Notes on the book: *Apostol, Modular Functions and Dirichlet Series in Number Theory, 2nd edition*

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Chapter 1: Elliptic functions

Exercise 1.1.

Given two pairs of complex numbers (ω_1, ω_2) and (ω_1', ω_2') with nonreal ratios ω_1/ω_2 and ω_1'/ω_2'. Prove that they generate the same set of periods if, and only if, there is a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integer entries and determinant ± 1 such that

$$\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix}.$$

Proof.

(1) ($\Longrightarrow$) Suppose (ω_1, ω_2) and (ω_1', ω_2') generate the same set of periods. In particular, there is a 2×2 matrix $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{M}_{2\times 2}(\mathbb{Z})$ (resp. $A' := \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathsf{M}_{2\times 2}(\mathbb{Z})$) such that

$$\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = A \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix}, \qquad \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} = A' \begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix}.$$

Hence it suffices to show $\det(A) = \pm 1$.

(2) Note that

$$\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = AA' \begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix}.$$

Hence

$$AA' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Take the determinant on the both sides to get

$$\det(A)\det(A') = 1.$$

Since $\det(\mathsf{M}_{2\times 2}(\mathbb{Z})) \subseteq \mathbb{Z}$, $\det(A) = \pm 1$.

(3) ($\Longleftarrow$) $\Omega(\omega_1', \omega_2') \subseteq \Omega(\omega_1, \omega_2)$ is obvious. Note that

$$\begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} = \underbrace{\frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}_{\in \mathsf{M}_{2\times 2}(\mathbb{Z})} \begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix}.$$

Thus $\Omega(\omega_1, \omega_2) \subseteq \Omega(\omega_1', \omega_2')$. Therefore $\Omega(\omega_1, \omega_2) = \Omega(\omega_1', \omega_2')$.

□

Supplement 1.1.1.

(Exercise I.1.1 in the textbook: *Jörgen Neukirch, Algebraic Number Theory.*) *$\alpha \in \mathbb{Z}[i]$ is a unit if and only if $N(\alpha) = 1$.*

Proof.

(1) ($\Longrightarrow$) Since α is a unit, there is $\beta \in \mathbb{Z}[i]$ such that $\alpha\beta = 1$. So $N(\alpha\beta) = N(1)$, or $N(\alpha)N(\beta) = 1$. Since the image of N is nonnegative integers, $N(\alpha) = 1$.

(2) ($\Longleftarrow$) $N(\alpha) = \alpha\overline{\alpha}$, or $1 = \alpha\overline{\alpha}$ since $N(\alpha) = 1$. That is, $\overline{\alpha} \in \mathbb{Z}[i]$ is the inverse of $\alpha \in \mathbb{Z}[i]$. (Or we solve the equation $N(\alpha) = a^2 + b^2 = 1$, and show that all four solutions (± 1 and $\pm i$) are units.)

(3) Conclusion: a unit $\alpha = a + bi$ of $\mathbb{Z}[i]$ is satisfying the equation $N(\alpha) = a^2 + b^2 = 1$ by (1)(2). That is, the only unit of $\mathbb{Z}[i]$ are ± 1 and $\pm i$.

$\square$

Exercise 1.2.

Let $S(0)$ denote the sum of the zeros of an elliptic function f in a period parallelogram, and let $S(\infty)$ denote the sum of the poles in the same parallelogram. Prove that $S(0) - S(\infty)$ is a period of f. (Hint: Integrate $z\frac{f'(z)}{f(z)}$.)

Proof.

(1) Similar to Theorem 1.8, the integral

$$\frac{1}{2\pi i}\int_C z\frac{f'(z)}{f(z)}$$

taken around the boundary C of a cell (no zeros or poles on its boundary) counts the difference between the sum of the zeros and the sum of the poles inside the cell, that is,

$$S(0) - S(\infty) = \frac{1}{2\pi i}\int_C z\frac{f'(z)}{f(z)}.$$

(The proof is similar to the proof of the argument principle.)

(2) Let C_1 be the path from 0 to ω_1, C_2 be the path from ω_1 to $\omega_1 + \omega_2$, C_3

be the path from $\omega_1+\omega_2$ to ω_2, and C_4 be the path from ω_2 to 0. Hence

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{C_1} z\frac{f'(z)}{f(z)} + \frac{1}{2\pi i}\int_{C_3} z\frac{f'(z)}{f(z)}\\
=&\frac{1}{2\pi i}\int_{C_1} z\frac{f'(z)}{f(z)} + \frac{1}{2\pi i}\int_{-C_1} (z+\omega_2)\frac{f'(z+\omega_2)}{f(z+\omega_2)}\\
=&\frac{1}{2\pi i}\int_{C_1} z\frac{f'(z)}{f(z)} - \frac{1}{2\pi i}\int_{C_1} (z+\omega_2)\frac{f'(z)}{f(z)}\\
=&-\omega_2\frac{1}{2\pi i}\int_{C_1} \frac{f'(z)}{f(z)}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{C_2} z\frac{f'(z)}{f(z)} + \frac{1}{2\pi i}\int_{C_4} z\frac{f'(z)}{f(z)}\\
=&\frac{1}{2\pi i}\int_{-C_4} (z+\omega_1)\frac{f'(z+\omega_1)}{f(z+\omega_1)} + \frac{1}{2\pi i}\int_{C_4} z\frac{f'(z)}{f(z)}\\
=&-\frac{1}{2\pi i}\int_{C_4} (z+\omega_1)\frac{f'(z)}{f(z)} + \frac{1}{2\pi i}\int_{C_4} z\frac{f'(z)}{f(z)}\\
=&-\omega_1\frac{1}{2\pi i}\int_{C_4} \frac{f'(z)}{f(z)}
\end{aligned}$$

Therefore

$$S(0)-S(\infty) = -\omega_1\frac{1}{2\pi i}\int_{C_4}\frac{f'(z)}{f(z)} - \omega_2\frac{1}{2\pi i}\int_{C_1}\frac{f'(z)}{f(z)}.$$

So it suffices to show that $\frac{1}{2\pi i}\int_{C_1}\frac{f'(z)}{f(z)} \in \mathbb{Z}$. (Other cases are similar.)

(3) By choosing one branch of log, we have

$$\begin{aligned}
\frac{1}{2\pi i}\int_{C_1}\frac{f'(z)}{f(z)} &= \frac{1}{2\pi i}\log\frac{f(\omega_1)}{f(0)}\\
&= \frac{1}{2\pi i}\log(1) && (f(\omega_1)=f(0))\\
&= \frac{1}{2\pi i}(2\pi i m) \text{ for some } m\in\mathbb{Z}\\
&= m\in\mathbb{Z}.
\end{aligned}$$

□

Exercise 1.3.

(a) *Prove that $\wp(u)=\wp(v)$ if, and only if, $u-v$ or $u+v$ is a period of $\wp$.*

(b) *Let* $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_m$ *be complex numbers such that none of the numbers* $\wp(a_i) - \wp(b_j)$ *is zero. Let*

$$f(z) = \frac{\prod_{k=1}^{n}[\wp(z) - \wp(a_k)]}{\prod_{r=1}^{m}[\wp(z) - \wp(b_r)]}.$$

Prove that f *is an even elliptic function with zeros at* $a_1, \ldots, a_n$ *and poles at* $b_1, \ldots, b_m$.

Proof of (a).

(1) Let Ω be the lattice generated by periods of $\wp$.

(2) ($\Longrightarrow$) It is equivalent to show that the equation $\wp(u) = \wp(v)$ in terms of u has exactly two roots in some period parallelogram. $u \equiv v \pmod{\Omega}$ is a root clearly and $u \equiv -v \pmod{\Omega}$ is also a root since $\wp$ is even. Since $\wp$ is an elliptic function of order 2 (Theorem 1.8), $u \equiv \pm v \pmod{\Omega}$ is the only two roots of $\wp(u) = \wp(v)$.

(3) ($\Longleftarrow$) Obvious.

$\square$

Proof of (b).

(1) Since $\wp$ is an even elliptic function, f is an even elliptic function too.

(2) f has zeros at $a_1, \ldots, a_n$ and poles at $b_1, \ldots, b_m$ (by construction and (a)).

