

## Chapter 4: Continuity

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**Exercise 4.1.** Suppose  $f$  is a real function define on  $\mathbb{R}^1$  which satisfies

$$\lim_{h \rightarrow 0} [f(x+h) - f(x-h)] = 0$$

for every  $x \in \mathbb{R}^1$ . Does this imply that  $f$  is continuous?

*Proof.*  $\lim_{h \rightarrow 0} [f(x+h) - f(x-h)] = 0$  holds if  $f$  is continuous. But the converse of this statement and is not true. For example, define  $f : \mathbb{R}^1 \rightarrow \mathbb{R}^1$  by

$$f(x) = \begin{cases} 1 & (x = 0), \\ 0 & (x \neq 0). \end{cases}$$

$f$  is not continuous at  $x = 0$  but

$$\lim_{h \rightarrow 0} [f(x+h) - f(x-h)] = 0$$

for any  $x \in \mathbb{R}^1$ . (The identity holds for  $x \neq 0$  since  $f$  is continuous on  $\mathbb{R}^1 - \{0\}$ . Besides,  $\lim_{h \rightarrow 0} [f(0+h) - f(0-h)] = \lim_{h \rightarrow 0} [0 - 0] = 0$ .)  $\square$

**Exercise 4.2.** If  $f$  is a continuous mapping of a metric space  $X$  into a metric space  $Y$ , prove that  $f(\overline{E}) \subseteq \overline{f(E)}$  for every set  $E \subseteq X$ . ( $\overline{E}$  denotes the closure of  $E$ .) Show, by an example, that  $f(\overline{E})$  can be a proper subset of  $\overline{f(E)}$ .

*Proof.*

(1) Since  $f$  is continuous and  $\overline{f(E)}$  is closed,  $f^{-1}(\overline{f(E)})$  is closed. Hence,

$$\begin{aligned} f^{-1}(\overline{f(E)}) &\supseteq f^{-1}(f(E)) && \text{(Monotonicity of } f^{-1}) \\ &\supseteq E, && \text{(Note in Theorem 4.14)} \\ \overline{E} &\subseteq f^{-1}(\overline{f(E)}), && \text{(Monotonicity of closure)} \\ f(\overline{E}) &\subseteq f(f^{-1}(\overline{f(E)})) && \text{(Monotonicity of } f) \\ &\subseteq \overline{f(E)}. && \text{(Note in Theorem 4.14)} \end{aligned}$$

(2) Let  $f : (0, \infty) \rightarrow \mathbb{R}$  be a continuous function defined by

$$f(x) = \frac{1}{x}.$$

Consider  $E = \mathbb{Z}^+ \subseteq (0, \infty)$ . Then  $f(E) = \left\{ \frac{1}{n} : n \in \mathbb{Z}^+ \right\}$ , and thus

$$\begin{aligned} f(\overline{E}) &= \left\{ \frac{1}{n} : n \in \mathbb{Z}^+ \right\}. \\ \overline{f(E)} &= \left\{ \frac{1}{n} : n \in \mathbb{Z}^+ \right\} \cup \{0\}. \end{aligned}$$

□

**Supplement (Inverse image).**

(1)  $E \subseteq f^{-1}[f(E)]$  for  $E \subseteq X$ .

$$\begin{aligned} \forall x \in E &\implies f(x) \in f(E) \\ &\iff x \in f^{-1}[f(E)]. \quad (\text{Definition of the inverse image}) \end{aligned}$$

□

(2)  $f[f^{-1}(E)] \subseteq E$  for  $E \subseteq Y$ .

$$\begin{aligned} \forall y \in f[f^{-1}(E)] &\iff \exists x \in f^{-1}(E) \text{ such that } y = f(x) \\ &\iff \exists x, f(x) \in E \text{ such that } y = f(x) \\ &\implies \exists x, y = f(x) \in E. \end{aligned}$$

□

**Supplement (Continuity).** Let  $f$  be a map from a topological space on  $X$  to a topological space on  $Y$ . Then, the following statements are equivalent:

- (1)  $f$  is continuous: For each  $x \in X$  and every neighborhood  $V$  of  $f(x)$ , there is a neighborhood  $U$  of  $x$  such that  $f(U) \subseteq V$ .
- (2) For every open set  $O$  in  $Y$ , the inverse image  $f^{-1}(O)$  is open in  $X$ .
- (3) For every closed set  $C$  in  $Y$ , the inverse image  $f^{-1}(C)$  is closed in  $X$ .
- (4)  $f(A)^\circ \subseteq f(A^\circ)$  for every subset  $A$  of  $X$ .
- (5)  $f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ$  for every subset  $B$  of  $Y$ .
- (6)  $f(\overline{A}) \subseteq \overline{f(A)}$  for every subset  $A$  of  $X$ .
- (7)  $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$  for every subset  $B$  of  $Y$ .

**Exercise 4.3.** Let  $f$  be a continuous real function on a metric space  $X$ . Let  $Z(f)$  (the zero set of  $f$ ) be the set of all  $p \in X$  at which  $f(p) = 0$ . Prove that  $Z(f)$  is closed.

*Proof (Corollary to Theorem 4.8).* Since  $f$  is continuous,  $f^{-1}(\{0\}) = Z(f)$  is closed in  $X$  for a closed subset  $\{0\}$  in  $\mathbb{R}^1$ .  $\square$

Denote the complement of any set  $E$  by  $\widetilde{E}$ .

*Proof (Theorem 4.8).* Consider the complement of  $Z(f)$  in  $X$ ,

$$\begin{aligned}\widetilde{Z(f)} &= \{x \in X : f(x) \neq 0\} \\ &= f^{-1}((-\infty, 0) \cup (0, \infty)).\end{aligned}$$

Since  $f$  is continuous,  $f^{-1}((-\infty, 0) \cup (0, \infty)) = \widetilde{Z(f)}$  is open in  $X$  for a open subset  $(-\infty, 0) \cup (0, \infty)$  in  $\mathbb{R}^1$ .  $\square$

*Proof (Definition 2.18(d)).* Given any limit point  $p$  of  $Z(f)$ . Show that  $f(p) = 0$  or  $p \in Z(f)$ . Since  $f$  is continuous, given any  $\varepsilon > 0$  there exists a  $\delta > 0$  such that  $|f(x) - f(p)| < \varepsilon$  for all  $x \in X$  for which  $d_X(x, p) < \delta$ . Since  $p$  is a limit point of  $Z(f)$ , for such  $\delta > 0$  we have a point  $q \neq p$  such that  $q \in Z(f)$ , or  $f(q) = 0$ . So  $|f(p)| < \varepsilon$  for any  $\varepsilon > 0$ .  $f(p) = 0$ .  $\square$

*Proof (Definition 2.18(f)).* Consider the complement of  $Z(f)$  in  $X$ ,

$$\widetilde{Z(f)} = \{x \in X : f(x) \neq 0\} = \{f > 0\} \cup \{f < 0\}$$

where  $\{f > 0\} = \{x \in X : f(x) > 0\}$  and  $\{f < 0\} = \{x \in X : f(x) < 0\}$ . It suffices to show  $\{f > 0\}$  is open. ( $\{f < 0\}$  is similar.) Given any point  $p$  of  $\{f > 0\}$  or  $f(p) > 0$ . Want to show  $p$  is an interior point of  $\{f > 0\}$ . Since  $f$  is continuous, given any  $\varepsilon = \frac{f(p)}{2} > 0$  there exists a  $\delta > 0$  such that  $|f(x) - f(p)| < \frac{f(p)}{2}$  for all  $x \in X$  for which  $d_X(x, p) < \delta$ . For such  $x$  with  $d_X(x, p) < \delta$  we have

$$\frac{1}{2}f(p) < f(x) < \frac{3}{2}f(p).$$

That is,  $N = \{x : d_X(x, p) < \delta\}$  is a neighborhood  $p$  such that  $N \subseteq \{f > 0\}$ .  $\square$

**Exercise 4.4.** Let  $f$  and  $g$  be continuous mappings of a metric space  $X$  into a metric space  $Y$ , and let  $E$  be a dense subset of  $X$ . Prove that  $f(E)$  is dense in  $f(X)$ . If  $g(p) = f(p)$  for all  $p \in E$ , prove that  $g(p) = f(p)$  for all  $p \in X$ . (In other words, a continuous mapping is determined by its values on a dense subset of its domain.)

*Proof.*

- (1) Show that  $f(E)$  is dense in  $f(X)$ . It suffices to show that every point  $y \in f(X) - f(E)$  is a limit point of  $f(E)$ . Since  $y \in f(X) - f(E)$ , there exists a point  $x \in X - E$  such that  $y = f(x)$ . Since  $E$  is dense in  $X$ , there exists a sequence  $\{x_n\}$  in  $E$  such that  $x_n \rightarrow x$  as  $n \rightarrow \infty$ . Let  $y_n = f(x_n) \in f(E)$ . Take limit and use the continuity of  $f$ ,  $y_n \rightarrow y$  as  $n \rightarrow \infty$ , or  $y$  is a limit point of  $f(E)$ .
- (2) Show that  $g(p) = f(p)$  for all  $p \in X$  if  $g(p) = f(p)$  for all  $p \in E$ . It suffices to show  $g(p) = f(p)$  for all  $p \in X - E$ . Given any  $p \in X - E$ , there exists a sequence  $\{p_n\}$  in  $E$  such that  $p_n \rightarrow p$  as  $n \rightarrow \infty$ . Notice that  $g(p_n) = f(p_n)$  by the assumption. Take limit and use the continuity of  $f$  and  $g$ ,  $g(p) = f(p)$  for  $p \in X - E$ .

□

**Exercise 4.5.** If  $f$  is a real continuous function defined on a closed set  $E \subseteq \mathbb{R}^1$ , prove that there exist continuous real function  $g$  on  $\mathbb{R}^1$  such that  $g(x) = f(x)$  for all  $x \in E$ . (Such functions  $g$  are called **continuous extensions** of  $f$  from  $E$  to  $\mathbb{R}^1$ .) Show that the result becomes false if the word “closed” is omitted. Extend the result to vector valued functions. (Hint: Let the graph of  $g$  be a straight line on each of the segments which constitute the complement of  $E$  (compare Exercise 2.29). The result remains true if  $\mathbb{R}^1$  is replaced by any metric space, but the proof is not so simple.)

