

Chapter 9: Functions of Several Variables

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Exercise 9.1. *If S is a nonempty subset of a vector space X , prove (as asserted in Section 9.1) that the span of S is a vector space.*

Denote the span of S by $\text{span}(S)$.

Proof.

- (1) Since $S \neq \emptyset$, there is $\mathbf{z} \in S$. So $1\mathbf{z} = \mathbf{z} \in \text{span}(S) \neq \emptyset$. (In fact, $\text{span}(S) \supseteq S$.)
- (2) If $\mathbf{x}, \mathbf{y} \in \text{span}(S)$, then there exist elements $\mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{y}_1, \dots, \mathbf{y}_n \in S$ and scalars $a_1, \dots, a_m, b_1, \dots, b_n$ such that

$$\mathbf{x} = a_1\mathbf{x}_1 + \dots + a_m\mathbf{x}_m,$$

$$\mathbf{y} = b_1\mathbf{y}_1 + \dots + b_n\mathbf{y}_n.$$

Then

$$\mathbf{x} + \mathbf{y} = a_1\mathbf{x}_1 + \dots + a_m\mathbf{x}_m + b_1\mathbf{y}_1 + \dots + b_n\mathbf{y}_n$$

is a linear combination of the elements of S . For any scalar c ,

$$c\mathbf{x} = (ca_1)\mathbf{x}_1 + \dots + (ca_m)\mathbf{x}_m$$

is again linear combination of the elements of S .

- (3) By (1)(2), $\text{span}(S)$ is a vector space.

□

Note. Any subspace of X that contains S must also contain $\text{span}(S)$.

Exercise 9.2. *Prove (as asserted in Section 9.6) that BA is linear if A and B are linear transformations. Prove also that A^{-1} is linear and invertible if A is invertible.*

Proof. Use the notation in Definitions 9.6.

- (1) *Show that BA is linear if A and B are linear transformations.* Let X, Y, Z be vector spaces, $A \in L(X, Y)$ and $B \in L(Y, Z)$.

(a) Given any $\mathbf{x}_1, \mathbf{x}_2 \in X$.

$$\begin{aligned}
 (BA)(\mathbf{x}_1 + \mathbf{x}_2) &= B(A(\mathbf{x}_1 + \mathbf{x}_2)) \\
 &= B(A\mathbf{x}_1 + A\mathbf{x}_2) && (A \text{ is a linear transformation}) \\
 &= B(A\mathbf{x}_1) + B(A\mathbf{x}_2) && (B \text{ is a linear transformation}) \\
 &= (BA)\mathbf{x}_1 + (BA)\mathbf{x}_2.
 \end{aligned}$$

(b) For any $\mathbf{x} \in X$ and scalar c ,

$$\begin{aligned}
 (BA)(c\mathbf{x}) &= B(A(c\mathbf{x})) \\
 &= B(cA\mathbf{x}) && (A \text{ is a linear transformation}) \\
 &= cB(A\mathbf{x}) && (B \text{ is a linear transformation}) \\
 &= c(BA)\mathbf{x}.
 \end{aligned}$$

By (a)(b), $BA \in L(X, Z)$.

(2) Show that A^{-1} is linear if A is invertible.

(a) Given any $\mathbf{y}_1, \mathbf{y}_2 \in X$. Since A is surjective, there exist $\mathbf{x}_1, \mathbf{x}_2 \in X$ such that

$$\begin{aligned}
 \mathbf{y}_1 &= A\mathbf{x}_1 \\
 \mathbf{y}_2 &= A\mathbf{x}_2.
 \end{aligned}$$

So

$$\begin{aligned}
 A^{-1}\mathbf{y}_1 &= A^{-1}(A\mathbf{x}_1) = \mathbf{x}_1 \\
 A^{-1}\mathbf{y}_2 &= A^{-1}(A\mathbf{x}_2) = \mathbf{x}_2
 \end{aligned}$$

