Chapter 1: Rings and Ideals

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Exercise 1.1. *Let x be a nilpotent element of A. Show that $1+x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit.*

Proof.

(1) Suppose $x^m = 0$ for some odd integer $m \geq 0$. Then
$$1 = 1 + x^m = (1+x)(1 - x + x^2 - \cdots + (-1)^{m-1}x^{m-1}),$$
or $1+x$ is a unit.

(2) If u is any unit and x is any nilpotent, $u + x = u \cdot (1 + u^{-1}x)$ is a product of two units (using that $u^{-1}x$ is nilpotent and applying (1)) and hence a unit again.

□

Proof (Proposition 1.9).

(1) *The nilradical is a subset of the Jacobson radical.*
   (a) The nilradical $\mathfrak{N}$ of A is the intersection of all the prime ideals of A by Proposition 1.8.
   (b) The Jacobson radical $\mathfrak{J}$ of A is the intersection of all the maximal ideals of A by definition.

(2) By Proposition 1.9, $x \in \mathfrak{J}$ if and only if $1 - xy$ is a unit in A for all $y \in A$. So $1 + x = 1 - (-x) \cdot 1$ is a unit in A since x is a nilpotent and $\mathfrak{J}$ is an ideal.

□

Exercise 1.2. *Let A be a ring and let $A[x]$ be the ring of polynomials in an indeterminate x, with coefficients in A. Let $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$. Prove that*

(i) *f is a unit in $A[x]$ if and only if a_0 is a unit in A and $a_1, ..., a_n$ are nilpotent. (Hint: If $b_0 + b_1x + \cdots + b_mx^m$ is the inverse of f, prove by induction on r that $a_n^{r+1}b_{m-r} = 0$. Hence show that a_n is nilpotent, and then use Exercise 1.1.)*

(ii) *f is nilpotent if and only if $a_0, a_1, ..., a_n$ are nilpotent.*

(iii) *f is a zero-divisor if and only if there exists $a \neq 0$ such that $af = 0$. (Hint: Choose a polynomial $g = b_0 + b_1 x + \cdots + b_m x^m$ of least degree m such that $fg = 0$. Then $a_n b_m = 0$, hence $a_n g = 0$ (because $a_n g$ annihilates f and has degree $< m$). Now show by induction that $a_{n-r} g = 0$ $(0 \leq r \leq n)$.)*

(iv) *f is said to be primitive if $(a_0, a_1, ..., a_n) = (1)$. Prove that if $f, g \in A[x]$, then fg is primitive if and only if f and g are primitive.*

Proof of (i).

(1) ($\Longleftarrow$) holds by Exercise 1.1.

(2) ($\Longrightarrow$) There exists the inverse g of f, say $g = b_0 + b_1 x + \cdots + b_m x^m$ satisfying $1 = fg$. Clearly, $1 = a_0 b_0$, or a_0 is a unit in A. Also,

$$\begin{aligned} 0 &= a_n b_m, \\ 0 &= a_n b_{m-1} + a_{n-1} b_m, \\ 0 &= a_n b_{m-2} + a_{n-1} b_{m-1} + a_{n-2} b_m, \\ &\cdots \end{aligned}$$

A direct computing shows that

$$\begin{aligned} 0 &= a_n^1 b_m, \\ 0 &= a_n(a_n b_{m-1} + a_{n-1} b_m) \\ &= a_n^2 b_{m-1} + a_{n-1} a_n b_m \\ &= a_n^2 b_{m-1}, \\ 0 &= a_n^2(a_n b_{m-2} + a_{n-1} b_{m-1} + a_{n-2} b_m) \\ &= a_n^3 b_{m-2} + a_{n-1} a_n^2 b_{m-1} + a_{n-2} a_n^2 b_m \\ &= a_n^3 b_{m-2}, \\ &\cdots \end{aligned}$$

So we might have $a_n^{r+1} b_{m-r} = 0$ for $r = 0, 1, 2, ..., m$.

(3) *Show that $a_n^{r+1} b_{m-r} = 0$ for $r = 0, 1, 2, ..., m$ by induction on r.*

(a) As $r = 0$, $a_n b_m = 0$ by comparing the coefficient of $fg = 1$ at x^{n+m}.

