

Chapter 1: Rings and Ideals

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Exercise 1.1. *Let x be a nilpotent element of A . Show that $1 + x$ is a unit of A . Deduce that the sum of a nilpotent element and a unit is a unit.*

Proof.

- (1) Suppose $x^m = 0$ for some odd integer $m \geq 0$. Then

$$1 = 1 + x^m = (1 + x)(1 - x + x^2 - \cdots + (-1)^{m-1}x^{m-1}),$$

or $1 + x$ is a unit.

- (2) If u is any unit and x is any nilpotent, $u + x = u \cdot (1 + u^{-1}x)$ is a product of two units (using that $u^{-1}x$ is nilpotent and applying (1)) and hence a unit again.

□

Proof (Proposition 1.9).

- (1) *The nilradical is a subset of the Jacobson radical.*

(a) The nilradical \mathfrak{N} of A is the intersection of all the prime ideals of A by Proposition 1.8.

(b) The Jacobson radical \mathfrak{J} of A is the intersection of all the maximal ideals of A by definition.

- (2) By Proposition 1.9, $x \in \mathfrak{J}$ if and only if $1 - xy$ is a unit in A for all $y \in A$. So $1 + x = 1 - (-x) \cdot 1$ is a unit in A since x is a nilpotent and \mathfrak{J} is an ideal.

□

Exercise 1.2. *Let A be a ring and let $A[x]$ be the ring of polynomials in an indeterminate x , with coefficients in A . Let $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$. Prove that*

- (i) *f is a unit in $A[x]$ if and only if a_0 is a unit in A and a_1, \dots, a_n are nilpotent. (Hint: If $b_0 + b_1x + \cdots + b_mx^m$ is the inverse of f , prove by induction on r that $a_n^{r+1}b_{m-r} = 0$. Hence show that a_n is nilpotent, and then use Exercise 1.1.)*

- (ii) f is nilpotent if and only if a_0, a_1, \dots, a_n are nilpotent.
- (iii) f is a zero-divisor if and only if there exists $a \neq 0$ such that $af = 0$. (Hint: Choose a polynomial $g = b_0 + b_1x + \dots + b_mx^m$ of least degree m such that $fg = 0$. Then $a_nb_m = 0$, hence $a_ng = 0$ (because a_ng annihilates f and has degree $< m$). Now show by induction that $a_{n-r}g = 0$ ($0 \leq r \leq n$).)
- (iv) f is said to be primitive if $(a_0, a_1, \dots, a_n) = (1)$. Prove that if $f, g \in A[x]$, then fg is primitive if and only if f and g are primitive.

Proof of (i).

- (1) (\Leftarrow) holds by Exercise 1.1.
- (2) (\Rightarrow) There exists the inverse g of f , say $g = b_0 + b_1x + \dots + b_mx^m$ satisfying $1 = fg$. Clearly, $1 = a_0b_0$, or a_0 is a unit in A . Also,

$$\begin{aligned} 0 &= a_nb_m, \\ 0 &= a_nb_{m-1} + a_{n-1}b_m, \\ 0 &= a_nb_{m-2} + a_{n-1}b_{m-1} + a_{n-2}b_m, \\ &\dots \end{aligned}$$

A direct computing shows that

$$\begin{aligned} 0 &= a_n^1b_m, \\ 0 &= a_n(a_nb_{m-1} + a_{n-1}b_m) \\ &= a_n^2b_{m-1} + a_{n-1}a_nb_m \\ &= a_n^2b_{m-1}, \\ 0 &= a_n^2(a_nb_{m-2} + a_{n-1}b_{m-1} + a_{n-2}b_m) \\ &= a_n^3b_{m-2} + a_{n-1}a_n^2b_{m-1} + a_{n-2}a_n^2b_m \\ &= a_n^3b_{m-2}, \\ &\dots \end{aligned}$$

So we might have $a_n^{r+1}b_{m-r} = 0$ for $r = 0, 1, 2, \dots, m$.

