

Solutions to the book: *Fulton, Algebraic Curves*

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Contents

Chapter 1: Affine Algebraic Sets	4
1.1. Algebraic Preliminaries	4
Problem 1.1.*	4
Problem 1.5.*	5
Problem 1.6.*	5
1.2. Affine Space and Algebraic Sets	6
Problem 1.8.*	6
Problem 1.9.	7
Problem 1.11.	7
Problem 1.15.*	8
1.3. The Ideal of a Set of Points	9
Problem 1.18.*	9
Problem PLACEHOLDER	10
1.4. The Hilbert Basis Theorem	10
1.5. Irreducible Components of an Algebraic Set	10
1.6. Algebraic Subsets of the Plane	10
1.7. Hilbert's Nullstellensatz	10
1.8. Modules; Finiteness Conditions	10
1.9. Integral Elements	10
1.10. Field Extensions	10
Chapter 2: Affine Varieties	11
2.1. Coordinate Rings	11
Problem PLACEHOLDER	11
2.2. Polynomial Maps	11
2.3. Coordinate Changes	11
2.4. Rational Functions and Local Rings	11
2.5. Discrete Valuation Rings	11
2.6. Forms	11

2.7. Direct Products of Rings	11
2.8. Operations with Ideals	11
2.9. Ideals with a Finite Number of Zeros	11
2.10. Quotient Modules and Exact Sequences	11
2.11. Free Modules	11
Chapter 3: Local Properties of Plane Curves	12
3.1. Multiple Points and Tangent Lines	12
Problem PLACEHOLDER	12
3.2. Multiplicities and Local Rings	12
3.3. Intersection Numbers	12
Chapter 4: Projective Varieties	13
4.1. Projective Space	13
Problem PLACEHOLDER	13
4.2. Projective Algebraic Sets	13
4.3. Affine and Projective Varieties	13
4.4. Multiprojective Space	13
Chapter 5: Projective Plane Curves	14
5.1. Definitions	14
Problem PLACEHOLDER	14
5.2. Linear Systems of Curves	14
5.3. Bézout's Theorem	14
5.4. Multiple Points	14
5.5. Max Noether's Fundamental Theorem	14
5.6. Applications of Noether's Theorem	14
Chapter 6: Varieties, Morphisms, and Rational Maps	15
6.1. The Zariski Topology	15
6.2. Varieties	15
6.3. Morphisms of Varieties	15
6.4. Products and Graphs	15
6.5. Algebraic Function Fields and Dimension of Varieties	15
6.6. Rational Maps	15
Chapter 7: Resolution of Singularities	16
7.1. Rational Maps of Curves	16
Problem PLACEHOLDER	16
7.2. Blowing up a Point in \mathbf{A}^2	16
7.3. Blowing up a Point in \mathbf{P}^2	16
7.4. Quadratic Transformations	16
7.5. Nonsingular Models of Curves	16

Chapter 8: Riemann-Roch Theorem	17
8.1. Divisors	17
Problem PLACEHOLDER	17
8.1. The Vector Spaces $L(D)$	17
8.1. Riemann's Theorem	17
8.1. Derivations and Differentials	17
8.1. Canonical Divisors	17
8.6. Riemann-Roch Theorem	17

Chapter 1: Affine Algebraic Sets

1.1. Algebraic Preliminaries

Problem 1.1.*

Let R be a domain.

- (a) If f, g are forms of degree r, s respectively in $R[x_1, \dots, x_n]$, show that fg is a form of degree $r + s$.
- (b) Show that any factor of a form in $R[x_1, \dots, x_n]$ is also a form.

Proof of (a).

- (1) Write

$$f = \sum_{(i)} a_{(i)} x^{(i)},$$
$$g = \sum_{(j)} b_{(j)} x^{(j)},$$

where $\sum_{(i)}$ is the summation over $(i) = (i_1, \dots, i_n)$ with $i_1 + \dots + i_n = r$ and $\sum_{(j)}$ is the summation over $(j) = (j_1, \dots, j_n)$ with $j_1 + \dots + j_n = s$.

- (2) Hence,

$$fg = \sum_{(i)} \sum_{(j)} a_{(i)} b_{(j)} x^{(i)} x^{(j)}$$
$$= \sum_{(i), (j)} a_{(i)} b_{(j)} x^{(k)}$$

where $(k) = (i_1 + j_1, \dots, i_n + j_n)$ with $(i_1 + j_1) + \dots + (i_n + j_n) = r + s$. Each $x^{(k)}$ is the form of degree $r + s$ and $a_{(i)} b_{(j)} \in R$. Hence fg is a form of degree $r + s$.

□

Proof of (b).