(by Definitions 9.4). Hence

$$\begin{aligned}
 A^{-1}(\mathbf{y}_1 + \mathbf{y}_2) &= A^{-1}(A\mathbf{x}_1 + A\mathbf{x}_2) \\
 &= A^{-1}(A(\mathbf{x}_1 + \mathbf{x}_2)) && (A \text{ is a linear transformation}) \\
 &= \mathbf{x}_1 + \mathbf{x}_2 && (\text{Definitions 9.4}) \\
 &= A^{-1}\mathbf{y}_1 + A^{-1}\mathbf{y}_2.
 \end{aligned}$$

(b) For any $\mathbf{y} \in X$ and scalar c , there is a corresponding $\mathbf{x} \in X$ such that $\mathbf{y} = A\mathbf{x}$ since A is surjective. So $A^{-1}\mathbf{y} = \mathbf{x}$ by Definition 9.4. Hence

$$\begin{aligned}
 A^{-1}(c\mathbf{y}) &= A^{-1}(cA\mathbf{x}) \\
 &= A^{-1}(A(c\mathbf{x})) && (A \text{ is a linear transformation}) \\
 &= c\mathbf{x} && (\text{Definitions 9.4}) \\
 &= cA^{-1}\mathbf{y}.
 \end{aligned}$$

By (a)(b), $A^{-1} \in L(X)$.

(3) *Show that A^{-1} is invertible if A is invertible.* It suffices to show that A^{-1} is injective and surjective.

(a) *Show that A^{-1} is injective.* Given any $\mathbf{y}_1, \mathbf{y}_2 \in X$. Since A is surjective, there exist $\mathbf{x}_1, \mathbf{x}_2 \in X$ such that

$$\begin{aligned}\mathbf{y}_1 &= A\mathbf{x}_1 \\ \mathbf{y}_2 &= A\mathbf{x}_2.\end{aligned}$$

Suppose $A^{-1}\mathbf{y}_1 = A^{-1}\mathbf{y}_2$. So $A^{-1}(A\mathbf{x}_1) = A^{-1}(A\mathbf{x}_2)$, or $\mathbf{x}_1 = \mathbf{x}_2$, or $\mathbf{y}_1 = A\mathbf{x}_1 = A\mathbf{x}_2 = \mathbf{y}_2$.

(b) *Show that A^{-1} is surjective.* For any $\mathbf{x} \in X$, there exists $A\mathbf{x} \in X$ such that $A^{-1}(A\mathbf{x}) = \mathbf{x}$ by Definitions 9.4.

□

Exercise 9.3. Assume $A \in L(X, Y)$ and $A\mathbf{x} = \mathbf{0}$ only when $\mathbf{x} = \mathbf{0}$. Prove that A is then 1-1.

Proof. Suppose $A\mathbf{x} = A\mathbf{y}$. Since A is a linear transformation, $A(\mathbf{x} - \mathbf{y}) = A\mathbf{x} - A\mathbf{y} = \mathbf{0}$. By assumption, $\mathbf{x} - \mathbf{y} = \mathbf{0}$ or $\mathbf{x} = \mathbf{y}$. □

Exercise 9.4. Prove (as asserted in Section 9.30) that null spaces and ranges of linear transformations are vector spaces.

Proof. Use the notation in Definitions 9.30. Suppose X, Y are vector spaces, and $A \in L(X, Y)$, as in Definition 9.6.

(1) *Show that $\mathcal{N}(A)$ is a vector space in X .*

(a) Note that $\mathbf{0} \in X$. Since $A\mathbf{0} = \mathbf{0}$, $\mathbf{0} \in \mathcal{N}(A) \neq \emptyset$.

