Chapter 8: Some Special Functions

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Supplement. Fourier coefficients in Definition 8.9.

(1) Write

$$f(x) = a_0 + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}$$

(as the textbook Rudin, Principles of Mathematical Analysis, Third Edition). Then

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx.$$
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx, n \in \mathbb{Z}^+.$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx, n \in \mathbb{Z}^+.$$

(2) One might write in one different form,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

The only difference between the new one and the old one is a_0, so a_0 should be

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx.$$

(3) Again, one might write in one different form,

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

Similarly, a_0 should be

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x)}{\sqrt{2}} dx.$$

(4) Recall $f(x) = \sum_{-N}^{N} c_n e^{inx}$ ($x \in \mathbb{R}$) where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

The relations among a_n, b_n of this textbook and c_n are

$$c_0 = a_0$$
$$c_n = \frac{1}{2}\left(a_n + ib_n\right), n \in \mathbb{Z}^+.$$

Supplement. Parseval's theorem 8.16.

(1) Given
$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx)), x \in \mathbb{R}.$$
Then
$$\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2dx = 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

(2) Given
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx)), x \in \mathbb{R}.$$
Then
$$\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

(3) Given
$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx)), x \in \mathbb{R}.$$
Then
$$\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2dx = a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

Exercise 8.1. *Define*
$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$
Prove that f has derivatives of all orders at $x = 0$, and that $f^{(n)}(0) = 0$ for $n = 1, 2, 3, ...$

$f(x)$ is an example of non-analytic smooth function, that is, infinitely differentiable functions are not necessarily analytic. In this exercise, we will show that Taylor series of f at the origin converges everywhere to the zero function. So the Taylor series does not equal $f(x)$ for $x \neq 0$. Consequently, f is not analytic at $x = 0$.

Claim 1.
$$\lim_{x \to 0} g(x) e^{-\frac{1}{x^2}} = 0$$
for any rational function $g(x) \in \mathbb{R}(x)$.

Proof. Write $g(x) = \frac{p(x)}{q(x)}$ for some $p(x), q(x) \in \mathbb{R}[x]$. Write $q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0$. $q(x)$ is not identically zero, that is, there exists the unique coefficient of the least power of x in $q(x)$ which is non-zero, say $b_M \neq 0$. Now write $g(x)$ as $g(x) = \frac{p(x)/x^M}{q(x)/x^M}$. The denominator of $g(x)$ tends to $b_M \neq 0$ as $x \to 0$. By the similar argument of Theorem 8.6(f) ($\lim_{x \to \infty} x^n e^{-x} = 0$ for any $n \in \mathbb{Z}$),
$$\frac{p(x)}{x^M} e^{-\frac{1}{x^2}} \to 0 \text{ as } x \to 0.$$
Hence, $\lim_{x \to 0} g(x) e^{-\frac{1}{x^2}} = 0$ for any $g(x) \in \mathbb{R}(x)$. □

Claim 2. *Given any real* $x \neq 0$
$$f^{(n)}(x) = g_n(x) e^{-\frac{1}{x^2}}$$
for some rational function $g(x) \in \mathbb{R}(x)$.

Proof. Say $g_0(x) = 1 \in \mathbb{R}(x)$. Notice that $\mathbb{R}(x)$ is a field and $g'(x) \in \mathbb{R}(x)$ for any $g(x) \in \mathbb{R}(x)$. (Write $g(x) = \frac{p(x)}{q(x)}$ for some $p(x), q(x) \in \mathbb{R}[x]$. Notice that $p'(x) \in \mathbb{R}[x]$ for any $p(x) \in \mathbb{R}[x]$.) Now we prove by mathematical induction. For $n = 1$, we have
$$\begin{aligned} f'(x) &= g_0'(x) e^{-\frac{1}{x^2}} + g_0(x) \cdot \left(-\frac{1}{x^2} \right)' e^{-\frac{1}{x^2}} \\ &= \left(g_0'(x) + g_0(x) \cdot \left(-\frac{1}{x^2} \right)' \right) e^{-\frac{1}{x^2}} \\ &= g_1(x) e^{-\frac{1}{x^2}} \end{aligned}$$
where $g_1(x) = g_0'(x) + g_0(x) \cdot (-\frac{1}{x^2})' \in \mathbb{R}(x)$. Now assume $n = k$ holds. For $n = k+1$, similar to $n = 1$, $f^{(k+1)}(x) = g_{k+1}(x) e^{-\frac{1}{x^2}}$ where $g_{k+1}(x) = g_k'(x) + g_k(x) \cdot (-\frac{1}{x^2})' \in \mathbb{R}(x)$. □