- (1) Given any form $f \in R[x_1, \dots, x_n]$, and write $f = gh$. It suffices to show that g is a form as well. (So does h .)
- (2) Write

$$g = g_0 + \dots + g_r, \quad h = h_0 + \dots + h_s$$

where $g_r \neq 0$ and $h_s \neq 0$. So

$$f = gh = g_0h_0 + \cdots + g_rh_s.$$

Since R is a domain, $R[x_1, \dots, x_n]$ is a domain and thus $g_rh_s \neq 0$. The maximality of r and s implies that $\deg f = r + s$. Therefore, by the maximality of $r + s$, $f = g_rh_s$, or $g = g_r$, or g is a form.

□

Problem 1.5.*

Let k be any field. Show that there are an infinitely number of irreducible monic polynomials in $k[x]$. (Hint: Suppose f_1, \dots, f_n were all of them, and factor $f_1 \cdots f_n + 1$ into irreducible factors.)

Proof (Due to Euclid).

- (1) If f_1, \dots, f_n were all irreducible monic polynomials, then we consider

$$g = f_1 \cdots f_n + 1 \in k[x].$$

So there is an irreducible monic polynomial $f = f_i$ dividing g for some i since

$$\deg g = \deg f_1 + \cdots + \deg f_n \geq 1.$$

- (2) However, f would divide the difference

$$g - f_1 \cdots f_{i-1}f_{i+1} \cdots f_n = 1,$$

contrary to $\deg f_i \geq 1$.

□

Problem 1.6.*

Show that any algebraically closed field is infinite. (Hint: The irreducible monic polynomials are $x - a$, $a \in k$.)

Proof (Due to Euclid).

- (1) Let k be an algebraically closed field. If a_1, \dots, a_n were all elements in k , then we consider a monic polynomials

$$f(x) = (x - a_1) \cdots (x - a_n) + 1 \in k[x].$$

- (2) Since k is algebraically closed, there is an element $a \in k$ such that $f(a) = 0$. By assumption, $a = a_i$ for some $1 \leq i \leq n$, and thus $f(a) = f(a_i) = 1$, contrary to the fact that a field is a commutative ring where $0 \neq 1$ and all nonzero elements are invertible.

□

1.2. Affine Space and Algebraic Sets

Problem 1.8.*

Show that the algebraic subsets of $\mathbf{A}^1(k)$ are just the finite subsets, together with $\mathbf{A}^1(k)$ itself.

Proof.

- (1) Show that $k[x]$ is a PID if k is a field.
- (a) Let I be an ideal of $k[x]$.
 - (b) If $I = \{0\}$ then $I = (0)$ and I is principal.
 - (c) If $I \neq \{0\}$, then take f to be a polynomial of minimal degree in I . It suffices to show that $I = (f)$. Clearly, $(f) \subseteq I$ since I is an ideal. Conversely, for any $g \in I$,

$$g(x) = f(x)h(x) + r(x)$$

for some $h, r \in k[x]$ with $r = 0$ or $\deg r < \deg f$. Now as

$$r = g - fh \in I,$$

$r = 0$ (otherwise contrary to the minimality of f), we have $g = fh \in (f)$ for all $g \in I$.

- (2) Let Y be an algebraic subset of $\mathbf{A}^1(k)$, say $Y = V(I)$ for some ideal I of $k[x]$. Since $k[x]$ is a PID, $I = (f)$ for some $f \in k[x]$.
- (a) If $f = 0$, then $I = (0)$ and $Y = V(0) = \mathbf{A}^1(k)$.
 - (b) If $f \neq 0$, then $f(x) = 0$ has finitely many roots in k , say $a_1, \dots, a_m \in k$. Hence,

$$Y = V(I) = V(f) = \{f(a) = 0 : a \in k\} = \{a_1, \dots, a_m\}$$

is a finite subsets of $\mathbf{A}^1(k)$.

By (a)(b), the result is established.

□

Notes.

- (1) By the Hilbert basis theorem, $k[x]$ is Noetherian as k is Noetherian. Hence, for any algebraic subset $Y = V(I)$ of $\mathbf{A}^1(k)$, we can write $I = (f_1, \dots, f_m)$. Note that

$$Y = V(I) = V(f_1) \cap \dots \cap V(f_m).$$

Now apply the same argument to get the same conclusion.

- (2) Suppose $k = \bar{k}$. $\mathbf{A}^1(k)$ is irreducible, because its only proper closed subsets are finite, yet it is infinite (because k is algebraically closed, hence infinite).

Problem 1.9.

If k is a finite field, show that every subset of $\mathbf{A}^n(k)$ is algebraic.

