cases} |x|^k & (k \text{ is odd}), \\ x|x|^{k-1} & (k > 0 \text{ is even}), \end{cases}$$

such that all $f^{(0)}(0) = \cdots = f^{(k-1)}(0) = 0$ exist but $f^{(k)}(0)$ does not exist.

Exercise 5.13.

Exercise 5.14. *Let f be a differentiable real function defined in (a, b). Prove that f is convex if and only if f' is monotonically increasing. Assume next $f''(x)$ exists for every $x \in (a, b)$, and prove that f is convex if and only if $f''(x) \geq 0$ for all $x \in (a, b)$.*

Proof.

(1) *Show that f' is monotonically increasing if f is convex.*

(a) Since f is convex, by definition (Exercise 4.23)

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y)$$

whenever $a < x < b$, $a < y < b$, $0 < \lambda < 1$.

(b) As $x \neq y$, we have

$$\begin{aligned} f(y) - f(x) &\geq \frac{f(x + \lambda(y - x)) - f(x)}{\lambda} \\ &= \frac{f(x + \lambda(y - x)) - f(x)}{\lambda(y - x)} \cdot (y - x) \end{aligned}$$

and let $\lambda \to 0$ to get

$$f(y) - f(x) \geq f'(x)(y - x)$$

(since $f'(x)$ exists). Similarly, we have

$$f(x) - f(y) \geq f'(y)(x - y).$$

(c) Given any $y > x$, we have

$$f'(y)(y - x) \geq f(y) - f(x) \geq f'(x)(y - x).$$

Hence $f'(y) \geq f'(x)$ whenever $y > x$, or f' is monotonically increasing.

(2) *Show that f is convex if f' is monotonically increasing.* Given any $y > x$ and any $0 < \lambda < 1$.

(a) By Theorem 5.10 (the mean value theorem), there is a point $x < \xi < y$ such that

$$f(y) - f(x) = f'(\xi)(y - x).$$

Since f' is monotonically increasing,

$$f'(y)(y - x) \geq f(y) - f(x) \geq f'(x)(y - x).$$

(b) Write $z = \lambda x + (1 - \lambda)y$. Hence

$$\begin{aligned} f(y) - f(z) &\geq f'(z)(y - z), \\ f(z) - f(x) &\leq f'(z)(z - x), \end{aligned}$$

or

$$\begin{aligned} f(y) &\geq f(z) + f'(z)(y - z), \\ f(x) &\geq f(z) + f'(z)(x - z), \end{aligned}$$

or

$$\begin{aligned} \lambda f(x) + (1 - \lambda) f(y) \geq &\lambda[f(z) + f'(z)(x - z)] \\ &+ (1 - \lambda)[f(z) + f'(z)(y - z)] \\ = &f(z) \\ = &f(\lambda x + (1 - \lambda)y). \end{aligned}$$

Hence f is convex.

(3) *Show that $f''(x) \geq 0$ if f is convex and f'' exists.* By (1), f' is monotonically increasing since f is convex. Given any $x \neq y$, we have

$$\frac{f'(y) - f'(x)}{y - x} \geq 0.$$

Let $y \to x$, we have $f''(x) \geq 0$ if f'' exists.

(4) *Show that f is convex if f'' exists and $f''(x) \geq 0$.* By Theorem 5.11(a), f' is monotonically increasing. By (2), f is convex.

$\square$

Exercise 5.15 (Landau-Kolmogorov inequality on the half-line). *Suppose $a \in \mathbb{R}^1$, f is a twice-differentiable real function on (a, ∞), and M_0, M_1, M_2 are the least upper bounds of $|f(x)|$, $|f'(x)|$, $|f''(x)|$, respectively, on (a, ∞). Prove that*

$$M_1^2 \leq 4M_0M_2.$$

(Hint: If $h > 0$, Taylor's theorem shows that

$$f'(x) = \frac{1}{2h}[f(x + 2h) - f(x)] - hf''(\xi)$$

for some $\xi \in (x, x+2h)$*. Hence*

$$|f'(x)| \leq hM_2 + \frac{M_0}{h}.)$$

To show that $M_1^2 = 4M_0M_2$ *can actually happen, take* $a = -1$*, define*

$$f(x) = \begin{cases} 2x^2 - 1 & (-1 < x < 0), \\ \frac{x^2-1}{x^2+1} & (0 \leq x < \infty), \end{cases}$$

and show that $M_0 = 1$*,* $M_1 = 4$*,* $M_2 = 4$*. Does* $M_1^2 \leq 4M_0M_2$ *hold for vector-valued functions too?*

Note.

(1) Write

$$M_1 \leq 2M_0^{\frac{1}{2}} M_2^{\frac{1}{2}}.$$

2 is called the Landau-Kolmogorov constant, which is the best possible by the above example.

(2) In general, suppose $a \in \mathbb{R}^1$, f is a nth differentiable real function on (a, ∞), and M_0, M_k, M_n are the least upper bounds of $|f(x)|$, $\left|f^{(k)}(x)\right|$, $\left|f^{(n)}(x)\right|$, respectively, on (a, ∞) where $1 \leq k < n$. Then

$$M_k \leq C(n,k) M_0^{1-\frac{k}{n}} M_n^{\frac{k}{n}}.$$

Proof.

(1) Consider some trivial cases.

