

# Notes on the book: *Apostol, Introduction to Analytic Number Theory*

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## Chapter 1: The Fundamental Theorem of Arithmetic

### Exercise 1.11.

Prove that  $n^4 + 4$  is composite if  $n > 1$ .

*Proof.*

$$n^4 + 4 = \underbrace{((n-1)^2 + 1)}_{>1} \underbrace{((n+1)^2 + 1)}_{>1}$$

since  $n > 1$ .  $\square$

### Exercise 1.15.

Prove that every  $n \geq 12$  is the sum of two composite numbers.

*Proof.* Write  $n = 2m$  (resp.  $n = 2m + 1$ ) where  $m \in \mathbb{Z}$ ,  $m \geq 6$ . Then  $n = 8 + 2(m-4)$  (resp.  $n = 9 + 2(m-4)$ ) is the sum of two composite numbers.  $\square$

### Exercise 1.16. (Mersenne primes)

Prove that if  $2^n - 1$  is prime, then  $n$  is prime.

*Proof.* Suppose  $n$  is a composite number, then we can write  $n = ab$  with  $a > 1$ ,  $b > 1$ . Hence

$$2^n - 1 = 2^{ab} - 1 = 2^{ab} - 1 = \underbrace{(2^a - 1)}_{>1} \underbrace{\{(2^a)^{b-1} + \dots + 1\}}_{>1}$$

is also a composite number.  $\square$

### Exercise 1.17. (Fermat primes)

Prove that if  $2^n + 1$  is prime, then  $n$  is a power of 2.

*Proof.* Write  $n = 2^a b$  where  $a$  is a nonnegative integer and  $b$  is odd. Suppose  $n$  is not a power of 2, then  $b > 1$ . Hence

$$2^n + 1 = 2^{2^a b} + 1 = \underbrace{(2^{2^a} + 1)}_{>1} \underbrace{\{2^{2^a(b-1)} - \dots + 1\}}_{>1}$$

is a composite number. (Note that  $1 < 2^{2^a(b-1)} < 2^n + 1$  implies that  $1 < (2^{2^a(b-1)} - \dots + 1) < 2^n + 1$  too.)  $\square$

**Exercise 1.30.**

If  $n > 1$  prove that the sum

$$\sum_{k=1}^n \frac{1}{k}$$

is not an integer.

*Proof.*

(1) (Reductio ad absurdum) Suppose

$$H := \sum_{k=1}^n \frac{1}{k}$$

were an integer.

(2) Let  $s$  be the largest integer such that  $2^s \leq n$ . So the integer number

$$\begin{aligned} 2^{s-1}H &= \sum_{k=1}^n \frac{2^{s-1}}{k} \\ &= 2^{s-1} + 2^{s-2} + \frac{2^{s-1}}{3} + 2^{s-3} + \frac{2^{s-1}}{5} + \frac{2^{s-2}}{3} + \dots + \frac{1}{2} + \dots \end{aligned}$$

has only one term of even denominators (as  $n > 1$ ) if we write all terms in irreducible fractions. That is,

$$2^{s-1}H = \frac{1}{2} + \frac{c}{d} \in \mathbb{Z}$$

where  $\frac{c}{d}$  is an irreducible fraction with odd  $d$ . Hence it suffices to show that  $2 \nmid d$  to get a contradiction.

(3) By

$$\frac{1}{2} + \frac{c}{d} = \frac{d+2c}{2d} \in \mathbb{Z}$$

we have  $d+2c = 2dd'$  for some  $d' \in \mathbb{Z}$ . Note that 2 is a prime. So  $2 \mid (d+2c)$  or  $2 \mid d$ , which is absurd.

$\square$

## Chapter 2: Arithmetical functions and Dirichlet multiplication

### Exercise 2.1.

Find all integers  $n$  such that

- (a)  $\varphi(n) = \frac{n}{2}$ ,
- (b)  $\varphi(n) = \varphi(2n)$ ,
- (c)  $\varphi(n) = 12$ .

*Proof of (a).*

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right) = \frac{n}{2}$$

(Theorem 2.4) implies that  $n = 2$ .  $\square$

*Proof of (b).*

- (1)  $\varphi(n) = \varphi(2n)$  implies that

$$n \prod_{p|n} \left(1 - \frac{1}{p}\right) = 2n \prod_{p|(2n)} \left(1 - \frac{1}{p}\right).$$

- (2) If  $2|n$ , then  $n = 2n$  or  $n = 0$ , which is absurd.
- (3) If  $2 \nmid n$ , then

$$n \prod_{p|n} \left(1 - \frac{1}{p}\right) = 2n \prod_{p|(2n)} \left(1 - \frac{1}{p}\right) = \underbrace{2n \left(1 - \frac{1}{2}\right)}_{=n} \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

is always true. Hence  $n$  is odd if  $\varphi(n) = \varphi(2n)$ .