(b) Suppose $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{N}(A)$. Then

$$\begin{aligned}A(\mathbf{x}_1 + \mathbf{x}_2) &= A\mathbf{x}_1 + A\mathbf{x}_2 && (A \text{ is a linear transformation}) \\ &= \mathbf{0} + \mathbf{0} && (\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{N}(A)) \\ &= \mathbf{0}.\end{aligned}$$

So $\mathbf{x}_1 + \mathbf{x}_2 \in \mathcal{N}(A)$.

(c) Suppose $\mathbf{x} \in \mathcal{N}(A)$ and c is a scalar. Then

$$\begin{aligned}A(c\mathbf{x}) &= cA\mathbf{x} && (A \text{ is a linear transformation}) \\ &= c\mathbf{0} && (\mathbf{x} \in \mathcal{N}(A)) \\ &= \mathbf{0}.\end{aligned}$$

So $c\mathbf{x} \in \mathcal{N}(A)$.

By (a)(b)(c), $\mathcal{N}(A)$ is a vector space.

(2) Show that $\mathcal{R}(A)$ is a vector space in Y .

(a) Note that $\mathbf{0} \in X$. So $A\mathbf{0} = \mathbf{0} \in \mathcal{R}(A) \neq \emptyset$.

(b) Suppose $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{R}(A)$. Then there exist $\mathbf{x}_1, \mathbf{x}_2 \in X$ such that $A\mathbf{x}_1 = \mathbf{y}_1$ and $A\mathbf{x}_2 = \mathbf{y}_2$. Hence

$$\begin{aligned}\mathbf{y}_1 + \mathbf{y}_2 &= A\mathbf{x}_1 + A\mathbf{x}_2 \\ &= A(\mathbf{x}_1 + \mathbf{x}_2) \quad (A \text{ is a linear transformation}).\end{aligned}$$

So $\mathbf{y}_1 + \mathbf{y}_2 \in \mathcal{R}(A)$.

(c) Suppose $\mathbf{y} \in \mathcal{R}(A)$ and c is a scalar. Then there exists $\mathbf{x} \in X$ such that $A\mathbf{x} = \mathbf{y}$. Hence

$$\begin{aligned}c\mathbf{y} &= cA\mathbf{x} \\ &= A(c\mathbf{x}) \quad (A \text{ is a linear transformation}).\end{aligned}$$

So $c\mathbf{y} \in \mathcal{R}(A)$.

By (a)(b)(c), $\mathcal{R}(A)$ is a vector space.

□

Exercise 9.5. Prove that to every $A \in L(\mathbb{R}^n, \mathbb{R}^1)$ corresponds a unique $\mathbf{y} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$. Prove also that $\|A\| = \|\mathbf{y}\|$. (Hint: Under certain conditions, equality holds in the Schwarz inequality.)

Proof.

(1) Recall that $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the standard basis of \mathbb{R}^n (Definitions 9.1).

Given any $\mathbf{x} \in \mathbb{R}^n$, write $\mathbf{x} = (x_1, \dots, x_n)$ as $\mathbf{x} = \sum x_j \mathbf{e}_j$.

(2) Show that \mathbf{y} exists. Since A is a linear transformation,

$$\begin{aligned}A\mathbf{x} &= A\left(\sum x_j \mathbf{e}_j\right) \\ &= \sum x_j A\mathbf{e}_j \\ &= (x_1, \dots, x_n) \cdot (A\mathbf{e}_1, \dots, A\mathbf{e}_n) \\ &= \mathbf{x} \cdot \sum (A\mathbf{e}_j) \mathbf{e}_j.\end{aligned}$$

Define $\mathbf{y} = \sum (A\mathbf{e}_j) \mathbf{e}_j \in \mathbb{R}^n$ so that $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$.

