

Chapter 4: Continuity

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Exercise 4.1. Suppose f is a real function define on \mathbb{R}^1 which satisfies

$$\lim_{h \rightarrow 0} [f(x+h) - f(x-h)] = 0$$

for every $x \in \mathbb{R}^1$. Does this imply that f is continuous?

Proof. $\lim_{h \rightarrow 0} [f(x+h) - f(x-h)] = 0$ holds if f is continuous. But the converse of this statement and is not true. For example, define $f : \mathbb{R}^1 \rightarrow \mathbb{R}^1$ by

$$f(x) = \begin{cases} 1 & (x = 0), \\ 0 & (x \neq 0). \end{cases}$$

f is not continuous at $x = 0$ but

$$\lim_{h \rightarrow 0} [f(x+h) - f(x-h)] = 0$$

for any $x \in \mathbb{R}^1$. (The identity holds for $x \neq 0$ since f is continuous on $\mathbb{R}^1 - \{0\}$. Besides, $\lim_{h \rightarrow 0} [f(0+h) - f(0-h)] = \lim_{h \rightarrow 0} [0 - 0] = 0$.) \square

Exercise 4.2. If f is a continuous mapping of a metric space X into a metric space Y , prove that $f(\overline{E}) \subseteq \overline{f(E)}$ for every set $E \subseteq X$. (\overline{E} denotes the closure of E .) Show, by an example, that $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$.

Proof.

(1) Since f is continuous and $\overline{f(E)}$ is closed, $f^{-1}(\overline{f(E)})$ is closed. Hence,

$$\begin{aligned} f^{-1}(\overline{f(E)}) &\supseteq f^{-1}(f(E)) && \text{(Monotonicity of } f^{-1}) \\ &\supseteq E, && \text{(Note in Theorem 4.14)} \\ \overline{E} &\subseteq f^{-1}(\overline{f(E)}), && \text{(Monotonicity of closure)} \\ f(\overline{E}) &\subseteq f(f^{-1}(\overline{f(E)})) && \text{(Monotonicity of } f) \\ &\subseteq \overline{f(E)}. && \text{(Note in Theorem 4.14)} \end{aligned}$$

(2) Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a continuous function defined by

$$f(x) = \frac{1}{x}.$$

Consider $E = \mathbb{Z}^+ \subseteq (0, \infty)$. Then $f(E) = \left\{ \frac{1}{n} : n \in \mathbb{Z}^+ \right\}$, and thus

$$\begin{aligned} f(\overline{E}) &= \left\{ \frac{1}{n} : n \in \mathbb{Z}^+ \right\}. \\ \overline{f(E)} &= \left\{ \frac{1}{n} : n \in \mathbb{Z}^+ \right\} \cup \{0\}. \end{aligned}$$

□

Supplement (Inverse image).

(1) $E \subseteq f^{-1}[f(E)]$ for $E \subseteq X$.

$$\begin{aligned} \forall x \in E &\implies f(x) \in f(E) \\ &\iff x \in f^{-1}[f(E)]. \quad (\text{Definition of the inverse image}) \end{aligned}$$

□

(2) $f[f^{-1}(E)] \subseteq E$ for $E \subseteq Y$.

$$\begin{aligned} \forall y \in f[f^{-1}(E)] &\iff \exists x \in f^{-1}(E) \text{ such that } y = f(x) \\ &\iff \exists x, f(x) \in E \text{ such that } y = f(x) \\ &\implies \exists x, y = f(x) \in E. \end{aligned}$$

□

Supplement (Continuity). Let f be a map from a topological space on X to a topological space on Y . Then, the following statements are equivalent:

- (1) f is continuous: For each $x \in X$ and every neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$.
- (2) For every open set O in Y , the inverse image $f^{-1}(O)$ is open in X .
- (3) For every closed set C in Y , the inverse image $f^{-1}(C)$ is closed in X .
- (4) $f(A)^\circ \subseteq f(A^\circ)$ for every subset A of X .
- (5) $f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ$ for every subset B of Y .
- (6) $f(\overline{A}) \subseteq \overline{f(A)}$ for every subset A of X .
- (7) $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for every subset B of Y .

Exercise 4.3. Let f be a continuous real function on a metric space X . Let $Z(f)$ (the zero set of f) be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.

Proof (Corollary to Theorem 4.8). Since f is continuous, $f^{-1}(\{0\}) = Z(f)$ is closed in X for a closed subset $\{0\}$ in \mathbb{R}^1 . \square

Denote the complement of any set E by \widetilde{E} .