$\square$

*Proof of (c).*

- (1) Show that the solutions of  $\varphi(n) = 12$  are  $n = 13, 26, 21, 28, 42, 36$ . Write  $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$  where  $p_1 < p_2 < \dots$ . Then

$$12 = \varphi(n) = \prod_{i=1}^r p_i^{\alpha_i-1} (p_i - 1).$$

(Theorem 2.5). It implies that  $p_i \in \{2, 3, 5, 7, 13\}$  if  $\alpha_i > 0$ . Consider all possible cases of the greatest prime divisor  $p_r$  of  $n$  as follows.

(2) If  $p_r = 13$ , then  $\alpha_r = 1$  since  $13 \nmid 12$ . So

$$12 = \varphi(n) = \underbrace{\varphi(13)}_{=12} \varphi\left(\frac{n}{13}\right)$$

or  $1 = \varphi\left(\frac{n}{13}\right)$ . Hence  $\frac{n}{13} = 1, 2$ . In this case  $n = 13, 26$ .

(3) If  $p_r = 7$ , then  $\alpha_r = 1$  since  $7 \nmid 12$ . So

$$12 = \varphi(n) = \underbrace{\varphi(7)}_{=6} \varphi\left(\frac{n}{7}\right)$$

or  $2 = \varphi\left(\frac{n}{7}\right)$ . Hence  $\frac{n}{7} = 3, 4, 6$ . In this case  $n = 21, 28, 42$ .

(5) If  $p_r = 5$ , then  $\alpha_r = 1$  since  $5 \nmid 12$ . So  $12 = \varphi(5)\varphi\left(\frac{n}{5}\right)$  or  $3 = \varphi\left(\frac{n}{5}\right)$ , which is impossible.

(6) If  $p_r = 3$ , then  $\alpha_r = 1, 2$ .  $\alpha_r = 1$  is impossible since  $3 \mid 12$ . So

$$12 = \varphi(n) = \underbrace{\varphi(3^2)}_{=6} \varphi\left(\frac{n}{3^2}\right)$$

or  $2 = \varphi\left(\frac{n}{3^2}\right)$ . Hence  $\frac{n}{3^2} = 4$ . (By assumption  $\frac{n}{3^2}$  cannot have any prime factor  $> 3$ .) In this case  $n = 36$ .

□

### Exercise 2.2.

For each of the following statements either give a proof or exhibit a counter example.

- (a) If  $(m, n) = 1$  then  $(\varphi(m), \varphi(n)) = 1$ .
- (b) If  $n$  is composite, then  $(n, \varphi(n)) > 1$ .
- (c) If the same primes divide  $m$  and  $n$ , then  $n\varphi(m) = m\varphi(n)$ .

*Proof of (a).* It is false since  $(5, 13) = 1$  and  $(\varphi(5), \varphi(13)) = (4, 12) = 4$ . □

*Proof of (b).* It is false since  $(15, \varphi(15)) = (15, 8) = 1$ . □

*Proof of (c).*

- (1) It is true.

(2) If the same primes divide  $m$  and  $n$ , then

$$\frac{\varphi(n)}{n} = \prod_{p|n} \left(1 - \frac{1}{p}\right) = \prod_{p|m} \left(1 - \frac{1}{p}\right) = \frac{\varphi(m)}{m}$$

(Theorem 2.4). Hence  $n\varphi(m) = m\varphi(n)$ .

□

### Exercise 2.3.

Prove that

$$\frac{n}{\varphi(n)} = \sum_{d|n} \frac{\mu(d)^2}{\varphi(d)}.$$

*Proof.*

(1) Note that  $fg$ ,  $f/g$  and  $f * g$  are multiplicative if  $f$  and  $g$  are multiplicative (Example 5 on page 34 and Theorem 2.14). Hence  $\frac{n}{\varphi(n)}$  and  $\sum_{d|n} \frac{\mu^2(d)}{\varphi(d)}$  are multiplicative. Hence it might assume that  $n = p^a$  for some prime  $p$  and integer  $a \geq 1$ . (The case  $n = 1$  is trivial.)

