

# Solutions to the book: *do Carmo, Differential Geometry of Curves and Surfaces*

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# Chapter 1: Curves

## 1-1. Introduction

Classical differential geometry: the study of local properties of curves and surfaces.

Global differential geometry: the study of the influence of the local properties on the behavior of the entire curve and surface.

*No exercises.*

## 1-2. Parametrized Curves

### Exercise 1-2.1.

Find a parametrized curve  $\alpha(t)$  whose trace is the circle  $x^2 + y^2 = 1$  such that  $\alpha(t)$  runs clockwise around the circle with  $\alpha(0) = (0, 1)$ .

*Proof.*  $\alpha(t) = (\sin t, \cos t)$ ,  $t \in \mathbb{R}$ .  $\square$

### Exercise 1-2.2.

Let  $\alpha(t)$  be a parametrized curve which does not pass through the origin. If  $\alpha(t_0)$  is the point of the trace of  $\alpha$  closest to the origin and  $\alpha'(t_0) \neq 0$ , show that the position vector  $\alpha(t_0)$  is orthogonal to  $\alpha'(t_0)$ .

*Proof.* Let  $f(t) = |\alpha(t)|^2 = \alpha(t) \cdot \alpha(t)$ .  $f(t)$  is differentiable and  $f(t)$  has a local minimum at a point  $t = t_0 \in I$ . So  $f'(t_0) = 0$ . [Theorem 5.8 in *W. Rudin, Principles of Mathematical Analysis, 3rd edition.*] Since

$$f'(t) = 2\alpha(t) \cdot \alpha'(t),$$

$f'(t_0) = 2\alpha(t_0) \cdot \alpha'(t_0) = 0$ , or  $\alpha(t_0) \cdot \alpha'(t_0) = 0$ . Since  $\alpha(t_0) \neq 0$  and  $\alpha'(t_0) \neq 0$ ,  $\alpha(t_0)$  is orthogonal to  $\alpha'(t_0)$ .  $\square$

**Exercise 1-2.3.**

A parametrized curve  $\alpha(t)$  has a property that its second derivative  $\alpha''(t)$  is identically zero. What can be said about  $\alpha$ ?

*Proof.*

- (1)  $\alpha(t)$  is a straight line.
- (2) Since  $\alpha''(t)$  is identically zero,  $\alpha'(t) = a$  is a constant. [Theorem 5.11 in *W. Rudin, Principles of Mathematical Analysis, 3rd edition.*] Define  $f(t) = \alpha(t) - at$  (on  $I$ ). Since  $f'(t) = \alpha'(t) - a = 0$ ,  $f(t) = \alpha(t) - at = b$  is a constant again. Therefore,  $\alpha(t) = at + b$ , which is a straight line (on  $I$ ).

□

**Exercise 1-2.4.**

Let  $\alpha : I \rightarrow \mathbb{R}^3$  be a parametrized curve and let  $v \in \mathbb{R}^3$  be a fixed vector. Assume that  $\alpha'(t)$  is orthogonal to  $v$  for all  $t \in I$  and that  $\alpha(0)$  is orthogonal to  $v$ . Prove that  $\alpha(t)$  is orthogonal to  $v$  for all  $t \in I$ .

Need to assume that  $\alpha(t) \neq 0$  for all  $t \in I$ .

*Proof.* Given any  $t \neq 0 \in I$ . (Nothing to do at  $t = 0$ .) Define  $f : I \rightarrow \mathbb{R}$  by  $f(t) = \alpha(t) \cdot v$ . By the mean value theorem, there exists a point  $\xi$  between 0 and  $t$  such that

$$f(t) - f(0) = f'(\xi)(t - 0),$$

where  $f'(t) = \alpha'(t) \cdot v + \alpha(t) \cdot v' = \alpha'(t) \cdot v$ . Note that  $f(0) = 0$  since  $\alpha(0)$  is orthogonal to  $v$ , and  $f'(\xi) = 0$  since  $\alpha'(\xi)$  is orthogonal to  $v$ . So the identity is reduced to

$$f(t) = 0,$$

or  $\alpha(t) \cdot v = 0$ , or  $\alpha(t)$  is orthogonal to  $v$ . □

**Exercise 1-2.5.**

Let  $\alpha : I \rightarrow \mathbb{R}^3$  be a parametrized curve, with  $\alpha'(t) \neq 0$  for all  $t \in I$ . Show that  $|\alpha(t)|$  is a nonzero constant if and only if  $\alpha(t)$  is orthogonal to  $\alpha'(t)$  for all  $t \in I$ .

The same trick in Exercise 1-2.2.

*Proof.* It is equivalent to show that  $|\alpha(t)|^2$  is a nonzero constant if and only if  $\alpha(t)$  is orthogonal to  $\alpha'(t)$  for all  $t \in I$ . Let

$$f(t) = |\alpha(t)|^2 = \alpha(t) \cdot \alpha(t).$$

Notice that  $\alpha'(t) \neq 0$ , and thus

$$\begin{aligned} & |\alpha(t)| \text{ is a nonzero constant} \\ \iff & f(t) = |\alpha(t)|^2 \text{ is a nonzero constant} \\ \iff & f'(t) = 0 \text{ and } f(t) \text{ is a nonzero constant} \\ \iff & \alpha(t) \cdot \alpha'(t) = 0 \text{ and } \alpha(t) \text{ is a nonzero constant} \\ \iff & \alpha(t) \text{ is orthogonal to } \alpha'(t) \text{ for all } t \in I. \end{aligned}$$

□

### 1-3. Regular Curves; Arc Length

#### Exercise 1-3.1.

Show that the tangent lines to the regular parametrized curve  $\alpha(t) = (3t, 3t^2, 2t^3)$  make a constant angle with the line  $y = 0, z = x$ .

*Proof.*  $\alpha'(t) = (3, 6t, 6t^2)$ . The line  $y = 0, z = x$  is  $\beta(t) = (1, 0, 1)$ . The cosine of the angle  $\theta$  between these two curves is

$$\begin{aligned} \cos \theta &= \frac{(3, 6t, 6t^2) \cdot (1, 0, 1)}{|(3, 6t, 6t^2)| |(1, 0, 1)|} \\ &= \frac{3 + 6t^2}{\sqrt{3^2 + (6t)^2 + (6t^2)^2} \sqrt{2}} \\ &= \frac{3 + 6t^2}{\sqrt{9 + 36t^2 + 36t^4} \sqrt{2}} \\ &= \frac{3 + 6t^2}{\sqrt{(3 + 6t^2)^2} \sqrt{2}} \\ &= \frac{1}{\sqrt{2}}. \end{aligned}$$

(Notice  $3 + 6t^2 > 0$  for all  $t \in \mathbb{R}$ .) That is, the angle between  $\alpha'$  and  $\beta$  is a constant ( $= \pi/4$ ). □

**Exercise 1-3.2. (Cycloid)**

A circular disk of radius 1 in the plane  $xy$  rolls without slipping along the  $x$  axis. The figure described by a point of the circumference of the disk is called a **cycloid** (Figure 1-7 in Mantredo P. do Carmo, *Differential Geometry of Curves and Surfaces*).