(b) For any $r > 0$, comparing the coefficient of $fg = 1$ at x^{n+m-r},

$$0 = a_n b_{m-r} + a_{n-1} b_{m-r+1} + \cdots + a_{n-r} b_m.$$

Multiplying by a_n^r on the both sides,

$$\begin{aligned} 0 &= a_n^{r+1} b_{m-r} + a_{n-1} a_n^r b_{m-r+1} + \cdots + a_{n-r} a_n^r b_m \\ &= a_n^{r+1} b_{m-r}. \end{aligned}$$

by the induction hypothesis.

(4) a_n *is a nilpotent.* Putting $r = m$ in $a_n^{r+1} b_{m-r} = 0$ and get $a_n^{m+1} b_0 = 0$. Notice that b_0 is a unit, $a_n^{m+1} = 0$, or a_n is a nilpotent.

(5) Consider $f - a_n x^n = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$, a polynomial $\in A[x]$ of degree $n-1$. Note that f is a unit and $a_n x^n$ is a nilpotent. By Exercise 1.1, $f - a_n x^n$ is a unit too. Applying the (2)(3)(4) again, a_{n-1} is a nilpotent as $n - 1 > 0$, that is, applying descending induction on n then yields the desired property.

□

Proof of (ii).

(1) ($\Longleftarrow$) holds since the nilradical of any ring is an ideal.

(2) ($\Longrightarrow$) $f^N = 0$ for some $N > 0$. So $0 = f^N = a_n^N x^{nN} + \cdots + a_0^N$. Comparing the coefficient in the leading term x^{nN} leads to $a_n^N = 0$, or a_n is a nilpotent.

(3) Consider $f - a_n x^n = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$, a polynomial $\in A[x]$ of degree $n-1$. Note that f and $a_n x^n$ are nilpotent. $f - a_n x^n$ is a nilpotent too. Similar to step (5) in the proof of (i), applying descending induction on n then yields the desired property.

□

Proof of (iii).

(1) ($\Longleftarrow$) holds trivially.

(2) ($\Longrightarrow$) Pick a polynomial $g = b_0 + b_1 x + \cdots + b_m x^m$ of least degree m such that $fg = 0$. Especially, $a_n b_m = 0$.

(3) Consider

$$\begin{aligned} a_n g &= a_n b_0 + \cdots + a_n b_{m-1} x^{m-1} + a_n b_m x^m \\ &= a_n b_0 + \cdots + a_n b_{m-1} x^{m-1} \end{aligned}$$

(since $a_n b_m = 0$). $a_n g$ is a polynomial over A of having degree strictly less than m. Notice that $f \cdot (a_n g) = a_n \cdot (fg) = 0$. By minimality of m, $a_n g = 0$.

(4) Induction on the degree n of f.

(a) As $n = 0$, $f = a_0$. There exists $b_m \neq 0$ such that $b_m f = b_m a_0 = 0$ by (2).

(b) For any zero-divisor f of degree n, there is a polynomial $g = b_0 + b_1 x + \cdots + b_m x^m$ of least degree m such that $fg = 0$. By (2)(3),

$$\begin{aligned} (f - a_n x^n) \cdot g &= fg - a_n x^n g \\ &= 0 - 0 \\ &= 0. \end{aligned}$$

That is, $f - a_n x^n$ is a zero-divisor of degree $n-1$. By the induction hypothesis, there exists $b_m \neq 0$ such that $b_m(f - a_n x^n) = 0$. So $b_m f = b_m(f - a_n x^n) + b_m a_n x^n = 0 + 0 = 0$.

(c) By (a)(b), ($\Longrightarrow$) holds by mathematical induction.

$\square$

Proof of (iv). Note that

(1) $f \notin \mathfrak{m}[x]$ for any maximal ideal $\mathfrak{m}$ of A if and only if f is primitive.

(2) For any maximal ideal $\mathfrak{m}$ of A, $A/\mathfrak{m}$ is a field (or an integral domain).

(3) $A[x]$ is an integral domain if A is an integral domain.

(4) $A[x]/\mathfrak{m}[x] \cong (A/\mathfrak{m})[x]$ as a ring isomorphism.

