

Solutions to the book: *do Carmo, Differential Geometry of Curves and Surfaces*

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Contents

| | |
|---|----------|
| Chapter 1: Curves | 3 |
| 1-1. Introduction | 3 |
| 1-2. Parametrized Curves | 3 |
| Exercise 1-2.1. | 3 |
| Exercise 1-2.2. | 3 |
| Exercise 1-2.3. | 4 |
| Exercise 1-2.4. | 4 |
| Exercise 1-2.5. | 4 |
| 1-3. Regular Curves; Arc Length | 5 |
| Exercise 1-3.1. | 5 |
| Exercise 1-3.2. (Cycloid) | 6 |
| Exercise 1-3.3. (Cisoid of Diocles) | 7 |
| Exercise 1-3.4. (Tractrix) | 8 |
| Exercise 1-3.5. (Folium of Descartes) | 9 |
| Exercise 1-3.6. (Logarithmic spiral) | 11 |
| Exercise 1-3.7. | 12 |
| Exercise 1-3.8. | 14 |
| Exercise 1-3.9. | 16 |
| Exercise 1-3.10. (Straight Lines as Shortest) | 17 |
| 1-4. The Vector Product in \mathbb{R}^3 | 19 |
| Exercise 1-4.1. | 19 |
| Exercise 1-4.2. | 19 |
| Exercise 1-4.3. | 20 |
| Exercise 1-4.4. | 21 |
| Exercise 1-4.5. | 21 |
| Exercise 1-4.6. | 22 |
| Exercise 1-4.7. | 22 |

| | |
|--|----|
| Exercise 1-4.8. | 23 |
| Exercise 1-4.9. | 23 |
| Exercise 1-4.10. (Oriented area) | 24 |
| Exercise 1-4.11. (Oriented volume) | 24 |
| Exercise 1-4.12. | 25 |
| Exercise 1-4.13. | 26 |
| Exercise 1-4.14. | 26 |
| 1-5. The Local Theory of Curves Parametrized by Arc Length | 27 |
| Exercise 1-5.2. | 27 |
| 1-6. The Local Canonical Form | 28 |
| 1-7. Global Properties of Plane Curves | 28 |

Chapter 1: Curves

1-1. Introduction

Classical differential geometry: the study of local properties of curves and surfaces.

Global differential geometry: the study of the influence of the local properties on the behavior of the entire curve and surface.

No exercises.

1-2. Parametrized Curves

Exercise 1-2.1.

Find a parametrized curve $\alpha(t)$ whose trace is the circle $x^2 + y^2 = 1$ such that $\alpha(t)$ runs clockwise around the circle with $\alpha(0) = (0, 1)$.

Proof. $\alpha(t) = (\sin t, \cos t)$, $t \in \mathbb{R}$. \square

Exercise 1-2.2.

Let $\alpha(t)$ be a parametrized curve which does not pass through the origin. If $\alpha(t_0)$ is the point of the trace of α closest to the origin and $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

Proof. Let $f(t) = |\alpha(t)|^2 = \alpha(t) \cdot \alpha(t)$. $f(t)$ is differentiable and $f(t)$ has a local minimum at a point $t = t_0 \in I$. So $f'(t_0) = 0$. [Theorem 5.8 in *W. Rudin, Principles of Mathematical Analysis, 3rd edition.*] Since

$$f'(t) = 2\alpha(t) \cdot \alpha'(t),$$

$f'(t_0) = 2\alpha(t_0) \cdot \alpha'(t_0) = 0$, or $\alpha(t_0) \cdot \alpha'(t_0) = 0$. Since $\alpha(t_0) \neq 0$ and $\alpha'(t_0) \neq 0$, $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$. \square

Exercise 1-2.3.

A parametrized curve $\alpha(t)$ has a property that its second derivative $\alpha''(t)$ is identically zero. What can be said about α ?

Proof.

- (1) $\alpha(t)$ is a straight line.
- (2) Since $\alpha''(t)$ is identically zero, $\alpha'(t) = a$ is a constant. [Theorem 5.11 in *W. Rudin, Principles of Mathematical Analysis, 3rd edition.*] Define $f(t) = \alpha(t) - at$ (on I). Since $f'(t) = \alpha'(t) - a = 0$, $f(t) = \alpha(t) - at = b$ is a constant again. Therefore, $\alpha(t) = at + b$, which is a straight line (on I).

□

Exercise 1-2.4.

Let $\alpha : I \rightarrow \mathbb{R}^3$ be a parametrized curve and let $v \in \mathbb{R}^3$ be a fixed vector. Assume that $\alpha'(t)$ is orthogonal to v for all $t \in I$ and that $\alpha(0)$ is orthogonal to v . Prove that $\alpha(t)$ is orthogonal to v for all $t \in I$.

Need to assume that $\alpha(t) \neq 0$ for all $t \in I$.

Proof. Given any $t \neq 0 \in I$. (Nothing to do at $t = 0$.) Define $f : I \rightarrow \mathbb{R}$ by $f(t) = \alpha(t) \cdot v$. By the mean value theorem, there exists a point ξ between 0 and t such that

$$f(t) - f(0) = f'(\xi)(t - 0),$$

where $f'(t) = \alpha'(t) \cdot v + \alpha(t) \cdot v' = \alpha'(t) \cdot v$. Note that $f(0) = 0$ since $\alpha(0)$ is orthogonal to v , and $f'(\xi) = 0$ since $\alpha'(\xi)$ is orthogonal to v . So the identity is reduced to

$$f(t) = 0,$$

or $\alpha(t) \cdot v = 0$, or $\alpha(t)$ is orthogonal to v . □

Exercise 1-2.5.

Let $\alpha : I \rightarrow \mathbb{R}^3$ be a parametrized curve, with $\alpha'(t) \neq 0$ for all $t \in I$. Show that $|\alpha(t)|$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.

The same trick in Exercise 1-2.2.

Proof. It is equivalent to show that $|\alpha(t)|^2$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$. Let

$$f(t) = |\alpha(t)|^2 = \alpha(t) \cdot \alpha(t).$$

Notice that $\alpha'(t) \neq 0$, and thus

$$\begin{aligned} & |\alpha(t)| \text{ is a nonzero constant} \\ \iff & f(t) = |\alpha(t)|^2 \text{ is a nonzero constant} \\ \iff & f'(t) = 0 \text{ and } f(t) \text{ is a nonzero constant} \\ \iff & \alpha(t) \cdot \alpha'(t) = 0 \text{ and } \alpha(t) \text{ is a nonzero constant} \\ \iff & \alpha(t) \text{ is orthogonal to } \alpha'(t) \text{ for all } t \in I. \end{aligned}$$

□

1-3. Regular Curves; Arc Length

Exercise 1-3.1.

