

# Solutions to the book: *H. L. Royden, Real Analysis, 3rd edition*

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## Contents

<b>Chapter 1: Set Theory</b>	<b>3</b>
§1.1: Introduction . . . . .	3
Problem 1.1. . . . .	3
Problem 1.2. . . . .	3
Problem 1.3. . . . .	3
Problem 1.4. . . . .	4
Problem 1.5. . . . .	4
§1.3: Unions, Intersections, and Complements . . . . .	5
Problem 1.9. . . . .	5
Problem 1.11. . . . .	5
Problem 1.14. . . . .	6
<b>Chapter 2: The Real Number System</b>	<b>7</b>
§2.1: Axioms for the Real Numbers . . . . .	7
Problem 2.1. . . . .	7
<b>Chapter 3: Lebesgue Measure</b>	<b>8</b>
§3.1: Introduction . . . . .	8
Problem 3.1. (Monotonicity) . . . . .	8
Problem 3.2. (Countable subadditivity) . . . . .	8
Problem 3.3. . . . .	9
Problem 3.4. (Counting measure) . . . . .	9
§3.2: Outer Measure . . . . .	10
Problem 3.5. . . . .	10
Supplement 3.5. . . . .	10
Problem 3.6. . . . .	11
Problem 3.7. (Translation invariant) . . . . .	12
Problem 3.8. . . . .	12

§3.3: Measurable Sets and Lebesgue Measure . . . . .	13
Problem 3.9. . . . .	13
Problem 3.10. . . . .	13
Problem 3.11. . . . .	14
Problem 3.12. . . . .	14
Problem 3.13. . . . .	15
§3.4: A Nonmeasurable Set . . . . .	16
§3.5: Measurable Functions . . . . .	16
§3.6: Littlewood's Three Principles . . . . .	16

# Chapter 1: Set Theory

## §1.1: Introduction

### Problem 1.1.

Show that  $\{x : x \neq x\} = \emptyset$ .

*Proof.* Every element  $x$  of  $\{x : x \neq x\}$  satisfying  $x \neq x$ , contrary to  $x = x$ . That is, there are no elements in  $\{x : x \neq x\}$ , or  $\{x : x \neq x\} = \emptyset$ .  $\square$

### Problem 1.2.

Show that if  $x \in \emptyset$ , then  $x$  is a green-eyed lion.

*Proof.*  $\emptyset \subseteq \{\text{a green-eyed lion}\}$ .  $\square$

### Problem 1.3.

Show that in general the sets  $X \times (Y \times Z)$  and  $(X \times Y) \times Z$  are different but that there is a natural correspondence between each of them and  $X \times Y \times Z$ .

*Proof.*

(1)

$$\begin{aligned} X \times (Y \times Z) &= \{\langle x, \langle y, z \rangle \rangle : x \in X, y \in Y, z \in Z\} \\ (X \times Y) \times Z &= \{\langle \langle x, y \rangle, z \rangle : x \in X, y \in Y, z \in Z\} \\ X \times Y \times Z &= \{\langle x, y, z \rangle : x \in X, y \in Y, z \in Z\} \end{aligned}$$

are all not the same.

(2) There is a natural correspondence

$$X \times (Y \times Z) \longleftrightarrow (X \times Y) \times Z \longleftrightarrow X \times Y \times Z$$

defined by

$$\langle x, \langle y, z \rangle \rangle \longleftrightarrow \langle \langle x, y \rangle, z \rangle \longleftrightarrow \langle x, y, z \rangle$$

for every  $x \in X, y \in Y, z \in Z$ .

$\square$

**Problem 1.4.**

Show that the well-ordering principle implies the principle of mathematical induction. (Hint: Consider the set  $\{n \in \mathbb{N} : P(n) \text{ is false}\}$ .)

*Proof (Hint).* Suppose that

- (1)  $P(n)$  be a proposition defined for each  $n \in \mathbb{N}$ ,
- (2)  $P(1)$  is true,
- (3)  $[P(n) \Rightarrow P(n+1)]$  is true.

