Notes on the book: P.J. Hilton and U. Stammbach, A Course in Homological Algebra

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Chapter I: Modules

§1. Modules

Exercise 1.1. (Diagram chasing)

Complete the proof of Lemma 1.1. Show moreover that α is surjective (resp. injective) if α' , α'' are surjective (resp. injective).

Lemma 1.1. Let $0 \to A' \to A \to A'' \to 0$ and $0 \to B' \to B \to B'' \to 0$ be two short exact sequences. Suppose that in the commutative diagram

$$0 \longrightarrow A' \stackrel{\mu}{\longrightarrow} A \stackrel{\varepsilon}{\longrightarrow} A'' \longrightarrow 0$$

$$\downarrow^{\alpha'} \qquad \downarrow^{\alpha} \qquad \downarrow^{\alpha''}$$

$$0 \longrightarrow B' \stackrel{\mu'}{\longrightarrow} B \stackrel{\varepsilon'}{\longrightarrow} B'' \longrightarrow 0$$

any two of the three homomorphisms α' , α , α'' are isomorphisms. Then the third is an isomorphism, too.

Proof (Diagram chasing).

- (1) Show that α is surjective if α' , α'' are surjective.
 - (a) Take any $b \in B$, it suffices to find $a \in A$ such that $\alpha a = b$.
 - (b) Consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varepsilon} & A'' \\ \downarrow^{\alpha} & & \downarrow^{\alpha''} \\ B & \xrightarrow{\varepsilon'} & B'' \end{array}$$

 $\varepsilon'b \in B'$. By the surjectivity of α'' , $\exists \, a'' \in A''$ such that $\alpha''a'' = \varepsilon'b$. By the surjectivity of ε , $\exists \, \overline{a} \in A$ such that $\varepsilon \overline{a} = a''$. Hence

$$\varepsilon'(b - \alpha \overline{a}) = \varepsilon'b - \varepsilon'\alpha \overline{a}$$

$$= \varepsilon'b - \alpha''\varepsilon \overline{a}$$

$$= \varepsilon'b - \alpha''a''$$

$$= \varepsilon'b - \varepsilon'b$$

$$= 0.$$
(The diagram commutes)

(c) Consider the short exact sequence

$$0 \longrightarrow B' \stackrel{\mu'}{\longrightarrow} B \stackrel{\varepsilon'}{\longrightarrow} B'' \longrightarrow 0$$
 As $\varepsilon'(b - \alpha \overline{a}) = 0$, $\exists b' \in B'$ such that $\mu'b' = b - \alpha \overline{a}$.

(d) Consider the commutative diagram

$$\begin{array}{ccc} A' \stackrel{\mu}{\longrightarrow} A \\ \downarrow^{\alpha'} & \downarrow^{\alpha} \\ B' \stackrel{\mu'}{\longrightarrow} B \end{array}$$

By the surjectivity of α' , $\exists a' \in A'$ such that $\alpha'a' = b'$. Hence

$$\alpha(\mu a' + \overline{a}) = \alpha \mu a' + \alpha \overline{a}$$

$$= \mu' \alpha' a' + \alpha \overline{a}$$
 (The diagram commutes)
$$= \mu' b' + \alpha \overline{a}$$

$$= (b - \alpha \overline{a}) + \alpha \overline{a}$$

$$= b.$$

Therefore, there exists $a := \mu a' + \overline{a}$ such that $\alpha a = b$.

(2) Show that α is injective if α' , α'' are injective.

- (a) It suffices to show that $\ker \alpha = 0$. Take $a \in \ker \alpha$. $(\alpha(a) = \alpha a = 0)$
- (b) Consider the commutative diagram

$$\begin{array}{ccc} A & \stackrel{\varepsilon}{\longrightarrow} A'' \\ \downarrow^{\alpha} & \downarrow^{\alpha''} \\ B & \stackrel{\varepsilon'}{\longrightarrow} B'' \end{array}$$

we have $0 = \varepsilon' \alpha a = \alpha'' \varepsilon a$. By the injectivity of α'' , $\varepsilon a = 0$.