*Proof.*

- (1) Every open set in  $\mathbb{R}^1$  is the union of an at most countable collection of disjoint segments (Exercise 2.29).
- (2) We need to construct a continuous real function on the complement of  $E$ . By (1), write  $\tilde{E} = \bigcup_{i \in \mathcal{C}} (a_i, b_i)$  where  $\mathcal{C}$  is at most countable and  $a_i < b_i$ . ( $a_i, b_i$  could be  $\pm\infty$ .) Define  $g(x)$  by

$$g(x) = \begin{cases} f(x) & (x \in E), \\ f(a_i) + \frac{f(b_i) - f(a_i)}{b_i - a_i}(x - a_i) & (x \in (a_i, b_i) : \text{finite interval}), \\ f(a_i) & (x \in (a_i, b_i) : a_i : \text{finite}, b_i = +\infty), \\ f(b_i) & (x \in (a_i, b_i) : a_i = -\infty, b_i : \text{finite}), \\ 0 & (x \in (a_i, b_i) : a_i = -\infty, b_i = +\infty). \end{cases}$$

Show that  $g$  is continuous in  $\mathbb{R}^1$ , or show that  $g(x)$  is continuous at  $x = p$  for any point  $p \in \mathbb{R}^1$ .

- (a) Given a point  $p \in \tilde{E}$ . There is an open interval  $I = (a_i, b_i)$  such that  $p \in I$ . Since the graph of  $g$  in an open interval  $I$  is a straight line,  $g$  is continuous at  $x = p$ .

- (b) Given an isolated point  $p \in E$ . There are two open intervals  $I = (a_i, b_i)$  and  $J = (a_j, b_j)$  such that  $b_i = p = a_j$ . So  $\lim_{x \rightarrow p^-} g(x) = \lim_{x \rightarrow p^+} g(x) = f(p)$  by the construction of  $g$ , which says  $g$  is continuous at  $x = p$ .
- (c) Given a limit point  $p \in E$ . So that  $g(p) = f(p)$ . Given  $\varepsilon > 0$ . Consider  $\lim_{x \rightarrow p^+} g(x)$  first. (The case  $\lim_{x \rightarrow p^-} g(x)$  is similar.)
- (i) For such  $\varepsilon > 0$ , there is a  $\delta' > 0$  such that

$$f(p) - \varepsilon < f(x) < f(p) + \varepsilon$$

whenever

$$x \in E \text{ and } p < x < p + \delta'.$$

Since  $p$  is a limit point of  $E$ , there is a point  $q \neq p$  such that  $|q - p| < \delta'$ . Might assume that  $q > p$ , and then retake  $\delta = \min\{\delta', q - p\} > 0$ . (If no such  $q$ ,  $\lim_{x \rightarrow p^+} g(x) = f(p)$  trivially.)

- (ii) For any  $x$  such that  $p < x < q$ , consider  $x \in E$  or else  $x \in \tilde{E}$ . As  $x \in E$ , nothing to do by (i).
- (iii) As  $x \in \tilde{E}$ , there exists an open interval  $I = (a_i, b_i)$  such that  $x \in I \subseteq (p, q)$ . Therefore,

$$f(a_i) \leq g(x) \leq f(b_i) \text{ or } f(a_i) \geq g(x) \geq f(b_i).$$

By (i),

$$\begin{aligned} f(p) - \varepsilon &< f(a_i) < f(p) + \varepsilon \text{ and} \\ f(p) - \varepsilon &< f(b_i) < f(p) + \varepsilon, \\ f(p) - \varepsilon &< f(a_i) \leq g(x) \leq f(b_i) < f(p) + \varepsilon \text{ or} \\ f(p) - \varepsilon &< f(b_i) \leq g(x) \leq f(a_i) < f(p) + \varepsilon. \end{aligned}$$

Hence, given  $\varepsilon > 0$  there is a  $\delta > 0$  such that  $|g(x) - g(p)| < \varepsilon$  whenever  $p < x < p + \delta$  (and  $x \in \mathbb{R}^1$ ), or  $\lim_{x \rightarrow p^+} g(x) = g(p)$ .

- (3) Consider  $f(x) = \log(x)$  in  $(0, \infty)$ . Since  $\lim_{x \rightarrow 0} f(x) = -\infty$ , we cannot find any real continuous function  $g$  defined on  $x = 0$ .
- (4) For a vector-valued function  $\mathbf{f} = (f_1, \dots, f_k)$ , with each  $f_i$  is continuous on a closed set  $E \subseteq \mathbb{R}^1$ , extend  $f_i$  to a continuous function  $g_i$  on  $\mathbb{R}^1$  as (2). Put  $\mathbf{g} = (g_1, \dots, g_k)$ . Clearly  $\mathbf{g}$  is an extension of  $\mathbf{f}$ . Besides,  $\mathbf{g}$  is continuous in  $\mathbb{R}^1$  by Theorem 4.10.

□

**Supplement (Tietze's Extension Theorem).** *If  $X$  is a normal topological space and  $f : A \rightarrow \mathbb{R}$  is a continuous map from a closed subset  $A$  of  $X$  into the real numbers carrying the standard topology, then there exists a continuous map  $g : X \rightarrow \mathbb{R}$  with  $g(a) = f(a)$  for all  $a \in A$ .*

**Exercise 4.6.** If  $f$  is defined on  $E$ , the graph of  $f$  is the set of points  $(x, f(x))$ , for  $x \in E$ . In particular, if  $E$  is a set of real numbers, and  $f$  is real-valued, the graph of  $f$  is a subset of the plane. Suppose  $E$  is compact, and prove that  $f$  is continuous on  $E$  if and only if its graph is compact.

*Proof.* Let  $G = \{(x, f(x)) : x \in E\}$  be the graph of  $f$ .

(1) ( $\implies$ ) Let  $\mathbf{f} : E \rightarrow G$  defined by

$$\mathbf{f}(x) = (x, f(x)).$$

$\mathbf{f}(E) = G$  exactly. Since  $f$  and  $x$  are continuous in  $E$ ,  $\mathbf{f}$  is continuous (Theorem 4.10). As  $E$  is compact,  $\mathbf{f}(E)$  is compact (Theorem 4.14).

(2) ( $\impliedby$ ) Let  $\pi : G \rightarrow E$  be a projection map defined by

$$\pi(x, f(x)) = x.$$

Notice that  $\pi \circ \mathbf{f} = \text{id}_E$  and  $\mathbf{f} \circ \pi = \text{id}_G$ . Besides,  $\pi$  is a continuous one-to-one mapping of a compact set  $G$  onto  $E$ . Then the inverse mapping  $\pi^{-1} = \mathbf{f}$  is a continuous mapping of  $E$  onto  $G$  (Theorem 4.17). So  $f$  is continuous (Theorem 4.10).

□

**Exercise 4.7.** If  $E \subseteq X$  and if  $f$  is a function defined on  $X$ , the **restriction** of  $f$  to  $E$  is the function  $g$  whose domain of definition is  $E$ , such that  $g(p) = f(p)$  for  $p \in E$ . Define  $f$  and  $g$  on  $\mathbb{R}^2$  by:

$$f(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0) \\ \frac{xy^2}{x^2+y^4} & \text{if } (x, y) \neq (0, 0), \end{cases}$$

$$g(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0) \\ \frac{xy^2}{x^2+y^6} & \text{if } (x, y) \neq (0, 0), \end{cases}$$

Prove that  $f$  is bounded on  $\mathbb{R}^2$ , that  $g$  is unbounded in every neighborhood of  $(0, 0)$ , and that  $f$  is not continuous at  $(0, 0)$ ; nevertheless, the restrictions of both  $f$  and  $g$  to every straight line in  $\mathbb{R}^2$  are continuous!

*Proof.*

(1) Show that  $f$  is bounded on  $\mathbb{R}^2$ .

$$\begin{aligned}
(|x| - |y^2|)^2 \geq 0 &\iff |x|^2 - 2|x||y^2| + |y^2|^2 \geq 0 \\
&\iff |x|^2 + |y^2|^2 \geq 2|x||y^2| \\
&\iff |x^2 + y^4| \geq 2|xy^2| \\
&\implies \frac{1}{2} \geq \left| \frac{xy^2}{x^2 + y^2} \right| \text{ whenever } (x, y) \neq (0, 0) \\
&\implies |f(x, y)| \leq \frac{1}{2} \text{ whenever } (x, y) \neq (0, 0).
\end{aligned}$$

Note that  $f(0, 0) = 0 \leq \frac{1}{2}$ . Hence  $f$  is bounded by  $\frac{1}{2}$  on  $\mathbb{R}^2$ .

(2) Show that  $g$  is unbounded in every neighborhood of  $\mathbb{R}^2$ . Consider a sequence  $\{\mathbf{p}_n\}_{n \geq 1} \subseteq \mathbb{R}^2$

$$\mathbf{p}_n = (x_n, y_n) = \left( \frac{1}{n^3}, \frac{1}{n} \right)$$

such that  $\mathbf{p}_n \neq \mathbf{0}$  and  $\lim \mathbf{p}_n = \mathbf{0}$ . Thus,

$$\lim_{n \rightarrow \infty} g(\mathbf{p}_n) = \lim_{n \rightarrow \infty} \frac{x_n y_n^2}{x_n^2 + y_n^6} = \lim_{n \rightarrow \infty} \frac{(\frac{1}{n^3})(\frac{1}{n})^2}{(\frac{1}{n^3})^2 + (\frac{1}{n})^6} = \lim_{n \rightarrow \infty} \frac{n}{2} = \infty.$$

Hence  $g$  is unbounded in every neighborhood of  $\mathbb{R}^2$ .

(3) Show that  $f$  is not continuous at  $(0, 0)$ . Consider a sequence  $\{\mathbf{p}_n\}_{n \geq 1} \subseteq \mathbb{R}^2$

$$\mathbf{p}_n = (x_n, y_n) = \left( \frac{1}{n^2}, \frac{1}{n} \right)$$

such that  $\mathbf{p}_n \neq \mathbf{0}$  and  $\lim \mathbf{p}_n = \mathbf{0}$ . Thus,

$$\lim_{n \rightarrow \infty} f(\mathbf{p}_n) = \lim_{n \rightarrow \infty} \frac{x_n y_n^2}{x_n^2 + y_n^4} = \lim_{n \rightarrow \infty} \frac{(\frac{1}{n^2})(\frac{1}{n})^2}{(\frac{1}{n^2})^2 + (\frac{1}{n})^4} = \frac{1}{2}.$$

So,  $\lim f(\mathbf{p}_n) = \frac{1}{2} \neq 0$ . By Theorem 4.6,  $f$  is not continuous at  $(0, 0)$ .

(4) The restrictions of  $f$  to every straight line in  $\mathbb{R}^2$  is continuous.

