

## Chapter 3: Numerical Sequences and Series

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**Exercise 3.1.** Prove that the convergence of  $\{s_n\}$  implies convergence of  $\{|s_n|\}$ . Is the converse true?

*Proof.*

- (1) Since  $\{s_n\}$  is convergent, there is  $s \in \mathbb{R}^1$  with the following property: given any  $\varepsilon > 0$ , there is  $N$  such that  $|s_n - s| < \varepsilon$  whenever  $n \geq N$ . So

$$||s_n| - |s|| \leq |s_n - s| < \varepsilon$$

(Exercise 1.13). That is,  $\{|s_n|\}$  converges to  $|s|$ .

- (2) The converse is not true by considering  $s_n = (-1)^{n+1}$ .

□

**Exercise 3.2.** Calculate  $\lim_{n \rightarrow \infty} (\sqrt{n^2 + n} - n)$ .

*Proof.*

$$\sqrt{n^2 + n} - n = \frac{n}{\sqrt{n^2 + n} + n} = \frac{1}{\sqrt{1 + \frac{1}{n}} + 1} \rightarrow \frac{1}{1 + 1} = \frac{1}{2}$$

as  $n \rightarrow \infty$ . □

*Proof ( $\varepsilon - N$  argument).* Let  $s_n = \sqrt{n^2 + n} - n$ . Show that the sequence  $\{s_n\}$  converges to  $s = \frac{1}{2}$ . Given any  $\varepsilon > 0$ , there is  $N > \frac{1}{\varepsilon}$  such that

$$\begin{aligned} |s_n - s| &= \left| (\sqrt{n^2 + n} - n) - \frac{1}{2} \right| = \left| \frac{1}{\sqrt{1 + \frac{1}{n}} + 1} - \frac{1}{2} \right| \\ &= \left| \frac{2 - \left( \sqrt{1 + \frac{1}{n}} + 1 \right)}{2 \left( \sqrt{1 + \frac{1}{n}} + 1 \right)} \right| = \left| \frac{1 - \sqrt{1 + \frac{1}{n}}}{2 \left( \sqrt{1 + \frac{1}{n}} + 1 \right)} \right| \\ &= \left| \frac{1 - \left( 1 - \frac{1}{n} \right)}{2 \left( \sqrt{1 + \frac{1}{n}} + 1 \right)^2} \right| = \left| \frac{-\frac{1}{n}}{2 \left( \sqrt{1 + \frac{1}{n}} + 1 \right)^2} \right| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon \end{aligned}$$

whenever  $n \geq N$ .  $\square$

**Exercise 3.3.** If  $s_1 = \sqrt{2}$  and

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} \quad (n = 1, 2, 3, \dots),$$

prove that  $\{s_n\}$  converges, and that  $s_n < 2$  for  $n = 1, 2, 3, \dots$ .

The convergence of  $\{s_n\}$  implies there is  $s \in \mathbb{R}$  such that  $s_n \rightarrow s$  where  $s = \sqrt{2 + \sqrt{s}}$  and  $\sqrt{2} < s \leq 2$ . WolframAlpha shows that

$$s = \frac{1}{3} \left( -1 + \sqrt[3]{\frac{1}{2}(79 - 3\sqrt{249})} + \sqrt[3]{\frac{1}{2}(79 + 3\sqrt{249})} \right).$$

*Proof (Theorem 3.14).*

(1) Show that  $\{s_n\}$  is increasing (by mathematical induction).

(a) Show that  $s_2 > s_1$ . In fact,

$$s_2 = \sqrt{2 + \sqrt{s_1}} = \sqrt{2 + \sqrt{\sqrt{2}}} < \sqrt{2} = s_1.$$

(a) Show that  $s_{n+1} > s_n$  if  $s_n > s_{n-1}$ .

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} > \sqrt{2 + \sqrt{s_{n-1}}} = s_n.$$

By mathematical induction,  $\{s_n\}$  is (strictly) increasing.

(2) Show that  $\{s_n\}$  is bounded (by mathematical induction).

(a) Show that  $s_1 \leq 2$ .  $\sqrt{2} \leq 2$ .

(a) Show that  $s_{n+1} \leq 2$  if  $s_n \leq 2$ .

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} \leq \sqrt{2 + \sqrt{2}} < 2.$$

By mathematical induction,  $\{s_n\}$  is bounded by 2.