Proof of Exercise 8.1. Prove by mathematical induction. For $n = 1$,
$$f'(0) = \lim_{t \to 0} \frac{e^{-\frac{1}{t^2}} - 0}{t} = 0.$$
(Use Claim 1.) Now assume $n = k$ holds. For $n = k + 1$,
$$f^{(k+1)}(0) = \lim_{t \to 0} \frac{f^{(k)}(t) - f^{(k)}(0)}{t} = \lim_{t \to 0} \frac{g_k(t) e^{-\frac{1}{t^2}} - 0}{t} = 0.$$

(Use Claim 1 and 2.) Thus, $f^{(n)}(0) = 0$ for $n \in \mathbb{Z}^+$. $\square$

Exercise 8.6. *Suppose $f(x)f(y) = f(x+y)$ for all real x and y.*
(a) Assuming that f is differentiable and not zero, prove that

$$f(x) = e^{cx}$$

where c is a constant.
(b) Prove the same thing, assuming only that f is continuous.

(b) implies (a). We prove (b) directly.

Proof of (b). Since $f(x)$ is not zero, there exists $x_0 \in \mathbb{R}$ such that $f(x_0) \neq 0$. So $f(0)f(x_0) = f(x_0)$, or $f(0) = 1$ by cancelling $f(x_0) \neq 0$.

Next, $f(\frac{n}{m}) = f(\frac{1}{m})^n$ for $m \in \mathbb{Z}$, $n \in \mathbb{Z}^+$. Since f is continuous at $x = 0$, f is positive in the neighborhood of $x = 0$. That is, there exists $N \in \mathbb{Z}^+$ such that $f(\frac{1}{m}) > 0$ whenever $|m| \geq N$. So, $f(\frac{n}{m}) = f(\frac{1}{m})^n > 0$. (Since $f(\frac{n}{m}) = f(\frac{kn}{km})$ for any $k \in \mathbb{Z}^+$, we can rescale m to km such that $|km| \geq N$.) That is, f is positive on $\mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and f is continuous on $\mathbb{R}$, f is positive on $\mathbb{R}$.

Now let $c = \log f(1)$ (which is well-defined since $f > 0$). We write $f(1)$ in the two ways. Firstly, $f(1) = f(\frac{n}{n}) = f(\frac{1}{n})^n$ where $n \in \mathbb{Z}^+$. Secondly, $f(1) = e^c = (e^{\frac{c}{n}})^n$. Since the positive n-th root is unique (Theorem 1.21), $f(\frac{1}{n}) = e^{\frac{c}{n}}$ for $n \in \mathbb{Z}^+$. By $f(x)f(-x) = f(0) = 1$ or $f(-x) = \frac{1}{f(x)}$, $f(-\frac{1}{n}) = \frac{1}{e^{\frac{c}{n}}} = e^{-\frac{c}{n}}$ for $n \in \mathbb{Z}^+$. Therefore,

$$f\left(\frac{1}{m}\right) = e^{\frac{c}{m}} \text{ where } m \in \mathbb{Z}.$$

By using $f(\frac{n}{m}) = f(\frac{1}{m})^n$ for $m \in \mathbb{Z}$, $n \in \mathbb{Z}^+$ again, $f(\frac{n}{m}) = e^{c\frac{n}{m}}$ where $m \in \mathbb{Z}, n \in \mathbb{Z}^+$, or

$$f(x) = e^{cx} \text{ where } x \in \mathbb{Q}.$$

Since $g(x) = f(x) - e^{cx}$ vanishes on a dense set of $\mathbb{Q}$ and g is continuous on $\mathbb{R}$, g vanishes on $\mathbb{R}$. Therefore, $f(x) = e^{cx}$ for $x \in \mathbb{R}$. $\square$

Supplement. Proof of (a).
Proof of (a). Since $f(x)$ is not zero, there exists $x_0 \in \mathbb{R}$ such that $f(x_0) \neq 0$. So $f(0)f(x_0) = f(x_0)$, or $f(0) = 1$ by cancelling $f(x_0) \neq 0$.