Proof (Theorem 4.8). Consider the complement of $Z(f)$ in X ,

$$\begin{aligned}\widetilde{Z(f)} &= \{x \in X : f(x) \neq 0\} \\ &= f^{-1}((-\infty, 0) \cup (0, \infty)).\end{aligned}$$

Since f is continuous, $f^{-1}((-\infty, 0) \cup (0, \infty)) = \widetilde{Z(f)}$ is open in X for a open subset $(-\infty, 0) \cup (0, \infty)$ in \mathbb{R}^1 . \square

Proof (Definition 2.18(d)). Given any limit point p of $Z(f)$. Show that $f(p) = 0$ or $p \in Z(f)$. Since f is continuous, given any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(p)| < \varepsilon$ for all $x \in X$ for which $d_X(x, p) < \delta$. Since p is a limit point of $Z(f)$, for such $\delta > 0$ we have a point $q \neq p$ such that $q \in Z(f)$, or $f(q) = 0$. So $|f(p)| < \varepsilon$ for any $\varepsilon > 0$. $f(p) = 0$. \square

Proof (Definition 2.18(f)). Consider the complement of $Z(f)$ in X ,

$$\widetilde{Z(f)} = \{x \in X : f(x) \neq 0\} = \{f > 0\} \cup \{f < 0\}$$

where $\{f > 0\} = \{x \in X : f(x) > 0\}$ and $\{f < 0\} = \{x \in X : f(x) < 0\}$. It suffices to show $\{f > 0\}$ is open. ($\{f < 0\}$ is similar.) Given any point p of $\{f > 0\}$ or $f(p) > 0$. Want to show p is an interior point of $\{f > 0\}$. Since f is continuous, given any $\varepsilon = \frac{f(p)}{2} > 0$ there exists a $\delta > 0$ such that $|f(x) - f(p)| < \frac{f(p)}{2}$ for all $x \in X$ for which $d_X(x, p) < \delta$. For such x with $d_X(x, p) < \delta$ we have

$$\frac{1}{2}f(p) < f(x) < \frac{3}{2}f(p).$$

That is, $N = \{x : d_X(x, p) < \delta\}$ is a neighborhood p such that $N \subseteq \{f > 0\}$. \square

Exercise 4.4. Let f and g be continuous mappings of a metric space X into a metric space Y , and let E be a dense subset of X . Prove that $f(E)$ is dense in $f(X)$. If $g(p) = f(p)$ for all $p \in E$, prove that $g(p) = f(p)$ for all $p \in X$. (In other words, a continuous mapping is determined by its values on a dense subset of its domain.)

Proof.

- (1) Show that $f(E)$ is dense in $f(X)$. It suffices to show that every point $y \in f(X) - f(E)$ is a limit point of $f(E)$. Since $y \in f(X) - f(E)$, there exists a point $x \in X - E$ such that $y = f(x)$. Since E is dense in X , there exists a sequence $\{x_n\}$ in E such that $x_n \rightarrow x$ as $n \rightarrow \infty$. Let $y_n = f(x_n) \in f(E)$. Take limit and use the continuity of f , $y_n \rightarrow y$ as $n \rightarrow \infty$, or y is a limit point of $f(E)$.
- (2) Show that $g(p) = f(p)$ for all $p \in X$ if $g(p) = f(p)$ for all $p \in E$. It suffices to show $g(p) = f(p)$ for all $p \in X - E$. Given any $p \in X - E$, there exists a sequence $\{p_n\}$ in E such that $p_n \rightarrow p$ as $n \rightarrow \infty$. Notice that $g(p_n) = f(p_n)$ by the assumption. Take limit and use the continuity of f and g , $g(p) = f(p)$ for $p \in X - E$.

□

Exercise 4.5. If f is a real continuous function defined on a closed set $E \subseteq \mathbb{R}^1$, prove that there exist continuous real function g on \mathbb{R}^1 such that $g(x) = f(x)$ for all $x \in E$. (Such functions g are called **continuous extensions** of f from E to \mathbb{R}^1 .) Show that the result becomes false if the word “closed” is omitted. Extend the result to vector valued functions. (Hint: Let the graph of g be a straight line on each of the segments which constitute the complement of E (compare Exercise 2.29). The result remains true if \mathbb{R}^1 is replaced by any metric space, but the proof is not so simple.)

Proof.

- (1) Every open set in \mathbb{R}^1 is the union of an at most countable collection of disjoint segments (Exercise 2.29).
- (2) We need to construct a continuous real function on the complement of E . By (1), write $\tilde{E} = \bigcup_{i \in \mathcal{C}} (a_i, b_i)$ where \mathcal{C} is at most countable and $a_i < b_i$. (a_i, b_i could be $\pm\infty$.) Define $g(x)$ by

$$g(x) = \begin{cases} f(x) & (x \in E), \\ f(a_i) + \frac{f(b_i) - f(a_i)}{b_i - a_i}(x - a_i) & (x \in (a_i, b_i) : \text{finite interval}), \\ f(a_i) & (x \in (a_i, b_i) : a_i : \text{finite}, b_i = +\infty), \\ f(b_i) & (x \in (a_i, b_i) : a_i = -\infty, b_i : \text{finite}), \\ 0 & (x \in (a_i, b_i) : a_i = -\infty, b_i = +\infty). \end{cases}$$

Show that g is continuous in \mathbb{R}^1 , or show that $g(x)$ is continuous at $x = p$ for any point $p \in \mathbb{R}^1$.

