Chapter 6: The Riemann-Stieltjes Integral

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Supplement. Another definition of Riemann-Stieltjes integral. *(Exercise 7.3, 7.4 of the book T. M. Apostol, Mathematical Analysis, Second Edition.) Let P be a partition of $[a, b]$. The norm of a partition P is the length of the largest subinterval $[x_{i-1}, x_i]$ of P and is denoted by $\|P\|$.*

We say $f \in \mathscr{R}(\alpha)$ if there exists $A \in \mathbb{R}$ having the property that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any partition P of $[a, b]$ with norm $\|P\| < \delta$ and for any choice of $t_i \in [x_{i-1}, x_i]$, we have $|\sum_{i=1}^n f(t_i)\Delta\alpha_i - A| < \varepsilon$.

Claim. *$f \in \mathscr{R}$ in the sense of Definition 6.2 implies that $f \in \mathscr{R}$ in the sense of this another definition.*

Proof of Claim. Let $A = \int f dx$, $M > 0$ be one upper bound of $|f|$ on $[a, b]$. Given $\varepsilon > 0$, there exists a partition $P_0 = \{a = x_0, x_1, \ldots, x_{N-1}, x_N = b\}$ such that $U(P_0, f) \leq A + \frac{\varepsilon}{2}$. Let $\delta = \frac{\varepsilon}{2MN} > 0$. Then for any partition P with norm $\|P\| < \delta$, write

$$U(P, f) = \sum_{i=1}^n M_i \Delta x_i = S_1 + S_2,$$

where S_1 is the sum of terms arising from those subintervals of P containing no point of P_0, S_2 is the sum of the remaining terms. Then

$$\begin{aligned} S_1 &\leq U(P_0, f) < A + \frac{\varepsilon}{2}, \\ S_2 &\leq NM\|P\| < NM\delta < \frac{\varepsilon}{2}. \end{aligned}$$

Therefore, $U(P, f) < A + \varepsilon$. Similarly, $L(P, f) > A - \varepsilon$ whenever $\|P\| < \delta'$. Hence, $|\sum_{i=1}^n f(t_i)\Delta x_i - A| < \varepsilon$ whenever $\|P\| < \min\{\delta, \delta'\}$. (Copy Apostol's hint and ensure $M > 0$. M in Apostol's hint might be zero if $f = 0$.) $\square$

This supplement will be used in computing $\int_0^\infty (\frac{\sin x}{x})^2 dx = \frac{\pi}{2}$ in Exercise 8.12.

Exercise 6.1. *Suppose α increases on $[a, b]$, $a \leq x_0 \leq b$, α is continuous at x_0, $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathscr{R}(\alpha)$ and that $\int f d\alpha = 0$.*

Given any partition $P = \{a = p_0, p_1, \ldots, p_{n-1}, p_n = b\}$, where $a = p_0 \leq p_1 \leq \cdots \leq p_{n-1} \leq p_n = b$. We might compute $L(P, f, \alpha)$ and $U(P, f, \alpha)$ by using ε-δ

argument since we are hinted by the condition that α is continuous. A function which is continuous at x_0 has a nice property near x_0 and this property would help us estimate $U(P, f, \alpha)$ near x_0. On the contrary, if both f and α are discontinuous at x_0, it might be $f \notin \mathscr{R}(\alpha)$. Besides, if f has too many points of discontinuity ($f(x) = 0$ if $x \in \mathbb{Q}$ and $f(x) = 1$ otherwise, for example), then f might not be Riemann-integrable on $[0, 1]$.

Claim 1. $L(P, f, \alpha) = 0$.

Proof of Claim 1. $m_i = 0$ since $\inf f(x) = 0$ on any subinterval of $[a, b]$. So $L(P, f, \alpha) = \sum m_i \Delta\alpha_i = 0$. Here we don't need the condition that α is continuous at x_0. □

Claim 2. *For any $\varepsilon > 0$, there exists a partition P such that $U(P, f, \alpha) < \varepsilon$.*

Proof of Claim 2. Say $x_0 \in [p_{i_0-1}, p_{i_0}]$ for some i_0. Then

$$M_i = \sup_{p_{i-1} \leq x \leq p_i} f(x) = \begin{cases} 0 & \text{if } i \neq i_0, \\ 1 & \text{if } i = i_0. \end{cases}$$

