Chapter 10: Integration of Differential Forms

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Exercise 10.1. ...

Proof.

  (1)

  (2)

□

Exercise 10.2. *For $i = 1, 2, 3, \ldots$, let $\varphi_i \in \mathscr{C}(\mathbb{R}^1)$ have support in $(2^{-i}, 2^{1-i})$, such that $\int \varphi_i = 1$. Put*

$$f(x, y) = \sum_{i=1}^{\infty} [\varphi_i(x) - \varphi_{i+1}(x)]\varphi_i(y)$$

Then f has compact support in $\mathbb{R}^2$, f is continuous except at $(0,0)$, and

$$\int dy \int f(x, y)dx = 0 \qquad but \qquad \int dx \int f(x, y)dy = 1.$$

Observe that f is unbounded in every neighborhood of $(0, 0)$.

Proof.

  (1)

  (2)

□

Exercise 10.3. ...

Proof.

  (1)

  (2)

□

Exercise 10.4. *For $(x, y) \in \mathbb{R}^2$, define*

$$\mathbf{F}(x, y) = (e^x \cos y - 1, e^x \sin y)$$

Prove that $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$, where

$$\mathbf{G}_1(x, y) = (e^x \cos y - 1, y)$$
$$\mathbf{G}_2(u, v) = (u, (1 + u) \tan v)$$

are primitive in some neighborhood of $(0, 0)$. Compute the Jacobians of $\mathbf{G}_1$, $\mathbf{G}_2$, $\mathbf{F}$ at $(0, 0)$. Define

$$\mathbf{H}_2(x, y) = (x, e^x \sin y)$$

and find

$$\mathbf{H}_1(u, v) = (h(u, v), v)$$

so that $\mathbf{F} = \mathbf{H}_1 \circ \mathbf{H}_2$ is in some neighborhood of $(0, 0)$.

Proof.

(1) By Definition 10.5,

$$\mathbf{G}_1(x, y) = (e^x \cos y - 1)\mathbf{e}_1 + y\mathbf{e}_2,$$
$$\mathbf{G}_2(u, v) = u\mathbf{e}_1 + ((1 + u) \tan v)\mathbf{e}_2$$

are primitive in some neighborhood of $(0, 0)$.

(2) *Show that $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$.* Given any $(x, y) \in \mathbb{R}^2$, we have

$$
\begin{aligned}
(\mathbf{G}_2 \circ \mathbf{G}_1)(x, y) &= \mathbf{G}_2(\mathbf{G}_1(x, y)) \\
&= \mathbf{G}_2(e^x \cos y - 1, y) \\
&= (e^x \cos y - 1, (1 + (e^x \cos y - 1)) \tan y) \\
&= (e^x \cos y - 1, e^x \sin y) \\
&= \mathbf{F}(x, y).
\end{aligned}
$$

(3) Since

$$J_{\mathbf{G}_1}(x, y) = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ 0 & 1 \end{bmatrix}$$

$$J_{\mathbf{G}_2}(x, y) = \begin{bmatrix} 1 & 0 \\ \tan y & (1 + x) \sec^2 y \end{bmatrix}$$

$$J_{\mathbf{F}}(x, y) = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix},$$

$$J_{\mathbf{G}_1}(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$J_{\mathbf{G}_2}(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$J_{\mathbf{F}}(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(4) Define $h(u,v) = \sqrt{e^{2u} - v^2} - 1$ on

$$B\left((0,0); \frac{1}{64}\right) \subseteq \mathbb{R}^2.$$

$h(u,v)$ is well-defined since $e^{2u} - v^2 > 0$ for all $(u,v) \in B\left((0,0); \frac{1}{64}\right)$.

(5) Given any $(x,y) \in \mathbb{R}^2$, we have

$$\begin{aligned}
(\mathbf{H}_1 \circ \mathbf{H}_2)(x,y) &= \mathbf{H}_1(\mathbf{H}_2(x,y)) \\
&= \mathbf{H}_1(x, e^x \sin y) \\
&= (\sqrt{e^{2x} - (e^x \sin y)^2} - 1, e^x \sin y) \\
&= (e^x \cos y - 1, e^x \sin y) \\
&= \mathbf{F}(x,y).
\end{aligned}$$

□

Exercise 10.5. *Formulate and prove an analogue of Theorem 10.8, in which K is a compact subset of an arbitrary metric space. (Replace the functions φ_i that occur in the proof of Theorem 10.8 by functions of the type constructed in Exercise 4.22.)*

Proof (Theorem 10.8).

(1) *(Partitions of unity.) Suppose K is a compact subset of a metric space X, and $\{V_\alpha\}$ is an open cover of K. Then there exist functions $\psi_1, \ldots, \psi_s \in \mathscr{C}(X)$ such that*

   (a) *$0 \le \psi_i \le 1$ for $1 \le i \le s$.*
   (b) *each ψ_i has its support in some V_α, and*
   (c) *$\psi_1(x) + \cdots + \psi_s(x) = 1$ for every $x \in K$.*

(2) *It is trivial that some $V_\alpha = X$ by taking $s = 1$ and $\psi_1(x) = 1 \in \mathscr{C}(X)$. Now we assume that all $V_\alpha \subsetneq X$.*

(3) *Associate with each $x \in K$ an index $\alpha(x)$ so that $x \in V_{\alpha(x)}$. Then there are open balls $B(x)$ and $W(x)$, centered at x, with*

$$x \in B(x) \subseteq \overline{B(x)} \subseteq W(x) \subseteq \overline{W(x)} \subseteq V_{\alpha(x)}$$

(Since $V_{\alpha(x)}$ is open, there exists $r > 0$ such that $B(x; r) \subseteq V_{\alpha(x)}$. Take $B(x) = B\left(x; \frac{r}{89}\right)$ and $W(x) = B\left(x; \frac{r}{64}\right)$.)

(4) Since K is compact, there are finitely many points $x_1, \ldots, x_s \in K$ such that

$$K \subseteq B(x_1) \cup \cdots \cup B(x_s).$$

Note that

  (a) $\overline{B(x_i)}$ is a nonempty closed set since $x_i \in B(x_i) \subseteq \overline{B(x_i)}$.

  (b) $X - W(x_i) \supseteq X - V_{\alpha(x_i)}$ is a nonempty closed set by the assumption in (2).

  (c) $\overline{B(x_i)} \cap (X - W(x_i)) \subseteq W(x_i) \cap (X - W(x_i)) = \varnothing.$

By Exercise 4.22, there is a function

$$\varphi_i(x) = \frac{\rho_{\overline{B(x_i)}}(x)}{\rho_{\overline{B(x_i)}}(x) + \rho_{X-W(x_i)}(x)} \in \mathscr{C}(X)$$

such that $\varphi_i(x) = 1$ on $\overline{B(x_i)}$, $\varphi_i(x) = 0$ outside $W(x_i)$, and $0 \le \varphi_i(x) \le 1$ on X for $1 \le i \le s$.

(5) Define $\psi_1 = \varphi_1 \in \mathscr{C}(X)$ and

$$\psi_{i+1} = (1 - \varphi_1) \cdots (1 - \varphi_i) \varphi_{i+1} \in \mathscr{C}(X)$$

for $1 \le i \le s - 1$. Properties (a) and (b) in (1) are clear. Also,

$$\psi_1(x) + \cdots + \psi_s(x) = 1 - (1 - \varphi_1(x)) \cdots (1 - \varphi_s(x))$$

by the construction of ψ_i. If $x \in K$, then $x \in B(x_i)$ for some i, hence $\varphi_i(x) = 1$, and the product $(1 - \varphi_1(x)) \cdots (1 - \varphi_s(x)) = 0$. This proves property (c) in (1).

$\square$

Exercise 10.6. *Strengthen the conclusion of Theorem 10.8 by showing that the functions ψ_i can be made differentiable, and even infinitely differentiable. (Use Exercise 8.1 in the construction of the auxiliary functions ψ_i.)*

Proof (Theorem 10.8).

(1) It is trivial that some $V_\alpha = \mathbb{R}^n$ by taking $s = 1$ and $\psi_1(\mathbf{x}) = 1 \in \mathscr{C}^\infty(\mathbb{R}^n)$. Now we assume that all $V_\alpha \subsetneq \mathbb{R}^n$.

