

## Chapter 8: Some Special Functions

*Author: Meng-Gen Tsai*

*Email: plover@gmail.com*

**Supplement.** Fourier coefficients in Definition 8.9.

(1) Write

$$f(x) = a_0 + \sum_{n=1}^N (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}$$

(as the textbook Rudin, Principles of Mathematical Analysis, Third Edition).

Then

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx.$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx, n \in \mathbb{Z}^+.$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx, n \in \mathbb{Z}^+.$$

(2) One might write in one different form,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^N (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

The only difference between the new one and the old one is  $a_0$ , so  $a_0$  should be

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx.$$

(3) Again, one might write in one different form,

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^N (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

Similarly,  $a_0$  should be

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x)}{\sqrt{2}} dx.$$

(4) Recall  $f(x) = \sum_{n=-N}^N c_n e^{inx}$  ( $x \in \mathbb{R}$ ) where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

The relations among  $a_n$ ,  $b_n$  of this textbook and  $c_n$  are

$$\begin{aligned} c_0 &= a_0 \\ c_n &= \frac{1}{2}(a_n + ib_n), n \in \mathbb{Z}^+. \end{aligned}$$

- (5) In some textbooks (Henryk Iwaniec, Topics in Classical Automorphic Forms), it is convenient to consider periodic functions  $f$  of period 1. Define

$$e(n) = e^{2\pi i x} = \cos(2\pi x) + i \sin(2\pi x).$$

Any periodic and piecewise continuous function  $f$  has the Fourier series representation

$$f(x) = \sum_{-\infty}^{\infty} a_n e(nx)$$

with coefficients given by

$$a_n = \int_0^1 f(x) e(-nx) dx.$$

Here is one exercise for this representation. *Show that the fractional part of  $x$ ,  $\{x\} = x - [x]$ , is given by*

$$\{x\} = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{\pi n}.$$

**Supplement.** Parseval's theorem 8.16.

- (1) Given

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

Then

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = 2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

- (2) Given

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

Then

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

(3) Given

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

Then

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

**Exercise 8.1.** Define

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

Prove that  $f$  has derivatives of all orders at  $x = 0$ , and that  $f^{(n)}(0) = 0$  for  $n = 1, 2, 3, \dots$

$f(x)$  is an example of non-analytic smooth function, that is, infinitely differentiable functions are not necessarily analytic. In this exercise, we will show that Taylor series of  $f$  at the origin converges everywhere to the zero function. So the Taylor series does not equal  $f(x)$  for  $x \neq 0$ . Consequently,  $f$  is not analytic at  $x = 0$ .

*Proof.*

(1) Show that

$$\lim_{x \rightarrow 0} g(x) e^{-\frac{1}{x^2}} = 0$$

for any rational function  $g(x) \in \mathbb{R}(x)$ .

- (a) Write  $g(x) = \frac{p(x)}{q(x)}$  for some  $p(x), q(x) \in \mathbb{R}[x]$ ,  $g(x) \neq 0$ .
- (b) Write  $q(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_0$ .  $q(x)$  is not identically zero, that is, there exists the unique coefficient of the least power of  $x$  in  $q(x)$  which is non-zero, say  $b_M \neq 0$ .
- (c) Thus,

$$g(x) = \frac{p(x)/x^M}{q(x)/x^M}.$$

The denominator of  $g(x)$  tends to  $b_M \neq 0$  as  $x \rightarrow 0$ . By the similar argument in Theorem 8.6(f), we have

$$\frac{p(x)}{x^M} e^{-\frac{1}{x^2}} \rightarrow 0 \text{ as } x \rightarrow 0.$$

Hence,  $\lim_{x \rightarrow 0} g(x) e^{-\frac{1}{x^2}} = 0$  for any  $g(x) \in \mathbb{R}(x)$ .

(2) Given any real  $x \neq 0$ , show that

$$f^{(n)}(x) = g_n(x) e^{-\frac{1}{x^2}}$$

for some rational function  $g(x) \in \mathbb{R}(x)$ .

- (a) Say  $g_0(x) = 1 \in \mathbb{R}(x)$ .
- (b)  $\mathbb{R}(x)$  is a field. Show that  $g'(x) \in \mathbb{R}(x)$  for any  $g(x) \in \mathbb{R}(x)$ . Write  $g(x) = \frac{p(x)}{q(x)}$  for some  $p(x), q(x) \in \mathbb{R}[x]$ ,  $q(x) \neq 0$ . Thus

$$g'(x) = \frac{p'(x)q(x) - p(x)q'(x)}{q(x)^2}.$$

The numerator of  $g'(x)$  is in  $\mathbb{R}[x]$  since the differentiation operator on  $\mathbb{R}[x]$  is closed in  $\mathbb{R}[x]$ . Also, the denominator of  $g'(x) = \frac{p'(x)q(x) - p(x)q'(x)}{q(x)^2} \neq 0$  since  $\mathbb{R}[x]$  is an integral domain. Therefore,  $g'(x) \in \mathbb{R}(x)$ .

- (c) Induction on  $n$ . For  $n = 1$ , we have

$$\begin{aligned} f'(x) &= g'_0(x)e^{-\frac{1}{x^2}} + g_0(x) \cdot \left(-\frac{1}{x^2}\right)' e^{-\frac{1}{x^2}} \\ &= \left(g'_0(x) + g_0(x) \cdot \left(-\frac{1}{x^2}\right)'\right) e^{-\frac{1}{x^2}} \\ &= g_1(x)e^{-\frac{1}{x^2}} \end{aligned}$$

where

$$g_1(x) = g'_0(x) + g_0(x) \cdot \left(-\frac{1}{x^2}\right)' \in \mathbb{R}(x).$$

Now assume that the conclusion holds for  $n = k$ . As  $n = k + 1$ , similar to the case  $n = 1$ ,

$$f^{(k+1)}(x) = g_{k+1}(x)e^{-\frac{1}{x^2}}$$

where

$$g_{k+1}(x) = g'_k(x) + g_k(x) \cdot \left(-\frac{1}{x^2}\right)' \in \mathbb{R}(x).$$

By induction, the conclusion is true.

- (3) Induction on  $n$ . For  $n = 1$ , by (1) we have

$$f'(0) = \lim_{t \rightarrow 0} \frac{e^{-\frac{1}{t^2}} - 0}{t} = 0.$$

Now assume that the statement holds for  $n = k$ . As  $n = k + 1$ , by (1)(2) we have

$$f^{(k+1)}(0) = \lim_{t \rightarrow 0} \frac{f^{(k)}(t) - f^{(k)}(0)}{t} = \lim_{t \rightarrow 0} \frac{g_k(t)e^{-\frac{1}{t^2}} - 0}{t} = 0.$$

Thus,  $f^{(n)}(0) = 0$  for  $n \in \mathbb{Z}^+$ .

□

**Exercise 8.2.** Let  $a_{ij}$  be the number in the  $i$ th row and  $j$ th column of the array

$$\begin{array}{ccccc} -1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & -1 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{2} & -1 & 0 & \cdots \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

so that

$$a_{ij} = \begin{cases} 0 & (i < j), \\ -1 & (i = j), \\ 2^{j-i} & (i > j). \end{cases}$$

Prove that

$$\sum_i \sum_j a_{ij} = -2, \quad \sum_j \sum_i a_{ij} = 0.$$

Also see Theorem 8.3.

*Proof (Brute-force).*

$$\begin{aligned} \sum_i \sum_j a_{ij} &= \sum_{i=1}^{\infty} \left( \sum_{j=i}^{\infty} a_{ij} + \sum_{j<i} a_{ij} \right) \\ &= \sum_{i=1}^{\infty} \left( -1 + \sum_{j=1}^{i-1} 2^{j-i} \right) \\ &= \sum_{i=1}^{\infty} (-1 + (1 - 2^{1-i})) \\ &= \sum_{i=1}^{\infty} -2^{1-i} \\ &= -2. \end{aligned}$$

$$\begin{aligned}
\sum_j \sum_i a_{ij} &= \sum_{j=1}^{\infty} \left( \sum_{i=j}^{\infty} a_{ij} + \sum_{i>j} a_{ij} \right) \\
&= \sum_{j=1}^{\infty} \left( -1 + \sum_{i=j+1}^{\infty} 2^{j-i} \right) \\
&= \sum_{j=1}^{\infty} (-1 + 1) \\
&= \sum_{j=1}^{\infty} 0 \\
&= 0.
\end{aligned}$$

□

**Exercise 8.3.** *Prove that*

$$\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$$

*if  $a_{ij} \geq 0$  for all  $i$  and  $j$  (the case  $+\infty = +\infty$  may occur).*

*Note.* It can be proved by Theorem 8.3 if both summations are finite.

*Proof.*

(1) Let  $\mathcal{F}(I)$  be the collection of all finite subsets of  $I$ .

(2) Let

$$s = \sup \left\{ \sum_{(i,j) \in E} a_{ij} : E \in \mathcal{F}(\mathbb{N}^2) \right\}$$

(the case  $s = +\infty$  may occur). *It suffices to show that  $\sum_i \sum_j a_{ij} = s$ .*

The case  $\sum_j \sum_i a_{ij} = s$  is similar, and thus  $\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$ .

(3) *Show that  $\sum_i \sum_j a_{ij} \geq s$ .* Given any  $E \in \mathcal{F}(\mathbb{N}^2)$ . It is clear that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \geq \sum_{(i,j) \in E} a_{ij}$$

(since  $a_{ij} \geq 0$ ). Thus,

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \geq \sup \left\{ \sum_{(i,j) \in E} a_{ij} : E \in \mathcal{F}(\mathbb{N}^2) \right\} = s.$$

- (4) *Show that  $\sum_i \sum_j a_{ij} \leq s$ . (Reductio ad absurdum) If  $\sum_i \sum_j a_{ij} > s$ , especially  $s < \infty$ , then there exists  $\varepsilon > 0$  such that*

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} > s + \varepsilon,$$

or

$$\sum_{i=1}^n \sum_{j=1}^{\infty} a_{ij} > s + \varepsilon$$

for some integer  $n$ . Consider two possible cases.