Show that the tangent lines to the regular parametrized curve $\alpha(t) = (3t, 3t^2, 2t^3)$ make a constant angle with the line $y = 0, z = x$.

Proof. $\alpha'(t) = (3, 6t, 6t^2)$. The line $y = 0, z = x$ is $\beta(t) = (1, 0, 1)$. The cosine of the angle θ between these two curves is

$$\begin{aligned} \cos \theta &= \frac{(3, 6t, 6t^2) \cdot (1, 0, 1)}{|(3, 6t, 6t^2)| |(1, 0, 1)|} \\ &= \frac{3 + 6t^2}{\sqrt{3^2 + (6t)^2 + (6t^2)^2} \sqrt{2}} \\ &= \frac{3 + 6t^2}{\sqrt{9 + 36t^2 + 36t^4} \sqrt{2}} \\ &= \frac{3 + 6t^2}{\sqrt{(3 + 6t^2)^2} \sqrt{2}} \\ &= \frac{1}{\sqrt{2}}. \end{aligned}$$

(Notice $3 + 6t^2 > 0$ for all $t \in \mathbb{R}$.) That is, the angle between α' and β is a constant ($= \pi/4$). □

Exercise 1-3.2. (Cycloid)

A circular disk of radius 1 in the plane xy rolls without slipping along the x axis. The figure described by a point of the circumference of the disk is called a **cycloid** (Figure 1-7 in Mantredo P. do Carmo, *Differential Geometry of Curves and Surfaces*).

- (a) Obtain a parametrized curve $\alpha : \mathbb{R} \rightarrow \mathbb{R}^2$ the trace of which is the cycloid and determine its singular points.
- (b) Compute the arc length of the cycloid corresponding to a complete rotation of the disk.

Proof of (a).

- (1) Since

$$\begin{cases} x = t - \sin t \\ y = 1 - \cos t, \end{cases}$$

we define $\alpha(t) = (t - \sin t, 1 - \cos t)$.

- (2) $\alpha'(t) = (1 - \cos t, \sin t)$. $\alpha'(t) = 0$ if and only if $t = 2n\pi$ where $n \in \mathbb{Z}$. That is, all singular points are $\alpha(2n\pi) = (2n\pi, 0)$ where $n \in \mathbb{Z}$.

□

Proof of (b). The arc length of the cycloid corresponding to a complete rotation of the disk is

$$\begin{aligned} \int_0^{2\pi} |\alpha'(t)| dt &= \int_0^{2\pi} \sqrt{(1 - \cos t)^2 + (\sin t)^2} dt \\ &= \int_0^{2\pi} \sqrt{2} \sqrt{1 - \cos t} dt \\ &= \int_0^{2\pi} 2 \sin \frac{t}{2} dt \\ &= \left[-4 \cos \frac{t}{2} \right]_{t=0}^{t=2\pi} \\ &= 8. \end{aligned}$$

□

Supplement. The cycloid is not an algebraic curve.

Exercise 1-3.3. (Cisoid of Diocles)

Let $0A = 2a$ be the diameter of a circle \mathbb{S}^1 and $0Y$ and AV be the tangents to \mathbb{S}^1 at 0 and A , respectively. A half-line r is drawn from 0 which meets the circle \mathbb{S}^1 at C and the line AV at B . On $0B$ mark off the segment $0p = CB$. If we rotate r about 0 , the point p will describe a curve called the **cisoid of Diocles**. By taking $0A$ as the x axis and $0Y$ as the y axis, prove that

(a) The tract of

$$\alpha(t) = \left(\frac{2at^2}{1+t^2}, \frac{2at^3}{1+t^2} \right), \quad t \in \mathbb{R},$$

is the cisoid of Diocles ($t = \tan \theta$; see Figure 1-8 in Mantredo P. do Carmo, *Differential Geometry of Curves and Surfaces*).

(b) The origin $(0, 0)$ is a singular point of the cisoid.

(c) As $t \rightarrow \infty$, $\alpha(t)$ approaches the line $x = 2a$, and $\alpha'(t) \rightarrow (0, 2a)$. Thus, as $t \rightarrow \infty$, the curve and its tangent approach the line $x = 2a$; we say that $x = 2a$ is an **asymptote** to the cisoid.

Proof of (a).

(1) The polar equations of the circle \mathbb{S}^1 and the half-line r is

$$\begin{aligned} r &= 2a \cos \theta, \\ r &= 2a \sec \theta, \end{aligned}$$

respectively.

(2) By construction, the polar equation of the cisoid is

$$r = 2a \sec \theta - 2a \cos \theta = 2a \frac{\sin^2 \theta}{\cos \theta} = 2a \sin \theta \tan \theta.$$

(3) Put $t = \tan \theta$, we have

$$\begin{aligned} x &= r \cos \theta = 2a \sin^2 \theta = \frac{2at^2}{1+t^2}, \\ y &= r \sin \theta = tx = \frac{2at^3}{1+t^2}. \end{aligned}$$

So

$$\alpha(t) = (x, y) = \left(\frac{2at^2}{1+t^2}, \frac{2at^3}{1+t^2} \right).$$

□

Supplement. The cisoid is an algebraic curve $= V((x^2 + y^2)x = 2ay^2)$.

Proof of (b). Note that $\alpha(0) = (0, 0)$ and

$$\alpha'(t) = \left(\frac{4at}{(t^2 + 1)^2}, \frac{2at^2(t^2 + 3)}{(t^2 + 1)^2} \right).$$

Hence $\alpha'(0) = (0, 0)$. That is, $(0, 0)$ is a singular point of the cissoid. (In fact, the origin is the unique singular point of the cissoid.) \square

Proof of (c).

(1) Note that

$$\begin{aligned} \lim_{t \rightarrow \pm\infty} x(t) &= \lim_{t \rightarrow \pm\infty} \frac{2at^2}{1 + t^2} = 2a, \\ \lim_{t \rightarrow \pm\infty} y(t) &= \lim_{t \rightarrow \pm\infty} \frac{2at^3}{1 + t^2} = \pm\infty. \end{aligned}$$

Hence, $\alpha(t)$ approaches the line $x = 2a$ as $t \rightarrow \pm\infty$.

(2) Similarly,

$$\begin{aligned} \lim_{t \rightarrow \pm\infty} x'(t) &= \lim_{t \rightarrow \pm\infty} \frac{4at}{(t^2 + 1)^2} = 0, \\ \lim_{t \rightarrow \pm\infty} y'(t) &= \lim_{t \rightarrow \pm\infty} \frac{2at^2(t^2 + 3)}{(t^2 + 1)^2} = 2a. \end{aligned}$$

Therefore, $\alpha'(t) \rightarrow (0, 2a)$ as $t \rightarrow \pm\infty$.