(2) Associate with each $\mathbf{x} \in K$ an index $\alpha(x)$ so that $\mathbf{x} \in V_{\alpha(x)}$. Then there are open n-cells $B(\mathbf{x})$ and $W(\mathbf{x})$ (Definition 10.1), centered at $\mathbf{x}$, with

$$\mathbf{x} \in B(\mathbf{x}) \subseteq \overline{B(\mathbf{x})} \subseteq W(\mathbf{x}) \subseteq \overline{W(\mathbf{x})} \subseteq V_{\alpha(\mathbf{x})}$$

(Since $V_{\alpha(\mathbf{x})}$ is open, there exists $r > 0$ such that $B(\mathbf{x}; r) \subseteq V_{\alpha(\mathbf{x})}$. Take

$$B(\mathbf{x}) = I\left(\mathbf{x}; \frac{r}{89\sqrt{n}}\right), \qquad W(\mathbf{x}) = I\left(\mathbf{x}; \frac{r}{64\sqrt{n}}\right)$$

where $I(\mathbf{p}; r)$ is the open n-cell centered at $\mathbf{p} = (p_1, \ldots, p_n)$ defined by

$$I(\mathbf{p}; r) = (p_1 - r, p_1 + r) \times \cdots \times (p_n - r, p_n + r) \subseteq \mathbb{R}^n.)$$

(3) Define

$$f(y) = \begin{cases} e^{-\frac{1}{y^2}} & (y > 0), \\ 0 & (y \leq 0). \end{cases}$$

$f(y) \in \mathscr{C}^\infty(\mathbb{R}^1)$ by applying the similar argument in Exercise 8.1.

(4) Given any $\mathbf{x} = (x_1, \ldots, x_n) \in K$ and construct $B(\mathbf{x})$ and $W(\mathbf{x})$ as in (2). Define

$$g_{x_j}(y_j) = \frac{f(y_j)}{f(y_j) + f\left(\frac{r}{64\sqrt{n}} - \frac{r}{89\sqrt{n}} - y_j\right)}$$

for $1 \leq j \leq n$. g_{x_j} is well-defined and $g_{x_j} \in \mathscr{C}^\infty(\mathbb{R}^1)$. So

$$g_{x_j}(y_j) = \begin{cases} 0 & \text{if } y_j \leq 0, \\ \text{strictly increasing} & \text{if } 0 \leq y_j \leq \frac{r}{64\sqrt{n}} - \frac{r}{89\sqrt{n}}, \\ 1 & \text{if } y_j \geq \frac{r}{64\sqrt{n}} - \frac{r}{89\sqrt{n}}. \end{cases}$$

Next, define

$$h_{x_j}(y_j) = g_{x_j}\left(y_j - x_j + \frac{r}{64\sqrt{n}}\right) g_{x_j}\left(x_j + \frac{r}{64\sqrt{n}} - y_j\right)$$

for $1 \leq j \leq n$. $h_{x_j} \in \mathscr{C}^\infty(\mathbb{R}^1)$. So

$$h_{x_j}(y_j) = \begin{cases} 0 & \text{if } y_j \leq x_j - \frac{r}{64\sqrt{n}}, \\ \text{strictly increasing} & \text{if } x_j - \frac{r}{64\sqrt{n}} \leq y_j \leq x_j - \frac{r}{89\sqrt{n}}, \\ 1 & \text{if } x_j - \frac{r}{89\sqrt{n}} \leq y_j \leq x_j + \frac{r}{89\sqrt{n}}, \\ \text{strictly decreasing} & \text{if } x_j + \frac{r}{89\sqrt{n}} \leq y_j \leq x_j + \frac{r}{64\sqrt{n}}, \\ 0 & \text{if } y_j \geq x_j + \frac{r}{64\sqrt{n}}. \end{cases}$$

Finally we define $\mathbf{h}_{\mathbf{x}} : \mathbb{R}^n \to \mathbb{R}^1$ by

$$\mathbf{h}_{\mathbf{x}}(\mathbf{y}) = \prod_{j=1}^{n} h_{x_j}(y_j)$$

where $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Hence, $\mathbf{h}_{\mathbf{x}} \in \mathscr{C}^\infty(\mathbb{R}^n)$ (Theorem 9.21). Also, $\mathbf{h}_{\mathbf{x}}(\mathbf{y}) = 1$ on $\overline{B(\mathbf{x})}$, $\mathbf{h}_{\mathbf{x}}(\mathbf{y}) = 0$ outside $W(\mathbf{x})$, and $0 \leq \mathbf{h}_{\mathbf{x}}(\mathbf{y}) \leq 1$.

(5) Since K is compact, there are finitely many points $\mathbf{x}_1, \ldots, \mathbf{x}_s \in K$ such that
$$K \subseteq B(\mathbf{x}_1) \cup \cdots \cup B(\mathbf{x}_s).$$

Take
$$\varphi_i(\mathbf{x}) = \mathbf{h}_{\mathbf{x}_i}(\mathbf{x}) \in \mathscr{C}^\infty(\mathbb{R}^n)$$

for $1 \le i \le s$.

(6) The rest are the same as the proof of Theorem 10.8 or Exercise 10.5.

$\square$

Exercise 10.7.

(a) *Show that the simplex Q^k is the smallest convex subset of $\mathbb{R}^k$ such that contains $\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_k$.*

(b) *Show that affine mappings take convex sets to convex sets.*

Proof of (a).

(1) *Show that Q^k contains $\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_k$.* Recall
$$Q^k = \{(x_1, \ldots, x_k) \in \mathbb{R}^k : x_1 + \cdots + x_k \le 1 \text{ and } x_1, \ldots, x_k \ge 0\}$$

(Example 10.14). Hence $\mathbf{0} = (0, \ldots, 0) \in Q^k$ and
$$\mathbf{e}_i = (0, \ldots, \underbrace{1}_{i\text{th coordinate}}, \ldots, 0) \in Q^k.$$

(2) *Show that Q^k is a convex subset of $\mathbb{R}^k$.* Given any $\mathbf{x} = (x_1, \ldots, x_k) \in Q^k$, $\mathbf{y} = (y_1, \ldots, y_k) \in Q^k$ and $0 < \lambda < 1$. Hence
$$\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} = (\lambda x_1 + (1 - \lambda)y_1, \ldots, \lambda x_k + (1 - \lambda)y_k) \in Q^k$$

since each $\lambda x_i + (1 - \lambda)y_i \ge 0$ and
$$\sum_{i=1}^k (\lambda x_i + (1 - \lambda)y_i) = \lambda \sum_{i=1}^k x_i + (1 - \lambda) \sum_{i=1}^k y_i \le \lambda + (1 - \lambda) = 1.$$

(3) *Given any convex set $E \subseteq \mathbb{R}^k$ containing $\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_k$. Show that $E \supseteq Q^k$.*

(a) Induction on k. Base case: $k = 1$. Given any $\mathbf{x} = (x_1) \in Q^1$. We have $0 \le x_1 \le 1$ by the definition of Q^1. So that $\mathbf{x} = x_1 \mathbf{e}_1 + (1 - x_1)\mathbf{0} \in E$ since $\mathbf{0}, \mathbf{e}_1 \in E$ and E is convex.

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(b) Inductive step: suppose the statement holds for $k = n$. Given any $\mathbf{x} = (x_1, \ldots, x_n, x_{n+1}) \in Q^{n+1}$. If $x_{n+1} = 1$, then $x_1 = \cdots = x_n = 0$ by the definition of Q^{n+1}. So $\mathbf{x} = \mathbf{e}_{n+1} \in E$ by the assumption of E. If $0 \leq x_{n+1} < 1$, then $x_1 + \cdots + x_n \leq 1 - x_{n+1}$ or

$$\frac{x_1}{1 - x_{n+1}} + \cdots + \frac{x_n}{1 - x_{n+1}} \leq 1.$$

So the point

$$\left(\frac{x_1}{1 - x_{n+1}}, \ldots, \frac{x_n}{1 - x_{n+1}} \right) \in Q^n,$$

or

$$\left(\frac{x_1}{1 - x_{n+1}}, \ldots, \frac{x_n}{1 - x_{n+1}}, 0 \right), \text{ say } \widehat{\mathbf{x}}, \in E$$

by the induction hypothesis. Note that $\mathbf{e}_{n+1} \in E$. Hence

$$\mathbf{x} = x_{n+1}\mathbf{e}_{n+1} + (1 - x_{n+1})\widehat{\mathbf{x}} \in E$$

by the convexity of E.

(c) Conclusion: Since both the base case and the inductive step have been proved as true, by mathematical induction the statement holds.

□

Proof of (b).

(1) Let $\mathbf{f}$ be an affine mapping that carries a vector space X into a vector space Y such that
$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{0}) + A\mathbf{x}$$
for some $A \in L(X, Y)$.