Since f is differentiable, for any $x \in \mathbb{R}$,

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x)f(h) - f(x)}{h} \\ &= f(x) \lim_{h \to 0} \frac{f(h) - 1}{h} \\ &= f(x) \lim_{h \to 0} \frac{f(h) - f(0)}{h} \\ &= f(x) f'(0). \end{aligned}$$

Let $c = f'(0)$ be a constant. Then $f'(x) = cf(x)$. So $f(x) = e^{cx}$ for $x \in \mathbb{R}$. (To see this, let $g(x) = \frac{f(x)}{e^{cx}}$ be well-defined on $\mathbb{R}$. $g(0) = 1$. $g'(x) = 0$ since $f'(x) = cf(x)$. So $g(x)$ is a constant, or $g(x) = 1$ since $g(0) = 1$. Therefore, $f(x) = e^{cx}$ on $\mathbb{R}$.) $\square$

Supplement. Cauchy's functional equation.

(1) *(Cauchy's functional equation.) Suppose $f(x) + f(y) = f(x + y)$ for all real x and y. Assuming that f is continuous, prove that $f(x) = cx$ where c is a constant.*

 Notice that we cannot let $g(x) = \log f(x)$ and apply Cauchy's functional equation on $g(x)$ to prove Exercise 8.6 since $f(x)$ is not necessary positive and thus $g(x) = \log f(x)$ might be meaningless. However, this wrong approach gives you some useful ideas such as you need to prove that $f(x)$ is positive first, and $f(x)$ should be equal to e^{cx} where $c = g(1) = \log f(1)$.

(2) *Suppose $f(xy) = f(x) + f(y)$ for all positive real x and y. Assuming that f is continuous, prove that $f(x) = c \log x$ where c is a constant.*

(3) *Suppose $f(xy) = f(x)f(y)$ for all positive real x and y. Assuming that f is continuous and positive, prove that $f(x) = x^c$ where c is a constant.*

(4) *Suppose $f(x + y) = f(x) + f(y) + xy$ for all real x and y. Assuming that f is continuous, prove that $f(x) = \frac{1}{2}x^2 + cx$ where c is a constant.*

(5) *(USA 2002.) Suppose $f(x^2 - y^2) = xf(x) - yf(y)$ for all real x and y. Assuming that f is continuous, prove that $f(x) = cx$ where c is a constant.*

Exercise 8.10. *Prove that $\sum \frac{1}{p}$ diverges; the sum extends over all primes.*

There are many proofs of this result. We provide some of them.

Proof (Due to hint). Given N.

Claim 1. *Show that* $\sum_{n\leq N}\frac{1}{n}\leq\prod_{p\leq N}\left(1-\frac{1}{p}\right)^{-1}$.

Proof of Claim 1. By the unique factorization theorem on $n\leq N$,

$$\sum_{n\leq N}\frac{1}{n}\leq\prod_{p\leq N}\left(1+\frac{1}{p}+\frac{1}{p^2}+\cdots\right)=\prod_{p\leq N}\left(1-\frac{1}{p}\right)^{-1}.$$

□

By Claim 1 and the fact that $\sum\frac{1}{n}$ diverges, there are infinitely many primes.

Claim 2. *Show that* $\prod_{p\leq N}\left(1-\frac{1}{p}\right)^{-1}\leq\exp\left(\sum_{p\leq N}\frac{2}{p}\right)$.

Proof of Claim 2. By applying the inequality $(1-x)^{-1}<e^{2x}$ where $x\in(0,\frac{1}{2}]$ on any prime p,

$$\left(1-\frac{1}{p}\right)^{-1}<\exp\left(\frac{2}{p}\right).$$

Now multiplying the inequality over all primes $p\leq N$ and noticing that $\exp(x)\cdot\exp(y)=\exp(x+y)$, we have

$$\prod_{p\leq N}\left(1-\frac{1}{p}\right)^{-1}\leq\exp\left(\sum_{p\leq N}\frac{2}{p}\right).$$

□

By Claim 1 and Claim 2,

$$\sum_{n\leq N}\frac{1}{n}\leq\exp\left(\sum_{p\leq N}\frac{2}{p}\right).$$

Since $\sum_{n\leq N}\frac{1}{n}$ diverges, the result holds. □

Proof (Due to Kenneth Ireland and Michael Rosen). The proof in Kenneth Ireland and Michael Rosen, A Classical Introduction to Modern Number Theory, Second Edition (Theorem 3 in Chapter 2) does not use the inequality $(1-x)^{-1}<e^{2x}$ ($x\in(0,\frac{1}{2}]$) directly. Instead, the authors take the logarithm on $(1-p^{-1})^{-1}$ and estimate it. (So the length of proof is longer than the proof due to hint.)