(3) Show that \mathbf{y} is unique. Suppose there exists some $\mathbf{z} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{x} \cdot \mathbf{z}$. So

$$\begin{aligned}0 &= A\mathbf{x} - A\mathbf{x} \\ &= \mathbf{x} \cdot \mathbf{y} - \mathbf{x} \cdot \mathbf{z} \\ &= \mathbf{x} \cdot (\mathbf{y} - \mathbf{z})\end{aligned}$$

for any $\mathbf{x} \in \mathbb{R}^n$. In particular, take $\mathbf{x} = \mathbf{y} - \mathbf{z} \in \mathbb{R}^n$ to get

$$0 = (\mathbf{y} - \mathbf{z}) \cdot (\mathbf{y} - \mathbf{z}) = |\mathbf{y} - \mathbf{z}|^2$$

or $\mathbf{y} - \mathbf{z} = \mathbf{0}$ or $\mathbf{y} = \mathbf{z}$.

(4) *Show that $\|A\| = |\mathbf{y}|$.* By the Schwarz inequality (Theorem 1.37(d)),

$$|A\mathbf{x}| = |\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}| \leq |\mathbf{y}|$$

as $|\mathbf{x}| \leq 1$. Take the sup over all $|\mathbf{x}| \leq 1$ to get

$$\|A\| \leq |\mathbf{y}|.$$

If $\mathbf{y} = \mathbf{0}$, then $\|A\| = |\mathbf{y}| = 0$. If $\mathbf{y} \neq \mathbf{0}$, then the equality holds when $\mathbf{x} = \frac{\mathbf{y}}{|\mathbf{y}|} \in \mathbb{R}^n$. (Here $|\mathbf{x}| = 1$.)

□

Exercise 9.6. *If $f(0,0) = 0$ and*

$$f(x,y) = \frac{xy}{x^2 + y^2} \quad \text{if } (x,y) \neq (0,0),$$

prove that $(D_1f)(x,y)$ and $(D_2f)(x,y)$ exist at every point of \mathbb{R}^2 , although f is not continuous at $(0,0)$.

Proof.

(1) *Show that*

$$(D_1f)(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0), \\ \frac{y(y^2 - x^2)}{(x^2 + y^2)^2} & \text{if } (x,y) \neq (0,0). \end{cases}$$

Write

$$\begin{aligned} (D_1f)(x,y) &= \lim_{t \rightarrow 0} \frac{f((x,y) + t(1,0)) - f(x,y)}{t} \\ &= \lim_{t \rightarrow 0} \frac{f(x+t,y) - f(x,y)}{t}. \end{aligned}$$

If $(x,y) = (0,0)$,

$$(D_1f)(0,0) = \lim_{t \rightarrow 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t \rightarrow 0} \frac{0 - 0}{t} = 0.$$

If $(x, y) \neq (0, 0)$,

$$\begin{aligned}
 (D_1 f)(x, y) &= \lim_{t \rightarrow 0} \frac{f(x+t, y) - f(x, y)}{t} \\
 &= \lim_{t \rightarrow 0} \frac{\frac{(x+t)y}{(x+t)^2 + y^2} - \frac{xy}{x^2 + y^2}}{t} \\
 &= \lim_{t \rightarrow 0} \frac{y(y^2 - x^2) - txy}{((x+t)^2 + y^2)(x^2 + y^2)} \\
 &= \frac{y(y^2 - x^2)}{(x^2 + y^2)^2}.
 \end{aligned}$$

(2) Show that

$$(D_2 f)(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0), \\ \frac{x(x^2 - y^2)}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0). \end{cases}$$

Similar to (1).

(3) Show that f is not continuous at $(0, 0)$. Note that

$$\lim_{n \rightarrow \infty} f\left(\frac{1}{n}, \frac{1}{n}\right) = \lim_{n \rightarrow \infty} \frac{\frac{1}{n} \cdot \frac{1}{n}}{\frac{1}{n^2} + \frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{1}{2} = \frac{1}{2}$$

and

$$\lim_{n \rightarrow \infty} f\left(\frac{1}{n}, 0\right) = \lim_{n \rightarrow \infty} \frac{0}{\frac{1}{n^2} + 0} = \lim_{n \rightarrow \infty} 0 = 0.$$

Hence the limit $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ does not exist.