So

$$U(P, f, \alpha) = \sum M_i \Delta\alpha_i = \Delta\alpha_{i_0}.$$

It is not true for any arbitrary α. (For example, α has a jump on $x = x_0$.) In fact, Exercise 6.3 shows this. Luckily, α is continuous at x_0. So for $\varepsilon > 0$, there exists $\delta > 0$ such that $|\alpha(x) - \alpha(x_0)| < \frac{\varepsilon}{2}$ whenever $|x - x_0| < \delta$ (and $x \in [a, b]$). Now we pick a nice partition

$$P = \{a, x_0 - \delta_1, x_0 + \delta_2, b\},$$

where $\delta_1 = \min\{\delta, x_0 - a\} \geq 0$ and $\delta_2 = \min\{\delta, b - x_0\} \geq 0$. (It is a trick about resizing "δ" to avoid considering the edge cases $x_0 = a$ or $x_0 = b$ or $a = b$.) Then $x_0 \in [x_0 - \delta_1, x_0 + \delta_2]$ and $\Delta\alpha$ on $[x_0 - \delta_1, x_0 + \delta_2]$ is

$$\begin{aligned} \alpha(x_0 + \delta_2) - \alpha(x_0 - \delta_1) &= (\alpha(x_0 + \delta_2) - \alpha(x_0)) + (\alpha(x_0) - \alpha(x_0 - \delta_1)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore, $U(P, f, \alpha) < \varepsilon$. □

Proof (Definition 6.2). By Claim 1 and 2 and notice that $U(P, f, \alpha) \geq 0$ for any partition P,

$$\overline{\int_a^b} f d\alpha = \inf U(P, f, \alpha) = 0,$$
$$\underline{\int_a^b} f d\alpha = \sup L(P, f, \alpha) = 0,$$

the inf and sup again being taken over all partitions. Hence $f \in \mathscr{R}(\alpha)$ and that $\int f d\alpha = 0$ by Definition 6.2. $\square$

Proof (Theorem 6.5). By Claim 1 and 2,

$$0 \leq U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$$

Hence $f \in \mathscr{R}(\alpha)$ by Theorem 6.5. Furthermore,

$$\int f d\alpha = \underline{\int_a^b} f d\alpha = \sup L(P, f, \alpha) = 0.$$

$\square$

Proof (Theorem 6.10). $f \in \mathscr{R}(\alpha)$ by Theorem 6.10. Thus, by Claim 1

$$\int f d\alpha = \underline{\int_a^b} f d\alpha = \sup L(P, f, \alpha) = 0.$$

$\square$

Exercise 6.2. *Suppose $f \geq 0$, f is continuous on $[a, b]$, and $\int_a^b f(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$. (Compare with Exercise 6.1.)*

For one application, see Exercise 7.20.

Proof. (Reductio ad absurdum) If there were $p \in [a, b]$ such that $f(p) > 0$. Since f is continuous on $[a, b]$, given $\varepsilon = \frac{1}{64} f(p) > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(p)| \leq \frac{1}{64} f(p) \text{ whenever } |x - p| \leq \delta, x \in [a, b].$$

Hence

$$f(x) \geq \frac{63}{64} f(p)$$

whenever $x \in E = [\max\{a, p - \delta\}, \min\{b, p + \delta\}] \subseteq [a, b]$. Note that the length of E is $|E| > 0$. So

$$0 = \int_a^b f(x)dx \geq \int_E f(x)dx \geq \int_E \frac{63}{64} f(p)dx = \frac{63}{64} f(p)|E| > 0,$$

which is absurd. $\square$

Note. (Lebesgue integral) *Let f be a nonnegative measurable function. Then $\int f = 0$ implies $f = 0$ a.e.*

Exercise 6.3.
PLACEHOLDER

Exercise 6.4. *If*

$$f(x) = \begin{cases} 0 & \text{for all irrational } x, \\ 1 & \text{for all rational } x, \end{cases}$$

prove that $f \not\in \mathscr{R}$ on $[a,b]$ for any $a < b$.

Proof. Given any partition

$$P = \{a = p_0, p_1, \dots, p_{n-1}, p_n = b\}$$

of $[a,b]$ where $a = p_0 \leq p_1 \leq \cdots \leq p_{n-1} \leq p_n = b$. Since $a < b$, we might assume that $a = p_0 < p_1 < \cdots < p_{n-1} < p_n = b$ by removing duplicated points. Since $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$ are dense in $\mathbb{R}$, we have

$$\begin{aligned} M_i &= \sup_{p_{i-1} \leq x \leq p_i} f(x) = 1, \\ m_i &= \inf_{p_{i-1} \leq x \leq p_i} f(x) = 0, \\ U(P,f) &= \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} \Delta x_i = b - a, \\ L(P,f) &= \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} 0 = 0. \end{aligned}$$

Since P is arbitrary,

$$\begin{aligned} \overline{\int_a^b} f dx &= \inf U(P,f) = b - a > 0, \\ \underline{\int_a^b} f dx &= \sup L(P,f) = 0. \end{aligned}$$

Hence $f \not\in \mathscr{R}$ on $[a,b]$ for any $a < b$. $\square$

Note.

(1) $f \in \mathscr{R}$ on $[a,b]$ iff $a = b$.

