

Solutions to the book: *Marcus, Number Fields*

Meng-Gen Tsai
plover@gmail.com

October 13, 2021

Contents

Chapter 1: A Special Case of Fermat's Conjecture	3
Exercise 1.1.	3
Exercise 1.2.	4
Exercise 1.3.	4
Supplement.	5
Exercise 1.4.	5
Exercise 1.5.	5
Exercise 1.6.	5
Exercise 1.7.	6
Exercise 1.8.	7
Exercise 1.9.	8
Exercise 1.10.	9
Exercise 1.11.	9
Exercise 1.12.	10
Exercise 1.13.	10
Exercise 1.14.	11
Exercise 1.15.	12
Exercise 1.16.	13
Exercise 1.17.	13
Exercise 1.18.	14
Exercise 1.19.	15
Exercise 1.20.	16
Exercise 1.21.	16
Exercise 1.22.	17
Exercise 1.23.	18
Exercise 1.24.	18
Exercise 1.25.	19
Exercise 1.26.	20
Exercise 1.27.	20

Exercise 1.28.	20
Exercise 1.29.	21
Exercise 1.30.	21
Exercise 1.31.	22
Exercise 1.32.	23
Chapter 2: Number Fields and Number Rings	24
Exercise 2.1.	24
Supplement. (Isomorphic as vector spaces)	25
Exercise 2.2.	25
Exercise 2.3.	27
Supplement.	28
Exercise 2.4.	29
Exercise 2.5.	30
Exercise 2.6.	32
Exercise 2.7.	32
Exercise 2.8.	33
Exercise 2.9.	35
Exercise 2.10.	36
Exercise 2.11.	36
Exercise 2.12. (Kummer's Lemma)	38
Exercise 2.13.	39
Exercise 2.14.	40
Supplement. (Exercise I.1.6 in Jürgen Neukirch, <i>Algebraic Number Theory</i>)	41
Supplement. (Exercise I.1.7 in Jürgen Neukirch, <i>Algebraic Number Theory</i>)	43
Exercise 2.15.	44
Exercise 2.19. (Vandermonde determinant)	45
Exercise 2.20.	47
Exercise 2.28.	48
Exercise 2.43.	50
Exercise 2.45.	51

Chapter 1: A Special Case of Fermat's Conjecture

Exercise 1.1-1.9: Define $N : \mathbb{Z}[i] \rightarrow \mathbb{Z}$ by $N(a + bi) = a^2 + b^2$.

Exercise 1.1.

Verify that for all $\alpha, \beta \in \mathbb{Z}[i]$, $N(\alpha\beta) = N(\alpha)N(\beta)$, either by direct computation or using the fact that $N(a + bi) = (a + bi)(a - bi)$. Conclude that if $\alpha \mid \gamma$ in $\mathbb{Z}[i]$, then $N(\alpha) \mid N(\gamma)$ in \mathbb{Z} .

Proof.

(1) *Direct computation.* Write $\alpha = a + bi, \beta = c + di$ where $a, b, c, d \in \mathbb{Z}$. Thus,

$$\begin{aligned} N(\alpha\beta) &= N((a + bi)(c + di)) \\ &= N((ac - bd) + (ad + bc)i) \\ &= (ac - bd)^2 + (ad + bc)^2 \\ &= (a^2c^2 - 2abcd + b^2d^2) + (a^2d^2 + 2abcd + b^2c^2) \\ &= a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2, \\ N(\alpha)N(\beta) &= N(a + bi)N(c + di) \\ &= (a^2 + b^2)(c^2 + d^2) \\ &= a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2. \end{aligned}$$

Therefore, $N(\alpha\beta) = N(\alpha)N(\beta)$. (Note that we also get the identity $(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$.)

(2) *Using the fact that $N(a + bi) = (a + bi)(a - bi)$, or $N(\alpha) = \alpha\bar{\alpha}$ for any $\alpha \in \mathbb{Z}[i]$.* Thus,

$$\begin{aligned} N(\alpha\beta) &= \alpha\beta\overline{\alpha\beta} \\ &= \alpha\beta\bar{\alpha}\bar{\beta} \\ &= \alpha\bar{\alpha}\beta\bar{\beta} \\ &= N(\alpha)N(\beta). \end{aligned}$$

(3) *Show that if $\alpha \mid \gamma$ in $\mathbb{Z}[i]$, then $N(\alpha) \mid N(\gamma)$ in \mathbb{Z} .* Write $\gamma = \alpha\beta$ for some $\beta \in \mathbb{Z}[i]$. So $N(\gamma) = N(\alpha)N(\beta) \in \mathbb{Z}$, or $N(\alpha) \mid N(\gamma)$ in \mathbb{Z} .

□

Exercise 1.2.

Let $\alpha \in \mathbb{Z}[i]$. Show that α is a unit iff $N(\alpha) = 1$. Conclude that the only units are ± 1 and $\pm i$.

Proof.

- (1) (\implies) Since α is a unit, there is $\beta \in \mathbb{Z}[i]$ such that $\alpha\beta = 1$. By Exercise 1.1, $N(\alpha\beta) = N(1)$, or $N(\alpha)N(\beta) = 1$. Since the image of N is nonnegative integers, $N(\alpha) = 1$.
- (2) (\impliedby) By Exercise 1.1, $N(\alpha) = \alpha\bar{\alpha}$, or $1 = \alpha\bar{\alpha}$ since $N(\alpha) = 1$. That is, $\bar{\alpha} \in \mathbb{Z}[i]$ is the inverse of $\alpha \in \mathbb{Z}[i]$. (Or by (1), we solve the equation $N(\alpha) = a^2 + b^2 = 1$, and show that all four solutions (± 1 and $\pm i$) are unit.)

Conclusion: a unit $\alpha = a+bi$ of $\mathbb{Z}[i]$ is satisfying the equation $N(\alpha) = a^2+b^2 = 1$ by (1)(2). That is, the only units of $\mathbb{Z}[i]$ are ± 1 and $\pm i$. \square

Exercise 1.3.

Let $\alpha \in \mathbb{Z}[i]$. Show that if $N(\alpha)$ is a prime in \mathbb{Z} then α is irreducible in $\mathbb{Z}[i]$. Show that the same conclusion holds if $N(\alpha) = p^2$, where p is a prime in \mathbb{Z} , $p \equiv 3 \pmod{4}$.

Proof.

- (1) Show that if $N(\alpha)$ is a prime in \mathbb{Z} then α is irreducible in $\mathbb{Z}[i]$. Write $\alpha = \beta\gamma$. Then $N(\alpha) = N(\beta)N(\gamma)$ is a prime in \mathbb{Z} . Since each integer prime is irreducible, $N(\beta) = 1$ or $N(\gamma) = 1$. So that β is unit or γ is unit by Exercise 1.2. Hence, α is irreducible.
- (2) Show that α is irreducible in $\mathbb{Z}[i]$ if $N(\alpha) = p^2$, where p is a prime in \mathbb{Z} , $p \equiv 3 \pmod{4}$. Assume $\alpha = \beta\gamma$ were not irreducible. Similar to (1), $N(\alpha) = N(\beta)N(\gamma) = p^2$. Since β and γ are proper factors of α ,

$$N(\beta) = N(\gamma) = p.$$

Since any square $a^2 \equiv 0, 1 \pmod{4}$, any $N(a+bi) = a^2 + b^2 \equiv 0, 1, 2 \pmod{4}$. Especially, $N(\beta) \equiv 0, 1, 2 \pmod{4}$, contrary to $N(\beta) = p \equiv 3 \pmod{4}$ by the assumption. Therefore, α is irreducible in $\mathbb{Z}[i]$.

\square

Supplement.

- (1) The prime 2 is reducible in $\mathbb{Z}[i]$ (Exercise 1.4).
- (2) Every prime $p \equiv 1 \pmod{4}$ is reducible in $\mathbb{Z}[i]$ (Exercise 1.8).

Exercise 1.4.

Show that $1 - i$ is irreducible in \mathbb{Z} and that $2 = u(1 - i)^2$ for some unit u .

Proof.

- (1) $1 - i$ is irreducible. Since $N(1 - i) = 2$ is a prime in \mathbb{Z} , $1 - i$ is irreducible by Problem 1.3.
- (2) $2 = i(1 - i)^2$ where i is unit in \mathbb{Z} .

□

Exercise 1.5.

Notice that $(2 + i)(2 - i) = 5 = (1 + 2i)(1 - 2i)$. How is this consistent with unique factorization?

Proof. Since $2 + i = i(1 - 2i)$ and $2 - i = (-i)(1 + 2i)$, the factorization is unique up to order and multiplication of primes by units. □

Exercise 1.6.

Show that every nonzero, non-unit Gaussian integer α is a product of irreducible elements, by induction on $N(\alpha)$.

Proof. Induction on $N(\alpha)$.

- (1) $n = 2$. Given $\alpha \in \mathbb{Z}[i]$ with $N(\alpha) = 2$. Since $N(\alpha) = 2$ is a prime in \mathbb{Z} , α is irreducible (Exercise 1.3).
- (2) Suppose the result holds for $n \leq k$. Given $\alpha \in \mathbb{Z}[i]$ with $N(\alpha) = k + 1$. There are only two possible cases.
 - (a) α is irreducible. Nothing to do.

- (b) α is reducible. Write $\alpha = \beta\gamma$ where neither factor is unit. Since $N(\alpha) = N(\beta)N(\gamma)$ and neither factor is unit,

$$2 \leq N(\beta), N(\gamma) \leq k.$$

By the induction hypothesis, each factor of α (β and γ) is a product of irreducible elements. So that α again is a product of irreducible elements.

In any cases, α is a product of irreducible elements.

By induction, the result is established. \square

Exercise 1.7.

Show that $\mathbb{Z}[i]$ is a principal ideal domain (PID); i.e., every ideal I is principal. (As shown in Appendix 1, this implies that $\mathbb{Z}[i]$ is a UFD.)

Suggestion: Take $\alpha \in I \setminus \{0\}$ such that $N(\alpha)$ is minimized, and consider the multiplies $\gamma\alpha$, $\gamma \in \mathbb{Z}[i]$; show that these are the vertices of an infinite family of squares which fill up the complex plane. (For example, one of the squares has vertices 0 , α , $i\alpha$, and $(1+i)\alpha$; all others are translates of this one.) Obviously I contains all $\gamma\alpha$; show by a geometric argument that if I contains anything else then minimality of $N(\alpha)$ would be contradicted.

Proof (without geometric intuition). Define N on $\mathbb{Q}[i]$ by $N(a + bi) = a^2 + b^2$ where $a + bi \in \mathbb{Q}[i]$ as usual.

- (1) Show that $\mathbb{Z}[i]$ is a Euclidean domain. Given $\alpha = a + bi \in \mathbb{Z}[i]$ and $\gamma = c + di \in \mathbb{Z}[i]$ with $\gamma \neq 0$. It suffices to show there exist δ and ρ such that the identity $\alpha = \gamma\delta + \rho$ holds and either $\rho = 0$ or $N(\rho) < N(\gamma)$.

- (a) Pick $\delta \in \mathbb{Z}[i]$. (Intuition: Pick the ‘integer part’ of $\frac{\alpha}{\gamma}$ as we did in integer numbers.) Write $\frac{\alpha}{\gamma} = r + si \in \mathbb{Q}[i]$. Then we pick $\delta = m + ni \in \mathbb{Z}[i]$ such that $|r - m| \leq \frac{1}{2}$ and $|s - n| \leq \frac{1}{2}$. Therefore,

$$\begin{aligned} N\left(\frac{\alpha}{\gamma} - \delta\right) &= (r - m)^2 + (s - n)^2 \\ &\leq \frac{1}{4} + \frac{1}{4} \\ &= \frac{1}{2}. \end{aligned}$$

- (b) Pick $\rho \in \mathbb{Z}[i]$. Clearly we can pick $\rho = \alpha - \gamma\delta \in \mathbb{Z}[i]$. Therefore,

$\rho = 0$ or

$$\begin{aligned}
 N(\rho) &= N(\alpha - \gamma\delta) \\
 &= N\left(\gamma\left(\frac{\alpha}{\gamma} - \delta\right)\right) \\
 &= N(\gamma)N\left(\frac{\alpha}{\gamma} - \delta\right) \\
 &\leq \frac{1}{2}N(\gamma) \\
 &< N(\gamma).
 \end{aligned}$$

(2) *Show that every Euclidean domain R is a PID.* Given any ideal I of R . Take $\alpha \in I \setminus \{0\}$ such that $N(\alpha)$ is minimized.

(a) $R\alpha \subseteq I$ clearly.

(b) Conversely, for any $\beta \in I$, there are $\delta, \rho \in R$ such that $\beta = \alpha\delta + \rho$, where either $\rho = 0$ or $N(\rho) < N(\alpha)$. Since $\rho = \beta - \alpha\delta \in I$, we cannot have $N(\rho) < N(\alpha)$ by the minimality of $N(\alpha)$. Therefore, $\rho = 0$ and $\beta = \alpha\delta \in R\alpha$, or $R\alpha \supseteq I$.

By (1)(2), $\mathbb{Z}[i]$ is a PID. \square

Exercise 1.8.

We will use the unique factorization in $\mathbb{Z}[i]$ to prove that every prime $p \equiv 1 \pmod{4}$ is a sum of two squares.

- (a) Use the fact that the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$ of integers mod p is cyclic to show that if $p \equiv 1 \pmod{4}$ then $n^2 \equiv -1 \pmod{p}$ for some $n \in \mathbb{Z}$.
- (b) Prove that p cannot be irreducible in $\mathbb{Z}[i]$. (Hint: $p \mid n^2 + 1 = (n+i)(n-i)$.)
- (c) Prove that p is a sum of two squares. (Hint: (b) shows that $p = (a + bi)(c + di)$ with neither factor a unit. Take norms.)

Proof of (a). Since the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$ of integers mod p is cyclic, $(\mathbb{Z}/p\mathbb{Z})^\times$ is generated by (a primitive root) $g \in \mathbb{Z}/p\mathbb{Z}$. $g^{p-1} = 1$, or

$$(g^{\frac{p-1}{2}} - 1)(g^{\frac{p-1}{2}} + 1) = 0$$

since p is odd. Since $\mathbb{Z}/p\mathbb{Z}$ is an integral domain, $g^{\frac{p-1}{2}} - 1 = 0$ or $g^{\frac{p-1}{2}} + 1 = 0$. g cannot satisfy $g^{\frac{p-1}{2}} - 1 = 0$ since g is a generator of $(\mathbb{Z}/p\mathbb{Z})^\times$. So,

$$g^{\frac{p-1}{2}} + 1 = 0.$$

Let $n = g^{\frac{p-1}{4}} \in \mathbb{Z}$ since $p \equiv 1 \pmod{4}$. So $n^2 + 1 = 0 \pmod{p}$. \square

Proof of (b). Since $n^2 + 1 \equiv 0 \pmod{p}$ by (a), $p \mid n^2 + 1 = (n+i)(n-i)$. If p were irreducible in $\mathbb{Z}[i]$, $p \mid (n+i)$ or $p \mid (n-i)$ by using the unique factorization in $\mathbb{Z}[i]$. Hence

$$\frac{n+i}{p} = \frac{n}{p} + \frac{1}{p}i \notin \mathbb{Z}[i], \frac{n-i}{p} = \frac{n}{p} - \frac{1}{p}i \notin \mathbb{Z}[i],$$

contrary to the assumption. Therefore, p is reducible in $\mathbb{Z}[i]$. \square

Proof of (c). Since p is reducible in $\mathbb{Z}[i]$ by (b), write $p = (a+bi)(c+di)$ with neither factor a unit. Take norms,

$$p^2 = N(p) = N(a+bi)N(c+di).$$

Since neither factor of p is unit, $N(a+bi) = p$, or $a^2 + b^2 = p$, or p is a sum of two squares. \square

Exercise 1.9.

Describe all irreducible elements in $\mathbb{Z}[i]$.

Notice that α is irreducible if and only if $\bar{\alpha}$ is irreducible. (Write $\alpha = \beta\gamma$, then $\bar{\alpha} = \bar{\beta}\bar{\gamma}$. Besides, $\bar{\bar{\alpha}} = \alpha$.)

Proof. Show that all irreducible elements in $\mathbb{Z}[i]$ (up to units) are

- (1) $1+i$.
- (2) $\pi = a+bi$ for each integer prime $p \equiv 1 \pmod{4}$ with $p = a^2 + b^2$.
- (3) p for each integer prime $p \equiv 3 \pmod{4}$.