Hence,  $\{s_n\}$  converges since  $\{s_n\}$  is increasing and bounded (Theorem 3.14).  $\square$

**Exercise 3.4.** Find the upper and lower limits of the sequences  $\{s_n\}$  defined by

$$s_1 = 0; s_{2m} = \frac{s_{2m-1}}{2}; s_{2m+1} = \frac{1}{2} + s_{2m}.$$

Write out the first few terms of  $\{s_n\}$ :

$$0, 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{3}{8}, \frac{7}{8}, \frac{7}{16}, \frac{15}{16}, \dots$$

It suggests us

$$s_{2m+1} = 1 - \frac{1}{2^m} \quad (m = 0, 1, 2, \dots),$$

$$s_{2m} = \frac{1}{2} - \frac{1}{2^m} \quad (m = 1, 2, 3, \dots).$$

*Proof.*

(1) *Show that*

$$s_{2m+1} = 1 - \frac{1}{2^m} \quad (m = 0, 1, 2, \dots),$$

$$s_{2m} = \frac{1}{2} - \frac{1}{2^m}. \quad (m = 1, 2, 3, \dots)$$

Apply mathematical induction.

(2) The upper limit is 1.

(3) The lower limit is  $\frac{1}{2}$ .

□

**Exercise 3.5.** For any two real sequences  $\{a_n\}$ ,  $\{b_n\}$ , prove that

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n$$

provided the sum of the right is not of the form  $\infty - \infty$ .

*Proof.* Write  $\alpha = \limsup_{n \rightarrow \infty} a_n$  and  $\beta = \limsup_{n \rightarrow \infty} b_n$ .

(1)  $\alpha = \infty$  and  $\beta = \infty$ . Nothing to do.

(2)  $\alpha = -\infty$  and  $\beta = -\infty$ . Since  $\alpha = -\infty < \infty$ , there exists  $M'$  such that  $a_n < M'$  for all  $n$ . For any real  $M$ ,  $a_n > M - M'$  for at most a finite number of values of  $n$  (Theorem 3.17(a)). Hence  $a_n + b_n > M$  for at most a finite number of values of  $n$ . Hence  $\limsup_{n \rightarrow \infty} (a_n + b_n) = -\infty$ , or

$$\limsup_{n \rightarrow \infty} (a_n + b_n) = \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n$$

in this case.

- (3)  $\alpha$  and  $\beta$  are finite. (Similar to the argument in Theorem 3.37.) Choose  $\alpha' > \alpha$  and  $\beta' > \beta$ . There is an integer  $N$  such that

$$\alpha' \geq a_n \text{ and } \beta' \geq b_n$$

whenever  $n \geq N$ . Hence

$$a_n + b_n \leq \alpha' + \beta'$$

whenever  $n \geq N$ . Take  $\limsup$  to get Hence

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \alpha' + \beta'.$$

Since the inequality is true for every  $\alpha' > \alpha$  and  $\beta' > \beta$ , we have

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n.$$

□

**Exercise 3.7.** Prove that the convergence of  $\sum a_n$  implies the convergence of

$$\sum \frac{\sqrt{a_n}}{n},$$

if  $a_n \geq 0$ .

*Proof (Cauchy's inequality).*

- (1) Show that  $\sum \frac{\sqrt{a_n}}{n}$  is bounded. For any  $k \in \mathbb{Z}^+$ ,

$$\begin{aligned} \left( \sum_{n=1}^k \frac{\sqrt{a_n}}{n} \right)^2 &\leq \left( \sum_{n=1}^k a_n \right) \left( \sum_{n=1}^k \frac{1}{n^2} \right) && \text{(Cauchy's inequality)} \\ &\leq \left( \sum_{n=1}^{\infty} a_n \right) \left( \sum_{n=1}^{\infty} \frac{1}{n^2} \right). && \left( \sum a_n, \sum \frac{1}{n^2} : \text{convergent} \right) \end{aligned}$$

Thus,  $\left( \sum_{n=1}^k \frac{\sqrt{a_n}}{n} \right)^2$  is bounded, or  $\sum_{n=1}^k \frac{\sqrt{a_n}}{n}$  is bounded.

- (2) Show that  $\sum_{n=1}^k \frac{\sqrt{a_n}}{n}$  is increasing. It is clear due to  $\frac{\sqrt{a_n}}{n} \geq 0$ .

By Theorem 3.14,  $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$  converges. □

*Proof (AM-GM inequality). Show that  $\sum \frac{\sqrt{a_n}}{n}$  is bounded.*

$$\begin{aligned} \frac{\sqrt{a_n}}{n} &\leq \frac{1}{2} \left( a_n + \frac{1}{n^2} \right) && \text{(AM-GM inequality)} \\ \sum_{n=1}^k \frac{\sqrt{a_n}}{n} &\leq \frac{1}{2} \left( \sum_{n=1}^k a_n + \sum_{n=1}^k \frac{1}{n^2} \right) \\ &\leq \frac{1}{2} \left( \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} \frac{1}{n^2} \right). && \left( \sum a_n, \sum \frac{1}{n^2} : \text{convergent} \right) \end{aligned}$$

Thus,  $\sum_{n=1}^k \frac{\sqrt{a_n}}{n}$  is bounded. The rest proof is the same as previous.  $\square$

**Exercise 3.12.** Suppose  $a_n > 0$  and  $\sum a_n$  converges. Put

$$r_n = \sum_{m=n}^{\infty} a_m.$$

(a) Prove that

$$\frac{a_m}{r_m} + \dots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$

if  $m < n$ , and deduce that  $\sum \frac{a_n}{r_n}$  diverges.

