

## Chapter 9: Functions of Several Variables

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**Exercise 9.1.** If  $S$  is a nonempty subset of a vector space  $X$ , prove (as asserted in Section 9.1) that the span of  $S$  is a vector space.

Denote the span of  $S$  by  $\text{span}(S)$ .

*Proof.*

- (1) Since  $S \neq \emptyset$ , there is  $\mathbf{z} \in S$ . So  $1\mathbf{z} = \mathbf{z} \in \text{span}(S) \neq \emptyset$ . (In fact,  $\text{span}(S) \supseteq S$ .)
- (2) If  $\mathbf{x}, \mathbf{y} \in \text{span}(S)$ , then there exist elements  $\mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{y}_1, \dots, \mathbf{y}_n \in S$  and scalars  $a_1, \dots, a_m, b_1, \dots, b_n$  such that

$$\mathbf{x} = a_1\mathbf{x}_1 + \dots + a_m\mathbf{x}_m,$$

$$\mathbf{y} = b_1\mathbf{y}_1 + \dots + b_n\mathbf{y}_n.$$

Then

$$\mathbf{x} + \mathbf{y} = a_1\mathbf{x}_1 + \dots + a_m\mathbf{x}_m + b_1\mathbf{y}_1 + \dots + b_n\mathbf{y}_n$$

is a linear combination of the elements of  $S$ . For any scalar  $c$ ,

$$c\mathbf{x} = (ca_1)\mathbf{x}_1 + \dots + (ca_m)\mathbf{x}_m$$

is again linear combination of the elements of  $S$ .

- (3) By (1)(2),  $\text{span}(S)$  is a vector space.

□

*Note.* Any subspace of  $X$  that contains  $S$  must also contain  $\text{span}(S)$ .

**Exercise 9.2.** Prove (as asserted in Section 9.6) that  $BA$  is linear if  $A$  and  $B$  are linear transformations. Prove also that  $A^{-1}$  is linear and invertible if  $A$  is invertible.

*Proof.* Use the notation in Definitions 9.6.

- (1) Show that  $BA$  is linear if  $A$  and  $B$  are linear transformations. Let  $X, Y, Z$  be vector spaces,  $A \in L(X, Y)$  and  $B \in L(Y, Z)$ .

(a) Given any  $\mathbf{x}_1, \mathbf{x}_2 \in X$ .

$$\begin{aligned}
(BA)(\mathbf{x}_1 + \mathbf{x}_2) &= B(A(\mathbf{x}_1 + \mathbf{x}_2)) \\
&= B(A\mathbf{x}_1 + A\mathbf{x}_2) && (A \text{ is a linear transformation}) \\
&= B(A\mathbf{x}_1) + B(A\mathbf{x}_2) && (B \text{ is a linear transformation}) \\
&= (BA)\mathbf{x}_1 + (BA)\mathbf{x}_2.
\end{aligned}$$

(b) For any  $\mathbf{x} \in X$  and scalar  $c$ ,

$$\begin{aligned}
(BA)(c\mathbf{x}) &= B(A(c\mathbf{x})) \\
&= B(cA\mathbf{x}) && (A \text{ is a linear transformation}) \\
&= cB(A\mathbf{x}) && (B \text{ is a linear transformation}) \\
&= c(BA)\mathbf{x}.
\end{aligned}$$

By (a)(b),  $BA \in L(X, Z)$ .

(2) Show that  $A^{-1}$  is linear if  $A$  is invertible.

(a) Given any  $\mathbf{y}_1, \mathbf{y}_2 \in X$ . Since  $A$  is surjective, there exist  $\mathbf{x}_1, \mathbf{x}_2 \in X$  such that

$$\begin{aligned}
\mathbf{y}_1 &= A\mathbf{x}_1 \\
\mathbf{y}_2 &= A\mathbf{x}_2.
\end{aligned}$$

So

$$\begin{aligned}
A^{-1}\mathbf{y}_1 &= A^{-1}(A\mathbf{x}_1) = \mathbf{x}_1 \\
A^{-1}\mathbf{y}_2 &= A^{-1}(A\mathbf{x}_2) = \mathbf{x}_2
\end{aligned}$$

(by Definitions 9.4). Hence

$$\begin{aligned}
A^{-1}(\mathbf{y}_1 + \mathbf{y}_2) &= A^{-1}(A\mathbf{x}_1 + A\mathbf{x}_2) \\
&= A^{-1}(A(\mathbf{x}_1 + \mathbf{x}_2)) && (A \text{ is a linear transformation}) \\
&= \mathbf{x}_1 + \mathbf{x}_2 && (\text{Definitions 9.4}) \\
&= A^{-1}\mathbf{y}_1 + A^{-1}\mathbf{y}_2.
\end{aligned}$$

(b) For any  $\mathbf{y} \in X$  and scalar  $c$ , there is a corresponding  $\mathbf{x} \in X$  such that  $\mathbf{y} = A\mathbf{x}$  since  $A$  is surjective. So  $A^{-1}\mathbf{y} = \mathbf{x}$  by Definition 9.4. Hence

$$\begin{aligned}
A^{-1}(c\mathbf{y}) &= A^{-1}(cA\mathbf{x}) \\
&= A^{-1}(A(c\mathbf{x})) && (A \text{ is a linear transformation}) \\
&= c\mathbf{x} && (\text{Definitions 9.4}) \\
&= cA^{-1}\mathbf{y}.
\end{aligned}$$

By (a)(b),  $A^{-1} \in L(X)$ .

(3) *Show that  $A^{-1}$  is invertible if  $A$  is invertible.* It suffices to show that  $A^{-1}$  is injective and surjective.

(a) *Show that  $A^{-1}$  is injective.* Given any  $\mathbf{y}_1, \mathbf{y}_2 \in X$ . Since  $A$  is surjective, there exist  $\mathbf{x}_1, \mathbf{x}_2 \in X$  such that

$$\begin{aligned}\mathbf{y}_1 &= A\mathbf{x}_1 \\ \mathbf{y}_2 &= A\mathbf{x}_2.\end{aligned}$$

Suppose  $A^{-1}\mathbf{y}_1 = A^{-1}\mathbf{y}_2$ . So  $A^{-1}(A\mathbf{x}_1) = A^{-1}(A\mathbf{x}_2)$ , or  $\mathbf{x}_1 = \mathbf{x}_2$ , or  $\mathbf{y}_1 = A\mathbf{x}_1 = A\mathbf{x}_2 = \mathbf{y}_2$ .

(b) *Show that  $A^{-1}$  is surjective.* For any  $\mathbf{x} \in X$ , there exists  $A\mathbf{x} \in X$  such that  $A^{-1}(A\mathbf{x}) = \mathbf{x}$  by Definitions 9.4.

□

**Exercise 9.3.** Assume  $A \in L(X, Y)$  and  $A\mathbf{x} = \mathbf{0}$  only when  $\mathbf{x} = \mathbf{0}$ . Prove that  $A$  is then 1-1.

*Proof.* Suppose  $A\mathbf{x} = A\mathbf{y}$ . Since  $A$  is a linear transformation,  $A(\mathbf{x} - \mathbf{y}) = A\mathbf{x} - A\mathbf{y} = \mathbf{0}$ . By assumption,  $\mathbf{x} - \mathbf{y} = \mathbf{0}$  or  $\mathbf{x} = \mathbf{y}$ . □

**Exercise 9.4.** Prove (as asserted in Section 9.30) that null spaces and ranges of linear transformations are vector spaces.

*Proof.* Use the notation in Definitions 9.30. Suppose  $X, Y$  are vector spaces, and  $A \in L(X, Y)$ , as in Definition 9.6.

(1) *Show that  $\mathcal{N}(A)$  is a vector space in  $X$ .*

(a) Note that  $\mathbf{0} \in X$ . Since  $A\mathbf{0} = \mathbf{0}$ ,  $\mathbf{0} \in \mathcal{N}(A) \neq \emptyset$ .

(b) Suppose  $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{N}(A)$ . Then

$$\begin{aligned}A(\mathbf{x}_1 + \mathbf{x}_2) &= A\mathbf{x}_1 + A\mathbf{x}_2 && (A \text{ is a linear transformation}) \\ &= \mathbf{0} + \mathbf{0} && (\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{N}(A)) \\ &= \mathbf{0}.\end{aligned}$$

So  $\mathbf{x}_1 + \mathbf{x}_2 \in \mathcal{N}(A)$ .

(c) Suppose  $\mathbf{x} \in \mathcal{N}(A)$  and  $c$  is a scalar. Then

$$\begin{aligned}A(c\mathbf{x}) &= cA\mathbf{x} && (A \text{ is a linear transformation}) \\ &= c\mathbf{0} && (\mathbf{x} \in \mathcal{N}(A)) \\ &= \mathbf{0}.\end{aligned}$$

So  $c\mathbf{x} \in \mathcal{N}(A)$ .

By (a)(b)(c),  $\mathcal{N}(A)$  is a vector space.

(2) Show that  $\mathcal{R}(A)$  is a vector space in  $Y$ .

(a) Note that  $\mathbf{0} \in X$ . So  $A\mathbf{0} = \mathbf{0} \in \mathcal{R}(A) \neq \emptyset$ .

(b) Suppose  $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{R}(A)$ . Then there exist  $\mathbf{x}_1, \mathbf{x}_2 \in X$  such that  $A\mathbf{x}_1 = \mathbf{y}_1$  and  $A\mathbf{x}_2 = \mathbf{y}_2$ . Hence

$$\begin{aligned}\mathbf{y}_1 + \mathbf{y}_2 &= A\mathbf{x}_1 + A\mathbf{x}_2 \\ &= A(\mathbf{x}_1 + \mathbf{x}_2) \quad (A \text{ is a linear transformation}).\end{aligned}$$

So  $\mathbf{y}_1 + \mathbf{y}_2 \in \mathcal{R}(A)$ .

(c) Suppose  $\mathbf{y} \in \mathcal{R}(A)$  and  $c$  is a scalar. Then there exists  $\mathbf{x} \in X$  such that  $A\mathbf{x} = \mathbf{y}$ . Hence

$$\begin{aligned}c\mathbf{y} &= cA\mathbf{x} \\ &= A(c\mathbf{x}) \quad (A \text{ is a linear transformation}).\end{aligned}$$

So  $c\mathbf{y} \in \mathcal{R}(A)$ .

By (a)(b)(c),  $\mathcal{R}(A)$  is a vector space.

□

**Exercise 9.5.** Prove that to every  $A \in L(\mathbb{R}^n, \mathbb{R}^1)$  corresponds a unique  $\mathbf{y} \in \mathbb{R}^n$  such that  $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$ . Prove also that  $\|A\| = \|\mathbf{y}\|$ . (Hint: Under certain conditions, equality holds in the Schwarz inequality.)

*Proof.*

(1) Recall that  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  is the standard basis of  $\mathbb{R}^n$  (Definitions 9.1).

Given any  $\mathbf{x} \in \mathbb{R}^n$ , write  $\mathbf{x} = (x_1, \dots, x_n)$  as  $\mathbf{x} = \sum x_j \mathbf{e}_j$ .

(2) Show that  $\mathbf{y}$  exists. Since  $A$  is a linear transformation,

$$\begin{aligned}A\mathbf{x} &= A\left(\sum x_j \mathbf{e}_j\right) \\ &= \sum x_j A\mathbf{e}_j \\ &= (x_1, \dots, x_n) \cdot (A\mathbf{e}_1, \dots, A\mathbf{e}_n) \\ &= \mathbf{x} \cdot \sum (A\mathbf{e}_j) \mathbf{e}_j.\end{aligned}$$

Define  $\mathbf{y} = \sum (A\mathbf{e}_j) \mathbf{e}_j \in \mathbb{R}^n$  so that  $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$ .