(3) By (1)(2), the curve and its tangent approach the line $x = 2a$ as $t \rightarrow \pm\infty$, or $x = 2a$ is an asymptote to the cissoid.

\square

Exercise 1-3.4. (Tractrix)

Let $\alpha : (0, \pi) \rightarrow \mathbb{R}^2$ be given by

$$\alpha(t) = \left(\sin t, \cos t + \log \tan \frac{t}{2} \right),$$

where t is the angle that the y axis makes with the vector $\alpha(t)$. The trace of α is called the **tractrix**. (Figure 1-9 in Mantredo P. do Carmo, *Differential Geometry of Curves and Surfaces*). Show that

- (a) α is a differentiable parametrized curve, regular except at $t = \frac{\pi}{2}$.
- (b) The length of the segment of the tangent of the tractrix between the point of tangency and the y axis is constantly equal to 1.

Proof of (a).

$$\begin{aligned}\alpha'(t) &= \left(\cos t, -\sin t + \frac{1}{\tan \frac{t}{2}} \frac{1}{\cos^2 \frac{t}{2}} \frac{1}{2} \right) \\ &= \left(\cos t, -\sin t + \frac{1}{2 \sin \frac{t}{2} \cos \frac{t}{2}} \right) \\ &= \left(\cos t, \frac{\cos^2 t}{\sin t} \right)\end{aligned}$$

exists. And $\alpha'(t) = 0$ if and only if $t = \frac{\pi}{2}$. That is, there is a unique singular point at $t = \frac{\pi}{2}$. \square

Proof of (b). The tangent line of the tractrix through the regular point t is parametrized by $\beta : \mathbb{R} \rightarrow \mathbb{R}^2$ which is defined by

$$\begin{aligned}\beta(u) &= u\alpha'(t) + \alpha(t) \\ &= \left(u \cos t + \sin t, u \frac{\cos^2 t}{\sin t} + \cos t + \log \tan \frac{t}{2} \right).\end{aligned}$$

By construction, this tangent line $\beta(u)$ meets the tractrix at $u = 0$, and meets the y -axis when $u \cos t + \sin t = 0$ or $u = -\tan t$. So the length of the segment is

$$\begin{aligned}|\beta(0) - \beta(-\tan t)| &= \sqrt{(-\tan t \cos t)^2 + \left(-\tan t \frac{\cos^2 t}{\sin t}\right)^2} \\ &= \sqrt{(\sin t)^2 + (\cos t)^2} \\ &= 1.\end{aligned}$$

\square

Exercise 1-3.5. (Folium of Descartes)

Let $\alpha : (-1, +\infty) \rightarrow \mathbb{R}^2$ be given by

$$\alpha(t) = \left(\frac{3at}{1+t^3}, \frac{3at^2}{1+t^3} \right).$$

Prove that:

- (a) For $t = 0$, α is tangent to the x axis.
- (b) As $t \rightarrow +\infty$, $\alpha(t) \rightarrow (0, 0)$ and $\alpha'(t) = (0, 0)$.
- (c) Take the curve the the opposite orientation. Now, as $t \rightarrow -1$, the curve and its tangent approach the line $x + y + a = 0$.

The figure obtained by completing the trace of α in such a way that it becomes symmetric relative the the line $y = x$ is called the **folium of Descartes** (See Figure 1-10 in Mantredo P. do Carmo, *Differential Geometry of Curves and Surfaces*).

Proof of (a). Note that

$$\alpha'(t) = \left(\frac{3a(1-2t^3)}{(1+t^3)^2}, \frac{3at(2-t^3)}{(1+t^3)^2} \right).$$

Hence, $\alpha'(0) = (3a, 0)$, or α is tangent to the x axis when $t = 0$. \square

Proof of (b).

(1)

$$\begin{aligned} \lim_{t \rightarrow +\infty} \alpha(t) &= \lim_{t \rightarrow +\infty} \left(\frac{3at}{1+t^3}, \frac{3at^2}{1+t^3} \right) \\ &= \left(\lim_{t \rightarrow +\infty} \frac{3at}{1+t^3}, \lim_{t \rightarrow +\infty} \frac{3at^2}{1+t^3} \right) \\ &= (0, 0). \end{aligned}$$

(2)

$$\begin{aligned} \lim_{t \rightarrow +\infty} \alpha'(t) &= \lim_{t \rightarrow +\infty} \left(\frac{3a(1-2t^3)}{(1+t^3)^2}, \frac{3at(2-t^3)}{(1+t^3)^2} \right) \\ &= \left(\lim_{t \rightarrow +\infty} \frac{3a(1-2t^3)}{(1+t^3)^2}, \lim_{t \rightarrow +\infty} \frac{3at(2-t^3)}{(1+t^3)^2} \right) \\ &= (0, 0). \end{aligned}$$

\square

Proof of (c).

(1) Note that

$$\begin{aligned} \lim_{t \rightarrow -1^+} \alpha(t) &= \lim_{t \rightarrow -1^+} \left(\frac{3at}{1+t^3}, \frac{3at^2}{1+t^3} \right) \\ &= \left(\lim_{t \rightarrow -1^+} \frac{3at}{1+t^3}, \lim_{t \rightarrow -1^+} \frac{3at^2}{1+t^3} \right) \\ &= (-\infty, +\infty) \end{aligned}$$

and

$$\begin{aligned} \lim_{t \rightarrow -1^+} (x(t) + y(t)) &= \lim_{t \rightarrow -1^+} \left(\frac{3at}{1+t^3} + \frac{3at^2}{1+t^3} \right) \\ &= \lim_{t \rightarrow -1^+} \frac{3at}{1-t+t^2} \\ &= -a. \end{aligned}$$

Therefore, as $t \rightarrow -1$, the curve approaches the line $x + y + a = 0$.

(2) Note that

$$\begin{aligned}\lim_{t \rightarrow -1^+} \frac{y'(t)}{x'(t)} &= \lim_{t \rightarrow -1^+} \frac{\frac{3a(1-2t^3)}{(1+t^3)^2}}{\frac{3at(2-t^3)}{(1+t^3)^2}} \\ &= \lim_{t \rightarrow -1^+} \frac{1-2t^3}{t(2-t^3)} \\ &= -1.\end{aligned}$$

Hence, as $t \rightarrow -1$, its tangent also approaches the line $x + y + a = 0$.

□

Exercise 1-3.6. (Logarithmic spiral)

Let $\alpha(t) = (ae^{bt} \cos t, ae^{bt} \sin t)$, $t \in \mathbb{R}$, a and b constants, $a > 0$, $b < 0$, be a parametrized curve.