Let α be any irreducible element in $\mathbb{Z}[i]$. Consider $N(\alpha) = \alpha\bar{\alpha}$. $N(\alpha) \neq 1$ since α is not unit. By the unique factorization theorem in \mathbb{Z} , $N(\alpha) \in \mathbb{Z}$ is a product of primes in \mathbb{Z} .

There are three possible cases.

- (a) $2 \mid N(\alpha)$. Write $(1+i)(1-i) \mid \alpha\bar{\alpha}$ in $\mathbb{Z}[i]$. Notice that $1+i$, $1-i$, α and $\bar{\alpha}$ are all irreducible (Exercise 1.4). By the unique factorization theorem in $\mathbb{Z}[i]$, $\alpha = 1+i$ (up to units).
- (b) $p \mid N(\alpha)$ for some prime $p \equiv 3 \pmod{4}$. Write $p \mid \alpha\bar{\alpha}$ in $\mathbb{Z}[i]$. Notice that p , α and $\bar{\alpha}$ are all irreducible (Exercise 1.3). By the unique factorization theorem in $\mathbb{Z}[i]$, $\alpha = p$ (up to units) or $\bar{\alpha} = p$ (up to units). So in any cases $\alpha = p$ (up to units). (Note that $\bar{p} = p$.)

- (c) $p \mid N(\alpha)$ for some prime $p \equiv 1 \pmod{4}$. For such p , there is an irreducible $\pi \in \mathbb{Z}[i]$ satisfying $p = \pi\bar{\pi}$ (Exercise 1.8). Now we write $\pi\bar{\pi} \mid \alpha\bar{\alpha}$ in $\mathbb{Z}[i]$. Notice that $\pi, \bar{\pi}, \alpha$ and $\bar{\alpha}$ are all irreducible. By the unique factorization theorem in $\mathbb{Z}[i]$, $\alpha = \pi$ or $\alpha = \bar{\pi}$. In any cases, $\alpha = a + bi$ for integer prime $p \equiv 1 \pmod{4}$ with $p = a^2 + b^2$.

□

Exercise 1.10 - 1.14: Let $\omega = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Define $N : \mathbb{Z}[\omega] \rightarrow \mathbb{Z}$ by $N(a + b\omega) = a^2 - ab + b^2$.

Exercise 1.10.

Show that if $a + b\omega$ is written in the form $u + vi$ where u and v are real, then $N(a + b\omega) = u^2 + v^2$.

Proof. By $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, write

$$a + b\omega = \left(a - \frac{1}{2}b\right) + \left(\frac{\sqrt{3}}{2}b\right)i.$$

Here $u = a - \frac{1}{2}b \in \mathbb{R}$ and $v = \frac{\sqrt{3}}{2}b \in \mathbb{R}$. Hence $u^2 + v^2 = (a - \frac{1}{2}b)^2 + (\frac{\sqrt{3}}{2}b)^2 = a^2 - ab + b^2 = N(a + b\omega)$. □

Exercise 1.11.

Show that for all $\alpha, \beta \in \mathbb{Z}[\omega]$, $N(\alpha\beta) = N(\alpha)N(\beta)$, either by direct computation or by using Exercise 1.10. Conclude that if $\alpha \mid \gamma$ in $\mathbb{Z}[\omega]$, then $N(\alpha) \mid N(\gamma)$ in \mathbb{Z} .

Proof.

- (1) *Direct computation.* Note that $1 + \omega + \omega^2 = 0$ or $\omega^2 = -1 - \omega$. Write $\alpha = a + b\omega, \beta = c + d\omega$ where $a, b, c, d \in \mathbb{Z}$. Thus,

$$\begin{aligned} N(\alpha\beta) &= N((a + b\omega)(c + d\omega)) \\ &= N(ac + (ad + bc)\omega + bd\omega^2) \\ &= N(ac + (ad + bc)\omega + bd(-1 - \omega)) \\ &= N((ac - bd) + (ad + bc - bd)\omega) \\ &= (ac - bd)^2 - (ac - bd)(ad + bc - bd) + (ad + bc - bd)^2 \\ &= (a^2 - ab + b^2)(c^2 - cd + d^2), \\ N(\alpha)N(\beta) &= N(a + b\omega)N(c + d\omega) \\ &= (a^2 - ab + b^2)(c^2 - cd + d^2). \end{aligned}$$

- (2) *Exercise 1.10.* The result is established by Exercise 1.10 and Exercise 1.1.
- (3) *Using the fact that $N(a+b\omega) = (a+b\omega)\overline{(a+b\omega)}$.* Similar to the argument of Exercise 1.1.
- (4) *Show that if $\alpha \mid \gamma$ in $\mathbb{Z}[\omega]$, then $N(\alpha) \mid N(\gamma)$ in \mathbb{Z} .* Similar to the argument of Exercise 1.1.

□

Exercise 1.12.

Let $\alpha \in \mathbb{Z}[\omega]$. Show that α is a unit iff $N(\alpha) = 1$, and find all units in $\mathbb{Z}[\omega]$. (There are six of them.)

Proof.

- (1) (\implies) Since α is a unit, there is $\beta \in \mathbb{Z}[\omega]$ such that $\alpha\beta = 1$. By Exercise 1.11, $N(\alpha\beta) = N(1)$, or $N(\alpha)N(\beta) = 1$. Since the image of N is nonnegative integers, $N(\alpha) = 1$.
- (2) (\impliedby) By Exercise 1.10, $N(\alpha) = \alpha\bar{\alpha}$, or $1 = \alpha\bar{\alpha}$ since $N(\alpha) = 1$. That is, $\bar{\alpha} \in \mathbb{Z}[\omega]$ is the inverse of $\alpha \in \mathbb{Z}[\omega]$.
- (3) By (1), we solve the equation $N(\alpha) = a^2 - ab + b^2 = 1$, or $4 = (2a-b)^2 + 3b^2$. There are 2 possible cases.
 - (a) $2a - b = \pm 1, b = \pm 1$.
 - (b) $2a - b = \pm 2, b = \pm 0$.

Solve these 6 pairs of equations yields the result $\pm 1, \pm\omega, \pm\omega^2$.

□

Exercise 1.13.

Show that $1 - \omega$ is irreducible in $\mathbb{Z}[\omega]$, and that $3 = u(1 - \omega)^2$ for some unit u .

3 is not irreducible in $\mathbb{Z}[\omega]$.

Proof.

- (1) $N(1 - \omega) = 3$ is an integer prime. Similar to the argument in Exercise 1.3, $1 - \omega$ is irreducible in $\mathbb{Z}[\omega]$.

- (2) Note that $1 + \omega + \omega^2 = 0$. So $(1 - \omega)^2 = 1 - 2\omega + \omega^2 = 3(-\omega)$, or $(-\omega^2)(1 - \omega)^2 = 3$. By Exercise 1.12, $-\omega^2$ is unit. Hence $3 = u(1 - \omega)^2$ for some unit $u = -\omega^2$.

□

Exercise 1.14.

Modify Exercise 1.7 to show that $\mathbb{Z}[\omega]$ is a PID, hence a UFD. Here the squares are replaced by parallelograms; one of them has vertices $0, \alpha, \omega\alpha, (\omega+1)\alpha$, and all others are translates of this one. Use Exercise 1.10 for the geometric argument at the end.

Similar to Exercise 1.7.

Proof (without geometric intuition). Define N on $\mathbb{Q}[\omega]$ by $N(a+b\omega) = a^2 - ab + b^2$ where $a + b\omega \in \mathbb{Q}[\omega]$ as usual.

- (1) Show that $\mathbb{Z}[\omega]$ is a Euclidean domain. Given $\alpha = a + b\omega \in \mathbb{Z}[\omega]$ and $\gamma = c + d\omega \in \mathbb{Z}[\omega]$ with $\gamma \neq 0$. It suffices to show there exist δ and ρ such that the identity $\alpha = \gamma\delta + \rho$ holds and either $\rho = 0$ or $N(\rho) < N(\gamma)$.
- (a) Pick $\delta \in \mathbb{Z}[\omega]$. (Intuition: Pick the ‘integer part’ of $\frac{\alpha}{\gamma}$ as we did in integer numbers.) Write $\frac{\alpha}{\gamma} = r + s\omega \in \mathbb{Q}[\omega]$. Then we pick $\delta = m + n\omega \in \mathbb{Z}[\omega]$ such that $|r - m| \leq \frac{1}{2}$ and $|s - n| \leq \frac{1}{2}$. Therefore,

$$\begin{aligned} N\left(\frac{\alpha}{\gamma} - \delta\right) &\leq |r - m|^2 + |r - m||s - n| + |s - n|^2 \\ &\leq \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \\ &= \frac{3}{4}. \end{aligned}$$

- (b) Pick $\rho \in \mathbb{Z}[\omega]$. Clearly we can pick $\rho = \alpha - \gamma\delta \in \mathbb{Z}[\omega]$. Therefore, $\rho = 0$ or

$$\begin{aligned} N(\rho) &= N(\alpha - \gamma\delta) \\ &= N\left(\gamma\left(\frac{\alpha}{\gamma} - \delta\right)\right) \\ &= N(\gamma)N\left(\frac{\alpha}{\gamma} - \delta\right) \\ &\leq \frac{3}{4}N(\gamma) \\ &< N(\gamma). \end{aligned}$$

(2) Show that every Euclidean domain R is a PID. Given any ideal I of R . Take $\alpha \in I \setminus \{0\}$ such that $N(\alpha)$ is minimized.

(a) $R\alpha \subseteq I$ clearly.

(b) Conversely, for any $\beta \in I$, there are $\delta, \rho \in R$ such that $\beta = \alpha\delta + \rho$, where either $\rho = 0$ or $N(\rho) < N(\alpha)$. Since $\rho = \beta - \alpha\delta \in I$, we cannot have $N(\rho) < N(\alpha)$ by the minimality of $N(\alpha)$. Therefore, $\rho = 0$ and $\beta = \alpha\delta \in R\alpha$, or $R\alpha \supseteq I$.

By (1)(2), $\mathbb{Z}[\omega]$ is a PID. \square

Exercise 1.15.

Here is a proof of Fermat's conjecture for $n = 4$: If $x^4 + y^4 = z^4$ has a solution in positive integers, then so does $x^4 + y^4 = w^2$. Let x, y, w be a solution with smallest possible w . Then x^2, y^2, w is a primitive Pythagorean triple. Assuming (without loss of generality) that x is odd, we can write

$$x^2 = m^2 - n^2, y^2 = 2mn, w = m^2 + n^2$$

with m and n are relatively prime positive integers, not both odd.

(a) Show that

$$x = r^2 - s^2, n = 2rs, m = r^2 + s^2$$

with r and s are relatively prime positive integers, not both odd.

(b) Show that r, s and m are pairwise relatively prime. Using $y^2 = 4rsm$, conclude that r, s and m are all squares, say a^2, b^2 and c^2 .

(c) Show that $a^4 + b^4 = c^2$, and that this contradicts minimality of w .

Proof of (a). Write $x^2 + n^2 = m^2$ by moving n^2 of $x^2 = m^2 - n^2$ to the left side. Notice that x is odd, and thus $x = r^2 - s^2, n = 2rs, m = r^2 + s^2$ with r and s are relatively prime positive integers, not both odd. \square

Proof of (b).

(1) It suffices to show that $(r, m) = 1$. By assumption, $(r, s) = 1$. So $(r, s) = 1 \Rightarrow (r, s^2) = 1 \Rightarrow (r, r^2 + s^2) = 1$ and note that $m = r^2 + s^2$ to get the result.

(2) $y^2 = 2mn = 2m(2rs) = 4rsm$ by (a). Since r, s and m are pairwise relatively prime, r, s and m are all squares.

□

Proof of (c). By (b), $r = a^2$, $s = b^2$, $m = c^2$. By (a), $m = r^2 + s^2$, or $c^2 = (a^2)^2 + (b^2)^2 = a^4 + b^4$. However, $w = m^2 + n^2 > m^2 > m = c^2 > c$, contrary to the minimality of w . □

Exercise 1.16-1.28: Let p be an odd prime, $\omega = e^{\frac{2\pi i}{p}}$.

Exercise 1.16.

Show that

$$(1 - \omega)(1 - \omega^2) \cdots (1 - \omega^{p-1}) = p$$

by considering equation $t^p - 1 = (t - 1)(t - \omega)(t - \omega^2) \cdots (t - \omega^{p-1})$.

Proof. Note that $t^p - 1 = (t - 1)(t^{p-1} + t^{p-2} + \cdots + t + 1)$. Cancel out $t - 1$ of Equation (2),

$$t^{p-1} + t^{p-2} + \cdots + t + 1 = (t - \omega)(t - \omega^2) \cdots (t - \omega^{p-1}).$$

Put $t = 1$ to get $p = (1 - \omega)(1 - \omega^2) \cdots (1 - \omega^{p-1})$. □

Exercise 1.17.

Let $x^p + y^p = z^p$. Suppose that $\mathbb{Z}[\omega]$ is a UFD and $\pi \mid x + y\omega$, and π is a prime in $\mathbb{Z}[\omega]$. Show that π does not divide any of the other factors on the left side of

$$(x + y)(x + y\omega)(x + y\omega^2) \cdots (x + y\omega^{p-1}) = z^p$$

by showing that if it did, then π would divide both z and yp (Hint: Use Exercise 1.16); but z and yp are relatively prime (assuming p divides none of x, y, z), hence $zm + ypn = 1$ for some $m, n \in \mathbb{Z}$. How is this a contradiction?

Proof. Write

$$z = u\pi_1^{e_1} \cdots \pi_m^{e_m}$$

where u is unit and π_k ($1 \leq k \leq m$) are distinct primes in $\mathbb{Z}[\omega]$ and $e_k \in \mathbb{Z}^+$ ($1 \leq k \leq m$). Since $\mathbb{Z}[\omega]$ is a UFD by assumption, the factorization of z is unique up to order and units.

(1) Show that $\pi \mid z$. Since $\pi \mid x + y\omega$, $\pi \mid z^p$. The factorization of z^p is

$$z^p = u^p \pi_1^{pe_1} \cdots \pi_m^{pe_m}.$$

u^p is unit, and $\pi \mid z^p$ implies that $\pi = \pi_k$ for some k , that is, $\pi \mid z$.

(2) Show that $\pi \mid yp$ if π were divide any of the other factors on the left side of $(x+y)(x+y\omega)(x+y\omega^2)\cdots(x+y\omega^{p-1}) = z^p$. Say $\pi \mid x+y\omega^k$ for some $k \neq 1$. So that $\pi \mid ((x+y\omega) - (x+y\omega^k))$, or $\pi \mid y(\omega - \omega^k)$. Since $k \neq 1$, there are two possible cases.

(a) $k > 1$. $\pi \mid y\omega(1 - \omega^{k-1})$. By Exercise 1.16, $\pi \mid y\omega p$, or $\pi \mid yp$ since ω is unit. (ω^{p-1} is the inverse of ω since $\omega \cdot \omega^{p-1} = 1$.)

(b) $k = 0$. $\pi \mid y(\omega - 1)$, or $\pi \mid y(1 - \omega)$. By Exercise 1.16, $\pi \mid yp$.

In any case, $\pi \mid yp$.

(3) Note that z and yp are integers, and they are relatively prime by the assumption that p divides none of x, y, z . Therefore, on \mathbb{Z} we have $zm + ypn = 1$ for some $m, n \in \mathbb{Z}$.

(4) $zm + ypn = 1$ is also true in $\mathbb{Z}[\omega]$. Therefore, by (1)(2) we have $\pi \mid (zm + ypn)$ or $\pi \mid 1$, or π is unit, contrary to the primality of π .

□

Exercise 1.18.

Use Exercise 1.17 to show that if $\mathbb{Z}[\omega]$ is a UFD then $x + y\omega = u\alpha^p$, $\alpha \in \mathbb{Z}[\omega]$, u a unit in $\mathbb{Z}[\omega]$.

Proof.