(3) Show that  $\mathbf{y}$  is unique. Suppose there exists some  $\mathbf{z} \in \mathbb{R}^n$  such that  $A\mathbf{x} = \mathbf{x} \cdot \mathbf{z}$ . So

$$\begin{aligned}0 &= A\mathbf{x} - A\mathbf{x} \\ &= \mathbf{x} \cdot \mathbf{y} - \mathbf{x} \cdot \mathbf{z} \\ &= \mathbf{x} \cdot (\mathbf{y} - \mathbf{z})\end{aligned}$$

for any  $\mathbf{x} \in \mathbb{R}^n$ . In particular, take  $\mathbf{x} = \mathbf{y} - \mathbf{z} \in \mathbb{R}^n$  to get

$$0 = (\mathbf{y} - \mathbf{z}) \cdot (\mathbf{y} - \mathbf{z}) = |\mathbf{y} - \mathbf{z}|^2$$

or  $\mathbf{y} - \mathbf{z} = \mathbf{0}$  or  $\mathbf{y} = \mathbf{z}$ .

(4) *Show that  $\|A\| = |\mathbf{y}|$ .* By the Schwarz inequality (Theorem 1.37(d)),

$$|A\mathbf{x}| = |\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}| \leq |\mathbf{y}|$$

as  $|\mathbf{x}| \leq 1$ . Take the sup over all  $|\mathbf{x}| \leq 1$  to get

$$\|A\| \leq |\mathbf{y}|.$$

If  $\mathbf{y} = \mathbf{0}$ , then  $\|A\| = |\mathbf{y}| = 0$ . If  $\mathbf{y} \neq \mathbf{0}$ , then the equality holds when  $\mathbf{x} = \frac{\mathbf{y}}{|\mathbf{y}|} \in \mathbb{R}^n$ . (Here  $|\mathbf{x}| = 1$ .)

□

**Exercise 9.6.** *If  $f(0,0) = 0$  and*

$$f(x,y) = \frac{xy}{x^2 + y^2} \quad \text{if } (x,y) \neq (0,0),$$

*prove that  $(D_1f)(x,y)$  and  $(D_2f)(x,y)$  exist at every point of  $\mathbb{R}^2$ , although  $f$  is not continuous at  $(0,0)$ .*

*Proof.*

(1) *Show that*

$$(D_1f)(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0), \\ \frac{y(y^2 - x^2)}{(x^2 + y^2)^2} & \text{if } (x,y) \neq (0,0). \end{cases}$$

Write

$$\begin{aligned} (D_1f)(x,y) &= \lim_{t \rightarrow 0} \frac{f((x,y) + t(1,0)) - f(x,y)}{t} \\ &= \lim_{t \rightarrow 0} \frac{f(x+t,y) - f(x,y)}{t}. \end{aligned}$$

If  $(x,y) = (0,0)$ ,

$$(D_1f)(0,0) = \lim_{t \rightarrow 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t \rightarrow 0} \frac{0 - 0}{t} = 0.$$

If  $(x, y) \neq (0, 0)$ ,

$$\begin{aligned}
 (D_1 f)(x, y) &= \lim_{t \rightarrow 0} \frac{f(x+t, y) - f(x, y)}{t} \\
 &= \lim_{t \rightarrow 0} \frac{\frac{(x+t)y}{(x+t)^2 + y^2} - \frac{xy}{x^2 + y^2}}{t} \\
 &= \lim_{t \rightarrow 0} \frac{y(y^2 - x^2) - txy}{((x+t)^2 + y^2)(x^2 + y^2)} \\
 &= \frac{y(y^2 - x^2)}{(x^2 + y^2)^2}.
 \end{aligned}$$

(2) Show that

$$(D_2 f)(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0), \\ \frac{x(x^2 - y^2)}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0). \end{cases}$$

Similar to (1).

(3) Show that  $f$  is not continuous at  $(0, 0)$ . Note that

$$\lim_{n \rightarrow \infty} f\left(\frac{1}{n}, \frac{1}{n}\right) = \lim_{n \rightarrow \infty} \frac{\frac{1}{n} \cdot \frac{1}{n}}{\frac{1}{n^2} + \frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{1}{2} = \frac{1}{2}$$

and

$$\lim_{n \rightarrow \infty} f\left(\frac{1}{n}, 0\right) = \lim_{n \rightarrow \infty} \frac{0}{\frac{1}{n^2} + 0} = \lim_{n \rightarrow \infty} 0 = 0.$$

Hence the limit  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  does not exist.

□

**Exercise 9.7.** Suppose that  $f$  is a real-valued function defined in an open set  $E \subseteq \mathbb{R}^n$ , and that the partial derivatives  $D_1 f, \dots, D_n f$  are bounded in  $E$ . Prove that  $f$  is continuous in  $E$ . (Hint: Proceed as in the proof of Theorem 9.21.)

*Proof.*

(1) Since  $D_j f$  is bounded in  $E$ , there is a real number  $M_j$  such that  $|D_j f| \leq M_j$  in  $E$ . Take  $M = \max_{1 \leq j \leq n} M_j$  so that  $|D_j f| \leq M$  in  $E$  for all  $1 \leq j \leq n$ .

(2) Fix  $\mathbf{x} \in E$  and  $\varepsilon > 0$ . Since  $E$  is open, there is an open neighborhood

$$B(\mathbf{x}; r) = \{\mathbf{x} + \mathbf{h} \in E : |\mathbf{h}| < r\} \subseteq E$$

with

$$0 < r < \frac{\varepsilon}{n(M+1)}.$$

- (3) Write  $\mathbf{h} = \sum h_j \mathbf{e}_j$ ,  $|\mathbf{h}| < r$ , put  $\mathbf{v}_0 = \mathbf{0}$ , and  $\mathbf{v}_k = h_1 \mathbf{e}_1 + \cdots + h_k \mathbf{e}_k$  for  $1 \leq k \leq n$ . Then

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = \sum_{j=1}^n [f(\mathbf{x} + \mathbf{v}_j) - f(\mathbf{x} + \mathbf{v}_{j-1})].$$

Since  $|\mathbf{v}_k| < r$  for  $1 \leq k \leq n$  and since  $B(\mathbf{x}; r)$  is convex, the open interval with end points  $\mathbf{x} + \mathbf{v}_{j-1}$  and  $\mathbf{x} + \mathbf{v}_j$  lie in  $B(\mathbf{x}; r)$ . Since  $\mathbf{v}_j = \mathbf{v}_{j-1} + h_j \mathbf{e}_j$ , the mean value theorem (Theorem 5.10) show that

$$f(\mathbf{x} + \mathbf{v}_j) - f(\mathbf{x} + \mathbf{v}_{j-1}) = h_j (D_j f)(\mathbf{x} + \mathbf{v}_{j-1} + \theta_j h_j \mathbf{e}_j)$$

for some  $\theta_j \in (0, 1)$ .

- (4) Note that  $|h_j| \leq |\mathbf{h}| < r < \frac{\varepsilon}{n(M+1)}$ . Hence

$$\begin{aligned} |f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x})| &\leq \sum_{j=1}^n |f(\mathbf{x} + \mathbf{v}_j) - f(\mathbf{x} + \mathbf{v}_{j-1})| \\ &= \sum_{j=1}^n |h_j| |(D_j f)(\mathbf{x} + \mathbf{v}_{j-1} + \theta_j h_j \mathbf{e}_j)| \\ &\leq \sum_{j=1}^n \frac{\varepsilon}{n(M+1)} \cdot M \\ &< \varepsilon \end{aligned}$$

as  $|\mathbf{h}| < r < \frac{\varepsilon}{n(M+1)}$ . Hence  $f$  is continuous at all  $\mathbf{x} \in E$ .

□

**Exercise 9.8.** Suppose that  $f$  is a differentiable real function in an open set  $E \subseteq \mathbb{R}^n$ , and that  $f$  has a local maximum at a point  $\mathbf{x} \in E$ . Prove that  $f'(\mathbf{x}) = 0$ .

*Proof (Theorem 5.8).*

- (1) Apply Theorem 5.8 to each  $D_j f$  for  $1 \leq j \leq n$ . Since  $f$  has a local maximum at a point  $\mathbf{x} \in E$ , there is an open neighborhood  $B(\mathbf{x}; r)$  of  $\mathbf{x}$  in  $E$  such that

$$f(\mathbf{y}) \leq f(\mathbf{x})$$

for all  $\mathbf{y} \in B(\mathbf{x}; r)$ . Therefore,

$$f(\mathbf{x} + t\mathbf{e}_j) \leq f(\mathbf{x})$$

for all  $|t| < r$  and  $1 \leq j \leq n$ , or  $t \mapsto f(\mathbf{x} + t\mathbf{e}_j)$  has a local maximum at a point  $t = 0 \in (-r, r)$ .

- (2) Since  $f$  is differentiable in  $E$ , each partial derivatives  $D_j f$  exist (Theorem 9.21). Hence Theorem 5.8 implies that  $(D_j f)(\mathbf{x}) = 0$  for all  $1 \leq j \leq n$ . So

$$f'(\mathbf{x}) = [(D_1 f)(\mathbf{x}) \cdots (D_n f)(\mathbf{x})] = [0 \cdots 0] = 0$$

(as the zero matrix).

□

**Exercise 9.9.** If  $\mathbf{f}$  is a differentiable mapping of a connected open set  $E \subseteq \mathbb{R}^n$ , and if  $\mathbf{f}'(\mathbf{x}) = 0$  for every  $\mathbf{x} \in E$ , prove that  $\mathbf{f}$  is a constant in  $E$ .

*Proof.*

- (1) Show that  $\mathbf{f}$  is **locally constant**. Given any  $\mathbf{x} \in E$ . Since  $E$  is open, there exists an open neighborhood  $B(\mathbf{x}; r)$  of  $\mathbf{x}$  such that  $B(\mathbf{x}; r) \subseteq E$  and  $r > 0$ . Corollary to Theorem 9.19 implies that  $\mathbf{f}$  is a constant on  $B(\mathbf{x}; r)$ , that is,  $\mathbf{f}$  is locally constant.
- (2) Show that  $\mathbf{f}$  is constant if  $\mathbf{f}$  is locally constant in a connected set  $E \subseteq \mathbb{R}^n$ . Might assume that  $E \neq \emptyset$ . (Otherwise there is nothing to do.) Take some  $\mathbf{x}_0 \in E$ .

(a) Let

$$U = \{\mathbf{y} \in E : \mathbf{f}(\mathbf{y}) = \mathbf{f}(\mathbf{x}_0)\}.$$

- (b)  $U$  is open since  $\mathbf{f}$  is locally constant (by (1)). (Take any  $\mathbf{y} \in U$ . Since  $\mathbf{f}$  is locally constant, there is an open neighborhood  $B(\mathbf{y}) \subseteq E$  of  $\mathbf{y}$  such that  $\mathbf{f}(\mathbf{z}) = \mathbf{f}(\mathbf{y}) = \mathbf{f}(\mathbf{x}_0)$  whenever  $\mathbf{z} \in B(\mathbf{y})$ . So that  $B(\mathbf{y}) \subseteq U$ , or  $U$  is open.)
- (c) Besides, since  $\mathbf{f}$  is continuous (Remarks 9.13(c)), the set  $U$  is closed. (The proof is the same as Proof (Definition 2.18(d)) in Exercise 4.3.)
- (d) So  $U$  is open and closed. Write  $E = U \cup (E - U)$ . Here  $U$  and  $E - U$  are both open and closed. Hence  $U \cap \overline{E - U} = U \cap (E - U) = \emptyset$  and  $\overline{U} \cap (E - U) = U \cap (E - U) = \emptyset$ . Note that  $\mathbf{x}_0 \in U \neq \emptyset$ . By the connectedness of  $E$ ,  $E - U = \emptyset$ , or  $E = U$ , or  $\mathbf{f}$  is constant on  $E$ .

