

## Chapter 6: The Riemann-Stieltjes Integral

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**Supplement.** Another definition of Riemann-Stieltjes integral. (*Exercise 7.3, 7.4 of the book T. M. Apostol, Mathematical Analysis, Second Edition.*) Let  $P$  be a partition of  $[a, b]$ . The norm of a partition  $P$  is the length of the largest subinterval  $[x_{i-1}, x_i]$  of  $P$  and is denoted by  $\|P\|$ .

We say  $f \in \mathcal{R}(\alpha)$  if there exists  $A \in \mathbb{R}$  having the property that for any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for any partition  $P$  of  $[a, b]$  with norm  $\|P\| < \delta$  and for any choice of  $t_i \in [x_{i-1}, x_i]$ , we have  $|\sum_{i=1}^n f(t_i)\Delta\alpha_i - A| < \varepsilon$ .

**Claim.**  $f \in \mathcal{R}$  in the sense of Definition 6.2 implies that  $f \in \mathcal{R}$  in the sense of this another definition.

*Proof of Claim.* Let  $A = \int f dx$ ,  $M > 0$  be one upper bound of  $|f|$  on  $[a, b]$ . Given  $\varepsilon > 0$ , there exists a partition  $P_0 = \{a = x_0, x_1, \dots, x_{N-1}, x_N = b\}$  such that  $U(P_0, f) \leq A + \frac{\varepsilon}{2}$ . Let  $\delta = \frac{\varepsilon}{2MN} > 0$ . Then for any partition  $P$  with norm  $\|P\| < \delta$ , write

$$U(P, f) = \sum_{i=1}^n M_i \Delta x_i = S_1 + S_2,$$

where  $S_1$  is the sum of terms arising from those subintervals of  $P$  containing no point of  $P_0$ ,  $S_2$  is the sum of the remaining terms. Then

$$S_1 \leq U(P_0, f) < A + \frac{\varepsilon}{2},$$

$$S_2 \leq NM\|P\| < NM\delta < \frac{\varepsilon}{2}.$$

Therefore,  $U(P, f) < A + \varepsilon$ . Similarly,  $L(P, f) > A - \varepsilon$  whenever  $\|P\| < \delta'$ . Hence,  $|\sum_{i=1}^n f(t_i)\Delta x_i - A| < \varepsilon$  whenever  $\|P\| < \min\{\delta, \delta'\}$ . (Copy Apostol's hint and ensure  $M > 0$ .  $M$  in Apostol's hint might be zero if  $f = 0$ .)  $\square$

This supplement will be used in computing  $\int_0^\infty (\frac{\sin x}{x})^2 dx = \frac{\pi}{2}$  in Exercise 8.12.

**Exercise 6.1.** Suppose  $\alpha$  increases on  $[a, b]$ ,  $a \leq x_0 \leq b$ ,  $\alpha$  is continuous at  $x_0$ ,  $f(x_0) = 1$ , and  $f(x) = 0$  if  $x \neq x_0$ . Prove that  $f \in \mathcal{R}(\alpha)$  and that  $\int f d\alpha = 0$ .

Given any partition  $P = \{a = p_0, p_1, \dots, p_{n-1}, p_n = b\}$ , where  $a = p_0 \leq p_1 \leq \dots \leq p_{n-1} \leq p_n = b$ . We might compute  $L(P, f, \alpha)$  and  $U(P, f, \alpha)$  by using  $\varepsilon$ - $\delta$

argument since we are hinted by the condition that  $\alpha$  is continuous. A function which is continuous at  $x_0$  has a nice property near  $x_0$  and this property would help us estimate  $U(P, f, \alpha)$  near  $x_0$ . On the contrary, if both  $f$  and  $\alpha$  are discontinuous at  $x_0$ , it might be  $f \notin \mathcal{R}(\alpha)$ . Besides, if  $f$  has too many points of discontinuity ( $f(x) = 0$  if  $x \in \mathbb{Q}$  and  $f(x) = 1$  otherwise, for example), then  $f$  might not be Riemann-integrable on  $[0, 1]$ .

**Claim 1.**  $L(P, f, \alpha) = 0$ .

*Proof of Claim 1.*  $m_i = 0$  since  $\inf f(x) = 0$  on any subinterval of  $[a, b]$ . So  $L(P, f, \alpha) = \sum m_i \Delta \alpha_i = 0$ . Here we don't need the condition that  $\alpha$  is continuous at  $x_0$ .  $\square$

**Claim 2.** For any  $\varepsilon > 0$ , there exists a partition  $P$  such that  $U(P, f, \alpha) < \varepsilon$ .

*Proof of Claim 2.* Say  $x_0 \in [p_{i_0-1}, p_{i_0}]$  for some  $i_0$ . Then

$$M_i = \sup_{p_{i-1} \leq x \leq p_i} f(x) = \begin{cases} 0 & \text{if } i \neq i_0, \\ 1 & \text{if } i = i_0. \end{cases}$$