$\square$

Exercise 1.4.

Prove that every even elliptic function f *is a rational function of* $\wp$*, where periods of* $\wp$ *are a subset of the periods of* f.

Proof.

(1) Nothing to do if f is constant. Let C be one period parallelogram of f and $\wp$. Let $\Omega(\omega_1, \omega_2)$ be the lattice generated by periods of $\wp$. Suppose f has zeros at $a_1, \ldots, a_n$ and poles at $b_1, \ldots, b_m$.

(2) Might assume that $\wp(z) - \wp(a_k)$ (resp. $\wp(z) - \wp(b_r)$) has a simple zero in a_k (resp. b_r) for all k, r. So the function

$$g(z) := f(z) \cdot \frac{\prod_{r=1}^{m}[\wp(z) - \wp(b_r)]^{\beta_r}}{\prod_{k=1}^{n}[\wp(z) - \wp(a_k)]^{\alpha_k}}$$

is an elliptic function with no zeros or poles in C where α_k (resp. β_r) is the order of the zero a_k (resp. the pole b_r). By Theorems 1.4 and 1.5, $g(z)$ is a constant. Hence

$$f(z) = C \cdot \frac{\prod_{k=1}^{n}[\wp(z) - \wp(a_k)]^{\alpha_k}}{\prod_{r=1}^{m}[\wp(z) - \wp(b_r)]^{\beta_r}}$$

for some constant $C \in \mathbb{C}$.

(3) Now we consider the case a_k (resp. b_r) is a zero of $\wp'(z)$. Since f is an even elliptic function, the order of a_k (resp. b_r) of f is even. Note that the order of a_k (resp. b_r) of $\wp(z) - \wp(a_k)$ (resp. $\wp(z) - \wp(b_r)$) is 2. Hence the function

$$g(z) := f(z) \cdot \frac{\prod_{\wp'(b_r)\neq 0}[\wp(z) - \wp(b_r)]^{\beta_r}}{\prod_{\wp'(a_k)\neq 0}[\wp(z) - \wp(a_k)]^{\alpha_k}} \cdot \frac{\prod_{\wp'(b_r)=0}[\wp(z) - \wp(b_r)]^{\frac{\beta_r}{2}}}{\prod_{\wp'(a_k)=0}[\wp(z) - \wp(a_k)]^{\frac{\alpha_k}{2}}}$$

is a constant too.

□

Supplement 1.4.1. (Divisor class group)

(Problem 8.6 in the textbook: *William Fulton, Algebraic Curves.*) *Let $D(X)$ be the group of divisors on X, $D_0(X)$ the subgroup consisting of divisors of degree zero, and $P(X)$ the subgroup of $D_0(X)$ consisting of divisors of rational functions. Let $C_0(X) = D_0(X)/P(X)$ be the quotient group. It is the* ***divisor class group*** *on X.*

(a) *If $X = \mathbf{P}^1$, then $C_0(X) = 0$.*

(b) *Let $X = C$ be a nonsingular cubic. Pick $P_0 \in C$, defining $\oplus$ on C. Show that the map from C to $C_0(X)$ that sends P to the residue class of the divisor $P - P_0$ is an isomorphism from $(C, \oplus)$ onto $C_0(X)$.*

Proof of (a).

(1) Given a divisor

$$D = \sum_{P \in X} n_P P \in D_0(X)$$

where $n_P \in \mathbb{Z}$ and $\sum_P n_P = 0$.

(2) Note that $\sum_P n_P = 0$. We can define a rational function $z \in k(X)$ by

$$z = \prod_{P=[a_P:b_P]\in X} (b_P x - a_P y)^{n_P}.$$

Hence $\mathrm{div}(z) = D \in P(X)$. Therefore $C_0(X) = D_0(X)/P(X) = 0$.

□

Proof of (b).

(1) Define $\alpha : (C, \oplus) \to C_0(X)$ by $P \mapsto [P - P_0]$.

(2) *Show that α is a group homomorphism.* If $P \oplus Q = R$, then

$$\begin{aligned} & P \oplus Q = R \\ \Longleftrightarrow\ & [P + Q] = [R + P_0] && \text{(Problem 8.3(c))} \\ \Longleftrightarrow\ & [P - P_0] + [Q - P_0] = [R - P_0] && \text{(Proposition 2)} \\ \Longleftrightarrow\ & \alpha(P) + \alpha(Q) = \alpha(R) = \alpha(P \oplus Q). \end{aligned}$$

(3) *Show that α is injective.*

$$\begin{aligned} \alpha(P) = 0 &\Longleftrightarrow [P - P_0] = 0 \\ &\Longleftrightarrow [P] = [P_0] && \text{(Proposition 2)} \\ &\Longleftrightarrow P = P_0. && \text{(Problem 8.3(a))} \end{aligned}$$

(4) *Show that α is surjective.* Given $[D] \in C_0(X)$ and we want to find a point $P \in C$ such that $\alpha(P) = [D]$. Write

$$D = (P_1 + \cdots + P_r) - (Q_1 + \cdots + Q_r)$$

for some $P_i, Q_i \in C$. So

$$\begin{aligned} [D] &= [P_1 - P_0] + \cdots + [P_r - P_0] - [Q_1 - P_0] - \cdots - [Q_r - P_0] \\ &= \alpha(P_1) + \cdots + \alpha(P_r) - \alpha(Q_1) - \cdots - \alpha(Q_r) \\ &= \alpha(P_1) + \cdots + \alpha(P_r) + \alpha(Q_1') + \cdots + \alpha(Q_r') \\ &= \alpha(P_1 \oplus \cdots \oplus P_r \oplus Q_1' \oplus \cdots \oplus Q_r'). \end{aligned}$$

where Q_i' is the inverse of Q_i in $(C, \oplus)$. Hence there is a point $P := P_1 \oplus \cdots \oplus P_r \oplus Q_1' \oplus \cdots \oplus Q_r' \in C$ such that $\alpha(P) = [D]$.

□

Exercise 1.5.

Prove that every elliptic function f can be expressed in the form

$$f(z) = R_1[\wp(z)] + \wp'(z) R_2[\wp(z)]$$

where R_1 and R_2 are rational functions and $\wp$ has the same set of periods as f.

Proof.

$$
\begin{aligned}
f(z) &= \underbrace{\frac{f(z)+f(-z)}{2}}_{\text{even}} + \wp'(z)\underbrace{\frac{f(z)-f(-z)}{2\wp'(z)}}_{\text{even}} \\
&= R_1[\wp(z)] + \wp'(z)R_2[\wp(z)] \text{ for some rational functions } R_1, R_2
\end{aligned}
$$

(by Exercise 1.4). □

Exercise 1.6.

Let f and g be two elliptic functions with the same set of periods. Prove that there exists a polynomial $P(x, y)$, not identically zero, such that

$$P[f(z), g(z)] = C$$

where C is a constant (depending on f and g but not on z).

Proof.

(1) By Exercise 1.5, we have

$$f(z) = R_1[\wp(z)] + \wp'(z)R_2[\wp(z)]$$

for some rational functions R_1, R_2 and $\wp$ has the same set of periods as f. By cleaning the denominators of R_1 and R_2, we might assume

$$S[\wp(z)]f(z) = R_1[\wp(z)] + \wp'(z)R_2[\wp(z)]$$

for some polynomials R_1, R_2, S.

(2) So

$$
\begin{aligned}
&\wp'(z)R_2[\wp(z)] = S[\wp(z)]f(z) - R_1[\wp(z)] \\
\Longrightarrow\ &\wp'(z)^2R_2[\wp(z)]^2 = (S[\wp(z)]f(z) - R_1[\wp(z)])^2 \\
\Longrightarrow\ &(4\wp(z)^3 - 60G_4\wp(z) - 140G_6)R_2[\wp(z)]^2 \\
&\quad = (S[\wp(z)]f(z) - R_1[\wp(z)])^2. && \text{(Theorem 1.12)} \\
\Longrightarrow\ &F(\wp(z), f(z)) = 0
\end{aligned}
$$

for some polynomials $F(x, y) \in \mathbb{C}[x, y]$. Note that $F(x, y)$ is of degree 2 if we view $F \in (\mathbb{C}[x])[y]$.