(1) Write $z = u\pi_1^{e_1} \cdots \pi_m^{e_m}$ as Exercise 1.17. So

$$z^p = u^p \pi_1^{pe_1} \cdots \pi_m^{pe_m}.$$

(2) Factorize $x + y\omega = vq_1^{f_1} \cdots q_n^{f_n}$, where v is unit and all q_h ($1 \leq h \leq n$) are distinct primes in $\mathbb{Z}[\omega]$ and $f_h \in \mathbb{Z}^+$. Since $\mathbb{Z}[\omega]$ is a UFD, for every $q_h \mid x + y\omega$, there is some $k(h)$ such that $q_h = \pi_{k(h)}$ and also $q_h^{f_h} = \pi_{k(h)}^{pe_{k(h)}}$ or $f_h = pe_{k(h)}$.

(3) Hence,

$$x + y\omega = v \left(\pi_{k(1)}^{e_{k(1)}} \cdots \pi_{k(n)}^{e_{k(n)}} \right)^p,$$

where $\alpha = \pi_{k(1)}^{e_{k(1)}} \cdots \pi_{k(n)}^{e_{k(n)}} \in \mathbb{Z}[\omega]$ and v is unit.

□

Exercise 1.19.

Dropping the assumption that $\mathbb{Z}[\omega]$ is a UFD but using the fact that ideals factor uniquely (up to order) into prime ideals, show that the principal ideal $(x + y\omega)$ has no prime ideal factor in common with any of the other principal ideals on the left side of the equation

$$(x + y)(x + y\omega)(x + y\omega^2) \cdots (x + y\omega^{p-1}) = (z)^p$$

in which all factors are interpreted as principal ideals. (Hint: Modify the proof of Exercise 1.17 appropriately, using the fact that if A is an ideal dividing another ideal B , then $A \supseteq B$.)

Proof. Write

$$(z) = \pi_1^{e_1} \cdots \pi_m^{e_m}$$

where π_k ($1 \leq k \leq m$) are distinct prime ideals of $\mathbb{Z}[\omega]$ and $e_k \in \mathbb{Z}^+$ ($1 \leq k \leq m$). By assumption that $\mathbb{Z}[\omega]$ is a Dedekind domain, the factorization of z is unique up to order.

- (1) Show that $\pi \mid (z)$. Since $\pi \mid (x + y\omega)$, $\pi \mid (z)^p$. The factorization of $(z)^p$ is

$$(z)^p = \pi_1^{pe_1} \cdots \pi_m^{pe_m}.$$

$\pi \mid (z)^p$ implies that $\pi = \pi_k$ for some k , that is, $\pi \mid (z)$.

- (2) Show that $\pi \mid (yp)$ if π were divide any of the other factors on the left side of $(x + y)(x + y\omega)(x + y\omega^2) \cdots (x + y\omega^{p-1}) = (z)^p$. Say $\pi \mid (x + y\omega^k)$ for some $k \neq 1$. So that $x + y\omega \in \pi$ and $x + y\omega^k \in \pi$, or $y(\omega - \omega^k) \in \pi$. Since $k \neq 1$, there are two possible cases.

- (a) $k > 1$. $y\omega(1 - \omega^{k-1}) \in \pi$. By Exercise 1.16, $y\omega p \in \pi$, or $yp \in \pi$ since ω is unit. (ω^{p-1} is the inverse of ω since $\omega \cdot \omega^{p-1} = 1$.)
(b) $k = 0$. $y(\omega - 1) \in \pi$, or $y(1 - \omega) \in \pi$. By Exercise 1.16, $yp \in \pi$.

In any case, $yp \in \pi$, or $\pi \mid (yp)$.

- (3) Note that z and yp are integers, and they are relatively prime by the assumption that p divides none of x, y, z . Therefore, on \mathbb{Z} we have $zm + ypn = 1$ for some $m, n \in \mathbb{Z}$.
(4) $zm + ypn = 1$ is also true in $\mathbb{Z}[\omega]$. Therefore, by (1)(2) we have $z \in \pi$ and $yp \in \pi$. So $zm + ypn \in \pi$ since π is an ideal. So $1 \in \pi$ or $\pi = (1)$, contrary to the primality of π .

□

Exercise 1.20.

Use Exercise 1.19 to show that $(x + y\omega) = I^p$ for some ideal I .

Proof.

- (1) Write $(z) = \pi_1^{e_1} \cdots \pi_m^{e_m}$ as Exercise 1.17. So

$$(z)^p = \pi_1^{pe_1} \cdots \pi_m^{pe_m}.$$

- (2) Factorize $(x + y\omega) = q_1^{f_1} \cdots q_n^{f_n}$, where every q_h ($1 \leq h \leq n$) are distinct prime ideals of $\mathbb{Z}[\omega]$ and $f_h \in \mathbb{Z}^+$. By assumption that $\mathbb{Z}[\omega]$ is a Dedekind domain, for every $q_h \mid (x + y\omega)$, there is some $k(h)$ such that $q_h = \pi_{k(h)}$ and also $q_h^{f_h} = \pi_{k(h)}^{pe_{k(h)}}$ or $f_h = pe_{k(h)}$.

- (3) Hence,

$$(x + y\omega) = \left(\pi_{k(1)}^{e_{k(1)}} \cdots \pi_{k(n)}^{e_{k(n)}} \right)^p,$$

where $I = \pi_{k(1)}^{e_{k(1)}} \cdots \pi_{k(n)}^{e_{k(n)}}$ is an ideal of $\mathbb{Z}[\omega]$.

□

Exercise 1.21.

Show that every number of $\mathbb{Q}[\omega]$ is uniquely representable in the form

$$a_0 + a_1\omega + a_2\omega^2 + \cdots + a_{p-2}\omega^{p-2}, a_i \in \mathbb{Q} \ \forall i$$

by show that ω is a root of the polynomial

$$f(t) = t^{p-1} + t^{p-2} + \cdots + t + 1$$

and that $f(t)$ is irreducible over \mathbb{Q} . (Hint: It is enough to show that $f(t+1)$ is irreducible, which can be established by Eisenstein's criterion. It helps to notice that $f(t+1) = \frac{(t+1)^p - 1}{t}$.)

Proof.

- (1) Given any number $\alpha \in \mathbb{Q}[\omega]$. Show that

$$\alpha = a_0 + a_1\omega + a_2\omega^2 + \cdots + a_{p-2}\omega^{p-2}, a_i \in \mathbb{Q} \ \forall i.$$

Since $\omega^p = 1$, we can write

$$\alpha = a'_0 + a'_1\omega + a'_2\omega^2 + \cdots + a'_{p-2}\omega^{p-2} + a'_{p-1}\omega^{p-1}, a_i \in \mathbb{Q} \ \forall i.$$

Note that $\omega^{p-1} + \omega^{p-2} + \cdots + \omega + 1 = 0$, and thus we can replace ω^{p-1} by $-\omega^{p-2} - \cdots - \omega - 1$.

- (2) Show that ω is a root of the polynomial $f(t) = t^{p-1} + t^{p-2} + \cdots + t + 1$.
 $f(\omega) = \omega^{p-1} + \omega^{p-2} + \cdots + \omega + 1 = 0$.
- (3) Show that $f(t)$ is irreducible over \mathbb{Q} . It suffices to show that $f(t+1)$ is irreducible over \mathbb{Q} . Write $(t-1)f(t) = t^p - 1$. So

$$\begin{aligned}
tf(t+1) &= (t+1)^p - 1 && \text{(Put } t \mapsto t+1) \\
&= \left(\sum_{k=0}^p \binom{p}{k} t^k \right) - 1 && \text{(Binomial theorem)} \\
&= \sum_{k=1}^p \binom{p}{k} t^k, \\
f(t+1) &= \sum_{k=1}^p \binom{p}{k} t^{k-1} \\
&= t^{p-1} + pt^{p-2} + \cdots + \frac{p(p-1)}{2}t + p.
\end{aligned}$$

By Eisenstein's criterion, $f(t+1)$ is irreducible over \mathbb{Q} .

- (4) To show the uniqueness, it suffices to show that the relation

$$0 = a_0 + a_1\omega + a_2\omega^2 + \cdots + a_{p-2}\omega^{p-2}$$

implies all $a_i = 0$. Say $g(t) = a_0 + a_1t + a_2t^2 + \cdots + a_{p-2}t^{p-2} \in \mathbb{Q}[t]$. Clearly $g(\omega) = 0$. By the minimality of $f(t)$, $g(t)$ is identical zero, or all $a_i = 0$.

□

Exercise 1.22.

Use Exercise 1.21 to show that if $\alpha \in \mathbb{Z}[\omega]$ and $p \mid \alpha$, then (writing $\alpha = a_0 + a_1\omega + \cdots + a_{p-2}\omega^{p-2}$, $a_i \in \mathbb{Z}$) all a_i are divisible by p .

Proof. Since $p \mid \alpha$, there is $\beta \in \mathbb{Z}[\omega]$ such that $\alpha = p\beta$. Write

$$\begin{aligned}
\alpha &= a_0 + a_1\omega + \cdots + a_{p-2}\omega^{p-2}, \\
\beta &= b_0 + b_1\omega + \cdots + b_{p-2}\omega^{p-2},
\end{aligned}$$

where $a_i, b_j \in \mathbb{Z}$. By $\alpha = p\beta$ and Exercise 1.21, $a_i = pb_i$ for every $1 \leq i \leq p-2$. So all a_i are divisible by p . □

Define congruence mod p for $\beta, \gamma \in \mathbb{Z}[\omega]$ as follows:

$$\beta \equiv \gamma \pmod{p} \text{ iff } \beta - \gamma = \delta p \text{ for some } \delta \in \mathbb{Z}[\omega].$$

(Equivalently, this is congruence mod the principal ideal $p\mathbb{Z}[\omega]$).

Exercise 1.23.

Show that if $\beta \equiv \gamma \pmod{p}$, then $\bar{\beta} \equiv \bar{\gamma} \pmod{p}$ where the bar denotes complex conjugation.

Proof.

(1) Show that $\bar{\delta} \in \mathbb{Z}[\omega]$ for any $\delta \in \mathbb{Z}[\omega]$. Write

$$\delta = a_0 + a_1\omega + \cdots + a_{p-1}\omega^{p-1}$$

where $a_0, \dots, a_{p-1} \in \mathbb{Z}$. Take the complex conjugation to get

$$\begin{aligned} \bar{\delta} &= \bar{a_0} + \bar{a_1} \cdot \bar{\omega} + \cdots + \bar{a_{p-1}} \cdot \bar{\omega}^{p-1} \\ &= a_0 + a_1\bar{\omega} + \cdots + a_{p-1}\bar{\omega}^{p-1} && \text{(Every } a_k \in \mathbb{Z}) \\ &= a_0 + a_1\omega^{p-1} + \cdots + a_{p-1}\omega \in \mathbb{Z}[\omega]. && (\omega^p = 1) \end{aligned}$$

(2)

$$\begin{aligned} \beta &\equiv \gamma \pmod{p} \\ \iff \beta - \gamma &= \delta p \text{ for some } \delta \in \mathbb{Z}[\omega] \\ \iff \bar{\beta} - \bar{\gamma} &= \bar{\delta} p \text{ for some } \delta \in \mathbb{Z}[\omega] && \text{(Complex conjugation)} \\ \iff \bar{\beta} - \bar{\gamma} &= \delta' p \text{ for some } \delta' \in \mathbb{Z}[\omega] && ((1)) \\ \iff \bar{\beta} &\equiv \bar{\gamma} \pmod{p} \end{aligned}$$

□

Exercise 1.24.

Show that $(\beta + \gamma)^p \equiv \beta^p + \gamma^p \pmod{p}$ and generalize this to sums of arbitrarily many terms by induction.

Proof.

(1) Binomial theorem gives us

$$(\beta + \gamma)^p = \sum_{k=0}^p \binom{p}{k} \beta^k \gamma^{p-k} = \beta^p + \gamma^p + \sum_{k=1}^{p-1} \binom{p}{k} \beta^k \gamma^{p-k}.$$

- (2) Note that every binomial coefficient $\binom{p}{k}$ is divided by p in \mathbb{Z} for $1 \leq k \leq p-1$. Also, every term $\beta^k \gamma^{p-k}$ is in $\mathbb{Z}[\omega]$. So $(\beta + \gamma)^p - \beta^p - \gamma^p = \delta p$ for some $\delta \in \mathbb{Z}[\omega]$. Hence the result holds.

- (3) In general,

$$\left(\sum_{k=1}^n \alpha_k \right)^p \equiv \sum_{k=1}^n \alpha_k^p \pmod{p}.$$

Induction by $(\alpha_1 + \alpha_2)^p \equiv \alpha_1^p + \alpha_2^p \pmod{p}$ and $\left(\sum_{k=1}^{n+1} \alpha_k \right)^p \equiv (\sum_{k=1}^n \alpha_k)^p + \alpha_{n+1}^p \equiv (\sum_{k=1}^n \alpha_k^p) + \alpha_{n+1}^p \equiv \sum_{k=1}^{n+1} \alpha_k^p \pmod{p}$.

□

Exercise 1.25.

Show that for all $\alpha \in \mathbb{Z}[\omega]$, α^p is congruent \pmod{p} to some $a \in \mathbb{Z}$. (Hint: Write α in terms of ω and use Exercise 1.24.)

Proof (Hint). Write

$$\alpha = a_0 + a_1\omega + \cdots + a_{p-1}\omega^{p-1}$$

where $a_0, \dots, a_{p-1} \in \mathbb{Z}$. By Exercise 1.24,

$$\begin{aligned} \alpha^p &\equiv a_0^p + (a_1\omega)^p + \cdots + (a_{p-1}\omega^{p-1})^p \\ &\equiv a_0^p + a_1^p\omega^p + \cdots + a_{p-1}^p(\omega^{p-1})^p \\ &\equiv a_0^p + a_1^p\omega^p + \cdots + a_{p-1}^p(\omega^p)^{p-1} \\ &\equiv a_0^p + a_1^p + \cdots + a_{p-1}^p. \end{aligned} \quad (\omega^p = 1)$$

Here $a_0^p + a_1^p + \cdots + a_{p-1}^p \in \mathbb{Z}$, and thus α^p is congruent \pmod{p} to some integer. □

Exercise 1.26-1.28: Now assume $p \geq 5$. We will show that if $x + y\omega = u\alpha^p \pmod{p}$, $\alpha \in \mathbb{Z}[\omega]$, u a unit in $\mathbb{Z}[\omega]$, x and y integers not divisible by p , then $x \equiv y \pmod{p}$. For this we will need the following result, proved by Kummer, on the units of $\mathbb{Z}[\omega]$:

Lemma: If u is a unit in $\mathbb{Z}[\omega]$ and \bar{u} is its complex conjugate, then u/\bar{u} is a power of ω . (For the proof, see Exercise 2.12.)

Exercise 1.26.

Show that $x + y\omega \equiv u\alpha^p \pmod{p}$ implies

$$x + y\omega \equiv (x + y\omega^{-1})\omega^k \pmod{p}$$

for some $k \in \mathbb{Z}$. (Use the Lemma on units and Exercise 1.23 and 1.25. Note that $\bar{\omega} = \omega^{-1}$.)

Proof (Hint).

$$\begin{aligned} x + y\omega &\equiv u\alpha^p \pmod{p} \\ \implies x + y\omega &\equiv ua \pmod{p} \text{ for some } a \in \mathbb{Z} && \text{(Exercise 1.25)} \\ \implies \overline{x + y\omega} &\equiv \bar{u}\bar{a} \pmod{p} && \text{(Exercise 1.23)} \\ \implies x + y\bar{\omega} &\equiv \bar{u}a \pmod{p} \\ \implies x + y\omega^{-1} &\equiv \bar{u}a \pmod{p} && (\bar{\omega} = \omega^{-1}) \\ \implies x + y\omega^{-1} &\equiv u\omega^{-k}a \pmod{p} \text{ for some } k \in \mathbb{Z} && \text{(Lemma)} \\ \implies ua &\equiv (x + y\omega^{-1})\omega^k \pmod{p} \\ \implies x + y\omega &\equiv (x + y\omega^{-1})\omega^k \pmod{p}. \end{aligned}$$

□

Exercise 1.27.

Use Exercise 1.22 to show that a contradiction results unless $k \equiv 1 \pmod{p}$. (Recall that $p \nmid xy$, $p \geq 5$, and $\omega^{p-1} + \omega^{p-2} + \dots + \omega + 1 = 0$.)