Proof.

- (1) Every subset of $\mathbf{A}^n(k)$ is finite since $|\mathbf{A}^n(k)| = |k|^n$ is finite.
- (2) Note that $V(x_1 - a_1, \dots, x_n - a_n) = \{(a_1, \dots, a_n)\} \subseteq \mathbf{A}^n(k)$ (property (5) in this section) and any finite union of algebraic sets is algebraic (property (4) in this section). Thus, every subset of $\mathbf{A}^n(k)$ is algebraic (by (1)).

□

Problem 1.11.

Show that the following are algebraic sets:

- (a) $\{(t, t^2, t^3) \in \mathbf{A}^3(k) : t \in k\}$;
- (b) $\{(\cos(t), \sin(t)) \in \mathbf{A}^2(\mathbb{R}) : t \in \mathbb{R}\}$;
- (c) *the set of points in $\mathbf{A}^2(\mathbb{R})$ whose polar coordinates (r, θ) satisfy the equation $r = \sin(\theta)$.*

Proof of (a).

- (1) The twisted cubic curve

$$Y = \{(t, t^2, t^3) \in \mathbf{A}^3(k) : t \in k\} = V(x^2 - y) \cap V(x^3 - z)$$

is algebraic. We say that Y is given by the parametric representation $x = t, y = t^2, z = t^3$.

- (2) The generators for the ideal $I(Y)$ are $x^2 - y$ and $x^3 - z$.
- (3) Y is an affine variety of dimension 1.
- (4) The affine coordinate ring $A(Y)$ is isomorphic to a polynomial ring in one variable over k .

□

Proof of (b). The circle

$$\{(\cos(t), \sin(t)) \in \mathbf{A}^2(\mathbb{R}) : t \in \mathbb{R}\} = V(x^2 - y^2 - 1)$$

is algebraic. □

Proof of (c). The circle

$$\{(r, \theta) : r = \sin(\theta)\} = V(x^2 + y^2 - y)$$

is algebraic again. □

Problem 1.15.*

Let $V \subseteq \mathbf{A}^n(k)$, $W \subseteq \mathbf{A}^m(k)$ be algebraic sets. Show that

$$V \times W = \{(a_1, \dots, a_n, b_1, \dots, b_m) : (a_1, \dots, a_n) \in V, (b_1, \dots, b_m) \in W\}$$

is an algebraic set in $\mathbf{A}^{n+m}(k)$. It is called the **product** of V and W .

Proof.

- (1) Write

$$\begin{aligned} V &= V(S_V) = \{a \in \mathbf{A}^n(k) : f(a) = 0 \forall f \in S_V\} \\ W &= V(S_W) = \{b \in \mathbf{A}^m(k) : g(b) = 0 \forall g \in S_W\}, \end{aligned}$$

where $S_V \subseteq k[x_1, \dots, x_n]$ and $S_W \subseteq k[y_1, \dots, y_m]$. It suffices to show that

$$V \times W = V(S),$$

where $S \subseteq k[x_1, \dots, x_n, y_1, \dots, y_m]$ is the union of S_V and S_W .

- (2) Here we can identify S_V with the subset of $k[x_1, \dots, x_n, y_1, \dots, y_m]$ by noting that

$$k[x_1, \dots, x_n] \hookrightarrow (k[y_1, \dots, y_m])[x_1, \dots, x_n] = k[x_1, \dots, x_n, y_1, \dots, y_m].$$

Here we regard k as a subring of $k[y_1, \dots, y_m]$. Similar treatment to S_W .

- (3) By construction, $V \times W \subseteq V(S)$. Conversely, given any $(a, b) \in V(S)$, we have $h(a, b) = 0$ for all $h \in S = S_V \cup S_W$ (by (2)). By construction, $f(a) = 0$ for all $f \in S_V$ since f only involve x_1, \dots, x_n . Hence, $a \in V$. Similarly, $b \in W$. Therefore, $(a, b) \in V \times W$.

□

1.3. The Ideal of a Set of Points

Problem 1.18.*

Let I be an ideal in a ring R . If $a^n \in I$, $b^m \in I$, show that $(a + b)^{n+m} \in I$. Show that $\text{Rad}(I)$ is an ideal, in fact a radical ideal. Show that any prime ideal is radical.

Proof.

- (1) Show that $(a + b)^{n+m} \in I$ if $a^n \in I$, $b^m \in I$. By the binomial theorem,

$$(a + b)^{n+m} = \sum_{i=0}^{n+m} a^i b^{n+m-i}.$$

For each term $a^i b^{n+m-i}$, either $i \geq n$ holds or $n + m - i \geq m$ holds, and thus $a^i b^{n+m-i} \in I$ (since $a^n \in I$, $b^m \in I$ and I is an ideal). Hence, the result is established.