- (a) Obtain a parametrized curve  $\alpha : \mathbb{R} \rightarrow \mathbb{R}^2$  the trace of which is the cycloid and determine its singular points.
- (b) Compute the arc length of the cycloid corresponding to a complete rotation of the disk.

*Proof of (a).*

- (1) Since

$$\begin{cases} x = t - \sin t \\ y = 1 - \cos t, \end{cases}$$

we define  $\alpha(t) = (t - \sin t, 1 - \cos t)$ .

- (2)  $\alpha'(t) = (1 - \cos t, \sin t)$ .  $\alpha'(t) = 0$  if and only if  $t = 2n\pi$  where  $n \in \mathbb{Z}$ . That is, all singular points are  $\alpha(2n\pi) = (2n\pi, 0)$  where  $n \in \mathbb{Z}$ .

□

*Proof of (b).* The arc length of the cycloid corresponding to a complete rotation of the disk is

$$\begin{aligned} \int_0^{2\pi} |\alpha'(t)| dt &= \int_0^{2\pi} \sqrt{(1 - \cos t)^2 + (\sin t)^2} dt \\ &= \int_0^{2\pi} \sqrt{2} \sqrt{1 - \cos t} dt \\ &= \int_0^{2\pi} 2 \sin \frac{t}{2} dt \\ &= \left[ -4 \cos \frac{t}{2} \right]_{t=0}^{t=2\pi} \\ &= 8. \end{aligned}$$

□

**Supplement.** The cycloid is not an algebraic curve.

**Exercise 1-3.3. (Cisoid of Diocles)**

Let  $0A = 2a$  be the diameter of a circle  $\mathbb{S}^1$  and  $0Y$  and  $AV$  be the tangents to  $\mathbb{S}^1$  at  $0$  and  $A$ , respectively. A half-line  $r$  is drawn from  $0$  which meets the circle  $\mathbb{S}^1$  at  $C$  and the line  $AV$  at  $B$ . On  $0B$  mark off the segment  $0p = CB$ . If we rotate  $r$  about  $0$ , the point  $p$  will describe a curve called the **cisoid of Diocles**. By taking  $0A$  as the  $x$  axis and  $0Y$  as the  $y$  axis, prove that

(a) The tract of

$$\alpha(t) = \left( \frac{2at^2}{1+t^2}, \frac{2at^3}{1+t^2} \right), \quad t \in \mathbb{R},$$

is the cisoid of Diocles ( $t = \tan \theta$ ; see Figure 1-8 in Mantredo P. do Carmo, *Differential Geometry of Curves and Surfaces*).

(b) The origin  $(0, 0)$  is a singular point of the cisoid.

(c) As  $t \rightarrow \infty$ ,  $\alpha(t)$  approaches the line  $x = 2a$ , and  $\alpha'(t) \rightarrow (0, 2a)$ . Thus, as  $t \rightarrow \infty$ , the curve and its tangent approach the line  $x = 2a$ ; we say that  $x = 2a$  is an **asymptote** to the cisoid.

*Proof of (a).*

(1) The polar equations of the circle  $\mathbb{S}^1$  and the half-line  $r$  is

$$\begin{aligned} r &= 2a \cos \theta, \\ r &= 2a \sec \theta, \end{aligned}$$

respectively.

(2) By construction, the polar equation of the cisoid is

$$r = 2a \sec \theta - 2a \cos \theta = 2a \frac{\sin^2 \theta}{\cos \theta} = 2a \sin \theta \tan \theta.$$

(3) Put  $t = \tan \theta$ , we have

$$\begin{aligned} x &= r \cos \theta = 2a \sin^2 \theta = \frac{2at^2}{1+t^2}, \\ y &= r \sin \theta = tx = \frac{2at^3}{1+t^2}. \end{aligned}$$

So

$$\alpha(t) = (x, y) = \left( \frac{2at^2}{1+t^2}, \frac{2at^3}{1+t^2} \right).$$

□

**Supplement.** The cisoid is an algebraic curve  $= V((x^2 + y^2)x = 2ay^2)$ .

*Proof of (b).* Note that  $\alpha(0) = (0, 0)$  and

$$\alpha'(t) = \left( \frac{4at}{(t^2 + 1)^2}, \frac{2at^2(t^2 + 3)}{(t^2 + 1)^2} \right).$$

Hence  $\alpha'(0) = (0, 0)$ . That is,  $(0, 0)$  is a singular point of the cissoid. (In fact, the origin is the unique singular point of the cissoid.)  $\square$

*Proof of (c).*

(1) Note that

$$\begin{aligned} \lim_{t \rightarrow \pm\infty} x(t) &= \lim_{t \rightarrow \pm\infty} \frac{2at^2}{1+t^2} = 2a, \\ \lim_{t \rightarrow \pm\infty} y(t) &= \lim_{t \rightarrow \pm\infty} \frac{2at^3}{1+t^2} = \pm\infty. \end{aligned}$$

Hence,  $\alpha(t)$  approaches the line  $x = 2a$  as  $t \rightarrow \pm\infty$ .

(2) Similarly,

$$\begin{aligned} \lim_{t \rightarrow \pm\infty} x'(t) &= \lim_{t \rightarrow \pm\infty} \frac{4at}{(t^2 + 1)^2} = 0, \\ \lim_{t \rightarrow \pm\infty} y'(t) &= \lim_{t \rightarrow \pm\infty} \frac{2at^2(t^2 + 3)}{(t^2 + 1)^2} = 2a. \end{aligned}$$

Therefore,  $\alpha'(t) \rightarrow (0, 2a)$  as  $t \rightarrow \pm\infty$ .

(3) By (1)(2), the curve and its tangent approach the line  $x = 2a$  as  $t \rightarrow \pm\infty$ , or  $x = 2a$  is an asymptote to the cissoid.

$\square$

### Exercise 1-3.4. (Tractrix)

Let  $\alpha : (0, \pi) \rightarrow \mathbb{R}^2$  be given by

$$\alpha(t) = \left( \sin t, \cos t + \log \tan \frac{t}{2} \right),$$

where  $t$  is the angle that the  $y$  axis makes with the vector  $\alpha(t)$ . The trace of  $\alpha$  is called the **tractrix**. (Figure 1-9 in Mantredo P. do Carmo, *Differential Geometry of Curves and Surfaces*). Show that

- (a)  $\alpha$  is a differentiable parametrized curve, regular except at  $t = \frac{\pi}{2}$ .
- (b) The length of the segment of the tangent of the tractrix between the point of tangency and the  $y$  axis is constantly equal to 1.



