

## Chapter 10: Integration of Differential Forms

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**Exercise 10.1.** Let  $H$  be a compact convex set in  $\mathbb{R}^k$ , with nonempty interior. Let  $f \in \mathcal{C}(H)$ , put  $f(\mathbf{x}) = 0$  in the complement of  $H$ , and define  $\int_H f$  as in Definition 10.3. Prove that  $\int_H f$  is independent of the order in which the  $k$  integrations are carried out. (Hint: Approximate  $f$  by functions that are continuous on  $\mathbb{R}^k$  and whose supports are in  $H$ , as was done in Example 10.4.)

*Proof.*

(1)

(2)

□

**Exercise 10.2.** For  $i = 1, 2, 3, \dots$ , let  $\varphi_i \in \mathcal{C}(\mathbb{R}^1)$  have support in  $(2^{-i}, 2^{1-i})$ , such that  $\int \varphi_i = 1$ . Put

$$f(x, y) = \sum_{i=1}^{\infty} [\varphi_i(x) - \varphi_{i+1}(x)] \varphi_i(y)$$

Then  $f$  has compact support in  $\mathbb{R}^2$ ,  $f$  is continuous except at  $(0, 0)$ , and

$$\int dy \int f(x, y) dx = 0 \quad \text{but} \quad \int dx \int f(x, y) dy = 1.$$

Observe that  $f$  is unbounded in every neighborhood of  $(0, 0)$ .

*Proof.*

(1)

(2)

□

**Exercise 10.3.**

- (a) If  $\mathbf{F}$  is as in Theorem 10.7, put  $\mathbf{A} = \mathbf{F}'(\mathbf{0})$ ,  $\mathbf{F}_1(\mathbf{x}) = \mathbf{A}^{-1}\mathbf{F}(\mathbf{x})$ . Then  $\mathbf{F}_1(\mathbf{0}) = \mathbf{I}$ . Show that

$$\mathbf{F}_1(\mathbf{x}) = \mathbf{G}_n \circ \mathbf{G}_{n-1} \circ \cdots \circ \mathbf{G}_1(\mathbf{x})$$

in some neighborhood of  $\mathbf{0}$ , for certain primitive mappings  $\mathbf{G}_1, \dots, \mathbf{G}_n$ . This gives another version of Theorem 10.7:

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}'(\mathbf{0})\mathbf{G}_n \circ \mathbf{G}_{n-1} \circ \dots \circ \mathbf{G}_1(\mathbf{x}).$$

- (b) Prove that the mapping  $(x, y) \mapsto (y, x)$  of  $\mathbb{R}^2$  onto  $\mathbb{R}^2$  is not the composition of any two primitive mappings, in any neighborhood of the origin. (This shows that the flips  $B_i$  cannot be omitted from the statement of Theorem 10.7.)

*Proof of (a).*

- (1) Suppose  $\mathbf{F}$  is a  $\mathcal{C}'$ -mapping of an open set  $E \subseteq \mathbb{R}^n$  into  $\mathbb{R}^n$ ,  $\mathbf{0} \in E$ ,  $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ , and  $\mathbf{F}'(\mathbf{0})$  is invertible.
- (2) Similar to the proof of Theorem 10.7. Put  $\mathbf{F}_1 = \mathbf{F}$ .
- (3) As  $m = 1$ , there is an open neighborhood  $V_1 \subseteq E$  of  $\mathbf{0}$  such that  $\mathbf{F}_1(\mathbf{0}) = (\mathbf{F}'(\mathbf{0}))^{-1}\mathbf{F}(\mathbf{0}) = \mathbf{0}$ ,  $\mathbf{F}'_1(\mathbf{0}) = \mathbf{I}$  is invertible, and

$$\mathbf{F}_1(\mathbf{x}) = \sum_{i=1}^n \alpha_i(\mathbf{x})\mathbf{e}_i,$$

where  $\alpha_1, \dots, \alpha_n$  are real  $\mathcal{C}'$ -functions in  $V_1$ . Hence

$$\mathbf{F}'_1(\mathbf{0})\mathbf{e}_1 = \sum_{i=1}^n (D_1\alpha_i)(\mathbf{0})\mathbf{e}_i.$$

Note that  $(D_1\alpha_1)(\mathbf{0}) = 1 \neq 0$ , and we might pick  $B_1 = \mathbf{I}$ . Thus we can define

$$\mathbf{G}_1(\mathbf{x}) = \mathbf{x} + [\alpha_1(\mathbf{x}) - x_1]\mathbf{e}_1 \quad (\mathbf{x} \in V_1).$$

Then  $\mathbf{G}_1 \in \mathcal{C}'(V_1)$ ,  $\mathbf{G}_1$  is primitive, and  $\mathbf{G}'_1(\mathbf{0}) = \mathbf{I}$  is invertible.

- (4) Now we make the induction hypothesis for  $1 \leq m \leq n-1$ .
- (5) Since  $\mathbf{G}'_m(\mathbf{0}) = \mathbf{I}$  is invertible, the inverse function theorem shows that there is an open set  $U_m$ , with  $\mathbf{0} \in U_m \subseteq V_m$ , such that  $\mathbf{G}_m$  is an injective mapping of  $U_m$  onto a neighborhood  $V_{m+1}$  of  $\mathbf{0}$ , in which  $\mathbf{G}_m^{-1} \in \mathcal{C}'(V_{m+1})$ . Define  $\mathbf{F}_{m+1}$  by

$$\mathbf{F}_{m+1}(\mathbf{y}) = \mathbf{F}_m \circ \mathbf{G}_m^{-1}(\mathbf{y}) \quad (\mathbf{y} \in V_{m+1}).$$

Then  $\mathbf{F}_{m+1} \in \mathcal{C}'(V_{m+1})$ ,  $\mathbf{F}_m(\mathbf{0}) = \mathbf{0}$ , and  $\mathbf{F}'_{m+1}(\mathbf{0}) = \mathbf{I}$  is invertible by the chain rule and the inverse function theorem. So

$$\mathbf{F}_{m+1}(\mathbf{x}) = P_m\mathbf{x} + \sum_{i=m+1}^n \alpha_i(\mathbf{x})\mathbf{e}_i,$$

where  $\alpha_1, \dots, \alpha_n$  are real  $\mathcal{C}'$ -functions in  $V_{m+1}$ . Hence

$$\mathbf{F}'_{m+1}(\mathbf{0})\mathbf{e}_{m+1} = \sum_{i=m+1}^n (D_{m+1}\alpha_i)(\mathbf{0})\mathbf{e}_i.$$

Note that  $(D_{m+1}\alpha_{m+1})(\mathbf{0}) = 1 \neq 0$ , and we might pick  $B_{m+1} = \mathbf{I}$ . Thus we can define

$$\mathbf{G}_{m+1}(\mathbf{x}) = \mathbf{x} + [\alpha_{m+1}(\mathbf{x}) - x_{m+1}]\mathbf{e}_{m+1} \quad (\mathbf{x} \in V_{m+1}).$$

Then  $\mathbf{G}_{m+1} \in \mathcal{C}'(V_{m+1})$ ,  $\mathbf{G}_{m+1}$  is primitive, and  $\mathbf{G}'_{m+1}(\mathbf{0}) = \mathbf{I}$  is invertible. Our induction hypothesis holds therefore with  $m+1$  in place of  $m$ .

(6) Note that

$$\mathbf{F}_m(\mathbf{x}) = \mathbf{F}_{m+1}(\mathbf{G}_m(\mathbf{x})) \quad (\mathbf{x} \in U_m).$$

If we apply this with  $m = 1, \dots, n-1$ , we successively obtain

$$\mathbf{F}_1 = \mathbf{F}_n \circ \mathbf{G}_{n-1} \circ \dots \circ \mathbf{G}_1$$

in some open neighborhood of  $\mathbf{0}$ . Note that  $\mathbf{F}_n$  is primitive since

$$\mathbf{F}_n(\mathbf{x}) = P_{n-1}\mathbf{x} + \alpha_n(\mathbf{x})\mathbf{e}_n.$$

This completes the proof.

□

*Proof of (b).*

(1) For  $(x, y) \in \mathbb{R}^2$ , define

$$\mathbf{F}(x, y) = (y, x).$$

(2) (Reductio ad absurdum) If  $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$  for some primitive mappings  $\mathbf{G}_i$  ( $i = 1, 2$ ) in some neighborhood  $V_i$  of the origin,  $\mathbf{G}_i(\mathbf{0}) = \mathbf{0}$  and  $\mathbf{G}'_i$  is invertible, then we may assume that

$$\mathbf{G}_1(x, y) = (x, g_1(x, y)) \quad \text{and} \quad \mathbf{G}_2(x, y) = (g_2(x, y), y).$$

Here the case  $\mathbf{G}_1(x, y) = (g_1(x, y), y)$  and  $\mathbf{G}_2(x, y) = (x, g_2(x, y))$  is similar to the above case. Besides,  $\mathbf{G}_1(x, y) = (x, g_1(x, y))$  and  $\mathbf{G}_2(x, y) = (x, g_2(x, y))$  implies that

$$\mathbf{G}_2 \circ \mathbf{G}_1(x, y) = (x, g_2(x, g_1(x, y))) \neq (y, x) = \mathbf{F}(x, y).$$

Same reason for  $\mathbf{G}_1(x, y) = (g_1(x, y), y)$  and  $\mathbf{G}_2(x, y) = (g_2(x, y), y)$ .

(3) Note that

$$\mathbf{F}'(\mathbf{0}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Since

$$\mathbf{F}'(\mathbf{0}) = \mathbf{G}'_2(\mathbf{G}_1(\mathbf{0}))\mathbf{G}'_1(\mathbf{0}) = \mathbf{G}'_2(\mathbf{0})\mathbf{G}'_1(\mathbf{0}),$$

we have

$$\begin{aligned} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} &= \begin{bmatrix} D_1g_2(0,0) & D_2g_2(0,0) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ D_1g_1(0,0) & D_2g_1(0,0) \end{bmatrix} \\ &= \begin{bmatrix} * & * \\ D_1g_1(0,0) & D_2g_1(0,0) \end{bmatrix}. \end{aligned}$$

So  $D_1g_1(0,0) = 1$  and  $D_2g_1(0,0) = 0$ , and thus  $\mathbf{G}'_1(\mathbf{0}) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$  is not invertible, which is absurd.