- (a) Given a point $p \in \tilde{E}$. There is an open interval $I = (a_i, b_i)$ such that $p \in I$. Since the graph of g in an open interval I is a straight line, g is continuous at $x = p$.

- (b) Given an isolated point $p \in E$. There are two open intervals $I = (a_i, b_i)$ and $J = (a_j, b_j)$ such that $b_i = p = a_j$. So $\lim_{x \rightarrow p^-} g(x) = \lim_{x \rightarrow p^+} g(x) = f(p)$ by the construction of g , which says g is continuous at $x = p$.
- (c) Given a limit point $p \in E$. So that $g(p) = f(p)$. Given $\varepsilon > 0$. Consider $\lim_{x \rightarrow p^+} g(x)$ first. (The case $\lim_{x \rightarrow p^-} g(x)$ is similar.)
- (i) For such $\varepsilon > 0$, there is a $\delta' > 0$ such that

$$f(p) - \varepsilon < f(x) < f(p) + \varepsilon$$

whenever

$$x \in E \text{ and } p < x < \delta'.$$

Since p is a limit point of E , there is a point $q \neq p$ such that $|q - p| < \delta'$. Might assume that $q > p$, and then retake $\delta = \min\{\delta', q - p\} > 0$. (If no such q , $\lim_{x \rightarrow p^+} g(x) = f(p)$ trivially.)

- (ii) For any x such that $p < x < q$, consider $x \in E$ or else $x \in \tilde{E}$. As $x \in E$, nothing to do by (i).
- (iii) As $x \in \tilde{E}$, there exists an open interval $I = (a_i, b_i)$ such that $x \in I \subseteq (p, q)$. Therefore,

$$f(a_i) \leq g(x) \leq f(b_i) \text{ or } f(a_i) \geq g(x) \geq f(b_i).$$

By (i),

$$\begin{aligned} f(p) - \varepsilon &< f(a_i) < f(p) + \varepsilon \text{ and} \\ f(p) - \varepsilon &< f(b_i) < f(p) + \varepsilon, \\ f(p) - \varepsilon &< f(a_i) \leq g(x) \leq f(b_i) < f(p) + \varepsilon \text{ or} \\ f(p) - \varepsilon &< f(b_i) \leq g(x) \leq f(a_i) < f(p) + \varepsilon. \end{aligned}$$

Hence, given $\varepsilon > 0$ there is a $\delta > 0$ such that $|g(x) - g(p)| < \varepsilon$ whenever $p < x < \delta$ (and $x \in \mathbb{R}^1$), or $\lim_{x \rightarrow p^+} g(x) = g(p)$.

- (3) Consider $f(x) = \log(x)$ in $(0, \infty)$. Since $\lim_{x \rightarrow 0} f(x) = -\infty$, we cannot find any real continuous function g defined on $x = 0$.
- (4) For a vector-valued function $\mathbf{f} = (f_1, \dots, f_k)$, with each f_i is continuous on a closed set $E \subseteq \mathbb{R}^1$, extend f_i to a continuous function g_i on \mathbb{R}^1 as (2). Put $\mathbf{g} = (g_1, \dots, g_k)$. Clearly \mathbf{g} is an extension of \mathbf{f} . Besides, \mathbf{g} is continuous in \mathbb{R}^1 by Theorem 4.10.

□

Supplement (Tietze's Extension Theorem). *If X is a normal topological space and $f : A \rightarrow \mathbb{R}$ is a continuous map from a closed subset A of X into the real numbers carrying the standard topology, then there exists a continuous map $g : X \rightarrow \mathbb{R}$ with $g(a) = f(a)$ for all $a \in A$.*

Exercise 4.6. If f is defined on E , the graph of f is the set of points $(x, f(x))$, for $x \in E$. In particular, if E is a set of real numbers, and f is real-valued, the graph of f is a subset of the plane. Suppose E is compact, and prove that f is continuous on E if and only if its graph is compact.

Proof. Let $G = \{(x, f(x)) : x \in E\}$ be the graph of f .

(1) (\implies) Let $\mathbf{f} : E \rightarrow G$ defined by

$$\mathbf{f}(x) = (x, f(x)).$$

$\mathbf{f}(E) = G$ exactly. Since f and x are continuous in E , \mathbf{f} is continuous (Theorem 4.10). As E is compact, $\mathbf{f}(E)$ is compact (Theorem 4.14).