*Proof of (a).*

$$\begin{aligned}\alpha'(t) &= \left( \cos t, -\sin t + \frac{1}{\tan \frac{t}{2}} \frac{1}{\cos^2 \frac{t}{2}} \frac{1}{2} \right) \\ &= \left( \cos t, -\sin t + \frac{1}{2 \sin \frac{t}{2} \cos \frac{t}{2}} \right) \\ &= \left( \cos t, \frac{\cos^2 t}{\sin t} \right)\end{aligned}$$

exists. And  $\alpha'(t) = 0$  if and only if  $t = \frac{\pi}{2}$ . That is, there is a unique singular point at  $t = \frac{\pi}{2}$ .  $\square$

*Proof of (b).* The tangent line of the tractrix through the regular point  $t$  is parametrized by  $\beta : \mathbb{R} \rightarrow \mathbb{R}^2$  which is defined by

$$\begin{aligned}\beta(u) &= u\alpha'(t) + \alpha(t) \\ &= \left( u \cos t + \sin t, u \frac{\cos^2 t}{\sin t} + \cos t + \log \tan \frac{t}{2} \right).\end{aligned}$$

By construction, this tangent line  $\beta(u)$  meets the tractrix at  $u = 0$ , and meets the  $y$ -axis when  $u \cos t + \sin t = 0$  or  $u = -\tan t$ . So the length of the segment is

$$\begin{aligned}|\beta(0) - \beta(-\tan t)| &= \sqrt{(-\tan t \cos t)^2 + \left(-\tan t \frac{\cos^2 t}{\sin t}\right)^2} \\ &= \sqrt{(\sin t)^2 + (\cos t)^2} \\ &= 1.\end{aligned}$$

$\square$

### Exercise 1-3.5. (Folium of Descartes)

Let  $\alpha : (-1, +\infty) \rightarrow \mathbb{R}^2$  be given by

$$\alpha(t) = \left( \frac{3at}{1+t^3}, \frac{3at^2}{1+t^3} \right).$$

*Prove that:*

- (a) For  $t = 0$ ,  $\alpha$  is tangent to the  $x$  axis.
- (b) As  $t \rightarrow +\infty$ ,  $\alpha(t) \rightarrow (0, 0)$  and  $\alpha'(t) = (0, 0)$ .
- (c) Take the curve the opposite orientation. Now, as  $t \rightarrow -1$ , the curve and its tangent approach the line  $x + y + a = 0$ .

The figure obtained by completing the trace of  $\alpha$  in such a way that it becomes symmetric relative the the line  $y = x$  is called the **folium of Descartes** (See Figure 1-10 in Mantredo P. do Carmo, *Differential Geometry of Curves and Surfaces*).

*Proof of (a).* Note that

$$\alpha'(t) = \left( \frac{3a(1-2t^3)}{(1+t^3)^2}, \frac{3at(2-t^3)}{(1+t^3)^2} \right).$$

Hence,  $\alpha'(0) = (3a, 0)$ , or  $\alpha$  is tangent to the  $x$  axis when  $t = 0$ .  $\square$

*Proof of (b).*

(1)

$$\begin{aligned} \lim_{t \rightarrow +\infty} \alpha(t) &= \lim_{t \rightarrow +\infty} \left( \frac{3at}{1+t^3}, \frac{3at^2}{1+t^3} \right) \\ &= \left( \lim_{t \rightarrow +\infty} \frac{3at}{1+t^3}, \lim_{t \rightarrow +\infty} \frac{3at^2}{1+t^3} \right) \\ &= (0, 0). \end{aligned}$$

(2)

$$\begin{aligned} \lim_{t \rightarrow +\infty} \alpha'(t) &= \lim_{t \rightarrow +\infty} \left( \frac{3a(1-2t^3)}{(1+t^3)^2}, \frac{3at(2-t^3)}{(1+t^3)^2} \right) \\ &= \left( \lim_{t \rightarrow +\infty} \frac{3a(1-2t^3)}{(1+t^3)^2}, \lim_{t \rightarrow +\infty} \frac{3at(2-t^3)}{(1+t^3)^2} \right) \\ &= (0, 0). \end{aligned}$$

$\square$

*Proof of (c).*

(1) Note that

$$\begin{aligned} \lim_{t \rightarrow -1^+} \alpha(t) &= \lim_{t \rightarrow -1^+} \left( \frac{3at}{1+t^3}, \frac{3at^2}{1+t^3} \right) \\ &= \left( \lim_{t \rightarrow -1^+} \frac{3at}{1+t^3}, \lim_{t \rightarrow -1^+} \frac{3at^2}{1+t^3} \right) \\ &= (-\infty, +\infty) \end{aligned}$$

and

$$\begin{aligned} \lim_{t \rightarrow -1^+} (x(t) + y(t)) &= \lim_{t \rightarrow -1^+} \left( \frac{3at}{1+t^3} + \frac{3at^2}{1+t^3} \right) \\ &= \lim_{t \rightarrow -1^+} \frac{3at}{1-t+t^2} \\ &= -a. \end{aligned}$$

Therefore, as  $t \rightarrow -1$ , the curve approaches the line  $x + y + a = 0$ .

(2) Note that

$$\begin{aligned}\lim_{t \rightarrow -1^+} \frac{y'(t)}{x'(t)} &= \lim_{t \rightarrow -1^+} \frac{\frac{3a(1-2t^3)}{(1+t^3)^2}}{\frac{3at(2-t^3)}{(1+t^3)^2}} \\ &= \lim_{t \rightarrow -1^+} \frac{1-2t^3}{t(2-t^3)} \\ &= -1.\end{aligned}$$

Hence, as  $t \rightarrow -1$ , its tangent also approaches the line  $x + y + a = 0$ .

□

### Exercise 1-3.6. (Logarithmic spiral)

Let  $\alpha(t) = (ae^{bt} \cos t, ae^{bt} \sin t)$ ,  $t \in \mathbb{R}$ ,  $a$  and  $b$  constants,  $a > 0$ ,  $b < 0$ , be a parametrized curve.

- (a) Show that as  $t \rightarrow +\infty$ ,  $\alpha(t)$  approaches the origin 0, spiraling around it (because of this, the trace of  $\alpha$  is called the **logarithmic spiral**; See Figure 1-11 in Mantredo P. do Carmo, *Differential Geometry of Curves and Surfaces*).
- (b) Show that  $\alpha'(t) \rightarrow (0, 0)$  as  $t \rightarrow +\infty$  and that

$$\lim_{t \rightarrow +\infty} \int_{t_0}^t |\alpha'(t)| dt$$

is finite; that is,  $\alpha$  has finite arc length in  $[t_0, \infty)$ .

*Proof of (a).*

(1) Note that

$$\lim_{t \rightarrow +\infty} x(t) = \lim_{t \rightarrow +\infty} \frac{\overbrace{a \cos t}^{\text{bounded}}}{\underbrace{e^{-bt}}_{\rightarrow +\infty}} = 0$$

and  $\lim_{t \rightarrow +\infty} y(t) = 0$  (by the similar argument). Hence  $\alpha(t)$  approaches the origin 0 as  $t \rightarrow +\infty$ .