(b) Prove that

$$\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$$

and deduce that  $\sum \frac{a_n}{\sqrt{r_n}}$  converges.

*Note.*

- (1) Each  $r_n$  is positive and finite (since  $a_n > 0$  and  $\sum a_n$  converges).
- (2)  $\{r_n\}$  is monotonic decreasing (since  $a_n > 0$ ).
- (3)  $\{r_n\}$  converges to 0 (since  $\sum a_n$  converges).

*Proof of (a).*

(1)

$$\begin{aligned}
\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} &> \frac{a_m}{r_m} + \cdots + \frac{a_n}{r_m} && (r_m > r_k \text{ for } k = m+1, \dots, n) \\
&= \frac{a_m + \cdots + a_n}{r_m} \\
&= \frac{r_m - r_{n+1}}{r_m} && (\text{Definition of } r_k) \\
&> \frac{r_m - r_n}{r_m} && (r_n > r_{n+1}) \\
&= 1 - \frac{r_n}{r_m}.
\end{aligned}$$

(2) (Reductio ad absurdum) If  $\sum \frac{a_n}{r_n}$  were converged, then given  $\varepsilon = \frac{1}{64} > 0$  there is an integer  $N$  such that

$$\left| \frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} \right| < \frac{1}{64} \text{ whenever } n \geq m \geq N$$

(Theorem 3.22). By (1), let  $m = N$  to get

$$1 - \frac{r_n}{r_N} < \frac{1}{64} \text{ whenever } n \geq N,$$

or

$$r_n > \frac{63}{64} r_N,$$

contrary to the assumption that  $\{r_n\}$  converges to 0 (since  $\sum a_n$  converges).

□

*Proof of (b).*

(1) Note that each  $r_n$  is positive and finite, and thus

$$\begin{aligned}
\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}}) &\iff \frac{r_n - r_{n+1}}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}}) \\
&\iff \frac{\sqrt{r_n} + \sqrt{r_{n+1}}}{\sqrt{r_n}} < 2 \\
&\iff \sqrt{r_n} + \sqrt{r_{n+1}} < 2\sqrt{r_n} \\
&\iff \sqrt{r_{n+1}} < \sqrt{r_n} \\
&\iff r_{n+1} < r_n.
\end{aligned}$$

The last statement holds since  $\{r_n\}$  is monotonic decreasing.

(2) (a) Each term  $\frac{a_n}{\sqrt{r_n}}$  of  $\sum \frac{a_n}{\sqrt{r_n}}$  is nonnegative.

(b) The partial sum

$$\sum_{k=1}^n \frac{a_k}{\sqrt{r_k}} < \sum_{k=1}^n 2(\sqrt{r_k} - \sqrt{r_{k+1}}) = 2(\sqrt{r_1} - \sqrt{r_{n+1}}) < 2\sqrt{r_1}$$

is bounded by  $2\sqrt{r_1}$ .

By (a)(b),  $\sum \frac{a_n}{\sqrt{r_n}}$  converges (Theorem 3.24).

□

**Exercise 3.20.** Suppose  $\{p_n\}$  is a Cauchy sequence in a metric space  $X$ , and some subsequence  $\{p_{n_i}\}$  converges to a point  $p \in X$ . Prove that the full sequence  $\{p_n\}$  converges to  $p$ .

*Proof.* Given any  $\varepsilon > 0$ .

- (1) Since  $\{p_n\}$  is a Cauchy sequence, there exists a positive integer  $N_1$  such that

$$d(p_n, p_m) < \frac{\varepsilon}{2} \text{ whenever } n, m \geq N_1.$$

- (2) Since the subsequence  $\{p_{n_i}\}$  converges to a point  $p \in X$ , there exists a positive integer  $N_2$  such that

$$d(p_{n_i}, p) < \frac{\varepsilon}{2} \text{ whenever } n_i \geq N_2.$$

- (3) Let  $N = \max\{N_1, N_2\}$  be a positive integer. So

$$\begin{aligned} d(p_n, p) &\leq d(p_n, p_{n_i}) + d(p_{n_i}, p) && \text{(Definition 2.15(c))} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \text{ whenever } n, n_i \geq N && ((1)(2)) \\ &= \varepsilon \text{ whenever } n \geq N. \end{aligned}$$

Hence the full sequence  $\{p_n\}$  converges to  $p$ .