(3) Similarly,

$$G(\wp(z), g(z)) = 0$$

for some polynomials $G(x, y) \in \mathbb{C}[x, y]$.

(4) Let $P = \mathrm{Res}_x(F, G)$ be the resultant of two polynomials F and G with respect t x to eliminate $\wp(z)$. Note that P is a nonzero polynomial (since F and G are nonzero) and $P[f(z), g(z)] = 0$. So P is our desired polynomial.

$\square$

Exercise 1.7.

The discriminant of the polynomial $f(x) = 4(x - x_1)(x - x_2)(x - x_3)$ is the product $16\{(x_2 - x_1)(x_3 - x_2)(x_3 - x_1)\}^2$. Prove that the discriminant of $f(x) = 4x^3 - ax - b$ is $a^3 - 27b^2$.

Proof.

(1) Since

$$f'(x) = 4(x - x_2)(x - x_3) + 4(x - x_1)(x - x_3) + 4(x - x_1)(x - x_2),$$

we have

$$\begin{aligned} f'(x_1) &= 4(x_1 - x_2)(x_1 - x_3), \\ f'(x_2) &= 4(x_2 - x_1)(x_2 - x_3), \\ f'(x_3) &= 4(x_3 - x_1)(x_3 - x_2). \end{aligned}$$

Hence

$$f'(x_1)f'(x_2)f'(x_3) = -4\mathrm{disc}(f)$$

where $\mathrm{disc}(f)$ is the discriminant of $f(x)$.

(2) As $f(x) = 4x^3 - ax - b$, we have $f'(x) = 12x^2 - a$. So

$$f'(x_1)f'(x_2)f'(x_3) = (12x_1^2 - a)(12x_2^2 - a)(12x_3^2 - a).$$

Note that

$$\begin{aligned} x_1x_2x_3 &= \frac{b}{4}, \\ x_1x_2 + x_2x_3 + x_3x_1 &= -\frac{a}{4}, \\ x_1 + x_2 + x_3 &= 0, \end{aligned}$$

we have

$$\begin{aligned} x_1^2x_2^2x_3^2 &= \frac{b^2}{4^2}, \\ x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_1^2 &= (x_1x_2 + x_2x_3 + x_3x_1)^2 - 2x_1x_2x_3(x_1 + x_2 + x_3) \\ &= \frac{a^2}{4^2}, \\ x_1^2 + x_2^2 + x_3^2 &= (x_1 + x_2 + x_3)^2 - 2(x_1x_2 + x_2x_3 + x_3x_1) \\ &= \frac{a}{2}. \end{aligned}$$

(3) Hence

$$\begin{aligned}
f'(x_1)f'(x_2)f'(x_3) &= (12x_1^2 - a)(12x_2^2 - a)(12x_3^2 - a) \\
&= 12^3(x_1^2x_2^2x_3^2) - 12^2a(x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_1^2) \\
&\quad + 12a^2(x_1^2 + x_2^2 + x_3^2) - a^3 \\
&= 12^3 \cdot \frac{b^2}{4^2} - 12^2a \cdot \frac{a^2}{4^2} + 12a^2 \cdot \frac{a}{2} - a^3 \\
&= -4(a^3 - 27b^2).
\end{aligned}$$

Therefore

$$\operatorname{disc}(4x^3 - ax - b) = a^3 - 27b^2.$$

□

Exercise 1.8.

The differential equation for $\wp$ *shows that* $\wp'(z) = 0$ *if* $z = \frac{\omega_1}{2}, \frac{\omega_2}{2}$ *or* $\frac{\omega_1+\omega_2}{2}$. *Show that*

$$\wp''\left(\frac{\omega_1}{2}\right) = 2(e_1 - e_2)(e_1 - e_3)$$

and obtain corresponding formulas for $\wp''\left(\frac{\omega_2}{2}\right)$ *and* $\wp''\left(\frac{\omega_1+\omega_2}{2}\right)$.

Proof.

(1) Differentiation of the equation

$$4\wp(z)^3 - g_2\wp(z) - g_3 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$$

in Theorem 1.14 to get

$$\begin{aligned}
12\wp(z)^2\wp'(z) - g_2\wp'(z) &= 4\wp'(z)(\wp(z) - e_2)(\wp(z) - e_3) \\
&\quad + 4\wp'(z)(\wp(z) - e_1)(\wp(z) - e_3) \\
&\quad + 4\wp'(z)(\wp(z) - e_1)(\wp(z) - e_2).
\end{aligned}$$

Since $\wp''(z) = 6\wp(z)^2 - \frac{g_2}{2}$, we have

$$\begin{aligned}
\wp''(z) &= 2(\wp(z) - e_2)(\wp(z) - e_3) \\
&\quad + 2(\wp(z) - e_1)(\wp(z) - e_3) \\
&\quad + 2(\wp(z) - e_1)(\wp(z) - e_2).
\end{aligned}$$

(2) Hence

$$\begin{aligned}
\wp''\left(\frac{\omega_1}{2}\right) &= 2(e_1 - e_2)(e_1 - e_3), \\
\wp''\left(\frac{\omega_2}{2}\right) &= 2(e_2 - e_1)(e_2 - e_3), \\
\wp''\left(\frac{\omega_1 + \omega_2}{2}\right) &= 2(e_3 - e_1)(e_3 - e_2).
\end{aligned}$$

□

Exercise 1.9.

According to Exercise 1.4, the function $\wp(2z)$ *is a rational function of* $\wp(z)$*. Prove that, in fact,*

$$\wp(2z) = \frac{\{\wp(z)^2 + \frac{1}{4}g_2\}^2 + 2g_3\wp(z)}{4\wp(z)^3 - g_2\wp(z) - g_3} = -2\wp(z) + \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2.$$

Proof.

(1) By $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$ and $\wp''(z) = 6\wp(z)^2 - \frac{1}{2}g_2$, we have

$$\begin{aligned}
&-2\wp(z) + \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2 \\
=& -2\wp(z) + \frac{1}{4}\cdot\frac{(6\wp(z)^2 - \frac{1}{2}g_2)^2}{4\wp(z)^3 - g_2\wp(z) - g_3} \\
=& \frac{-2\wp(z)[4\wp(z)^3 - g_2\wp(z) - g_3] + \frac{1}{4}(6\wp(z)^2 - \frac{1}{2}g_2)^2}{4\wp(z)^3 - g_2\wp(z) - g_3} \\
=& \frac{\wp(z)^4 + \frac{1}{2}g_2\wp(z)^2 + \frac{1}{16}g_2^2 + 2g_3\wp(z)}{4\wp(z)^3 - g_2\wp(z) - g_3} \\
=& \frac{\{\wp(z)^2 + \frac{1}{4}g_2\}^2 + 2g_3\wp(z)}{4\wp(z)^3 - g_2\wp(z) - g_3}.
\end{aligned}$$

So it suffices to show that $\wp(2z) = -2\wp(z) + \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2$.

(2) Let Ω be the lattice generated by periods of $\wp$. Suppose the addition theorem of $\wp$ holds, that is,

$$\wp(u) + \wp(v) + \wp(u+v) = \frac{1}{4}\left(\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\right)^2$$

with $u, v, u+v \not\equiv 0 \pmod{\Omega}$. Then letting v to u, we have

$$\begin{aligned}
\wp(2u) &= \lim_{v\to u}\wp(u+v) \\
&= \lim_{v\to u}\left\{-\wp(u) - \wp(v) + \frac{1}{4}\left(\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\right)^2\right\} \\
&= -2\wp(u) + \frac{1}{4}\lim_{v\to u}\frac{\wp'(v) - \wp'(u)}{\wp(v) - \wp(u)} \\
&= -2\wp(u) + \frac{1}{4}\left(\frac{\wp''(u)}{\wp'(u)}\right)^2.
\end{aligned}$$

The last equality is followed by L'Hospital's rule. So it suffices to show the addition theorem of $\wp$ is true.