- (2) Show that $\text{Rad}(I)$ is an ideal.

- (a) $0 \in \text{Rad}(I)$ since $0 = 0^1 \in I$ for any ideal in R .
- (b) $(a + b)^{n+m} \in I$ if $a^n \in I$, $b^m \in I$ by (1).
- (c) $(-a)^{2n} = (a^n)^2 \in I$ if $a^n \in I$ (since I is an ideal).
- (d) $(ra)^n = r^n a^n \in I$ if $a^n \in I$ and $r \in R$ (since I is an ideal and R is commutative).

- (3) Show that $\text{Rad}(\text{Rad}(I)) = \text{Rad}(I)$. It suffices to show $\text{Rad}(\text{Rad}(I)) \subseteq \text{Rad}(I)$. Given any $a \in \text{Rad}(\text{Rad}(I))$. By definition $a^n \in \text{Rad}(I)$ for some positive integer n . Again by definition $(a^n)^m = a^{nm} \in I$ for some positive integer m . As nm is a positive integer, $a \in \text{Rad}(I)$.

- (4) Show that every prime ideal \mathfrak{p} is radical. Given any $a \in \text{Rad}(\mathfrak{p})$, that is, $a^n \in \mathfrak{p}$ for some positive integer. Write $a^n = aa^{n-1}$ if $n > 1$. By the primality of \mathfrak{p} , $a \in \mathfrak{p}$ or $a^{n-1} \in \mathfrak{p}$. If $a \in \mathfrak{p}$, we are done. If $a^{n-1} \in \mathfrak{p}$, we continue this descending argument until the power of a is equal to 1. Hence \mathfrak{p} is radical.

□

Problem PLACEHOLDER

PLACEHOLDER

Proof.

(1) PLACEHOLDER

1.4. The Hilbert Basis Theorem

1.5. Irreducible Components of an Algebraic Set

1.6. Algebraic Subsets of the Plane

1.7. Hilbert's Nullstellensatz

1.8. Modules; Finiteness Conditions

1.9. Integral Elements

1.10. Field Extensions

Chapter 2: Affine Varieties

2.1. Coordinate Rings

Problem PLACEHOLDER

PLACEHOLDER

Proof.

(1) PLACEHOLDER

2.2. Polynomial Maps

2.3. Coordinate Changes

2.4. Rational Functions and Local Rings

2.5. Discrete Valuation Rings

2.6. Forms

2.7. Direct Products of Rings

2.8. Operations with Ideals

2.9. Ideals with a Finite Number of Zeros

2.10. Quotient Modules and Exact Sequences

2.11. Free Modules

Chapter 3: Local Properties of Plane Curves

3.1. Multiple Points and Tangent Lines

Problem PLACEHOLDER

PLACEHOLDER

Proof.

(1) PLACEHOLDER

3.2. Multiplicities and Local Rings

3.3. Intersection Numbers

Chapter 4: Projective Varieties

4.1. Projective Space

Problem PLACEHOLDER

PLACEHOLDER

Proof.

- (1) PLACEHOLDER

4.2. Projective Algebraic Sets

4.3. Affine and Projective Varieties

4.4. Multiprojective Space

Chapter 5: Projective Plane Curves

5.1. Definitions

Problem PLACEHOLDER

PLACEHOLDER

Proof.

- (1) PLACEHOLDER

5.2. Linear Systems of Curves

5.3. Bézout's Theorem

5.4. Multiple Points

5.5. Max Noether's Fundamental Theorem

5.6. Applications of Noether's Theorem

Chapter 6: Varieties, Morphisms, and Rational Maps

6.1. The Zariski Topology

6.2. Varieties

6.3. Morphisms of Varieties

6.4. Products and Graphs

6.5. Algebraic Function Fields and Dimension of Varieties

6.6. Rational Maps

Chapter 7: Resolution of Singularities

7.1. Rational Maps of Curves

Problem PLACEHOLDER

PLACEHOLDER

Proof.

(1) PLACEHOLDER

7.2. Blowing up a Point in A^2

7.3. Blowing up a Point in P^2

7.4. Quadratic Transformations

7.5. Nonsingular Models of Curves

Chapter 8: Riemann-Roch Theorem

8.1. Divisors

Problem PLACEHOLDER

PLACEHOLDER

Proof.

(1) PLACEHOLDER

8.2. The Vector Spaces $L(D)$

8.3. Riemann's Theorem

8.4. Derivations and Differentials

8.5. Canonical Divisors

8.6. Riemann-Roch Theorem