(2) Given any convex subset C of X. To show that $\mathbf{f}(C)$ is convex, it suffices to show that
$$\lambda \mathbf{y}_1 + (1 - \lambda)\mathbf{y}_2 \in \mathbf{f}(C)$$
for any $\mathbf{y}_1, \mathbf{y}_2 \in \mathbf{f}(C)$ and $0 < \lambda < 1$. Write $\mathbf{y}_1 = \mathbf{f}(\mathbf{x}_1)$, $\mathbf{y}_2 = \mathbf{f}(\mathbf{x}_2)$ for some $\mathbf{x}_1, \mathbf{x}_2 \in C$. Note that $\lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2 \in C$ by the convexity of C. Hence

$$
\begin{aligned}
&\mathbf{f}(\lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2) \\
=&\mathbf{f}(\mathbf{0}) + A(\lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2) \\
=&\mathbf{f}(\mathbf{0}) + \lambda A\mathbf{x}_1 + (1 - \lambda)A\mathbf{x}_2 && (A \in L(X, Y)) \\
=&\lambda(\mathbf{f}(\mathbf{0}) + A\mathbf{x}_1) + (1 - \lambda)(\mathbf{f}(\mathbf{0}) + A\mathbf{x}_2) \\
=&\lambda\mathbf{f}(\mathbf{x}_1) + (1 - \lambda)\mathbf{f}(\mathbf{x}_2) \\
=&\lambda\mathbf{y}_1 + (1 - \lambda)\mathbf{y}_2 \in \mathbf{f}(C).
\end{aligned}
$$

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□

Exercise 10.8. *Let H be the parallelogram in $\mathbb{R}^2$ whose vertices are $(1, 1)$, $(3, 2)$, $(4, 5)$, $(2, 4)$. Find the affine map T which sends $(0, 0)$ to $(1, 1)$, $(1, 0)$ to $(3, 2)$, $(1, 1)$ to $(4, 5)$, $(0, 1)$ to $(2, 4)$. Show that $J_T = 5$. Use T to convert the integral*

$$\alpha = \int_H e^{x-y} dx dy$$

to an integral over I^2 and thus compute α.

Proof.

(1) By Affine simplexes 10.26,

$$T(\mathbf{x}) = T(\mathbf{0}) + A\mathbf{x},$$

where $A \in L(\mathbb{R}^2, \mathbb{R}^2)$, say $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Note that $T : \begin{bmatrix} 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus

$$T : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 + ax + by \\ 1 + cx + dy \end{bmatrix}.$$

(2) By $T : (1, 0) \mapsto (3, 2)$ and $T : (0, 1) \mapsto (2, 4)$, we can solve A as

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

It is easy to verify such

$$T : \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{x}} \mapsto \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{T(\mathbf{0})} + \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{x}} = \begin{bmatrix} 1 + 2x + y \\ 1 + x + 3y \end{bmatrix}$$

satisfying our requirement.

(3)

$$J_T = \det \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = 5.$$

(4)

$$\int_H e^{x-y} dx dy = \int_{[0,1]^2} e^{(1+2u+v)-(1+u+3v)} |J_T| du dv$$

$$= 5 \int_{[0,1]^2} e^{u-2v} du dv$$

$$= 5 \left\{ \int_0^1 e^u du \right\} \left\{ \int_0^1 e^{-2v} dv \right\} \qquad \text{(Theorem 10.2)}$$

$$= \frac{5}{2}(e-1)(1-e^{-2}).$$

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□

Exercise 10.9. *...*

Proof.

  (1)

  (2)

□

Exercise 10.10. *...*

Proof.

  (1)

  (2)

□

Exercise 10.11. *...*

Proof.

  (1)

  (2)

□

Exercise 10.12. *...*

Proof.

  (1)

  (2)

□

Exercise 10.13. *...*

Proof.

(1)

(2)

□

Exercise 10.14 (Levi-Civita symbol). *Prove $\varepsilon(j_1, \ldots, j_k) = s(j_1, \ldots, j_k)$, where*

$$s(j_1, \ldots, j_k) = \prod_{p<q} \operatorname{sgn}(j_q - j_p).$$

It is usually to define the Levi-Civita symbol by

$$\varepsilon(j_1, \ldots, j_k) = \begin{cases} 1 & \text{if } (j_1, \cdots, j_k) \text{ is an even permutation of } J, \\ -1 & \text{if } (j_1, \cdots, j_k) \text{ is an odd permutation of } J, \\ 0 & \text{otherwise} \end{cases}$$

(Basic k-forms 10.14). Thus, it is the sign of the permutation in the case of a permutation, and zero otherwise. So $\varepsilon(j_1, \ldots, j_k)$ is equivalent to an explicit expression $s(j_1, \ldots, j_k) = \prod_{p<q} \operatorname{sgn}(j_q - j_p)$.

Proof.

(1) Induction on k. Base case: *Show that $\varepsilon(j_1, j_2) = s(j_1, j_2)$.* Since

$$\varepsilon(j_1, j_2) = \begin{cases} 1 & \text{if } j_1 < j_2 \\ -1 & \text{if } j_1 > j_2, \end{cases}$$

$\varepsilon(j_1, j_2) = \operatorname{sgn}(j_2 - j_1) = s(j_1, j_2)$.

(2) Inductive step: *Show that for any $s \geq 2$, if $\varepsilon(j_1, \ldots, j_s) = s(j_1, \ldots, j_s)$ holds, then $\varepsilon(j_1, \ldots, j_{s+1}) = s(j_1, \ldots, j_{s+1})$ also holds.*

$$\varepsilon(j_1, \ldots, j_{s+1}) = \varepsilon(j_1, \ldots, j_s) \prod_{\substack{1 \leq p \leq s \\ q=s+1}} \operatorname{sgn}(j_q - j_p)$$

$$= s(j_1, \ldots, j_s) \prod_{\substack{1 \leq p \leq s \\ q=s+1}} \operatorname{sgn}(j_q - j_p)$$

$$= \prod_{1 \leq p < q \leq s} \operatorname{sgn}(j_q - j_p) \prod_{\substack{1 \leq p \leq s \\ q=s+1}} \operatorname{sgn}(j_q - j_p)$$

$$= \prod_{1 \leq p < q \leq s+1} \operatorname{sgn}(j_q - j_p)$$

$$= s(j_1, \ldots, j_{s+1}).$$

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(3) Conclusion: Since both the base case and the inductive step have been proved as true, by mathematical induction the statement holds for every integer $k \geq 2$.

□

Exercise 10.15. *If ω and λ are k- and m-forms, respectively, prove that*

$$\omega \wedge \lambda = (-1)^{km} \lambda \wedge \omega.$$

Proof.

(1) Write

$$\omega = \sum_I b_I(\mathbf{x}) dx_I, \qquad \lambda = \sum_J c_J(\mathbf{x}) dx_J$$

in the stardard presentations, where I and J range over all increasing k-indices and over all increasing m-indices taken from the set $\{1, \ldots, n\}$.

(2) *Show that $dx_I \wedge dx_J = (-1)^{km} dx_J \wedge dx_I$.*

$$\begin{aligned}
dx_I \wedge dx_J &= dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_J \\
&= (-1)^m dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}} \wedge dx_J \wedge dx_{i_k} \\
&= (-1)^{2m} dx_{i_1} \wedge \cdots \wedge dx_{i_{k-2}} \wedge dx_J \wedge dx_{i_{k-1}} \wedge dx_{i_k} \\
&\cdots \\
&= (-1)^{km} dx_J \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&= (-1)^{km} dx_J \wedge dx_I.
\end{aligned}$$

(3)

$$\begin{aligned}
\omega \wedge \lambda &= \sum_{I,J} b_I(\mathbf{x}) c_J(\mathbf{x}) dx_I \wedge dx_J \\
&= (-1)^{km} \sum_{J,I} c_J(\mathbf{x}) b_I(\mathbf{x}) dx_J \wedge dx_I \\
&= (-1)^{km} \lambda \wedge \omega.
\end{aligned}$$

□

Exercise 10.16. *If $k \geq 2$ and $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_k]$ is an oriented affine k-simplex, prove that $\partial^2 \sigma = 0$, directly from the definition of the boundary operator ∂. Deduce from this that $\partial^2 \Psi = 0$ for every chain Ψ. (Hint: For orientation, do it first for $k = 2$, $k = 3$. In general, if $i < j$, let σ_{ij} be the $(k-2)$-simplex obtained by deleting $\mathbf{p}_i$ and $\mathbf{p}_j$ from σ. Show that each σ_{ij} occurs twice in $\partial^2 \sigma$, with opposite sign.)*

Proof (Brute-force).

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(1) Write the boundary of the oriented affine k-simplex $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_k]$ as

$$\partial\sigma = \sum_{i=0}^{k}(-1)^i[\mathbf{p}_0, \ldots, \widehat{\mathbf{p}_i}, \ldots, \mathbf{p}_k]$$

where where the oriented $(k-1)$-simplex $[\mathbf{p}_0, \ldots, \widehat{\mathbf{p}_i}, \ldots, \mathbf{p}_k]$ is obtained by deleting σ's i-th vertex (Boundaries 10.29).