Proof. Exercise 1.26 shows

$$x + y\omega \equiv (x + y\omega^{-1})\omega^k \pmod{p}.$$

Multiply ω on the both sides to get $x\omega + y\omega^2 \equiv y\omega^k + x\omega^{k+1} \pmod{p}$, or

$$p \mid (x\omega + y\omega^2 - y\omega^k - x\omega^{k+1}).$$

If k were satisfying $k \not\equiv 1 \pmod{p}$, then by Exercise 1.22 and $p \geq 5$ we have $p \mid x$ or $p \mid y$, contrary to the assumption that x and y are integers not divisible by p . □

Exercise 1.28.

Finally, show $x \equiv y \pmod{p}$.

Proof. In the argument of Exercise 1.27 we have

$$p \mid ((x - y)\omega + (y - x)\omega^2)$$

by replacing $k = 1$. By Exercise 1.22 and $p \geq 5$, $x - y$ is divisible by p , or $x \equiv y \pmod{p}$ as integers. \square

Exercise 1.29.

Let $\omega = \exp(\frac{2\pi i}{23})$. Verify that the product

$$(1 + \omega^2 + \omega^4 + \omega^5 + \omega^6 + \omega^{10} + \omega^{11})(1 + \omega + \omega^5 + \omega^6 + \omega^7 + \omega^9 + \omega^{11})$$

is divisible by 2 in $\mathbb{Z}[\omega]$, although neither factor is. It can be shown (Exercise 3.17) that 2 is an irreducible element in $\mathbb{Z}[\omega]$; it follows that $\mathbb{Z}[\omega]$ cannot be a UFD.

Proof. Note that $\sum_{k=0}^{22} \omega^k = 0$. So

$$\begin{aligned} & (1 + \omega^2 + \omega^4 + \omega^5 + \omega^6 + \omega^{10} + \omega^{11})(1 + \omega + \omega^5 + \omega^6 + \omega^7 + \omega^9 + \omega^{11}) \\ &= 2(\omega^5 + \omega^6 + \omega^7 + \omega^9 + \omega^{10} + 3\omega^{11} + \omega^{12} + \omega^{13} + \omega^{15} + \omega^{16} + \omega^{17}) \end{aligned}$$

is divisible by 2 in $\mathbb{Z}[\omega]$, although neither factor is. \square

Exercise 1.30-1.32: R is an integral domain (commutative ring with 1 and no zero divisors).

Exercise 1.30.

Show that two ideals in R are isomorphic as R -modules iff they are in the same ideal class.

Proof. Given any two ideals A, B in an commutative integral domain R .

- (1) (\implies) Let $\varphi : A \rightarrow B$ be an R -module isomorphism. Given any nonzero $\alpha \in A$, we have

$$\begin{aligned} \varphi(\alpha)A &= \{\varphi(\alpha)a : a \in A\} \\ &= \{\varphi(\alpha a) : a \in A\} && (\varphi \text{ is a homomorphism}) \\ &= \{\alpha\varphi(a) : a \in A\} && (\varphi \text{ is a homomorphism}) \\ &= \{\alpha b : b \in B\} && (\varphi \text{ is an isomorphism}) \\ &= \alpha B. \end{aligned}$$

Notice that $\varphi(\alpha) \neq 0$ since $\alpha \neq 0$ and φ is injective. Therefore, $A \sim B$.

(2) (\Leftarrow) Given $A \sim B$, there are nonzero $\alpha, \beta \in R$ such that $\alpha A = \beta B$. Define a map $\varphi : A \rightarrow B$ by $\varphi(a) = b$ if $\alpha a = \beta b$.

(a) φ is well-defined.

(i) *Existence of b .* Since $\alpha a \in \alpha A = \beta B$, there is $b \in B$ such that $\alpha a = \beta b$.

(ii) *Uniqueness of b .* If $\alpha a = \beta b_1 = \beta b_2$, $\beta(b_1 - b_2) = 0$. Since R is an integral domain and $\beta \neq 0$, $b_1 - b_2 = 0$ or $b_1 = b_2$.

(b) φ is an R -module homomorphism.

(i) Show that $\varphi(a_1 + a_2) = \varphi(a_1) + \varphi(a_2)$. Write $\varphi(a_1) = b_1$ and $\varphi(a_2) = b_2$.

$$\begin{aligned} \varphi(a_1) = b_1 \text{ and } \varphi(a_2) = b_2 & \\ \implies \alpha a_1 = \beta b_1 \text{ and } \alpha a_2 = \beta b_2 & \quad (\text{Definition of } \varphi) \\ \implies \alpha a_1 + \alpha a_2 = \beta b_1 + \beta b_2 & \quad (\text{Add together}) \\ \implies \alpha(a_1 + a_2) = \beta(b_1 + b_2) & \\ \implies \varphi(a_1 + a_2) = b_1 + b_2 = \varphi(a_1) + \varphi(a_2). & \quad (\text{Definition of } \varphi) \end{aligned}$$

(ii) Show that $\varphi(ra) = r\varphi(a)$. Write $\varphi(a) = b$.

$$\begin{aligned} \varphi(a) = b & \implies \alpha a = \beta b & (\text{Definition of } \varphi) \\ \implies r\alpha a = r\beta b & & (\text{Multiply } r) \\ \implies \alpha(ra) = \beta(rb) & & (R \text{ is commutative}) \\ \implies \varphi(ra) = rb = r\varphi(a). & & (\text{Definition of } \varphi) \end{aligned}$$

(c) φ is injective. Given $\varphi(a) = 0$. Then $\alpha a = \beta b = \beta 0 = 0$. Since R is an integral domain and $\alpha \neq 0$, $a = 0$.

(d) φ is surjective. Given any $b \in B$. $\beta b \in \beta B = \alpha A$. There is $a \in A$ such that $\beta b = \alpha a$. Such a satisfies $\varphi(a) = b$.

Therefore, $\varphi : A \rightarrow B$ is an R -module isomorphism.

□

Exercise 1.31.

Show that if A is an ideal in R and if αA is principal for some nonzero $\alpha \in R$, then A is principal. Conclude that the principal ideals form an ideal class.

Proof.

(1) Write $\alpha A = (b)$ for some $b \in \alpha A$. That is, there is $a \in A$ such that

$$b = \alpha a.$$

- (2) *Show that $A = (a)$ is principal.* $(a) \subseteq A$ holds trivially since $a \in A$ and A is an ideal. Given any $x \in A$, $\alpha x \in \alpha A = (b)$, and thus there is $y \in R$ such that $\alpha x = by$. Replace b by $b = \alpha a$ to get $\alpha x = \alpha ay$ or

$$\alpha(x - ay) = 0.$$

Since $\alpha \neq 0$ and R is an integral domain, $x - ay = 0$ or $x = ay \in (a)$ or $A \subseteq (a)$. Hence $A = (a)$ is principal.

- (3) *Show that the principal ideals form an ideal class.* Given any $A = (a) \neq 0$ and $B = (b) \neq 0$, we have $bA = aB = (ab)$ for $a, b \in R$ or $A \sim B$.

□

Exercise 1.32.

Show that the ideal classes in R form a group iff for every ideal A there is an ideal B such that AB is principal.

Note. The Picard group of the spectrum of a Dedekind domain is its ideal class group.

Proof. Let $[A]$ be the ideal class representing by a nonzero ideal A of R . Let

$$\text{Pic}(R) = \{[A] : A \text{ is an ideal of } R\}$$

be the set of all ideal classes. Define the operation $\cdot : \text{Pic}(R) \times \text{Pic}(R) \rightarrow \text{Pic}(R)$ by $[A] \cdot [B] \mapsto [AB]$.

- (1) *(Closure) Show that the operation $[A] \cdot [B] \mapsto [AB]$ is well-defined.* Trivial due to the definition of the ideal class. Note that $[A] \cdot [B] = [B] \cdot [A]$ by the commutativity of R .
- (2) *(Associativity) Show that $([A] \cdot [B]) \cdot [C] = [A] \cdot ([B] \cdot [C])$.* Trivial due to the definition of the ideal class.
- (3) *(Identity element) Show that the non-zero principal ideals form the ideal class $[1]$.* Exercise 1.30 and note that (1) is principal too.
- (4) *Show that the set $\text{Pic}(R)$ forms an (abelian) group with $[1]$ as the identity element if and only if every $[A]$ has an inverse in $\text{Pic}(R)$.* By (1)(2)(3), the set $\text{Pic}(R)$ forms an (abelian) group iff every element has an inverse element. The conclusion is established.

□

Chapter 2: Number Fields and Number Rings

Exercise 2.1.

- (a) Show that every number field of degree 2 over \mathbb{Q} is one of the quadratic fields $\mathbb{Q}[\sqrt{m}]$, $m \in \mathbb{Z}$.
- (b) Show that the fields $\mathbb{Q}[\sqrt{m}]$, m squarefree, are pairwise distinct. (Hint: Consider the equation $\sqrt{m} = a + b\sqrt{n}$; use this to show that they are in fact pairwise non-isomorphic).

Proof of (a). Let $f(x) = ax^2 + bx + c$, $a, b, c \in \mathbb{Z}$ ($a \neq 0$) and assume f is irreducible over \mathbb{Q} . Let α be a root of $f(x)$. So

$$\alpha = \frac{-b \pm \sqrt{m}}{2a}$$

where $m = b^2 - 4ac \in \mathbb{Z}$. Therefore,

$$\mathbb{Q}[\alpha] = \mathbb{Q}\left[\frac{-b \pm \sqrt{m}}{2a}\right] = \mathbb{Q}[\sqrt{m}].$$

□

Proof of (b). Show that $\mathbb{Q}[\sqrt{m}]$ and $\mathbb{Q}[\sqrt{n}]$ are not isomorphic as fields if m and n are squarefree and $m \neq n$. Reductio ad absurdum.

- (1) If $\varphi : \mathbb{Q}[\sqrt{m}] \rightarrow \mathbb{Q}[\sqrt{n}]$ were an isomorphism as fields, then φ is an identity map on \mathbb{Q} , and

$$\begin{aligned} \varphi(\sqrt{m}) &= a + b\sqrt{n} \text{ for some } a, b \in \mathbb{Q} \\ \implies \varphi(\sqrt{m})\varphi(\sqrt{m}) &= (a + b\sqrt{n})^2 \\ \implies \varphi(\sqrt{m}\sqrt{m}) &= (a + b\sqrt{n})^2 \\ \implies \varphi(m) &= a^2 + nb^2 + 2ab\sqrt{n} \\ \implies m &= a^2 + nb^2 + 2ab\sqrt{n}. \end{aligned}$$

If $2ab \neq 0$, then $\sqrt{n} = \frac{m - a^2 - nb^2}{2ab} \in \mathbb{Q}$, contrary to the assumption that n is squarefree. Hence $2ab = 0$.

- (2) $a = 0$. Write $b = \frac{r}{s} \in \mathbb{Q}$ where $r, s \in \mathbb{Z}$ and $(r, s) = 1$. So

$$ms^2 = nr^2.$$

Hence

$$\begin{aligned} b \neq 0 &\implies s^2 > 0 \text{ and } r^2 > 0 \\ &\implies m \text{ and } n \text{ have the same sign} \\ &\implies (\exists \text{ prime } p \mid m, p \nmid n) \text{ or } (\exists \text{ prime } q \mid n, q \nmid m) \text{ since } m \neq n. \end{aligned}$$

(a) *There is a prime $p \mid m$ but $p \nmid n$.*

$$\begin{aligned}
p \mid m &\implies \text{Write } m = pm_1 \text{ for some } m_1 \in \mathbb{Z} \\
&\implies (pm_1)s^2 = nr^2 && (ms^2 = nr^2) \\
&\implies p \mid nr^2 \\
&\implies p \mid r^2 && (p \nmid n \text{ by assumption}) \\
&\implies p \mid r && (p \text{ is a prime}) \\
&\implies \text{Write } r = pr_1 \text{ for some } r_1 \in \mathbb{Z} \\
&\implies (pm_1)s^2 = n(pr_1)^2 && (ms^2 = nr^2) \\
&\implies m_1s^2 = npr_1^2 \\
&\implies p \mid m_1s^2 \\
&\implies p \mid m_1 && ((r, s) = 1 \text{ and } p \mid r) \\
&\implies \text{Write } m_1 = pm_2 \text{ for some } m_2 \in \mathbb{Z} \\
&\implies m = p^2m_2,
\end{aligned}$$

contrary to the assumption that m is squarefree.

(b) *There is a prime $q \mid n$ but $q \nmid m$.* Similar to (a).

(3) $b = 0$. $m = a^2$. Write $a = \frac{r}{s} \in \mathbb{Q}$ where $r, s \in \mathbb{Z}$ and $(r, s) = 1$. Hence $ms^2 = r^2$. Similar to the argument in (2).

(4) By (2)(3), no such isomorphism φ , that is, $\mathbb{Q}[\sqrt{m}]$ and $\mathbb{Q}[\sqrt{n}]$ are not isomorphic as fields.

□

Supplement. (Isomorphic as vector spaces)

Show that $\mathbb{Q}[\sqrt{m}]$ and $\mathbb{Q}[\sqrt{n}]$ are isomorphic as \mathbb{Q} -vector spaces.

Proof. $[\mathbb{Q}[\sqrt{m}] : \mathbb{Q}] = [\mathbb{Q}[\sqrt{n}] : \mathbb{Q}] = 2$. There is a natural map $\varphi : \mathbb{Q}[\sqrt{m}] \rightarrow \mathbb{Q}[\sqrt{n}]$ defined by $\varphi(a + b\sqrt{m}) = a + b\sqrt{n}$. Clearly φ is well-defined, linear, injective and surjective. □

Exercise 2.2.

Let I be the ideal generated by 2 and $1 + \sqrt{-3}$ in the ring $\mathbb{Z}[\sqrt{-3}] = \{a + b\sqrt{-3} : a, b \in \mathbb{Z}\}$. Show that $I \neq (2)$ but $I^2 = 2I$. Conclude that ideals in $\mathbb{Z}[\sqrt{-3}]$ do not factor uniquely into prime ideals. Show moreover that I is the unique prime ideal containing (2) and conclude that (2) is not a product of prime ideals.

Proof.

(1) Show that $I \neq (2)$.

(a) Show that $I \supseteq (2)$. $2 \in (2, 1 + \sqrt{-3}) = I$.

(b) Show that $I \not\subseteq (2)$. Consider $1 + \sqrt{-3} \in I$. (Reductio ad absurdum)
If $1 + \sqrt{-3}$ were in (2) , then there exists $a + b\sqrt{-3}$ such that

$$1 + \sqrt{-3} = 2(a + b\sqrt{-3}) = 2a + 2b\sqrt{-3}.$$

Thus, $a = \frac{1}{2}$ and $b = \frac{1}{2}$, which is absurd.

(2) Show that $I^2 = 2I$.

(a) Show that $I^2 \supseteq 2I$. Since $2 \in (2, 1 + \sqrt{-3}) = I$, $2I \subseteq I^2$.

(b) Show that $I^2 \subseteq 2I$. All elements of I^2 are generated by

$$2 \cdot 2, 2(1 + \sqrt{-3}) \text{ and } (1 + \sqrt{-3})^2.$$

Clearly, $2 \cdot 2, 2(1 + \sqrt{-3}) \in 2I$. Besides,

$$(1 + \sqrt{-3})^2 = -2 + 2\sqrt{-3} = 2(-(1) + \sqrt{-3}) \in 2I.$$

Hence $I^2 \subseteq 2I$.

(3) Show that ideals in $\mathbb{Z}[\sqrt{-3}]$ do not factor uniquely into prime ideals. It is followed by $I^2 = 2I$ and $I \neq (2)$.

(4) Show that I is the unique prime ideal containing (2) .

(a) Show that $I = (2, 1 + \sqrt{-3})$ is a prime ideal containing (2) . Note that

$$\mathbb{Z}[\sqrt{-3}]/(2) = (\mathbb{Z}/2\mathbb{Z})[\sqrt{-3}] = \{0, 1, \sqrt{-3}, 1 + \sqrt{-3}\}$$

and

$$I/(2) = (1 + \sqrt{-3})$$

is an ideal of $\mathbb{Z}[\sqrt{-3}]/(2)$. So

$$\mathbb{Z}[\sqrt{-3}]/I = (\mathbb{Z}[\sqrt{-3}]/(2))/(I/(2)) = \{0, 1\} = \mathbb{Z}/2\mathbb{Z}$$

is an integral domain. Hence I is a prime ideal containing (2) .

(b) Suppose I' is a prime ideal containing (2) . Similar to part (a),

$$\begin{aligned} \mathbb{Z}[\sqrt{-3}]/I' &= (\mathbb{Z}[\sqrt{-3}]/(2))/(I'/(2)) \\ &= \{0, 1, \sqrt{-3}, 1 + \sqrt{-3}\}/(I'/(2)) \end{aligned}$$

must be an integral domain.