□

Exercise 9.7. Suppose that f is a real-valued function defined in an open set $E \subseteq \mathbb{R}^n$, and that the partial derivatives $D_1 f, \dots, D_n f$ are bounded in E . Prove that f is continuous in E . (Hint: Proceed as in the proof of Theorem 9.21.)

Proof.

(1) Since $D_j f$ is bounded in E , there is a real number M_j such that $|D_j f| \leq M_j$ in E . Take $M = \max_{1 \leq j \leq n} M_j$ so that $|D_j f| \leq M$ in E for all $1 \leq j \leq n$.

(2) Fix $\mathbf{x} \in E$ and $\varepsilon > 0$. Since E is open, there is an open neighborhood

$$B(\mathbf{x}; r) = \{\mathbf{x} + \mathbf{h} \in E : |\mathbf{h}| < r\} \subseteq E$$

with

$$0 < r < \frac{\varepsilon}{n(M+1)}.$$

- (3) Write $\mathbf{h} = \sum h_j \mathbf{e}_j$, $|\mathbf{h}| < r$, put $\mathbf{v}_0 = \mathbf{0}$, and $\mathbf{v}_k = h_1 \mathbf{e}_1 + \cdots + h_k \mathbf{e}_k$ for $1 \leq k \leq n$. Then

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = \sum_{j=1}^n [f(\mathbf{x} + \mathbf{v}_j) - f(\mathbf{x} + \mathbf{v}_{j-1})].$$

Since $|\mathbf{v}_k| < r$ for $1 \leq k \leq n$ and since $B(\mathbf{x}; r)$ is convex, the open interval with end points $\mathbf{x} + \mathbf{v}_{j-1}$ and $\mathbf{x} + \mathbf{v}_j$ lie in $B(\mathbf{x}; r)$. Since $\mathbf{v}_j = \mathbf{v}_{j-1} + h_j \mathbf{e}_j$, the mean value theorem (Theorem 5.10) show that

$$f(\mathbf{x} + \mathbf{v}_j) - f(\mathbf{x} + \mathbf{v}_{j-1}) = h_j (D_j f)(\mathbf{x} + \mathbf{v}_{j-1} + \theta_j h_j \mathbf{e}_j)$$

for some $\theta_j \in (0, 1)$.

- (4) Note that $|h_j| \leq |\mathbf{h}| < r < \frac{\varepsilon}{n(M+1)}$. Hence

$$\begin{aligned} |f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x})| &\leq \sum_{j=1}^n |f(\mathbf{x} + \mathbf{v}_j) - f(\mathbf{x} + \mathbf{v}_{j-1})| \\ &= \sum_{j=1}^n |h_j| |(D_j f)(\mathbf{x} + \mathbf{v}_{j-1} + \theta_j h_j \mathbf{e}_j)| \\ &\leq \sum_{j=1}^n \frac{\varepsilon}{n(M+1)} \cdot M \\ &< \varepsilon \end{aligned}$$

as $|\mathbf{h}| < r < \frac{\varepsilon}{n(M+1)}$. Hence f is continuous at all $\mathbf{x} \in E$.

□

Exercise 9.8. Suppose that f is a differentiable real function in an open set $E \subseteq \mathbb{R}^n$, and that f has a local maximum at a point $\mathbf{x} \in E$. Prove that $f'(\mathbf{x}) = 0$.

Proof (Theorem 5.8).

- (1) Apply Theorem 5.8 to each $D_j f$ for $1 \leq j \leq n$. Since f has a local maximum at a point $\mathbf{x} \in E$, there is an open neighborhood $B(\mathbf{x}; r)$ of \mathbf{x} in E such that

$$f(\mathbf{y}) \leq f(\mathbf{x})$$

for all $\mathbf{y} \in B(\mathbf{x}; r)$. Therefore,

$$f(\mathbf{x} + t\mathbf{e}_j) \leq f(\mathbf{x})$$

for all $|t| < r$ and $1 \leq j \leq n$, or $t \mapsto f(\mathbf{x} + t\mathbf{e}_j)$ has a local maximum at a point $t = 0 \in (-r, r)$.