- (a) Show that as $t \rightarrow +\infty$, $\alpha(t)$ approaches the origin 0, spiraling around it (because of this, the trace of α is called the **logarithmic spiral**; See Figure 1-11 in Mantredo P. do Carmo, *Differential Geometry of Curves and Surfaces*).
- (b) Show that $\alpha'(t) \rightarrow (0, 0)$ as $t \rightarrow +\infty$ and that

$$\lim_{t \rightarrow +\infty} \int_{t_0}^t |\alpha'(t)| dt$$

is finite; that is, α has finite arc length in $[t_0, \infty)$.

Proof of (a).

(1) Note that

$$\lim_{t \rightarrow +\infty} x(t) = \lim_{t \rightarrow +\infty} \frac{\overbrace{a \cos t}^{\text{bounded}}}{\underbrace{e^{-bt}}_{\rightarrow +\infty}} = 0$$

and $\lim_{t \rightarrow +\infty} y(t) = 0$ (by the similar argument). Hence $\alpha(t)$ approaches the origin 0 as $t \rightarrow +\infty$.

- (2) $\alpha(t) = (ae^{bt} \cos t, ae^{bt} \sin t)$ is moving in counter-clockwise on a circle path and sweeping out a length ae^{bt} as t is moving from t_0 to $+\infty$. Note that $t \mapsto ae^{bt}$ is decreasing strictly (as t is moving from t_0 to $+\infty$). Hence α spiraling around the origin.

□

Proof of (b).

(1) Note that

$$\alpha'(t) = (ae^{bt} \underbrace{(b \cos t - \sin t)}_{\text{bounded}}, ae^{bt} \underbrace{(b \sin t + \cos t)}_{\text{bounded}}).$$

As $t \rightarrow +\infty$, $\alpha'(t) \rightarrow (0, 0)$.

(2) As

$$\begin{aligned} \int_{t_0}^{+\infty} |\alpha'(t)| dt &= \int_{t_0}^{+\infty} ae^{bt} \sqrt{b^2 + 1} dt \\ &= \left[\frac{a}{b} e^{bt} \sqrt{b^2 + 1} \right]_{t=t_0}^{t=+\infty} \\ &= -\frac{a}{b} e^{bt_0} \sqrt{b^2 + 1} \\ &< +\infty, \end{aligned}$$

α has finite arc length in $[t_0, \infty)$.

□

Exercise 1-3.7.

A map $\alpha : I \rightarrow \mathbb{R}^3$ is called a **curve of class \mathcal{C}^k** if each of the coordinate functions in the expression $\alpha(t) = (x(t), y(t), z(t))$ has continuous derivatives up to order k . If α is merely continuous, we say that α is of class \mathcal{C}^0 . A curve α is called **simple** if the map α is one-to-one. Thus, the curve $\alpha(t) = (t^3 - 4t, t^2 - 4)$ ($t \in \mathbb{R}$) is not simple.

Let $\alpha : I \rightarrow \mathbb{R}^3$ be a simple curve of class \mathcal{C}^0 . We say that α has a **weak tangent** at $t = t_0 \in I$ if the line determined by $\alpha(t_0 + h)$ and $\alpha(t_0)$ has a limit position when $h \rightarrow 0$. We say that α has a **strong tangent** at $t = t_0 \in I$ if the line determined by $\alpha(t_0 + h)$ and $\alpha(t_0 + k)$ has a limit position when $h, k \rightarrow 0$. Show that

- (a) $\alpha(t) = (t^3, t^2)$, $t \in \mathbb{R}$, has a weak tangent but not a strong tangent at $t = 0$.
- (b) If $\alpha : I \rightarrow \mathbb{R}^3$ is of class \mathcal{C}^1 and regular at $t = t_0$, then it has a strong tangent at $t = t_0$.
- (c) The curve given by

$$\alpha(t) = \begin{cases} (t^2, t^2), & t \geq 0, \\ (t^2, -t^2), & t \leq 0, \end{cases}$$

is of class \mathcal{C}^1 but not of class \mathcal{C}^2 . Draw a sketch of the curve and its tangent vectors.

Proof of (a).

- (1) Note that $\alpha(0) = (0, 0)$ and $\alpha(h) = (h^3, h^2)$. The line passing $\alpha(0)$ and $\alpha(h)$ is

$$\begin{aligned} (x - 0)(h^2 - 0) - (y - 0)(h^3 - 0) &= 0 \\ \iff x - hy &= 0. \end{aligned}$$

As $h \rightarrow 0$, the line has a limit position $x = 0$. Therefore, $\alpha(t)$ has a weak tangent.

- (2) The line passing $\alpha(h)$ and $\alpha(k)$ is

$$\begin{aligned} (x - k^2)(h^2 - k^2) - (y - k^3)(h^3 - k^3) &= 0 \\ \iff (x - k^2)(h + k) - (y - k^3)(h^2 + hk + k^2) &= 0. \end{aligned}$$

As $h \rightarrow 0$, the line has a limit position

$$\begin{aligned} (x - k^2) - (y - k^3)k &= 0 \\ \iff x - ky + k^4 - k^2 &= 0. \end{aligned}$$

As $k \rightarrow 0$, the line has a limit position $x = 0$.

- (3) On the other hand, as $h = -k$ we have $y - k^3 = 0$. As $k \rightarrow 0$, the line has a limit position $y = 0$, contrary to (2). Therefore, $\alpha(t)$ has a strong tangent.

□

Proof of (b).

- (1) The line L passing $\alpha(t_0 + h)$ and $\alpha(t_0 + k)$ is

$$\begin{aligned} x(s) &= x(t_0) + \frac{x(t_0 + h) - x(t_0 + k)}{h - k}s, \\ y(s) &= y(t_0) + \frac{y(t_0 + h) - y(t_0 + k)}{h - k}s, \\ z(s) &= z(t_0) + \frac{z(t_0 + h) - z(t_0 + k)}{h - k}s. \end{aligned}$$

- (2) By the mean value theorem,

$$\frac{x(t_0 + h) - x(t_0 + k)}{h - k} = x'(t_0 + \xi)$$

for some ξ between h and k . Since $\alpha \in \mathcal{C}^1$, $x(t) \in \mathcal{C}^1$. Hence

$$\begin{aligned} \lim_{h,k \rightarrow 0} \frac{x(t_0+h) - x(t_0+k)}{h-k} &= \lim_{h,k \rightarrow 0} x'(t_0 + \xi) \\ &= \lim_{\xi \rightarrow 0} x'(t_0 + \xi) \\ &= x'(t_0). \end{aligned}$$

Similarly, we have $\lim_{h,k \rightarrow 0} \frac{y(t_0+h) - y(t_0+k)}{h-k} = y'(t_0)$ and $\lim_{h,k \rightarrow 0} \frac{z(t_0+h) - z(t_0+k)}{h-k} = z'(t_0)$. Since α is regular, $\lim_{h,k \rightarrow 0} L$ is a non degenerate line

$$\begin{aligned} x(s) &= x(t_0) + x'(t_0)s, \\ y(s) &= y(t_0) + y'(t_0)s, \\ z(s) &= z(t_0) + z'(t_0)s \end{aligned}$$

and thus $\lim_{h,k \rightarrow 0} L$ is a strong tangent at $t = t_0$.

