

## Chapter 11: The Lebesgue Theory

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**Exercise 11.1.** If  $f \geq 0$  and  $\int_E f d\mu = 0$ , prove that  $f(x) = 0$  almost everywhere on  $E$ . (Hint: Let  $E_n$  be the subset of  $E$  on which  $f(x) > \frac{1}{n}$ . Write  $A = \bigcup E_n$ . Then  $\mu(A) = 0$  if and only if  $\mu(E_n) = 0$  for every  $n$ .)

Might assume that  $f$  is measurable on  $E$ .

*Proof (Hint).*

- (1) Define  $A = \{x \in E : f(x) > 0\}$ . So  $f(x) = 0$  almost everywhere on  $E$  if and only if  $\mu(A) = 0$ .

- (2) Define

$$E_n = \left\{x \in E : f(x) > \frac{1}{n}\right\}$$

for  $n = 1, 2, 3, \dots$ . Note that  $E_1 \subseteq E_2 \subseteq E_3 \subseteq \dots$  and

$$A = \bigcup_{n=1}^{\infty} E_n.$$

Since  $\mu$  is a measure,

$$\lim_{n \rightarrow \infty} \mu(E_n) = \mu(A)$$

(Theorem 11.3).

- (3) (Reductio ad absurdum) If  $\mu(A) > 0$ , there is an integer  $N$  such that  $\mu(E_n) \geq \frac{\mu(A)}{2}$  whenever  $n \geq N$  (by (2)). In particular, take  $n = N$  to get

$$\begin{aligned} \int_E f d\mu &\geq \int_{E_N} f d\mu && (\mu \text{ is a measure and } E_N \subseteq E) \\ &\geq \frac{1}{N} \cdot \mu(E_N) && (\text{Remarks 11.23(b)}) \\ &\geq \frac{1}{N} \cdot \frac{\mu(A)}{2} \\ &> 0, \end{aligned}$$

contrary to the assumption that  $\int_E f d\mu = 0$ .

□

*Note.* Compare to Exercise 6.2.

**Exercise 11.2.** *If  $\int_A f d\mu = 0$  for every measurable subset  $A$  of a measurable set  $E$ , then  $f(x) = 0$  almost everywhere on  $E$ .*

Might assume that  $f$  is measurable on  $E$ .

*Proof.*

- (1) Define

$$A = \{x \in E : f(x) \geq 0\} \quad \text{and} \quad B = \{x \in E : f(x) \leq 0\}.$$

$A$  and  $B$  are measurable subsets of a measurable set  $E$  since  $f$  is measurable.

- (2) Apply Exercise 11.1 to the fact that  $f \geq 0$  on  $A$  (by construction) and  $\int_A f d\mu = 0$  (by assumption), we have  $f(x) = 0$  almost everywhere on  $A$ .
- (3) Similarly, apply Exercise 11.1 to the fact that  $-f \geq 0$  on  $B$  and  $\int_B (-f) d\mu = -\int_B f d\mu = 0$ , we have  $f(x) = 0$  almost everywhere on  $B$ .
- (4) As  $E = A \cup B$ ,  $f(x) = 0$  almost everywhere on  $E$  by (2)(3).

□

**Exercise 11.3.** *If  $\{f_n\}$  is a sequence of measurable functions, prove that the set of points  $x$  at which  $\{f_n(x)\}$  converges is measurable.*

*Proof.*

- (1) It suffices to show that

$$E = \{x : \{f_n(x)\} \text{ is convergent} \} = \{x : \{f_n(x)\} \text{ is Cauchy} \}$$

is measurable (since  $\mathbb{R}^1$  is complete).

- (2) Write

$$E = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n,m \geq N} \left\{ x : |f_n(x) - f_m(x)| \leq \frac{1}{k} \right\}$$

Since  $\{f_n\}$  is a sequence of measurable functions,  $x \mapsto |f_n(x) - f_m(x)|$  is measurable (Theorem 11.16 and Theorem 11.18). Hence

$$\left\{ x : |f_n(x) - f_m(x)| \leq \frac{1}{k} \right\}$$

is measurable (Theorem 11.15). Therefore  $E$  is measurable.

□

**Exercise 11.4.** If  $f \in \mathcal{L}(\mu)$  on  $E$  and  $g$  is bounded and measurable on  $E$ , then  $fg \in \mathcal{L}(\mu)$  on  $E$ .

*Proof (Theorem 11.27).*

- (1)  $fg$  is measurable since both  $f$  and  $g$  are measurable (Theorem 11.18).
- (2)  $|g| \leq M$  for some real  $M \in \mathbb{R}^1$  by the boundedness of  $g$ . Hence

$$|fg| \leq M|f|$$

on  $E$ .