- (a) The line  $L_\infty = \{(0, y) : y \in \mathbb{R}\}$ . Hence  $f|_{L_\infty}(x, y) = 0$  for all  $(x, y) \in L_\infty$  (including  $(0, 0) \in L_\infty$ ). Therefore  $f|_{L_\infty}$  is continuous.
- (b) The line  $L_\alpha = \{(x, \alpha x) : x \in \mathbb{R}\}$  for some  $\alpha \in \mathbb{R}$ .  $f|_{L_\alpha}(x, y)$  is continuous on  $L_\alpha - \{(0, 0)\}$ .

$$f|_{L_\alpha}(x, y) = f|_{L_\alpha}(x, \alpha x) = \begin{cases} 0 & \text{if } (x, y) = (0, 0) \\ \frac{\alpha^2 x}{1 + \alpha^4 x^2} & \text{if } (x, y) \neq (0, 0). \end{cases}$$

So

$$\lim_{(x, y) \rightarrow (0, 0)} f|_{L_\alpha}(x, y) = \lim_{x \rightarrow 0} \frac{\alpha^2 x}{1 + \alpha^4 x^2} = 0 = f(0, 0),$$

- or  $f|_{L_\alpha}(x, y)$  is continuous at  $(0, 0)$ . Therefore,  $f|_{L_\alpha}(x, y)$  is continuous on  $L_\alpha$ .
- (c) *The line  $L$  not passing  $(0, 0)$ .* It is clear since  $f(x, y)$  is continuous on  $\mathbb{R}^2 - \{(0, 0)\}$ .
- (5) *The restrictions of  $g$  to every straight line in  $\mathbb{R}^2$  is continuous.* Similar to (4).
- (a) *The line  $L_\infty = \{(0, y) : y \in \mathbb{R}\}$ .* Hence  $g|_{L_\infty}(x, y) = 0$  for all  $(x, y) \in L_\infty$  (including  $(0, 0) \in L_\infty$ ). Therefore  $g|_{L_\infty}$  is continuous.
- (b) *The line  $L_\alpha = \{(x, \alpha x) : x \in \mathbb{R}\}$  for some  $\alpha \in \mathbb{R}$ .*  $g|_{L_\alpha}(x, y)$  is continuous on  $L_\alpha - \{(0, 0)\}$ .

$$g|_{L_\alpha}(x, y) = g|_{L_\alpha}(x, \alpha x) = \begin{cases} 0 & \text{if } (x, y) = (0, 0) \\ \frac{\alpha^2 x}{1 + \alpha^6 x^4} & \text{if } (x, y) \neq (0, 0). \end{cases}$$

So

$$\lim_{(x, y) \rightarrow (0, 0)} g|_{L_\alpha}(x, y) = \lim_{x \rightarrow 0} \frac{\alpha^2 x}{1 + \alpha^6 x^4} = 0 = g(0, 0),$$

or  $g|_{L_\alpha}(x, y)$  is continuous at  $(0, 0)$ . Therefore,  $g|_{L_\alpha}(x, y)$  is continuous on  $L_\alpha$ .

- (c) *The line  $L$  not passing  $(0, 0)$ .* It is clear since  $g(x, y)$  is continuous on  $\mathbb{R}^2 - \{(0, 0)\}$ .

□

**Exercise 4.8.** Let  $f$  be a real uniformly continuous function on the bounded set  $E$  in  $\mathbb{R}$ . Prove that  $f$  is bounded on  $E$ . Show that the conclusion is false if boundedness of  $E$  is omitted from the hypothesis.

The conclusion is false if boundedness of  $E$  is omitted from the hypothesis. For example,  $f(x) = x$  on  $\mathbb{R}$  is uniformly continuous on  $\mathbb{R}$  but  $f(\mathbb{R}) = \mathbb{R}$  is unbounded.

*Proof (Brute-force).*

- (1) Since  $f : E \rightarrow \mathbb{R}$  is uniformly continuous, given any  $\varepsilon > 0$ , there is  $\delta > 0$  such that  $|f(x) - f(y)| < \varepsilon$  whenever  $|x - y| < \delta$ . In particular, pick  $\varepsilon = 1$ .
- (2) By the boundedness of  $E$ , there is  $M > 0$  such that  $|x| < M$  for all  $x \in E$ .
- (3) For such  $\delta > 0$ , we construct a covering of  $E \subseteq \mathbb{R}$ . Construct a special collection  $\mathcal{C}$  of intervals

$$I_a = \left[ \frac{\delta}{2}a, \frac{\delta}{2}(a+1) \right]$$



where  $a \in \mathbb{Z}$  satisfying

$$|a| < \frac{2M}{\delta} + 1.$$

By construction,  $\mathcal{C}$  is a finite covering of  $E$ .

- (4) For every interval  $I_a$  of the collection  $\mathcal{C}$ , pick a point  $x_a \in E \cap I_a$  if possible. This process will terminate eventually since  $\mathcal{C}$  is finite. Collect these representative points as  $\mathcal{D} = \{x_a\}$ . Notice that  $\mathcal{D}$  is finite again.
- (5) Now for any point  $x \in E$ ,  $x$  lies in some  $I_a$  containing  $x_a$ . Both  $x$  and  $x_a$  are in the same interval and their distance satisfies

$$|x - x_a| \leq \frac{\delta}{2} < \delta$$

and thus by (1)

$$|f(x) - f(x_a)| < 1, \text{ or } |f(x)| < 1 + |f(x_a)|.$$

- (6) Let

$$M = 1 + \max_{x_a \in \mathcal{D}} |f(x_a)|.$$

So given any  $x \in E$ ,  $|f(x)| < M$ .

□

*Proof (Heine-Borel Theorem).* Heine-Borel theorem provides the finiteness property to construct the boundedness property of  $f$ .

- (1) Let  $E$  be a bounded subset of a metric space  $X$ . Show that the closure of  $E$  in  $X$  is also bounded in  $X$ .  $E$  is bounded if  $E \subseteq B_X(a; r)$  for some  $r > 0$  and some  $a \in X$ . (The ball  $B_X(a; r)$  is defined to be the set of all  $x \in X$  such that  $d_X(x, a) < r$ .) Take the closure on the both sides,

$$\overline{E} \subseteq \overline{B_X(a; r)} = \{x \in X : d_X(x, a) \leq r\} \subseteq B_X(a; 2r),$$

or  $\overline{E}$  is bounded.

- (2) Since  $f : E \rightarrow \mathbb{R}$  is uniformly continuous, given any  $\varepsilon > 0$ , there is  $\delta > 0$  such that  $|f(x) - f(y)| < \varepsilon$  whenever  $|x - y| < \delta$ . In particular, pick  $\varepsilon = 1$ .
- (3) For such  $\delta > 0$ , we construct an open covering of  $\overline{E} \subseteq \mathbb{R}$ . Pick a collection  $\mathcal{C}$  of open balls  $B(a; \delta) \subseteq \mathbb{R}$  where  $a$  runs over all elements of  $E$ .  $\mathcal{C}$  covers  $\overline{E}$  (by the definition of accumulation points). Since  $\overline{E}$  is closed and bounded (by applying (1) on the boundedness of  $E$ ),  $\overline{E}$  is compact (Heine-Borel theorem). That is, there is a finite subcollection  $\mathcal{C}'$  of  $\mathcal{C}$  also covers  $\overline{E}$ , say

$$\mathcal{C}' = \{B(a_1; \delta), B(a_2; \delta), \dots, B(a_m; \delta)\}.$$

- (4) Given any  $x \in E \subseteq \overline{E}$ , there is some  $a_i \in E$  ( $1 \leq i \leq m$ ) such that  $x \in B(a_i; \delta)$ . In such ball,  $|x - a_i| < \delta$ . By (2),  $|f(x) - f(a_i)| < 1$ , or  $|f(x)| < 1 + |f(a_i)|$ . Almost done. Notice that  $a_i$  depends on  $x$ , and thus we might use finiteness of  $\{a_1, a_2, \dots, a_m\}$  to remove dependence of  $a_i$ .

- (5) Let

$$M = 1 + \max_{1 \leq i \leq m} |f(a_i)|.$$

So given any  $x \in E$ ,  $|f(x)| < M$ .

□

**Supplement.** Exercise about considering the closure. (Problem 3.5 in H. L. Royden, Real Analysis, 3rd Edition.) Let  $A = \mathbb{Q} \cap [0, 1]$ , and let  $\{I_n\}$  be a finite collection of open intervals covering  $A$ . Then  $\sum l(I_n) \geq 1$ .

*Proof.*

$$\begin{aligned} 1 = m^*[0, 1] &= m^*\overline{A} \leq m^*\left(\overline{\bigcup I_n}\right) = m^*\left(\bigcup \overline{I_n}\right) \\ &\leq \sum m^*(\overline{I_n}) = \sum l(\overline{I_n}) = \sum l(I_n). \end{aligned}$$

□

**Exercise 4.9.** Show that the requirement in the definition of uniform continuity can be rephrased as follows, in terms of diameters of sets: To every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that  $\text{diam} f(E) < \varepsilon$  for all  $E \subseteq X$  with  $\text{diam} E < \delta$ .

*Proof.*

- (1) ( $\implies$ ) Given  $\varepsilon > 0$ . By Definition 4.18, there exists a  $\delta > 0$  such that

$$d(f(p), f(q)) < \frac{\varepsilon}{64}$$

for all  $p$  and  $q$  in  $X$  for which  $d(p, q) < \delta$ . Let  $E$  be any subset of  $X$  satisfying  $\text{diam} E < \delta$ . Then for any  $p, q \in E$ ,

$$d(p, q) \leq \text{diam} E < \delta.$$

So that

$$d(f(p), f(q)) < \frac{\varepsilon}{64},$$

or  $\frac{\varepsilon}{64}$  is an upper bound of  $S = \{d(f(p), f(q)) : p, q \in E\}$ . Hence

$$\text{diam} f(E) = \sup S \leq \frac{\varepsilon}{64} < \varepsilon.$$

(Here we pick " $\frac{\varepsilon}{64}$ " instead of  $\varepsilon$  since we want to get " $\text{diam} f(E) < \varepsilon$ " instead of  $\text{diam} f(E) \leq \varepsilon$ .)

- (2) ( $\Leftarrow$ ) Easy. Given  $\varepsilon > 0$  there exists a  $\delta > 0$  such that  $\text{diam}f(E) < \varepsilon$  for all  $E \subseteq X$  with  $\text{diam}E < \delta$ . In particular, for any  $p, q \in X$  with  $d(p, q) < \delta$ , we can take  $E = \{p, q\} \subseteq X$  and its diameter

$$\text{diam}E = d(p, q) < \delta.$$

So that

$$d(f(p), f(q)) = \text{diam}f(E) < \varepsilon,$$

or Definition 4.18 holds.