(2)

$$\frac{p^a}{\varphi(p^a)} = \frac{p^a}{p^a - p^{a-1}} = \frac{p}{p-1}.$$

(3)

$$\begin{aligned} \sum_{d|p^a} \frac{\mu(d)^2}{\varphi(d)} &= \frac{\mu(1)^2}{\varphi(1)} + \frac{\mu(p)^2}{\varphi(p)} + \overbrace{\frac{\mu(p^2)^2}{\varphi(p^2)}}^{=0} + \cdots + \overbrace{\frac{\mu(p^a)^2}{\varphi(p^a)}}^{=0} \\ &= 1 + \frac{1}{p-1} + 0 + \cdots + 0 \\ &= \frac{p}{p-1}. \end{aligned}$$

(4) Or apply Theorems 2.4 and 2.18 to get

$$\begin{aligned} \sum_{d|n} \frac{\mu(d)^2}{\varphi(d)} &= \prod_{p|n} \left(1 - \frac{\mu(p)}{\varphi(p)}\right) \\ &= \prod_{p|n} \left(1 - \frac{-1}{p-1}\right) \\ &= \prod_{p|n} \frac{p}{p-1} \\ &= \frac{n}{\varphi(n)}. \end{aligned}$$

□

**Supplement 2.3.1. (Chinese remainder theorem)**

(Exercise I.3.5 in the textbook: *Jürgen Neukirch, Algebraic Number Theory*.)  
*The quotient ring  $\mathcal{O}/\mathfrak{a}$  of a Dedekind domain by an ideal  $\mathfrak{a} \neq 0$  is a principal ideal domain. (Hint: For  $\mathfrak{a} = \mathfrak{p}^n$  the only proper ideals of  $\mathcal{O}/\mathfrak{a}$  are given by  $\mathfrak{p}/\mathfrak{p}^n, \dots, \mathfrak{p}^{n-1}/\mathfrak{p}^n$ . Choose  $\pi \in \mathfrak{p} \setminus \mathfrak{p}^2$  and show that  $\mathfrak{p}^\nu = \mathcal{O}\pi^\nu + \mathfrak{p}^n$ .)*

*Proof.*

- (1) By the Chinese remainder theorem, it suffices to show the case  $\mathfrak{a} = \mathfrak{p}^n$  where  $\mathfrak{p}$  is prime.
- (2) There is a natural correspondence between

$$\{\text{ideals of } \mathcal{O}/\mathfrak{p}^n\} \longleftrightarrow \{\text{ideals of } \mathcal{O} \text{ containing } \mathfrak{p}^n\}.$$

Hence the proper ideals of  $\mathcal{O}/\mathfrak{p}^n$  are given by  $\mathfrak{p}/\mathfrak{p}^n, \dots, \mathfrak{p}^{n-1}/\mathfrak{p}^n$ .

- (3) Similar to Exercise I.3.4, choose  $\pi \in \mathfrak{p} \setminus \mathfrak{p}^2$  and thus  $\mathfrak{p}^\nu = \mathcal{O}\pi^\nu + \mathfrak{p}^n$  ( $\nu = 1, \dots, n-1$ ) since they have the same prime factorization. Hence  $\mathfrak{p}^\nu/\mathfrak{p}^n = (\pi^\nu + \mathfrak{p}^n)$  is principal.

□

**Exercise 2.4.**

*Prove that  $\varphi(n) > \frac{n}{6}$  for all  $n$  with at most 8 distinct prime factors.*

*Proof.*

- (1)

$$\begin{aligned} \varphi(n) &= n \prod_{p|n} \left(1 - \frac{1}{p}\right) && \text{(Theorem 2.4)} \\ &\geq n \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{5}\right) \left(1 - \frac{1}{7}\right) \\ &\quad \left(1 - \frac{1}{11}\right) \left(1 - \frac{1}{13}\right) \left(1 - \frac{1}{17}\right) \left(1 - \frac{1}{19}\right) \\ &= \frac{55296}{323323} n \\ &> \frac{n}{6}. \end{aligned}$$

(2) The conclusion does not hold if  $n$  has more than 9 distinct prime factors.

□

**Exercise 2.6.**

Prove that

$$\sum_{d^2|n} \mu(d) = \mu(n)^2$$

and, more generally

$$\sum_{d^k|n} \mu(d) = \begin{cases} 0 & \text{if } m^k|n \text{ for some } m > 1, \\ 1 & \text{otherwise.} \end{cases}$$

The last sum is extended over all positive divisors  $d$  of  $n$  whose  $k$ th power also divide  $n$ .