Consider the set

$$S = \{n \in \mathbb{N} : P(n) \text{ is false}\} \subseteq \mathbb{N}.$$

Want to show  $S$  is empty, or the principle of mathematical induction holds. If  $S$  were nonempty, by the well-ordering principle  $S$  has a smallest element  $m$ .  $m$  cannot be 1 by (2). Say  $m > 1$ . Therefore,  $m-1 \in \mathbb{N}$  and  $P(m-1)$  is true by the minimality of  $m$ . By (3),  $P((m-1)+1) = P(m)$  is true, which is absurd.  $\square$

**Problem 1.5.**

Use mathematical induction to establish that the well-ordering principle. (Hint: Given a set  $S$  of positive integers, let  $P(n)$  be the proposition ‘If  $n \in S$ , then  $S$  has a least element’.)

*Proof (Modified hint).*

- (1) Given a set  $S$  of positive integers, let  $P(n)$  be the proposition ‘If  $m \in S$  for some  $m \leq n$ , then  $S$  has a least element’. Want to show  $P(n)$  is true for all  $n \in \mathbb{N}$ .
  - (a)  $P(1)$  is true. For  $m \in S$  with  $m \leq n = 1$ , or  $m = 1$  by the minimality of  $1 \in \mathbb{N}$ ,  $S$  has a least element 1 ( $m$  itself) in  $\mathbb{N}$ .
  - (b) Suppose  $P(n)$  is true. If  $n+1 \in S$ , then there are only two possible cases.
    - (i) There is a positive integer  $m \in S$  less than  $n+1$ . So  $n \geq m \in S$ . Since  $P(n)$  is true,  $S$  has a least element.
    - (ii) There is no positive integer  $m \in S$  less than  $n+1$ . In this case  $n+1$  is the least element in  $S$ .

In any cases (i)(ii),  $S$  has a least element, or  $P(n+1)$  is true.

By mathematical induction,  $P(n)$  is true for all  $n \in \mathbb{N}$ .

- (2) *Show that the well-ordering principle holds.* Let  $T$  be a nonempty subset of  $\mathbb{N}$ , so there exists a positive integer  $k \in T$ . Notice that  $P(k)$  is true by (1), thus  $T$  has a least element since  $k \leq k$ .

□

### §1.3: Unions, Intersections, and Complements

#### Problem 1.9.

Show that  $A \subseteq B \Leftrightarrow A \cap B = A \Leftrightarrow A \cup B = B$ .

*Proof.*

$$(1) A \subseteq B \Leftrightarrow A \cap B = A.$$

(a) ( $\Rightarrow$ ) It suffices to show  $A \cap B \supseteq A$ . For any  $x \in A$ ,  $x \in B$  by  $A \subseteq B$ , so  $x \in A \cap B$ , so  $A \cap B \supseteq A$ .

(b) ( $\Leftarrow$ )  $A = A \cap B \subseteq B$ .

$$(2) A \subseteq B \Leftrightarrow A \cup B = B.$$

(a) ( $\Rightarrow$ ) It suffices to show  $A \cup B \subseteq B$ . For any  $x \in A \cup B$ ,  $x \in A$  or  $x \in B$ . By  $A \subseteq B$ ,  $x \in B$  or  $x \in B$ .  $x \in B$ , so  $A \cup B \subseteq B$ .

(b) ( $\Leftarrow$ )  $A \subseteq A \cup B = B$ .

□

#### Problem 1.11.

Show that  $A \subseteq B \Leftrightarrow \tilde{B} \subseteq \tilde{A}$ .

*Proof.*

$$\begin{aligned} A \subseteq B &\Leftrightarrow x \in A \Rightarrow x \in B \\ &\Leftrightarrow x \notin B \Rightarrow x \notin A \\ &\Leftrightarrow \tilde{B} \subseteq \tilde{A}. \end{aligned}$$

□

**Problem 1.14.**

*Show that*

$$B \cap \left[ \bigcup_{A \in \mathcal{C}} A \right] = \bigcup_{A \in \mathcal{C}} (B \cap A).$$

*Proof.*

$$\begin{aligned} x \in B \cap \left[ \bigcup_{A \in \mathcal{C}} A \right] &\iff x \in B \text{ and } x \in \bigcup_{A \in \mathcal{C}} A \\ &\iff x \in B \text{ and } x \in A \text{ for some } A \in \mathcal{C} \\ &\iff x \in B \cap A \text{ for some } A \in \mathcal{C} \\ &\iff x \in \bigcup_{A \in \mathcal{C}} (B \cap A). \end{aligned}$$

□

## Chapter 2: The Real Number System

### §2.1: Axioms for the Real Numbers

#### Problem 2.1.

Show that  $1 \in P$ .

