

Chapter 9: Functions of Several Variables

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Exercise 9.1. If S is a nonempty subset of a vector space X , prove (as asserted in Section 9.1) that the span of S is a vector space.

Denote the span of S by $\text{span}(S)$.

Proof.

- (1) Since $S \neq \emptyset$, there is $\mathbf{z} \in S$. So $1\mathbf{z} = \mathbf{z} \in \text{span}(S) \neq \emptyset$. (In fact, $\text{span}(S) \supseteq S$.)
- (2) If $\mathbf{x}, \mathbf{y} \in \text{span}(S)$, then there exist elements $\mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{y}_1, \dots, \mathbf{y}_n \in S$ and scalars $a_1, \dots, a_m, b_1, \dots, b_n$ such that

$$\begin{aligned}\mathbf{x} &= a_1\mathbf{x}_1 + \cdots + a_m\mathbf{x}_m, \\ \mathbf{y} &= b_1\mathbf{y}_1 + \cdots + b_n\mathbf{y}_n.\end{aligned}$$

Then

$$\mathbf{x} + \mathbf{y} = a_1\mathbf{x}_1 + \cdots + a_m\mathbf{x}_m + b_1\mathbf{y}_1 + \cdots + b_n\mathbf{y}_n$$

is a linear combination of the elements of S . For any scalar c ,

$$c\mathbf{x} = (ca_1)\mathbf{x}_1 + \cdots + (ca_m)\mathbf{x}_m$$

is again linear combination of the elements of S .

- (3) By (1)(2), $\text{span}(S)$ is a vector space.

□

Note. Any subspace of X that contains S must also contain $\text{span}(S)$.

Exercise 9.2. Prove (as asserted in Section 9.6) that BA is linear if A and B are linear transformations. Prove also that A^{-1} is linear and invertible if A is invertible.

Proof. Use the notation in Definitions 9.6.

- (1) Show that BA is linear if A and B are linear transformations. Let X, Y, Z be vector spaces, $A \in L(X, Y)$ and $B \in L(Y, Z)$.

(a) Given any $\mathbf{x}_1, \mathbf{x}_2 \in X$.

$$\begin{aligned}
 (BA)(\mathbf{x}_1 + \mathbf{x}_2) &= B(A(\mathbf{x}_1 + \mathbf{x}_2)) \\
 &= B(A\mathbf{x}_1 + A\mathbf{x}_2) && (A \text{ is a linear transformation}) \\
 &= B(A\mathbf{x}_1) + B(A\mathbf{x}_2) && (B \text{ is a linear transformation}) \\
 &= (BA)\mathbf{x}_1 + (BA)\mathbf{x}_2.
 \end{aligned}$$

(b) For any $\mathbf{x} \in X$ and scalar c ,

$$\begin{aligned}
 (BA)(c\mathbf{x}) &= B(A(c\mathbf{x})) \\
 &= B(cA\mathbf{x}) && (A \text{ is a linear transformation}) \\
 &= cB(A\mathbf{x}) && (B \text{ is a linear transformation}) \\
 &= c(BA)\mathbf{x}.
 \end{aligned}$$

By (a)(b), $BA \in L(X, Z)$.

(2) Show that A^{-1} is linear if A is invertible.

(a) Given any $\mathbf{y}_1, \mathbf{y}_2 \in X$. Since A is surjective, there exist $\mathbf{x}_1, \mathbf{x}_2 \in X$ such that

$$\begin{aligned}
 \mathbf{y}_1 &= A\mathbf{x}_1 \\
 \mathbf{y}_2 &= A\mathbf{x}_2.
 \end{aligned}$$

So

$$\begin{aligned}
 A^{-1}\mathbf{y}_1 &= A^{-1}(A\mathbf{x}_1) = \mathbf{x}_1 \\
 A^{-1}\mathbf{y}_2 &= A^{-1}(A\mathbf{x}_2) = \mathbf{x}_2
 \end{aligned}$$

(by Definitions 9.4). Hence

$$\begin{aligned}
 A^{-1}(\mathbf{y}_1 + \mathbf{y}_2) &= A^{-1}(A\mathbf{x}_1 + A\mathbf{x}_2) \\
 &= A^{-1}(A(\mathbf{x}_1 + \mathbf{x}_2)) && (A \text{ is a linear transformation}) \\
 &= \mathbf{x}_1 + \mathbf{x}_2 && (\text{Definitions 9.4}) \\
 &= A^{-1}\mathbf{y}_1 + A^{-1}\mathbf{y}_2.
 \end{aligned}$$