(c) Consider the short exact sequence

$$0 \longrightarrow A' \stackrel{\mu}{\longrightarrow} A \stackrel{\varepsilon}{\longrightarrow} A'' \longrightarrow 0$$

As $\varepsilon a = 0$, $\exists a' \in A'$ such that $\mu a' = a$.

(d) Consider the commutative diagram

$$A' \xrightarrow{\mu} A$$

$$\downarrow^{\alpha'} \qquad \downarrow^{\alpha}$$

$$B' \xrightarrow{\mu'} B$$

 $0 = \alpha a = \alpha \mu a' = \mu' \alpha' a'$. By the injectivity of $\mu' \alpha'$, a' = 0. Therefore, $a = \mu a' = 0$.

(3) Suppose α is surjective. Show that α'' is surjective.

- (a) Take any $b'' \in B''$, it suffices to find $a'' \in A''$ such that $\alpha''a'' = b''$.
- (b) Consider the commutative diagram

$$\begin{array}{ccc} A & \stackrel{\varepsilon}{\longrightarrow} & A'' \\ \downarrow^{\alpha} & & \downarrow^{\alpha''} \\ B & \stackrel{\varepsilon'}{\longrightarrow} & B'' \end{array}$$

By the surjectivity of ε' , $\exists b \in B$ such that $\varepsilon'b = b''$. By the surjectivity of α , $\exists a \in A$ such that $\alpha a = b$. Take $a'' := \varepsilon a \in A''$. Hence

$$\alpha''a'' = \alpha'' \varepsilon a$$

 $= \varepsilon' \alpha a$ (The diagram commutes)
 $= \varepsilon' b$
 $= b''$.

- (4) Suppose α' is surjective and α is injective. Show that α'' is injective.
 - (a) It suffices to show that $\ker \alpha'' = 0$. Take $a'' \in \ker \alpha''$. $(\alpha''(a'') = \alpha''a'' = 0.)$
 - (b) Consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varepsilon} & A'' \\ \downarrow^{\alpha} & & \downarrow^{\alpha''} \\ B & \xrightarrow{\varepsilon'} & B'' \end{array}$$

By the surjectivity of ε , $\exists a \in A$ such that $\varepsilon a = a''$. So

$$0 = \alpha'' a''$$

$$= \alpha'' \varepsilon a$$

$$= \varepsilon' \alpha a.$$
 (The diagram commutes)

(c) Consider the short exact sequence

$$0 \longrightarrow B' \xrightarrow{\mu'} B \xrightarrow{\varepsilon'} B'' \longrightarrow 0$$

As $\varepsilon'(\alpha a) = 0$, $\exists b' \in B'$ such that $\mu'b' = \alpha a$.

(d) Consider the commutative diagram

$$A' \xrightarrow{\mu} A$$

$$\downarrow^{\alpha'} \qquad \downarrow^{\alpha}$$

$$B' \xrightarrow{\mu'} B$$

By surjectivity of α' , $\exists a' \in A'$ such that $\alpha'a' = b'$. So

$$\begin{aligned} \alpha a &= \mu' b' \\ &= \mu' \alpha' a' \\ &= \alpha \mu a'. \end{aligned} \qquad \text{(The diagram commutes)}$$

By the injectivity of α , $a = \mu a'$. Hence

$$a'' = \varepsilon a = \varepsilon \mu a' = 0.$$

Therefore $\ker \alpha'' = 0$.