(c) Since $\{0, 1, \sqrt{-3}, 1 + \sqrt{-3}\}$ is not an integral domain, $I'/(2) \neq (0)$ or $I' \neq (2)$. Also, $I'/(2) \neq \{0, 1, \sqrt{-3}, 1 + \sqrt{-3}\}$ implies that $I'/(2) \neq (1) = (\sqrt{-3})$. Therefore we must have $I'/(2) = (1 + \sqrt{-3})$. Here the existence is guaranteed by part (a).

- (5) Show that (2) is not a product of prime ideals. (Reductio ad absurdum)
 Suppose (2) were a product of prime ideals. By part (4), we might write $(2) = I^n$ for some positive integer n . Since $I \neq (2)$ and $I^2 = 2I$,

$$(2) = (2)I^{n-1} \subseteq (2)I.$$

for some $n \geq 2$.

- (6) Take $2 \in (2) \subseteq (2)I$. Write

$$2 = 2a_1 + \cdots + 2a_k = 2 \underbrace{(a_1 + \cdots + a_k)}_{:=a \in I}$$

where $a_1, \dots, a_k \in I$. We take the norm of the both sides to get $N(a) = 1$. a is a unit in $\mathbb{Z}[\sqrt{-3}]$. $I = \mathbb{Z}[\sqrt{-3}]$, which is absurd. Therefore (2) is not a product of prime ideals.

□

Exercise 2.3.

Complete the proof of Corollary 2, Theorem 2.1.

Corollary 2: Let m be a squarefree integer. The set of algebraic integers in the quadratic field $\mathbb{Q}[\sqrt{m}]$ is

$$\begin{aligned} & \{a + b\sqrt{m} : a, b \in \mathbb{Z}\} \text{ if } m \equiv 2, 3 \pmod{4}, \\ & \left\{ \frac{a + b\sqrt{m}}{2} : a, b \in \mathbb{Z}, a \equiv b \pmod{2} \right\} \text{ if } m \equiv 1 \pmod{4}. \end{aligned}$$

Proof.

- (1) Let $\alpha = r + s\sqrt{m}$, $r, s \in \mathbb{Q}$. If $s \neq 0$, then the monic irreducible polynomial over \mathbb{Q} having α as a root is

$$x^2 - 2rx + r^2 - ms^2.$$

Thus α is an algebraic integer iff $2r$ and $r^2 - ms^2$ are both integers.

- (2) Hence $4(r^2 - ms^2) = (2r)^2 - m(2s)^2 \in \mathbb{Z}$. $m(2s)^2 \in \mathbb{Z}$ since $2r \in \mathbb{Z}$. Hence $2s \in \mathbb{Z}$ since m is squarefree. Let $a = 2r, b = 2s \in \mathbb{Z}$. Then $a^2 - mb^2 = 4(r^2 - ms^2) \equiv 0 \pmod{4}$. Note that a square $\equiv 0, 1 \pmod{4}$ and thus we consider the following two cases.

(3) If $m \equiv 1 \pmod{4}$, then

$$\begin{aligned} a^2 - mb^2 &\equiv a^2 - b^2 \pmod{4} \\ \implies a \text{ and } b \text{ has the same parity} \\ \implies \alpha = r + s\sqrt{m} &= \frac{a + b\sqrt{m}}{2}, a, b \in \mathbb{Z}, a \equiv b \pmod{2}. \end{aligned}$$

(4) If $m \equiv 2, 3 \pmod{4}$, then

$$\begin{aligned} a^2 - mb^2 &\equiv a^2 + 2b^2 \text{ or } a^2 + b^2 \pmod{4} \\ \implies \text{both } a \text{ and } b \text{ are even} \\ \implies \text{both } r \text{ and } s \text{ are rational integers} \\ \implies \alpha = r + s\sqrt{m}, r, s \in \mathbb{Z}. \end{aligned}$$

□

Supplement.

(Exercise I.2.4 in [Jürgen Neukirch, *Algebraic Number Theory*].) Let D be a squarefree rational integer $\neq 0, 1$ and d the discriminant of the quadratic number field $K = \mathbb{Q}(\sqrt{D})$. Show that

$$d = \begin{cases} D & \text{if } D \equiv 1 \pmod{4}, \\ 4D & \text{if } D \equiv 2, 3 \pmod{4}. \end{cases}$$

and that an integral basis of K is given by $\{1, \sqrt{D}\}$ in the second case, by $\left\{1, \frac{1+\sqrt{D}}{2}\right\}$ in the first case, and by $\left\{1, \frac{d+\sqrt{d}}{2}\right\}$ in both case.

Proof.

- (1) The Galois group of $K|\mathbb{Q}$ has two elements, the identity and an automorphism sending \sqrt{D} to $-\sqrt{D}$.
- (2) Note that $\alpha \in \mathcal{O}_K$ iff $\text{Tr}_{K|\mathbb{Q}}(\alpha), N_{K|\mathbb{Q}}(\alpha) \in \mathbb{Z}$ (by noting that the equation $x^2 - \text{Tr}_{K|\mathbb{Q}}(\alpha)x + N_{K|\mathbb{Q}}(\alpha) = 0$ has a root $x = \alpha$). So given $\alpha = x + y\sqrt{D} \in \mathcal{O}_K$, we have

$$\begin{aligned} \text{Tr}_{K|\mathbb{Q}}(\alpha) &= 2x \in \mathbb{Z}, \\ N_{K|\mathbb{Q}}(\alpha) &= x^2 - Dy^2 \in \mathbb{Z}. \end{aligned}$$

- (3) So $4(x^2 - Dy^2) = (2x)^2 - D(2y)^2 \in \mathbb{Z}$. So $D(2y)^2 \in \mathbb{Z}$ since $2x \in \mathbb{Z}$. So $2y \in \mathbb{Z}$ since D is squarefree $\neq 0, 1$. Let $r = 2x, s = 2y$. Then $r^2 - Ds^2 = 4(x^2 - Dy^2) \equiv 0 \pmod{4}$. Note that a square $\equiv 0, 1 \pmod{4}$ and thus we consider the following two cases.

(4) If $D \equiv 1 \pmod{4}$, then

$$\begin{aligned}
& r^2 - Ds^2 \equiv r^2 - s^2 \pmod{4} \\
& \implies r \text{ and } s \text{ has the same parity} \\
& \implies \mathcal{O}_K = \left\{ \frac{r + s\sqrt{D}}{2} : r \equiv s \pmod{2} \right\} \\
& \implies \mathcal{O}_K = \left\{ \frac{r-s}{2} + s \cdot \frac{1+\sqrt{D}}{2} : r \equiv s \pmod{2} \right\} \\
& \implies \mathcal{O}_K = \mathbb{Z} + \mathbb{Z} \frac{1+\sqrt{D}}{2}.
\end{aligned}$$

So $\left\{1, \frac{1+\sqrt{D}}{2}\right\}$ is an integral basis of K . Hence

$$d = \begin{vmatrix} 1 & \frac{1+\sqrt{D}}{2} \\ 1 & \frac{1-\sqrt{D}}{2} \end{vmatrix}^2 = D.$$

(5) If $D \equiv 2, 3 \pmod{4}$, then

$$\begin{aligned}
& r^2 - Ds^2 \equiv r^2 + 2s^2 \text{ or } r^2 + s^2 \pmod{4} \\
& \implies \text{both } r \text{ and } s \text{ are even} \\
& \implies \text{both } x \text{ and } y \text{ are rational integers} \\
& \implies \mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\sqrt{D}.
\end{aligned}$$

So $\{1, \sqrt{D}\}$ is an integral basis of K . Hence

$$d = \begin{vmatrix} 1 & \sqrt{D} \\ 1 & -\sqrt{D} \end{vmatrix}^2 = 4D.$$

(6) By (4)(5), $\left\{1, \frac{d+\sqrt{d}}{2}\right\}$ is an integral basis of K for any case.

□

Exercise 2.4.

Suppose a_0, \dots, a_{n-1} are algebraic integers and α is a complex number satisfying

$$\alpha^n + a_{n-1}\alpha^{n-1} + \dots + a_1\alpha + a_0 = 0.$$

Show that the ring $\mathbb{Z}[a_0, \dots, a_{n-1}, \alpha]$ has a finitely generated additive group. (Hint: Consider the products $a_0^{m_0} a_1^{m_1} \dots a_{n-1}^{m_{n-1}} \alpha^m$ and show that only finitely many values of the exponents are needed.) Conclude that α is an algebraic

integer.

Proof. Let $V = \mathbb{Z}[a_0, \dots, a_{n-1}, \alpha]$. Let n_k be the degree of the algebraic integer a_k where $0 \leq k \leq n-1$.

- (1) *Show that V is finitely generated as an additive subgroup of \mathbb{C} . It suffices to show that V is generated by*

$$a_0^{m_0} a_1^{m_1} \dots a_{n-1}^{m_{n-1}} \alpha^m$$

where $0 \leq m_k < n_k$ and $0 \leq m < n$. Given any $x \in V$, x is a finite sum of the product $a_0^{m_0} a_1^{m_1} \dots a_{n-1}^{m_{n-1}} \alpha^m$ with $m_k \geq 0$ and $m \geq 0$.

If $m \geq n$, replace α^m by

$$\begin{aligned} \alpha^m &= \alpha^{m-n} \alpha^n \\ &= \alpha^{m-n} (-a_{n-1} \alpha^{n-1} - \dots - a_1 \alpha - a_0) \\ &= -a_{n-1} \alpha^{m-1} - \dots - a_1 \alpha^{m-n+1} - a_0 \alpha^{m-n}. \end{aligned}$$

Repeat this process to reduce the degree of α^m less than n . Therefore, we can write x as a finite sum of the product $a_0^{m'_0} a_1^{m'_1} \dots a_{n-1}^{m'_{n-1}} \alpha^{m'}$ with $m'_k \geq 0$ and $0 \leq m' < n$.

Once the degree of α^m is reduced, continue to reduce the degree of each $a_k^{m'_k}$ without affecting other a_h ($h \neq k$) and α . Now replace $a_k^{m'_k}$ by

$$a_k^{m'_k} = \sum_{i=0}^{n_k-1} b_{k,i} a_k^i$$

where $b_{k,i} \in \mathbb{Z}$. Therefore, we can write x as a finite sum of the product $a_0^{m''_0} a_1^{m''_1} \dots a_{n-1}^{m''_{n-1}} \alpha^{m'}$ with $0 \leq m''_k < n_k$ and $0 \leq m' < n$.

- (4) *Show that α is an algebraic integer.* Since $\alpha \in V$, $\alpha V \subseteq V$. Thus α is an algebraic integer (Theorem 2.2).

□

Exercise 2.5.

Show that if f is any polynomials over $\mathbb{Z}/p\mathbb{Z}$ (p a prime) then $f(x^p) = (f(x))^p$. (Suggestion: Use induction on the number of terms.)

Proof.

(1) Let

$$\binom{p}{k} = \frac{p!}{k!(p-k)!}$$

be a binomial coefficient. If $1 \leq k \leq p-1$, show that p divides $\binom{p}{k}$.

(a) If $1 \leq k \leq p-1$, then $p \nmid k!$ and $p \nmid (p-k)!$ since p is a prime.

(b) Write $a = \frac{p!}{k!(p-k)!} \in \mathbb{Z}$. Hence,

$$\begin{aligned} a = \frac{p!}{k!(p-k)!} &\iff p! = ak!(p-k)! \\ &\implies p \mid p! \text{ or } p \mid ak!(p-k)! \\ &\implies p \mid a \text{ by (a).} \end{aligned}$$

Hence p divides $\binom{p}{k}$ if $1 \leq k \leq p-1$.

(2) Note that $a^p = a \in \mathbb{Z}/p\mathbb{Z}$ for all $a \in \mathbb{Z}/p\mathbb{Z}$.

(3) Write

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}/p\mathbb{Z}[x].$$

Induction on n .

(a) $n = 0$. So $f(x) = a_0$, and thus $f(x)^p = a_0^p = a_0$ by (2).

(b) $n = 1$. By $f(x) = a_1 x + a_0$,

$$\begin{aligned} f(x)^p &= (a_1 x + a_0)^p \\ &= a_1^p x^p + \sum_{k=1}^{p-1} \binom{p}{k} (a_1 x)^k a_0^{p-k} + a_0^p \quad (\text{Binomial theorem}) \\ &= a_1^p x^p + a_0^p \quad ((1)) \\ &= a_1 x^p + a_0 \quad ((2)) \\ &= f(x^p). \end{aligned}$$

(c) If the statement holds for $n-1$, then

$$\begin{aligned} f(x)^p &= (a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0)^p \\ &= [a_n x^n + (a_{n-1} x^{n-1} + \cdots + a_1 x + a_0)]^p \\ &= (a_n x^n)^p + (a_{n-1} x^{n-1} + \cdots + a_1 x + a_0)^p \quad (\text{Same as (b)}) \\ &= a_n (x^p)^n + (a_{n-1} x^{n-1} + \cdots + a_1 x + a_0)^p \quad ((2)) \\ &= a_n (x^p)^n + a_{n-1} (x^p)^{n-1} + \cdots + a_1 x^p + a_0 \quad (\text{Induction hypothesis}) \\ &= f(x^p). \end{aligned}$$

The inductive step is established.

By induction, $f(x)^p = f(x^p)$ holds for any $n \geq 0$.

□

Exercise 2.6.

Show that if f and g are polynomials over a field K and $f^2 \mid g$ in $K[x]$, then $f \mid g'$. (Hint: Write $g = f^2h$ and differentiate.)

Proof (Hint). Since $f^2 \mid g$ in $K[x]$, there exists $h \in K[x]$ such $g = f^2h$. Differentiate to get $g' = 2ff'h + f^2h' = f(2f'h + fh')$, or $f \mid g'$ in $K[x]$. □

Exercise 2.7.

Complete the proof of Corollary 2, Theorem 2.3.

Corollary 2: The galois group of $\mathbb{Q}[\omega]$ over \mathbb{Q} is isomorphic to the multiplicative group of integer $(\bmod m)$

$$(\mathbb{Z}/m\mathbb{Z})^* = \{k : 1 \leq k \leq m, (k, m) = 1\}.$$

For each $k \in (\mathbb{Z}/m\mathbb{Z})^*$, the corresponding automorphism in the galois group sends ω to ω^k (and hence $g(\omega) \rightarrow g(\omega^k)$ for each $g \in \mathbb{Z}[x]$).

Proof.

- (1) An automorphism of $\mathbb{Q}[\omega]$ is uniquely determined by the image of ω , and Theorem 2.3 shows that ω can be sent to any of the ω^k , $(k, m) = 1$. (Clearly it can't be sent anywhere else.) This established the one-to-one correspondence between the galois group and the multiplicative group of integer $(\bmod m)$, say

$$\alpha : \text{Gal}(\mathbb{Q}[\omega]/\mathbb{Q}) \rightarrow (\mathbb{Z}/m\mathbb{Z})^*.$$

- (2) The composition of automorphisms corresponds to multiplication $(\bmod m)$ in the natural way. That is, if $\sigma, \tau \in \text{Gal}(\mathbb{Q}[\omega]/\mathbb{Q})$ with $\sigma(\omega) = \omega^k$ and $\tau(\omega) = \omega^h$, then

$$(\sigma\tau)(\omega) = \sigma(\omega^h) = \omega^{kh} \xrightarrow{\alpha} kh.$$

Hence α is a group homomorphism.

□

Exercise 2.8.

- (a) Let $\omega = e^{\frac{2\pi i}{p}}$, p an odd prime. Show that $\mathbb{Q}[\omega]$ contains \sqrt{p} if $p \equiv 1 \pmod{4}$, and $\sqrt{-p}$ if $p \equiv 3 \pmod{4}$. (Hint: Recall that we have shown that $\text{disc}(\omega) = \pm p^{p-2}$ with $+$ holding iff $p \equiv 1 \pmod{4}$.) Express $\sqrt{-3}$ and $\sqrt{5}$ as polynomials in the appropriate ω .
- (b) Show that the eighth cyclotomic field contains $\sqrt{2}$.
- (c) Show that every quadratic field is contained in a cyclotomic field: In fact, $\mathbb{Q}[\sqrt{m}]$ is contained in the d -th cyclotomic field, where $d = \text{disc}(\mathcal{O}_{\mathbb{Q}[\sqrt{m}]})$. (More generally, Kronecker and Weber proved that every abelian extension of \mathbb{Q} (normal with abelian Galois group) is contained in a cyclotomic field. See the chapter 4 exercises. Hilbert and others investigated the abelian extensions of an arbitrary number field; their results are known as **class field theory**, which will be discussed in chapter 8.)

