

## Chapter 11: The Lebesgue Theory

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**Exercise 11.1.** If  $f \geq 0$  and  $\int_E f d\mu = 0$ , prove that  $f(x) = 0$  almost everywhere on  $E$ . (Hint: Let  $E_n$  be the subset of  $E$  on which  $f(x) > \frac{1}{n}$ . Write  $A = \bigcup E_n$ . Then  $\mu(A) = 0$  if and only if  $\mu(E_n) = 0$  for every  $n$ .)

Might assume that  $f$  is measurable on  $E$ .

*Proof (Hint).*

(1) Define  $A = \{x \in E : f(x) > 0\}$ . So  $f(x) = 0$  almost everywhere on  $E$  if and only if  $\mu(A) = 0$ .

(2) Define

$$E_n = \left\{ x \in E : f(x) > \frac{1}{n} \right\}$$

for  $n = 1, 2, 3, \dots$ . Note that  $E_1 \subseteq E_2 \subseteq E_3 \subseteq \dots$  and

$$A = \bigcup_{n=1}^{\infty} E_n.$$

Since  $\mu$  is a measure,

$$\lim_{n \rightarrow \infty} \mu(E_n) = \mu(A)$$

(Theorem 11.3).

(3) (Reductio ad absurdum) If  $\mu(A) > 0$ , there is an integer  $N$  such that  $\mu(E_n) \geq \frac{\mu(A)}{2}$  whenever  $n \geq N$  (by (2)). In particular, take  $n = N$  to get

$$\begin{aligned} \int_E f d\mu &\geq \int_{E_N} f d\mu && (\mu \text{ is a measure and } E_N \subseteq E) \\ &\geq \frac{1}{N} \cdot \mu(E_N) && (\text{Remarks 11.23(b)}) \\ &\geq \frac{1}{N} \cdot \frac{\mu(A)}{2} \\ &> 0, \end{aligned}$$

contrary to the assumption that  $\int_E f d\mu = 0$ .

□

*Note.* Compare to Exercise 6.2.

**Exercise 11.2.** *If  $\int_A f d\mu = 0$  for every measurable subset  $A$  of a measurable set  $E$ , then  $f(x) = 0$  almost everywhere on  $E$ .*

Might assume that  $f$  is measurable on  $E$ .

*Proof.*

- (1) Define

$$A = \{x \in E : f(x) \geq 0\} \quad \text{and} \quad B = \{x \in E : f(x) \leq 0\}.$$

$A$  and  $B$  are measurable subsets of a measurable set  $E$  since  $f$  is measurable.

- (2) Apply Exercise 11.1 to the fact that  $f \geq 0$  on  $A$  (by construction) and  $\int_A f d\mu = 0$  (by assumption), we have  $f(x) = 0$  almost everywhere on  $A$ .
- (3) Similarly, apply Exercise 11.1 to the fact that  $-f \geq 0$  on  $B$  and  $\int_B (-f) d\mu = -\int_B f d\mu = 0$ , we have  $f(x) = 0$  almost everywhere on  $B$ .
- (4) As  $E = A \cup B$ ,  $f(x) = 0$  almost everywhere on  $E$  by (2)(3).

□

**Exercise 11.3.** *If  $\{f_n\}$  is a sequence of measurable functions, prove that the set of points  $x$  at which  $\{f_n(x)\}$  converges is measurable.*

*Proof.*

- (1) It suffices to show that

$$E = \{x : \{f_n(x)\} \text{ is convergent}\} = \{x : \{f_n(x)\} \text{ is Cauchy}\}$$

is measurable (since  $\mathbb{R}^1$  is complete).

- (2) Write

$$E = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n,m \geq N} \left\{ x : |f_n(x) - f_m(x)| \leq \frac{1}{k} \right\}$$

Since  $\{f_n\}$  is a sequence of measurable functions,  $x \mapsto |f_n(x) - f_m(x)|$  is measurable (Theorem 11.16 and Theorem 11.18). Hence

$$\left\{ x : |f_n(x) - f_m(x)| \leq \frac{1}{k} \right\}$$

is measurable (Theorem 11.15). Therefore  $E$  is measurable.

□

**Exercise 11.4.** If  $f \in \mathcal{L}(\mu)$  on  $E$  and  $g$  is bounded and measurable on  $E$ , then  $fg \in \mathcal{L}(\mu)$  on  $E$ .

*Proof (Theorem 11.27).*

- (1)  $fg$  is measurable since both  $f$  and  $g$  are measurable (Theorem 11.18).
- (2)  $|g| \leq M$  for some real  $M \in \mathbb{R}^1$  by the boundedness of  $g$ . Hence

$$|fg| \leq M|f|$$

on  $E$ .

- (3) To apply Theorem 11.27, it suffices to show that  $M|f| \in \mathcal{L}(\mu)$  on  $E$ . Theorem 11.26 implies that  $|f| \in \mathcal{L}(\mu)$  if  $f \in \mathcal{L}(\mu)$ . And Remarks 11.23(d) implies that  $M|f| \in \mathcal{L}(\mu)$  if  $|f| \in \mathcal{L}(\mu)$ .

□

*Note (Riemann integral).* If  $f \in \mathcal{R}$  on  $[a, b]$  and  $g$  is bounded and measurable on  $[a, b]$ , then  $fg$  might be not Riemann integrable.