(b) For any $\mathbf{y} \in X$ and scalar c , there is a corresponding $\mathbf{x} \in X$ such that $\mathbf{y} = A\mathbf{x}$ since A is surjective. So $A^{-1}\mathbf{y} = \mathbf{x}$ by Definition 9.4. Hence

$$\begin{aligned}
 A^{-1}(c\mathbf{y}) &= A^{-1}(cA\mathbf{x}) \\
 &= A^{-1}(A(c\mathbf{x})) && (A \text{ is a linear transformation}) \\
 &= c\mathbf{x} && (\text{Definitions 9.4}) \\
 &= cA^{-1}\mathbf{y}.
 \end{aligned}$$

By (a)(b), $A^{-1} \in L(X)$.

(3) *Show that A^{-1} is invertible if A is invertible.* It suffices to show that A^{-1} is injective and surjective.

(a) *Show that A^{-1} is injective.* Given any $\mathbf{y}_1, \mathbf{y}_2 \in X$. Since A is surjective, there exist $\mathbf{x}_1, \mathbf{x}_2 \in X$ such that

$$\begin{aligned}\mathbf{y}_1 &= A\mathbf{x}_1 \\ \mathbf{y}_2 &= A\mathbf{x}_2.\end{aligned}$$

Suppose $A^{-1}\mathbf{y}_1 = A^{-1}\mathbf{y}_2$. So $A^{-1}(A\mathbf{x}_1) = A^{-1}(A\mathbf{x}_2)$, or $\mathbf{x}_1 = \mathbf{x}_2$, or $\mathbf{y}_1 = A\mathbf{x}_1 = A\mathbf{x}_2 = \mathbf{y}_2$.

(b) *Show that A^{-1} is surjective.* For any $\mathbf{x} \in X$, there exists $A\mathbf{x} \in X$ such that $A^{-1}(A\mathbf{x}) = \mathbf{x}$ by Definitions 9.4.

□

Exercise 9.3. Assume $A \in L(X, Y)$ and $A\mathbf{x} = \mathbf{0}$ only when $\mathbf{x} = \mathbf{0}$. Prove that A is then 1-1.

Proof. Suppose $A\mathbf{x} = A\mathbf{y}$. Since A is a linear transformation, $A(\mathbf{x} - \mathbf{y}) = A\mathbf{x} - A\mathbf{y} = \mathbf{0}$. By assumption, $\mathbf{x} - \mathbf{y} = \mathbf{0}$ or $\mathbf{x} = \mathbf{y}$. □

Exercise 9.4. Prove (as asserted in Section 9.30) that null spaces and ranges of linear transformations are vector spaces.

Proof. Use the notation in Definitions 9.30. Suppose X, Y are vector spaces, and $A \in L(X, Y)$, as in Definition 9.6.

(1) *Show that $\mathcal{N}(A)$ is a vector space in X .*

(a) Note that $\mathbf{0} \in X$. Since $A\mathbf{0} = \mathbf{0}$, $\mathbf{0} \in \mathcal{N}(A) \neq \emptyset$.

(b) Suppose $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{N}(A)$. Then

$$\begin{aligned}A(\mathbf{x}_1 + \mathbf{x}_2) &= A\mathbf{x}_1 + A\mathbf{x}_2 && (A \text{ is a linear transformation}) \\ &= \mathbf{0} + \mathbf{0} && (\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{N}(A)) \\ &= \mathbf{0}.\end{aligned}$$

So $\mathbf{x}_1 + \mathbf{x}_2 \in \mathcal{N}(A)$.

(c) Suppose $\mathbf{x} \in \mathcal{N}(A)$ and c is a scalar. Then

$$\begin{aligned}A(c\mathbf{x}) &= cA\mathbf{x} && (A \text{ is a linear transformation}) \\ &= c\mathbf{0} && (\mathbf{x} \in \mathcal{N}(A)) \\ &= \mathbf{0}.\end{aligned}$$

So $c\mathbf{x} \in \mathcal{N}(A)$.

By (a)(b)(c), $\mathcal{N}(A)$ is a vector space.