Proof of (a).

- (1) Recall that we have shown that

$$\text{disc}(\omega) = \prod_{1 \leq r < s \leq p} (\omega_r - \omega_s)^2 = (-1)^{\frac{p-1}{2}} p^{p-2} = (-1)^{\frac{p-1}{2}} p \cdot p^{p-3}$$

where $\omega_1 = \omega, \dots, \omega_p$ are the conjugates of ω over \mathbb{Q} . Hence

$$\prod_{1 \leq r < s \leq p} (\omega_r - \omega_s) = \pm \sqrt{(-1)^{\frac{p-1}{2}} p \cdot p^{\frac{p-3}{2}}} \in \mathbb{Q}[\omega].$$

Note that $p^{\frac{p-3}{2}} \in \mathbb{Q}$ as $p \geq 3$ is odd and \pm is unrelated as $\mathbb{Q}[\omega]$ is a field. Therefore

$$\sqrt{(-1)^{\frac{p-1}{2}} p} \in \mathbb{Q}[\omega].$$

- (2) Express $\sqrt{-3}$ as polynomials in the appropriate ω . Take $\omega = e^{\frac{2\pi i}{3}}$. A direct computing shows that

$$\begin{aligned} \prod_{1 \leq r < s \leq 3} (\omega_r - \omega_s) &= \prod_{1 \leq r < s \leq 3} (\omega^r - \omega^s) \\ &= (1 - \omega)(1 - \omega^2)(\omega - \omega^2) \\ &= 3(-\omega^2 + \omega) \\ &= 3\sqrt{-3}. \end{aligned}$$

Hence $\sqrt{-3} = -\omega^2 + \omega$.

- (3) Express $\sqrt{5}$ as polynomials in the appropriate ω . Take $\omega = e^{\frac{2\pi i}{5}}$. A direct computing shows that

$$\begin{aligned}\prod_{1 \leq r < s \leq 5} (\omega_r - \omega_s) &= \prod_{1 \leq r < s \leq 5} (\omega^r - \omega^s) \\ &= 3(\omega - \omega^2) \\ &= -25(\omega^4 - \omega^3 - \omega^2 + \omega) \\ &= -25\sqrt{5}.\end{aligned}$$

Hence $\sqrt{5} = \omega^4 - \omega^3 - \omega^2 + \omega$.

- (4) (Another proof) The quadratic Gauss sum shows that

$$\sum_{n=0}^{p-1} e^{\frac{2\pi i n^2}{p}} = \sqrt{(-1)^{\frac{p-1}{2}} p}.$$

So $\sqrt{-3} = 2\omega_3 + 1$ and $\sqrt{5} = 2\omega_5^4 + 2\omega_5 + 1$.

□

Proof of (b).

- (1) A root of eighth unity is $\omega = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$.
(2) Hence

$$\omega + \omega^{-1} = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{-2}}{2} \right) + \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{-2}}{2} \right) = \sqrt{2} \in \mathbb{Q}[\omega].$$

□

Proof of (c).

- (1) Note that $\mathbb{Q}[\omega_a, \omega_b] = \mathbb{Q}[\omega_{ab}]$ if $a, b \in \mathbb{Z}$ are relatively prime. Might assume that m is squarefree since $\mathbb{Q}[\sqrt{ab^2}] = \mathbb{Q}[\sqrt{a}]$. Consider the following four cases.
(2) Suppose $m > 0$ and $2 \nmid m$. Write

$$m = p_1 \cdots p_r \cdot q_1 \cdots q_s$$

as a product of distinct primes where $p_i \equiv 1 \pmod{4}$ and $q_j \equiv 3 \pmod{4}$. Part (a) shows that

$$\sqrt{p_1}, \dots, \sqrt{p_r}, \sqrt{-q_1}, \dots, \sqrt{-q_s} \in \mathbb{Q}[\omega_{p_1}, \dots, \omega_{p_r}, \omega_{q_1}, \dots, \omega_{q_s}].$$

So $\sqrt{(-1)^s m} \in \mathbb{Q}[\omega_m]$. If s is even, then $\sqrt{m} \in \mathbb{Q}[\omega_m]$ or $\mathbb{Q}[\sqrt{m}] \subseteq \mathbb{Q}[\omega_m]$. If s is odd, then $\sqrt{m} \in \mathbb{Q}[\omega_m, \omega_4] = \mathbb{Q}[\omega_{4m}]$ (since $\sqrt{-1} \in \mathbb{Q}[\omega_4]$). In any case, $\mathbb{Q}[\sqrt{m}]$ is contained in the d -th cyclotomic field, where $d = \text{disc}(\mathcal{O}_{\mathbb{Q}[\sqrt{m}]})$. (See Supplement to Exercise 2.3.)

(3) Suppose $m < 0$ and $2 \nmid m$. Similar to (2).

(4) Suppose $m > 0$ and $2 \mid m$. Write

$$m = 2 \cdot p_1 \cdots p_r \cdot q_1 \cdots q_s$$

as a product of distinct primes where $p_i \equiv 1 \pmod{4}$ and $q_j \equiv 3 \pmod{4}$. Parts (a)(b) show that

$$\sqrt{2}, \sqrt{p_1}, \dots, \sqrt{p_r}, \sqrt{-q_1}, \dots, \sqrt{-q_s} \in \mathbb{Q}[\omega_8, \omega_{p_1}, \dots, \omega_{p_r}, \omega_{q_1}, \dots, \omega_{q_s}].$$

So $\sqrt{(-1)^s m} \in \mathbb{Q}[\omega_{4m}]$. Note that $\sqrt{(-1)^s} \in \mathbb{Q}[\omega_4] \subseteq \mathbb{Q}[\omega_{4m}]$. Hence $\sqrt{m} \in \mathbb{Q}[\omega_{4m}]$ is contained in the d -th cyclotomic field, where $d = 4m = \text{disc}(\mathcal{O}_{\mathbb{Q}[\sqrt{m}]})$.

(5) Suppose $m < 0$ and $2 \mid m$. Same as (4).

□

Exercise 2.9.

With notation as in the proof of Corollary 3, Theorem 2.3, show that there exist integers u and v such that $e^{\frac{2\pi i}{r}} = \omega^u \theta^v$.

Proof.

(1) Recall $\omega = e^{\frac{2\pi i}{m}}$, $\theta = e^{\frac{2\pi i}{k}}$ and r is the least common multiple of k and m .

(2) As r is the least common multiple of k and m , there exist coprime integers a and b such that $r = am = bk$. As $(a, b) = 1$, there exist integers u and v such that $au + bv = 1$.

(3) Hence,

$$\begin{aligned} \omega^u \theta^v &= e^{\frac{2\pi i u}{m}} \cdot e^{\frac{2\pi i v}{k}} \\ &= e^{\frac{2\pi i a u}{r}} \cdot e^{\frac{2\pi i b v}{r}} \\ &= e^{\frac{2\pi i (a u + b v)}{r}} \\ &= e^{\frac{2\pi i}{r}}. \end{aligned}$$

□

Exercise 2.10.

Complete the proof of Corollary 3 to Theorem 2.3, by showing if m is even, $m \mid r$, and $\varphi(r) \leq \varphi(m)$, then $r = m$.

Proof.

- (1) Since m is even, write the unique factorization of m as

$$m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

where $p_1 = 2$, all $\alpha_i \geq 1$ ($1 \leq i \leq k$), and all p_i ($1 \leq i \leq k$) are distinct prime numbers.

- (2) Since $m \mid r$, write $r = mm_1$ for some $m_1 \in \mathbb{Z}$. Thus we can write the unique factorization of r as

$$r = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k} q_1^{\gamma_1} \cdots q_h^{\gamma_h}$$

where all $\beta_i \geq \alpha_i \geq 1$ ($1 \leq i \leq k$) and all p_i ($1 \leq i \leq k$) and q_j ($1 \leq j \leq h$) are distinct prime numbers. Here h might be zero if $m_1 = 1$, and all $q_j \mid m_1$ but $q_j \nmid m$.

- (3) Thus,

$$\begin{aligned} \varphi(m) &= m \left(1 - \frac{1}{2}\right) \cdots \left(1 - \frac{1}{p_k}\right) \\ \varphi(r) &= mm_1 \left(1 - \frac{1}{2}\right) \cdots \left(1 - \frac{1}{p_k}\right) \left(1 - \frac{1}{q_1}\right) \cdots \left(1 - \frac{1}{q_h}\right) \\ &= \varphi(m)m_1 \left(1 - \frac{1}{q_1}\right) \cdots \left(1 - \frac{1}{q_h}\right) \\ &\geq \varphi(m)(q_1 \cdots q_h) \left(1 - \frac{1}{q_1}\right) \cdots \left(1 - \frac{1}{q_h}\right) \\ &\geq \varphi(m)(q_1 - 1) \cdots (q_h - 1). \end{aligned}$$

- (4) Since all $q_j \neq 2$ ($1 \leq j \leq h$), $q_j - 1 > 1$. Hence by (3) and assumption that $\varphi(r) \leq \varphi(m)$, $h = 0$ or $m_1 = 1$ or $r = m$.

□

Exercise 2.11.

- (a) Suppose all roots of a monic polynomial $f \in \mathbb{Q}[x]$ has absolute value 1. Show that the coefficient of x^r has absolute value $\leq \binom{n}{r}$, where n is the degree of f and $\binom{n}{r}$ is the binomial coefficient.

- (b) Show that there are only finitely many algebraic integers α of fixed degree n , all of whose conjugates (including α) have absolute value 1. (Note: If you don't use Theorem 2.1, your proof is probably wrong.)
- (c) Show that α must be a root of 1. (Show that its powers are restricted to a finite set.)

Proof of (a).

(1) Write $f(x) = (x - \alpha_1) \cdots (x - \alpha_n)$ where $\alpha_i \in \mathbb{C}$, $|\alpha_i| = 1$ for $i = 1, 2, \dots, n$.

(2) So

$$f(x) = x^n - s_1 x^{n-1} + s_2 x^{n-2} + \cdots + (-1)s_n$$

where

$$s_r = \sum_{1 \leq j_1 < \cdots < j_r \leq n} \alpha_{j_1} \cdots \alpha_{j_r} \in \mathbb{C}.$$

Let $c_r = (-1)^r s_{n-r}$ be the coefficient of x^r .

(3)

$$\begin{aligned} |c_r| &= |(-1)^r s_{n-r}| \\ &= \left| \sum_{1 \leq j_1 < \cdots < j_{n-r} \leq n} \alpha_{j_1} \cdots \alpha_{j_{n-r}} \right| \\ &\leq \sum_{1 \leq j_1 < \cdots < j_{n-r} \leq n} |\alpha_{j_1} \cdots \alpha_{j_{n-r}}| \\ &= \sum_{1 \leq j_1 < \cdots < j_{n-r} \leq n} |\alpha_{j_1}| \cdots |\alpha_{j_{n-r}}| \\ &= \sum_{1 \leq j_1 < \cdots < j_{n-r} \leq n} 1 \\ &= \binom{n}{n-r} \\ &= \binom{n}{r}. \end{aligned}$$

□

Proof of (b).

- (1) Let f be an irreducible monic polynomial over \mathbb{Z} of degree n such that $f(\alpha) = 0$. So f is irreducible over \mathbb{Q} (Theorem 2.1), and thus all the conjugates of α (including α) are roots of f .

- (2) By (a), all the coefficient of x^r has absolute value $\leq \binom{n}{r}$. Since all the coefficient of x^r are integers, there are finitely many irreducible monic polynomials $f \in \mathbb{Z}[x]$ such that $f(\alpha) = 0$ with $|\alpha| = 1$.
- (3) For each such f , there are only finitely many roots. Therefore, there are only finitely many such algebraic integers α .

□

Proof of (c).

- (1) If $\alpha_1, \dots, \alpha_n$ are the roots of f of degree n over \mathbb{Q} , then for every $r \in \mathbb{Z}^+$, $\alpha_1^r, \dots, \alpha_n^r$ are all the roots of some monic polynomial f_r of degree n over \mathbb{Q} (Fundamental theorem of symmetric polynomials).
- (2) Now we consider the powers of α . All the powers of α (α^r) are algebraic integers (Theorem 2.2), and of degree at most n . (Let $g \in \mathbb{Z}[x]$ be the minimal polynomial of α^r over \mathbb{Q} . By (1), $f_r(\alpha^r) = 0$, and thus $g \mid f_r$. Hence $\deg(g) \leq \deg(f_r) = n$.)
- (3) By (b), the powers of α are restricted to a finite set, say $\alpha^r = \alpha^s$ for some $s > r \geq 1$. So $\alpha^{s-r} = 1$ with $s - r \geq 1$. That is, α is a root of unity.

□

Exercise 2.12. (Kummer's Lemma)

Now we can prove Kummer's lemma on units in the p -th cyclotomic field, as stated before Exercise 1.26: Let $\omega = e^{\frac{2\pi i}{p}}$, p an odd prime, and suppose u is a unit in $\mathbb{Z}[\omega]$.

- (a) Show that u/\bar{u} is a root of 1. (Use Exercise 2.11(c) above and observe that complex conjugation is a member of the Galois group of $\mathbb{Z}[\omega]$ over \mathbb{Q} .) Conclude that $u/\bar{u} = \pm \omega^k$ for some k .
- (b) Show that the + sign holds: Assuming $u/\bar{u} = -\omega^k$, we have $u^p = -\bar{u}^p$; show that this implies that u^p is divisible by p in $\mathbb{Z}[\omega]$. (Use Exercise 1.23 and 1.25) But this is impossible since u^p is a unit.

Proof of (a). Write $\alpha = u/\bar{u}$. Then

$$\begin{aligned}
 |\alpha| = 1 &\implies \alpha \text{ is a root of unity} && \text{(Exercise 2.11)} \\
 &\implies \alpha \text{ is a } 2p\text{-th root of unity} && \text{(Corollary 3 to Theorem 2.3)} \\
 &\implies \alpha = \pm \omega^k \text{ for some } k \in \mathbb{Z}
 \end{aligned}$$

□

Proof of (b). (Reductio ad absurdum) Assume that $u/\bar{u} = -\omega^k$, then

$$\begin{aligned} u/\bar{u} = -\omega^k &\implies (u/\bar{u})^p = (-\omega^k)^p \\ &\implies u^p/\bar{u}^p = (-1)^p \omega^{pk} = -1 \quad (p \text{ is odd}) \\ &\implies u^p = -\bar{u}^p = -\overline{u^p} \end{aligned}$$

By Exercise 1.25, $u^p \equiv a \pmod{p}$ for some $a \in \mathbb{Z}$. By Exercise 1.23, $\bar{u}^p \equiv \bar{a} \equiv a \pmod{p}$. Thus

$$\begin{aligned} u^p = -\bar{u}^p &\implies a \equiv -a \pmod{p} \\ &\implies 2a \equiv 0 \pmod{p} \\ &\implies a \equiv 0 \pmod{p} \quad (p \text{ is odd}) \end{aligned}$$

or $u^p \equiv 0 \pmod{p}$, contradicts the assumption that u is a unit. Hence $u/\bar{u} = \omega^k$ for some k . \square

Exercise 2.13.

Show that 1 and -1 are the only units in the ring $\mathbb{A} \cap \mathbb{Q}[\sqrt{m}]$, m squarefree, $m < 0$, $m \neq -1, -3$. What if $m = -1$ or -3 ?

Proof.

- (1) Let $K = \mathbb{Q}[\sqrt{m}]$ and $\mathcal{O}_K = \mathbb{A} \cap \mathbb{Q}[\sqrt{m}]$. Define a norm N on K by

$$N(a + b\sqrt{m}) = (a + b\sqrt{m})(a - b\sqrt{m}) = a^2 + |m|b^2.$$

- (2) Corollary 2 to Theorem 2.1:

$$\mathcal{O}_K = \begin{cases} \{a + b\sqrt{m} : a, b \in \mathbb{Z}\} & (m \equiv 2, 3 \pmod{4}), \\ \left\{ \frac{a+b\sqrt{m}}{2} : a, b \in \mathbb{Z}, a \equiv b \pmod{2} \right\} & (m \equiv 1 \pmod{4}). \end{cases}$$

Clearly, N maps \mathcal{O}_K to nonnegative integers. That is, u is a unit in \mathcal{O}_K if and only if $N(u) = 1$ (by the fact that $N(u) = u\bar{u}$).