*Note.* The only subsets of a connected set  $E$  which are both open and closed are  $E$  and  $\emptyset$ .

□

**Exercise 9.10.** If  $f$  is a real function defined in a convex open set  $E \subseteq \mathbb{R}^n$ , such that  $(D_1 f)(\mathbf{x}) = 0$  for every  $\mathbf{x} \in E$ , prove that  $f(\mathbf{x})$  depends only on  $x_2, \dots, x_n$ . Show that the convexity of  $E$  can be replaced by a weaker condition, but that some condition is required. For example, if  $n = 2$  and  $E$  is shaped like



a horseshoe, the statement may be false.

*Proof.*

- (1) It suffices to show that

$$f(a, x_2, \dots, x_n) = f(b, x_2, \dots, x_n)$$

whenever  $\mathbf{x} = (a, x_2, \dots, x_n) \in E$  and  $\mathbf{y} = (b, x_2, \dots, x_n) \in E$  if  $(D_1 f)(\mathbf{x}) = 0$  in the convex open set  $E$ .

- (2) Might assume that  $a < b$ . Since  $g : t \mapsto f(t, x_2, \dots, x_n)$  is a real continuous function on  $[a, b]$  (by the openness of  $E$ ) and differentiable in  $(a, b)$  (by the existence of  $D_1 f$ ),

$$g(b) - g(a) = (b - a)g'(\xi)$$

for some  $\xi \in (a, b)$ . Note that

$$g'(\xi) = (D_1 f)(\xi, x_2, \dots, x_n) = 0$$

by assumption.  $g(b) = g(a)$  or  $f(a, x_2, \dots, x_n) = f(b, x_2, \dots, x_n)$ .

- (3) (2) shows that the convexity of  $E$  can be replaced by a weaker condition that  $E \subseteq \mathbb{R}^n$  is convex in the first coordinate, say  $E$  is open and

$$\lambda \mathbf{x} + (1 - \lambda) \mathbf{y} = (\lambda a + (1 - \lambda)b, x_2, \dots, x_n) \in E$$

whenever  $\mathbf{x} = (a, x_2, \dots, x_n) \in E$ ,  $\mathbf{y} = (b, x_2, \dots, x_n) \in E$ , and  $0 < \lambda < 1$ .

- (4) Show that the convexity of  $E$  or some weaker condition is required. Define  $f(x, y) = \operatorname{sgn}(x)$  on  $E = \{(x, y) \in \mathbb{R}^2 : x \neq 0\}$ .  $E$  is open and  $(D_1 f)(x, y) = 0$  in  $E$ . Note that  $f(1989, 0) = 1$  and  $f(-64, 0) = -1$ , and thus  $f(x, y)$  does not depend only on  $y = 0$ .

□

**Exercise 9.11.** If  $f$  and  $g$  are differentiable real functions in  $\mathbb{R}^n$ , prove that

$$\nabla(fg) = f\nabla g + g\nabla f$$

and that

$$\nabla\left(\frac{1}{f}\right) = -\frac{1}{f^2}\nabla f$$

whenever  $f \neq 0$ .

*Proof.* Recall Example 9.18:

$$(\nabla(f))(\mathbf{x}) = \sum_{i=1}^n (D_i f)(\mathbf{x}) \mathbf{e}_i.$$

(1) Show that  $\nabla(fg) = f\nabla g + g\nabla f$ . For any  $\mathbf{x} \in \mathbb{R}^n$ ,

$$\begin{aligned}
(\nabla(fg))(\mathbf{x}) &= \sum_{i=1}^n (D_i(fg))(\mathbf{x}) \mathbf{e}_i \\
&= \sum_{i=1}^n (g(D_i f) + f(D_i g))(\mathbf{x}) \mathbf{e}_i && \text{(Theorem 5.3(b))} \\
&= \sum_{i=1}^n [g(\mathbf{x})(D_i f)(\mathbf{x}) + f(\mathbf{x})(D_i g)(\mathbf{x})] \mathbf{e}_i \\
&= g(\mathbf{x}) \sum_{i=1}^n (D_i f)(\mathbf{x}) \mathbf{e}_i + f(\mathbf{x}) \sum_{i=1}^n (D_i g)(\mathbf{x}) \mathbf{e}_i \\
&= g(\mathbf{x})(\nabla f)(\mathbf{x}) + f(\mathbf{x})(\nabla g)(\mathbf{x}) \\
&= (f\nabla g + g\nabla f)(\mathbf{x}).
\end{aligned}$$

(2) Show that

$$\nabla \left( \frac{1}{f} \right) = -\frac{1}{f^2} \nabla f$$

whenever  $f \neq 0$ . Note that  $\nabla(1) = 0$  since

$$\nabla(1)(\mathbf{x}) = \sum (D_i 1)(\mathbf{x}) \mathbf{e}_i = \sum (0)(\mathbf{x}) \mathbf{e}_i = \sum 0 \mathbf{e}_i = 0.$$

Hence as  $f \neq 0$ , we have

$$\begin{aligned}
0 &= \nabla(1) \\
&= \nabla \left( f \frac{1}{f} \right) && (f \neq 0) \\
&= f \nabla \left( \frac{1}{f} \right) + \frac{1}{f} \nabla f && ((1)),
\end{aligned}$$

$$\text{or } \nabla \left( \frac{1}{f} \right) = -\frac{1}{f^2} \nabla f.$$

□

**Exercise 9.12.** Fix two real numbers  $a$  and  $b$ ,  $0 < a < b$ . Define a mapping  $\mathbf{f} = (f_1, f_2, f_3)$  of  $\mathbb{R}^2$  into  $\mathbb{R}^3$  by

$$\begin{aligned}
f_1(s, t) &= (b + a \cos s) \cos t \\
f_2(s, t) &= (b + a \cos s) \sin t \\
f_3(s, t) &= a \sin s.
\end{aligned}$$

Describe the range  $K$  of  $\mathbf{f}$ . (It is a certain compact subset of  $\mathbb{R}^3$ .)

- (a) Show that there are exactly 4 points  $\mathbf{p} \in K$  such that

$$(\nabla f_1)(\mathbf{f}^{-1}(\mathbf{p})) = \mathbf{0}.$$

Find these points.

- (b) Determine the set of all  $\mathbf{q} \in K$  such that

$$(\nabla f_3)(\mathbf{f}^{-1}(\mathbf{q})) = \mathbf{0}.$$

- (c) Show that one of the point  $\mathbf{p}$  found in part (a) corresponds to a local maximum of  $f_1$ , one corresponds to a local minimum, and that the other two are neither (they are so-called “saddle points”). Which of the points  $\mathbf{q}$  found in part (b) corresponds to maxima or minima?
- (d) Let  $\lambda$  be an irrational real number, and define  $\mathbf{g}(t) = \mathbf{f}(t, \lambda t)$ . Prove that  $\mathbf{g}$  is a one-to-one mapping of  $\mathbb{R}^1$  onto a dense subset of  $K$ . Prove that

$$|\mathbf{g}'(t)|^2 = a^2 + \lambda^2(b + a \cos t)^2.$$

*Proof.*

- (1)  $K$  is a torus, where
- (a)  $s, t$  are angles which make a full circle (so that their values start and end at the same point).
  - (b)  $b$  is the distance from the center of the tube to the center of the torus.
  - (c)  $a$  is the radius of the tube.
- (2) Show that  $K$  is compact. Since  $\sin$  and  $\cos$  are periodic (with period  $2\pi$ ),  $K = \mathbf{f}([0, 2\pi]^2)$  is compact by the compactness of  $[0, 2\pi]^2$  and the continuity of  $\mathbf{f}$  (Theorem 4.14).

□

*Proof of (a).*

- (1)

$$\begin{aligned} (\nabla f_1)(\mathbf{x}) &= (D_1 f_1)(\mathbf{x})\mathbf{e}_1 + (D_2 f_1)(\mathbf{x})\mathbf{e}_2 \\ &= ((D_1 f_1)(s, t), (D_2 f_1)(s, t)) \\ &= (-a \sin s \cos t, -(b + a \cos t) \sin t) \end{aligned}$$

So  $(\nabla f_1)(\mathbf{x}) = \mathbf{0}$  if and only if

$$\begin{aligned} 0 &= -a \sin s \cos t, \\ 0 &= -(b + a \cos t) \sin t. \end{aligned}$$

- (2) Note that  $b + a \cos t > 0$  for any  $b > a > 0$  and  $t \in \mathbb{R}^1$ . Hence  $(\nabla f_1)(\mathbf{x}) = \mathbf{0}$  if and only if  $\sin t = \sin s = 0$ . Therefore,  $\mathbf{p} = (\pm(b \pm a), 0, 0)$ , or there are exactly 4 points  $\mathbf{p} = (b + a, 0, 0)$ ,  $(b - a, 0, 0)$ ,  $(-b - a, 0, 0)$ , or  $(-b + a, 0, 0) \in K$ .

□

*Proof of (b).*

(1)

$$\begin{aligned} (\nabla f_3)(\mathbf{x}) &= (D_1 f_3)(\mathbf{x})\mathbf{e}_1 + (D_2 f_3)(\mathbf{x})\mathbf{e}_2 \\ &= ((D_1 f_3)(s, t), (D_2 f_3)(s, t)) \\ &= (a \cos s, 0) \end{aligned}$$

So  $(\nabla f_1)(\mathbf{x}) = \mathbf{0}$  if and only if  $\cos s = 0$  (since  $a > 0$ ).

- (2) Therefore,  $\mathbf{q} = (b \cos t, b \sin t, \pm a)$ .

□

*Proof of (c).*

- (1) Since  $-1 \leq \cos s \leq 1$  and  $-1 \leq \cos t \leq 1$ ,  $-b - a \leq f_1(s, t) \leq b + a$ .

- (a)  $(b + a, 0, 0)$  corresponds to a local maximum of  $f_1$ .
- (b)  $(-b - a, 0, 0)$  corresponds to a local minimum of  $f_1$ .
- (c)  $(b - a, 0, 0)$  and  $(-b + a, 0, 0)$  are saddle points by considering any open neighborhood of  $(s, t)$  at which  $\cos s = \pm 1$  and  $\cos t = \mp 1$ .

- (2) Since  $-1 \leq \sin s \leq 1$ ,  $-a \leq f_3(s, t) \leq a$ .

- (a)  $(b \cos t, b \sin t, a)$  corresponds to a local maximum of  $f_3$ .
- (b)  $(b \cos t, b \sin t, -a)$  corresponds to a local minimum of  $f_3$ .

□

*Proof of (d).*

(1)

$$\mathbf{g}(t) = \mathbf{f}(t, \lambda t) = ((b + a \cos t) \cos(\lambda t), (b + a \cos t) \sin(\lambda t), a \sin t).$$

- (2) Show that  $\mathbf{g}$  is a one-to-one mapping of  $\mathbb{R}^1$ . It suffices to show that  $\mathbf{g}(t) = \mathbf{g}(s)$  implies  $t = s$ .

(a) By  $\mathbf{g}(t) = \mathbf{g}(s)$ ,

$$(b + a \cos t) \cos(\lambda t) = (b + a \cos s) \cos(\lambda s), \quad (\text{I})$$

$$(b + a \cos t) \sin(\lambda t) = (b + a \cos s) \sin(\lambda s), \quad (\text{II})$$

$$a \sin t = a \sin s. \quad (\text{III})$$

(I) and (II) imply that  $\cos t = \cos s$  (since  $b > a > 0$ ). (III) implies that  $\sin t = \sin s$ . Hence

$$t = s + 2n\pi$$

for some integer  $n$ .