*Proof.* By the field axioms,

- (a)  $1 \in \mathbb{R}$  such that  $1 \neq 0$ .
- (b)  $-1 \in \mathbb{R}$ .
- (c)  $(-1) \cdot (-1) = 1$ .

By the axioms of order,  $1 = 0$  or  $1 \in P$  or  $-1 \in P$ . Consider three possible cases,

- (1)  $1 = 0$ , contrary to the field axioms  $1 \neq 0$ .
- (2)  $1 \in P$ .
- (3)  $-1 \in P$ . By the axioms of order,  $(-1)(-1) \in P$ . Since  $(-1)(-1) = 1$  by the field axioms,  $1 \in P$ . By the axioms of order,  $-1 \notin P$ , contrary to  $-1 \in P$ .

By (1)(2)(3),  $1 \in P$ .  $\square$

Applying the similar argument to  $\sqrt{-1}$ , we get  $\sqrt{-1} \notin \mathbb{R}$  as our expectation.

## Chapter 3: Lebesgue Measure

### §3.1: Introduction

#### Problem 3.1. (Monotonicity)

If  $A$  and  $B$  are two sets in  $\mathfrak{M}$  with  $A \subseteq B$ , then  $mA \leq mB$ . This property is called monotonicity.

*Proof.* Write

$$B = B \cap X = B \cap (A \cup \tilde{A}) = (B \cap A) \cup (B \cap \tilde{A}) = A \cup (B - A).$$

Here  $B \cap A = A$  comes from  $A \subseteq B$  (Problem 1.9). Notice that  $A$  and  $B - A$  are disjoint. Since  $m$  is a countably additive measure ( $m$  is nonnegative) on a  $\sigma$ -algebra  $\mathfrak{M}$ ,

$$mB = mA + m(B - A) \geq mA.$$

□

#### Problem 3.2. (Countable subadditivity)

Let  $\langle E_n \rangle$  be any sequence of sets in  $\mathfrak{M}$ . Then  $m(\bigcup E_n) \leq \sum mE_n$ . (Hint: Use Proposition 1.2) This property of a measure is called countable subadditivity.

As the argument in Problem 3.1.

*Proof.* Since  $\langle E_n \rangle$  is a sequence of sets in  $\sigma$ -algebra  $\mathfrak{M}$ , by Proposition 1.2 and its proof, there is a sequence  $\langle F_n \rangle$  of sets in  $\sigma$ -algebra  $\mathfrak{M}$  such that all  $F_n$  are pairwise disjoint,  $F_n \subseteq E_n$ , and

$$\bigcup E_n = \bigcup F_n.$$

Since  $m$  is a countably additive measure on a  $\sigma$ -algebra  $\mathfrak{M}$ ,

$$m\left(\bigcup E_n\right) = m\left(\bigcup F_n\right) = \sum mF_n \geq \sum mE_n.$$

The last inequality holds by applying Problem 3.1 on  $F_n \subseteq E_n$  for any  $n$ . □



**Problem 3.3.**

If there is a set  $A$  in  $\mathfrak{M}$  such that  $mA < \infty$ , then  $m\emptyset = 0$ .

*Proof.* For such  $A$ , write  $A = A \cup \emptyset$ .  $A$  and  $\emptyset$  are disjoint. Since  $m$  is a countably additive measure on a  $\sigma$ -algebra  $\mathfrak{M}$ ,

$$mA = mA + m\emptyset.$$

Since  $mA < \infty$ , we can cancel out  $mA$  on the both sides to get  $m\emptyset = 0$ .  $\square$

**Problem 3.4. (Counting measure)**

Let  $nE$  be  $\infty$  for an infinite set  $E$  and be equal to the number of elements of  $E$  for a finite set. Show that  $n$  is a countably additive set function that is translation invariant and defined for all sets of real numbers. This measure is called the **counting measure**.

