Chapter 10: Integration of Differential Forms

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Exercise 10.1. *Let H be a compact convex set in $\mathbb{R}^k$, with nonempty interior. Let $f \in \mathscr{C}(H)$, put $f(\mathbf{x}) = 0$ in the complement of H, and define $\int_H f$ as in Definition 10.3. Prove that $\int_H f$ is independent of the order in which the k integrations are carried out. (Hint: Approximate f by functions that are continuous on $\mathbb{R}^k$ and whose supports are in H, as was done in Example 10.4.)*

Proof.

(1)

(2)

□

Exercise 10.2. *For $i = 1, 2, 3, \ldots$, let $\varphi_i \in \mathscr{C}(\mathbb{R}^1)$ have support in $(2^{-i}, 2^{1-i})$, such that $\int \varphi_i = 1$. Put*

$$f(x, y) = \sum_{i=1}^{\infty} [\varphi_i(x) - \varphi_{i+1}(x)]\varphi_i(y)$$

Then f has compact support in $\mathbb{R}^2$, f is continuous except at $(0, 0)$, and

$$\int dy \int f(x, y) dx = 0 \qquad \text{but} \qquad \int dx \int f(x, y) dy = 1.$$

Observe that f is unbounded in every neighborhood of $(0, 0)$.

Proof.

(1) If $f, g : \mathbb{R}^n \to \mathbb{R}^m$ are two functions, then

  (a) $\mathrm{supp}(fg) \subseteq \mathrm{supp}(f) \cap \mathrm{supp}(g)$.
  (b) $\mathrm{supp}(f + g) \subseteq \mathrm{supp}(f) \cup \mathrm{supp}(g)$.

(2) Note that $f(x, y)$ is well-defined on $\mathbb{R}^2$ since only finitely many terms are nonzero for each fixed point $(x, y) \in \mathbb{R}^2$ (by (1)). Besides,

$$\begin{aligned}
&\mathrm{supp}([\varphi_i(x) - \varphi_{i+1}(x)]\varphi_i(y)) \\
\subseteq\ &\{(x, y) : x \in \mathrm{supp}(\varphi_i) \cup \mathrm{supp}(\varphi_{i+1}), y \in \mathrm{supp}(\varphi_i)\} \\
\subseteq\ &\{(x, y) : x \in (2^{-i}, 2^{-i+1}) \cup (2^{-i-1}, 2^{-i}), y \in (2^{-i}, 2^{-i+1})\} \\
\subseteq\ &\{(x, y) : x \in (0, 1), y \in (0, 1)\}
\end{aligned}$$

for all $i = 1, 2, 3, \ldots$. So $\mathrm{supp}(f) \subseteq (0,1)^2$, or $\mathrm{supp}(f)$ is bounded. As $\mathrm{supp}(f)$ is closed (by definition), $\mathrm{supp}(f)$ is compact (Theorem 2.41).

(3) *Show that $f(x,y)$ is not continuous at $(0,0)$.*

(a) Note that $f(0,0) = 0$ since $(0,0) \not\in \mathrm{supp}(f) \subseteq (0,1)^2$. It suffices to show that there exists a sequence $\{(t_n, t_n)\}$ in $\mathbb{R}^2$ such that $(t_n, t_n) \neq (0,0)$, $\lim_{n\to\infty}(t_n, t_n) = (0,0)$ but $\lim_{n\to\infty} f(t_n, t_n)$ does not converge to 0 (Theorem 4.2).

(b) For any $n = 1, 2, 3, \ldots$,

$$1 = \int \varphi_n = \int_{2^{-n}}^{2^{-n+1}} \varphi(t)dt \leq 2^{-n} \sup_{t \in [2^{-n}, 2^{-n+1}]} \varphi(t),$$

or $\sup_{t\in[2^{-n},2^{-n+1}]} \varphi(t) \geq 2^n$. By the continunity of φ_n, there exists $t_n \in [2^{-n}, 2^{-n+1}]$ such that $\varphi_n(t_n) \geq 2^n$ (Theorem 4.16).

(c) We construct $\{(t_n, t_n)\}$ in $\mathbb{R}^2$ by (b) for all $n = 1, 2, 3, \ldots$. Clearly, $(t_n, t_n) \neq (0,0)$ and $\lim_{n\to\infty}(t_n, t_n) = (0,0)$. However,

$$f(t_n, t_n) = [\varphi_n(t_n) - \varphi_{n+1}(t_n)]\varphi_n(t_n) = \varphi_n(t_n)^2 \geq 2^{2n}$$

does not converge to 0 as $n \to \infty$.

(4) *Show that $f(x,y)$ is continuous at $\mathbf{x}_0 = (x_0, y_0) \neq (0,0)$.* Consider an open neighborhood $B(\mathbf{x}_0; r)$ of $\mathbf{x}_0$ with $r = \frac{\|\mathbf{x}_0\|}{64} > 0$. Hence,

$$f(x,y)|_{B(\mathbf{x}_0;r)} = \sum_{i=1}^{N} [\varphi_i(x) - \varphi_{i+1}(x)]\varphi_i(y)$$

is the sum of finitely many terms where $N = \log_2 \frac{89}{\|\mathbf{x}_0\|} \geq 1$ (since $[\varphi_i(x) - \varphi_{i+1}(x)]\varphi_i(y) = 0$ on $B(\mathbf{x}_0; r)$ whenever $i \geq N$). Therefore, $f(x,y)|_{B(\mathbf{x}_0;r)}$ is continuous by the continuity of φ_i.

(5) *Show that $\int dy \int f(x,y)dx = 0$.* For any fixed y, there is a positive integer $N(y)$ such that $\varphi_{N(y)+1}(y) = \varphi_{N(y)+2}(y) = \ldots = 0$ and

$$f(x,y) = \sum_{i=1}^{N(y)} [\varphi_i(x) - \varphi_{i+1}(x)]\varphi_i(y).$$

So

$$\begin{aligned}\int f(x,y)dx &= \int \sum_{i=1}^{N(y)} [\varphi_i(x) - \varphi_{i+1}(x)]\varphi_i(y)dx \\ &= \sum_{i=1}^{N(y)} \varphi_i(y) \int [\varphi_i(x) - \varphi_{i+1}(x)]dx \\ &= \sum_{i=1}^{N(y)} \varphi_i(y) \left(\int \varphi_i(x)dx - \int \varphi_{i+1}(x)dx \right) \\ &= \sum_{i=1}^{N(y)} \varphi_i(y)\,(1-1) \\ &= 0,\end{aligned}$$

and thus

$$\int dy \int f(x,y)dx = \int 0dy = 0.$$

(6) *Show that* $\int dx \int f(x,y)dy = 0$. For any fixed x, there is a positive integer $N(x)$ such that $\varphi_{N(x)+1}(x) = \varphi_{N(x)+2}(x) = \ldots = 0$ and

$$f(x,y) = \sum_{i=1}^{N(x)} [\varphi_i(x) - \varphi_{i+1}(x)]\varphi_i(y).$$

So

$$\begin{aligned}\int f(x,y)dy &= \int \sum_{i=1}^{N(x)} [\varphi_i(x) - \varphi_{i+1}(x)]\varphi_i(y)dy \\ &= \sum_{i=1}^{N(x)} [\varphi_i(x) - \varphi_{i+1}(x)] \int \varphi_i(y)dy \\ &= \sum_{i=1}^{N(x)} [\varphi_i(x) - \varphi_{i+1}(x)] \\ &= \varphi_1(x),\end{aligned}$$

and thus

$$\int dx \int f(x,y)dy = \int \varphi_1(x)dx = 1.$$

□

Exercise 10.3.

