

Chapter 8: Some Special Functions

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Supplement. Fourier coefficients in Definition 8.9.

(1) Write

$$f(x) = a_0 + \sum_{n=1}^N (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}$$

(as the textbook Rudin, Principles of Mathematical Analysis, Third Edition).

Then

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx.$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx, n \in \mathbb{Z}^+.$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx, n \in \mathbb{Z}^+.$$

(2) One might write in one different form,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^N (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

The only difference between the new one and the old one is a_0 , so a_0 should be

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx.$$

(3) Again, one might write in one different form,

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^N (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

Similarly, a_0 should be

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x)}{\sqrt{2}} dx.$$

(4) Recall $f(x) = \sum_{n=-N}^N c_n e^{inx}$ ($x \in \mathbb{R}$) where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

The relations among a_n , b_n of this textbook and c_n are

$$\begin{aligned} c_0 &= a_0 \\ c_n &= \frac{1}{2}(a_n + ib_n), n \in \mathbb{Z}^+. \end{aligned}$$

- (5) In some textbooks (Henryk Iwaniec, Topics in Classical Automorphic Forms), it is convenient to consider periodic functions f of period 1. Define

$$e(n) = e^{2\pi i x} = \cos(2\pi x) + i \sin(2\pi x).$$

Any periodic and piecewise continuous function f has the Fourier series representation

$$f(x) = \sum_{-\infty}^{\infty} a_n e(nx)$$

with coefficients given by

$$a_n = \int_0^1 f(x) e(-nx) dx.$$

Here is one exercise for this representation. *Show that the fractional part of x , $\{x\} = x - [x]$, is given by*

$$\{x\} = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{\pi n}.$$

Supplement. Parseval's theorem 8.16.

- (1) Given

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

Then

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = 2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

- (2) Given

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

Then

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

(3) Given

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

Then

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

Exercise 8.1. Define

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

Prove that f has derivatives of all orders at $x = 0$, and that $f^{(n)}(0) = 0$ for $n = 1, 2, 3, \dots$

$f(x)$ is an example of non-analytic smooth function, that is, infinitely differentiable functions are not necessarily analytic. In this exercise, we will show that Taylor series of f at the origin converges everywhere to the zero function. So the Taylor series does not equal $f(x)$ for $x \neq 0$. Consequently, f is not analytic at $x = 0$.

Proof.

(1) Show that

$$\lim_{x \rightarrow 0} g(x) e^{-\frac{1}{x^2}} = 0$$

for any rational function $g(x) \in \mathbb{R}(x)$.

- (a) Write $g(x) = \frac{p(x)}{q(x)}$ for some $p(x), q(x) \in \mathbb{R}[x]$, $g(x) \neq 0$.
- (b) Write $q(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_0$. $q(x)$ is not identically zero, that is, there exists the unique coefficient of the least power of x in $q(x)$ which is non-zero, say $b_M \neq 0$.
- (c) Thus,

$$g(x) = \frac{p(x)/x^M}{q(x)/x^M}.$$

The denominator of $g(x)$ tends to $b_M \neq 0$ as $x \rightarrow 0$. By the similar argument in Theorem 8.6(f), we have

$$\frac{p(x)}{x^M} e^{-\frac{1}{x^2}} \rightarrow 0 \text{ as } x \rightarrow 0.$$

Hence, $\lim_{x \rightarrow 0} g(x) e^{-\frac{1}{x^2}} = 0$ for any $g(x) \in \mathbb{R}(x)$.

(2) Given any real $x \neq 0$, show that

$$f^{(n)}(x) = g_n(x) e^{-\frac{1}{x^2}}$$

for some rational function $g(x) \in \mathbb{R}(x)$.

- (a) Say $g_0(x) = 1 \in \mathbb{R}(x)$.
- (b) $\mathbb{R}(x)$ is a field. Show that $g'(x) \in \mathbb{R}(x)$ for any $g(x) \in \mathbb{R}(x)$. Write $g(x) = \frac{p(x)}{q(x)}$ for some $p(x), q(x) \in \mathbb{R}[x]$, $q(x) \neq 0$. Thus

$$g'(x) = \frac{p'(x)q(x) - p(x)q'(x)}{q(x)^2}.$$

The numerator of $g'(x)$ is in $\mathbb{R}[x]$ since the differentiation operator on $\mathbb{R}[x]$ is closed in $\mathbb{R}[x]$. Also, the denominator of $g'(x) = \frac{p'(x)q(x) - p(x)q'(x)}{q(x)^2} \neq 0$ since $\mathbb{R}[x]$ is an integral domain. Therefore, $g'(x) \in \mathbb{R}(x)$.

- (c) Induction on n . For $n = 1$, we have

$$\begin{aligned} f'(x) &= g'_0(x)e^{-\frac{1}{x^2}} + g_0(x) \cdot \left(-\frac{1}{x^2}\right)' e^{-\frac{1}{x^2}} \\ &= \left(g'_0(x) + g_0(x) \cdot \left(-\frac{1}{x^2}\right)'\right) e^{-\frac{1}{x^2}} \\ &= g_1(x)e^{-\frac{1}{x^2}} \end{aligned}$$

where

$$g_1(x) = g'_0(x) + g_0(x) \cdot \left(-\frac{1}{x^2}\right)' \in \mathbb{R}(x).$$

Now assume that the conclusion holds for $n = k$. As $n = k + 1$, similar to the case $n = 1$,

$$f^{(k+1)}(x) = g_{k+1}(x)e^{-\frac{1}{x^2}}$$

where

$$g_{k+1}(x) = g'_k(x) + g_k(x) \cdot \left(-\frac{1}{x^2}\right)' \in \mathbb{R}(x).$$

By induction, the conclusion is true.

- (3) Induction on n . For $n = 1$, by (1) we have

$$f'(0) = \lim_{t \rightarrow 0} \frac{e^{-\frac{1}{t^2}} - 0}{t} = 0.$$

Now assume that the statement holds for $n = k$. As $n = k + 1$, by (1)(2) we have

$$f^{(k+1)}(0) = \lim_{t \rightarrow 0} \frac{f^{(k)}(t) - f^{(k)}(0)}{t} = \lim_{t \rightarrow 0} \frac{g_k(t)e^{-\frac{1}{t^2}} - 0}{t} = 0.$$

Thus, $f^{(n)}(0) = 0$ for $n \in \mathbb{Z}^+$.

□

Exercise 8.2. Let a_{ij} be the number in the i th row and j th column of the array

$$\begin{array}{ccccc} -1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & -1 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{2} & -1 & 0 & \cdots \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

so that

$$a_{ij} = \begin{cases} 0 & (i < j), \\ -1 & (i = j), \\ 2^{j-i} & (i > j). \end{cases}$$

Prove that

$$\sum_i \sum_j a_{ij} = -2, \quad \sum_j \sum_i a_{ij} = 0.$$

Also see Theorem 8.3.

Proof (Brute-force).

$$\begin{aligned} \sum_i \sum_j a_{ij} &= \sum_{i=1}^{\infty} \left(\sum_{j=i}^{\infty} a_{ij} + \sum_{j<i} a_{ij} \right) \\ &= \sum_{i=1}^{\infty} \left(-1 + \sum_{j=1}^{i-1} 2^{j-i} \right) \\ &= \sum_{i=1}^{\infty} (-1 + (1 - 2^{1-i})) \\ &= \sum_{i=1}^{\infty} -2^{1-i} \\ &= -2. \end{aligned}$$

$$\begin{aligned}
\sum_j \sum_i a_{ij} &= \sum_{j=1}^{\infty} \left(\sum_{i=j}^{\infty} a_{ij} + \sum_{i>j} a_{ij} \right) \\
&= \sum_{j=1}^{\infty} \left(-1 + \sum_{i=j+1}^{\infty} 2^{j-i} \right) \\
&= \sum_{j=1}^{\infty} (-1 + 1) \\
&= \sum_{j=1}^{\infty} 0 \\
&= 0.
\end{aligned}$$

□

Exercise 8.3. *Prove that*

$$\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$$

if $a_{ij} \geq 0$ for all i and j (the case $+\infty = +\infty$ may occur).

Note. It can be proved by Theorem 8.3 if both summations are finite.

Proof.

(1) Let $\mathcal{F}(I)$ be the collection of all finite subsets of I .

(2) Let

$$s = \sup \left\{ \sum_{(i,j) \in E} a_{ij} : E \in \mathcal{F}(\mathbb{N}^2) \right\}$$

(the case $s = +\infty$ may occur). *It suffices to show that $\sum_i \sum_j a_{ij} = s$.*

The case $\sum_j \sum_i a_{ij} = s$ is similar, and thus $\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$.

(3) *Show that $\sum_i \sum_j a_{ij} \geq s$.* Given any $E \in \mathcal{F}(\mathbb{N}^2)$. It is clear that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \geq \sum_{(i,j) \in E} a_{ij}$$

(since $a_{ij} \geq 0$). Thus,

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \geq \sup \left\{ \sum_{(i,j) \in E} a_{ij} : E \in \mathcal{F}(\mathbb{N}^2) \right\} = s.$$

- (4) Show that $\sum_i \sum_j a_{ij} \leq s$. (Reductio ad absurdum) If $\sum_i \sum_j a_{ij} > s$, especially $s < \infty$, then there exists $\varepsilon > 0$ such that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} > s + \varepsilon,$$

or

$$\sum_{i=1}^n \sum_{j=1}^{\infty} a_{ij} > s + \varepsilon$$

for some integer n . Consider two possible cases.

- (a) If there is some $1 \leq i_0 \leq n$ such that

$$\sum_{j=1}^{\infty} a_{i_0 j} = \infty,$$

then there is some m such that

$$\sum_{j=1}^m a_{i_0 j} > s.$$

For $E = \{(i_0, 1), \dots, (i_0, m)\} \in \mathcal{F}(\mathbb{N}^2)$,

$$\sum_{(i,j) \in E} a_{ij} = \sum_{j=1}^m a_{i_0 j} > s,$$

contrary to the supremum of s .

- (b) Otherwise, for each $1 \leq i \leq n$ we have

$$\sum_{j=1}^{\infty} a_{ij} < \infty,$$

or there exists some m_i such that

$$\sum_{j=1}^{m_i} a_{ij} > \sum_{j=1}^{\infty} a_{ij} - \frac{\varepsilon}{n}.$$

For $E = \bigcup_{1 \leq i \leq n} \{(i, 1), \dots, (i, m_i)\} \in \mathcal{F}(\mathbb{N}^2)$,

$$\begin{aligned}
\sum_{(i,j) \in E} a_{ij} &= \sum_{i=1}^n \sum_{j=1}^{m_i} a_{ij} \\
&> \sum_{i=1}^n \left(\sum_{j=1}^{\infty} a_{ij} - \frac{\varepsilon}{n} \right) \\
&= \sum_{i=1}^n \sum_{j=1}^{\infty} a_{ij} - \sum_{i=1}^n \frac{\varepsilon}{n} \\
&> s + \varepsilon - \varepsilon \\
&= s,
\end{aligned}$$

contrary to the supremum of s .

Therefore, $\sum_i \sum_j a_{ij} \leq s$.

- (5) By (3)(4), $\sum_i \sum_j a_{ij} = s$. Similarly, $\sum_j \sum_i a_{ij} = s$. Hence, $\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$ (including the case $+\infty = +\infty$).

□

Exercise 8.4. *Prove the following limit relations:*

(a) $\lim_{x \rightarrow 0} \frac{b^x - 1}{x} = \log b \quad (b > 0).$

(b) $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1.$

(c) $\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e.$

(d) $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x.$

Proof of (a).

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{b^x - 1}{x} &= \lim_{x \rightarrow 0} \frac{\exp(x \log b) - 1}{x} \\
&= \left. \frac{d}{dx} \exp(x \log b) \right|_{x=0} \\
&= \exp(x \log b) \cdot \log b \Big|_{x=0} \\
&= \log b.
\end{aligned}$$

□

Proof of (b).