□

**Exercise 10.4.** For  $(x, y) \in \mathbb{R}^2$ , define

$$\mathbf{F}(x, y) = (e^x \cos y - 1, e^x \sin y)$$

Prove that  $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$ , where

$$\begin{aligned} \mathbf{G}_1(x, y) &= (e^x \cos y - 1, y) \\ \mathbf{G}_2(u, v) &= (u, (1 + u) \tan v) \end{aligned}$$

are primitive in some neighborhood of  $(0, 0)$ . Compute the Jacobians of  $\mathbf{G}_1$ ,  $\mathbf{G}_2$ ,  $\mathbf{F}$  at  $(0, 0)$ . Define

$$\mathbf{H}_2(x, y) = (x, e^x \sin y)$$

and find

$$\mathbf{H}_1(u, v) = (h(u, v), v)$$

so that  $\mathbf{F} = \mathbf{H}_1 \circ \mathbf{H}_2$  is in some neighborhood of  $(0, 0)$ .

*Proof.*

(1) By Definition 10.5,

$$\begin{aligned} \mathbf{G}_1(x, y) &= (e^x \cos y - 1)\mathbf{e}_1 + y\mathbf{e}_2, \\ \mathbf{G}_2(u, v) &= u\mathbf{e}_1 + ((1 + u) \tan v)\mathbf{e}_2 \end{aligned}$$

are primitive in some neighborhood of  $(0, 0)$ .

(2) Show that  $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$ . Given any  $(x, y) \in \mathbb{R}^2$ , we have

$$\begin{aligned}
(\mathbf{G}_2 \circ \mathbf{G}_1)(x, y) &= \mathbf{G}_2(\mathbf{G}_1(x, y)) \\
&= \mathbf{G}_2(e^x \cos y - 1, y) \\
&= (e^x \cos y - 1, (1 + (e^x \cos y - 1)) \tan y) \\
&= (e^x \cos y - 1, e^x \sin y) \\
&= \mathbf{F}(x, y).
\end{aligned}$$

(3) Since

$$\begin{aligned}
J_{\mathbf{G}_1}(x, y) &= \det \begin{bmatrix} e^x \cos y & -e^x \sin y \\ 0 & 1 \end{bmatrix} = e^x \cos y \\
J_{\mathbf{G}_2}(x, y) &= \det \begin{bmatrix} 1 & 0 \\ \tan y & (1 + x) \sec^2 y \end{bmatrix} = (1 + x) \sec^2 y \\
J_{\mathbf{F}}(x, y) &= \det \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix} = e^{2x},
\end{aligned}$$

$$J_{\mathbf{G}_1}(0, 0) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$

$$J_{\mathbf{G}_2}(0, 0) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$

$$J_{\mathbf{F}}(0, 0) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1.$$

(4) Define  $h(u, v) = \sqrt{e^{2u} - v^2} - 1$  on

$$B\left((0, 0); \frac{1}{64}\right) \subseteq \mathbb{R}^2.$$

$h(u, v)$  is well-defined since  $e^{2u} - v^2 > 0$  for all  $(u, v) \in B((0, 0); \frac{1}{64})$ .

(5) Given any  $(x, y) \in \mathbb{R}^2$ , we have

$$\begin{aligned}
(\mathbf{H}_1 \circ \mathbf{H}_2)(x, y) &= \mathbf{H}_1(\mathbf{H}_2(x, y)) \\
&= \mathbf{H}_1(x, e^x \sin y) \\
&= (\sqrt{e^{2x} - (e^x \sin y)^2} - 1, e^x \sin y) \\
&= (e^x \cos y - 1, e^x \sin y) \\
&= \mathbf{F}(x, y).
\end{aligned}$$

□

**Exercise 10.5.** Formulate and prove an analogue of Theorem 10.8, in which  $K$  is a compact subset of an arbitrary metric space. (Replace the functions  $\varphi_i$  that occur in the proof of Theorem 10.8 by functions of the type constructed in Exercise 4.22.)

*Proof (Theorem 10.8).*

- (1) (Partitions of unity.) Suppose  $K$  is a compact subset of a metric space  $X$ , and  $\{V_\alpha\}$  is an open cover of  $K$ . Then there exist functions  $\psi_1, \dots, \psi_s \in \mathcal{C}(X)$  such that

- (a)  $0 \leq \psi_i \leq 1$  for  $1 \leq i \leq s$ .
- (b) each  $\psi_i$  has its support in some  $V_\alpha$ , and
- (c)  $\psi_1(x) + \dots + \psi_s(x) = 1$  for every  $x \in K$ .

- (2) It is trivial that some  $V_\alpha = X$  by taking  $s = 1$  and  $\psi_1(x) = 1 \in \mathcal{C}(X)$ . Now we assume that all  $V_\alpha \subsetneq X$ .

- (3) Associate with each  $x \in K$  an index  $\alpha(x)$  so that  $x \in V_{\alpha(x)}$ . Then there are open balls  $B(x)$  and  $W(x)$ , centered at  $x$ , with

$$x \in B(x) \subseteq \overline{B(x)} \subseteq W(x) \subseteq \overline{W(x)} \subseteq V_{\alpha(x)}$$

(Since  $V_{\alpha(x)}$  is open, there exists  $r > 0$  such that  $B(x; r) \subseteq V_{\alpha(x)}$ . Take  $B(x) = B(x; \frac{r}{89})$  and  $W(x) = B(x; \frac{r}{64})$ .)

- (4) Since  $K$  is compact, there are finitely many points  $x_1, \dots, x_s \in K$  such that

$$K \subseteq B(x_1) \cup \dots \cup B(x_s).$$

Note that

- (a)  $\overline{B(x_i)}$  is a nonempty closed set since  $x_i \in B(x_i) \subseteq \overline{B(x_i)}$ .
- (b)  $X - W(x_i) \supseteq X - V_{\alpha(x_i)}$  is a nonempty closed set by the assumption in (2).
- (c)  $\overline{B(x_i)} \cap (X - W(x_i)) \subseteq W(x_i) \cap (X - W(x_i)) = \emptyset$ .

By Exercise 4.22, there is a function

$$\varphi_i(x) = \frac{\rho_{\overline{B(x_i)}}(x)}{\rho_{\overline{B(x_i)}}(x) + \rho_{X - W(x_i)}(x)} \in \mathcal{C}(X)$$

such that  $\varphi_i(x) = 1$  on  $\overline{B(x_i)}$ ,  $\varphi_i(x) = 0$  outside  $W(x_i)$ , and  $0 \leq \varphi_i(x) \leq 1$  on  $X$  for  $1 \leq i \leq s$ .

- (5) Define  $\psi_1 = \varphi_1 \in \mathcal{C}(X)$  and

$$\psi_{i+1} = (1 - \varphi_1) \cdots (1 - \varphi_i) \varphi_{i+1} \in \mathcal{C}(X)$$

for  $1 \leq i \leq s-1$ . Properties (a) and (b) in (1) are clear. Also,

$$\psi_1(x) + \cdots + \psi_s(x) = 1 - (1 - \varphi_1(x)) \cdots (1 - \varphi_s(x))$$

by the construction of  $\psi_i$ . If  $x \in K$ , then  $x \in B(x_i)$  for some  $i$ , hence  $\varphi_i(x) = 1$ , and the product  $(1 - \varphi_1(x)) \cdots (1 - \varphi_s(x)) = 0$ . This proves property (c) in (1).

□

**Exercise 10.6.** *Strengthen the conclusion of Theorem 10.8 by showing that the functions  $\psi_i$  can be made differentiable, and even infinitely differentiable. (Use Exercise 8.1 in the construction of the auxiliary functions  $\psi_i$ .)*

*Proof (Theorem 10.8).*

- (1) It is trivial that some  $V_\alpha = \mathbb{R}^n$  by taking  $s = 1$  and  $\psi_1(\mathbf{x}) = 1 \in \mathcal{C}^\infty(\mathbb{R}^n)$ . Now we assume that all  $V_\alpha \subsetneq \mathbb{R}^n$ .
- (2) Associate with each  $\mathbf{x} \in K$  an index  $\alpha(x)$  so that  $\mathbf{x} \in V_{\alpha(x)}$ . Then there are open  $n$ -cells  $B(\mathbf{x})$  and  $W(\mathbf{x})$  (Definition 10.1), centered at  $\mathbf{x}$ , with

$$\mathbf{x} \in B(\mathbf{x}) \subseteq \overline{B(\mathbf{x})} \subseteq W(\mathbf{x}) \subseteq \overline{W(\mathbf{x})} \subseteq V_{\alpha(\mathbf{x})}$$

(Since  $V_{\alpha(\mathbf{x})}$  is open, there exists  $r > 0$  such that  $B(\mathbf{x}; r) \subseteq V_{\alpha(\mathbf{x})}$ . Take

$$B(\mathbf{x}) = I\left(\mathbf{x}; \frac{r}{89\sqrt{n}}\right), \quad W(\mathbf{x}) = I\left(\mathbf{x}; \frac{r}{64\sqrt{n}}\right)$$

where  $I(\mathbf{p}; r)$  is the open  $n$ -cell centered at  $\mathbf{p} = (p_1, \dots, p_n)$  defined by

$$I(\mathbf{p}; r) = (p_1 - r, p_1 + r) \times \cdots \times (p_n - r, p_n + r) \subseteq \mathbb{R}^n.$$

- (3) Define

$$f(y) = \begin{cases} e^{-\frac{1}{y^2}} & (y > 0), \\ 0 & (y \leq 0). \end{cases}$$

$f(y) \in \mathcal{C}^\infty(\mathbb{R}^1)$  by applying the similar argument in Exercise 8.1.

- (4) Given any  $\mathbf{x} = (x_1, \dots, x_n) \in K$  and construct  $B(\mathbf{x})$  and  $W(\mathbf{x})$  as in (2). Define

$$g_{x_j}(y_j) = \frac{f(y_j)}{f(y_j) + f\left(\frac{r}{64\sqrt{n}} - \frac{r}{89\sqrt{n}} - y_j\right)}$$

for  $1 \leq j \leq n$ .  $g_{x_j}$  is well-defined and  $g_{x_j} \in \mathcal{C}^\infty(\mathbb{R}^1)$ . So

$$g_{x_j}(y_j) = \begin{cases} 0 & \text{if } y_j \leq 0, \\ \text{strictly increasing} & \text{if } 0 \leq y_j \leq \frac{r}{64\sqrt{n}} - \frac{r}{89\sqrt{n}}, \\ 1 & \text{if } y_j \geq \frac{r}{64\sqrt{n}} - \frac{r}{89\sqrt{n}}. \end{cases}$$

Next, define

$$h_{x_j}(y_j) = g_{x_j} \left( y_j - x_j + \frac{r}{64\sqrt{n}} \right) g_{x_j} \left( x_j + \frac{r}{64\sqrt{n}} - y_j \right)$$

for  $1 \leq j \leq n$ .  $h_{x_j} \in \mathcal{C}^\infty(\mathbb{R}^1)$ . So

$$h_{x_j}(y_j) = \begin{cases} 0 & \text{if } y_j \leq x_j - \frac{r}{64\sqrt{n}}, \\ \text{strictly increasing} & \text{if } x_j - \frac{r}{64\sqrt{n}} \leq y_j \leq x_j - \frac{r}{89\sqrt{n}}, \\ 1 & \text{if } x_j - \frac{r}{89\sqrt{n}} \leq y_j \leq x_j + \frac{r}{89\sqrt{n}}, \\ \text{strictly decreasing} & \text{if } x_j + \frac{r}{89\sqrt{n}} \leq y_j \leq x_j + \frac{r}{64\sqrt{n}}, \\ 0 & \text{if } y_j \geq x_j + \frac{r}{64\sqrt{n}}. \end{cases}$$

Finally we define  $\mathbf{h}_{\mathbf{x}} : \mathbb{R}^n \rightarrow \mathbb{R}^1$  by

$$\mathbf{h}_{\mathbf{x}}(\mathbf{y}) = \prod_{j=1}^n h_{x_j}(y_j)$$

where  $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ . Hence,  $\mathbf{h}_{\mathbf{x}} \in \mathcal{C}^\infty(\mathbb{R}^n)$  (Theorem 9.21). Also,  $\mathbf{h}_{\mathbf{x}}(\mathbf{y}) = 1$  on  $B(\mathbf{x})$ ,  $\mathbf{h}_{\mathbf{x}}(\mathbf{y}) = 0$  outside  $W(\mathbf{x})$ , and  $0 \leq \mathbf{h}_{\mathbf{x}}(\mathbf{y}) \leq 1$ .