- (3) To apply Theorem 11.27, it suffices to show that  $M|f| \in \mathcal{L}(\mu)$  on  $E$ . Theorem 11.26 implies that  $|f| \in \mathcal{L}(\mu)$  if  $f \in \mathcal{L}(\mu)$ . And Remarks 11.23(d) implies that  $M|f| \in \mathcal{L}(\mu)$  if  $|f| \in \mathcal{L}(\mu)$ .

□

*Note (Riemann integral).* If  $f \in \mathcal{R}$  on  $[a, b]$  and  $g$  is bounded and measurable on  $[a, b]$ , then  $fg$  might be not Riemann integrable.

**Exercise 11.5.** Put

$$g(x) = \begin{cases} 0 & (0 \leq x \leq \frac{1}{2}), \\ 1 & (\frac{1}{2} < x \leq 1), \end{cases}$$

and

$$\begin{aligned} f_{2k}(x) &= g(x) & (0 \leq x \leq 1), \\ f_{2k+1}(x) &= g(1-x) & (0 \leq x \leq 1). \end{aligned}$$

Show that

$$\liminf_{n \rightarrow \infty} f_n(x) = 0 \quad (0 \leq x \leq 1),$$

but

$$\int_0^1 f_n(x) dx = \frac{1}{2}.$$

(Compare with the Fatou's theorem.)

*Proof.*

- (1) Show that  $\liminf_{n \rightarrow \infty} f_n(x) = 0$ . Note that

$$g(1-x) = \begin{cases} 1 & (0 \leq x < \frac{1}{2}), \\ 0 & (\frac{1}{2} < x \leq 1). \end{cases}$$

Since  $f_n(x) \geq 0$  by definition,  $\liminf_{n \rightarrow \infty} f_n(x) \geq 0$ . Since  $f_{2k}(0) = f_{2k+1}(1) = 0$  for all positive integers  $k$ ,  $\liminf_{n \rightarrow \infty} f_n(x) \leq 0$ . Therefore the result is established.

(2) Show that  $\int_0^1 f_n(x) dx = \frac{1}{2}$ . Since

$$\begin{aligned}\int_0^1 f_{2k}(x) dx &= \int_0^1 g(x) dx = \frac{1}{2}, \\ \int_0^1 f_{2k+1}(x) dx &= \int_0^1 g(1-x) dx = \frac{1}{2},\end{aligned}$$

in any case  $\int_0^1 f_n(x) dx = \frac{1}{2}$  for all positive integers  $n$ .

(3) This example shows that we may have the strict inequality in the Fatou's theorem.

□

**Supplement (Similar exercise).** Consider the sequence  $\{f_n\}$  defined by  $f_n(x) = 1$  if  $n \leq x < n+1$ , with  $f_n(x) = 0$  otherwise. Show that we may have the strict inequality in the Fatou's theorem.

**Exercise 11.6.** Let

$$f_n(x) = \begin{cases} \frac{1}{n} & (|x| \leq n), \\ 0 & (|x| > n). \end{cases}$$

Then  $f_n(x) \rightarrow 0$  uniformly on  $\mathbb{R}^1$ , but

$$\int_{-\infty}^{\infty} f_n(x) dx = 2 \quad (n = 1, 2, 3, \dots).$$

(We write  $\int_{-\infty}^{\infty}$  in place of  $\int_{\mathbb{R}^1}$ .) Thus uniform convergence does not imply dominated convergence in the sense of Theorem 11.32. However, on sets of finite measure, uniformly convergent sequences of bounded functions do satisfy Theorem 11.32.

*Proof.*

(1) Show that  $f_n(x) \rightarrow 0$  uniformly on  $\mathbb{R}^1$ . Given any  $\varepsilon > 0$ , there is an integer  $N > \frac{1}{\varepsilon}$  such that

$$|f_n(x) - 0| \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

whenever  $n \geq N$  and  $x \in \mathbb{R}^1$ . Hence  $f_n(x) \rightarrow 0$  uniformly.

(2) Show that  $\int_{-\infty}^{\infty} f_n(x)dx = 2$ .

$$\int_{-\infty}^{\infty} f_n(x)dx = \int_{-n}^n \frac{1}{n}dx = 2.$$

(3) By (1)(2),

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n(x)dx \neq \int_{-\infty}^{\infty} \lim_{n \rightarrow \infty} f_n(x)dx$$

suggests that the Lebesgue's dominated convergence theorem (Theorem 11.32) does not hold in this case. In fact, if there were  $g \in \mathcal{L}$  such that  $|f_n(x)| \leq g(x)$ , then

$$\begin{aligned} \int_{-\infty}^{\infty} g(x)dx &\geq \int_0^{\infty} g(x)dx && \text{(Theorem 11.24)} \\ &= \sum_{n=1}^{\infty} \int_{n-1}^n g(x)dx && \text{(Theorem 11.24)} \\ &\geq \sum_{n=1}^{\infty} \int_{n-1}^n |f_n(x)|dx \\ &= \sum_{n=1}^{\infty} \int_{n-1}^n \frac{1}{n}dx \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \\ &= \infty, \end{aligned}$$

which is absurd.