*Proof.*

- (1) Write  $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r} q_1^{\beta_1} \cdots q_s^{\beta_s}$  where  $\alpha_i \geq 2$  and  $\beta_j = 1$ . The proof is similar to Theorem 2.1.
- (2) If  $p_1^{\alpha_1} \cdots p_r^{\alpha_r} = 1$ , then  $\sum_{d^2|n} \mu(d) = \mu(1) = 1$ .
- (3) If  $p_1^{\alpha_1} \cdots p_r^{\alpha_r} > 1$ , then

$$\begin{aligned} \sum_{d^2|n} \mu(d) &= \mu(1) + \mu(p_1) + \cdots + \mu(p_r) \\ &\quad + \mu(p_1 p_2) + \cdots + \mu(p_{r-1} p_r) + \cdots + \mu(p_1 \cdots p_r) \\ &= 1 + \binom{r}{1}(-1) + \binom{r}{2}(-1)^2 + \cdots + \binom{r}{r}(-1)^r \\ &= (1 - 1)^r \\ &= 0. \end{aligned}$$

- (4) By (2)(3),  $\sum_{d^2|n} \mu(d) = \mu(n)^2$ . Besides, we have

$$\sum_{d^k|n} \mu(d) = \begin{cases} 0 & \text{if } m^k|n \text{ for some } m > 1, \\ 1 & \text{otherwise} \end{cases}$$

by the same argument as (1)(2)(3).

□



**Exercise 2.7.**

Let  $\mu(p, d)$  denote the value of the Möbius function at the gcd of  $p$  and  $d$ . Prove that for every prime  $p$  we have

$$\sum_{d|n} \mu(d) \mu(p, d) = \begin{cases} 1 & \text{if } n = 1, \\ 2 & \text{if } n = p^a, a \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.*

- (1) It suffices to show that  $\mu(p, n)$  is multiplicative. If so, then

$$h(n) := \sum_{d|n} \mu(d) \mu(p, d)$$

is also multiplicative by taking  $f(n) := \mu(n) \mu(p, n)$  and  $g(n) := 1$  in Theorem 2.14.

- (2) A direct calculation shows that  $h(1) = 1$  (or by Theorem 2.12) and

$$\begin{aligned} h(p^a) &= \mu(1) \mu(p, 1) + \mu(p) \mu(p, p) = 1 \cdot 1 + (-1) \cdot (-1) = 2, \\ h(q^b) &= \mu(1) \mu(p, 1) + \mu(q) \mu(p, q) = 1 \cdot 1 + (-1) \cdot 1 = 0 \end{aligned}$$

where  $q \neq p$  and  $a, b \geq 1$ . Hence (1) and Theorem 2.13 show that

$$h(n) = \begin{cases} 1 & \text{if } n = 1, \\ 2 & \text{if } n = p^a, a \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

- (3) Show that  $\mu(p, n)$  is multiplicative. Suppose  $(m, n) = 1$ . There are two possible cases:  $p \nmid mn$  and  $p | mn$ .

- (a) If  $p \nmid mn$ , then all  $\mu(p, mn), \mu(p, m), \mu(p, n)$  are equal to  $\mu(1) = 1$ .
- (b) If  $p | mn$ , then  $p | m$  or  $p | n$ . Note that  $(m, n) = 1$  and thus  $p$  cannot be a common divisor of  $m, n$ . Hence  $\mu(p, mn) = \mu(p) = -1$  and  $\mu(p, m) \mu(p, n) = \mu(p) \mu(1) = -1$ .

In any case  $\mu(p, mn) = \mu(p, m) \mu(p, n)$  if  $(m, n) = 1$ .

□

**Exercise 2.9.**

If  $x$  is real,  $x \geq 1$ , let  $\varphi(x, n)$  denote the number of positive integers  $\leq x$  that are relatively prime to  $n$ . [Note that  $\varphi(n, n) = \varphi(n)$ .] Prove that

$$\varphi(x, n) = \sum_{d|n} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor, \quad \sum_{d|n} \varphi\left(\frac{x}{d}, \frac{n}{d}\right) = [x].$$

*Proof.*

- (1) Show that  $\varphi(x, n) = \sum_{d|n} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor$ . Similar to the proof of Theorem 2.3.  $\varphi(x, n)$  can be written in the form

$$\varphi(x, n) = \sum_{1 \leq k \leq x} \left[ \frac{1}{(n, k)} \right],$$

where now  $k$  runs through all integers  $\leq x$ . Now we use Theorem 2.1 with  $n$  replaced by  $(n, k)$  to obtain

$$\varphi(x, n) = \sum_{1 \leq k \leq x} \sum_{d|(n, k)} \mu(d) = \sum_{1 \leq k \leq x} \sum_{\substack{d|n \\ d|k}} \mu(d).$$