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} &= \left. \frac{d}{dx} \log(1+x) \right|_{x=0} \\ &= \left. \frac{1}{x+1} \right|_{x=0} \\ &= 1.\end{aligned}$$

□

Proof of (c).

$$\begin{aligned}\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} &= \lim_{x \rightarrow 0} \exp\left(\frac{\log(1+x)}{x}\right) \\ &= \exp\left(\lim_{x \rightarrow 0} \frac{\log(1+x)}{x}\right) \\ &= \exp(1) \\ &= e.\end{aligned}$$

□

Proof of (d).

$$\begin{aligned}\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n &= \lim_{n \rightarrow \infty} \left(\left(1 + \frac{x}{n}\right)^{\frac{n}{x}}\right)^x \\ &= \left(\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^{\frac{n}{x}}\right)^x \\ &= \left(\lim_{y \rightarrow 0} (1+y)^{\frac{1}{y}}\right)^x \\ &= \exp(x).\end{aligned}$$

□

Exercise 8.5. Find the following limits

(a) $\lim_{x \rightarrow 0} \frac{e - (1+x)^{\frac{1}{x}}}{x}.$

(b) $\lim_{n \rightarrow \infty} \frac{n}{\log n} \left[n^{\frac{1}{n}} - 1 \right].$

(c) $\lim_{x \rightarrow 0} \frac{\tan x - x}{x(1 - \cos x)}.$

(d) $\lim_{x \rightarrow 0} \frac{x - \sin x}{\tan x - x}.$

Proof of (a). By L'Hospital's rule (Theorem 5.13),

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{e - (1+x)^{\frac{1}{x}}}{x} &= \lim_{x \rightarrow 0} \frac{-(1+x)^{\frac{1}{x}} \cdot \frac{\frac{x}{x+1} - \log(x+1)}{x^2}}{1} \\
&= \lim_{x \rightarrow 0} \left(-(1+x)^{\frac{1}{x}} \cdot \frac{\frac{x}{x+1} - \log(x+1)}{x^2} \right) \\
&= - \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} \cdot \lim_{x \rightarrow 0} \frac{\frac{x}{x+1} - \log(x+1)}{x^2} \\
&= -e \cdot \lim_{x \rightarrow 0} \frac{\frac{x}{x+1} - \log(x+1)}{x^2} \quad (\text{Exercise 8.4(c)}) \\
&= -e \cdot \lim_{x \rightarrow 0} \frac{-\frac{x}{(x+1)^2}}{2x} \\
&= e \cdot \lim_{x \rightarrow 0} \frac{1}{2(x+1)^2} \\
&= e \cdot \frac{1}{2} \\
&= \frac{e}{2}.
\end{aligned}$$

Here

$$\begin{aligned}
\frac{d}{dx} \left(e - (1+x)^{\frac{1}{x}} \right) &= \frac{d}{dx} \left(e - \exp \left(\frac{\log(x+1)}{x} \right) \right) \\
&= - \exp \left(\frac{1}{x} \log(x+1) \right) \cdot \frac{\frac{1}{x+1} \cdot x - \log(x+1) \cdot 1}{x^2} \\
&= -(1+x)^{\frac{1}{x}} \cdot \frac{\frac{x}{x+1} - \log(x+1)}{x^2},
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dx} \left(\frac{x}{x+1} - \log(x+1) \right) &= \frac{(x+1) - x}{(x+1)^2} - \frac{1}{x+1} \\
&= -\frac{x}{(x+1)^2}.
\end{aligned}$$

□

Proof of (b).

(1) Let $x = \frac{\log n}{n}$. Note that $\lim_{n \rightarrow \infty} \frac{\log n}{n} = 0$.

(2)

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{n}{\log n} \left[n^{\frac{1}{n}} - 1 \right] &= \lim_{n \rightarrow \infty} \frac{n}{\log n} \left[\exp \left(\frac{\log n}{n} \right) - 1 \right] \\ &= \lim_{x \rightarrow 0} \frac{\exp(x) - 1}{x} \\ &= \frac{d}{dx} \exp(x) \Big|_{x=0} \\ &= \exp(x) \Big|_{x=0} \\ &= 1.\end{aligned} \tag{1)}$$

□

Proof of (c) (L'Hospital's rule). By L'Hospital's rule (Theorem 5.13) three times,

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\tan x - x}{x(1 - \cos x)} &= \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{1 - \cos x + x \sin x} \\ &= \lim_{x \rightarrow 0} \frac{2 \sec x (\tan x \sec x)}{\sin x + \sin x + x \cos x} \\ &= \lim_{x \rightarrow 0} \frac{2 \tan x \sec^2 x}{2 \sin x + x \cos x} \\ &= \lim_{x \rightarrow 0} \frac{2[\sec^2 x \sec^2 x + \tan x \cdot 2 \sec x (\tan x \sec x)]}{2 \cos x + \cos x - x \sin x} \\ &= \lim_{x \rightarrow 0} \frac{2 \sec^4 x + 2 \sec^2 x \tan^2 x}{3 \cos x - x \sin x} \\ &= \frac{2}{3}.\end{aligned}$$

□

Proof of (c) (Taylor series). Since

$$\begin{aligned}\cos x &= 1 - \frac{x^2}{2} + O(x^4) \\ \tan x &= x + \frac{x^3}{3} + O(x^5),\end{aligned}$$

we have

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x(1 - \cos x)} = \lim_{x \rightarrow 0} \frac{\frac{x^3}{3} + O(x^5)}{\frac{x^3}{2} + O(x^5)} = \frac{2}{3}.$$

□

Proof of (d) (L'Hospital's rule). By L'Hospital's rule (Theorem 5.13) three times,

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{x - \sin x}{\tan x - x} &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{\sec^2 x - 1} \\
&= \lim_{x \rightarrow 0} \frac{\sin x}{2 \sec x (\tan x \sec x)} \\
&= \lim_{x \rightarrow 0} \frac{\sin x}{2 \tan x \sec^2 x} \\
&= \lim_{x \rightarrow 0} \frac{\cos x}{2 \tan x \sec^2 x} \\
&= \lim_{x \rightarrow 0} \frac{\cos x}{2 [\sec^2 x \sec^2 x + \tan x \cdot 2 \sec x (\tan x \sec x)]} \\
&= \lim_{x \rightarrow 0} \frac{\cos x}{2 \sec^4 x + 2 \sec^2 x \tan^2 x} \\
&= \frac{1}{2}.
\end{aligned}$$

□

Proof of (d) (Taylor series). Since

$$\begin{aligned}
\sin x &= x - \frac{x^3}{6} + O(x^5) \\
\tan x &= x + \frac{x^3}{3} + O(x^5),
\end{aligned}$$

we have

$$\lim_{x \rightarrow 0} \frac{x - \sin x}{\tan x - x} = \lim_{x \rightarrow 0} \frac{\frac{x^3}{6} + O(x^5)}{\frac{x^3}{3} + O(x^5)} = \frac{1}{2}.$$

□

Exercise 8.6. Suppose $f(x)f(y) = f(x+y)$ for all real x and y .

(a) Assuming that f is differentiable and not zero, prove that

$$f(x) = e^{cx}$$

where c is a constant.

(b) Prove the same thing, assuming only that f is continuous.

Part (b) implies part (a). We prove part (b) directly.

Proof of (b).

- (1) Since $f(x)$ is not zero, there exists $x_0 \in \mathbb{R}$ such that $f(x_0) \neq 0$. So $f(0)f(x_0) = f(x_0)$, or $f(0) = 1$ by cancelling $f(x_0) \neq 0$.
- (2) Next, $f(\frac{n}{m}) = f(\frac{1}{m})^n$ for $m \in \mathbb{Z}$, $n \in \mathbb{Z}^+$. Since f is continuous at $x = 0$, f is positive in the neighborhood of $x = 0$. That is, there exists $N \in \mathbb{Z}^+$ such that $f(\frac{1}{m}) > 0$ whenever $|m| \geq N$. So, $f(\frac{n}{m}) = f(\frac{1}{m})^n > 0$. (Since $f(\frac{n}{m}) = f(\frac{kn}{km})$ for any $k \in \mathbb{Z}^+$, we can rescale m to km such that $|km| \geq N$.) That is, f is positive on \mathbb{Q} . Since \mathbb{Q} is dense in \mathbb{R} and f is continuous on \mathbb{R} , f is positive on \mathbb{R} .
- (3) Now let $c = \log f(1)$ (which is well-defined since $f > 0$). We write $f(1)$ in the two ways. Firstly, $f(1) = f(\frac{n}{n}) = f(\frac{1}{n})^n$ where $n \in \mathbb{Z}^+$. Secondly, $f(1) = e^c = (e^{\frac{c}{n}})^n$. Since the positive n -th root is unique (Theorem 1.21), $f(\frac{1}{n}) = e^{\frac{c}{n}}$ for $n \in \mathbb{Z}^+$. By $f(x)f(-x) = f(0) = 1$ or $f(-x) = \frac{1}{f(x)}$, $f(-\frac{1}{n}) = \frac{1}{e^{\frac{c}{n}}} = e^{-\frac{c}{n}}$ for $n \in \mathbb{Z}^+$. Therefore,

$$f\left(\frac{1}{m}\right) = e^{\frac{c}{m}} \text{ where } m \in \mathbb{Z}.$$

- (4) By using $f(\frac{n}{m}) = f(\frac{1}{m})^n$ for $m \in \mathbb{Z}$, $n \in \mathbb{Z}^+$ again, $f(\frac{n}{m}) = e^{c\frac{n}{m}}$ where $m \in \mathbb{Z}, n \in \mathbb{Z}^+$, or

$$f(x) = e^{cx} \text{ where } x \in \mathbb{Q}.$$

Since $g(x) = f(x) - e^{cx}$ vanishes on a dense set of \mathbb{Q} and g is continuous on \mathbb{R} , g vanishes on \mathbb{R} . Therefore, $f(x) = e^{cx}$ for $x \in \mathbb{R}$.

□

Supplement. *Proof of (a).*

- (1) Since $f(x)$ is not zero, there exists $x_0 \in \mathbb{R}$ such that $f(x_0) \neq 0$. So $f(0)f(x_0) = f(x_0)$, or $f(0) = 1$ by cancelling $f(x_0) \neq 0$.
- (2) Since f is differentiable, for any $x \in \mathbb{R}$,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x)f(h) - f(x)}{h} \\ &= f(x) \lim_{h \rightarrow 0} \frac{f(h) - 1}{h} \\ &= f(x) \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} \\ &= f(x)f'(0). \end{aligned}$$

Let $c = f'(0)$ be a constant. Then $f'(x) = cf(x)$. So $f(x) = e^{cx}$ for $x \in \mathbb{R}$. (To see this, let $g(x) = \frac{f(x)}{e^{cx}}$ be well-defined on \mathbb{R} . $g(0) = 1$. $g'(x) = 0$ since $f'(x) = cf(x)$. So $g(x)$ is a constant, or $g(x) = 1$ since $g(0) = 1$. Therefore, $f(x) = e^{cx}$ on \mathbb{R} .)

□

Supplement. Cauchy's functional equation.

- (1) *(Cauchy's functional equation.) Suppose $f(x) + f(y) = f(x + y)$ for all real x and y . Assuming that f is continuous, prove that $f(x) = cx$ where c is a constant.*

Notice that we cannot let $g(x) = \log f(x)$ and apply Cauchy's functional equation on $g(x)$ to prove Exercise 8.6 since $f(x)$ is not necessarily positive and thus $g(x) = \log f(x)$ might be meaningless. However, this wrong approach gives you some useful ideas such as you need to prove that $f(x)$ is positive first, and $f(x)$ should be equal to e^{cx} where $c = g(1) = \log f(1)$.

- (2) *Suppose $f(xy) = f(x) + f(y)$ for all positive real x and y . Assuming that f is continuous, prove that $f(x) = c \log x$ where c is a constant.*
- (3) *Suppose $f(xy) = f(x)f(y)$ for all positive real x and y . Assuming that f is continuous and positive, prove that $f(x) = x^c$ where c is a constant.*
- (4) *Suppose $f(x + y) = f(x) + f(y) + xy$ for all real x and y . Assuming that f is continuous, prove that $f(x) = \frac{1}{2}x^2 + cx$ where c is a constant.*
- (5) *(USA 2002.) Suppose $f(x^2 - y^2) = xf(x) - yf(y)$ for all real x and y . Assuming that f is continuous, prove that $f(x) = cx$ where c is a constant.*

Supplement. Show that the only automorphism of \mathbb{Q} is the identity.

Proof. Given any $\sigma \in \text{Aut}(\mathbb{Q})$.

- (1) *Show that $\sigma(1) = 1$.* Since $1^2 = 1$, $\sigma(1)\sigma(1) = \sigma(1)$. $\sigma(1) = 0$ or 1 . There are only two possible cases.

- (a) Assume that $\sigma(1) = 0$. So

$$\sigma(a) = \sigma(a \cdot 1) = \sigma(a) \cdot \sigma(1) = \sigma(a) \cdot 0 = 0$$

for any $a \in \mathbb{Q}$. That is, $\sigma = 0 \in \text{Aut}(\mathbb{Q})$, which is absurd.

- (b) Therefore, $\sigma(1) = 1$.

- (2) *Show that $\sigma(n) = n$ for all $n \in \mathbb{Z}^+$.* Write $n = 1 + 1 + \cdots + 1$ (n times 1). Applying the additivity of σ , we have

$$\sigma(n) = \sigma(1) + \sigma(1) + \cdots + \sigma(1) = 1 + 1 + \cdots + 1 = n.$$

(Might use induction on n to eliminate \cdots symbols.)

- (3) *Show that $\sigma(n) = n$ for all $n \in \mathbb{Z}$.* By the additivity of σ , $\sigma(-n) = -\sigma(n) = -n$ for $n \geq 0$. The result is established.