(2) Show that $\mathcal{R}(A)$ is a vector space in Y .

(a) Note that $\mathbf{0} \in X$. So $A\mathbf{0} = \mathbf{0} \in \mathcal{R}(A) \neq \emptyset$.

(b) Suppose $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{R}(A)$. Then there exist $\mathbf{x}_1, \mathbf{x}_2 \in X$ such that $A\mathbf{x}_1 = \mathbf{y}_1$ and $A\mathbf{x}_2 = \mathbf{y}_2$. Hence

$$\begin{aligned}\mathbf{y}_1 + \mathbf{y}_2 &= A\mathbf{x}_1 + A\mathbf{x}_2 \\ &= A(\mathbf{x}_1 + \mathbf{x}_2) \quad (A \text{ is a linear transformation}).\end{aligned}$$

So $\mathbf{y}_1 + \mathbf{y}_2 \in \mathcal{R}(A)$.

(c) Suppose $\mathbf{y} \in \mathcal{R}(A)$ and c is a scalar. Then there exists $\mathbf{x} \in X$ such that $A\mathbf{x} = \mathbf{y}$. Hence

$$\begin{aligned}c\mathbf{y} &= cA\mathbf{x} \\ &= A(c\mathbf{x}) \quad (A \text{ is a linear transformation}).\end{aligned}$$

So $c\mathbf{y} \in \mathcal{R}(A)$.

By (a)(b)(c), $\mathcal{R}(A)$ is a vector space.

□

Exercise 9.5. Prove that to every $A \in L(\mathbb{R}^n, \mathbb{R}^1)$ corresponds a unique $\mathbf{y} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$. Prove also that $\|A\| = \|\mathbf{y}\|$. (Hint: Under certain conditions, equality holds in the Schwarz inequality.)

Proof.

(1) Recall that $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the standard basis of \mathbb{R}^n (Definitions 9.1).

Given any $\mathbf{x} \in \mathbb{R}^n$, write $\mathbf{x} = (x_1, \dots, x_n)$ as $\mathbf{x} = \sum x_j \mathbf{e}_j$.

(2) Show that \mathbf{y} exists. Since A is a linear transformation,

$$\begin{aligned}A\mathbf{x} &= A\left(\sum x_j \mathbf{e}_j\right) \\ &= \sum x_j A\mathbf{e}_j \\ &= (x_1, \dots, x_n) \cdot (A\mathbf{e}_1, \dots, A\mathbf{e}_n) \\ &= \mathbf{x} \cdot \sum (A\mathbf{e}_j) \mathbf{e}_j.\end{aligned}$$

Define $\mathbf{y} = \sum (A\mathbf{e}_j) \mathbf{e}_j \in \mathbb{R}^n$ so that $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$.

(3) Show that \mathbf{y} is unique. Suppose there exists some $\mathbf{z} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{x} \cdot \mathbf{z}$. So

$$\begin{aligned}0 &= A\mathbf{x} - A\mathbf{x} \\ &= \mathbf{x} \cdot \mathbf{y} - \mathbf{x} \cdot \mathbf{z} \\ &= \mathbf{x} \cdot (\mathbf{y} - \mathbf{z})\end{aligned}$$

for any $\mathbf{x} \in \mathbb{R}^n$. In particular, take $\mathbf{x} = \mathbf{y} - \mathbf{z} \in \mathbb{R}^n$ to get

$$0 = (\mathbf{y} - \mathbf{z}) \cdot (\mathbf{y} - \mathbf{z}) = |\mathbf{y} - \mathbf{z}|^2$$

or $\mathbf{y} - \mathbf{z} = \mathbf{0}$ or $\mathbf{y} = \mathbf{z}$.

(4) *Show that $\|A\| = |\mathbf{y}|$.* By the Schwarz inequality (Theorem 1.37(d)),

$$|A\mathbf{x}| = |\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}| \leq |\mathbf{y}|$$

as $|\mathbf{x}| \leq 1$. Take the sup over all $|\mathbf{x}| \leq 1$ to get

$$\|A\| \leq |\mathbf{y}|.$$

If $\mathbf{y} = \mathbf{0}$, then $\|A\| = |\mathbf{y}| = 0$. If $\mathbf{y} \neq \mathbf{0}$, then the equality holds when $\mathbf{x} = \frac{\mathbf{y}}{|\mathbf{y}|} \in \mathbb{R}^n$. (Here $|\mathbf{x}| = 1$.)

