Chapter 15: Bernoulli Numbers

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Supplement. *Prove that*

$$x \cot x = 1 - 2 \sum_{n=1}^{\infty} \frac{x^2}{n^2\pi^2 - x^2}.$$

Proof.

(1) *Show that*

$$\cot x = \frac{1}{2^n} \sum_{k=0}^{2^n-1} \cot \frac{x+k\pi}{2^n}$$

for all integers $n \geq 1$. Notice that

$$\begin{aligned} \cot(x+\pi) &= \cot x, \\ \cot\left(x + \frac{\pi}{2}\right) &= -\tan x, \\ \cot x &= \frac{1}{2}\left(\cot\frac{x}{2} - \tan\frac{x}{2}\right). \end{aligned}$$

Use mathematical induction. The case $n = 1$ is the same as the note. Assume the case $n = m$ holds. For $n = m+1$,

$$\begin{aligned} \sum_{k=0}^{2^{m+1}-1} \cot \frac{x+k\pi}{2^{m+1}} &= \sum_{k=0}^{2^m-1} \cot \frac{x+k\pi}{2^{m+1}} + \sum_{k=2^m}^{2^{m+1}-1} \cot \frac{x+k\pi}{2^{m+1}} \\ &= \sum_{k=0}^{2^m-1} \cot \frac{x+k\pi}{2^{m+1}} + \sum_{k=0}^{2^m-1} \cot \frac{x+(2^m+k)\pi}{2^{m+1}} \\ &= \sum_{k=0}^{2^m-1} \cot \frac{x+k\pi}{2^{m+1}} + \sum_{k=0}^{2^m-1} \cot \left(\frac{x+k\pi}{2^{m+1}} + \frac{\pi}{2}\right) \\ &= \sum_{k=0}^{2^m-1} \left(\cot \frac{x+k\pi}{2^{m+1}} - \tan \frac{x+k\pi}{2^{m+1}}\right) \\ &= \sum_{k=0}^{2^m-1} \left(\cot \frac{x+k\pi}{2^{m+1}} - \tan \frac{x+k\pi}{2^{m+1}}\right) \\ &= 2 \sum_{k=0}^{2^m-1} \cot \frac{x+k\pi}{2^m}. \end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{2^{m+1}} \sum_{k=0}^{2^{m+1}-1} \cot \frac{x+k\pi}{2^{m+1}} &= \frac{1}{2^{m+1}} \cdot 2 \sum_{k=0}^{2^m-1} \cot \frac{x+k\pi}{2^m} \\
&= \frac{1}{2^m} \sum_{k=0}^{2^m-1} \cot \frac{x+k\pi}{2^m} \\
&= \cot x.
\end{aligned}$$

(2) By rearranging the index of summation of the identity in (1), we have

$$x \cot x = \frac{x}{2^n} \cot \frac{x}{2^n} - \frac{x}{2^n} \tan \frac{x}{2^n} + \sum_{k=1}^{2^{n-1}-1} \frac{x}{2^n} \left(\cot \frac{x+k\pi}{2^n} + \cot \frac{x-k\pi}{2^n} \right)$$

for all integers $n \geq 1$.

(3) Notice that $\lim_{x \to 0} x \cot x = 1$. Let $n \to \infty$, the result is established.

□

Exercise 15.6. *For $m \geq 3$, show $|B_{2m+2}| > |B_{2m}|$. (Hint: Use Theorem 2.)*

Proof. By Theorem 2,

$$2\zeta(2m) = (-1)^{m+1} \frac{(2\pi)^{2m}}{(2m)!} B_{2m}.$$

Thus,

$$\frac{|B_{2m+2}|}{|B_{2m}|} = \frac{\zeta(2m+2)}{\zeta(2m)} \cdot \frac{(2m+2)(2m+1)}{(2\pi)^2}.$$

(1) The first term is

$$\frac{\zeta(2m+2)}{\zeta(2m)} > \frac{1}{\zeta(6)} = \frac{945}{\pi^6}.$$

(2) The second term is

$$\frac{(2m+2)(2m+1)}{(2\pi)^2} \geq \frac{8 \cdot 7}{(2\pi)^2} = \frac{14}{\pi^2}.$$

By (1)(2),

$$\frac{|B_{2m+2}|}{|B_{2m}|} > \frac{945}{\pi^6} \cdot \frac{14}{\pi^2} > 1,$$

or $|B_{2m+2}| > |B_{2m}|$. □