- (5) By (3)(4), α'' is an isomorphism if both α' and α are isomorphisms.
- (6) Suppose α is surjective and α'' is injective. Show that α' is surjective.
 - (a) Take any $b' \in B'$, it suffices to find $a' \in A'$ such that $\alpha' a' = b'$. Let $b := \mu' b' \in B$ and note that $\varepsilon' b = 0$ by the exactness of

$$0 \to B^\prime \to B \to B^{\prime\prime} \to 0.$$

(b) Consider the commutative diagram

$$A \xrightarrow{\varepsilon} A''$$

$$\downarrow^{\alpha} \qquad \downarrow^{\alpha''}$$

$$B \xrightarrow{\varepsilon'} B''$$

By the surjectivity of α , $\exists a \in A$ such that $\alpha a = b$. So

$$0 = \varepsilon' b$$

$$= \varepsilon' \alpha a$$

$$= \alpha'' \varepsilon a.$$
 (The diagram commutes)

By the injectivity of α'' , $\varepsilon a = 0$.

(c) Consider the short exact sequence

$$0 \longrightarrow A' \stackrel{\mu}{\longrightarrow} A \stackrel{\varepsilon}{\longrightarrow} A'' \longrightarrow 0$$

As $\varepsilon a = 0$, $\exists a' \in A'$ such that $\mu a' = a$.

(d) Consider the commutative diagram

$$\begin{array}{ccc} A' \stackrel{\mu}{\longrightarrow} A \\ \downarrow^{\alpha'} & \downarrow^{\alpha} \\ B' \stackrel{\mu'}{\rightarrowtail} B \end{array}$$

Note that

$$\mu'(\alpha'a') = \mu'\alpha'a'$$

$$= \alpha\mu a' \qquad \text{(The diagram commutes)} = \alpha a$$

$$= b$$

$$= \mu'b'.$$

By the injectivity of μ' , $b' = \alpha' a'$ for some $a' \in A'$.

- (7) Suppose α is injective. Show that α' is injective.
 - (a) It suffices to show that $\ker \alpha' = 0$. Take $a' \in \ker \alpha'$. $(\alpha'(a') = \alpha'a' = 0.)$
 - (b) Consider the commutative diagram

$$A' \xrightarrow{\mu} A$$

$$\downarrow^{\alpha'} \qquad \downarrow^{\alpha}$$

$$B' \xrightarrow{\mu'} B$$

Note that

$$0 = \mu' 0$$

= $\mu' \alpha' a'$
= $\alpha \mu a'$. (The diagram commutes)

The injectivity of $\alpha\mu$ shows that a'=0.

(8) By (6)(7), α' is an isomorphism if both α and α'' are isomorphisms.

Exercise 1.2. (Five lemma)

Show that, given a commutative diagram

$$\cdots \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow A_4 \longrightarrow A_5 \longrightarrow \cdots$$

$$\downarrow^{\varphi_1} \qquad \downarrow^{\varphi_2} \qquad \downarrow^{\varphi_3} \qquad \downarrow^{\varphi_4} \qquad \downarrow^{\varphi_5}$$

$$\cdots \longrightarrow B_1 \longrightarrow B_2 \longrightarrow B_3 \longrightarrow B_4 \longrightarrow B_5 \longrightarrow \cdots$$

with exact rows, in which φ_1 , φ_2 , φ_4 , φ_5 are isomorphisms, then φ_3 is also an isomorphism. Can we weaken the hypotheses in a reasonable way?

One reasonable hypotheses:

- (a) If φ_1 is surjective and φ_2, φ_4 is injective, then φ_3 is injective.
- (b) If φ_5 is injective and φ_2, φ_4 is surjective, then φ_3 is surjective.

Proof of (a).

(1) Write

$$\cdots \longrightarrow A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} A_4 \xrightarrow{\alpha_4} A_5 \longrightarrow \cdots$$

$$\downarrow^{\varphi_1} \qquad \downarrow^{\varphi_2} \qquad \downarrow^{\varphi_3} \qquad \downarrow^{\varphi_4} \qquad \downarrow^{\varphi_5}$$

$$\cdots \longrightarrow B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} B_4 \xrightarrow{\beta_4} B_5 \longrightarrow \cdots$$

Take $a \in \ker(\varphi_3)$ and then we need to show a = 0.