- (3) Show that $a_n^{r+1}b_{m-r} = 0$ for $r = 0, 1, 2, \dots, m$ by induction on r .
 - (a) As $r = 0$, $a_nb_m = 0$ by comparing the coefficient of $fg = 1$ at x^{n+m} .
 - (b) For any $r > 0$, comparing the coefficient of $fg = 1$ at x^{n+m-r} ,

$$0 = a_nb_{m-r} + a_{n-1}b_{m-r+1} + \dots + a_{n-r}b_m.$$

Multiplying by a_n^r on the both sides,

$$\begin{aligned} 0 &= a_n^{r+1}b_{m-r} + a_{n-1}a_n^rb_{m-r+1} + \dots + a_{n-r}a_n^rb_m \\ &= a_n^{r+1}b_{m-r}. \end{aligned}$$

by the induction hypothesis.

- (4) a_n is a nilpotent. Putting $r = m$ in $a_n^{r+1}b_{m-r} = 0$ and get $a_n^{m+1}b_0 = 0$. Notice that b_0 is a unit, $a_n^{m+1} = 0$, or a_n is a nilpotent.
- (5) Consider $f - a_n x^n = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$, a polynomial $\in A[x]$ of degree $n-1$. Note that f is a unit and $a_n x^n$ is a nilpotent. By Exercise 1.1, $f - a_n x^n$ is a unit too. Applying the (2)(3)(4) again, a_{n-1} is a nilpotent as $n-1 > 0$, that is, applying descending induction on n then yields the desired property.

□

Proof of (ii).

- (1) (\Leftarrow) holds since the nilradical of any ring is an ideal.
- (2) (\Rightarrow) $f^N = 0$ for some $N > 0$. So $0 = f^N = a_n^N x^{nN} + \cdots + a_0^N$. Comparing the coefficient in the leading term x^{nN} leads to $a_n^N = 0$, or a_n is a nilpotent.
- (3) Consider $f - a_n x^n = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$, a polynomial $\in A[x]$ of degree $n-1$. Note that f and $a_n x^n$ are nilpotent. $f - a_n x^n$ is a nilpotent too. Similar to step (5) in the proof of (i), applying descending induction on n then yields the desired property.

□

Proof of (iii).

- (1) (\Leftarrow) holds trivially.
- (2) (\Rightarrow) Pick a polynomial $g = b_0 + b_1 x + \cdots + b_m x^m$ of least degree m such that $fg = 0$. Especially, $a_n b_m = 0$.
- (3) Consider

$$\begin{aligned} a_n g &= a_n b_0 + \cdots + a_n b_{m-1} x^{m-1} + a_n b_m x^m \\ &= a_n b_0 + \cdots + a_n b_{m-1} x^{m-1} \end{aligned}$$

(since $a_n b_m = 0$). $a_n g$ is a polynomial over A of having degree strictly less than m . Notice that $f \cdot (a_n g) = a_n \cdot (fg) = 0$. By minimality of m , $a_n g = 0$.

- (4) Induction on the degree n of f .
- (a) As $n = 0$, $f = a_0$. There exists $b_m \neq 0$ such that $b_m f = b_m a_0 = 0$ by (2).
- (b) For any zero-divisor f of degree n , there is a polynomial $g = b_0 + b_1 x + \cdots + b_m x^m$ of least degree m such that $fg = 0$. By (2)(3),

$$\begin{aligned} (f - a_n x^n) \cdot g &= fg - a_n x^n g \\ &= 0 - 0 \\ &= 0. \end{aligned}$$

That is, $f - a_n x^n$ is a zero-divisor of degree $n - 1$. By the induction hypothesis, there exists $b_m \neq 0$ such that $b_m(f - a_n x^n) = 0$. So $b_m f = b_m(f - a_n x^n) + b_m a_n x^n = 0 + 0 = 0$.

(c) By (a)(b), (\implies) holds by mathematical induction.