For a fixed divisor  $d$  of  $n$  we must sum over all those  $k$  in the range  $1 \leq k \leq x$  which are multiples of  $d$ . If we write  $k = qd$  then  $1 \leq k \leq x$  if and only if  $1 \leq q \leq \left\lfloor \frac{x}{d} \right\rfloor$ . Hence the last sum for  $\varphi(x, n)$  can be written as

$$\varphi(x, n) = \sum_{d|n} \sum_{1 \leq q \leq \left\lfloor \frac{x}{d} \right\rfloor} \mu(d) = \sum_{d|n} \mu(d) \sum_{1 \leq q \leq \left\lfloor \frac{x}{d} \right\rfloor} 1 = \sum_{d|n} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor.$$

- (2) Show that  $\sum_{d|n} \varphi\left(\frac{x}{d}, \frac{n}{d}\right) = [x]$ . Similar to the proof of Theorem 2.2. Let  $S$  denote the set  $\{1, 2, \dots, [x]\}$ . We distribute the integers of  $S$  into disjoint sets as follows. For each divisor  $d$  of  $n$ , let

$$A(d) = \{k : (k, n) = d, 1 \leq k \leq x\}.$$

That is,  $A(d)$  contains those elements of  $S$  which have the gcd  $d$  with  $n$ . The sets  $A(d)$  form a disjoint collection whose union is  $S$ . Therefore if  $f(d)$  denotes the number of integers in  $A(d)$  we have

$$\sum_{d|n} f(d) = [x].$$

But  $(k, n) = d$  if and only if  $\left(\frac{k}{d}, \frac{n}{d}\right) = 1$ , and  $0 < k \leq x$  if and only if  $0 < \frac{k}{d} \leq \frac{x}{d}$ . Therefore, if we let  $q = \frac{k}{d}$ , there is a one-to-one correspondence between the elements in  $A(d)$  and those integers  $q$  satisfying  $0 < q \leq \frac{x}{d}$ ,  $\left(q, \frac{n}{d}\right) = 1$ . The number of such  $q$  is  $\varphi\left(\frac{x}{d}, \frac{n}{d}\right)$ . Hence  $f(d) = \varphi\left(\frac{x}{d}, \frac{n}{d}\right)$  and thus

$$\sum_{d|n} \varphi\left(\frac{x}{d}, \frac{n}{d}\right) = [x].$$

□

In Exercise 2.10, 2.11 and 2.12,  $d(n)$  denotes the number of positive divisors of  $n$ .

**Exercise 2.10.**

Prove that  $\prod_{t|n} t = n^{\frac{d(n)}{2}}$ .

*Proof.*

(1) Note that  $d(1) = 1$  and

$$d(p_1^{\alpha_1} \cdots p_r^{\alpha_r}) = (\alpha_1 + 1) \cdots (\alpha_r + 1) = d(p_1^{\alpha_1}) \cdots d(p_r^{\alpha_r}).$$

Hence  $d(n)$  is multiplicative (Theorem 2.13).

(2) Show that  $\prod_{t|n} t = n^{\frac{d(n)}{2}}$ .  $n = 1$  is trivial. Assume  $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r} > 1$ . Then  $t|n$  if and only if  $t = p_1^{x_1} \cdots p_r^{x_r}$  with  $0 \leq x_i \leq \alpha_i$  ( $i = 1, \dots, r$ ). So

$$\begin{aligned} \prod_{t|n} t &= \prod_{\substack{0 \leq x_1 \leq \alpha_1 \\ \vdots \\ 0 \leq x_r \leq \alpha_r}} p_1^{x_1} \cdots p_r^{x_r} \\ &= p_1^{(0+1+\cdots+\alpha_1)(\alpha_2+1)\cdots(\alpha_r+1)} \cdots p_r^{(\alpha_1+1)\cdots(\alpha_{r-1}+1)(0+1+\cdots+\alpha_r)} \\ &= p_1^{\frac{\alpha_1(\alpha_1+1)}{2} \cdot (\alpha_2+1)\cdots(\alpha_r+1)} \cdots p_r^{(\alpha_1+1)\cdots(\alpha_{r-1}+1) \cdot \frac{\alpha_r(\alpha_r+1)}{2}} \\ &= p_1^{\alpha_1 \frac{d(n)}{2}} \cdots p_r^{\alpha_r \frac{d(n)}{2}} \\ &= (p_1^{\alpha_1} \cdots p_r^{\alpha_r})^{\frac{d(n)}{2}} \\ &= n^{\frac{d(n)}{2}}. \end{aligned}$$

□

**Exercise 2.11.**

Prove that  $d(n)$  is odd if, and only if,  $n$  is a square.