*Proof.*

- (1) Show that  $n$  is a countably additive set function. Note that  $n$  is defined on any subset of real numbers since the finiteness is defined on any subset of real numbers. Suppose  $\langle E_m \rangle$  is a sequence of disjoint sets of real numbers. We need to show that  $n(\bigcup E_m) = \sum nE_m$ .
- (2) If  $E_m$  is infinite for some  $m = k$ , then  $\bigcup E_m$  is also infinite. Hence,  $n(\bigcup E_m) = \infty$ , and  $\sum nE_m \geq nE_k = \infty \implies \sum nE_m = \infty$ .
- (3) Suppose all  $E_n$  are finite. Note that  $\bigcup E_m$  is infinite if and only if all but finitely many  $E_m \neq \emptyset$  if and only if  $\sum nE_m = \infty$ . Besides, if  $\bigcup E_m$  is finite, then all but finitely many  $E_m = \emptyset$  and thus

$$n\left(\bigcup_m E_m\right) = \sum_{E_m \neq \emptyset} nE_m = \sum_m nE_m < \infty.$$

- (4) Since

$$\begin{aligned} n(E + y) &= n(\{x + y : x \in E\}) \\ &= \text{the number of elements } x \in E \\ &= n(E), \end{aligned}$$

$n$  is translation invariant.

$\square$

### §3.2: Outer Measure

#### Problem 3.5.

Let  $A$  be the set of rational numbers between 0 and 1, and let  $\{I_n\}$  be a finite collection of open intervals covering  $A$ . Then  $\sum \ell(I_n) \geq 1$ .

*Idea.* If  $\{I_n\}$  is a covering of  $[0, 1]$  then we are done since the length of  $[0, 1]$  is 1. However,  $\{I_n\}$  only covers  $A$  and not necessarily covers  $[0, 1]$ . (For example,  $\{I_n\} = \left\{ \left(-89, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, 64\right) \right\}$  covers  $A$  but not  $\frac{1}{\sqrt{2}}$ .) Hence, it is natural to consider the closure of  $A$  and the closure of  $I_n$ . Now  $\{\overline{I_n}\}$  is a (closed) covering of  $\overline{A} = [0, 1]$ .

*Proof.*

$$\begin{aligned}
 1 &= m^*[0, 1] && \text{(Proposition 3.1)} \\
 &= m^*\overline{A} && (A \text{ is dense in } [0, 1]) \\
 &\leq m^*\left(\overline{\bigcup I_n}\right) && \text{(Proposition 2.10)} \\
 &= m^*\left(\bigcup \overline{I_n}\right) && \text{(Proposition 2.10)} \\
 &\leq \sum m^*(\overline{I_n}) && \text{(Proposition 3.2)} \\
 &= \sum \ell(\overline{I_n}) && \text{(Proposition 3.1)} \\
 &= \sum \ell(I_n). && \text{(Definition of length)}
 \end{aligned}$$

□

#### Supplement 3.5.

Exercise about considering the closure. (Exercise 4.52 in the textbook: *T. M. Apostol, Mathematical Analysis, 2nd edition.*) Assume that  $f$  is uniformly continuous on a bounded set  $S$  in  $\mathbb{R}^n$ . Prove that  $f$  must be bounded on  $S$ .

*Proof.*

- (1) Since  $f : S \rightarrow T$  is uniformly continuous, given any  $\varepsilon > 0$ , there is  $\delta > 0$  such that  $d_T(f(x), f(y)) < \varepsilon$  whenever  $d_S(x, y) < \delta$ . Choose  $\varepsilon = 1 > 0$ .
- (2) For such  $\delta > 0$ , construct an open covering of  $\overline{S} \subseteq \mathbb{R}^n$ . Pick a collection  $\mathcal{F}$  of open balls  $B(a; \delta) \subseteq \mathbb{R}^n$  where  $a$  runs over all elements of  $S$ .  $\mathcal{F}$  covers  $\overline{S}$  (by the definition of accumulation points). Since  $\overline{S}$  is closed and bounded (since  $S$  is bounded),  $\overline{S}$  is compact. So there is a finite subcollection  $\mathcal{F}'$  of