(2) (\impliedby) Let $\pi : G \rightarrow E$ be a projection map defined by

$$\pi(x, f(x)) = x.$$

Notice that $\pi \circ \mathbf{f} = \text{id}_E$ and $\mathbf{f} \circ \pi = \text{id}_G$. Besides, π is a continuous one-to-one mapping of a compact set G onto E . Then the inverse mapping $\pi^{-1} = \mathbf{f}$ is a continuous mapping of E onto G (Theorem 4.17). So f is continuous (Theorem 4.10).

□

Exercise 4.7. If $E \subseteq X$ and if f is a function defined on X , the **restriction** of f to E is the function g whose domain of definition is E , such that $g(p) = f(p)$ for $p \in E$. Define f and g on \mathbb{R}^2 by:

$$f(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0) \\ \frac{xy^2}{x^2+y^4} & \text{if } (x, y) \neq (0, 0), \end{cases}$$

$$g(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0) \\ \frac{xy^2}{x^2+y^6} & \text{if } (x, y) \neq (0, 0), \end{cases}$$

Prove that f is bounded on \mathbb{R}^2 , that g is unbounded in every neighborhood of $(0, 0)$, and that f is not continuous at $(0, 0)$; nevertheless, the restrictions of both f and g to every straight line in \mathbb{R}^2 are continuous!

Proof.

(1) Show that f is bounded on \mathbb{R}^2 .

$$\begin{aligned}
(|x| - |y^2|)^2 \geq 0 &\iff |x|^2 - 2|x||y^2| + |y^2|^2 \geq 0 \\
&\iff |x|^2 + |y^2|^2 \geq 2|x||y^2| \\
&\iff |x^2 + y^4| \geq 2|xy^2| \\
&\implies \frac{1}{2} \geq \left| \frac{xy^2}{x^2 + y^2} \right| \text{ whenever } (x, y) \neq (0, 0) \\
&\implies |f(x, y)| \leq \frac{1}{2} \text{ whenever } (x, y) \neq (0, 0).
\end{aligned}$$

Note that $f(0, 0) = 0 \leq \frac{1}{2}$. Hence f is bounded by $\frac{1}{2}$ on \mathbb{R}^2 .

(2) Show that g is unbounded in every neighborhood of \mathbb{R}^2 . Consider a sequence $\{\mathbf{p}_n\}_{n \geq 1} \subseteq \mathbb{R}^2$

$$\mathbf{p}_n = (x_n, y_n) = \left(\frac{1}{n^3}, \frac{1}{n} \right)$$

such that $\mathbf{p}_n \neq \mathbf{0}$ and $\lim \mathbf{p}_n = \mathbf{0}$. Thus,

$$\lim_{n \rightarrow \infty} g(\mathbf{p}_n) = \lim_{n \rightarrow \infty} \frac{x_n y_n^2}{x_n^2 + y_n^6} = \lim_{n \rightarrow \infty} \frac{(\frac{1}{n^3})(\frac{1}{n})^2}{(\frac{1}{n^3})^2 + (\frac{1}{n})^6} = \lim_{n \rightarrow \infty} \frac{n}{2} = \infty.$$

Hence g is unbounded in every neighborhood of \mathbb{R}^2 .

(3) Show that f is not continuous at $(0, 0)$. Consider a sequence $\{\mathbf{p}_n\}_{n \geq 1} \subseteq \mathbb{R}^2$

$$\mathbf{p}_n = (x_n, y_n) = \left(\frac{1}{n^2}, \frac{1}{n} \right)$$

such that $\mathbf{p}_n \neq \mathbf{0}$ and $\lim \mathbf{p}_n = \mathbf{0}$. Thus,

$$\lim_{n \rightarrow \infty} f(\mathbf{p}_n) = \lim_{n \rightarrow \infty} \frac{x_n y_n^2}{x_n^2 + y_n^4} = \lim_{n \rightarrow \infty} \frac{(\frac{1}{n^2})(\frac{1}{n})^2}{(\frac{1}{n^2})^2 + (\frac{1}{n})^4} = \frac{1}{2}.$$

So, $\lim f(\mathbf{p}_n) = \frac{1}{2} \neq 0$. By Theorem 4.6, f is not continuous at $(0, 0)$.

(4) The restrictions of f to every straight line in \mathbb{R}^2 is continuous.