□

Proof of (c).

(1) Since

$$\alpha'(t) = \begin{cases} (2t, 2t), & t \geq 0, \\ (2t, -2t), & t \leq 0, \end{cases}$$

α is of class \mathcal{C}^1 .

(2) Since

$$\alpha''(t) = \begin{cases} (2, 2), & t > 0, \\ \text{undefined}, & t = 0 \\ (2, -2), & t < 0, \end{cases}$$

α is not of class \mathcal{C}^2 .

(Skip drawing a sketch of the curve and its tangent vectors.) □

Exercise 1-3.8.

Let $\alpha : I \rightarrow \mathbb{R}^3$ be a differentiable curve and let $[a, b] \subseteq I$ be a closed interval. For every partition

$$a = t_0 < t_1 < \cdots < t_n = b$$

of $[a, b]$, consider the sum

$$\sum_{i=1}^n |\alpha(t_i) - \alpha(t_{i-1})| = l(\alpha, P),$$

where P stands for the given partition. The norm $|P|$ of a partition P is defined as

$$|P| = \max(t_i - t_{i-1}), i = 1, \dots, n.$$

Geometrically, $l(\alpha, P)$ is the length of a polygon inscribed in $\alpha([a, b])$ with vertices in $\alpha(t_i)$ (see Figure 1-3 in Mantredo P. do Carmo, *Differential Geometry of Curves and Surfaces*). The point of the exercise is to show that the arc length of $\alpha([a, b])$ is, in some sense, a limit of lengths of inscribed polygons. Prove that given $\varepsilon > 0$ there exists $\delta > 0$ such that if $|P| < \delta$ then

$$\left| \int_a^b |\alpha'(t)| dt - l(\alpha, P) \right| < \varepsilon.$$

Assume that $\alpha'(t)$ is continuous.

Proof. Given $\varepsilon > 0$.

- (1) Since $\alpha'(t)$ is continuous on a compact set $[a, b]$, $\alpha'(t)$ is uniformly continuous, that is, there there exists $\delta > 0$ such that

$$|\alpha'(s) - \alpha'(t)| < \frac{\varepsilon}{2(b-a)} \text{ whenever } |s - t| < \delta.$$

- (2) Let $P = \{a = t_0, t_1, \dots, t_n = b\}$ be a partition of $[a, b]$, with $\Delta t_i = t_i - t_{i-1} < \delta$ for all $i = 1, \dots, n$. If $t_{i-1} \leq t \leq t_i$, it follows that

$$|\alpha'(t_i)| - \frac{\varepsilon}{2(b-a)} \leq |\alpha'(t)| \leq |\alpha'(t_i)| + \frac{\varepsilon}{2(b-a)}.$$

Hence,

$$\begin{aligned} & \int_{t_{i-1}}^{t_i} |\alpha'(t)| dt \\ & \geq |\alpha'(t_i)| \Delta t_i - \frac{\varepsilon}{2(b-a)} \Delta t_i \\ & = \left| \int_{t_{i-1}}^{t_i} [\alpha'(t) + \alpha'(t_i) - \alpha'(t)] dt \right| - \frac{\varepsilon}{2(b-a)} \Delta t_i \\ & \geq \left| \int_{t_{i-1}}^{t_i} \alpha'(t) dt \right| - \left| \int_{t_{i-1}}^{t_i} [\alpha'(t_i) - \alpha'(t)] dt \right| - \frac{\varepsilon}{2(b-a)} \Delta t_i \\ & \geq |\alpha(t_i) - \alpha(t_{i-1})| - \frac{\varepsilon}{b-a} \Delta t_i \end{aligned}$$

and

$$\begin{aligned}
& \int_{t_{i-1}}^{t_i} |\alpha'(t)| dt \\
& \leq |\alpha'(t_i)| \Delta t_i + \frac{\varepsilon}{2(b-a)} \Delta t_i \\
& = \left| \int_{t_{i-1}}^{t_i} [\alpha'(t) + \alpha'(t_i) - \alpha'(t)] dt \right| + \frac{\varepsilon}{2(b-a)} \Delta t_i \\
& \leq \left| \int_{t_{i-1}}^{t_i} \alpha'(t) dt \right| + \left| \int_{t_{i-1}}^{t_i} [\alpha'(t_i) - \alpha'(t)] dt \right| + \frac{\varepsilon}{2(b-a)} \Delta t_i \\
& \leq |\alpha(t_i) - \alpha(t_{i-1})| + \frac{\varepsilon}{b-a} \Delta t_i.
\end{aligned}$$

(3) If we add these inequalities, we obtain

$$l(\alpha, P) - \varepsilon \leq \int_a^b |\alpha'(t)| dt \leq l(\alpha, P) + \varepsilon.$$

□

Exercise 1-3.9.

- (a) Let $\alpha : I \rightarrow \mathbb{R}^3$ be a curve of class \mathcal{C}^0 (compare Exercise 1-3.7). Use the approximation by polygons described in Exercise 1-3.8 to give a reasonable definition of arc length of α .
- (b) (A Nonrectifiable Curve.) The following example shows that, with any reasonable definition, the arc length of a \mathcal{C}^0 curve in a closed interval may be unbounded. Let $\alpha : [0, 1] \rightarrow \mathbb{R}^2$ be given as $\alpha(t) = (t, t \sin(\frac{\pi}{t}))$ if $t \neq 0$, and $\alpha(0) = (0, 0)$. Show, geometrically, that the arc length of the portion of the curve corresponding to $\frac{1}{n+1} \leq t \leq \frac{1}{n}$ is at least $\frac{2}{n+\frac{1}{2}}$. Use this to show that the length of curve in the interval $\frac{1}{N} \leq t \leq 1$ is greater than $2 \sum_{n=1}^{N-1} \frac{1}{n+1}$, and thus it tends to infinity as $N \rightarrow \infty$.

Proof of (a). Define

$$l(\alpha) = \sup \{l(\alpha, P) : P \text{ is a partition of } [a, b]\}.$$

□

Note. (Theorem 6.17 in Tom. M. Apostol, *Mathematical Analysis*, 2nd edition.). α is rectifiable if and only if α is of bounded variation on $[a, b]$.

Proof of (b).

(1) Consider a partition $P = \left\{ \frac{1}{n+1}, \frac{1}{n+\frac{1}{2}}, \frac{1}{n} \right\}$ of $\left[\frac{1}{n+1}, \frac{1}{n} \right]$. So that $\alpha(\frac{1}{n+1}) = \alpha(\frac{1}{n}) = 0$ and $\alpha(\frac{1}{n+\frac{1}{2}}) = \pm 1$.