(a) *If* $\mathbf{F}$ *is as in Theorem 10.7, put* $\mathbf{A} = \mathbf{F}'(\mathbf{0})$, $\mathbf{F}_1(\mathbf{x}) = \mathbf{A}^{-1}\mathbf{F}(\mathbf{x})$. *Then* $\mathbf{F}_1(\mathbf{0}) = \mathbf{I}$. *Show that*

$$\mathbf{F}_1(\mathbf{x}) = \mathbf{G}_n \circ \mathbf{G}_{n-1} \circ \cdots \circ \mathbf{G}_1(\mathbf{x})$$

in some neighborhood of $\mathbf{0}$, *for certain primitive mappings* $\mathbf{G}_1, \ldots, \mathbf{G}_n$. *This gives another version of Theorem 10.7:*

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}'(\mathbf{0})\mathbf{G}_n \circ \mathbf{G}_{n-1} \circ \cdots \circ \mathbf{G}_1(\mathbf{x}).$$

(b) *Prove that the mapping* $(x, y) \mapsto (y, x)$ *of* $\mathbb{R}^2$ *onto* $\mathbb{R}^2$ *is not the composition of any two primitive mappings, in any neighborhood of the origin. (This shows that the flips* B_i *cannot be omitted from the statement of Theorem 10.7.)*

Proof of (a).

(1) Suppose $\mathbf{F}$ is a $\mathscr{C}'$-mapping of an open set $E \subseteq \mathbb{R}^n$ into $\mathbb{R}^n$, $\mathbf{0} \in E$, $\mathbf{F}(\mathbf{0}) = \mathbf{0}$, and $\mathbf{F}'(\mathbf{0})$ is invertible.

(2) Similar to the proof of Theorem 10.7. Put $\mathbf{F}_1 = \mathbf{F}$.

(3) As $m = 1$, there is an open neighborhood $V_1 \subseteq E$ of $\mathbf{0}$ such that $\mathbf{F}_1(\mathbf{0}) = (\mathbf{F}'(\mathbf{0}))^{-1}\mathbf{F}(\mathbf{0}) = \mathbf{0}$, $\mathbf{F}_1'(\mathbf{0}) = \mathbf{I}$ is invertible, and

$$\mathbf{F}_1(\mathbf{x}) = \sum_{i=1}^{n} \alpha_i(\mathbf{x})\mathbf{e}_i,$$

where $\alpha_1, \ldots, \alpha_n$ are real $\mathscr{C}'$-functions in V_1. Hence

$$\mathbf{F}_1'(\mathbf{0})\mathbf{e}_1 = \sum_{i=1}^{n} (D_1\alpha_i)(\mathbf{0})\mathbf{e}_i.$$

Note that $(D_1\alpha_1)(\mathbf{0}) = 1 \neq 0$, and we might pick $B_1 = \mathbf{I}$. Thus we can define

$$\mathbf{G}_1(\mathbf{x}) = \mathbf{x} + [\alpha_1(\mathbf{x}) - x_1]\mathbf{e}_1 \qquad (\mathbf{x} \in V_1).$$

Then $\mathbf{G}_1 \in \mathscr{C}'(V_1)$, $\mathbf{G}_1$ is primitive, and $\mathbf{G}_1'(\mathbf{0}) = \mathbf{I}$ is invertible.

(4) Now we make the induction hypothesis for $1 \leq m \leq n - 1$.

(5) Since $\mathbf{G}_m'(\mathbf{0}) = \mathbf{I}$ is invertible, the inverse function theorem shows that there is an open set U_m, with $\mathbf{0} \in U_m \subseteq V_m$, such that $\mathbf{G}_m$ is an injective mapping of U_m onto a neighborhood V_{m+1} of $\mathbf{0}$, in which $\mathbf{G}_m^{-1} \in \mathscr{C}'(V_{m+1})$. Define $\mathbf{F}_{m+1}$ by

$$\mathbf{F}_{m+1}(\mathbf{y}) = \mathbf{F}_m \circ \mathbf{G}_m^{-1}(\mathbf{y}) \qquad (\mathbf{y} \in V_{m+1}).$$

Then $\mathbf{F}_{m+1} \in \mathscr{C}'(V_{m+1})$, $\mathbf{F}_m(\mathbf{0}) = \mathbf{0}$, and $\mathbf{F}'_{m+1}(\mathbf{0}) = \mathbf{I}$ is invertible by the chain rule and the inverse function theorem. So

$$\mathbf{F}_{m+1}(\mathbf{x}) = P_m\mathbf{x} + \sum_{i=m+1}^{n} \alpha_i(\mathbf{x})\mathbf{e}_i,$$

where $\alpha_1, \ldots, \alpha_n$ are real $\mathscr{C}'$-functions in V_{m+1}. Hence

$$\mathbf{F}'_{m+1}(\mathbf{0})\mathbf{e}_{m+1} = \sum_{i=m+1}^{n} (D_{m+1}\alpha_i)(\mathbf{0})\mathbf{e}_i.$$

Note that $(D_{m+1}\alpha_{m+1})(\mathbf{0}) = 1 \neq 0$, and we might pick $B_{m+1} = \mathbf{I}$. Thus we can define

$$\mathbf{G}_{m+1}(\mathbf{x}) = \mathbf{x} + [\alpha_{m+1}(\mathbf{x}) - x_{m+1}]\mathbf{e}_{m+1} \qquad (\mathbf{x} \in V_{m+1}).$$

Then $\mathbf{G}_{m+1} \in \mathscr{C}'(V_{m+1})$, $\mathbf{G}_{m+1}$ is primitive, and $\mathbf{G}'_{m+1}(\mathbf{0}) = \mathbf{I}$ is invertible. Our induction hypothesis holds therefore with $m+1$ in place of m.

(6) Note that

$$\mathbf{F}_m(\mathbf{x}) = \mathbf{F}_{m+1}(\mathbf{G}_m(\mathbf{x})) \qquad (\mathbf{x} \in U_m).$$

If we apply this with $m = 1, \ldots, n-1$, we successively obtain

$$\mathbf{F}_1 = \mathbf{F}_n \circ \mathbf{G}_{n-1} \circ \cdots \circ \mathbf{G}_1$$

in some open neighborhood of $\mathbf{0}$. Note that $\mathbf{F}_n$ is primitive since

$$\mathbf{F}_n(\mathbf{x}) = P_{n-1}\mathbf{x} + \alpha_n(\mathbf{x})\mathbf{e}_n.$$

This completes the proof.

$\square$

Proof of (b).