So

$$U(P, f, \alpha) = \sum M_i \Delta \alpha_i = \Delta \alpha_{i_0}.$$

It is not true for any arbitrary  $\alpha$ . (For example,  $\alpha$  has a jump on  $x = x_0$ .) In fact, Exercise 6.3 shows this. Luckily,  $\alpha$  is continuous at  $x_0$ . So for  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $|\alpha(x) - \alpha(x_0)| < \frac{\varepsilon}{2}$  whenever  $|x - x_0| < \delta$  (and  $x \in [a, b]$ ). Now we pick a nice partition

$$P = \{a, x_0 - \delta_1, x_0 + \delta_2, b\},$$

where  $\delta_1 = \min\{\delta, x_0 - a\} \geq 0$  and  $\delta_2 = \min\{\delta, b - x_0\} \geq 0$ . (It is a trick about resizing “ $\delta$ ” to avoid considering the edge cases  $x_0 = a$  or  $x_0 = b$  or  $a = b$ .) Then  $x_0 \in [x_0 - \delta_1, x_0 + \delta_2]$  and  $\Delta \alpha$  on  $[x_0 - \delta_1, x_0 + \delta_2]$  is

$$\begin{aligned} \alpha(x_0 + \delta_2) - \alpha(x_0 - \delta_1) &= (\alpha(x_0 + \delta_2) - \alpha(x_0)) + (\alpha(x_0) - \alpha(x_0 - \delta_1)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore,  $U(P, f, \alpha) < \varepsilon$ .  $\square$

*Proof (Definition 6.2).* By Claim 1 and 2 and notice that  $U(P, f, \alpha) \geq 0$  for any

partition  $P$ ,

$$\begin{aligned}\int_a^{\bar{b}} f d\alpha &= \inf U(P, f, \alpha) = 0, \\ \int_a^{\underline{b}} f d\alpha &= \sup L(P, f, \alpha) = 0,\end{aligned}$$

the inf and sup again being taken over all partitions. Hence  $f \in \mathcal{R}(\alpha)$  and that  $\int f d\alpha = 0$  by Definition 6.2.  $\square$

*Proof (Theorem 6.5).* By Claim 1 and 2,

$$0 \leq U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$$

Hence  $f \in \mathcal{R}(\alpha)$  by Theorem 6.5. Furthermore,

$$\int f d\alpha = \int_a^{\bar{b}} f d\alpha = \sup L(P, f, \alpha) = 0.$$

$\square$

*Proof (Theorem 6.10).*  $f \in \mathcal{R}(\alpha)$  by Theorem 6.10. Thus, by Claim 1

$$\int f d\alpha = \int_a^{\bar{b}} f d\alpha = \sup L(P, f, \alpha) = 0.$$

$\square$

**Exercise 6.2.** Suppose  $f \geq 0$ ,  $f$  is continuous on  $[a, b]$ , and  $\int_a^b f(x) dx = 0$ . Prove that  $f(x) = 0$  for all  $x \in [a, b]$ . (Compare with Exercise 6.1.)

For one application, see Exercise 7.20.

*Proof.* (Reductio ad absurdum) If there were  $p \in [a, b]$  such that  $f(p) > 0$ . Since  $f$  is continuous on  $[a, b]$ , given  $\varepsilon = \frac{1}{64}f(p) > 0$  there exists  $\delta > 0$  such that

$$|f(x) - f(p)| \leq \frac{1}{64}f(p) \text{ whenever } |x - p| \leq \delta, x \in [a, b].$$

Hence

$$f(x) \geq \frac{63}{64}f(p)$$

whenever  $x \in E = [\max\{a, p - \delta\}, \min\{b, p + \delta\}] \subseteq [a, b]$ . Note that the length of  $E$  is  $|E| > 0$ . So

$$0 = \int_a^b f(x) dx \geq \int_E f(x) dx \geq \int_E \frac{63}{64}f(p) dx = \frac{63}{64}f(p)|E| > 0,$$

which is absurd.  $\square$

*Note.* (Lebesgue integral) Let  $f$  be a nonnegative measurable function. Then  $\int f = 0$  implies  $f = 0$  a.e.

**Exercise 6.3.**  
PLACEHOLDER

**Exercise 6.4.** If

$$f(x) = \begin{cases} 0 & \text{for all irrational } x, \\ 1 & \text{for all rational } x, \end{cases}$$

prove that  $f \notin \mathcal{R}$  on  $[a, b]$  for any  $a < b$ .

*Proof.* Given any partition

$$P = \{a = p_0, p_1, \dots, p_{n-1}, p_n = b\}$$

of  $[a, b]$  where  $a = p_0 \leq p_1 \leq \dots \leq p_{n-1} \leq p_n = b$ . Since  $a < b$ , we might assume that  $a = p_0 < p_1 < \dots < p_{n-1} < p_n = b$  by removing duplicated points. Since  $\mathbb{Q}$  and  $\mathbb{R} - \mathbb{Q}$  are dense in  $\mathbb{R}$ , we have

$$\begin{aligned} M_i &= \sup_{p_{i-1} \leq x \leq p_i} f(x) = 1, \\ m_i &= \inf_{p_{i-1} \leq x \leq p_i} f(x) = 0, \\ U(P, f) &= \sum_{i=1}^n M_i \Delta x_i = \sum_{i=1}^n \Delta x_i = b - a, \\ L(P, f) &= \sum_{i=1}^n m_i \Delta x_i = \sum_{i=1}^n 0 = 0. \end{aligned}$$

Since  $P$  is arbitrary,

$$\begin{aligned} \int_a^b f dx &= \inf U(P, f) = b - a > 0, \\ \int_a^b f dx &= \sup L(P, f) = 0. \end{aligned}$$

Hence  $f \notin \mathcal{R}$  on  $[a, b]$  for any  $a < b$ .  $\square$

*Note.*

(1)  $f \in \mathcal{R}$  on  $[a, b]$  iff  $a = b$ .