- (5) Since  $K$  is compact, there are finitely many points  $\mathbf{x}_1, \dots, \mathbf{x}_s \in K$  such that

$$K \subseteq B(\mathbf{x}_1) \cup \dots \cup B(\mathbf{x}_s).$$

Take

$$\varphi_i(\mathbf{x}) = \mathbf{h}_{\mathbf{x}_i}(\mathbf{x}) \in \mathcal{C}^\infty(\mathbb{R}^n)$$

for  $1 \leq i \leq s$ .

- (6) The rest are the same as the proof of Theorem 10.8 or Exercise 10.5.

□

### Exercise 10.7.

- (a) Show that the simplex  $Q^k$  is the smallest convex subset of  $\mathbb{R}^k$  such that contains  $\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_k$ .
- (b) Show that affine mappings take convex sets to convex sets.

*Proof of (a).*

- (1) Show that  $Q^k$  contains  $\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_k$ . Recall

$$Q^k = \{(x_1, \dots, x_k) \in \mathbb{R}^k : x_1 + \dots + x_k \leq 1 \text{ and } x_1, \dots, x_k \geq 0\}$$

(Example 10.14). Hence  $\mathbf{0} = (0, \dots, 0) \in Q^k$  and

$$\mathbf{e}_i = (0, \dots, \underbrace{1}_{i\text{th coordinate}}, \dots, 0) \in Q^k.$$



- (2) Show that  $Q^k$  is a convex subset of  $\mathbb{R}^k$ . Given any  $\mathbf{x} = (x_1, \dots, x_k) \in Q^k$ ,  $\mathbf{y} = (y_1, \dots, y_k) \in Q^k$  and  $0 < \lambda < 1$ . Hence

$$\lambda \mathbf{x} + (1 - \lambda) \mathbf{y} = (\lambda x_1 + (1 - \lambda) y_1, \dots, \lambda x_k + (1 - \lambda) y_k) \in Q^k$$

since each  $\lambda x_i + (1 - \lambda) y_i \geq 0$  and

$$\sum_{i=1}^k (\lambda x_i + (1 - \lambda) y_i) = \lambda \sum_{i=1}^k x_i + (1 - \lambda) \sum_{i=1}^k y_i \leq \lambda + (1 - \lambda) = 1.$$

- (3) Given any convex set  $E \subseteq \mathbb{R}^k$  containing  $\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_k$ . Show that  $E \supseteq Q^k$ .

- (a) Induction on  $k$ . Base case:  $k = 1$ . Given any  $\mathbf{x} = (x_1) \in Q^1$ . We have  $0 \leq x_1 \leq 1$  by the definition of  $Q^1$ . So that  $\mathbf{x} = x_1 \mathbf{e}_1 + (1 - x_1) \mathbf{0} \in E$  since  $\mathbf{0}, \mathbf{e}_1 \in E$  and  $E$  is convex.
- (b) Inductive step: suppose the statement holds for  $k = n$ . Given any  $\mathbf{x} = (x_1, \dots, x_n, x_{n+1}) \in Q^{n+1}$ . If  $x_{n+1} = 1$ , then  $x_1 = \dots = x_n = 0$  by the definition of  $Q^{n+1}$ . So  $\mathbf{x} = \mathbf{e}_{n+1} \in E$  by the assumption of  $E$ . If  $0 \leq x_{n+1} < 1$ , then  $x_1 + \dots + x_n \leq 1 - x_{n+1}$  or

$$\frac{x_1}{1 - x_{n+1}} + \dots + \frac{x_n}{1 - x_{n+1}} \leq 1.$$

So the point

$$\left( \frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}} \right) \in Q^n,$$

or

$$\left( \frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}}, 0 \right), \text{ say } \hat{\mathbf{x}}, \in E$$

by the induction hypothesis. Note that  $\mathbf{e}_{n+1} \in E$ . Hence

$$\mathbf{x} = x_{n+1} \mathbf{e}_{n+1} + (1 - x_{n+1}) \hat{\mathbf{x}} \in E$$

by the convexity of  $E$ .

- (c) Conclusion: Since both the base case and the inductive step have been proved as true, by mathematical induction the statement holds.

□

*Proof of (b).*

- (1) Let  $\mathbf{f}$  be an affine mapping that carries a vector space  $X$  into a vector space  $Y$  such that

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{0}) + A\mathbf{x}$$

for some  $A \in L(X, Y)$ .

- (2) Given any convex subset  $C$  of  $X$ . To show that  $\mathbf{f}(C)$  is convex, it suffices to show that

$$\lambda \mathbf{y}_1 + (1 - \lambda) \mathbf{y}_2 \in \mathbf{f}(C)$$

for any  $\mathbf{y}_1, \mathbf{y}_2 \in \mathbf{f}(C)$  and  $0 < \lambda < 1$ . Write  $\mathbf{y}_1 = \mathbf{f}(\mathbf{x}_1)$ ,  $\mathbf{y}_2 = \mathbf{f}(\mathbf{x}_2)$  for some  $\mathbf{x}_1, \mathbf{x}_2 \in C$ . Note that  $\lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2 \in C$  by the convexity of  $C$ . Hence

$$\begin{aligned} & \mathbf{f}(\lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2) \\ &= \mathbf{f}(\mathbf{0}) + A(\lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2) \\ &= \mathbf{f}(\mathbf{0}) + \lambda A \mathbf{x}_1 + (1 - \lambda) A \mathbf{x}_2 & (A \in L(X, Y)) \\ &= \lambda(\mathbf{f}(\mathbf{0}) + A \mathbf{x}_1) + (1 - \lambda)(\mathbf{f}(\mathbf{0}) + A \mathbf{x}_2) \\ &= \lambda \mathbf{f}(\mathbf{x}_1) + (1 - \lambda) \mathbf{f}(\mathbf{x}_2) \\ &= \lambda \mathbf{y}_1 + (1 - \lambda) \mathbf{y}_2 \in \mathbf{f}(C). \end{aligned}$$

□

**Exercise 10.8.** Let  $H$  be the parallelogram in  $\mathbb{R}^2$  whose vertices are  $(1, 1)$ ,  $(3, 2)$ ,  $(4, 5)$ ,  $(2, 4)$ . Find the affine map  $T$  which sends  $(0, 0)$  to  $(1, 1)$ ,  $(1, 0)$  to  $(3, 2)$ ,  $(1, 1)$  to  $(4, 5)$ ,  $(0, 1)$  to  $(2, 4)$ . Show that  $J_T = 5$ . Use  $T$  to convert the integral

$$\alpha = \int_H e^{x-y} dx dy$$

to an integral over  $I^2$  and thus compute  $\alpha$ .

*Proof.*

- (1) By Affine simplexes 10.26,

$$T(\mathbf{x}) = T(\mathbf{0}) + A\mathbf{x},$$

where  $A \in L(\mathbb{R}^2, \mathbb{R}^2)$ , say  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Note that  $T : \begin{bmatrix} 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . Thus

$$T : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 + ax + by \\ 1 + cx + dy \end{bmatrix}.$$

- (2) By  $T : (1, 0) \mapsto (3, 2)$  and  $T : (0, 1) \mapsto (2, 4)$ , we can solve  $A$  as

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

It is easy to verify such

$$T : \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{x}} \mapsto \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{T(\mathbf{0})} + \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{x}} = \begin{bmatrix} 1 + 2x + y \\ 1 + x + 3y \end{bmatrix}$$

satisfying our requirement.

(3)

$$J_T = \det \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = 5.$$

(4) By Example 10.4 and Theorem 10.9, we have

$$\begin{aligned} \int_H e^{x-y} dx dy &= \int_{I^2} e^{(1+2u+v)-(1+u+3v)} |J_T| du dv \\ &= 5 \int_{I^2} e^{u-2v} du dv \\ &= 5 \left\{ \int_0^1 e^u du \right\} \left\{ \int_0^1 e^{-2v} dv \right\} \quad (\text{Theorem 10.2}) \\ &= \frac{5}{2} (e-1)(1-e^{-2}). \end{aligned}$$

□

**Exercise 10.9.** Define  $(x, y) = T(r, \theta)$  on the rectangle

$$0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi$$

by the equations

$$x = r \cos \theta, \quad y = r \sin \theta.$$

Show that  $T$  maps this rectangle onto the closed disc  $D$  with center at  $(0, 0)$  and radius  $a$ , that  $T$  is one-to-one in the interior of the rectangle, and that  $J_T(r, \theta) = r$ . If  $f \in \mathcal{C}(D)$ , prove the formula for integration in polar coordinates:

$$\int_D f(x, y) dx dy = \int_0^a \int_0^{2\pi} f(T(r, \theta)) r dr d\theta.$$

(Hint: Let  $D_0$  be the interior of  $D$ , minus the interval from  $(0, 0)$  to  $(0, a)$ . As it stands, Theorem 10.9 applies to continuous functions  $f$  whose support lies in  $D_0$ . To remove this restriction, proceed as in Example 10.4.)

