Chapter 4: Continuity

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Exercise 4.1. *Suppose f is a real function define on $\mathbb{R}^1$ which satisfies*

$$\lim_{h\to 0}[f(x+h) - f(x-h)] = 0$$

for every $x \in \mathbb{R}^1$. Does this imply that f is continuous?

Proof. $\lim_{h\to 0}[f(x+h)-f(x-h)] = 0$ holds if f is continuous. But the converse of this statement and is not true. For example, define $f : \mathbb{R}^1 \to \mathbb{R}^1$ by

$$f(x) = \begin{cases} 1 & (x = 0), \\ 0 & (x \neq 0). \end{cases}$$

f is not continuous at $x = 0$ but

$$\lim_{h\to 0}[f(x+h) - f(x-h)] = 0$$

for any $x \in \mathbb{R}^1$. (The identity holds for $x \neq 0$ since f is continuous on $\mathbb{R}^1 - \{0\}$. Besides, $\lim_{h\to 0}[f(0+h) - f(0-h)] = \lim_{h\to 0}[0-0] = 0$.) $\square$

Exercise 4.2. *If f is a continuous mapping of a metric space X into a metric space Y, prove that $f(\overline{E}) \subseteq \overline{f(E)}$ for every set $E \subseteq X$. ($\overline{E}$ denotes the closure of E.) Show, by an example, that $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$.*

Proof.

(1) Since f is continuous and $\overline{f(E)}$ is closed, $f^{-1}(\overline{f(E)})$ is closed. Hence,

$$\begin{aligned} f^{-1}(\overline{f(E)}) &\supseteq f^{-1}(f(E)) && \text{(Monotonicity of } f^{-1}) \\ &\supseteq E, && \text{(Note in Theorem 4.14)} \\ \overline{E} &\subseteq f^{-1}(\overline{f(E)}), && \text{(Monotonicity of closure)} \\ f(\overline{E}) &\subseteq f(f^{-1}(\overline{f(E)})) && \text{(Monotonicity of } f) \\ &\subseteq \overline{f(E)}. && \text{(Note in Theorem 4.14)} \end{aligned}$$

(2) Let $f : (0, \infty) \to \mathbb{R}$ be a continuous function defined by

$$f(x) = \frac{1}{x}.$$

Consider $E = \mathbb{Z}^+ \subseteq (0, \infty)$. Then $f(E) = \left\{\frac{1}{n} : n \in \mathbb{Z}^+\right\}$, and thus

$$f(\overline{E}) = \left\{\frac{1}{n} : n \in \mathbb{Z}^+\right\}.$$
$$\overline{f(E)} = \left\{\frac{1}{n} : n \in \mathbb{Z}^+\right\} \bigcup \{0\}.$$

$\square$

Supplement (Inverse image).

(1) *$E \subseteq f^{-1}[f(E)]$ for $E \subseteq X$.*

$$\begin{aligned} \forall\, x \in E &\Longrightarrow f(x) \in f(E) \\ &\Longleftrightarrow x \in f^{-1}[f(E)]. \qquad \text{(Definition of the inverse image)} \end{aligned}$$

$\square$

(2) *$f[f^{-1}(E)] \subseteq E$ for $E \subseteq Y$.*

$$\begin{aligned} \forall\, y \in f[f^{-1}(E)] &\Longleftrightarrow \exists\, x \in f^{-1}(E) \text{ such that } y = f(x) \\ &\Longleftrightarrow \exists\, x, f(x) \in E \text{ such that } y = f(x) \\ &\Longrightarrow \exists\, x, y = f(x) \in E. \end{aligned}$$

$\square$

Supplement (Continuity). *Let f be a map from a topological space on X to a topological space on Y. Then, the following statements are equivalent:*

(1) *f is continuous: For each $x \in X$ and every neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$.*

(2) *For every open set O in Y, the inverse image $f^{-1}(O)$ is open in X.*

(3) *For every closed set C in Y, the inverse image $f^{-1}(C)$ is closed in X.*

(4) *$f(A)^\circ \subseteq f(A^\circ)$ for every subset A of X.*

(5) *$f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ$ for every subset B of Y.*

(6) *$f(\overline{A}) \subseteq \overline{f(A)}$ for every subset A of X.*

(7) *$\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for every subset B of Y.*

Exercise 4.3. *Let f be a continuous real function on a metric space X. Let $Z(f)$ (the zero set of f) be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.*

Proof (Corollary to Theorem 4.8). Since f is continuous, $f^{-1}(\{0\}) = Z(f)$ is closed in X for a closed subset $\{0\}$ in $\mathbb{R}^1$. $\square$

Denote the complement of any set E by $\widetilde{E}$.