(a) If $M_0 = 0$, then $f(x) = 0$ on $(a, +\infty)$. So that $f'(x) = f''(x) = 0$ on $(a, +\infty)$, or $M_1 = M_2 = 0$. The inequality holds.

(b) If $M_2 = 0$, then $f''(x) = 0$ on $(a, +\infty)$. So that $f'(x) = \alpha$ for some constant $\alpha \in \mathbb{R}^1$ (Theorem 5.11(b)), and $f(x) = \alpha x + \beta$ for some constant $\beta \in \mathbb{R}^1$ (by applying Theorem 5.11(b) to $x \mapsto f(x) - \alpha x$). Hence $M_1 = |\alpha|$ and

$$M_0 = \begin{cases} +\infty & (\alpha \neq 0), \\ |\beta| & (\alpha = 0). \end{cases}$$

In any case, the inequality holds.

(c) If $M_0 = +\infty$ and $M_2 \neq 0$, there is nothing to do.

(d) If $M_2 = +\infty$ and $M_0 \neq 0$, there is nothing to do.

(2) By (1), we suppose that $0 < M_0 < +\infty$ and $0 < M_2 < +\infty$. Given $x \in (a, +\infty)$ and $h > 0$. By Taylor's theorem (Theorem 5.15):

$$f(x+2h) = f(x) + 2hf'(x) + 2h^2 f''(\xi)$$

for some $\xi \in (x, x+2h) \subseteq (a, +\infty)$. Thus

$$\begin{aligned} 2h|f'(x)| &\leq |f(x+2h)| + |f(x)| + 2h^2|f''(\xi)| \\ &\leq 2M_0 + 2h^2 M_2, \\ |f'(x)| &\leq \frac{M_0}{h} + hM_2 \end{aligned}$$

holds for all $h > 0$. In particular, take

$$h = \sqrt{\frac{M_0}{M_2}}$$

to get

$$|f'(x)| \leq 2\sqrt{M_0 M_2}.$$

Thus $2\sqrt{M_0 M_2}$ is an upper bound of $|f'(x)|$ for all $x \in (a, +\infty)$. Hence

$$M_1 \leq 2\sqrt{M_0 M_2}$$

or

$$M_1^2 \leq 4M_0 M_2.$$

(3) *Define*

$$f(x) = \begin{cases} 2x^2 - 1 & (-1 < x < 0), \\ \frac{x^2-1}{x^2+1} & (0 \leq x < \infty). \end{cases}$$

Show that $M_0 = 1$, $M_1 = 4$, $M_2 = 4$. Similar to Exercise 5.12,

$$f'(x) = \begin{cases} 4x & (-1 < x \leq 0), \\ \frac{4x}{(x^2+1)^2} & (0 \leq x < \infty). \end{cases}$$

(Here $\lim_{x \to 0+} f'(x) = 0$ and $\lim_{x \to 0-} f'(x) = 0$. So $f'(0) = 0$ by Exercise 5.9.) Also,

$$f''(x) = \begin{cases} 4 & (-1 < x \leq 0), \\ \frac{-12x^2+4}{(x^2+1)^3} & (0 \leq x < \infty). \end{cases}$$

(Here $\lim_{x \to 0+} f''(x) = 4$ and $\lim_{x \to 0-} f''(x) = 4$. So $f''(0) = 4$ by Exercise 5.9.) Hence, $M_0 = 1$, $M_1 = 4$, $M_2 = 4$.

(4) *Given*

$$\mathbf{f}(x) = (f_1(x), \ldots, f_k(x))$$

be a twice-differentiable vector-valued function from (a,∞) *to* $\mathbb{R}^k$*. and* M_0*,* M_1*,* M_2 *are the least upper bounds of* $|\mathbf{f}(x)|$*,* $|\mathbf{f}'(x)|$*,* $|\mathbf{f}''(x)|$*, respectively, on* (a,∞)*. Show that*

$$M_1^2 \leq 4M_0M_2.$$

Similar to (1), we suppose that $0 < M_0 < +\infty$ and $0 < M_2 < +\infty$. Given any $\mathbf{v} = (v_1, \ldots, v_k) \in \mathbb{R}^k$, $\mathbf{v} \cdot \mathbf{f}$ is a twice-differentiable real function on (a,∞). Similar to (2), Given $x \in (a, +\infty)$ and $h > 0$. By Taylor's theorem (Theorem 5.15):

$$(\mathbf{v} \cdot \mathbf{f})(x+2h) = (\mathbf{v} \cdot \mathbf{f})(x) + 2h(\mathbf{v} \cdot \mathbf{f})'(x) + 2h^2(\mathbf{v} \cdot \mathbf{f})''(\xi)$$

for some $\xi \in (x, x+2h) \subseteq (a, +\infty)$. Thus by the Schwarz inequality (Theorem 1.35)

$$\begin{aligned} 2h|(\mathbf{v} \cdot \mathbf{f})'(x)| &\leq |(\mathbf{v} \cdot \mathbf{f})(x+2h)| + |(\mathbf{v} \cdot \mathbf{f})(x)| + 2h^2|(\mathbf{v} \cdot \mathbf{f})''(\xi)| \\ &\leq |\mathbf{v}||\mathbf{f}(x+2h)| + |\mathbf{v}||\mathbf{f}(x)| + 2h^2|\mathbf{v}||\mathbf{f}''(\xi)| \\ &\leq (2M_0 + 2h^2M_2)|\mathbf{v}|, \\ |(\mathbf{v} \cdot \mathbf{f})'(x)| &\leq \left(\frac{M_0}{h} + hM_2\right)|\mathbf{v}| \end{aligned}$$

holds for any $\mathbf{v}$ and $h > 0$. In particular, we take

$$\mathbf{v} = \mathbf{f}'(y)$$

and

$$h = \sqrt{\frac{M_0}{M_2}}$$

to get

$$|\mathbf{f}'(x) \cdot \mathbf{f}'(y)| \leq 2\sqrt{M_0M_2}|\mathbf{f}'(y)| \leq 2M_1\sqrt{M_0M_2}.$$