(2) (Problem 4.1 in *H. L. Royden, Real Analysis, 3rd edition.*) *Construct a sequence* $\{f_n\}$ *of nonnegative, Riemann integrable functions such that* f_n *increases monotonically to* f*. What does this imply about changing the order of integration and the limiting process?* (Since $\mathbb{Q}$ is countable, write

$$\mathbb{Q} = \{r_1, r_2, \ldots\}.$$

Define

$$f_n(x) = \begin{cases} 0 & \text{if } x \not\in \{r_1, \ldots, r_n\} , \\ 1 & \text{if } x \in \{r_1, \ldots, r_n\} . \end{cases}$$

By construction, f_n increases monotonically to f pointwise. Note that $f_n \to f$ not uniformly. Also, $\int_a^b f_n(x)dx = 0$ by using the same argument in Theorem 6.10. Therefore, $\lim_{n\to\infty} \int_a^b f_n(x)dx = 0$ but $\int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx$ does not exist.)

Exercise 6.5.
PLACEHOLDER

Exercise 6.6.
PLACEHOLDER

Exercise 6.7. *Suppose* f *is a real function on* $(0, 1]$ *and* $f \in \mathscr{R}$ *on* $[c, 1]$ *for every* $c > 0$*. Define*

$$\int_0^1 f(x)dx = \lim_{c\to 0} \int_c^1 f(x)dx$$

if this limit exists (and is finite).

(a) *If* $f \in \mathscr{R}$ *on* $[0, 1]$*, show that this definition of the integral agrees with the old one.*

(b) *Construct a function such that the above limit exists, although it fails to exist with* $|f|$ *in place of* f*.*

Proof of (a).

(1) Since $f \in \mathscr{R}$ on $[0, 1]$, f is bounded or $|f| \leq M$ for some real M.

(2) For any $0 < c < 1$, we have

$$\begin{aligned} \left| \int_0^1 f(x)dx - \int_c^1 f(x)dx \right| &= \left| \int_0^c f(x)dx \right| && \text{(Theorem 6.12(c))} \\ &\leq Mc. && \text{(Theorem 6.12(d))} \end{aligned}$$

(3) Given any $\varepsilon > 0$, there exists $\delta = \frac{\varepsilon}{M+1} > 0$ such that

$$\left| \int_0^c f(x)dx - \int_0^1 f(x)dx \right| \leq Mc < M\delta = M \cdot \frac{\varepsilon}{M+1} < \varepsilon$$

whenever $0 < c < \delta$. Hence $\lim_{c \to 0} \int_0^c f(x)dx = \int_0^1 f(x)dx$.

$\square$

Proof of (b)(Construct by nonabsolutely convergent series).

(1) Given any nonabsolutely (conditionally) convergent series $\sum_{k=1}^n a_k$ (take $\sum \frac{(-1)^n}{n}$ for example and then see Remark 3.46), we define f on $(0, 1]$ by

$$f(x) = 2^n a_n$$

if $\frac{1}{2^n} < x \leq \frac{1}{2^{n-1}}$ as $n = 1, 2, \ldots$.

(2) By construction,

$$\int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} f(x)dx = \left(\frac{1}{2^{n-1}} - \frac{1}{2^n} \right) 2^n a_n = a_n.$$

and thus

$$\int_{\frac{1}{2^n}}^1 f(x)dx = \int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} f(x)dx + \cdots + \int_{\frac{1}{2}}^1 f(x)dx = \sum_{k=1}^n a_k.$$

(3) Given any $\varepsilon > 0$. Since $\sum a_n$ is convergent, there exists a common integer N such that

$$|a_n| \leq \frac{\varepsilon}{89}$$

and

$$\left| \sum_{k=1}^n a_k - A \right| \leq \frac{\varepsilon}{64}$$

for some real A whenever $n \geq N$ (Definition 3.21 and Theorem 3.23). Therefore, for any $0 < c \leq \frac{1}{2^N}$, say $\frac{1}{2^{n+1}} < c \leq \frac{1}{2^n} \leq \frac{1}{2^N}$ for some $n \geq N$, we have

$$\begin{aligned}
\left| \int_c^1 f(x)dx - A \right| &= \left| \int_c^{\frac{1}{2^n}} f(x)dx + \int_{\frac{1}{2^n}}^1 f(x)dx - A \right| \\
&\leq \left| \left(\frac{1}{2^n} - c \right) 2^{n+1} a_{n+1} \right| + \left| \sum_{k=1}^n a_k - A \right| \\
&\leq |a_{n+1}| + \left| \sum_{k=1}^n a_k - A \right| \\
&\leq \frac{\varepsilon}{89} + \frac{\varepsilon}{64} \\
&\leq \varepsilon.
\end{aligned}$$

Hence, $\lim_{c \to 0} \int_c^1 f(x)dx = A$ exists.

(4) Since

$$\int_{\frac{1}{2^n}}^1 |f(x)|dx = \int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} |f(x)|dx + \cdots + \int_{\frac{1}{2}}^1 |f(x)|dx = \sum_{k=1}^n |a_k| \to \infty$$

as $n \to \infty$, $\lim_{c \to 0} \int_c^1 f(x)dx$ does not exist.

