Chapter 5: Differentiation

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Exercise 5.1. *Let f be defined for all real x, and suppose that*

$$|f(x) - f(y)| \leq (x-y)^2$$

for all real x and y. Prove that f is a constant.

Proof.

(1) Write

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq |x - y|$$

if $x \neq y$.

(2) Given any $y \in \mathbb{R}$,

$$\left| \frac{f(x) - f(y)}{x - y} \right| \to 0 \quad \text{as} \quad x \to y,$$

or $|f'(y)| = 0$.

(3) Or using ε-δ argument. Fix $y \in \mathbb{R}$. Given any $\varepsilon > 0$, there exists $\delta = \varepsilon > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} - 0 \right| \leq |x - y| < \delta = \varepsilon$$

whenever $|x - y| < \delta$. That is, $|f'(y)| = 0$.

(4) So $f'(y) = 0$ for any $y \in \mathbb{R}$. By Theorem 5.11 (b), f is a constant.

□

Exercise 5.2. *Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b), and let g be its inverse function. Prove that g is differentiable, and that*

$$g'(f(x)) = \frac{1}{f'(x)} \qquad (a < x < b).$$

Proof. Let $E = (a, b)$.

(1) Theorem 5.10 implies that for any $a < p < q < b$ there exists $\xi \in (p, q)$ such that
$$f(p) - f(q) = (p - q)f'(\xi).$$
Since $\xi \in (p, q) \subseteq E$, by assumption $f'(\xi) > 0$. Hence $f(p) - f(q) = (p - q)f'(\xi) < 0$ (here $p - q < 0$), or
$$f(p) < f(q)$$
if $p < q$. Therefore, f is strictly increasing in (a, b).

(2) *Show that f is one-to-one in E if f is strictly increasing in E.* If $f(p) = f(q)$, then it cannot be $p > q$ or $p < q$ ((1)). So that $p = q$, or f is injective.

(3) *Show that g is well-defined.* Theorem 5.2 and Theorem 4.17.

(4) *Show that $g'(f(x)) = \frac{1}{f'(x)}$.* Given $y \in f(E)$, say $y = f(x)$ for some $x \in E$. Given any $s \in f(E)$ with $s \neq y$. Here $s = f(t)$ for some $t \in E$ and $t \neq x$.
$$\begin{aligned}\lim_{s \to y} \frac{g(s) - g(y)}{s - y} &= \lim_{f(t) \to f(x)} \frac{g(f(t)) - g(f(x))}{f(t) - f(x)} \\ &= \lim_{t \to x} \frac{t - x}{f(t) - f(x)} \\ &= \lim_{t \to x} \frac{1}{\frac{f(t) - f(x)}{t - x}} \\ &= \frac{1}{f'(x)}. && (f' > 0)\end{aligned}$$
Here $s \to y$ if and only if $t \to x$ since both f and g are continuous and one-to-one. Hence g is differentiable and $g'(f(x)) = \frac{1}{f'(x)}$.

□

Exercise 5.3. *Suppose g is a real function on $\mathbb{R}^1$, with bounded derivative (say $|g'| \leq M$). Fix $\varepsilon > 0$, and define $f(x) = x + \varepsilon g(x)$. Prove that f is one-to-one if ε is small enough. (A set of admissible values of ε can be determined which depends only on M.)*

Proof.

(1) Note that $f'(x) = 1 + \varepsilon g'(x)$ (Theorem 5.3). Since $|g'| \leq M$,
$$1 - \varepsilon M \leq f'(x) \leq 1 + \varepsilon M.$$

(2) Pick
$$\varepsilon = \frac{1}{M + 1} > 0.$$

Thus,

$$f'(x) \geq \frac{1}{M+1} > 0.$$

By Exercise 5.2, $f(x)$ is strictly increasing in $\mathbb{R}$ or one-to-one in $\mathbb{R}$.

$\square$

Exercise 5.4. *If*

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$

where $C_0, ..., C_n$ are real constants, prove that the equation

$$C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n = 0$$

has at least one real root between 0 *and* 1*.*

Proof. Let

$$g(x) = C_0 x + \frac{C_1}{2} x^2 + \cdots + \frac{C_{n-1}}{n} x^n + \frac{C_n}{n+1} x^{n+1} \in \mathbb{R}[x].$$

Then $g(0) = g(1) = 0$, and $g'(x) = C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n$. By the mean value theorem (Theorem 5.10), there exists a point $\xi \in (0, 1)$ at which

$$g(1) - g(0) = g'(\xi)(1 - 0),$$

or $g'(\xi) = 0$. That is, there exists a real root $x = \xi$ between 0 and 1 at which $C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n = 0$. $\square$