- (2) (Problem 4.1 in *H. L. Royden, Real Analysis, 3rd edition.*) Construct a sequence  $\{f_n\}$  of nonnegative, Riemann integrable functions such that  $f_n$  increases monotonically to  $f$ . What does this imply about changing the order of integration and the limiting process? (Since  $\mathbb{Q}$  is countable, write

$$\mathbb{Q} = \{r_1, r_2, \dots\}.$$

Define

$$f_n(x) = \begin{cases} 0 & \text{if } x \notin \{r_1, \dots, r_n\}, \\ 1 & \text{if } x \in \{r_1, \dots, r_n\}. \end{cases}$$

By construction,  $f_n$  increases monotonically to  $f$  pointwise. Note that  $f_n \rightarrow f$  not uniformly. Also,  $\int_a^b f_n(x)dx = 0$  by using the same argument in Theorem 6.10. Therefore,  $\lim_{n \rightarrow \infty} \int_a^b f_n(x)dx = 0$  but  $\int_a^b \lim_{n \rightarrow \infty} f_n(x)dx = \int_a^b f(x)dx$  does not exist.)

*Note.* (Lebesgue integral)  $f$  is Lebesgue integrable.

**Exercise 6.5.** Suppose  $f$  is a bounded real function on  $[a, b]$ , and  $f^2 \in \mathcal{R}$  on  $[a, b]$ . Does it follow that  $f \in \mathcal{R}$ ? Does the answer change if we assume that  $f^3 \in \mathcal{R}$ ?

*Proof.*

- (1) Show that  $f^2 \in \mathcal{R}$  on  $[a, b]$  does not imply that  $f \in \mathcal{R}$  (unless  $f \geq 0$  on  $[a, b]$ ). Similar to Exercise 6.4, define

$$f(x) = \begin{cases} -1 & \text{for all irrational } x, \\ 1 & \text{for all rational } x. \end{cases}$$

$f^2 = 1 \in \mathcal{R}$  on  $[a, b]$  but  $f \notin \mathcal{R}$  on  $[a, b]$  for any  $a < b$ . (The proof for the “unless” part is similar to (2).)

- (2) Show that  $f^3 \in \mathcal{R}$  on  $[a, b]$  implies that  $f \in \mathcal{R}$ . Let  $\phi(x) = x^{\frac{1}{3}}$  on  $\mathbb{R}$ . By Theorem 6.11,  $f(x) = \phi(f(x)^3) \in \mathcal{R}$ .

□

*Note.* (Lebesgue integral) Suppose that  $f^2$  is Lebesgue integrable. Does it follow that  $f$  is Lebesgue integrable? Does the answer change if we assume that  $f^3$  is Lebesgue integrable? Both answers are no.

**Exercise 6.6.**

PLACEHOLDER

**Exercise 6.7.** Suppose  $f$  is a real function on  $(0, 1]$  and  $f \in \mathcal{R}$  on  $[c, 1]$  for every  $c > 0$ . Define

$$\int_0^1 f(x)dx = \lim_{c \rightarrow 0} \int_c^1 f(x)dx$$

if this limit exists (and is finite).

- (a) If  $f \in \mathcal{R}$  on  $[0, 1]$ , show that this definition of the integral agrees with the old one.
- (b) Construct a function such that the above limit exists, although it fails to exist with  $|f|$  in place of  $f$ .

*Proof of (a).*

- (1) Since  $f \in \mathcal{R}$  on  $[0, 1]$ ,  $f$  is bounded or  $|f| \leq M$  for some real  $M$ .
- (2) For any  $0 < c < 1$ , we have

$$\begin{aligned} \left| \int_0^1 f(x)dx - \int_c^1 f(x)dx \right| &= \left| \int_0^c f(x)dx \right| && \text{(Theorem 6.12(c))} \\ &\leq Mc. && \text{(Theorem 6.12(d))} \end{aligned}$$

- (3) Given any  $\varepsilon > 0$ , there exists  $\delta = \frac{\varepsilon}{M+1} > 0$  such that

$$\left| \int_0^c f(x)dx - \int_0^1 f(x)dx \right| \leq Mc < M\delta = M \cdot \frac{\varepsilon}{M+1} < \varepsilon$$

whenever  $0 < c < \delta$ . Hence  $\lim_{c \rightarrow 0} \int_c^1 f(x)dx = \int_0^1 f(x)dx$ .

□

*Proof of (b)(Construct by nonabsolutely convergent series).*

- (1) Given any nonabsolutely (conditionally) convergent series  $\sum_{k=1}^n a_k$  (take  $\sum \frac{(-1)^n}{n}$  for example and then see Remark 3.46), we define  $f$  on  $(0, 1]$  by

$$f(x) = 2^n a_n$$

if  $\frac{1}{2^n} < x \leq \frac{1}{2^{n-1}}$  as  $n = 1, 2, \dots$

- (2) By construction,

$$\int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} f(x)dx = \left( \frac{1}{2^{n-1}} - \frac{1}{2^n} \right) 2^n a_n = a_n.$$

and thus

$$\int_{\frac{1}{2^n}}^1 f(x)dx = \int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} f(x)dx + \cdots + \int_{\frac{1}{2}}^1 f(x)dx = \sum_{k=1}^n a_k.$$