- (2) Since f is differentiable in E , each partial derivatives $D_j f$ exist (Theorem 9.21). Hence Theorem 5.8 implies that $(D_j f)(\mathbf{x}) = 0$ for all $1 \leq j \leq n$. So

$$f'(\mathbf{x}) = [(D_1 f)(\mathbf{x}) \cdots (D_n f)(\mathbf{x})] = [0 \cdots 0] = 0$$

(as the zero matrix).

□

Exercise 9.9. ...

Proof.

(1)

(2)

□

Exercise 9.10. ...

Proof.

(1)

(2)

□

Exercise 9.11. If f and g are differentiable real functions in \mathbb{R}^n , prove that

$$\nabla(fg) = f\nabla g + g\nabla f$$

and that

$$\nabla \left(\frac{1}{f} \right) = -\frac{1}{f^2} \nabla f$$

whenever $f \neq 0$.

Proof. Recall Example 9.18:

$$(\nabla(f))(\mathbf{x}) = \sum_{i=1}^n (D_i f)(\mathbf{x}) \mathbf{e}_i.$$

(1) Show that $\nabla(fg) = f\nabla g + g\nabla f$. For any $\mathbf{x} \in \mathbb{R}^n$,

$$\begin{aligned}
(\nabla(fg))(\mathbf{x}) &= \sum_{i=1}^n (D_i(fg))(\mathbf{x}) \mathbf{e}_i \\
&= \sum_{i=1}^n (g(D_i f) + f(D_i g))(\mathbf{x}) \mathbf{e}_i && \text{(Theorem 5.3(b))} \\
&= \sum_{i=1}^n [g(\mathbf{x})(D_i f)(\mathbf{x}) + f(\mathbf{x})(D_i g)(\mathbf{x})] \mathbf{e}_i \\
&= g(\mathbf{x}) \sum_{i=1}^n (D_i f)(\mathbf{x}) \mathbf{e}_i + f(\mathbf{x}) \sum_{i=1}^n (D_i g)(\mathbf{x}) \mathbf{e}_i \\
&= g(\mathbf{x})(\nabla f)(\mathbf{x}) + f(\mathbf{x})(\nabla g)(\mathbf{x}) \\
&= (f\nabla g + g\nabla f)(\mathbf{x}).
\end{aligned}$$

(2) Show that

$$\nabla \left(\frac{1}{f} \right) = -\frac{1}{f^2} \nabla f$$

whenever $f \neq 0$. Note that $\nabla(1) = 0$ since

$$\nabla(1)(\mathbf{x}) = \sum (D_i 1)(\mathbf{x}) \mathbf{e}_i = \sum (0)(\mathbf{x}) \mathbf{e}_i = \sum 0 \mathbf{e}_i = 0.$$

Hence as $f \neq 0$, we have

$$\begin{aligned}
0 &= \nabla(1) \\
&= \nabla \left(f \frac{1}{f} \right) && (f \neq 0) \\
&= f \nabla \left(\frac{1}{f} \right) + \frac{1}{f} \nabla f && ((1)),
\end{aligned}$$

$$\text{or } \nabla \left(\frac{1}{f} \right) = -\frac{1}{f^2} \nabla f.$$

□

Exercise 9.12. ...

Proof.

(1)

(2)

□

Exercise 9.13. Suppose \mathbf{f} is a differentiable mapping of \mathbb{R}^1 into \mathbb{R}^3 such that $|\mathbf{f}(t)| = 1$ for every t . Prove that $\mathbf{f}'(t) \cdot \mathbf{f}(t) = 0$. Interpret this result geometrically.