- (a) If there is some  $1 \leq i_0 \leq n$  such that

$$\sum_{j=1}^{\infty} a_{i_0 j} = \infty,$$

then there is some  $m$  such that

$$\sum_{j=1}^m a_{i_0 j} > s.$$

For  $E = \{(i_0, 1), \dots, (i_0, m)\} \in \mathcal{F}(\mathbb{N}^2)$ ,

$$\sum_{(i,j) \in E} a_{ij} = \sum_{j=1}^m a_{i_0 j} > s,$$

contrary to the supremum of  $s$ .

- (b) Otherwise, for each  $1 \leq i \leq n$  we have

$$\sum_{j=1}^{\infty} a_{ij} < \infty,$$

or there exists some  $m_i$  such that

$$\sum_{j=1}^{m_i} a_{ij} > \sum_{j=1}^{\infty} a_{ij} - \frac{\varepsilon}{n}.$$

For  $E = \bigcup_{1 \leq i \leq n} \{(i, 1), \dots, (i, m_i)\} \in \mathcal{F}(\mathbb{N}^2)$ ,

$$\begin{aligned}
\sum_{(i,j) \in E} a_{ij} &= \sum_{i=1}^n \sum_{j=1}^{m_i} a_{ij} \\
&> \sum_{i=1}^n \left( \sum_{j=1}^{\infty} a_{ij} - \frac{\varepsilon}{n} \right) \\
&= \sum_{i=1}^n \sum_{j=1}^{\infty} a_{ij} - \sum_{i=1}^n \frac{\varepsilon}{n} \\
&> s + \varepsilon - \varepsilon \\
&= s,
\end{aligned}$$

contrary to the supremum of  $s$ .

Therefore,  $\sum_i \sum_j a_{ij} \leq s$ .

- (5) By (3)(4),  $\sum_i \sum_j a_{ij} = s$ . Similarly,  $\sum_j \sum_i a_{ij} = s$ . Hence,  $\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$  (including the case  $+\infty = +\infty$ ).

□

**Exercise 8.4.** *Prove the following limit relations:*

(a)  $\lim_{x \rightarrow 0} \frac{b^x - 1}{x} = \log b \quad (b > 0).$

(b)  $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1.$

(c)  $\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e.$

(d)  $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x.$

*Proof of (a).*

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{b^x - 1}{x} &= \lim_{x \rightarrow 0} \frac{\exp(x \log b) - 1}{x} \\
&= \left. \frac{d}{dx} \exp(x \log b) \right|_{x=0} \\
&= \exp(x \log b) \cdot \log b \Big|_{x=0} \\
&= \log b.
\end{aligned}$$

□



*Proof of (b).*

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} &= \left. \frac{d}{dx} \log(1+x) \right|_{x=0} \\ &= \left. \frac{1}{x+1} \right|_{x=0} \\ &= 1.\end{aligned}$$

□

*Proof of (c).*

$$\begin{aligned}\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} &= \lim_{x \rightarrow 0} \exp\left(\frac{\log(1+x)}{x}\right) \\ &= \exp\left(\lim_{x \rightarrow 0} \frac{\log(1+x)}{x}\right) \\ &= \exp(1) \\ &= e.\end{aligned}$$

□

*Proof of (d).*

$$\begin{aligned}\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n &= \lim_{n \rightarrow \infty} \left(\left(1 + \frac{x}{n}\right)^{\frac{n}{x}}\right)^x \\ &= \left(\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^{\frac{n}{x}}\right)^x \\ &= \left(\lim_{y \rightarrow 0} (1+y)^{\frac{1}{y}}\right)^x \\ &= \exp(x).\end{aligned}$$

□

**Exercise 8.5.** *Find the following limits*

(a)  $\lim_{x \rightarrow 0} \frac{e - (1+x)^{\frac{1}{x}}}{x}.$

(b)  $\lim_{n \rightarrow \infty} \frac{n}{\log n} \left[ n^{\frac{1}{n}} - 1 \right].$

(c)  $\lim_{x \rightarrow 0} \frac{\tan x - x}{x(1 - \cos x)}.$

(d)  $\lim_{x \rightarrow 0} \frac{x - \sin x}{\tan x - x}.$

*Proof of (a).* By L'Hospital's rule (Theorem 5.13),

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{e - (1+x)^{\frac{1}{x}}}{x} &= \lim_{x \rightarrow 0} \frac{-(1+x)^{\frac{1}{x}} \cdot \frac{\frac{x}{x+1} - \log(x+1)}{x^2}}{1} \\
&= \lim_{x \rightarrow 0} \left( -(1+x)^{\frac{1}{x}} \cdot \frac{\frac{x}{x+1} - \log(x+1)}{x^2} \right) \\
&= - \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} \cdot \lim_{x \rightarrow 0} \frac{\frac{x}{x+1} - \log(x+1)}{x^2} \\
&= -e \cdot \lim_{x \rightarrow 0} \frac{\frac{x}{x+1} - \log(x+1)}{x^2} \quad (\text{Exercise 8.4(c)}) \\
&= -e \cdot \lim_{x \rightarrow 0} \frac{-\frac{x}{(x+1)^2}}{2x} \\
&= e \cdot \lim_{x \rightarrow 0} \frac{1}{2(x+1)^2} \\
&= e \cdot \frac{1}{2} \\
&= \frac{e}{2}.
\end{aligned}$$

Here

$$\begin{aligned}
\frac{d}{dx} \left( e - (1+x)^{\frac{1}{x}} \right) &= \frac{d}{dx} \left( e - \exp \left( \frac{\log(x+1)}{x} \right) \right) \\
&= - \exp \left( \frac{1}{x} \log(x+1) \right) \cdot \frac{\frac{1}{x+1} \cdot x - \log(x+1) \cdot 1}{x^2} \\
&= -(1+x)^{\frac{1}{x}} \cdot \frac{\frac{x}{x+1} - \log(x+1)}{x^2},
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dx} \left( \frac{x}{x+1} - \log(x+1) \right) &= \frac{(x+1) - x}{(x+1)^2} - \frac{1}{x+1} \\
&= -\frac{x}{(x+1)^2}.
\end{aligned}$$

□

*Proof of (b).*

(1) Let  $x = \frac{\log n}{n}$ . Note that  $\lim_{n \rightarrow \infty} \frac{\log n}{n} = 0$ .

(2)

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{n}{\log n} \left[ n^{\frac{1}{n}} - 1 \right] &= \lim_{n \rightarrow \infty} \frac{n}{\log n} \left[ \exp \left( \frac{\log n}{n} \right) - 1 \right] \\ &= \lim_{x \rightarrow 0} \frac{\exp(x) - 1}{x} \\ &= \frac{d}{dx} \exp(x) \Big|_{x=0} \\ &= \exp(x) \Big|_{x=0} \\ &= 1.\end{aligned} \tag{1)}$$

□

*Proof of (c) (L'Hospital's rule).* By L'Hospital's rule (Theorem 5.13) three times,

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\tan x - x}{x(1 - \cos x)} &= \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{1 - \cos x + x \sin x} \\ &= \lim_{x \rightarrow 0} \frac{2 \sec x (\tan x \sec x)}{\sin x + \sin x + x \cos x} \\ &= \lim_{x \rightarrow 0} \frac{2 \tan x \sec^2 x}{2 \sin x + x \cos x} \\ &= \lim_{x \rightarrow 0} \frac{2[\sec^2 x \sec^2 x + \tan x \cdot 2 \sec x (\tan x \sec x)]}{2 \cos x + \cos x - x \sin x} \\ &= \lim_{x \rightarrow 0} \frac{2 \sec^4 x + 2 \sec^2 x \tan^2 x}{3 \cos x - x \sin x} \\ &= \frac{2}{3}.\end{aligned}$$

□

*Proof of (c) (Taylor series).* Since

$$\begin{aligned}\cos x &= 1 - \frac{x^2}{2} + O(x^4) \\ \tan x &= x + \frac{x^3}{3} + O(x^5),\end{aligned}$$

we have

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x(1 - \cos x)} = \lim_{x \rightarrow 0} \frac{\frac{x^3}{3} + O(x^5)}{\frac{x^3}{2} + O(x^5)} = \frac{2}{3}.$$

□

*Proof of (d) (L'Hospital's rule).* By L'Hospital's rule (Theorem 5.13) three times,

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{x - \sin x}{\tan x - x} &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{\sec^2 x - 1} \\
&= \lim_{x \rightarrow 0} \frac{\sin x}{2 \sec x (\tan x \sec x)} \\
&= \lim_{x \rightarrow 0} \frac{\sin x}{2 \tan x \sec^2 x} \\
&= \lim_{x \rightarrow 0} \frac{\cos x}{2 \tan x \sec^2 x} \\
&= \lim_{x \rightarrow 0} \frac{\cos x}{2 [\sec^2 x \sec^2 x + \tan x \cdot 2 \sec x (\tan x \sec x)]} \\
&= \lim_{x \rightarrow 0} \frac{\cos x}{2 \sec^4 x + 2 \sec^2 x \tan^2 x} \\
&= \frac{1}{2}.
\end{aligned}$$

□

*Proof of (d) (Taylor series).* Since

$$\begin{aligned}
\sin x &= x - \frac{x^3}{6} + O(x^5) \\
\tan x &= x + \frac{x^3}{3} + O(x^5),
\end{aligned}$$

we have

$$\lim_{x \rightarrow 0} \frac{x - \sin x}{\tan x - x} = \lim_{x \rightarrow 0} \frac{\frac{x^3}{6} + O(x^5)}{\frac{x^3}{3} + O(x^5)} = \frac{1}{2}.$$

□

**Exercise 8.6.** Suppose  $f(x)f(y) = f(x+y)$  for all real  $x$  and  $y$ .

(a) Assuming that  $f$  is differentiable and not zero, prove that

$$f(x) = e^{cx}$$

where  $c$  is a constant.