- (3) Given any  $\varepsilon > 0$ . Since  $\sum a_n$  is convergent, there exists a common integer  $N$  such that

$$|a_n| \leq \frac{\varepsilon}{89}$$

and

$$\left| \sum_{k=1}^n a_k - A \right| \leq \frac{\varepsilon}{64}$$

for some real  $A$  whenever  $n \geq N$  (Definition 3.21 and Theorem 3.23). Therefore, for any  $0 < c \leq \frac{1}{2^N}$ , say  $\frac{1}{2^{n+1}} < c \leq \frac{1}{2^n} \leq \frac{1}{2^N}$  for some  $n \geq N$ , we have

$$\begin{aligned} \left| \int_c^1 f(x)dx - A \right| &= \left| \int_c^{\frac{1}{2^n}} f(x)dx + \int_{\frac{1}{2^n}}^1 f(x)dx - A \right| \\ &\leq \left| \left( \frac{1}{2^n} - c \right) 2^{n+1} a_{n+1} \right| + \left| \sum_{k=1}^n a_k - A \right| \\ &\leq |a_{n+1}| + \left| \sum_{k=1}^n a_k - A \right| \\ &\leq \frac{\varepsilon}{89} + \frac{\varepsilon}{64} \\ &\leq \varepsilon. \end{aligned}$$

Hence,  $\lim_{c \rightarrow 0} \int_c^1 f(x)dx = A$  exists.

- (4) Since

$$\int_{\frac{1}{2^n}}^1 |f(x)|dx = \int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} |f(x)|dx + \cdots + \int_{\frac{1}{2}}^1 |f(x)|dx = \sum_{k=1}^n |a_k| \rightarrow \infty$$

as  $n \rightarrow \infty$ ,  $\lim_{c \rightarrow 0} \int_c^1 f(x)dx$  does not exist.

□

**Exercise 6.8.**  
PLACEHOLDER

**Exercise 6.9.**

PLACEHOLDER

**Exercise 6.10.** Let  $p$  and  $q$  be positive real integers such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Prove the following statements.

(a) If  $u \geq 0$  and  $v \geq 0$ , then

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

Equality holds if and only if  $u^p = v^q$ .

(b) If  $f \in \mathcal{R}(\alpha)$ ,  $g \in \mathcal{R}(\alpha)$ ,  $f \geq 0$ ,  $g \geq 0$ , and

$$\int_a^b f^p d\alpha = \int_a^b g^q d\alpha = 1,$$

then

$$\int_a^b fg d\alpha \leq 1.$$

(c) If  $f$  and  $g$  are complex functions in  $\mathcal{R}(\alpha)$ , then

$$\left| \int_a^b fg d\alpha \right| \leq \left\{ \int_a^b |f|^p d\alpha \right\}^{\frac{1}{p}} \left\{ \int_a^b |g|^q d\alpha \right\}^{\frac{1}{q}}.$$

This is **Hölder's inequality**. When  $p = q = 2$  it is usually called the Schwarz inequality. (Note that Theorem 1.35 is a very special case of this.)

(d) Show that Hölder's inequality is also true for the "improper" integrals described in Exercise 6.7 and 6.8.

*Proof of (a) (Young's inequality).*

(1)  $u = 0$  or  $v = 0$  is nothing to do. For  $u > 0$  and  $v > 0$ , we give some different proofs.

(2) First proof.

$$\begin{aligned} uv &= \exp(\log(uv)) \\ &= \exp\left(\frac{1}{p} \log(u^p) + \frac{1}{q} \log(v^q)\right) \\ &\leq \frac{1}{p} \exp(\log(u^p)) + \frac{1}{q} \exp(\log(v^q)) \quad (\text{Convexity of } \exp(x)) \\ &= \frac{u^p}{p} + \frac{v^q}{q}. \end{aligned}$$



Here the convexity of  $\exp(x)$  can be derived by the fact that  $(\exp(x))'' > 0$  and Exercise 5.14. The fact that the equality holds if and only if  $u^p = v^q$  is derived from the strictly convexity of  $\exp(x)$  additionally. (For the details about the exponential and logarithmic functions, might see Chapter 8.)

(3) Second proof.

$$\begin{aligned}\log\left(\frac{u^p}{p} + \frac{v^q}{q}\right) &\geq \frac{1}{p}\log(u^p) + \frac{1}{q}\log(v^q) \quad (\text{Concavity of } \log(x)) \\ &= \log(u) + \log(v) \\ &= \log(uv).\end{aligned}$$

Since  $\log(x)$  increases monotonically ( $(\log(x))' = \frac{1}{x} > 0$  if  $x > 0$ ),  $\frac{u^p}{p} + \frac{v^q}{q} \geq uv$  (or take the exponential function to get the same conclusion). Here the concavity of  $\log(x)$  can be derived by the fact that  $(\log(x))'' < 0$  and a statement that  $f''(x) \leq 0$  if and only if  $f$  is concave. The fact that the equality holds if and only if  $u^p = v^q$  is derived from the strictly concavity of  $\log(x)$  additionally. (The proof is analogous to Exercise 5.14.)