$\mathcal{F}$  also covers  $\overline{S}$ , say

$$\mathcal{F}' = \{B(a_1; \delta), B(a_2; \delta), \dots, B(a_m; \delta)\}.$$

- (3) Given any  $x \in S \subseteq \overline{S}$ , there is some  $a_i \in S$  ( $1 \leq i \leq m$ ) such that  $x \in B(a_i; \delta)$ . In such ball,  $d_S(x, a_i) < \delta$ . By (1),  $\|f(x) - f(a_i)\| < 1$ , or  $\|f(x)\| < 1 + \|f(a_i)\|$ . Therefore, for any  $x \in S$ ,

$$\|f(x)\| < 1 + \max_{1 \leq i \leq m} \|f(a_i)\|.$$

□

### Problem 3.6.

*Prove Proposition 5: Given any set  $A$  and any  $\varepsilon > 0$ , there is an open set  $O$  such that  $A \subseteq O$  and  $m^*O \leq m^*A + \varepsilon$ . There is a  $G \in G_\delta$  such that  $m^*G = m^*A$ .*

*Proof.*

- (1) *Use the definition of the outer measure.* By the definition of  $m^*$ , for such  $\varepsilon > 0$  there exists a countable collection  $\{I_n\}$  of open intervals that covers  $A$  and

$$m^*A + \varepsilon \geq \sum \ell(I_n).$$

- (2) *Construct an open set  $O$ .* Let  $O = \bigcup I_n \supseteq A$  which is the union of any collection of open sets  $I_n$ . By Proposition 2.7,  $O$  is open.

- (3) *Show that  $m^*O \leq m^*A + \varepsilon$ .* By Proposition 3.2 and 3.1,

$$m^*O = m^*\left(\bigcup I_n\right) \leq \sum m^*I_n = \sum \ell(I_n) \leq m^*A + \varepsilon.$$

Therefore, given any set  $A$  and any  $\varepsilon > 0$ , there is an open set  $O$  such that  $A \subseteq O$  and  $m^*O \leq m^*A + \varepsilon$ .

- (4) *Construct  $G \in G_\delta$  in a natural way.* Given any  $n \in \mathbb{N}$ , there exists an open set  $O_n$  such that  $O_n \supseteq A$  and  $m^*O_n \leq m^*A + \frac{1}{n}$ . Let

$$G = \bigcap_{n=1}^{\infty} O_n \in G_\delta.$$

- (5) *Show that  $m^*G = m^*A$ .*

(a) Since  $A \subseteq O_n$  for any  $n \in \mathbb{N}$ ,  $A \subseteq \bigcap_{n=1}^{\infty} O_n = G$ . Thus  $m^*A \leq m^*G$ .

(b) Since  $O_n \supseteq \bigcap_{n=1}^{\infty} O_n = G$  for any  $n \in \mathbb{N}$ ,

$$m^*A + \frac{1}{n} \geq m^*O_n \geq m^*G$$

for any  $n \in \mathbb{N}$ . Since  $n \in \mathbb{N}$  is arbitrary,  $m^*A \geq m^*G$ .

By (a)(b),  $m^*A = m^*G$ .

□

**Problem 3.7. (Translation invariant)**

*Prove that  $m^*$  is translation invariant.*

*Proof.* Given  $E \in \mathfrak{M}$  and  $y \in \mathbb{R}$ .

- (1)  $m^*(E+y) \leq m^*E$ . Let  $\{I_n\}$  of open intervals that cover  $E$ . Then  $\{I_n+y\}$  of open intervals that cover  $E+y$ . Notice that the definition of  $m^*$  and  $\ell(I_n+y) = \ell(I_n)$ , then

$$m^*(E+y) \leq \sum \ell(I_n+y) = \sum \ell(I_n).$$

Take the infimum of all such sum  $\sum \ell(I_n)$ ,  $m^*(E+y) \leq m^*E$ .

- (2)  $m^*(E) \leq m^*(E+y)$ . Similar to (1).