*Proof.*

(1)

(2)

□

**Exercise 10.10.** Let  $a \rightarrow \infty$  in Exercise 10.9 and prove that

$$\int_{\mathbb{R}^2} f(x, y) dx dy = \int_0^\infty \int_0^{2\pi} f(T(r, \theta)) r dr d\theta,$$

for continuous functions  $f$  that decrease sufficiently rapidly as  $|x| + |y| \rightarrow \infty$ .  
 (Find a more precise formulation.) Apply this to

$$f(x, y) = \exp(-x^2 - y^2)$$

to derive formula

$$\int_{-\infty}^{\infty} e^{-s^2} ds = \sqrt{\pi}.$$

*Proof.*

(1)

(2)

□

**Exercise 10.11.** ...

*Proof.*

(1)

(2)

□

**Exercise 10.12.** ...

*Proof.*

(1)

(2)

□

**Exercise 10.13.** Let  $r_1, \dots, r_k$  be nonnegative integers, and prove that

$$\int_{Q^k} x_1^{r_1} \cdots x_k^{r_k} d\mathbf{x} = \frac{r_1! \cdots r_k!}{(k + r_1 + \cdots + r_k)!}$$

(Hint: Use Exercise 10.12, Theorems 10.9 and 8.20.) Note that the special case  $r_1 = \cdots = r_k = 0$  shows that the volume of  $Q^k$  is  $\frac{1}{k!}$ .

*Proof.*

(1) Define  $T : I^k$  onto  $Q^k$  as in Exercise 10.12, and  $f : Q^k \rightarrow \mathbb{R}^1$  by

$$f(\mathbf{x}) = f(x_1, \dots, x_k) = x_1^{r_1} \cdots x_k^{r_k} = \prod_{i=1}^k x_i^{r_i}.$$

(2) By Exercise 10.12, Example 10.4 and Theorems 10.9, we have

$$\begin{aligned} \int_{Q^k} x_1^{r_1} \cdots x_k^{r_k} d\mathbf{x} &= \int_{Q^k} f(\mathbf{x}) d\mathbf{x} \\ &= \int_{I^k} f(T(\mathbf{u})) |J_T(\mathbf{u})| d\mathbf{u} \\ &= \int_{I^k} \prod_{i=1}^k \left( u_i \prod_{j=1}^{i-1} (1 - u_j) \right)^{r_i} \prod_{i=1}^k (1 - u_i)^{k-i} d\mathbf{u} \\ &= \int_{I^k} \prod_{i=1}^k u_i^{r_i} (1 - u_i)^{k-i+\sum_{j=i+1}^k r_j} d\mathbf{u} \\ &= \prod_{i=1}^k \int_0^1 u_i^{r_i} (1 - u_i)^{k-i+\sum_{j=i+1}^k r_j} du_i && \text{(Theorem 10.2)} \\ &= \prod_{i=1}^k \frac{r_i! \left( k - i + \sum_{j=i+1}^k r_j \right)!}{\left( k - i + 1 + \sum_{j=i}^k r_j \right)!} && \text{(Theorem 8.20)} \\ &= \frac{r_1! \cdots r_k!}{(k + r_1 + \cdots + r_k)!}. \end{aligned}$$

□

**Exercise 10.14 (Levi-Civita symbol).** Prove  $\varepsilon(j_1, \dots, j_k) = s(j_1, \dots, j_k)$ , where

$$s(j_1, \dots, j_k) = \prod_{p < q} \text{sgn}(j_q - j_p).$$

It is usually to define the Levi-Civita symbol by

$$\varepsilon(j_1, \dots, j_k) = \begin{cases} 1 & \text{if } (j_1, \dots, j_k) \text{ is an even permutation of } J, \\ -1 & \text{if } (j_1, \dots, j_k) \text{ is an odd permutation of } J, \\ 0 & \text{otherwise} \end{cases}$$

(Basic  $k$ -forms 10.14). Thus, it is the sign of the permutation in the case of a permutation, and zero otherwise. So  $\varepsilon(j_1, \dots, j_k)$  is equivalent to an explicit expression  $s(j_1, \dots, j_k) = \prod_{p < q} \text{sgn}(j_q - j_p)$ .

*Proof.*

(1) Induction on  $k$ . Base case: Show that  $\varepsilon(j_1, j_2) = s(j_1, j_2)$ . Since

$$\varepsilon(j_1, j_2) = \begin{cases} 1 & \text{if } j_1 < j_2 \\ -1 & \text{if } j_1 > j_2, \end{cases}$$

$$\varepsilon(j_1, j_2) = \operatorname{sgn}(j_2 - j_1) = s(j_1, j_2).$$

(2) Inductive step: Show that for any  $s \geq 2$ , if  $\varepsilon(j_1, \dots, j_s) = s(j_1, \dots, j_s)$  holds, then  $\varepsilon(j_1, \dots, j_{s+1}) = s(j_1, \dots, j_{s+1})$  also holds.

$$\begin{aligned} \varepsilon(j_1, \dots, j_{s+1}) &= \varepsilon(j_1, \dots, j_s) \prod_{\substack{1 \leq p \leq s \\ q=s+1}} \operatorname{sgn}(j_q - j_p) \\ &= s(j_1, \dots, j_s) \prod_{\substack{1 \leq p \leq s \\ q=s+1}} \operatorname{sgn}(j_q - j_p) \\ &= \prod_{1 \leq p < q \leq s} \operatorname{sgn}(j_q - j_p) \prod_{\substack{1 \leq p \leq s \\ q=s+1}} \operatorname{sgn}(j_q - j_p) \\ &= \prod_{1 \leq p < q \leq s+1} \operatorname{sgn}(j_q - j_p) \\ &= s(j_1, \dots, j_{s+1}). \end{aligned}$$

(3) Conclusion: Since both the base case and the inductive step have been proved as true, by mathematical induction the statement holds for every integer  $k \geq 2$ .

□

**Exercise 10.15.** If  $\omega$  and  $\lambda$  are  $k$ - and  $m$ -forms, respectively, prove that

$$\omega \wedge \lambda = (-1)^{km} \lambda \wedge \omega.$$

*Proof.*

(1) Write

$$\omega = \sum_I b_I(\mathbf{x}) dx_I, \quad \lambda = \sum_J c_J(\mathbf{x}) dx_J$$

in the standard presentations, where  $I$  and  $J$  range over all increasing  $k$ -indices and over all increasing  $m$ -indices taken from the set  $\{1, \dots, n\}$ .

(2) Show that  $dx_I \wedge dx_J = (-1)^{km} dx_J \wedge dx_I$ .

$$\begin{aligned}
dx_I \wedge dx_J &= dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_J \\
&= (-1)^m dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}} \wedge dx_J \wedge dx_{i_k} \\
&= (-1)^{2m} dx_{i_1} \wedge \cdots \wedge dx_{i_{k-2}} \wedge dx_J \wedge dx_{i_{k-1}} \wedge dx_{i_k} \\
&\dots \\
&= (-1)^{km} dx_J \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&= (-1)^{km} dx_J \wedge dx_I.
\end{aligned}$$

(3)

$$\begin{aligned}
\omega \wedge \lambda &= \sum_{I,J} b_I(\mathbf{x}) c_J(\mathbf{x}) dx_I \wedge dx_J \\
&= (-1)^{km} \sum_{J,I} c_J(\mathbf{x}) b_I(\mathbf{x}) dx_J \wedge dx_I \\
&= (-1)^{km} \lambda \wedge \omega.
\end{aligned}$$

□

**Exercise 10.16.** If  $k \geq 2$  and  $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_k]$  is an oriented affine  $k$ -simplex, prove that  $\partial^2 \sigma = 0$ , directly from the definition of the boundary operator  $\partial$ . Deduce from this that  $\partial^2 \Psi = 0$  for every chain  $\Psi$ . (Hint: For orientation, do it first for  $k = 2$ ,  $k = 3$ . In general, if  $i < j$ , let  $\sigma_{ij}$  be the  $(k-2)$ -simplex obtained by deleting  $\mathbf{p}_i$  and  $\mathbf{p}_j$  from  $\sigma$ . Show that each  $\sigma_{ij}$  occurs twice in  $\partial^2 \sigma$ , with opposite sign.)

*Proof (Brute-force).*

- (1) Write the boundary of the oriented affine  $k$ -simplex  $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_k]$  as

$$\partial \sigma = \sum_{i=0}^k (-1)^i [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k]$$

where where the oriented  $(k-1)$ -simplex  $[\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k]$  is obtained by deleting  $\sigma$ 's  $i$ -th vertex (Boundaries 10.29).

(2)

$$\begin{aligned}
\partial^2 \sigma &= \partial \left( \sum_i (-1)^i [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k] \right) \\
&= \sum_i (-1)^i \partial [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k] \\
&= \sum_{j < i} (-1)^i (-1)^j [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_j, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k] \\
&\quad + \sum_{j > i} (-1)^i (-1)^{j-1} [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \widehat{\mathbf{p}}_j, \dots, \mathbf{p}_k] \\
&= \sum_{j < i} (-1)^{i+j} [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_j, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k] \\
&\quad - \sum_{j > i} (-1)^{i+j} [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \widehat{\mathbf{p}}_j, \dots, \mathbf{p}_k].
\end{aligned}$$

The latter two summations cancel since after switching  $i$  and  $j$  in the second sum. Therefore  $\partial^2 \sigma = 0$ .

(3) The boundary of a chain is the linear combination of boundaries of the simplices in the chain. Write  $\Psi = \sum_{i=1}^r \sigma_i$ , where  $\sigma_i$  is an oriented affine simplex. Then

$$\partial^2 \Psi = \partial \left( \partial \sum \sigma_i \right) = \partial \left( \sum \partial \sigma_i \right) = \sum \partial^2 \sigma_i = \sum 0 = 0$$

for any affine chain  $\Psi$ .

□

**Exercise 10.17.** Put  $J^2 = \tau_1 + \tau_2$ , where

$$\tau_1 = [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2], \quad \tau_2 = -[\mathbf{0}, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1].$$

Explain why it is reasonable to call  $J^2$  the positively oriented unit square in  $\mathbb{R}^2$ . Show that  $\partial J^2$  is the sum of 4 oriented affine 1-simplexes. Find these. What is  $\partial(\tau_1 - \tau_2)$ ?

*Proof.*

(1) Note that the unit square  $I^2 \in \mathbb{R}^2$  is the union of  $\tau_1(Q^2)$  and  $\tau_2(Q^2)$ , where

$$\begin{aligned}
\tau_1(\mathbf{u}) &= ([\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2])(\mathbf{u}) \\
&= \mathbf{0} + \alpha_1 \mathbf{e}_1 + \alpha_2 (\mathbf{e}_1 + \mathbf{e}_2) \\
&= \mathbf{0} + (\alpha_1 + \alpha_2) \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 \\
&= \mathbf{0} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{u}
\end{aligned}$$



and

$$\begin{aligned}
\tau_2(\mathbf{u}) &= (-[\mathbf{0}, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1])(\mathbf{u}) \\
&= ([\mathbf{0}, \mathbf{e}_2 + \mathbf{e}_1, \mathbf{e}_2])(\mathbf{u}) \\
&= \mathbf{0} + \alpha_1(\mathbf{e}_1 + \mathbf{e}_2) + \alpha_2\mathbf{e}_2 \\
&= \mathbf{0} + \alpha_1\mathbf{e}_1 + (\alpha_1 + \alpha_2)\mathbf{e}_2 \\
&= \mathbf{0} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{u}
\end{aligned}$$

where  $\mathbf{u} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 \in \mathbb{R}^2$  (as in Equation (78)). Both  $\tau_1$  and  $\tau_2$  have Jacobian  $1 > 0$ , or positively oriented (Affine simplexes 10.26). So it is reasonable to call  $J^2$  the positively oriented unit square in  $\mathbb{R}^2$ .