Proof (Theorem 4.8). Consider the complement of $Z(f)$ in X,

$$\begin{aligned} \widetilde{Z(f)} &= \{x \in X : f(x) \neq 0\} \\ &= f^{-1}((-\infty, 0) \cup (0, \infty)). \end{aligned}$$

Since f is continuous, $f^{-1}((-\infty, 0) \cup (0, \infty)) = \widetilde{Z(f)}$ is open in X for a open subset $(-\infty, 0) \cup (0, \infty)$ in $\mathbb{R}^1$. $\square$

Proof (Definition 2.18(d)). Given any limit point p of $Z(f)$. *Show that $f(p) = 0$ or $p \in Z(f)$.* Since f is continuous, given any $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(p)| < \epsilon$ for all $x \in X$ for which $d_X(x, p) < \delta$. Since p is a limit point of $Z(f)$, for such $\delta > 0$ we have a point $q \neq p$ such that $q \in Z(f)$, or $f(q) = 0$. So $|f(p)| < \epsilon$ for any $\epsilon > 0$. $f(p) = 0$. $\square$

Proof (Definition 2.18(f)). Consider the complement of $Z(f)$ in X,

$$\widetilde{Z(f)} = \{x \in X : f(x) \neq 0\} = \{f > 0\} \cup \{f < 0\}$$

where $\{f > 0\} = \{x \in X : f(x) > 0\}$ and $\{f < 0\} = \{x \in X : f(x) < 0\}$. It suffices to show $\{f > 0\}$ is open. ($\{f < 0\}$ is similar.) Given any point p of $\{f > 0\}$ or $f(p) > 0$. *Want to show p is an interior point of $\{f > 0\}$.* Since f is continuous, given any $\epsilon = \frac{f(p)}{2} > 0$ there exists a $\delta > 0$ such that $|f(x) - f(p)| < \frac{f(p)}{2}$ for all $x \in X$ for which $d_X(x, p) < \delta$. For such x with $d_X(x, p) < \delta$ we have

$$\frac{1}{2} f(p) < f(x) < \frac{3}{2} f(p).$$

That is, $N = \{x : d_X(x, p) < \delta\}$ is a neighborhood p such that $N \subseteq \{f > 0\}$. $\square$

Exercise 4.4. *Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that $f(E)$ is dense in $f(X)$. If $g(p) = f(p)$ for all $p \in E$, prove that $g(p) = f(p)$ for all $p \in X$. (In other words, a continuous mapping is determined by its values on a dense subset of its domain.)*

Exercise 4.5. *If f is a real continuous function defined on a closed set $E \subset \mathbb{R}^1$, prove that there exist continuous real function g on $\mathbb{R}^1$ such that $g(x) = f(x)$ for* *all $x \in E$. (Such functions g are called **continuous extensions** of f from E to $\mathbb{R}^1$.) Show that the result becomes false if the word "closed" is omitted. Extend the result to vector valued functions. (Hint: Let the graph of g be a straight line on each of the segments which constitute the complement of E (compare Exercise 2.29). The result remains true if $\mathbb{R}^1$ is replaced by any metric space, but the proof is not so simple.)*

Supplement (Tietze's Extension Theorem). *If X is a normal topological space and $f : A \to \mathbb{R}$ is a continuous map from a closed subset A of X into the real numbers carrying the standard topology, then there exists a continuous map $g : X \to \mathbb{R}$ with $g(a) = f(a)$ for all $a \in A$.*

Exercise 4.23. *A real-valued function f defined in (a, b) is said to be **convex** if*

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y)$$

whenever $a < x < b$, $a < y < b$, $0 < \lambda < 1$. Prove that every convex function is continuous. Prove that every increasing convex function of a convex function is convex. (For example, if f is convex, so is e^f.)

If f is convex in (a, b) and if $a < s < t < u < b$, show that

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

Proof.