By (1)(2),  $m^*(E+y) = m^*E$ , that is,  $m^*$  is translation invariant. □

**Problem 3.8.**

*Prove that if  $m^*A = 0$ , then  $m^*(A \cup B) = m^*B$ .*

*Proof.*

- (1)  $m^*(A \cup B) \geq m^*B$  since  $A \cup B \supseteq B$  and the definition of  $m^*$ . (Any covering of  $A \cup B$  by open intervals is also a covering of  $B$  so that the latter infimum is taken over a larger collection than the former.)
- (2)  $m^*(A \cup B) \leq m^*B$ . By Proposition 3.2,

$$m^*(A \cup B) \leq m^*A + m^*B = 0 + m^*B = m^*B.$$

By (1)(2),  $m^*(A \cup B) = m^*B$ . □

### §3.3: Measurable Sets and Lebesgue Measure

**Problem 3.9.**

Show that if  $E$  is a measurable set, then each translate  $E + y$  of  $E$  is also measurable.

*Proof.*

- (1)  $E$  is measurable if and only if for each set  $A$ , each  $y \in \mathbb{R}$ ,

$$m^*(A + y) = m^*((A + y) \cap E) + m^*((A + y) \cap \widetilde{E}).$$

- (a)  $(\implies)$   $E$  is measurable and  $A + y$  is a set (for any set  $A$  and  $y \in \mathbb{R}$ ).

- (b)  $(\impliedby)$   $A = (A - y) + y$  for any set  $A$  and  $y \in \mathbb{R}$ .

- (2) For any set  $E$  and  $y \in \mathbb{R}$ ,  $\widetilde{E + y} = \widetilde{E} + y$  by the definition of translation.

- (3) For any sets  $E_1, E_2$  and  $y \in \mathbb{R}$ ,  $(E_1 \cap E_2) + y = (E_1 + y) \cap (E_2 + y)$  by the definition of translation.

- (4) For each set  $A$  and  $y \in \mathbb{R}$ ,

$$\begin{aligned} & m^*((A + y) \cap (E + y)) + m^*((A + y) \cap \widetilde{(E + y)}) \\ &= m^*((A + y) \cap (E + y)) + m^*((A + y) \cap (\widetilde{E} + y)) && ((2)) \\ &= m^*((A \cap E) + y) + m^*((A \cap \widetilde{E}) + y) && ((3)) \\ &= m^*(A \cap E) + m^*(A \cap \widetilde{E}) && (\text{Problem 3.7}) \\ &= m^*A && (\text{Measurability of } E) \\ &= m^*(A + y). && (\text{Problem 3.7}) \end{aligned}$$

By (1),  $E + y$  is measurable.

□

**Problem 3.10.**

Show that if  $E_1$  and  $E_2$  are measurable, then

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = mE_1 + mE_2.$$

*Proof.* Since the collection  $\mathfrak{M}$  of measurable sets is a  $\sigma$ -algebra (Theorem 3.10) and  $m$  is countable additive (Proposition 3.13),

$$\begin{aligned} m(E_1 \cup E_2) + m(E_1 \cap E_2) &= \left( m(E_1) + m(E_2 \cap \widetilde{E_1}) \right) + m(E_2 \cap E_1) \\ &= m(E_1) + \left( m(E_2 \cap \widetilde{E_1}) + m(E_2 \cap E_1) \right) \\ &= m(E_1) + m(E_2). \end{aligned}$$

( $E_1$  and  $E_2 \cap \widetilde{E_1}$  are disjoint.  $E_2 \cap \widetilde{E_1}$  and  $E_2 \cap E_1$  are disjoint too.)  $\square$

**Problem 3.11.**

Show that the condition  $mE_1 < \infty$  is necessary in Proposition 3.14 by giving a decreasing sequence  $\langle E_n \rangle$  of measurable sets with  $\emptyset = \bigcap E_n$  and  $mE_n = \infty$  for each  $n$ .

*Proof.* Set

$$E_n = (n, \infty)$$

for each  $n \in \mathbb{N}$ .

- (1)  $\langle E_n \rangle$  is a decreasing sequence of measurable sets.  $E_n \supseteq E_{n+1}$  by definition. Besides, each  $E_n$  is measurable by Lemma 3.11.
- (2)  $\bigcap E_n = \emptyset$ . For each  $x \in \mathbb{R}$ ,  $x \notin E_1$  if  $x \leq 1$ ;  $x \notin E_{[x]}$  if  $x \geq 1$  where  $x \mapsto [x]$  is the floor function.
- (3)  $mE_n = \infty$  for each  $n$ . The length of each  $E_n$  is  $\infty$  (Proposition 3.1).