(b) Again, (I) and (II) imply that

$$\cos(\lambda t) = \cos(\lambda s) \quad \text{and} \quad \sin(\lambda t) = \sin(\lambda s).$$

Hence

$$\lambda t = \lambda s + 2m\pi$$

for some integer  $m$ . By assumption that  $t = s + 2n\pi$ , we have  $m = n\lambda$ . Since  $\lambda$  is irrational,  $m = n = 0$ . Therefore  $t = s$  holds.

(3) Show that  $\mathbf{g}(\mathbb{R}^1)$  is dense in  $K$ . Note that  $\mathbf{f}([0, 2\pi]^2) = K$ . Use the notations  $\{x\}$  in Exercise 4.16. It suffices to show that the set

$$\left\{ \left( 2\pi \left\{ \frac{t}{2\pi} \right\}, 2\pi \left\{ \frac{\lambda t}{2\pi} \right\} \right) : t \in \mathbb{R}^1 \right\}$$

is dense in  $[0, 2\pi]^2$  (Exercise 4.4), or to show that

$$\{(\{t\}, \{\lambda t\}) : t \in \mathbb{R}^1\}$$

is dense in  $[0, 1]^2$ , which is the conclusion of Exercise 4.25(b).

(4) Show that  $|\mathbf{g}'(t)|^2 = a^2 + \lambda^2(b + a \cos t)^2$ . By

$$\begin{aligned} \mathbf{g}'(t) = & \begin{pmatrix} -a \sin t \cos(\lambda t) - \lambda(b + a \cos t) \sin(\lambda t), \\ -a \sin t \sin(\lambda t) + \lambda(b + a \cos t) \cos(\lambda t), \\ a \cos t \end{pmatrix}, \end{aligned}$$

$$\begin{aligned}
|\mathbf{g}'(t)|^2 &= \mathbf{g}'(t) \cdot \mathbf{g}'(t) \\
&= (-a \sin t \cos(\lambda t) - \lambda(b + a \cos t) \sin(\lambda t))^2 \\
&\quad + (-a \sin t \sin(\lambda t) + \lambda(b + a \cos t) \cos(\lambda t))^2 + (a \cos t)^2 \\
&= \underbrace{a^2 \sin^2 t \cos^2(\lambda t) + a^2 \cos^2 t}_{=a^2} \\
&\quad + \underbrace{\lambda^2(b + a \cos t)^2 \sin^2(\lambda t) + \lambda^2(b + a \cos t)^2 \cos^2(\lambda t)}_{=\lambda^2(b+a \cos t)^2} \\
&\quad + 2a\lambda \sin t \cos(\lambda t) \lambda(b + a \cos t) \sin(\lambda t) \\
&\quad - 2a\lambda \sin t \sin(\lambda t) \lambda(b + a \cos t) \cos(\lambda t) \\
&= a^2 + \lambda^2(b + a \cos t)^2.
\end{aligned}$$

□

**Exercise 9.13.** Suppose  $\mathbf{f}$  is a differentiable mapping of  $\mathbb{R}^1$  into  $\mathbb{R}^3$  such that  $|\mathbf{f}(t)| = 1$  for every  $t$ . Prove that  $\mathbf{f}'(t) \cdot \mathbf{f}(t) = 0$ . Interpret this result geometrically.

*Proof.*

- (1) Write  $\mathbf{f} = (f_1, f_2, f_3)$  as a vector-valued function. By Remarks 5.16,  $\mathbf{f}$  is differentiable if and only if each  $f_1, f_2, f_3$  is differentiable. So  $\mathbf{f}' = (f'_1, f'_2, f'_3)$ . Hence

$$\begin{aligned}
|\mathbf{f}(t)| &= 1 \text{ for every } t \\
\iff \mathbf{f}(t) \cdot \mathbf{f}(t) &= 1 \\
\iff f_1(t)^2 + f_2(t)^2 + f_3(t)^2 &= 1 \\
\implies 2f_1(t)f'_1(t) + 2f_2(t)f'_2(t) + 2f_3(t)f'_3(t) &= 0 \\
\iff f_1(t)f'_1(t) + f_2(t)f'_2(t) + f_3(t)f'_3(t) &= 0 \\
\iff (f_1(t), f_2(t), f_3(t)) \cdot (f'_1(t), f'_2(t), f'_3(t)) &= 0 \\
\iff \mathbf{f}(t) \cdot \mathbf{f}'(t) = \mathbf{f}'(t) \cdot \mathbf{f}(t) &= 0.
\end{aligned}$$

- (2) The vector  $\mathbf{f}'(t)$  is called the **tangent vector** (or **velocity vector**) of  $\mathbf{f}$  at  $t$ . Geometrically, given any mapping  $\mathbf{f}$  lying on the sphere  $S^2$ , its tangent vector at  $t$  is lying on the tangent plane of  $S^2$  at  $t$ .

□

**Exercise 9.14.** Define  $f(0, 0) = 0$  and

$$f(x, y) = \frac{x^3}{x^2 + y^2} \quad \text{if } (x, y) \neq (0, 0).$$

- (a) Prove that  $D_1f$  and  $D_2f$  are bounded functions in  $\mathbb{R}^2$ . (Hence  $f$  is continuous.)
- (b) Let  $\mathbf{u}$  be any unit vector in  $\mathbb{R}^2$ . Show that the directional derivative  $(D_{\mathbf{u}}f)(0,0)$  exists, and that its absolute value is at most 1.
- (c) Let  $\gamma$  be a differentiable mapping of  $\mathbb{R}^1$  into  $\mathbb{R}^2$  (in other words,  $\gamma$  is a differentiable curve in  $\mathbb{R}^2$ ), with  $\gamma(t) = (0,0)$  and  $\gamma'(t) \neq (0,0)$  for any  $t \in \mathbb{R}^1$ . Put  $g(t) = f(\gamma(t))$  and prove that  $g$  is differentiable for every  $t \in \mathbb{R}^1$ . If  $\gamma \in \mathcal{C}'$ , prove that  $g \in \mathcal{C}'$ .
- (d) In spite of this, prove that  $f$  is not differentiable at  $(0,0)$ .

*Proof of (a).*

- (1) Show that

$$(D_1f)(x,y) = \begin{cases} 1 & \text{if } (x,y) = (0,0), \\ \frac{x^2(x^2+3y^2)}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0). \end{cases}$$

If  $(x,y) = (0,0)$ ,

$$(D_1f)(0,0) = \lim_{t \rightarrow 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t \rightarrow 0} \frac{t-0}{t} = 1.$$

If  $(x,y) \neq (0,0)$ ,

$$\begin{aligned} (D_1f)(x,y) &= \lim_{t \rightarrow 0} \frac{f(x+t,y) - f(x,y)}{t} \\ &= \lim_{t \rightarrow 0} \frac{\frac{(x+t)^3}{(x+t)^2+y^2} - \frac{x^3}{x^2+y^2}}{t} \\ &= \lim_{t \rightarrow 0} \frac{x^2(x^2+3y^2) + tx(2x^2+3y^2) + t^2(x^2+y^2)}{((x+t)^2+y^2)(x^2+y^2)} \\ &= \frac{x^2(x^2+3y^2)}{(x^2+y^2)^2}. \end{aligned}$$

(Or differentiate directly.)

- (2) Show that  $(D_1f)(x,y)$  is bounded. It suffices to show that  $(D_1f)(x,y)$  is bounded if  $(x,y) \neq (0,0)$ . Write  $x = r \cos \theta$  and  $y = r \sin \theta$  in the polar coordinates. (Here  $r > 0$ .) Hence

$$(D_1f)(x,y) = \frac{x^2(x^2+3y^2)}{(x^2+y^2)^2} = \cos^2 \theta (\cos^2 \theta + 3 \sin^2 \theta)$$

is bounded by  $1 \cdot (1+3) = 4$ .

(3) Show that

$$(D_2f)(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0), \\ \frac{-2x^3y}{(x^2+y^2)^2} & \text{if } (x, y) \neq (0, 0). \end{cases}$$

If  $(x, y) = (0, 0)$ ,

$$(D_2f)(0, 0) = \lim_{t \rightarrow 0} \frac{f(0, t) - f(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{0 - 0}{t} = 0.$$

If  $(x, y) \neq (0, 0)$ ,

$$\begin{aligned} (D_2f)(x, y) &= \lim_{t \rightarrow 0} \frac{f(x, y+t) - f(x, y)}{t} \\ &= \lim_{t \rightarrow 0} \frac{\frac{x^3}{x^2+(y+t)^2} - \frac{x^3}{x^2+y^2}}{t} \\ &= \lim_{t \rightarrow 0} \frac{-2x^3y - tx^3}{(x^2 + (y+t)^2)(x^2 + y^2)} \\ &= \frac{-2x^3y}{(x^2 + y^2)^2}. \end{aligned}$$

(Or differentiate directly.)

(4) Show that  $(D_2f)(x, y)$  is bounded. Similar to (2).

(5) Show that  $f$  is continuous. Apply Exercise 9.7 to (2)(4).

□

*Proof of (b).*

(1) Write  $\mathbf{u} = (u_1, u_2)$ . The formula

$$(D_{\mathbf{u}}f)(0, 0) = (D_1f)(0, 0)u_1 + (D_2f)(0, 0)u_2 = u_1$$

might be false since we don't know if  $f$  is differentiable or not. Actually, we will show that  $(D_{\mathbf{u}}f)(0, 0) = u_1^3 \neq u_1$ .

(2)

$$\begin{aligned} (D_{\mathbf{u}}f)(0, 0) &= \lim_{t \rightarrow 0} \frac{f(tu_1, tu_2) - f(0, 0)}{t} \\ &= \lim_{t \rightarrow 0} \frac{\frac{t^3 u_1^3}{t^2 u_1^2 + t^2 u_2^2} - 0}{t} \\ &= \lim_{t \rightarrow 0} u_1^3 \quad (|\mathbf{u}| = 1) \\ &= u_1^3. \end{aligned}$$

Also  $|(D_{\mathbf{u}}f)(0, 0)| = |u_1|^3 \leq 1$  since  $|\mathbf{u}| = 1$ .



□

*Proof of (c).*

(1) Given any  $t \in \mathbb{R}^1$ .

$$g'(t) = \lim_{x \rightarrow t} \frac{g(x) - g(t)}{x - t} = \lim_{x \rightarrow t} \frac{f(\gamma(x)) - f(\gamma(t))}{x - t}.$$

Write  $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ .

(2) Suppose that  $\gamma(t) \neq (0, 0)$ . Since  $\gamma$  is differentiable,  $\gamma$  is continuous. So there exists an open neighborhood  $B(t) \subseteq \mathbb{R}^1$  of  $t$  such that  $\gamma(x) \neq (0, 0)$  whenever  $x \in B(t)$ . Hence

$$\begin{aligned} g'(t) &= \lim_{x \rightarrow t} \frac{\frac{\gamma_1(x)^3}{\gamma_1(x)^2 + \gamma_2(x)^2} - \frac{\gamma_1(t)^3}{\gamma_1(t)^2 + \gamma_2(t)^2}}{x - t} \\ &= \frac{d}{dt} \left( \frac{\gamma_1(t)^3}{\gamma_1(t)^2 + \gamma_2(t)^2} \right) \\ &= \frac{3\gamma_1(t)^2 \gamma_1'(t)}{\gamma_1(t)^2 + \gamma_2(t)^2} - \frac{\gamma_1(t)^3 (2\gamma_1(t) \gamma_1'(t) + 2\gamma_2(t) \gamma_2'(t))}{(\gamma_1(t)^2 + \gamma_2(t)^2)^2}. \end{aligned}$$

exists since  $\gamma_1$  and  $\gamma_2$  are differentiable.