For any $a = \frac{n}{m} \in \mathbb{Q}$ ($m, n \in \mathbb{Z}$, $n \neq 0$), applying the multiplication of σ on $am = n$, that is, $\sigma(a)\sigma(m) = \sigma(n)$. By (3), we have $\sigma(a)m = n$, or

$$\sigma(a) = \frac{m}{n} = a$$

provided $n \neq 0$, or σ is the identity. \square

Exercise 8.7. If $0 < x < \frac{\pi}{2}$, prove that

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1.$$

Proof.

(1) Let

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

be a continuous function on $[0, \frac{\pi}{2}]$ (since $\lim_{x \rightarrow 0+} f(x) = 1$). So

$$f'(x) = \frac{x \cos x - \sin x}{x^2} < 0$$

on $(0, \frac{\pi}{2})$ since $\tan x > x$ on $(0, \frac{\pi}{2})$.

(2) Show that $\frac{\sin x}{x} < 1$ on $(0, \frac{\pi}{2})$. Given any $x \in (0, \frac{\pi}{2})$, there exists $\xi_1 \in (0, x)$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(\xi_1) < 0$$

by the mean value theorem (Theorem 5.10). So $f(x) < f(0) = 1$, or $\frac{\sin x}{x} < 1$.

(3) Show that $\frac{\sin x}{x} > \frac{2}{\pi}$ on $(0, \frac{\pi}{2})$. Given any $x \in (0, \frac{\pi}{2})$, there exists $\xi_2 \in (0, x)$ such that

$$\frac{f(\frac{\pi}{2}) - f(x)}{\frac{\pi}{2} - x} = f'(\xi_2) < 0$$

by the mean value theorem (Theorem 5.10). So $f(x) > f(\frac{\pi}{2}) = \frac{2}{\pi}$, or $\frac{\sin x}{x} > \frac{2}{\pi}$.

\square

Exercise 8.8. For $n = 0, 1, 2, \dots$, and x real, prove that

$$|\sin(nx)| \leq n|\sin x|.$$

Note that this inequality may be false for other values of n . For instance,

$$\left| \sin\left(\frac{1}{2}\pi\right) \right| > \frac{1}{2} |\sin \pi|.$$

Proof. Induction on n .

(1) Note that

$$\sin(a + b) = \sin a \cos b + \cos a \sin b$$

for any $a, b \in \mathbb{R}$.

(2) $n = 0, 1$ are clearly true.

(3) Assume the induction hypothesis that for the single case $n = k$ holds, meaning

$$|\sin(kx)| \leq k |\sin x|$$

is true. It follows that

$$\begin{aligned} |\sin((k+1)x)| &= |\sin(kx) \cos x + \cos(kx) \sin x| && ((1)) \\ &\leq |\sin(kx)| |\cos x| + |\cos(kx)| |\sin x| && (\text{Triangle inequality}) \\ &\leq |\sin(kx)| + |\sin x| && (|\cos(\cdot)| \leq 1) \\ &\leq k |\sin x| + |\sin x| && (\text{Induction hypothesis}) \\ &\leq (k+1) |\sin x|. \end{aligned}$$

□

Exercise 8.9 (The Euler-Mascheroni constant).

(a) Put $s_N = 1 + \frac{1}{2} + \cdots + \frac{1}{N}$. Prove that

$$\lim_{N \rightarrow \infty} (s_N - \log N)$$

exists. (The limit, often denoted by γ , is called Euler's constant. Its numerical value is $0.5772\dots$. It is not known whether γ is rational or not.)

(b) Roughly how large must m be so that $N = 10^m$ satisfies $s_N > 100$?

Proof of (a) (Theorem 3.14).

(1) Note that

$$\begin{aligned}
& \frac{1}{1 + \frac{1}{n}} \leq \frac{1}{x} \leq 1 \text{ for } x \in \left[1, 1 + \frac{1}{n}\right] \\
& \Rightarrow \int_1^{1 + \frac{1}{n}} \frac{dx}{1 + \frac{1}{n}} \leq \int_1^{1 + \frac{1}{n}} \frac{dx}{x} \leq \int_1^{1 + \frac{1}{n}} dx \quad (\text{Theorem 6.12(b)}) \\
& \Rightarrow \frac{1}{n+1} \leq \int_1^{1 + \frac{1}{n}} \frac{dx}{x} \leq \frac{1}{n} \\
& \Rightarrow \frac{1}{n+1} \leq \log \left(1 + \frac{1}{n}\right) \leq \frac{1}{n}. \quad (\text{Equation (39) on page 180})
\end{aligned}$$

(2) Define

$$\gamma_n = s_n - \log n.$$

It suffices to show that $\{\gamma_n\}$ is monotonic and bounded (Theorem 3.14).

(3) Show that $\{\gamma_n\}$ is decreasing.

$$\begin{aligned}
\gamma_{n+1} - \gamma_n &= (s_{n+1} - \log(n+1)) - (s_n - \log n) \\
&= (s_{n+1} - s_n) - (\log(n+1) - \log n) \\
&= \frac{1}{n+1} - \log \left(\frac{n+1}{n}\right) \\
&= \frac{1}{n+1} - \log \left(1 + \frac{1}{n}\right) \\
&\leq 0. \quad ((1))
\end{aligned}$$

Note. $\gamma_n \leq \dots \leq \gamma_1 = 1$ for all $n = 1, 2, 3, \dots$

(4) Show that $\gamma_n \geq 0$ for all $n = 1, 2, 3, \dots$ Since

$$\begin{aligned}
\log n &= \sum_{k=1}^{n-1} (\log(k+1) - \log k) \\
&= \sum_{k=1}^{n-1} \log \frac{k+1}{k} \\
&= \sum_{k=1}^{n-1} \log \left(1 + \frac{1}{k}\right) \\
&\leq \sum_{k=1}^{n-1} \frac{1}{k} \quad ((1)) \\
&= s_{n-1},
\end{aligned}$$

we have

$$\gamma_n = s_n - \log n \geq s_n - s_{n-1} = \frac{1}{n} > 0.$$

By (3)(4), $\{\gamma_n\}$ converges to $\lim_{N \rightarrow \infty} (s_N - \log N) = \gamma$. \square

Supplement. Show that if $f \geq 0$ on $[0, \infty)$ and f is monotonically decreasing, and if

$$c_n = \sum_{k=1}^n f(k) - \int_1^n f(x) dx,$$

then $\lim_{n \rightarrow \infty} c_n$ exists. (Exercise 10 of Section 5.2 in the textbook: *R Creighton Buck, Advanced Calculus, 3rd edition*. See page 235.) If this exercise is true, we can get the existence of γ by taking $f(x) = \frac{1}{x}$.

(1) Note that

$$f(n+1) \leq \int_n^{n+1} f(x) dx \leq f(n).$$

(2) Show that $\{c_n\}$ is decreasing.

$$c_{n+1} - c_n = f(n+1) - \int_n^{n+1} f(x) dx \leq 0.$$

(3) Show that $c_n \geq 0$. Since $f(k) \geq \int_k^{k+1} f(x) dx$,

$$\begin{aligned} \sum_{k=1}^n f(k) &\geq \sum_{k=1}^n \int_k^{k+1} f(x) dx \\ &= \int_1^{n+1} f(x) dx \\ &\geq \int_1^n f(x) dx. \end{aligned} \quad (f \geq 0)$$

So that $c_n = \sum_{k=1}^n f(k) - \int_1^n f(x) dx \geq 0$.

(4) By (2)(3), $\{c_n\}$ converges (Theorem 3.14).

\square

Proof of (a) (Limit comparison test). Inspired by this paper: *Philippe Flajolet and Ilan Vardi, Zeta Function Expansions of Classical Constants*.

(1) Rewrite

$$\gamma_n + \log n - \log(n+1) = \sum_{k=1}^n \left(\frac{1}{k} - \log \left(1 + \frac{1}{k} \right) \right)$$

(similar to the argument in (a)(4)(Theorem 3.14)). Let

$$c_k = \frac{1}{k} - \log \left(1 + \frac{1}{k} \right).$$

(2) Show that

$$\lim_{k \rightarrow \infty} \frac{c_k}{\frac{1}{k^2}} = \frac{1}{2}.$$

In fact,

$$\begin{aligned} & \lim_{k \rightarrow \infty} \frac{c_k}{\frac{1}{k^2}} \\ &= \lim_{x \rightarrow 0} \frac{x - \log(1+x)}{x^2} \quad \left(\text{Put } x = \frac{1}{k}\right) \\ &= \lim_{x \rightarrow 0} \frac{1 - \frac{1}{1+x}}{2x} \quad (\text{L'Hospital's rule}) \\ &= \lim_{x \rightarrow 0} \frac{1}{2(x+1)} \\ &= \frac{1}{2}. \end{aligned}$$

(3) By limit comparison test or comparison test, $\sum c_k$ converges since $\sum \frac{1}{k^2}$ converges. Also,

$$\lim_{n \rightarrow \infty} \log n - \log(n+1) = 0.$$

Therefore, $\lim_{n \rightarrow \infty} \gamma_n$ exists.

□

Note. This proof is based on **limit comparison test** (Theorem 8.21) in this textbook: *Tom. M. Apostol, Mathematical Analysis, 2nd edition*. It is easy to prove by the original comparison test.

Proof of (a) (Comparison test).

(1) Note that

$$0 \leq x - \log(x+1) \leq \frac{x^2}{2}$$

for all $x \geq 0$.

(2) Write

$$c_n = \frac{1}{n} - \log\left(1 + \frac{1}{n}\right).$$

as in the the proof of (a) (Limit comparison test). By (1),

$$|c_n| \leq \frac{1}{2n^2}$$

for all $n = 1, 2, \dots$. Hence, by the comparison test (Theorem 3.25(a)), $\sum c_n$ converges since $\sum \frac{1}{n^2}$ converges (to $\frac{\pi^2}{6}$). Use the same argument in the proof of (a) (Limit comparison test), since

$$\gamma_n + \log n - \log(n+1) = \sum c_n \text{ and } \lim_{n \rightarrow \infty} \log n - \log(n+1) = 0,$$

we have the existence of $\lim \gamma_n = \gamma$.

□

Proof of (a) (Uniformly convergence of $\sum \frac{x}{n(x+n)}$). (One example to Exercise 7 of Section 6.2 in the textbook: *R Creighton Buck, Advanced Calculus, 3rd edition*. See pages 270 to 271.)

(1) Let

$$f_n(x) = \frac{x}{n(x+n)} = \frac{1}{n} - \frac{1}{x+n}$$

defined on $E = [0, 1]$.

(2) Note that

$$|f_n(x)| \leq \frac{1}{n^2}$$

for all $x \in [0, 1]$. Since $\sum \frac{1}{n^2}$ converges, $\sum f_n$ converges uniformly on $[0, 1]$ (Theorem 7.10).

(3) Corollary to Theorem 7.16 implies that

$$\begin{aligned} \int_0^1 \sum_{n=1}^{\infty} \frac{x}{n(x+n)} dx &= \sum_{n=1}^{\infty} \int_0^1 \frac{x}{n(x+n)} dx \\ &= \sum_{n=1}^{\infty} \int_0^1 \left(\frac{1}{n} - \frac{1}{x+n} \right) dx \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \log \frac{n+1}{n} \right) \\ &= \lim_{N \rightarrow \infty} \left(\sum_{n=1}^N \frac{1}{n} - \log(N+1) \right) \\ &= \lim_{N \rightarrow \infty} (s_N - \log(N+1)) \end{aligned}$$

exists. Since $\lim_{N \rightarrow \infty} (\log(N+1) - \log N) = 0$,

$$\begin{aligned} \gamma &= \lim_{N \rightarrow \infty} (s_N - \log N) \\ &= \lim_{N \rightarrow \infty} (s_N - \log(N+1)) + \lim_{N \rightarrow \infty} (\log(N+1) - \log N) \end{aligned}$$

exists.

□

Proof of (a) (Existence of $\int_1^{\infty} \frac{\{x\}}{x^2} dx$).

- (1) Define $\{x\} = x - [x]$ where $[x]$ is the greatest integer $\leq x$ (Exercise 6.16).
Show that

$$\int_1^\infty \frac{\{x\}}{x^2} dx < \infty.$$

Use the similar argument in Exercise 6.16(b). Since $\frac{\{x\}}{x^2} \leq \frac{1}{x^2}$ on $[1, \infty)$ and $\int_1^\infty \frac{1}{x^2} dx = 1$ exists, the result is established (Theorem 6.12(b)).