(2) The commutative diagram

$$\begin{array}{ccc} A_3 & \xrightarrow{\alpha_3} & A_4 \\ \downarrow^{\varphi_3} & & \downarrow^{\varphi_4} \\ B_3 & \xrightarrow{\beta_3} & B_4 \end{array}$$

implies that $0 = \beta_3 0 = \beta_3 \varphi_3 a = \varphi_4 \alpha_3 a$. The injectivity of φ_4 implies that $\alpha_3 a = 0$.

(3) The exact sequence

$$\cdots \longrightarrow A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} A_4 \longrightarrow \cdots$$

shows that $a \in \ker(\alpha_3) = \operatorname{im}(\alpha_2)$. So there exists $a_2 \in A_2$ such that $\alpha_2 a_2 = a$.

(4) The commutative diagram

$$A_{2} \xrightarrow{\alpha_{2}} A_{3}$$

$$\downarrow \varphi_{2} \qquad \qquad \downarrow \varphi_{3}$$

$$B_{2} \xrightarrow{\beta_{2}} B_{3}$$

implies that $0 = \varphi_3 a = \varphi_3 \alpha_2 a_2 = \beta_2 \varphi_2 a_2$.

(5) The exact sequence

$$\cdots \longrightarrow B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \longrightarrow \cdots$$

shows that $\varphi_2 a_2 \in \ker(\beta_2) = \operatorname{im}(\beta_1)$. So there exists $b_1 \in B_1$ such that $\varphi_2 a_2 = \beta_1 b_1$.

(6) Consider the commutative diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\alpha_1} & A_2 \\ & & & & & & \downarrow \varphi_2 \\ & & & & & & \downarrow \varphi_2 \\ B_1 & \xrightarrow{\beta_1} & B_2 \end{array}$$

The surjectivity of φ_i implies that $\exists a_1 \in A_1$ such that $\varphi_1 a_1 = b_1$. Hence the commutative diagram implies that $\varphi_2(\alpha_1 a_1) = \varphi_2 \alpha_1 a_1 = \beta_1 \varphi_1 a_1 = \beta_1 b_1 = \varphi_2 a_2$. The injectivity of φ_2 implies that $\alpha_1 a_1 = a_2$.

(7) The exact sequence

$$\cdots \longrightarrow A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \longrightarrow \cdots$$

shows that $a = \alpha_2 a_2 = \alpha_2 \alpha_1 a_1 = 0$. Therefore φ_3 is injective.

Proof of (b).

- (1) Take any $b \in B_3$, it suffices to find $a \in A$ such that $\varphi_3 a = b$.
- (2) Let $b_4 := \beta_3 b \in B_4$. The exact sequence

$$\cdots \longrightarrow B_3 \xrightarrow{\beta_3} B_4 \xrightarrow{\beta_4} B_5 \longrightarrow \cdots$$

shows that $\beta_4 b_4 = \beta_4(\beta_3 b) = 0$.

(3) Look at the commutative diagram

$$\begin{array}{c} A_4 \stackrel{\alpha_4}{\longrightarrow} A_5 \\ \downarrow^{\varphi_4} & \downarrow^{\varphi_5} \\ B_4 \stackrel{\beta_4}{\longrightarrow} B_5 \end{array}$$

By the surjectivity of φ_4 , $\exists a_4 \in A_4$ such that $\varphi_4 a_4 = b_4$. So the commutative diagram says that $0 = \beta_4 b_4 = \beta_4 \varphi_4 a_4 = \varphi_5 \alpha_4 a_4$. By the injectivity of φ_5 , $\alpha_4 a_4 = 0$.

(4) The exact sequence

$$\cdots \longrightarrow A_3 \xrightarrow{\alpha_3} A_4 \xrightarrow{\alpha_4} A_5 \longrightarrow \cdots$$

shows that $a_4 \in \ker(\alpha_4) = \operatorname{im}(\alpha_3)$. So there exists $a_3 \in A_3$ such that $\alpha_3 a_3 = a_4$.