(3) Suppose that  $\gamma(t) = (0, 0)$  and thus  $\gamma'(t) \neq (0, 0)$ . So

$$g'(t) = \lim_{x \rightarrow t} \frac{f(\gamma(x))}{x - t}$$

Note that  $\gamma(x) \neq (0, 0)$  in some open neighborhood of  $t$  since

$$\lim_{\substack{x \rightarrow t \\ \gamma(x) = (0, 0)}} \frac{\gamma(x) - \gamma(t)}{x - t} = (0, 0),$$

contrary to the assumption that  $\gamma'(t) \neq (0, 0)$ . Note that  $\gamma_1(t) = \gamma_2(t) = 0$ . So

$$\begin{aligned} g'(t) &= \lim_{x \rightarrow t} \frac{f(\gamma(x))}{x - t} \\ &= \lim_{x \rightarrow t} \frac{\gamma_1(x)^3}{\gamma_1(x)^2 + \gamma_2(x)^2} \cdot \frac{1}{x - t} \\ &= \lim_{x \rightarrow t} \frac{(\gamma_1(x) - \gamma_1(t))^3}{(\gamma_1(x) - \gamma_1(t))^2 + (\gamma_2(x) - \gamma_2(t))^2} \cdot \frac{1}{x - t} \\ &= \lim_{x \rightarrow t} \frac{\left( \frac{\gamma_1(x) - \gamma_1(t)}{x - t} \right)^3}{\left( \frac{\gamma_1(x) - \gamma_1(t)}{x - t} \right)^2 + \left( \frac{\gamma_2(x) - \gamma_2(t)}{x - t} \right)^2} \\ &= \frac{\gamma_1'(t)^3}{\gamma_1'(t)^2 + \gamma_2'(t)^2} \end{aligned}$$

since  $\gamma'(t) \neq (0, 0)$ .

(4) By (2)(3),  $g'(t)$  exists and

$$g'(t) = \begin{cases} \frac{3\gamma_1(t)^2\gamma_1'(t)}{\gamma_1(t)^2 + \gamma_2(t)^2} - \frac{\gamma_1(t)^3(2\gamma_1(t)\gamma_1'(t) + 2\gamma_2(t)\gamma_2'(t))}{(\gamma_1(t)^2 + \gamma_2(t)^2)^2} & \text{if } \gamma(t) \neq (0, 0), \\ \frac{\gamma_1'(t)^3}{\gamma_1'(t)^2 + \gamma_2'(t)^2} & \text{if } \gamma(t) = (0, 0). \end{cases}$$

(5) Now suppose  $\gamma \in \mathcal{C}'$ . To show  $g' \in \mathcal{C}'$ , it suffices to show that

$$\lim_{x \rightarrow t} g'(x) = g'(t)$$

if  $\gamma(t) = (0, 0)$  since  $g'(t)$  is always continuous if  $\gamma(t) \neq (0, 0)$ . Here all  $\gamma_1, \gamma_2, \gamma_1', \gamma_2'$  are continuous and  $\gamma_1(t)^2 + \gamma_2(t)^2 \neq 0$  by assumption. So

$$\begin{aligned} & \lim_{x \rightarrow t} \frac{3\gamma_1(x)^2\gamma_1'(x)}{\gamma_1(x)^2 + \gamma_2(x)^2} \\ &= \lim_{x \rightarrow t} \frac{3 \left( \frac{\gamma_1(x) - \gamma_1(t)}{x - t} \right)^2 \gamma_1'(x)}{\left( \frac{\gamma_1(x) - \gamma_1(t)}{x - t} \right)^2 + \left( \frac{\gamma_2(x) - \gamma_2(t)}{x - t} \right)^2} \\ &= \frac{3\gamma_1'(t)^2 \cdot \gamma_1'(t)}{\gamma_1'(t)^2 + \gamma_2'(t)^2} \\ &= \frac{3\gamma_1'(t)^3}{\gamma_1'(t)^2 + \gamma_2'(t)^2} \end{aligned}$$

and similarly

$$\begin{aligned} & \lim_{x \rightarrow t} \frac{\gamma_1(t)^3(2\gamma_1(t)\gamma_1'(t) + 2\gamma_2(t)\gamma_2'(t))}{(\gamma_1(t)^2 + \gamma_2(t)^2)^2} \\ &= \lim_{x \rightarrow t} \frac{\left( \frac{\gamma_1(x) - \gamma_1(t)}{x - t} \right)^3 \left( 2\frac{\gamma_1(x) - \gamma_1(t)}{x - t} \gamma_1'(t) + 2\frac{\gamma_2(x) - \gamma_2(t)}{x - t} \gamma_2'(t) \right)}{\left( \left( \frac{\gamma_1(x) - \gamma_1(t)}{x - t} \right)^2 + \left( \frac{\gamma_2(x) - \gamma_2(t)}{x - t} \right)^2 \right)^2} \\ &= \frac{\gamma_1'(t)^3 \cdot (2\gamma_1'(t)\gamma_1'(t) + 2\gamma_2'(t)\gamma_2'(t))}{(\gamma_1'(t)^2 + \gamma_2'(t)^2)^2} \\ &= \frac{2\gamma_1'(t)^3}{\gamma_1'(t)^2 + \gamma_2'(t)^2}. \end{aligned}$$

Hence

$$\lim_{x \rightarrow t} g'(x) = \frac{3\gamma_1'(t)^3}{\gamma_1'(t)^2 + \gamma_2'(t)^2} - \frac{2\gamma_1'(t)^3}{\gamma_1'(t)^2 + \gamma_2'(t)^2} = g'(t).$$

□

*Proof of (d).* (Reductio ad absurdum) If  $f$  were differentiable, then

$$(D_{\mathbf{u}}f)(0, 0) = (D_1f)(0, 0)u_1 + (D_2f)(0, 0)u_2 = u_1$$

(Formula (40) in Chapter 9), contrary to (b) if we take  $\mathbf{u} = \left(\frac{1}{64}, \frac{\sqrt{4095}}{64}\right)$ .  $\square$

**Exercise 9.15.** Define  $f(0, 0) = 0$ , and put

$$f(x, y) = x^2 + y^2 - 2x^2y - \frac{4x^6y^2}{(x^4 + y^2)^2}$$

if  $(x, y) \neq (0, 0)$ .

(a) Prove, for all  $(x, y) \in \mathbb{R}^2$ , that

$$4x^4y^2 \leq (x^4 + y^2)^2.$$

Conclude that  $f$  is continuous.

(b) For  $0 \leq \theta \leq 2\pi$ ,  $-\infty < t < \infty$ , define

$$g_\theta(t) = f(t \cos \theta, t \sin \theta).$$

Show that  $g_\theta(0) = 0$ ,  $g'_\theta(0) = 0$ ,  $g''_\theta(0) = 2$ . Each  $g_\theta$  has therefore a strict local minimum at  $t = 0$ . In other words, the restriction of  $f$  to each line through  $(0, 0)$  has a strict local minimum at  $(0, 0)$ .

(c) Show that  $(0, 0)$  is nevertheless not a local minimum for  $f$ , since  $f(x, x^2) = -x^4$ .

*Proof of (a).*

(1) Since  $t^2 \geq 0$  for all  $t \in \mathbb{R}^1$ ,

$$(x^4 + y^2)^2 - 4x^4y^2 = (x^4 - y^2)^2 \geq 0.$$

Hence  $4x^4y^2 \leq (x^4 + y^2)^2$ .

(2)  $f(x, y)$  is continuous at  $(x, y) \neq (0, 0)$ . Besides,

$$\begin{aligned} |f(x, y)| &= \left| x^2 + y^2 - 2x^2y - \frac{4x^6y^2}{(x^4 + y^2)^2} \right| \\ &\leq |x^2| + |y^2| + |2x^2y| + |x^2| \left| \frac{4x^4y^2}{(x^4 + y^2)^2} \right| \\ &\leq |x^2| + |y^2| + |2x^2y| + |x^2|. \end{aligned}$$

Hence  $|x^2| + |y^2| + |2x^2y| + |x^2| \rightarrow 0$  as  $(x, y) \rightarrow (0, 0)$ , or

$$\lim_{(x, y) \rightarrow (0, 0)} |f(x, y)| = 0 = f(0, 0),$$

or  $\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = f(0, 0)$ , or  $f(x, y)$  is continuous at  $(0, 0)$ .

□

*Proof of (b).*

(1)

$$g_\theta(t) = \begin{cases} t^2 - 2t^3 \cos^2 \theta \sin \theta - \frac{4t^4 \cos^6 \theta \sin^2 \theta}{(t^2 \cos^4 \theta + \sin^2 \theta)^2} & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

(Note that  $\frac{4t^4 \cos^6 \theta \sin^2 \theta}{(t^2 \cos^4 \theta + \sin^2 \theta)^2}$  is undefined as  $t = 0$  and  $\sin \theta = 0$ .)

(2)  $g_\theta(0) = 0$  by definition.

(3) Show that  $g'_\theta(0) = 0$  for any  $\theta \in [0, 2\pi]$ . If  $\sin \theta \neq 0$  ( $\theta \neq 0, \pi, 2\pi$ ), then

$$\begin{aligned} g'_\theta(0) &= \lim_{t \rightarrow 0} \frac{t^2 - 2t^3 \cos^2 \theta \sin \theta - \frac{4t^4 \cos^6 \theta \sin^2 \theta}{(t^2 \cos^4 \theta + \sin^2 \theta)^2} - 0}{t} \\ &= \lim_{t \rightarrow 0} \left( t - 2t^2 \cos^2 \theta \sin \theta - \frac{4t^3 \cos^6 \theta \sin^2 \theta}{(t^2 \cos^4 \theta + \sin^2 \theta)^2} \right) \\ &= 0. \end{aligned}$$

If  $\sin \theta = 0$ , then

$$g'_\theta(0) = \lim_{t \rightarrow 0} \frac{t^2 - 0}{t} = \lim_{t \rightarrow 0} t = 0.$$

(4) Combine (3) and a direct calculation for the case  $t \neq 0$ , we have

$$g'_\theta(t) = \begin{cases} 2t - 6t^2 \cos^2 \theta \sin \theta - \frac{16t^3 \cos^6 \theta \sin^4 \theta}{(t^2 \cos^4 \theta + \sin^2 \theta)^3} & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

(5) Show that  $g''_\theta(0) = 2$  for any  $\theta \in [0, 2\pi]$ . If  $\sin \theta \neq 0$  ( $\theta \neq 0, \pi, 2\pi$ ), then

$$\begin{aligned} g''_\theta(0) &= \lim_{t \rightarrow 0} \frac{2t - 6t^2 \cos^2 \theta \sin \theta - \frac{16t^3 \cos^6 \theta \sin^4 \theta}{(t^2 \cos^4 \theta + \sin^2 \theta)^3} - 0}{t} \\ &= \lim_{t \rightarrow 0} \left( t - 6t \cos^2 \theta \sin \theta - \frac{16t^2 \cos^6 \theta \sin^4 \theta}{(t^2 \cos^4 \theta + \sin^2 \theta)^3} \right) \\ &= 2. \end{aligned}$$

If  $\sin \theta = 0$ , then

$$g''_\theta(0) = \lim_{t \rightarrow 0} \frac{2t - 0}{t} = \lim_{t \rightarrow 0} 2 = 2.$$

(6) Since  $g''_\theta(0) > 0$  and  $g'_\theta(0) = 0$ ,  $g_\theta$  has a strict local minimum at  $t = 0$ . As  $\theta$  is fixed,  $f$  is restricted to some line through  $(0, 0)$ . Hence, such restriction of  $f$  has a strict local minimum at  $t = 0$ .

□

*Proof of (c).* Since  $f(x, x^2) = -x^4 \leq 0 = f(0, 0)$  in any open neighborhood of  $(0, 0)$ ,  $f(0, 0) = 0$  cannot be a local minimum for  $f$ . □

**Exercise 9.16.** Show that the continuity of  $\mathbf{f}'$  at the point  $\mathbf{a}$  is needed in the inverse function theorem, even in the case  $n = 1$ : If

$$f(t) = t + 2t^2 \sin \frac{1}{t}$$

for  $t \neq 0$ , and  $f(0) = 0$ , then  $f'(0) = 1$ ,  $f'$  is bounded in  $(-1, 1)$ , but  $f$  is not one-to-one in any neighborhood of 0.