- (2) Show that

$$\int_1^N \frac{[x]}{x^2} dx = s_N - 1.$$

Use the similar argument in Exercise 6.16(a),

$$\begin{aligned} \int_1^N \frac{[x]}{x^2} dx &= \sum_{k=1}^{N-1} \int_k^{k+1} \frac{[x]}{x^2} dx \\ &= \sum_{k=1}^{N-1} \int_k^{k+1} \frac{k}{x^2} dx \\ &= \sum_{k=1}^{N-1} \int_k^{k+1} \frac{k}{x^2} dx \\ &= \sum_{k=1}^{N-1} \frac{1}{k+1} \\ &= \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} \\ &= s_N - 1. \end{aligned}$$

Supplement (Euler's summation formula). (Theorem 7.13 in the textbook: Tom. M. Apostol, *Mathematical Analysis*, 2nd edition.) If f has a continuous derivative f' on $[a, b]$, then we have

$$\sum_{a < n \leq b} f(n) = \int_a^b f(x) dx + \int_a^b f'(x) \{x\} dx + f(a)\{a\} - f(b)\{b\},$$

where $\sum_{a < n \leq b}$ means the sum from $n = [a] + 1$ to $n = [b]$. When a and b are integers, this becomes

$$\sum_{n=a}^b f(n) = \int_a^b f(x) dx + \int_a^b f'(x) \left(\{x\} - \frac{1}{2} \right) dx + \frac{f(a) + f(b)}{2}.$$

By taking $f(x) = \frac{1}{x}$ we can get the same result.

(3) Show that

$$\int_1^N \frac{\{x\}}{x^2} dx = \log N - s_N + 1 = 1 - \gamma_N.$$

In fact,

$$\begin{aligned} \int_1^N \frac{\{x\}}{x^2} dx &= \int_1^N \frac{x - [x]}{x^2} dx \\ &= \int_1^N \frac{1}{x} dx - \int_1^N \frac{[x]}{x^2} dx \\ &= \log N - (s_N - 1) \\ &= \log N - s_N + 1 \\ &= 1 - \gamma_N. \end{aligned}$$

(4) Since

$$\lim_{N \rightarrow \infty} \int_1^N \frac{\{x\}}{x^2} dx = \int_1^\infty \frac{\{x\}}{x^2} dx$$

exists (by (1)), $\gamma = \lim \gamma_N$ exists.

□

Proof of (b). By $s_n - \log n > 0$ in (a)(4)(Theorem 3.14), it suffices to choose $N = 10^m$ such that $s_N \geq \log(N+1) > 100$, or

$$m > \frac{\log(\exp(100) - 1)}{\log 10},$$

or choose m satisfying

$$m > \frac{100}{\log 10} > \frac{\log(\exp(100) - 1)}{\log 10},$$

or $m = 44$. □

Note. The exact value of N is

$$15092688622113788323693563264538101449859497 \approx 1.509 \times 10^{43}.$$

Exercise 8.10. Prove that $\sum \frac{1}{p}$ diverges; the sum extends over all primes.

There are many proofs of this result. We provide some of them.

Proof (Due to hint). Given N .

(1) Show that

$$\sum_{n \leq N} \frac{1}{n} \leq \prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1}.$$

By the unique factorization theorem on $n \leq N$,

$$\sum_{n \leq N} \frac{1}{n} \leq \prod_{p \leq N} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1}.$$

(2) By (1) and the fact that $\sum \frac{1}{n}$ diverges, there are infinitely many primes.

(3) Show that

$$\prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1} \leq \exp \left(\sum_{p \leq N} \frac{2}{p} \right).$$

By applying the inequality $(1 - x)^{-1} < e^{2x}$ where $x \in (0, \frac{1}{2}]$ on any prime p ,

$$\left(1 - \frac{1}{p}\right)^{-1} < \exp \left(\frac{2}{p} \right).$$

Now multiplying the inequality over all primes $p \leq N$ and noticing that $\exp(x) \cdot \exp(y) = \exp(x + y)$, we have

$$\prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1} \leq \exp \left(\sum_{p \leq N} \frac{2}{p} \right).$$

(4) By (1)(3),

$$\sum_{n \leq N} \frac{1}{n} \leq \exp \left(\sum_{p \leq N} \frac{2}{p} \right).$$

Since $\sum_{n \leq N} \frac{1}{n}$ diverges, the result holds.

□

Proof (Due to Kenneth Ireland and Michael Rosen). The proof in Kenneth Ireland and Michael Rosen, A Classical Introduction to Modern Number Theory, Second Edition (Theorem 3 in Chapter 2) does not use the inequality $(1 - x)^{-1} < e^{2x}$ ($x \in (0, \frac{1}{2}]$) directly. Instead, the authors take the logarithm on $(1 - p^{-1})^{-1}$ and estimate it. (So the length of proof is longer than the proof due to hint.)

That is,

$$\begin{aligned}
-\log(1 - p^{-1}) &= \sum_{n=1}^{\infty} \frac{p^{-n}}{n} \\
&= \frac{1}{p} + \sum_{n=2}^{\infty} \frac{p^{-n}}{n} \\
&< \frac{1}{p} + \sum_{n=2}^{\infty} p^{-n} \\
&= \frac{1}{p} + \frac{p^{-2}}{1 - p^{-1}} \\
&< \frac{1}{p} + 2 \cdot \frac{1}{p^2}.
\end{aligned}$$

Now we sum over all primes $p \leq N$,

$$\log \left(\prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1} \right) < \sum_{p \leq N} \frac{1}{p} + 2 \sum_{p \leq N} \frac{1}{p^2}.$$

So

$$\log \sum_{n \leq N} \frac{1}{n} < \sum_{p \leq N} \frac{1}{p} + 2 \sum_{p \leq N} \frac{1}{p^2}.$$

Notice that $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{p^2}$ converges (since $\sum \frac{1}{n^2}$ converges). Therefore, $\sum \frac{1}{p}$ diverges. \square

Proof (Due to I. Niven). It is an exercise in Kenneth Ireland and Michael Rosen, A Classical Introduction to Modern Number Theory, Second Edition. See Exercise 27 in Chapter 2.

- (1) Show that $\sum' \frac{1}{n}$, the sum being over square free integers, diverges. For any positive integers n , we can write $n = a^2 b$ where $a \in \mathbb{Z}^+$ and b is a square free integer. Given N ,

$$\sum_{n \leq N} \frac{1}{n} \leq \left(\sum_{a=1}^{\infty} \frac{1}{a^2} \right) \left(\sum'_{b \leq N} \frac{1}{b} \right).$$

Notice that $\sum_{a=1}^{\infty} \frac{1}{a^2}$ converges. Since $\sum_{n \leq N} \frac{1}{n} \rightarrow \infty$ as $N \rightarrow \infty$, $\sum'_{b \leq N} \frac{1}{b} \rightarrow \infty$ as $N \rightarrow \infty$.

- (2) Show that

$$\prod_{p \leq N} \left(1 + \frac{1}{p}\right) \rightarrow \infty \text{ as } N \rightarrow \infty.$$

By the unique factorization theorem on $n \leq N$,

$$\prod_{p \leq N} \left(1 + \frac{1}{p}\right) \geq \sum'_{n \leq N} \frac{1}{n}.$$

Since $\sum'_{n \leq N} \frac{1}{n} \rightarrow \infty$ as $N \rightarrow \infty$ by (1), the conclusion is established.

(3) By applying the inequality $e^x > 1 + x$ on any prime p ,

$$\exp\left(\frac{1}{p}\right) > 1 + \frac{1}{p}.$$

Now multiplying the inequality over all primes $p \leq N$ and noticing that $\exp(x) \cdot \exp(y) = \exp(x + y)$, we have

$$\exp\left(\sum_{p \leq N} \frac{1}{p}\right) > \prod_{p \leq N} \left(1 + \frac{1}{p}\right).$$

By (2), $\exp\left(\sum_{p \leq N} \frac{1}{p}\right) \rightarrow \infty$ as $N \rightarrow \infty$, or $\sum_{p \leq N} \frac{1}{p} \rightarrow \infty$ as $N \rightarrow \infty$.

□

Exercise 8.11. Suppose $f \in \mathcal{R}$ on $[0, A]$ for all $A < \infty$, and $f(x) \rightarrow 1$ as $x \rightarrow +\infty$. Prove that

$$\lim_{t \rightarrow 0} t \int_0^\infty e^{-tx} f(x) dx = 1 \quad (t > 0),$$

It is similar to Exercise 3.14(a).

Proof. Given any $\varepsilon > 0$.

(1) The integral $\int_0^\infty e^{-tx} f(x) dx$ is well-defined. (It suffices to show that $\int_0^\infty e^{-tx} f(x) dx$ converges absolutely in the sense of Exercise 6.8. It is quite easy since $f(x) \rightarrow 1$ as $x \rightarrow +\infty$ and well-behavior of $\int_{A_0}^\infty e^{-tx} f(x) dx$ for any $A_0 > 0$.)

(2) Note that

$$t \int_0^\infty e^{-tx} dx = 1$$

for any $t > 0$.

(3) Since $f(x) \rightarrow 1$ as $x \rightarrow +\infty$, there is $A_0 > 0$ such that

$$|f(x) - 1| < \frac{\varepsilon}{64} \text{ whenever } x \geq A_0.$$

(4) Since $f \in \mathcal{R}$ on $[0, A_0]$, f is bounded on $[0, A_0]$, or $|f| \leq M$ on $[0, A_0]$ for some M (Theorem 6.7(c)).

(5) As $t > 0$,

$$\begin{aligned} & \left| \left(t \int_0^\infty e^{-tx} f(x) dx \right) - 1 \right| \\ &= \left| t \int_0^\infty e^{-tx} (f(x) - 1) dx \right| \end{aligned} \quad ((2))$$

$$\leq t \int_0^\infty e^{-tx} |f(x) - 1| dx \quad ((1) \text{ with Theorem 6.13})$$

$$\begin{aligned} &= t \int_0^{A_0} e^{-tx} |f(x) - 1| dx + t \int_{A_0}^\infty e^{-tx} |f(x) - 1| dx \\ &\leq t \int_0^{A_0} (M + 1) dx + t \int_{A_0}^\infty e^{-tx} |f(x) - 1| dx \end{aligned} \quad ((3) \text{ and } e^{-tx} \leq 1)$$

$$\leq t \int_0^{A_0} (M + 1) dx + t \int_{A_0}^\infty e^{-tx} \frac{\varepsilon}{64} dx \quad ((4))$$

$$\begin{aligned} &= t A_0 (M + 1) + \exp(-A_0 t) \frac{\varepsilon}{64} \\ &\leq t A_0 (M + 1) + \frac{\varepsilon}{64}. \end{aligned} \quad (e^{-tx} \leq 1)$$

Since t is arbitrary, take $t = \frac{\varepsilon}{89 A_0 (M + 1)} > 0$ to get

$$\left| \left(t \int_0^\infty e^{-tx} f(x) dx \right) - 1 \right| < \frac{\varepsilon}{89} + \frac{\varepsilon}{64} < \varepsilon,$$

or

$$\lim_{t \rightarrow 0^+} t \int_0^\infty e^{-tx} f(x) dx = 1.$$

□

Exercise 8.12. Suppose $0 < \delta < \pi$,

$$f(x) = \begin{cases} 1 & \text{if } |x| \leq \delta, \\ 0 & \text{if } \delta < |x| \leq \pi, \end{cases}$$

and $f(x + 2\pi) = f(x)$ for all x .

(a) Compute the Fourier coefficients of f .

(b) Compute that

$$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2} \quad (0 < \delta < \pi).$$

(c) Deduce from Parseval's theorem that

$$\sum_{n=1}^{\infty} \frac{(\sin(n\delta))^2}{n^2\delta} = \frac{\pi - \delta}{2}.$$

(d) Let $\delta \rightarrow 0$ and prove that

$$\int_0^{\infty} \left(\frac{\sin x}{x} \right)^2 dx = \frac{\pi}{2}.$$

(e) Put $\delta = \frac{\pi}{2}$ in (c). What do you get?

It is a centered square pulse around $x = 0$ with shift δ . Besides, $f(x)$ is an even function.

Proof of (a).

$$\begin{aligned} c_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \\ &= \frac{1}{2\pi} \int_{-\delta}^{\delta} dx \\ &= \frac{\delta}{\pi}. \end{aligned}$$

For $0 \neq n \in \mathbb{Z}$,

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \\ &= \frac{1}{2\pi} \int_{-\delta}^{\delta} e^{-inx} dx \\ &= \frac{1}{2\pi} \cdot \frac{2 \sin(n\delta)}{n} \\ &= \frac{\sin(n\delta)}{n\pi}. \end{aligned}$$

□

Supplement. Find a_n and b_n of this textbook.