*Proof.*  $n = 1$  is trivial. Assume  $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r} > 1$ . Then

$$\begin{aligned} d(n) &= (\alpha_1 + 1) \cdots (\alpha_r + 1) \text{ is odd} && \text{(Exercise 2.10)} \\ \iff \alpha_1 + 1, \dots, \alpha_r + 1 &\text{ are odd} \\ \iff \alpha_1, \dots, \alpha_r &\text{ are even} \\ \iff n &\text{ is a square.} \end{aligned}$$

□

**Exercise 2.12.**

Prove that  $\sum_{t|n} d(t)^3 = \left(\sum_{t|n} d(t)\right)^2$ .

*Proof.*

(1) Exercise 2.10 shows that  $d(n)$  is multiplicative. Similar to the proof of Exercise 2.7, both  $f(n) := \sum_{t|n} d(t)^3$  and  $g(n) := \left(\sum_{t|n} d(t)\right)^2$  are multiplicative. So it suffices to show that  $f(p^a) = g(p^a)$  (Theorem 2.13).

(2) A direct calculation shows that

$$\begin{aligned} f(p^a) &= \sum_{t|p^a} d(t)^3 \\ &= d(1)^3 + d(p)^3 + \cdots + d(p^a)^3 \\ &= 1^3 + 2^3 + \cdots + (a+1)^3 \\ &= \left(\frac{(a+1)(a+2)}{2}\right)^2 \end{aligned}$$

and

$$\begin{aligned} g(p^a) &= \left(\sum_{t|p^a} d(t)\right)^2 \\ &= (d(1) + d(p) + \cdots + d(p^a))^2 \\ &= (1 + 2 + \cdots + (a+1))^2 \\ &= \left(\frac{(a+1)(a+2)}{2}\right)^2 \end{aligned}$$

are equal.

□

## Chapter 3: Average of arithmetical functions

### Exercise 3.1.

Use Euler's summation formula to deduce the following for  $x \geq 2$ :

- (a)  $\sum_{n \leq x} \frac{\log n}{n} = \frac{1}{2}(\log x)^2 + A + O\left(\frac{\log x}{x}\right)$ , where  $A$  is a constant.
- (b)  $\sum_{2 \leq n \leq x} \frac{1}{n \log n} = \log \log x + B + O\left(\frac{1}{x \log x}\right)$ , where  $B$  is a constant.

*Proof of (a).*

- (1) Similar to the proof of Theorem 3.2. We take  $f(t) = \frac{\log t}{t}$  in Euler's summation formula to obtain

$$\begin{aligned} \sum_{n \leq x} \frac{\log n}{n} &= \int_1^x \frac{\log t}{t} dt + \int_1^x (t - [t]) \frac{1 - \log t}{t^2} dt \\ &\quad + \frac{\log x}{x}([x] - x) - \underbrace{\frac{\log(1)}{1}([1] - 1)}_{=0} \\ &= \frac{1}{2}(\log x)^2 + \int_1^x (t - [t]) \frac{1 - \log t}{t^2} dt + O\left(\frac{\log x}{x}\right) \\ &= \frac{1}{2}(\log x)^2 + \int_1^\infty (t - [t]) \frac{1 - \log t}{t^2} dt \\ &\quad - \int_x^\infty (t - [t]) \frac{1 - \log t}{t^2} dt + O\left(\frac{\log x}{x}\right). \end{aligned}$$

- (2) The improper integral  $\int_1^\infty (t - [t]) \frac{1 - \log t}{t^2} dt$  exists since it is dominated by  $\int_1^e \frac{1 - \log t}{t^2} dt + \int_e^\infty \frac{\log t - 1}{t^2} dt = 2e^{-1}$ .
- (3) Might assume that  $x \geq e$ . So

$$0 \leq - \int_x^\infty (t - [t]) \frac{1 - \log t}{t^2} dt \leq \int_x^\infty \frac{\log t - 1}{t^2} dt = \frac{\log x}{x}.$$

- (4) Therefore

$$\sum_{n \leq x} \frac{\log n}{n} = \frac{1}{2}(\log x)^2 + A + O\left(\frac{\log x}{x}\right)$$

where  $A = \int_1^\infty (t - [t]) \frac{1 - \log t}{t^2} dt$  is a constant.