**Exercise 11.5.** Put

$$g(x) = \begin{cases} 0 & (0 \leq x \leq \frac{1}{2}), \\ 1 & (\frac{1}{2} < x \leq 1), \end{cases}$$

and

$$\begin{aligned} f_{2k}(x) &= g(x) & (0 \leq x \leq 1), \\ f_{2k+1}(x) &= g(1-x) & (0 \leq x \leq 1). \end{aligned}$$

Show that

$$\liminf_{n \rightarrow \infty} f_n(x) = 0 \quad (0 \leq x \leq 1),$$

but

$$\int_0^1 f_n(x) dx = \frac{1}{2}.$$

(Compare with the Fatou's theorem.)

*Proof.*

- (1) Show that  $\liminf_{n \rightarrow \infty} f_n(x) = 0$ . Note that

$$g(1-x) = \begin{cases} 1 & (0 \leq x < \frac{1}{2}), \\ 0 & (\frac{1}{2} < x \leq 1). \end{cases}$$

Since  $f_n(x) \geq 0$  by definition,  $\liminf_{n \rightarrow \infty} f_n(x) \geq 0$ . Since  $f_{2k}(0) = f_{2k+1}(1) = 0$  for all positive integers  $k$ ,  $\liminf_{n \rightarrow \infty} f_n(x) \leq 0$ . Therefore the result is established.

(2) Show that  $\int_0^1 f_n(x) dx = \frac{1}{2}$ . Since

$$\begin{aligned}\int_0^1 f_{2k}(x) dx &= \int_0^1 g(x) dx = \frac{1}{2}, \\ \int_0^1 f_{2k+1}(x) dx &= \int_0^1 g(1-x) dx = \frac{1}{2},\end{aligned}$$

in any case  $\int_0^1 f_n(x) dx = \frac{1}{2}$  for all positive integers  $n$ .

(3) This example shows that we may have the strict inequality in the Fatou's theorem.

□

**Supplement (Similar exercise).** Consider the sequence  $\{f_n\}$  defined by  $f_n(x) = 1$  if  $n \leq x < n+1$ , with  $f_n(x) = 0$  otherwise. Show that we may have the strict inequality in the Fatou's theorem.

**Exercise 11.6.** Let

$$f_n(x) = \begin{cases} \frac{1}{n} & (|x| \leq n), \\ 0 & (|x| > n). \end{cases}$$

Then  $f_n(x) \rightarrow 0$  uniformly on  $\mathbb{R}^1$ , but

$$\int_{-\infty}^{\infty} f_n(x) dx = 2 \quad (n = 1, 2, 3, \dots).$$

(We write  $\int_{-\infty}^{\infty}$  in place of  $\int_{\mathbb{R}^1}$ .) Thus uniform convergence does not imply dominated convergence in the sense of Theorem 11.32. However, on sets of finite measure, uniformly convergent sequences of bounded functions do satisfy Theorem 11.32.

*Proof.*

(1) Show that  $f_n(x) \rightarrow 0$  uniformly on  $\mathbb{R}^1$ . Given any  $\varepsilon > 0$ , there is an integer  $N > \frac{1}{\varepsilon}$  such that

$$|f_n(x) - 0| \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

whenever  $n \geq N$  and  $x \in \mathbb{R}^1$ . Hence  $f_n(x) \rightarrow 0$  uniformly.

(2) Show that  $\int_{-\infty}^{\infty} f_n(x)dx = 2$ .

$$\int_{-\infty}^{\infty} f_n(x)dx = \int_{-n}^n \frac{1}{n}dx = 2.$$

(3) By (1)(2),

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n(x)dx \neq \int_{-\infty}^{\infty} \lim_{n \rightarrow \infty} f_n(x)dx$$

suggests that the Lebesgue's dominated convergence theorem (Theorem 11.32) does not hold in this case. In fact, if there were  $g \in \mathcal{L}$  such that  $|f_n(x)| \leq g(x)$ , then

$$\begin{aligned} \int_{-\infty}^{\infty} g(x)dx &\geq \int_0^{\infty} g(x)dx && \text{(Theorem 11.24)} \\ &= \sum_{n=1}^{\infty} \int_{n-1}^n g(x)dx && \text{(Theorem 11.24)} \\ &\geq \sum_{n=1}^{\infty} \int_{n-1}^n |f_n(x)|dx \\ &= \sum_{n=1}^{\infty} \int_{n-1}^n \frac{1}{n}dx \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \\ &= \infty, \end{aligned}$$

which is absurd.

(4) Show that on sets of finite measure, uniformly convergent sequences of bounded functions  $\{f_n\}$  do satisfy Theorem 11.32.

(a) Since  $\{f_n\}$  is uniformly convergent,  $\{f_n\}$  is uniformly bounded (Exercise 7.1), or there exists a real number  $M$  such that

$$|f_n(x)| \leq M$$

for all positive integer  $n$  and  $x \in E$ .

(b) Define  $g(x) = M$  on  $E$ . It is clear that

$$\int_E g(x)dx = M\mu(E) < +\infty.$$

Now we can apply the Lebesgue's dominated convergence theorem (Theorem 11.32) to get

$$\lim_{n \rightarrow \infty} \int_E f_n d\mu = \int_E \lim_{n \rightarrow \infty} f_n d\mu.$$

□

**Exercise 11.7.** Find a necessary and sufficient condition that  $f \in \mathcal{R}(\alpha)$  on  $[a, b]$ . (Hint: Consider Example 11.6(b) and Theorem 11.33.)