By (a), $a_0 = \frac{\delta}{\pi}$, $a_n = \frac{2 \sin(n\delta)}{n\pi}$, $b_n = 0$ for $n \in \mathbb{Z}^+$. Surely, we can compute a_n

and b_n ($n > 0$) directly. Since $f(x)$ is an even function, $b_n = 0$. And

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \\ &= \frac{2}{\pi} \int_0^{\delta} \cos(nx) dx \\ &= \frac{2 \sin(n\delta)}{n\pi}. \end{aligned}$$

Proof of (b). Given $x = 0$, there are constants $\delta' = \delta > 0$ and $M = 1 < \infty$ such that

$$|f(0+t) - f(0)| \leq M|t|$$

for all $t \in (-\delta', \delta')$. By Theorem 8.14,

$$\sum_{-\infty}^{\infty} c_n = f(0).$$

Notice that $c_{-n} = c_n$ for $n \in \mathbb{Z}^+$, so

$$\begin{aligned} \frac{\delta}{\pi} + 2 \sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n\pi} &= 1 \\ \sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} &= \frac{\pi - \delta}{2}. \end{aligned}$$

□

We can also use the expression a_n and b_n to prove the same thing. Besides, taking $\delta = 1$ yields

$$\sum_{n=1}^{\infty} \frac{\sin n}{n} = \frac{\pi - 1}{2}.$$

Proof of (c). Since $f(x)$ is a Riemann-integrable function with period 2π , by Parseval's theorem

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{-\infty}^{\infty} |c_n|^2.$$

So

$$\frac{\delta}{\pi} = \frac{\delta^2}{\pi^2} + 2 \sum_{n=1}^{\infty} \frac{(\sin(n\delta))^2}{n^2 \pi^2},$$

or

$$\sum_{n=1}^{\infty} \frac{(\sin(n\delta))^2}{n^2 \delta} = \frac{\pi - \delta}{2}.$$

□

Notices that

$$\sum_{n=1}^{\infty} \frac{(\sin n)^2}{n^2} = \frac{\pi - 1}{2}$$

as $\delta = 1$.

Proof of (d). Given $\varepsilon > 0$. By Exercise 6.8,

$$\int_0^{\infty} \left(\frac{\sin x}{x} \right)^2 dx$$

exists. So there exists $b > 0$ such that

$$\left| \int_0^b \left(\frac{\sin x}{x} \right)^2 dx - \int_0^{\infty} \left(\frac{\sin x}{x} \right)^2 dx \right| < \frac{\varepsilon}{4}$$

By Supplement in Chapter 6, there exists $\delta > 0$ such that for any partition $P_m = \{0, \frac{b}{m}, \frac{2b}{m}, \dots, \frac{(m-1)b}{m}, b\}$ of $[0, b]$ with $\|P\| = \frac{b}{m} < \delta$, or $m > \frac{b}{\delta}$, we have

$$\begin{aligned} \left| \sum_{n=1}^m \frac{(\sin(n \frac{b}{m}))^2}{(n \frac{b}{m})^2} \cdot \frac{b}{m} - \int_0^b \left(\frac{\sin x}{x} \right)^2 dx \right| &< \frac{\varepsilon}{4}, \\ \left| \sum_{n=1}^m \frac{(\sin(n \frac{b}{m}))^2}{n^2 \frac{b}{m}} - \int_0^b \left(\frac{\sin x}{x} \right)^2 dx \right| &< \frac{\varepsilon}{4}. \end{aligned}$$

For simplicity we resize δ to $\delta < \pi$ to make $0 < \frac{b}{m} < \delta < \pi$. Besides, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, there exists $N > 0$ such that

$$\left| \sum_{n=1}^{\infty} \frac{(\sin(n \frac{b}{m}))^2}{n^2 \frac{b}{m}} - \sum_{n=1}^m \frac{(\sin(n \frac{b}{m}))^2}{n^2 \frac{b}{m}} \right| < \frac{\varepsilon}{4}$$

whenever $m \geq N$. By (c),

$$\left| \frac{\pi - \frac{b}{m}}{2} - \sum_{n=1}^m \frac{(\sin(n \frac{b}{m}))^2}{n^2 \frac{b}{m}} \right| < \frac{\varepsilon}{4}$$

whenever $m \geq N$. Last, it is easy to get

$$\left| \frac{\pi}{2} - \frac{\pi - \frac{b}{m}}{2} \right| < \frac{\varepsilon}{4}$$

whenever $m > \frac{2b}{\varepsilon}$. Now we have

$$\left| \frac{\pi}{2} - \int_0^{\infty} \left(\frac{\sin x}{x} \right)^2 dx \right| < \varepsilon$$

whenever $m > \max(\frac{b}{\delta}, N, \frac{2b}{\varepsilon})$. Since ε is arbitrary, $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$. \square

Proof of (e).

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

Write

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^2} &= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \\ &= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}, \end{aligned}$$

so

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{6}.$$

\square

Exercise 8.13. Put $f(x) = x$ if $0 \leq x < 2\pi$, and apply Parseval's theorem to conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi}{6}.$$

Proof.

$$\begin{aligned} c_0 &= \frac{1}{2\pi} \int_0^{2\pi} x dx \\ &= \pi, \end{aligned}$$

For $n \neq 0$,

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_0^{2\pi} x e^{-inx} dx \\ &= \frac{1}{2\pi} \left(\left[-\frac{1}{in} x e^{-inx} \right]_{x=0}^{x=2\pi} - \int_0^{2\pi} -\frac{1}{in} e^{-inx} dx \right) \\ &= \frac{i}{n}. \end{aligned}$$

Since $f(x)$ is a Riemann-integrable function with period 2π , by Parseval's theorem

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

So

$$\frac{1}{2\pi} \cdot \frac{(2\pi)^3}{3} = \pi^2 + 2 \sum_{n=1}^{\infty} \frac{1}{n^2},$$

or

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

□

Supplement. Put $f(x) = x^n$ if $n \in \mathbb{Z}^+$ and $0 \leq x < 2\pi$. Might get

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!}.$$

Exercise 8.14. PLACEHOLDER.

Exercise 8.15. With the Dirichlet kernel D_n as defined by

$$D_n(x) = \sum_{k=-n}^n \exp(ikx) = \frac{\sin(n + \frac{1}{2})x}{\sin(\frac{x}{2})},$$

put the **Fejér kernel**

$$K_N(x) = \frac{1}{N+1} \sum_{n=0}^N D_n(x).$$

Prove that

$$K_N(x) = \frac{1}{N+1} \cdot \frac{1 - \cos(N+1)x}{1 - \cos x}$$

and that

- (a) $K_N \geq 0$,
- (b) $\frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(x) dx = 1$,
- (c) $K_N(x) \leq \frac{1}{N+1} \cdot \frac{2}{1 - \cos \delta}$ if $0 < \delta \leq |x| \leq \pi$.

If $s_N = s_N(f; x)$ is the N th partial sum of the Fourier series of f , consider the arithmetic means

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_N}{N+1}.$$

Prove that

$$s_N(f; x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) K_N(t) dt,$$

and hence prove **Fejér's theorem**:

If f is continuous, with period 2π , then $\sigma_N(f; x) \rightarrow f(x)$ uniformly on $[-\pi, \pi]$.

(Hint: Use properties (a), (b), (c) to proceed as in Theorem 7.26.)

Proof of $K_N(x) = \frac{1}{N+1} \cdot \frac{1 - \cos(N+1)x}{1 - \cos x}$. Since

$$\begin{aligned} (1 - \cos x)K_N(x) &= 2 \left(\sin \frac{x}{2} \right)^2 \frac{1}{N+1} \sum_{n=0}^N \frac{\sin\left(n + \frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)} \\ &= \frac{1}{N+1} \sum_{n=0}^N 2 \sin \frac{x}{2} \sin\left(n + \frac{1}{2}\right)x \\ &= \frac{1}{N+1} \sum_{n=0}^N (\cos(nx) - \cos(n+1)x) \\ &= \frac{1 - \cos(N+1)x}{N+1}, \\ K_N(x) &= \frac{1}{N+1} \cdot \frac{1 - \cos(N+1)x}{1 - \cos x} \end{aligned}$$

if $x \neq 2k\pi$ for $k \in \mathbb{Z}$. \square

Proof of (a). It is clear since $\cos x \leq 1$ for all $x \in \mathbb{R}$. Or we may write

$$K_N(x) = \frac{1}{N+1} \left(\frac{\sin \frac{(N+1)x}{2}}{\sin \frac{x}{2}} \right)^2 \geq 0.$$

\square

Proof of (b). By the definition of $D_n(x)$,

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(x) dx &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{N+1} \sum_{n=0}^N D_n(x) dx \\ &= \frac{1}{N+1} \sum_{n=0}^N \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(x) dx \\ &= \frac{1}{N+1} \sum_{n=0}^N 1 \\ &= 1. \end{aligned}$$

\square

Proof of (c). Since $\cos x$ is bounded by 1 and monotonically decreasing on $(0, \pi]$,

$$\begin{aligned} K_N(x) &= \frac{1}{N+1} \cdot \frac{1 - \cos(N+1)x}{1 - \cos x} \\ &\leq \frac{1}{N+1} \cdot \frac{2}{1 - \cos \delta}. \end{aligned}$$

□

Proof of $s_N(f; x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)K_N(t)dt$.

$$\begin{aligned} \sigma_N(f; x) &= \frac{1}{N+1} \sum_{n=0}^N s_N(f; x) \\ &= \frac{1}{N+1} \sum_{n=0}^N \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)D_N(t)dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) \left(\frac{1}{N+1} \sum_{n=0}^N D_N(t) \right) dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)K_N(t)dt. \end{aligned}$$

□

Proof of Fejér's theorem. Given any $\varepsilon > 0$.

(1)

$$\begin{aligned} |\sigma_N(f; x) - f(x)| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)K_N(t)dt - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)K_N(t)dt \right| \\ &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x-t) - f(x))K_N(t)dt \right| \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x-t) - f(x)|K_N(t)dt. \end{aligned}$$

(2) Since f is continuous on a compact set $[-\pi, \pi]$, f is continuous uniformly. For such $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(y) - f(x)| < \frac{\varepsilon}{2}$$

whenever $x, y \in [-\pi, \pi]$ and $|y - x| < \delta$.

(3) Since f is continuous on a compact set $[-\pi, \pi]$, f is bounded on $[-\pi, \pi]$, say $M = \sup |f(x)|$.

(4) Therefore,

$$\begin{aligned}
& |\sigma_N(f; x) - f(x)| \\
& \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x-t) - f(x)| K_N(t) dt \\
& = \frac{1}{2\pi} \int_{-\pi}^{-\delta} |f(x-t) - f(x)| K_N(t) dt \\
& \quad + \frac{1}{2\pi} \int_{-\delta}^{\delta} |f(x-t) - f(x)| K_N(t) dt \\
& \quad + \frac{1}{2\pi} \int_{\delta}^{\pi} |f(x-t) - f(x)| K_N(t) dt \\
& \leq \frac{1}{2\pi} \int_{-\pi}^{-\delta} 2M \cdot \frac{1}{N+1} \cdot \frac{2}{1-\cos \delta} dt \\
& \quad + \frac{1}{2\pi} \int_{-\delta}^{\delta} \frac{\varepsilon}{2} K_N(t) dt \\
& \quad + \frac{1}{2\pi} \int_{\delta}^{\pi} 2M \cdot \frac{1}{N+1} \cdot \frac{2}{1-\cos \delta} dt \\
& = \frac{4M(\pi-\delta)}{(N+1)(1-\cos \delta)\pi} + \frac{\varepsilon}{2} \cdot \frac{1}{2\pi} \int_{-\delta}^{\delta} K_N(t) dt \\
& \leq \frac{4M(\pi-\delta)}{(N+1)(1-\cos \delta)\pi} + \frac{\varepsilon}{2}.
\end{aligned}$$

(5) Since N is arbitrary, we can take an integer $N > \frac{4M(\pi-\delta)}{(1-\cos \delta)\pi\varepsilon} - 1$ so that

$$\begin{aligned}
|\sigma_N(f; x) - f(x)| & \leq \frac{4M(\pi-\delta)}{(N+1)(1-\cos \delta)\pi} + \frac{\varepsilon}{2} \\
& < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
& = \varepsilon.
\end{aligned}$$

Therefore, the conclusion holds.

□

Exercise 8.16. Prove a pointwise version of Fejér's theorem: If $f \in \mathcal{R}$ and $f(x+)$, $f(x-)$ exist for some x , then

$$\lim_{N \rightarrow \infty} \sigma_N(f; x) = \frac{1}{2}[f(x+) + f(x-)].$$

Proof. Given any $\varepsilon > 0$.

(1) Since $K_N(-t) = K_N(t)$, we have

$$\sigma_N(f; x) = \frac{1}{2\pi} \int_0^\pi f(x-t)K_N(t)dt + \frac{1}{2\pi} \int_0^\pi f(x+t)K_N(t)dt$$

and

$$\frac{1}{2\pi} \int_0^\pi K_N(t)dt = \frac{1}{2}.$$

(2) Since $f \in \mathcal{B}$, f is bounded on $[-\pi, \pi]$, say $M = \sup |f(x)|$.