- (2)  $\alpha(t) = (ae^{bt} \cos t, ae^{bt} \sin t)$  is moving in counter-clockwise on a circle path and sweeping out a length  $ae^{bt}$  as  $t$  is moving from  $t_0$  to  $+\infty$ . Note that  $t \mapsto ae^{bt}$  is decreasing strictly (as  $t$  is moving from  $t_0$  to  $+\infty$ ). Hence  $\alpha$  spiraling around the origin.

□

*Proof of (b).*

(1) Note that

$$\alpha'(t) = (ae^{bt} \underbrace{(b \cos t - \sin t)}_{\text{bounded}}, ae^{bt} \underbrace{(b \sin t + \cos t)}_{\text{bounded}}).$$

As  $t \rightarrow +\infty$ ,  $\alpha'(t) \rightarrow (0, 0)$ .

(2) As

$$\begin{aligned} \int_{t_0}^{+\infty} |\alpha'(t)| dt &= \int_{t_0}^{+\infty} ae^{bt} \sqrt{b^2 + 1} dt \\ &= \left[ \frac{a}{b} e^{bt} \sqrt{b^2 + 1} \right]_{t=t_0}^{t=+\infty} \\ &= -\frac{a}{b} e^{bt_0} \sqrt{b^2 + 1} \\ &< +\infty, \end{aligned}$$

$\alpha$  has finite arc length in  $[t_0, \infty)$ .

□

### Exercise 1-3.7.

A map  $\alpha : I \rightarrow \mathbb{R}^3$  is called a **curve of class  $\mathcal{C}^k$**  if each of the coordinate functions in the expression  $\alpha(t) = (x(t), y(t), z(t))$  has continuous derivatives up to order  $k$ . If  $\alpha$  is merely continuous, we say that  $\alpha$  is of class  $\mathcal{C}^0$ . A curve  $\alpha$  is called **simple** if the map  $\alpha$  is one-to-one. Thus, the curve  $\alpha(t) = (t^3 - 4t, t^2 - 4)$  ( $t \in \mathbb{R}$ ) is not simple.

Let  $\alpha : I \rightarrow \mathbb{R}^3$  be a simple curve of class  $\mathcal{C}^0$ . We say that  $\alpha$  has a **weak tangent** at  $t = t_0 \in I$  if the line determined by  $\alpha(t_0 + h)$  and  $\alpha(t_0)$  has a limit position when  $h \rightarrow 0$ . We say that  $\alpha$  has a **strong tangent** at  $t = t_0 \in I$  if the line determined by  $\alpha(t_0 + h)$  and  $\alpha(t_0 + k)$  has a limit position when  $h, k \rightarrow 0$ . Show that

- (a)  $\alpha(t) = (t^3, t^2)$ ,  $t \in \mathbb{R}$ , has a weak tangent but not a strong tangent at  $t = 0$ .
- (b) If  $\alpha : I \rightarrow \mathbb{R}^3$  is of class  $\mathcal{C}^1$  and regular at  $t = t_0$ , then it has a strong tangent at  $t = t_0$ .
- (c) The curve given by

$$\alpha(t) = \begin{cases} (t^2, t^2), & t \geq 0, \\ (t^2, -t^2), & t \leq 0, \end{cases}$$

is of class  $\mathcal{C}^1$  but not of class  $\mathcal{C}^2$ . Draw a sketch of the curve and its tangent vectors.

*Proof of (a).*

- (1) Note that  $\alpha(0) = (0, 0)$  and  $\alpha(h) = (h^3, h^2)$ . The line passing  $\alpha(0)$  and  $\alpha(h)$  is

$$\begin{aligned} (x - 0)(h^2 - 0) - (y - 0)(h^3 - 0) &= 0 \\ \iff x - hy &= 0. \end{aligned}$$

As  $h \rightarrow 0$ , the line has a limit position  $x = 0$ . Therefore,  $\alpha(t)$  has a weak tangent.

- (2) The line passing  $\alpha(h)$  and  $\alpha(k)$  is

$$\begin{aligned} (x - k^2)(h^2 - k^2) - (y - k^3)(h^3 - k^3) &= 0 \\ \iff (x - k^2)(h + k) - (y - k^3)(h^2 + hk + k^2) &= 0. \end{aligned}$$

As  $h \rightarrow 0$ , the line has a limit position

$$\begin{aligned} (x - k^2) - (y - k^3)k &= 0 \\ \iff x - ky + k^4 - k^2 &= 0. \end{aligned}$$

As  $k \rightarrow 0$ , the line has a limit position  $x = 0$ .

- (3) On the other hand, as  $h = -k$  we have  $y - k^3 = 0$ . As  $k \rightarrow 0$ , the line has a limit position  $y = 0$ , contrary to (2). Therefore,  $\alpha(t)$  has a strong tangent.

□

*Proof of (b).*

- (1) The line  $L$  passing  $\alpha(t_0 + h)$  and  $\alpha(t_0 + k)$  is

$$\begin{aligned} x(s) &= x(t_0) + \frac{x(t_0 + h) - x(t_0 + k)}{h - k}s, \\ y(s) &= y(t_0) + \frac{y(t_0 + h) - y(t_0 + k)}{h - k}s, \\ z(s) &= z(t_0) + \frac{z(t_0 + h) - z(t_0 + k)}{h - k}s. \end{aligned}$$

- (2) By the mean value theorem,

$$\frac{x(t_0 + h) - x(t_0 + k)}{h - k} = x'(t_0 + \xi)$$

for some  $\xi$  between  $h$  and  $k$ . Since  $\alpha \in \mathcal{C}^1$ ,  $x(t) \in \mathcal{C}^1$ . Hence

$$\begin{aligned} \lim_{h,k \rightarrow 0} \frac{x(t_0+h) - x(t_0+k)}{h-k} &= \lim_{h,k \rightarrow 0} x'(t_0 + \xi) \\ &= \lim_{\xi \rightarrow 0} x'(t_0 + \xi) \\ &= x'(t_0). \end{aligned}$$

Similarly, we have  $\lim_{h,k \rightarrow 0} \frac{y(t_0+h) - y(t_0+k)}{h-k} = y'(t_0)$  and  $\lim_{h,k \rightarrow 0} \frac{z(t_0+h) - z(t_0+k)}{h-k} = z'(t_0)$ . Since  $\alpha$  is regular,  $\lim_{h,k \rightarrow 0} L$  is a non degenerate line

$$\begin{aligned} x(s) &= x(t_0) + x'(t_0)s, \\ y(s) &= y(t_0) + y'(t_0)s, \\ z(s) &= z(t_0) + z'(t_0)s \end{aligned}$$

and thus  $\lim_{h,k \rightarrow 0} L$  is a strong tangent at  $t = t_0$ .

□

*Proof of (c).*

(1) Since

$$\alpha'(t) = \begin{cases} (2t, 2t), & t \geq 0, \\ (2t, -2t), & t \leq 0, \end{cases}$$

$\alpha$  is of class  $\mathcal{C}^1$ .

(2) Since

$$\alpha''(t) = \begin{cases} (2, 2), & t > 0, \\ \text{undefined}, & t = 0 \\ (2, -2), & t < 0, \end{cases}$$

$\alpha$  is not of class  $\mathcal{C}^2$ .