(3) *Let* $u+v+w=0$*, with* $u,v,w \not\equiv 0 \pmod{\Omega}$*. Show that*

$$\begin{vmatrix} \wp(u) & \wp'(u) & 1 \\ \wp(v) & \wp'(v) & 1 \\ \wp(w) & \wp'(w) & 1 \end{vmatrix} = 0.$$

Consider the elliptic function

$$f(z) := \wp'(z) - \underbrace{\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}}_{:=A}\wp(z) - \underbrace{\frac{\wp(u)\wp'(v)-\wp(v)\wp'(u)}{\wp(u)-\wp(v)}}_{:=B}.$$

f has exactly 3 zeros in a period parallelogram as f has order 3. Note that f has a pole at 0 of order 3. By Exercise 1.2, the sum of the zeros is equal to the sum of poles in a period parallelogram. Since u and v are zeros of f (by verifying $f(u)=f(v)=0$ directly), the third zero must be $-u-v=w$. Hence there is a line

$$y = Ax + B$$

passing through 3 points $(\wp(u),\wp'(u))$, $(\wp(v),\wp'(v))$ and $(\wp(w),\wp'(w))$. So the determinant is zero.

(4) Now we are going to remove the term $\wp'(w)$ to prove the addition theorem of $\wp$. By Theorem 1.12, we have the system of equations

$$\begin{cases} y = Ax + B, \\ y^2 = 4x^3 - g_2 x - g_3, \end{cases}$$

where $(x,y) = (\wp(z),\wp'(z))$. Hence

$$\begin{aligned}
&(Ax+B)^2 = 4x^3 - g_2x - g_3 \\
\iff\, & 4x^3 - A^2x^2 - (2AB+g_2)x - (B^2+g_3) = 0 \\
\implies & \text{sum of three roots of } x \text{ is } \frac{A^2}{4} \\
\implies & \wp(u)+\wp(v)+\wp(w) = \frac{1}{4}\left(\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right)^2 \\
\implies & \wp(-u-v) = -\wp(u)-\wp(v)+\frac{1}{4}\left(\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right)^2 \\
\implies & \wp(u+v) = -\wp(u)-\wp(v)+\frac{1}{4}\left(\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right)^2. \qquad (\wp\text{: even})
\end{aligned}$$

So the addition theorem of $\wp$ is established.

$\square$

Note.

(1) In the proof, part (4) is similar to defining an addition $\oplus$ on a nonsingular cubic E in $\mathbf{P}^2(k)$. It is equivalent to defining the divisor class group on E. See Problem 8.6 in the textbook: *William Fulton, Algebraic Curves.*

(2) If $E \in \mathbf{P}^2(\mathbb{C})$ is the elliptic curve corresponding to the lattice Ω, then there is an isomorphism

$$\alpha : \mathbb{C}/\Omega \longrightarrow E : y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3$$

defined by

$$\alpha(z) = \begin{cases} [\wp(z) : \wp'(z) : 1] & \text{if } z \neq 0 \in \Omega, \\ [0 : 1 : 0] & \text{if } z = 0 \in \Omega, \end{cases}$$

such that α is both analytic (as a mapping of complex manifolds) and algebraic (as a homomorphism of groups).

Exercise 1.10.

Let ω_1 and ω_2 be complex numbers with nonreal ratio. Let $f(z)$ be an entire function and assume there are constants a and b such that

$$f(z + \omega_1) = af(z), \qquad f(z + \omega_2) = bf(z),$$

for all z. Prove that $f(z) = Ae^{Bz}$, where A and B are constant.

Proof.

(1) Might assume that $a \neq 0$ and $b \neq 0$ (otherwise $f = 0$ on $\mathbb{C}$).

(2) Define

$$g(z) := \frac{f(z)}{e^{Bz}}.$$

It suffices to show g is a constant. Note that $g(z)$ is entire (since f and $e^{Bz} \neq 0$ are entire). By Theorem 1.4, it suffices to show g is doubly periodic, that is, to show

$$g(z + \omega_1) = g(z) \text{ and } g(z + \omega_2) = g(z)$$

for suitable B.

(3) Note that

$$
\begin{aligned}
& g(z+\omega_1) = g(z) \text{ and } g(z+\omega_2) = g(z) \\
\Longleftrightarrow\ & \frac{a}{e^{B\omega_1}} \cdot g(z) = g(z) \text{ and } \frac{b}{e^{B\omega_2}} \cdot g(z) = g(z) \\
\Longleftrightarrow\ & e^{B\omega_1} = a \text{ and } e^{B\omega_2} = b \\
\Longleftrightarrow\ & \exists\, B \text{ such that } e^{B\omega_1} = a \text{ and } e^{B\omega_2} = b.
\end{aligned}
$$

Take B such that $e^{B(\omega_1-\omega_2)} = \frac{a}{b}$ (since $\frac{a}{b}$ is well-defined, $\omega_1 - \omega_2 \neq 0$, and $z \mapsto \exp(z)$ is a onto map from $\mathbb{C}$ to $\mathbb{C} \smallsetminus \{0\}$). Hence g is doubly periodic.

□

Exercise 1.11.

If $k \geq 2$ and $\tau \in H$ prove that the Eisenstein series

$$G_{2k}(\tau) = \sum_{(m,n)\neq(0,0)} (m+n\tau)^{-2k}$$

has the Fourier expansion

$$G_{2k}(\tau) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) e^{2\pi i n\tau}.$$

Proof.

(1) Let $q = e^{2\pi i\tau}$. Similar to Lemma 1.3 on page 19, we have

$$(2k-1)! \sum_{m=-\infty}^{+\infty} \frac{1}{(\tau+m)^{2k}} = (2\pi i)^{2k} \sum_{r=1}^{\infty} r^{2k-1} q^r.$$

(2) Similar to Theorem 1.18, we have

$$\begin{aligned}
G_{2k}(\tau) &= \sum_{(m,n)\neq(0,0)} (m+n\tau)^{-2k} \\
&= \sum_{\substack{m=-\infty \\ m\neq 0 (n=0)}}^{+\infty} m^{-2k} + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{+\infty} ((m+n\tau)^{-2k} + (m-n\tau)^{-2k}) \\
&= 2\zeta(2k) + 2\sum_{n=1}^{\infty} \sum_{m=-\infty}^{+\infty} (m+n\tau)^{-2k} \\
&= 2\zeta(2k) + 2\sum_{n=1}^{\infty} \frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{r=1}^{\infty} r^{2k-1} q^{nr} \\
&= 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \underbrace{\sum_{d|n} d^{2k-1}}_{=\sigma_{2k-1}(n)} q^n.
\end{aligned}$$

In the last double sum we collect together those terms for which nr is constant.

□

Exercise 1.12.

Refer to Exercise 1.11. If $\tau \in H$ prove that

$$G_{2k}\left(-\frac{1}{\tau}\right) = \tau^{2k} G_{2k}(\tau)$$

and deduce that

$$\begin{aligned}
G_{2k}\left(\frac{i}{2}\right) &= (-4)^k G_{2k}(2i) && \text{for all } k \geq 2, \\
G_{2k}(i) &= 0 && \text{if } k \text{ is odd}, \\
G_{2k}(e^{\frac{2\pi i}{3}}) &= 0 && \text{if } k \not\equiv 0 \pmod 3.
\end{aligned}$$

Proof.

(1)

$$\begin{aligned} G_{2k}\left(-\frac{1}{\tau}\right) &= \sum_{(m,n)\neq(0,0)} \left(m - \frac{n}{\tau}\right)^{-2k} \\ &= \tau^{2k} \sum_{(m,n)\neq(0,0)} (\tau m - n)^{-2k} \\ &= \tau^{2k} G_{2k}(\tau). \end{aligned}$$

(2) Let $\tau = 2i$. We have $G_{2k}\left(\frac{i}{2}\right) = (-4)^k G_{2k}(2i)$.

(3) Let $\tau = i$. We have $G_{2k}(i) = (-1)^k G_{2k}(i)$. Hence $G_{2k}(i) = 0$ if k is odd.