$\square$

Exercise 6.8.
PLACEHOLDER

Exercise 6.9.
PLACEHOLDER

Exercise 6.10. *Let p and q be positive real integers such that*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Prove the following statements.

(a) *If $u \geq 0$ and $v \geq 0$, then*

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

Equality holds if and only if $u^p = v^q$.

(b) *If $f \in \mathscr{R}(\alpha)$, $g \in \mathscr{R}(\alpha)$, $f \geq 0$, $g \geq 0$, and*

$$\int_a^b f^p d\alpha = \int_a^b g^q d\alpha = 1,$$

then

$$\int_a^b fg d\alpha \leq 1.$$

(c) *If f and g are complex functions in $\mathscr{R}(\alpha)$, then*

$$\left| \int_a^b fg d\alpha \right| \leq \left\{ \int_a^b |f|^p d\alpha \right\}^{\frac{1}{p}} \left\{ \int_a^b |g|^q d\alpha \right\}^{\frac{1}{q}}.$$

*This is **Hölder's inequality**. When $p = q = 2$ it is usually called the Schwarz inequality. (Note that Theorem 1.35 is a very special case of this.)*

(d) *Show that Hölder's inequality is also true for the "improper" integrals described in Exercise 6.7 and 6.8.*

Proof of (a)(Young's inequality).

(1) $u = 0$ or $v = 0$ is nothing to do. For $u > 0$ and $v > 0$, we give some different proofs.

(2) First proof.

$$\begin{aligned} uv &= \exp(\log(uv)) \\ &= \exp\left(\frac{1}{p}\log(u^p) + \frac{1}{q}\log(v^q)\right) \\ &\leq \frac{1}{p}\exp(\log(u^p)) + \frac{1}{q}\exp(\log(v^q)) \qquad \text{(Convexity of } \exp(x)) \\ &= \frac{u^p}{p} + \frac{v^q}{q}. \end{aligned}$$

Here the convexity of $\exp(x)$ can be derived by the fact that $(\exp(x))'' > 0$ and Exercise 5.14. The fact that the equality holds if and only if $u^p = v^q$ is derived from the strictly convexity of $\exp(x)$ additionally. (For the details about the exponential and logarithmic functions, might see Chapter 8.)

(3) Second proof.

$$\begin{aligned} \log\left(\frac{u^p}{p} + \frac{v^q}{q}\right) &\geq \frac{1}{p}\log(u^p) + \frac{1}{q}\log(v^q) \qquad \text{(Concavity of } \log(x)) \\ &= \log(u) + \log(v) \\ &= \log(uv). \end{aligned}$$

Since $\log(x)$ increases monotonically ($(\log(x))' = \frac{1}{x} > 0$ if $x > 0$), $\frac{u^p}{p} + \frac{v^q}{q} \geq uv$ (or take the exponential function to get the same conclusion). Here the concavity of $\log(x)$ can be derived by the fact that $(\log(x))'' < 0$ and a statement that $f''(x) \leq 0$ if and only if f is concave. The fact that the equality holds if and only if $u^p = v^q$ is derived from the strictly concavity of $\log(x)$ additionally. (The proof is analogous to Exercise 5.14.)

(4) Third proof. *Suppose that $f : [0, \infty) \to [0, \infty)$ is a strictly increasing continuous function such that $f(0) = 0$ and $\lim_{x\to\infty} f(x) = \infty$. Then*

$$uv \leq \int_0^u f(x)dx + \int_0^v f^{-1}(x)dx$$

for every $u, v \geq 0$, and equality occurs if and only if $v = f(u)$. Define

$$F(x) = -xf(x) + \int_0^x f(t)dt + \int_0^{f(x)} f^{-1}(t)dt.$$

By Theorem 6.20 (the fundamental theorem of calculus) and Theorem 5.5 (chain rule),

$$F'(x) = -(f(x) + xf'(x)) + f(x) + f'(x)f^{-1}(f(x)) = 0.$$

Hence $F(x)$ is a constant on $(0, u)$ (Theorem 5.11(b)). Note that $F(x)$ is continuous on $[0, u]$ and $F(0) = 0$, so $F(x) = 0$ on $[0, u]$ or

$$\int_0^x f(t)dt + \int_0^{f(x)} f^{-1}(t)dt = xf(x).$$

Take $x = u$ to get

$$\int_0^u f(x)dx + \int_0^{f(u)} f^{-1}(x)dx = uf(u).$$

Hence

$$\begin{aligned}
&\int_0^u f(x)dx + \int_0^v f^{-1}(x)dx - uv \\
=&\int_0^u f(x)dx + \int_0^{f(u)} f^{-1}(x)dx + \int_{f(u)}^v f^{-1}(x)dx - uv \\
=&uf(u) + \int_{f(u)}^v f^{-1}(x)dx - uv \\
=&\int_{f(u)}^v [f^{-1}(x) - f^{-1}(f(u))]dx \\
\geq&0.
\end{aligned}$$

The last inequality holds since f is strictly increasing and thus f^{-1} is strictly increasing too. Besides, the equality holds if and only if $f(u) = v$. Now the conclusion holds by taking $f(x) = x^{p-1}$ in

$$uv \leq \int_0^u f(x)dx + \int_0^v f^{-1}(x)dx$$

and the equality holds if and only if $u^p = v^q$.