Hence,

$$\begin{aligned} f, g: \text{ primitive} &\Longleftrightarrow f, g \notin \mathfrak{m}[x] \text{ for any maximal ideal } \mathfrak{m} \\ &\Longleftrightarrow f, g \neq 0 \text{ in } (A/\mathfrak{m})[x] \text{ for any maximal ideal } \mathfrak{m} \\ &\Longleftrightarrow fg \neq 0 \text{ in } (A/\mathfrak{m})[x] \text{ for any maximal ideal } \mathfrak{m} \\ &\Longleftrightarrow fg \notin \mathfrak{m}[x] \text{ for any maximal ideal } \mathfrak{m} \\ &\Longleftrightarrow fg: \text{ primitive.} \end{aligned}$$

$\square$

Exercise 1.4. *In the ring $A[x]$, the Jacobson radical is equal to the nilradical.*

Proof.

(1) The nilradical $\mathfrak{N}$ is a subset of the Jacobson radical $\mathfrak{J}$. It suffices to show that $\mathfrak{J} \subseteq \mathfrak{N}$.

(2) Given any $f \in \mathfrak{J}$. By Proposition 1.9, $f \in \mathfrak{J}$ if and only if $1 - fy$ is a unit in $A[x]$ for all $y \in A[x]$. Especially, pick $y = x \in A[x]$ and then $1 - xf$ is a unit in $A[x]$.

(3) By Exercise 1.2 (i), all coefficients of f are nilpotent. By Exercise 1.2 (ii), f is nilpotent, or $f \in \mathfrak{N}$.

$\square$

Exercise 1.7. *Let A be a ring in which every element satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal in A is maximal.*

Proof. It suffices to show that *for any prime ideal $\mathfrak{p}$ in A, $A/\mathfrak{p}$ is a field.*

(1) Take any $0 \neq \overline{x} \in A/\mathfrak{p}$, which is represented by $x \in A - \mathfrak{p}$. By assumption there exists $n \geq 2$ such that $x^n = x$. So $\overline{x}^n = \overline{x}$ or $\overline{x}(\overline{x}^{n-1} - 1) = 0$.

(2) Since $\mathfrak{p}$ is prime, $A/\mathfrak{p}$ is a integral domain. That is, $\overline{x} = 0$ (impossible) or $\overline{x}^{n-1} - 1 = 0$. Write $\overline{x} \cdot \overline{x}^{n-2} = 1$ in $A/\mathfrak{p}$. So $\overline{x}^{n-2}$ is an inverse of $\overline{x} \neq 0$ in $A/\mathfrak{p}$, which implies that $A/\mathfrak{p}$ is a field (since $\overline{x}$ is arbitrary).

(3) $A/\mathfrak{p}$ is a field if and only if $\mathfrak{p}$ is maximal.

□

Exercise 1.8. *Let A be a ring $\neq 0$. Show that the set of prime ideals of A has minimal elements with respect to inclusion.*

Similar to Theorem 1.3.

Proof (Zorn's Lemma).

(1) Let Σ be the set of all prime ideals of A.

(2) Order Σ by $\supseteq$, that is, $\mathfrak{p} \leq \mathfrak{q}$ if $\mathfrak{p} \supseteq \mathfrak{q}$.

(3) Σ is not empty, since every ring $A \neq 0$ has at least one maximal ideal (or prime ideal) (Theorem 1.3).

(4) To apply Zorn's lemma we must show that every chain in Σ has a lower bound in Σ; let then $(\mathfrak{p}_\alpha)$ be a chain of prime ideals in Σ, so that for each pair of indices α, β we have either $\mathfrak{p}_\alpha \subseteq \mathfrak{p}_\beta$ or $\mathfrak{p}_\beta \subseteq \mathfrak{p}_\alpha$. Let $\mathfrak{p} = \bigcap_\alpha \mathfrak{p}_\alpha$.

(5) *Show that $\mathfrak{p}$ is a prime ideal.* Clearly $\mathfrak{p}$ is an ideal. Given any $xy \in \mathfrak{p}$ and $x \notin \mathfrak{p}$. So xy is in all prime ideals $\mathfrak{p}_\alpha$. By assumption $x \notin \mathfrak{p}$, there is some β such that $x \notin \mathfrak{p}_\beta$, or $x \notin \mathfrak{p}_\alpha$ whenever $\alpha \geq \beta$. So $y \in \mathfrak{p}_\alpha$ whenever $\alpha \geq \beta$. Since $y \in \mathfrak{p}_\beta$, $y \in \mathfrak{p}_\gamma$ whenever $\beta \geq \gamma$. Therefore, $y \in \mathfrak{p}_\alpha$ for all α, or $y \in \mathfrak{p}$, or $\mathfrak{p}$ is prime.