□

Proof of (iv). Note that

- (1) $f \notin \mathfrak{m}[x]$ for any maximal ideal \mathfrak{m} of A if and only if f is primitive.
- (2) For any maximal ideal \mathfrak{m} of A , A/\mathfrak{m} is a field (or an integral domain).
- (3) $A[x]$ is an integral domain if A is an integral domain.
- (4) $A[x]/\mathfrak{m}[x] \cong (A/\mathfrak{m})[x]$ as a ring isomorphism.

Hence,

$$\begin{aligned}
 f, g : \text{primitive} &\iff f, g \notin \mathfrak{m}[x] \text{ for any maximal ideal } \mathfrak{m} \\
 &\iff f, g \neq 0 \text{ in } (A/\mathfrak{m})[x] \text{ for any maximal ideal } \mathfrak{m} \\
 &\iff fg \neq 0 \text{ in } (A/\mathfrak{m})[x] \text{ for any maximal ideal } \mathfrak{m} \\
 &\iff fg \notin \mathfrak{m}[x] \text{ for any maximal ideal } \mathfrak{m} \\
 &\iff fg : \text{primitive}.
 \end{aligned}$$

□

Exercise 1.4. *In the ring $A[x]$, the Jacobson radical is equal to the nilradical.*

Proof.

- (1) The nilradical \mathfrak{N} is a subset of the Jacobson radical \mathfrak{J} . It suffices to show that $\mathfrak{J} \subseteq \mathfrak{N}$.
- (2) Given any $f \in \mathfrak{J}$. By Proposition 1.9, $f \in \mathfrak{J}$ if and only if $1 - fy$ is a unit in $A[x]$ for all $y \in A[x]$. Especially, pick $y = x \in A[x]$ and then $1 - xf$ is a unit in $A[x]$.
- (3) By Exercise 1.2 (i), all coefficients of f are nilpotent. By Exercise 1.2 (ii), f is nilpotent, or $f \in \mathfrak{N}$.

□

Exercise 1.7. *Let A be a ring in which every element satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal in A is maximal.*

Proof. It suffices to show that for any prime ideal \mathfrak{p} in A , A/\mathfrak{p} is a field.

- (1) Take any $0 \neq \bar{x} \in A/\mathfrak{p}$, which is represented by $x \in A - \mathfrak{p}$. By assumption there exists $n \geq 2$ such that $x^n = x$. So $\bar{x}^n = \bar{x}$ or $\bar{x}(\bar{x}^{n-1} - 1) = 0$.
- (2) Since \mathfrak{p} is prime, A/\mathfrak{p} is a integral domain. That is, $\bar{x} = 0$ (impossible) or $\bar{x}^{n-1} - 1 = 0$. Write $\bar{x} \cdot \bar{x}^{n-2} = 1$ in A/\mathfrak{p} . So \bar{x}^{n-2} is an inverse of $\bar{x} \neq 0$ in A/\mathfrak{p} , which implies that A/\mathfrak{p} is a field (since \bar{x} is arbitrary).
- (3) A/\mathfrak{p} is a field if and only if \mathfrak{p} is maximal.

□

The prime spectrum of a ring

Exercise 1.15. Let A be a ring and let X be the set of all prime ideals of A . For each subset E of A , let $V(E)$ denote the set of all prime ideals of A which contain E . Prove that

- (i) if \mathfrak{a} is the ideal generated by E , then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$.
- (ii) $V(0) = X$, $V(1) = \emptyset$.
- (iii) if $(E_i)_{i \in I}$ is any family of subsets of A , then

$$V\left(\bigcup_{i \in I} E_i\right) = \bigcap_{i \in I} V(E_i).$$

- (iv) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ for any ideals $\mathfrak{a}, \mathfrak{b}$ of A .

The results show that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. The resulting topology is called the Zariski topology. The topological space X is called the prime spectrum of A , and is written $\text{Spec}(A)$.

Note that if $E_1 \subseteq E_2$, then $V(E_1) \supseteq V(E_2)$.

Proof of (i).