(b) Prove the same thing, assuming only that  $f$  is continuous.

Part (b) implies part (a). We prove part (b) directly.

*Proof of (b).*

- (1) Since  $f(x)$  is not zero, there exists  $x_0 \in \mathbb{R}$  such that  $f(x_0) \neq 0$ . So  $f(0)f(x_0) = f(x_0)$ , or  $f(0) = 1$  by cancelling  $f(x_0) \neq 0$ .
- (2) Next,  $f(\frac{n}{m}) = f(\frac{1}{m})^n$  for  $m \in \mathbb{Z}$ ,  $n \in \mathbb{Z}^+$ . Since  $f$  is continuous at  $x = 0$ ,  $f$  is positive in the neighborhood of  $x = 0$ . That is, there exists  $N \in \mathbb{Z}^+$  such that  $f(\frac{1}{m}) > 0$  whenever  $|m| \geq N$ . So,  $f(\frac{n}{m}) = f(\frac{1}{m})^n > 0$ . (Since  $f(\frac{n}{m}) = f(\frac{kn}{km})$  for any  $k \in \mathbb{Z}^+$ , we can rescale  $m$  to  $km$  such that  $|km| \geq N$ .) That is,  $f$  is positive on  $\mathbb{Q}$ . Since  $\mathbb{Q}$  is dense in  $\mathbb{R}$  and  $f$  is continuous on  $\mathbb{R}$ ,  $f$  is positive on  $\mathbb{R}$ .
- (3) Now let  $c = \log f(1)$  (which is well-defined since  $f > 0$ ). We write  $f(1)$  in the two ways. Firstly,  $f(1) = f(\frac{n}{n}) = f(\frac{1}{n})^n$  where  $n \in \mathbb{Z}^+$ . Secondly,  $f(1) = e^c = (e^{\frac{c}{n}})^n$ . Since the positive  $n$ -th root is unique (Theorem 1.21),  $f(\frac{1}{n}) = e^{\frac{c}{n}}$  for  $n \in \mathbb{Z}^+$ . By  $f(x)f(-x) = f(0) = 1$  or  $f(-x) = \frac{1}{f(x)}$ ,  $f(-\frac{1}{n}) = \frac{1}{f(\frac{1}{n})} = e^{-\frac{c}{n}}$  for  $n \in \mathbb{Z}^+$ . Therefore,

$$f\left(\frac{1}{m}\right) = e^{\frac{c}{m}} \text{ where } m \in \mathbb{Z}.$$

- (4) By using  $f(\frac{n}{m}) = f(\frac{1}{m})^n$  for  $m \in \mathbb{Z}$ ,  $n \in \mathbb{Z}^+$  again,  $f(\frac{n}{m}) = e^{c\frac{n}{m}}$  where  $m \in \mathbb{Z}, n \in \mathbb{Z}^+$ , or

$$f(x) = e^{cx} \text{ where } x \in \mathbb{Q}.$$

Since  $g(x) = f(x) - e^{cx}$  vanishes on a dense set of  $\mathbb{Q}$  and  $g$  is continuous on  $\mathbb{R}$ ,  $g$  vanishes on  $\mathbb{R}$ . Therefore,  $f(x) = e^{cx}$  for  $x \in \mathbb{R}$ .

□

**Supplement.** *Proof of (a).*

- (1) Since  $f(x)$  is not zero, there exists  $x_0 \in \mathbb{R}$  such that  $f(x_0) \neq 0$ . So  $f(0)f(x_0) = f(x_0)$ , or  $f(0) = 1$  by cancelling  $f(x_0) \neq 0$ .
- (2) Since  $f$  is differentiable, for any  $x \in \mathbb{R}$ ,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x)f(h) - f(x)}{h} \\ &= f(x) \lim_{h \rightarrow 0} \frac{f(h) - 1}{h} \\ &= f(x) \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} \\ &= f(x)f'(0). \end{aligned}$$

Let  $c = f'(0)$  be a constant. Then  $f'(x) = cf(x)$ . So  $f(x) = e^{cx}$  for  $x \in \mathbb{R}$ . (To see this, let  $g(x) = \frac{f(x)}{e^{cx}}$  be well-defined on  $\mathbb{R}$ .  $g(0) = 1$ .  $g'(x) = 0$  since  $f'(x) = cf(x)$ . So  $g(x)$  is a constant, or  $g(x) = 1$  since  $g(0) = 1$ . Therefore,  $f(x) = e^{cx}$  on  $\mathbb{R}$ .)

□

**Supplement.** Cauchy's functional equation.

- (1) (*Cauchy's functional equation.*) Suppose  $f(x) + f(y) = f(x + y)$  for all real  $x$  and  $y$ . Assuming that  $f$  is continuous, prove that  $f(x) = cx$  where  $c$  is a constant.

Notice that we cannot let  $g(x) = \log f(x)$  and apply Cauchy's functional equation on  $g(x)$  to prove Exercise 8.6 since  $f(x)$  is not necessarily positive and thus  $g(x) = \log f(x)$  might be meaningless. However, this wrong approach gives you some useful ideas such as you need to prove that  $f(x)$  is positive first, and  $f(x)$  should be equal to  $e^{cx}$  where  $c = g(1) = \log f(1)$ .

- (2) Suppose  $f(xy) = f(x) + f(y)$  for all positive real  $x$  and  $y$ . Assuming that  $f$  is continuous, prove that  $f(x) = c \log x$  where  $c$  is a constant.
- (3) Suppose  $f(xy) = f(x)f(y)$  for all positive real  $x$  and  $y$ . Assuming that  $f$  is continuous and positive, prove that  $f(x) = x^c$  where  $c$  is a constant.
- (4) Suppose  $f(x + y) = f(x) + f(y) + xy$  for all real  $x$  and  $y$ . Assuming that  $f$  is continuous, prove that  $f(x) = \frac{1}{2}x^2 + cx$  where  $c$  is a constant.
- (5) (*USA 2002.*) Suppose  $f(x^2 - y^2) = xf(x) - yf(y)$  for all real  $x$  and  $y$ . Assuming that  $f$  is continuous, prove that  $f(x) = cx$  where  $c$  is a constant.

**Supplement.** Show that the only automorphism of  $\mathbb{Q}$  is the identity.

*Proof.* Given any  $\sigma \in \text{Aut}(\mathbb{Q})$ .

- (1) Show that  $\sigma(1) = 1$ . Since  $1^2 = 1$ ,  $\sigma(1)\sigma(1) = \sigma(1)$ .  $\sigma(1) = 0$  or  $1$ . There are only two possible cases.

- (a) Assume that  $\sigma(1) = 0$ . So

$$\sigma(a) = \sigma(a \cdot 1) = \sigma(a) \cdot \sigma(1) = \sigma(a) \cdot 0 = 0$$

for any  $a \in \mathbb{Q}$ . That is,  $\sigma = 0 \in \text{Aut}(\mathbb{Q})$ , which is absurd.

- (b) Therefore,  $\sigma(1) = 1$ .

- (2) Show that  $\sigma(n) = n$  for all  $n \in \mathbb{Z}^+$ . Write  $n = 1 + 1 + \cdots + 1$  ( $n$  times 1). Applying the additivity of  $\sigma$ , we have

$$\sigma(n) = \sigma(1) + \sigma(1) + \cdots + \sigma(1) = 1 + 1 + \cdots + 1 = n.$$

(Might use induction on  $n$  to eliminate  $\cdots$  symbols.)

- (3) Show that  $\sigma(n) = n$  for all  $n \in \mathbb{Z}$ . By the additivity of  $\sigma$ ,  $\sigma(-n) = -\sigma(n) = -n$  for  $n \geq 0$ . The result is established.

For any  $a = \frac{n}{m} \in \mathbb{Q}$  ( $m, n \in \mathbb{Z}$ ,  $n \neq 0$ ), applying the multiplication of  $\sigma$  on  $am = n$ , that is,  $\sigma(a)\sigma(m) = \sigma(n)$ . By (3), we have  $\sigma(a)m = n$ , or

$$\sigma(a) = \frac{m}{n} = a$$

provided  $n \neq 0$ , or  $\sigma$  is the identity.  $\square$

**Exercise 8.7.** If  $0 < x < \frac{\pi}{2}$ , prove that

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1.$$

*Proof.*

(1) Let

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

be a continuous function on  $[0, \frac{\pi}{2}]$  (since  $\lim_{x \rightarrow 0+} f(x) = 1$ ). So

$$f'(x) = \frac{x \cos x - \sin x}{x^2} < 0$$

on  $(0, \frac{\pi}{2})$  since  $\tan x > x$  on  $(0, \frac{\pi}{2})$ .

(2) Show that  $\frac{\sin x}{x} < 1$  on  $(0, \frac{\pi}{2})$ . Given any  $x \in (0, \frac{\pi}{2})$ , there exists  $\xi_1 \in (0, x)$  such that

$$\frac{f(x) - f(0)}{x - 0} = f'(\xi_1) < 0$$

by the mean value theorem (Theorem 5.10). So  $f(x) < f(0) = 1$ , or  $\frac{\sin x}{x} < 1$ .