Exercise 5.5. *Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.*

Proof. Given any $x > 0$. Since f is differentiable for every $x > 0$, f is differentiable on $[x, x+1]$. By Theorem 5.2 and Theorem 5.10 (the mean value theorem), there is a point $\xi \in (x, x+1)$ at which

$$f(x+1) - f(x) = [(x+1) - x] f'(\xi)$$

or

$$g(x) = f'(\xi).$$

As $x \to +\infty$, $\xi \to +\infty$. Hence

$$\lim_{x \to +\infty} g(x) = \lim_{\xi \to +\infty} f'(\xi) = 0.$$

□

Exercise 5.6. *Suppose*

(a) f *is continuous for* $x \geq 0$,

(b) $f'(x)$ *exists for* $x > 0$,

(c) $f(0) = 0$,

(d) f' *is monotonically increasing.*

Put

$$g(x) = \frac{f(x)}{x} \qquad (x > 0)$$

and prove that g *is monotonically increasing.*

Proof.

(1) It suffices to show that $g'(x) \geq 0$ for $x > 0$ (Theorem 5.11(a)), that is, to show that
$$g'(x) = \frac{xf'(x) - f(x)}{x^2} \geq 0 \qquad (x > 0),$$
or
$$xf'(x) - f(x) \geq 0 \qquad (x > 0)$$
since $x^2 > 0$ for all nonzero x.

(2) Given $x > 0$. By (a)(b), we apply the mean value theorem (Theorem 5.10) on f to get
$$f(x) - f(0) = (x - 0)f'(\xi)$$
for some $\xi \in (0, x)$. By (c),
$$f(x) = xf'(\xi).$$
By (d),
$$f(x) = xf'(\xi) \leq xf'(x).$$
Hence $xf'(x) - f(x) \geq 0$, or g is monotonically increasing.

□

Note. g is increasing strictly if f is increasing strictly.

Exercise 5.7. *Suppose* $f'(x)$, $g'(x)$ *exist,* $g'(x) \neq 0$, *and* $f(x) = g(x) = 0$. *Prove that*

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}.$$

(This holds also for complex functions.)

Proof.

$$\begin{aligned}
\frac{f'(t)}{g'(t)} &= \frac{\lim_{t \to x} \frac{f(t)-f(x)}{t-x}}{\lim_{t \to x} \frac{g(t)-g(x)}{t-x}} \\
&= \lim_{t \to x} \frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}} && \text{(Both limits exist and } g' \neq 0) \\
&= \lim_{t \to x} \frac{f(t)}{g(t)}. && (f(x) = g(x) = 0)
\end{aligned}$$

This proof is also true for complex functions. □

Exercise 5.8.

Exercise 5.9. *Let f be a continuous real function on $\mathbb{R}^1$, of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$. Dose it follow that $f'(0)$ exists?*

Proof.

(1) *Show that $f'(0) = 3$.* It suffices to show that

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 3.$$

Write $F(x) = f(x) - f(0)$ and $G(x) = x - 0$ on $\mathbb{R}^1$. So that

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{F(x)}{G(x)} = 0.$$

(2) Note that

$$\lim_{x \to 0} \frac{F'(x)}{G'(x)} = \lim_{x \to 0} \frac{f'(x)}{1} = 3.$$

(3) Since f is continuous on $\mathbb{R}^1$, F is continuous on $\mathbb{R}^1$. Hence

$$\lim_{x \to 0} F(x) = F(\lim_{x \to 0} x) = F(0) = 0.$$

Also, G is continuous on $\mathbb{R}^1$ implies that

$$\lim_{x \to 0} G(x) = G(\lim_{x \to 0} x) = G(0) = 0.$$

(4) Apply L'Hospital's rule (Theorem 5.13) to (2)(3), we have

$$\lim_{x \to 0} \frac{F(x)}{G(x)} = 3,$$

or $f'(0) = 3$.

□

Exercise 5.10.
Exercise 5.11.
Exercise 5.12. *If $f(x) = |x|^3$, compute $f'(x)$, $f''(x)$ for all real x, and show that $f^{(3)}(0)$ does not exist.*

Proof.

(1) Write
$$f(x) = \begin{cases} x^3 & (x \geq 0), \\ -x^3 & (x \leq 0). \end{cases}$$

(2) *Show that $f'(x) = 3x|x|$.* It is trivial that
$$f'(x) = \begin{cases} 3x^2 & (x > 0), \\ -3x^2 & (x < 0). \end{cases}$$
Note that
$$\lim_{x \to 0} f'(x) = 0.$$
Apply the same argument in Exercise 5.9, we have
$$f'(0) = 0.$$
Hence f' exists and $f'(x) = 3x|x|$ for any $x \in \mathbb{R}$.