(3) Therefore,

$$\begin{aligned} & \left| \frac{1}{2\pi} \int_0^\pi f(x-t)K_N(t)dt - \frac{1}{2}f(x-) \right| \\ &= \left| \frac{1}{2\pi} \int_0^\pi (f(x-t) - f(x-))K_N(t)dt \right| \\ &\leq \frac{1}{2\pi} \int_0^\pi |f(x-t) - f(x-)|K_N(t)dt. \end{aligned}$$

Since $f(x-)$ exists, for fixed $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(y) - f(x-)| < \frac{\varepsilon}{2}$$

whenever $y \in (x - \delta, x) \cap [-\pi, \pi]$. Hence,

$$\begin{aligned} & \left| \frac{1}{2\pi} \int_0^\pi f(x-t)K_N(t)dt - \frac{1}{2}f(x-) \right| \\ &\leq \frac{1}{2\pi} \int_0^\pi |f(x-t) - f(x-)|K_N(t)dt \\ &= \frac{1}{2\pi} \int_0^\delta |f(x-t) - f(x-)|K_N(t)dt \\ &\quad + \frac{1}{2\pi} \int_\delta^\pi |f(x-t) - f(x-)|K_N(t)dt \\ &\leq \frac{1}{2\pi} \int_0^\delta \frac{\varepsilon}{2} K_N(t)dt + \frac{1}{2\pi} \int_\delta^\pi 2M \cdot \frac{1}{N+1} \cdot \frac{2}{1-\cos\delta} dt \\ &= \frac{\varepsilon}{2} \cdot \frac{1}{2\pi} \int_0^\delta K_N(t)dt + \frac{2M(\pi-\delta)}{(N+1)(1-\cos\delta)\pi} \\ &\leq \frac{\varepsilon}{4} + \frac{2M(\pi-\delta)}{(N+1)(1-\cos\delta)\pi}. \end{aligned}$$

(4) Since N is arbitrary, we can take an integer $N_1 > \frac{8M(\pi-\delta)}{(1-\cos \delta)\pi\varepsilon} - 1$ such that

$$\begin{aligned} \left| \frac{1}{2\pi} \int_0^\pi f(x-t)K_n(t)dt - \frac{1}{2}f(x-) \right| &\leq \frac{\varepsilon}{4} + \frac{2M(\pi-\delta)}{(n+1)(1-\cos \delta)\pi} \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &= \frac{\varepsilon}{2} \end{aligned}$$

whenever $n \geq N_1$. Similarly, we can take an integer N_2 such that

$$\begin{aligned} \left| \frac{1}{2\pi} \int_0^\pi f(x+t)K_n(t)dt - \frac{1}{2}f(x+) \right| &\leq \frac{\varepsilon}{4} + \frac{2M(\pi-\delta)}{(n+1)(1-\cos \delta)\pi} \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &= \frac{\varepsilon}{2}. \end{aligned}$$

whenever $n \geq N_2$.

(5) Hence,

$$\begin{aligned} &\left| \sigma_n(f; x) - \frac{1}{2}[f(x+) + f(x-)] \right| \\ &\leq \left| \frac{1}{2\pi} \int_0^\pi f(x-t)K_n(t)dt - \frac{1}{2}f(x-) \right| \\ &\quad + \left| \frac{1}{2\pi} \int_0^\pi f(x+t)K_n(t)dt - \frac{1}{2}f(x+) \right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

whenever $n \geq \max\{N_1, N_2\}$. Hence, $\lim \sigma_n(f; x) = \frac{1}{2}[f(x+) + f(x-)]$.

□

Supplement. Poisson's equation. (Theorem 1 of Section 2.2 in the textbook: *Lawrence C. Evans, Partial Differential Equations.*) Let the fundamental solution of Laplace's equation be

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log |x| & (n=2) \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & (n \geq 3), \end{cases}$$

where $x \in \mathbb{R}^n$, $x \neq 0$. Let

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)dy.$$

Then $-\Delta u = f$ in \mathbb{R}^n . Note that $\Phi(x)$ blows up at 0. To calculate $\Delta u(x)$, we need to isolate this singularity inside a small ball, say $B(0; \varepsilon)$. Therefore,

$$\Delta u(x) = \int_{B(0; \varepsilon)} \Phi(y) \Delta_x f(x - y) dy + \int_{\mathbb{R}^n - B(0; \varepsilon)} \Phi(y) \Delta_x f(x - y) dy,$$

and we can continue estimating two integrals individually as the textbook did.

Exercise 8.17. PLACEHOLDER.

Exercise 8.18. PLACEHOLDER.

Exercise 8.19. Suppose f is a continuous function on \mathbb{R} , $f(x + 2\pi) = f(x)$, and $\frac{\alpha}{\pi}$ is irrational. Prove that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(x + n\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt$$

for every x . (Hint: Do it first for $f(x) = \exp(ikx)$.)

Proof (Hint). Given any $\varepsilon > 0$.

(1) Do it first for $f(x) = \exp(ikx)$. Note that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(ikx) dt = \begin{cases} 1 & (k = 0), \\ 0 & (k \neq 0). \end{cases}$$

(a) $k = 0$ is nothing to do.

(b) Suppose $k \neq 0$.

$$\begin{aligned} \frac{1}{N} \sum_{n=1}^N f(x + n\alpha) &= \frac{1}{N} \sum_{n=1}^N \exp(ik(x + n\alpha)) \\ &= \frac{1}{N} \sum_{n=1}^N \exp(ikx) \exp(ik\alpha n) \\ &= \frac{1}{N} \exp(ikx) \cdot \frac{\exp(ik\alpha) - \exp(ik\alpha(N + 1))}{1 - \exp(ik\alpha)} \\ &= \exp(ik(x + \alpha)) \left[\frac{1}{N} \cdot \frac{1 - \exp(ik\alpha N)}{1 - \exp(ik\alpha)} \right] \\ &= f(x + \alpha) \frac{1}{N} \frac{1 - \exp(ik\alpha N)}{1 - \exp(ik\alpha)} \rightarrow 0 \end{aligned}$$

as $N \rightarrow \infty$ since $\exp(iy)$ is bounded ($y \in \mathbb{R}$). (Note that the denominator $1 - \exp(ik\alpha) \neq 0$ since $k \neq 0$ and $\frac{\alpha}{\pi}$ is irrational.)

By (a)(b),

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(x + n\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt.$$

for $f(x) = \exp(ikx)$ and any $x \in \mathbb{R}$.

(2) Therefore,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(x + n\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt$$

is also true for trigonometric polynomials $f(x)$.

(3) By Theorem 8.15, there is a trigonometric polynomial

$$P(x) = \sum_{n=-N_1}^{N_1} c_n \exp(inx)$$

such that

$$|P(x) - f(x)| < \frac{\varepsilon}{89}.$$

By (2), there is an integer N_2 such that

$$\left| \frac{1}{N} \sum_{n=1}^N P(x + n\alpha) - \frac{1}{2\pi} \int_{-\pi}^{\pi} P(t) dt \right| < \frac{\varepsilon}{64}$$

whenever $N \geq N_2$. Therefore,

$$\begin{aligned}
& \left| \frac{1}{N} \sum_{n=1}^N f(x + n\alpha) - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt \right| \\
& \leq \left| \frac{1}{N} \sum_{n=1}^N f(x + n\alpha) - \frac{1}{N} \sum_{n=1}^N P(x + n\alpha) \right| \\
& \quad + \left| \frac{1}{N} \sum_{n=1}^N P(x + n\alpha) - \frac{1}{2\pi} \int_{-\pi}^{\pi} P(t) dt \right| \\
& \quad + \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} P(t) dt - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt \right| \\
& \leq \frac{1}{N} \sum_{n=1}^N |f(x + n\alpha) - P(x + n\alpha)| \\
& \quad + \left| \frac{1}{N} \sum_{n=1}^N P(x + n\alpha) - \frac{1}{2\pi} \int_{-\pi}^{\pi} P(t) dt \right| \\
& \quad + \frac{1}{2\pi} \int_{-\pi}^{\pi} |P(t) - f(t)| dt \\
& < \frac{1}{N} \sum_{n=1}^N \frac{\varepsilon}{89} + \frac{\varepsilon}{64} + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\varepsilon}{89} dt \\
& = \frac{\varepsilon}{89} + \frac{\varepsilon}{64} + \frac{\varepsilon}{89} \\
& < \varepsilon
\end{aligned}$$

whenever $N \geq N_2$. Hence

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(x + n\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt$$

is also true for continuous function $f(x)$ (with period 2π).

□

Exercise 8.20. The following simple computation yields a good approximation to Stirling's formula. For $m = 1, 2, 3, \dots$, define

$$f(x) = (m + 1 - x) \log m + (x - m) \log(m + 1)$$

if $m \leq x \leq m + 1$, and define

$$g(x) = \frac{x}{m} - 1 + \log m$$

if $m - \frac{1}{2} \leq x < m + \frac{1}{2}$. Draw the graphs of f and g . Note that $f(x) \leq \log x \leq g(x)$ if $x \geq 1$ and that

$$\int_1^n f(x)dx = \log(n!) - \frac{1}{2} \log n > -\frac{1}{8} + \int_1^n g(x)dx.$$

Integrate $\log x$ over $[1, n]$. Conclude that

$$\frac{7}{8} < \log(n!) - \left(n + \frac{1}{2}\right) \log n + n < 1$$

for $n = 2, 3, 4, \dots$ (Note: $\log \sqrt{2\pi} \approx 0.918 \dots$) Thus

$$e^{\frac{7}{8}} < \frac{n!}{\left(\frac{n}{e}\right)^n \sqrt{n}} < e.$$

Proof.

- (1) Omit the graphs of f and g . Note that the concavity of $\log(x)$ implies that $f(x) \leq \log(x)$. Here the equality holds if and only if $x \in \mathbb{Z}^+$. Besides, since $g(x)$ is the tangent line at $(x, \log x)$ whenever $x \in \mathbb{Z}^+$, $g(x) \geq \log(x)$ and the equality holds if and only if $x \in \mathbb{Z}^+$.

(2)

$$\begin{aligned} \int_1^n f(x)dx &= \sum_{m=1}^{n-1} \int_m^{m+1} f(x)dx \\ &= \sum_{m=1}^{n-1} \int_m^{m+1} (m+1-x) \log m + (x-m) \log(m+1) dx \\ &= \sum_{m=1}^{n-1} \int_m^{m+1} (\log(m+1) - \log m)x + (m+1) \log m - m \log(m+1) dx \\ &= \sum_{m=1}^{n-1} (\log(m+1) - \log m) \left(\frac{(m+1)^2 - m^2}{2} \right) + (m+1) \log m - m \log(m+1) \\ &= \sum_{m=1}^{n-1} \log m + \frac{1}{2} \sum_{m=1}^{n-1} (\log(m+1) - \log m) \\ &= \log((n-1)!) + \frac{1}{2} \log n \\ &= \log(n!) - \frac{1}{2} \log n. \end{aligned}$$

(3) Write

$$\int_1^n g(x)dx = \left(\sum_{m=1}^n \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} g(x)dx \right) - \int_{\frac{1}{2}}^1 g(x)dx - \int_n^{n+\frac{1}{2}} g(x)dx.$$

(a)

$$\begin{aligned}\sum_{m=1}^n \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} g(x) dx &= \sum_{m=1}^n \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} \left(\frac{x}{m} - 1 + \log m \right) dx \\ &= \sum_{m=1}^n \log m \\ &= \log(n!).\end{aligned}$$

(b)

$$\int_{\frac{1}{2}}^1 g(x) dx = \int_{\frac{1}{2}}^1 (x - 1 + \log 1) dx = -\frac{1}{8}.$$

(c)

$$\int_n^{n+\frac{1}{2}} g(x) dx = \int_{\frac{1}{2}}^1 \left(\frac{x}{n} - 1 + \log n \right) dx = \frac{1}{2} \log n - \frac{1}{8n}.$$

By (a)(b)(c),

$$\int_1^n g(x) dx = \log(n!) - \frac{1}{2} \log n + \frac{1}{8} \left(1 - \frac{1}{n}\right) < \log(n!) - \frac{1}{2} \log n + \frac{1}{8}.$$

(4) Since $f(x) \leq \log x \leq g(x)$ and the equality holds if and only if $x \in \mathbb{Z}^+$ (by (1)),

$$\int_1^n f(x) dx \leq \int_1^n \log x dx \leq \int_1^n g(x) dx$$

for all $n = 1, 2, 3, \dots$. The equality holds if and only if $n = 1$. Hence by (2)(3)

$$\log(n!) - \frac{1}{2} \log n \leq n \log n - n + 1 \leq \log(n!) - \frac{1}{2} \log n + \frac{1}{8}.$$

Arrange the inequality to get

$$\frac{7}{8} < \log(n!) - \left(n + \frac{1}{2}\right) \log n + n \leq 1$$

for $n = 1, 2, 3, \dots$. Note that the equality holds if and only if $n = 1$. Therefore

$$\frac{7}{8} < \log(n!) - \left(n + \frac{1}{2}\right) \log n + n < 1$$

for $n = 2, 3, \dots$

(5) Exponentiate to get

$$\exp\left(\frac{7}{8}\right) < \exp\left[\log(n!) - \left(n + \frac{1}{2}\right) \log n + n\right] < \exp(1),$$

or

$$e^{\frac{7}{8}} < \frac{\exp(\log(n!)) \exp(n)}{\exp[(n + \frac{1}{2}) \log n]} < e,$$

or $e^{\frac{7}{8}} < \frac{n!}{(\frac{n}{e})^n \sqrt{n}} < e$ (since $\exp(x)$ is a strictly increasing function of x).