- (3) If $m \equiv 2, 3 \pmod{4}$ and $u = a + b\sqrt{m} \in \mathcal{O}_K$ is a unit ($a, b \in \mathbb{Z}$), then

$$N(u) = 1 = a^2 + |m|b^2.$$

- (a) $m = -1$ or $|m| = 1$. $1 = a^2 + b^2$ or $(a, b) = (\pm 1, 0), (0, \pm 1)$. Hence all units in \mathcal{O}_K are

$$\pm 1, \pm \sqrt{-1}.$$

- (b) $m < -1$ or $|m| > 1$. $1 = a^2 + |m|b^2$ implies that $b^2 = 0$. Hence all units in \mathcal{O}_K are ± 1 .

- (4) If $m \equiv 1 \pmod{4}$ and $u = \frac{a+b\sqrt{m}}{2} \in \mathcal{O}_K$ is a unit ($a, b \in \mathbb{Z}, a \equiv b \pmod{2}$), then $N(u) = 1 = (\frac{a}{2})^2 + |m|(\frac{b}{2})^2$ or

$$4 = a^2 + |m|b^2.$$

- (a) $m = -3$ or $|m| = 3$. $4 = a^2 + 3b^2$ or $(a, b) = (\pm 2, 0), (\pm 1, \pm 1)$. Hence all units in \mathcal{O}_K are

$$\pm 1, \frac{\pm 1 \pm \sqrt{-3}}{2}.$$

- (b) $m < -3$ or $|m| > 3$. $4 = a^2 + |m|b^2$ implies that $b^2 = 0$. Hence all units in \mathcal{O}_K are ± 1 .

- (5) By (3)(4), all units in \mathcal{O}_K are

$$\begin{cases} \pm 1 & (m \neq -1, -3), \\ \pm 1, \pm \sqrt{-1} & (m = -1), \\ \pm 1, \frac{\pm 1 \pm \sqrt{-3}}{2} & (m = -3). \end{cases}$$

□

Exercise 2.14.

Show that $1 + \sqrt{2}$ is a unit in $\mathbb{Z}[\sqrt{2}]$. Use the powers of $1 + \sqrt{2}$ to generate infinitely many solutions to the diophantine equation $a^2 - 2b^2 = \pm 1$. (It will be shown in Chapter 5 that all units in $\mathbb{Z}[\sqrt{2}]$ are of the form $\pm(1 + \sqrt{2})^k$, $k \in \mathbb{Z}$.)

Might assume to find nonnegative solutions to the Pell's equation $a^2 - 2b^2 = \pm 1$.

Proof.

- (1) Show that $1 + \sqrt{2}$ is a unit in $\mathbb{Z}[\sqrt{2}]$. There is $-1 + \sqrt{2} \in \mathbb{Z}[\sqrt{2}]$ such that

$$(1 + \sqrt{2})(-1 + \sqrt{2}) = 1 \in \mathbb{Z}[\sqrt{2}].$$

Hence $1 + \sqrt{2}$ is a unit.

- (2) $N(a + b\sqrt{2}) = |a^2 - 2b^2|$ is a norm on $\mathbb{Z}[\sqrt{2}]$. To prove this, use the same argument as Exercise 1.1 and note that

$$N(a + b\sqrt{2}) = |(a + b\sqrt{2})(a - b\sqrt{2})|.$$

- (3) By (1)(2), all $(1+\sqrt{2})^k$ with $k \geq 0$ are distinct solutions to the diophantine equation $a^2 - 2b^2 = \pm 1$. Explicitly, let

$$\begin{aligned}(a_0, b_0) &= (1, 0), \\(a_1, b_1) &= (1, 1), \\(a_2, b_2) &= (3, 2), \\(a_3, b_3) &= (7, 5), \\&\dots \\(a_k, b_k) &= (a_{k-1} + 2b_{k-1}, a_{k-1} + b_{k-1}), \\&\dots\end{aligned}$$

Note that all (a_k, b_k) are distinct and satisfying $a_k^2 - 2b_k^2 = \pm 1$. Hence we get infinitely many solutions to the Pell's equation $a^2 - 2b^2 = \pm 1$.

Note. Suppose that all units in $\mathbb{Z}[\sqrt{2}]$ are of the form $\pm(1+\sqrt{2})^k$, $k \in \mathbb{Z}$. Note that $(1+\sqrt{2})^k = (-1+\sqrt{2})^{-k}$. Thus we can find all nonnegative solutions to the Pell's equation $a^2 - 2b^2 = \pm 1$ are exactly the same as (3). \square

Supplement. (Exercise I.1.6 in Jürgen Neukirch, *Algebraic Number Theory*)

Show that the ring $\mathbb{Z}[\sqrt{d}] = \mathbb{Z} + \mathbb{Z}\sqrt{d}$, for any squarefree rational integer $d > 1$, has infinitely many units.

Proof. The proof is quoted from Proposition 17.5.2 in the book: Ireland and Rosen, *A Classical Introduction to Modern Number Theory*, 2nd Ed.

- (1) Define the norm of $z = x + y\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$ by $N(z) = z\bar{z}$ or

$$N(x + y\sqrt{d}) = \underbrace{(x + y\sqrt{d})}_{=: z} \underbrace{(x - y\sqrt{d})}_{=: \bar{z}} = x^2 - dy^2.$$

Note that a norm is multiplicative. Similar to Exercise I.1.1, $\alpha \in \mathbb{Z}[\sqrt{d}]$ is a unit if and only if $N(\alpha) = \pm 1$.

- (2) To show $\mathbb{Z}[\sqrt{d}]$ has infinitely many units, it suffices to show the equation $x^2 - dy^2 = 1$ has infinitely many (x, y) solutions.
- (3) If ξ is irrational then there are infinitely many rational numbers $\frac{x}{y}$, $(x, y) = 1$ such that $\left| \frac{x}{y} - \xi \right| < \frac{1}{y^2}$. It is followed by the pigeonhole principle.
- (4) If d is a positive squarefree integer then there is a constant $M := 2\sqrt{d} + 1$ such that $|x^2 - dy^2| < M$ has infinitely many solutions over \mathbb{Z} . Write $x^2 - dy^2 = (x + y\sqrt{d})(x - y\sqrt{d})$. By part (3), there exist infinitely many

pairs of relatively prime integers (x, y) , $y > 0$ satisfying $|x - y\sqrt{d}| < \frac{1}{y}$.
Hence

$$\begin{aligned} |x^2 - dy^2| &= |x + y\sqrt{d}| |x - y\sqrt{d}| \\ &\leq (|x - y\sqrt{d}| + 2y\sqrt{d}) |x - y\sqrt{d}| \\ &\leq 2\sqrt{d} + 1. \end{aligned}$$

- (5) By part (4), there is an integer m such that $x^2 - dy^2 = m$ for infinitely many solutions over \mathbb{Z} . Here $m \neq 0$. We might assume $x, y > 0$ and x components of solutions are distinct.
- (6) The pigeonhole principle shows that there are two distinct solutions (x_1, y_1) , (x_2, y_2) with $x_1 \neq x_2$ such that

$$x_1 \equiv x_2 \pmod{|m|}, \quad y_1 \equiv y_2 \pmod{|m|}.$$

Let $\alpha = x_1 - y_1\sqrt{d}$, $\beta = x_2 + y_2\sqrt{d}$ and $\gamma = \alpha\beta$. Hence

$$\begin{aligned} \gamma &= (x_1 - y_1\sqrt{d})(x_2 + y_2\sqrt{d}) \\ &= \underbrace{(x_1x_2 - dy_1y_2)}_{\equiv 0 \pmod{|m|}} + \underbrace{(x_1y_2 - x_2y_1)}_{\equiv 0 \pmod{|m|}} \sqrt{d} \\ &:= m(u + v\sqrt{d}) \end{aligned}$$

for some $u + v\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$. Taking norms of $\gamma = \alpha\beta$ gives $N(\gamma) = N(\alpha)N(\beta)$ or

$$m^2(u + v\sqrt{d}) = m^2.$$

Hence $u + v\sqrt{d} = 1$. By construction of x_1, x_2 , $v \neq 0$. Therefore the equation $x^2 - dy^2 = 1$ has one solution with $x, y > 0$.

- (7) By part (6), we might take a unit $\varepsilon = x + y\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$ with $x, y > 0$. Note that $\varepsilon \geq 1 + \sqrt{d} > 1$ (over the ordered field \mathbb{R}). Hence there are infinitely many units

$$\varepsilon, \varepsilon^2, \varepsilon^3, \dots$$

in $\mathbb{Z}[\sqrt{d}]$.

□

Note. Furthermore, show that there is a unit ε such that every unit has the form $\pm\varepsilon^n$, $n \in \mathbb{Z}$.

Proof.

- (1) By the well-ordering principle, there is a unit $\varepsilon = x_1 + y_1\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$ such that $x_1, y_1 > 0$ and (x_1, y_1) is the smallest solution of $x^2 - dy^2 = \pm 1$ with $x, y > 0$.

- (2) Now given any unit $\varepsilon' = x + y\sqrt{d}$, $x, y > 0$, it suffices to show that there is a positive integer n such that $\varepsilon' = \varepsilon^n$.
- (3) (Reductio ad absurdum) If not, there were a positive integer n such that $\varepsilon^n < \varepsilon' < \varepsilon^{n+1}$. Hence $1 < \varepsilon^{-n}\varepsilon' < \varepsilon$. Say $\varepsilon^{-n}\varepsilon' := x' + y'\sqrt{d}$. As $\varepsilon^{-n}\varepsilon' > 1 > 0$, the inverse is satisfying $x' - y'\sqrt{d} > 0$. Hence $x' > 0$.
- (4) As the inverse is satisfying $x' - y'\sqrt{d} < 1$, $y' \geq 0$. Note that $y' \neq 0$ (since $\varepsilon > 1$). Hence the existence of $\varepsilon^{-n}\varepsilon'$ contradicts the minimality of ε .
- (5) Now suppose a unit $\varepsilon' = x + y\sqrt{d}$ is of the form $x > 0$, $y < 0$. Then $\varepsilon'^{-1} = x - y\sqrt{d} = \varepsilon^n$ for some positive integer n by (2)(3)(4). Hence $\varepsilon' = \varepsilon^{-n}$ for some positive integer n . Other two cases of $\varepsilon' = x + y\sqrt{d}$ are similar. Therefore, every unit has the form $\pm\varepsilon^n$, $n \in \mathbb{Z}$.

□

Supplement. (Exercise I.1.7 in Jürgen Neukirch, *Algebraic Number Theory*)

Show that the ring $\mathbb{Z}[\sqrt{2}] = \mathbb{Z} + \mathbb{Z}\sqrt{2}$ is euclidean. Show furthermore that its units are given by $\pm(1 + \sqrt{2})^n$, $n \in \mathbb{Z}$, and determine its prime elements.

Proof.

- (1) Show that $\mathbb{Z}[\sqrt{2}]$ is euclidean with respect to the function $N : \mathbb{Z}[\sqrt{2}] \rightarrow \mathbb{N} \cup \{0\}$, $\alpha \mapsto \alpha\bar{\alpha}$. For $\alpha, \beta \neq 0 \in \mathbb{Z}[\sqrt{2}]$, one has to find $\gamma, \rho \in \mathbb{Z}[\sqrt{2}]$ such that

$$\alpha = \gamma\beta + \rho, \quad N(\rho) < N(\beta).$$

- (2) Extend the norm function N to $\mathbb{Q}[\sqrt{2}]$. Write

$$\frac{\alpha}{\beta} = x + y\sqrt{2} \in \mathbb{Q}[\sqrt{2}].$$

Take $\gamma = u + v\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$ such that u, v are satisfying $|u - x| \leq \frac{1}{2}$, $|v - y| \leq \frac{1}{2}$. Now take $\rho = \alpha - \gamma\beta$.

- (3) Hence,

$$N\left(\frac{\alpha}{\beta} - \gamma\right) = (u - x)^2 + 2(v - y)^2 \leq \left(\frac{1}{2}\right)^2 + 2 \cdot \left(\frac{1}{2}\right)^2 < 1$$

and thus

$$N(\rho) = N(\alpha - \gamma\beta) = N(\beta)N\left(\frac{\alpha}{\beta} - \gamma\right) < N(\beta).$$

- (4) Show that its units are given by $\pm(1 + \sqrt{2})^n$, $n \in \mathbb{Z}$. $\varepsilon = 1 + \sqrt{2} \in \mathbb{Z}[\sqrt{2}]$ is a unit such that $(1, 1)$ is the smallest solution of $x^2 - 2y^2 = \pm 1$ with $x, y > 0$. By the note in Exercise I.1.6, all units are given by $\pm(1 + \sqrt{2})^n$, $n \in \mathbb{Z}$.
- (5) For all prime numbers $p \neq 2$, one has $p = a^2 - 2b^2$ ($a, b \in \mathbb{Z}$) if and only if $p \equiv 1, 7 \pmod{8}$. Similar to the proof of Proposition I.1.1, it suffices to show that a prime number $p \equiv 1, 7 \pmod{8}$ of \mathbb{Z} does not remain a prime element in the ring $\mathbb{Z}[\sqrt{2}]$. (Reductio ad absurdum) Note that the congruence

$$2 \equiv x^2 \pmod{p}$$

admits a solution (by the law of quadratic reciprocity). Thus we have $p \mid x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2})$. Hence $\frac{x}{p} \pm \frac{\sqrt{2}}{p} \in \mathbb{Z}[\sqrt{2}]$, which is absurd.

- (6) The prime element π of $\mathbb{Z}[\sqrt{2}]$, up to associated elements, are given as follows.
- (i) $\pi = \sqrt{2}$,
 - (ii) $\pi = a + \sqrt{2}b$ with $a^2 - 2b^2 = p$, $p \equiv 1, 7 \pmod{8}$,
 - (iii) $\pi = p$, $p \equiv 3, 5 \pmod{8}$.

Here, p denotes a prime number of \mathbb{Z} . The proof is exactly the same as Theorem I.1.4.

□

Exercise 2.15.

- (a) Show that $\mathbb{Z}[\sqrt{-5}]$ contains no element whose norm is 2 or 3.
- (b) Verify that $2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$ is an example of non-unique factorization in the number ring $\mathbb{Z}[\sqrt{-5}]$.

Proof of (a). Since $N(a + b\sqrt{-5}) = a^2 + 5b^2 \equiv a^2 \equiv 0, 1, 4 \pmod{5}$, there is no element whose norm is 2 or 3. □

Proof of (b).

- (1) Show that $2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$.

$$2 \cdot 3 = 6 \text{ and } (1 + \sqrt{-5})(1 - \sqrt{-5}) = 6.$$

- (2) *Show that 2 is irreducible.* Suppose $2 = \alpha\beta$ where $\alpha, \beta \in \mathbb{Z}[\sqrt{-5}]$. Take norm to get

$$\begin{aligned} N(2) = N(\alpha)N(\beta) &\implies 4 = N(\alpha)N(\beta) \\ &\implies N(\alpha) = 1 \text{ or } N(\beta) = 1 \\ &\implies \alpha \text{ or } \beta \text{ is unit.} \end{aligned} \quad ((1))$$

- (3) *Show that 3 is irreducible.* Similar to (2).

- (4) *Show that $1 \pm \sqrt{-5}$ is irreducible.* Since $N(1 \pm \sqrt{-5}) = 2$ is prime, $1 \pm \sqrt{-5}$ is irreducible.

Hence 6 has a non-unique factorization in the number ring $\mathbb{Z}[\sqrt{-5}]$. \square

Exercise 2.19. (Vandermonde determinant)

Let R be a commutative ring and fix elements $a_1, a_2, \dots \in R$. We will prove by induction that the Vandermonde determinant

$$\begin{vmatrix} 1 & a_1 & \cdots & a_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & \cdots & a_n^{n-1} \end{vmatrix}$$

is equal to the product $\prod_{1 \leq r < s \leq n} (a_s - a_r)$. Assuming that the result holds for some n , consider the determinant

$$\begin{vmatrix} 1 & a_1 & \cdots & a_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & \cdots & a_n^n \\ 1 & a_{n+1} & \cdots & a_{n+1}^n \end{vmatrix}.$$

Show that this is equal to

$$\begin{vmatrix} 1 & a_1 & \cdots & f(a_1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & \cdots & f(a_n) \\ 1 & a_{n+1} & \cdots & f(a_{n+1}) \end{vmatrix}$$

for any monic polynomial f over R of degree n . Then choose f cleverly so that the determinant is easily calculated.