That is,

$$
\begin{aligned}
-\log(1-p^{-1}) &= \sum_{n=1}^{\infty} \frac{p^{-n}}{n} \\
&= \frac{1}{p} + \sum_{n=2}^{\infty} \frac{p^{-n}}{n} \\
&< \frac{1}{p} + \sum_{n=2}^{\infty} p^{-n} \\
&= \frac{1}{p} + \frac{p^{-2}}{1-p^{-1}} \\
&< \frac{1}{p} + 2 \cdot \frac{1}{p^2}.
\end{aligned}
$$

Now we sum over all primes $p \le N$,

$$\log \left(\prod_{p \le N} \left(1 - \frac{1}{p}\right)^{-1} \right) < \sum_{p \le N} \frac{1}{p} + 2 \sum_{p \le N} \frac{1}{p^2}.$$

So

$$\log \sum_{n \le N} \frac{1}{n} < \sum_{p \le N} \frac{1}{p} + 2 \sum_{p \le N} \frac{1}{p^2}.$$

Notice that $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{p^2}$ converges (since $\sum \frac{1}{n^2}$ converges). Therefore, $\sum \frac{1}{p}$ diverges. $\square$

Proof (Due to I. Niven). It is an exercise in Kenneth Ireland and Michael Rosen, A Classical Introduction to Modern Number Theory, Second Edition. See Exercise 27 in Chapter 2.

Claim 1. *Show that $\sum' \frac{1}{n}$, the sum being over square free integers, diverges.*
Proof of Claim 1. For any positive integers n, we can write $n = a^2 b$ where $a \in \mathbb{Z}^+$ and b is a square free integer. Given N,

$$\sum_{n \le N} \frac{1}{n} \le \left(\sum_{a=1}^{\infty} \frac{1}{a^2} \right) \left(\sum_{b \le N}{}' \frac{1}{b} \right).$$

Notices that $\sum_{a=1}^{\infty} \frac{1}{a^2}$ converges. Since $\sum_{n \le N} \frac{1}{n} \to \infty$ as $N \to \infty$, $\sum_{b \le N}' \frac{1}{b} \to \infty$ as $N \to \infty$. $\square$

Claim 2. *Show that $\prod_{p \le N}(1 + \frac{1}{p}) \to \infty$ as $N \to \infty$.*
Proof of Claim 2. By the unique factorization theorem on $n \le N$,

$$\prod_{p \le N} \left(1 + \frac{1}{p}\right) \ge \sum_{n \le N}{}' \frac{1}{n}.$$

Since ${\sum_{n\le N}}' \frac{1}{n} \to \infty$ as $N \to \infty$ (Claim 1), the conclusion is established. $\square$

By applying the inequality $e^x > 1 + x$ on any prime p,

$$\exp\left(\frac{1}{p}\right) > 1 + \frac{1}{p}.$$

Now multiplying the inequality over all primes $p \le N$ and noticing that $\exp(x) \cdot \exp(y) = \exp(x+y)$, we have

$$\exp\left(\sum_{p\le N} \frac{1}{p}\right) > \prod_{p\le N} \left(1 + \frac{1}{p}\right).$$

By Claim 2, $\exp\left(\sum_{p\le N} \frac{1}{p}\right) \to \infty$ as $N \to \infty$, or $\sum_{p\le N} \frac{1}{p} \to \infty$ as $N \to \infty$. $\square$

Exercise 8.12. *Suppose* $0 < \delta < \pi$*,*

$$f(x) = \begin{cases} 1 & \text{if } |x| \le \delta, \\ 0 & \text{if } \delta < |x| \le \pi, \end{cases}$$

and $f(x + 2\pi) = f(x)$ *for all* x*.*
(a) Compute the Fourier coefficients of f*.*
(b) Compute that

$$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2} \qquad (0 < \delta < \pi).$$

(c) Deduce from Parseval's theorem that

$$\sum_{n=1}^{\infty} \frac{(\sin(n\delta))^2}{n^2\delta} = \frac{\pi - \delta}{2}.$$

(d) Let $\delta \to 0$ *and prove that*

$$\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}.$$

(e) Put $\delta = \frac{\pi}{2}$ *in (c). What do you get?*

It is a centered square pulse around $x = 0$ with shift δ. Besides, $f(x)$ is an even function.