$\square$

**Problem 3.12.**

Let  $\langle E_n \rangle$  be a sequence of disjoint measurable sets and  $A$  any set. Then  $m^*(A \cap \bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m^*(A \cap E_i)$ .

*Proof.*

- (1)  $A \cap \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} (A \cap E_i)$  (Problem 1.14).
- (2)  $m^*(\bigcup_{i=1}^{\infty} (A \cap E_i)) \leq \sum_{i=1}^{\infty} m^*(A \cap E_i)$  by the subadditivity of  $m^*$  (Proposition 3.2).

(3) By Lemma 3.9,

$$m^* \left( \bigcup_{i=1}^n (A \cap E_i) \right) = \sum_{i=1}^n m^*(A \cap E_i)$$

for any  $n \in \mathbb{N}$ . Since  $\bigcup_{i=1}^\infty (A \cap E_i) \supseteq \bigcup_{i=1}^n (A \cap E_i)$ ,  $m^* (\bigcup_{i=1}^\infty (A \cap E_i)) \geq m^* (\bigcup_{i=1}^n (A \cap E_i))$  by the monotonicity of  $m^*$ . Thus,

$$m^* \left( \bigcup_{i=1}^\infty (A \cap E_i) \right) \geq \sum_{i=1}^n m^*(A \cap E_i)$$

for any  $n \in \mathbb{N}$ . Since  $\sum_{i=1}^n m^*(A \cap E_i)$  is bounded and increasing (by the non-negativity of  $m^*$ ),

$$m^* \left( \bigcup_{i=1}^\infty (A \cap E_i) \right) \geq \sum_{i=1}^\infty m^*(A \cap E_i).$$

By (2)(3),  $m^* (A \cap \bigcup_{i=1}^\infty E_i) = \sum_{i=1}^\infty m^*(A \cap E_i)$ .  $\square$

### Problem 3.13.

*Prove Proposition 15: Let  $E$  be a given set. The the following five statements are equivalent:*

- (i)  $E$  is measurable.
- (ii) Given  $\varepsilon > 0$ , there is an open set  $O \supseteq E$  with  $m^*(O - E) < \varepsilon$ .
- (iii) Given  $\varepsilon > 0$ , there is an open set  $F \subseteq E$  with  $m^*(E - F) < \varepsilon$ .
- (iv) There is an  $G$  in  $G_\delta$  with  $G \supseteq E$ ,  $m^*(G - E) = 0$ .
- (v) There is an  $F$  in  $F_\sigma$  with  $F \subseteq E$ ,  $m^*(E - F) = 0$ .

*If  $m^*E$  is finite, the above statements are equivalent to:*

- (vi) Given  $\varepsilon > 0$ , there is a finite union  $U$  of open intervals such that

$$m^*(U \Delta E) < \varepsilon.$$

*(Hints:*

- (a) Show that for  $m^*E < \infty$ , (i)  $\Rightarrow$  (ii)  $\Leftrightarrow$  (vi) (cf. Proposition 5).
- (b) Use (a) to show that for arbitrary sets  $E$ , (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (vi)  $\Rightarrow$  (i).
- (c) Use (b) to show that (i)  $\Rightarrow$  (iii).

*Proof.*

- (1) *Show that for  $m^*E < \infty$ ,  $(i) \Rightarrow (ii)$ .* Given  $\varepsilon > 0$ , there is a countable collections  $\{I_n\}$  of open intervals that cover  $E$  such that

$$\sum \ell(I_n) < m^*E + \varepsilon = mE + \varepsilon$$

by the definition of the outer measure and measurable sets. Take  $O = \bigcup I_n$  be an open set which contains  $E$ . Hence,

$$\begin{aligned} m^*(O - E) &= m(O - E) && (O, E: \text{measurable}) \\ &= mO - mE \\ &= m\left(\bigcup I_n\right) - mE \\ &\leq \sum \ell(I_n) - mE \\ &< \varepsilon. \end{aligned}$$

(2)

□

### §3.4: A Nonmeasurable Set

### §3.5: Measurable Functions

### §3.6: Littlewood's Three Principles