Proof.

(1) Let

$$V_n = \begin{pmatrix} 1 & a_1 & \cdots & a_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & \cdots & a_n^{n-1} \end{pmatrix}$$

be the Vandermonde matrix. We will apply the induction to show that $\det(V_n) = \prod_{1 \leq r < s \leq n} (a_s - a_r)$.

(2) Nothing to do for $n = 1, 2$. Now Assuming that the result holds for some n , consider the determinant

$$\det(V_{n+1}) = \begin{vmatrix} 1 & a_1 & \cdots & a_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & \cdots & a_n^n \\ 1 & a_{n+1} & \cdots & a_{n+1}^n \end{vmatrix}.$$

(3) *Show that*

$$\det(V_{n+1}) = \begin{vmatrix} 1 & a_1 & \cdots & f(a_1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & \cdots & f(a_n) \\ 1 & a_{n+1} & \cdots & f(a_{n+1}) \end{vmatrix}$$

for any monic polynomial f over R of degree n . Note that $\det(V_{n+1})$ is unchanged by adding a multiple of one column of V_{n+1} to another column of V_{n+1} . In particular, we add a multiple of the i -th column of V_{n+1} to the last column of V_{n+1} for $i = 1, 2, \dots, n$. Then we obtain the equation

$$\det(V_{n+1}) = \begin{vmatrix} 1 & a_1 & \cdots & f(a_1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & \cdots & f(a_n) \\ 1 & a_{n+1} & \cdots & f(a_{n+1}) \end{vmatrix}.$$

(4) In particular, we take

$$f(x) = (x - a_1)(x - a_2) \cdots (x - a_n).$$

Therefore

$$\begin{aligned}
\det(V_{n+1}) &= \begin{vmatrix} 1 & a_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & \cdots & 0 \\ 1 & a_{n+1} & \cdots & \prod_{1 \leq r \leq n} (a_{n+1} - a_r) \end{vmatrix} \\
&= (-1)^{(n+1)+(n+1)} \prod_{1 \leq r \leq n} (a_{n+1} - a_r) \begin{vmatrix} 1 & a_1 & \cdots & a_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & \cdots & a_n^{n-1} \end{vmatrix} \\
&= \prod_{1 \leq r \leq n} (a_{n+1} - a_r) \prod_{1 \leq r < s \leq n} (a_s - a_r) \\
&= \prod_{1 \leq r < s \leq n+1} (a_s - a_r).
\end{aligned}$$

By induction, the result is established.

□

Exercise 2.20.

Let f be a monic irreducible polynomial over a number field K and let α be one of its roots in \mathbb{C} . Show that $f'(\alpha) = \prod_{\beta \neq \alpha} (\alpha - \beta)$ with the product taken over all roots $\beta \neq \alpha$. (Hint: Write $f(x) = (x - \alpha)g(x)$.)

Proof.

- (1) Note that f has no repeated roots in \mathbb{C} by the irreducibility of f . So we can write

$$f(x) = (x - \alpha)g(x) = (x - \alpha) \prod_{\beta \neq \alpha} (x - \beta).$$

- (2) So

$$f'(x) = g(x) + (x - \alpha)g'(x)$$

by the Leibniz rule. Take $x = \alpha$ to get

$$f'(\alpha) = g(\alpha) = \prod_{\beta \neq \alpha} (\alpha - \beta).$$

□

Exercise 2.28.

Let $f(x) = x^3 + ax + b$, a and $b \in \mathbb{Z}$, and assume f is irreducible over \mathbb{Q} . Let α be a root of f .

(a) Show that $f'(\alpha) = -\frac{2a\alpha+3b}{\alpha}$.

(b) Show that $2a\alpha + 3b$ is a root of

$$\left(\frac{x-3b}{2a}\right)^3 + a\left(\frac{x-3b}{2a}\right) + b.$$

Use this to find $N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(2a\alpha + 3b)$.

(c) Show that $\text{disc}(\alpha) = -(4a^3 + 27b^2)$.

(d) Suppose $\alpha^3 = \alpha + 1$. Prove that $\{1, \alpha, \alpha^2\}$ is an integral basis for $\mathbb{A} \cap \mathbb{Q}[\alpha]$.
(See Exercise 2.27(e).) Do the same if $\alpha^3 + \alpha = 1$.

Proof of (a).

(1) Show that $\alpha \neq 0$. If α were 0, then $f(\alpha) = f(0) = b$. So $f(x) = x^3 + ax = x(x^2 + a)$ is reducible, contrary to the irreducibility of f .

(2) Since α be a root of f , $f(\alpha) = 0$, or $\alpha^3 + a\alpha + b = 0$, or $\alpha^3 = -a\alpha - b$.

(3)

$$\begin{aligned} f'(x) = 3x^2 + a &\implies f'(\alpha) = 3\alpha^2 + a \\ &\iff \alpha f'(\alpha) = 3\alpha^3 + a\alpha & (\alpha \neq 0) \\ &\iff \alpha f'(\alpha) = 3(-a\alpha - b) + a\alpha & (\alpha^3 = -a\alpha - b) \\ &\iff \alpha f'(\alpha) = -2a\alpha - 3b. \end{aligned}$$

So $f'(\alpha) = -\frac{2a\alpha+3b}{\alpha}$.

□

Proof of (b).

(1) Since $\alpha^3 + a\alpha + b = 0$,

$$\left(\frac{(2a\alpha + 3b) - 3b}{2a}\right)^3 + a\left(\frac{(2a\alpha + 3b) - 3b}{2a}\right) + b = 0.$$

That is, $2a\alpha + 3b$ is a root of $\left(\frac{x-3b}{2a}\right)^3 + a\left(\frac{x-3b}{2a}\right) + b$.

- (2) $N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(2a\alpha + 3b)$ is the product of three roots of $\left(\frac{x-3b}{2a}\right)^3 + a\left(\frac{x-3b}{2a}\right) + b$.
Hence,

$$\begin{aligned} N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(2a\alpha + 3b) &= (2a)^3 \left[\left(\frac{-3b}{2a}\right)^3 + a \cdot \frac{-3b}{2a} + b \right] \\ &= 8a^3 \left[\frac{-27b^3}{8a^3} - \frac{b}{2} \right] \\ &= -27b^3 - 4a^3b. \end{aligned}$$

□

Proof of (c).

$$\begin{aligned} \text{disc}(\alpha) &= (-1)^{\frac{n(n-1)}{2}} N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(f'(\alpha)) && \text{(Theorem 2.8)} \\ &= -N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]} \left(-\frac{2a\alpha + 3b}{\alpha} \right) && (n = 3 \text{ and (a)}) \\ &= \frac{N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(2a\alpha + 3b)}{N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(\alpha)} \\ &= \frac{-27b^3 - 4a^3b}{b} && ((b)) \\ &= -27b^2 - 4a^3. \end{aligned}$$

□

Proof of (d).

- (1) (a) $\alpha^3 = \alpha + 1$, or $\alpha^3 - \alpha - 1 = 0$.
 (b) $f(x) = x^3 - x - 1$ is irreducible over \mathbb{Q} since $f(x)$ is irreducible over $\mathbb{Z}/3\mathbb{Z}$.
 (c) $\text{disc}(\alpha) = -23$ (by (c)).
 (d) Since $\text{disc}(\alpha)$ is squarefree, the result is established (Exercise 2.27(e)).
- (2) (a) $\alpha^3 + \alpha = 1$, or $\alpha^3 + \alpha - 1 = 0$.
 (b) $f(x) = x^3 + x - 1$ is irreducible over \mathbb{Q} since $f(x)$ is irreducible over $\mathbb{Z}/2\mathbb{Z}$.
 (c) $\text{disc}(\alpha) = -31$ (by (c)).
 (d) Since $\text{disc}(\alpha)$ is squarefree, the result is established (Exercise 2.27(e)).

□

Exercise 2.43.

Let $f(x) = x^5 + ax + b$, a and $b \in \mathbb{Z}$, and assume f is irreducible over \mathbb{Q} . Let α be a root of f .

- (a) Show that $\text{disc}(\alpha) = 4^4 a^5 + 5^4 b^4$. (Suggestion: See Exercise 2.28.)
- (b) Suppose $\alpha^5 = \alpha + 1$. Prove that $\mathbb{A} \cap \mathbb{Q}[\alpha] = \mathbb{Z}[\alpha]$. ($x^5 - x - 1$ is irreducible over \mathbb{Q} ; this can be shown by reducing (mod 3).)
- (c)
- (d)

Proof of (a) (Exercise 2.28).

- (1) Show that $f'(\alpha) = -\frac{4a\alpha+5b}{\alpha}$.

- (a) Show that $\alpha \neq 0$. If α were 0, then $f(\alpha) = f(0) = b$. So $f(x) = x^5 + ax = x(x^4 + a)$ is reducible, contrary to the irreducibility of f .
- (b) Since α be a root of f , $f(\alpha) = 0$, or $\alpha^5 + a\alpha + b = 0$, or $\alpha^5 = -a\alpha - b$.
- (c)

$$\begin{aligned}
 f'(x) = 5x^4 + a &\implies f'(\alpha) = 5\alpha^4 + a \\
 &\iff \alpha f'(\alpha) = 5\alpha^5 + a\alpha & (\alpha \neq 0) \\
 &\iff \alpha f'(\alpha) = 5(-a\alpha - b) + a\alpha & (\alpha^5 = -a\alpha - b) \\
 &\iff \alpha f'(\alpha) = -4a\alpha - 5b.
 \end{aligned}$$

$$\text{So } f'(\alpha) = -\frac{4a\alpha+5b}{\alpha}.$$

- (2) Show that $4a\alpha + 5b$ is a root of

$$\left(\frac{x-5b}{4a}\right)^5 + a\left(\frac{x-5b}{4a}\right) + b.$$

Use this to show that $N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(4a\alpha + 5b) = -4^4 a^5 b - 5^5 b^5$.

- (a) Since $\alpha^5 + a\alpha + b = 0$,

$$\left(\frac{(4a\alpha + 5b) - 5b}{4a}\right)^5 + a\left(\frac{(4a\alpha + 5b) - 5b}{4a}\right) + b = 0.$$

That is, $4a\alpha + 5b$ is a root of $\left(\frac{x-5b}{4a}\right)^5 + a\left(\frac{x-5b}{4a}\right) + b$.

- (b) $N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(4a\alpha + 5b)$ is the product of 5 roots of $\left(\frac{x-5b}{4a}\right)^5 + a\left(\frac{x-5b}{4a}\right) + b$.
Hence,

$$\begin{aligned} N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(4a\alpha + 5b) &= (4a)^5 \left[\left(\frac{-5b}{4a}\right)^5 + a \cdot \frac{-5b}{4a} + b \right] \\ &= 4^5 a^5 \left[\frac{-5^5 b^5}{4^5 a^5} - \frac{b}{4} \right] \\ &= -5^5 b^5 - 4^4 a^5 b. \end{aligned}$$

- (3) Show that $\text{disc}(\alpha) = 4^4 a^5 + 5^4 b^4$.

$$\begin{aligned} \text{disc}(\alpha) &= (-1)^{\frac{n(n-1)}{2}} N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(f'(\alpha)) && \text{(Theorem 2.8)} \\ &= N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]} \left(-\frac{4a\alpha + 5b}{\alpha} \right) && (n = 5 \text{ and (1)}) \\ &= -\frac{N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(4a\alpha + 5b)}{N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(\alpha)} \\ &= -\frac{-4^4 a^5 b - 5^5 b^5}{b} && ((2)) \\ &= 4^4 a^5 + 5^4 b^4. \end{aligned}$$

□

Proof of (b) (Exercise 2.28).

- (1) $\alpha^5 = \alpha + 1$, or $\alpha^5 - \alpha - 1 = 0$.
- (2) $f(x) = x^5 - x - 1$ is irreducible over \mathbb{Q} since $f(x)$ is irreducible over $\mathbb{Z}/3\mathbb{Z}$.
- (3) $\text{disc}(\alpha) = 881$ (by (a)).
- (4) Since $\text{disc}(\alpha)$ is squarefree (a prime number), the result is established (Exercise 2.27(e)).

□

Exercise 2.45.

Obtain a formula for $\text{disc}(\alpha)$ if α is a root of an irreducible polynomial $x^n + ax + b$ over \mathbb{Q} . Do the same for $x^n + ax^{n-1} + b$.

Assume that $n \geq 2$.

Proof of $x^n + ax + b$ (Exercise 2.28).

(1) Show that $f'(\alpha) = -\frac{(n-1)a\alpha+nb}{\alpha}$.

- (a) Show that $\alpha \neq 0$. If α were 0, then $f(\alpha) = f(0) = b$. So $f(x) = x^n + ax = x(x^{n-1} + a)$ is reducible, contrary to the irreducibility of f .
- (b) Since α be a root of f , $f(\alpha) = 0$, or $\alpha^n + a\alpha + b = 0$, or $\alpha^n = -a\alpha - b$.
- (c)

$$\begin{aligned} f'(x) = nx^{n-1} + a &\implies f'(\alpha) = n\alpha^{n-1} + a \\ &\iff \alpha f'(\alpha) = n\alpha^n + a\alpha \quad (\alpha \neq 0) \\ &\iff \alpha f'(\alpha) = n(-a\alpha - b) + a\alpha \quad (\alpha^n = -a\alpha - b) \\ &\iff \alpha f'(\alpha) = -(n-1)a\alpha - nb. \end{aligned}$$

$$\text{So } f'(\alpha) = -\frac{(n-1)a\alpha+nb}{\alpha}.$$

(2) Let $\beta = (n-1)a\alpha + nb$. Show that β is a root of

$$\left(\frac{x-nb}{(n-1)a}\right)^n + a\left(\frac{x-nb}{(n-1)a}\right) + b.$$

Use this to show that

$$N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(\beta) = -(n-1)^{n-1}a^n b + (-1)^n n^n b^n.$$

- (a) Since $\alpha^n + a\alpha + b = 0$,

$$\left(\frac{\beta-nb}{(n-1)a}\right)^n + a\left(\frac{\beta-nb}{(n-1)a}\right) + b = 0.$$

That is, β is a root of $\left(\frac{x-nb}{(n-1)a}\right)^n + a\left(\frac{x-nb}{(n-1)a}\right) + b$.

- (b) $N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(\beta)$ is the product of n roots of $\left(\frac{x-nb}{(n-1)a}\right)^n + a\left(\frac{x-nb}{(n-1)a}\right) + b$.

Hence,

$$\begin{aligned} N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(\beta) &= ((n-1)a)^n \left[\left(\frac{-nb}{(n-1)a}\right)^n + a \cdot \frac{-nb}{(n-1)a} + b \right] \\ &= (n-1)^n a^n \left[\frac{(-1)^n n^n b^n}{(n-1)^n a^n} - \frac{b}{n-1} \right] \\ &= (-1)^n n^n b^n - (n-1)^{n-1} a^n b. \end{aligned}$$

(3) Show that $\text{disc}(\alpha) = (-1)^{\frac{(n-1)(n-2)}{2}}(n-1)^{n-1}a^n + (-1)^{\frac{n(n-1)}{2}}n^nb^{n-1}$.

$$\text{disc}(\alpha) = (-1)^{\frac{n(n-1)}{2}} N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(f'(\alpha)) \quad (\text{Theorem 2.8})$$

$$= (-1)^{\frac{n(n-1)}{2}} N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]} \left(-\frac{(n-1)a\alpha + nb}{\alpha} \right) \quad ((1))$$

$$= (-1)^{\frac{n(n-1)}{2}} (-1)^n \frac{N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}((n-1)a\alpha + nb)}{N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(\alpha)}$$

$$= (-1)^{\frac{n(n-1)}{2}} (-1)^n \frac{-(n-1)^{n-1}a^nb + (-1)^n n^n b^n}{b} \quad ((2))$$

$$= (-1)^{\frac{(n-1)(n-2)}{2}} (n-1)^{n-1}a^n + (-1)^{\frac{n(n-1)}{2}} n^nb^{n-1}.$$

□