(4) Show that on sets of finite measure, uniformly convergent sequences of bounded functions  $\{f_n\}$  do satisfy Theorem 11.32.

(a) Since  $\{f_n\}$  is uniformly convergent,  $\{f_n\}$  is uniformly bounded (Exercise 7.1), or there exists a real number  $M$  such that

$$|f_n(x)| \leq M$$

for all positive integer  $n$  and  $x \in E$ .

(b) Define  $g(x) = M$  on  $E$ . It is clear that

$$\int_E g(x)dx = M\mu(E) < +\infty.$$

Now we can apply the Lebesgue's dominated convergence theorem (Theorem 11.32) to get

$$\lim_{n \rightarrow \infty} \int_E f_n d\mu = \int_E \lim_{n \rightarrow \infty} f_n d\mu.$$

□

**Exercise 11.7.** ...

*Proof.*

(1)

(2)

□

**Exercise 11.8.** If  $f \in \mathcal{R}$  on  $[a, b]$  and if  $F(x) = \int_a^x f(t)dt$ , prove that  $F'(x) = f(x)$  almost everywhere on  $[a, b]$ .

*Proof.*

(1) Theorem 6.20 implies that  $F'(x_0) = f(x_0)$  if  $f$  is continuous at  $x_0 \in [a, b]$ .

(2) Since  $f \in \mathcal{R}$  on  $[a, b]$ ,  $f$  is bounded on  $[a, b]$ . Theorem 11.33 implies that  $f$  is continuous almost everywhere on  $[a, b]$ .

By (1)(2),  $F'(x) = f(x)$  almost everywhere on  $[a, b]$ . □

**Exercise 11.9.** Prove that the function  $F$  given by

$$F(x) = \int_a^x f dt \quad (a \leq x \leq b)$$

(where  $f \in \mathcal{L}$  on  $[a, b]$ ) is continuous on  $[a, b]$ .

*Proof.*

(1) Let  $f \in \mathcal{L}$  on  $E$ . Show that given any  $\varepsilon > 0$  there is a  $\delta > 0$  such that

$$\int_A f d\mu < \varepsilon$$

whenever  $A \subseteq E$  with  $\mu(A) < \delta$ .

(a) Define  $f_n(x) = \min\{f(x), n\}$  on  $E$  for  $n = 1, 2, 3, \dots$ . Then  $\{f_n\}$  is a sequence of measurable functions such that

$$0 \leq f_1(x) \leq f_2(x) \leq \dots$$

Also,  $f_n \rightarrow f$ . Then by the Lebesgue's monotone convergence theorem (Theorem 11.28),

$$\lim_{n \rightarrow \infty} \int_E f_n d\mu = \int_E f d\mu.$$

(b) For such  $\varepsilon > 0$ , there is an integer  $N \geq 1$  such that

$$\int_E (f - f_N) d\mu < \frac{\varepsilon}{2}.$$

Choose  $\delta > 0$  so that  $\delta < \frac{\varepsilon}{2N}$ . If  $\mu(A) < \delta$ , we have

$$\begin{aligned} \int_A f d\mu &= \int_A (f - f_N) d\mu + \int_A f_N d\mu \\ &\leq \int_E (f - f_N) d\mu + N\mu(A) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

(2) Apply (1) to  $f^+$  and  $f^-$  on  $E = [a, b]$ . Given any  $\varepsilon > 0$ , there is a common  $\delta > 0$  such that

$$\left| \int_x^y f^+ dt \right| < \frac{\varepsilon}{2} \quad \text{and} \quad \left| \int_x^y f^- dt \right| < \frac{\varepsilon}{2}$$

whenever  $|y - x| < \delta$ . So

$$|F(y) - F(x)| \leq \left| \int_x^y f^+ dt \right| + \left| \int_x^y f^- dt \right| < \varepsilon$$

whenever  $|y - x| < \delta$ . Hence  $F$  is uniformly continuous. (In fact,  $F$  is absolutely continuous by the same argument.)

□

*Note.* Compare to Theorem 6.20.