(1) *Show that* $\frac{f(t)-f(s)}{t-s} \leq \frac{f(u)-f(s)}{u-s} \leq \frac{f(u)-f(t)}{u-t}$. Since

$$\begin{aligned} t &= \frac{t - s}{u - s} u + \left(1 - \frac{t - s}{u - s}\right) s \\ &= \left(1 - \frac{u - t}{u - s}\right) u + \frac{u - t}{u - s} s \end{aligned}$$

and $0 < \frac{t-s}{u-s}, \frac{u-t}{u-s} < 1$, by the convexity of f we have

$$\begin{aligned} f(t) &\leq \frac{t - s}{u - s} f(u) + \left(1 - \frac{t - s}{u - s}\right) f(s), \\ f(t) &\leq \left(1 - \frac{u - t}{u - s}\right) f(u) + \frac{u - t}{u - s} f(s). \end{aligned}$$

It is equivalent to

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

□

(2) *If x, y, x', y' are points of (a,b) with $x \leq x' < y'$ and $x < y \leq y'$, then the chord over (x', y') has larger slope than the chord over (x, y); that is,*

$$\frac{f(y)-f(x)}{y-x} \leq \frac{f(y')-f(x')}{y'-x'}.$$

It is a corollary to (1).

(3) *Show that f is continuous.* Let $[c,d] \subseteq (a,b)$. Then by (2),

$$\frac{f(c)-f(a)}{c-a} \leq \frac{f(y)-f(x)}{y-x} \leq \frac{f(b)-f(d)}{b-d}$$

for x, y in $[c,d]$. Thus $|f(y) - f(x)| \leq M|y-x|$ in $[c,d]$ (where $M = \max\left(|\frac{f(c)-f(a)}{c-a}|, |\frac{f(b)-f(d)}{b-d}|\right)$), and so f is absolutely continuous on each closed subinterval of (a,b). Especially, f is continuous.

(4) *Let f be a convex function, g be an increasing convex function, and $h = g \circ f$. Show that h is convex.*

$$\begin{aligned}
f(\lambda x + (1-\lambda)y) &\leq \lambda f(x) + (1-\lambda) f(y), && \text{(Convexity of } f\text{)} \\
g(f(\lambda x + (1-\lambda)y)) &\leq g(\lambda f(x) + (1-\lambda) f(y)) && \text{(Increasing of } g\text{)} \\
&\leq \lambda g(f(x)) + (1-\lambda) g(f(y)), && \text{(Convexity of } g\text{)} \\
h(\lambda x + (1-\lambda)y) &\leq \lambda h(x) + (1-\lambda) h(y).
\end{aligned}$$

□

Exercise 4.24. *Assume that f is a continuous real function defined in (a,b) such that*

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$$

for all $x, y \in (a,b)$. Prove that f is convex.

Proof.

(1) *Show that*

$$f\left(\frac{x_1 + \cdots + x_n}{n}\right) \leq \frac{f(x_1) + \cdots + f(x_n)}{n}$$

whenever $a < x_i < b$ $(1 \leq i \leq n)$. Apply Cauchy induction and use the same argument in proving the AM-GM inequality. As $n = 1, 2$, the inequality holds by assumption. Suppose $n = 2^k$ $(k \geq 1)$ the inequality holds. As $n = 2^{k+1}$,

$$\begin{aligned}
&f\left(\frac{x_1+\cdots+x_{2^{k+1}}}{2^{k+1}}\right)\\
=&f\left(\frac{1}{2}\left(\frac{x_1+\cdots+x_{2^k}}{2^k}+\frac{x_{2^k+1}+\cdots+x_{2^{k+1}}}{2^k}\right)\right)\\
\leq&\frac{1}{2}\left(f\left(\frac{x_1+\cdots+x_{2^k}}{2^k}\right)+f\left(\frac{x_{2^k+1}+\cdots+x_{2^{k+1}}}{2^k}\right)\right)\\
\leq&\frac{1}{2}\left(\frac{f(x_1)+\cdots+f(x_{2^k})}{2^k}+\frac{f(x_{2^k+1})+\cdots+f(x_{2^{k+1}})}{2^k}\right)\\
=&\frac{f(x_1)+\cdots+f(x_{2^k})+f(x_{2^k+1})+\cdots+f(x_{2^{k+1}})}{2^{k+1}}\\
=&\frac{f(x_1)+\cdots+f(x_{2^{k+1}})}{2^{k+1}}.
\end{aligned}$$

As n is not a power of 2, then it is certainly less than some natural power of 2, say $n < 2^m$ for some m. Let

$$x_{n+1} = \cdots = x_{2^m} = \frac{x_1+\cdots+x_n}{n} = \alpha.$$

Then by the induction hypothesis,

$$\begin{aligned}
f(\alpha) = f\left(\frac{x_1+\cdots+x_n+\alpha+\cdots+\alpha}{2^m}\right)&\\
\leq \frac{f(x_1)+\cdots+f(x_n)+f(\alpha)+\cdots+f(\alpha)}{2^m}&\\
\leq \frac{f(x_1)+\cdots+f(x_n)+(2^m-n)f(\alpha)}{2^m}&,\\
2^m f(\alpha) \leq f(x_1)+\cdots+f(x_n)+(2^m-n)f(\alpha)&,\\
nf(\alpha) \leq f(x_1)+\cdots+f(x_n)&,
\end{aligned}$$

or $f\left(\frac{1}{n}(x_1+\cdots+x_n)\right) \leq \frac{1}{n}(f(x_1)+\cdots f(x_n))$.