- (1) Show that $V(E) = V(\mathfrak{a})$.
 - (a) Show that $V(E) \subseteq V(\mathfrak{a})$. Given any $\mathfrak{p} \in V(E)$, $\mathfrak{p} \supseteq E$. For any $a \in \mathfrak{a}$, since \mathfrak{a} is generated by E , we can write a as a finite sum $a = \sum \alpha\beta$ where $\alpha \in A$ and $\beta \in E$. Since $E \subseteq \mathfrak{p}$, all $\beta \in \mathfrak{p}$. Since \mathfrak{p} is an ideal, $a = \sum \alpha\beta \in \mathfrak{p}$. That is, $\mathfrak{p} \supseteq \mathfrak{a}$, or $\mathfrak{p} \in V(\mathfrak{a})$.
 - (b) $V(E) \supseteq V(\mathfrak{a})$ since $\mathfrak{a} \supseteq E$.
- (2) Show that $V(\mathfrak{a}) = V(r(\mathfrak{a}))$.

(a) Show that $V(\mathfrak{a}) \subseteq V(r(\mathfrak{a}))$. Given any $\mathfrak{p} \in V(\mathfrak{a})$,

$$\begin{aligned} \mathfrak{p} \in V(\mathfrak{a}) &\implies \mathfrak{p} \supseteq \mathfrak{a} \\ &\implies \mathfrak{p} \supseteq \text{the intersection of the primes ideals } \mathfrak{p} \supseteq \mathfrak{a} \\ &\implies \mathfrak{p} \supseteq r(\mathfrak{a}) \text{ (by Proposition 1.14)} \\ &\implies \mathfrak{p} \in V(r(\mathfrak{a})). \end{aligned}$$

(b) $V(\mathfrak{a}) \supseteq V(r(\mathfrak{a}))$ since $r(\mathfrak{a}) \supseteq \mathfrak{a}$.

□

Proof of (ii).

(1) $V(1) = \emptyset$ since no prime ideal contains 1 by definition.

(2) $V(0) = X$ since 0 is in every ideal (especially in every prime ideal).

□

Proof of (iii).

$$\begin{aligned} \mathfrak{p} \in V\left(\bigcup_{i \in I} E_i\right) &\iff \mathfrak{p} \supseteq \bigcup_{i \in I} E_i \\ &\iff \mathfrak{p} \supseteq E_i \text{ for all } i \in I \\ &\iff \mathfrak{p} \in V(E_i) \text{ for all } i \in I \\ &\iff \mathfrak{p} \in \bigcap_{i \in I} V(E_i). \end{aligned}$$

□

Lemma. For any $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$, $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$.

Proof of Lemma.

(1) If $\mathfrak{p} \supseteq \mathfrak{a}$. We are done.

(2) If $\mathfrak{p} \not\supseteq \mathfrak{a}$, there exists $a \in \mathfrak{a} - \mathfrak{p}$. So for any $b \in \mathfrak{b}$, $b \in \mathfrak{p}$ since $ab \in \mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$ and \mathfrak{p} is a prime ideal, that is, $\mathfrak{p} \supseteq \mathfrak{b}$.

By (1)(2), $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$. □

Proof of (iv).

(1) Show that $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$.

(a) $V(\mathfrak{a} \cap \mathfrak{b}) \subseteq V(\mathfrak{a}\mathfrak{b})$ since $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b}$.

(b) *Show that $V(\mathfrak{a} \cap \mathfrak{b}) \supseteq V(\mathfrak{a}\mathfrak{b})$.* Given any $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$, $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$. By Lemma, $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$. Notice that $\mathfrak{a} \supseteq \mathfrak{a} \cap \mathfrak{b}$ and $\mathfrak{b} \supseteq \mathfrak{a} \cap \mathfrak{b}$. In any case, $\mathfrak{p} \supseteq \mathfrak{a} \cap \mathfrak{b}$, $\mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b})$.