(4) Let $\tau = e^{\frac{\pi i}{3}}$. We have $G_{2k}(e^{\frac{2\pi i}{3}}) = e^{\frac{2k\pi i}{3}} G_{2k}(e^{\frac{\pi i}{3}})$. Since

$$e^{\frac{2\pi i}{3}} = -1 + e^{\frac{\pi i}{3}}$$

and each Eisenstein series is a periodic function of τ of period 1, we have $G_{2k}(e^{\frac{2\pi i}{3}}) = G_{2k}(e^{\frac{\pi i}{3}})$. So $G_{2k}(e^{\frac{2\pi i}{3}}) = e^{\frac{2k\pi i}{3}} G_{2k}(e^{\frac{2\pi i}{3}})$. Therefore $G_{2k}(e^{\frac{2\pi i}{3}}) = 0$ if $k \not\equiv 0 \pmod 3$.

□

Exercise 1.13.

Ramanujan's tau function $\tau(n)$ is defined by the Fourier expansion

$$\Delta(\tau) = (2\pi)^{12} \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n \tau},$$

derived in Theorem 1.19. Prove that

$$\tau(n) = 8000\{(\sigma_3 \circ \sigma_3) \circ \sigma_3\}(n) - 147(\sigma_5 \circ \sigma_5)(n),$$

where $f \circ g$ denotes the Cauchy product of two sequences,

$$(f \circ g)(n) = \sum_{k=0}^{n} f(k) g(n-k),$$

and $\sigma_\alpha(n) = \sum_{d \mid n} d^\alpha$ for $n \geq 1$, with $\sigma_3(0) = \frac{1}{240}$, $\sigma_5(0) = -\frac{1}{504}$. (Hint: Theorem 1.18.)

Proof.

(1) Let $q = e^{2\pi i\tau}$. Write

$$g_2(\tau) = \frac{4\pi^4}{3}\left\{1 + 240\sum_{k=1}^{\infty}\sigma_3(k)q^k\right\} = \frac{4\pi^4}{3}\left\{240\sum_{k=0}^{\infty}\sigma_3(k)q^k\right\},$$
$$g_3(\tau) = \frac{8\pi^6}{27}\left\{1 - 504\sum_{k=1}^{\infty}\sigma_5(k)q^k\right\} = \frac{8\pi^6}{27}\left\{-504\sum_{k=0}^{\infty}\sigma_5(k)q^k\right\}$$

(Theorem 1.18).

(2) Similar to the proof of Theorem 1.19,

$$\begin{aligned}\Delta(\tau) &= g_2(\tau)^3 - 27g_3(\tau)^2 \\ &= \frac{64\pi^{12}}{27}\left\{\left(240\sum_{k=0}^{\infty}\sigma_3(k)q^k\right)^3 - \left(-504\sum_{k=0}^{\infty}\sigma_5(k)q^k\right)^2\right\} \\ &= (2\pi)^{12}\left\{8000\left(\sum_{k=0}^{\infty}\sigma_3(k)q^k\right)^3 - 147\left(\sum_{k=0}^{\infty}\sigma_5(k)q^k\right)^2\right\} \\ &= (2\pi)^{12}\sum_{n=0}^{\infty}\left\{8000\{(\sigma_3\circ\sigma_3)\circ\sigma_3\}(n) - 147(\sigma_5\circ\sigma_5)(n)\right\}q^n \\ &= (2\pi)^{12}\sum_{n=1}^{\infty}\left\{8000\{(\sigma_3\circ\sigma_3)\circ\sigma_3\}(n) - 147(\sigma_5\circ\sigma_5)(n)\right\}q^n.\end{aligned}$$

(Here $8000\{(\sigma_3\circ\sigma_3)\circ\sigma_3\}(0) - 147(\sigma_5\circ\sigma_5)(0) = 0$.)

(3) Therefore

$$\tau(n) = 8000\{(\sigma_3\circ\sigma_3)\circ\sigma_3\}(n) - 147(\sigma_5\circ\sigma_5)(n)$$

for $n \geq 1$.

□

Exercise 1.14. (Lambert series)

*A series of the form $\sum_{n=1}^{\infty} f(n)\frac{x^n}{1-x^n}$ is called a **Lambert series**. Assuming absolute convergence, prove that*

$$\sum_{n=1}^{\infty} f(n)\frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} F(n)x^n,$$

where

$$F(n) = \sum_{d\mid n} f(d).$$

Apply this result to obtain the following formulas, valid for $|x| < 1$.

(a)
$$\sum_{n=1}^{\infty} \frac{\mu(n)x^n}{1-x^n} = x.$$

(b)
$$\sum_{n=1}^{\infty} \frac{\varphi(n)x^n}{1-x^n} = \frac{x}{(1-x)^2}.$$

(c)
$$\sum_{n=1}^{\infty} \frac{n^\alpha x^n}{1-x^n} = \sum_{n=1}^{\infty} \sigma_\alpha(n)x^n.$$

(d)
$$\sum_{n=1}^{\infty} \frac{\lambda(n)x^n}{1-x^n} = \sum_{n=1}^{\infty} x^{n^2}.$$

(e) *Use the result in (c) to express $g_2(\tau)$ and $g_3(\tau)$ in terms of Lambert series in $x = e^{2\pi i\tau}$.*

Note. In (a), $\mu(n)$ is the Möbius function; In (b), $\varphi(n)$ is Euler's totient; and in (d), $\lambda(n)$ is Liouville's function.

Proof. Similar to the proof of Exercise 1.11.
$$\begin{aligned}\sum_{n=1}^{\infty} f(n)\frac{x^n}{1-x^n} &= \sum_{n=1}^{\infty} f(n)\sum_{r=1}^{\infty} x^{rn} \\ &= \sum_{n=1}^{\infty}\sum_{r=1}^{\infty} f(n)x^{rn} \\ &= \sum_{n=1}^{\infty} \underbrace{\left(\sum_{d|n} f(d)\right)}_{=F(n)} x^n.\end{aligned}$$

□

Proof of (a). Theorem 2.1 in the textbook: *T. M. Apostol, Introduction to Analytic Number Theory* shows that
$$F(n) := \sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

Hence
$$\sum_{n=1}^{\infty} \mu(n)\frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} F(n)x^n = x.$$

□

Proof of (b). Theorem 2.2 in the textbook: *T. M. Apostol, Introduction to Analytic Number Theory* shows that $F(n) := \sum_{d|n} \varphi(d) = n$. Hence

$$\sum_{n=1}^{\infty} \varphi(n) \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} F(n) x^n = \sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}.$$

□

Proof of (c). Since

$$F(n) := \sum_{d|n} d^{\alpha} = \sigma_{\alpha}(n),$$

we have

$$\sum_{n=1}^{\infty} n^{\alpha} \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} F(n) x^n = \sum_{n=1}^{\infty} \sigma_{\alpha}(n) x^n.$$

□

Proof of (d). Theorem 2.19 in the textbook: *T. M. Apostol, Introduction to Analytic Number Theory* shows that

$$F(n) := \sum_{d|n} \lambda(d) = \begin{cases} 1 & \text{if } n \text{ is a square,} \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\sum_{n=1}^{\infty} \lambda(n) \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} F(n) x^n = \sum_{n=1}^{\infty} x^{n^2}.$$

□

Proof of (e).

(1) Let $q = x = e^{2\pi i \tau}$.

$$\begin{aligned} g_2(\tau) &= \frac{4\pi^4}{3} \left\{ 1 + 240 \sum_{k=1}^{\infty} \sigma_3(k) q^k \right\} && \text{(Theorem 1.18)} \\ &= \frac{4\pi^4}{3} \left\{ 1 + 240 \sum_{k=1}^{\infty} \frac{k^3 q^k}{1-q^k} \right\} && \text{((c)).} \end{aligned}$$

(2) Similarly,

$$\begin{aligned} g_3(\tau) &= \frac{8\pi^6}{27} \left\{ 1 - 504 \sum_{k=1}^{\infty} \sigma_5(k) q^k \right\} && \text{(Theorem 1.18)} \\ &= \frac{8\pi^6}{27} \left\{ 1 - 504 \sum_{k=1}^{\infty} \frac{k^5 q^k}{1-q^k} \right\} && \text{((c)).} \end{aligned}$$

$\square$

Note.

(1)

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)x^n}{1-x^n} = \sum_{n=1}^{\infty} \log(n)x^n,$$

where $\Lambda(n)$ is von Mangoldt function.