*Proof.*

(1) Show that

$$f'(t) = \begin{cases} 1 + 4t \sin \frac{1}{t} - 2 \cos \frac{1}{t} & \text{if } t \neq 0, \\ 1 & \text{if } t = 0. \end{cases}$$

It suffices to show that  $f'(0) = 1$ . In fact,

$$f'(0) = \lim_{t \rightarrow 0} \frac{t + 2t^2 \sin \frac{1}{t} - 0}{t - 0} = \lim_{t \rightarrow 0} \left( 1 + 2t \sin \frac{1}{t} \right) = 1$$

(since  $\sin \frac{1}{t}$  is bounded and  $2t \rightarrow 0$  as  $t \rightarrow 0$ ).

*Note.*  $f'(t)$  is not continuous at  $t = 0$ .

(2) Show that  $f'$  is bounded in  $(-1, 1)$ .

$$|f'(t)| \leq 1 + 4|t| \left| \sin \frac{1}{t} \right| + 2 \left| \cos \frac{1}{t} \right| \leq 1 + 4 + 2 = 7$$

if  $t \neq 0$ . Hence  $f'$  is bounded by 7 in  $(-1, 1)$ .

(3) Show that  $f$  is not one-to-one in any neighborhood of 0. Take

$$x_n = \frac{1}{2n\pi} \quad \text{and} \quad y_n = \frac{1}{2n\pi + \pi}$$

for  $n = 1, 2, 3, \dots$ . So that

$$f'(x_n) = -1 < 0 \quad \text{and} \quad f'(y_n) = 3 > 0.$$

Since  $f'(t)$  is continuous if  $t \neq 0$ , there exists  $\xi_n \in (y_n, x_n)$  such that  $f'(\xi_n) = 0$  (Theorem 4.23). Then Theorem 5.11 implies that  $f$  has a local maximum at  $\xi_n$ , that is,  $f$  is not one-to-one in the interval  $[y_n, x_n]$  (by applying Theorem 4.23 again). Since  $x_n \rightarrow 0$  and  $y_n \rightarrow 0$  as  $n \rightarrow \infty$ ,  $f$  is not one-to-one in any neighborhood of 0.

□

**Exercise 9.17.** Let  $\mathbf{f} = (f_1, f_2)$  be the mapping of  $\mathbb{R}^2$  into  $\mathbb{R}^2$  given by

$$f_1(x, y) = e^x \cos y, \quad f_2(x, y) = e^x \sin y.$$

- (a) What is the range of  $\mathbf{f}$ ?
- (b) Show that the Jacobian of  $\mathbf{f}$  is not zero at any point of  $\mathbb{R}^2$ . Thus every point of  $\mathbb{R}^2$  has a neighborhood in which  $\mathbf{f}$  is one-to-one. Nevertheless,  $\mathbf{f}$  is not one-to-one on  $\mathbb{R}^2$ .
- (c) Put  $\mathbf{a} = (0, \frac{\pi}{3})$ ,  $\mathbf{b} = \mathbf{f}(\mathbf{a})$ , let  $\mathbf{g}$  be the continuous inverse of  $\mathbf{f}$ , defined in a neighborhood of  $\mathbf{b}$  such that  $\mathbf{g}(\mathbf{b}) = \mathbf{a}$ . Find an explicit formula for  $\mathbf{g}$ , compute  $\mathbf{f}'(\mathbf{a})$  and  $\mathbf{g}'(\mathbf{b})$ , and verify the formula

$$\mathbf{g}'(\mathbf{y}) = \{\mathbf{f}'(\mathbf{g}(\mathbf{y}))\}^{-1}.$$

- (d) What are the images under  $\mathbf{f}$  of lines parallel to the coordinate axes?

*Proof of (a).*

- (1) The range of  $\mathbf{f}$  is  $\mathbb{R}^2 - \{(0, 0)\}$ .
- (2) If  $(a, b) \neq (0, 0)$ , then  $\mathbf{f} : (\log \sqrt{a^2 + b^2}, \text{atan2}(b, a)) \mapsto (a, b)$  where

$$\text{atan2}(b, a) = \begin{cases} \arctan\left(\frac{b}{a}\right) & \text{if } a > 0, \\ \arctan\left(\frac{b}{a}\right) + \pi & \text{if } a < 0 \text{ and } b \geq 0, \\ \arctan\left(\frac{b}{a}\right) - \pi & \text{if } a < 0 \text{ and } b < 0, \\ \frac{\pi}{2} & \text{if } a = 0 \text{ and } b > 0, \\ -\frac{\pi}{2} & \text{if } a = 0 \text{ and } b < 0. \end{cases}$$

(Or apply Theorem 8.7(d).)

- (3) If  $(a, b) = (0, 0)$ , then for any  $(x, y) \in \mathbb{R}^2$  we have  $f_1(x, y)^2 + f_2(x, y)^2 = e^{2x} \neq 0$ . So that there is no  $(x, y)$  such that  $\mathbf{f} : (x, y) \mapsto (0, 0)$ .

□

*Proof of (b).*

- (1)

$$[\mathbf{f}'(x, y)] = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}.$$

So  $\mathbf{f}'$  is continuous and

$$J_{\mathbf{f}}(x, y) = \det \mathbf{f}'(x, y) = e^{2x} \neq 0.$$

- (2) Since  $J_{\mathbf{f}}(x, y) \neq 0$ ,  $\mathbf{f}'(x, y)$  is invertible (Theorem 9.36). So the inverse function theorem (Theorem 9.24) implies that there exists an open neighborhood  $B(x, y)$  of  $(x, y)$  such that  $\mathbf{f}$  is injective on  $B(x, y)$ .

- (3) Note that

$$\mathbf{f}(0, 0) = \mathbf{f}(0, 2\pi) = (1, 0).$$

So that  $\mathbf{f}$  is not injective on the whole  $\mathbb{R}^2$ . (Injectivity of  $\mathbf{f}$  is a local property.)

□

*Proof of (c).*

- (1) If  $\mathbf{a} = (0, \frac{\pi}{3})$ , then  $\mathbf{b} = \mathbf{f}(\mathbf{a}) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ .

- (2) Similar to (2) in the proof of (a), define  $\mathbf{g} : E \rightarrow \mathbb{R}^2$  by

$$\mathbf{g}(x, y) = \left( \log \sqrt{x^2 + y^2}, \arctan \left( \frac{y}{x} \right) \right).$$

where  $E$  is some open neighborhood of the point  $\mathbf{b} \in \mathbb{R}^2$  described in (b). So  $\mathbf{g}$  is a continuous inverse of  $\mathbf{f}$ .

- (3) Since

$$[\mathbf{f}'(x, y)] = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix},$$

$$[\mathbf{f}'(\mathbf{a})] = [\mathbf{f}'(0, \frac{\pi}{3})] = \begin{bmatrix} e^0 \cos \frac{\pi}{3} & -e^0 \sin \frac{\pi}{3} \\ e^0 \sin \frac{\pi}{3} & e^0 \cos \frac{\pi}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

- (4) Since

$$[\mathbf{g}'(x, y)] = \begin{bmatrix} \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix},$$

$$[\mathbf{g}'(\mathbf{b})] = \left[ \mathbf{g}' \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) \right] = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

Here we can see  $[\mathbf{f}'(\mathbf{a})][\mathbf{g}'(\mathbf{b})] = [\mathbf{g}'(\mathbf{b})][\mathbf{f}'(\mathbf{a})] = 1$ .

- (5)

$$\begin{aligned} [\mathbf{g}'(\mathbf{y})] &= [\mathbf{g}'(\mathbf{f}(\mathbf{x}))] \\ &= [\mathbf{g}'(e^x \cos y, e^x \sin y)] \\ &= \begin{bmatrix} \frac{e^x \cos y}{e^{2x}} & \frac{e^x \sin y}{e^{2x}} \\ \frac{-e^x \sin y}{e^{2x}} & \frac{e^x \cos y}{e^{2x}} \end{bmatrix} \\ &= \begin{bmatrix} e^{-x} \cos y & e^{-x} \sin y \\ -e^{-x} \sin y & e^{-x} \cos y \end{bmatrix}, \end{aligned}$$

and

$$[\mathbf{f}'(\mathbf{g}(\mathbf{y}))] = [\mathbf{f}'(\mathbf{x})] = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}.$$

Note that

$$\begin{bmatrix} e^{-x} \cos y & e^{-x} \sin y \\ -e^{-x} \sin y & e^{-x} \cos y \end{bmatrix} \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix} = 1.$$

Therefore

$$\mathbf{g}'(\mathbf{y}) = \{\mathbf{f}'(\mathbf{g}(\mathbf{y}))\}^{-1}$$

on  $\mathbf{g}(E)$ .

□

*Proof of (d).*

- (1) The case  $L_r = \{(x, y) \in \mathbb{R}^2 : x = r\}$  parallel to  $y$ -axis where  $r \in \mathbb{R}^1$  is constant. The image under  $\mathbf{f}$  is

$$\begin{aligned} \mathbf{f}(L_r) &= \{(e^r \cos y, e^r \sin y) \in \mathbb{R}^2 : y \in \mathbb{R}^1\} \\ &= \{(s, t) \in \mathbb{R}^2 : s^2 + t^2 = (e^r)^2\}, \end{aligned}$$

a circle which is centered at the origin  $(0, 0) \in \mathbb{R}^2$  with radius  $e^r > 0$ .

- (2) The case  $L_\theta = \{(x, y) \in \mathbb{R}^2 : y = \theta\}$  parallel to  $x$ -axis where  $\theta \in \mathbb{R}^1$  is constant. The image under  $\mathbf{f}$  is

$$\begin{aligned} \mathbf{f}(L_\theta) &= \{(e^x \cos \theta, e^x \sin \theta) \in \mathbb{R}^2 : x \in \mathbb{R}^1\} \\ &= \{(y \cos \theta, y \sin \theta) \in \mathbb{R}^2 : y > 0\}, \end{aligned}$$

which is a ray from the origin  $(0, 0)$  (not included) to the infinity passing through a point  $(\cos \theta, \sin \theta)$  in the unit circle.

□

### Exercise 9.18. ...

*Proof.*

(1)

(2)

□



**Exercise 9.19.** Show that the system of equations

$$\begin{aligned} 3x + y - z + u^2 &= 0 \\ x - y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0 \end{aligned}$$

can be solved for  $x, y, u$  in terms of  $z$ ; for  $x, z, u$  in terms of  $y$ ; for  $y, z, u$  in terms of  $x$ ; but not for  $x, y, z$  in terms of  $u$ .

*Proof (Brute-force).*

(1) Denote

$$3x + y - z + u^2 = 0 \quad (\text{I})$$

$$x - y + 2z + u = 0 \quad (\text{II})$$

$$2x + 2y - 3z + 2u = 0 \quad (\text{III})$$

So (I) - 3(II) implies that

$$4y + u(u - 3) = 7z, \quad (\text{IV})$$

and (III) - 2(II) implies that

$$4y = 7z. \quad (\text{V})$$

By (IV)(V), we have  $u(u - 3) = 0$ . Hence  $u = 0$  or  $u = 3$  in any case.