□

**Exercise 4.10.** Complete the details of the following alternative proof of Theorem 4.19 (Let  $f$  be a continuous mapping of a compact metric space  $X$  into a metric space  $Y$ . Then  $f$  is uniformly continuous on  $X$ ): If  $f$  is not uniformly continuous, then for some  $\varepsilon > 0$  there are sequences  $\{p_n\}, \{q_n\}$  in  $X$  such that  $d_X(p_n, q_n) \rightarrow 0$  but  $d_Y(f(p_n), f(q_n)) > \varepsilon$ . Use Theorem 2.37 to obtain a contradiction.

*Proof.*

- (1) (Reductio ad absurdum) If  $f$  were not uniformly continuous, then for some  $\varepsilon > 0$  there are sequences  $\{p_n\}, \{q_n\}$  in  $X$  such that  $d_X(p_n, q_n) \rightarrow 0$  but  $d_Y(f(p_n), f(q_n)) > \varepsilon$ .
- (2) By Theorem 2.37, there is a subsequence  $\{p_{n_k}\}$  of  $\{p_n\}$  such that  $\{p_{n_k}\}$  converges to  $p \in X$ . Similar argument to  $\{q_n\}$ , we have a subsequence  $\{q_{n'_k}\}$  of  $\{q_n\}$  converging to  $q \in X$ .
- (3) Since

$$d_X(p, q) \leq d_X(p, p_{n_k}) + d_X(p_{n_k}, q_{n'_k}) + d_X(q_{n'_k}, q) \rightarrow 0$$

(by assumption and (2)) and  $d_X(p, q)$  is a constant,  $d_X(p, q) = 0$  or  $p = q$ .

- (4) Since  $f$  is continuous,

$$\lim_{k \rightarrow \infty} f(p_{n_k}) = f(p) = f(q) = \lim_{k \rightarrow \infty} f(q_{n'_k})$$

or  $d_Y(f(p_{n_k}), f(q_{n'_k})) \rightarrow 0$ , contrary to the assumption.

□

**Exercise 4.11.** Suppose  $f$  is a uniformly continuous mapping of a metric space  $X$  into a metric space  $Y$  and prove that  $\{f(x_n)\}$  is a Cauchy sequence in  $Y$  for every Cauchy sequence  $\{x_n\}$  in  $X$ . Use this result to give an alternative proof

of the theorem stated in Exercise 4.13.

An alternative proof of Exercise 4.13 will be in Exercise 4.13 itself.

*Proof (Definition 4.18).* Given any Cauchy sequence  $\{x_n\}$  in  $X$ .

- (1) Given any  $\varepsilon > 0$ . Since  $f$  is uniformly continuous, there exists a  $\delta > 0$  such that

$$d_Y(f(p), f(q)) < \varepsilon$$

for all  $p$  and  $q$  in  $X$  for which  $d_X(p, q) < \delta$ .

- (2) Since  $\{x_n\}$  is Cauchy in  $X$ , for such  $\delta > 0$  there is an integer  $N$  such that

$$d_X(x_n, x_m) < \delta$$

whenever  $n, m \geq N$ .

- (3) By (1)(2),

$$d_Y(f(x_n), f(x_m)) < \varepsilon$$

whenever  $n, m \geq N$ . Hence  $\{f(x_n)\}$  is Cauchy in  $Y$ .

□

*Proof (Exercise 4.9).* Given any Cauchy sequence  $\{x_n\}$  in  $X$ .

- (1) Given any  $\varepsilon > 0$ . Since  $f$  is uniformly continuous, there exists a  $\delta > 0$  such that  $\text{diam} f(E) < \varepsilon$  for all  $E \subseteq X$  with  $\text{diam} E < \delta$ .

- (2) Since  $\{x_n\}$  is Cauchy in  $X$ , for such  $\delta > 0$  there is an integer  $N$  such that

$$d_X(x_n, x_m) < \frac{\delta}{64}$$

whenever  $n, m \geq N$ .

- (3) Consider  $E = \{x_N, x_{N+1}, \dots\}$ . By (2),  $\text{diam} E \leq \frac{\delta}{64} < \delta$ . By (1),

$$d_Y(f(x_n), f(x_m)) \leq \text{diam} f(E) < \varepsilon$$

whenever  $n, m \geq N$ . Hence  $\{f(x_n)\}$  is Cauchy in  $Y$ .

□

**Exercise 4.12.** A uniformly continuous function of a uniformly continuous function is uniformly continuous. State this more precisely and prove it.

Statement (similar to Theorem 4.7): suppose  $X, Y, Z$  are metric space,  $E \subseteq X$ ,  $f$  maps  $E$  into  $Y$ ,  $g$  maps the range of  $f$ ,  $f(E)$ , into  $Z$ , and  $h$  is the mapping of  $E$  into  $Z$  defined by

$$h(x) = g(f(x)) \quad (x \in E).$$

If  $f$  is uniformly continuous on  $E$  and  $g$  is uniformly continuous on  $f(E)$ , then  $h$  is uniformly continuous on  $E$ .

*Proof.*

- (1) Given  $\varepsilon > 0$ . Since  $g$  is uniformly continuous on  $f(E)$ , there exists  $\eta > 0$  such that

$$d_Z(g(f(p)), g(f(q))) < \varepsilon \quad \text{if} \quad d_Y(f(p), f(q)) < \eta \quad \text{and} \quad f(p), f(q) \in f(E).$$

- (2) Since  $f$  is uniformly continuous on  $E$ , there exists  $\delta > 0$  such that

$$d_Y(f(p), f(q)) < \eta \quad \text{if} \quad d_X(p, q) < \delta \quad \text{and} \quad p, q \in E.$$

- (3) By (1)(2),

$$d_Z(h(p), h(q)) = d_Z(g(f(p)), g(f(q))) < \varepsilon$$

if  $d_X(p, q) < \delta$  and  $p, q \in E$ . Hence  $h$  is uniformly continuous on  $E$ .

□

**Exercise 4.13.** Let  $E$  be a dense subset of a metric space  $X$ , and let  $f$  be a uniformly continuous real function defined on  $E$ . Prove that  $f$  has a continuous extension from  $E$  to  $X$  (see Exercise 4.5 for terminology). (Uniqueness follows from Exercise 4.4.) (Hint: For each  $p \in X$  and each positive integer  $n$ , let  $V_n(p)$  be the set of all  $q \in E$  with  $d(p, q) < \frac{1}{n}$ . Use Exercise 4.9 to show that the intersection of the closures of the sets  $f(V_1(p)), f(V_2(p)), \dots$ , consists of a single point, say  $g(p)$ , of  $\mathbb{R}^1$ . Prove that the function  $g$  so define on  $X$  is the desired extension of  $f$ .) Could the range space  $\mathbb{R}^1$  be replaced by  $\mathbb{R}^k$ ? By any compact metric space? By any complete metric space? By any metric space?

*Proof (Hint).* We prove the case that the range metric space is complete.

- (1) Given any  $p \in X$ . We will extend  $f$  on  $x = p$ . For any positive integer  $n$ , let  $V_n(p)$  be the set of all  $q \in E$  with  $d(p, q) < \frac{1}{n}$ .
- (2) Show that  $\overline{f(V_n(p))} \supseteq \overline{f(V_{n+1}(p))}$ . By construction,  $V_n(p) \supseteq V_{n+1}(p)$ . Thus  $f(V_n(p)) \supseteq f(V_{n+1}(p))$  and

$$\overline{f(V_n(p))} \supseteq \overline{f(V_{n+1}(p))}.$$

- (3) Show that  $\lim_{n \rightarrow \infty} \text{diam} \overline{f(V_n(p))} = 0$ .

(a) Since  $E$  is dense in  $X$ ,  $V_n(p) \neq \emptyset$  and thus

$$f(V_n(p)) \neq \emptyset.$$

Especially,

$$\overline{f(V_n(p))} \supseteq f(V_n(p)) \neq \emptyset.$$

Hence  $\text{diam} V_n(p)$  and  $\text{diam} f(V_n(p))$  are well-defined.

(b) By the definition of  $V_n(p)$  or  $0 \leq \text{diam} V_n(p) \leq \frac{2}{n}$ ,

$$\lim_{n \rightarrow \infty} \text{diam} V_n(p) = 0.$$

(c) By the uniform continuity of  $f$  (Exercise 4.9),

$$\lim_{n \rightarrow \infty} \text{diam} f(V_n(p)) = 0.$$

(d) Since  $\text{diam} \overline{f(V_n(p))} = \text{diam} f(V_n(p))$  (Theorem 3.10(a)),

$$\lim_{n \rightarrow \infty} \text{diam} \overline{f(V_n(p))} = 0.$$

(4) *Show that there is an integer  $N$  such that  $\overline{f(V_n(p))}$  is closed and bounded whenever  $n \geq N$ .*

(a) (Closeness.) Each  $\overline{f(V_n(p))}$  is closed.

(b) (Boundedness.) Since  $\lim_{n \rightarrow \infty} \text{diam} \overline{f(V_n(p))} = 0$  by (3), there is an integer  $N$  such that

$$\text{diam} \overline{f(V_n(p))} \leq \frac{1}{89}$$

whenever  $n \geq N$ . By the definition of diameters of  $\overline{f(V_n(p))}$ , each  $\overline{f(V_n(p))}$  is bounded by  $\frac{1}{64}$  whenever  $n \geq N$ .

(c) *Note.* If we apply Exercise 4.8 instead, we need extra efforts to generalize Exercise 4.8 to different range spaces for answering the following questions.

(5) By (2)(3)(4) and Exercise 3.21,

$$\bigcap_{n=N}^{\infty} \overline{f(V_n(p))}$$

or

$$\bigcap_{n=1}^{\infty} \overline{f(V_n(p))}$$

consists of exactly one point, say  $g(p)$ . This point  $g(p)$  is an extension of  $f$  at  $x = p$ . Clearly,  $g(p) = f(p)$  if  $p \in E$ .

(6) Define

$$g(p) = \begin{cases} \bigcap_{n=1}^{\infty} \overline{f(V_n(p))} = f(p) & (p \in E), \\ \bigcap_{n=1}^{\infty} \overline{f(V_n(p))} & (p \notin E). \end{cases}$$

Show that  $g$  is uniformly continuous.

- (a) Given any  $\varepsilon > 0$ . Since  $f$  is uniformly continuous on  $E$ , there exists  $\delta > 0$  such that

$$d(f(p), f(q)) < \frac{\varepsilon}{3} < \varepsilon$$

whenever  $d(p, q) < \delta$  and  $p, q \in E$ . We will show that such  $\delta$  also holds for  $g$ . Now given any  $p, q \in X$  with  $d(p, q) < \delta$ .