**Exercise 11.10.** If  $\mu(X) < +\infty$  and  $f \in \mathcal{L}^2(\mu)$  on  $X$ , prove that  $f \in \mathcal{L}$  on  $X$ . If

$$\mu(X) = +\infty,$$

this is false. For instance, if

$$f(x) = \frac{1}{1 + |x|},$$

then  $f^2 \in \mathcal{L}$  on  $\mathbb{R}^1$ , but  $f \notin \mathcal{L}$  on  $\mathbb{R}^1$ .

*Proof.*

(1) Since  $\mu(X) < +\infty$ ,  $1 \in \mathcal{L}^2(\mu)$  on  $X$ . By Theorem 11.35,  $f \in \mathcal{L}(\mu)$ , and

$$\int_X |f| d\mu \leq \|f\| \|1\|.$$

- (2) Show that  $f^2 \in \mathcal{L}$  on  $\mathbb{R}^1$ . To apply Theorem 11.33, we might restrict the measure space  $X = \mathbb{R}^1$  to some interval  $[a, b]$ . Then apply the Lebesgue's monotone convergence theorem (Theorem 11.28) to get the conclusion.

(a) Write

$$f(x)^2 = \left( \frac{1}{1+|x|} \right)^2 = \frac{1}{1+2|x|+x^2} \leq \frac{1}{1+x^2}.$$

By Theorem 11.27, it suffices to show that  $\frac{1}{1+x^2} \in \mathcal{L}$  on  $\mathbb{R}^1$ .

(b) Consider the sequence  $\{f_n\}$  defined by

$$f_n(x) = \frac{1}{1+x^2} \chi_{[-n,n]}(x).$$

(Here  $\chi_{[-n,n]} = K_{[-n,n]}$  is the characteristic function of  $[-n, n]$  defined in Definition 11.19.) By construction,

$$0 \leq f_1(x) \leq f_2(x) \leq \cdots \quad (x \in \mathbb{R}^1)$$

and

$$f_n(x) \rightarrow \frac{1}{1+x^2} \quad (x \in \mathbb{R}^1).$$

(c) Hence

$$\begin{aligned} \int_{\mathbb{R}^1} \frac{1}{1+x^2} dx &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^1} f_n(x) dx && \text{(Theorem 11.28)} \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^1} \frac{1}{1+x^2} \chi_{[-n,n]}(x) dx \\ &= \lim_{n \rightarrow \infty} \int_{-n}^n \frac{1}{1+x^2} dx \\ &= \lim_{n \rightarrow \infty} \mathcal{R} \int_{-n}^n \frac{1}{1+x^2} dx && \text{(Theorem 11.33)} \\ &= \lim_{n \rightarrow \infty} 2 \arctan(n) \\ &= \pi < \infty. \end{aligned}$$

- (4) Show that  $f \notin \mathcal{L}$  on  $\mathbb{R}^1$ .

(a) Consider the sequence  $\{f_n\}$  defined by

$$f_n(x) = f(x) \chi_{[-n,n]}(x) = \frac{1}{1+|x|} \chi_{[-n,n]}(x).$$

By construction,

$$0 \leq f_1(x) \leq f_2(x) \leq \cdots \quad (x \in \mathbb{R}^1)$$

and

$$f_n(x) \rightarrow f(x) \quad (x \in \mathbb{R}^1).$$



(b) Hence

$$\begin{aligned}\int_{\mathbb{R}^1} f(x)dx &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^1} f_n(x)dx && \text{(Theorem 11.28)} \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^1} \frac{1}{1+|x|} \chi_{[-n,n]}(x)dx \\ &= \lim_{n \rightarrow \infty} \int_{-n}^n \frac{1}{1+|x|} dx \\ &= \lim_{n \rightarrow \infty} \mathcal{R} \int_{-n}^n \frac{1}{1+|x|} dx && \text{(Theorem 11.33)} \\ &= \lim_{n \rightarrow \infty} 2 \log(n+1) \\ &= \infty,\end{aligned}$$

or  $f \notin \mathcal{L}$  on  $\mathbb{R}^1$ .

□

*Note.* Compare to Exercise 6.5.

**Exercise 11.11.** ...

*Proof.*

(1)

(2)

□

**Exercise 11.12.** ...

*Proof.*

(1)

(2)

□

**Exercise 11.13.** ...

*Proof.*

(1)

(2)

□

**Exercise 11.14. ...**

*Proof.*

(1)

(2)

□

**Exercise 11.15. ...**

*Proof.*

(1)

(2)

□

**Exercise 11.16. ...**

*Proof.*

(1)

(2)

□

**Exercise 11.17. ...**

*Proof.*

(1)

(2)

□

**Exercise 11.18. ...**

*Proof.*

(1)

(2)

□