(2) Show that (I)(II)(III) can be solved for  $x, y, u$  in terms of  $z$ . (V) implies that  $y = \frac{7z}{4}$ . Hence

$$(x, y, u) = \left( -\frac{z}{4}, \frac{7z}{4}, 0 \right), \left( -\frac{z}{4} - 3, \frac{7z}{4}, 3 \right).$$

(3) Show that (I)(II)(III) can be solved for  $x, z, u$  in terms of  $y$ .

$$(x, z, u) = \left( -\frac{y}{7}, \frac{4y}{7}, 0 \right), \left( -\frac{y}{7} - 3, \frac{4y}{7}, 3 \right).$$

(4) Show that (I)(II)(III) can be solved for  $y, z, u$  in terms of  $x$ .

$$(y, z, u) = (-7x, -4x, 0), (-7x - 21, -4x - 12, 3).$$

(5) Show that (I)(II)(III) can not be solved for  $x, y, z$  in terms of  $u$ . Actually,

$$(x, y, z) = (-t - u, 7t, 4t)$$

for all  $t \in \mathbb{R}^1$ .

□

*Proof (The implicit function theorem).*

- (1) Define  $\mathbf{f}$  be a  $\mathcal{C}'$ -mapping of  $\mathbb{R}^{3+1}$  into  $\mathbb{R}^3$  by

$$\mathbf{f}(x, y, z, u) = (3x + y - z + u^2, x - y + 2z + u, 2x + 2y - 3z + 2u).$$

Note that  $\mathbf{f}(0, 0, 0, 0) = \mathbf{0}$  and  $\mathbf{f}(-3, 0, 0, 3) = \mathbf{0}$ .

- (2) Since

$$[\mathbf{f}'(x, y, z, u)] = \begin{bmatrix} 3 & 1 & -1 & 2u \\ 1 & -1 & 2 & 1 \\ 2 & 2 & -3 & 2 \end{bmatrix},$$

$\mathbf{f}'$  is continuous,

$$[\mathbf{f}'(0, 0, 0, 0)] = \begin{bmatrix} 3 & 1 & -1 & 0 \\ 1 & -1 & 2 & 1 \\ 2 & 2 & -3 & 2 \end{bmatrix},$$

and

$$[\mathbf{f}'(-3, 0, 0, 3)] = \begin{bmatrix} 3 & 1 & -1 & 6 \\ 1 & -1 & 2 & 1 \\ 2 & 2 & -3 & 2 \end{bmatrix}.$$

- (3) The submatrix

$$[\mathbf{f}'(0, 0, 0, 0)]_x = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & -3 & 2 \end{bmatrix}$$

is invertible since its determinant is  $3 \neq 0$ . By the implicit function theorem (Theorem 9.28), the system can be solved for  $y, z, u$  in terms of  $x$ . Similar arguments to  $[\mathbf{f}'(0, 0, 0, 0)]_y$ ,  $[\mathbf{f}'(0, 0, 0, 0)]_z$ ,  $[\mathbf{f}'(-3, 0, 0, 3)]_x$ ,  $[\mathbf{f}'(-3, 0, 0, 3)]_y$ , and  $[\mathbf{f}'(-3, 0, 0, 3)]_z$ .

- (4) Note that  $[\mathbf{f}'(0, 0, 0, 0)]_u$  and  $[\mathbf{f}'(-3, 0, 0, 3)]_u$  are not invertible, we cannot apply the implicit function theorem (Theorem 9.28). We need to show by brute-force in this case.

□

**Exercise 9.20.** Take  $n = m = 1$  in the implicit function theorem, and interpret the theorem (as well as its proof) graphically.

**Implicit function theorem (for  $n = m = 1$ ).** Let  $f(x, y)$  be a  $\mathcal{C}'$ -mapping of an open set  $E \subseteq \mathbb{R}^2$  into  $\mathbb{R}$ , such that  $f(a, b) = 0$  for some point  $(a, b) \in E$ . Assume that

$$D_1 f(a, b) \neq 0.$$

Then there exist open sets  $U \subseteq E$  and  $W \subseteq \mathbb{R}^1$ , with  $(a, b) \in U$  and  $b \in W$ , having the following property:

To every  $y \in W$  corresponds a unique  $x$  such that

$$(x, y) \in U \quad \text{and} \quad f(x, y) = 0.$$

If this  $x$  is defined to be  $g(y)$ , then  $g$  is a  $\mathcal{C}'$ -mapping of  $W$  into  $\mathbb{R}^1$ ,  
 $g(b) = a$ ,

$$f(g(y), y) = 0 \quad (y \in W),$$

and

$$g'(b) = -\frac{D_2 f(a, b)}{D_1 f(a, b)}.$$

*Proof.*

- (1) In the notations of Exercise 4.6, define the graph of  $f$  by the set

$$S = \{(x, y) \in E : f(x, y) = 0\}.$$

- (2) Consider the graph  $S$ . As  $D_1 f(a, b) \neq 0$  and  $f(x, y) \in \mathcal{C}'$ , there are an open neighborhood  $U \subseteq E$  of  $(a, b)$  and an open neighborhood  $W$  of  $b$  such that  $x \mapsto f(x, y)$  is strictly monotonic whenever  $y \in W$ . “Graphically” by the monotony of  $f(x, y)$ , for any fixed  $y$  there is a unique  $x$  such that  $f(x, y) = 0$ .

- (3) “Graphically” the tangent line passing through  $(a, b)$  is

$$D_1 f(a, b)(x - a) + D_2 f(a, b)(y - b) = 0.$$

Thus  $g'(b) = -\frac{D_2 f(a, b)}{D_1 f(a, b)}$  if  $D_1 f(a, b) \neq 0$ .

□

**Exercise 9.21.** Define  $f$  in  $\mathbb{R}^2$  by

$$f(x, y) = 2x^3 - 3x^2 + 2y^3 + 3y^2.$$

- (a) Find the four points in  $\mathbb{R}^2$  at which the gradient of  $f$  is zero. Show that  $f$  has exactly one local maximum and one local minimum in  $\mathbb{R}^1$ .
- (b) Let  $S$  be the set of all  $(x, y) \in \mathbb{R}^2$  at which  $f(x, y) = 0$ . Find those points of  $S$  that have no neighborhoods in which the equation  $f(x, y) = 0$  can be solved for  $y$  in terms of  $x$  (or for  $x$  in terms of  $y$ ). Describe  $S$  as precisely as you can.

*Proof of (a).*

- (1)

$$(\nabla f)(x, y) = ((D_1 f)(x, y), (D_2 f)(x, y)) = (6x(x - 1), 6y(y + 1)).$$

So  $(\nabla f)(x, y) = 0$  if and only if  $(x, y) = (0, 0), (0, -1), (1, 0), (1, -1)$ .

- (2)  $x \mapsto 2x^3 - 3x^2$  have one local maximum at  $x = 0$  and one local minimum at  $x = 1$ .  $y \mapsto 2y^3 + 3y^2$  have one local maximum at  $y = -1$  and one local minimum at  $y = 0$ . Hence  $f : (x, y) \mapsto 2x(2x^3 - 3x^2) + (2y^3 + 3y^2)$  have one local maximum at  $(x, y) = (0, -1)$  and one local minimum at  $(x, y) = (1, 0)$ . Other two points  $(0, 0)$  and  $(1, -1)$  are saddle points.

□

*Proof of (b).*

- (1) By definition,

$$\begin{aligned} S &= \{f(x, y) = 0\} \\ &= \{(x + y)(2x^2 - 2xy - 3x + 2y^2 + 3y) = 0\} \\ &= \{x + y = 0\} \cup \{2x^2 - 2xy - 3x + 2y^2 + 3y = 0\}, \end{aligned}$$

which is a union of a line  $L = \{x + y = 0\}$  and an ellipse  $E = \{2x^2 - 2xy - 3x + 2y^2 + 3y = 0\}$ . The intersection of  $L \cap E$  is  $\{(0, 0), (1, -1)\}$ , and it suggested that  $f(x, y) = 0$  cannot be solved for  $y$  in terms of  $x$  (or for  $x$  in terms of  $y$ ) on  $L \cap E = \{(0, 0), (1, -1)\}$ .

- (2) By (1) in the proof of (a) and the implicit function theorem (Theorem 9.28),  $f(x, y) = 0$  can be solved for  $y$  in terms of  $x$  (or for  $x$  in terms of  $y$ ) whenever  $x \notin \{0, 1\}$  and  $y \notin \{0, -1\}$ . Now we consider the rest cases, say

$$(x, y) \in T = \left\{ (0, 0), (1, -1), \left(0, -\frac{3}{2}\right), \left(1, \frac{1}{2}\right), \left(\frac{3}{2}, 0\right), \left(-\frac{1}{2}, -1\right) \right\}.$$

- (3)  $T \cap L = \{(0, 0), (1, -1)\}$  and  $T \cap E = T$ .

- (4) As  $(x, y) \in T - T \cap L = \left\{ \left(0, -\frac{3}{2}\right), \left(1, \frac{1}{2}\right), \left(\frac{3}{2}, 0\right), \left(-\frac{1}{2}, -1\right) \right\}$ ,  $f(x, y) = 0$  if and only if  $2x^2 - 2xy - 3x + 2y^2 + 3y = 0$ . Hence

$$\begin{aligned} x &= \frac{1}{4} \left( 2y - 3 \pm \sqrt{-3(2y - 1)(2y + 3)} \right), \\ y &= \frac{1}{4} \left( 2x - 3 \pm \sqrt{-3(2x + 1)(2x - 3)} \right). \end{aligned}$$

As  $(x, y) \in \left\{ \left(\frac{3}{2}, 0\right), \left(-\frac{1}{2}, -1\right) \right\}$ , we can write  $y$  in terms of  $x$  but cannot write  $x$  in terms of  $y$ . As  $(x, y) \in \left\{ \left(0, -\frac{3}{2}\right), \left(1, \frac{1}{2}\right) \right\}$ , we can write  $x$  in terms of  $y$  but cannot write  $y$  in terms of  $x$ .

- (5) As  $(x, y) \in T \cap L = \{(0, 0), (1, -1)\}$ ,

$$\begin{cases} x = -y \\ x = \frac{1}{4} \left( 2y - 3 \pm \sqrt{-3(2y - 1)(2y + 3)} \right) \end{cases}$$

or

$$\begin{cases} y = -x \\ y = \frac{1}{4} \left( 2x - 3 \pm \sqrt{-3(2x+1)(2x-3)} \right) \end{cases}.$$

Both expressions are not unique, and thus cannot be solved for  $y$  in terms of  $x$  (or for  $x$  in terms of  $y$ ).

□

**Exercise 9.22.** *Given a similar discussion for*

$$f(x, y) = 2x^3 + 6xy^2 - 3x^2 + 3y^2.$$

*Outline.*

- (a) *Find the two points in  $\mathbb{R}^2$  at which the gradient of  $f$  is zero. Show that  $f$  has exactly one saddle point and one local minimum in  $\mathbb{R}^1$ .*
- (b) *Let  $S$  be the set of all  $(x, y) \in \mathbb{R}^2$  at which  $f(x, y) = 0$ . Find those points of  $S$  that have no neighborhoods in which the equation  $f(x, y) = 0$  can be solved for  $y$  in terms of  $x$  (or for  $x$  in terms of  $y$ ). Describe  $S$  as precisely as you can.*

*Proof of (a).*

(1)

(2)

□

*Proof of (b).*

- (1)  $S$  is a folium of Descartes with a double point at the origin and asymptote  $x + \frac{1}{2} = 0$ .

(2)

□

**Exercise 9.23.** ...

*Proof.*

(1)

(2)

□

**Exercise 9.24.** For  $(x, y) \neq (0, 0)$ , define  $\mathbf{f} = (f_1, f_2)$  by

$$f_1(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \quad f_2(x, y) = \frac{xy}{x^2 + y^2}.$$

Compute the rank of  $\mathbf{f}'(x, y)$ , and find the range of  $\mathbf{f}$ .