(2)

$$\begin{aligned}
\partial^2\sigma =&\partial\left(\sum_i(-1)^i[\mathbf{p}_0, \ldots, \widehat{\mathbf{p}_i}, \ldots, \mathbf{p}_k]\right)\\
=&\sum_i(-1)^i\partial[\mathbf{p}_0, \ldots, \widehat{\mathbf{p}_i}, \ldots, \mathbf{p}_k]\\
=&\sum_{j<i}(-1)^i(-1)^j[\mathbf{p}_0, \ldots, \widehat{\mathbf{p}_j}, \ldots, \widehat{\mathbf{p}_i}, \ldots, \mathbf{p}_k]\\
&+\sum_{j>i}(-1)^i(-1)^{j-1}[\mathbf{p}_0, \ldots, \widehat{\mathbf{p}_i}, \ldots, \widehat{\mathbf{p}_j}, \ldots, \mathbf{p}_k]\\
=&\sum_{j<i}(-1)^{i+j}[\mathbf{p}_0, \ldots, \widehat{\mathbf{p}_j}, \ldots, \widehat{\mathbf{p}_i}, \ldots, \mathbf{p}_k]\\
&-\sum_{j>i}(-1)^{i+j}[\mathbf{p}_0, \ldots, \widehat{\mathbf{p}_i}, \ldots, \widehat{\mathbf{p}_j}, \ldots, \mathbf{p}_k].
\end{aligned}$$

The latter two summations cancel since after switching i and j in the second sum. Therefore $\partial^2\sigma = 0$.

(3) The boundary of a chain is the linear combination of boundaries of the simplices in the chain. Write $\Psi = \sum_{i=1}^{r}\sigma_i$. where σ_i is an oriented affine simplex. Then

$$\partial^2\Psi = \partial\left(\partial\sum\sigma_i\right) = \partial\left(\sum\partial\sigma_i\right) = \sum\partial^2\sigma_i = \sum 0 = 0$$

for any affine chain Ψ.

□

Exercise 10.17. *Put* $J^2 = \tau_1 + \tau_2$, *where*

$$\tau_1 = [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2], \qquad \tau_2 = -[\mathbf{0}, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1].$$

Explain why it is reasonable to call J^2 the positively oriented unit square in $\mathbb{R}^2$. Show that ∂J^2 is the sum of 4 oriented affine 1-simplexes. Find these. What is $\partial(\tau_1 - \tau_2)$?

Proof.

(1) Note that the unit square $I^2 \in \mathbb{R}^2$ is the union of $\tau_1(Q^2)$ and $\tau_2(Q_2)$, where

$$
\begin{aligned}
\tau_1(\mathbf{u}) &= ([\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2])(\mathbf{u}) \\
&= \mathbf{0} + \alpha_1 \mathbf{e}_1 + \alpha_2(\mathbf{e}_1 + \mathbf{e}_2) \\
&= \mathbf{0} + (\alpha_1 + \alpha_2)\mathbf{e}_1 + \alpha_2 \mathbf{e}_2 \\
&= \mathbf{0} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{u}
\end{aligned}
$$

and

$$
\begin{aligned}
\tau_2(\mathbf{u}) &= (-[\mathbf{0}, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1])(\mathbf{u}) \\
&= ([\mathbf{0}, \mathbf{e}_2 + \mathbf{e}_1, \mathbf{e}_2])(\mathbf{u}) \\
&= \mathbf{0} + \alpha_1(\mathbf{e}_1 + \mathbf{e}_2) + \alpha_2 \mathbf{e}_2 \\
&= \mathbf{0} + \alpha_1 \mathbf{e}_1 + (\alpha_1 + \alpha_2)\mathbf{e}_2 \\
&= \mathbf{0} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{u}
\end{aligned}
$$

where $\mathbf{u} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 \in \mathbb{R}^2$ (as in Equation (78)). Both τ_1 and τ_2 have Jacobian $1 > 0$, or positively oriented (Affine simplexes 10.26). So it is reasonable to call J^2 the positively oriented unit square in $\mathbb{R}^2$.

(2)

$$
\begin{aligned}
\partial \tau_1 &= [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] - [\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{0}, \mathbf{e}_1], \\
\partial \tau_2 &= [\mathbf{e}_2 + \mathbf{e}_1, \mathbf{e}_2] - [\mathbf{0}, \mathbf{e}_2] + [\mathbf{0}, \mathbf{e}_2 + \mathbf{e}_1] \\
&= [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2] + [\mathbf{e}_2, \mathbf{0}] + [\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2].
\end{aligned}
$$

(3) By (2),

$$
\partial J^2 = \partial \tau_1 + \partial \tau_2 = [\mathbf{0}, \mathbf{e}_1] + [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2] + [\mathbf{e}_2, \mathbf{0}],
$$

which is the positively oriented boundary of I^2.

(4) By (2),

$$
\begin{aligned}
\partial(\tau_1 - \tau_2) =& \partial \tau_1 - \partial \tau_2 \\
=& [\mathbf{0}, \mathbf{e}_1] + [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{0}] \\
& + [\mathbf{0}, \mathbf{e}_2] + [\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{0}].
\end{aligned}
$$

□

Exercise 10.18. *Consider the oriented affine 3-simplex*

$$
\sigma_1 = [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]
$$

in $\mathbb{R}^3$. Show that σ_1 (regarded as a linear transformation) has determinant 1. Thus σ_1 is positively oriented.

Let $\sigma_2, \ldots, \sigma_6$ be five other oriented 3-simplexes, obtained as follows: There are five permutations (i_1, i_2, i_3) of $(1, 2, 3)$, distinct from $(1, 2, 3)$. Associate with each (i_1, i_2, i_3) the simplex

$$s(i_1, i_2, i_3)[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}]$$

where s is the sign that occurs in the definition of the determinant. (This is how τ_2 was obtained from τ_1 in Exercise 10.17.) Show that $\sigma_2, \ldots, \sigma_6$ are positively oriented.

Put $J^3 = \sigma_1 + \cdots + \sigma_6$. Then J^3 may be called the positively oriented unit cube in $\mathbb{R}^3$. Show that ∂J^3 is the sum of 12 oriented affine 2-simplexes. (These 12 triangles cover the surface of the unit cube I^3.)

Show that $\mathbf{x} = (x_1, x_2, x_3)$ is in the range of σ_1 if and only if $0 \leq x_3 \leq x_2 \leq x_1 \leq 1$.

Show that the range of $\sigma_1, \ldots, \sigma_6$ have disjoint interiors, and that their union covers I^3. (Compared with Exercise 10.13; note that $3! = 6$.)

Proof.

(1) *Show that σ_1 (regarded as a linear transformation) has determinant 1.* Given any $\mathbf{u} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3 \in \mathbb{R}^3$, we have

$$\begin{aligned} \sigma_1(\mathbf{u}) &= ([\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3])(\mathbf{u}) \\ &= \mathbf{0} + \alpha_1 \mathbf{e}_1 + \alpha_2(\mathbf{e}_1 + \mathbf{e}_2) + \alpha_3(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) \\ &= \mathbf{0} + (\alpha_1 + \alpha_2 + \alpha_3)\mathbf{e}_1 + (\alpha_2 + \alpha_3)\mathbf{e}_2 + \alpha_3 \mathbf{e}_3 \\ &= \mathbf{0} + \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{say } A} \mathbf{u}. \end{aligned}$$

So

$$\det(A) = \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = 1.$$

(2) *Show that $\sigma_2, \ldots, \sigma_6$ are positively oriented.* Define the permutation matrix $P_{(i_1, i_2, i_3)}$ corresponding to a permutation (i_1, i_2, i_3) of $(1, 2, 3)$ by

$$P_{(i_1, i_2, i_3)} = \begin{bmatrix} \mathbf{e}_{i_1} & \mathbf{e}_{i_2} & \mathbf{e}_{i_3} \end{bmatrix}.$$

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For example,

$$P_{(2,3,1)} = \begin{bmatrix} \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Note that the sign $s(i_1, i_2, i_3)$ of the permutation (i_1, i_2, i_3) is exactly the same as the determinant of the permutation matrix $P_{(i_1, i_2, i_3)}$. Define a permutation $(j_1, j_2, 3)$ of $(1, 2, 3)$ (for swapping the first and the second coordinates of $\mathbf{u}$) by

$$(j_1, j_2, 3) = \begin{cases} (1, 2, 3) & \text{if } s(i_1, i_2, i_3) = 1, \\ (2, 1, 3) & \text{if } s(i_1, i_2, i_3) = -1. \end{cases}$$