Note that x and y are arbitrary (in $(a, +\infty)$). In particular, we take $x = y$ to get

$$|\mathbf{f}'(x)|^2 \leq 2M_1\sqrt{M_0M_2}.$$

Thus $2M_1\sqrt{M_0M_2}$ is an upper bound of $|\mathbf{f}'(x)|^2$ for all $x \in (a, +\infty)$. Hence

$$M_1^2 \leq 2M_1\sqrt{M_0M_2}$$

or

$$M_1^2 \leq 4M_0M_2.$$

□

Supplement (Landau-Kolmogorov inequality on the real line). *Suppose* f *is a twice-differentiable real function on* $(-\infty, +\infty)$*, and* M_0*,* M_1*,* M_2 *are the* *least upper bounds of* $|f(x)|$, $|f'(x)|$, $|f''(x)|$, *respectively, on* $(-\infty, +\infty)$. *Prove that*

$$M_1^2 \leq 2M_0M_2.$$

Proof.

(1) Similar to (1) in Landau-Kolmogorov inequality on the half-line, we suppose that $0 < M_0 < +\infty$ and $0 < M_2 < +\infty$.

(2) Similar to (2) in Landau-Kolmogorov inequality on the half-line. Given $x \in \mathbb{R}^1$ and $h > 0$. By Taylor's theorem (Theorem 5.15):

$$f(x+2h) = f(x) + 2hf'(x) + 2h^2 f''(\xi_1) \qquad \text{(I)}$$
$$f(x-2h) = f(x) - 2hf'(x) + 2h^2 f''(\xi_2) \qquad \text{(II)}$$

for some $\xi_1 \in (x, x+2h)$ and $\xi_2 \in (x, x-2h)$. So (I) subtracts (II):

$$f(x+2h) - f(x-2h) = 4hf'(x) + 2h^2 f''(\xi_1) - 2h^2 f''(\xi_2).$$

Thus

$$\begin{aligned} 4h|f'(x)| &\leq |f(x+2h)| + |f(x-2h)| + 2h^2|f''(\xi_1)| + 2h^2|f''(\xi_2)| \\ &\leq 2M_0 + 4h^2 M_2, \\ |f'(x)| &\leq \frac{M_0}{2h} + hM_2 \end{aligned}$$

holds for all $h > 0$. In particular, take

$$h = \sqrt{\frac{M_0}{2M_2}}$$

to get

$$|f'(x)| \leq \sqrt{2M_0M_2}.$$

Thus $\sqrt{2M_0M_2}$ is an upper bound of $|f'(x)|$ for all $x \in \mathbb{R}^1$. Hence

$$M_1 \leq \sqrt{2M_0M_2}$$

or

$$M_1^2 \leq 2M_0M_2.$$

□

Note.

(1) Write

$$M_1 \leq \sqrt{2}M_0^{\frac{1}{2}}M_2^{\frac{1}{2}}.$$

$\sqrt{2}$ is called the Landau-Kolmogorov constant, which is the best possible.

(2) In general, suppose f is a nth differentiable real function on $\mathbb{R}^1$, and M_0, M_k, M_n are the least upper bounds of $|f(x)|$, $\left|f^{(k)}(x)\right|$, $\left|f^{(n)}(x)\right|$, respectively, on $\mathbb{R}^1$ where $1 \leq k < n$. Then

$$M_k \leq C(n,k) M_0^{1-\frac{k}{n}} M_n^{\frac{k}{n}}.$$

Exercise 5.16. *Suppose f is twice-differentiable on $(0,\infty)$, f'' is bounded on $(0,\infty)$, and $f(x) \to 0$ as $x \to \infty$. Prove that $f'(x) \to 0$ as $x \to \infty$. (Hint: Let $a \to \infty$ in Exercise 5.15.)*

Proof.

(1) Write $|f''| \leq M$ for some real M since f'' is bounded on $(0,\infty)$.

(2) Given any $a > 0$. As in Exercise 5.15, define M_0, M_1, M_2 are the least upper bounds of $|f(x)|$, $|f'(x)|$, $|f''(x)|$ on (a,∞). Note that $M_2 \leq M$ for any $a > 0$ (by (1)). So that

$$M_1^2 \leq 4M_0M_2 \leq 4MM_0$$

for any $a > 0$.

(3) By assumption, $M_0 \to 0$ as $a \to \infty$. (So given any $\varepsilon > 0$, there exists a real A such that

$$0 \leq M_0 < \frac{\varepsilon}{4M+1}$$

whenever $a \geq A$. Hence

$$M_1^2 \leq 4MM_0 \leq 4M \cdot \frac{\varepsilon}{4M+1} < \varepsilon.$$

whenever $a \geq A$.) Therefore $M_1^2 \to 0$ as $a \to \infty$, or $f'(x) \to 0$ as $x \to \infty$.