(2) *Show that $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.*

(a) *Show that $V(\mathfrak{a}\mathfrak{b}) \subseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$.* Given any $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$, $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$. By Lemma, $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$, $\mathfrak{p} \in V(\mathfrak{a})$ or $\mathfrak{p} \in V(\mathfrak{b})$, $\mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$.

(b) *Show that $V(\mathfrak{a}\mathfrak{b}) \supseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$.* Given any $\mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$, $\mathfrak{p} \in V(\mathfrak{a})$ or $\mathfrak{p} \in V(\mathfrak{b})$, $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$. Notice that $\mathfrak{a} \supseteq \mathfrak{a}\mathfrak{b}$ and $\mathfrak{b} \supseteq \mathfrak{a}\mathfrak{b}$. In any cases, $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$, or $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$.

□

Exercise 1.17. For each $f \in A$, let X_f denote the complement of $V(f)$ in $X = \text{Spec}(A)$. The sets X_f are open. Show that they form a basis of open sets for the Zariski topology, and that

(i) $X_f \cap X_g = X_{fg}$.

(ii) $X_f = \emptyset \iff f$ is nilpotent.

(iii) $X_f = X \iff f$ is a unit.

(iv) $X_f = X_g \iff r((f)) = r((g))$.

(v) X is quasi-compact (compact), that is, every open covering of X has a finite subcovering.

(vi) More generally, each X_f is quasi-compact.

(vii) An open subset of X is quasi-compact if and only if it is a finite union of sets X_f .

The sets X_f are called basic open sets of $X = \text{Spec}(A)$.

(Hint: To prove (v), remark that it is enough to consider a covering of X by basic open sets X_{f_i} ($i \in I$). Show that the f_i generate the unit ideal and hence that there is an equation of the form

$$1 = \sum_{i \in J} g_i f_i \quad (g_i \in A)$$

where J is some finite subset of I . Then the X_{f_i} ($i \in J$) cover X .)

Proof of basis. It is equivalent to Exercise 1.15 (iii). Given any open set O in X . Write $O = X - V(\mathfrak{a})$ for some ideal \mathfrak{a} of A . Since

$$V(\mathfrak{a}) = V\left(\bigcup_{f \in \mathfrak{a}} (f)\right) = \bigcap_{f \in \mathfrak{a}} V(f),$$

we have

$$O = X - V(\mathfrak{a}) = X - \bigcap_{f \in \mathfrak{a}} V(f) = \bigcup_{f \in \mathfrak{a}} (X - V(f)) = \bigcup_{f \in \mathfrak{a}} X_f,$$

or any open set is a union of basic open sets. \square

Proof of (i). $X_f \cap X_g = X_{fg} \iff V(f) \cup V(g) = V(fg)$ holds by Exercise 1.15 (iv). \square

Proof of (ii).

$$\begin{aligned} X_f = \emptyset &\iff V(f) = X \\ &\iff f \in \mathfrak{p} \text{ for all prime ideal } \mathfrak{p} \text{ of } A \\ &\iff f \in \mathfrak{N}, \text{ the nilradical of } A \text{ (Proposition 1.8)} \\ &\iff f \text{ is nilpotent (Proposition 1.7)} \end{aligned}$$

\square

Proof of (ii)(Using (iv)).

$$\begin{aligned} X_f = \emptyset &\iff X_f = X_0 && \text{(Exercise 15(ii))} \\ &\iff r(f) = r(0) && \text{((iv))} \\ &\iff f \in r(f) = r(0) \\ &\iff f^m = 0 \text{ for some } m > 0 \\ &\iff f \text{ is nilpotent} \end{aligned}$$

\square

Proof of (iii).

$$\begin{aligned} X_f = X &\iff V(f) = \emptyset \\ &\iff f \notin \mathfrak{p} \text{ for all prime ideal } \mathfrak{p} \text{ of } A \\ &\iff f \text{ is unit (Corollary 1.5)} \end{aligned}$$

\square

Proof of (iii)(Using (iv)).

$$\begin{aligned} X_f = X &\iff X_f = X_1 && \text{(Exercise 15(ii))} \\ &\iff r(f) = r(1) && \text{((iv))} \\ &\iff f \in r(f) = r(1) \\ &\iff f^m = 1 \text{ for some } m > 0 \\ &\iff f \text{ is unit} \end{aligned}$$

□

Proof of (iv).