□

Proof of (b). Every integral is well-defined (Theorem 6.11 and Theorem 6.13(a)). Let $u = f \geq 0$ and $v = g \geq 0$ in (a). Integrate both sides of the inequality

$$fg \leq \frac{f^p}{p} + \frac{g^q}{q}$$

to get

$$\begin{aligned}
\int_a^b fg d\alpha &\leq \int_a^b \left(\frac{f^p}{p} + \frac{g^q}{q} \right) d\alpha && \text{(Theorem 6.12(b))} \\
&= \int_a^b \frac{f^p}{p} d\alpha + \int_a^b \frac{g^q}{q} d\alpha && \text{(Theorem 6.12(a))} \\
&= \frac{1}{p} \int_a^b f^p d\alpha + \frac{1}{q} \int_a^b g^q d\alpha && \text{(Theorem 6.12(a))} \\
&= \frac{1}{p} + \frac{1}{q} && \text{(Assumption)} \\
&= 1.
\end{aligned}$$

The equality holds if $f^p = g^q$. Note that the equality does not hold only if $f^p = g^q$. (Consider α is constant on some subinterval $[c,d] \subsetneq [a,b]$.) Luckily, it is true for the additional assumption that $\alpha(x) = x$ and f, g are continuous on $[a,b]$. $\square$

Proof of (c). There are three possible cases.

(1) The case $\left\{ \int_a^b |f|^p d\alpha \right\}^{\frac{1}{p}} = 0$. So $\int_a^b |f|^p d\alpha = 0$.

(a) *Show that $\int_a^b |f| d\alpha = 0$ if $\int_a^b |f|^p d\alpha = 0$.* (Reductio ad absurdum) If $\int_a^b |f| d\alpha = A > 0$, then given $\varepsilon = \frac{A}{2} > 0$, there exists a partition $P_0 = \{a = x_0 \leq \cdots \leq x_n = b\}$ such that

$$\sum_{i=0}^n m_i \Delta\alpha_i > \frac{A}{2},$$

where $m_i = \inf_{x \in [x_{i-1}, x_i]} |f|$ and $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$. By the pigeonhole principle, there exists $1 \leq i_0 \leq n$ such that

$$L(P_0, |f|, \alpha) = m_{i_0} \Delta\alpha_{i_0} > \frac{A}{2n} > 0.$$

Especially, $m_{i_0} > 0$ and $\Delta\alpha_{i_0} > 0$. Now we consider $L(P, |f|^p, \alpha)$. Hence

$$L(P_0, |f|^p, \alpha) = \sum_{i=0}^n m_i^p \Delta\alpha_i \geq m_{i_0}^p \Delta\alpha_{i_0} > 0,$$

or

$$\underline{\int_a^b} |f| d\alpha = \sup L(P, f, \alpha) \geq m_{i_0}^p \Delta\alpha_{i_0} > 0,$$

which is absurd.

(b) *Show that* $\int_a^b |fg| d\alpha = 0$ *if* $\int_a^b |f| d\alpha = 0$. Since $g \in \mathscr{R}(\alpha)$, $|g|$ is bounded by some real M on $[a,b]$, that is, $|g(x)| \leq M$. Hence

$$0 \leq \int_a^b |fg| d\alpha \leq \int_a^b M|f| d\alpha = M \int_a^b |f| d\alpha = 0.$$

Therefore $\int_a^b |fg| d\alpha = 0$.

By (a)(b), $\int_a^b |fg| d\alpha = 0$ and thus Hölder's inequality holds for this case.

(2) The case $\left\{ \int_a^b |g|^q d\alpha \right\}^{\frac{1}{q}} = 0$. Similar to (1).

(3) If both $\left\{ \int_a^b |f|^p d\alpha \right\}^{\frac{1}{p}} > 0$ and $\left\{ \int_a^b |g|^q d\alpha \right\}^{\frac{1}{q}} > 0$, then we apply (b) to

$$F(x) = \frac{|f(x)|}{\left\{ \int_a^b |f(x)|^p d\alpha \right\}^{\frac{1}{p}}} \qquad \text{and} \qquad G(x) = \frac{|g(x)|}{\left\{ \int_a^b |g(x)|^q d\alpha \right\}^{\frac{1}{q}}}.$$

Here $F(x) \geq 0$ and $G(x) \geq 0$ are well-defined and Riemann integrable. Thus the conclusion holds. The equality holds if $F(x)^p = G(x)^q$ or

$$\frac{|f|^p}{\int_a^b |f|^p d\alpha} = \frac{|g|^q}{\int_a^b |g|^q d\alpha}.$$

Note that the equality does not hold only if $\frac{|f|^p}{\int_a^b |f|^p d\alpha} = \frac{|g|^q}{\int_a^b |g|^q d\alpha}$. Luckily, it is true for the additional assumption that $\alpha(x) = x$ and f, g are continuous on $[a,b]$.