□

Exercise 5.17. *Suppose f is a real, three times differentiable function on $[-1,1]$, such that*

$$f(-1) = 0, \qquad f(0) = 0, \qquad f(1) = 1, \qquad f'(0) = 0.$$

Prove that $f^{(3)}(x) \geq 3$ for some $x \in (-1,1)$. Note that equality holds for $\frac{1}{2}(x^3+x^2)$. (Hint: Use Theorem 5.15, with $\alpha = 0$ and $\beta = \pm 1$, to show that there exist $s \in (0,1)$ and $t \in (-1,0)$ such that

$$f^{(3)}(s) + f^{(3)}(t) = 6.)$$

We can drop the assumption that $f(0) = 0$ actually.

Proof (Hint).

(1) Use the Taylor's theorem (Theorem 5.15), with $\alpha = 0$ and $\beta = \pm 1$,

$$f(1) = f(0) + f'(0) + \frac{f''(0)}{2} + \frac{f'''(s)}{6} \tag{I}$$

$$f(-1) = f(0) - f'(0) + \frac{f''(0)}{2} - \frac{f'''(t)}{6} \tag{II}$$

for some $s \in (0, 1)$ and $t \in (-1, 0)$.

(2) (I) subtracts (II) implies that

$$f(1) - f(-1) = 2f'(0) + \frac{f'''(s)}{6} + \frac{f'''(t)}{6}.$$

By assumption, $f(-1) = 0$, $f(1) = 1$ and $f'(0) = 0$. Hence

$$f^{(3)}(s) + f^{(3)}(t) = 6$$

for some $s \in (0, 1)$ and $t \in (-1, 0)$. So either $f^{(3)}(s) \geq 3$ or $f^{(3)}(t) \geq 3$ for some $s, t \in (-1, 1)$.

□

Exercise 5.18.
Exercise 5.19.
Exercise 5.20.
Exercise 5.21. *Let E be a closed subset of $\mathbb{R}^1$. We saw in Exercise 4.22, that there is a real continuous function f on $\mathbb{R}^1$ whose zero set is E. Is it possible, for each closed set E, to find such an f which is differentiable on $\mathbb{R}^1$, or one which is n times differentiable, or even one which has derivatives of all orders on $\mathbb{R}^1$?*

Exercise 5.22.
Exercise 5.23.
Exercise 5.24.
Exercise 5.25. *Suppose f is twice differentiable on $[a, b]$, $f(a) < 0$, $f(b) > 0$, $f'(x) \geq \delta > 0$, and $0 \leq f''(x) \leq M$ for all $x \in [a, b]$. Let ξ be the unique point in (a, b) at which $f(\xi) = 0$. Complete the details in the following outline of **Newton's method** for computing ξ.*

(a) *Choose $x_1 \in (\xi, b)$, and define $\{x_n\}$ by*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Interpret this geometrically, in terms of a tangent to the graph of f.

(b) *Prove that $x_{n+1} < x_n$ and that*

$$\lim_{n \to \infty} x_n = \xi.$$

(c) *Use Taylor's theorem to show that*

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

for some $t_n \in (\xi, x_n)$.

(d) *(Quadratic convergence) If* $A = \frac{M}{2\delta}$, *deduce that*

$$0 \le x_{n+1} - \xi \le \frac{1}{A}[A(x_1 - \xi)]^{2^n}.$$

(Compare with Exercise 3.16 and 3.18.)

(e) *Show that Newton's method amounts to finding a fixed point of the function* g *defined by*

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

How does $g'(x)$ *behave for* x *near* ξ*?*

(f) *Put* $f(x) = x^{\frac{1}{3}}$ *on* $(-\infty, +\infty)$ *and try Newton's method. What happens?*

Proof of (a) (Wikipedia). The equation of the tangent line to the curve $y = f(x)$ at $x = x_n$ is

$$y = f'(x_n)(x - x_n) + f(x_n).$$

The x-intercept of this line (the value of x which makes $y = 0$) is taken as the next approximation, x_{n+1}, to the root, so that the equation of the tangent line is satisfied when $(x, y) = (x_{n+1}, 0)$:

$$0 = f'(x_n)(x - x_n) + f(x_n).$$

Solving for x_{n+1} gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

□

Proof of (b).

(1) *Show that* $x_n \ge \xi$ *for all* n. Induction on n.

  (a) $n = 1$ is clearly true: $x_1 > \xi$ by assumption.

  (b) Assume the induction hypothesis that for the single case $n = k$ holds. By the mean value theorem (Theorem 5.10), there is a point $\xi_k \in (\xi, x_k)$

$$f(x_k) - f(\xi) = f'(\xi_k)(x_k - \xi),$$

  or

$$f(x_k) = f'(\xi_k)(x_k - \xi)$$

(since $f(\xi) = 0$). Since $f'' \geq 0$, f' is monotonically increasing (Theorem 5.11(a)). Hence $f'(\xi_k) \leq f'(x_k)$ and thus

$$f(x_k) = f'(\xi_k)(x_k - \xi) \leq f'(x_k)(x_k - \xi).$$

Since $f'(x_k) > 0$ by assumption,

$$\xi \leq x_k - \frac{f(x_k)}{f'(x_k)} = x_{k+1}.$$

(c) Since both the base case in (a) and the inductive step in (b) have been proved as true, by mathematical induction $x_n \geq \xi$ for all n.

