Chapter 3: Numerical Sequences and Series

Author: Meng-Gen Tsai
Email: plover@gmail.com

Exercise 3.1. *Prove that the convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?*

Proof.

(1) Since $\{s_n\}$ is convergent, there is $s \in \mathbb{R}^1$ with the following property: given any $\varepsilon > 0$, there is N such that $|s_n - s| < \varepsilon$ whenever $n \geq N$. So

$$||s_n| - |s|| \leq |s_n - s| < \varepsilon$$

(Exercise 1.13). That is, $\{|s_n|\}$ converges to $|s|$.

(2) The converse is not true by considering $s_n = (-1)^{n+1}$.

$\square$

Exercise 3.2 *Calculate* $\lim_{n\to\infty}(\sqrt{n^2+n} - n)$.

Proof.

$$\sqrt{n^2+n} - n = \frac{n}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+\frac{1}{n}}+1} \to \frac{1}{1+1} = \frac{1}{2}$$

as $n \to \infty$. $\square$

Proof ($\varepsilon - N$ argument). Let $s_n = \sqrt{n^2+n} - n$. *Show that the sequence* $\{s_n\}$ *converges to* $s = \frac{1}{2}$. Given any $\varepsilon > 0$, there is $N > \frac{1}{\varepsilon}$ such that

$$\begin{aligned}
|s_n - s| &= \left|(\sqrt{n^2+n} - n) - \frac{1}{2}\right| = \left|\frac{1}{\sqrt{1+\frac{1}{n}}+1} - \frac{1}{2}\right| \\
&= \left|\frac{2 - \left(\sqrt{1+\frac{1}{n}}+1\right)}{2\left(\sqrt{1+\frac{1}{n}}+1\right)}\right| = \left|\frac{1 - \sqrt{1+\frac{1}{n}}}{2\left(\sqrt{1+\frac{1}{n}}+1\right)}\right| \\
&= \left|\frac{1 - \left(1 - \frac{1}{n}\right)}{2\left(\sqrt{1+\frac{1}{n}}+1\right)^2}\right| = \left|\frac{-\frac{1}{n}}{2\left(\sqrt{1+\frac{1}{n}}+1\right)^2}\right| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon
\end{aligned}$$

wheneven $n \geq N$. $\square$

Exercise 3.3 *If* $s_1 = \sqrt{2}$ *and*

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} \ (n = 1, 2, 3, ...),$$

prove that $\{s_n\}$ *converges, and that* $s_n < 2$ *for* $n = 1, 2, 3, ...$

The convergence of $\{s_n\}$ implies there is $s \in \mathbb{R}$ such that $s_n \to s$ where $s = \sqrt{2 + \sqrt{s}}$ and $\sqrt{2} < s \leq 2$. WolframAlpha shows that

$$s = \frac{1}{3}\left(-1 + \sqrt[3]{\frac{1}{2}(79 - 3\sqrt{249})} + \sqrt[3]{\frac{1}{2}(79 + 3\sqrt{249})}\right).$$

Proof (Theorem 3.14).

(1) *Show that* $\{s_n\}$ *is increasing (by mathematical induction).*

(a) *Show that* $s_2 > s_1$. In fact,

$$s_2 = \sqrt{2 + \sqrt{s_1}} = \sqrt{2 + \sqrt{\sqrt{2}}} < \sqrt{2} = s_1.$$

(a) *Show that* $s_{n+1} > s_n$ *if* $s_n > s_{n-1}$.

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} > \sqrt{2 + \sqrt{s_{n-1}}} = s_n.$$

By mathematical induction, $\{s_n\}$ is (strictly) increasing.

(2) *Show that* $\{s_n\}$ *is bounded (by mathematical induction).*

(a) *Show that* $s_1 \leq 2$. $\sqrt{2} \leq 2$.

(a) *Show that* $s_{n+1} \leq 2$ *if* $s_n \leq 2$.

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} \leq \sqrt{2 + \sqrt{2}} < 2.$$

By mathematical induction, $\{s_n\}$ is bounded by 2.

Hence, $\{s_n\}$ converges since $\{s_n\}$ is increasing and bounded (Theorem 3.14). $\square$

Exercise 3.4 *Find the upper and lower limits of the sequences* $\{s_n\}$ *defined by*

$$s_1 = 0; s_{2m} = \frac{s_{2m-1}}{2}; s_{2m+1} = \frac{1}{2} + s_{2m}.$$

Write out the first few terms of $\{s_n\}$:

$$0, 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{3}{8}, \frac{7}{8}, \frac{7}{16}, \frac{15}{16}, ...$$

It suggests us

$$\begin{aligned} s_{2m+1} &= 1 - \frac{1}{2^m} \ \ (m = 0, 1, 2, ...), \\ s_{2m} &= \frac{1}{2} - \frac{1}{2^m} \ \ (m = 1, 2, 3, ...). \end{aligned}$$

Proof.

(1) *Show that*

$$\begin{aligned} s_{2m+1} &= 1 - \frac{1}{2^m} \ \ (m = 0, 1, 2, ...), \\ s_{2m} &= \frac{1}{2} - \frac{1}{2^m}. \ \ (m = 1, 2, 3, ...) \end{aligned}$$

Apply mathematical induction.

(2) The upper limit is 1.

(3) The lower limit is $\frac{1}{2}$.

$\square$

Exercise 3.7 *Prove that the convergence of* $\sum a_n$ *implies the convergence of*

$$\sum \frac{\sqrt{a_n}}{n},$$

if $a_n \geq 0$.

Proof (Cauchy's inequatity).

(1) *Show that* $\sum \frac{\sqrt{a_n}}{n}$ *is bounded.* For any $k \in \mathbb{Z}^+$,

$$\begin{aligned} \left(\sum_{n=1}^{k} \frac{\sqrt{a_n}}{n} \right)^2 &\leq \left(\sum_{n=1}^{k} a_n \right) \left(\sum_{n=1}^{k} \frac{1}{n^2} \right) && \text{(Cauchy's inequatity)} \\ &\leq \left(\sum_{n=1}^{\infty} a_n \right) \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right). && \left(\sum a_n, \sum \frac{1}{n^2} \text{: convergent} \right) \end{aligned}$$

Thus, $\left(\sum_{n=1}^{k} \frac{\sqrt{a_n}}{n} \right)^2$ is bounded, or $\sum_{n=1}^{k} \frac{\sqrt{a_n}}{n}$ is bounded.

(2) *Show that* $\sum_{n=1}^{k} \frac{\sqrt{a_n}}{n}$ *is increasing.* It is clear due to $\frac{\sqrt{a_n}}{n} \geq 0$.

By Theorem 3.14, $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges. $\square$

Proof (AM-GM inequality). Show that $\sum \frac{\sqrt{a_n}}{n}$ *is bounded.*

$$\begin{aligned}
\frac{\sqrt{a_n}}{n} \leq& \frac{1}{2}\left(a_n + \frac{1}{n^2}\right) && \text{(AM-GM inequality)} \\
\sum_{n=1}^{k} \frac{\sqrt{a_n}}{n} \leq& \frac{1}{2}\left(\sum_{n=1}^{k} a_n + \sum_{n=1}^{k} \frac{1}{n^2}\right) \\
\leq& \frac{1}{2}\left(\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} \frac{1}{n^2}\right). && \left(\sum a_n, \sum \frac{1}{n^2}\text{: convergent}\right)
\end{aligned}$$

Thus, $\sum_{n=1}^{k} \frac{\sqrt{a_n}}{n}$ is bounded. The rest proof is the same as previous. $\square$