By (1)(2)(3), in any case the equality holds if

$$|f|^p \int_a^b |g|^q d\alpha = |g|^q \int_a^b |f|^p d\alpha.$$

In addition, if $\alpha(x) = x$ and f, g are continuous on $[a,b]$, then the equality holds if and only if

$$|f|^p \int_a^b |g|^q d\alpha = |g|^q \int_a^b |f|^p d\alpha.$$

□

Proof of (d).

(1) *Suppose* f *and* g *are real functions on* $(0,1]$ *and* $f, g \in \mathscr{R}$ *on* $[c,1]$ *for every* $c > 0$. *Show that*

$$\left| \int_0^1 fg dx \right| \leq \left\{ \int_0^1 |f|^p dx \right\}^{\frac{1}{p}} \left\{ \int_0^1 |g|^q dx \right\}^{\frac{1}{q}}.$$

Here $\int_0^1$ *is one improper integral defined in Exercise 6.7.*

(a) By (c), we have

$$\left|\int_c^1 fgdx\right| \leq \left\{\int_c^1 |f|^p dx\right\}^{\frac{1}{p}} \left\{\int_c^1 |g|^q dx\right\}^{\frac{1}{q}}$$

for any $c \in (0,1]$. Here every integral is well-defined (Theorem 6.11 and Theorem 6.13).

(b) Since every integral is ≥ 0, by taking the limit in the right hand side we have

$$\begin{aligned}\left|\int_c^1 fgdx\right| &\leq \left\{\int_c^1 |f|^p dx\right\}^{\frac{1}{p}} \left\{\int_c^1 |g|^q dx\right\}^{\frac{1}{q}} \\ &\leq \left\{\int_0^1 |f|^p dx\right\}^{\frac{1}{p}} \left\{\int_0^1 |g|^q dx\right\}^{\frac{1}{q}}.\end{aligned}$$

It is possible that $\left\{\int_0^1 |f|^p dx\right\}^{\frac{1}{p}} = \infty$ or $\left\{\int_0^1 |g|^q dx\right\}^{\frac{1}{q}} = \infty$.

(c) Now $\left|\int_c^1 fgdx\right|$ is bounded by $\left\{\int_0^1 |f|^p dx\right\}^{\frac{1}{p}} \left\{\int_0^1 |g|^q dx\right\}^{\frac{1}{q}}$. Take limit to get

$$\left|\int_0^1 fgdx\right| \leq \left\{\int_0^1 |f|^p dx\right\}^{\frac{1}{p}} \left\{\int_0^1 |g|^q dx\right\}^{\frac{1}{q}}$$

even if some limit is divergent.

(2) *Suppose f and g are real functions on $[a,b]$ and $f, g \in \mathscr{R}$ on $[a,b]$ for every $b > a$ where a is fixed. Show that*

$$\left|\int_a^\infty fgdx\right| \leq \left\{\int_a^\infty |f|^p dx\right\}^{\frac{1}{p}} \left\{\int_a^\infty |g|^q dx\right\}^{\frac{1}{q}}.$$

Here $\int_a^\infty$ is one improper integral defined in Exercise 6.8. Same as (1).

□

Exercise 6.11. *Let α be a fixed increasing function on $[a,b]$. For $u \in \mathscr{R}(\alpha)$, define*

$$\|u\|_2 = \left\{\int_a^b |u|^2 d\alpha\right\}^{\frac{1}{2}}.$$

Suppose $f, g, h \in \mathscr{R}(\alpha)$, and prove the triangle inequality

$$\|f-h\|_2 \leq \|f-g\|_2 + \|g-h\|_2$$

as a consequence of the Schwarz inequality, as in the proof of Theorem 1.37.

Proof.

(1) By Exercise 6.10(c) with $p = q = 2$, we have

$$\begin{aligned}\int_a^b |f-g||g-h|d\alpha &= \left|\int_a^b |f-g||g-h|d\alpha\right| \\ &\leq \left\{\int_a^b |f-g|^2 dx\right\}^{\frac{1}{2}} \left\{\int_a^b |g-h|^2 dx\right\}^{\frac{1}{2}} \\ &= \|f-g\|_2 \|g-h\|_2.\end{aligned}$$

Every integral is well-defined (Theorem 6.12 and Theorem 6.13 (or Theorem 6.11)).