(2)

$$\begin{aligned}
\partial\tau_1 &= [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] - [\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{0}, \mathbf{e}_1], \\
\partial\tau_2 &= [\mathbf{e}_2 + \mathbf{e}_1, \mathbf{e}_2] - [\mathbf{0}, \mathbf{e}_2] + [\mathbf{0}, \mathbf{e}_2 + \mathbf{e}_1] \\
&= [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2] + [\mathbf{e}_2, \mathbf{0}] + [\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2].
\end{aligned}$$

(3) By (2),

$$\partial J^2 = \partial\tau_1 + \partial\tau_2 = [\mathbf{0}, \mathbf{e}_1] + [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2] + [\mathbf{e}_2, \mathbf{0}],$$

which is the positively oriented boundary of  $I^2$ .

(4) By (2),

$$\begin{aligned}
\partial(\tau_1 - \tau_2) &= \partial\tau_1 - \partial\tau_2 \\
&= [\mathbf{0}, \mathbf{e}_1] + [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{0}] \\
&\quad + [\mathbf{0}, \mathbf{e}_2] + [\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{0}].
\end{aligned}$$

□

**Exercise 10.18.** Consider the oriented affine 3-simplex

$$\sigma_1 = [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]$$

in  $\mathbb{R}^3$ . Show that  $\sigma_1$  (regarded as a linear transformation) has determinant 1. Thus  $\sigma_1$  is positively oriented.

Let  $\sigma_2, \dots, \sigma_6$  be five other oriented 3-simplexes, obtained as follows: There are five permutations  $(i_1, i_2, i_3)$  of  $(1, 2, 3)$ , distinct from  $(1, 2, 3)$ . Associate with each  $(i_1, i_2, i_3)$  the simplex

$$s(i_1, i_2, i_3)[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}]$$

where  $s$  is the sign that occurs in the definition of the determinant. (This is how  $\tau_2$  was obtained from  $\tau_1$  in Exercise 10.17.) Show that  $\sigma_2, \dots, \sigma_6$  are positively oriented.

Put  $J^3 = \sigma_1 + \dots + \sigma_6$ . Then  $J^3$  may be called the positively oriented unit cube in  $\mathbb{R}^3$ . Show that  $\partial J^3$  is the sum of 12 oriented affine 2-simplexes. (These 12 triangles cover the surface of the unit cube  $I^3$ .)

Show that  $\mathbf{x} = (x_1, x_2, x_3)$  is in the range of  $\sigma_1$  if and only if  $0 \leq x_3 \leq x_2 \leq x_1 \leq 1$ .

Show that the range of  $\sigma_1, \dots, \sigma_6$  have disjoint interiors, and that their union covers  $I^3$ . (Compared with Exercise 10.13; note that  $3! = 6$ .)

*Proof.*

- (1) Show that  $\sigma_1$  (regarded as a linear transformation) has determinant 1. Given any  $\mathbf{u} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3 \in \mathbb{R}^3$ , we have

$$\begin{aligned} \sigma_1(\mathbf{u}) &= ([\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3])(\mathbf{u}) \\ &= \mathbf{0} + \alpha_1 \mathbf{e}_1 + \alpha_2(\mathbf{e}_1 + \mathbf{e}_2) + \alpha_3(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) \\ &= \mathbf{0} + (\alpha_1 + \alpha_2 + \alpha_3)\mathbf{e}_1 + (\alpha_2 + \alpha_3)\mathbf{e}_2 + \alpha_3 \mathbf{e}_3 \\ &= \mathbf{0} + \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{say } A} \mathbf{u}. \end{aligned}$$

So

$$\det(A) = \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = 1.$$

- (2) Show that  $\sigma_2, \dots, \sigma_6$  are positively oriented. Define the permutation matrix  $P_{(i_1, i_2, i_3)}$  corresponding to a permutation  $(i_1, i_2, i_3)$  of  $(1, 2, 3)$  by

$$P_{(i_1, i_2, i_3)} = [\mathbf{e}_{i_1} \quad \mathbf{e}_{i_2} \quad \mathbf{e}_{i_3}].$$

For example,

$$P_{(2,3,1)} = [\mathbf{e}_2 \quad \mathbf{e}_3 \quad \mathbf{e}_1] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Note that the sign  $s(i_1, i_2, i_3)$  of the permutation  $(i_1, i_2, i_3)$  is exactly the same as the determinant of the permutation matrix  $P_{(i_1, i_2, i_3)}$ . Define a

permutation  $(j_1, j_2, 3)$  of  $(1, 2, 3)$  (for swapping the first and the second coordinates of  $\mathbf{u}$ ) by

$$(j_1, j_2, 3) = \begin{cases} (1, 2, 3) & \text{if } s(i_1, i_2, i_3) = 1, \\ (2, 1, 3) & \text{if } s(i_1, i_2, i_3) = -1. \end{cases}$$

Write

$$\sigma_{(i_1, i_2, i_3)} = s(i_1, i_2, i_3)[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}].$$

(So that  $\sigma_1 = \sigma_{(1,2,3)}$ .) Hence,

$$\begin{aligned} & \sigma_{(i_1, i_2, i_3)}(\mathbf{u}) \\ &= \mathbf{0} + \alpha_{j_1} \mathbf{e}_{i_1} + \alpha_{j_2}(\mathbf{e}_{i_1} + \mathbf{e}_{i_2}) + \alpha_3(\mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}) \\ &= \mathbf{0} + (\alpha_{j_1} + \alpha_{j_2} + \alpha_3) \mathbf{e}_{i_1} + (\alpha_{j_2} + \alpha_3) \mathbf{e}_{i_2} + \alpha_3 \mathbf{e}_{i_3} \\ &= \mathbf{0} + P_{(i_1, i_2, i_3)} A P_{(j_1, j_2, 3)} \mathbf{u} \end{aligned}$$

where  $\mathbf{u} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3 \in \mathbb{R}^3$ . For example,

$$P_{(2,3,1)} A P_{(1,2,3)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

So

$$\begin{aligned} \det(P_{(i_1, i_2, i_3)} A P_{(j_1, j_2, 3)}) &= \det(P_{(i_1, i_2, i_3)}) \det(A) \det(P_{(j_1, j_2, 3)}) \\ &= s(i_1, i_2, i_3) \cdot 1 \cdot s(i_1, i_2, i_3) \\ &= 1. \end{aligned}$$

(3) Show that  $\partial J^3$  is the sum of 12 oriented affine 2-simplexes. Note that

$$\begin{aligned} \sum_{(i_1, i_2, i_3)} \sigma_{(i_1, i_2, i_3)} &= \sum_{\substack{(i_1, i_2, i_3) \\ i_1 > i_2}} \sigma_{(i_1, i_2, i_3)} + \sum_{\substack{(i_1, i_2, i_3) \\ i_1 < i_2}} \sigma_{(i_1, i_2, i_3)} \\ &= \sum_{\substack{(i_1, i_2, i_3) \\ i_1 > i_2}} s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\ &\quad + \sum_{\substack{(i_1, i_2, i_3) \\ i_2 > i_1}} -s(i_2, i_1, i_3) [\mathbf{0}, \mathbf{e}_{i_2} + \mathbf{e}_{i_1}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\ &= \mathbf{0} \end{aligned}$$

and

$$\begin{aligned}
\sum_{(i_1, i_2, i_3)} \sigma_{(i_1, i_2, i_3)} &= \sum_{\substack{(i_1, i_2, i_3) \\ i_2 > i_3}} \sigma_{(i_1, i_2, i_3)} + \sum_{\substack{(i_1, i_2, i_3) \\ i_2 < i_3}} \sigma_{(i_1, i_2, i_3)} \\
&= \sum_{\substack{(i_1, i_2, i_3) \\ i_2 > i_3}} s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\
&\quad + \sum_{\substack{(i_1, i_2, i_3) \\ i_3 > i_2}} -s(i_1, i_3, i_2) [\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\
&= \mathbf{0}.
\end{aligned}$$

So

$$\begin{aligned}
\partial J^3 &= \sum_{(i_1, i_2, i_3)} \partial \sigma_{(i_1, i_2, i_3)} \\
&= \sum_{(i_1, i_2, i_3)} s(i_1, i_2, i_3) [\mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}] \\
&\quad - s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}] \\
&\quad + s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}] \\
&\quad - s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}] \\
&= \sum_{(i_1, i_2, i_3)} s(i_1, i_2, i_3) [\mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\
&\quad - \underbrace{\sum_{(i_1, i_2, i_3)} s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]}_{=0} \\
&\quad + \underbrace{\sum_{(i_1, i_2, i_3)} s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]}_{=0} \\
&\quad - \sum_{(i_1, i_2, i_3)} s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}].
\end{aligned}$$

Thus,

$$\begin{aligned}
\partial J^3 &= \sum_{(i_1, i_2, i_3)} s(i_1, i_2, i_3) [\mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\
&\quad - \sum_{(i_1, i_2, i_3)} s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}]
\end{aligned}$$

is the sum of 12 oriented affine 2-simplexes. (Note that  $3! = 6$ .)

- (4) Show that  $\mathbf{x} = (x_1, x_2, x_3)$  is in the range of  $\sigma_1$  if and only if  $0 \leq x_3 \leq x_2 \leq x_1 \leq 1$ .

- (a) By (1),  $\mathbf{x}$  is in the range of  $\sigma_1$  if and only if  $\mathbf{x} = A\mathbf{u}$  for  $\mathbf{u} = (u_1, u_2, u_3) \in Q^3$ , or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 + u_2 + u_3 \\ u_2 + u_3 \\ u_3 \end{bmatrix}.$$

- (b) Since  $\mathbf{u} = (u_1, u_2, u_3) \in Q^3$ ,  $u_1 + u_2 + u_3 \leq 1$  and  $u_1, u_2, u_3 \geq 0$ . Hence  $0 \leq u_3 \leq u_2 + u_3 \leq u_1 + u_2 + u_3 \leq 1$  or  $0 \leq x_3 \leq x_2 \leq x_1 \leq 1$ .  
(c) Conversely, if  $0 \leq x_3 \leq x_2 \leq x_1 \leq 1$ , we define

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_3 \end{bmatrix}.$$

Clearly,  $\mathbf{v} \in Q^3$ .