(Skip drawing a sketch of the curve and its tangent vectors.) □

### Exercise 1-3.8.

Let  $\alpha : I \rightarrow \mathbb{R}^3$  be a differentiable curve and let  $[a, b] \subseteq I$  be a closed interval. For every partition

$$a = t_0 < t_1 < \cdots < t_n = b$$

of  $[a, b]$ , consider the sum

$$\sum_{i=1}^n |\alpha(t_i) - \alpha(t_{i-1})| = l(\alpha, P),$$

where  $P$  stands for the given partition. The norm  $|P|$  of a partition  $P$  is defined as

$$|P| = \max(t_i - t_{i-1}), i = 1, \dots, n.$$

Geometrically,  $l(\alpha, P)$  is the length of a polygon inscribed in  $\alpha([a, b])$  with vertices in  $\alpha(t_i)$  (see Figure 1-3 in Mantredo P. do Carmo, *Differential Geometry of Curves and Surfaces*). The point of the exercise is to show that the arc length of  $\alpha([a, b])$  is, in some sense, a limit of lengths of inscribed polygons. Prove that given  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $|P| < \delta$  then

$$\left| \int_a^b |\alpha'(t)| dt - l(\alpha, P) \right| < \varepsilon.$$

Assume that  $\alpha'(t)$  is continuous.

*Proof.* Given  $\varepsilon > 0$ .

- (1) Since  $\alpha'(t)$  is continuous on a compact set  $[a, b]$ ,  $\alpha'(t)$  is uniformly continuous, that is, there there exists  $\delta > 0$  such that

$$|\alpha'(s) - \alpha'(t)| < \frac{\varepsilon}{2(b-a)} \text{ whenever } |s - t| < \delta.$$

- (2) Let  $P = \{a = t_0, t_1, \dots, t_n = b\}$  be a partition of  $[a, b]$ , with  $\Delta t_i = t_i - t_{i-1} < \delta$  for all  $i = 1, \dots, n$ . If  $t_{i-1} \leq t \leq t_i$ , it follows that

$$|\alpha'(t_i)| - \frac{\varepsilon}{2(b-a)} \leq |\alpha'(t)| \leq |\alpha'(t_i)| + \frac{\varepsilon}{2(b-a)}.$$

Hence,

$$\begin{aligned} & \int_{t_{i-1}}^{t_i} |\alpha'(t)| dt \\ & \geq |\alpha'(t_i)| \Delta t_i - \frac{\varepsilon}{2(b-a)} \Delta t_i \\ & = \left| \int_{t_{i-1}}^{t_i} [\alpha'(t) + \alpha'(t_i) - \alpha'(t)] dt \right| - \frac{\varepsilon}{2(b-a)} \Delta t_i \\ & \geq \left| \int_{t_{i-1}}^{t_i} \alpha'(t) dt \right| - \left| \int_{t_{i-1}}^{t_i} [\alpha'(t_i) - \alpha'(t)] dt \right| - \frac{\varepsilon}{2(b-a)} \Delta t_i \\ & \geq |\alpha(t_i) - \alpha(t_{i-1})| - \frac{\varepsilon}{b-a} \Delta t_i \end{aligned}$$

and

$$\begin{aligned}
& \int_{t_{i-1}}^{t_i} |\alpha'(t)| dt \\
& \leq |\alpha'(t_i)| \Delta t_i + \frac{\varepsilon}{2(b-a)} \Delta t_i \\
& = \left| \int_{t_{i-1}}^{t_i} [\alpha'(t) + \alpha'(t_i) - \alpha'(t)] dt \right| + \frac{\varepsilon}{2(b-a)} \Delta t_i \\
& \leq \left| \int_{t_{i-1}}^{t_i} \alpha'(t) dt \right| + \left| \int_{t_{i-1}}^{t_i} [\alpha'(t_i) - \alpha'(t)] dt \right| + \frac{\varepsilon}{2(b-a)} \Delta t_i \\
& \leq |\alpha(t_i) - \alpha(t_{i-1})| + \frac{\varepsilon}{b-a} \Delta t_i.
\end{aligned}$$

(3) If we add these inequalities, we obtain

$$l(\alpha, P) - \varepsilon \leq \int_a^b |\alpha'(t)| dt \leq l(\alpha, P) + \varepsilon.$$

□

### Exercise 1-3.9.

- (a) Let  $\alpha : I \rightarrow \mathbb{R}^3$  be a curve of class  $\mathcal{C}^0$  (compare Exercise 1-3.7). Use the approximation by polygons described in Exercise 1-3.8 to give a reasonable definition of arc length of  $\alpha$ .
- (b) (A Nonrectifiable Curve.) The following example shows that, with any reasonable definition, the arc length of a  $\mathcal{C}^0$  curve in a closed interval may be unbounded. Let  $\alpha : [0, 1] \rightarrow \mathbb{R}^2$  be given as  $\alpha(t) = (t, t \sin(\frac{\pi}{t}))$  if  $t \neq 0$ , and  $\alpha(0) = (0, 0)$ . Show, geometrically, that the arc length of the portion of the curve corresponding to  $\frac{1}{n+1} \leq t \leq \frac{1}{n}$  is at least  $\frac{2}{n+\frac{1}{2}}$ . Use this to show that the length of curve in the interval  $\frac{1}{N} \leq t \leq 1$  is greater than  $2 \sum_{n=1}^{N-1} \frac{1}{n+1}$ , and thus it tends to infinity as  $N \rightarrow \infty$ .

*Proof of (a).* Define

$$l(\alpha) = \sup \{ l(\alpha, P) : P \text{ is a partition of } [a, b] \}.$$

□

*Note.* (Theorem 6.17 in Tom. M. Apostol, *Mathematical Analysis*, 2nd edition.).  $\alpha$  is rectifiable if and only if  $\alpha$  is of bounded variation on  $[a, b]$ .

*Proof of (b).*



(1) Consider a partition  $P = \left\{ \frac{1}{n+1}, \frac{1}{n+\frac{1}{2}}, \frac{1}{n} \right\}$  of  $\left[ \frac{1}{n+1}, \frac{1}{n} \right]$ . So that  $\alpha(\frac{1}{n+1}) = \alpha(\frac{1}{n}) = 0$  and  $\alpha(\frac{1}{n+\frac{1}{2}}) = \pm 1$ .