(2) Thus,

$$\begin{aligned}
 & \text{The arc length of the portion of } \alpha \text{ over } \left[\frac{1}{n+1}, \frac{1}{n} \right] \\
 & \geq \text{The sum of each length of the individual chords} \\
 & = \sqrt{\left(\frac{1}{n+\frac{1}{2}} - \frac{1}{n+1} \right)^2 + \left(\frac{1}{n+\frac{1}{2}} \right)^2} \\
 & \quad + \sqrt{\left(\frac{1}{n} - \frac{1}{n+\frac{1}{2}} \right)^2 + \left(\frac{1}{n+\frac{1}{2}} \right)^2} \\
 & \geq \frac{2}{n+\frac{1}{2}}.
 \end{aligned}$$

(3) So

$$\begin{aligned}
 & \text{The arc length of } \alpha \text{ over } \left[\frac{1}{N}, 1 \right] \\
 & = \sum_{n=1}^{N-1} \left\{ \text{The arc length of } \alpha \text{ over } \left[\frac{1}{n+1}, \frac{1}{n} \right] \right\} \\
 & \geq \sum_{n=1}^{N-1} \frac{2}{n+\frac{1}{2}} \\
 & > 2 \sum_{n=1}^{N-1} \frac{1}{n+1}.
 \end{aligned}$$

It tends to infinity as $N \rightarrow \infty$, or α is nonrectifiable.

□

Exercise 1-3.10. (Straight Lines as Shortest)

Let $\alpha : I \rightarrow \mathbb{R}^3$ be a parametrized curve. Let $[a, b] \subseteq I$ and set $\alpha(a) = p$, $\alpha(b) = q$.

(a) Show that, for any constant vector v , $|v| = 1$,

$$(q - p) \cdot v = \int_a^b \alpha'(t) \cdot v dt \leq \int_a^b |\alpha'(t)| dt.$$

(b) Set

$$v = \frac{q-p}{|q-p|}$$

and show that

$$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)| dt;$$

that is, the curve of shortest length from $\alpha(a)$ to $\alpha(b)$ is the straight line joining these points.

Assume $p \neq q$ (otherwise $v = \frac{q-p}{|q-p|}$ is meaningless).

Proof of (a). Let $f(t) = \alpha(t) \cdot v$ defined on I . By the fundamental theorem of calculus,

$$\int_a^b f'(t) dt = f(b) - f(a).$$

Since $f'(t) = \alpha'(t) \cdot v$,

$$(\alpha(b) - \alpha(a)) \cdot v = \int_a^b \alpha'(t) \cdot v dt.$$

Therefore,

$$\begin{aligned} (q-p) \cdot v &= \int_a^b \alpha'(t) \cdot v dt \\ &\leq \int_a^b |\alpha'(t) \cdot v| dt \\ &\leq \int_a^b |\alpha'(t)| |v| dt \\ &= \int_a^b |\alpha'(t)| dt. \end{aligned}$$

□

Proof of (b). $|v| = \frac{|q-p|}{|q-p|} = 1$. So,

$$\begin{aligned} (q-p) \cdot \frac{q-p}{|q-p|} &\leq \int_a^b |\alpha'(t)| dt, \\ |q-p| &\leq \int_a^b |\alpha'(t)| dt. \end{aligned}$$

□

1-4. The Vector Product in \mathbb{R}^3

Exercise 1-4.1.

Check whether the following bases are positive:

- (a) The basis $\{(1, 3), (4, 2)\}$ in \mathbb{R}^2 .
- (b) The basis $\{(1, 3, 5), (2, 3, 7), (4, 8, 3)\}$ in \mathbb{R}^3 .

Proof of (a). Write $u = (1, 3)$ and $v = (4, 2)$. Then

$$\det(u, v) = \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = -10 < 0.$$

Thus $\{u, v\}$ is negative w.r.t. the natural order basis $\{e_1 = (1, 0), e_2 = (0, 1)\}$. \square

Proof of (b). Write $u = (1, 3, 5)$, $v = (2, 3, 7)$, $w = (4, 8, 3)$. Then

$$\det(u, v, w) = \begin{vmatrix} 1 & 3 & 5 \\ 2 & 3 & 7 \\ 4 & 8 & 3 \end{vmatrix} = 39 > 0.$$

Thus $\{u, v, w\}$ is positive w.r.t. the natural order basis $\{e_1, e_2, e_3\}$. \square

Exercise 1-4.2.

A plane P contained in \mathbb{R}^3 is given by the equation $ax+by+cz+d=0$. Show that the vector $v = (a, b, c)$ is perpendicular to the plane and that $|d|/\sqrt{a^2+b^2+c^2}$ measures the distance from the plane to the origin $(0, 0, 0)$.

Say v is a normal vector of E .

In general, the distance from the plane E to any point $(x_0, y_0, z_0) \in \mathbb{R}^3$ is

$$\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Proof.

- (1) To show $v = (a, b, c)$ is perpendicular to the plane, it suffices to show that $v \cdot u = 0$ for any vector u lying on the plane E . Write $u = \overrightarrow{PQ}$ where $P = (x_1, y_1, z_1) \in E$ and $Q = (x_2, y_2, z_2) \in E$. Hence $u = (x_2 - x_1, y_2 -$

$$y_1, z_2 - z_1).$$

$$\begin{aligned} v \cdot u &= (a, b, c) \cdot (x_2 - x_1, y_2 - y_1, z_2 - z_1) \\ &= a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) \\ &= (ax_2 + by_2 + cz_2) - (ax_1 + by_1 + cz_1) \\ &= (-d) - (-d) \\ &= 0. \end{aligned}$$

- (2) Pick any point $(x_1, y_1, z_1) \in E$. The distance from the plane E to the point (x_0, y_0, z_0) is

$$\begin{aligned} & \left| (x_1 - x_0, y_1 - y_0, z_1 - z_0) \cdot \frac{v}{|v|} \right| \\ &= \left| (x_1 - x_0, y_1 - y_0, z_1 - z_0) \cdot \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}} \right| \\ &= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|-d - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}. \end{aligned}$$

□

Exercise 1-4.3.

Determine the angle of intersection of the two planes $5x + 3y + 2z - 4 = 0$ and $3x + 4y - 7z = 0$.

Proof.

- (1) The angle of intersection of the two planes is equal to a angle between two normal vectors of planes.
- (2) Let
 - (a) the angle of intersection of the two planes be θ .
 - (b) the normal vector of $5x + 3y + 2z - 4 = 0$ be $n_1 = (5, 3, 2)$.
 - (c) the normal vector of $3x + 4y - 7z = 0$ be $n_2 = (3, 4, -7)$.