(1) Show that $X_f \subseteq X_g \iff r((f)) \subseteq r((g))$. Actually,

$$\begin{aligned}
X_f \subseteq X_g &\implies V(f) \supseteq V(g) \\
&\implies \{\mathfrak{p} \in \text{Spec}(A) : \mathfrak{p} \supseteq (f)\} \supseteq \{\mathfrak{p} \in \text{Spec}(A) : \mathfrak{p} \supseteq (g)\} \\
&\implies \bigcap_{(f) \subseteq \mathfrak{p} \in \text{Spec}(A)} \mathfrak{p} \subseteq \bigcap_{(g) \subseteq \mathfrak{p} \in \text{Spec}(A)} \mathfrak{p} \\
&\stackrel{1.14}{\implies} r(f) \subseteq r(g) \\
&\implies V(r(f)) \supseteq V(r(g)) \\
&\implies V(f) \supseteq V(g) \\
&\implies X_f \subseteq X_g.
\end{aligned}$$

(2) By (1),

$$\begin{aligned}
X_f \subseteq X_g &\iff r((f)) \subseteq r((g)), \\
X_f \supseteq X_g &\iff r((f)) \supseteq r((g)).
\end{aligned}$$

Hence,

$$X_f = X_g \iff r((f)) = r((g)).$$

□

Proof of (v). Notice that it is enough to consider a covering of X by basic open sets $X_{f_i} (i \in I)$.

(1) Since X is covered by $X_{f_i} (i \in I)$,

$$\begin{aligned}
X = \bigcup_{i \in I} X_{f_i} &\implies X - V(1) = \bigcup_{i \in I} (X - V(f_i)) \\
&\implies V(1) = \bigcap_{i \in I} V(f_i) \\
&\implies V(1) = V\left(\sum_{i \in I} f_i\right) \\
&\implies r(1) = r\left(\sum_{i \in I} f_i\right).
\end{aligned}$$

Hence, $1 \in r(1) = r\left(\sum_{i \in I} f_i\right)$ can be expressed as

$$1 = 1^m = \sum_{j \in J} g_j f_j$$

where J is a finite subset of I and $g_j \in A$. That is, $(1) = \sum_{j \in J} f_j$.

- (2) Hence, $V(1) = V\left(\sum_{j \in J} f_j\right)$. Therefore, X is covered by finite subcovering $\{X_{f_j}\}(j \in J)$.

□

Proof of (v)(Using (vi)). Since $X = X_1$, X is quasi-compact by (vi). □

Proof of (vi). Notice that it is enough to consider a covering of X_f by basic open sets $X_{f_i}(i \in I)$.

- (1) Since X_f is covered by $X_{f_i}(i \in I)$,

$$\begin{aligned} X_f = \bigcup_{i \in I} X_{f_i} &\implies X - V(f) = \bigcup_{i \in I} (X - V(f_i)) \\ &\implies V(f) = \bigcap_{i \in I} V(f_i) \\ &\implies V(f) = V\left(\sum_{i \in I} f_i\right) \\ &\implies r(f) = r\left(\sum_{i \in I} f_i\right). \end{aligned}$$

Hence, $f \in r(f) = r\left(\sum_{i \in I} f_i\right)$ can be expressed as

$$f^m = \sum_{j \in J} g_j f_j$$

where J is a finite subset of I and $g_j \in A$. That is, $f^m \in \sum_{j \in J} f_j$.

- (2) Show that $V\left(\sum_{j \in J} f_j\right) = V(f)$.

- (a) (\subseteq) For any prime ideal $\mathfrak{p} \supseteq \sum_{j \in J} f_j$, $f^m \in \mathfrak{p}$ or $f \in \mathfrak{p}$ (since \mathfrak{p} is prime). So $\mathfrak{p} \supseteq (f)$, or $V\left(\sum_{j \in J} f_j\right) \subseteq V(f)$.