(1) For $(x, y) \in \mathbb{R}^2$, define

$$\mathbf{F}(x, y) = (y, x).$$

(2) (Reductio ad absurdum) If $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$ for some primitive mappings $\mathbf{G}_i$ $(i = 1, 2)$ in some neighborhood V_i of the origin, $\mathbf{G}_i(\mathbf{0}) = \mathbf{0}$ and $\mathbf{G}'_i$ is invertible, then we may assume that

$$\mathbf{G}_1(x, y) = (x, g_1(x, y)) \qquad \text{and} \qquad \mathbf{G}_2(x, y) = (g_2(x, y), y).$$

Here the case $\mathbf{G}_1(x, y) = (g_1(x, y), y)$ and $\mathbf{G}_2(x, y) = (x, g_2(x, y))$ is similar to the above case. Besides, $\mathbf{G}_1(x, y) = (x, g_1(x, y))$ and $\mathbf{G}_2(x, y) = (x, g_2(x, y))$ implies that

$$\mathbf{G}_2 \circ \mathbf{G}_1(x, y) = (x, g_2(x, g_1(x, y))) \neq (y, x) = \mathbf{F}(x, y).$$

Same reason for $\mathbf{G}_1(x, y) = (g_1(x, y), y)$ and $\mathbf{G}_2(x, y) = (g_2(x, y), y)$.

(3) Note that

$$\mathbf{F}'(\mathbf{0}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Since

$$\mathbf{F}'(\mathbf{0}) = \mathbf{G}_2'(\mathbf{G}_1(\mathbf{0}))\mathbf{G}_1'(\mathbf{0}) = \mathbf{G}_2'(\mathbf{0})\mathbf{G}_1'(\mathbf{0}),$$

we have

$$\begin{aligned} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} &= \begin{bmatrix} D_1 g_2(0,0) & D_2 g_2(0,0) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ D_1 g_1(0,0) & D_2 g_1(0,0) \end{bmatrix} \\ &= \begin{bmatrix} * & * \\ D_1 g_1(0,0) & D_2 g_1(0,0) \end{bmatrix}. \end{aligned}$$

So $D_1 g_1(0,0) = 1$ and $D_2 g_1(0,0) = 0$, and thus $\mathbf{G}_1'(\mathbf{0}) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ is not invertible, which is absurd.

□

Exercise 10.4. *For* $(x, y) \in \mathbb{R}^2$*, define*

$$\mathbf{F}(x, y) = (e^x \cos y - 1, e^x \sin y)$$

Prove that $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$*, where*

$$\begin{aligned} \mathbf{G}_1(x, y) &= (e^x \cos y - 1, y) \\ \mathbf{G}_2(u, v) &= (u, (1 + u) \tan v) \end{aligned}$$

are primitive in some neighborhood of $(0, 0)$*. Compute the Jacobians of* $\mathbf{G}_1$*,* $\mathbf{G}_2$*,* $\mathbf{F}$ *at* $(0, 0)$*. Define*

$$\mathbf{H}_2(x, y) = (x, e^x \sin y)$$

and find

$$\mathbf{H}_1(u, v) = (h(u, v), v)$$

so that $\mathbf{F} = \mathbf{H}_1 \circ \mathbf{H}_2$ *is in some neighborhood of* $(0, 0)$*.*

Proof.

(1) By Definition 10.5,

$$\begin{aligned} \mathbf{G}_1(x, y) &= (e^x \cos y - 1)\mathbf{e}_1 + y\mathbf{e}_2, \\ \mathbf{G}_2(u, v) &= u\mathbf{e}_1 + ((1 + u) \tan v)\mathbf{e}_2 \end{aligned}$$

are primitive in some neighborhood of $(0, 0)$.

(2) *Show that* $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$. Given any $(x, y) \in \mathbb{R}^2$, we have

$$\begin{aligned}
(\mathbf{G}_2 \circ \mathbf{G}_1)(x, y) &= \mathbf{G}_2(\mathbf{G}_1(x, y)) \\
&= \mathbf{G}_2(e^x \cos y - 1, y) \\
&= (e^x \cos y - 1, (1 + (e^x \cos y - 1)) \tan y) \\
&= (e^x \cos y - 1, e^x \sin y) \\
&= \mathbf{F}(x, y).
\end{aligned}$$

(3) Since

$$\begin{aligned}
J_{\mathbf{G}_1}(x, y) &= \det \begin{bmatrix} e^x \cos y & -e^x \sin y \\ 0 & 1 \end{bmatrix} = e^x \cos y \\
J_{\mathbf{G}_2}(x, y) &= \det \begin{bmatrix} 1 & 0 \\ \tan y & (1 + x) \sec^2 y \end{bmatrix} = (1 + x) \sec^2 y \\
J_{\mathbf{F}}(x, y) &= \det \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix} = e^{2x},
\end{aligned}$$

$$\begin{aligned}
J_{\mathbf{G}_1}(0, 0) &= \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \\
J_{\mathbf{G}_2}(0, 0) &= \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \\
J_{\mathbf{F}}(0, 0) &= \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1.
\end{aligned}$$

(4) Define $h(u, v) = \sqrt{e^{2u} - v^2} - 1$ on

$$B\left((0, 0); \frac{1}{64}\right) \subseteq \mathbb{R}^2.$$

$h(u, v)$ is well-defined since $e^{2u} - v^2 > 0$ for all $(u, v) \in B\left((0, 0); \frac{1}{64}\right)$.

(5) Given any $(x, y) \in \mathbb{R}^2$, we have

$$\begin{aligned}
(\mathbf{H}_1 \circ \mathbf{H}_2)(x, y) &= \mathbf{H}_1(\mathbf{H}_2(x, y)) \\
&= \mathbf{H}_1(x, e^x \sin y) \\
&= (\sqrt{e^{2x} - (e^x \sin y)^2} - 1, e^x \sin y) \\
&= (e^x \cos y - 1, e^x \sin y) \\
&= \mathbf{F}(x, y).
\end{aligned}$$

$\square$

Exercise 10.5. *Formulate and prove an analogue of Theorem 10.8, in which K is a compact subset of an arbitrary metric space. (Replace the functions φ_i that occur in the proof of Theorem 10.8 by functions of the type constructed in Exercise 4.22.)*

Proof (Theorem 10.8).

(1) *(Partitions of unity.) Suppose K is a compact subset of a metric space X, and $\{V_\alpha\}$ is an open cover of K. Then there exist functions $\psi_1, \dots, \psi_s \in \mathscr{C}(X)$ such that*

   (a) $0 \leq \psi_i \leq 1$ *for* $1 \leq i \leq s$.

   (b) *each ψ_i has its support in some V_α, and*

   (c) $\psi_1(x) + \cdots + \psi_s(x) = 1$ *for every* $x \in K$.

(2) It is trivial that some $V_\alpha = X$ by taking $s = 1$ and $\psi_1(x) = 1 \in \mathscr{C}(X)$. Now we assume that all $V_\alpha \subsetneq X$.

(3) Associate with each $x \in K$ an index $\alpha(x)$ so that $x \in V_{\alpha(x)}$. Then there are open balls $B(x)$ and $W(x)$, centered at x, with

$$x \in B(x) \subseteq \overline{B(x)} \subseteq W(x) \subseteq \overline{W(x)} \subseteq V_{\alpha(x)}$$

(Since $V_{\alpha(x)}$ is open, there exists $r > 0$ such that $B(x; r) \subseteq V_{\alpha(x)}$. Take $B(x) = B\left(x; \frac{r}{89}\right)$ and $W(x) = B\left(x; \frac{r}{64}\right)$.)

(4) Since K is compact, there are finitely many points $x_1, \dots, x_s \in K$ such that

$$K \subseteq B(x_1) \cup \cdots \cup B(x_s).$$

Note that

   (a) $\overline{B(x_i)}$ is a nonempty closed set since $x_i \in B(x_i) \subseteq \overline{B(x_i)}$.