(3) Show that  $\frac{\sin x}{x} > \frac{2}{\pi}$  on  $(0, \frac{\pi}{2})$ . Given any  $x \in (0, \frac{\pi}{2})$ , there exists  $\xi_2 \in (0, x)$  such that

$$\frac{f(\frac{\pi}{2}) - f(x)}{\frac{\pi}{2} - x} = f'(\xi_2) < 0$$

by the mean value theorem (Theorem 5.10). So  $f(x) > f(\frac{\pi}{2}) = \frac{2}{\pi}$ , or  $\frac{\sin x}{x} > \frac{2}{\pi}$ .

$\square$

**Exercise 8.8.** For  $n = 0, 1, 2, \dots$ , and  $x$  real, prove that

$$|\sin(nx)| \leq n|\sin x|.$$

Note that this inequality may be false for other values of  $n$ . For instance,

$$\left| \sin\left(\frac{1}{2}\pi\right) \right| > \frac{1}{2} |\sin \pi|.$$

*Proof.* Induction on  $n$ .

(1) Note that

$$\sin(a + b) = \sin a \cos b + \cos a \sin b$$

for any  $a, b \in \mathbb{R}$ .

(2)  $n = 0, 1$  are clearly true.

(3) Assume the induction hypothesis that for the single case  $n = k$  holds, meaning

$$|\sin(kx)| \leq k |\sin x|$$

is true. It follows that

$$\begin{aligned} |\sin((k+1)x)| &= |\sin(kx) \cos x + \cos(kx) \sin x| && ((1)) \\ &\leq |\sin(kx)| |\cos x| + |\cos(kx)| |\sin x| && (\text{Triangle inequality}) \\ &\leq |\sin(kx)| + |\sin x| && (|\cos(\cdot)| \leq 1) \\ &\leq k |\sin x| + |\sin x| && (\text{Induction hypothesis}) \\ &\leq (k+1) |\sin x|. \end{aligned}$$

□

### Exercise 8.9 (The Euler-Mascheroni constant).

(a) Put  $s_N = 1 + \frac{1}{2} + \cdots + \frac{1}{N}$ . Prove that

$$\lim_{N \rightarrow \infty} (s_N - \log N)$$

exists. (The limit, often denoted by  $\gamma$ , is called Euler's constant. Its numerical value is  $0.5772\dots$ . It is not known whether  $\gamma$  is rational or not.)

(b) Roughly how large must  $m$  be so that  $N = 10^m$  satisfies  $s_N > 100$ ?

*Proof of (a) (Theorem 3.14).*



(1) Note that

$$\begin{aligned}
& \frac{1}{1 + \frac{1}{n}} \leq \frac{1}{x} \leq 1 \text{ for } x \in \left[1, 1 + \frac{1}{n}\right] \\
& \Rightarrow \int_1^{1 + \frac{1}{n}} \frac{dx}{1 + \frac{1}{n}} \leq \int_1^{1 + \frac{1}{n}} \frac{dx}{x} \leq \int_1^{1 + \frac{1}{n}} dx \quad (\text{Theorem 6.12(b)}) \\
& \Rightarrow \frac{1}{n+1} \leq \int_1^{1 + \frac{1}{n}} \frac{dx}{x} \leq \frac{1}{n} \\
& \Rightarrow \frac{1}{n+1} \leq \log \left(1 + \frac{1}{n}\right) \leq \frac{1}{n}. \quad (\text{Equation (39) on page 180})
\end{aligned}$$

(2) Define

$$\gamma_n = s_n - \log n.$$

It suffices to show that  $\{\gamma_n\}$  is monotonic and bounded (Theorem 3.14).

(3) Show that  $\{\gamma_n\}$  is decreasing.

$$\begin{aligned}
\gamma_{n+1} - \gamma_n &= (s_{n+1} - \log(n+1)) - (s_n - \log n) \\
&= (s_{n+1} - s_n) - (\log(n+1) - \log n) \\
&= \frac{1}{n+1} - \log \left(\frac{n+1}{n}\right) \\
&= \frac{1}{n+1} - \log \left(1 + \frac{1}{n}\right) \\
&\leq 0. \quad ((1))
\end{aligned}$$

Note.  $\gamma_n \leq \dots \leq \gamma_1 = 1$  for all  $n = 1, 2, 3, \dots$

(4) Show that  $\gamma_n \geq 0$  for all  $n = 1, 2, 3, \dots$  Since

$$\begin{aligned}
\log n &= \sum_{k=1}^{n-1} (\log(k+1) - \log k) \\
&= \sum_{k=1}^{n-1} \log \frac{k+1}{k} \\
&= \sum_{k=1}^{n-1} \log \left(1 + \frac{1}{k}\right) \\
&\leq \sum_{k=1}^{n-1} \frac{1}{k} \quad ((1)) \\
&= s_{n-1},
\end{aligned}$$

we have

$$\gamma_n = s_n - \log n \geq s_n - s_{n-1} = \frac{1}{n} > 0.$$

By (3)(4),  $\{\gamma_n\}$  converges to  $\lim_{N \rightarrow \infty} (s_N - \log N) = \gamma$ .  $\square$

**Supplement.** Show that if  $f \geq 0$  on  $[0, \infty)$  and  $f$  is monotonically decreasing, and if

$$c_n = \sum_{k=1}^n f(k) - \int_1^n f(x) dx,$$

then  $\lim_{n \rightarrow \infty} c_n$  exists. (Exercise 10 of Section 5.2 in the textbook: *R Creighton Buck, Advanced Calculus, 3rd edition*. See page 235.) If this exercise is true, we can get the existence of  $\gamma$  by taking  $f(x) = \frac{1}{x}$ .

(1) Note that

$$f(n+1) \leq \int_n^{n+1} f(x) dx \leq f(n).$$

(2) Show that  $\{c_n\}$  is decreasing.

$$c_{n+1} - c_n = f(n+1) - \int_n^{n+1} f(x) dx \leq 0.$$

(3) Show that  $c_n \geq 0$ . Since  $f(k) \geq \int_k^{k+1} f(x) dx$ ,

$$\begin{aligned} \sum_{k=1}^n f(k) &\geq \sum_{k=1}^n \int_k^{k+1} f(x) dx \\ &= \int_1^{n+1} f(x) dx \\ &\geq \int_1^n f(x) dx. \end{aligned} \quad (f \geq 0)$$

So that  $c_n = \sum_{k=1}^n f(k) - \int_1^n f(x) dx \geq 0$ .

(4) By (2)(3),  $\{c_n\}$  converges (Theorem 3.14).

$\square$

*Proof of (a) (Limit comparison test).* Inspired by this paper: *Philippe Flajolet and Ilan Vardi, Zeta Function Expansions of Classical Constants*.

(1) Rewrite

$$\gamma_n + \log n - \log(n+1) = \sum_{k=1}^n \left( \frac{1}{k} - \log \left( 1 + \frac{1}{k} \right) \right)$$

(similar to the argument in (a)(4)(Theorem 3.14)). Let

$$c_k = \frac{1}{k} - \log \left( 1 + \frac{1}{k} \right).$$

(2) Show that

$$\lim_{k \rightarrow \infty} \frac{c_k}{\frac{1}{k^2}} = \frac{1}{2}.$$

In fact,

$$\begin{aligned} & \lim_{k \rightarrow \infty} \frac{c_k}{\frac{1}{k^2}} \\ &= \lim_{x \rightarrow 0} \frac{x - \log(1+x)}{x^2} \quad \left(\text{Put } x = \frac{1}{k}\right) \\ &= \lim_{x \rightarrow 0} \frac{1 - \frac{1}{1+x}}{2x} \quad (\text{L'Hospital's rule}) \\ &= \lim_{x \rightarrow 0} \frac{1}{2(x+1)} \\ &= \frac{1}{2}. \end{aligned}$$

(3) By limit comparison test or comparison test,  $\sum c_k$  converges since  $\sum \frac{1}{k^2}$  converges. Also,

$$\lim_{n \rightarrow \infty} \log n - \log(n+1) = 0.$$

Therefore,  $\lim_{n \rightarrow \infty} \gamma_n$  exists.

□

*Note.* This proof is based on **limit comparison test** (Theorem 8.21) in this textbook: *Tom. M. Apostol, Mathematical Analysis, 2nd edition*. It is easy to prove by the original comparison test.

*Proof of (a) (Comparison test).*

(1) Note that

$$0 \leq x - \log(x+1) \leq \frac{x^2}{2}$$

for all  $x \geq 0$ .

(2) Write

$$c_n = \frac{1}{n} - \log\left(1 + \frac{1}{n}\right).$$

as in the the proof of (a) (Limit comparison test). By (1),

$$|c_n| \leq \frac{1}{2n^2}$$

for all  $n = 1, 2, \dots$ . Hence, by the comparison test (Theorem 3.25(a)),  $\sum c_n$  converges since  $\sum \frac{1}{n^2}$  converges (to  $\frac{\pi^2}{6}$ ). Use the same argument in the proof of (a) (Limit comparison test), since

$$\gamma_n + \log n - \log(n+1) = \sum c_n \text{ and } \lim_{n \rightarrow \infty} \log n - \log(n+1) = 0,$$

we have the existence of  $\lim \gamma_n = \gamma$ .

□

*Proof of (a) (Uniformly convergence of  $\sum \frac{x}{n(x+n)}$ ).* (One example to Exercise 7 of Section 6.2 in the textbook: *R Creighton Buck, Advanced Calculus, 3rd edition*. See pages 270 to 271.)

(1) Let

$$f_n(x) = \frac{x}{n(x+n)} = \frac{1}{n} - \frac{1}{x+n}$$

defined on  $E = [0, 1]$ .