(2) Similar to Exercise 1.15, we have a similar formula for (a)

$$\sum_{n=1}^{\infty} \frac{\mu(n)x^n}{1+x^n} = x - 2x^2$$

by noting that

$$\sum_{n=1}^{\infty} \frac{f(n)x^n}{1+x^n} = \sum_{n=1}^{\infty} \frac{f(n)x^n}{1-x^n} - 2\sum_{n=1}^{\infty} \frac{f(n)x^{2n}}{1-x^{2n}}.$$

Exercise 1.15.

Let

$$G(x) = \sum_{n=1}^{\infty} \frac{n^5 x^n}{1-x^n},$$

and let

$$F(x) = \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^5 x^n}{1+x^n}.$$

(a) *Prove that* $F(x) = G(x) - 34G(x^2) + 64(x^4)$.

(b) *Prove that*

$$\sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^5}{1+e^{n\pi}} = \frac{31}{504}.$$

(c) *Use Theorem 12.17 in the textbook: T. M. Apostol, Introduction to Analytic Number Theory to prove the more general result*

$$\sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^{4k+1}}{1+e^{n\pi}} = \frac{2^{4k+1}-1}{8k+4} B_{4k+2}.$$

Proof of (a).

(1) Consider the general case. *Let*

$$G(x) = \sum_{n=1}^{\infty} \frac{n^{4k+1}x^n}{1-x^n},$$

and let

$$F(x) = \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^{4k+1}x^n}{1+x^n}.$$

Show that $F(x) = G(x) - (2^{4k+1}+2)G(x^2) + 2^{4k+2}G(x^4)$.

(2) The identity

$$\sum_{n=1}^{\infty} \frac{x^n}{1+x^n} = \sum_{n=1}^{\infty} \frac{x^n}{1-x^n} - 2\sum_{n=1}^{\infty} \frac{x^{2n}}{1-x^{2n}}$$

is always true. Hence $H(x) := \sum_{n=1}^{\infty} \frac{n^{4k+1}x^n}{1+x^n} = G(x) - 2G(x^2)$.

(3) Note that

$$\begin{aligned}
H(x) &= \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^{4k+1}x^n}{1+x^n} + \sum_{\substack{n=1 \\ (n \text{ even})}}^{\infty} \frac{n^{4k+1}x^n}{1+x^n} \\
&= F(x) + \sum_{n=1}^{\infty} \frac{(2n)^{4k+1}x^{2n}}{1+x^{2n}} \\
&= F(x) + 2^{4k+1}\sum_{n=1}^{\infty} \frac{n^{4k+1}x^{2n}}{1+x^{2n}} \\
&= F(x) + 2^{4k+1}H(x^2).
\end{aligned}$$

Hence

$$\begin{aligned}
F(x) &= H(x) - 2^{4k+1}H(x^2) \\
&= [G(x) - 2G(x^2)] - 2^{4k+1}[G(x^2) - 2G(x^4)] \\
&= G(x) - (2^{4k+1}+2)G(x^2) + 2^{4k+2}G(x^4).
\end{aligned}$$

□

Proof of (b). Take $k = 1$ in part (c), we have

$$\sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^5}{1+e^{n\pi}} = \frac{31}{12} \cdot \frac{1}{42} = \frac{31}{504}.$$

□

Proof of (c).

(1) Let $q = e^{2\pi i\tau}$. So

$$\begin{aligned}
G_{4k+2}(\tau) &= 2\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!}\sum_{n=1}^{\infty}\sigma_{4k+1}(n)q^n && \text{(Exercise 1.11)} \\
&= 2\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!}G(q) && \text{(Exercise 1.14(c))}
\end{aligned}$$

Hence

$$\begin{aligned}
&G_{4k+2}(\tau) - (2^{4k+1}+2)G_{4k+2}(2\tau) + 2^{4k+2}G_{4k+2}(4\tau) \\
= &\left[2\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!}G(q)\right] \\
&- (2^{4k+1}+2)\left[2\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!}G(q^2)\right] \\
&+ 2^{4k+2}\left[2\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!}G(q^4)\right] \\
= &(1 - (2^{4k+1}+2) + 2^{4k+2}) \cdot 2\zeta(4k+2) \\
&+ \frac{2(2\pi i)^{4k+2}}{(4k+1)!}[G(q) - (2^{4k+1}+2)G(q^2) + 2^{4k+2}G(q^4)] \\
= &(2^{4k+2}-2)\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!}F(q).
\end{aligned}$$

(2) By taking $\tau = \frac{i}{2}$, we have

$$F(q) = F(e^{-\pi}) = \sum_{\substack{n=1 \\ (n\text{ odd})}}^{\infty} \frac{n^{4k+1}}{1+e^{n\pi}}$$

and

$$\begin{aligned}
&G_{4k+2}(\tau) - (2^{4k+1}+2)G_{4k+2}(2\tau) + 2^{4k+2}G_{4k+2}(4\tau) \\
= &G_{4k+2}\left(\frac{i}{2}\right) - (2^{4k+1}+2)G_{4k+2}(i) + 2^{4k+2}G_{4k+2}(2i) \\
= &(-4)^{2k+1}G_{4k+2}(2i) - (2^{4k+1}+2)\cdot 0 + 2^{4k+2}G_{4k+2}(2i) \\
= &0.
\end{aligned}$$

(Exercise 1.12). Hence

$$0 = (2^{4k+2}-2)\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!}\sum_{\substack{n=1 \\ (n\text{ odd})}}^{\infty} \frac{n^{4k+1}}{1+e^{n\pi}}.$$

(3) Theorem 12.17 in the textbook: *T. M. Apostol, Introduction to Analytic Number Theory* shows that

$$\zeta(4k+2) = (-1)^{2k+1+1}\frac{(2\pi)^{4k+2}B_{4k+2}}{2(4k+2)!} = \frac{(2\pi)^{4k+2}B_{4k+2}}{2(4k+2)!}.$$

Hence

$$\sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^{4k+1}}{1+e^{n\pi}} = \frac{2^{4k+1}-1}{8k+4} B_{4k+2}.$$

□

Chapter 2: The modular group and modular functions

In these exercise, Γ denotes the modular group, S and T denote its generators $S(\tau) = -\frac{1}{\tau}$, $T(\tau) = \tau + 1$, and I denotes the identity element.

Exercise 2.2.

Find the smallest integer $n > 0$ such that $(ST)^n = I$.

Proof.

(1) $n = 3$.

(2) Write
$$ST = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$
So
$$(ST)^2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix},$$
$$(ST)^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = I \in \Gamma.$$
Here we identify each matrix with its negative, since both of them represent the same transformation.

□

Exercise 2.4.

Determine all elements A of Γ which leave i fixed.

Proof.

(1) *Show that $A = I, S \in \Gamma$ fixes i.* Say $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ satisfying $Ai = i$. Then $Ai = \frac{ai+b}{ci+d} = i$ implies that $ai + b = ci^2 + di = di - c$ or $b = -c$ and $a = d$. Hence $1 = \det(A) = ad - bc = a^2 + b^2$. Therefore $(a, b) = (\pm 1, 0)$ or $(0, \pm 1)$.

(2) The case $a = \pm 1$ and $b = 0$. $A = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (by identifying each matrix with its negative).

(3) The case $b = \pm 1$ and $a = 0$. $A = S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (by identifying each matrix with its negative).

□

Exercise 2.5.

Determine all elements A of Γ which leave $\rho = e^{\frac{2\pi i}{3}}$ fixed.

Proof.

(1) *Show that $A = I, S \in \Gamma$ fixes ρ.* Say $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ satisfying $A\rho = \rho$. Then $A\rho = \frac{a\rho+b}{c\rho+d} = \rho$ implies that $a\rho + b = c\rho^2 + d\rho = c(-\rho - 1) + d\rho = (-c + d)\rho - c$ or $a = -c + d$ and $b = -c$.

(2) So $1 = \det(A) = ad - bc = (-c + d)d + c^2 = c^2 - cd + d^2$, or $1 = \left(c - \frac{d}{2}\right)^2 + \frac{3}{4}d^2$, or $4 = (2c - d)^2 + 3d^2$. Hence $(c, d) = (\pm 1, 0)$, $(0, \pm 1)$.

