Chapter 2: Some Basic Notions of Set Theory

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Exercise 2.6. *Let $f : S \to T$ be a function. If A and B are arbitrary subsets of S, prove that*

$$f(A \cup B) = f(A) \cap f(B) \text{ and } f(A \cap B) \subseteq f(A) \cup f(B).$$

Generalize to arbitrary unions and intersections.

Generalization. Let $f : S \to T$ be a function. If $\mathscr{F}$ is an arbitrary collection of sets, then

$$f\left(\bigcup_{A \in \mathscr{F}} A\right) = \bigcap_{A \in \mathscr{F}} f(A) \text{ and } f\left(\bigcap_{A \in \mathscr{F}} A\right) \subseteq \bigcup_{A \in \mathscr{F}} f(A).$$

Note. $f(A \cap B)$ might not be equal to $f(A) \cup f(B)$. For example, let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$. Then for any nonempty disjoint subsets A and B, we have $\varnothing = f(A \cap B) \not\supseteq f(A) \cup f(B) = \{0\}$.

Proof.

(1)

$$\begin{aligned}
\forall\, y \in f\left(\bigcup_{A \in \mathscr{F}} A\right) &\Longleftrightarrow \exists\, x \in \bigcup_{A \in \mathscr{F}} A \text{ such that } f(x) = y \\
&\Longleftrightarrow \exists\, x \in A \text{ for some } A \in \mathscr{F} \text{ such that } f(x) = y \\
&\Longleftrightarrow \exists\, A \in \mathscr{F} \text{ such that } y \in f(A) \\
&\Longleftrightarrow \forall\, y \in \bigcap_{A \in \mathscr{F}} f(A)
\end{aligned}$$

(2)

$$
\begin{aligned}
\forall\, y \in f\left(\bigcap_{A\in\mathscr{F}} A\right) &\Longleftrightarrow \exists\, x \in \bigcap_{A\in\mathscr{F}} A \text{ such that } f(x)=y\\
&\Longleftrightarrow \exists\, x \text{ in all } A\in\mathscr{F} \text{ such that } f(x)=y\\
&\qquad (x \text{ not depending on } A)\\
&\Longrightarrow \forall\, A\in\mathscr{F}, \exists\, x\in A \text{ such that } f(x)=y\\
&\qquad (x \text{ depending on } A)\\
&\Longleftrightarrow \forall\, A\in\mathscr{F}, y\in f(A)\\
&\Longleftrightarrow \forall\, y \in \bigcup_{A\in\mathscr{F}} f(A).
\end{aligned}
$$

□

Exercise 2.15. *A real number is called algebraic if it is a root of an algebraic equation $f(x)=0$, where $a_0+a_1x+\cdots+a_nx^n=0$ is a polynomial with integer coefficients. Prove that the set of all polynomials with integer coefficients is countable and deduce that the set of algebraic numbers is also countable.*

Might assume $a_n \neq 0$.

For example, all rational numbers are algebraic since $p=\frac{\alpha}{\beta}$ (where $\alpha,\beta\in\mathbb{Z}$) is a root of $\beta x-\alpha=0$.

Besides, $x=\sqrt{2}+\sqrt{3}$ is algebraic since $x^4-10x^2+1=0$. In fact, $x=\pm\sqrt{2}+\pm\sqrt{3}$ are also algebraic since $x^4-10x^2+1=(x-\sqrt{2}-\sqrt{3})(x+\sqrt{2}-\sqrt{3})(x-\sqrt{2}+\sqrt{3})(x+\sqrt{2}+\sqrt{3})$.

Note. *Countable set* in the sense of Tom M. Apostol is equivalent to *at most countable set* in the sense of Walter Rudin.

Lemma. *The set of all polynomials over $\mathbb{Z}$ is countable implies that the set of algebraic numbers is countable.*

Proof of Lemma. By definition, we write the set of algebraic numbers as

$$S=\bigcup_{f(x)\in\mathbb{Z}[x]}\{\alpha\in\mathbb{R} : f(\alpha)=0\}.$$

Since each polynomial of degree n has at most n roots, $\{\alpha\in\mathbb{R} : f(\alpha)=0\}$ is finite (or countable) for each given $f(x)\in\mathbb{Z}[x]$. So S is a countable union (by assumption) of countable sets, and hence countable by Theorem 2.27. □

Now we show that *the set of all polynomials over $\mathbb{Z}$ is countable.*

Proof (Walter Rudin). For every positive integer N there are only finitely many equations with $n + |a_0| + |a_1| + \cdots + |a_n| = N$. Write

$$P_N = \{f(x) \in \mathbb{Z}[x] : n + |a_0| + |a_1| + \cdots + |a_n| = N\}$$

where $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ with $a_n \neq 0$, and

$$P = \bigcup_{N=1}^{\infty} P_N.$$

P is the set of all polynomials over $\mathbb{Z}$.

Each P_N is finite (or countable) for given N (since the equation $n + |a_0| + |a_1| + \cdots + |a_n| = N$ has finitely many solutions $(n, a_0, a_1, ..., a_n) \in \mathbb{Z}^{n+2}$). So P is a countable union of countable sets, and hence countable by Theorem 2.27. $\square$

Proof (Theorem 2.18).

(1) *$\mathbb{Z}^N$ is countable for any integer $N > 0$.* Induction on N and apply the same argument of Theorem 2.18.

(2) *The set of all polynomials over $\mathbb{Z}$ is countable.* Let

$$P_n = \{f \in \mathbb{Z}[x] : \deg f = n\},$$

and

$$P = \bigcup_{n=1}^{\infty} P_n = \mathbb{Z}[x].$$

Claim: P_n is countable. Define a one-to-one map $\varphi_n : P_n \to \mathbb{Z}^{n+1}$ by

$$\varphi_n(a_0 + a_1 x + \cdots + a_n x^n) = (a_0, a_1, ..., a_n).$$

By (1) and Theorem 2.16, P_n is countable. Now P is a countable union of countable sets, and hence countable by Theorem 2.27.

$\square$

Proof (Unique factorization theorem).

(1) *The set of prime numbers is countable.* Write all primes in the ascending order as $p_1, p_2, ..., p_n, ...$ where $p_1 = 2, p_2 = 3, ..., p_{10001} = 104743, ...$ (See ProjectEuler 7: 10001st prime. Use sieve of Eratosthenes to get p_{10001}.)

(2) *The set of all polynomials over $\mathbb{Z}$ is countable.* Let

$$P_n = \{f \in \mathbb{Z}[x] : \deg f = n\},$$

and

$$P = \bigcup_{n=1}^{\infty} P_n = \mathbb{Z}[x].$$

Claim: P_n is countable. Define a map $\varphi_n : P_n \to \mathbb{Z}^+$ by

$$\varphi_n(a_0 + a_1 x + \cdots + a_n x^n) = p_1^{\psi(a_0)} p_2^{\psi(a_1)} \cdots p_{n+1}^{\psi(a_n)},$$

where ψ is a one-to-one correspondence from $\mathbb{Z}$ to $\mathbb{Z}^+$. By the unique factorization theorem, φ_n is one-to-one. So P_n is countable by Theorem 2.16. Now P is a countable union of countable sets, and hence countable by Theorem 2.27.

□