Write

$$\sigma_{(i_1, i_2, i_3)} = s(i_1, i_2, i_3)[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}].$$

(So that $\sigma_1 = \sigma_{(1,2,3)}$.) Hence,

$$
\begin{aligned}
&\sigma_{(i_1,i_2,i_3)}(\mathbf{u}) \\
&= \mathbf{0} + \alpha_{j_1}\mathbf{e}_{i_1} + \alpha_{j_2}(\mathbf{e}_{i_1} + \mathbf{e}_{i_2}) + \alpha_3(\mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}) \\
&= \mathbf{0} + (\alpha_{j_1} + \alpha_{j_2} + \alpha_3)\mathbf{e}_{i_1} + (\alpha_{j_2} + \alpha_3)\mathbf{e}_{i_2} + \alpha_3\mathbf{e}_{i_3} \\
&= \mathbf{0} + P_{(i_1,i_2,i_3)} A P_{(j_1,j_2,3)}\mathbf{u}
\end{aligned}
$$

where $\mathbf{u} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3 \in \mathbb{R}^3$. For example,

$$P_{(2,3,1)} A P_{(1,2,3)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

So

$$
\begin{aligned}
\det\big(P_{(i_1,i_2,i_3)} A P_{(j_1,j_2,3)}\big) &= \det\big(P_{(i_1,i_2,i_3)}\big) \det(A) \det\big(P_{(j_1,j_2,3)}\big) \\
&= s(i_1, i_2, i_3) \cdot 1 \cdot s(i_1, i_2, i_3) \\
&= 1.
\end{aligned}
$$

(3) *Show that ∂J^3 is the sum of 12 oriented affine 2-simplexes.* Note that

$$
\begin{aligned}
\sum_{(i_1,i_2,i_3)} \sigma_{(i_1,i_2,i_3)} &= \sum_{\substack{(i_1,i_2,i_3) \\ i_1 > i_2}} \sigma_{(i_1,i_2,i_3)} + \sum_{\substack{(i_1,i_2,i_3) \\ i_1 < i_2}} \sigma_{(i_1,i_2,i_3)} \\
&= \sum_{\substack{(i_1,i_2,i_3) \\ i_1 > i_2}} s(i_1,i_2,i_3)[\mathbf{0}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\
&\quad + \sum_{\substack{(i_1,i_2,i_3) \\ i_2 > i_1}} -s(i_2,i_1,i_3)[\mathbf{0}, \mathbf{e}_{i_2} + \mathbf{e}_{i_1}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\
&= \mathbf{0}
\end{aligned}
$$

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and

$$\sum_{(i_1,i_2,i_3)} \sigma_{(i_1,i_2,i_3)} = \sum_{\substack{(i_1,i_2,i_3)\\ i_2>i_3}} \sigma_{(i_1,i_2,i_3)} + \sum_{\substack{(i_1,i_2,i_3)\\ i_2<i_3}} \sigma_{(i_1,i_2,i_3)}$$

$$= \sum_{\substack{(i_1,i_2,i_3)\\ i_2>i_3}} s(i_1,i_2,i_3)[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]$$

$$+ \sum_{\substack{(i_1,i_2,i_3)\\ i_3>i_2}} -s(i_1,i_3,i_2)[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]$$

$$=\mathbf{0}.$$

So

$$\partial J^3 = \sum_{(i_1,i_2,i_3)} \partial\sigma_{(i_1,i_2,i_3)}$$

$$= \sum_{(i_1,i_2,i_3)} s(i_1,i_2,i_3)[\mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}]$$

$$- s(i_1,i_2,i_3)[\mathbf{0}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}]$$
$$+ s(i_1,i_2,i_3)[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}]$$
$$- s(i_1,i_2,i_3)[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}]$$

$$= \sum_{(i_1,i_2,i_3)} s(i_1,i_2,i_3)[\mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]$$

$$\underbrace{- \sum_{(i_1,i_2,i_3)} s(i_1,i_2,i_3)[\mathbf{0}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]}_{=\mathbf{0}}$$

$$\underbrace{+ \sum_{(i_1,i_2,i_3)} s(i_1,i_2,i_3)[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]}_{=\mathbf{0}}$$

$$- \sum_{(i_1,i_2,i_3)} s(i_1,i_2,i_3)[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}].$$

Thus,

$$\partial J^3 = \sum_{(i_1,i_2,i_3)} s(i_1,i_2,i_3)[\mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]$$

$$- \sum_{(i_1,i_2,i_3)} s(i_1,i_2,i_3)[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}]$$

is the sum of 12 oriented affine 2-simplexes. (Note that $3! = 6$.)

(4) *Show that* $\mathbf{x} = (x_1, x_2, x_3)$ *is in the range of* σ_1 *if and only if* $0 \leq x_3 \leq x_2 \leq x_1 \leq 1$.

(a) By (1), $\mathbf{x}$ is in the range of σ_1 if and only if $\mathbf{x} = A\mathbf{u}$ for $\mathbf{u} = (u_1, u_2, u_3) \in Q^3$, or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 + u_2 + u_3 \\ u_2 + u_3 \\ u_3 \end{bmatrix}.$$

(b) Since $\mathbf{u} = (u_1, u_2, u_3) \in Q^3$, $u_1 + u_2 + u_3 \leq 1$ and $u_1, u_2, u_3 \geq 0$. Hence $0 \leq u_3 \leq u_2 + u_3 \leq u_1 + u_2 + u_3 \leq 1$ or $0 \leq x_3 \leq x_2 \leq x_1 \leq 1$.

(c) Conversely, if $0 \leq x_3 \leq x_2 \leq x_1 \leq 1$, we define

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_3 \end{bmatrix}.$$

Clearly, $\mathbf{v} \in Q^3$.

(5) *Show that the range of $\sigma_1, \ldots, \sigma_6$ have disjoint interiors, and that their union covers I^3. Similar to (4).* By (2), $\mathbf{x} = P_{(i_1, i_2, i_3)} A P_{(j_1, j_2, 3)} \mathbf{u}$, or $P_{(i_1, i_2, i_3)^{-1}} \mathbf{x} = A P_{(j_1, j_2, 3)} \mathbf{u}$, or

$$\begin{bmatrix} x_{i_1} \\ x_{i_2} \\ x_{i_3} \end{bmatrix} = \begin{bmatrix} u_1 + u_2 + u_3 \\ u_{j_2} + u_3 \\ u_3 \end{bmatrix}.$$

In any case, we always have $0 \leq u_3 \leq u_{j_2} + u_3 \leq u_1 + u_2 + u_3 \leq 1$. Hence $\mathbf{x} = (x_1, x_2, x_3)$ is in the range of $\sigma_{(i_1, i_2, i_3)}$ if and only if

$$0 \leq x_{i_3} \leq x_{i_2} \leq x_{i_1} \leq 1.$$

The interior of $\sigma_{(i_1, i_2, i_3)}$ is

$$\{\mathbf{x} \in \mathbb{R}^3 : 0 < x_{i_3} < x_{i_2} < x_{i_1} < 1\},$$

and thus the range of $\sigma_1, \ldots, \sigma_6$ have disjoint interiors. Also, any $\mathbf{x} \in I^3$ has the relation

$$0 \leq x_{i_3} \leq x_{i_2} \leq x_{i_1} \leq 1$$

for some permutation (i_1, i_2, i_3) of $(1, 2, 3)$. Hence

$$I^3 = \bigcup_{(i_1, i_2, i_3)} \sigma_{(i_1, i_2, i_3)}(Q^3) = \bigcup_{i=1}^{6} \sigma_i(Q^3).$$

□

Exercise 10.19. *Let J^2 and J^3 be as in Exercise 10.17 and Exercise 10.18. Define*

$$
\begin{aligned}
B_{01}(u,v) &= (0,u,v), & B_{11}(u,v) &= (1,u,v),\\
B_{02}(u,v) &= (u,0,v), & B_{12}(u,v) &= (u,1,v),\\
B_{03}(u,v) &= (u,v,0), & B_{13}(u,v) &= (u,v,1).
\end{aligned}
$$

These are affine, and map $\mathbb{R}^2$ into $\mathbb{R}^3$. Put $\beta_{ri} = B_{ri}(J^2)$, for $r = 0,1$, $i = 1,2,3$. Each β_{ri} is an affine-oriented 2-chain. (See Section 10.30.) Verify that

$$
\partial J^3 = \sum_{i=1}^{3}(-1)^i(\beta_{0i} - \beta_{1i}),
$$

in agreement with Exercise 10.18.)

Proof.