(5) Let $\bar{b} = b - \varphi_3 a_3 \in B_3$. The commutative diagram

$$A_{3} \xrightarrow{\alpha_{3}} A_{4}$$

$$\downarrow^{\varphi_{3}} \qquad \downarrow^{\varphi_{4}}$$

$$B_{3} \xrightarrow{\beta_{3}} B_{4}$$

implies that $\beta_3 \overline{b} = \beta_3 b - \beta_3 \varphi_3 a_3 = \beta_3 b - \varphi_4 \alpha_3 a_3 = \beta_3 b - \varphi_4 a_4 = \beta_3 b - b_4 = \beta_3 b - \beta_3 b = 0$. So $\overline{b} \in \ker(\beta_3)$.

(6) The exact sequence

$$\cdots \longrightarrow B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} B_4 \longrightarrow \cdots$$

shows that $\bar{b} \in \ker(\beta_3) = \operatorname{im}(\beta_2)$. Hence $\exists b_2 \in B_2$ such that $\bar{b} = \beta_2 b_2$.

(7) Look at the commutative diagram

$$\begin{array}{c} A_2 \xrightarrow{\alpha_2} A_3 \\ \downarrow^{\varphi_2} & \downarrow^{\varphi_3} \\ B_2 \xrightarrow{\beta_2} B_3 \end{array}$$

The surjectivity of φ_2 implies that $\exists a_2 \in A_2$ such that $b_2 = \varphi_2 a_2$. Let $a := \alpha_2 a_2 + a_3$. Hence

$$\varphi_3(a) = \varphi_3 \alpha_2 a_2 + \varphi_3 a_3$$

$$= \beta_2 \varphi_2 a_2 + \varphi_3 a_3$$

$$= \beta_2 b_2 + \varphi_3 a_3$$

$$= \overline{b} + \varphi_3 a_3$$

$$= (b - \varphi_3 a_3) + \varphi_3 a_3$$

$$= b.$$
(The diagram commutes)

Exercise 1.4.

Show that the abelian group A admits the structure of a $\mathbb{Z}/(m)$ -module if and only if mA = 0.

Proof.

(1) (\Longrightarrow) It suffices to show that ma = 0 for all $a \in A$. Let $\Lambda = \mathbb{Z}/(m)$.

$$ma = \underbrace{a + \cdots + a}_{m \text{ times}}$$

$$= \underbrace{1_{\Lambda}a + \cdots + 1_{\Lambda}a}_{m \text{ times}} \qquad (Axiom M3)$$

$$= \underbrace{(1_{\Lambda} + \cdots + 1_{\Lambda})a}_{m \text{ times}} \qquad (Axiom M1)$$

$$= 0_{\Lambda}a \qquad (char(\Lambda) = m)$$

$$= 0. \qquad (Axiom M1)$$

(2) (\iff) Write $\overline{\lambda} \in \Lambda := \mathbb{Z}/(m)$ where $\lambda \in \mathbb{Z}$ and $\overline{\lambda}$ is the residue class of λ in Λ . Define $\omega : \Lambda \to \operatorname{End}(A, A)$ by

$$\omega(\overline{\lambda})(a) = \lambda a$$

for all $a \in A$ and $\overline{\lambda} \in \Lambda$. ω is well-defined since mA = 0. Note that all four module axioms hold for A (as a Λ -module).

§2. The Group of Homomorphisms

Exercise 2.1.

Show that in the setting of Theorem 2.1 $\varepsilon_* = \operatorname{Hom}(A, \varepsilon)$ is not, in general, surjective even if ε is. (Hint: Take $\Lambda = \mathbb{Z}$, $A = \mathbb{Z}/(n)$, the integers mod n, and the short exact sequence $\mathbb{Z} \stackrel{\mu}{\rightarrowtail} \mathbb{Z} \stackrel{\varepsilon}{\longrightarrow} \mathbb{Z}/(n)$ where μ is multiplication by n.)