(2) Thus,

$$\begin{aligned} & \text{The arc length of the portion of } \alpha \text{ over } \left[ \frac{1}{n+1}, \frac{1}{n} \right] \\ & \geq \text{The sum of each length of the individual chords} \\ & = \sqrt{\left( \frac{1}{n+\frac{1}{2}} - \frac{1}{n+1} \right)^2 + \left( \frac{1}{n+\frac{1}{2}} \right)^2} \\ & \quad + \sqrt{\left( \frac{1}{n} - \frac{1}{n+\frac{1}{2}} \right)^2 + \left( \frac{1}{n+\frac{1}{2}} \right)^2} \\ & \geq \frac{2}{n+\frac{1}{2}}. \end{aligned}$$

(3) So

$$\begin{aligned} & \text{The arc length of } \alpha \text{ over } \left[ \frac{1}{N}, 1 \right] \\ & = \sum_{n=1}^{N-1} \left\{ \text{The arc length of } \alpha \text{ over } \left[ \frac{1}{n+1}, \frac{1}{n} \right] \right\} \\ & \geq \sum_{n=1}^{N-1} \frac{2}{n+\frac{1}{2}} \\ & > 2 \sum_{n=1}^{N-1} \frac{1}{n+1}. \end{aligned}$$

It tends to infinity as  $N \rightarrow \infty$ , or  $\alpha$  is nonrectifiable.

□

### Exercise 1-3.10. (Straight Lines as Shortest)

Let  $\alpha : I \rightarrow \mathbb{R}^3$  be a parametrized curve. Let  $[a, b] \subseteq I$  and set  $\alpha(a) = p$ ,  $\alpha(b) = q$ .

(a) Show that, for any constant vector  $v$ ,  $|v| = 1$ ,

$$(q - p) \cdot v = \int_a^b \alpha'(t) \cdot v dt \leq \int_a^b |\alpha'(t)| dt.$$

(b) Set

$$v = \frac{q-p}{|q-p|}$$

and show that

$$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)| dt;$$

that is, the curve of shortest length from  $\alpha(a)$  to  $\alpha(b)$  is the straight line joining these points.

Assume  $p \neq q$  (otherwise  $v = \frac{q-p}{|q-p|}$  is meaningless).

*Proof of (a).* Let  $f(t) = \alpha(t) \cdot v$  defined on  $I$ . By the fundamental theorem of calculus,

$$\int_a^b f'(t) dt = f(b) - f(a).$$

Since  $f'(t) = \alpha'(t) \cdot v$ ,

$$(\alpha(b) - \alpha(a)) \cdot v = \int_a^b \alpha'(t) \cdot v dt.$$

Therefore,

$$\begin{aligned} (q-p) \cdot v &= \int_a^b \alpha'(t) \cdot v dt \\ &\leq \int_a^b |\alpha'(t) \cdot v| dt \\ &\leq \int_a^b |\alpha'(t)| |v| dt \\ &= \int_a^b |\alpha'(t)| dt. \end{aligned}$$

□

*Proof of (b).*  $|v| = \frac{|q-p|}{|q-p|} = 1$ . So,

$$\begin{aligned} (q-p) \cdot \frac{q-p}{|q-p|} &\leq \int_a^b |\alpha'(t)| dt, \\ |q-p| &\leq \int_a^b |\alpha'(t)| dt. \end{aligned}$$

□

## 1-4. The Vector Product in $\mathbb{R}^3$

### Exercise 1-4.1.

Check whether the following bases are positive:

- (a) The basis  $\{(1, 3), (4, 2)\}$  in  $\mathbb{R}^2$ .
- (b) The basis  $\{(1, 3, 5), (2, 3, 7), (4, 8, 3)\}$  in  $\mathbb{R}^3$ .

*Proof of (a).* Write  $u = (1, 3)$  and  $v = (4, 2)$ . Then

$$\det(u, v) = \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = -10 < 0.$$

Thus  $\{u, v\}$  is negative w.r.t. the natural order basis  $\{e_1 = (1, 0), e_2 = (0, 1)\}$ .  $\square$

*Proof of (b).* Write  $u = (1, 3, 5)$ ,  $v = (2, 3, 7)$ ,  $w = (4, 8, 3)$ . Then

$$\det(u, v, w) = \begin{vmatrix} 1 & 3 & 5 \\ 2 & 3 & 7 \\ 4 & 8 & 3 \end{vmatrix} = 39 > 0.$$

Thus  $\{u, v, w\}$  is positive w.r.t. the natural order basis  $\{e_1, e_2, e_3\}$ .  $\square$

### Exercise 1-4.2.

A plane  $P$  contained in  $\mathbb{R}^3$  is given by the equation  $ax+by+cz+d=0$ . Show that the vector  $v = (a, b, c)$  is perpendicular to the plane and that  $|d|/\sqrt{a^2+b^2+c^2}$  measures the distance from the plane to the origin  $(0, 0, 0)$ .

Say  $v$  is a normal vector of  $E$ .

In general, the distance from the plane  $E$  to any point  $(x_0, y_0, z_0) \in \mathbb{R}^3$  is

$$\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

*Proof.*

- (1) To show  $v = (a, b, c)$  is perpendicular to the plane, it suffices to show that  $v \cdot u = 0$  for any vector  $u$  lying on the plane  $E$ . Write  $u = \overrightarrow{PQ}$  where  $P = (x_1, y_1, z_1) \in E$  and  $Q = (x_2, y_2, z_2) \in E$ . Hence  $u = (x_2 - x_1, y_2 -$

$$y_1, z_2 - z_1).$$

$$\begin{aligned} v \cdot u &= (a, b, c) \cdot (x_2 - x_1, y_2 - y_1, z_2 - z_1) \\ &= a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) \\ &= (ax_2 + by_2 + cz_2) - (ax_1 + by_1 + cz_1) \\ &= (-d) - (-d) \\ &= 0. \end{aligned}$$

- (2) Pick any point  $(x_1, y_1, z_1) \in E$ . The distance from the plane  $E$  to the point  $(x_0, y_0, z_0)$  is

$$\begin{aligned} & \left| (x_1 - x_0, y_1 - y_0, z_1 - z_0) \cdot \frac{v}{|v|} \right| \\ &= \left| (x_1 - x_0, y_1 - y_0, z_1 - z_0) \cdot \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}} \right| \\ &= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|-d - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}. \end{aligned}$$

□

### Exercise 1-4.3.

Determine the angle of intersection of the two planes  $5x + 3y + 2z - 4 = 0$  and  $3x + 4y - 7z = 0$ .

*Proof.*

- (1) The angle of intersection of the two planes is equal to a angle between two normal vectors of planes.
- (2) Let
  - (a) the angle of intersection of the two planes be  $\theta$ .
  - (b) the normal vector of  $5x + 3y + 2z - 4 = 0$  be  $n_1 = (5, 3, 2)$ .
  - (c) the normal vector of  $3x + 4y - 7z = 0$  be  $n_2 = (3, 4, -7)$ .

(3) Hence,

$$\cos \theta = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{13}{2\sqrt{703}}.$$

$$\theta = \cos^{-1} \left( \frac{13}{2\sqrt{703}} \right).$$

□

#### Exercise 1-4.4.

Given two planes  $a_ix + b_iy + c_iz + d_i = 0$ ,  $i = 1, 2$ , prove that a necessary and sufficient condition for them to be parallel is

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2},$$

where the convention is made that if a denominator is zero, the corresponding numerator is also zero (we say that two planes are parallel if they either coincide or do not intersect).