*Proof.*

- (1) Defines the regular measure  $\mu$  by

$$\begin{aligned}\mu([a, b)) &= \alpha(b-) - \alpha(a-) \\ \mu([a, b]) &= \alpha(b+) - \alpha(a-) \\ \mu((a, b]) &= \alpha(b+) - \alpha(a+) \\ \mu((a, b)) &= \alpha(b-) - \alpha(a+)\end{aligned}$$

where  $\alpha : \mathbb{R}^1 \rightarrow \mathbb{R}^1$  might be defined by

$$\alpha(x) = \begin{cases} \alpha(a) & \text{if } x < a, \\ \alpha(x) & \text{if } a \leq x \leq b, \\ \alpha(b) & \text{if } x > b. \end{cases}$$

(Example 11.6(b)).

- (2) Suppose  $f$  is bounded on  $[a, b]$ . Then  $f \in \mathcal{R}(\alpha)$  on  $[a, b]$  if and only if  $f$  and  $\alpha$  satisfy both properties (I) and (II) below.

(I)  $f$  and  $\alpha$  cannot be both left-discontinuous, or both right-discontinuous at same point.

(II)  $f$  is continuous almost everywhere with respect to  $\mu$  on  $[a, b] - D_\alpha$  where  $D_\alpha$  is the set of discontinuities of  $\alpha$  on  $[a, b]$ .

- (3) Similar to Theorem 11.33. By Definition 6.2 and Theorem 6.4 there is a sequence  $\{P_k\}$  of partitions of  $[a, b]$ , such that  $P_{k+1}$  is a refinement of  $P_k$ , such that the distance between adjacent points of  $P_k$  is less than  $\frac{1}{k}$ , and such that

$$\lim_{k \rightarrow \infty} L(P_k, f, \alpha) = \mathcal{R} \int f d\alpha, \quad \lim_{k \rightarrow \infty} \bar{L}(P_k, f, \alpha) = \mathcal{R} \int f d\alpha.$$

(In this proof, all integrals are taken over  $[a, b]$ .)

- (4) If  $P_k = \{x_0, x_1, \dots, x_n\}$ , with  $x_0 = a$ ,  $x_n = b$ , define

$$L_k(a) = U_k(a) = f(a);$$

put  $U_k(x) = M_i$  and  $L_k(x) = m_i$  for  $x_{i-1} < x \leq x_i$ ,  $1 \leq i \leq n$ , using the notation introduced in Definition 6.1. Then

$$L(P_k, f, \alpha) = \int L_k d\mu, \quad U(P_k, f, \alpha) = \int U_k d\mu$$

and

$$L_1(x) \leq L_2(x) \leq \cdots \leq f(x) \leq \cdots \leq U_2(x) \leq U_1(x)$$

for all  $x \in [a, b]$ , since  $P_{k+1}$  refines  $P_k$ .

(5) So there exist

$$L(x) = \lim_{k \rightarrow \infty} L_k(x), \quad U(x) = \lim_{k \rightarrow \infty} U_k(x).$$

Observe that  $L$  and  $U$  are bounded  $\mu$ -measurable function on  $[a, b]$ , that

$$L(x) \leq f(x) \leq U(x) \quad (a \leq x \leq b),$$

and that

$$\begin{aligned} \int L d\mu &= \lim \int L_k d\mu = \lim L(P_k, f, \alpha) = \mathcal{R} \int f d\alpha, \\ \int U d\mu &= \lim \int U_k d\mu = \lim U(P_k, f, \alpha) = \mathcal{R} \int f d\alpha \end{aligned}$$

by the Lebesgue's monotone convergence theorem (Theorem 11.28).

(6) So  $f \in \mathcal{R}(\alpha)$  on  $[a, b]$  if and only if  $\int L d\mu = \int U d\mu$  if and only if

(III)  $L(x) = U(x)$  almost everywhere with respect to  $\mu$

by Exercise 11.1 and the fact that  $U(x) - L(x) \geq 0$ .

(7) Show that  $f \in \mathcal{R}(\alpha)$  on  $[a, b]$  implies the property (I). It is independent of the Lebesgue theory.

(a) Suppose both  $f$  and  $\alpha$  are discontinuous from the right at  $x = c$ ; that is, suppose that there exists an  $\varepsilon > 0$  such that for every  $\delta > 0$  there are values of  $x, y \in (c, c + \delta) \subseteq [a, b]$  for which

$$|f(x) - f(c)| \geq \sqrt{\varepsilon}, \quad |\alpha(y) - \alpha(c)| \geq \sqrt{\varepsilon}.$$

(b) Let  $P = \{x_0 < \cdots < x_n\}$  be any partition of  $[a, b]$  containing  $c$ , say  $c = x_{i-1}$  for some  $i = 1, \dots, n$ . Then

$$\begin{aligned} U(P, f, \alpha) - L(P, f, \alpha) &= \sum_{j=1}^n (M_j - m_j)(\alpha(x_j) - \alpha(x_{j-1})) \\ &\geq (M_i - m_i)(\alpha(x_i) - \alpha(x_{i-1})). \end{aligned}$$

Take  $\delta = x_i - x_{i-1}$ .  $x_i = x_{i-1} + \delta = c + \delta$ . Then

$$\alpha(x_i) - \alpha(x_{i-1}) = \alpha(c + \delta) - \alpha(c) \geq \alpha(y) - \alpha(c) \geq \sqrt{\varepsilon}$$

(by the monotonicity of  $\alpha$ ). Besides,

$$M_i - m_i \geq |f(x) - f(c)| \geq \sqrt{\varepsilon}.$$

Hence,

$$U(P, f, \alpha) - L(P, f, \alpha) \geq \varepsilon.$$

Therefore, Theorem 6.6 implies that  $f \notin \mathcal{R}(\alpha)$  on  $[a, b]$ .

