

Notes on the book:
Ash, Probability and Measure Theory,
2nd edition

Meng-Gen Tsai
plover@gmail.com

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Chapter 1: Fundamentals of Measure and Integration Theory

Problem 1.1.1.

Establish formulas (1)-(5).

Formulas.

- (1) If $A_n \uparrow A$, then $A_n^c \downarrow A^c$; If $A_n \downarrow A$, then $A_n^c \uparrow A^c$.
 (2)

$$\bigcup_{i=1}^n A_i = A_1 \cup (A_1^c \cap A_2) \cup (A_1^c \cap A_2^c \cap A_3) \cup \dots \cup (A_1^c \cap \dots \cap A_{n-1}^c \cap A_n).$$

- (3) Furthermore,

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} (A_1^c \cap \dots \cap A_{n-1}^c \cap A_n).$$

- (4) If the A_n form an increasing sequence, then

$$\bigcup_{i=1}^n A_i = A_1 \cup (A_2 - A_1) \cup \dots \cup (A_n - A_{n-1}).$$

- (5) If the A_n form an increasing sequence, then

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} (A_n - A_{n-1})$$

(take A_0 as the empty set).

Proof of Formula (1).

- (1) Suppose that $A_n \uparrow A$ is an increasing sequence of sets with limit A . Then $A_1 \subset A_2 \subset \dots$ and $\bigcup_{n=1}^{\infty} A_n = A$. So $A_1^c \supset A_2^c \supset \dots$ and

$$\bigcap_n A_n^c = \left(\bigcup_n A_n \right)^c = A^c$$

by the De Morgan laws. Hence $A_n \uparrow A$ implies that $A_n^c \downarrow A^c$.

- (2) Conversely, suppose that $A_n \downarrow A$ is an decreasing sequence of sets with limit A . Then $A_1 \supset A_2 \supset \cdots$ and $\bigcap_{n=1}^{\infty} A_n = A$. So $A_1^c \subset A_2^c \subset \cdots$ and

$$\bigcup_n A_n^c = \left(\bigcap_n A_n \right)^c = A^c$$

by the De Morgan laws. Hence $A_n \downarrow A$ implies that $A_n^c \uparrow A^c$.

□

Proof of Formula (2).

- (1) Set

$$B_i = A_1^c \cap \cdots \cap A_{i-1}^c \cap A_i$$

for $i = 1, \dots, n$. Observe that $B_1 = A_1$. So it is equivalent to show that

$$\bigcup_{i=1}^n A_i = \bigcup_{i=1}^n B_i.$$

- (2) Since each B_i is a subset of A_i , $\bigcup_{i=1}^n A_i \supset \bigcup_{i=1}^n B_i$.
(3) Conversely, given any $x \in \bigcup_{i=1}^n A_i$. $x \in A_j$ for some j . Now take the minimal value of j such that $x \in A_j$. The minimality of j implies that $x \notin A_1, A_2, \dots, A_{j-1}$. Hence

$$x \in A_1^c \cap \cdots \cap A_{j-1}^c \cap A_j = B_j \subset \bigcup_{i=1}^n B_i.$$

Therefore, $\bigcup_{i=1}^n A_i \subset \bigcup_{i=1}^n B_i$.

- (4) By (2)(3), $\bigcup_{i=1}^n A_i$ and $\bigcup_{i=1}^n B_i$ are equal.

□

Proof of Formula (3). Same as the proof of formula (2) since the minimality of j described in part (3) exists. □

Proof of Formula (4).

- (1) As A_n form an increasing sequence, $A_1 \subset A_2 \subset \cdots$ or $A_1^c \supset A_2^c \supset \cdots$.
Hence

$$A_1^c \cap \cdots \cap A_{i-1}^c = A_{i-1}^c.$$

Therefore, B_i is reduced to

$$B_i = A_1^c \cap \cdots \cap A_{i-1}^c \cap A_i = A_{i-1}^c \cap A_i = A_i - A_{i-1}.$$

(2) Now formula (2) becomes

$$\bigcup_{i=1}^n A_i = \bigcup_{i=1}^n B_i = \bigcup_{i=1}^n (A_i - A_{i-1}).$$

□

Proof of Formula (5). Note that $B_n = A_n - A_{n-1}$ in the proof of formula (4). Formula (3) becomes $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} (A_n - A_{n-1})$. □

Problem 1.1.2.

Define sets of real numbers as follows. Let $A_n = (-\frac{1}{n}, 1]$ if n is odd, and $A_n = (-1, \frac{1}{n}]$ if n is even. Find $\limsup_n A_n$ and $\liminf_n A_n$.

Proof.

(1) Write

$$\begin{aligned} \bigcup_{k=n}^{\infty} A_k &= \left(\bigcup_{k=\lfloor \frac{n}{2} \rfloor}^{\infty} A_{2k+1} \right) \cup \left(\bigcup_{k=\lfloor \frac{n+1}{2} \rfloor}^{\infty} A_{2k} \right) \\ &= \left(\bigcup_{k=\lfloor \frac{n}{2} \rfloor}^{\infty} \left(-\frac{1}{2k+1}, 1 \right] \right) \cup \left(\bigcup_{k=\lfloor \frac{n+1}{2} \rfloor}^{\infty} \left(-1, \frac{1}{2k} \right] \right) \\ &= \left(-\frac{1}{2\lfloor \frac{n}{2} \rfloor + 1}, 1 \right] \cup \left(-1, \frac{1}{2\lfloor \frac{n+1}{2} \rfloor} \right] \\ &= (-1, 1] \end{aligned}$$

for each k . Hence

$$\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \bigcap_{n=1}^{\infty} (-1, 1] = (-1, 1].$$

(2) Similarly, for each k we have

$$\begin{aligned} \bigcap_{k=n}^{\infty} A_k &= \left(\bigcap_{k=\lfloor \frac{n}{2} \rfloor}^{\infty} A_{2k+1} \right) \cap \left(\bigcap_{k=\lfloor \frac{n+1}{2} \rfloor}^{\infty} A_{2k} \right) \\ &= \left(\bigcap_{k=\lfloor \frac{n}{2} \rfloor}^{\infty} \left(-\frac{1}{2k+1}, 1 \right] \right) \cap \left(\bigcap_{k=\lfloor \frac{n+1}{2} \rfloor}^{\infty} \left(-1, \frac{1}{2k} \right] \right) \\ &= [0, 1] \cup (-1, 0] \\ &= \{0\}. \end{aligned}$$

Hence

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} \{0\} = \{0\}.$$

□