   (b) $X - W(x_i) \supseteq X - V_{\alpha(x_i)}$ is a nonempty closed set by the assumption in (2).

   (c) $\overline{B(x_i)} \cap (X - W(x_i)) \subseteq W(x_i) \cap (X - W(x_i)) = \varnothing$.

By Exercise 4.22, there is a function

$$\varphi_i(x) = \frac{\rho_{\overline{B(x_i)}}(x)}{\rho_{\overline{B(x_i)}}(x) + \rho_{X - W(x_i)}(x)} \in \mathscr{C}(X)$$

such that $\varphi_i(x) = 1$ on $\overline{B(x_i)}$, $\varphi_i(x) = 0$ outside $W(x_i)$, and $0 \leq \varphi_i(x) \leq 1$ on X for $1 \leq i \leq s$.

(5) Define $\psi_1 = \varphi_1 \in \mathscr{C}(X)$ and

$$\psi_{i+1} = (1 - \varphi_1) \cdots (1 - \varphi_i)\varphi_{i+1} \in \mathscr{C}(X)$$

for $1 \leq i \leq s-1$. Properties (a) and (b) in (1) are clear. Also,

$$\psi_1(x) + \cdots + \psi_s(x) = 1 - (1 - \varphi_1(x)) \cdots (1 - \varphi_s(x))$$

by the construction of ψ_i. If $x \in K$, then $x \in B(x_i)$ for some i, hence $\varphi_i(x) = 1$, and the product $(1 - \varphi_1(x)) \cdots (1 - \varphi_s(x)) = 0$. This proves property (c) in (1).

□

Exercise 10.6. *Strengthen the conclusion of Theorem 10.8 by showing that the functions ψ_i can be made differentiable, and even infinitely differentiable. (Use Exercise 8.1 in the construction of the auxiliary functions ψ_i.)*

Proof (Theorem 10.8).

(1) It is trivial that some $V_\alpha = \mathbb{R}^n$ by taking $s = 1$ and $\psi_1(\mathbf{x}) = 1 \in \mathscr{C}^\infty(\mathbb{R}^n)$. Now we assume that all $V_\alpha \subsetneq \mathbb{R}^n$.

(2) Associate with each $\mathbf{x} \in K$ an index $\alpha(x)$ so that $\mathbf{x} \in V_{\alpha(x)}$. Then there are open n-cells $B(\mathbf{x})$ and $W(\mathbf{x})$ (Definition 10.1), centered at $\mathbf{x}$, with

$$\mathbf{x} \in B(\mathbf{x}) \subseteq \overline{B(\mathbf{x})} \subseteq W(\mathbf{x}) \subseteq \overline{W(\mathbf{x})} \subseteq V_{\alpha(\mathbf{x})}$$

(Since $V_{\alpha(\mathbf{x})}$ is open, there exists $r > 0$ such that $B(\mathbf{x}; r) \subseteq V_{\alpha(\mathbf{x})}$. Take

$$B(\mathbf{x}) = I\left(\mathbf{x}; \frac{r}{89\sqrt{n}}\right), \qquad W(\mathbf{x}) = I\left(\mathbf{x}; \frac{r}{64\sqrt{n}}\right)$$

where $I(\mathbf{p}; r)$ is the open n-cell centered at $\mathbf{p} = (p_1, \ldots, p_n)$ defined by

$$I(\mathbf{p}; r) = (p_1 - r, p_1 + r) \times \cdots \times (p_n - r, p_n + r) \subseteq \mathbb{R}^n.)$$

(3) Define

$$f(y) = \begin{cases} e^{-\frac{1}{y^2}} & (y > 0), \\ 0 & (y \leq 0). \end{cases}$$

$f(y) \in \mathscr{C}^\infty(\mathbb{R}^1)$ by applying the similar argument in Exercise 8.1.

(4) Given any $\mathbf{x} = (x_1, \ldots, x_n) \in K$ and construct $B(\mathbf{x})$ and $W(\mathbf{x})$ as in (2). Define

$$g_{x_j}(y_j) = \frac{f(y_j)}{f(y_j) + f\left(\frac{r}{64\sqrt{n}} - \frac{r}{89\sqrt{n}} - y_j\right)}$$

for $1 \leq j \leq n$. g_{x_j} is well-defined and $g_{x_j} \in \mathscr{C}^\infty(\mathbb{R}^1)$. So

$$g_{x_j}(y_j) = \begin{cases} 0 & \text{if } y_j \leq 0, \\ \text{strictly increasing} & \text{if } 0 \leq y_j \leq \frac{r}{64\sqrt{n}} - \frac{r}{89\sqrt{n}}, \\ 1 & \text{if } y_j \geq \frac{r}{64\sqrt{n}} - \frac{r}{89\sqrt{n}}. \end{cases}$$

Next, define

$$h_{x_j}(y_j) = g_{x_j}\left(y_j - x_j + \frac{r}{64\sqrt{n}}\right) g_{x_j}\left(x_j + \frac{r}{64\sqrt{n}} - y_j\right)$$

for $1 \leq j \leq n$. $h_{x_j} \in \mathscr{C}^\infty(\mathbb{R}^1)$. So

$$h_{x_j}(y_j) = \begin{cases} 0 & \text{if } y_j \leq x_j - \frac{r}{64\sqrt{n}}, \\ \text{strictly increasing} & \text{if } x_j - \frac{r}{64\sqrt{n}} \leq y_j \leq x_j - \frac{r}{89\sqrt{n}}, \\ 1 & \text{if } x_j - \frac{r}{89\sqrt{n}} \leq y_j \leq x_j + \frac{r}{89\sqrt{n}}, \\ \text{strictly decreasing} & \text{if } x_j + \frac{r}{89\sqrt{n}} \leq y_j \leq x_j + \frac{r}{64\sqrt{n}}, \\ 0 & \text{if } y_j \geq x_j + \frac{r}{64\sqrt{n}}. \end{cases}$$

Finally we define $\mathbf{h}_\mathbf{x} : \mathbb{R}^n \to \mathbb{R}^1$ by

$$\mathbf{h}_\mathbf{x}(\mathbf{y}) = \prod_{j=1}^{n} h_{x_j}(y_j)$$

where $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Hence, $\mathbf{h}_\mathbf{x} \in \mathscr{C}^\infty(\mathbb{R}^n)$ (Theorem 9.21). Also, $\mathbf{h}_\mathbf{x}(\mathbf{y}) = 1$ on $\overline{B(\mathbf{x})}$, $\mathbf{h}_\mathbf{x}(\mathbf{y}) = 0$ outside $W(\mathbf{x})$, and $0 \leq \mathbf{h}_\mathbf{x}(\mathbf{y}) \leq 1$.

(5) Since K is compact, there are finitely many points $\mathbf{x}_1, \ldots, \mathbf{x}_s \in K$ such that

$$K \subseteq B(\mathbf{x}_1) \cup \cdots \cup B(\mathbf{x}_s).$$

Take

$$\varphi_i(\mathbf{x}) = \mathbf{h}_{\mathbf{x}_i}(\mathbf{x}) \in \mathscr{C}^\infty(\mathbb{R}^n)$$

for $1 \leq i \leq s$.

(6) The rest are the same as the proof of Theorem 10.8 or Exercise 10.5.

$\square$

Exercise 10.7.

(a) *Show that the simplex Q^k is the smallest convex subset of $\mathbb{R}^k$ such that contains $\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_k$.*

(b) *Show that affine mappings take convex sets to convex sets.*

Proof of (a).