(2) Note that

$$|f_n(x)| \leq \frac{1}{n^2}$$

for all  $x \in [0, 1]$ . Since  $\sum \frac{1}{n^2}$  converges,  $\sum f_n$  converges uniformly on  $[0, 1]$  (Theorem 7.10).

(3) Corollary to Theorem 7.16 implies that

$$\begin{aligned} \int_0^1 \sum_{n=1}^{\infty} \frac{x}{n(x+n)} dx &= \sum_{n=1}^{\infty} \int_0^1 \frac{x}{n(x+n)} dx \\ &= \sum_{n=1}^{\infty} \int_0^1 \left( \frac{1}{n} - \frac{1}{x+n} \right) dx \\ &= \sum_{n=1}^{\infty} \left( \frac{1}{n} - \log \frac{n+1}{n} \right) \\ &= \lim_{N \rightarrow \infty} \left( \sum_{n=1}^N \frac{1}{n} - \log(N+1) \right) \\ &= \lim_{N \rightarrow \infty} (s_N - \log(N+1)) \end{aligned}$$

exists. Since  $\lim_{N \rightarrow \infty} (\log(N+1) - \log N) = 0$ ,

$$\begin{aligned} \gamma &= \lim_{N \rightarrow \infty} (s_N - \log N) \\ &= \lim_{N \rightarrow \infty} (s_N - \log(N+1)) + \lim_{N \rightarrow \infty} (\log(N+1) - \log N) \end{aligned}$$

exists.

□

*Proof of (a) (Existence of  $\int_1^{\infty} \frac{\{x\}}{x^2} dx$ ).*

- (1) Define  $\{x\} = x - [x]$  where  $[x]$  is the greatest integer  $\leq x$  (Exercise 6.16).  
Show that

$$\int_1^\infty \frac{\{x\}}{x^2} dx < \infty.$$

Use the similar argument in Exercise 6.16(b). Since  $\frac{\{x\}}{x^2} \leq \frac{1}{x^2}$  on  $[1, \infty)$  and  $\int_1^\infty \frac{1}{x^2} dx = 1$  exists, the result is established (Theorem 6.12(b)).

- (2) Show that

$$\int_1^N \frac{[x]}{x^2} dx = s_N - 1.$$

Use the similar argument in Exercise 6.16(a),

$$\begin{aligned} \int_1^N \frac{[x]}{x^2} dx &= \sum_{k=1}^{N-1} \int_k^{k+1} \frac{[x]}{x^2} dx \\ &= \sum_{k=1}^{N-1} \int_k^{k+1} \frac{k}{x^2} dx \\ &= \sum_{k=1}^{N-1} \int_k^{k+1} \frac{k}{x^2} dx \\ &= \sum_{k=1}^{N-1} \frac{1}{k+1} \\ &= \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} \\ &= s_N - 1. \end{aligned}$$

**Supplement (Euler's summation formula).** (Theorem 7.13 in the textbook: Tom. M. Apostol, *Mathematical Analysis*, 2nd edition.) If  $f$  has a continuous derivative  $f'$  on  $[a, b]$ , then we have

$$\sum_{a < n \leq b} f(n) = \int_a^b f(x) dx + \int_a^b f'(x) \{x\} dx + f(a)\{a\} - f(b)\{b\},$$

where  $\sum_{a < n \leq b}$  means the sum from  $n = [a] + 1$  to  $n = [b]$ . When  $a$  and  $b$  are integers, this becomes

$$\sum_{n=a}^b f(n) = \int_a^b f(x) dx + \int_a^b f'(x) \left( \{x\} - \frac{1}{2} \right) dx + \frac{f(a) + f(b)}{2}.$$

By taking  $f(x) = \frac{1}{x}$  we can get the same result.

(3) Show that

$$\int_1^N \frac{\{x\}}{x^2} dx = \log N - s_N + 1 = 1 - \gamma_N.$$

In fact,

$$\begin{aligned} \int_1^N \frac{\{x\}}{x^2} dx &= \int_1^N \frac{x - [x]}{x^2} dx \\ &= \int_1^N \frac{1}{x} dx - \int_1^N \frac{[x]}{x^2} dx \\ &= \log N - (s_N - 1) \\ &= \log N - s_N + 1 \\ &= 1 - \gamma_N. \end{aligned}$$

(4) Since

$$\lim_{N \rightarrow \infty} \int_1^N \frac{\{x\}}{x^2} dx = \int_1^\infty \frac{\{x\}}{x^2} dx$$

exists (by (1)),  $\gamma = \lim \gamma_N$  exists.

□

*Proof of (b).* By  $s_n - \log n > 0$  in (a)(4)(Theorem 3.14), it suffices to choose  $N = 10^m$  such that  $s_N \geq \log(N+1) > 100$ , or

$$m > \frac{\log(\exp(100) - 1)}{\log 10},$$

or choose  $m$  satisfying

$$m > \frac{100}{\log 10} > \frac{\log(\exp(100) - 1)}{\log 10},$$

or  $m = 44$ . □

*Note.* The exact value of  $N$  is

$$15092688622113788323693563264538101449859497 \approx 1.509 \times 10^{43}.$$

**Exercise 8.10.** Prove that  $\sum \frac{1}{p}$  diverges; the sum extends over all primes.

There are many proofs of this result. We provide some of them.

*Proof (Due to hint).* Given  $N$ .

(1) Show that

$$\sum_{n \leq N} \frac{1}{n} \leq \prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1}.$$

By the unique factorization theorem on  $n \leq N$ ,

$$\sum_{n \leq N} \frac{1}{n} \leq \prod_{p \leq N} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1}.$$

(2) By (1) and the fact that  $\sum \frac{1}{n}$  diverges, there are infinitely many primes.

(3) Show that

$$\prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1} \leq \exp \left( \sum_{p \leq N} \frac{2}{p} \right).$$

By applying the inequality  $(1 - x)^{-1} < e^{2x}$  where  $x \in (0, \frac{1}{2}]$  on any prime  $p$ ,

$$\left(1 - \frac{1}{p}\right)^{-1} < \exp \left( \frac{2}{p} \right).$$

Now multiplying the inequality over all primes  $p \leq N$  and noticing that  $\exp(x) \cdot \exp(y) = \exp(x + y)$ , we have

$$\prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1} \leq \exp \left( \sum_{p \leq N} \frac{2}{p} \right).$$

(4) By (1)(3),

$$\sum_{n \leq N} \frac{1}{n} \leq \exp \left( \sum_{p \leq N} \frac{2}{p} \right).$$

Since  $\sum_{n \leq N} \frac{1}{n}$  diverges, the result holds.

□

*Proof (Due to Kenneth Ireland and Michael Rosen).* The proof in Kenneth Ireland and Michael Rosen, A Classical Introduction to Modern Number Theory, Second Edition (Theorem 3 in Chapter 2) does not use the inequality  $(1 - x)^{-1} < e^{2x}$  ( $x \in (0, \frac{1}{2}]$ ) directly. Instead, the authors take the logarithm on  $(1 - p^{-1})^{-1}$  and estimate it. (So the length of proof is longer than the proof due to hint.)

That is,

$$\begin{aligned}
-\log(1 - p^{-1}) &= \sum_{n=1}^{\infty} \frac{p^{-n}}{n} \\
&= \frac{1}{p} + \sum_{n=2}^{\infty} \frac{p^{-n}}{n} \\
&< \frac{1}{p} + \sum_{n=2}^{\infty} p^{-n} \\
&= \frac{1}{p} + \frac{p^{-2}}{1 - p^{-1}} \\
&< \frac{1}{p} + 2 \cdot \frac{1}{p^2}.
\end{aligned}$$

Now we sum over all primes  $p \leq N$ ,

$$\log \left( \prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1} \right) < \sum_{p \leq N} \frac{1}{p} + 2 \sum_{p \leq N} \frac{1}{p^2}.$$

So

$$\log \sum_{n \leq N} \frac{1}{n} < \sum_{p \leq N} \frac{1}{p} + 2 \sum_{p \leq N} \frac{1}{p^2}.$$

Notice that  $\sum \frac{1}{n}$  diverges and  $\sum \frac{1}{p^2}$  converges (since  $\sum \frac{1}{n^2}$  converges). Therefore,  $\sum \frac{1}{p}$  diverges.  $\square$

*Proof (Due to I. Niven).* It is an exercise in Kenneth Ireland and Michael Rosen, A Classical Introduction to Modern Number Theory, Second Edition. See Exercise 27 in Chapter 2.

- (1) Show that  $\sum' \frac{1}{n}$ , the sum being over square free integers, diverges. For any positive integers  $n$ , we can write  $n = a^2 b$  where  $a \in \mathbb{Z}^+$  and  $b$  is a square free integer. Given  $N$ ,

$$\sum_{n \leq N} \frac{1}{n} \leq \left( \sum_{a=1}^{\infty} \frac{1}{a^2} \right) \left( \sum'_{b \leq N} \frac{1}{b} \right).$$

Notice that  $\sum_{a=1}^{\infty} \frac{1}{a^2}$  converges. Since  $\sum_{n \leq N} \frac{1}{n} \rightarrow \infty$  as  $N \rightarrow \infty$ ,  $\sum'_{b \leq N} \frac{1}{b} \rightarrow \infty$  as  $N \rightarrow \infty$ .

- (2) Show that

$$\prod_{p \leq N} \left(1 + \frac{1}{p}\right) \rightarrow \infty \text{ as } N \rightarrow \infty.$$



By the unique factorization theorem on  $n \leq N$ ,

$$\prod_{p \leq N} \left(1 + \frac{1}{p}\right) \geq \sum'_{n \leq N} \frac{1}{n}.$$

Since  $\sum'_{n \leq N} \frac{1}{n} \rightarrow \infty$  as  $N \rightarrow \infty$  by (1), the conclusion is established.