Theorem 2.1. Let $B' \stackrel{\mu}{\rightarrowtail} B \stackrel{\varepsilon}{\longrightarrow} B''$ be an exact sequence of Λ -modules. For every Λ -module A the induced sequence

$$0 \longrightarrow \operatorname{Hom}_{\Lambda}(A, B') \stackrel{\mu_*}{\longrightarrow} \operatorname{Hom}_{\Lambda}(A, B) \stackrel{\varepsilon_*}{\longrightarrow} \operatorname{Hom}_{\Lambda}(A, B'')$$

is exact.

Proof.

(1) Consider

$$\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/(n),\mathbb{Z}) \stackrel{\varepsilon_*}{\longrightarrow} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/(n),\mathbb{Z}/(n)).$$

Note that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/(n),\mathbb{Z}/(n)) \cong \mathbb{Z}/(n)$ is not trivial. So to prove that ε_* is not surjective, it suffices to show that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/(n),\mathbb{Z}) = 0$.

(2) Show that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/(n), \mathbb{Z}) = 0$. Suppose $\alpha \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/(n), \mathbb{Z})$. Given any $a \in \mathbb{Z}/(n)$. So na = 0 by the Lagrange's theorem in group theory. So

$$0 = \alpha(0) = \alpha(na) = n\alpha(a) \in \mathbb{Z}.$$

So $\alpha(a) = 0 \in \mathbb{Z}$. Hence α is a zero map.

Exercise 2.2.

Prove Theorem 2.2. Show that $\mu^* = \operatorname{Hom}_{\Lambda}(\mu, B)$ is not, in general, surjective even if μ is injective. (Hint: Take $\Lambda = \mathbb{Z}$, $B = \mathbb{Z}/(n)$, the integers mod n, and

the short exact sequence $\mathbb{Z} \stackrel{\mu}{\longrightarrow} \mathbb{Z} \stackrel{\varepsilon}{\longrightarrow} \mathbb{Z}/(n)$ where μ is multiplication by n.)

Theorem 2.2. Let $A' \xrightarrow{\mu} A \xrightarrow{\varepsilon} A''$ be an exact sequence of Λ-modules. For every Λ-module B the induced sequence

$$0 \longrightarrow \operatorname{Hom}_{\Lambda}(A'', B) \xrightarrow{\varepsilon^*} \operatorname{Hom}_{\Lambda}(A, B) \xrightarrow{\mu^*} \operatorname{Hom}_{\Lambda}(A', B)$$

is exact.

Proof of Theorem 2.2.

(1) Show that ε^* is injective. Take $\alpha \in \ker(\varepsilon^*) \subseteq \operatorname{Hom}_{\Lambda}(A'', B)$. It suffices to show that $\alpha a'' = 0$ for all $a'' \in A''$. By the surjectivity of ε , there exists $a \in A$ such that $\varepsilon a = a''$. Hence

$$\alpha a'' = \alpha \varepsilon a = (\varepsilon^*(\alpha))(a) = (0)(a) = 0.$$

- (2) Show that $\operatorname{im}(\varepsilon^*) \subseteq \ker(\mu^*)$. A map in $\operatorname{im}(\varepsilon^*)$ is of the form $\alpha \varepsilon$. Plainly, $\varepsilon \mu \alpha$ is a zero map, since $\varepsilon \mu$ already is.
- (3) Show that $\ker(\mu^*) \subseteq \operatorname{im}(\varepsilon^*)$. Consider the diagram

$$A' \xrightarrow{\mu} A \xrightarrow{\varepsilon} A''$$

$$\downarrow^{\alpha}_{\mu} \exists \beta$$

We have to show that if $\mu^*\alpha=\alpha\mu$ is the zero map, then α is of the form $\varepsilon^*\beta=\beta\varepsilon$ for some $\beta:A''\to B$. But if $\alpha\mu=0$, $\ker(\alpha)\supseteq \operatorname{im}(\mu)=\ker(\varepsilon)$. Since ε is surjective, α gives rise to a (unique) map $\beta:A''\to B$ such that $\alpha=\beta\varepsilon$. In brief,