(3) The case $c = \pm 1$ and $d = 0$. $A = S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (by identifying each matrix with its negative).

(4) The case $c = 0$ and $d = \pm 1$. $A = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (by identifying each matrix with its negative).

□

Quadratic forms and the modular group

*The following exercises relate quadratic forms and the modular group Γ. We consider quadratic forms $Q(x, y) = ax^2 + bxy + cy^2$ in x and y with real coefficients a, b, c. The number $d = 4ac - b^2$ is called the **discriminant** of $Q(x, y)$.*

Exercise 2.6.

If x and y are subjected to unimodular transformation, say

$$x = \alpha x' + \beta y', \qquad y = \gamma x' + \delta y', \qquad \textit{where } \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma, \qquad (*)$$

prove that $Q(x,y)$ gets transformed to a quadratic form $Q_1(x',y')$ having the same discriminant. Two forms $Q(x,y)$ and $Q_1(x',y')$ so related are called ***equivalent****. This equivalence relation separates all forms into equivalence classes. The forms in a given class has the same discriminant, and they represent the same integers. That is, if $Q(x,y) = n$ for some pair of integers x and y, then $Q_1(x',y') = n$ for the pair of integers x' and y' givin by (*).*

Proof.

(1) Write

$$Q(x,y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Thus the discriminant of $Q(x,y)$ is $4\det \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$.

(2) Hence

$$\begin{aligned} Q_1(x',y') &= Q(\alpha x' + \beta y', \gamma x' + \delta y') \\ &= \begin{pmatrix} \alpha x' + \beta y' & \gamma x' + \delta y' \end{pmatrix} \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \begin{pmatrix} \alpha x' + \beta y' \\ \gamma x' + \delta y' \end{pmatrix} \\ &= \begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}. \end{aligned}$$

Thus the discriminant of $Q_1(x',y')$ is

$$\begin{aligned} & 4\det \left(\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right) \\ &= 4\det \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \det \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \\ &= 4(\underbrace{\alpha\delta - \beta\gamma}_{=\pm 1})^2 \det \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \\ &= 4\det \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}, \end{aligned}$$

which is the same as the discriminant of $Q(x,y)$.

□

Congruence subgroups

The modular group Γ has many subgroups of special interest in number theory. The following exercises deal with a class of subgroups call ***congruence*** *subgroups.*

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

be two unimodular matrices. (In this discussion we do not identify a matrix with its negative.) If n *is a positive integer write*

$$A \equiv B \pmod{n} \text{ whenever } a \equiv \alpha,\ b \equiv \beta,\ c \equiv \gamma \text{ and } d \equiv \delta \pmod{n}.$$

This defines an equivalence relation with the property that

$$A_1 \equiv A_2 \pmod{n} \text{ and } B_1 \equiv B_2 \pmod{n}$$

implies

$$A_1 B_1 \equiv A_2 B_2 \pmod{n} \text{ and } A_1^{-1} \equiv A_2^{-1} \pmod{n}.$$

Hence

$$A \equiv B \pmod{n} \text{ if, and only if, } AB^{-1} \equiv I \pmod{n},$$

where I *is the identity matrix. We denote by* $\Gamma^{(n)}$ *the set of all matrices in* Γ *congruent modulo* n *to the identity. This is called the* ***congruence subgroup of level*** n*.*

Prove each of the following statements:

Exercise 2.11.

$\Gamma^{(n)}$ *is a subgroup of* Γ*. Moreover, if* $B \in \Gamma^{(n)}$ *then* $A^{-1}BA \in \Gamma^{(n)}$ *for every* A *in* Γ*. That is,* $\Gamma^{(n)}$ *is a normal subgroup of* Γ*.*

Proof.

(1) *Show that* $\Gamma^{(n)}$ *is a subgroup of* Γ*.* $\Gamma^{(n)} \neq \varnothing$ since $I \in \Gamma^{(n)}$. Suppose $A, B \in \Gamma^{(n)}$, that is, $A \equiv I \pmod{n}$ and $B \equiv I \pmod{n}$. Hence $AB^{-1} \equiv II^{-1} \equiv I \pmod{n}$ or $AB^{-1} \in \Gamma^{(n)}$.

(2) *Show that* $\Gamma^{(n)}$ *is normal in* Γ*.* Note that

$$A^{-1}BA \equiv A^{-1}IA \equiv A^{-1}A \equiv I \pmod{n}$$

for every $B \in \Gamma^{(n)}$ and A in Γ. Hence $A^{-1}BA \in \Gamma^{(n)}$.

$\square$

Exercise 2.12.

The quotient group $\Gamma/\Gamma^{(n)}$ is finite. That is, there exist a finite number of elements of Γ, say $A_1, \ldots, A_k$, such that every B in Γ is representable in the form

$$B = A_i B^{(n)} \text{ where } 1 \leq i \leq k \text{ and } B^{(n)} \in \Gamma^{(n)}.$$

The smallest such k is called the index of $\Gamma^{(n)}$ in Γ.

Proof.

(1) Consider the exact sequence

$$1 \to \Gamma^{(n)} \to SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/n\mathbb{Z}) \to 1.$$

The surjectivity of the residue class map is proved in Exercise 2.14.

(2) Hence $\Gamma/\Gamma^{(n)} \cong SL_2(\mathbb{Z}/n\mathbb{Z})$ is a finite group.

□

Exercise 2.13.

The index of $\Gamma^{(n)}$ in Γ is the number of equivalence classes of matrices modulo n.

Proof. The index is the number of all cosets of $\Gamma/\Gamma^{(n)} = |SL_2(\mathbb{Z}/n\mathbb{Z})|$ (by Exercise 2.12). □

The following exercises determine an explicit formula for the index.

Exercise 2.14.

Given integers a, b, c, d with $ad - bc \equiv 1 \pmod{n}$, there exist integers $\alpha, \beta, \gamma, \delta$ such that $\alpha \equiv a$, $\beta \equiv b$, $\gamma \equiv c$ and $\delta \equiv d \pmod{n}$ with $\alpha\delta - \beta\gamma = 1$.

It is equivalent to show that the residue class map

$$SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/n\mathbb{Z})$$

is surjective.

Proof.

(1) Might assume $a \neq 0$. Suppose $a = 0$, we can lift

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & -a \\ d & -c \end{pmatrix}$$

from $SL_2(\mathbb{Z}/n\mathbb{Z})$ to $SL_2(\mathbb{Z})$ (where $b \neq 0$) by the following proof, say

$$\begin{pmatrix} \beta & -\alpha \\ \delta & -\gamma \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Thus

$$\begin{pmatrix} \beta & -\alpha \\ \delta & -\gamma \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

is our desired.

(2) Since $ad - bc \equiv 1 \pmod{n}$, there is an integer $s \in \mathbb{Z}$ such that

$$ad - bc + sn = 1.$$

Note that $a \neq 0$ and $\gcd(a, b, n) = 1$. Take

$$t = \prod_{\substack{p \mid a \\ p \nmid b}} p$$

where p is a prime factor of a. (We take $t = 1$ if $a = \pm 1$.)

(3) Hence $(a, b + tn) = 1$ by the construction of t and $\gcd(a, b, n) = 1$. So 1 is a linear combination of a and $b + tn$. In particular, there exist $u, v \in \mathbb{Z}$ such that

$$ua - v(b + tn) = s + tc.$$

Define

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a & b + tn \\ c + vn & d + un \end{pmatrix}.$$

(4) Therefore $\alpha \equiv a$, $\beta \equiv b$, $\gamma \equiv c$ and $\delta \equiv d \pmod{n}$ and

$$\begin{aligned} \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &= \det \begin{pmatrix} a & b + tn \\ c + vn & d + un \end{pmatrix} \\ &= a(d + un) - (b + tn)(c + vn) \\ &= \underbrace{(ad - bc + sn)}_{=1} + \underbrace{(au - (b + tn)v - s - ct)}_{=0} n \\ &= 1. \end{aligned}$$

□

Exercise 2.15.