□

**Exercise 3.21.** Prove the following analogue of Theorem 3.10(b): If  $\{E_n\}$  is a sequence of closed and bounded sets in a complete metric space  $X$ , if  $E_n \supseteq E_{n+1}$ , and if

$$\lim_{n \rightarrow \infty} \text{diam}(E_n) = 0,$$

then  $\bigcap_{n=1}^{\infty} E_n$  consists of exactly one point.

Assume  $E_n \neq \emptyset$ . It is unnecessary to assume that  $E_n$  is bounded since we have the condition that  $\lim_{n \rightarrow \infty} \text{diam}(E_n) = 0$ .

*Note.* Every compact metric space is complete, but complete spaces need not be compact. In fact, a metric space is compact if and only if it is complete and totally bounded.

*Proof.*

- (1) Pick  $p_n \in E_n$  for  $n = 1, 2, \dots$
- (2) Show that  $\{p_n\}$  is a Cauchy sequence. Given any  $\varepsilon > 0$ . There is a positive integer  $N$  such that  $\text{diam}(E_n) < \varepsilon$  whenever  $n \geq N$ . Especially,

$$\text{diam}(E_N) < \varepsilon.$$

As  $m, n \geq N$ ,  $p_m \in E_m \subseteq E_N$  and  $p_n \in E_n \subseteq E_N$ . By the definition of the diameter of  $E_N$ ,

$$d(p_m, p_n) \leq \text{diam}(E_N) < \varepsilon \text{ whenever } m, n \geq N.$$

- (3) Since  $X$  is complete,  $\{p_n\}$  converges to a point  $p \in X$ .
- (4) Show that  $p \in \bigcap_{n=1}^{\infty} E_n$ . (Reductio ad absurdum) If there were some  $n$  such that  $p \notin E_n$ . Consider the subsequence

$$p_n, p_{n+1}, p_{n+2}, \dots$$

Note that all  $p_n, p_{n+1}, \dots$  are in  $E_n$ . By (3), it converges to  $p$ . Thus  $p$  is a limit point of  $E_n$ . Since  $E_n$  is closed,  $p \in E_n$ , which is absurd.

- (5) Show that  $\bigcap_{n=1}^{\infty} E_n = \{p\}$ . (Reductio ad absurdum) If there were  $q \in \bigcap_{n=1}^{\infty} E_n$  with  $q \neq p$ , then  $d(p, q) > 0$  (Definition 2.15(a)). It implies that

$$\text{diam}(E_n) \geq d(p, q) > 0 \text{ for all } n,$$

contrary to  $\lim_{n \rightarrow \infty} \text{diam}(E_n) = 0$ .

□

**Exercise 3.22 (Baire category theorem).** Suppose  $X$  is a complete metric space, and  $\{G_n\}$  is a sequence of dense open subsets of  $X$ . Prove Baire's theorem, namely, that  $\bigcap_{n=1}^{\infty} G_n$  is not empty. (In fact, it is dense in  $X$ .) (Hint: Find a shrinking sequence of neighborhoods  $E_n$  such that  $\overline{E_n} \subseteq G_n$ , and apply Exercise 3.21.)

*Proof.* Given any open set  $G_0$  in  $X$ , will show that

$$\bigcap_{n=0}^{\infty} G_n \neq \emptyset.$$



- (1) Since  $G_1$  is dense,  $G_0 \cap G_1$  is nonempty. Take any one point  $p_1$  in the open set  $G_0 \cap G_1$ , then there exists a closed neighborhood

$$V_1 = \{q \in X : d(q, p_1) < r_1\}$$

of  $p_1$  with  $r_1 < 1$  such that

$$V_1 \subseteq G_0 \cap G_1.$$

Take  $U_1 \subseteq E_1 \subseteq V_1$  such that

$$\begin{aligned} E_1 &= \left\{q \in X : d(q, p_1) \leq \frac{r_1}{64}\right\} \subseteq V_1, \\ U_1 &= \left\{q \in X : d(q, p_1) < \frac{r_1}{89}\right\} \subseteq E_1. \end{aligned}$$

- (2) Suppose  $V_n, E_n, U_n$  have been constructed, take any one point  $p_{n+1}$  in the open set  $U_n \cap G_{n+1}$ , there exists an open neighborhood

$$V_{n+1} = \{q \in X : d(q, p_{n+1}) < r_{n+1}\}$$

of  $p_{n+1}$  with  $r_{n+1} < \frac{1}{n+1}$  such that

$$V_{n+1} \subseteq U_n \cap G_{n+1}.$$

Take  $U_1 \subseteq E_1 \subseteq V_1$  such that

$$\begin{aligned} E_{n+1} &= \left\{q \in X : d(q, p_{n+1}) \leq \frac{r_{n+1}}{64}\right\} \subseteq V_{n+1}, \\ U_{n+1} &= \left\{q \in X : d(q, p_{n+1}) < \frac{r_{n+1}}{89}\right\} \subseteq E_{n+1}. \end{aligned}$$