(2) *Show that $x_{n+1} < x_n$ for all n.*

(a) Since $f' > 0$, $f'(x_n) > 0$ for all n.

(b) Since $f' > 0$, f is strictly increasing (Theorem 5.10). Hence $f(x_n) > f(\xi) = 0$ for all n (by (1)).

(c) By (a)(b), $\frac{f(x_n)}{f'(x_n)} > 0$ or

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} < x_n.$$

(3) By Theorem 3.14, $\{x_n\}$ converges to some real number $\zeta \geq \xi$. Note that f and f' are continuous by the existence of f'' (Theorem 5.2), we have

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n - \frac{f(\lim_{n\to\infty} x_n)}{f'(\lim_{n\to\infty} x_n)}$$

provided $f' \neq 0$ (Theorem 4.9 and Theorem 4.4). Hence

$$\zeta = \zeta - \frac{f(\zeta)}{f'(\zeta)}$$

or $f(\zeta) = 0$. By the uniqueness of ξ, $\zeta = \xi$ or $\lim x_n = \xi$ as desired.

$\square$

Proof of (c). By Taylor's theorem (Theorem 5.15),

$$f(\xi) = f(x_n) + f'(x_n)(\xi - x_n) + \frac{f''(t_n)}{2}(\xi - x_n)^2$$

for some $t_n \in (\xi, x_n)$. Note that $f(\xi) = 0$, $f'(x_n) \neq 0$ and $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, we have the desired result. $\square$

Proof of (d). Clearly, $0 \leq x_{n+1} - \xi$ for all n (by (b)). Besides, by (c)

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

Note that $f'' \leq M$ and $f' \geq \delta > 0$ by assumption, and thus

$$x_{n+1} - \xi \leq \frac{M}{2\delta}(x_n - \xi)^2 = A(x_n - \xi)^2.$$

By induction,

$$x_{n+1} - \xi \leq \frac{1}{A}(A(x_1 - \xi))^{2^n}.$$

$\square$

Note. Compare with Exercise 3.16 and Exercise 3.18. Might assume that $p > 1$.

(1) Fix a positive number α. Let $f(x) = x^p - \alpha$ on $E = (a, b)$ where $a = \frac{1}{2}\alpha^{\frac{1}{p}}$ and

$$b = \begin{cases} 2\alpha^{\frac{1}{p}} & (p = 2), \\ \left(\frac{2(p-1)}{p}\right)^{\frac{1}{p-2}} \alpha^{\frac{1}{p}} & (p > 2). \end{cases}$$

$E = (a, b)$ is well-defined since $a < b$. Besides, $\xi = \alpha^{\frac{1}{p}} \in E = (a, b)$.

(2) By construction,

$$f(a) < 0 \text{ and } f(b) > 0.$$

By $f'(x) = px^{p-1}$ and $f''(x) = p(p-1)x^{p-2}$,

$$\begin{aligned} f'(x) &\geq pa^{p-1} > 0, \\ 0 \leq f''(x) &\leq p(p-1)b^{p-2}. \end{aligned}$$

on E. Write

$$\begin{aligned} \delta &= pa^{p-1} = \frac{p}{2^{p-1}}\alpha^{\frac{p-1}{p}}, \\ M &= p(p-1)b^{p-2} = 2(p-1)^2\alpha^{\frac{p-2}{p}}. \end{aligned}$$

(3) Hence the Newton's method works for $f(x) = x^p - \alpha$. That is, as we define $\{x_n\}$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{p-1}{p}x_n + \frac{\alpha}{p}x_n^{-p+1},$$

we have $\lim x_n = \xi = \alpha^{\frac{1}{p}}$. And

$$0 \leq x_{n+1} - \xi \leq \frac{1}{A}(A(x_1 - \xi))^{2^n}.$$

Here

$$A = \frac{M}{2\delta} = \frac{2^{p-1}(p-1)^2}{p\alpha^{\frac{1}{p}}}.$$

(4) Note that
$$\beta = \frac{p\alpha^{\frac{1}{p}}}{(p-1)^2} \neq \frac{p\alpha^{\frac{1}{p}}}{2^{p-1}(p-1)^2} = \frac{1}{A}.$$
where β is defined in the proof of Exercise 3.18. Note that $f'(x_n) \geq f'(\xi)$ (since f' is monotonically increasing and all $x_n \geq \xi$), and thus A can be chosen by a better estimation:
$$A = \frac{M}{2f'(\xi)} = \frac{(p-1)^2}{p\alpha^{\frac{1}{p}}} = \frac{1}{\beta}.$$
Now it is exactly the same as Exercise 3.16 and Exercise 3.18.

Proof of (e).

(1) Define $g(x) = x - \frac{f(x)}{f'(x)}$ on $[a, b]$. $g(\xi) = \xi$ if and only if $f(\xi) = 0$.

(2) By the construction of g, g is differentiable and
$$g'(x) = 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2}.$$

(3) Hence
$$|g'(x)| \leq \left|\frac{f(x)f''(x)}{f'(x)^2}\right| = \frac{|f(x)||f''(x)|}{|f'(x)|^2} \leq \frac{M}{\delta^2}|f(x)|.$$
As $x \to \xi$, $|f(x)| \to 0$. Therefore, $|g'(x)| \to 0$ or $g'(x) \to 0$ as $x \to \xi$.