(c) The argument is similar if  $c$  is a common discontinuity from the left.

(8) Show that (III) implies (II).

(a) Show that  $f$  is continuous at  $x \in [a, b] - D_\alpha$  if  $U(x) = L(x)$  and  $x \notin \bigcup_{k=1}^{\infty} P_k$ . (Reductio ad absurdum) If  $f$  were not continuous at  $x$ , then there exists an  $\varepsilon > 0$  such that there is a sequence  $\{y_m\}$  in  $[a, b]$  such that  $|y_m - x| < \frac{1}{m}$  but

$$|f(y_m) - f(x)| > \varepsilon$$

for  $m = 1, 2, 3, \dots$  (Theorem 4.2).

(b) Given any  $P_k$ . Since  $x \notin P_k$ ,  $x \in (x_{i-1}, x_i)$  for some  $i$ . Since  $(x_{i-1}, x_i)$  is open, there exists an integer  $N_k$  such that  $y_m \in (x_{i-1}, x_i)$  whenever  $m \geq N_k$ . Hence,

$$U_k(x) - L_k(x) = M_i - m_i \geq |f(y_m) - f(x)| > \varepsilon.$$

Take the limit to get

$$U(x) - L(x) \geq \varepsilon,$$

which is absurd. Therefore, the statement in (a) is proved.

(c) Now it suffices to show that both sets

$$E_1 = \{x \in [a, b] - D_\alpha : L(x) \neq U(x)\}$$

$$E_2 = \left\{x \in [a, b] - D_\alpha : x \in \bigcup_{k=1}^{\infty} P_k\right\}$$

are  $\mu$ -measure zero.  $E_1$  is  $\mu$ -measure zero by (III).  $E_2$  is  $\mu$ -measure zero since  $E_2$  is countable and defined on  $[a, b] - D_\alpha$ .

Therefore, (II) is established.

(9) Show that (I)(II) implies (III). Use the notation in (8).

(a) It suffices to show that

$$\begin{aligned} & \{x \in [a, b] : L(x) \neq U(x)\} \\ &= \underbrace{\{x \in [a, b] - D_\alpha : L(x) \neq U(x)\}}_{=E_1} \bigcup \underbrace{\{x \in D_\alpha : L(x) \neq U(x)\}}_{=E_3} \end{aligned}$$

is  $\mu$ -measure zero.



- (b) Note that  $E_2$  is  $\mu$ -measure zero. Hence (II) and (8)(a) imply that  $E_1$  is  $\mu$ -measure zero. So it suffices to show that  $E_3$  is  $\mu$ -measure zero. In fact, we will show that  $E_3 = \emptyset$ , or  $L(x) = U(x)$  whenever  $x \in D_\alpha$ .
- (c) Write

$$D_\alpha = \{y_1, \dots, y_m, \dots\}$$

since  $D_\alpha$  is at most countable (Theorem 4.30). (Set  $y_m = y_{m+1} = \dots$  if  $D_\alpha$  is finite.) Define a refinement of  $P_k$  by

$$P_k \bigcup \{y_1, \dots, y_k\}$$

and use the same symbol  $P_k$  to denote this refinement.

- (d) Given any  $x \in D_\alpha$ . Suppose  $\alpha$  is discontinuous from the right at  $x$ . By the construction in (c), there is an integer  $N_1$  such that  $x = x_{i-1}$  is in some subinterval  $[x_{i-1}, x_i]$  of  $P_k$  whenever  $k \geq N_1$ .
- (e) Note that  $f$  is continuous from the right at  $x$  by (I). Given an  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that

$$|f(y) - f(x)| \leq \frac{\varepsilon}{2}$$

whenever  $y \in [x, x + \delta)$ . So

$$|f(y) - f(z)| \leq |f(y) - f(x)| + |f(x) - f(z)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever  $y, z \in [x, x + \delta)$ .

- (f) Take an integer  $N_2 > \frac{1}{\delta}$  such that for any any subinterval  $[x_{i-1}, x_i]$  of  $P_k$  we have  $[x_{i-1}, x_i] \subseteq [x, x + \delta)$  whenever  $k \geq N_2$ .
- (g) Take  $N = \max\{N_1, N_2\}$ . As  $k \geq N$ ,  $[x_{i-1}, x_i] \subseteq [x, x + \delta)$  and

$$U_k(x) - L_k(x) = M_i - m_i = \sup_{y, z \in [x_{i-1}, x_i]} |f(y) - f(z)| \leq \varepsilon.$$

Take the limit to get

$$U(x) - L(x) \leq \varepsilon.$$

Since  $\varepsilon$  is arbitrary,  $L(x) = U(x)$ .

- (h) The argument is similar if  $\alpha$  is discontinuous from the left at  $x$ .

□

**Exercise 11.8.** If  $f \in \mathcal{R}$  on  $[a, b]$  and if  $F(x) = \int_a^x f(t)dt$ , prove that  $F'(x) = f(x)$  almost everywhere on  $[a, b]$ .

*Proof.*

- (1) Theorem 6.20 implies that  $F'(x_0) = f(x_0)$  if  $f$  is continuous at  $x_0 \in [a, b]$ .

- (2) Since  $f \in \mathcal{R}$  on  $[a, b]$ ,  $f$  is bounded on  $[a, b]$ . Theorem 11.33 implies that  $f$  is continuous almost everywhere on  $[a, b]$ .