□

Exercise 1.9. *Let $\mathfrak{a}$ be an ideal $\neq (1)$ in a ring A. Show that $\mathfrak{a} = r(\mathfrak{a}) \Longleftrightarrow \mathfrak{a}$ is an intersection of prime ideals.*

Proof.

(1) $(\Longrightarrow)$. By Proposition 1.14, $\mathfrak{a} = r(\mathfrak{a})$ is the intersection of the prime ideals which contain $\mathfrak{a}$.

(2) ($\Longleftarrow$).

$$\begin{aligned}
\mathfrak{a} &= \bigcap \{\mathfrak{p} \in \text{some subset of } \operatorname{Spec}(A)\} \\
&= \bigcap \{\mathfrak{p} \in \text{some subset of } \operatorname{Spec}(A) : \mathfrak{p} \supseteq \mathfrak{a}\} \\
&\supseteq \bigcap \{\mathfrak{p} \in \operatorname{Spec}(A) : \mathfrak{p} \supseteq \mathfrak{a}\} \\
&= r(\mathfrak{a}) \\
&\supseteq \mathfrak{a}.
\end{aligned}$$

$\square$

The prime spectrum of a ring

Exercise 1.15. *Let A be a ring and let X be the set of all prime ideals of A. For each subset E of A, let $V(E)$ denote the set of all prime ideals of A which contain E. Prove that*

(i) *if $\mathfrak{a}$ is the ideal generated by E, then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$.*

(ii) $V(0) = X$, $V(1) = \varnothing$.

(iii) *if $(E_i)_{i \in I}$ is any family of subsets of A, then*

$$V\left(\bigcup_{i \in I} E_i\right) = \bigcap_{i \in I} V(E_i).$$

(iv) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ *for any ideals $\mathfrak{a}$, $\mathfrak{b}$ of A.*

The results show that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. The resulting topology is called the Zariski topology. The topological space X is called the prime spectrum of A, and is written $\operatorname{Spec}(A)$.

Note that if $E_1 \subseteq E_2$, then $V(E_1) \supseteq V(E_2)$.

Proof of (i).

(1) *Show that $V(E) = V(\mathfrak{a})$.*

   (a) *Show that $V(E) \subseteq V(\mathfrak{a})$.* Given any $\mathfrak{p} \in V(E)$, $\mathfrak{p} \supseteq E$. For any $a \in \mathfrak{a}$, since $\mathfrak{a}$ is generated by E, we can write a as a finite sum $a = \sum \alpha\beta$ where $\alpha \in A$ and $\beta \in E$. Since $E \subseteq \mathfrak{p}$, all $\beta \in \mathfrak{p}$. Since $\mathfrak{p}$ is an ideal, $a = \sum \alpha\beta \in \mathfrak{p}$. That is, $\mathfrak{p} \supseteq \mathfrak{a}$, or $\mathfrak{p} \in V(\mathfrak{a})$.

   (b) $V(E) \supseteq V(\mathfrak{a})$ since $\mathfrak{a} \supseteq E$.

(2) *Show that $V(\mathfrak{a}) = V(r(\mathfrak{a}))$.*

(a) *Show that* $V(\mathfrak{a}) \subseteq V(r(\mathfrak{a}))$. Given any $\mathfrak{p} \in V(\mathfrak{a})$,

$$\begin{aligned}
\mathfrak{p} \in V(\mathfrak{a}) &\Longrightarrow \mathfrak{p} \supseteq \mathfrak{a} \\
&\Longrightarrow \mathfrak{p} \supseteq \text{the intersection of the primes ideals } \mathfrak{p} \supseteq \mathfrak{a} \\
&\Longrightarrow \mathfrak{p} \supseteq r(\mathfrak{a}) \text{ (by Proposition 1.14)} \\
&\Longrightarrow \mathfrak{p} \in V(r(\mathfrak{a})).
\end{aligned}$$

(b) $V(\mathfrak{a}) \supseteq V(r(\mathfrak{a}))$ since $r(\mathfrak{a}) \supseteq \mathfrak{a}$.

□

Proof of (ii).

(1) $V(1) = \varnothing$ since no prime ideal contains 1 by definition.