- (a) The line $L_\infty = \{(0, y) : y \in \mathbb{R}\}$. Hence $f|_{L_\infty}(x, y) = 0$ for all $(x, y) \in L_\infty$ (including $(0, 0) \in L_\infty$). Therefore $f|_{L_\infty}$ is continuous.
- (b) The line $L_\alpha = \{(x, \alpha x) : x \in \mathbb{R}\}$ for some $\alpha \in \mathbb{R}$. $f|_{L_\alpha}(x, y)$ is continuous on $L_\alpha - \{(0, 0)\}$.

$$f|_{L_\alpha}(x, y) = f|_{L_\alpha}(x, \alpha x) = \begin{cases} 0 & \text{if } (x, y) = (0, 0) \\ \frac{\alpha^2 x}{1 + \alpha^4 x^2} & \text{if } (x, y) \neq (0, 0). \end{cases}$$

So

$$\lim_{(x, y) \rightarrow (0, 0)} f|_{L_\alpha}(x, y) = \lim_{x \rightarrow 0} \frac{\alpha^2 x}{1 + \alpha^4 x^2} = 0 = f(0, 0),$$

- or $f|_{L_\alpha}(x, y)$ is continuous at $(0, 0)$. Therefore, $f|_{L_\alpha}(x, y)$ is continuous on L_α .
- (c) *The line L not passing $(0, 0)$.* It is clear since $f(x, y)$ is continuous on $\mathbb{R}^2 - \{(0, 0)\}$.
- (5) *The restrictions of g to every straight line in \mathbb{R}^2 is continuous.* Similar to (4).
- (a) *The line $L_\infty = \{(0, y) : y \in \mathbb{R}\}$.* Hence $g|_{L_\infty}(x, y) = 0$ for all $(x, y) \in L_\infty$ (including $(0, 0) \in L_\infty$). Therefore $g|_{L_\infty}$ is continuous.
- (b) *The line $L_\alpha = \{(x, \alpha x) : x \in \mathbb{R}\}$ for some $\alpha \in \mathbb{R}$.* $g|_{L_\alpha}(x, y)$ is continuous on $L_\alpha - \{(0, 0)\}$.

$$g|_{L_\alpha}(x, y) = g|_{L_\alpha}(x, \alpha x) = \begin{cases} 0 & \text{if } (x, y) = (0, 0) \\ \frac{\alpha^2 x}{1 + \alpha^6 x^4} & \text{if } (x, y) \neq (0, 0). \end{cases}$$

So

$$\lim_{(x, y) \rightarrow (0, 0)} g|_{L_\alpha}(x, y) = \lim_{x \rightarrow 0} \frac{\alpha^2 x}{1 + \alpha^6 x^4} = 0 = g(0, 0),$$

or $g|_{L_\alpha}(x, y)$ is continuous at $(0, 0)$. Therefore, $g|_{L_\alpha}(x, y)$ is continuous on L_α .

- (c) *The line L not passing $(0, 0)$.* It is clear since $g(x, y)$ is continuous on $\mathbb{R}^2 - \{(0, 0)\}$.

□

Exercise 4.8. Let f be a real uniformly continuous function on the bounded set E in \mathbb{R} . Prove that f is bounded on E . Show that the conclusion is false if boundedness of E is omitted from the hypothesis.

The conclusion is false if boundedness of E is omitted from the hypothesis. For example, $f(x) = x$ on \mathbb{R} is uniformly continuous on \mathbb{R} but $f(\mathbb{R}) = \mathbb{R}$ is unbounded.

Proof (Brute-force).

- (1) Since $f : E \rightarrow \mathbb{R}$ is uniformly continuous, given any $\varepsilon > 0$, there is $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$. In particular, pick $\varepsilon = 1$.
- (2) By the boundedness of E , there is $M > 0$ such that $|x| < M$ for all $x \in E$.
- (3) For such $\delta > 0$, we construct a covering of $E \subseteq \mathbb{R}$. Construct a special collection \mathcal{C} of intervals

$$I_a = \left[\frac{\delta}{2}a, \frac{\delta}{2}(a + 1) \right]$$

where $a \in \mathbb{Z}$ satisfying

$$|a| < \frac{2M}{\delta} + 1.$$

By construction, \mathcal{C} is a finite covering of E .

- (4) For every interval I_a of the collection \mathcal{C} , pick a point $x_a \in E \cap I_a$ if possible. This process will terminate eventually since \mathcal{C} is finite. Collect these representative points as $\mathcal{D} = \{x_a\}$. Notice that \mathcal{D} is finite again.
- (5) Now for any point $x \in E$, x lies in some I_a containing x_a . Both x and x_a are in the same interval and their distance satisfies

$$|x - x_a| \leq \frac{\delta}{2} < \delta$$

and thus by (1)

$$|f(x) - f(x_a)| < 1, \text{ or } |f(x)| < 1 + |f(x_a)|.$$

- (6) Let

$$M = 1 + \max_{x_a \in \mathcal{D}} |f(x_a)|.$$

So given any $x \in E$, $|f(x)| < M$.

□

Proof (Heine-Borel Theorem). Heine-Borel theorem provides the finiteness property to construct the boundedness property of f .

- (1) Let E be a bounded subset of a metric space X . Show that the closure of E in X is also bounded in X . E is bounded if $E \subseteq B_X(a; r)$ for some $r > 0$ and some $a \in X$. (The ball $B_X(a; r)$ is defined to the set of all $x \in X$ such that $d_X(x, a) < r$.) Take the closure on the both sides,

$$\overline{E} \subseteq \overline{B_X(a; r)} = \{x \in X : d_X(x, a) \leq r\} \subseteq B_X(a; 2r),$$

or \overline{E} is bounded.