(4) Third proof. Suppose that  $f : [0, \infty) \rightarrow [0, \infty)$  is a strictly increasing continuous function such that  $f(0) = 0$  and  $\lim_{x \rightarrow \infty} f(x) = \infty$ . Then

$$uv \leq \int_0^u f(x)dx + \int_0^v f^{-1}(x)dx$$

for every  $u, v \geq 0$ , and equality occurs if and only if  $v = f(u)$ . Define

$$F(x) = -xf(x) + \int_0^x f(t)dt + \int_0^{f(x)} f^{-1}(t)dt.$$

By Theorem 6.20 (the fundamental theorem of calculus) and Theorem 5.5 (chain rule),

$$F'(x) = -(f(x) + xf'(x)) + f(x) + f'(x)f^{-1}(f(x)) = 0.$$

Hence  $F(x)$  is a constant on  $(0, u)$  (Theorem 5.11(b)). Note that  $F(x)$  is continuous on  $[0, u]$  and  $F(0) = 0$ , so  $F(x) = 0$  on  $[0, u]$  or

$$\int_0^x f(t)dt + \int_0^{f(x)} f^{-1}(t)dt = xf(x).$$

Take  $x = u$  to get

$$\int_0^u f(x)dx + \int_0^{f(u)} f^{-1}(x)dx = uf(u).$$

Hence

$$\begin{aligned}
& \int_0^u f(x)dx + \int_0^v f^{-1}(x)dx - uv \\
&= \int_0^u f(x)dx + \int_0^{f(u)} f^{-1}(x)dx + \int_{f(u)}^v f^{-1}(x)dx - uv \\
&= uf(u) + \int_{f(u)}^v f^{-1}(x)dx - uv \\
&= \int_{f(u)}^v [f^{-1}(x) - f^{-1}(f(u))]dx \\
&\geq 0.
\end{aligned}$$

The last inequality holds since  $f$  is strictly increasing and thus  $f^{-1}$  is strictly increasing too. Besides, the equality holds if and only if  $f(u) = v$ . Now the conclusion holds by taking  $f(x) = x^{p-1}$  in

$$uv \leq \int_0^u f(x)dx + \int_0^v f^{-1}(x)dx$$

and the equality holds if and only if  $u^p = v^q$ .

□

*Proof of (b).* Every integral is well-defined (Theorem 6.11 and Theorem 6.13(a)). Let  $u = f \geq 0$  and  $v = g \geq 0$  in (a). Integrate both sides of the inequality

$$fg \leq \frac{f^p}{p} + \frac{g^q}{q}$$

to get

$$\begin{aligned}
\int_a^b fg d\alpha &\leq \int_a^b \left( \frac{f^p}{p} + \frac{g^q}{q} \right) d\alpha && \text{(Theorem 6.12(b))} \\
&= \int_a^b \frac{f^p}{p} d\alpha + \int_a^b \frac{g^q}{q} d\alpha && \text{(Theorem 6.12(a))} \\
&= \frac{1}{p} \int_a^b f^p d\alpha + \frac{1}{q} \int_a^b g^q d\alpha && \text{(Theorem 6.12(a))} \\
&= \frac{1}{p} + \frac{1}{q} && \text{(Assumption)} \\
&= 1.
\end{aligned}$$

The equality holds if  $f^p = g^q$ . Note that the equality does not hold only if  $f^p = g^q$ . (Consider  $\alpha$  is constant on some subinterval  $[c, d] \subsetneq [a, b]$ .) Luckily, it is true for the additional assumption that  $\alpha(x) = x$  and  $f, g$  are continuous on  $[a, b]$ . □

*Proof of (c).* There are three possible cases.

(1) The case  $\left\{ \int_a^b |f|^p d\alpha \right\}^{\frac{1}{p}} = 0$ . So  $\int_a^b |f|^p d\alpha = 0$ .

(a) Show that  $\int_a^b |f| d\alpha = 0$  if  $\int_a^b |f|^p d\alpha = 0$ . (Reductio ad absurdum)  
 If  $\int_a^b |f| d\alpha = A > 0$ , then given  $\varepsilon = \frac{A}{2} > 0$ , there exists a partition  $P_0 = \{a = x_0 \leq \dots \leq x_n = b\}$  such that

$$\sum_{i=0}^n m_i \Delta\alpha_i > \frac{A}{2},$$

where  $m_i = \inf_{x \in [x_{i-1}, x_i]} |f|$  and  $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$ . By the pigeonhole principle, there exists  $1 \leq i_0 \leq n$  such that

$$L(P_0, |f|, \alpha) = m_{i_0} \Delta\alpha_{i_0} > \frac{A}{2n} > 0.$$

Especially,  $m_{i_0} > 0$  and  $\Delta\alpha_{i_0} > 0$ . Now we consider  $L(P, |f|^p, \alpha)$ . Hence

$$L(P_0, |f|^p, \alpha) = \sum_{i=0}^n m_i^p \Delta\alpha_i \geq m_{i_0}^p \Delta\alpha_{i_0} > 0,$$

or

$$\int_a^b |f| d\alpha = \sup L(P, f, \alpha) \geq m_{i_0}^p \Delta\alpha_{i_0} > 0,$$

which is absurd.

(b) Show that  $\int_a^b |fg| d\alpha = 0$  if  $\int_a^b |f| d\alpha = 0$ . Since  $g \in \mathcal{R}(\alpha)$ ,  $|g|$  is bounded by some real  $M$  on  $[a, b]$ , that is,  $|g(x)| \leq M$ . Hence

$$0 \leq \int_a^b |fg| d\alpha \leq \int_a^b M |f| d\alpha = M \int_a^b |f| d\alpha = 0.$$

Therefore  $\int_a^b |fg| d\alpha = 0$ .

By (a)(b),  $\int_a^b |fg| d\alpha = 0$  and thus Hölder's inequality holds for this case.

(2) The case  $\left\{ \int_a^b |g|^q d\alpha \right\}^{\frac{1}{q}} = 0$ . Similar to (1).