By (1)(2),  $F'(x) = f(x)$  almost everywhere on  $[a, b]$ .  $\square$

**Exercise 11.9.** Prove that the function  $F$  given by

$$F(x) = \int_a^x f dt \quad (a \leq x \leq b)$$

(where  $f \in \mathcal{L}$  on  $[a, b]$ ) is continuous on  $[a, b]$ .

*Proof.*

- (1) Let  $f \in \mathcal{L}$  on  $E$ . Show that given any  $\varepsilon > 0$  there is a  $\delta > 0$  such that

$$\int_A f d\mu < \varepsilon$$

whenever  $A \subseteq E$  with  $\mu(A) < \delta$ .

- (a) Define  $f_n(x) = \min\{f(x), n\}$  on  $E$  for  $n = 1, 2, 3, \dots$ . Then  $\{f_n\}$  is a sequence of measurable functions such that

$$0 \leq f_1(x) \leq f_2(x) \leq \dots$$

Also,  $f_n \rightarrow f$ . Then by the Lebesgue's monotone convergence theorem (Theorem 11.28),

$$\lim_{n \rightarrow \infty} \int_E f_n d\mu = \int_E f d\mu.$$

- (b) For such  $\varepsilon > 0$ , there is an integer  $N \geq 1$  such that

$$\int_E (f - f_N) d\mu < \frac{\varepsilon}{2}.$$

Choose  $\delta > 0$  so that  $\delta < \frac{\varepsilon}{2N}$ . If  $\mu(A) < \delta$ , we have

$$\begin{aligned} \int_A f d\mu &= \int_A (f - f_N) d\mu + \int_A f_N d\mu \\ &\leq \int_E (f - f_N) d\mu + N\mu(A) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

- (2) Apply (1) to  $f^+$  and  $f^-$  on  $E = [a, b]$ . Given any  $\varepsilon > 0$ , there is a common  $\delta > 0$  such that

$$\left| \int_x^y f^+ dt \right| < \frac{\varepsilon}{2} \quad \text{and} \quad \left| \int_x^y f^- dt \right| < \frac{\varepsilon}{2}$$

whenever  $|y - x| < \delta$ . So

$$|F(y) - F(x)| \leq \left| \int_x^y f^+ dt \right| + \left| \int_x^y f^- dt \right| < \varepsilon$$

whenever  $|y - x| < \delta$ . Hence  $F$  is uniformly continuous. (In fact,  $F$  is absolutely continuous by the same argument.)

□

*Note.* Compare to Theorem 6.20.

**Exercise 11.10.** If  $\mu(X) < +\infty$  and  $f \in \mathcal{L}^2(\mu)$  on  $X$ , prove that  $f \in \mathcal{L}$  on  $X$ . If

$$\mu(X) = +\infty,$$

this is false. For instance, if

$$f(x) = \frac{1}{1 + |x|},$$

then  $f^2 \in \mathcal{L}$  on  $\mathbb{R}^1$ , but  $f \notin \mathcal{L}$  on  $\mathbb{R}^1$ .

*Proof.*

- (1) Since  $\mu(X) < +\infty$ ,  $1 \in \mathcal{L}^2(\mu)$  on  $X$ . By Theorem 11.35,  $f \in \mathcal{L}(\mu)$ , and

$$\int_X |f| d\mu \leq \|f\| \|1\|.$$

- (2) Show that  $f^2 \in \mathcal{L}$  on  $\mathbb{R}^1$ . To apply Theorem 11.33, we might restrict the measure space  $X = \mathbb{R}^1$  to some interval  $[a, b]$ . Then apply the Lebesgue's monotone convergence theorem (Theorem 11.28) to get the conclusion.

- (a) Write

$$f(x)^2 = \left( \frac{1}{1 + |x|} \right)^2 = \frac{1}{1 + 2|x| + x^2} \leq \frac{1}{1 + x^2}.$$

By Theorem 11.27, it suffices to show that  $\frac{1}{1+x^2} \in \mathcal{L}$  on  $\mathbb{R}^1$ .

- (b) Consider the sequence  $\{f_n\}$  defined by

$$f_n(x) = \frac{1}{1 + x^2} \chi_{[-n, n]}(x).$$

(Here  $\chi_{[-n,n]} = K_{[-n,n]}$  is the characteristic function of  $[-n, n]$  defined in Definition 11.19.) By construction,

$$0 \leq f_1(x) \leq f_2(x) \leq \cdots \quad (x \in \mathbb{R}^1)$$

and

$$f_n(x) \rightarrow \frac{1}{1+x^2} \quad (x \in \mathbb{R}^1).$$

(c) Hence

$$\begin{aligned} \int_{\mathbb{R}^1} \frac{1}{1+x^2} dx &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^1} f_n(x) dx && \text{(Theorem 11.28)} \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^1} \frac{1}{1+x^2} \chi_{[-n,n]}(x) dx \\ &= \lim_{n \rightarrow \infty} \int_{-n}^n \frac{1}{1+x^2} dx \\ &= \lim_{n \rightarrow \infty} \mathcal{R} \int_{-n}^n \frac{1}{1+x^2} dx && \text{(Theorem 11.33)} \\ &= \lim_{n \rightarrow \infty} 2 \arctan(n) \\ &= \pi < \infty. \end{aligned}$$

(4) Show that  $f \notin \mathcal{L}$  on  $\mathbb{R}^1$ .