- (b) (\supseteq)

$$\sum_{j \in J} f_j \subseteq \sum_{i \in I} f_i \implies V\left(\sum_{j \in J} f_j\right) \supseteq V\left(\sum_{i \in I} f_i\right) = V(f).$$

- (3) Therefore, X_f is covered by finite subcovering $\{X_{f_j}\}(j \in J)$.

□

Proof of (vi)(Using (v)). Exercise 3.21 (i) shows that X_f is the spectrum of A_f . By (v), X_f is quasi-compact. □

Proof of (vii).

- (1) (\implies) Given an open subset O . Since X_f form a basis of open sets,

$$O = \bigcup_{f \in \mathfrak{a}} X_f \text{ for some ideal } \mathfrak{a} \text{ of } A$$

Especially, $\{X_f\}_{f \in \mathfrak{a}}$ is an open covering of O . Since O is quasi-compact, there exists a finite subcovering $\{X_f\}_{f \in J}$ of O , where J is a finite subset of \mathfrak{a} (as a set). That is, $O = \bigcup_{f \in J} X_f$ is a finite union of sets X_f .

- (2) (\impliedby) Since X_f is quasi-compact, any finite union of quasi-compact sets is quasi-compact again.

□

Exercise 1.19. A topological space X is said to be irreducible if $X \neq \emptyset$ and if every pair of non-empty open sets in X intersect, or equivalently if every non-empty open set is dense in X . Show that $\text{Spec}(A)$ is irreducible if and only if the nilradical of A is a prime ideal.

Exercise 1.20. Let X be a topological space.

- (i) If Y is an irreducible subspace of X , then the closure \overline{Y} of Y in X is irreducible.

Proof of (i).

- (1) Y is irreducible if and only if Y cannot be represented as the union of two proper closed subspaces.

$$\begin{aligned} & \forall \text{ nonempty open sets } U_1 \text{ and } U_2, U_1 \cap U_2 \neq \emptyset \\ \iff & \forall \text{ nonempty open sets } U_1 \text{ and } U_2, X - (U_1 \cap U_2) \neq X \\ \iff & \forall \text{ nonempty open sets } U_1 \text{ and } U_2, (X - U_1) \cup (X - U_2) \neq X \\ \iff & \forall \text{ proper closed sets } Y_1 \text{ and } Y_2, Y_1 \cup Y_2 \neq X \\ \iff & \nexists \text{ proper closed sets } Y_1 \text{ and } Y_2, Y_1 \cup Y_2 = X. \end{aligned}$$

- (2) If \overline{Y} were reducible, there are two closed set Y_1 and Y_2 such that

$$\overline{Y} \subseteq Y_1 \cup Y_2, \overline{Y} \not\subseteq Y_i (i = 1, 2).$$

- (a) $Y \subseteq \overline{Y} \subseteq Y_1 \cup Y_2$.
 (b) $Y \not\subseteq Y_i (i = 1, 2)$. If not, $Y \subseteq Y_i$ for some i . Take closure to get $\overline{Y} \subseteq \overline{Y_i} = Y_i$ (since Y_i is closed), contrary to the assumption.

By (a)(b), Y is reducible, which is absurd.

□

Supplement. (*Exercise I.1.6 in Robin Hartshorne, Algebraic Geometry.*) Any nonempty open subset of an irreducible topological space is dense and irreducible. If Y is a subset of a topological space X , which is irreducible in its induced topology, then the closure \overline{Y} is also irreducible.

Here we use the definition of irreducibility given by Hartshorne.

Definition. A nonempty subset Y of a topological space X is irreducible if it cannot be expressed as the union $Y = Y_1 \cup Y_2$ of two proper subsets, each one of which is closed in Y . The empty set is not considered to be irreducible.

The proof is the same as Exercise 1.20(i).