(3) If both  $\left\{ \int_a^b |f|^p d\alpha \right\}^{\frac{1}{p}} > 0$  and  $\left\{ \int_a^b |g|^q d\alpha \right\}^{\frac{1}{q}} > 0$ , then we apply (b) to

$$F(x) = \frac{|f(x)|}{\left\{ \int_a^b |f(x)|^p d\alpha \right\}^{\frac{1}{p}}} \quad \text{and} \quad G(x) = \frac{|g(x)|}{\left\{ \int_a^b |g(x)|^q d\alpha \right\}^{\frac{1}{q}}}.$$

Here  $F(x) \geq 0$  and  $G(x) \geq 0$  are well-defined and Riemann integrable. Thus the conclusion holds. The equality holds if  $F(x)^p = G(x)^q$  or

$$\frac{|f|^p}{\int_a^b |f|^p d\alpha} = \frac{|g|^q}{\int_a^b |g|^q d\alpha}.$$

Note that the equality does not hold only if  $\frac{|f|^p}{\int_a^b |f|^p d\alpha} = \frac{|g|^q}{\int_a^b |g|^q d\alpha}$ . Luckily, it is true for the additional assumption that  $\alpha(x) = x$  and  $f, g$  are continuous on  $[a, b]$ .

By (1)(2)(3), in any case the equality holds if

$$|f|^p \int_a^b |g|^q d\alpha = |g|^q \int_a^b |f|^p d\alpha.$$

In addition, if  $\alpha(x) = x$  and  $f, g$  are continuous on  $[a, b]$ , then the equality holds if and only if

$$|f|^p \int_a^b |g|^q d\alpha = |g|^q \int_a^b |f|^p d\alpha.$$

□

*Proof of (d).*

- (1) Suppose  $f$  and  $g$  are real functions on  $(0, 1]$  and  $f, g \in \mathcal{R}$  on  $[c, 1]$  for every  $c > 0$ . Show that

$$\left| \int_0^1 f g dx \right| \leq \left\{ \int_0^1 |f|^p dx \right\}^{\frac{1}{p}} \left\{ \int_0^1 |g|^q dx \right\}^{\frac{1}{q}}.$$

Here  $\int_0^1$  is one improper integral defined in Exercise 6.7.

- (a) By (c), we have

$$\left| \int_c^1 f g dx \right| \leq \left\{ \int_c^1 |f|^p dx \right\}^{\frac{1}{p}} \left\{ \int_c^1 |g|^q dx \right\}^{\frac{1}{q}}$$

for any  $c \in (0, 1]$ . Here every integral is well-defined (Theorem 6.11 and Theorem 6.13).

- (b) Since every integral is  $\geq 0$ , by taking the limit in the right hand side we have

$$\begin{aligned} \left| \int_c^1 f g dx \right| &\leq \left\{ \int_c^1 |f|^p dx \right\}^{\frac{1}{p}} \left\{ \int_c^1 |g|^q dx \right\}^{\frac{1}{q}} \\ &\leq \left\{ \int_0^1 |f|^p dx \right\}^{\frac{1}{p}} \left\{ \int_0^1 |g|^q dx \right\}^{\frac{1}{q}}. \end{aligned}$$

It is possible that  $\left\{ \int_0^1 |f|^p dx \right\}^{\frac{1}{p}} = \infty$  or  $\left\{ \int_0^1 |g|^q dx \right\}^{\frac{1}{q}} = \infty$ .

- (c) Now  $\left| \int_c^1 f g dx \right|$  is bounded by  $\left\{ \int_0^1 |f|^p dx \right\}^{\frac{1}{p}} \left\{ \int_0^1 |g|^q dx \right\}^{\frac{1}{q}}$ . Take limit to get

$$\left| \int_0^1 f g dx \right| \leq \left\{ \int_0^1 |f|^p dx \right\}^{\frac{1}{p}} \left\{ \int_0^1 |g|^q dx \right\}^{\frac{1}{q}}$$

even if some limit is divergent.

- (2) Suppose  $f$  and  $g$  are real functions on  $[a, b]$  and  $f, g \in \mathcal{R}$  on  $[a, b]$  for every  $b > a$  where  $a$  is fixed. Show that

$$\left| \int_a^\infty f g dx \right| \leq \left\{ \int_a^\infty |f|^p dx \right\}^{\frac{1}{p}} \left\{ \int_a^\infty |g|^q dx \right\}^{\frac{1}{q}}.$$

Here  $\int_a^\infty$  is one improper integral defined in Exercise 6.8. Same as (1).

□

**Exercise 6.11.** Let  $\alpha$  be a fixed increasing function on  $[a, b]$ . For  $u \in \mathcal{R}(\alpha)$ , define

$$\|u\|_2 = \left\{ \int_a^b |u|^2 d\alpha \right\}^{\frac{1}{2}}.$$

Suppose  $f, g, h \in \mathcal{R}(\alpha)$ , and prove the triangle inequality

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2$$

as a consequence of the Schwarz inequality, as in the proof of Theorem 1.37.

*Proof.*

- (1) By Exercise 6.10(c) with  $p = q = 2$ , we have

$$\begin{aligned} \int_a^b |f - g| |g - h| d\alpha &= \left| \int_a^b |f - g| |g - h| d\alpha \right| \\ &\leq \left\{ \int_a^b |f - g|^2 d\alpha \right\}^{\frac{1}{2}} \left\{ \int_a^b |g - h|^2 d\alpha \right\}^{\frac{1}{2}} \\ &= \|f - g\|_2 \|g - h\|_2. \end{aligned}$$

Every integral is well-defined (Theorem 6.12 and Theorem 6.13 (or Theorem 6.11)).