- (3) Note that

- (a)  $E_n$  is closed and nonempty (since  $p_n \in E_n$ ).
- (b)  $\lim_{n \rightarrow \infty} \text{diam}(E_n) = 0$  (since  $\text{diam}(E_n) \leq 2 \cdot \frac{r_n}{64} < r_n < \frac{1}{n}$ .)
- (c)  $E_1 \supseteq E_2 \supseteq \cdots$  (since  $E_{n+1} \subseteq V_{n+1} \subseteq U_n \cap G_{n+1} \subseteq U_n \subseteq E_n$ ).

Since  $X$  is complete, by Exercise 3.21,

$$\bigcap_{n=1}^{\infty} E_n = \{p\}$$

for some  $p \in X$ .

(4) Hence

$$\begin{aligned}
p \in \bigcap_{n=1}^{\infty} E_n &\iff p \in E_n \text{ for all } n = 1, 2, 3, \dots \\
&\implies p \in E_1 \subseteq G_0 \cap G_1 \text{ and } p \in E_{n+1} \subseteq U_n \cap G_{n+1} \subseteq G_{n+1} \\
&\implies p \in G_0 \cap G_1 \cap \dots = \bigcap_{n=0}^{\infty} G_n \\
&\implies \bigcap_{n=0}^{\infty} G_n \neq \emptyset.
\end{aligned}$$

□

**Exercise 3.23.** Suppose  $\{p_n\}$  and  $\{q_n\}$  are Cauchy sequences in a metric space  $X$ . Show that the sequence  $\{d(p_n, q_n)\}$  converges. (Hint: For any  $m, n$ ,

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n);$$

it follows that

$$|d(p_n, q_n) - d(p_m, q_m)|$$

is small if  $m$  and  $n$  are large.)

*Proof.* Given any  $\varepsilon > 0$ .

(1) Since  $\{p_n\}$  and  $\{q_n\}$  are Cauchy sequences, there exists  $N$  such that

$$d(p_n, p_m) < \frac{\varepsilon}{2} \text{ and } d(q_m, q_n) < \frac{\varepsilon}{2}$$

whenever  $m, n \geq N$ .

(2) Note that

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n).$$

It follows that

$$|d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_m, q_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus  $\{d(p_n, q_n)\}$  is a Cauchy sequence in  $\mathbb{R}^1$  (not in  $X$ ).

(3) Since  $\mathbb{R}^1$  is a complete metric space,  $\{d(p_n, q_n)\}$  converges.

□

**Exercise 3.24.** Let  $X$  be a metric space.

- (a) Call two Cauchy sequences  $\{p_n\}, \{q_n\}$  in  $X$  equivalent if

$$\lim_{n \rightarrow \infty} d(p_n, q_n) = 0.$$

Prove that this is an equivalence relation.

- (b) Let  $X^*$  be the set of all equivalence classes so obtained. If  $P \in X^*$ ,  $Q \in X^*$ ,  $\{p_n\} \in P$ ,  $\{q_n\} \in Q$ , define

$$\Delta(P, Q) = \lim_{n \rightarrow \infty} d(p_n, q_n);$$

by Exercise 3.23, this limit exists. Show that the number  $\Delta(P, Q)$  is unchanged if  $\{p_n\}$  and  $\{q_n\}$  are replaced by equivalent sequences, and hence that  $\Delta$  is a distance function in  $X^*$ .

- (c) Prove that the resulting metric space  $X^*$  is complete.  
 (d) For each  $p \in X$ , there is a Cauchy sequence all of whose terms are  $p$ ; let  $P_p$  be the element of  $X^*$  which contains this sequence. Prove that

$$\Delta(P_p, P_q) = d(p, q)$$

for all  $p, q \in X$ . In other words, the mapping  $\varphi$  defined by  $\varphi(p) = P_p$  is an isometry (i.e., a distance-preserving mapping) of  $X$  into  $X^*$ .

- (e) Prove that  $\varphi(X)$  is dense in  $X^*$ , and that  $\varphi(X) = X^*$  if  $X$  is complete. By (d), we may identify  $X$  and  $\varphi(X)$  and thus regard  $X$  as embedded in the complete metric space  $X^*$ . We call  $X^*$  the **completion** of  $X$ .

*Proof of (a).* Given Cauchy sequences  $\{p_n\}, \{q_n\}, \{r_n\}$  in  $X$ .

- (1) (*Reflexivity*)

$$\lim_{n \rightarrow \infty} d(p_n, q_n) = \lim_{n \rightarrow \infty} 0 = 0$$

by the reflexivity of the metric function  $d$ .