(a) Consider the sequence  $\{f_n\}$  defined by

$$f_n(x) = f(x) \chi_{[-n,n]}(x) = \frac{1}{1+|x|} \chi_{[-n,n]}(x).$$

By construction,

$$0 \leq f_1(x) \leq f_2(x) \leq \cdots \quad (x \in \mathbb{R}^1)$$

and

$$f_n(x) \rightarrow f(x) \quad (x \in \mathbb{R}^1).$$

(b) Hence

$$\begin{aligned} \int_{\mathbb{R}^1} f(x) dx &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^1} f_n(x) dx && \text{(Theorem 11.28)} \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^1} \frac{1}{1+|x|} \chi_{[-n,n]}(x) dx \\ &= \lim_{n \rightarrow \infty} \int_{-n}^n \frac{1}{1+|x|} dx \\ &= \lim_{n \rightarrow \infty} \mathcal{R} \int_{-n}^n \frac{1}{1+|x|} dx && \text{(Theorem 11.33)} \\ &= \lim_{n \rightarrow \infty} 2 \log(n+1) \\ &= \infty, \end{aligned}$$

or  $f \notin \mathcal{L}$  on  $\mathbb{R}^1$ .

□

*Note.* Compare to Exercise 6.5.

**Exercise 11.11.** If  $f, g \in \mathcal{L}(\mu)$  on  $X$ , defined the distance between  $f$  and  $g$  by

$$\int_X |f - g| d\mu.$$

Prove that  $\mathcal{L}(\mu)$  is a complete metric space.

*Proof.*

(1) Define

$$\|f - g\|_1 = \int_X |f - g| d\mu.$$

Thus  $\|f - g\|_1 = 0$  if and only if  $f = g$  almost everywhere on  $X$  (Exercise 11.1). As in Remark 11.37, we identify two functions to be equivalent if they are equal almost everywhere.

(2) Show that  $\mathcal{L}(\mu)$  is a metric space.

- (a) By definition,  $\|f - g\|_1 \geq 0$ . Besides,  $\|f - g\|_1 = 0$  if and only if  $f = g$  almost everywhere by (1).
- (b)  $\|f - g\|_1 = \|g - f\|_1$  since  $|f(x) - g(x)| = |g(x) - f(x)|$  for all  $x \in X$ .
- (c) Since  $|f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)|$  for all  $x \in X$ , Remarks 11.23(c) and Theorem 11.29 imply that

$$\|f - g\|_1 \leq \|f - h\|_1 + \|h - g\|_1.$$

(3) Show that  $\mathcal{L}(\mu)$  is complete. Similar to the proof of Theorem 11.42.

- (a) Let  $\{f_n\}$  be a Cauchy sequence in  $\mathcal{L}(\mu)$ , show that there exists a function  $f \in \mathcal{L}(\mu)$  such that  $\{f_n\}$  converges to  $f \in \mathcal{L}(\mu)$ .
- (b) Since  $\{f_n\}$  is a Cauchy sequence, we can find a sequence  $\{n_k\}$ ,  $k = 1, 2, 3, \dots$ , such that

$$\|f_{n_k} - f_{n_{k+1}}\|_1 = \int_X |f_{n_k} - f_{n_{k+1}}| d\mu < \frac{1}{2^k} \quad (k = 1, 2, 3, \dots).$$

Hence

$$\sum_{k=1}^{\infty} \int_X |f_{n_k} - f_{n_{k+1}}| d\mu \leq \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 < +\infty.$$

- (c) By Theorem 11.30, we may interchange the summation and integration to get

$$\int_X \sum_{k=1}^{\infty} |f_{n_k} - f_{n_{k+1}}| d\mu < +\infty,$$

or

$$\sum_{k=1}^{\infty} |f_{n_k}(x) - f_{n_{k+1}}(x)| = \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)| < +\infty$$

almost everywhere on  $X$ .

- (d) Since the  $k$ th partial sum of the series

$$\sum_{k=1}^{\infty} (f_{n_{k+1}}(x) - f_{n_k}(x))$$

which converges almost everywhere on  $X$  (Theorem 3.45), is

$$f_{n_{k+1}}(x) - f_{n_1}(x),$$

we see that the equation

$$f(x) = \lim_{k \rightarrow \infty} f_{n_k}(x)$$

defines  $f(x)$  for almost all  $x \in X$ , and it does not matter how we define  $f(x)$  at the remaining points of  $X$ .

- (e) We shall now show that this function  $f$  has the desired properties. Let  $\varepsilon > 0$  be given, and choose  $N$  such that

$$\|f_n - f_m\|_1 \leq \varepsilon$$

whenever  $n, m \geq N$ . If  $n_k > N$ , Fatou's theorem shows that

$$\|f - f_{n_k}\|_1 \leq \liminf_{i \rightarrow \infty} \|f_{n_i} - f_{n_k}\|_1 \leq \varepsilon.$$

Thus  $f - f_{n_k} \in \mathcal{L}(\mu)$ , and since  $f = (f - f_{n_k}) + f_{n_k} \in \mathcal{L}(\mu)$ , we see that  $f \in \mathcal{L}(\mu)$ . Also, since  $\varepsilon$  is arbitrary,

$$\lim_{k \rightarrow \infty} \|f - f_{n_k}\|_1 = 0.$$

- (f) Finally, the inequality

$$\|f - f_n\|_1 \leq \|f - f_{n_k}\|_1 + \|f_{n_k} - f_n\|_1$$

shows that  $\{f_n\}$  converges to  $f \in \mathcal{L}(\mu)$ ; for if we take  $n$  and  $n_k$  large enough, each of the two terms can be made arbitrary small.