(1) *Show that Q^k contains $\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_k$.* Recall

$$Q^k = \{(x_1, \ldots, x_k) \in \mathbb{R}^k : x_1 + \cdots + x_k \leq 1 \text{ and } x_1, \ldots, x_k \geq 0\}$$

(Example 10.14). Hence $\mathbf{0} = (0, \ldots, 0) \in Q^k$ and

$$\mathbf{e}_i = (0, \ldots, \underbrace{1}_{i\text{th coordinate}}, \ldots, 0) \in Q^k.$$

(2) *Show that Q^k is a convex subset of $\mathbb{R}^k$.* Given any $\mathbf{x} = (x_1, \dots, x_k) \in Q^k$, $\mathbf{y} = (y_1, \dots, y_k) \in Q^k$ and $0 < \lambda < 1$. Hence

$$\lambda \mathbf{x} + (1-\lambda)\mathbf{y} = (\lambda x_1 + (1-\lambda)y_1, \dots, \lambda x_k + (1-\lambda)y_k) \in Q^k$$

since each $\lambda x_i + (1-\lambda)y_i \geq 0$ and

$$\sum_{i=1}^{k} (\lambda x_i + (1-\lambda)y_i) = \lambda \sum_{i=1}^{k} x_i + (1-\lambda) \sum_{i=1}^{k} y_i \leq \lambda + (1-\lambda) = 1.$$

(3) *Given any convex set $E \subseteq \mathbb{R}^k$ containing $\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_k$. Show that $E \supseteq Q^k$.*

(a) Induction on k. Base case: $k = 1$. Given any $\mathbf{x} = (x_1) \in Q^1$. We have $0 \leq x_1 \leq 1$ by the definition of Q^1. So that $\mathbf{x} = x_1 \mathbf{e}_1 + (1-x_1)\mathbf{0} \in E$ since $\mathbf{0}, \mathbf{e}_1 \in E$ and E is convex.

(b) Inductive step: suppose the statement holds for $k = n$. Given any $\mathbf{x} = (x_1, \dots, x_n, x_{n+1}) \in Q^{n+1}$. If $x_{n+1} = 1$, then $x_1 = \cdots = x_n = 0$ by the definition of Q^{n+1}. So $\mathbf{x} = \mathbf{e}_{n+1} \in E$ by the assumption of E. If $0 \leq x_{n+1} < 1$, then $x_1 + \cdots + x_n \leq 1 - x_{n+1}$ or

$$\frac{x_1}{1 - x_{n+1}} + \cdots + \frac{x_n}{1 - x_{n+1}} \leq 1.$$

So the point

$$\left(\frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}} \right) \in Q^n,$$

or

$$\left(\frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}}, 0 \right), \text{ say } \widehat{\mathbf{x}}, \in E$$

by the induction hypothesis. Note that $\mathbf{e}_{n+1} \in E$. Hence

$$\mathbf{x} = x_{n+1}\mathbf{e}_{n+1} + (1 - x_{n+1})\widehat{\mathbf{x}} \in E$$

by the convexity of E.

(c) Conclusion: Since both the base case and the inductive step have been proved as true, by mathematical induction the statement holds.

□

Proof of (b).

(1) Let $\mathbf{f}$ be an affine mapping that carries a vector space X into a vector space Y such that

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{0}) + A\mathbf{x}$$

for some $A \in L(X, Y)$.

(2) Given any convex subset C of X. To show that $\mathbf{f}(C)$ is convex, it suffices to show that

$$\lambda \mathbf{y}_1 + (1-\lambda)\mathbf{y}_2 \in \mathbf{f}(C)$$

for any $\mathbf{y}_1, \mathbf{y}_2 \in \mathbf{f}(C)$ and $0 < \lambda < 1$. Write $\mathbf{y}_1 = \mathbf{f}(\mathbf{x}_1)$, $\mathbf{y}_2 = \mathbf{f}(\mathbf{x}_2)$ for some $\mathbf{x}_1, \mathbf{x}_2 \in C$. Note that $\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \in C$ by the convexity of C. Hence

$$\begin{aligned}
&\mathbf{f}(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \\
=&\mathbf{f}(\mathbf{0}) + A(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \\
=&\mathbf{f}(\mathbf{0}) + \lambda A\mathbf{x}_1 + (1-\lambda)A\mathbf{x}_2 && (A \in L(X,Y)) \\
=&\lambda(\mathbf{f}(\mathbf{0}) + A\mathbf{x}_1) + (1-\lambda)(\mathbf{f}(\mathbf{0}) + A\mathbf{x}_2) \\
=&\lambda \mathbf{f}(\mathbf{x}_1) + (1-\lambda)\mathbf{f}(\mathbf{x}_2) \\
=&\lambda \mathbf{y}_1 + (1-\lambda)\mathbf{y}_2 \in \mathbf{f}(C).
\end{aligned}$$

□

Exercise 10.8. *Let H be the parallelogram in $\mathbb{R}^2$ whose vertices are $(1,1)$, $(3,2)$, $(4,5)$, $(2,4)$. Find the affine map T which sends $(0,0)$ to $(1,1)$, $(1,0)$ to $(3,2)$, $(1,1)$ to $(4,5)$, $(0,1)$ to $(2,4)$. Show that $J_T = 5$. Use T to convert the integral*

$$\alpha = \int_H e^{x-y} dx\, dy$$

to an integral over I^2 and thus compute α.

Proof.

(1) By Affine simplexes 10.26,

$$T(\mathbf{x}) = T(\mathbf{0}) + A\mathbf{x},$$

where $A \in L(\mathbb{R}^2, \mathbb{R}^2)$, say $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Note that $T : \begin{bmatrix} 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus

$$T : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 + ax + by \\ 1 + cx + dy \end{bmatrix}.$$

(2) By $T : (1,0) \mapsto (3,2)$ and $T : (0,1) \mapsto (2,4)$, we can solve A as

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

It is easy to verify such

$$T : \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{x}} \mapsto \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{T(\mathbf{0})} + \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{x}} = \begin{bmatrix} 1 + 2x + y \\ 1 + x + 3y \end{bmatrix}$$

satisfying our requirement.

(3)
$$J_T = \det \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = 5.$$

(4) By Example 10.4 and Theorem 10.9, we have

$$\begin{aligned}
\int_H e^{x-y} dx\, dy &= \int_{I^2} e^{(1+2u+v)-(1+u+3v)} |J_T| du\, dv \\
&= 5 \int_{I^2} e^{u-2v} du\, dv \\
&= 5 \left\{ \int_0^1 e^u du \right\} \left\{ \int_0^1 e^{-2v} dv \right\} && \text{(Theorem 10.2)} \\
&= \frac{5}{2}(e-1)(1-e^{-2}).
\end{aligned}$$

□

Exercise 10.9. *Define $(x, y) = T(r, \theta)$ one the rectangle*
$$0 \le r \le a, \qquad 0 \le \theta \le 2\pi$$
by the equations
$$x = r\cos\theta, \qquad y = r\sin\theta.$$
Show that T maps this rectangle onto the closed disc D with center at $(0, 0)$ and radius a, that T is one-to-one in the interior of the rectangle, and that $J_T(r, \theta) = r$. If $f \in \mathscr{C}(D)$, prove the formula for integration in polar coordinates:
$$\int_D f(x, y) dx\, dy = \int_0^a \int_0^{2\pi} f(T(r, \theta)) r dr\, d\theta.$$

(Hint: Let D_0 be the interior of D, minus the interval from $(0, 0)$ to $(a, 0)$. As it stands, Theorem 10.9 applies to continuous functions f whose support lies in D_0. To remove this restriction, proceed as in Example 10.4.)