- (2) (*Symmetry*)

$$\lim_{n \rightarrow \infty} d(p_n, q_n) = \lim_{n \rightarrow \infty} d(q_n, p_n) = 0$$

by the symmetry of the metric function  $d$ .

- (3) (*Transitivity*) Suppose that  $\lim_{n \rightarrow \infty} d(p_n, q_n) = \lim_{n \rightarrow \infty} d(q_n, r_n) = 0$ . By the triangle inequality of the metric function  $d$ , we have

$$0 \leq d(p_n, r_n) \leq d(p_n, q_n) + d(q_n, r_n).$$

Take limit to get

$$\begin{aligned}
0 &\leq \lim_{n \rightarrow \infty} d(p_n, r_n) \\
&\leq \lim_{n \rightarrow \infty} (d(p_n, q_n) + d(q_n, r_n)) \\
&= \lim_{n \rightarrow \infty} d(p_n, q_n) + \lim_{n \rightarrow \infty} d(q_n, r_n) \\
&= 0
\end{aligned}$$

or  $\lim_{n \rightarrow \infty} d(p_n, r_n) = 0$ .

□

*Proof of (b).*

(1) *Show that  $\Delta$  is well-defined.* Given any  $\{p_n\}, \{p'_n\} \in P$  and  $\{q_n\}, \{q'_n\} \in Q$ .

(a)  $\lim_{n \rightarrow \infty} d(p_n, p'_n) = 0$  since  $\{p_n\}$  and  $\{p'_n\}$  are in the same equivalence class.

(b)  $\lim_{n \rightarrow \infty} d(q_n, q'_n) = 0$  (similar to (a)).

(c) *Show that  $\lim_{n \rightarrow \infty} d(p_n, q_n) \leq \lim_{n \rightarrow \infty} d(p'_n, q'_n)$ .* Since  $d(p_n, q_n) \leq d(p_n, p'_n) + d(p'_n, q'_n) + d(q'_n, q_n)$ , take limit to get

$$\begin{aligned}
\lim_{n \rightarrow \infty} d(p_n, q_n) &\leq \lim_{n \rightarrow \infty} (d(p_n, p'_n) + d(p'_n, q'_n) + d(q'_n, q_n)) \\
&= \lim_{n \rightarrow \infty} d(p_n, p'_n) + \lim_{n \rightarrow \infty} d(p'_n, q'_n) + \lim_{n \rightarrow \infty} d(q'_n, q_n) \\
&= 0 + \lim_{n \rightarrow \infty} d(p'_n, q'_n) + 0 \\
&= \lim_{n \rightarrow \infty} d(p'_n, q'_n)
\end{aligned}$$

since (a)(b).

(d) *Show that  $\lim_{n \rightarrow \infty} d(p_n, q_n) \geq \lim_{n \rightarrow \infty} d(p'_n, q'_n)$ .* Similar to (c).

By (c)(d),  $\lim_{n \rightarrow \infty} d(p_n, q_n) = \lim_{n \rightarrow \infty} d(p'_n, q'_n)$ , or  $\Delta(P, Q)$  is well-defined.

(2) *Show that  $\Delta$  is a metric.*

(a) *Show that  $\Delta(P, Q) > 0$  if  $P \neq Q$ ;  $\Delta(P, P) = 0$ .* It is the definition of  $\Delta$ .

(b) *Show that  $\Delta(P, Q) = \Delta(Q, P)$ .* Similar to the argument in (a)(2).

(c) *Show that  $\Delta(P, Q) \leq \Delta(P, R) + \Delta(R, Q)$ .* Similar to the argument in (a)(3).

□

*Proof of (c). Show that  $\{P_k\}_{k=1}^\infty$  converges to  $P$  in  $(X^*, \Delta)$  for any given Cauchy sequence  $\{P_k\}$ .*

- (1) Take a Cauchy sequence  $\{p_n^{(k)}\}_{n=1}^\infty$  to represent  $P_k$  for each  $k$ . We will construct a Cauchy sequence  $\{p_k\}$  in  $(X, d)$  such that  $\{P_k\}$  converges to  $P$  which is the equivalent class of  $\{p_k\}$ .
- (2) For each  $k$ , there exists  $N_k$  such that

$$d(p_m^{(k)}, p_n^{(k)}) < \frac{1}{k} \text{ whenever } m, n \geq N_k.$$

Especially,

$$d(p_m^{(k)}, p_{N_k}^{(k)}) < \frac{1}{k} \text{ whenever } m \geq N_k.$$

Let  $p_k = p_{N_k}^{(k)}$  and collect all  $p_k$  as  $\{p_k\}_{k=1}^\infty$ .