- (b) Since  $\text{diam} f(V_n(p)) = \text{diam} \overline{f(V_n(p))}$  and  $\lim_{n \rightarrow \infty} \text{diam} \overline{f(V_n(p))} = 0$  (whether  $p \in E$  or not), there is an integer  $N_1$  such that

$$\text{diam} f(V_n(p)) < \frac{\varepsilon}{3}$$

whenever  $n \geq N_1$ . Similarly, there is an integer  $N_2$  such that

$$\text{diam} f(V_n(q)) < \frac{\varepsilon}{3}$$

whenever  $n \geq N_2$ .

- (c) Take an integer  $N_3$  satisfying

$$N_3 > \frac{4}{\delta - d(p, q)} > 0.$$

For any  $p' \in V_n(p) \neq \emptyset$  and  $q' \in V_n(q) \neq \emptyset$  as  $n \geq N_3$ , we have

$$\begin{aligned} d(p', q') &\leq d(p', p) + d(p, q) + d(q, q') \\ &\leq \frac{2}{n} + d(p, q) + \frac{2}{n} \\ &\leq \frac{2}{N_3} + d(p, q) + \frac{2}{N_3} \\ &< \frac{2(\delta - d(p, q))}{4} + d(p, q) + \frac{2(\delta - d(p, q))}{4} \\ &= \delta. \end{aligned}$$

- (d) Take  $N = \max\{N_1, N_2, N_3\}$ . For any  $p' \in V_N(p)$  and  $q' \in V_N(p)$ , we have

$$\begin{aligned} d(g(p), f(p')) &\leq \text{diam} f(V_N(p)) < \frac{\varepsilon}{3}, \\ d(f(p'), f(q')) &< \frac{\varepsilon}{3}, \\ d(f(q'), g(q)) &\leq \text{diam} f(V_N(q)) < \frac{\varepsilon}{3}. \end{aligned}$$

Hence

$$\begin{aligned} d(g(p), g(q)) &\leq d(g(p), f(p')) + d(f(p'), f(q')) + d(f(q'), g(q)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

(7) Show that the range space  $\mathbb{R}^1$  cannot be replaced by any metric space.

- (a) Take  $X = \mathbb{R}$  and  $Y = \mathbb{Q}$  with the Euclidean metric. Let  $E = \mathbb{Q}$  be a dense subset of  $X = \mathbb{R}$ . Define  $f : E \rightarrow Y$  by

$$f(x) = x.$$

- (b)  $f$  is uniformly continuous on  $E$ .  
(c) (Reductio ad absurdum) If  $f$  were having a continuous extension  $g$  on  $X$ , then

$$\lim_{n \rightarrow \infty} g(p_n) = g(p)$$

for any sequence  $\{p_n\}$  in  $X$  such that  $p_n \neq p$  and  $\lim_{n \rightarrow \infty} p_n = p$ .

- (d) In particular, for some rational sequence  $\{p_n\}$  in  $E = \mathbb{Q}$  converging to  $\sqrt{2} \in X$ , we have

$$\lim_{n \rightarrow \infty} g(p_n) = \lim_{n \rightarrow \infty} f(p_n) = \lim_{n \rightarrow \infty} p_n = \sqrt{2} = g(p) \in \mathbb{Q},$$

which is absurd.

□

*Proof (Exercise 4.11).* We prove the case that the range metric space is complete.

- (1) Given any  $p \in X$ . We will extend  $f$  on  $x = p$ . Since  $E$  is dense in  $X$ , there exists a sequence  $\{p_n\}$  in  $E$  converging to  $p$  (whether  $p \in E$  or not).  
(2) Since  $E$  is dense in  $X$ , there exists a sequence  $\{p_n\}$  in  $E$  converging to  $p$  (whether  $p \in E$  or not). Hence  $\{p_n\}$  is Cauchy in  $E$  (Theorem 3.11(a)). Since  $f$  is uniformly continuous,  $\{f(p_n)\}$  is Cauchy. Since the range space is complete,  $\{f(p_n)\}$  converges to a point, say  $g(p)$ . This point  $g(p)$  is an extension of  $f$  at  $x = p$ . Clearly,  $g(p) = f(p)$  if  $p \in E$ .  
(3) Show that  $g(p)$  is well-defined. If  $\{p'_n\}$  is another sequence in  $E$  converging to  $p$ , we construct a new sequence  $\{p''_n\}$  based on  $\{p_n\}$  and  $\{p'_n\}$  by

$$p''_n = \begin{cases} p_{\frac{n+1}{2}} & (n \equiv 1 \pmod{2}), \\ p'_{\frac{n}{2}} & (n \equiv 0 \pmod{2}). \end{cases}$$

Clearly  $\{p''_n\}$  also converges to  $p$ . So  $\{f(p''_n)\}$  converges to a single point. Note that  $\{f(p_n)\}$  and  $\{f(p'_n)\}$  are two subsequences of  $\{f(p''_n)\}$ , and thus both subsequences converge to the same point.



(4) Define

$$g(p) = \begin{cases} f(p) & (p \in E), \\ \lim_{n \rightarrow \infty} f(p_n) & (p \notin E) \end{cases}$$

where  $\{p_n\}$  is any sequence in  $E$  converging to  $p$ . Show that  $g$  is uniformly continuous.

- (a) Given any  $\varepsilon > 0$ . Since  $f$  is uniformly continuous on  $E$ , there exists  $\delta > 0$  such that

$$d(f(p), f(q)) < \frac{\varepsilon}{3} < \varepsilon$$

whenever  $d(p, q) < \delta$  and  $p, q \in E$ . We will show that such  $\delta$  also holds for  $g$ . Now given any  $p, q \in X$  with  $d(p, q) < \delta$ .

- (b) By (2), there exists a sequence  $\{p_n\}$  in  $E$  such that  $\lim p_n = p$ . Take an integer  $N_1$  such that

$$d(p_n, p) < \frac{\delta - d(p, q)}{2}$$

whenever  $n \geq N_1$ . Similarly, there exists a sequence  $\{q_n\}$  in  $E$  such that  $\lim q_n = q$ . Take an integer  $N_2$  such that

$$d(q_n, q) < \frac{\delta - d(p, q)}{2}$$

whenever  $n \geq N_2$ . Therefore,

$$\begin{aligned} d(p_n, q_n) &\leq d(p_n, p) + d(p, q) + d(q, q_n) \\ &< \frac{\delta - d(p, q)}{2} + d(p, q) + \frac{\delta - d(p, q)}{2} \\ &= \delta. \end{aligned}$$

whenever  $n \geq N_1$  and  $n \geq N_2$ .

- (c) Since  $\lim f(p_n) = g(p)$ , there is an integer  $N_3$  such that

$$d(f(p_n), g(p)) < \frac{\varepsilon}{3}$$

whenever  $n \geq N_3$ . Similarly, since  $\lim f(q_n) = g(q)$ , there is an integer  $N_4$  such that

$$d(f(q_n), g(q)) < \frac{\varepsilon}{3}$$

whenever  $n \geq N_4$ .

- (d) Take  $N = \max\{N_1, N_2, N_3, N_4\}$ , we have

$$\begin{aligned} d(g(p), f(p_N)) &< \frac{\varepsilon}{3}, \\ d(f(p_N), f(q_N)) &< \frac{\varepsilon}{3}, \\ d(f(q_N), g(q)) &< \frac{\varepsilon}{3}. \end{aligned}$$

Hence

$$\begin{aligned} d(g(p), g(q)) &\leq d(g(p), f(p_N)) + d(f(p_N), f(q_N)) + d(f(q_N), g(q)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

□

**Exercise 4.14 (Brouwer's fixed-point theorem).** Let  $I = [0, 1]$  be the closed unit interval. Suppose  $f$  is continuous mapping of  $I$  into  $I$ . Prove that  $f(x) = x$  for at least one  $x \in I$ .

*Proof (Theorem 4.23).* Let  $g(x) = f(x) - x$  in  $I$ .

- (1)  $g(0) = 0$ . Take  $x = 0$ .
- (2)  $g(1) = 0$ . Take  $x = 1$ .
- (3) Suppose  $g(0) \neq 0$  ( $f(0) \neq 0$ ) and  $g(1) \neq 0$  ( $f(1) \neq 1$ ). Since  $f : I \rightarrow I$ ,  $f(0) > 0$  and  $f(1) < 1$ . That is,  $g(0) > 0$  and  $g(1) < 0$ . Applying the intermediate value theorem (Theorem 4.23), there is a point in  $\xi \in (0, 1)$  such that  $g(\xi) = 0$ . That is,  $f(\xi) = \xi$  for some  $\xi \in (0, 1)$ .

In any case, the conclusion holds. □

**Supplement.** Brouwer's fixed-point theorem.

- (1) In the  $\mathbb{R}^1$ , see Exercise 4.14 itself.
- (2) In the  $\mathbb{R}^2$ , see Exercise 8.29.
- (3) In the  $\mathbb{R}^n$ , every continuous function from a closed ball of a Euclidean space  $\mathbb{R}^n$  into itself has a fixed point (without proof).
- (4) In a Banach space, Schauder fixed-point theorem.

**Exercise 4.15.** Call a mapping of  $X$  into  $Y$  **open** if  $f(V)$  is an open set in  $Y$  whenever  $V$  is an open set in  $X$ . Prove that every continuous open mapping of  $\mathbb{R}^1$  into  $\mathbb{R}^1$  is monotonic.

In fact,  $f$  is strictly monotonic.

*Proof.*

- (1) (Reductio ad absurdum) If  $f$  were not strictly monotonic, then there exist  $a < c < b \in \mathbb{R}^1$  such that

$$f(a) \leq f(c) \geq f(b)$$

or

$$f(a) \geq f(c) \leq f(b).$$

- (2) In any case,  $f$  is a real continuous function on a compact set  $[a, b]$ . By Theorem 4.16, there exists  $p, q \in [a, b]$  such that

$$M = \sup_{x \in [a, b]} f(x) = f(p),$$

$$m = \inf_{x \in [a, b]} f(x) = f(q).$$

- (3) As  $f(a) \leq f(c) \geq f(b)$ , we consider where  $f$  reaches its maximum value  $M$  (by (2)).

(a)  $f(a) = M$  or  $f(b) = M$ . Since  $f(a) \leq f(c) \geq f(b)$ , by the maximality of  $M$ ,  $f(c) = M$  or  $M \in f((a, b))$ .

(b)  $f(a) < M$  and  $f(b) < M$ . Hence  $M \in f((a, b))$  clearly.