*Proof.*

(1)

$$[\mathbf{f}'(x, y)] = \begin{bmatrix} \frac{4xy^2}{(x^2+y^2)^2} & \frac{4x^2y}{(x^2+y^2)^2} \\ \frac{-y(x^2-y^2)}{(x^2+y^2)^2} & \frac{x(x^2-y^2)}{(x^2+y^2)^2} \end{bmatrix}.$$

(2) Show that  $\text{rank}([\mathbf{f}'(x, y)]) \neq 2$ . It is equivalent to show that  $\det[\mathbf{f}'(x, y)] = 0$ . Actually,

$$\det[\mathbf{f}'(x, y)] = \frac{4xy^2}{(x^2+y^2)^2} \cdot \frac{x(x^2-y^2)}{(x^2+y^2)^2} - \frac{4x^2y}{(x^2+y^2)^2} \cdot \frac{-y(x^2-y^2)}{(x^2+y^2)^2} = 0.$$

(3) Show that  $\text{rank}([\mathbf{f}'(x, y)]) \neq 0$ .

$$\begin{aligned} [\mathbf{f}'(x, y)] \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} \frac{4xy^2}{(x^2+y^2)^2} & \frac{4x^2y}{(x^2+y^2)^2} \\ \frac{-y(x^2-y^2)}{(x^2+y^2)^2} & \frac{x(x^2-y^2)}{(x^2+y^2)^2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{4xy^2}{(x^2+y^2)^2} \\ \frac{-y(x^2-y^2)}{(x^2+y^2)^2} \end{bmatrix} \\ &\neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

for all  $(x, y) \in \mathbb{R}^2 - \{(0, 0)\}$ .

(4) Since the rank of  $\mathbf{f}'$  is the dimension of the subspace  $\mathcal{R}(\mathbf{f}')$  in  $\mathbb{R}^2$ ,  $\text{rank}([\mathbf{f}'(x, y)]) = 0, 1, 2$ . By (2)(3),  $\text{rank}([\mathbf{f}'(x, y)]) = 1$ .

(5) Show that the range of  $\mathbf{f}$  is an ellipse

$$E = \{(s, t) \in \mathbb{R}^2 : s^2 + 4t^2 = 1\}.$$

(a) Clearly,  $(f_1(x, y), f_2(x, y)) \in E$ .

(b) Conversely, for any  $(s, t) \in E$  write

$$s = \cos \theta \quad \text{and} \quad t = \frac{1}{2} \sin \theta$$

for some unique  $\theta \in [0, 2\pi)$  (Theorem 8.7(d)). By the tangent half-angle formula,

$$s = \cos \theta = \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}} \quad \text{and} \quad t = \frac{1}{2} \sin \theta = \frac{\tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}.$$

Thus, there exists a point  $(1, \tan \frac{\theta}{2}) \in \mathbb{R}^2$  such that

$$f : \left(1, \tan \frac{\theta}{2}\right) \mapsto (s, t) \in E.$$

- (c) Or we can do a linear projection from a given point  $P = (1, 0)$ , say for any  $\lambda \in \mathbb{R}^1$  we define a line through  $P$  with slope  $-\lambda$  meeting  $E$  in a further point

$$Q_\lambda = \left( \frac{\lambda^2 - 1}{\lambda^2 + 1}, \frac{\lambda}{\lambda^2 + 1} \right).$$

Might define  $Q_\infty = P$ . Graphically and informally,

$$\{Q_\lambda : \lambda \in \mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}\} = E.$$

Therefore,  $f(1, 0) = P$  and  $f(\lambda, 1) \in E - \{P\}$ .

By (a)(b), the range of  $\mathbf{f}$  is exactly the same as an ellipse  $E$ .

□

#### Exercise 9.25. ...

*Proof.*

(1)

(2)

□

#### Exercise 9.26. ...

*Proof.*

(1)

(2)

□

**Exercise 9.27.** Put  $f(0, 0) = 0$ , and

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$$

if  $(x, y) \neq (0, 0)$ . Prove that

- (a)  $f, D_1f, D_2f$  are continuous in  $\mathbb{R}^2$ .
- (b)  $D_{12}f$  and  $D_{21}f$  exist at every point of  $\mathbb{R}^2$ , and are continuous except at  $(0, 0)$ .
- (c)  $(D_{12}f)(0, 0) = 1$ , and  $(D_{21}f)(0, 0) = -1$ .

*Proof of (a).*

- (1) Show that  $f$  is continuous in  $\mathbb{R}^2$ .

- (a) Clearly,  $f(x, y)$  is continuous if  $(x, y) \neq (0, 0)$ . So it suffices to show that

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = f(0, 0) = 0.$$

- (b) Write  $x = r \cos \theta$  and  $y = r \sin \theta$  in the polar coordinates. (Here  $r > 0$ .) Hence

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} f(x, y) &= \lim_{(x,y) \rightarrow (0,0)} \frac{xy(x^2 - y^2)}{x^2 + y^2} \\ &= \lim_{r \rightarrow 0} r^2 \cos \theta \sin \theta (\cos^2 \theta - \sin^2 \theta) \\ &= 0 \end{aligned}$$

since  $\cos \theta \sin \theta (\cos^2 \theta - \sin^2 \theta)$  is bounded by 2.

- (2) Show that  $D_1f$  is continuous in  $\mathbb{R}^2$ .

- (a)  $(x, y) \neq (0, 0)$  implies that

$$(D_1f)(x, y) = \frac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2}.$$

Besides,

$$\begin{aligned} (D_1f)(0, 0) &= \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x - 0} \\ &= \lim_{x \rightarrow 0} \frac{0}{x} \\ &= 0. \end{aligned}$$



In summary,

$$(D_1f)(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0) \\ \frac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0). \end{cases}$$

- (b) Clearly,  $(D_1f)(x, y)$  is continuous if  $(x, y) \neq (0, 0)$ . So it suffices to show that

$$\lim_{(x, y) \rightarrow (0, 0)} (D_1f)(x, y) = (D_1f)(0, 0) = 0.$$

- (c) Similar to (1)(b). Write  $x = r \cos \theta$  and  $y = r \sin \theta$  in the polar coordinates. (Here  $r > 0$ .) Hence

$$\begin{aligned} \lim_{(x, y) \rightarrow (0, 0)} (D_1f)(x, y) &= \lim_{(x, y) \rightarrow (0, 0)} \frac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2} \\ &= \lim_{r \rightarrow 0} r(\cos^4 \theta \sin \theta + 4 \cos^2 \theta \sin^3 \theta - \sin^5 \theta) \\ &= 0 \end{aligned}$$

since  $\cos^4 \theta \sin \theta + 4 \cos^2 \theta \sin^3 \theta - \sin^5 \theta$  is bounded by 6.

- (3) Similar to (2). *Show that  $D_2f$  is continuous in  $\mathbb{R}^2$ .*

- (a)  $(x, y) \neq (0, 0)$  implies that

$$(D_2f)(x, y) = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2 + y^2)^2}.$$

Besides,

$$\begin{aligned} (D_2f)(0, 0) &= \lim_{y \rightarrow 0} \frac{f(0, y) - f(0, 0)}{y - 0} \\ &= \lim_{y \rightarrow 0} \frac{0}{y} \\ &= 0. \end{aligned}$$

In summary,

$$(D_2f)(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0) \\ \frac{x^5 - 4x^3y^2 - xy^4}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0). \end{cases}$$

- (b) Clearly,  $(D_2f)(x, y)$  is continuous if  $(x, y) \neq (0, 0)$ . So it suffices to show that

$$\lim_{(x, y) \rightarrow (0, 0)} (D_2f)(x, y) = (D_2f)(0, 0) = 0.$$

- (c) Similar to (1)(b). Write  $x = r \cos \theta$  and  $y = r \sin \theta$  in the polar coordinates. (Here  $r > 0$ .) Hence

$$\begin{aligned}\lim_{(x,y) \rightarrow (0,0)} (D_2 f)(x, y) &= \lim_{(x,y) \rightarrow (0,0)} \frac{x^5 - 4x^3 y^2 - xy^4}{(x^2 + y^2)^2} \\ &= \lim_{r \rightarrow 0} r (\cos^5 \theta - 4 \cos^3 \theta \sin^2 \theta - \cos \theta \sin^4 \theta) \\ &= 0\end{aligned}$$

since  $\cos^5 \theta - 4 \cos^3 \theta \sin^2 \theta - \cos \theta \sin^4 \theta$  is bounded by 6.

□

*Proof of (b).*

- (1) Show that  $D_{12}f$  exists at every point of  $\mathbb{R}^2$ .

- (a)  $(x, y) \neq (0, 0)$  implies that

$$(D_{12}f)(x, y) = (D_1 D_2 f)(x, y) = \frac{x^6 + 9x^4 y^2 - 9x^2 y^4 - y^6}{(x^2 + y^2)^3}.$$

- (b) Besides,

$$\begin{aligned}(D_{12}f)(0, 0) &= \lim_{x \rightarrow 0} \frac{(D_2 f)(x, 0) - (D_2 f)(0, 0)}{x - 0} \\ &= \lim_{x \rightarrow 0} \frac{x}{x} \\ &= 1.\end{aligned}$$

In summary,

$$(D_{12}f)(x, y) = \begin{cases} 1 & \text{if } (x, y) = (0, 0) \\ \frac{x^6 + 9x^4 y^2 - 9x^2 y^4 - y^6}{(x^2 + y^2)^3} & \text{if } (x, y) \neq (0, 0). \end{cases}$$

- (2) Show that  $D_{12}f$  is continuous except at  $(0, 0)$ .

- (a) Clearly,  $(D_{12}f)(x, y)$  is continuous if  $(x, y) \neq (0, 0)$ . So it suffices to show that

$$\lim_{(x,y) \rightarrow (0,0)} (D_{12}f)(x, y)$$

does not exist.

- (b) Take

$$\mathbf{p}_n = \left( \frac{1}{n}, 0 \right) \quad \text{and} \quad \mathbf{q}_n = \left( 0, \frac{1}{n} \right)$$

for  $n = 1, 2, 3, \dots$ . So  $\lim \mathbf{p}_n = \lim \mathbf{q}_n = \mathbf{0}$ ,

$$\lim (D_{12}f)(\mathbf{p}_n) = 1 \quad \text{and} \quad \lim (D_{12}f)(\mathbf{q}_n) = -1.$$

Hence  $\lim_{(x,y) \rightarrow (0,0)} (D_{12}f)(x, y)$  does not exist.

(3) Show that  $D_{21}f$  exists at every point of  $\mathbb{R}^2$ . Similar to (1).

(a)  $(x, y) \neq (0, 0)$  implies that

$$(D_{21}f)(x, y) = (D_2D_1f)(x, y) = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2 + y^2)^3},$$

which is the same as  $(D_{12}f)(x, y)$ .

(b) Besides,

$$\begin{aligned}(D_{21}f)(0, 0) &= \lim_{y \rightarrow 0} \frac{(D_1f)(0, y) - (D_1f)(0, 0)}{y - 0} \\ &= \lim_{y \rightarrow 0} \frac{-y}{y} \\ &= -1.\end{aligned}$$

In summary,

$$(D_{21}f)(x, y) = \begin{cases} -1 & \text{if } (x, y) = (0, 0) \\ \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2 + y^2)^3} & \text{if } (x, y) \neq (0, 0). \end{cases}$$

(4) Show that  $D_{21}f$  is continuous except at  $(0, 0)$ . Exactly the same as (2) since  $(D_{21}f)(x, y) = (D_{12}f)(x, y)$  if  $(x, y) \neq (0, 0)$ .

□

*Proof of (c).* See (2)(4) in the proof of (b). □

**Exercise 9.28.** ...

*Proof.*

(1)

(2)

□

**Exercise 9.29.** ...

*Proof.*

(1)

(2)

□

**Exercise 9.30. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.31. ...**

*Proof.*

(1)

(2)

□