(3) By applying the inequality  $e^x > 1 + x$  on any prime  $p$ ,

$$\exp\left(\frac{1}{p}\right) > 1 + \frac{1}{p}.$$

Now multiplying the inequality over all primes  $p \leq N$  and noticing that  $\exp(x) \cdot \exp(y) = \exp(x + y)$ , we have

$$\exp\left(\sum_{p \leq N} \frac{1}{p}\right) > \prod_{p \leq N} \left(1 + \frac{1}{p}\right).$$

By (2),  $\exp\left(\sum_{p \leq N} \frac{1}{p}\right) \rightarrow \infty$  as  $N \rightarrow \infty$ , or  $\sum_{p \leq N} \frac{1}{p} \rightarrow \infty$  as  $N \rightarrow \infty$ .

□

**Exercise 8.11.** Suppose  $f \in \mathcal{R}$  on  $[0, A]$  for all  $A < \infty$ , and  $f(x) \rightarrow 1$  as  $x \rightarrow +\infty$ . Prove that

$$\lim_{t \rightarrow 0} t \int_0^\infty e^{-tx} f(x) dx = 1 \quad (t > 0),$$

It is similar to Exercise 3.14(a).

*Proof.* Given any  $\varepsilon > 0$ .

(1) The integral  $\int_0^\infty e^{-tx} f(x) dx$  is well-defined. (It suffices to show that  $\int_0^\infty e^{-tx} f(x) dx$  converges absolutely in the sense of Exercise 6.8. It is quite easy since  $f(x) \rightarrow 1$  as  $x \rightarrow +\infty$  and well-behavior of  $\int_{A_0}^\infty e^{-tx} f(x) dx$  for any  $A_0 > 0$ .)

(2) Note that

$$t \int_0^\infty e^{-tx} dx = 1$$

for any  $t > 0$ .

(3) Since  $f(x) \rightarrow 1$  as  $x \rightarrow +\infty$ , there is  $A_0 > 0$  such that

$$|f(x) - 1| < \frac{\varepsilon}{64} \text{ whenever } x \geq A_0.$$

(4) Since  $f \in \mathcal{R}$  on  $[0, A_0]$ ,  $f$  is bounded on  $[0, A_0]$ , or  $|f| \leq M$  on  $[0, A_0]$  for some  $M$  (Theorem 6.7(c)).

(5) As  $t > 0$ ,

$$\begin{aligned} & \left| \left( t \int_0^\infty e^{-tx} f(x) dx \right) - 1 \right| \\ &= \left| t \int_0^\infty e^{-tx} (f(x) - 1) dx \right| \end{aligned} \quad ((2))$$

$$\leq t \int_0^\infty e^{-tx} |f(x) - 1| dx \quad ((1) \text{ with Theorem 6.13})$$

$$\begin{aligned} &= t \int_0^{A_0} e^{-tx} |f(x) - 1| dx + t \int_{A_0}^\infty e^{-tx} |f(x) - 1| dx \\ &\leq t \int_0^{A_0} (M + 1) dx + t \int_{A_0}^\infty e^{-tx} |f(x) - 1| dx \end{aligned} \quad ((3) \text{ and } e^{-tx} \leq 1)$$

$$\leq t \int_0^{A_0} (M + 1) dx + t \int_{A_0}^\infty e^{-tx} \frac{\varepsilon}{64} dx \quad ((4))$$

$$\begin{aligned} &= t A_0 (M + 1) + \exp(-A_0 t) \frac{\varepsilon}{64} \\ &\leq t A_0 (M + 1) + \frac{\varepsilon}{64}. \end{aligned} \quad (e^{-tx} \leq 1)$$

Since  $t$  is arbitrary, take  $t = \frac{\varepsilon}{89 A_0 (M + 1)} > 0$  to get

$$\left| \left( t \int_0^\infty e^{-tx} f(x) dx \right) - 1 \right| < \frac{\varepsilon}{89} + \frac{\varepsilon}{64} < \varepsilon,$$

or

$$\lim_{t \rightarrow 0^+} t \int_0^\infty e^{-tx} f(x) dx = 1.$$

□

**Exercise 8.12.** Suppose  $0 < \delta < \pi$ ,

$$f(x) = \begin{cases} 1 & \text{if } |x| \leq \delta, \\ 0 & \text{if } \delta < |x| \leq \pi, \end{cases}$$

and  $f(x + 2\pi) = f(x)$  for all  $x$ .

(a) Compute the Fourier coefficients of  $f$ .

(b) Compute that

$$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2} \quad (0 < \delta < \pi).$$

(c) Deduce from Parseval's theorem that

$$\sum_{n=1}^{\infty} \frac{(\sin(n\delta))^2}{n^2\delta} = \frac{\pi - \delta}{2}.$$

(d) Let  $\delta \rightarrow 0$  and prove that

$$\int_0^{\infty} \left( \frac{\sin x}{x} \right)^2 dx = \frac{\pi}{2}.$$

(e) Put  $\delta = \frac{\pi}{2}$  in (c). What do you get?

It is a centered square pulse around  $x = 0$  with shift  $\delta$ . Besides,  $f(x)$  is an even function.

*Proof of (a).*

$$\begin{aligned} c_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \\ &= \frac{1}{2\pi} \int_{-\delta}^{\delta} dx \\ &= \frac{\delta}{\pi}. \end{aligned}$$

For  $0 \neq n \in \mathbb{Z}$ ,

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \\ &= \frac{1}{2\pi} \int_{-\delta}^{\delta} e^{-inx} dx \\ &= \frac{1}{2\pi} \cdot \frac{2 \sin(n\delta)}{n} \\ &= \frac{\sin(n\delta)}{n\pi}. \end{aligned}$$

□

**Supplement.** Find  $a_n$  and  $b_n$  of this textbook.

By (a),  $a_0 = \frac{\delta}{\pi}$ ,  $a_n = \frac{2 \sin(n\delta)}{n\pi}$ ,  $b_n = 0$  for  $n \in \mathbb{Z}^+$ . Surely, we can compute  $a_n$

and  $b_n$  ( $n > 0$ ) directly. Since  $f(x)$  is an even function,  $b_n = 0$ . And

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \\ &= \frac{2}{\pi} \int_0^{\delta} \cos(nx) dx \\ &= \frac{2 \sin(n\delta)}{n\pi}. \end{aligned}$$

*Proof of (b).* Given  $x = 0$ , there are constants  $\delta' = \delta > 0$  and  $M = 1 < \infty$  such that

$$|f(0+t) - f(0)| \leq M|t|$$

for all  $t \in (-\delta', \delta')$ . By Theorem 8.14,

$$\sum_{-\infty}^{\infty} c_n = f(0).$$

Notice that  $c_{-n} = c_n$  for  $n \in \mathbb{Z}^+$ , so

$$\begin{aligned} \frac{\delta}{\pi} + 2 \sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n\pi} &= 1 \\ \sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} &= \frac{\pi - \delta}{2}. \end{aligned}$$

□

We can also use the expression  $a_n$  and  $b_n$  to prove the same thing. Besides, taking  $\delta = 1$  yields

$$\sum_{n=1}^{\infty} \frac{\sin n}{n} = \frac{\pi - 1}{2}.$$

*Proof of (c).* Since  $f(x)$  is a Riemann-integrable function with period  $2\pi$ , by Parseval's theorem

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{-\infty}^{\infty} |c_n|^2.$$

So

$$\frac{\delta}{\pi} = \frac{\delta^2}{\pi^2} + 2 \sum_{n=1}^{\infty} \frac{(\sin(n\delta))^2}{n^2 \pi^2},$$

or

$$\sum_{n=1}^{\infty} \frac{(\sin(n\delta))^2}{n^2 \delta} = \frac{\pi - \delta}{2}.$$

□

Notices that

$$\sum_{n=1}^{\infty} \frac{(\sin n)^2}{n^2} = \frac{\pi - 1}{2}$$

as  $\delta = 1$ .

*Proof of (d).* Given  $\varepsilon > 0$ . By Exercise 6.8,

$$\int_0^{\infty} \left( \frac{\sin x}{x} \right)^2 dx$$

exists. So there exists  $b > 0$  such that

$$\left| \int_0^b \left( \frac{\sin x}{x} \right)^2 dx - \int_0^{\infty} \left( \frac{\sin x}{x} \right)^2 dx \right| < \frac{\varepsilon}{4}$$

By Supplement in Chapter 6, there exists  $\delta > 0$  such that for any partition  $P_m = \{0, \frac{b}{m}, \frac{2b}{m}, \dots, \frac{(m-1)b}{m}, b\}$  of  $[0, b]$  with  $\|P\| = \frac{b}{m} < \delta$ , or  $m > \frac{b}{\delta}$ , we have

$$\begin{aligned} \left| \sum_{n=1}^m \frac{(\sin(n \frac{b}{m}))^2}{(n \frac{b}{m})^2} \cdot \frac{b}{m} - \int_0^b \left( \frac{\sin x}{x} \right)^2 dx \right| &< \frac{\varepsilon}{4}, \\ \left| \sum_{n=1}^m \frac{(\sin(n \frac{b}{m}))^2}{n^2 \frac{b}{m}} - \int_0^b \left( \frac{\sin x}{x} \right)^2 dx \right| &< \frac{\varepsilon}{4}. \end{aligned}$$