- (2) Since

$$\begin{aligned} \|f - h\|_2^2 &= \int_a^b |f - h|^2 d\alpha \\ &\leq \int_a^b (|f - g| + |g - h|)^2 d\alpha && \text{(Triangle inequality)} \\ &= \int_a^b (|f - g|^2 + 2|f - g||g - h| + |g - h|^2) d\alpha \\ &= \int_a^b |f - g|^2 d\alpha + 2 \int_a^b |f - g| |g - h| d\alpha + \int_a^b |g - h|^2 d\alpha \\ &\leq \|f - g\|_2^2 + 2\|f - g\|_2 \|g - h\|_2 + \|g - h\|_2^2 && ((1)) \\ &= (\|f - g\|_2 + \|g - h\|_2)^2, \end{aligned}$$

we have

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2.$$

Here every integral is well-defined (Theorem 6.12 and Theorem 6.13 (or Theorem 6.11)).

□

**Exercise 6.12.** With the notations of Exercise 6.11, suppose  $f \in \mathcal{R}(\alpha)$  and  $\varepsilon > 0$ . Prove that there exists a continuous function  $g$  on  $[a, b]$  such that  $\|f - g\|_2 < \varepsilon$ . (Hint: Let  $P = \{a = x_0 \leq \dots \leq x_n = b\}$  be a suitable partition of  $[a, b]$ , define

$$g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i)$$

if  $x_{i-1} \leq t \leq x_i$ .)

*Proof.* Given  $\varepsilon > 0$ .

- (1) There are some real numbers  $m$  and  $M$  such that  $m \leq f(x) \leq M$  if  $x \in [a, b]$  since  $f \in \mathcal{R}(\alpha)$  or  $f$  is bounded on  $[a, b]$ . By Theorem 6.6, there exists a partition  $P = \{a = x_0 \leq \dots \leq x_n = b\}$  such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \frac{\varepsilon^2}{M - m + 1}.$$

Here

$$U(P, f, \alpha) = \sum_{i=1}^n M_i \Delta \alpha_i \text{ where } M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x)$$

$$L(P, f, \alpha) = \sum_{i=1}^n m_i \Delta \alpha_i \text{ where } m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x).$$

- (2) For such partition  $P$ , define  $g$  on  $[a, b]$  by

$$g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i)$$

if  $x_{i-1} \leq t \leq x_i$ . So that

$$\begin{aligned} |f(t) - g(t)| &= \left| \left( \frac{x_i - t}{\Delta x_i} + \frac{t - x_{i-1}}{\Delta x_i} \right) f(t) - \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i) \right| \\ &= \left| \frac{x_i - t}{\Delta x_i} (f(t) - f(x_{i-1})) + \frac{t - x_{i-1}}{\Delta x_i} (f(t) - f(x_i)) \right| \\ &\leq \frac{x_i - t}{\Delta x_i} |f(t) - f(x_{i-1})| + \frac{t - x_{i-1}}{\Delta x_i} |f(t) - f(x_i)| \\ &\leq \frac{x_i - t}{\Delta x_i} (M_i - m_i) + \frac{t - x_{i-1}}{\Delta x_i} (M_i - m_i) \\ &= M_i - m_i \end{aligned}$$

if  $x_{i-1} \leq t \leq x_i$ . Especially,

$$|f(t) - g(t)| \leq M - m$$

if  $a \leq t \leq b$ .

- (3) Note that the integral  $\int_a^b |f - g|^2 d\alpha$  is well-defined (Theorem 6.8, Theorem 6.11 and Theorem 6.12). So that

$$\begin{aligned} \int_a^b |f - g|^2 d\alpha &= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f - g|^2 d\alpha \\ &\leq \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (M - m)(M_i - m_i) d\alpha \\ &= (M - m) \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (M_i - m_i) \Delta\alpha_i \\ &= (M - m)[U(P, f, \alpha) - L(P, f, \alpha)] \\ &\leq (M - m) \cdot \frac{\varepsilon^2}{M - m + 1} \\ &< \varepsilon^2. \end{aligned}$$

Hence,

$$\|f - g\|_2 = \left\{ \int_a^b |f - g|^2 d\alpha \right\}^{\frac{1}{2}} < \varepsilon.$$

□

*Note.*

- (1) Apply the same argument we can prove the following statement:

*Suppose  $f \in \mathcal{R}(\alpha)$  and  $\varepsilon > 0$ . Prove that there exists a continuous function  $g$  on  $[a, b]$  such that  $\int_a^b |f - g| d\alpha < \varepsilon$ .*

- (2) (Lebesgue integral)

- (a) *Let  $f$  be Lebesgue integrable over  $E$ . Then, given  $\varepsilon > 0$ , there is a simple function  $\varphi$  such that*

$$\int_E |f - \varphi| < \varepsilon.$$

- (b) *Under the same hypothesis there is a step function  $\psi$  such that*

$$\int_E |f - \psi| < \varepsilon.$$

- (c) Under the same hypothesis there is a continuous function  $g$  vanishing outside a finite interval such that

$$\int_E |f - g| < \varepsilon.$$

**Exercise 6.13.**  
PLACEHOLDER

**Exercise 6.14.**  
PLACEHOLDER

**Exercise 6.15.** Suppose  $f$  is a real, continuously differentiable function on  $[a, b]$ ,  $f(a) = f(b) = 0$ , and

$$\int_a^b f(x)^2 dx = 1.$$

Prove that

$$\int_a^b x f(x) f'(x) dx = -\frac{1}{2}$$

and that

$$\int_a^b [f'(x)]^2 dx \int_a^b x^2 f(x)^2 dx > \frac{1}{4}.$$

*Proof.* Every integral is well-defined (Theorem 4.9 and Theorem 6.8).