(3) Hence,

$$\cos \theta = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{13}{2\sqrt{703}}.$$

$$\theta = \cos^{-1} \left(\frac{13}{2\sqrt{703}} \right).$$

□

Exercise 1-4.4.

Given two planes $a_ix + b_iy + c_iz + d_i = 0$, $i = 1, 2$, prove that a necessary and sufficient condition for them to be parallel is

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2},$$

where the convention is made that if a denominator is zero, the corresponding numerator is also zero (we say that two planes are parallel if they either coincide or do not intersect).

Proof.

(1) Write

$$E_i : a_ix + b_iy + c_iz + d_i = 0.$$

By Exercise 1-4.2, the vector (a_i, b_i, c_i) is perpendicular to the plane E_i .

(2) Hence,

$$E_1 \text{ is parallel to } E_2 \iff (a_1, b_1, c_1) \text{ is parallel to } (a_2, b_2, c_2)$$

$$\iff \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

(where the convention is made that if a denominator is zero, the corresponding numerator is also zero).

□

Exercise 1-4.5.

Show that the equation of a plane passing through three noncolinear points $p_i = (x_i, y_i, z_i)$, $i = 1, 2, 3$ is given by

$$(p - p_1) \wedge (p - p_2) \cdot (p - p_3) = 0.$$

where $p = (x, y, z)$ is an arbitrary point of the plane and $p - p_1$, for instance, means the vector $(x - x_1, y - y_1, z - z_1)$.

Proof.

- (1) By Exercise 1-4.11(a), the volume V of a parallelepiped generated by $p - p_1, p - p_2, p - p_3 \in \mathbb{R}^3$ is given by

$$V = (p - p_1) \wedge (p - p_2) \cdot (p - p_3).$$

- (2) Since all vectors $p - p_1, p - p_2, p - p_3$ are lying on the same plane, $V = 0$. Therefore, the equation of a plane is $(p - p_1) \wedge (p - p_2) \cdot (p - p_3) = 0$.

□

Exercise 1-4.6.

Given two nonparallel planes $a_i x + b_i y + c_i z + d_i = 0$, $i = 1, 2$, show that their line of intersection may be parametrized as

$$x - x_0 = u_1 t, \quad y - y_0 = u_2 t, \quad z - z_0 = u_3 t,$$

where (x_0, y_0, z_0) belongs to the intersection and $u = (u_1, u_2, u_3)$ is the vector product $u = v_1 \wedge v_2$, $v_i = (a_i, b_i, c_i)$, $i = 1, 2$.

Proof.

- (1) Suppose that the line of intersection is

$$L : x - x_0 = u'_1 t, \quad y - y_0 = u'_2 t, \quad z - z_0 = u'_3 t$$

where (x_0, y_0, z_0) belongs to the intersection.

- (2) By Exercise 1-4.2, the vector $v_i = (a_i, b_i, c_i)$ is perpendicular to the plane $E_i : a_i x + b_i y + c_i z + d_i = 0$. Hence v_i is perpendicular to $u' = (u'_1, u'_2, u'_3)$. Since $v_i \cdot (v_1 \wedge v_2) = 0$, we may choose $u' = v_1 \wedge v_2 = u$.

□

Exercise 1-4.7.

Prove that a necessary and sufficient condition for the plane

$$ax + by + cz + d = 0$$

and the line

$$x - x_0 = u_1 t, \quad y - y_0 = u_2 t, \quad z - z_0 = u_3 t$$

to be parallel is

$$au_1 + bu_2 + cu_3 = 0.$$

Proof. Write

$$\begin{aligned} E : ax + by + cz + d &= 0 \\ L : x - x_0 &= u_1 t, \quad y - y_0 = u_2 t, \quad z - z_0 = u_3 t. \end{aligned}$$

By Exercise 1-4.2, the vector (a, b, c) is perpendicular to the plane E . Hence,

$$\begin{aligned} E \text{ is parallel to } L &\iff (a, b, c) \text{ is perpendicular to } L \\ &\iff (a, b, c) \text{ is perpendicular to } u = (u_1, u_2, u_3) \\ &\iff 0 = (a, b, c) \cdot (u_1, u_2, u_3) = au_1 + bu_2 + cu_3. \end{aligned}$$

□

Exercise 1-4.8.

Prove that the distance ρ between the nonparallel lines

$$\begin{aligned} x - x_0 &= u_1 t, & y - y_0 &= u_2 t, & z - z_0 &= u_3 t, \\ x - x_0 &= v_1 t, & y - y_0 &= v_2 t, & z - z_0 &= v_3 t \end{aligned}$$

is given by

$$\rho = \frac{|(u \wedge v) \cdot r|}{|u \wedge v|}$$

where $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$, $r = (x_0 - x_1, y_0 - y_1, z_0 - z_1)$.

Proof.

$$\rho = |r| |\cos \angle(u \wedge v, r)| = |r| \frac{|(u \wedge v) \cdot r|}{|u \wedge v| |r|} = \frac{|(u \wedge v) \cdot r|}{|u \wedge v|}.$$

It is well-defined ($|u \wedge v| > 0$) since two lines are nonparallel. □

Exercise 1-4.9.

Determine the angle of intersection of the plane $3x + 4y + 7z + 8 = 0$ and the line $x - 2 = 3t$, $y - 3 = 5t$, $z - 5 = 9t$.

Proof.

- (1) The angle of intersection is equal to $\pi/2$ minus an acute angle θ between the normal vector of the plane and the direction vector of a line.
- (2)

$$\cos \theta = \frac{(3, 4, 7) \cdot (3, 5, 9)}{|(3, 4, 7)| |(3, 5, 9)|} = \frac{92}{\sqrt{74} \sqrt{115}}.$$

Hence, the angle of intersection is

$$\pi/2 - \arccos\left(\frac{92}{\sqrt{74}\sqrt{115}}\right).$$

□

Exercise 1-4.10. (Oriented area)

The natural orientation of \mathbb{R}^2 makes it possible to associate a sign to the area A of a parallelogram generated by two linearly independent vectors $u, v \in \mathbb{R}^2$, and write $u = u_1e_1 + u_2e_2$, $v = v_1e_1 + v_2e_2$. Observe the matrix relation

$$\begin{bmatrix} u \cdot u & u \cdot v \\ v \cdot u & v \cdot v \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$$

and conclude that

$$A^2 = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2.$$

Since the last determinant has the same sign as the basis $\{u, v\}$, we can say that A is positive or negative according to whether the orientation of $\{u, v\}$ is positive or negative. This is called the **oriented area** in \mathbb{R}^2 .