□

Proof of (f).

(1) It is clearly that $f(x) = 0$ if and only if $x = 0$. Write $\xi = 0$.

(2) Note that
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = -2x_n,$$
or
$$x_n = (-2)^{n-1}x_1$$
for any $x_1 \in (\xi, \infty)$ where $n = 1, 2, 3, \ldots$. Hence, the sequence $\{x_n\}$ does not converge for any choice of $x_1 \in (\xi, \infty)$. In this case we cannot find ξ satisfying $f(\xi) = 0$ by Newton's method.

(3) In fact,
$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} \to 0 \text{ as } x \to \pm\infty.$$
Hence such $\delta > 0$ satisfying $f'(x) \geq \delta > 0$ does not exist.

$\square$

Exercise 5.26. *Suppose f is differentiable on $[a,b]$, $f(a) = 0$, and there is a real number A such that $|f'(x)| \leq A|f(x)|$ on $[a,b]$. Prove that $f(x) = 0$ for all $x \in [a,b]$. (Hint: Fix $x_0 \in [a,b]$, let*

$$M_0 = \sup|f(x)|, \qquad M_1 = \sup|f'(x)|$$

for $a \leq x \leq x_0$. For any such x,

$$|f(x)| \leq M_1(x_0 - a) \leq A(x_0 - a)M_0.$$

Hence $M_0 = 0$ if $A(x_0 - a) < 1$. That is, $f = 0$ on $[a, x_0]$. Proceed.)

Proof (Hint).

(1) If $A = 0$, then $f'(x) = 0$ or $f(x)$ is constant on $[a,b]$ (Theorem 5.11(b)). Since $f(a) = 0$, $f(x) = 0$ on $[a,b]$.

(2) Suppose that $A > 0$. Fix $x_0 \in [a,b]$, let

$$M_0 = \sup|f(x)|, \qquad M_1 = \sup|f'(x)|$$

for $a \leq x \leq x_0$. Since $|f'(x)| \leq A|f(x)|$ on $[a,b]$,

$$|f'(x)| \leq A|f(x)| \leq AM_0.$$

Since AM_0 is an upper bound for $|f'(x)|$,

$$M_1 \leq AM_0.$$

(3) Given any $x \in [a, x_0]$. Since f is differentiable on $[a, x_0] \subseteq [a,b]$, by the mean value theorem (Theorem 5.10), there is $\xi \in (a,x)$ such that

$$f(x) - f(a) = f'(\xi)(x - a).$$

Note that $f(a) = 0$ by assumption. So that

$$\begin{aligned}
|f(x)| &= |f'(\xi)|(x - a) \\
&\leq M_1(x - a) && \text{(Definition of } M_1\text{)} \\
&\leq AM_0(x - a) && \text{((2))} \\
&\leq AM_0(x_0 - a). && (x \in [a, x_0])
\end{aligned}$$

Since $AM_0(x_0 - a)$ is an upper bound for $|f(x)|$,

$$M_0 \leq AM_0(x_0 - a).$$

Take

$$x_0 = \min\left\{\frac{1}{2A} + a, b\right\}$$

so that $M_0 \leq AM_0(x_0 - a) \leq \frac{M_0}{2}$. $M_0 = 0$ or $f(x) = 0$ on $[a, x_0]$.

(4) Take a partition
$$P = \{a = x_{-1}, x_0, \dots, x_n = b\}$$
of $[a,b]$ such that each subinterval $[x_{i-1}, x_i]$ satisfying $\Delta x_i = x_i - x_{i-1} < \frac{1}{2A}$. By (3), $f(x) = 0$ on $[x_{-1}, x_0]$. Apply the same argument in (3), $f(x) = 0$ on $[x_0, x_1]$. Continue this process, $f(x) = 0$ on each subinterval and thus on the whole interval $[a,b]$.

□

Note. It holds for vector-valued functions too:

> *Suppose* $\mathbf{f}$ *is a vector-valued differentiable function on* $[a,b]$, $f(a) = 0$, *and there is a real number* A *such that* $|\mathbf{f}'(x)| \leq A|\mathbf{f}(x)|$ *on* $[a,b]$. *Prove that* $\mathbf{f}(x) = 0$ *for all* $x \in [a,b]$.

The proof is similar except using Theorem 5.19 ($|\mathbf{f}(b) - \mathbf{f}(a)| \leq (b-a)|\mathbf{f}'(x)|$) in addition.

Exercise 5.27. *Let* ϕ *be a real function defined on a rectangle* R *in the plane, given by* $a \leq x \leq b$, $\alpha \leq y \leq \beta$. *A* ***solution*** *of the initial-value problem*
$$y' = \phi(x,y), \qquad y(a) = c \qquad (\alpha \leq c \leq \beta)$$
is, by definition, a differentiable function f *on* $[a,b]$ *such that* $f(a) = c$, $\alpha \leq f(x) \leq \beta$, *and*
$$f'(x) = \phi(x, f(x)) \qquad (a \leq x \leq b)$$
Prove that such a problem has at most one solution if there is a constant A *such that*
$$|\phi(x,y_2) - \phi(x,y_1)| \leq A|y_2 - y_1|$$
whenever $(x,y_1) \in R$ *and* $(x,y_2) \in R$. *(Hint: Apply Exercise 26 to the difference of two solutions.) Note that this uniqueness theorem does not hold for the initial-value problem*
$$y' = y^{\frac{1}{2}}, \qquad y(0) = 0,$$
which has two solutions: $f(x) = 0$ *and* $f(x) = \frac{x^2}{4}$. *Find all other solutions.*

Proof (Hint).