Proof. Define $E = [0, a] \times [0, 2\pi]$.

(1) *Show that T maps E onto D.* Given any $(x, y) \in D$.

(a) It is equivalent to solve (r, θ) from
$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$
in terms of (x, y). Let L be the closed interval from $(0, 0)$ to $(a, 0)$, say $L = \{(r, 0) \in \mathbb{R}^2 : 0 \le r \le a\} \subseteq D$.

(b) If $(x,y) \in L$, say $(x,y) = (r,0)$ for some $r \in [0,a]$, then there exists $(r,0)$ or $(r,2\pi)$ such that $T(r,0) = T(r,2\pi) = (r,0)$. (Therefore, T is not one-to-one on L.)

(c) If $(x,y) \in D - L$, then there is $r = (x^2+y^2)^{\frac{1}{2}}$ in $[0,a]$ (since $(x,y) \in D = \{(x,y) \in \mathbb{R}^2 : (x^2+y^2)^{\frac{1}{2}} \leq a\}$). Define

$$\theta = \begin{cases} \arccos\left(\frac{x}{r}\right) & \text{if } y \geq 0, \\ 2\pi - \arccos\left(\frac{x}{r}\right) & \text{if } y < 0. \end{cases}$$

It is well-defined since $r \neq 0$. Besides, $\theta \in [0,2\pi]$ and $T(r,\theta) = (x,y)$.

(2) *Show that T is one-to-one in the interior of the rectangle E.* Suppose $(r_1,\theta_1),(r_2,\theta_2) \in \mathrm{int}(E) = (0,a) \times (0,2\pi)$ and $T(r_1,\theta_1) = T(r_2,\theta_2) = (x,y) \in D$. Then

$$\begin{aligned} x &= r_1\cos\theta_1 = r_2\cos\theta_2, \\ y &= r_1\sin\theta_1 = r_2\sin\theta_2. \end{aligned}$$

Note that $r_1^2 = r_2^2 = x^2+y^2$ and $r_1, r_2 > 0$, we have $r_1 = r_2$. Solve $\cos\theta_1 = \cos\theta_2$ and $\sin\theta_1 = \sin\theta_2$ to get $\theta_1 = \theta_2 + 2m\pi$ for all $m \in \mathbb{Z}$. Here m must be zero since $\theta_1, \theta_2 \in (0,2\pi)$. Therefore, $(r_1,\theta_1) = (r_2,\theta_2)$.

(3) $T(\mathrm{int}(E)) = D_0 = \mathrm{int}(D) - L$ (by (1)(2)).

(4) *Show that $J_T(r,\theta) = r$.*

$$J_T(r,\theta) = \det\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r.$$

(5) *If $f \in \mathscr{C}(D)$, show that*

$$\int_D f(x,y)dx\,dy$$

is well-defined. Similar to Example 10.4.

(a) Extend f to a function on $I^2 = [-a,a]^2$ by setting $f(x,y) = 0$ off D, and define

$$\int_D f = \int_{I^2} f.$$

Since f may be discontinuous on I^2, the existence of the integral $\int_{I^2} f$. We also wish to show that this integral is independent of the order in which the 2 integrations are carried out.

(b) To do this, suppose $0 < \delta < 1$, put

$$\varphi_\delta(t) = \begin{cases} 1 & \text{if } t \leq 1-\delta, \\ \frac{1-t}{\delta} & \text{if } 1-\delta \leq t \leq 1, \\ 0 & \text{if } t \geq 1, \end{cases}$$

and define

$$F_\delta(x,y) = \varphi_\delta\left(\frac{\sqrt{x^2+y^2}}{a}\right) f(x,y).$$

Then $F_\delta \in \mathscr{C}(I^2)$ (or $\mathscr{C}(\mathbb{R}^2)$).

(c) For each $x \in [-a,a]$, the set of all y such that $F_\delta(x,y) \neq f(x,y)$ is contained a union of two segment whose length does not exceed

$$a\sqrt{1^2-(1-\delta)^2} = a\sqrt{2\delta-\delta^2} < a\sqrt{2\delta}.$$

Since $0 \leq \varphi_\delta \leq 1$, it follows that

$$\left|\int_{-a}^{a} F_\delta(x,y)dy - \int_{-a}^{a} f(x,y)dy\right| \leq 2a\sqrt{2\delta}\|f\|$$

where $\|f\| = \max_{(x,y)\in I^2}|f(x,y)|$. So the sequence of continuous function

$$\left\{\int_{-a}^{a} F_\delta(x,y)dy\right\} \to \int_{-a}^{a} f(x,y)dy := g(x)$$

uniformly as $\delta \to 0$. ($\delta = \frac{1}{n}$ for example.) So $g(x) \in \mathscr{C}([-a,a])$, and the further integrations present no problem, that is, $\int_{I^2} f$ is existed.

(d) Moreover,

$$\left|\int_{I^2} F_\delta(x,y)dxdy - \int_{I^2} f(x,y)dxdy\right| \leq 4a^2\sqrt{2\delta}\|f\|$$

It is true, regardless of the order in which the 2 single integrations are carried out. Since $F_\delta \in \mathscr{C}(I^2)$, $\int F_\delta$ is unaffected by any change in this order. Hence the inequality shows that the same is true of $\int f$.

(6) *Show that*

$$\int_D f(x,y)dx\,dy = \int_0^a \int_0^{2\pi} f(T(r,\theta))r dr\,d\theta.$$

(a) Note that T is a one-to-one $\mathscr{C}'$-mapping of an open set D_0 such that $J_T(r,\theta) = r \neq 0$ for all $(r,\theta) \in \mathrm{int}(E)$. To apply Theorem 10.9, we will leverage Example 10.4 again.

(b) Given any $\min\left\{\frac{a}{89}, \frac{\pi}{64}\right\} > \delta > 0$. Define

$$E_\delta = [\delta, a-\delta] \times [\delta, 2\pi-\delta] \subseteq \mathrm{int}(E)$$

and $D_\delta = T(E_\delta)$. Similar to Example 10.4, let φ_δ be a continuous function on $\mathbb{R}^2$ such that $\varphi_\delta(x,y) = 1$ on D_δ and $\varphi_\delta(x,y) = 0$ off $D_{\frac{\delta}{2}}$. Consider $f_\delta(x,y) = \varphi_\delta(x,y)f(x,y)$ on $\mathbb{R}^2$. By construction, f_δ is a continuous function on $\mathbb{R}^2$ whose support is compact and lies in $D_{\frac{\delta}{2}} \subseteq D_0$. Hence, by Theorem 10.9

$$\int_{\mathbb{R}^2} f_\delta = \int_{D_{\frac{\delta}{2}}} f_\delta = \int_{\frac{\delta}{2}}^{a-\frac{\delta}{2}} \int_{\frac{\delta}{2}}^{2\pi-\frac{\delta}{2}} f(T(r,\theta))rdrd\theta.$$