(1) A direct calculation shows that

$$
\begin{aligned}
B_{01}(\tau_1) - B_{11}(\tau_1) &= [\mathbf{0}, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3] - [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]\\
B_{02}(\tau_1) - B_{12}(\tau_1) &= [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_3] - [\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]\\
B_{03}(\tau_1) - B_{13}(\tau_1) &= [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] - [\mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]\\
B_{01}(\tau_2) - B_{11}(\tau_2) &= -[\mathbf{0}, \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3] + [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]\\
B_{02}(\tau_2) - B_{12}(\tau_2) &= -[\mathbf{0}, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_3] + [\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]\\
B_{03}(\tau_2) - B_{13}(\tau_2) &= -[\mathbf{0}, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3].
\end{aligned}
$$

(2) To express the formula in (1) clearly, we define

$$
\omega_{(i_1,i_2,i_3)} = [\mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] - [\mathbf{0}, \mathbf{e}_{i_2}, \mathbf{e}_{i_2} + \mathbf{e}_{i_3}],
$$

and thus

$$
\begin{aligned}
-(B_{01}(\tau_1) - B_{11}(\tau_1)) &= s(1,2,3)\omega_{(1,2,3)}\\
B_{02}(\tau_1) - B_{12}(\tau_1) &= s(2,1,3)\omega_{(2,1,3)}\\
-(B_{03}(\tau_1) - B_{13}(\tau_1)) &= s(3,1,2)\omega_{(3,1,2)}\\
-(B_{01}(\tau_2) - B_{11}(\tau_2)) &= s(1,3,2)\omega_{(1,3,2)}\\
B_{02}(\tau_2) - B_{12}(\tau_2) &= s(2,3,1)\omega_{(2,3,1)}\\
-(B_{03}(\tau_2) - B_{13}(\tau_2)) &= s(3,2,1)\omega_{(3,2,1)}.
\end{aligned}
$$

(3) Note that

$$
\begin{aligned}
\beta_{0i} - \beta_{1i} &= B_{0i}(J^2) - B_{1i}(J^2)\\
&= B_{0i}(\tau_1 + \tau_2) - B_{1i}(\tau_1 + \tau_2)\\
&= B_{0i}(\tau_1) + B_{0i}(\tau_2) - B_{1i}(\tau_1) - B_{1i}(\tau_2)\\
&= (B_{0i}(\tau_1) - B_{1i}(\tau_1)) + (B_{0i}(\tau_2) - B_{1i}(\tau_2)).
\end{aligned}
$$

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Thus,

$$\sum_{i=1}^{3}(-1)^i(\beta_{0i} - \beta_{1i})$$

$$=\sum_{i=1}^{3}(-1)^i(B_{0i}(\tau_1) - B_{1i}(\tau_1)) + \sum_{i=1}^{3}(-1)^i(B_{0i}(\tau_2) - B_{1i}(\tau_2))$$

$$=\sum_{(i_1,i_2,i_3)} s(i_1, i_2, i_3)\omega_{(i_1,i_2,i_3)}$$

$$=\sum_{(i_1,i_2,i_3)} s(i_1, i_2, i_3)[\mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]$$

$$-\sum_{(i_1,i_2,i_3)} s(i_1, i_2, i_3)[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}]$$

$$=\partial J^3.$$

□

Exercise 10.20. *State conditions under which the formula*

$$\int_{\Phi} f\,d\omega = \int_{\partial\Phi} f\omega - \int_{\Phi} (df) \wedge \omega$$

is valid, and show that it generalizes the formula for integration by parts. (Hint: $d(f\omega) = (df) \wedge \omega + f\,d\omega$.)

Proof.

(1) *If*

    (a) Φ *is a k-chain of class* $\mathscr{C}''$ *in an open set* $V \subseteq \mathbb{R}^m$,

    (b) ω *is a* $(k-1)$*-form of class* $\mathscr{C}'$ *in* V,

    (c) f *is a 0-form of class* $\mathscr{C}'$ *in* V,

    then

$$\int_{\Phi} f\,d\omega = \int_{\partial\Phi} f\omega - \int_{\Phi} (df) \wedge \omega$$

(2) Theorem 10.20(a) implies that

$$d(f\omega) = (df) \wedge \omega + f\,d\omega.$$

(3) The Stokes' theorem (Theorem 10.33) shows that

$$\int_{\Phi} d(f\omega) = \int_{\partial\Phi} f\omega.$$

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Hence

$$\int_\Phi f d\omega = \int_\Phi d(f\omega) - \int_\Phi (df) \wedge \omega = \int_{\partial\Phi} f\omega - \int_\Phi (df) \wedge \omega.$$

(4) Define $\Phi : Q^1 = [0,1] \to [a,b]$ by

$$\Phi(\alpha) = a + \alpha(b-a).$$

Φ is a 1-simplex of class $\mathscr{C}''$ in an open set $V \supseteq [a,b]$. Also,

$$\partial\Phi = [b] - [a].$$

Let $\omega = g$ be a 0-form of class $\mathscr{C}'(V)$.

(5) Note that

$$\int_\Phi f d\omega = \int_\Phi f dg = \int_0^1 f(\Phi(t))g'(\Phi(t))\Phi'(t)dt = \int_a^b f(u)g'(u)du,$$

$$\int_{\partial\Phi} f\omega = \int_{[b]} fg + \int_{-[a]} fg = f(b)g(b) + (-1)f(a)f(a),$$

$$\int_\Phi (df) \wedge \omega = \int_\Phi (df)g = \int_0^1 f'(\Phi(t))g(\Phi(t))\Phi'(t)dt = \int_a^b f'(u)g(u)du.$$

Hence

$$\int_a^b f(u)g'(u)du = f(b)g(b) - f(a)f(a) - \int_a^b f'(u)g(u)du,$$

which is the same as the integration by parts (Theorem 6.22).

$\square$

Exercise 10.21. *As in Example 10.36, consider the 1-form*

$$\eta = \frac{x\,dy - y\,dx}{x^2 + y^2}$$

in $\mathbb{R}^2 - \{\mathbf{0}\}$.

(a) *Carry out the computation that leads to*

$$\int_\gamma \eta = 2\pi \neq 0,$$

and prove that $d\eta = 0$.

(b)

(c) *Take $\Gamma(t) = (a\cos t, b\sin t)$ where $a > 0$, $b > 0$ are fixed. Use part (b) to show that*

$$\int_0^{2\pi} \frac{ab}{a^2\cos^2 t + b^2\sin^2 t}\,dt = 2\pi.$$

(d)

(e)

(f) *If Γ is any closed $\mathscr{C}'$-curve in $\mathbb{R}^2 - \{\mathbf{0}\}$, prove that*

$$\frac{1}{2\pi}\int_\Gamma \eta = \mathrm{Ind}(\Gamma).$$

(See Exercise 8.23 for the definition of the index of a curve.)

Proof of (a).

(1)

$$
\begin{aligned}
\int_\gamma \eta &= \int_0^{2\pi} \frac{(r\cos t)d(r\sin t) - (r\sin t)d(r\cos t)}{(r\cos t)^2 + (r\sin t)^2} \\
&= \int_0^{2\pi} \frac{(r\cos t)(r\cos t) - (r\sin t)(-r\sin t)}{(r\cos t)^2 + (r\sin t)^2}\,dt \\
&= \int_0^{2\pi} dt \\
&= 2\pi.
\end{aligned}
$$

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(2)

$$d\eta = d\left(\frac{xdy - ydx}{x^2 + y^2}\right)$$

$$= d\left(\frac{x}{x^2 + y^2}\right) \wedge dy + \frac{x}{x^2 + y^2} \wedge d^2 y$$

$$\quad - d\left(\frac{y}{x^2 + y^2}\right) \wedge dx - \frac{y}{x^2 + y^2} \wedge d^2 x$$

$$= d\left(\frac{x}{x^2 + y^2}\right) \wedge dy - d\left(\frac{y}{x^2 + y^2}\right) \wedge dx \qquad (d^2 = 0)$$

$$= \left\{D_1\left(\frac{x}{x^2 + y^2}\right) dx + D_2\left(\frac{y}{x^2 + y^2}\right) dy\right\} \wedge dy$$

$$\quad - \left\{D_1\left(\frac{x}{x^2 + y^2}\right) dx + D_2\left(\frac{y}{x^2 + y^2}\right) dy\right\} \wedge dx$$

$$= D_1\left(\frac{x}{x^2 + y^2}\right) dx \wedge dy \qquad (dy \wedge dy = 0)$$

$$\quad - D_2\left(\frac{y}{x^2 + y^2}\right) dy \wedge dx \qquad (dx \wedge dx = 0)$$

$$= 0$$

□

Note.