Proof.

$$\begin{aligned} A^2 &= \begin{vmatrix} u \cdot u & u \cdot v \\ v \cdot u & v \cdot v \end{vmatrix} \\ &= \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \\ &= \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2 \end{aligned}$$

since $\det A^T = \det A$. □

Exercise 1-4.11. (Oriented volume)

(a) Show that the volume V of a parallelepiped generated by three linearly independent vectors $u, v, w \in \mathbb{R}^3$ is given by $V = |(u \wedge v) \cdot w|$, and introduce an **oriented volume** in \mathbb{R}^3 .

(b) Prove that

$$V^2 = \begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix}.$$

Proof of (a).

- (1) We can calculate the volume V by multiplying the area of the base $|u \wedge v|$ and the height h .
- (2) Note that

$$h = |w| |\cos \angle(u \wedge v, w)| = |w| \frac{|(u \wedge v) \cdot w|}{|u \wedge v| |w|} = \frac{|(u \wedge v) \cdot w|}{|u \wedge v|}.$$

It is well-defined since u and v are linearly independent. Therefore,

$$V = |u \wedge v| h = |(u \wedge v) \cdot w|.$$

- (3) The oriented volume is defined by

$$(u \wedge v) \cdot w.$$

□

Proof of (b). Recall $(u \wedge v) \cdot w = \det(u, v, w)$. Also note that $\det((u, v, w)^T) = \det(u, v, w)$. So

$$\begin{aligned} \begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix} &= \det((u, v, w) \cdot (u, v, w)^T) \\ &= \det(u, v, w) \det((u, v, w)^T) \\ &= \det(u, v, w)^2 \\ &= ((u \wedge v) \cdot w)^2 \\ &= V^2. \end{aligned}$$

□

Exercise 1-4.12.

Given the vectors $v \neq 0$ and w , show that there exists a vector u such that $u \wedge v = w$ if and only if v is perpendicular to w . Is this vector u uniquely determined? If not, what is the most general solution?

Proof.

- (1) Suppose that there exists a vector u such that $u \wedge v = w$. So

$$v \cdot w = w \cdot v = (u \wedge v) \cdot v = 0$$

or v is perpendicular to w .

- (2) Suppose that v is perpendicular to w . Linear algebra says that there exists a vector u such that $\{\tilde{u}, v, w\}$ is a basis of \mathbb{R}^3 . Note that $\tilde{u} \wedge v$ is parallel to w , say $\tilde{u} \wedge v = cw$ for some nonzero constant $c \in \mathbb{R}$. Take $u = c^{-1}\tilde{u}$ and thus

$$u \wedge v = c^{-1}\tilde{u} \wedge v = c^{-1}cw = w.$$

(Note that $\{u, v, w\}$ is also a basis of \mathbb{R}^3 .)

- (3) Such vector u is not uniquely determined. Let L be a line passing the point $u = (u_1, u_2, u_3)$ and parallel to the vector v . By the definition of vector product, for any point $p = (p_1, p_2, p_3) \in L$ we have $p \wedge v = w$ as a vector.

□

Exercise 1-4.13.

Let $u(t) = (u_1(t), u_2(t), u_3(t))$ and $v(t) = (v_1(t), v_2(t), v_3(t))$ be differentiable maps from the interval (a, b) into \mathbb{R}^3 . If the derivatives $u'(t)$ and $v'(t)$ satisfy the conditions

$$u'(t) = au(t) + bv(t), \quad v'(t) = cu(t) - av(t),$$

where a , b , and c are constants, show that $u(t) \wedge v(t)$ is a constant vector.

Proof. Since

$$\begin{aligned} \frac{d}{dt}(u(t) \wedge v(t)) &= u'(t) \wedge v(t) + u(t) \wedge v'(t) \\ &= (au(t) + bv(t)) \wedge v(t) + u(t) \wedge (cu(t) - av(t)) \\ &= au(t) \wedge v(t) + u(t) \wedge (-av(t)) \\ &= a(u(t) \wedge v(t)) + (-a)(u(t) \wedge v(t)) \\ &= (0, 0, 0), \end{aligned}$$

$u(t) \wedge v(t)$ is a constant vector. □

Exercise 1-4.14.

Find all unit vectors which are perpendicular to the vector $(2, 2, 1)$ and parallel to the plane determined by the points $(0, 0, 0)$, $(1, -2, 1)$, $(-1, 1, 1)$.

Proof.

- (1) Let E be the plane determined by the points $p_0 = (0, 0, 0)$, $p_1 = (1, -2, 1)$, $p_2 = (-1, 1, 1)$. The normal vector of E is

$$(p_1 - p_0) \wedge (p_2 - p_0) = (1, -2, 1) \wedge (-1, 1, 1) = (-3, -2, -1).$$

- (2) All unit vectors which are perpendicular to $(2, 2, 1)$ and parallel to E are all unit vectors which are perpendicular to $(2, 2, 1)$ and $(-3, -2, -1)$. Note that

$$(2, 2, 1) \wedge (-3, -2, -1) = (0, -1, 2).$$

Hence, the desired unit vectors are

$$\pm \left(0, -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right).$$

□

1-5. The Local Theory of Curves Parametrized by Arc Length

Exercise 1-5.2.

Show that the torsion τ of α is given by

$$\tau(s) = -\frac{\alpha'(s) \wedge \alpha''(s) \cdot \alpha'''(s)}{|\kappa(s)|^2}.$$

Proof.

- (1) Take inner product $n(s)$ to the definition of torsion $\tau(s)n(s) = b'(s)$, we have

$$\tau(s) = b'(s) \cdot n(s).$$

Since $b'(s) = t(s) \wedge n'(s)$, we have to compute $n'(s)$ first.

- (2) Compute $n'(s)$.

$$n'(s) = \frac{d}{ds} \left(\frac{\alpha''(s)}{\kappa(s)} \right) = \frac{\alpha'''(s)}{\kappa(s)} - \frac{\alpha''(s)\kappa'(s)}{\kappa(s)^2}.$$

(3) By (1)(2),

$$\begin{aligned}
\tau(s) &= b'(s) \cdot n(s) \\
&= (t(s) \wedge n'(s)) \cdot n(s) \\
&= \left(\alpha'(s) \wedge \left(\frac{\alpha'''(s)}{\kappa(s)} - \frac{\alpha''(s)\kappa'(s)}{\kappa(s)^2} \right) \right) \cdot \frac{\alpha''(s)}{\kappa(s)} \\
&= \left(\alpha'(s) \wedge \frac{\alpha'''(s)}{\kappa(s)} \right) \cdot \frac{\alpha''(s)}{\kappa(s)} \\
&= \frac{\alpha'(s) \wedge \alpha'''(s) \cdot \alpha''(s)}{|\kappa(s)|^2},
\end{aligned}$$

or

$$\tau(s) = \frac{\alpha'(s) \wedge \alpha'''(s) \cdot \alpha''(s)}{\alpha''(s)^2}.$$

□

1-6. The Local Canonical Form

1-7. Global Properties of Plane Curves