- (2) Since $f : E \rightarrow \mathbb{R}$ is uniformly continuous, given any $\varepsilon > 0$, there is $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$. In particular, pick $\varepsilon = 1$.
- (3) For such $\delta > 0$, we construct an open covering of $\overline{E} \subseteq \mathbb{R}$. Pick a collection \mathcal{C} of open balls $B(a; \delta) \subseteq \mathbb{R}$ where a runs over all elements of E . \mathcal{C} covers \overline{E} (by the definition of accumulation points). Since \overline{E} is closed and bounded (by applying (1) on the boundedness of E), \overline{E} is compact (Heine-Borel theorem). That is, there is a finite subcollection \mathcal{C}' of \mathcal{C} also covers \overline{E} , say

$$\mathcal{C}' = \{B(a_1; \delta), B(a_2; \delta), \dots, B(a_m; \delta)\}.$$

- (4) Given any $x \in E \subseteq \overline{E}$, there is some $a_i \in E$ ($1 \leq i \leq m$) such that $x \in B(a_i; \delta)$. In such ball, $|x - a_i| < \delta$. By (2), $|f(x) - f(a_i)| < 1$, or $|f(x)| < 1 + |f(a_i)|$. Almost done. Notice that a_i depends on x , and thus we might use finiteness of $\{a_1, a_2, \dots, a_m\}$ to remove dependence of a_i .

- (5) Let

$$M = 1 + \max_{1 \leq i \leq m} |f(a_i)|.$$

So given any $x \in E$, $|f(x)| < M$.

□

Supplement. Exercise about considering the closure. (Problem 3.5 in H. L. Royden, Real Analysis, 3rd Edition.) Let $A = \mathbb{Q} \cap [0, 1]$, and let $\{I_n\}$ be a finite collection of open intervals covering A . Then $\sum l(I_n) \geq 1$.

Proof.

$$\begin{aligned} 1 = m^*[0, 1] &= m^*\overline{A} \leq m^*\left(\overline{\bigcup I_n}\right) = m^*\left(\bigcup \overline{I_n}\right) \\ &\leq \sum m^*(\overline{I_n}) = \sum l(\overline{I_n}) = \sum l(I_n). \end{aligned}$$

□

Exercise 4.14. Let $I = [0, 1]$ be the closed unit interval. Suppose f is continuous mapping of I into I . Prove that $f(x) = x$ for at least one $x \in I$.

Proof (Theorem 4.23). Let $g(x) = f(x) - x$ in I .

- (1) $g(0) = 0$. Take $x = 0$.
- (2) $g(1) = 0$. Take $x = 1$.
- (3) Suppose $g(0) \neq 0$ ($f(0) \neq 0$) and $g(1) \neq 0$ ($f(1) \neq 1$). Since $f : I \rightarrow I$, $f(0) > 0$ and $f(1) < 1$. That is, $g(0) > 0$ and $g(1) < 0$. Applying the intermediate value theorem (Theorem 4.23), there is a point in $\xi \in (0, 1)$ such that $g(\xi) = 0$. That is, $f(\xi) = \xi$ for some $\xi \in (0, 1)$.

In any case, the conclusion holds. □

Exercise 4.16. Let $[x]$ denote the largest integer contained in x , this is, $[x]$ is a integer such that $x - 1 < [x] \leq x$; and let $(x) = x - [x]$ denote the fractional part of x . What discontinuities do the function $[x]$ and (x) have?

Proof.

- (1) The function $[x]$ only has discontinuities at $x \in \mathbb{Z}$.
- (a) For any $p \notin \mathbb{Z}$, there is an integer n such that $n < p < n + 1$. Given any $\varepsilon > 0$, there is a $\delta = \min\{p - n, (n + 1) - p\} > 0$ such that $|[x] - [p]| < \varepsilon$ whenever $|x - p| < \delta$. In fact, $|x - p| < \delta$ is equivalent to $n < x < n + 1$ and therefore $[x] - [p] = |n - n| = 0 < \varepsilon$.
- (b) For any $p \in \mathbb{Z}$, $\lim_{x \rightarrow p^+} [x] = p$ and $\lim_{x \rightarrow p^-} [x] = p - 1$.
- (2) The function (x) only has discontinuities at $x \in \mathbb{Z}$.
- (a) Since $[x]$ is continuous on $\mathbb{R} - \mathbb{Z}$ and x is continuous on \mathbb{R} , especially on $\mathbb{R} - \mathbb{Z}$, $(x) = x - [x]$ is continuous on $\mathbb{R} - \mathbb{Z}$.
- (b) For any $p \in \mathbb{Z}$, $\lim_{x \rightarrow p^+} (x) = 0$ and $\lim_{x \rightarrow p^-} (x) = 1$.