□

*Proof of (b).*

(1) We take  $f(t) = \frac{1}{t \log t}$  in Euler's summation formula to obtain

$$\begin{aligned}
\sum_{2 \leq n \leq x} \frac{1}{n \log n} &= \int_2^x \frac{1}{t \log t} dt + \int_2^x -(t - [t]) \frac{\log t + 1}{t^2 (\log t)^2} dt \\
&\quad + \frac{1}{x \log x} ([x] - x) - \underbrace{\frac{1}{2 \cdot \log(2)} ([2] - 2)}_{=0} \\
&= \log \log x - \log \log 2 - \int_2^x (t - [t]) \frac{\log t + 1}{t^2 (\log t)^2} dt \\
&\quad + O\left(\frac{1}{x \log x}\right) \\
&= \log \log x - \log \log 2 - \int_2^\infty (t - [t]) \frac{\log t + 1}{t^2 (\log t)^2} dt \\
&\quad + \int_x^\infty (t - [t]) \frac{\log t + 1}{t^2 (\log t)^2} dt + O\left(\frac{1}{x \log x}\right).
\end{aligned}$$

(2) The improper integral  $\int_2^\infty (t - [t]) \frac{\log t + 1}{t^2 (\log t)^2} dt$  exists since it is dominated by  $\int_2^\infty \frac{\log t + 1}{t^2 (\log t)^2} dt = \frac{1}{2 \log 2} < \infty$ .

(3)

$$0 \leq \int_x^\infty (t - [t]) \frac{\log t + 1}{t^2 (\log t)^2} dt \leq \int_x^\infty \frac{\log t + 1}{t^2 (\log t)^2} dt = \frac{1}{x \log x}.$$

(4) Therefore

$$\sum_{2 \leq n \leq x} \frac{1}{n \log n} = \log \log x + B + O\left(\frac{1}{x \log x}\right)$$

where  $B = -\log \log 2 - \int_2^\infty (t - [t]) \frac{\log t + 1}{t^2 (\log t)^2} dt$  is a constant.

□

### Exercise 3.2.

If  $x \geq 2$  prove that

$$\sum_{n \leq x} \frac{d(n)}{n} = \frac{1}{2} (\log x)^2 + 2C \log x + O(1),$$

where  $C$  is Euler's constant.

*Proof.* Similar to the proof of Theorem 3.3, we have

$$\sum_{n \leq x} \frac{d(n)}{n} = \sum_{n \leq x} \frac{1}{n} \sum_{d|n} 1 = \sum_{\substack{q, d \\ qd \leq x}} \frac{1}{qd} = \sum_{d \leq x} \frac{1}{d} \sum_{q \leq \frac{x}{d}} \frac{1}{q}.$$

Now we use Theorem 3.2(a) to obtain

$$\sum_{q \leq \frac{x}{d}} \frac{1}{q} = \log \frac{x}{d} + C + O\left(\frac{d}{x}\right) = \log x - \log d + C + O\left(\frac{d}{x}\right).$$

Using this along with Theorem 3.2(a) and Exercise 3.1 we find

$$\begin{aligned} \sum_{n \leq x} \frac{d(n)}{n} &= \sum_{d \leq x} \frac{1}{d} \left\{ \log x - \log d + C + O\left(\frac{d}{x}\right) \right\} \\ &= (\log x + C) \sum_{d \leq x} \frac{1}{d} - \sum_{d \leq x} \frac{\log d}{d} + \sum_{d \leq x} O\left(\frac{1}{x}\right) \\ &= (\log x + C) \left\{ \log x + C + O\left(\frac{1}{x}\right) \right\} \\ &\quad - \left\{ \frac{1}{2}(\log x)^2 + A + O\left(\frac{\log x}{x}\right) \right\} + O(1) \\ &= (\log x)^2 + 2C \log x - \frac{1}{2}(\log x)^2 + O(1) \\ &= \frac{1}{2}(\log x)^2 + 2C \log x + O(1). \end{aligned}$$

□

### Exercise 3.3.

If  $x \geq 2$  and  $\alpha > 0$ ,  $\alpha \neq 1$ , prove that

$$\sum_{n \leq x} \frac{d(n)}{n^\alpha} = \frac{x^{1-\alpha} \log x}{1-\alpha} + \zeta(\alpha)^2 + O(x^{1-\alpha}).$$