(2) Since

$$\begin{aligned}\|f-h\|_2^2 &= \int_a^b |f-h|^2 d\alpha \\ &\leq \int_a^b (|f-g| + |g-h|)^2 d\alpha && \text{(Triangle inequality)} \\ &= \int_a^b (|f-g|^2 + 2|f-g||g-h| + |g-h|^2) d\alpha \\ &= \int_a^b |f-g|^2 d\alpha + 2\int_a^b |f-g||g-h|d\alpha + \int_a^b |g-h|^2 d\alpha \\ &\leq \|f-g\|_2^2 + 2\|f-g\|_2\|g-h\|_2 + \|g-h\|_2^2 && ((1)) \\ &= (\|f-g\|_2 + \|g-h\|_2)^2,\end{aligned}$$

we have

$$\|f-h\|_2 \leq \|f-g\|_2 + \|g-h\|_2.$$

Here every integral is well-defined (Theorem 6.12 and Theorem 6.13 (or Theorem 6.11)).

$\square$

Exercise 6.12. *With the notations of Exercise 6.11, suppose $f \in \mathscr{R}(\alpha)$ and $\varepsilon > 0$. Prove that there exists a continuous function g on $[a, b]$ such that $\|f-g\|_2 < \varepsilon$. (Hint: Let $P = \{a = x_0 \leq \cdots \leq x_n = b\}$ be a suitable partition of $[a, b]$, define*

$$g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i)$$

if $x_{i-1} \leq t \leq x_i$.)

Proof. Given $\varepsilon > 0$.

(1) There are some real numbers m and M such that $m \leq f(x) \leq M$ if $x \in [a,b]$ since $f \in \mathscr{R}(\alpha)$ or f is bounded on $[a,b]$. By Theorem 6.6, there exists a partition $P = \{a = x_0 \leq \cdots \leq x_n = b\}$ such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \frac{\varepsilon^2}{M - m + 1}.$$

Here

$$\begin{aligned} U(P,f,\alpha) &= \sum_{i=1}^{n} M_i \Delta\alpha_i \text{ where } M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x) \\ L(P,f,\alpha) &= \sum_{i=1}^{n} m_i \Delta\alpha_i \text{ where } m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x). \end{aligned}$$

(2) For such partition P, define g on $[a,b]$ by

$$g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i)$$

if $x_{i-1} \leq t \leq x_i$. So that

$$\begin{aligned} |f(t) - g(t)| &= \left| \left(\frac{x_i - t}{\Delta x_i} + \frac{t - x_{i-1}}{\Delta x_i} \right) f(t) - \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i) \right| \\ &= \left| \frac{x_i - t}{\Delta x_i} (f(t) - f(x_{i-1})) + \frac{t - x_{i-1}}{\Delta x_i} (f(t) - f(x_i)) \right| \\ &\leq \frac{x_i - t}{\Delta x_i} |f(t) - f(x_{i-1})| + \frac{t - x_{i-1}}{\Delta x_i} |f(t) - f(x_i)| \\ &\leq \frac{x_i - t}{\Delta x_i} (M_i - m_i) + \frac{t - x_{i-1}}{\Delta x_i} (M_i - m_i) \\ &= M_i - m_i \end{aligned}$$

if $x_{i-1} \leq t \leq x_i$. Especially,

$$|f(t) - g(t)| \leq M - m$$

if $a \leq t \leq b$.

(3) Note that the integral $\int_a^b |f - g|^2 d\alpha$ is well-defined (Theorem 6.8, Theorem

6.11 and Theorem 6.12). So that

$$
\begin{aligned}
\int_a^b |f-g|^2 d\alpha =& \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f-g|^2 d\alpha \\
\leq& \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} (M-m)(M_i - m_i) d\alpha \\
=& (M-m) \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} (M_i - m_i) \Delta\alpha_i \\
=& (M-m)[U(P,f,\alpha) - L(P,f,\alpha)] \\
\leq& (M-m) \cdot \frac{\varepsilon^2}{M-m+1} \\
<& \varepsilon^2.
\end{aligned}
$$

Hence,

$$
\|f-g\|_2 = \left\{ \int_a^b |f-g|^2 d\alpha \right\}^{\frac{1}{2}} < \varepsilon.
$$

□

Note.

(1) Apply the same argument we can prove the following statement:

Suppose $f \in \mathscr{R}(\alpha)$ and $\varepsilon > 0$. Prove that there exists a continuous function g on $[a,b]$ such that $\int_a^b |f-g| d\alpha < \varepsilon$.

(2) (Lebesgue integral)

(a) *Let f be Lebesgue integrable over E. Then, given $\varepsilon > 0$, there is a simple function φ such that*

$$
\int_E |f-\varphi| < \varepsilon.
$$

(b) *Under the same hypothesis there is a step function ψ such that*

$$
\int_E |f-\psi| < \varepsilon.
$$

(c) *Under the same hypothesis there is a continuous function g vanishing outside a finite interval such that*

$$
\int_E |f-g| < \varepsilon.
$$

Exercise 6.13.
PLACEHOLDER

Exercise 6.14.
PLACEHOLDER

Exercise 6.15. *Suppose f is a real, continuously differentiable function on $[a,b]$, $f(a) = f(b) = 0$, and*

$$\int_a^b f(x)^2 dx = 1.$$

Prove that

$$\int_a^b x f(x) f'(x) dx = -\frac{1}{2}$$

and that

$$\int_a^b [f'(x)]^2 dx \int_a^b x^2 f(x)^2 dx > \frac{1}{4}.$$

Proof. Every integral is well-defined (Theorem 4.9 and Theorem 6.8).