(1) η is closed and locally exact, that is, $\eta = dt$ on $\mathbb{R}^2 - L$ where L is a half-line issuing from $\mathbf{0}$. η is not exact since $\int_\gamma \eta = 2\pi \neq 0$.

(2) *(Poincaré's Lemma for 1-form.)* Let $\omega = \sum a_i dx_i$ be defined in an open set $U \subseteq \mathbb{R}^n$. Then $d\omega = 0$ if and only if for each $p \in U$ there is a neighborhood $V \subseteq U$ of p and a differentiable function $f : V \to \mathbb{R}^1$ with $df = \omega$ (i.e., ω is locally exact).

Proof of (b).

(1)

(2)

□

Proof of (c).

(1) Γ satisfies all conditions described in (b). So

$$\int_\Gamma \eta = 2\pi.$$

(2) A direct calculation shows that

$$2\pi = \int_\Gamma \eta = \int_\Gamma \frac{xdy - ydx}{x^2 + y^2}$$

$$= \int_0^{2\pi} \frac{a\cos(t)d(b\sin(t)) - b\sin(t)d(a\cos(t))}{(a\cos(t))^2 + (b\sin(t))^2}$$

$$= \int_0^{2\pi} \frac{ab(\cos^2 t + \sin^2 t)}{a^2\cos^2 t + b^2\sin^2 t}$$

$$= \int_0^{2\pi} \frac{ab}{a^2\cos^2 t + b^2\sin^2 t}.$$

□

Proof of (d).

(1)

(2)

□

Proof of (e).

(1)

(2)

□

Proof of (f).

(1)

(2)

□

Exercise 10.22. *As in Example 10.37, define ζ in $\mathbb{R}^3 - \{0\}$ by*

$$\zeta = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{r^3}$$

where $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$,

Proof of (a).

(1)

(2)

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□

Proof of (b).

  (1)

  (2)

□

Proof of (c).

  (1)

  (2)

□

Proof of (d).

  (1)

  (2)

□

Proof of (e).

  (1)

  (2)

□

Proof of (f).

  (1)

  (2)

□

Proof of (g).

  (1)

  (2)

□

Exercise 10.23. ...

Proof of (a).

(1)

(2)

□

Proof of (b).

(1)

(2)

□

Proof of (c).

(1)

(2)

□

Proof of (d).

(1)

(2)

□

Exercise 10.24. *Let $\omega = \sum a_i(\mathbf{x})dx_i$ be a 1-form of class $\mathscr{C}''$ in a convex open set $E \subseteq \mathbb{R}^n$. Assume $d\omega = 0$ and prove that ω is exact in E, by completing the following outline:*

Fix $\mathbf{p} \in E$. Define

$$f(\mathbf{x}) = \int_{[\mathbf{p},\mathbf{x}]} \omega \qquad (\mathbf{x} \in E).$$

Apply Stokes' theorem to affine-oriented 2-simplexs $[\mathbf{p},\mathbf{x},\mathbf{y}]$ in E. Deduce that

$$f(\mathbf{y}) - f(\mathbf{x}) = \sum_{i=1}^{n}(y_i - x_i) \int_0^1 a_i((1-t)\mathbf{x} + t\mathbf{y})dt$$

for $\mathbf{x} \in E$, $\mathbf{y} \in E$. Hence $(D_i f)(\mathbf{x}) = a_i(\mathbf{x})$.

Proof.

  (1) Fix $\mathbf{p} \in E$. Define

$$f(\mathbf{x}) = \int_{[\mathbf{p},\mathbf{x}]} \omega \qquad (\mathbf{x} \in E).$$

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(2) Given any $\mathbf{x} \in E$, $\mathbf{y} \in E$, and $\mathbf{x} \neq \mathbf{y}$. The affine-oriented 2-simplexs $\Psi = [\mathbf{p}, \mathbf{x}, \mathbf{y}]$ is in E by the convexity of E. (If E is open but not convex, we can show that $\omega = df$ **locally** as the note in Exercise 10.21(a). That is why we say that ω is locally exact. The proof is exactly the same.)

(3) Note that
$$\partial \Psi = \partial[\mathbf{p}, \mathbf{x}, \mathbf{y}] = [\mathbf{x}, \mathbf{y}] - [\mathbf{p}, \mathbf{y}] + [\mathbf{p}, \mathbf{x}].$$
The Stokes' theorem (Theorem 10.33) implies that
$$\int_{\Psi} d\omega = \int_{\partial\Psi} \omega \iff \int_{\Psi} 0 = \int_{[\mathbf{x},\mathbf{y}]} \omega - \int_{[\mathbf{p},\mathbf{y}]} \omega + \int_{[\mathbf{p},\mathbf{x}]} \omega$$
$$\iff 0 = \int_{[\mathbf{x},\mathbf{y}]} \omega - f(\mathbf{y}) + f(\mathbf{x})$$
$$\iff f(\mathbf{y}) - f(\mathbf{x}) = \int_{[\mathbf{x},\mathbf{y}]} \omega.$$

(4) Define $\gamma : [0,1] \to E$ by
$$\gamma(t) = \mathbf{x} + t(\mathbf{y} - \mathbf{x})$$
$$= \sum_{i=1}^{n} x_i + t(y_i - x_i)$$
(where $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$). Hence $[0,1]$ is the parameter domain of $[\mathbf{x}, \mathbf{y}]$ with respect to γ. So
$$\int_{[\mathbf{x},\mathbf{y}]} \omega = \int_0^1 \sum_{i=1}^{n} a_i(\gamma(t)) \frac{\partial(x_i + t(y_i - x_i))}{\partial t} dt$$
$$= \int_0^1 \sum_{i=1}^{n} a_i(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))(y_i - x_i) dt$$
$$= \sum_{i=1}^{n} (y_i - x_i) \int_0^1 a_i(\mathbf{x} + t(\mathbf{y} - \mathbf{x})) dt.$$
Thus,
$$f(\mathbf{y}) - f(\mathbf{x}) = \sum_{i=1}^{n} (y_i - x_i) \int_0^1 a_i(\mathbf{x} + t(\mathbf{y} - \mathbf{x})) dt.$$

(5) Note that
$$f(\mathbf{x} + h\mathbf{e}_j) - f(\mathbf{x}) = \sum_{i=1}^{n} ((x_i + h\delta_{ij}) - x_i) \int_0^1 a_i(\mathbf{x} + t((\mathbf{x} + h\mathbf{e}_j) - \mathbf{x})) dt$$
$$= \sum_{i=1}^{n} h\delta_{ij} \int_0^1 a_i(\mathbf{x} + th\mathbf{e}_j) dt$$
$$= h \int_0^1 a_j(\mathbf{x} + th\mathbf{e}_j) dt.$$

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(Here δ_{ij} is the Kronecker delta.) So

$$(D_j f)(\mathbf{x}) = \lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{e}_j) - f(\mathbf{x})}{h}$$

$$= \lim_{h \to 0} \int_0^1 a_j(\mathbf{x} + t h\mathbf{e}_j) dt$$

$$= \int_0^1 a_j(\mathbf{x}) dt \qquad\qquad (a_j \in \mathscr{C}'')$$

$$= a_j(\mathbf{x}).$$

Thus,

$$df = \sum_{j=1}^n (D_j f)(\mathbf{x}) dx_j = \sum_{j=1}^n a_j(\mathbf{x}) dx_j = \omega,$$

or ω is exact in E.

$\square$

Exercise 10.25. *Assume ω is a 1-form in an open set $E \subseteq \mathbb{R}^n$ such that*

$$\int_\gamma \omega = 0$$

for every closed curve γ in E, of class $\mathscr{C}'$. Prove that ω is exact in E, by imitating part of the argument sketched in Exercise 10.24.

Proof.

(1) *Assume that E is a **connected** open subset of $\mathbb{R}^n$. Show that ω is exact in E if $\int_\gamma \omega = 0$ for every closed curve γ in E, of class $\mathscr{C}'$.*

(2) Fix $\mathbf{p} \in E$. Define

$$f(\mathbf{x}) = \int_{[\mathbf{p}, \mathbf{x}]} \omega \qquad (\mathbf{x} \in E).$$

It is well-defined since E is connected and $\int_\gamma \omega = 0$ for every closed curve γ in E.