(2) $V(0) = X$ since 0 is in every ideal (especially in every prime ideal).

□

Proof of (iii).

$$\begin{aligned}
\mathfrak{p} \in V\left(\bigcup_{i \in I} E_i\right) &\Longleftrightarrow \mathfrak{p} \supseteq \bigcup_{i \in I} E_i \\
&\Longleftrightarrow \mathfrak{p} \supseteq E_i \text{ for all } i \in I \\
&\Longleftrightarrow \mathfrak{p} \in V(E_i) \text{ for all } i \in I \\
&\Longleftrightarrow \mathfrak{p} \in \bigcap_{i \in I} V(E_i).
\end{aligned}$$

□

Lemma. *For any* $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$, $\mathfrak{p} \supseteq \mathfrak{a}$ *or* $\mathfrak{p} \supseteq \mathfrak{b}$.

Proof of Lemma.

(1) If $\mathfrak{p} \supseteq \mathfrak{a}$. We are done.

(2) If $\mathfrak{p} \not\supseteq \mathfrak{a}$, there exists $a \in \mathfrak{a} - \mathfrak{p}$. So for any $b \in \mathfrak{b}$, $b \in \mathfrak{p}$ since $ab \in \mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$ and $\mathfrak{p}$ is a prime ideal, that is, $\mathfrak{p} \supseteq \mathfrak{b}$.

By (1)(2), $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$. □

Proof of (iv).

(1) *Show that* $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$.

(a) $V(\mathfrak{a} \cap \mathfrak{b}) \subseteq V(\mathfrak{a}\mathfrak{b})$ since $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b}$.

(b) *Show that* $V(\mathfrak{a} \cap \mathfrak{b}) \supseteq V(\mathfrak{ab})$. Given any $\mathfrak{p} \in V(\mathfrak{ab})$, $\mathfrak{p} \supseteq \mathfrak{ab}$. By Lemma, $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$. Notice that $\mathfrak{a} \supseteq \mathfrak{a} \cap \mathfrak{b}$ and $\mathfrak{b} \supseteq \mathfrak{a} \cap \mathfrak{b}$. In any case, $\mathfrak{p} \supseteq \mathfrak{a} \cap \mathfrak{b}$, $\mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b})$.

(2) *Show that* $V(\mathfrak{ab}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.

(a) *Show that* $V(\mathfrak{ab}) \subseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$. Given any $\mathfrak{p} \in V(\mathfrak{ab})$, $\mathfrak{p} \supseteq \mathfrak{ab}$. By Lemma, $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$, $\mathfrak{p} \in V(\mathfrak{a})$ or $\mathfrak{p} \in V(\mathfrak{b})$, $\mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$.

(b) *Show that* $V(\mathfrak{ab}) \supseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$. Given any $\mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$, $\mathfrak{p} \in V(\mathfrak{a})$ or $\mathfrak{p} \in V(\mathfrak{b})$, $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$. Notice that $\mathfrak{a} \supseteq \mathfrak{ab}$ and $\mathfrak{b} \supseteq \mathfrak{ab}$. In any cases, $\mathfrak{p} \supseteq \mathfrak{ab}$, or $\mathfrak{p} \in V(\mathfrak{ab})$.

$\square$

Exercise 1.17. *For each $f \in A$, let X_f denote the complement of $V(f)$ in $X = \operatorname{Spec}(A)$. The sets X_f are open. Show that they form a basis of open sets for the Zariski topology, and that*

(i) $X_f \cap X_g = X_{fg}$.

(ii) $X_f = \varnothing \Longleftrightarrow f$ *is nilpotent.*

(iii) $X_f = X \Longleftrightarrow f$ *is a unit.*

(iv) $X_f = X_g \Longleftrightarrow r((f)) = r((g))$.

(v) *X is quasi-compact (compact), that is, every open covering of X has a finite subcovering.*

(vi) *More generally, each X_f is quasi-compact.*

(vii) *An open subset of X is quasi-compact if and only if it is a finite union of sets X_f.*

The sets X_f are called basic open sets of $X = \operatorname{Spec}(A)$.

(Hint: To prove (v), remark that it is enough to consider a covering of X by basic open sets $X_{f_i} (i \in I)$. Show that the f_i generate the unit ideal and hence that there is an equation of the form

$$1 = \sum_{i \in J} g_i f_i \quad (g_i \in A)$$

where J is some finite subset of I. Then the $X_{f_i} (i \in J)$ cover X.)