□

Exercise 8.21 (Norm of Dirichlet kernel). *Let*

$$L_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)| dt \quad (n = 1, 2, 3, \dots).$$

Prove that there exists a constant $C > 0$ such that

$$L_n > C \log n \quad (n = 1, 2, 3, \dots),$$

or, more precisely, that the sequence

$$\left\{ L_n - \frac{4}{\pi^2} \log n \right\}$$

is bounded.

Proof.

(1) Write

$$\begin{aligned} L_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)| dt \\ &= \frac{1}{\pi} \int_0^{\pi} |D_n(t)| dt && (D_n(-t) = D_n(t)) \\ &= \frac{1}{\pi} \int_0^{\pi} \frac{|\sin(n + \frac{1}{2})t|}{\sin(\frac{t}{2})} dt. && (\sin(\frac{t}{2}) \geq 0 \text{ on } [0, \pi]) \end{aligned}$$

(2) So,

$$\begin{aligned} L_n &= \frac{1}{\pi} \int_0^{\pi} \frac{|\sin(n + \frac{1}{2})t|}{\sin(\frac{t}{2})} dt \\ &= \frac{1}{\pi} \int_0^{\pi} \left| \sin\left(n + \frac{1}{2}\right)t \right| \left(\frac{1}{\sin(\frac{t}{2})} - \frac{1}{\frac{t}{2}} + \frac{1}{\frac{t}{2}} \right) dt \\ &= \underbrace{\frac{1}{\pi} \int_0^{\pi} \left| \sin\left(n + \frac{1}{2}\right)t \right| \left(\frac{1}{\sin(\frac{t}{2})} - \frac{1}{\frac{t}{2}} \right) dt}_{:= I_n} + \underbrace{\frac{2}{\pi} \int_0^{\pi} \frac{|\sin(n + \frac{1}{2})t|}{t} dt}_{:= J_n}. \end{aligned}$$

- (3) Show that I_n is uniformly bounded. Note that $f(x) = \frac{1}{\sin(x)} - \frac{1}{x}$ is bounded (since $\lim_{x \rightarrow 0} f(x) = 0$ by using L'Hospital's rule twice). Also, $|\sin(n + \frac{1}{2})t| \leq 1$ for any n . Hence

$$0 \leq I_n < \sup(f(x)) = \frac{2}{\pi}.$$

- (4) Show that $J_n - \frac{4}{\pi^2} \log n$ is uniformly bounded. Since

$$\begin{aligned} J_n &= \frac{2}{\pi} \int_0^\pi \frac{|\sin(n + \frac{1}{2})t|}{t} dt \\ &= \frac{2}{\pi} \int_0^{(n+\frac{1}{2})\pi} \frac{|\sin x|}{x} dx, \quad (\text{Let } x = (n + \frac{1}{2})t) \end{aligned}$$

we have

$$\underbrace{\frac{2}{\pi} \sum_{k=0}^{n-1} \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x} dx}_{:=J_n^{(1)}} \leq J_n \leq \underbrace{\frac{2}{\pi} \sum_{k=0}^n \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x} dx}_{:=J_n^{(2)}}.$$

So

$$\begin{aligned} J_n^{(1)} &\geq \frac{2}{\pi} \sum_{k=0}^{n-1} \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{(k+1)\pi} dx \\ &= \frac{2}{\pi} \sum_{k=0}^{n-1} \frac{2}{(k+1)\pi} \quad \left(\int_0^\pi |\sin x| dx = 0 \right) \\ &\geq \frac{4}{\pi^2} \log n, \quad (\text{Exercise 8.9}) \end{aligned}$$

and

$$\begin{aligned} J_n^{(2)} &= \frac{2}{\pi} \int_0^\pi \frac{|\sin x|}{x} dx + \frac{2}{\pi} \sum_{k=1}^n \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x} dx \\ &\leq \frac{2}{\pi} \int_0^\pi \frac{|\sin x|}{x} dx + \frac{2}{\pi} \sum_{k=1}^n \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{k\pi} dx \\ &= \frac{2}{\pi} \int_0^\pi \frac{|\sin x|}{x} dx + \frac{2}{\pi} \sum_{k=1}^n \frac{2}{k\pi} \\ &\leq \frac{2}{\pi} \int_0^\pi \frac{|\sin x|}{x} dx + \frac{4}{\pi^2} (\log n + 1) \\ &= \frac{4}{\pi^2} \log n + \frac{4}{\pi^2} + \frac{2}{\pi} \int_0^\pi \frac{|\sin x|}{x} dx. \end{aligned}$$

Hence,

$$0 \leq J_n - \frac{4}{\pi^2} \log n \leq \frac{4}{\pi^2} + \frac{2}{\pi} \int_0^\pi \frac{|\sin x|}{x} dx.$$

(5) By (3)(4),

$$0 \leq L_n - \frac{4}{\pi^2} \log n \leq \frac{2}{\pi} + \frac{4}{\pi^2} + \frac{2}{\pi} \int_0^\pi \frac{|\sin x|}{x} dx.$$

□

Exercise 8.22 (Newton's generalized binomial theorem). If α is a real and $-1 < x < 1$, prove Newton's binomial theorem

$$(1+x)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n.$$

(Hint: Denote the right side by $f(x)$. Prove that the series converges. Prove that

$$(1+x)f'(x) = \alpha f(x)$$

and solve this differential equation.) Show also that

$$(1-x)^{-\alpha} = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} x^n$$

if $-1 < x < 1$ and $\alpha > 0$.

Proof.

(1) Let

$$f(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$

where $\binom{\alpha}{n}$ is defined by

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

(2) Show that $\binom{\alpha-1}{n} + \binom{\alpha-1}{n-1} = \binom{\alpha}{n}$.

$$\begin{aligned} \binom{\alpha-1}{n} + \binom{\alpha-1}{n-1} &= \frac{(\alpha-1)\cdots(\alpha-n+1)(\alpha-n)}{n!} + \frac{(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!} \\ &= \frac{(\alpha-1)\cdots(\alpha-n+1)}{n!} [(\alpha-n) + n] \\ &= \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} \\ &= \binom{\alpha}{n}. \end{aligned}$$

(3) Show that $f(x)$ converges. Write $c_n = \binom{\alpha}{n}$. Since

$$\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\alpha - n}{n + 1} \right| = 1,$$

we have

$$\lim_{n \rightarrow \infty} \sqrt[n]{|c_n|} = \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| = 1$$

(Theorem 3.37) and thus the radius of convergence is 1. $f(x)$ converges if $|x| < 1$.

(4) Show that $(1+x)f'(x) = \alpha f(x)$. By Theorem 8.1,

$$\begin{aligned} f'(x) &= \sum_{n=0}^{\infty} \binom{\alpha}{n} n x^{n-1} \\ &= \sum_{n=1}^{\infty} \binom{\alpha}{n} n x^{n-1} \\ &= \sum_{n=1}^{\infty} \alpha \binom{\alpha-1}{n-1} x^{n-1} \\ &= \sum_{n=0}^{\infty} \alpha \binom{\alpha-1}{n} x^n. \end{aligned}$$

Besides,

$$x f'(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} n x^n = \sum_{n=0}^{\infty} \alpha \binom{\alpha-1}{n-1} x^n.$$

Hence,

$$\begin{aligned} (1+x)f'(x) &= \sum_{n=0}^{\infty} \alpha \binom{\alpha-1}{n} x^n + \sum_{n=0}^{\infty} \alpha \binom{\alpha-1}{n-1} x^n \\ &= \alpha \sum_{n=0}^{\infty} \left[\binom{\alpha-1}{n} + \binom{\alpha-1}{n-1} \right] x^n \\ &= \alpha \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \\ &= \alpha f(x). \end{aligned} \tag{2}$$

(5) Solve the differential equation $(1+x)f'(x) = \alpha f(x)$. Given any $1 > \varepsilon > 0$. Use the notations in Exercise 5.27. Let

$$\phi(x, y) = \frac{\alpha y}{1+x}$$

defined on $[-1 + \varepsilon, 1 - \varepsilon] \times \mathbb{R}$. Let

$$g(x) = (1+x)^\alpha$$

defined on $[-1 + \varepsilon, 1 - \varepsilon]$. Thus,

$$g'(x) = \alpha(1+x)^{\alpha-1} = \frac{\alpha(1+x)^\alpha}{1+x} = \frac{\alpha g(x)}{1+x} = \phi(x, g(x))$$

and $g(0) = 1$. (Clearly, $f'(x) = \phi(x, f(x))$ and $f(0) = 1$.) To show $f(x) = g(x)$, it suffices to show that there is a constant A such that

$$|\phi(x, g(x)) - \phi(x, f(x))| \leq A|g(x) - f(x)|$$

whenever $(x, f(x)) \in \mathbb{R}$ and $(x, g(x)) \in \mathbb{R}$. In fact,

$$\begin{aligned} |\phi(x, g(x)) - \phi(x, f(x))| &= \left| \frac{\alpha g(x)}{1+x} - \frac{\alpha f(x)}{1+x} \right| \\ &= \frac{\alpha}{1+x} |g(x) - f(x)| \\ &\leq \frac{\alpha}{\varepsilon} |g(x) - f(x)|. \end{aligned}$$

(Here $A = \frac{\alpha}{\varepsilon}$ is a constant.) By Exercise 5.27, $f(x) = g(x)$ on $[-1 + \varepsilon, 1 - \varepsilon]$ for any $1 > \varepsilon > 0$. So $f(x) = g(x)$ on $(-1, 1)$, or

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = (1+x)^\alpha$$

if $x \in (-1, 1)$.

(6) Show that

$$(1-x)^{-\alpha} = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} x^n$$

if $-1 < x < 1$ and $\alpha > 0$. In fact,

$$\begin{aligned} (1-x)^{-\alpha} &= \sum_{n=0}^{\infty} \binom{-\alpha}{n} (-x)^n \\ &= \sum_{n=0}^{\infty} \frac{(-\alpha)(-\alpha-1)\cdots(-\alpha-n+1)}{n!} (-1)^n x^n \\ &= \sum_{n=0}^{\infty} \frac{(\alpha)(\alpha+1)\cdots(\alpha+n-1)}{n!} x^n \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} x^n. \end{aligned}$$

□

Exercise 8.23. Let γ be a continuously differentiable **closed** curve in the complex plane, with parameter interval $[a, b]$, and assume that $\gamma(t) \neq 0$ for every $t \in [a, b]$. Define the **index** of γ to be

$$\text{Ind}(\gamma) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t)} dt.$$

Prove that $\text{Ind}(\gamma)$ is always an integer. (Hint: There exists φ on $[a, b]$ with $\varphi' = \frac{\gamma'}{\gamma}$, $\varphi(a) = 0$. Hence $\gamma \exp(-\varphi)$ is constant. Since $\gamma(a) = \gamma(b)$ it follows that $\exp(\varphi(b)) = \exp(\varphi(a)) = 1$. Note that $\varphi(b) = 2\pi i \text{Ind}(\gamma)$.) Compute $\text{Ind}(\gamma)$ when $\gamma(t) = \exp(it)$, $a = 0$, $b = 2\pi$. Explain why $\text{Ind}(\gamma)$ is often called the **winding number** of γ around 0.

Proof (Hint).

- (1) Show that $\text{Ind}(\gamma)$ is always an integer. Define

$$\varphi(x) = \int_a^x \frac{\gamma'(t)}{\gamma(t)} dt$$

if $x \in [a, b]$.

- (a) Show that $\varphi(x)$ is well-defined. Since γ is continuously differentiable with $\gamma(t) \neq 0$ on $[a, b]$, $\frac{\gamma'(t)}{\gamma(t)}$ is continuous on $[a, b]$. Hence $\varphi(x)$ is well-defined.
- (b) Show that $\varphi' = \frac{\gamma'}{\gamma}$ and $\varphi(a) = 0$. By Theorem 6.20, $\varphi(x)$ is continuous. Furthermore, $\varphi(x)$ is differentiable on $[a, b]$ and $\varphi'(x) = \frac{\gamma'(x)}{\gamma(x)}$. By the definition of φ , $\varphi(a) = 0$.
- (c) Show that $\gamma \exp(-\varphi)$ is constant. Write $f(x) = \gamma(x) \exp(-\varphi(x))$.

$$\begin{aligned} f'(x) &= \gamma'(x) \exp(-\varphi(x)) + \gamma(x)(-\varphi'(x)) \exp(-\varphi(x)) \\ &= (\gamma'(x) - \gamma(x)\varphi'(x)) \exp(-\varphi(x)) \\ &= 0. \end{aligned}$$

Hence $f = \gamma \exp(-\varphi)$ is constant (Theorem 5.11(b)).