For simplicity we resize  $\delta$  to  $\delta < \pi$  to make  $0 < \frac{b}{m} < \delta < \pi$ . Besides, since  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges, there exists  $N > 0$  such that

$$\left| \sum_{n=1}^{\infty} \frac{(\sin(n \frac{b}{m}))^2}{n^2 \frac{b}{m}} - \sum_{n=1}^m \frac{(\sin(n \frac{b}{m}))^2}{n^2 \frac{b}{m}} \right| < \frac{\varepsilon}{4}$$

whenever  $m \geq N$ . By (c),

$$\left| \frac{\pi - \frac{b}{m}}{2} - \sum_{n=1}^m \frac{(\sin(n \frac{b}{m}))^2}{n^2 \frac{b}{m}} \right| < \frac{\varepsilon}{4}$$

whenever  $m \geq N$ . Last, it is easy to get

$$\left| \frac{\pi}{2} - \frac{\pi - \frac{b}{m}}{2} \right| < \frac{\varepsilon}{4}$$

whenever  $m > \frac{2b}{\varepsilon}$ . Now we have

$$\left| \frac{\pi}{2} - \int_0^{\infty} \left( \frac{\sin x}{x} \right)^2 dx \right| < \varepsilon$$

whenever  $m > \max(\frac{b}{\delta}, N, \frac{2b}{\varepsilon})$ . Since  $\varepsilon$  is arbitrary,  $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$ .  $\square$

*Proof of (e).*

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

Write

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^2} &= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \\ &= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}, \end{aligned}$$

so

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{6}.$$

$\square$

**Exercise 8.13.** Put  $f(x) = x$  if  $0 \leq x < 2\pi$ , and apply Parseval's theorem to conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi}{6}.$$

*Proof.*

$$\begin{aligned} c_0 &= \frac{1}{2\pi} \int_0^{2\pi} x dx \\ &= \pi, \end{aligned}$$

For  $n \neq 0$ ,

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_0^{2\pi} x e^{-inx} dx \\ &= \frac{1}{2\pi} \left( \left[ -\frac{1}{in} x e^{-inx} \right]_{x=0}^{x=2\pi} - \int_0^{2\pi} -\frac{1}{in} e^{-inx} dx \right) \\ &= \frac{i}{n}. \end{aligned}$$

Since  $f(x)$  is a Riemann-integrable function with period  $2\pi$ , by Parseval's theorem

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

So

$$\frac{1}{2\pi} \cdot \frac{(2\pi)^3}{3} = \pi^2 + 2 \sum_{n=1}^{\infty} \frac{1}{n^2},$$

or

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

□

**Supplement.** Put  $f(x) = x^n$  if  $n \in \mathbb{Z}^+$  and  $0 \leq x < 2\pi$ . Might get

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!}.$$

**Exercise 8.14.** PLACEHOLDER.

**Exercise 8.15.** With the Dirichlet kernel  $D_n$  as defined by

$$D_n(x) = \sum_{k=-n}^n \exp(ikx) = \frac{\sin(n + \frac{1}{2})x}{\sin(\frac{x}{2})},$$

put the **Fejér kernel**

$$K_N(x) = \frac{1}{N+1} \sum_{n=0}^N D_n(x).$$

Prove that

$$K_N(x) = \frac{1}{N+1} \cdot \frac{1 - \cos(N+1)x}{1 - \cos x}$$

and that

- (a)  $K_N \geq 0$ ,
- (b)  $\frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(x) dx = 1$ ,
- (c)  $K_N(x) \leq \frac{1}{N+1} \cdot \frac{2}{1 - \cos \delta}$  if  $0 < \delta \leq |x| \leq \pi$ .

If  $s_N = s_N(f; x)$  is the  $N$ th partial sum of the Fourier series of  $f$ , consider the arithmetic means

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_N}{N+1}.$$

Prove that

$$s_N(f; x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) K_N(t) dt,$$

and hence prove **Fejér's theorem**:

If  $f$  is continuous, with period  $2\pi$ , then  $\sigma_N(f; x) \rightarrow f(x)$  uniformly on  $[-\pi, \pi]$ .

(Hint: Use properties (a), (b), (c) to proceed as in Theorem 7.26.)

*Proof of  $K_N(x) = \frac{1}{N+1} \cdot \frac{1 - \cos(N+1)x}{1 - \cos x}$ .* Since

$$\begin{aligned} (1 - \cos x)K_N(x) &= 2 \left( \sin \frac{x}{2} \right)^2 \frac{1}{N+1} \sum_{n=0}^N \frac{\sin\left(n + \frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)} \\ &= \frac{1}{N+1} \sum_{n=0}^N 2 \sin \frac{x}{2} \sin\left(n + \frac{1}{2}\right)x \\ &= \frac{1}{N+1} \sum_{n=0}^N (\cos(nx) - \cos(n+1)x) \\ &= \frac{1 - \cos(N+1)x}{N+1}, \\ K_N(x) &= \frac{1}{N+1} \cdot \frac{1 - \cos(N+1)x}{1 - \cos x} \end{aligned}$$

if  $x \neq 2k\pi$  for  $k \in \mathbb{Z}$ .  $\square$

*Proof of (a).* It is clear since  $\cos x \leq 1$  for all  $x \in \mathbb{R}$ . Or we may write

$$K_N(x) = \frac{1}{N+1} \left( \frac{\sin \frac{(N+1)x}{2}}{\sin \frac{x}{2}} \right)^2 \geq 0.$$

$\square$

*Proof of (b).* By the definition of  $D_n(x)$ ,

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(x) dx &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{N+1} \sum_{n=0}^N D_n(x) dx \\ &= \frac{1}{N+1} \sum_{n=0}^N \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(x) dx \\ &= \frac{1}{N+1} \sum_{n=0}^N 1 \\ &= 1. \end{aligned}$$

$\square$



*Proof of (c).* Since  $\cos x$  is bounded by 1 and monotonically decreasing on  $(0, \pi]$ ,

$$\begin{aligned} K_N(x) &= \frac{1}{N+1} \cdot \frac{1 - \cos(N+1)x}{1 - \cos x} \\ &\leq \frac{1}{N+1} \cdot \frac{2}{1 - \cos \delta}. \end{aligned}$$

□

*Proof of  $s_N(f; x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)K_N(t)dt$ .*

$$\begin{aligned} \sigma_N(f; x) &= \frac{1}{N+1} \sum_{n=0}^N s_N(f; x) \\ &= \frac{1}{N+1} \sum_{n=0}^N \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)D_N(t)dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) \left( \frac{1}{N+1} \sum_{n=0}^N D_N(t) \right) dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)K_N(t)dt. \end{aligned}$$

□

*Proof of Fejér's theorem.* Given any  $\varepsilon > 0$ .

(1)

$$\begin{aligned} |\sigma_N(f; x) - f(x)| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)K_N(t)dt - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)K_N(t)dt \right| \\ &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x-t) - f(x))K_N(t)dt \right| \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x-t) - f(x)|K_N(t)dt. \end{aligned}$$

(2) Since  $f$  is continuous on a compact set  $[-\pi, \pi]$ ,  $f$  is continuous uniformly. For such  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$|f(y) - f(x)| < \frac{\varepsilon}{2}$$

whenever  $x, y \in [-\pi, \pi]$  and  $|y - x| < \delta$ .

(3) Since  $f$  is continuous on a compact set  $[-\pi, \pi]$ ,  $f$  is bounded on  $[-\pi, \pi]$ , say  $M = \sup |f(x)|$ .

(4) Therefore,

$$\begin{aligned}
& |\sigma_N(f; x) - f(x)| \\
& \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x-t) - f(x)| K_N(t) dt \\
& = \frac{1}{2\pi} \int_{-\pi}^{-\delta} |f(x-t) - f(x)| K_N(t) dt \\
& \quad + \frac{1}{2\pi} \int_{-\delta}^{\delta} |f(x-t) - f(x)| K_N(t) dt \\
& \quad + \frac{1}{2\pi} \int_{\delta}^{\pi} |f(x-t) - f(x)| K_N(t) dt \\
& \leq \frac{1}{2\pi} \int_{-\pi}^{-\delta} 2M \cdot \frac{1}{N+1} \cdot \frac{2}{1-\cos \delta} dt \\
& \quad + \frac{1}{2\pi} \int_{-\delta}^{\delta} \frac{\varepsilon}{2} K_N(t) dt \\
& \quad + \frac{1}{2\pi} \int_{\delta}^{\pi} 2M \cdot \frac{1}{N+1} \cdot \frac{2}{1-\cos \delta} dt \\
& = \frac{4M(\pi-\delta)}{(N+1)(1-\cos \delta)\pi} + \frac{\varepsilon}{2} \cdot \frac{1}{2\pi} \int_{-\delta}^{\delta} K_N(t) dt \\
& \leq \frac{4M(\pi-\delta)}{(N+1)(1-\cos \delta)\pi} + \frac{\varepsilon}{2}.
\end{aligned}$$

(5) Since  $N$  is arbitrary, we can take an integer  $N > \frac{4M(\pi-\delta)}{(1-\cos \delta)\pi\varepsilon} - 1$  so that

$$\begin{aligned}
|\sigma_N(f; x) - f(x)| & \leq \frac{4M(\pi-\delta)}{(N+1)(1-\cos \delta)\pi} + \frac{\varepsilon}{2} \\
& < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
& = \varepsilon.
\end{aligned}$$

Therefore, the conclusion holds.

□

**Exercise 8.16.** Prove a pointwise version of Fejér's theorem: If  $f \in \mathcal{R}$  and  $f(x+)$ ,  $f(x-)$  exist for some  $x$ , then

$$\lim_{N \rightarrow \infty} \sigma_N(f; x) = \frac{1}{2}[f(x+) + f(x-)].$$

*Proof.* Given any  $\varepsilon > 0$ .

(1) Since  $K_N(-x) = K_N(x)$ , we have

$$\sigma_N(f; x) = \frac{1}{2\pi} \int_0^\pi f(x-t)K_N(t)dt + \frac{1}{2\pi} \int_0^\pi f(x+t)K_N(t)dt$$

and

$$\frac{1}{2\pi} \int_0^\pi K_N(t)dt = \frac{1}{2}.$$

(2) Since  $f \in \mathcal{B}$ ,  $f$  is bounded on  $[-\pi, \pi]$ , say  $M = \sup |f(x)|$ .