Proof of basis. It is equivalent to Exercise 1.15 (iii). Given any open set O in X. Write $O = X - V(\mathfrak{a})$ for some ideal $\mathfrak{a}$ of A. Since

$$V(\mathfrak{a}) = V\left(\bigcup_{f \in \mathfrak{a}} (f) \right) = \bigcap_{f \in \mathfrak{a}} V(f),$$

we have

$$O = X - V(\mathfrak{a}) = X - \bigcap_{f\in\mathfrak{a}} V(f) = \bigcup_{f\in\mathfrak{a}} (X - V(f)) = \bigcup_{f\in\mathfrak{a}} X_f,$$

or any open set is a union of basic open sets. □

Proof of (i). $X_f \cap X_g = X_{fg} \iff V(f) \cup V(g) = V(fg)$ holds by Exercise 1.15 (iv). □

Proof of (ii).

$$\begin{aligned} X_f = \varnothing &\iff V(f) = X \\ &\iff f \in \mathfrak{p} \text{ for all prime ideal } \mathfrak{p} \text{ of } A \\ &\iff f \in \mathfrak{N}, \text{the nilradical of } A \text{ (Proposition 1.8)} \\ &\iff f \text{ is nilpotent (Proposition 1.7)} \end{aligned}$$

□

Proof of (ii)(Using (iv)).

$$\begin{aligned} X_f = \varnothing &\iff X_f = X_0 && \text{(Exercise 15(ii))} \\ &\iff r(f) = r(0) && \text{((iv))} \\ &\iff f \in r(f) = r(0) \\ &\iff f^m = 0 \text{ for some } m > 0 \\ &\iff f \text{ is nilpotent} \end{aligned}$$

□

Proof of (iii).

$$\begin{aligned} X_f = X &\iff V(f) = \varnothing \\ &\iff f \notin \mathfrak{p} \text{ for all prime ideal } \mathfrak{p} \text{ of } A \\ &\iff f \text{ is unit (Corollary 1.5)} \end{aligned}$$

□

Proof of (iii)(Using (iv)).

$$\begin{aligned} X_f = X &\iff X_f = X_1 && \text{(Exercise 15(ii))} \\ &\iff r(f) = r(1) && \text{((iv))} \\ &\iff f \in r(f) = r(1) \\ &\iff f^m = 1 \text{ for some } m > 0 \\ &\iff f \text{ is unit} \end{aligned}$$

$\square$

Proof of (iv).

(1) *Show that* $X_f \subseteq X_g \Longleftrightarrow r((f)) \subseteq r((g))$. Actually,

$$\begin{aligned}
X_f \subseteq X_g &\Longrightarrow V(f) \supseteq V(g) \\
&\Longrightarrow \{\mathfrak{p} \in \mathrm{Spec}(A) : \mathfrak{p} \supseteq (f)\} \supseteq \{\mathfrak{p} \in \mathrm{Spec}(A) : \mathfrak{p} \supseteq (g)\} \\
&\Longrightarrow \bigcap_{(f)\subseteq\mathfrak{p}\in\mathrm{Spec}(A)} \mathfrak{p} \subseteq \bigcap_{(g)\subseteq\mathfrak{p}\in\mathrm{Spec}(A)} \mathfrak{p} \\
&\overset{1.14}{\Longrightarrow} r(f) \subseteq r(g) \\
&\Longrightarrow V(r(f)) \supseteq V(r(g)) \\
&\Longrightarrow V(f) \supseteq V(g) \\
&\Longrightarrow X_f \subseteq X_g.
\end{aligned}$$

(2) By (1),

$$\begin{aligned}
X_f \subseteq X_g &\Longleftrightarrow r((f)) \subseteq r((g)), \\
X_f \supseteq X_g &\Longleftrightarrow r((f)) \supseteq r((g)).
\end{aligned}$$

Hence,

$$X_f = X_g \Longleftrightarrow r((f)) = r((g)).$$

$\square$

Proof of (v). Notice that it is enough to consider a covering of X by basic open sets $X_{f_i} (i \in I)$.

