Notes on the book:
Robin Hartshorne, Algebraic Geometry

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Chapter I: Varieties

Exercise I.1.6.

Any nonempty open subset of an irreducible topological space is dense and irreducible. If Y is a subset of a topological space X, which is irreducible in its induced topology, then the closure $\overline{Y}$ is also irreducible.

Proof.

(1) *Show that any nonempty open subset of an irreducible topological space is dense.* It suffices to show that $U_1 \cap U_2 \neq \varnothing$ for any nonempty open subsets of an irreducible topological space.

$$\begin{aligned}
&\forall \text{ nonempty open sets } U_1 \text{ and } U_2, U_1 \cap U_2 \neq \varnothing \\
\Longleftrightarrow &\forall \text{ nonempty open sets } U_1 \text{ and } U_2, X - (U_1 \cap U_2) \neq X \\
\Longleftrightarrow &\forall \text{ nonempty open sets } U_1 \text{ and } U_2, (X - U_1) \cup (X - U_2) \neq X \\
\Longleftrightarrow &\forall \text{ proper closed sets } Y_1 \text{ and } Y_2, Y_1 \cup Y_2 \neq X \\
\Longleftrightarrow &\nexists \text{ proper closed sets } Y_1 \text{ and } Y_2, Y_1 \cup Y_2 = X.
\end{aligned}$$

(2) *Show that any nonempty open subset of an irreducible topological space is irreducible.* Given any open subset U of an irreducible topological space X. Write $U \subseteq Y_1 \cup Y_2$ where Y_1 and Y_2 are closed in X.

$$\begin{aligned}
&U \subseteq Y_1 \cup Y_2 \\
\Longrightarrow &\overline{U} \subseteq \overline{Y_1 \cup Y_2} \\
\Longrightarrow &X \subseteq Y_1 \cup Y_2 && (U \text{ is dense, } Y_1 \cup Y_2 \text{ is closed}) \\
\Longrightarrow &Y_1 = X \supseteq U \text{ or } Y_2 = X \supseteq U && (X \text{ is irreducible}) \\
\Longrightarrow &U \text{ is irreducible.}
\end{aligned}$$

(3) *Show that if Y is a subset of a topological space X, which is irreducible (in its induced topology), then the closure $\overline{Y}$ is also irreducible.* (Reductio ad absurdum) If $\overline{Y}$ were reducible, there are two closed sets Y_1 and Y_2 such that

$$\overline{Y} \subseteq Y_1 \cup Y_2, \overline{Y} \not\subseteq Y_i (i = 1, 2).$$

(a) $Y \subseteq \overline{Y} \subseteq Y_1 \cup Y_2$.

(b) $Y \not\subseteq Y_i (i = 1, 2)$. If not, $Y \subseteq Y_i$ for some i. Take closure to get $\overline{Y} \subseteq \overline{Y_i} = Y_i$ (since Y_i is closed), contrary to the assumption.

By (a)(b), Y is reducible, which is absurd.

□

Chapter II: Schemes

Exercise II.1.1. (Constant presheaf)

Let A be an abelian group, and define the ***constant presheaf*** *associated to A on the topological space X to be the presheaf $U \mapsto A$ for all $U \neq \varnothing$, with restriction maps the identity. Show that the constant sheaf $\mathscr{A}$ defined in the text is the sheaf associated to this presheaf.*

Proof.

(1) Let $\mathscr{F}$ be the constant presheaf.

(2) Let $\theta : \mathscr{F} \to \mathscr{A}$ be a morphism consists of a morphism of abelian groups $\theta(U) : \mathscr{F}(U) = A \to \mathscr{A}(U)$ for each open set $U \subseteq X$ such that $\theta(U)(a) = f_a : x \mapsto a$ for each element $x \in U$. (It is well-defined.)

(3) Given any sheaf $\mathscr{G}$ and any morphism $\varphi : \mathscr{F} \to \mathscr{G}$, it suffices to find a morphism $\psi : \mathscr{A} \to \mathscr{G}$ such that $\varphi = \psi \circ \theta$.

(4) Given an open set $U \subseteq X$. Suppose $f \in \mathscr{A}(U)$ is a continuous maps of U into A. Since A is equipped with the discrete topology, f is locally constant, that is,

$$f(V_i) = a_i$$

where each V_i is a connected component of U. (In particular, $\{V_i\}$ is an open covering of U.)

(5) Now

$$s_i := \varphi(V_i)(a_i) \in \mathscr{G}(V_i)$$

is defined. Since $\mathscr{G}$ is a sheaf and all V_i are disjoint, there is a $s \in \mathscr{G}(U)$ such that $s|_{V_i} = s_i$ for each i. Now we define $\psi(U)$ by

$$\psi(U)(f) = s.$$

Thus ψ is a morphism and $\varphi = \psi \circ \theta$ by construction.

□