(3) Therefore,

$$\begin{aligned} & \left| \frac{1}{2\pi} \int_0^\pi f(x-t)K_N(t)dt - \frac{1}{2}f(x-) \right| \\ &= \left| \frac{1}{2\pi} \int_0^\pi (f(x-t) - f(x-))K_N(t)dt \right| \\ &\leq \frac{1}{2\pi} \int_0^\pi |f(x-t) - f(x-)|K_N(t)dt. \end{aligned}$$

Since  $f(x-)$  exists, for fixed  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$|f(y) - f(x-)| < \frac{\varepsilon}{2}$$

whenever  $y \in (x - \delta, x) \cap [-\pi, \pi]$ . Hence,

$$\begin{aligned} & \left| \frac{1}{2\pi} \int_0^\pi f(x-t)K_N(t)dt - \frac{1}{2}f(x-) \right| \\ &\leq \frac{1}{2\pi} \int_0^\pi |f(x-t) - f(x-)|K_N(t)dt \\ &= \frac{1}{2\pi} \int_0^\delta |f(x-t) - f(x-)|K_N(t)dt \\ &\quad + \frac{1}{2\pi} \int_\delta^\pi |f(x-t) - f(x-)|K_N(t)dt \\ &\leq \frac{1}{2\pi} \int_0^\delta \frac{\varepsilon}{2} K_N(t)dt + \frac{1}{2\pi} \int_\delta^\pi 2M \cdot \frac{1}{N+1} \cdot \frac{2}{1-\cos\delta} dt \\ &= \frac{\varepsilon}{2} \cdot \frac{1}{2\pi} \int_0^\delta K_N(t)dt + \frac{2M(\pi-\delta)}{(N+1)(1-\cos\delta)\pi} \\ &\leq \frac{\varepsilon}{4} + \frac{2M(\pi-\delta)}{(N+1)(1-\cos\delta)\pi}. \end{aligned}$$

(4) Since  $N$  is arbitrary, we can take an integer  $N_1 > \frac{8M(\pi-\delta)}{(1-\cos \delta)\pi\varepsilon} - 1$  such that

$$\begin{aligned} \left| \frac{1}{2\pi} \int_0^\pi f(x-t)K_n(t)dt - \frac{1}{2}f(x-) \right| &\leq \frac{\varepsilon}{4} + \frac{2M(\pi-\delta)}{(n+1)(1-\cos \delta)\pi} \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &= \frac{\varepsilon}{2} \end{aligned}$$

whenever  $n \geq N_1$ . Similarly, we can take an integer  $N_2$  such that

$$\begin{aligned} \left| \frac{1}{2\pi} \int_0^\pi f(x+t)K_n(t)dt - \frac{1}{2}f(x+) \right| &\leq \frac{\varepsilon}{4} + \frac{2M(\pi-\delta)}{(n+1)(1-\cos \delta)\pi} \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &= \frac{\varepsilon}{2}. \end{aligned}$$

whenever  $n \geq N_2$ .

(5) Hence,

$$\begin{aligned} &\left| \sigma_N(f; x) - \frac{1}{2}[f(x+) + f(x-)] \right| \\ &\leq \left| \frac{1}{2\pi} \int_0^\pi f(x-t)K_n(t)dt - \frac{1}{2}f(x-) \right| \\ &\quad + \left| \frac{1}{2\pi} \int_0^\pi f(x+t)K_n(t)dt - \frac{1}{2}f(x+) \right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

whenever  $n \geq \max\{N_1, N_2\}$ . Hence,  $\lim \sigma_N(f; x) = \frac{1}{2}[f(x+) + f(x-)]$ .

□

**Supplement.** Poisson's equation. (Theorem 1 of Section 2.2 in the textbook: *Lawrence C. Evans, Partial Differential Equations.*) Let the fundamental solution of Laplace's equation be

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log |x| & (n=2) \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & (n \geq 3), \end{cases}$$

where  $x \in \mathbb{R}^n$ ,  $x \neq 0$ . Let

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)dy.$$

Then  $-\Delta u = -f$  in  $\mathbb{R}^n$ . Note that  $\Phi(x)$  blows up at 0. To calculate  $\Delta u(x)$ , we need to isolate this singularity inside a small ball, say  $B(0; \varepsilon)$ . Therefore,

$$\Delta u(x) = \int_{B(0; \varepsilon)} \Phi(y) \Delta_x f(x - y) dy + \int_{\mathbb{R}^n - B(0; \varepsilon)} \Phi(y) \Delta_x f(x - y) dy,$$

and we can continue estimating two integrals individually as the textbook did.

**Exercise 8.17.** PLACEHOLDER.

**Exercise 8.18.** PLACEHOLDER.

**Exercise 8.19.** PLACEHOLDER.

**Exercise 8.20.** PLACEHOLDER.

**Exercise 8.21.** PLACEHOLDER.

**Exercise 8.22.** PLACEHOLDER.

**Exercise 8.23.** PLACEHOLDER.

**Exercise 8.24.** PLACEHOLDER.

**Exercise 8.25.** PLACEHOLDER.

**Exercise 8.26.** PLACEHOLDER.

**Exercise 8.27.** PLACEHOLDER.

**Exercise 8.28.** PLACEHOLDER.

**Exercise 8.29.** PLACEHOLDER.

**Exercise 8.30.** Use Stirling's formula to prove that

$$\lim_{x \rightarrow \infty} \frac{\Gamma(x+c)}{x^c \Gamma(x)} = 1$$

for every real constant  $c$ .

*Proof.* By Stirling's formula,

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\Gamma(x+c)}{\left(\frac{x+c-1}{e}\right)^{x+c-1} \sqrt{2\pi(x+c-1)}} &= 1 \\ \lim_{x \rightarrow \infty} \frac{\Gamma(x)}{\left(\frac{x-1}{e}\right)^{x-1} \sqrt{2\pi(x-1)}} &= 1, \end{aligned}$$

we have

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\Gamma(x+c)}{x^c \Gamma(x)} &= \lim_{x \rightarrow \infty} \frac{\Gamma(x+c)}{x^c \Gamma(x)} \\ &\quad \times \lim_{x \rightarrow \infty} \frac{\left(\frac{x+c-1}{e}\right)^{x+c-1} \sqrt{2\pi(x+c-1)}}{\Gamma(x+c)} \\ &\quad \times \lim_{x \rightarrow \infty} \frac{\Gamma(x)}{\left(\frac{x-1}{e}\right)^{x-1} \sqrt{2\pi(x-1)}} \\ &= \lim_{x \rightarrow \infty} \frac{\left(\frac{x+c-1}{e}\right)^{x+c-1} \sqrt{2\pi(x+c-1)}}{x^c \left(\frac{x-1}{e}\right)^{x-1} \sqrt{2\pi(x-1)}} \\ &= \lim_{x \rightarrow \infty} \frac{\left(\frac{x+c-1}{e}\right)^c \left(\frac{x+c-1}{e}\right)^{x-1} \sqrt{\frac{x+c-1}{x-1}}}{x^c \left(\frac{x-1}{e}\right)^{x-1}} \\ &= \frac{1}{e^c} \cdot e^c \cdot 1 \\ &= 1 \end{aligned}$$

since

(1)

$$\lim_{x \rightarrow \infty} \frac{\left(\frac{x+c-1}{e}\right)^c}{x^c} = \frac{1}{e^c} \lim_{x \rightarrow \infty} \left(\frac{x+c-1}{x}\right)^c = \frac{1}{e^c}.$$

(2)

$$\lim_{x \rightarrow \infty} \frac{\left(\frac{x+c-1}{e}\right)^{x-1}}{\left(\frac{x-1}{e}\right)^{x-1}} = \lim_{x \rightarrow \infty} \left(\frac{x+c-1}{x-1}\right)^{x-1} = \lim_{x \rightarrow \infty} \left(1 + \frac{c}{x-1}\right)^{x-1} = e^c.$$

(3) and

$$\lim_{x \rightarrow \infty} \sqrt{\frac{x+c-1}{x-1}} = \lim_{x \rightarrow \infty} \sqrt{1 + \frac{c}{x-1}} = 1.$$

□

**Exercise 8.31.** In the proof of Theorem 7.26 it was shown that

$$\int_{-1}^1 (1-x^2)^n dx \geq \frac{4}{3\sqrt{n}}$$

for  $n = 1, 2, 3, \dots$ . Use Theorem 8.20 and Exercise 8.30 to show the more precise result

$$\lim_{n \rightarrow \infty} \sqrt{n} \int_{-1}^1 (1-x^2)^n dx = \sqrt{\pi}.$$

*Proof.*

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sqrt{n} \int_{-1}^1 (1-x^2)^n dx \\ &= \lim_{n \rightarrow \infty} \sqrt{n} \int_0^1 u^{-\frac{1}{2}} (1-u)^n dx && (u = x^2) \\ &= \lim_{n \rightarrow \infty} \sqrt{n} \frac{\Gamma(\frac{1}{2}) \Gamma(n+1)}{\Gamma(n + \frac{3}{2})} && (\text{Theorem 8.20}) \\ &= \Gamma\left(\frac{1}{2}\right) \lim_{n \rightarrow \infty} \frac{n^{\frac{1}{2}} \Gamma(n+1)}{\Gamma(n + \frac{3}{2})} \\ &= \Gamma\left(\frac{1}{2}\right) && (\text{Exercise 8.30}) \\ &= \sqrt{\pi}. && (\text{Some consequences 8.21}) \end{aligned}$$

□