- (5) Show that the range of  $\sigma_1, \dots, \sigma_6$  have disjoint interiors, and that their union covers  $I^3$ . Similar to (4). By (2),  $\mathbf{x} = P_{(i_1, i_2, i_3)} A P_{(j_1, j_2, 3)} \mathbf{u}$ , or  $P_{(i_1, i_2, i_3)}^{-1} \mathbf{x} = A P_{(j_1, j_2, 3)} \mathbf{u}$ , or

$$\begin{bmatrix} x_{i_1} \\ x_{i_2} \\ x_{i_3} \end{bmatrix} = \begin{bmatrix} u_1 + u_2 + u_3 \\ u_{j_2} + u_3 \\ u_3 \end{bmatrix}.$$

In any case, we always have  $0 \leq u_3 \leq u_{j_2} + u_3 \leq u_1 + u_2 + u_3 \leq 1$ . Hence  $\mathbf{x} = (x_1, x_2, x_3)$  is in the range of  $\sigma_{(i_1, i_2, i_3)}$  if and only if

$$0 \leq x_{i_3} \leq x_{i_2} \leq x_{i_1} \leq 1.$$

The interior of  $\sigma_{(i_1, i_2, i_3)}$  is

$$\{\mathbf{x} \in \mathbb{R}^3 : 0 < x_{i_3} < x_{i_2} < x_{i_1} < 1\},$$

and thus the range of  $\sigma_1, \dots, \sigma_6$  have disjoint interiors. Also, any  $\mathbf{x} \in I^3$  has the relation

$$0 \leq x_{i_3} \leq x_{i_2} \leq x_{i_1} \leq 1$$

for some permutation  $(i_1, i_2, i_3)$  of  $(1, 2, 3)$ . Hence

$$I^3 = \bigcup_{(i_1, i_2, i_3)} \sigma_{(i_1, i_2, i_3)}(Q^3) = \bigcup_{i=1}^6 \sigma_i(Q^3).$$

□

**Exercise 10.19.** Let  $J^2$  and  $J^3$  be as in Exercise 10.17 and Exercise 10.18. Define

$$\begin{aligned} B_{01}(u, v) &= (0, u, v), & B_{11}(u, v) &= (1, u, v), \\ B_{02}(u, v) &= (u, 0, v), & B_{12}(u, v) &= (u, 1, v), \\ B_{03}(u, v) &= (u, v, 0), & B_{13}(u, v) &= (u, v, 1). \end{aligned}$$

These are affine, and map  $\mathbb{R}^2$  into  $\mathbb{R}^3$ . Put  $\beta_{ri} = B_{ri}(J^2)$ , for  $r = 0, 1$ ,  $i = 1, 2, 3$ . Each  $\beta_{ri}$  is an affine-oriented 2-chain. (See Section 10.30.) Verify that

$$\partial J^3 = \sum_{i=1}^3 (-1)^i (\beta_{0i} - \beta_{1i}),$$

in agreement with Exercise 10.18.)

*Proof.*

(1) A direct calculation shows that

$$\begin{aligned} B_{01}(\tau_1) - B_{11}(\tau_1) &= [\mathbf{0}, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3] - [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\ B_{02}(\tau_1) - B_{12}(\tau_1) &= [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_3] - [\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\ B_{03}(\tau_1) - B_{13}(\tau_1) &= [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] - [\mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\ B_{01}(\tau_2) - B_{11}(\tau_2) &= -[\mathbf{0}, \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3] + [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\ B_{02}(\tau_2) - B_{12}(\tau_2) &= -[\mathbf{0}, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_3] + [\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\ B_{03}(\tau_2) - B_{13}(\tau_2) &= -[\mathbf{0}, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]. \end{aligned}$$

(2) To express the formula in (1) clearly, we define

$$\omega_{(i_1, i_2, i_3)} = [\mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] - [\mathbf{0}, \mathbf{e}_{i_2}, \mathbf{e}_{i_2} + \mathbf{e}_{i_3}],$$

and thus

$$\begin{aligned} -(B_{01}(\tau_1) - B_{11}(\tau_1)) &= s(1, 2, 3)\omega_{(1, 2, 3)} \\ B_{02}(\tau_1) - B_{12}(\tau_1) &= s(2, 1, 3)\omega_{(2, 1, 3)} \\ -(B_{03}(\tau_1) - B_{13}(\tau_1)) &= s(3, 1, 2)\omega_{(3, 1, 2)} \\ -(B_{01}(\tau_2) - B_{11}(\tau_2)) &= s(1, 3, 2)\omega_{(1, 3, 2)} \\ B_{02}(\tau_2) - B_{12}(\tau_2) &= s(2, 3, 1)\omega_{(2, 3, 1)} \\ -(B_{03}(\tau_2) - B_{13}(\tau_2)) &= s(3, 2, 1)\omega_{(3, 2, 1)}. \end{aligned}$$

(3) Note that

$$\begin{aligned} \beta_{0i} - \beta_{1i} &= B_{0i}(J^2) - B_{1i}(J^2) \\ &= B_{0i}(\tau_1 + \tau_2) - B_{1i}(\tau_1 + \tau_2) \\ &= B_{0i}(\tau_1) + B_{0i}(\tau_2) - B_{1i}(\tau_1) - B_{1i}(\tau_2) \\ &= (B_{0i}(\tau_1) - B_{1i}(\tau_1)) + (B_{0i}(\tau_2) - B_{1i}(\tau_2)). \end{aligned}$$

Thus,

$$\begin{aligned}
& \sum_{i=1}^3 (-1)^i (\beta_{0i} - \beta_{1i}) \\
&= \sum_{i=1}^3 (-1)^i (B_{0i}(\tau_1) - B_{1i}(\tau_1)) + \sum_{i=1}^3 (-1)^i (B_{0i}(\tau_2) - B_{1i}(\tau_2)) \\
&= \sum_{(i_1, i_2, i_3)} s(i_1, i_2, i_3) \omega_{(i_1, i_2, i_3)} \\
&= \sum_{(i_1, i_2, i_3)} s(i_1, i_2, i_3) [\mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\
&\quad - \sum_{(i_1, i_2, i_3)} s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}] \\
&= \partial J^3.
\end{aligned}$$

□

**Exercise 10.20.** *State conditions under which the formula*

$$\int_{\Phi} f d\omega = \int_{\partial\Phi} f\omega - \int_{\Phi} (df) \wedge \omega$$

*is valid, and show that it generalizes the formula for integration by parts. (Hint:  $d(f\omega) = (df) \wedge \omega + f d\omega$ .)*

*Proof.*

(1) *If*

- (a)  $\Phi$  is a  $k$ -chain of class  $\mathcal{C}''$  in an open set  $V \subseteq \mathbb{R}^m$ ,
- (b)  $\omega$  is a  $(k-1)$ -form of class  $\mathcal{C}'$  in  $V$ ,
- (c)  $f$  is a 0-form of class  $\mathcal{C}'$  in  $V$ ,

*then*

$$\int_{\Phi} f d\omega = \int_{\partial\Phi} f\omega - \int_{\Phi} (df) \wedge \omega$$

(2) Theorem 10.20(a) implies that

$$d(f\omega) = (df) \wedge \omega + f d\omega.$$

(3) The Stokes' theorem (Theorem 10.33) shows that

$$\int_{\Phi} d(f\omega) = \int_{\partial\Phi} f\omega.$$

Hence

$$\int_{\Phi} f d\omega = \int_{\Phi} d(f\omega) - \int_{\Phi} (df) \wedge \omega = \int_{\partial\Phi} f\omega - \int_{\Phi} (df) \wedge \omega.$$

(4) Define  $\Phi : Q^1 = [0, 1] \rightarrow [a, b]$  by

$$\Phi(\alpha) = a + \alpha(b - a).$$

$\Phi$  is a 1-simplex of class  $\mathcal{C}''$  in an open set  $V \supseteq [a, b]$ . Also,

$$\partial\Phi = [b] - [a].$$

Let  $\omega = g$  be a 0-form of class  $\mathcal{C}'(V)$ .

(5) Note that

$$\begin{aligned} \int_{\Phi} f d\omega &= \int_{\Phi} f dg = \int_0^1 f(\Phi(t))g'(\Phi(t))\Phi'(t)dt = \int_a^b f(u)g'(u)du, \\ \int_{\partial\Phi} f\omega &= \int_{[b]} fg + \int_{-[a]} fg = f(b)g(b) + (-1)f(a)g(a), \\ \int_{\Phi} (df) \wedge \omega &= \int_{\Phi} (df)g = \int_0^1 f'(\Phi(t))g(\Phi(t))\Phi'(t)dt = \int_a^b f'(u)g(u)du. \end{aligned}$$

Hence

$$\int_a^b f(u)g'(u)du = f(b)g(b) - f(a)g(a) - \int_a^b f'(u)g(u)du,$$

which is the same as the integration by parts (Theorem 6.22).

□

**Exercise 10.21.** *As in Example 10.36, consider the 1-form*

$$\eta = \frac{xdy - ydx}{x^2 + y^2}$$

*in  $\mathbb{R}^2 - \{\mathbf{0}\}$ .*

(a) *Carry out the computation that leads to*

$$\int_{\gamma} \eta = 2\pi \neq 0,$$

*and prove that  $d\eta = 0$ .*

(b)



- (c) Take  $\Gamma(t) = (a \cos t, b \sin t)$  where  $a > 0$ ,  $b > 0$  are fixed. Use part (b) to show that

$$\int_0^{2\pi} \frac{ab}{a^2 \cos^2 t + b^2 \sin^2 t} dt = 2\pi.$$

- (d) Show that

$$\eta = d \left( \arctan \frac{y}{x} \right)$$

in any convex open set in which  $x \neq 0$ , and that

$$\eta = d \left( -\arctan \frac{x}{y} \right)$$

in any convex open set in which  $y \neq 0$ . Explain why this justifies the notation  $\eta = d\theta$ , in spite of the fact that  $\eta$  is not exact in  $\mathbb{R}^2 - \{0\}$ .

- (e) Show that (b) can be derived from (d).

- (f) If  $\Gamma$  is any closed  $\mathcal{C}'$ -curve in  $\mathbb{R}^2 - \{0\}$ , prove that

$$\frac{1}{2\pi} \int_{\Gamma} \eta = \text{Ind}(\Gamma).$$

(See Exercise 8.23 for the definition of the index of a curve.)