If $\gcd(m,n) = 1$ *and* $A \in \Gamma$ *there exists* $\overline{A} \in \Gamma$ *such that*

$$\overline{A} \equiv A \pmod{n}, \qquad \overline{A} \equiv I \pmod{m}.$$

Proof.

(1) Suppose

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(\mathbb{Z}/mn\mathbb{Z}).$$

(2) First we solve α in the system of equations

$$\begin{cases} \alpha \equiv a \pmod{n} \\ \alpha \equiv 1 \pmod{m} \end{cases}$$

The chinese remainder theorem guarantees that α exists. Similarly, β, γ and δ exist.

(3) Note that $\det(B) \equiv 1 \pmod{n}$ and $\det(B) \equiv 1 \pmod{m}$. Hence $\det(B) \equiv 1 \pmod{mn}$ by the chinese remainder theorem. That is, $B \in \Gamma^{(mn)}$. By Exercise 2.14, we can lift $B \in \Gamma^{(mn)}$ to some $\overline{A} \in \Gamma$.

□

Supplement 2.15.1. (Chinese remainder theorem)

(Exercise I.3.5 in the textbook: *Jörgen Neukirch, Algebraic Number Theory.*) *The quotient ring* $\mathcal{O}/\mathfrak{a}$ *of a Dedekind domain by an ideal* $\mathfrak{a} \neq 0$ *is a principal ideal domain. (Hint: For* $\mathfrak{a} = \mathfrak{p}^n$ *the only proper ideals of* $\mathcal{O}/\mathfrak{a}$ *are given by* $\mathfrak{p}/\mathfrak{p}^n, \ldots, \mathfrak{p}^{n-1}/\mathfrak{p}^n$. *Choose* $\pi \in \mathfrak{p} \smallsetminus \mathfrak{p}^2$ *and show that* $\mathfrak{p}^\nu = \mathcal{O}\pi^\nu + \mathfrak{p}^n$*.)*

Proof.

(1) By the Chinese remainder theorem, it suffices to show the case $\mathfrak{a} = \mathfrak{p}^n$ where $\mathfrak{p}$ is prime.

(2) There is a natural correspondence between

$$\{\text{ideals of } \mathcal{O}/\mathfrak{p}^n\} \longleftrightarrow \{\text{ideals of } \mathcal{O} \text{ containing } \mathfrak{p}^n\}.$$

Hence the proper ideals of $\mathcal{O}/\mathfrak{p}^n$ are given by $\mathfrak{p}/\mathfrak{p}^n, \ldots, \mathfrak{p}^{n-1}/\mathfrak{p}^n$.

(3) Similar to Exercise I.3.4, choose $\pi \in \mathfrak{p} \smallsetminus \mathfrak{p}^2$ and thus $\mathfrak{p}^\nu = \mathcal{O}\pi^\nu + \mathfrak{p}^n$ ($\nu = 1, \ldots, n-1$) since they have the same prime factorization. Hence $\mathfrak{p}^\nu/\mathfrak{p}^n = (\pi^\nu + \mathfrak{p}^n)$ is principal.

□

Exercise 2.16.

Let $f(n)$ denote the number of equivalence classes of matrices modulo n. The f is a multiplicative function.

Proof.

(1) Exercise 2.20 shows everything.

(2) Or use the same proof in Exercise 2.15. Suppose $\gcd(m, n) = 1$ and it is equivalent to show $f(mn) = f(m)f(n)$. Define a natural group homomorphism

$$\alpha : SL_2(\mathbb{Z}/mn\mathbb{Z}) \to SL_2(\mathbb{Z}/m\mathbb{Z}) \times SL_2(\mathbb{Z}/m\mathbb{Z}).$$

α is well-defined. So it suffices to show that α is an isomorphism.

(3) Both the injectivity and the surjectivity are guaranteed the chinese remainder theorem. Hence α is isomorphic.

□

Exercise 2.17.

If a, b, n are integers with $n \geq 1$ and $\gcd(a, b, n) = 1$ the congruence

$$ax - by \equiv 1 \pmod{n}$$

has exactly n solutions, distinct congruent modulo n. (A solution is an ordered pair (x, y) of integers.)

Proof.

(1) Write $sa - tb + un = 1$ for some $s, t, u \in \mathbb{Z}$ since $\gcd(a, b, n) = 1$. Hence

$$\begin{aligned} ax - by \equiv 1 \pmod{n} &\Longleftrightarrow ax - by \equiv sa - tb + un \pmod{n} \\ &\Longleftrightarrow a(x - s) \equiv b(y - b) \pmod{n}. \end{aligned}$$

Hence it is equivalent to show that

$$ax \equiv by \pmod{n}$$

has exactly n solutions (upto modulo n).

(2) Start with a fixed y. The linear congruence equation $ax \equiv by \pmod{n}$ is solvable iff $g := \gcd(a, n) \mid (by)$ iff $g \mid y$ (since $\gcd(a, b, n) = \gcd(g, b) = 1$). If so, then x has exactly g solutions (upto modulo n).

(3) Note that there are $\frac{n}{g}$ possible choices of y satisfying $g \mid y$ (upto modulo n), that is, $y = \nu g$ for $1 \leq \nu \leq \frac{n}{g}$. So there are exactly $g \cdot \frac{n}{g} = n$ solutions.

□

Exercise 2.18.

For each prime p the number of solutions, distinct modulo p^r, of all possible congruences of the form

$$ax - by \equiv 1 \pmod{p^r}, \text{ where } \gcd(a, b, p) = 1$$

is equal to $f(p^r)$.

Proof. Note that $\gcd(a, b, p^r) = \gcd(a, b, p) = 1$. So the number of is exactly the same as $|SL_2(\mathbb{Z}/p^r\mathbb{Z})| = f(p^r)$. □

Exercise 2.19.

If p is the number of pairs of integers (a, b), incongruent modulo p^r, which satisfy the condition $\gcd(a, b, p) = 1$ is $p^{2r-2}(p^2 - 1)$.

Proof.

(1) The number is

$$\sum_{d \mid p^r} \mu(d) \left(\frac{p^r}{d}\right)^2 = p^{2r} \sum_{d \mid p^r} \frac{\mu(d)}{d^2} = p^{2r} \left(1 - \frac{1}{p^2}\right) = p^{2r-2}(p^2 - 1)$$

by the definition of the Möbius function μ.

(2) In particular, $f(p^r) = p^r \cdot p^{2r-2}(p^2 - 1) = p^{3r-2}(p^2 - 1)$.

□

Exercise 2.20.

$f(n) = n^3 \sum_{d|n} \frac{\mu(d)}{d^2}$, *where* μ *is the Möbius function.*

Proof.

(1)

$$\begin{aligned} f(n) &= |SL_2(\mathbb{Z}/n\mathbb{Z})| \\ &= n|\{(a,b) \pmod{n} : \gcd(a,b,n) = 1\}| && \text{(Exercise 2.17)} \\ &= n \sum_{d|n} \mu(d) \left(\frac{n}{d}\right)^2 && \text{(Inclusion-exclusion principle)} \\ &= n^3 \sum_{d|n} \frac{\mu(d)}{d^2}. \end{aligned}$$

(2) Since $n \mapsto \frac{1}{n^2}$ is multiplicative, Theorem 2.18 in the textbook: *T. M. Apostol, Introduction to Analytic Number Theory* shows that

$$\sum_{d|n} \frac{\mu(d)}{d^2} = \prod_{p|n} \left(1 - \frac{1}{p^2}\right).$$

Hence we can also write

$$f(n) = n^3 \sum_{d|n} \frac{\mu(d)}{d^2} = n^3 \prod_{p|n} \left(1 - \frac{1}{p^2}\right).$$

(3) In particular, f is a multiplicative function (Exercise 2.16).

(4) Or we can use Exercises 2.16 and 2.19 to show

$$f(n) = n^3 \prod_{p|n} \left(1 - \frac{1}{p^2}\right) = n^3 \sum_{d|n} \frac{\mu(d)}{d^2}.$$

$\square$

Note. See "ProjectEuler 193: Squarefree Numbers" for the same trick. The answer should be

$$\sum_{d=1}^{\sqrt{n}} \mu(d) \left\lfloor \frac{n}{d^2} \right\rfloor.$$