*Proof.*

(1) Similar to Exercise 3.2.

$$\sum_{n \leq x} \frac{d(n)}{n^\alpha} = \sum_{n \leq x} \frac{1}{n^\alpha} \sum_{d|n} 1 = \sum_{\substack{q, d \\ qd \leq x}} \frac{1}{q^\alpha d^\alpha} = \sum_{d \leq x} \frac{1}{d^\alpha} \sum_{q \leq \frac{x}{d}} \frac{1}{q^\alpha}.$$

Now we use Theorem 3.2(b) to obtain

$$\sum_{q \leq \frac{x}{d}} \frac{1}{q^\alpha} = \frac{1}{d^{1-\alpha}} \cdot \frac{x^{1-\alpha}}{1-\alpha} + \zeta(\alpha) + O\left(\frac{d^\alpha}{x^\alpha}\right).$$

Using this along with Theorem 3.2 we find

$$\begin{aligned}
\sum_{n \leq x} \frac{d(n)}{n^\alpha} &= \sum_{d \leq x} \frac{1}{d^\alpha} \left\{ \frac{1}{d^{1-\alpha}} \cdot \frac{x^{1-\alpha}}{1-\alpha} + \zeta(\alpha) + O\left(\frac{d^\alpha}{x^\alpha}\right) \right\} \\
&= \frac{x^{1-\alpha}}{1-\alpha} \sum_{d \leq x} \frac{1}{d} + \zeta(\alpha) \sum_{d \leq x} \frac{1}{d^\alpha} + \sum_{d \leq x} O(x^{-\alpha}) \\
&= \frac{x^{1-\alpha}}{1-\alpha} \{ \log x + C + O(x^{-1}) \} \\
&\quad + \zeta(\alpha) \left\{ \frac{x^{1-\alpha}}{1-\alpha} + \zeta(\alpha) + O(x^{-\alpha}) \right\} + O(x^{1-\alpha}) \\
&= \frac{x^{1-\alpha} \log x}{1-\alpha} + \zeta(\alpha)^2 + O(x^{1-\alpha}).
\end{aligned}$$

□

**Exercise 3.5.**

If  $x \geq 1$  prove that:

- (a)  $\sum_{n \leq x} \varphi(n) = \frac{1}{2} \sum_{n \leq x} \mu(n) \left[ \frac{x}{n} \right]^2 + \frac{1}{2}.$
- (b)  $\sum_{n \leq x} \frac{\varphi(n)}{n} = \sum_{n \leq x} \frac{\mu(n)}{n} \left[ \frac{x}{n} \right].$

These formulas, together with those in Exercise 3.4, show that, for  $x \geq 2$ ,

$$\sum_{n \leq x} \varphi(n) = \frac{1}{2} \frac{x^2}{\zeta(2)} + O(x \log x), \quad \sum_{n \leq x} \frac{\varphi(n)}{n} = \frac{x}{\zeta(2)} + O(\log x).$$

The last two formulas are trivial and we omit the proof.



*Proof of (a).* Same as the proof of Theorem 3.7.

$$\begin{aligned}
\sum_{n \leq x} \varphi(n) &= \sum_{n \leq x} \sum_{d|n} \mu(d) \frac{n}{d} \\
&= \sum_{\substack{q, d \\ qd \leq x}} \mu(d) q \\
&= \sum_{d \leq x} \mu(d) \sum_{q \leq \frac{x}{d}} q \\
&= \sum_{d \leq x} \mu(d) \frac{1}{2} \left[ \frac{x}{d} \right] \left( 1 + \left[ \frac{x}{d} \right] \right) \\
&= \frac{1}{2} \sum_{d \leq x} \mu(d) \left[ \frac{x}{d} \right]^2 + \frac{1}{2} \sum_{d \leq x} \mu(d) \left[ \frac{x}{d} \right] \\
&= \frac{1}{2} \sum_{d \leq x} \mu(d) \left[ \frac{x}{d} \right]^2 + \frac{1}{2} \quad (\text{Theorem 3.12})
\end{aligned}$$

□

*Proof of (b).*

(1)

$$\begin{aligned}
\sum_{n \leq x} \frac{\varphi(n)}{n} &= \sum_{n \leq x} \sum_{d|n} \frac{\mu(d)}{d} \quad (\text{Theorem 2.3}) \\
&= \sum_{n \leq x} \frac{\mu(n)}{n} \left[ \frac{x}{n} \right]. \quad (\text{Theorem 3.11})
\end{aligned}$$

□