In any case,  $M \in f((a, b))$ . Note that  $f((a, b))$  is open since  $f$  is an open mapping and  $(a, b)$  is open.

Since  $M$  is in an open set  $f((a, b))$ , there exists an open neighborhood  $B(M; r) \subseteq f((a, b))$  where  $r > 0$ . Hence

$$M + \frac{r}{64} \in B(M; r) \subseteq f((a, b)),$$

contrary to the maximality of  $M$ .

- (4) As  $f(a) \geq f(c) \leq f(b)$ , we consider where  $f$  reaches its minimum value  $m$  (by (2)). Similar to (3), we can reach a contradiction again.

- (5) By (3)(4), (1) is absurd, and thus  $f$  is strictly monotonic.

□

**Exercise 4.16.** Let  $[x]$  denote the largest integer contained in  $x$ , this is,  $[x]$  is a integer such that  $x - 1 < [x] \leq x$ ; and let  $(x) = x - [x]$  denote the fractional part of  $x$ . What discontinuities do the function  $[x]$  and  $(x)$  have?

*Proof.*

- (1) The function  $[x]$  only has discontinuities at  $x \in \mathbb{Z}$ .

- (a) For any  $p \notin \mathbb{Z}$ , there is an integer  $n$  such that  $n < p < n + 1$ . Given any  $\varepsilon > 0$ , there is a  $\delta = \min\{p - n, (n + 1) - p\} > 0$  such that  $|[x] - [p]| < \varepsilon$  whenever  $|x - p| < \delta$ . In fact,  $|x - p| < \delta$  is equivalent to  $n < x < n + 1$  and therefore  $|[x] - [p]| = |n - n| = 0 < \varepsilon$ .
  - (b) For any  $p \in \mathbb{Z}$ ,  $\lim_{x \rightarrow p^+} [x] = p$  and  $\lim_{x \rightarrow p^-} [x] = p - 1$ .
- (2) The function  $(x)$  only has discontinuities at  $x \in \mathbb{Z}$ .
- (a) Since  $[x]$  is continuous on  $\mathbb{R} - \mathbb{Z}$  and  $x$  is continuous on  $\mathbb{R}$ , especially on  $\mathbb{R} - \mathbb{Z}$ ,  $(x) = x - [x]$  is continuous on  $\mathbb{R} - \mathbb{Z}$ .
  - (b) For any  $p \in \mathbb{Z}$ ,  $\lim_{x \rightarrow p^+} (x) = 0$  and  $\lim_{x \rightarrow p^-} (x) = 1$ .

□

**Exercise 4.17.** Let  $f$  be a real function defined on  $(a, b)$ . Prove that the set of points at which  $f$  has a simple discontinuity is at most countable. (Hint: Let  $E$  be the set on which  $f(x-) < f(x+)$ . With each point  $x$  of  $E$ , associate a triple  $(p, q, r)$  of rational numbers such that

- (a)  $f(x-) < p < f(x+)$ ,
- (b)  $a < q < t < x$  implies  $f(t) < p$ ,
- (c)  $x < t < r < b$  implies  $f(t) > p$ .

The set of all such triples is countable. Show that each triple is associated with at most one point of  $E$ . Deal similarly with the other possible types of simple discontinuities.)

*Proof (Hint).*

- (1) By Definition 4.26, it suffices to show that each following set is at most countable and thus the union of four sets are at most countable (Corollary to Theorem 2.12).
  - (a)  $E_1 = \{x : f(x+) > f(x-)\}$ ,
  - (b)  $E_2 = \{x : f(x+) < f(x-)\}$ ,
  - (c)  $E_3 = \{x : f(x) > f(x+) = f(x-)\}$ ,
  - (d)  $E_4 = \{x : f(x) < f(x+) = f(x-)\}$ ,
- (2) Show that with each point  $x$  of  $E_1$ , we can associate a triple  $(p, q, r)$  of rational numbers such that
  - (a)  $f(x-) < p < f(x+)$ ,
  - (b)  $a < q < t < x$  implies  $f(t) < p$ ,
  - (c)  $x < t < r < b$  implies  $f(t) > p$ .

- (i) For (a), as  $(f(x-), f(x+))$  is a nonempty open set in  $\mathbb{R}$  and  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , there exists a rational number  $p \in (f(x-), f(x+))$  or  $f(x-) < p < f(x+)$ .
- (ii) For (b), since  $f(x-)$  exists, for  $\varepsilon = p - f(x-) > 0$  there exists  $\delta > 0$  such that

$$f(x-) - \varepsilon < f(t) < f(x-) + \varepsilon \quad \text{whenever} \quad a < x - \delta < t < x.$$

So

$$f(t) < p \quad \text{whenever} \quad a < x - \delta < t < x.$$

Since  $(x - \delta, x)$  is a nonempty open set in  $\mathbb{R}$  and  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , there exists a rational number  $q \in (x - \delta, x)$  or  $a < x - \delta < q < x$ . Hence if  $a < q < t < x$  we have  $f(t) < p$ .

- (iii) For (c), similar to (ii). Since  $f(x+)$  exists, for  $\varepsilon = f(x+) - p > 0$  there exists  $\delta > 0$  such that

$$f(x+) - \varepsilon < f(t) < f(x+) + \varepsilon \quad \text{whenever} \quad x < t < x + \delta < b.$$

So

$$f(t) > p \quad \text{whenever} \quad x < t < x + \delta < b.$$

Since  $(x, x + \delta)$  is a nonempty open set in  $\mathbb{R}$  and  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , there exists a rational number  $r \in (x, x + \delta)$  or  $x < r < x + \delta < b$ . Hence if  $x < t < r < b$  we have  $f(t) > p$ .

- (3) Show that each triple in (2) is associated with at most one point of  $E_1$ . (Reductio ad absurdum) If there were another  $x' > x$  such that

- (a)  $f(x'-) < p < f(x'+)$ ,
- (b)  $a < q < t < x'$  implies  $f(t) < p$ ,
- (c)  $x' < t < r < b$  implies  $f(t) > p$ ,

then there exists  $t_0 \in \mathbb{R}$  satisfying  $x < t_0 < x'$  such that  $f(t_0) > p$  (property (c) of  $x$ ) and  $f(t_0) < p$  (property (b) of  $x'$ ), which is absurd. The case  $x' < x$  is similar.

- (4) By (2)(3), the set  $E_1$  is a subset of a countable set

$$\{(p, q, r) : (p, q, r) \in \mathbb{Q}^3\},$$

which is at most countable (Theorem 2.8 and Corollary to Theorem 2.12). Similarly,  $E_2$  is at most countable.

- (5) Similar to (2), for each point  $x$  of  $E_3$ , we can associate a triple  $(p, q, r)$  of rational numbers such that

- (a)  $f(x) > p > f(x+) = f(x-)$ ,
- (b)  $a < q < t < x$  implies  $f(t) < p$ ,

(c)  $x < t < r < b$  implies  $f(t) < p$ .

(6) Show that each triple in (5) is associated with at most one point of  $E_3$ .  
(Reductio ad absurdum) If there were another  $x' > x$  such that

(a)  $f(x') > p > f(x'+) = f(x'-)$ ,

(b)  $a < q < t < x'$  implies  $f(t) < p$ ,

(c)  $x' < t < r < b$  implies  $f(t) < p$ ,

then  $f(x) < p$  (property (b) of  $x'$ ), which is absurd. The case  $x' < x$  is similar.

(7) By (5)(6), the set  $E_3$  is a subset of a countable set

$$\{(p, q, r) : (p, q, r) \in \mathbb{Q}^3\},$$

which is at most countable (Theorem 2.8 and Corollary to Theorem 2.12). Similarly,  $E_4$  is at most countable.

(8) By (4)(7), all  $E_j$  ( $j = 1, 2, 3, 4$ ) are at most countable. Hence the set of points at which  $f$  has a simple discontinuity is at most countable.

□

**Exercise 4.18 (Thomae's function).** Every rational  $x$  can be written in the form  $x = \frac{m}{n}$ , where  $n > 0$ , and  $m$  and  $n$  are integers without any common divisors. When  $x = 0$ , we take  $n = 1$ . Consider the function  $f$  defined on  $\mathbb{R}^1$  by

$$f(x) = \begin{cases} 0 & (x \text{ irrational}), \\ \frac{1}{n} & (x = \frac{m}{n}). \end{cases}$$

Prove that  $f$  is continuous at every irrational point, and that  $f$  has a simple discontinuity at every rational point.

*Proof.*

(1) Show that  $f$  has period 1.

(a) As  $x$  is irrational,  $x + 1$  is irrational too. Hence  $f(x + 1) = 0 = f(x)$ .

(b) As  $x = \frac{m}{n} \in \mathbb{Q}$ , where  $n > 0$ , and  $m$  and  $n$  are integers without any common divisors.

$$x + 1 = \frac{m + n}{n} \in \mathbb{Q},$$

where  $n > 0$ , and  $\gcd(m + n, n) = \gcd(m, n) = 1$ . Hence  $f(x + 1) = \frac{1}{n} = f(x)$ .

In any case,  $f(x + 1) = f(x)$  for any  $x \in \mathbb{R}$ .

(2) Show that  $f(p+) = f(p-) = 0$  for any  $p \in \mathbb{R}$ .

- (a) By (1), we might assume  $p \in [0, 1]$ . For the edge cases  $f(0-)$  and  $f(1+)$ , note that  $f(0-) = f(1-)$  and  $f(1+) = f(0+)$ .  
(b) Now given any point  $p \in [0, 1]$ . Given  $\varepsilon > 0$ , it suffices to find  $\delta > 0$  such that

$$|f(x) - 0| < \varepsilon \quad \text{whenever} \quad p < x < p + \delta.$$

(The case  $f(p-) = 0$  is similar.)

- (c) For such  $\varepsilon > 0$ , there is an integer  $N \geq s$  such that

$$\frac{1}{N} < \varepsilon.$$

Hence

$$E = \left\{ \frac{m}{n} \in [0, 1] : n \leq N \text{ where } \frac{m}{n} \text{ is defined as this exercise} \right\} - \{p\}$$

is finite since  $m$  and  $n$  are satisfying the relation  $0 \leq m \leq n \leq N$  and there are only finitely many pairs of  $(m, n)$ .

- (d) Now we can take

$$\delta = \min\{|q - p| : q \in E\} > 0.$$

As  $p < x < p + \delta$ ,  $|f(x) - 0| = 0 < \varepsilon$  if  $x$  is irrational;

$$|f(x) - 0| < \frac{1}{N} < \varepsilon$$

if  $x$  is rational. (Note that  $x \notin E$  by the construction of  $\delta$  and thus the denominator of  $x$  is  $> N$ .) In any case,  $|f(x) - 0| < \varepsilon$ .