- (3) Show that  $\{p_k\}$  is a Cauchy sequence in  $(X, d)$ . Note that for any  $k$ , we have

$$\begin{aligned} d(p_m, p_n) &= d(p_{N_m}^{(m)}, p_{N_n}^{(n)}) \\ &\leq d(p_{N_m}^{(m)}, p_k^{(m)}) + d(p_k^{(m)}, p_k^{(n)}) + d(p_k^{(n)}, p_{N_n}^{(n)}). \end{aligned}$$

Let  $k \rightarrow \infty$ , we have

$$\begin{aligned} d(p_m, p_n) &\leq \limsup_{k \rightarrow \infty} \left[ d(p_{N_m}^{(m)}, p_k^{(m)}) + d(p_k^{(m)}, p_k^{(n)}) + d(p_k^{(n)}, p_{N_n}^{(n)}) \right] \\ &\leq \frac{1}{m} + \Delta(P_m, P_n) + \frac{1}{n} \end{aligned}$$

for any  $m, n$  (by (2)). Let  $m, n \rightarrow \infty$ , we establish the result (since  $\{P_k\}$  is Cauchy).

- (4) Show that  $\{P_k\}$  converges to  $P \ni \{p_k\}$ . Given any  $\varepsilon > 0$ . Since  $\{p_k\}$  is Cauchy (3), there is  $N > \frac{2}{\varepsilon}$  such that

$$d(p_m, p_n) < \frac{\varepsilon}{2} \text{ whenever } m, n \geq N.$$

Note that

$$\begin{aligned} d(p_n^{(k)}, p_n) &= d(p_n^{(k)}, p_{N_n}^{(n)}) \\ &\leq d(p_n^{(k)}, p_{N_k}^{(k)}) + d(p_{N_k}^{(k)}, p_{N_n}^{(n)}). \end{aligned}$$

For any  $k \geq N$ , let  $n \rightarrow \infty$  to get

$$\begin{aligned}
\Delta(P_k, P) &= \lim_{n \rightarrow \infty} d(p_n^{(k)}, p_n) \\
&\leq \limsup_{n \rightarrow \infty} d(p_n^{(k)}, p_{N_k}^{(k)}) + \limsup_{n \rightarrow \infty} d(p_{N_k}^{(k)}, p_{N_n}^{(n)}) \\
&< \frac{1}{k} + \frac{\varepsilon}{2} \\
&\leq \frac{1}{N} + \frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&< \varepsilon.
\end{aligned}$$

Hence,  $(X^*, \Delta)$  is complete.  $\square$

*Proof of (d).*

- (1) Define  $\{p_n\}$  by  $p_n = p$  ( $n = 1, 2, \dots$ ) for any  $p \in X$ .
- (2) Show that  $\{p_n\}$  is a Cauchy sequence.  $d(p_m, p_n) = d(p, p) = 0$ .
- (3) Take  $\{p\} \in P_p$  and  $\{q\} \in P_q$ . Then

$$\Delta(P_p, P_q) = \lim_{n \rightarrow \infty} d(p_n, q_n) = \lim_{n \rightarrow \infty} d(p, q) = d(p, q).$$

$\square$

*Proof of (e).*

- (1) Show that  $\varphi(X)$  is dense in  $X^*$ . Given any  $P \in X^*$ , any  $\{p_n\} \in P$  and any  $\varepsilon > 0$ . Since  $\{p_n\}$  is Cauchy, there is  $N$  such that

$$d(p_m, p_n) < \frac{\varepsilon}{64} \text{ whenever } m, n \geq N.$$

Note that  $p_N \in X$ . Pick  $\{p_N\} \in P_{p_N} = \varphi(p_N) \in \varphi(X)$ . So

$$\Delta(P, P_{p_N}) = \lim_{n \rightarrow \infty} d(p_n, p_N) \leq \frac{\varepsilon}{64} < \varepsilon.$$

Hence  $\varphi(X)$  is dense in  $X^*$ .

- (2) Show that  $\varphi(X) = X^*$  if  $X$  is complete. Given any  $P \in X^* \ni \{p_n\}$ . Since  $X$  is complete, a Cauchy sequence  $\{p_n\}$  converges to  $p \in X$ . Pick  $\{p\} \in P_p = \varphi(p) \in \varphi(X)$ . So

$$\Delta(P, P_p) = \lim_{n \rightarrow \infty} d(p_n, p) = 0,$$

or  $P = P_p$ , or  $\varphi(X) = X^*$ .

□

**Exercise 3.25.** Let  $X$  be the metric space whose points are rational numbers, with the metric  $d(x, y) = |x - y|$ . What is the completion of this space? (Compare Exercise 3.24.)