*Proof of (a).*

(1)

$$\begin{aligned} \int_{\gamma} \eta &= \int_0^{2\pi} \frac{(r \cos t)d(r \sin t) - (r \sin t)d(r \cos t)}{(r \cos t)^2 + (r \sin t)^2} \\ &= \int_0^{2\pi} \frac{(r \cos t)(r \cos t) - (r \sin t)(-r \sin t)}{(r \cos t)^2 + (r \sin t)^2} dt \\ &= \int_0^{2\pi} dt \\ &= 2\pi. \end{aligned}$$

(2)

$$\begin{aligned}
d\eta &= d\left(\frac{xdy - ydx}{x^2 + y^2}\right) \\
&= d\left(\frac{x}{x^2 + y^2}\right) \wedge dy + \frac{x}{x^2 + y^2} \wedge d^2y \\
&\quad - d\left(\frac{y}{x^2 + y^2}\right) \wedge dx - \frac{y}{x^2 + y^2} \wedge d^2x \\
&= d\left(\frac{x}{x^2 + y^2}\right) \wedge dy - d\left(\frac{y}{x^2 + y^2}\right) \wedge dx \quad (d^2 = 0) \\
&= \left\{D_1\left(\frac{x}{x^2 + y^2}\right)dx + D_2\left(\frac{y}{x^2 + y^2}\right)dy\right\} \wedge dy \\
&\quad - \left\{D_1\left(\frac{x}{x^2 + y^2}\right)dx + D_2\left(\frac{y}{x^2 + y^2}\right)dy\right\} \wedge dx \\
&= D_1\left(\frac{x}{x^2 + y^2}\right)dx \wedge dy \quad (dy \wedge dy = 0) \\
&\quad - D_2\left(\frac{y}{x^2 + y^2}\right)dy \wedge dx \quad (dx \wedge dx = 0) \\
&= 0
\end{aligned}$$

□

*Note.*

- (1)  $\eta$  is closed and locally exact, that is,  $\eta = dt$  on  $\mathbb{R}^2 - L$  where  $L$  is a half-line issuing from  $\mathbf{0}$ .  $\eta$  is not exact since  $\int_\gamma \eta = 2\pi \neq 0$ .
- (2) (*Poincaré's Lemma for 1-form.*) Let  $\omega = \sum a_i dx_i$  be defined in an open set  $U \subseteq \mathbb{R}^n$ . Then  $d\omega = 0$  if and only if for each  $p \in U$  there is a neighborhood  $V \subseteq U$  of  $p$  and a differentiable function  $f : V \rightarrow \mathbb{R}^1$  with  $df = \omega$  (i.e.,  $\omega$  is locally exact).

*Proof of (b).*

(1)

(2)

□

*Proof of (c).*

- (1)  $\Gamma$  satisfies all conditions described in (b). So

$$\int_\Gamma \eta = 2\pi.$$

(2) A direct calculation shows that

$$\begin{aligned}
2\pi &= \int_{\Gamma} \eta = \int_{\Gamma} \frac{xdy - ydx}{x^2 + y^2} \\
&= \int_0^{2\pi} \frac{a \cos(t) d(b \sin(t)) - b \sin(t) d(a \cos(t))}{(a \cos(t))^2 + (b \sin(t))^2} \\
&= \int_0^{2\pi} \frac{ab(\cos^2 t + \sin^2 t)}{a^2 \cos^2 t + b^2 \sin^2 t} \\
&= \int_0^{2\pi} \frac{ab}{a^2 \cos^2 t + b^2 \sin^2 t}.
\end{aligned}$$

□

*Proof of (d).*

(1) In any convex open set in which  $x \neq 0$ , we have

$$\begin{aligned}
d\left(\arctan \frac{y}{x}\right) &= \left(D_1 \arctan \frac{y}{x}\right) dx + \left(D_2 \arctan \frac{y}{x}\right) dy \\
&= -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy \\
&= \eta.
\end{aligned}$$

(2) In any convex open set in which  $y \neq 0$ , we have

$$\begin{aligned}
d\left(-\arctan \frac{x}{y}\right) &= \left(D_1 \left(-\arctan \frac{x}{y}\right)\right) dx + \left(D_2 \left(-\arctan \frac{x}{y}\right)\right) dy \\
&= -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy \\
&= \eta.
\end{aligned}$$

(3) By (1)(2),  $\eta$  is locally exact. Note that  $\theta_1 = \arctan \frac{y}{x}$  and  $\theta_2 = -\arctan \frac{x}{y}$  cannot be patched together to defined a global 0-form  $\theta$  on  $\mathbb{R}^2 - \{\mathbf{0}\}$ .

□

*Proof of (e).*

(1)

(2)

□

*Proof of (f).*

(1)

(2)

□

**Exercise 10.22.** As in Example 10.37, define  $\zeta$  in  $\mathbb{R}^3 - \{\mathbf{0}\}$  by

$$\zeta = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{r^3}$$

where  $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$ ,

*Proof of (a).*

(1)

(2)

□

*Proof of (b).*

(1)

(2)

□

*Proof of (c).*

(1)

(2)

□

*Proof of (d).*

(1)

(2)

□

*Proof of (e).*

(1)

(2)

□

*Proof of (f).*

(1)

(2)

□

*Proof of (g).*

(1)

(2)

□

**Exercise 10.23. ...**

*Proof of (a).*

(1)

(2)

□

*Proof of (b).*

(1)

(2)

□

*Proof of (c).*

(1)

(2)

□

*Proof of (d).*

(1)

(2)

□

**Exercise 10.24.** Let  $\omega = \sum a_i(\mathbf{x})dx_i$  be a 1-form of class  $\mathcal{C}''$  in a convex open set  $E \subseteq \mathbb{R}^n$ . Assume  $d\omega = 0$  and prove that  $\omega$  is exact in  $E$ , by completing the following outline:

Fix  $\mathbf{p} \in E$ . Define

$$f(\mathbf{x}) = \int_{[\mathbf{p}, \mathbf{x}]} \omega \quad (\mathbf{x} \in E).$$

Apply Stokes' theorem to affine-oriented 2-simplexes  $[\mathbf{p}, \mathbf{x}, \mathbf{y}]$  in  $E$ . Deduce that

$$f(\mathbf{y}) - f(\mathbf{x}) = \sum_{i=1}^n (y_i - x_i) \int_0^1 a_i((1-t)\mathbf{x} + t\mathbf{y}) dt$$

for  $\mathbf{x} \in E, \mathbf{y} \in E$ . Hence  $(D_i f)(\mathbf{x}) = a_i(\mathbf{x})$ .

*Proof.*

(1) Fix  $\mathbf{p} \in E$ . Define

$$f(\mathbf{x}) = \int_{[\mathbf{p}, \mathbf{x}]} \omega \quad (\mathbf{x} \in E).$$

(2) Given any  $\mathbf{x} \in E, \mathbf{y} \in E$ , and  $\mathbf{x} \neq \mathbf{y}$ . The affine-oriented 2-simplex  $\Psi = [\mathbf{p}, \mathbf{x}, \mathbf{y}]$  is in  $E$  by the convexity of  $E$ . (If  $E$  is open but not convex, we can show that  $\omega = df$  **locally** as the note in Exercise 10.21(a). That is why we say that  $\omega$  is locally exact. The proof is exactly the same.)

(3) Note that

$$\partial\Psi = \partial[\mathbf{p}, \mathbf{x}, \mathbf{y}] = [\mathbf{x}, \mathbf{y}] - [\mathbf{p}, \mathbf{y}] + [\mathbf{p}, \mathbf{x}].$$

The Stokes' theorem (Theorem 10.33) implies that

$$\begin{aligned} \int_{\Psi} d\omega &= \int_{\partial\Psi} \omega \iff \int_{\Psi} 0 = \int_{[\mathbf{x}, \mathbf{y}]} \omega - \int_{[\mathbf{p}, \mathbf{y}]} \omega + \int_{[\mathbf{p}, \mathbf{x}]} \omega \\ &\iff 0 = \int_{[\mathbf{x}, \mathbf{y}]} \omega - f(\mathbf{y}) + f(\mathbf{x}) \\ &\iff f(\mathbf{y}) - f(\mathbf{x}) = \int_{[\mathbf{x}, \mathbf{y}]} \omega. \end{aligned}$$

(4) Define  $\gamma : [0, 1] \rightarrow E$  by

$$\begin{aligned} \gamma(t) &= \mathbf{x} + t(\mathbf{y} - \mathbf{x}) \\ &= \sum_{i=1}^n x_i + t(y_i - x_i) \end{aligned}$$

(where  $\mathbf{x} = (x_1, \dots, x_n)$  and  $\mathbf{y} = (y_1, \dots, y_n)$ ). Hence  $[0, 1]$  is the parameter domain of  $[\mathbf{x}, \mathbf{y}]$  with respect to  $\gamma$ . So

$$\begin{aligned}\int_{[\mathbf{x}, \mathbf{y}]} \omega &= \int_0^1 \sum_{i=1}^n a_i(\gamma(t)) \frac{\partial(x_i + t(y_i - x_i))}{\partial t} dt \\ &= \int_0^1 \sum_{i=1}^n a_i(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))(y_i - x_i) dt \\ &= \sum_{i=1}^n (y_i - x_i) \int_0^1 a_i(\mathbf{x} + t(\mathbf{y} - \mathbf{x})) dt.\end{aligned}$$

Thus,

$$f(\mathbf{y}) - f(\mathbf{x}) = \sum_{i=1}^n (y_i - x_i) \int_0^1 a_i(\mathbf{x} + t(\mathbf{y} - \mathbf{x})) dt.$$

(5) Note that

$$\begin{aligned}f(\mathbf{x} + h\mathbf{e}_j) - f(\mathbf{x}) &= \sum_{i=1}^n ((x_i + h\delta_{ij}) - x_i) \int_0^1 a_i(\mathbf{x} + t((\mathbf{x} + h\mathbf{e}_j) - \mathbf{x})) dt \\ &= \sum_{i=1}^n h\delta_{ij} \int_0^1 a_i(\mathbf{x} + t h\mathbf{e}_j) dt \\ &= h \int_0^1 a_j(\mathbf{x} + t h\mathbf{e}_j) dt.\end{aligned}$$

(Here  $\delta_{ij}$  is the Kronecker delta.) So

$$\begin{aligned}(D_j f)(\mathbf{x}) &= \lim_{h \rightarrow 0} \frac{f(\mathbf{x} + h\mathbf{e}_j) - f(\mathbf{x})}{h} \\ &= \lim_{h \rightarrow 0} \int_0^1 a_j(\mathbf{x} + t h\mathbf{e}_j) dt \\ &= \int_0^1 a_j(\mathbf{x}) dt \quad (a_j \in \mathcal{C}'') \\ &= a_j(\mathbf{x}).\end{aligned}$$

Thus,

$$df = \sum_{j=1}^n (D_j f)(\mathbf{x}) dx_j = \sum_{j=1}^n a_j(\mathbf{x}) dx_j = \omega,$$

or  $\omega$  is exact in  $E$ .