(1) Suppose f_1 and f_2 are two solutions of that problem. Define $f = f_1 - f_2$. f is differentiable on $[a,b]$, $f(a) = f_1(a) - f_2(a) = c - c = 0$. And
$$\begin{aligned} |f'(x)| &= |f_1'(x) - f_2'(x)| \\ &= |\phi(x, f_1(x)) - \phi(x, f_2(x))| \\ &\leq A|f_1(x) - f_2(x)| \end{aligned}$$
on $[a,b]$. By Exercise 5.26, $f(x) = 0$ on $[a,b]$, or $f_1(x) = f_2(x)$ on $[a,b]$.

(2) *The initial-value problem*

$$y' = y^{\frac{1}{2}}, \qquad y(0) = 0,$$

which has two solutions: $f(x) = 0$ and $f(x) = \frac{x^2}{4}$. Find all other solutions.

Note. It does not exist a real A such that $|\phi(x, y_2) - \phi(x, y_1)| \leq A|y_2 - y_1|$ in this initial-value problem.

(a) Clearly, $f(x) = 0$ and $f(x) = \frac{x^2}{4}$ are two solutions for the initial-value problem.

(b) Suppose $f(x) \neq 0$ on $[0, \infty)$. Since $f'(x) = f(x)^{\frac{1}{2}}$, $f(x) \geq 0$. Since $f(x)$ is continuous (Theorem 5.2), the set

$$E = \{x \in [0, \infty) : f(x) > 0\}$$

is open in $\mathbb{R}^1$ (Theorem 4.8). By Exercise 2.29 we write E as the union of an at most countable collection of disjoint segments, say

$$E = \bigcup_{(a_i, b_i) \subseteq [0, \infty)} (a_i, b_i)$$

where all (a_i, b_i) segments are disjoint. Note that E is nonempty.

(c) For any segment (a_i, b_i), define $g(x) = f(x)^{\frac{1}{2}}$ on (a_i, b_i). (Clearly, $g(a_i) = f(a_i) = 0$ by the definition of E.) Thus

$$g'(x) = \frac{1}{2} f(x)^{-\frac{1}{2}} f'(x) = \frac{1}{2}.$$

Hence

$$g(x) = \frac{1}{2}x + c$$

for some constant $c \in \mathbb{R}^1$. So

$$f(x) = g(x)^2 = \left(\frac{1}{2}x + c\right)^2.$$

$f(a_i) = 0$ implies that $c = -\frac{a_i}{2}$. Hence

$$f(x) = \frac{1}{4}(x - a_i)^2$$

on (a_i, b_i).

(d) By (c), if $b_i < 0$ is defined as a real number, then $f(b_i) = 0$ by definition of E. Note that

$$\lim_{x \to b_i -} f(x) = \frac{1}{4}(b_i - a_i)^2 > 0,$$

which is absurd. Hence $b_i = \infty$ and thus E is of the form

$$E = (a, \infty) \qquad (a \geq 0).$$

Therefore,

$$f(x) = \begin{cases} 0 & (0 \leq x \leq a), \\ \frac{1}{4}(x-a)^2 & (x > a \geq 0). \end{cases}$$

$\square$

Exercise 5.28. *Formulate and prove an analogous uniqueness theorem for systems of differential equations of the form*

$$y_j' = \phi_j(x, y_1, \ldots, y_k), \qquad y_j(a) = c_j \qquad (j = 1, \ldots, k)$$

Note that this can be rewritten in the form

$$\mathbf{y}' = \boldsymbol{\phi}(x, \mathbf{y}), \qquad \mathbf{y}(a) = \mathbf{c}$$

where $\mathbf{y} = (y_1, \ldots, y_k)$ *ranges over a* k*-cell,* $\boldsymbol{\phi}$ *is the mapping of a* $(k+1)$*-cell into the Euclidean* k*-space whose components are the function* $\phi_1, \ldots, \phi_k$*, and* $\mathbf{c}$ *is the vector* $(c_1, \ldots, c_k)$*. Use Exercise 5.26, for vector-valued functions.*

Proof.

(1) *A **solution** of the initial-value problem*

$$\mathbf{y}' = \boldsymbol{\phi}(x, \mathbf{y}), \qquad \mathbf{y}(a) = \mathbf{c}$$

is, by definition, a differentiable function $\mathbf{f}$ *on* $[a, b]$ *such that* $\mathbf{f}(a) = \mathbf{c}$*, and*

$$\mathbf{f}'(x) = \boldsymbol{\phi}(x, \mathbf{f}(x)) \qquad (a \leq x \leq b).$$

Then this problem has at most one solution if there is a constant A *such that*

$$|\boldsymbol{\phi}(x, \mathbf{y}_2) - \boldsymbol{\phi}(x, \mathbf{y}_1)| \leq A|\mathbf{y}_2 - \mathbf{y}_1|$$

whenever $(x, \mathbf{y}_1) \in R$ *and* $(x, \mathbf{y}_2) \in R$ *where* R *is a* $(k+1)$*-cell defined by*

$$R = [a, b] \times [\alpha_1, \beta_1] \times \cdots \times [\alpha_k, \beta_k].$$