(1) By Theorem 6.22 (integration by parts),

$$\int_a^b x \left(\frac{f(x)^2}{2} \right)' dx = \left[x \cdot \frac{f(x)^2}{2} \right]_{x=a}^{x=b} - \int_a^b \frac{f(x)^2}{2} dx,$$

or

$$\int_a^b x f(x) f'(x) dx = \left[b \cdot \frac{f(b)^2}{2} - a \cdot \frac{f(a)^2}{2} \right] - \frac{1}{2} \int_a^b f(x)^2 dx = -\frac{1}{2}.$$

(2) By Exercise 6.10(c),

$$\int_a^b [f'(x)]^2 dx \int_a^b x^2 f(x)^2 dx \geq \left(\int_a^b x f(x) f'(x) dx \right)^2 = \frac{1}{4}.$$

(3) (Reductio ad absurdum) If the equality were holding, then by Exercise 6.10(c)

$$(f'(x))^2 \int_a^b x^2 f(x)^2 dx = x^2 f(x)^2 \int_a^b [f'(x)]^2 dx$$

on $[a,b]$ (since x, $f(x)$ and $f'(x)$ are continuous on $[a,b]$).

(a) *Show that both integrals are nonzero.* (Reductio ad absurdum) If $\int_a^b x^2 f(x)^2 dx = 0$, then $x^2 f(x)^2 = 0$ or $xf(x) = 0$ on $[a, b]$ (Exercise 6.2). So that

$$\int_a^b xf(x)f'(x)dx = 0 \neq -\frac{1}{2},$$

which is absure. Similarly, $\int_a^b [f'(x)]^2 dx \neq 0$.

(b) By (a), we write

$$C = \left\{ \frac{\int_a^b [f'(x)]^2 dx}{\int_a^b x^2 f(x)^2 dx} \right\}^{\frac{1}{2}} > 0$$

be a positive constant. Hence

$$f'(x) = \pm Cxf(x).$$

Here the sign "$\pm$" is not necessary unchanged on $[a, b]$. Luckily, we can show that the sign "$\pm$" is unchanged on some subinterval of $[a, b]$.

(c) To find such subinterval of $[a, b]$, we consider the zero set $Z(f')$ and $Z(xf)$ on $[a, b]$. Since $f'(x) = \pm Cxf(x)$ with $C > 0$, we have

$$Z(f') = Z(xf).$$

Note that $Z(f') = Z(xf)$ is closed (Exercise 4.3) and not equal to $[a, b]$ (by applying the same argument in (a)). Hence the complement of $Z(f') = Z(xf)$ is open and nonempty, which can be written as the union of an at most countable collection of disjoint segments (Exercise 2.29).

(d) Consider any nonempty open interval in (c), say

$$(c, d) \subseteq [a, b].$$

By construction, $f'(x) \neq 0$ for all $x \in (c, d)$. Since $f'(x)$ is continuous, by Theorem 4.23 there are only two mutually exclusive possible cases:

(i) $f'(x) > 0$ for all $x \in (c, d)$,
(ii) $f'(x) < 0$ for all $x \in (c, d)$.

Similar result for $xf(x)$. Therefore, the sign "$\pm$" of $f'(x) = \pm Cxf(x)$ are unchanged on (c, d), that is,

(i) $f'(x) = Cxf(x)$ for all $x \in (c, d)$,
(ii) $f'(x) = -Cxf(x)$ for all $x \in (c, d)$,

(e) Suppose $f'(x) = Cxf(x)$ on (c, d). Since $f'(x)$ and $xf(x)$ are both vanishing at $x = c$ and $x = d$, $f'(x) = Cxf(x)$ at $x = c$ and $x = d$. So

$$f'(x) = Cxf(x) \text{ if } x \in [c, d].$$

Define

$$\phi(x,y) = Cxy$$

be a real function on $R = [c,d] \times \mathbb{R}$. And consider the initial-value problem

$$y' = \phi(x,y) \qquad \text{with} \qquad y(c) = 0.$$

Then

$$|\phi(x,y_2) - \phi(x,y_1)| = Cx|y_2 - y_1| \leq A|y_2 - y_1|$$

where $A = C \cdot \max\{|c|, |d|\}$ is a constant. By Exercise 5.27, this initial-value problem has at most one solution. Clearly, $y = f(x) = 0$ on $[c,d]$ is one solution of this initial-value problem, contrary to the construction of $[c,d]$. Similar result for the case $f'(x) = -Cxf(x)$.

Therefore, the equality does not hold.

□

Exercise 6.16.
PLACEHOLDER

Exercise 6.17.
PLACEHOLDER

Exercise 6.18.
PLACEHOLDER

Exercise 6.19.
PLACEHOLDER