

## Chapter 5: Differentiation

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**Exercise 5.1.** Let  $f$  be defined for all real  $x$ , and suppose that

$$|f(x) - f(y)| \leq (x - y)^2$$

for all real  $x$  and  $y$ . Prove that  $f$  is a constant.

*Proof.*

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq |x - y|$$

for  $x \neq y$ . Given any  $y \in \mathbb{R}$ ,  $\left| \frac{f(x) - f(y)}{x - y} \right| \rightarrow 0$  as  $x \rightarrow y$ , or  $|f'(y)| = 0$ . (Or using  $\epsilon$ - $\delta$  argument. Fix  $y \in \mathbb{R}$ . Given any  $\epsilon > 0$ , there exists  $\delta = \epsilon > 0$  such that

$$\left| \frac{f(x) - f(y)}{x - y} - 0 \right| \leq |x - y| < \delta = \epsilon$$

whenever  $|x - y| < \delta$ . That is,  $|f'(y)| = 0$ .) So  $f'(y) = 0$  for any  $y \in \mathbb{R}$ . By Theorem 5.11 (b),  $f$  is a constant.  $\square$

**Exercise 5.4.** If

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$

where  $C_0, \dots, C_n$  are real constants, prove that the equation

$$C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n = 0$$

has at least one real root between 0 and 1.

*Proof.* Let

$$g(x) = C_0x + \frac{C_1}{2}x^2 + \cdots + \frac{C_{n-1}}{n}x^n + \frac{C_n}{n+1}x^{n+1} \in \mathbb{R}[x].$$

Then  $g(0) = g(1) = 0$ , and  $g'(x) = C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n$ . By the mean value theorem (Theorem 5.10), there exists a point  $\xi \in (0, 1)$  at which

$$g(1) - g(0) = g'(\xi)(1 - 0),$$

or  $g'(\xi) = 0$ . That is, there exists a real root  $x = \xi$  between 0 and 1 at which  $C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n = 0$ .  $\square$