- (3) Since  $f(x) = 0$  if  $x \in \mathbb{R} - \mathbb{Q}$  and  $f(x) \neq 0$  if  $x \in \mathbb{Q}$ ,  $f$  is continuous at every irrational point and  $f$  has a simple discontinuity at every rational point (by (2)).

□

*Note.*

- (1)  $f$  is nowhere differentiable.

- (a) For any rational number, this follows from discontinuity.  
(b) For any irrational number  $x$ , given  $\varepsilon = \frac{1}{64} > 0$ , it suffices to show that for any  $\delta > 0$ , there exists  $x < t < x + \delta$  such that

$$\frac{f(t) - f(x)}{t - x} > \frac{1}{64}.$$

- (c) It is clear to find  $t$  in  $\mathbb{Q}$ . For such  $\delta > 0$ , there is an integer  $n$  such that

$$n > \frac{1}{\delta} > 0.$$

For such  $n > 0$ , there is an integer  $m$  such that

$$m - 1 < nx < m.$$

( $m - 1 \leq nx < m$  is true trivially and notice that the equality does not hold since  $x$  is irrational.) Take

$$t = \frac{m}{n}.$$

So that

$$x < t < x + \frac{1}{n} < x + \delta.$$

- (d) Let  $\gcd(m, n) = g \geq 1$ . Then

$$f(t) = \frac{1}{\frac{n}{g}} = \frac{g}{n} \geq \frac{1}{n}$$

and thus

$$\frac{f(t) - f(x)}{t - x} > \frac{\frac{1}{n}}{\frac{1}{n}} = 1 > \frac{1}{64}.$$

□

- (2)  $f \in \mathcal{R}$  on any bounded interval  $[a, b]$  and  $\int_a^b f dx = 0$ . To prove this, use the similar argument in Exercise 6.6.

**Exercise 4.19.** Suppose  $f$  is a real function with domain  $\mathbb{R}^1$  which has the intermediate value property: If  $f(a) < c < f(b)$ , then  $f(x) = c$  for some  $x$  between  $a$  and  $b$ . Suppose also, for every rational  $r$ , that the set of all  $x$  with  $f(x) = r$  is closed. Prove that  $f$  is continuous. (Hint: If  $x_n \rightarrow x_0$  but  $f(x_n) > r > f(x_0)$  for some  $r$  and all  $n$ , then  $f(t_n) = r$  for some  $t_n$  between  $x_0$  and  $x_n$ ; thus  $t_n \rightarrow x_0$ . Find a contradiction. (N. J. Fine, Amer. Math. Monthly, vol. 73, 1966, p.782.))

*Proof.*

- (1) (Reductio ad absurdum) If  $f$  were not continuous at  $x = x_0$ , then there exists a sequence  $\{x_n\}$  in  $\mathbb{R}^1$  such that  $\{f(x_n)\}$  in  $\mathbb{R}^1$  not converging to  $f(x_0)$ . So there exists  $\varepsilon_0 > 0$  and a subsequence  $\{x_{n'}\}$  of  $\{x_n\}$  such that

$$|f(x_{n'}) - f(x_0)| \geq \varepsilon_0$$



for all  $n'$ . Hence

$$f(x_{n'}) \geq f(x_0) + \varepsilon_0 \quad \text{or} \quad f(x_{n'}) \leq f(x_0) - \varepsilon_0$$

for all  $n'$ .

- (2) Since there are infinitely many  $\{x_{n'}\}$ , there are only two possible cases:
- (a) There is a subsequence  $\{x_{n''}\}$  of  $\{x_{n'}\}$  such that  $f(x_{n''}) \geq f(x_0) + \varepsilon_0$  for all  $n''$ .
  - (b) There is a subsequence  $\{x_{n''}\}$  of  $\{x_{n'}\}$  such that  $f(x_{n''}) \leq f(x_0) - \varepsilon_0$  for all  $n''$ .

In any case, we may assume that there exists  $\varepsilon_0 > 0$  and a sequence  $\{x_n\}$  such that

$$f(x_n) \geq f(x_0) + \varepsilon_0$$

for all  $n$ . (The case that  $f(x_n) \leq f(x_0) - \varepsilon_0$  is similar.)

- (3) Pick any rational number  $r \in (f(x_0), f(x_0) + \varepsilon_0)$  since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ . (In fact,  $\mathbb{Q}$  can be replaced by any dense subset of  $\mathbb{R}$  in this exercise.) Since

$$f(x_n) \geq f(x_0) + \varepsilon_0 > r > f(x_0),$$

by the intermediate value property, there exists some  $t_n$  between  $x_0$  and  $x_n$  such that  $f(t_n) = r$ .

- (4) Let  $E = \{t : f(t) = r\}$  be a closed set (since  $r \in \mathbb{Q}$  and the assumption). Hence  $t_n \in E$  for all  $n$  (by (3)).
- (5)  $\lim t_n = x_0$  since  $\lim x_n = x_0$ . Hence  $x_0$  is a limit point of  $E$ . (Note that  $x_0 \notin E$  since  $f(x_0) < r$  (by (3)).) Since  $E$  is closed,  $x_0 \in E$  or  $f(x_0) = r$ , which is absurd.

□

**Supplement.** Suppose  $f$  is a differentiable real function with domain  $\mathbb{R}^1$ . Suppose for every rational  $r$ , that the set of all  $x$  with  $f'(x) = r$  is closed. Prove that  $f'$  is continuous. Theorem 5.12 (Darboux's theorem) and Exercise 4.19 itself implies all.

**Exercise 4.20.** If  $E$  is a nonempty subset of a metric space  $X$ , define the distance from  $x \in X$  to  $E$  by

$$\rho_E(x) = \inf_{z \in E} d(x, z).$$

- (a) Prove that  $\rho_E(x) = 0$  if and only if  $x \in \overline{E}$ .

(b) Prove that  $\rho_E(x)$  is a uniformly continuous function on  $X$ , by showing that

$$|\rho_E(x) - \rho_E(y)| \leq d(x, y)$$

for all  $x \in X, y \in X$ . (Hint:  $\rho_E(x) \leq d(x, z) \leq d(x, y) + d(y, z)$ , so that

$$\rho_E(x) \leq d(x, y) + \rho_E(y).$$

*Proof of (a).*

$$\begin{aligned} \rho_E(x) > 0 &\iff \inf_{z \in E} d(x, z) > 0 \\ &\iff \exists r > 0 \text{ such that } B(x; r) \cap E = \emptyset \\ &\iff x \notin \overline{E}. \end{aligned}$$

□

*Proof of (b) (Hint).* Given any  $x \in X, y \in X$ . For any  $z \in E$ , we have

$$\rho_E(x) \leq d(x, z) \leq d(x, y) + d(y, z).$$

$\rho_E(x) - d(x, y)$  is a lower bound of the set  $\{d(y, z) : z \in E\}$ . So that

$$\rho_E(x) \leq d(x, y) + \rho_E(y).$$

Similarly,

$$\rho_E(y) \leq d(x, y) + \rho_E(x).$$

Hence

$$|\rho_E(x) - \rho_E(y)| \leq d(x, y),$$

or  $\rho_E(x)$  is uniformly continuous on  $X$  (by taking  $\delta = \varepsilon > 0$  in Definition 4.18).

□

**Exercise 4.21.** Suppose  $K$  and  $F$  are disjoint sets in a metric space  $X$ .  $K$  is compact.  $F$  is closed. Prove that there exists  $\delta > 0$  such that  $d(p, q) > \delta$  if  $p \in K, q \in F$ . (Hint:  $\rho_F$  is a continuous positive function on  $K$ .) Show that the conclusion may fail for two disjoint closed sets if neither is compact.

*Proof.*

(1) Define  $\rho_F : K \rightarrow \mathbb{R}$  by

$$\rho_F(x) = \inf_{q \in F} d(x, q).$$

(2)  $K \cap F = \emptyset$  and  $F = \overline{F}$  is closed,  $\rho_F(x) > 0$  (Exercise 4.20(a)).

- (3) Since  $\rho_F(x)$  is continuous (Exercise 4.20(b)) on a compact set  $K$ , there exists  $\xi \in K$  such that

$$\rho_F(\xi) = \inf_{p \in K} \rho_F(p)$$

(Theorem 4.16). By (2),  $\rho_F(\xi) > 0$ . Write

$$\rho_F(\xi) = \delta > 0.$$

Hence

$$\begin{aligned} \rho_F(\xi) &= \inf_{p \in K} \rho_F(p) = \delta > 0 \\ \iff \rho_F(p) &= \inf_{q \in F} d(p, q) \geq \delta > 0 \quad \forall p \in K \\ \iff d(p, q) &\geq \delta > 0 \quad \forall p \in K \quad \forall q \in F. \end{aligned}$$

- (4) Show that the conclusion may fail for two disjoint closed sets if neither is compact. Let  $X = \mathbb{R}$  equipped with the Euclidean metric,

$$K = \{n : n \in \mathbb{Z}^+\},$$

and

$$F = \left\{ n + \frac{1}{64^n} : n \in \mathbb{Z}^+ \right\}.$$

Both  $K$  and  $F$  are closed and neither is compact. And for any positive integer  $n$ , we have

$$d\left(n, n + \frac{1}{64^n}\right) = \frac{1}{64^n} \rightarrow 0$$

as  $n \rightarrow \infty$ . (Here  $n \in K$  and  $n + \frac{1}{64^n} \in F$ .)

□

**Exercise 4.22.** Let  $A$  and  $B$  be disjoint nonempty closed sets in a metric space  $X$ , and define

$$f(p) = \frac{\rho_A(p)}{\rho_A(p) + \rho_B(p)}, \quad (p \in X).$$

Show that  $f$  is a continuous function on  $X$  whose range lies in  $[0, 1]$ , that  $f(p) = 0$  precisely on  $A$  and  $f(p) = 1$  precisely on  $B$ . This establishes a converse of Exercise 4.3: Every closed set  $A$  contained in  $X$  is  $Z(f)$  for some continuous real  $f$  on  $X$ . Setting

$$V = f^{-1}\left(\left[0, \frac{1}{2}\right)\right), \quad W = f^{-1}\left(\left(\frac{1}{2}, 1\right]\right),$$

show that  $V$  and  $W$  are open and disjoint, and that  $A$  is contained in  $V$ ,  $B$  is contained in  $W$ . (Thus pairs of disjoint closed set in a metric space can be

covered by pairs of disjoint open sets. This property of metric spaces is called normality.)