(c) Since $f \in \mathscr{C}(D)$ and Exercise 6.7,

$$\lim_{\delta \to 0} \int_{\frac{\delta}{2}}^{a-\frac{\delta}{2}} \int_{\frac{\delta}{2}}^{2\pi-\frac{\delta}{2}} f(T(r,\theta)) r dr d\theta = \int_0^a \int_0^{2\pi} f(T(r,\theta)) r dr d\theta.$$

Therefore, it suffices to show that

$$\lim_{\delta \to 0} \int_{\mathbb{R}^2} f_\delta = \int_{\mathbb{R}^2} f.$$

(d) Note that $\int_{\mathbb{R}^2} f = \int_D f$ (by (5)) and $\int_{\mathbb{R}^2} f_\delta = \int_{D_{\frac{\delta}{2}}} f_\delta = \int_D f_\delta$ (by (b)). So

$$\left| \int_{\mathbb{R}^2} f - \int_{\mathbb{R}^2} f_\delta \right| = \left| \int_D f - \int_D f_\delta \right|.$$

To estimate the difference between $\int_D f$ and $\int_D f_\delta$, we notice that f and f_δ are coincide on D_δ by construction. Given any $(x, y) \in D - D_\delta$. Fix y, the set of all x such that $f(x, y) \neq f_\delta(x, y)$ is contained a union of two segment whose length does not exceed $\sqrt{a^2 - (a-\delta)^2 \cos^2 \delta}$. Similarly as in (5), we have

$$\left| \int_D f - \int_D f_\delta \right| \leq 4a\sqrt{a^2 - (a-\delta)^2 \cos^2 \delta} \|f\|.$$

Hence, $\lim \int_D f_\delta = \int_D f$.

$\square$

Exercise 10.10. *Let $a \to \infty$ in Exercise 10.9 and prove that*

$$\int_{\mathbb{R}^2} f(x, y) dx\, dy = \int_0^\infty \int_0^{2\pi} f(T(r,\theta)) r dr\, d\theta,$$

for continuous functions f that decrease sufficiently rapidly as $|x| + |y| \to \infty$. (Find a more precise formulation.) Apply this to

$$f(x, y) = \exp\left(-x^2 - y^2\right)$$

to derive formula

$$\int_{-\infty}^{\infty} e^{-s^2} ds = \sqrt{\pi}.$$

Proof.

(1)

(2)

$\square$

Exercise 10.11. *Define* $(u, v) = T(s, t)$ *on the strip*

$$0 < s < \infty, \qquad 0 < t < 1$$

by setting $u = s - st$, $v = st$. *Show that* T *is a* 1-1 *mapping of the strip onto the positive quadrant* Q *in* $\mathbb{R}^2$. *Show that* $J_T(s, t) = s$. *For* $x > 0$, $y > 0$, *integrate*

$$u^{x-1}e^{-u}v^{y-1}e^{-v}$$

over Q, *use Theorem 10.9 to convert the integral to one over the strip, and derive*

$$\int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

in this way. (For this application, Theorem 10.9 has to be extended so as to cover certain improper integrals. Provide this extension.)

Proof.

(1)

(2)

$\square$

Exercise 10.12. *Let* I^k *be the set of all* $\mathbf{u} = (u_1, \ldots, u_k) \in \mathbb{R}^k$ *with* $0 \leq u_i \leq 1$ *for all* i; *let* Q^k *be the set of all* $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$ *with* $x_i \geq 0$, $\sum x_i \leq 1$. *(*I^k *is the unit cube;* Q^k *is the standard simplex in* $\mathbb{R}^k$*.) Define* $\mathbf{x} = T(\mathbf{u})$ *by*

$$\begin{aligned}
x_1 &= u_1 \\
x_2 &= (1-u_1)u_2 \\
&\cdots \\
x_k &= (1-u_1)\cdots(1-u_{k-1})u_k.
\end{aligned}$$

Show that

$$\sum_{i=1}^{k} x_i = 1 - \prod_{i=1}^{k}(1-u_i).$$

Show that T *maps* I^k *onto* Q^k, *that* T *is* 1-1 *in the interior of* I^k, *and that its inverse* S *is defined in the interior of* Q^k *by* $u_1 = x_1$ *and*

$$u_i = \frac{x_i}{1 - x_1 - \cdots - x_{i-1}}$$

for $i = 2, \ldots, k$. *Show that*

$$J_T(\mathbf{u}) = (1-u_1)^{k-1}(1-u_2)^{k-2}\cdots(1-u_{k-1}),$$

and
$$J_S(\mathbf{x}) = [(1-x_1)(1-x_1-x_2)\cdots(1-x_1-\cdots-x_{k-1})]^{-1}.$$

Proof.

(1) *Show that*
$$\sum_{i=1}^{m} x_i = 1 - \prod_{i=1}^{m}(1-u_i)$$
for all $1 \leq m \leq k$. Induction on m. Base case: $x_1 = 1-(1-u_1)$. Inductive step: Suppose the case $m = h$ is true. Consider the the case $m = h+1$:
$$\begin{aligned}
\sum_{i=1}^{h+1} x_i &= \left(\sum_{i=1}^{h} x_i\right) + x_{h+1} \\
&= 1 - \prod_{i=1}^{h}(1-u_i) + x_{h+1} && \text{(Induction hypothesis)} \\
&= 1 - \prod_{i=1}^{h}(1-u_i) + u_{h+1}\prod_{i=1}^{h}(1-u_i) && \text{(Definition of } x_{h+1}\text{)} \\
&= 1 - (1-u_{h+1})\prod_{i=1}^{h}(1-u_i) \\
&= 1 - \prod_{i=1}^{h+1}(1-u_i).
\end{aligned}$$
Since both the base case and the inductive step have been proved as true, by mathematical induction the statement is established.

(2) *Show that T maps I^k onto Q^k.* Given any $\mathbf{x} = (x_1, \ldots, x_k) \in Q^k$. It is equivalent to solve $\mathbf{u} = (u_1, \ldots, u_k)$ from
$$\begin{aligned}
x_1 &= u_1 \\
x_2 &= (1-u_1)u_2 \\
&\cdots \\
x_k &= (1-u_1)\cdots(1-u_{k-1})u_k
\end{aligned}$$
in terms of $\mathbf{x} = (x_1, \ldots, x_k)$. It is clear that $u_1 = x_1$ and
$$u_i = \begin{cases} x_i(1-x_1-\cdots-x_{i-1})^{-1} & \text{if } x_1+\cdots+x_{i-1} \neq 1, \\ 0 & \text{if } x_1+\cdots+x_{i-1} = 1. \end{cases}$$
for $i = 2, \ldots, k$. (If $x_1+\cdots+x_{i-1} \neq 1$, by (1) we have
$$\prod_{j=1}^{i-1}(1-u_j) = 1 - \sum_{j=1}^{i-1} x_i \neq 0$$

and thus

$$u_i = x_i \left\{ \prod_{j=1}^{i-1} (1-u_j) \right\}^{-1} = x_i (1 - x_1 - \cdots - x_{i-1})^{-1}.$$

If $x_1 + \cdots + x_{i-1} = 1$, then $x_i = \cdots = x_k = 0$. We may take $u_i = 0$ to set the expression $x_i = (1-u_1)\cdots(1-u_{i-1})u_i$ to zero.) Note that the solution $\mathbf{u} \in I^k$ is well-defined by construction, or $T(I^k) = Q^k$.