(2) Hence,

$$f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y)$$

for any rational λ in $(0,1)$. (Given any positive integers $p<q$, put $n=q$, $x_1=\cdots=x_p=x$ and $x_{p+1}=\cdots=x_n=y$ in (1).)

(3) Given any real $\lambda\in(0,1)$, there is a sequence of rational numbers $\{r_n\}\subseteq(0,1)$ such that $r_n\to\lambda$. By (2),

$$f(r_n x+(1-r_n)y) \leq r_n f(x)+(1-r_n)f(y)$$

for any rational r_n in $(0,1)$. Taking limit on the both sides and using the continuity of f, we have

$$f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y).$$

□

Proof (Reductio ad absurdum). If f were not convex, then there is a subinterval $[c, d] \subseteq (a, b)$ such that

$$\frac{f(d) - f(c)}{d - c} < \frac{f(x_0) - f(c)}{x_0 - c}$$

for some $x_0 \in [c, d]$. Let

$$g(x) = f(x) - f(c) - \frac{f(d) - f(c)}{d - c}(x - c)$$

for $x \in [c, d]$. Therefore,

(1) $g(x)$ is continuous and midpoint convex.

(2) $g(c) = g(d) = 0$.

(3) Let $M = \sup\{g(x) : x \in [c, d]\}$. $\infty > M > 0$ due to the continuity of g and the existence of x_0. And let $\xi = \inf\{x \in [c, d] : g(x) = M\}$. By the continuity of g, $g(\xi) = M$. $\xi \in (c, d)$ by (2).

(4) Since (c, d) is open, there is $h > 0$ such that $(\xi - h, \xi + h) \subseteq (c, d)$. By the minimality of ξ and M, $g(\xi - h) < g(\xi)$ and $g(\xi + h) \leq g(\xi)$.

Therefore,

$$\begin{aligned}
g(\xi - h) + g(\xi + h) &< 2g(\xi), \\
\frac{g(\xi - h) + g(\xi + h)}{2} &< g(h) \\
&= g\left(\frac{(\xi - h) + (\xi + h)}{2}\right),
\end{aligned}$$

contrary to the midpoint convexity of g. □

The result becomes false if "continuity of f" is omitted.

Exercise 4.25. *If $A \subset \mathbb{R}^k$ and $B \subset \mathbb{R}^k$, define $A + B$ to be the set of all sums $\mathbf{x} + \mathbf{y}$ with $\mathbf{x} \in A$, $\mathbf{y} \in B$.*

(a) *If K is compact and C is closed in $\mathbb{R}^k$, prove that $K + C$ is closed. (Hint: Take $\mathbf{z} \notin K + C$, put $F = \mathbf{z} - C$, the set of all $\mathbf{z} - \mathbf{y}$ with $\mathbf{y} \in C$. Then K and F are disjoint. Choose δ as in Exercise 4.21. Show that the open ball with center $\mathbf{z}$ and radius δ does not intersect $K + C$.)*

(b) *Let α be an irrational real number. Let C_1 be the set of all integers, let C_2 be the set of all $n\alpha$ with $n \in C_1$. Show that C_1 and C_2 are closed subsets of $\mathbb{R}^1$ whose sum $C_1 + C_2$ is not closed, by showing that $C_1 + C_2$ is a countable dense subset of $\mathbb{R}^1$.*

Exercise 4.26. *Suppose X, Y, Z are metric spaces, and Y is compact. Let f map X into Y, let g be a continuous one-to-one mapping of Y into Z, and put $h(x) = g(f(x))$ for $x \in X$.*

Prove that f is uniformly continuous if h is uniformly continuous. (Hint: g^{-1} has compact domain $g(Y)$, and $f(x) = g^{-1}(h(x))$.)

Prove also that f is continuous if h is continuous.

Show (by modifying Example 4.21, or by finding a different example) that the compactness of Y cannot be omitted from the hypotheses, even when X and Z are compact.