- (a) Define β by $a'' \mapsto \alpha(a)$ where $a \in A$ satisfying $\varepsilon(a) = a''$. The existence of a is guaranteed by the surjectivity of ε .
- (b) β is well-defined since $\ker(\alpha) \supseteq \ker(\varepsilon)$.
- (c) β is a homomorphism since both α, ε are homomorphisms.

Proof.

(1) Show that $\mu^* = \operatorname{Hom}_{\Lambda}(\mu, B)$ is not, in general, surjective even if μ is injective. Consider

$$\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}/(n)) \xrightarrow{\mu^*} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}/(n)).$$

It suffices to show that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}/(n)) \cong \mathbb{Z}/(n)$ canonically. If so, the homomorphism μ^* maps each $\alpha \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}/(n))$ to the zero map in $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}/(n))$, which means μ^* is not surjective.

(2) Show that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/(n)) \cong \mathbb{Z}/(n)$. Take $\alpha \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/(n))$. Note that $\mathbb{Z} = (1)$. So α is uniquely determined by $\alpha(1)$. Conversely, each element $a \in \mathbb{Z}/(n)$ determines a unique homomorphism $\alpha : \mathbb{Z} \to \mathbb{Z}/(n)$ by $\alpha(1) = a$. Hence there is a group isomorphism

$$\Phi: \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/(n)) \to \mathbb{Z}/(n)$$

such that $\Phi: \alpha \mapsto \alpha(1)$. (It is easy to verify that Φ is a group homomorphism.)

Exercise 2.6.

Compute $\operatorname{Hom}(\mathbb{Z},\mathbb{Z}/(n))$, $\operatorname{Hom}(\mathbb{Z}/(m),\mathbb{Z}/(n))$, $\operatorname{Hom}(\mathbb{Z}/(m),\mathbb{Z})$, $\operatorname{Hom}(\mathbb{Q},\mathbb{Z})$, $\operatorname{Hom}(\mathbb{Q},\mathbb{Q})$. Here "Hom" means " $\operatorname{Hom}_{\mathbb{Z}}$ " and \mathbb{Q} is the group of rationals.

Proof.

(1) Show that $\operatorname{Hom}(\mathbb{Z},\mathbb{Z}/(n)) \cong \mathbb{Z}/(n)$. Each $\alpha \in \operatorname{Hom}(\mathbb{Z},\mathbb{Z}/(n))$ is uniquely determined by $\alpha(1) \in \mathbb{Z}/(n)$. Conversely, each element $a \in \mathbb{Z}/(n)$ determines a unique homomorphism $\alpha : \mathbb{Z} \to \mathbb{Z}/(n)$ by $\alpha(1) = a$. Hence there is a group isomorphism

$$\Phi: \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}/(n)) \to \mathbb{Z}/(n).$$

(2) Show that $\operatorname{Hom}(\mathbb{Z}/(m),\mathbb{Z}/(n)) \cong \mathbb{Z}/(m,n)$. Define a map

$$\Phi: \operatorname{Hom}(\mathbb{Z}/(m), \mathbb{Z}/(n)) \to \mathbb{Z}/(m, n)$$

by mapping $\alpha \in \operatorname{Hom}(\mathbb{Z}/(m),\mathbb{Z}/(n))$ to $\overline{\alpha(1)}$ where $\overline{\alpha(1)}$ is the residue class of $\alpha(1) \in \mathbb{Z}/(n)$ in $\mathbb{Z}/(m,n)$. Φ is well-defined. Φ is a group homomorphism. Φ is surjective and injective.