□

Exercise 4.23. A real-valued function f defined in (a, b) is said to be **convex** if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

whenever $a < x < b$, $a < y < b$, $0 < \lambda < 1$. Prove that every convex function is continuous. Prove that every increasing convex function of a convex function is convex. (For example, if f is convex, so is e^f .)

If f is convex in (a, b) and if $a < s < t < u < b$, show that

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

Proof.

- (1) Show that $\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}$. Since

$$\begin{aligned} t &= \frac{t - s}{u - s}u + \left(1 - \frac{t - s}{u - s}\right)s \\ &= \left(1 - \frac{u - t}{u - s}\right)u + \frac{u - t}{u - s}s \end{aligned}$$

and $0 < \frac{t - s}{u - s}, \frac{u - t}{u - s} < 1$, by the convexity of f we have

$$\begin{aligned} f(t) &\leq \frac{t - s}{u - s}f(u) + \left(1 - \frac{t - s}{u - s}\right)f(s), \\ f(t) &\leq \left(1 - \frac{u - t}{u - s}\right)f(u) + \frac{u - t}{u - s}f(s). \end{aligned}$$

It is equivalent to

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

□

- (2) If x, y, x', y' are points of (a, b) with $x \leq x' < y'$ and $x < y \leq y'$, then the chord over (x', y') has larger slope than the chord over (x, y) ; that is,

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(y') - f(x')}{y' - x'}.$$

It is a corollary to (1).

- (3) Show that f is continuous. Let $[c, d] \subseteq (a, b)$. Then by (2),

$$\frac{f(c) - f(a)}{c - a} \leq \frac{f(y) - f(x)}{y - x} \leq \frac{f(b) - f(d)}{b - d}$$

for x, y in $[c, d]$. Thus $|f(y) - f(x)| \leq M|y - x|$ in $[c, d]$ (where $M = \max\left(\left|\frac{f(c)-f(a)}{c-a}\right|, \left|\frac{f(b)-f(d)}{b-d}\right|\right)$), and so f is absolutely continuous on each closed subinterval of (a, b) . Especially, f is continuous.

- (4) Let f be a convex function, g be an increasing convex function, and $h = g \circ f$. Show that h is convex.

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y), \quad (\text{Convexity of } f)$$

$$g(f(\lambda x + (1 - \lambda)y)) \leq g(\lambda f(x) + (1 - \lambda)f(y)) \quad (\text{Increasing of } g)$$

$$\leq \lambda g(f(x)) + (1 - \lambda)g(f(y)), \quad (\text{Convexity of } g)$$

$$h(\lambda x + (1 - \lambda)y) \leq \lambda h(x) + (1 - \lambda)h(y).$$

□

Exercise 4.24. Assume that f is a continuous real function defined in (a, b) such that

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2}$$

for all $x, y \in (a, b)$. Prove that f is convex.

Proof.

- (1) Show that

$$f\left(\frac{x_1 + \cdots + x_n}{n}\right) \leq \frac{f(x_1) + \cdots + f(x_n)}{n}$$

whenever $a < x_i < b$ ($1 \leq i \leq n$). Apply Cauchy induction and use the same argument in proving the AM-GM inequality. As $n = 1, 2$, the inequality holds by assumption. Suppose $n = 2^k$ ($k \geq 1$) the inequality holds. As $n = 2^{k+1}$,

$$\begin{aligned}
& f\left(\frac{x_1 + \cdots + x_{2^{k+1}}}{2^{k+1}}\right) \\
&= f\left(\frac{1}{2}\left(\frac{x_1 + \cdots + x_{2^k}}{2^k} + \frac{x_{2^k+1} + \cdots + x_{2^{k+1}}}{2^k}\right)\right) \\
&\leq \frac{1}{2}\left(f\left(\frac{x_1 + \cdots + x_{2^k}}{2^k}\right) + f\left(\frac{x_{2^k+1} + \cdots + x_{2^{k+1}}}{2^k}\right)\right) \\
&\leq \frac{1}{2}\left(\frac{f(x_1) + \cdots + f(x_{2^k})}{2^k} + \frac{f(x_{2^k+1}) + \cdots + f(x_{2^{k+1}})}{2^k}\right) \\
&= \frac{f(x_1) + \cdots + f(x_{2^k}) + f(x_{2^k+1}) + \cdots + f(x_{2^{k+1}})}{2^{k+1}} \\
&= \frac{f(x_1) + \cdots + f(x_{2^{k+1}})}{2^{k+1}}.
\end{aligned}$$