(3) Given any $\mathbf{x} \in E$, $\mathbf{y} \in E$, and $\mathbf{x} \neq \mathbf{y}$. Let

$$\gamma = [\mathbf{x}, \mathbf{y}] - [\mathbf{p}, \mathbf{y}] + [\mathbf{p}, \mathbf{x}]$$

be a closed curve in E. Hence,

$$0 = \int_\gamma \omega \qquad\qquad \text{(Assumption)}$$

$$= \int_{[\mathbf{x},\mathbf{y}]} \omega - \int_{[\mathbf{p},\mathbf{y}]} \omega + \int_{[\mathbf{p},\mathbf{x}]} \omega$$

$$= \int_{[\mathbf{x},\mathbf{y}]} \omega - f(\mathbf{y}) + f(\mathbf{x}).$$

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So

$$f(\mathbf{y}) - f(\mathbf{x}) = \int_{[\mathbf{x},\mathbf{y}]} \omega$$

(4) Similar to (4)(5) in the proof of Exercise 10.24, we have $df = \omega$. So the statement in (1) is proved. In general, we can define each f_α on each connected component E_α (which is open) of E such that $df_\alpha = \omega$ on E_α. Take

$$f|_{E_\alpha} = f_\alpha$$

on E. Hence, $df = \omega$ on the whole E.

□

Exercise 10.26. *Assume ω is a 1-form in $\mathbb{R}^3 - \{\mathbf{0}\}$, of class $\mathscr{C}'$ and $d\omega = 0$. Prove that ω is exact in $\mathbb{R}^3 - \{\mathbf{0}\}$. (Hint: Every closed continuously differentiable curve in $\mathbb{R}^3 - \{\mathbf{0}\}$ is the boundary of a 2-surface in $\mathbb{R}^3 - \{\mathbf{0}\}$. Apply Stokes' theorem and Exercise 10.25.)*

Proof.

(1) Let $E = \mathbb{R}^3 - \{\mathbf{0}\}$. By Exercise 10.25, it suffices to show that

$$\int_\gamma \omega = 0$$

for every closed curve γ in E, of class $\mathscr{C}'$.

(2) Intuitively, every closed continuously differentiable curve in $\mathbb{R}^3 - \{\mathbf{0}\}$ is the boundary of a 2-surface in $\mathbb{R}^3 - \{\mathbf{0}\}$. So there is some 2-surface Ψ such that $\partial\Psi = \gamma$. The Stokes' theorem (Theorem 10.33) implies that

$$\int_\gamma \omega = \int_{\partial\Psi} \omega = \int_\Psi d\omega = \int_\Psi 0 = 0.$$

□

Exercise 10.27. ...

Proof.

(1)

(2)

□

Exercise 10.28. *Fix $b > a > 0$, define*

$$\Phi(r, \theta) = (r \cos \theta, r \sin \theta)$$

for $a \le r \le b$, $0 \le \theta \le 2\pi$. (The range of Φ is an annulus in $\mathbb{R}^2$.) Put $\omega = x^3 dy$, and compute both

$$\int_\Phi d\omega \qquad and \qquad \int_{\partial\Phi} \omega$$

to verify that they are equal.

Proof.

(1) Note that
$$\frac{\partial(x, y)}{\partial(r, \theta)} = \det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r.$$

So

$$\int_\Phi d\omega = \int_\Phi 3x^2 dx \wedge dy \qquad\qquad (dy \wedge dy = 0)$$

$$= \int_{[a,b]\times[0,2\pi]} 3(r\cos\theta)^2 \frac{\partial(x,y)}{\partial(r,\theta)} dr d\theta$$

$$= \int_a^b \int_0^{2\pi} 3r^3(\cos\theta)^2 dr d\theta$$

$$= \frac{3\pi}{4}(b^4 - a^4).$$

(2) Similar to Exercise 10.21(b), write

$$\partial\Phi = \Gamma - \gamma,$$

where $\Gamma(t) = (b\cos t, b\sin t)$ on $[0, 2\pi]$ and $\gamma(t) = (a\cos t, a\sin t)$ on $[0, 2\pi]$. Hence

$$\int_{\partial\Phi} \omega = \int_\Gamma \omega - \int_\gamma \omega$$

$$= \int_\Gamma x^3 dy - \int_\gamma x^3 dy$$

$$= \int_{[0,2\pi]} (b\cos\theta)^3 \frac{\partial y}{\partial\theta} d\theta - \int_{[0,2\pi]} (a\cos\theta)^3 \frac{\partial y}{\partial\theta} d\theta$$

$$= \int_0^{2\pi} b^4(\cos\theta)^4 d\theta - \int_0^{2\pi} a^4(\cos\theta)^4 d\theta$$

$$= \frac{3\pi}{4}(b^4 - a^4).$$

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(3)
$$\int_\Phi d\omega = \int_{\partial\Phi} \omega = \frac{3\pi}{4}(b^4 - a^4).$$

□

Exercise 10.29. ...

Proof.

(1)

(2)

□

Exercise 10.30. *If* **N** *is the vector given by*

$$\mathbf{N} = (\alpha_2\beta_3 - \alpha_3\beta_2)\mathbf{e}_1 + (\alpha_3\beta_1 - \alpha_1\beta_3)\mathbf{e}_2 + (\alpha_1\beta_2 - \alpha_2\beta_1)\mathbf{e}_3$$

(Equation (135)), prove that

$$\det \begin{bmatrix} \alpha_1 & \beta_1 & \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_2 & \beta_2 & \alpha_3\beta_1 - \alpha_1\beta_3 \\ \alpha_3 & \beta_3 & \alpha_1\beta_2 - \alpha_2\beta_1 \end{bmatrix} = |\mathbf{N}|^2$$

Also, verify

$$\mathbf{N} \cdot (T\mathbf{e}_1) = \mathbf{N} \cdot (T\mathbf{e}_2)$$

(Equation (137)).

Proof.

(1) By Laplace's expansion along the third column,

$$\det \begin{bmatrix} \alpha_1 & \beta_1 & \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_2 & \beta_2 & \alpha_3\beta_1 - \alpha_1\beta_3 \\ \alpha_3 & \beta_3 & \alpha_1\beta_2 - \alpha_2\beta_1 \end{bmatrix}$$

$$= (-1)^{1+3}(\alpha_2\beta_3 - \alpha_3\beta_2) \det \begin{bmatrix} \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix}$$

$$+ (-1)^{2+3}(\alpha_3\beta_1 - \alpha_1\beta_3) \det \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_3 & \beta_3 \end{bmatrix}$$

$$+ (-1)^{3+3}(\alpha_1\beta_2 - \alpha_2\beta_1) \det \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix}$$

$$= (\alpha_2\beta_3 - \alpha_3\beta_2)^2 + (\alpha_3\beta_1 - \alpha_1\beta_3)^2 + (\alpha_1\beta_2 - \alpha_2\beta_1)^2$$

$$= |\mathbf{N}|^2.$$

(2)

$$\begin{aligned}
\mathbf{N} \cdot (T\mathbf{e}_1) &= (\alpha_2\beta_3 - \alpha_3\beta_2, \alpha_3\beta_1 - \alpha_1\beta_3, \alpha_1\beta_2 - \alpha_2\beta_1) \cdot (\alpha_1, \alpha_2, \alpha_3) \\
&= (\alpha_2\beta_3 - \alpha_3\beta_2)\alpha_1 + (\alpha_3\beta_1 - \alpha_1\beta_3)\alpha_2 + (\alpha_1\beta_2 - \alpha_2\beta_1))\alpha_3 \\
&= (\alpha_3\alpha_2 - \alpha_2\alpha_3)\beta_1 + (\alpha_1\alpha_3 - \alpha_3\alpha_1)\beta_2 + (\alpha_2\alpha_1 - \alpha_1\alpha_2)\beta_3 \\
&= 0.
\end{aligned}$$

(3)

$$\begin{aligned}
\mathbf{N} \cdot (T\mathbf{e}_2) &= (\alpha_2\beta_3 - \alpha_3\beta_2, \alpha_3\beta_1 - \alpha_1\beta_3, \alpha_1\beta_2 - \alpha_2\beta_1) \cdot (\beta_1, \beta_2, \beta_3) \\
&= (\alpha_2\beta_3 - \alpha_3\beta_2)\beta_1 + (\alpha_3\beta_1 - \alpha_1\beta_3)\beta_2 + (\alpha_1\beta_2 - \alpha_2\beta_1))\beta_3 \\
&= (\beta_2\beta_3 - \beta_3\beta_2)\alpha_1 + (\beta_3\beta_1 - \beta_1\beta_3)\alpha_2 + (\beta_1\beta_2 - \beta_2\beta_1)\alpha_3 \\
&= 0.
\end{aligned}$$

□

Exercise 10.31. ...

Proof of (a).

  (1)

  (2)

□

Proof of (b).

  (1)

  (2)

□

Proof of (c).

  (1)

  (2)

□

Proof of (d).

  (1)

  (2)

□

Exercise 10.32. ...

Proof.

  (1)

  (2)

□