(1) Since X is covered by $X_{f_i} (i \in I)$,

$$\begin{aligned}
X = \bigcup_{i\in I} X_{f_i} &\Longrightarrow X - V(1) = \bigcup_{i\in I} (X - V(f_i)) \\
&\Longrightarrow V(1) = \bigcap_{i\in I} V(f_i) \\
&\Longrightarrow V(1) = V\left(\sum_{i\in I} f_i\right) \\
&\Longrightarrow r(1) = r\left(\sum_{i\in I} f_i\right).
\end{aligned}$$

Hence, $1 \in r(1) = r\left(\sum_{i\in I} f_i\right)$ can be expressed as

$$1 = 1^m = \sum_{j\in J} g_j f_j$$

where J is a finite subset of I and $g_j \in A$. That is, $(1) = \sum_{j\in J} f_j$.

(2) Hence, $V(1) = V\left(\sum_{j\in J} f_j\right)$. Therefore, X is covered by finite subcovering $\{X_{f_j}\}(j \in J)$.

$\square$

Proof of (v)(Using (vi)). Since $X = X_1$, X is quasi-compact by (vi). $\square$

Proof of (vi). Notice that it is enough to consider a covering of X_f by basic open sets $X_{f_i}(i \in I)$.

(1) Since X_f is covered by $X_{f_i}(i \in I)$,

$$\begin{aligned}
X_f = \bigcup_{i\in I} X_{f_i} &\Longrightarrow X - V(f) = \bigcup_{i\in I}(X - V(f_i)) \\
&\Longrightarrow V(f) = \bigcap_{i\in I} V(f_i) \\
&\Longrightarrow V(f) = V\left(\sum_{i\in I} f_i\right) \\
&\Longrightarrow r(f) = r\left(\sum_{i\in I} f_i\right).
\end{aligned}$$

Hence, $f \in r(f) = r\left(\sum_{i\in I} f_i\right)$ can be expressed as

$$f^m = \sum_{j\in J} g_j f_j$$

where J is a finite subset of I and $g_j \in A$. That is, $f^m \in \sum_{j\in J} f_j$.

(2) *Show that* $V\left(\sum_{j\in J} f_j\right) = V(f)$.

(a) $(\subseteq)$ For any prime ideal $\mathfrak{p} \supseteq \sum_{j\in J} f_j$, $f^m \in \mathfrak{p}$ or $f \in \mathfrak{p}$ (since $\mathfrak{p}$ is prime). So $\mathfrak{p} \supseteq (f)$, or $V\left(\sum_{j\in J} f_j\right) \subseteq V(f)$.

(b) $(\supseteq)$

$$\sum_{j\in J} f_j \subseteq \sum_{i\in I} f_i \Longrightarrow V\left(\sum_{j\in J} f_j\right) \supseteq V\left(\sum_{i\in I} f_i\right) = V(f).$$

(3) Therefore, X_f is covered by finite subcovering $\{X_{f_j}\}(j \in J)$.

$\square$

Proof of (vi)(Using (v)). Exercise 3.21 (i) shows that X_f is the spectrum of A_f. By (v), X_f is quasi-compact. $\square$

Proof of (vii).

(1) ($\Longrightarrow$) Given an open subset O. Since X_f form a basis of open sets,

$$O = \bigcup_{f \in \mathfrak{a}} X_f \text{ for some ideal } \mathfrak{a} \text{ of } A$$

Especially, $\{X_f\}_{f \in \mathfrak{a}}$ is an open covering of O. Since O is quasi-compact, there exists a finite subcovering $\{X_f\}_{f \in J}$ of O, where J is a finite subset of $\mathfrak{a}$ (as a set). That is, $O = \bigcup_{f \in J} X_f$ is a finite union of sets X_f.

(2) ($\Longleftarrow$) Since X_f is quasi-compact, any finite union of quasi-compact sets is quasi-compact again.