As n is not a power of 2, then it is certainly less than some natural power of 2, say $n < 2^m$ for some m . Let

$$x_{n+1} = \cdots = x_{2^m} = \frac{x_1 + \cdots + x_n}{n} = \alpha.$$

Then by the induction hypothesis,

$$\begin{aligned}
f(\alpha) &= f\left(\frac{x_1 + \cdots + x_n + \alpha + \cdots + \alpha}{2^m}\right) \\
&\leq \frac{f(x_1) + \cdots + f(x_n) + f(\alpha) + \cdots + f(\alpha)}{2^m} \\
&\leq \frac{f(x_1) + \cdots + f(x_n) + (2^m - n)f(\alpha)}{2^m}, \\
2^m f(\alpha) &\leq f(x_1) + \cdots + f(x_n) + (2^m - n)f(\alpha), \\
nf(\alpha) &\leq f(x_1) + \cdots + f(x_n),
\end{aligned}$$

$$\text{or } f\left(\frac{1}{n}(x_1 + \cdots + x_n)\right) \leq \frac{1}{n}(f(x_1) + \cdots + f(x_n)).$$

(2) Hence,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for any rational λ in $(0, 1)$. (Given any positive integers $p < q$, put $n = q$, $x_1 = \cdots = x_p = x$ and $x_{p+1} = \cdots = x_n = y$ in (1).)

(3) Given any real $\lambda \in (0, 1)$, there is a sequence of rational numbers $\{r_n\} \subseteq (0, 1)$ such that $r_n \rightarrow \lambda$. By (2),

$$f(r_n x + (1 - r_n)y) \leq r_n f(x) + (1 - r_n)f(y)$$

for any rational r_n in $(0, 1)$. Taking limit on the both sides and using the continuity of f , we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

□

Proof (Reductio ad absurdum). If f were not convex, then there is a subinterval $[c, d] \subseteq (a, b)$ such that

$$\frac{f(d) - f(c)}{d - c} < \frac{f(x_0) - f(c)}{x_0 - c}$$

for some $x_0 \in [c, d]$. Let

$$g(x) = f(x) - f(c) - \frac{f(d) - f(c)}{d - c}(x - c)$$

for $x \in [c, d]$. Therefore,

- (1) $g(x)$ is continuous and midpoint convex.
- (2) $g(c) = g(d) = 0$.
- (3) Let $M = \sup\{g(x) : x \in [c, d]\}$. $\infty > M > 0$ due to the continuity of g and the existence of x_0 . And let $\xi = \inf\{x \in [c, d] : g(x) = M\}$. By the continuity of g , $g(\xi) = M$. $\xi \in (c, d)$ by (2).
- (4) Since (c, d) is open, there is $h > 0$ such that $(\xi - h, \xi + h) \subseteq (c, d)$. By the minimality of ξ and M , $g(\xi - h) < g(\xi)$ and $g(\xi + h) \leq g(\xi)$.

Therefore,

$$\begin{aligned} g(\xi - h) + g(\xi + h) &< 2g(\xi), \\ \frac{g(\xi - h) + g(\xi + h)}{2} &< g(\xi) \\ &= g\left(\frac{(\xi - h) + (\xi + h)}{2}\right), \end{aligned}$$

contrary to the midpoint convexity of g . □

The result becomes false if “continuity of f ” is omitted.

Exercise 4.25. If $A \subset \mathbb{R}^k$ and $B \subset \mathbb{R}^k$, define $A + B$ to be the set of all sums $\mathbf{x} + \mathbf{y}$ with $\mathbf{x} \in A$, $\mathbf{y} \in B$.

- (a) If K is compact and C is closed in \mathbb{R}^k , prove that $K + C$ is closed. (Hint: Take $\mathbf{z} \notin K + C$, put $F = \mathbf{z} - C$, the set of all $\mathbf{z} - \mathbf{y}$ with $\mathbf{y} \in C$. Then K and F are disjoint. Choose δ as in Exercise 4.21. Show that the open ball with center \mathbf{z} and radius δ does not intersect $K + C$.)

- (b) Let α be an irrational real number. Let C_1 be the set of all integers, let C_2 be the set of all $n\alpha$ with $n \in C_1$. Show that C_1 and C_2 are closed subsets of \mathbb{R}^1 whose sum $C_1 + C_2$ is not closed, by showing that $C_1 + C_2$ is a countable dense subset of \mathbb{R}^1 .

Proof. TODO.

Exercise 4.26. Suppose X, Y, Z are metric spaces, and Y is compact. Let f map X into Y , let g be a continuous one-to-one mapping of Y into Z , and put $h(x) = g(f(x))$ for $x \in X$.

Prove that f is uniformly continuous if h is uniformly continuous. (Hint: g^{-1} has compact domain $g(Y)$, and $f(x) = g^{-1}(h(x))$.)

Prove also that f is continuous if h is continuous.

Show (by modifying Example 4.21, or by finding a different example) that the compactness of Y cannot be omitted from the hypotheses, even when X and Z are compact.

Proof. TODO.