□

**Exercise 11.12.** *Suppose*

- (a)  $|f(x, y)| \leq 1$  if  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ .
- (b) for fixed  $x$ ,  $f(x, y)$  is a continuous function of  $y$ .
- (c) for fixed  $y$ ,  $f(x, y)$  is a continuous function of  $x$ .

Put

$$g(x) = \int_0^1 f(x, y) dy \quad (0 \leq x \leq 1).$$

Is  $g$  continuous?

*Proof.*

- (1) Show that  $g$  is continuous.
- (2) Let  $\{x_n\}$  be a sequence in  $[0, 1]$  such that  $x_n \neq x$  and  $\lim x_n = x$ . It suffices to show that

$$\begin{aligned} \lim_{n \rightarrow \infty} g(x_n) &= \lim_{n \rightarrow \infty} \int_0^1 f(x_n, y) dy \\ &= \int_0^1 \lim_{n \rightarrow \infty} f(x_n, y) dy \\ &= \int_0^1 f(x, y) dy \\ &= g(x) \end{aligned}$$

(Theorem 4.2). Since  $\lim_{n \rightarrow \infty} f(x_n, y) = f(x, y)$  for any fixed  $y$  (by (c)), it suffices to show that

$$\lim_{n \rightarrow \infty} \int_0^1 f(x_n, y) dy = \int_0^1 \lim_{n \rightarrow \infty} f(x_n, y) dy.$$

- (3) Define  $\{f_n\}$  by  $f_n(y) = f(x_n, y)$ .  $f_n(y)$  is a continuous function of  $y$  for every fixed  $n$  (by (b)). Thus  $f_n(y)$  is measurable (Example 11.14). Besides,  $|f_n(y)| \leq 1$  and  $1 \in \mathcal{L}$  on  $[0, 1]$  (by (a)). The Lebesgue's dominated convergence theorem (Theorem 11.32) implies that

$$\lim_{n \rightarrow \infty} \int_0^1 f(x_n, y) dy = \int_0^1 \lim_{n \rightarrow \infty} f(x_n, y) dy.$$

□

**Supplement (Similar exercise).** *Suppose*

- (a)  $|f(x, y)| \leq g(y)$  if  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ , where  $g \in \mathcal{L}$  on  $[0, 1]$ .
- (b) for fixed  $x$ ,  $f(x, y)$  is a measurable function of  $y$ .
- (c) for fixed  $y$ ,  $f(x, y)$  is a continuous function of  $x$ .

Show that

$$h(x) = \int_0^1 f(x, y) dy \quad (0 \leq x \leq 1).$$

is continuous.

**Exercise 11.13.** Consider the functions

$$f_n(x) = \sin(nx) \quad (n = 1, 2, 3, \dots, -\pi \leq x \leq \pi)$$

as points of  $\mathcal{L}^2$ . Prove that the set of these points is closed and bounded, but not compact.

*Proof.* Define  $E = \{f_n\}$  as a set in  $\mathcal{L}^2$ .

- (1) Show that  $E$  is bounded. Note that

$$\|f_n\|_2 = \left( \int_{-\pi}^{\pi} \sin^2(nx) dx \right)^{\frac{1}{2}} = \sqrt{\pi}$$

for all  $n$  (Definition 8.10). So  $E$  is bounded by  $\sqrt{\pi}$ .

- (2) Show that  $E$  is closed. It suffices to show that  $E$  has no limit points.

$$\begin{aligned} \|f_n - f_m\|_2 &= \left( \int_{-\pi}^{\pi} (\sin(nx) - \sin(mx))^2 dx \right)^{\frac{1}{2}} \\ &= \left( \int_{-\pi}^{\pi} \sin^2(nx) - 2 \sin(nx) \sin(mx) + \sin^2(mx) dx \right)^{\frac{1}{2}} \\ &= (\pi + 0 + \pi)^{\frac{1}{2}} \\ &= \sqrt{2\pi} \end{aligned}$$

for all  $n \neq m$  (Definition 8.10). So all points of  $E$  are isolated.

- (3) Show that  $E$  is not compact.

- (a) Take a collection

$$\mathcal{G} = \{G_n = B(f_n; 1)\}$$

of open subsets ( $n = 1, 2, 3, \dots$ ).

- (b)  $\mathcal{G}$  is an open covering of  $E \subseteq \mathcal{L}^2$  since  $f_n \in G_n$  for each  $n = 1, 2, 3, \dots$



(c) Show that there is no finite subcoverings of  $\mathcal{G}$ . (Reductio ad absurdum)

If

$$\mathcal{G}' = \{G_{n_1}, G_{n_2}, \dots, G_{n_k}\}$$

were a finite subcovering of  $\mathcal{G}$  with  $n_1 < n_2 < \dots < n_k$ , then  $f_{n_k+1}$  is not in any open sets from  $\mathcal{G}'$  (by (2)), which is absurd.

□

**Exercise 11.14.** Prove that a complex function  $f$  is measurable if and only if  $f^{-1}(V)$  is measurable for every open set  $V$  in the plane.

*Proof.*

(1)

(2)

□

**Exercise 11.15.** Let  $\mathcal{R}$  be the ring of all elementary subsets of  $(0, 1]$ . If  $0 < a \leq b \leq 1$ , define

$$\phi([a, b]) = \phi([a, b)) = \phi((a, b]) = \phi((a, b)) = b - a,$$

but define

$$\phi((0, b)) = \phi((0, b]) = 1 + b$$

if  $0 < b \leq 1$ . Show that this gives an additive set function  $\phi$  on  $\mathcal{R}$ , which is not regular and which cannot be extended to a countably additive set function on a  $\sigma$ -ring.