$\square$

Exercise 1.18. *For psychological reasons it is sometimes convenient to denote a prime ideal of A by a letter such as x or y when thinking of it as a point of $X = \operatorname{Spec}(A)$. When thinking of x as a prime ideal of A, we denote it by $\mathfrak{p}_x$ (logically, of course, it is the same thing). Show that*

(i) *The set $\{x\}$ is closed (we say that x is a "closed point") in* $\operatorname{Spec}(A) \Longleftrightarrow$ *$\mathfrak{p}_x$ is maximal;*

Exercise 1.19. *A topological space X is said to be irreducible if $X \neq \varnothing$ and if every pair of non-empty open sets in X intersect, or equivalently if every non-empty open set is dense in X. Show that* $\operatorname{Spec}(A)$ *is irreducible if and only if the nilradical of A is a prime ideal.*

Proof. Use the notations in Proposition 1.7 and Exercise 1.17.

$$\begin{aligned}
&\operatorname{Spec}(A) \text{ is irreducible} \\
\Longleftrightarrow\ & X_f \cap X_g \neq \varnothing \text{ for any nonempty } X_f, X_g \in \operatorname{Spec}(A) \\
\Longleftrightarrow\ & X_{fg} \neq \varnothing \text{ for any nonempty } X_f, X_g \in \operatorname{Spec}(A) && \text{(Exercise 1.17 (i))} \\
\Longleftrightarrow\ & fg \notin \mathfrak{N} \text{ for any } f, g \notin \mathfrak{N} && \text{(Exercise 1.17 (ii))} \\
\Longleftrightarrow\ & \mathfrak{N} \text{ is prime.}
\end{aligned}$$

$\square$

Exercise 1.20. *Let X be a topological space.*

(i) *If Y is an irreducible subspace of X, then the closure $\overline{Y}$ of Y in X is irreducible.*

Proof of (i).

(1) *Y is irreducible if and only if Y cannot be represented as the union of two proper closed subspaces.*

$$\begin{aligned}
&\forall \text{ nonempty open sets } U_1 \text{ and } U_2, U_1 \cap U_2 \neq \varnothing \\
\Longleftrightarrow &\forall \text{ nonempty open sets } U_1 \text{ and } U_2, X - (U_1 \cap U_2) \neq X \\
\Longleftrightarrow &\forall \text{ nonempty open sets } U_1 \text{ and } U_2, (X - U_1) \cup (X - U_2) \neq X \\
\Longleftrightarrow &\forall \text{ proper closed sets } Y_1 \text{ and } Y_2, Y_1 \cup Y_2 \neq X \\
\Longleftrightarrow &\nexists \text{ proper closed sets } Y_1 \text{ and } Y_2, Y_1 \cup Y_2 = X.
\end{aligned}$$

(2) If $\overline{Y}$ were reducible, there are two closed set Y_1 and Y_2 such that

$$\overline{Y} \subseteq Y_1 \cup Y_2, \overline{Y} \not\subseteq Y_i (i = 1, 2).$$

(a) $Y \subseteq \overline{Y} \subseteq Y_1 \cup Y_2$.

(b) $Y \not\subseteq Y_i (i = 1, 2)$. If not, $Y \subseteq Y_i$ for some i. Take closure to get $\overline{Y} \subseteq \overline{Y_i} = Y_i$ (since Y_i is closed), contrary to the assumption.

By (a)(b), Y is reducible, which is absurd.

□

Supplement. *(Exercise I.1.6 in Robin Hartshorne, Algebraic Geometry.) Any nonempty open subset of an irreducible topological space is dense and irreducible. If Y is a subset of a topological space X, which is irreducible in its induced topology, then the closure $\overline{Y}$ is also irreducible.*

Here we use the definition of irreducibility given by Hartshorne.

Definition. A nonempty subset Y of a topological space X is irreducible if it cannot be expressed as the union $Y = Y_1 \cup Y_2$ of two proper subsets, each one of which is closed in Y. The empty set is not considered to be irreducible.

The proof is the same as Exercise 1.20(i) except that any nonempty open subset of an irreducible topological space is irreducible.

Proof (Irreducibility of open subsets). Given any open subset U of an irreducible topological space X. Write $U \subseteq Y_1 \cup Y_2$ where Y_1 and Y_2 are closed in X.

$$\begin{aligned}
U \subseteq Y_1 \cup Y_2 \Longrightarrow &\overline{U} \subseteq \overline{Y_1 \cup Y_2} \\
\Longrightarrow &X \subseteq Y_1 \cup Y_2 && (U \text{ is dense, } Y_1 \cup Y_2 \text{ is closed}) \\
\Longrightarrow &Y_1 = X \supseteq U \text{ or } Y_2 = X \supseteq U && (X \text{ is irreducible}) \\
\Longrightarrow &U \text{ is irreducible.}
\end{aligned}$$

□