Proof of (a).

$$\begin{aligned}
c_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \\
&= \frac{1}{2\pi}\int_{-\delta}^{\delta} dx \\
&= \frac{\delta}{\pi}.
\end{aligned}$$

For $0 \neq n \in \mathbb{Z}$,

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \\
&= \frac{1}{2\pi}\int_{-\delta}^{\delta} e^{-inx}dx \\
&= \frac{1}{2\pi}\cdot\frac{2\sin(n\delta)}{n} \\
&= \frac{\sin(n\delta)}{n\pi}.
\end{aligned}$$

□

Supplement. Find a_n and b_n of this textbook.
By (a), $a_0 = \frac{\delta}{\pi}$, $a_n = \frac{2\sin(n\delta)}{n\pi}$, $b_n = 0$ for $n \in \mathbb{Z}^+$. Surely, we can compute a_n and b_n $(n > 0)$ directly. Since $f(x)$ is an even function, $b_n = 0$. And

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \\
&= \frac{2}{\pi}\int_{0}^{\delta} \cos(nx)dx \\
&= \frac{2\sin(n\delta)}{n\pi}.
\end{aligned}$$

Proof of (b). Given $x = 0$, there are constants $\delta' = \delta > 0$ and $M = 1 < \infty$ such that

$$|f(0+t) - f(0)| \leq M|t|$$

for all $t \in (-\delta', \delta')$. By Theorem 8.14,

$$\sum_{-\infty}^{\infty} c_n = f(0).$$

Notice that $c_{-n} = c_n$ for $n \in \mathbb{Z}^+$, so

$$\begin{aligned}
\frac{\delta}{\pi} + 2\sum_{n=1}^{\infty}\frac{\sin(n\delta)}{n\pi} &= 1 \\
\sum_{n=1}^{\infty}\frac{\sin(n\delta)}{n} &= \frac{\pi - \delta}{2}.
\end{aligned}$$

$\square$

We can also use the expression a_n and b_n to prove the same thing. Besides, taking $\delta = 1$ yields

$$\sum_{n=1}^{\infty} \frac{\sin n}{n} = \frac{\pi - 1}{2}.$$

Proof of (c). Since $f(x)$ is a Riemann-integrable function with period 2π, by Parseval's theorem

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{-\infty}^{\infty} |c_n|^2.$$

So

$$\frac{\delta}{\pi} = \frac{\delta^2}{\pi^2} + 2 \sum_{n=1}^{\infty} \frac{(\sin(n\delta))^2}{n^2 \pi^2},$$

or

$$\sum_{n=1}^{\infty} \frac{(\sin(n\delta))^2}{n^2 \delta} = \frac{\pi - \delta}{2}.$$

$\square$

Notices that

$$\sum_{n=1}^{\infty} \frac{(\sin n)^2}{n^2} = \frac{\pi - 1}{2}$$

as $\delta = 1$.

Proof of (d). TODO. $\square$

Proof of (e).

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

Write

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^2} &= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \\ &= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}, \end{aligned}$$

so

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{6}.$$

□

Exercise 8.13. *Put $f(x) = x$ if $0 \le x < 2\pi$, and apply Parseval's theorem to conclude that*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi}{6}.$$

Proof.

$$\begin{aligned} c_0 &= \frac{1}{2\pi} \int_0^{2\pi} x dx \\ &= \pi, \end{aligned}$$

For $n \neq 0$,

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_0^{2\pi} x e^{-inx} dx \\ &= \frac{1}{2\pi} \left(\left[-\frac{1}{in} x e^{-inx} \right]_{x=0}^{x=2\pi} - \int_0^{2\pi} -\frac{1}{in} e^{-inx} dx \right) \\ &= \frac{i}{n}. \end{aligned}$$

Since $f(x)$ is a Riemann-integrable function with period 2π, by Parseval's theorem

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{-\infty}^{\infty} |c_n|^2.$$

So

$$\frac{1}{2\pi} \cdot \frac{(2\pi)^3}{3} = \pi^2 + 2 \sum_{n=1}^{\infty} \frac{1}{n^2},$$

or

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

□

Supplement. *Put $f(x) = x^k$ if $k \in \mathbb{Z}^+$ and $0 \le x < 2\pi$. Might show that*

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = r_k \pi^{2k}, r_k \in \mathbb{Q}.$$