*Proof.*

(1) Write

$$E_i : a_ix + b_iy + c_iz + d_i = 0.$$

By Exercise 1-4.2, the vector  $(a_i, b_i, c_i)$  is perpendicular to the plane  $E_i$ .

(2) Hence,

$$E_1 \text{ is parallel to } E_2 \iff (a_1, b_1, c_1) \text{ is parallel to } (a_2, b_2, c_2)$$

$$\iff \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

(where the convention is made that if a denominator is zero, the corresponding numerator is also zero).

□

#### Exercise 1-4.5.

Show that the equation of a plane passing through three noncolinear points  $p_i = (x_i, y_i, z_i)$ ,  $i = 1, 2, 3$  is given by

$$(p - p_1) \wedge (p - p_2) \cdot (p - p_3) = 0.$$

where  $p = (x, y, z)$  is an arbitrary point of the plane and  $p - p_1$ , for instance, means the vector  $(x - x_1, y - y_1, z - z_1)$ .

*Proof.*

- (1) By Exercise 1-4.11(a), the volume  $V$  of a parallelepiped generated by  $p - p_1, p - p_2, p - p_3 \in \mathbb{R}^3$  is given by

$$V = (p - p_1) \wedge (p - p_2) \cdot (p - p_3).$$

- (2) Since all vectors  $p - p_1, p - p_2, p - p_3$  are lying on the same plane,  $V = 0$ . Therefore, the equation of a plane is  $(p - p_1) \wedge (p - p_2) \cdot (p - p_3) = 0$ .

□

#### Exercise 1-4.6.

Given two nonparallel planes  $a_i x + b_i y + c_i z + d_i = 0$ ,  $i = 1, 2$ , show that their line of intersection may be parametrized as

$$x - x_0 = u_1 t, \quad y - y_0 = u_2 t, \quad z - z_0 = u_3 t,$$

where  $(x_0, y_0, z_0)$  belongs to the intersection and  $u = (u_1, u_2, u_3)$  is the vector product  $u = v_1 \wedge v_2$ ,  $v_i = (a_i, b_i, c_i)$ ,  $i = 1, 2$ .

*Proof.*

- (1) Suppose that the line of intersection is

$$L : x - x_0 = u'_1 t, \quad y - y_0 = u'_2 t, \quad z - z_0 = u'_3 t$$

where  $(x_0, y_0, z_0)$  belongs to the intersection.

- (2) By Exercise 1-4.2, the vector  $v_i = (a_i, b_i, c_i)$  is perpendicular to the plane  $E_i : a_i x + b_i y + c_i z + d_i = 0$ . Hence  $v_i$  is perpendicular to  $u' = (u'_1, u'_2, u'_3)$ . Since  $v_i \cdot (v_1 \wedge v_2) = 0$ , we may choose  $u' = v_1 \wedge v_2 = u$ .

□

#### Exercise 1-4.7.

Prove that a necessary and sufficient condition for the plane

$$ax + by + cz + d = 0$$

and the line

$$x - x_0 = u_1 t, \quad y - y_0 = u_2 t, \quad z - z_0 = u_3 t$$

to be parallel is

$$au_1 + bu_2 + cu_3 = 0.$$

*Proof.* Write

$$\begin{aligned} E : ax + by + cz + d &= 0 \\ L : x - x_0 &= u_1 t, \quad y - y_0 = u_2 t, \quad z - z_0 = u_3 t. \end{aligned}$$

By Exercise 1-4.2, the vector  $(a, b, c)$  is perpendicular to the plane  $E$ . Hence,

$$\begin{aligned} E \text{ is parallel to } L &\iff (a, b, c) \text{ is perpendicular to } L \\ &\iff (a, b, c) \text{ is perpendicular to } u = (u_1, u_2, u_3) \\ &\iff 0 = (a, b, c) \cdot (u_1, u_2, u_3) = au_1 + bu_2 + cu_3. \end{aligned}$$

□

#### Exercise 1-4.8.

*Prove that the distance  $\rho$  between the nonparallel lines*

$$\begin{aligned} x - x_0 &= u_1 t, & y - y_0 &= u_2 t, & z - z_0 &= u_3 t, \\ x - x_0 &= v_1 t, & y - y_0 &= v_2 t, & z - z_0 &= v_3 t \end{aligned}$$

*is given by*

$$\rho = \frac{|(u \wedge v) \cdot r|}{|u \wedge v|}$$

*where  $u = (u_1, u_2, u_3)$ ,  $v = (v_1, v_2, v_3)$ ,  $r = (x_0 - x_1, y_0 - y_1, z_0 - z_1)$ .*

*Proof.*

$$\rho = |r| |\cos \angle(u \wedge v, r)| = |r| \frac{|(u \wedge v) \cdot r|}{|u \wedge v| |r|} = \frac{|(u \wedge v) \cdot r|}{|u \wedge v|}.$$

It is well-defined ( $|u \wedge v| > 0$ ) since two lines are nonparallel. □

#### Exercise 1-4.9.

*Determine the angle of intersection of the plane  $3x + 4y + 7z + 8 = 0$  and the line  $x - 2 = 3t$ ,  $y - 3 = 5t$ ,  $z - 5 = 9t$ .*

*Proof.*

- (1) The angle of intersection is equal to  $\pi/2$  minus an acute angle  $\theta$  between the normal vector of the plane and the direction vector of a line.
- (2)

$$\cos \theta = \frac{(3, 4, 7) \cdot (3, 5, 9)}{|(3, 4, 7)| |(3, 5, 9)|} = \frac{92}{\sqrt{74} \sqrt{115}}.$$

Hence, the angle of intersection is

$$\pi/2 - \arccos\left(\frac{92}{\sqrt{74}\sqrt{115}}\right).$$

□

#### Exercise 1-4.10. (Oriented area)

The natural orientation of  $\mathbb{R}^2$  makes it possible to associate a sign to the area  $A$  of a parallelogram generated by two linearly independent vectors  $u, v \in \mathbb{R}^2$ , and write  $u = u_1e_1 + u_2e_2$ ,  $v = v_1e_1 + v_2e_2$ . Observe the matrix relation

$$\begin{bmatrix} u \cdot u & u \cdot v \\ v \cdot u & v \cdot v \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$$

and conclude that

$$A^2 = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2.$$

Since the last determinant has the same sign as the basis  $\{u, v\}$ , we can say that  $A$  is positive or negative according to whether the orientation of  $\{u, v\}$  is positive or negative. This is called the **oriented area** in  $\mathbb{R}^2$ .

*Proof.*

$$\begin{aligned} A^2 &= \begin{vmatrix} u \cdot u & u \cdot v \\ v \cdot u & v \cdot v \end{vmatrix} \\ &= \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \\ &= \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2 \end{aligned}$$

since  $\det A^T = \det A$ . □

#### Exercise 1-4.11. (Oriented volume)

(a) Show that the volume  $V$  of a parallelepiped generated by three linearly independent vectors  $u, v, w \in \mathbb{R}^3$  is given by  $V = |(u \wedge v) \cdot w|$ , and introduce an **oriented volume** in  $\mathbb{R}^3$ .