*Proof.*

(1)

(2)

□

**Exercise 4.23.** A real-valued function  $f$  defined in  $(a, b)$  is said to be **convex** if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

whenever  $a < x < b$ ,  $a < y < b$ ,  $0 < \lambda < 1$ . Prove that every convex function is continuous. Prove that every increasing convex function of a convex function is convex. (For example, if  $f$  is convex, so is  $e^f$ .)

If  $f$  is convex in  $(a, b)$  and if  $a < s < t < u < b$ , show that

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

*Proof.*

(1) Show that  $\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}$ . Since

$$\begin{aligned} t &= \frac{t - s}{u - s}u + \left(1 - \frac{t - s}{u - s}\right)s \\ &= \left(1 - \frac{u - t}{u - s}\right)u + \frac{u - t}{u - s}s \end{aligned}$$

and  $0 < \frac{t - s}{u - s}, \frac{u - t}{u - s} < 1$ , by the convexity of  $f$  we have

$$\begin{aligned} f(t) &\leq \frac{t - s}{u - s}f(u) + \left(1 - \frac{t - s}{u - s}\right)f(s), \\ f(t) &\leq \left(1 - \frac{u - t}{u - s}\right)f(u) + \frac{u - t}{u - s}f(s). \end{aligned}$$

It is equivalent to

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

□

- (2) If  $x, y, x', y'$  are points of  $(a, b)$  with  $x \leq x' < y'$  and  $x < y \leq y'$ , then the chord over  $(x', y')$  has larger slope than the chord over  $(x, y)$ ; that is,

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(y') - f(x')}{y' - x'}.$$

It is a corollary to (1).

- (3) Show that  $f$  is continuous. Let  $[c, d] \subseteq (a, b)$ . Then by (2),

$$\frac{f(c) - f(a)}{c - a} \leq \frac{f(y) - f(x)}{y - x} \leq \frac{f(b) - f(d)}{b - d}$$

for  $x, y$  in  $[c, d]$ . Thus  $|f(y) - f(x)| \leq M|y - x|$  in  $[c, d]$  (where  $M = \max\left(\left|\frac{f(c)-f(a)}{c-a}\right|, \left|\frac{f(b)-f(d)}{b-d}\right|\right)$ ), and so  $f$  is absolutely continuous on each closed subinterval of  $(a, b)$ . Especially,  $f$  is continuous.

- (4) Let  $f$  be a convex function,  $g$  be an increasing convex function, and  $h = g \circ f$ . Show that  $h$  is convex.

$$\begin{aligned} f(\lambda x + (1 - \lambda)y) &\leq \lambda f(x) + (1 - \lambda)f(y), && \text{(Convexity of } f) \\ g(f(\lambda x + (1 - \lambda)y)) &\leq g(\lambda f(x) + (1 - \lambda)f(y)) && \text{(Increasing of } g) \\ &\leq \lambda g(f(x)) + (1 - \lambda)g(f(y)), && \text{(Convexity of } g) \\ h(\lambda x + (1 - \lambda)y) &\leq \lambda h(x) + (1 - \lambda)h(y). \end{aligned}$$

□

**Exercise 4.24.** Assume that  $f$  is a continuous real function defined in  $(a, b)$  such that

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2}$$

for all  $x, y \in (a, b)$ . Prove that  $f$  is convex.

*Proof.*

- (1) Show that

$$f\left(\frac{x_1 + \cdots + x_n}{n}\right) \leq \frac{f(x_1) + \cdots + f(x_n)}{n}$$

whenever  $a < x_i < b$  ( $1 \leq i \leq n$ ). Apply Cauchy induction and use the same argument in proving the AM-GM inequality. As  $n = 1, 2$ , the inequality holds by assumption. Suppose  $n = 2^k$  ( $k \geq 1$ ) the inequality

holds. As  $n = 2^{k+1}$ ,

$$\begin{aligned}
& f\left(\frac{x_1 + \cdots + x_{2^{k+1}}}{2^{k+1}}\right) \\
&= f\left(\frac{1}{2}\left(\frac{x_1 + \cdots + x_{2^k}}{2^k} + \frac{x_{2^k+1} + \cdots + x_{2^{k+1}}}{2^k}\right)\right) \\
&\leq \frac{1}{2}\left(f\left(\frac{x_1 + \cdots + x_{2^k}}{2^k}\right) + f\left(\frac{x_{2^k+1} + \cdots + x_{2^{k+1}}}{2^k}\right)\right) \\
&\leq \frac{1}{2}\left(\frac{f(x_1) + \cdots + f(x_{2^k})}{2^k} + \frac{f(x_{2^k+1}) + \cdots + f(x_{2^{k+1}})}{2^k}\right) \\
&= \frac{f(x_1) + \cdots + f(x_{2^k}) + f(x_{2^k+1}) + \cdots + f(x_{2^{k+1}})}{2^{k+1}} \\
&= \frac{f(x_1) + \cdots + f(x_{2^{k+1}})}{2^{k+1}}.
\end{aligned}$$

As  $n$  is not a power of 2, then it is certainly less than some natural power of 2, say  $n < 2^m$  for some  $m$ . Let

$$x_{n+1} = \cdots = x_{2^m} = \frac{x_1 + \cdots + x_n}{n} = \alpha.$$

Then by the induction hypothesis,

$$\begin{aligned}
f(\alpha) &= f\left(\frac{x_1 + \cdots + x_n + \alpha + \cdots + \alpha}{2^m}\right) \\
&\leq \frac{f(x_1) + \cdots + f(x_n) + f(\alpha) + \cdots + f(\alpha)}{2^m} \\
&\leq \frac{f(x_1) + \cdots + f(x_n) + (2^m - n)f(\alpha)}{2^m}, \\
2^m f(\alpha) &\leq f(x_1) + \cdots + f(x_n) + (2^m - n)f(\alpha), \\
nf(\alpha) &\leq f(x_1) + \cdots + f(x_n),
\end{aligned}$$

$$\text{or } f\left(\frac{1}{n}(x_1 + \cdots + x_n)\right) \leq \frac{1}{n}(f(x_1) + \cdots + f(x_n)).$$

(2) Hence,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for any rational  $\lambda$  in  $(0, 1)$ . (Given any positive integers  $p < q$ , put  $n = q$ ,  $x_1 = \cdots = x_p = x$  and  $x_{p+1} = \cdots = x_n = y$  in (1).)

(3) Given any real  $\lambda \in (0, 1)$ , there is a sequence of rational numbers  $\{r_n\} \subseteq (0, 1)$  such that  $r_n \rightarrow \lambda$ . By (2),

$$f(r_n x + (1 - r_n)y) \leq r_n f(x) + (1 - r_n)f(y)$$

for any rational  $r_n$  in  $(0, 1)$ . Taking limit on the both sides and using the continuity of  $f$ , we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

□

*Proof (Reductio ad absurdum).* If  $f$  were not convex, then there is a subinterval  $[c, d] \subseteq (a, b)$  such that

$$\frac{f(d) - f(c)}{d - c} < \frac{f(x_0) - f(c)}{x_0 - c}$$

for some  $x_0 \in [c, d]$ . Let

$$g(x) = f(x) - f(c) - \frac{f(d) - f(c)}{d - c}(x - c)$$

for  $x \in [c, d]$ . Therefore,

- (1)  $g(x)$  is continuous and midpoint convex.
- (2)  $g(c) = g(d) = 0$ .
- (3) Let  $M = \sup\{g(x) : x \in [c, d]\}$ .  $\infty > M > 0$  due to the continuity of  $g$  and the existence of  $x_0$ . And let  $\xi = \inf\{x \in [c, d] : g(x) = M\}$ . By the continuity of  $g$ ,  $g(\xi) = M$ .  $\xi \in (c, d)$  by (2).
- (4) Since  $(c, d)$  is open, there is  $h > 0$  such that  $(\xi - h, \xi + h) \subseteq (c, d)$ . By the minimality of  $\xi$  and  $M$ ,  $g(\xi - h) < g(\xi)$  and  $g(\xi + h) \leq g(\xi)$ .

Therefore,

$$\begin{aligned} g(\xi - h) + g(\xi + h) &< 2g(\xi), \\ \frac{g(\xi - h) + g(\xi + h)}{2} &< g(\xi) \\ &= g\left(\frac{(\xi - h) + (\xi + h)}{2}\right), \end{aligned}$$

contrary to the midpoint convexity of  $g$ . □

The result becomes false if “continuity of  $f$ ” is omitted.

**Exercise 4.25.** If  $A \subset \mathbb{R}^k$  and  $B \subset \mathbb{R}^k$ , define  $A + B$  to be the set of all sums  $\mathbf{x} + \mathbf{y}$  with  $\mathbf{x} \in A$ ,  $\mathbf{y} \in B$ .

- (a) If  $K$  is compact and  $C$  is closed in  $\mathbb{R}^k$ , prove that  $K + C$  is closed. (Hint: Take  $\mathbf{z} \notin K + C$ , put  $F = \mathbf{z} - C$ , the set of all  $\mathbf{z} - \mathbf{y}$  with  $\mathbf{y} \in C$ . Then  $K$  and  $F$  are disjoint. Choose  $\delta$  as in Exercise 4.21. Show that the open ball with center  $\mathbf{z}$  and radius  $\delta$  does not intersect  $K + C$ .)
- (b) Let  $\alpha$  be an irrational real number. Let  $C_1$  be the set of all integers, let  $C_2$  be the set of all  $n\alpha$  with  $n \in C_1$ . Show that  $C_1$  and  $C_2$  are closed subsets of  $\mathbb{R}^1$  whose sum  $C_1 + C_2$  is not closed, by showing that  $C_1 + C_2$  is a countable dense subset of  $\mathbb{R}^1$ .

*Proof.*

(1)

(2)

□

**Exercise 4.26.** Suppose  $X, Y, Z$  are metric spaces, and  $Y$  is compact. Let  $f$  map  $X$  into  $Y$ , let  $g$  be a continuous one-to-one mapping of  $Y$  into  $Z$ , and put  $h(x) = g(f(x))$  for  $x \in X$ .

*Prove that  $f$  is uniformly continuous if  $h$  is uniformly continuous. (Hint:  $g^{-1}$  has compact domain  $g(Y)$ , and  $f(x) = g^{-1}(h(x))$ .)*

*Prove also that  $f$  is continuous if  $h$  is continuous.*

*Show (by modifying Example 4.21, or by finding a different example) that the compactness of  $Y$  cannot be omitted from the hypotheses, even when  $X$  and  $Z$  are compact.*

*Proof.*

(1)

(2)

□