□

Exercise 9.6. *If $f(0,0) = 0$ and*

$$f(x,y) = \frac{xy}{x^2 + y^2} \quad \text{if } (x,y) \neq (0,0),$$

prove that $(D_1f)(x,y)$ and $(D_2f)(x,y)$ exist at every point of \mathbb{R}^2 , although f is not continuous at $(0,0)$.

Proof.

(1) *Show that*

$$(D_1f)(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0), \\ \frac{y(y^2 - x^2)}{(x^2 + y^2)^2} & \text{if } (x,y) \neq (0,0). \end{cases}$$

Write

$$\begin{aligned} (D_1f)(x,y) &= \lim_{t \rightarrow 0} \frac{f((x,y) + t(1,0)) - f(x,y)}{t} \\ &= \lim_{t \rightarrow 0} \frac{f(x+t,y) - f(x,y)}{t}. \end{aligned}$$

If $(x,y) = (0,0)$,

$$(D_1f)(0,0) = \lim_{t \rightarrow 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t \rightarrow 0} \frac{0 - 0}{t} = 0.$$

If $(x, y) \neq (0, 0)$,

$$\begin{aligned}
 (D_1 f)(x, y) &= \lim_{t \rightarrow 0} \frac{f(x+t, y) - f(x, y)}{t} \\
 &= \lim_{t \rightarrow 0} \frac{\frac{(x+t)y}{(x+t)^2 + y^2} - \frac{xy}{x^2 + y^2}}{t} \\
 &= \lim_{t \rightarrow 0} \frac{y(y^2 - x^2) - txy}{((x+t)^2 + y^2)(x^2 + y^2)} \\
 &= \frac{y(y^2 - x^2)}{(x^2 + y^2)^2}.
 \end{aligned}$$

(2) *Show that*

$$(D_2 f)(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0), \\ \frac{x(x^2 - y^2)}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0). \end{cases}$$

Similar to (1).

(3) *Show that f is not continuous at $(0, 0)$.* Take $(x_n, y_n) = (\frac{1}{n}, \frac{1}{n})$ for $n = 1, 2, 3, \dots$. Then

$$\lim_{n \rightarrow \infty} f(x_n, y_n) = \lim_{n \rightarrow \infty} \frac{\frac{1}{n} \cdot \frac{1}{n}}{\frac{1}{n^2} + \frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{1}{2} = \frac{1}{2} \neq 0 = f(0, 0).$$

□

Exercise 9.7. ...

Proof.

(1)

(2)

□

Exercise 9.8. ...

Proof.

(1)

(2)

□

Exercise 9.9. ...

Proof.

(1)

(2)

□

Exercise 9.10. ...

Proof.

(1)

(2)

□

Exercise 9.11. ...

Proof.

(1)

(2)

□

Exercise 9.12. ...

Proof.

(1)

(2)

□

Exercise 9.13. ...

Proof.

(1)

(2)

□

Exercise 9.14. ...

Proof.

(1)

(2)

□

Exercise 9.15. ...

Proof.

(1)

(2)

□

Exercise 9.16. ...

Proof.

(1)

(2)

□

Exercise 9.17. ...

Proof.

(1)

(2)

□

Exercise 9.18. ...

Proof.

(1)

(2)

□

Exercise 9.19. ...

Proof.

(1)

(2)

□

Exercise 9.20. ...

Proof.

(1)

(2)

□

Exercise 9.21. ...

Proof.

(1)

(2)

□

Exercise 9.22. ...

Proof.

(1)

(2)

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Exercise 9.23. ...

Proof.

(1)

(2)

□

Exercise 9.24. ...

Proof.

(1)

(2)

□

Exercise 9.25. ...

Proof.

(1)

(2)

□

Exercise 9.26. ...

Proof.

(1)

(2)

□

Exercise 9.27. ...

Proof.

(1)

(2)

□

Exercise 9.28. ...

Proof.

(1)

(2)

□

Exercise 9.29. ...

Proof.

(1)

(2)

□

Exercise 9.30. ...

Proof.

(1)

(2)

□

Exercise 9.31. ...

Proof.

(1)

(2)

□