*Proof.*

(1) Define  $\phi : \mathcal{R} \rightarrow \mathbb{R} \cup \{\pm\infty\}$  by

$$\phi(A) = \sum_{i=1}^n \phi(I_i)$$

where  $A$  is a finite number of disjoint intervals  $I_1, \dots, I_n$  (Definition 11.4).

(2) Show that  $\phi$  is an additive set function. Given any two elementary sets  $A, B \in \mathcal{R}$  with  $A \cap B = \emptyset$ . By Definition 11.4,

$$A = \bigcup_{i=1}^n I_i, \quad B = \bigcup_{j=1}^m J_j$$

where  $I_i \cap J_j = \emptyset$  for all  $1 \leq i \leq n$  and  $1 \leq j \leq m$  (since  $A \cap B = \emptyset$ ). Hence,

$$\begin{aligned}\phi(A \cup B) &= \phi\left(\left\{\bigcup_{i=1}^n I_i\right\} \cup \left\{\bigcup_{j=1}^m J_j\right\}\right) \\ &= \sum_{i=1}^n \phi(I_i) + \sum_{j=1}^m \phi(J_j) \\ &= \phi(A) + \phi(B).\end{aligned}$$

(3) Show that  $\phi$  is not countably additive. Write

$$(0, 1] = \bigcup_{i=1}^{\infty} A_i$$

where  $A_i = (2^{-i}, 2^{-i+1}]$ . Note that  $A_i \cap A_j = \emptyset$  if  $i \neq j$ . So

$$\sum_{i=1}^{\infty} \phi(A_i) = \sum_{i=1}^{\infty} 2^{-i} = 1 \neq 2 = \phi((0, 1]).$$

(4) Show that  $\phi$  is not regular.

(a) Given any closed set  $F \in \mathcal{R}$ . Show that  $\phi(F) \leq 1$ . Write  $F = \bigcup_{i=1}^n I_i$  where each  $I_i$  are disjoint by Definition 11.4. Here every  $I_i$  is never of the form  $(0, b]$  or  $(0, b)$  where  $b > 0$ . (Otherwise 0 is a limit point of  $F$ , or  $0 \in \overline{F} = F$ , which is absurd.) Hence  $\phi(F) = \sum \phi(I_i) \leq 1$ .

(b) Take  $A = (0, \frac{1}{2}] \in \mathcal{R}$  and  $\varepsilon = \frac{1}{64} > 0$ . Then for every closed set  $F \in \mathcal{R}$ , we have

$$\phi(A) = \frac{3}{2} > 1 + \frac{1}{64} \geq \phi(F) + \varepsilon.$$

That is,  $\phi$  cannot be regular.

□

**Exercise 11.16.** Suppose  $\{n_k\}$  is an increasing sequence of positive integers and  $E$  is the set of all  $x \in (-\pi, \pi)$  at which  $\{\sin(n_k x)\}$  converges. Prove that  $m(E) = 0$ . (Hint: For every  $A \subseteq E$ ,

$$\int_A \sin(n_k x) dx \rightarrow 0,$$

and

$$2 \int_A (\sin(n_k x))^2 dx = \int_A (1 - \cos(2n_k x)) dx \rightarrow m(A)$$

as  $k \rightarrow \infty$ .)

*Proof.*

(1)

(2)

□

**Exercise 11.17.** Suppose  $E \subseteq (-\pi, \pi)$ ,  $m(E) > 0$ ,  $\delta > 0$ . Use the Bessel inequality to prove that there are at most finitely many integers  $n$  such that  $\sin(nx) \geq \delta$  for all  $x \in E$ .

*Proof.*

(1)

(2)

□

**Exercise 11.18.** Suppose  $f \in \mathcal{L}^2(\mu)$ ,  $g \in \mathcal{L}^2(\mu)$ . Prove that

$$\left| \int f \bar{g} d\mu \right|^2 = \int |f|^2 d\mu \int |g|^2 d\mu$$

if and only if  $f(x) = 0$  almost everywhere or there is a constant  $c$  such that  $g(x) = cf(x)$  almost everywhere. (Compare Theorem 11.35.)

*Proof.*

- (1) Since  $g \in \mathcal{L}^2(\mu)$ ,  $\bar{g} \in \mathcal{L}^2(\mu)$ . Theorem 11.35 implies that  $f\bar{g} \in \mathcal{L}^2(\mu)$ , and

$$\left\{ \int |f\bar{g}| d\mu \right\}^2 \leq \int |f|^2 d\mu \int |g|^2 d\mu.$$

With Theorem 11.26, we have

$$\left| \int f \bar{g} d\mu \right|^2 \leq \left\{ \int |f\bar{g}| d\mu \right\}^2 \leq \int |f|^2 d\mu \int |g|^2 d\mu.$$

Thus

$$\left| \int f \bar{g} d\mu \right|^2 = \int |f|^2 d\mu \int |g|^2 d\mu$$

if and only if

$$\left\{ \int |f\bar{g}| d\mu \right\}^2 = \int |f|^2 d\mu \int |g|^2 d\mu \text{ and } \left| \int f \bar{g} d\mu \right| = \int |f\bar{g}| d\mu.$$

(2)

□

*Note.* Compare Exercise 1.15 and Exercise 6.10.