(b) Prove that

$$V^2 = \begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix}.$$



*Proof of (a).*

- (1) We can calculate the volume  $V$  by multiplying the area of the base  $|u \wedge v|$  and the height  $h$ .
- (2) Note that

$$h = |w| |\cos \angle(u \wedge v, w)| = |w| \frac{|(u \wedge v) \cdot w|}{|u \wedge v| |w|} = \frac{|(u \wedge v) \cdot w|}{|u \wedge v|}.$$

It is well-defined since  $u$  and  $v$  are linearly independent. Therefore,

$$V = |u \wedge v| h = |(u \wedge v) \cdot w|.$$

- (3) The oriented volume is defined by

$$(u \wedge v) \cdot w.$$

□

*Proof of (b).* Recall  $(u \wedge v) \cdot w = \det(u, v, w)$ . Also note that  $\det((u, v, w)^T) = \det(u, v, w)$ . So

$$\begin{aligned} \begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix} &= \det((u, v, w) \cdot (u, v, w)^T) \\ &= \det(u, v, w) \det((u, v, w)^T) \\ &= \det(u, v, w)^2 \\ &= ((u \wedge v) \cdot w)^2 \\ &= V^2. \end{aligned}$$

□

#### **Exercise 1-4.12.**

Given the vectors  $v \neq 0$  and  $w$ , show that there exists a vector  $u$  such that  $u \wedge v = w$  if and only if  $v$  is perpendicular to  $w$ . Is this vector  $u$  uniquely determined? If not, what is the most general solution?

*Proof.*

- (1) Suppose that there exists a vector  $u$  such that  $u \wedge v = w$ . So

$$v \cdot w = w \cdot v = (u \wedge v) \cdot v = 0$$

or  $v$  is perpendicular to  $w$ .

- (2) Suppose that  $v$  is perpendicular to  $w$ . Linear algebra says that there exists a vector  $u$  such that  $\{\tilde{u}, v, w\}$  is a basis of  $\mathbb{R}^3$ . Note that  $\tilde{u} \wedge v$  is parallel to  $w$ , say  $\tilde{u} \wedge v = cw$  for some nonzero constant  $c \in \mathbb{R}$ . Take  $u = c^{-1}\tilde{u}$  and thus

$$u \wedge v = c^{-1}\tilde{u} \wedge v = c^{-1}cw = w.$$

(Note that  $\{u, v, w\}$  is also a basis of  $\mathbb{R}^3$ .)

- (3) Such vector  $u$  is not uniquely determined. Let  $L$  be a line passing the point  $u = (u_1, u_2, u_3)$  and parallel to the vector  $v$ . By the definition of vector product, for any point  $p = (p_1, p_2, p_3) \in L$  we have  $p \wedge v = w$  as a vector.

□

#### Exercise 1-4.13.

Let  $u(t) = (u_1(t), u_2(t), u_3(t))$  and  $v(t) = (v_1(t), v_2(t), v_3(t))$  be differentiable maps from the interval  $(a, b)$  into  $\mathbb{R}^3$ . If the derivatives  $u'(t)$  and  $v'(t)$  satisfy the conditions

$$u'(t) = au(t) + bv(t), \quad v'(t) = cu(t) - av(t),$$

where  $a$ ,  $b$ , and  $c$  are constants, show that  $u(t) \wedge v(t)$  is a constant vector.

*Proof.* Since

$$\begin{aligned} \frac{d}{dt}(u(t) \wedge v(t)) &= u'(t) \wedge v(t) + u(t) \wedge v'(t) \\ &= (au(t) + bv(t)) \wedge v(t) + u(t) \wedge (cu(t) - av(t)) \\ &= au(t) \wedge v(t) + u(t) \wedge (-av(t)) \\ &= a(u(t) \wedge v(t)) + (-a)(u(t) \wedge v(t)) \\ &= (0, 0, 0), \end{aligned}$$

$u(t) \wedge v(t)$  is a constant vector. □

#### Exercise 1-4.14.

Find all unit vectors which are perpendicular to the vector  $(2, 2, 1)$  and parallel to the plane determined by the points  $(0, 0, 0)$ ,  $(1, -2, 1)$ ,  $(-1, 1, 1)$ .

*Proof.*

- (1) Let  $E$  be the plane determined by the points  $p_0 = (0, 0, 0)$ ,  $p_1 = (1, -2, 1)$ ,  $p_2 = (-1, 1, 1)$ . The normal vector of  $E$  is

$$(p_1 - p_0) \wedge (p_2 - p_0) = (1, -2, 1) \wedge (-1, 1, 1) = (-3, -2, -1).$$

- (2) All unit vectors which are perpendicular to  $(2, 2, 1)$  and parallel to  $E$  are all unit vectors which are perpendicular to  $(2, 2, 1)$  and  $(-3, -2, -1)$ . Note that such vector is in the direction of

$$(2, 2, 1) \wedge (-3, -2, -1) = (0, -1, 2).$$

Hence, the desired unit vectors are

$$\pm \left( 0, -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right).$$

□

## 1-5. The Local Theory of Curves Parametrized by Arc Length

### Exercise 1-5.2.

Show that the torsion  $\tau$  of  $\alpha$  is given by

$$\tau(s) = -\frac{\alpha'(s) \wedge \alpha''(s) \cdot \alpha'''(s)}{|\kappa(s)|^2}.$$

*Proof.*

- (1) Take inner product  $n(s)$  to the definition of torsion  $\tau(s)n(s) = b'(s)$ , we have

$$\tau(s) = b'(s) \cdot n(s).$$

Since  $b'(s) = t(s) \wedge n'(s)$ , we have to compute  $n'(s)$  first.

- (2) Compute  $n'(s)$ .

$$n'(s) = \frac{d}{ds} \left( \frac{\alpha''(s)}{\kappa(s)} \right) = \frac{\alpha'''(s)}{\kappa(s)} - \frac{\alpha''(s)\kappa'(s)}{\kappa(s)^2}.$$

(3) By (1)(2),

$$\begin{aligned}
\tau(s) &= b'(s) \cdot n(s) \\
&= (t(s) \wedge n'(s)) \cdot n(s) \\
&= \left( \alpha'(s) \wedge \left( \frac{\alpha'''(s)}{\kappa(s)} - \frac{\alpha''(s)\kappa'(s)}{\kappa(s)^2} \right) \right) \cdot \frac{\alpha''(s)}{\kappa(s)} \\
&= \left( \alpha'(s) \wedge \frac{\alpha'''(s)}{\kappa(s)} \right) \cdot \frac{\alpha''(s)}{\kappa(s)} \\
&= \frac{\alpha'(s) \wedge \alpha'''(s) \cdot \alpha''(s)}{|\kappa(s)|^2},
\end{aligned}$$

or

$$\tau(s) = \frac{\alpha'(s) \wedge \alpha'''(s) \cdot \alpha''(s)}{\alpha''(s)^2}.$$

□

## 1-6. The Local Canonical Form

## 1-7. Global Properties of Plane Curves