Proof.

- (1) Write $\mathbf{f} = (f_1, f_2, f_3)$ as a vector-valued function. By Remarks 5.16, \mathbf{f} is differentiable if and only if each f_1, f_2, f_3 is differentiable. So $\mathbf{f}' = (f'_1, f'_2, f'_3)'$. Hence

$$\begin{aligned} |\mathbf{f}(t)| &= 1 \text{ for every } t \\ \iff \mathbf{f}(t) \cdot \mathbf{f}(t) &= 1 \\ \iff f_1(t)^2 + f_2(t)^2 + f_3(t)^2 &= 1 \\ \implies 2f_1(t)f'_1(t) + 2f_2(t)f'_2(t) + 2f_3(t)f'_3(t) &= 0 \\ \iff f_1(t)f'_1(t) + f_2(t)f'_2(t) + f_3(t)f'_3(t) &= 0 \\ \iff (f_1(t), f_2(t), f_3(t)) \cdot (f'_1(t), f'_2(t), f'_3(t)) &= 0 \\ \iff \mathbf{f}(t) \cdot \mathbf{f}'(t) = \mathbf{f}'(t) \cdot \mathbf{f}(t) &= 0. \end{aligned}$$

- (2) The vector $\mathbf{f}'(t)$ is called the **tangent vector** (or **velocity vector**) of \mathbf{f} at t . Geometrically, given any mapping \mathbf{f} lying on the sphere S^2 , its tangent vector at t is lying on the tangent plane of S^2 at t .

□

Exercise 9.14. Define $f(0, 0) = 0$ and

$$f(x, y) = \frac{x^3}{x^2 + y^2} \quad \text{if } (x, y) \neq (0, 0).$$

- (a) Prove that D_1f and D_2f are bounded functions in \mathbb{R}^2 . (Hence f is continuous.)
- (b) Let \mathbf{u} be any unit vector in \mathbb{R}^2 . Show that the directional derivative $(D_{\mathbf{u}}f)(0, 0)$ exists, and that its absolute value is at most 1.
- (c)
- (d) In spite of this, prove that f is not differentiable at $(0, 0)$.

Proof of (a).

(1) *Show that*

$$(D_1 f)(x, y) = \begin{cases} 1 & \text{if } (x, y) = (0, 0), \\ \frac{x^2(x^2+3y^2)}{(x^2+y^2)^2} & \text{if } (x, y) \neq (0, 0). \end{cases}$$

If $(x, y) = (0, 0)$,

$$(D_1 f)(0, 0) = \lim_{t \rightarrow 0} \frac{f(t, 0) - f(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{t - 0}{t} = 1.$$

If $(x, y) \neq (0, 0)$,

$$\begin{aligned} (D_1 f)(x, y) &= \lim_{t \rightarrow 0} \frac{f(x+t, y) - f(x, y)}{t} \\ &= \lim_{t \rightarrow 0} \frac{\frac{(x+t)^3}{(x+t)^2+y^2} - \frac{x^3}{x^2+y^2}}{t} \\ &= \lim_{t \rightarrow 0} \frac{x^2(x^2+3y^2) + tx(2x^2+3y^2) + t^2(x^2+y^2)}{((x+t)^2+y^2)(x^2+y^2)} \\ &= \frac{x^2(x^2+3y^2)}{(x^2+y^2)^2}. \end{aligned}$$

(Or differentiate directly.)

(2) *Show that $(D_1 f)(x, y)$ is bounded.* It suffices to show that $(D_1 f)(x, y)$ is bounded if $(x, y) \neq (0, 0)$. Write $x = r \cos \theta$ and $y = r \sin \theta$ in the polar coordinates. (Here $r > 0$.) Hence

$$(D_1 f)(x, y) = \frac{x^2(x^2+3y^2)}{(x^2+y^2)^2} = \cos^2 \theta (\cos^2 \theta + 3 \sin^2 \theta)$$

is bounded by $1 \cdot (1 + 3) = 4$.