□

**Exercise 10.25.** Assume  $\omega$  is a 1-form in an open set  $E \subseteq \mathbb{R}^n$  such that

$$\int_{\gamma} \omega = 0$$

for every closed curve  $\gamma$  in  $E$ , of class  $\mathcal{C}'$ . Prove that  $\omega$  is exact in  $E$ , by imitating part of the argument sketched in Exercise 10.24.

*Proof.*

- (1) Assume that  $E$  is a **connected** open subset of  $\mathbb{R}^n$ . Show that  $\omega$  is exact in  $E$  if  $\int_{\gamma} \omega = 0$  for every closed curve  $\gamma$  in  $E$ , of class  $\mathcal{C}'$ .

- (2) Fix  $\mathbf{p} \in E$ . Define

$$f(\mathbf{x}) = \int_{[\mathbf{p}, \mathbf{x}]} \omega \quad (\mathbf{x} \in E).$$

It is well-defined since  $E$  is connected and  $\int_{\gamma} \omega = 0$  for every closed curve  $\gamma$  in  $E$ .

- (3) Given any  $\mathbf{x} \in E$ ,  $\mathbf{y} \in E$ , and  $\mathbf{x} \neq \mathbf{y}$ . Let

$$\gamma = [\mathbf{x}, \mathbf{y}] - [\mathbf{p}, \mathbf{y}] + [\mathbf{p}, \mathbf{x}]$$

be a closed curve in  $E$ . Hence,

$$\begin{aligned} 0 &= \int_{\gamma} \omega && \text{(Assumption)} \\ &= \int_{[\mathbf{x}, \mathbf{y}]} \omega - \int_{[\mathbf{p}, \mathbf{y}]} \omega + \int_{[\mathbf{p}, \mathbf{x}]} \omega \\ &= \int_{[\mathbf{x}, \mathbf{y}]} \omega - f(\mathbf{y}) + f(\mathbf{x}). \end{aligned}$$

So

$$f(\mathbf{y}) - f(\mathbf{x}) = \int_{[\mathbf{x}, \mathbf{y}]} \omega$$

- (4) Similar to (4)(5) in the proof of Exercise 10.24, we have  $df = \omega$ . So the statement in (1) is proved. In general, we can define each  $f_{\alpha}$  on each connected component  $E_{\alpha}$  (which is open) of  $E$  such that  $df_{\alpha} = \omega$  on  $E_{\alpha}$ . Take

$$f|_{E_{\alpha}} = f_{\alpha}$$

on  $E$ . Hence,  $df = \omega$  on the whole  $E$ .

□



**Exercise 10.26.** Assume  $\omega$  is a 1-form in  $\mathbb{R}^3 - \{\mathbf{0}\}$ , of class  $\mathcal{C}^1$  and  $d\omega = 0$ . Prove that  $\omega$  is exact in  $\mathbb{R}^3 - \{\mathbf{0}\}$ . (Hint: Every closed continuously differentiable curve in  $\mathbb{R}^3 - \{\mathbf{0}\}$  is the boundary of a 2-surface in  $\mathbb{R}^3 - \{\mathbf{0}\}$ . Apply Stokes' theorem and Exercise 10.25.)

*Proof.*

- (1) Let  $E = \mathbb{R}^3 - \{\mathbf{0}\}$ . By Exercise 10.25, it suffices to show that

$$\int_{\gamma} \omega = 0$$

for every closed curve  $\gamma$  in  $E$ , of class  $\mathcal{C}^1$ .

- (2) Intuitively, every closed continuously differentiable curve in  $\mathbb{R}^3 - \{\mathbf{0}\}$  is the boundary of a 2-surface in  $\mathbb{R}^3 - \{\mathbf{0}\}$ . So there is some 2-surface  $\Psi$  such that  $\partial\Psi = \gamma$ . The Stokes' theorem (Theorem 10.33) implies that

$$\int_{\gamma} \omega = \int_{\partial\Psi} \omega = \int_{\Psi} d\omega = \int_{\Psi} 0 = 0.$$

□

**Exercise 10.27.** ...

*Proof.*

- (1)

- (2)

□

**Exercise 10.28.** Fix  $b > a > 0$ , define

$$\Phi(r, \theta) = (r \cos \theta, r \sin \theta)$$

for  $a \leq r \leq b$ ,  $0 \leq \theta \leq 2\pi$ . (The range of  $\Phi$  is an annulus in  $\mathbb{R}^2$ .) Put  $\omega = x^3 dy$ , and compute both

$$\int_{\Phi} d\omega \quad \text{and} \quad \int_{\partial\Phi} \omega$$

to verify that they are equal.

*Proof.*

(1) Note that

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \det \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix} = r.$$

So

$$\begin{aligned} \int_{\Phi} d\omega &= \int_{\Phi} 3x^2 dx \wedge dy && (dy \wedge dy = 0) \\ &= \int_{[a, b] \times [0, 2\pi]} 3(r \cos \theta)^2 \frac{\partial(x, y)}{\partial(r, \theta)} dr d\theta \\ &= \int_a^b \int_0^{2\pi} 3r^3 (\cos \theta)^2 dr d\theta \\ &= \frac{3\pi}{4} (b^4 - a^4). \end{aligned}$$

(2) Similar to Exercise 10.21(b), write

$$\partial\Phi = \Gamma - \gamma,$$

where  $\Gamma(t) = (b \cos t, b \sin t)$  on  $[0, 2\pi]$  and  $\gamma(t) = (a \cos t, a \sin t)$  on  $[0, 2\pi]$ .

Hence

$$\begin{aligned} \int_{\partial\Phi} \omega &= \int_{\Gamma} \omega - \int_{\gamma} \omega \\ &= \int_{\Gamma} x^3 dy - \int_{\gamma} x^3 dy \\ &= \int_{[0, 2\pi]} (b \cos \theta)^3 \frac{\partial y}{\partial \theta} d\theta - \int_{[0, 2\pi]} (a \cos \theta)^3 \frac{\partial y}{\partial \theta} d\theta \\ &= \int_0^{2\pi} b^4 (\cos \theta)^4 d\theta - \int_0^{2\pi} a^4 (\cos \theta)^4 d\theta \\ &= \frac{3\pi}{4} (b^4 - a^4). \end{aligned}$$

(3)

$$\int_{\Phi} d\omega = \int_{\partial\Phi} \omega = \frac{3\pi}{4} (b^4 - a^4).$$

□

### Exercise 10.29. ...

*Proof.*

(1)

(2)

□

**Exercise 10.30.** If  $\mathbf{N}$  is the vector given by

$$\mathbf{N} = (\alpha_2\beta_3 - \alpha_3\beta_2)\mathbf{e}_1 + (\alpha_3\beta_1 - \alpha_1\beta_3)\mathbf{e}_2 + (\alpha_1\beta_2 - \alpha_2\beta_1)\mathbf{e}_3$$

(Equation (135)), prove that

$$\det \begin{bmatrix} \alpha_1 & \beta_1 & \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_2 & \beta_2 & \alpha_3\beta_1 - \alpha_1\beta_3 \\ \alpha_3 & \beta_3 & \alpha_1\beta_2 - \alpha_2\beta_1 \end{bmatrix} = |\mathbf{N}|^2$$

Also, verify

$$\mathbf{N} \cdot (T\mathbf{e}_1) = \mathbf{N} \cdot (T\mathbf{e}_2)$$

(Equation (137)).

*Proof.*

(1) By Laplace's expansion along the third column,

$$\begin{aligned} & \det \begin{bmatrix} \alpha_1 & \beta_1 & \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_2 & \beta_2 & \alpha_3\beta_1 - \alpha_1\beta_3 \\ \alpha_3 & \beta_3 & \alpha_1\beta_2 - \alpha_2\beta_1 \end{bmatrix} \\ &= (-1)^{1+3}(\alpha_2\beta_3 - \alpha_3\beta_2) \det \begin{bmatrix} \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} \\ & \quad + (-1)^{2+3}(\alpha_3\beta_1 - \alpha_1\beta_3) \det \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_3 & \beta_3 \end{bmatrix} \\ & \quad + (-1)^{3+3}(\alpha_1\beta_2 - \alpha_2\beta_1) \det \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix} \\ &= (\alpha_2\beta_3 - \alpha_3\beta_2)^2 + (\alpha_3\beta_1 - \alpha_1\beta_3)^2 + (\alpha_1\beta_2 - \alpha_2\beta_1)^2 \\ &= |\mathbf{N}|^2. \end{aligned}$$

(2)

$$\begin{aligned} \mathbf{N} \cdot (T\mathbf{e}_1) &= (\alpha_2\beta_3 - \alpha_3\beta_2, \alpha_3\beta_1 - \alpha_1\beta_3, \alpha_1\beta_2 - \alpha_2\beta_1) \cdot (\alpha_1, \alpha_2, \alpha_3) \\ &= (\alpha_2\beta_3 - \alpha_3\beta_2)\alpha_1 + (\alpha_3\beta_1 - \alpha_1\beta_3)\alpha_2 + (\alpha_1\beta_2 - \alpha_2\beta_1)\alpha_3 \\ &= (\alpha_3\alpha_2 - \alpha_2\alpha_3)\beta_1 + (\alpha_1\alpha_3 - \alpha_3\alpha_1)\beta_2 + (\alpha_2\alpha_1 - \alpha_1\alpha_2)\beta_3 \\ &= 0. \end{aligned}$$

(3)

$$\begin{aligned} \mathbf{N} \cdot (T\mathbf{e}_2) &= (\alpha_2\beta_3 - \alpha_3\beta_2, \alpha_3\beta_1 - \alpha_1\beta_3, \alpha_1\beta_2 - \alpha_2\beta_1) \cdot (\beta_1, \beta_2, \beta_3) \\ &= (\alpha_2\beta_3 - \alpha_3\beta_2)\beta_1 + (\alpha_3\beta_1 - \alpha_1\beta_3)\beta_2 + (\alpha_1\beta_2 - \alpha_2\beta_1)\beta_3 \\ &= (\beta_2\beta_3 - \beta_3\beta_2)\alpha_1 + (\beta_3\beta_1 - \beta_1\beta_3)\alpha_2 + (\beta_1\beta_2 - \beta_2\beta_1)\alpha_3 \\ &= 0. \end{aligned}$$

□

**Exercise 10.31. ...**

*Proof of (a).*

(1)

(2)

□

*Proof of (b).*

(1)

(2)

□

*Proof of (c).*

(1)

(2)

□

*Proof of (d).*

(1)

(2)

□

**Exercise 10.32. ...**

*Proof.*

(1)

(2)

□