(3) *Show that $f''(x) = 6|x|$.* Similar to (2).
$$f''(x) = \begin{cases} 6x & (x > 0), \\ -6x & (x < 0). \end{cases}$$
Note that
$$\lim_{x \to 0} f''(x) = 0.$$
Apply the same argument in Exercise 5.9, we have
$$f''(0) = 0.$$
Hence f'' exists and $f''(x) = 6|x|$ for any $x \in \mathbb{R}$.

(4) *Show that $f^{(3)}(0)$ does not exist.*
$$f'''(x) = \begin{cases} 6 & (x > 0), \\ -6 & (x < 0). \end{cases}$$
There are some proofs for showing that $f^{(3)}(0)$ does not exist.

(a) Since

$$\lim_{t \to 0+} \frac{f''(t) - f''(0)}{t - 0} = \lim_{t \to 0+} \frac{6t}{t} = 6$$

and

$$\lim_{t \to 0-} \frac{f''(t) - f''(0)}{t - 0} = \lim_{t \to 0-} \frac{-6t}{t} = -6,$$

$f^{(3)}(0)$ does not exist.

(b) (Reductio ad absurdum) If f were differentiable on $\mathbb{R}^1$, then

$$\lim_{t \to 0+} f'''(t) = 6$$

and

$$\lim_{t \to 0-} f'''(t) = -6,$$

or f''' has a simple discontinuity at $x = 0$, contrary to Corollary to Theorem 5.12.

$\square$

Note. We can construct one real function $f = |x|^k$ on $\mathbb{R}^1$ such that all $f^{(0)}(0) = \cdots = f^{(k-1)}(0) = 0$ exist but $f^{(k)}(0)$ does not exist.

Exercise 5.13.

Exercise 5.14. *Let f be a differentiable real function defined in (a, b). Prove that f is convex if and only if f' is monotonically increasing. Assume next $f''(x)$ exists for every $x \in (a, b)$, and prove that f is convex if and only if $f''(x) \geq 0$ for all $x \in (a, b)$.*

Proof.

(1) *Show that f' is monotonically increasing if f is convex.*

(a) Since f is convex, by definition (Exercise 4.23)

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y)$$

whenever $a < x < b$, $a < y < b$, $0 < \lambda < 1$.

(b) As $x \neq y$, we have

$$\begin{aligned} f(y) - f(x) &\geq \frac{f(x + \lambda(y - x)) - f(x)}{\lambda} \\ &= \frac{f(x + \lambda(y - x)) - f(x)}{\lambda(y - x)} \cdot (y - x) \end{aligned}$$

and let $\lambda \to 0$ to get

$$f(y) - f(x) \geq f'(x)(y - x)$$

(since $f'(x)$ exists). Similarly, we have

$$f(x) - f(y) \geq f'(y)(x - y).$$

(c) Given any $y > x$, we have

$$f'(y)(y - x) \geq f(y) - f(x) \geq f'(x)(y - x).$$

Hence $f'(y) \geq f'(x)$ whenever $y > x$, or f' is monotonically increasing.

(2) *Show that f is convex if f' is monotonically increasing.* Given any $y > x$ and any $0 < \lambda < 1$.

(a) By Theorem 5.10 (the mean value theorem), there is a point $x < \xi < y$ such that

$$f(y) - f(x) = f'(\xi)(y - x).$$

Since f' is monotonically increasing,

$$f'(y)(y - x) \geq f(y) - f(x) \geq f'(x)(y - x).$$

(b) Write $z = \lambda x + (1 - \lambda)y$. Hence

$$\begin{aligned} f(y) - f(z) &\geq f'(z)(y - z), \\ f(z) - f(x) &\leq f'(z)(z - x), \end{aligned}$$

or

$$\begin{aligned} f(y) &\geq f(z) + f'(z)(y - z), \\ f(x) &\geq f(z) + f'(z)(x - z), \end{aligned}$$

or

$$\begin{aligned} \lambda f(x) + (1 - \lambda)f(y) \geq& \lambda[f(z) + f'(z)(x - z)] \\ &+ (1 - \lambda)[f(z) + f'(z)(y - z)] \\ =& f(z) \\ =& f(\lambda x + (1 - \lambda)y). \end{aligned}$$

Hence f is convex.

(3) *Show that $f''(x) \geq 0$ if f is convex and f'' exists.* By (1), f' is monotonically increasing since f is convex. Given any $x \neq y$, we have

$$\frac{f'(y) - f'(x)}{y - x} \geq 0.$$

Let $y \to x$, we have $f''(x) \geq 0$ if f'' exists.

(4) *Show that f is convex if f'' exists and $f''(x) \geq 0$.* By Theorem 5.11(a), f' is monotonically increasing. By (2), f is convex.

□