(2) Similar to Exercise 5.27, Suppose $\mathbf{f}_1$ and $\mathbf{f}_2$ are two solutions of that problem. Define $\mathbf{f} = \mathbf{f}_1 - \mathbf{f}_2$. $\mathbf{f}$ is differentiable on $[a, b]$, $\mathbf{f}(a) = \mathbf{f}_1(a) - \mathbf{f}_2(a) = \mathbf{c} - \mathbf{c} = 0$. And

$$\begin{aligned} |\mathbf{f}'(x)| &= |\mathbf{f}_1'(x) - \mathbf{f}_2'(x)| \\ &= |\boldsymbol{\phi}(x, \mathbf{f}_1(x)) - \boldsymbol{\phi}(x, \mathbf{f}_2(x))| \\ &\leq A|\mathbf{f}_1(x) - \mathbf{f}_2(x)| \end{aligned}$$

on $[a, b]$. By Note in Exercise 5.26, $\mathbf{f}(x) = 0$ on $[a, b]$, or $\mathbf{f}_1(x) = \mathbf{f}_2(x)$ on $[a, b]$.

□

Exercise 5.29. *Specialize Exercise 5.28 by considering the system*

$$y_j' = y_{j+1} \qquad (j = 1, \ldots, k-1),$$
$$y_k' = f(x) - \sum_{j=1}^{k} g_j(x) y_j$$

where $f, g_1, \ldots, g_k$ *are continuous real functions on* $[a, b]$*, and derive a uniqueness theorem for solutions of the equation*

$$y^{(k)} + g_k(x) y^{(k-1)} + \cdots + g_2(x) y' + g_1(x) y = f(x),$$

subject to initial conditions

$$y(a) = c_1, \qquad y'(a) = c_1, \qquad \ldots, \qquad y^{(k-1)}(a) = c_k.$$

Proof.

(1) Write

$$\begin{aligned}
\mathbf{y} &= (y_1, \ldots, y_k) \\
&= \left(y, y', y'', \ldots, y^{(k-1)}\right), \\
\boldsymbol{\phi}(x, \mathbf{y}) &= \left(y_2, y_3, \ldots, y_{k-1}, f(x) - \sum_{j=1}^{k} g_j(x) y_j\right) \\
&= \left(y', y'', \ldots, y^{(k-1)}, f(x) - \sum_{j=1}^{k} g_j(x) y^{(j-1)}\right), \\
\mathbf{c} &= (c_1, \ldots, c_k).
\end{aligned}$$

So that

$$\mathbf{y}' = \boldsymbol{\phi}(x, \mathbf{y}), \qquad \mathbf{y}(a) = \mathbf{c}$$

where $\mathbf{y}$ ranges over a k-cell R.

(2) *To show that the problem has at most one solution, by Exercise 5.28 it suffices to show that there is a constant* A *such that*

$$|\boldsymbol{\phi}(x, \mathbf{y}) - \boldsymbol{\phi}(x, \mathbf{z})| \leq A |\mathbf{y} - \mathbf{z}|$$

whenever $(x, \mathbf{y}) \in R$ *and* $(x, \mathbf{z}) \in R$.

(3) Since all g_j $(1 \leq j \leq k)$ are real continuous functions on a compact set $[a, b]$, all g_j are bounded (Theorem 4.15), say $|g_j| \leq M$ on $[a, b]$ for some $M_j \in \mathbb{R}^1$ $(1 \leq j \leq k)$.

(4) Write $\mathbf{y} = (y_1, \dots, y_k)$ and $\mathbf{z} = (z_1, \dots, z_k)$. So

$$\begin{aligned}
&|\boldsymbol{\phi}(x,\mathbf{y}) - \boldsymbol{\phi}(x,\mathbf{z})|^2 \\
=&\left|\left(y_2 - z_2, y_3 - z_3, \dots, y_{k-1} - z_{k-1}, -\sum_{j=1}^{k} g_j(x)(y_j - z_j)\right)\right|^2 \\
=&\sum_{j=2}^{k-1}(y_j - z_j)^2 + \left(-\sum_{j=1}^{k} g_j(x)(y_j - z_j)\right)^2 \\
\leq&\sum_{j=2}^{k-1}(y_j - z_j)^2 + \sum_{j=1}^{k} g_j(x)^2 \sum_{j=1}^{k}(y_j - z_j)^2 && \text{(Theorem 1.35)} \\
\leq&\sum_{j=2}^{k-1}(y_j - z_j)^2 + \sum_{j=1}^{k} M_j^2 \sum_{j=1}^{k}(y_j - z_j)^2 && \text{((3))} \\
\leq&\sum_{j=1}^{k}(y_j - z_j)^2 + \sum_{j=1}^{k} M_j^2 \sum_{j=1}^{k}(y_j - z_j)^2 && (x^2 \geq 0 \, \forall x \in \mathbb{R}^1) \\
\leq&\left(1 + \sum_{j=1}^{k} M_j^2\right) |\mathbf{y} - \mathbf{z}|^2.
\end{aligned}$$

Hence $|\boldsymbol{\phi}(x,\mathbf{y}) - \boldsymbol{\phi}(x,\mathbf{z})| \leq A|\mathbf{y} - \mathbf{z}|$ for some $A = \left(1 + \sum_{j=1}^{k} M_j^2\right)^{\frac{1}{2}}$.

□