- (1) By Theorem 6.22 (integration by parts),

$$\int_a^b x \left( \frac{f(x)^2}{2} \right)' dx = \left[ x \cdot \frac{f(x)^2}{2} \right]_{x=a}^{x=b} - \int_a^b \frac{f(x)^2}{2} dx,$$

or

$$\int_a^b x f(x) f'(x) dx = \left[ b \cdot \frac{f(b)^2}{2} - a \cdot \frac{f(a)^2}{2} \right] - \frac{1}{2} \int_a^b f(x)^2 dx = -\frac{1}{2}.$$

- (2) By Exercise 6.10(c),

$$\int_a^b [f'(x)]^2 dx \int_a^b x^2 f(x)^2 dx \geq \left( \int_a^b x f(x) f'(x) dx \right)^2 = \frac{1}{4}.$$



- (3) (Reductio ad absurdum) If the equality were holding, then by Exercise 6.10(c)

$$(f'(x))^2 \int_a^b x^2 f(x)^2 dx = x^2 f(x)^2 \int_a^b [f'(x)]^2 dx$$

on  $[a, b]$  (since  $x$ ,  $f(x)$  and  $f'(x)$  are continuous on  $[a, b]$ ).

- (a) *Show that both integrals are nonzero.* (Reductio ad absurdum) If  $\int_a^b x^2 f(x)^2 dx = 0$ , then  $x^2 f(x)^2 = 0$  or  $xf(x) = 0$  on  $[a, b]$  (Exercise 6.2). So that

$$\int_a^b xf(x)f'(x)dx = 0 \neq -\frac{1}{2},$$

which is absurd. Similarly,  $\int_a^b [f'(x)]^2 dx \neq 0$ .

- (b) By (a), we write

$$C = \left\{ \frac{\int_a^b [f'(x)]^2 dx}{\int_a^b x^2 f(x)^2 dx} \right\}^{\frac{1}{2}} > 0$$

be a positive constant. Hence

$$f'(x) = \pm Cxf(x).$$

Here the sign “ $\pm$ ” is not necessary unchanged on  $[a, b]$ . Luckily, we can show that the sign “ $\pm$ ” is unchanged on some subinterval of  $[a, b]$ .

- (c) To find such subinterval of  $[a, b]$ , we consider the zero set  $Z(f')$  and  $Z(xf)$  on  $[a, b]$ . Since  $f'(x) = \pm Cxf(x)$  with  $C > 0$ , we have

$$Z(f') = Z(xf).$$

Note that  $Z(f') = Z(xf)$  is closed (Exercise 4.3) and not equal to  $[a, b]$  (by applying the same argument in (a)). Hence the complement of  $Z(f') = Z(xf)$  is open and nonempty, which can be written as the union of an at most countable collection of disjoint segments (Exercise 2.29).

- (d) Consider any nonempty open interval in (c), say

$$(c, d) \subseteq [a, b].$$

By construction,  $f'(x) \neq 0$  for all  $x \in (c, d)$ . Since  $f'(x)$  is continuous, by Theorem 4.23 there are only two mutually exclusive possible cases:

- (i)  $f'(x) > 0$  for all  $x \in (c, d)$ ,
- (ii)  $f'(x) < 0$  for all  $x \in (c, d)$ .

Similar result for  $xf(x)$ . Therefore, the sign “ $\pm$ ” of  $f'(x) = \pm Cxf(x)$  are unchanged on  $(c, d)$ , that is,

- (i)  $f'(x) = Cxf(x)$  for all  $x \in (c, d)$ ,
- (ii)  $f'(x) = -Cxf(x)$  for all  $x \in (c, d)$ ,
- (e) Suppose  $f'(x) = Cxf(x)$  on  $(c, d)$ . Since  $f'(x)$  and  $xf(x)$  are both vanishing at  $x = c$  and  $x = d$ ,  $f'(x) = Cxf(x)$  at  $x = c$  and  $x = d$ . So

$$f'(x) = Cxf(x) \text{ if } x \in [c, d].$$

Define

$$\phi(x, y) = Cxy$$

be a real function on  $R = [c, d] \times \mathbb{R}$ . And consider the initial-value problem

$$y' = \phi(x, y) \quad \text{with} \quad y(c) = 0.$$

Then

$$|\phi(x, y_2) - \phi(x, y_1)| = Cx|y_2 - y_1| \leq A|y_2 - y_1|$$

where  $A = C \cdot \max\{|c|, |d|\}$  is a constant. By Exercise 5.27, this initial-value problem has at most one solution. Clearly,  $y = f(x) = 0$  on  $[c, d]$  is one solution of this initial-value problem, contrary to the construction of  $[c, d]$ . Similar result for the case  $f'(x) = -Cxf(x)$ .

Therefore, the equality does not hold.

□

**Exercise 6.16.**  
PLACEHOLDER

**Exercise 6.17.**  
PLACEHOLDER

**Exercise 6.18.**  
PLACEHOLDER

**Exercise 6.19.**  
PLACEHOLDER