*Proof.* By Exercise 3.24, we can identify one completion  $(X^*, \Delta)$  with  $(\mathbb{R}, |\cdot|)$  (Theorem 3.11(c) and Theorem 1.20(b)). □

**Supplement (Uniqueness of completion).** Show that a completion of a metric space is unique up to isometry.

*Outline.* Suppose there are two completions  $\{\varphi_i, (X_i^*, d_i^*)\}$  ( $i = 1, 2$ ) of  $(X, d)$ . Let

$$\psi = \varphi_2 \circ \varphi_1^{-1} : \varphi_1(X) \rightarrow \varphi_2(X)$$

be an isometry from  $\varphi_1(X)$  into  $\varphi_2(X)$ . The sets  $\varphi_i(X)$  ( $i = 1, 2$ ) are dense in  $X_i^*$ . So we can extend  $\psi$  (continuously) to a map  $\psi : X_1^* \rightarrow X_2^*$ .

*Proof.*

- (1) Given any  $P \in X_1^*$ , there is a Cauchy sequence  $\{P_{p_n}\} = \{\varphi_1(p_n)\}$  in  $\varphi_1(X)$  converging to  $P$ . Define  $\psi(P)$  by

$$\psi(P) = \lim_{n \rightarrow \infty} \psi(P_{p_n}).$$

- (2) Show that  $\psi$  is well-defined. Note that

$$\begin{aligned} \Delta_2(\psi(P_{p_m}), \psi(P_{p_n})) &= \Delta_2(\psi(\varphi_1(p_m)), \psi(\varphi_1(p_n))) \\ &= \Delta_2(\varphi_2(p_m), \varphi_2(p_n)) \\ &= d(p_m, p_n) && (\varphi_2 \text{ is isometric}) \\ &= \Delta_1(\varphi_1(p_m), \varphi_1(p_n)) && (\varphi_1 \text{ is isometric}) \\ &= \Delta_1(P_{p_m}, P_{p_n}). \end{aligned}$$

So  $\{\psi(P_{p_n})\}$  is a Cauchy sequence in  $\varphi_2(X)$  if (and only if)  $\{P_{p_n}\}$  is a Cauchy sequence in  $\varphi_1(X)$ . Since  $X_2^*$  is complete,  $\{\psi(P_{p_n})\}$  converges to  $\psi(P)$ . The limit  $\psi(P)$  is uniquely determined since  $\Delta_2$  is a metric function.

- (3) Since  $\psi$  is an isometry from  $\varphi_1(X)$  into  $\varphi_2(X)$ ,

$$\psi^{-1} = \varphi_1 \circ \varphi_2^{-1} : \varphi_2(X) \rightarrow \varphi_1(X)$$

is an isometry from  $\varphi_2(X)$  into  $\varphi_1(X)$ . Besides,  $\psi^{-1} \circ \psi = 1_{\varphi_1(X)}$  and  $\psi \circ \psi^{-1} = 1_{\varphi_2(X)}$ .

- (4) *Show that  $\psi$  is surjective.* Given any  $Q \in X_2^*$ , there is a Cauchy sequence  $\{P_{q_n}\} = \{\varphi_2(q_n)\}$  in  $\varphi_2(X)$  converging to  $Q$ . Define

$$P_{p_n} = \psi^{-1}(P_{q_n}) \in \varphi_1(X).$$

$\psi(P_{p_n}) = 1_{\varphi_2(X)}(P_{q_n}) = P_{q_n}$ . Besides, similar to argument in (2),  $\{P_{p_n}\}$  is a Cauchy sequence in  $\varphi_1(X)$ . Since  $X_1^*$  is complete,  $\{P_{p_n}\}$  converges to  $P \in X_1^*$ . It is easy to verify that  $\psi(P) = Q$ .

- (5) *Show that  $\psi$  is injective.* Given any  $P \in X_1^*$  and  $Q \in X_1^*$ , there are Cauchy sequences

$$\{P_{p_n}\} = \{\varphi_1(p_n)\} \rightarrow P \text{ and } \{P_{q_n}\} = \{\varphi_1(q_n)\} \rightarrow Q.$$

So

$$\begin{aligned} \psi(P) = \psi(Q) &\implies \lim_{n \rightarrow \infty} \psi(P_{p_n}) = \lim_{n \rightarrow \infty} \psi(P_{q_n}) \\ &\implies 0 = \lim_{n \rightarrow \infty} \Delta_2(\psi(P_{p_n}), \psi(P_{q_n})) \\ &\implies 0 = \lim_{n \rightarrow \infty} \Delta_2(\psi(\varphi_1(p_n)), \psi(\varphi_1(q_n))) \\ &\implies 0 = \lim_{n \rightarrow \infty} \Delta_2(\varphi_2(p_n), \varphi_2(q_n)) \\ &\implies 0 = \lim_{n \rightarrow \infty} d(p_n, q_n). \end{aligned} \quad (\varphi_2 \text{ is isometric})$$

Thus  $\{p_n\} \in P$  and  $\{q_n\} \in Q$  in the same equivalence class. Thus  $P = Q$ .

□