(3) *Show that*

$$(D_2 f)(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0), \\ \frac{-2x^3 y}{(x^2+y^2)^2} & \text{if } (x, y) \neq (0, 0). \end{cases}$$

If $(x, y) = (0, 0)$,

$$(D_2 f)(0, 0) = \lim_{t \rightarrow 0} \frac{f(0, t) - f(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{0 - 0}{t} = 0.$$

If $(x, y) \neq (0, 0)$,

$$\begin{aligned} (D_2 f)(x, y) &= \lim_{t \rightarrow 0} \frac{f(x, y+t) - f(x, y)}{t} \\ &= \lim_{t \rightarrow 0} \frac{\frac{x^3}{x^2+(y+t)^2} - \frac{x^3}{x^2+y^2}}{t} \\ &= \lim_{t \rightarrow 0} \frac{-2x^3 y - tx^3}{(x^2+(y+t)^2)(x^2+y^2)} \\ &= \frac{-2x^3 y}{(x^2+y^2)^2}. \end{aligned}$$

(Or differentiate directly.)

(4) Show that $(D_2f)(x, y)$ is bounded. Similar to (2).

(5) Show that f is continuous. Apply Exercise 9.7 to (2)(4).

□

Proof of (b).

(1) Write $\mathbf{u} = (u_1, u_2)$. The formula

$$(D_{\mathbf{u}}f)(0, 0) = (D_1f)(0, 0)u_1 + (D_2f)(0, 0)u_2 = u_1$$

might be false since we don't know if f is differentiable or not. Actually, we will show that $(D_{\mathbf{u}}f)(0, 0) = u_1^3 \neq u_1$.

(2)

$$\begin{aligned}(D_{\mathbf{u}}f)(0, 0) &= \lim_{t \rightarrow 0} \frac{f(tu_1, tu_2) - f(0, 0)}{t} \\ &= \lim_{t \rightarrow 0} \frac{\frac{t^3 u_1^3}{t^2 u_1^2 + t^2 u_2^2} - 0}{t} \\ &= \lim_{t \rightarrow 0} u_1^3 \quad (|\mathbf{u}| = 1) \\ &= u_1^3.\end{aligned}$$

Also $|(D_{\mathbf{u}}f)(0, 0)| = |u_1|^3 \leq 1$ since $|\mathbf{u}| = 1$.

□

Proof of (d). (Reductio ad absurdum) If f were differentiable, then

$$(D_{\mathbf{u}}f)(0, 0) = (D_1f)(0, 0)u_1 + (D_2f)(0, 0)u_2 = u_1$$

(Formula (40) in Chapter 9), contrary to (b). □

Exercise 9.15. ...

Proof.

(1)

(2)

□

Exercise 9.16. ...

Proof.

(1)

(2)

□

Exercise 9.17. ...

Proof.

(1)

(2)

□

Exercise 9.18. ...

Proof.

(1)

(2)

□

Exercise 9.19. ...

Proof.

(1)

(2)

□

Exercise 9.20. ...

Proof.

(1)

(2)

□

Exercise 9.21. ...

Proof.

(1)

(2)

□

Exercise 9.22. ...

Proof.

(1)

(2)

□

Exercise 9.23. ...

Proof.

(1)

(2)

□

Exercise 9.24. ...

Proof.

(1)

(2)

□

Exercise 9.25. ...

Proof.

(1)

(2)

□

Exercise 9.26. ...

Proof.

(1)

(2)

□

Exercise 9.27. ...

Proof.

(1)

(2)

□

Exercise 9.28. ...

Proof.

(1)

(2)

□

Exercise 9.29. ...

Proof.

(1)

(2)

□

Exercise 9.30. ...

Proof.

(1)

(2)

□

Exercise 9.31. ...

Proof.

(1)

(2)

□