- (d) Show that $\text{Ind}(\gamma) \in \mathbb{Z}$. By (c),

$$\begin{aligned} \gamma(b) \exp(-\varphi(b)) &= \gamma(a) \exp(-\varphi(a)) \\ \implies \exp(-\varphi(b)) &= \exp(-\varphi(a)) && (\gamma \text{ is closed}) \\ \implies \exp(\varphi(b)) &= \exp(\varphi(a)) \\ \implies \exp(2\pi i \text{Ind}(\gamma)) &= \exp(0) = 1 && ((b)) \\ \implies 2\pi i \text{Ind}(\gamma) &= 2\pi i n \text{ for some } n \in \mathbb{Z} && (\text{Theorem 8.7}) \\ \implies \text{Ind}(\gamma) &= n \text{ for some } n \in \mathbb{Z}. \end{aligned}$$

- (2) Compute $\text{Ind}(\gamma)$ when $\gamma(t) = \exp(int)$, $a = 0$, $b = 2\pi$.

$$\begin{aligned}\text{Ind}(\gamma) &= \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t)} dt \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{in \exp(int)}{\exp(int)} dt \\ &= \frac{1}{2\pi i} \int_0^{2\pi} i n dt \\ &= n.\end{aligned}$$

- (3) Explain why $\text{Ind}(\gamma)$ is often called the **winding number** of γ around 0. As (2) suggested, $\text{Ind}(\gamma)$ is an integer representing the total number of times that curve travels counterclockwise around 0. That's why we might say $\text{Ind}(\gamma)$ is the winding number.

□

Exercise 8.24. Let γ be as in Exercise 8.23, and assume in addition that the range of γ does not intersect the negative real axis. Prove that $\text{Ind}(\gamma) = 0$. (Hint: For $0 \leq c < \infty$, $\text{Ind}(\gamma + c)$ is a continuous integer-valued function of c . Also, $\text{Ind}(\gamma) \rightarrow 0$ as $c \rightarrow \infty$.)

Proof (Hint).

- (1) Let $f(t, c) = \gamma(t) + c$ defined on $E = [a, b] \times [0, \infty)$.
 (2) Define $E_K = [a, b] \times [0, K]$ for some $K \geq 0$. Show that

$$0 < m_K \leq |f(E_K)| \leq M_K$$

for some m_K, M_K . Especially, take $K = 0$ to get

$$0 < |f(t, 0)| = |\gamma(t)| \leq M_0$$

and thus

$$|f(t, c)| = |\gamma(t) + c| \geq c - |\gamma(t)| > 2M_0 - M_0 = M_0$$

whenever $c > 2M_0$.

- (a) Since f is a continuous mapping of compact metric space E_K into \mathbb{R}^2 , $f(E_K)$ is bounded (Theorem 4.15). Hence $0 \leq m_K \leq |f(E_K)| \leq M_K$ for some m_K and M_K .
 (b) Note that $f(t, c) \neq 0$ since the range of γ does not intersect the negative real axis. Hence, $m_K \neq 0$ (Theorem 4.16).

(3) *Show that*

$$m = \inf_{(t,c) \in E} |f(t,c)| > 0.$$

(a) By (2), there exists $M_0 > 0$ such that

$$|f(t,c)| \geq M_0 \text{ on } E - E_{2M_0}.$$

(b) For such $K = 2M_0$, we apply (2) again to get that there exists $m_K = m_{2M_0} > 0$ such that

$$|f(t,c)| \geq m_{2M_0} \text{ on } E_{2M_0}.$$

(c) By (a)(b),

$$|f(t,c)| > \min\{M_0, m_{2M_0}\} > 0 \text{ on } E.$$

(4) *Show that $\text{Ind}(\gamma + c)$ is uniformly continuous.* Since $[a, b]$ is compact and γ' is continuous, $|\gamma'| \leq M$ for some $M > 0$. Hence for any $c_1, c_2 \in [0, \infty)$, we have

$$\begin{aligned} & |\text{Ind}(\gamma + c_1) - \text{Ind}(\gamma + c_2)| \\ &= \left| \frac{1}{2\pi i} \int_a^b \frac{(\gamma(t) + c_1)'}{\gamma(t) + c_1} dt - \frac{1}{2\pi i} \int_a^b \frac{(\gamma(t) + c_2)'}{\gamma(t) + c_2} dt \right| \\ &= \left| \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)(c_2 - c_1)}{(\gamma(t) + c_1)(\gamma(t) + c_2)} dt \right| \\ &\leq \frac{1}{2\pi} \int_a^b \frac{|\gamma'(t)|}{|(\gamma(t) + c_1)(\gamma(t) + c_2)|} |c_1 - c_2| dt \\ &\leq \frac{(b-a)M}{2\pi m^2} |c_1 - c_2|. \end{aligned}$$

Hence, $\text{Ind}(\gamma + c)$ is uniformly continuous on $[0, \infty)$.

(5) *Show that $\text{Ind}(\gamma) = 0$.* Since $\text{Ind}(\gamma + c)$ is continuous (by (4)) and $\text{Ind}(\gamma + c) \in \mathbb{Z}$ (by Exercise 8.23), $\text{Ind}(\gamma + c)$ is constant. *It suffices to show that $\text{Ind}(\gamma + c_0) = 0$ for some $c_0 \in [0, \infty)$.* In fact,

$$\begin{aligned} |\text{Ind}(\gamma + c)| &= \left| \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t) + c} dt \right| \\ &\leq \frac{1}{2\pi} \int_a^b \frac{|\gamma'(t)|}{|\gamma(t) + c|} dt \\ &\leq \frac{1}{2\pi} \int_a^b \frac{|\gamma'(t)|}{|\gamma(t)| + c} dt \\ &\leq \frac{(b-a)M}{2\pi(m+c)}. \end{aligned}$$

Let $c \rightarrow \infty$ to get $|\text{Ind}(\gamma + c)| \rightarrow 0$. Hence, $\text{Ind}(\gamma + c_0) = 0$ for some $c_0 \in [0, \infty)$.

□

Exercise 8.25. Suppose γ_1 and γ_2 are curves as in Exercise 8.23, and

$$|\gamma_1(t) - \gamma_2(t)| < |\gamma_1(t)| \quad (a \leq t \leq b).$$

Prove that $\text{Ind}(\gamma_1) = \text{Ind}(\gamma_2)$. (Hint: Put $\gamma = \frac{\gamma_2}{\gamma_1}$. Then $|1 - \gamma| < 1$, hence $\text{Ind}(\gamma) = 0$, by Exercise 8.24. Also,

$$\frac{\gamma'}{\gamma} = \frac{\gamma_2'}{\gamma_2} - \frac{\gamma_1'}{\gamma_1}.)$$

Proof (Hint).

(1) Put $\gamma = \frac{\gamma_2}{\gamma_1}$. It is well-defined since $\gamma_1(t) \neq 0$. Then

$$\frac{\gamma'}{\gamma} = \frac{\frac{\gamma_1' \gamma_2 - \gamma_1 \gamma_2'}{\gamma_1^2}}{\frac{\gamma_2}{\gamma_1}} = \frac{\gamma_1' \gamma_2 - \gamma_1 \gamma_2'}{\gamma_1 \gamma_2} = \frac{\gamma_2'}{\gamma_2} - \frac{\gamma_1'}{\gamma_1}.$$

(2) Since

$$|1 - \gamma| = \left| 1 - \frac{\gamma_2}{\gamma_1} \right| = \left| \frac{\gamma_1 - \gamma_2}{\gamma_1} \right| = \frac{|\gamma_1 - \gamma_2|}{|\gamma_1|} < 1,$$

the range of γ does not intersect the negative real axis. (If not, there were some $t_0 \in [a, b]$ such that $\gamma(t_0) \leq 0$. Thus $|1 - \gamma(t_0)| \geq 1$, contrary to the assumption.) By Exercise 8.24, $\text{Ind}(\gamma) = 0$.

(3) Since

$$\begin{aligned} \text{Ind}(\gamma) &= \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t)} dt \\ &= \frac{1}{2\pi i} \int_a^b \left(\frac{\gamma_2'(t)}{\gamma_2(t)} - \frac{\gamma_1'(t)}{\gamma_1(t)} \right) dt \\ &= \frac{1}{2\pi i} \int_a^b \frac{\gamma_2'(t)}{\gamma_2(t)} dt - \frac{1}{2\pi i} \int_a^b \frac{\gamma_1'(t)}{\gamma_1(t)} dt \\ &= \text{Ind}(\gamma_2) - \text{Ind}(\gamma_1), \end{aligned} \tag{1}$$

$\text{Ind}(\gamma) = 0$ implies $\text{Ind}(\gamma_1) = \text{Ind}(\gamma_2)$.

PLACEHOLDER.

Exercise 8.26. PLACEHOLDER.

Exercise 8.27. PLACEHOLDER.

Exercise 8.28. PLACEHOLDER.

Exercise 8.29. PLACEHOLDER.

Exercise 8.30. Use Stirling's formula to prove that

$$\lim_{x \rightarrow \infty} \frac{\Gamma(x+c)}{x^c \Gamma(x)} = 1$$

for every real constant c .

Proof. By Stirling's formula,

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\Gamma(x+c)}{\left(\frac{x+c-1}{e}\right)^{x+c-1} \sqrt{2\pi(x+c-1)}} &= 1 \\ \lim_{x \rightarrow \infty} \frac{\Gamma(x)}{\left(\frac{x-1}{e}\right)^{x-1} \sqrt{2\pi(x-1)}} &= 1, \end{aligned}$$

we have

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\Gamma(x+c)}{x^c \Gamma(x)} &= \lim_{x \rightarrow \infty} \frac{\Gamma(x+c)}{x^c \Gamma(x)} \\ &\quad \times \lim_{x \rightarrow \infty} \frac{\left(\frac{x+c-1}{e}\right)^{x+c-1} \sqrt{2\pi(x+c-1)}}{\Gamma(x+c)} \\ &\quad \times \lim_{x \rightarrow \infty} \frac{\Gamma(x)}{\left(\frac{x-1}{e}\right)^{x-1} \sqrt{2\pi(x-1)}} \\ &= \lim_{x \rightarrow \infty} \frac{\left(\frac{x+c-1}{e}\right)^{x+c-1} \sqrt{2\pi(x+c-1)}}{x^c \left(\frac{x-1}{e}\right)^{x-1} \sqrt{2\pi(x-1)}} \\ &= \lim_{x \rightarrow \infty} \frac{\left(\frac{x+c-1}{e}\right)^c \left(\frac{x+c-1}{e}\right)^{x-1} \sqrt{\frac{x+c-1}{x-1}}}{x^c \left(\frac{x-1}{e}\right)^{x-1}} \\ &= \frac{1}{e^c} \cdot e^c \cdot 1 \\ &= 1 \end{aligned}$$

since

(1)

$$\lim_{x \rightarrow \infty} \frac{\left(\frac{x+c-1}{e}\right)^c}{x^c} = \frac{1}{e^c} \lim_{x \rightarrow \infty} \left(\frac{x+c-1}{x}\right)^c = \frac{1}{e^c}.$$

(2)

$$\lim_{x \rightarrow \infty} \frac{\left(\frac{x+c-1}{e}\right)^{x-1}}{\left(\frac{x-1}{e}\right)^{x-1}} = \lim_{x \rightarrow \infty} \left(\frac{x+c-1}{x-1}\right)^{x-1} = \lim_{x \rightarrow \infty} \left(1 + \frac{c}{x-1}\right)^{x-1} = e^c.$$

(3) and

$$\lim_{x \rightarrow \infty} \sqrt{\frac{x+c-1}{x-1}} = \lim_{x \rightarrow \infty} \sqrt{1 + \frac{c}{x-1}} = 1.$$

□

Exercise 8.31. In the proof of Theorem 7.26 it was shown that

$$\int_{-1}^1 (1-x^2)^n dx \geq \frac{4}{3\sqrt{n}}$$

for $n = 1, 2, 3, \dots$. Use Theorem 8.20 and Exercise 8.30 to show the more precise result

$$\lim_{n \rightarrow \infty} \sqrt{n} \int_{-1}^1 (1-x^2)^n dx = \sqrt{\pi}.$$

Proof.

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sqrt{n} \int_{-1}^1 (1-x^2)^n dx \\ &= \lim_{n \rightarrow \infty} \sqrt{n} \int_0^1 u^{-\frac{1}{2}} (1-u)^n dx && (u = x^2) \\ &= \lim_{n \rightarrow \infty} \sqrt{n} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma(n+1)}{\Gamma\left(n + \frac{3}{2}\right)} && (\text{Theorem 8.20}) \\ &= \Gamma\left(\frac{1}{2}\right) \lim_{n \rightarrow \infty} \frac{n^{\frac{1}{2}} \Gamma(n+1)}{\Gamma\left(n + \frac{3}{2}\right)} \\ &= \Gamma\left(\frac{1}{2}\right) && (\text{Exercise 8.30}) \\ &= \sqrt{\pi}. && (\text{Some consequences 8.21}) \end{aligned}$$

□