- (3) Show that $\operatorname{Hom}(\mathbb{Z}/(m),\mathbb{Z})=0$. See part (2) in the proof of Exercise 2.1.
- (4) Show that $\operatorname{Hom}(\mathbb{Q}, \mathbb{Z}) = 0$. (Reductio ad absurdum) Suppose there were a non zero map $\alpha : \mathbb{Q} \to \mathbb{Z}$. So $\exists a \in \mathbb{Q}$ such that $\alpha(a) = N \neq 0$. Note that

$$\alpha(a) = \alpha \left(\underbrace{\frac{a}{n} + \dots + \frac{a}{n}}_{n \text{ times}}\right) = \underbrace{\alpha\left(\frac{a}{n}\right) + \dots + \alpha\left(\frac{a}{n}\right)}_{n \text{ times}} = n\alpha\left(\frac{a}{n}\right)$$

for all integers n. As $\alpha\left(\frac{a}{n}\right) \in \mathbb{Z}$, $n \mid \alpha(a)$ for all $n \in \mathbb{Z}$, which is absurd.

(5) Show that $\operatorname{Hom}(\mathbb{Q}, \mathbb{Q}) \cong \mathbb{Q}$. Note that each $\alpha \in \operatorname{Hom}(\mathbb{Q}, \mathbb{Q})$ is uniquely determined by $\alpha(1) \in \mathbb{Q}$. $(\alpha(r) = r\alpha(1))$ by the similar argument in (4) and part (2) in the proof of Exercise 2.1.) Conversely, each element $a \in \mathbb{Q}$ determines a unique homomorphism $\alpha : \mathbb{Q} \to \mathbb{Q}$ by $\alpha(1) = a$. Hence there is a group isomorphism

$$\Phi: \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}) \to \mathbb{Q}.$$

§3. Sums and Products

Exercise 3.1.

Show that there is a canonical map $\sigma: \bigoplus_j A_j \to \prod_i A_j$.

Proof.

- (1) Define $\sigma: (a_j)_{j \in J} \mapsto (a_j)_{j \in J}$.
- (2) σ is well-defined since there are no restrictions on $\sigma((a_j)_{j\in J})$ though $(a_j)_{j\in J}\in \oplus_j A_j$ has one restriction on $(a_j)_{j\in J}$ (say $a_j\neq 0$ for only a finite number of subscripts).
- (3) σ is a Λ -module homomorphism and σ is injective.

Exercise 3.6.

Show that $\mathbb{Z}/(m) \oplus \mathbb{Z}/(n) = \mathbb{Z}/(mn)$ if and only if m and n are mutually prime.

Proof.

(1) (\Longrightarrow) Given any $g := (g_1, g_2) \in \mathbb{Z}/(m) \oplus \mathbb{Z}/(n) \cong \mathbb{Z}/(mn)$. As $mg_1 = 0 \in \mathbb{Z}/(m)$ and $ng_2 = 0 \in \mathbb{Z}/(n)$,

$$lcm(m, n)g = (lcm(m, n)g_1, lcm(m, n)g_2) = (0, 0).$$

So $\operatorname{lcm}(m, n)$ is divisible by the order of $\mathbb{Z}/(mn)$, that is, $mn|\operatorname{lcm}(m, n)$. Hence $\gcd(m, n) = 1$ since $mn = \operatorname{lcm}(m, n)\gcd(m, n)$. (2) \iff Define $\alpha: \mathbb{Z} \to \mathbb{Z}/(m) \oplus \mathbb{Z}/(n)$ by

$$\alpha: g \mapsto (g+(m), g+(n)).$$

 α is a group homomorphism. The Chinese remainder theorem and $\gcd(m,n)=1$ shows that α is surjective. The kernel of α is (mn). Hence α induces a group isomorphism

$$\overline{\alpha}: \mathbb{Z}/(mn) \to \mathbb{Z}/(m) \oplus \mathbb{Z}/(n).$$