(3) *Show that T is 1-1 in the interior of I^k.* Suppose $T(\mathbf{u}) = T(\mathbf{v}) = \mathbf{x}$ with $\mathbf{u}, \mathbf{v} \in \mathrm{int}(I^k)$. Then we consider the following equation:

$$\begin{aligned}
x_1 &= u_1 = v_1 \\
x_2 &= (1-u_1)u_2 = (1-v_1)v_2 \\
&\cdots \\
x_k &= (1-u_1)\cdots(1-u_{k-1})u_k = (1-v_1)\cdots(1-v_{k-1})v_k.
\end{aligned}$$

By (1),

$$\mathbf{x} \in \mathrm{int}(Q^k) = \left\{ (x_1, \ldots, x_k) \in \mathbb{R}^k : x_i > 0, \sum x_i < 1 \right\}.$$

Hence,

$$\begin{aligned}
u_1 &= v_1 = x_1 \\
u_2 &= v_1 = x_2(1-x_1)^{-1} \\
&\cdots \\
u_k &= v_k = x_k(1 - x_1 - \cdots - x_{k-1})^{-1}.
\end{aligned}$$

Here all $(1-x_1)^{-1}, \ldots, (1 - x_1 - \cdots - x_i)^{-1}$ are well-defined since $\mathbf{x} \in \mathrm{int}(Q^k)$. Therefore, T is injective on $\mathrm{int}(I^k)$.

(4) By (2)(3), T maps $\mathrm{int}(I^k)$ onto $\mathrm{int}(Q^k)$. That is, given any $\mathbf{x} = (x_1, \ldots, x_k) \in \mathrm{int}(Q^k)$, we can pick

$$\begin{aligned}
u_1 &= x_1 \\
u_i &= x_i(1 - x_1 - \cdots - x_{i-1})^{-1} \qquad (i = 2, \ldots, k)
\end{aligned}$$

such that $\mathbf{u} \in \mathrm{int}(I^k)$ and $T(\mathbf{u}) = \mathbf{x}$.

(5) Note that $T(\mathbf{u}) = (u_1, (1-u_1)u_2, \ldots, (1-u_1)\cdots(1-u_{k-1})u_k)$ on $\mathrm{int}(I^k)$. So

$$T'(\mathbf{u}) = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
* & (1-u_1) & 0 & \cdots & 0 \\
* & * & \prod_{i=1}^{2}(1-u_i) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
* & * & * & \cdots & \prod_{i=1}^{k-1}(1-u_i)
\end{bmatrix}$$

is a lower triangular matrix. Hence,

$$\begin{aligned}
J_T(\mathbf{u}) &= \det T'(\mathbf{u}) \\
&= 1 \cdot (1-u_1) \cdot \prod_{i=1}^{2}(1-u_i) \cdots \prod_{i=1}^{k-1}(1-u_i) \\
&= \prod_{i=1}^{k-1}(1-u_i)^{k-i}.
\end{aligned}$$

(6) Similar to (5). $S(\mathbf{x}) = (x_1, x_2(1-x_1)^{-1}, \ldots, x_k(1-x_1-\cdots-x_{k-1})^{-1})$ on $\mathrm{int}(Q^k)$. So

$$S'(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ * & (1-x_1)^{-1} & 0 & \cdots & 0 \\ * & * & (1-x_1-x_2)^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & (1-x_1-\cdots-x_{k-1})^{-1} \end{bmatrix}$$

is a lower triangular matrix. Hence,

$$\begin{aligned}
J_S(\mathbf{x}) &= \det S'(\mathbf{x}) \\
&= 1 \cdot (1-x_1)^{-1} \cdot (1-x_1-x_2)^{-1} \cdots (1-x_1-\cdots-x_{k-1})^{-1} \\
&= [(1-x_1)(1-x_1-x_2)\cdots(1-x_1-\cdots-x_{k-1})]^{-1}.
\end{aligned}$$

□

Exercise 10.13. *Let $r_1, \ldots, r_k$ be nonnegative integers, and prove that*

$$\int_{Q^k} x_1^{r_1} \cdots x_k^{r_k} d\mathbf{x} = \frac{r_1! \cdots r_k!}{(k + r_1 + \cdots + r_k)!}$$

(Hint: Use Exercise 10.12, Theorems 10.9 and 8.20.) Note that the special case $r_1 = \cdots = r_k = 0$ shows that the volume of Q^k is $\frac{1}{k!}$.

Proof.

(1) Define $T : I^k$ onto Q^k as in Exercise 10.12, and $f : Q^k \to \mathbb{R}^1$ by

$$f(\mathbf{x}) = f(x_1, \ldots, x_k) = x_1^{r_1} \cdots x_k^{r_k} = \prod_{i=1}^{k} x_i^{r_i}.$$

(2) By Exercise 10.12, Example 10.4 and Theorems 10.9, we have

$$\begin{aligned}
\int_{Q^k} x_1^{r_1} \cdots x_k^{r_k} d\mathbf{x} &= \int_{Q^k} f(\mathbf{x}) d\mathbf{x} \\
&= \int_{I^k} f(T(\mathbf{u})) |J_T(\mathbf{u})| d\mathbf{u} \\
&= \int_{I^k} \prod_{i=1}^{k} \left(u_i \prod_{j=1}^{i-1} (1-u_j) \right)^{r_i} \prod_{i=1}^{k} (1-u_i)^{k-i} d\mathbf{u} \\
&= \int_{I^k} \prod_{i=1}^{k} u_i^{r_i} (1-u_i)^{k-i+\sum_{j=i+1}^{k} r_j} d\mathbf{u} \\
&= \prod_{i=1}^{k} \int_0^1 u_i^{r_i} (1-u_i)^{k-i+\sum_{j=i+1}^{k} r_j} du_i && \text{(Theorem 10.2)} \\
&= \prod_{i=1}^{k} \frac{r_i! \left(k-i+\sum_{j=i+1}^{k} r_j \right)!}{\left(k-i+1+\sum_{j=i}^{k} r_j \right)!} && \text{(Theorem 8.20)} \\
&= \frac{r_1! \cdots r_k!}{(k+r_1+\cdots+r_k)!}.
\end{aligned}$$

□

Exercise 10.14 (Levi-Civita symbol). *Prove* $\varepsilon(j_1, \ldots, j_k) = s(j_1, \ldots, j_k)$, *where*

$$s(j_1, \ldots, j_k) = \prod_{p<q} \operatorname{sgn}(j_q - j_p).$$

It is usually to define the Levi-Civita symbol by

$$\varepsilon(j_1, \ldots, j_k) = \begin{cases} 1 & \text{if } (j_1, \cdots, j_k) \text{ is an even permutation of } J, \\ -1 & \text{if } (j_1, \cdots, j_k) \text{ is an odd permutation of } J, \\ 0 & \text{otherwise} \end{cases}$$

(Basic k-forms 10.14). Thus, it is the sign of the permutation in the case of a permutation, and zero otherwise. So $\varepsilon(j_1, \ldots, j_k)$ is equivalent to an explicit expression $s(j_1, \ldots, j_k) = \prod_{p<q} \operatorname{sgn}(j_q - j_p)$.

Proof.

(1) Induction on k. Base case: *Show that* $\varepsilon(j_1, j_2) = s(j_1, j_2)$. Since

$$\varepsilon(j_1, j_2) = \begin{cases} 1 & \text{if } j_1 < j_2 \\ -1 & \text{if } j_1 > j_2, \end{cases}$$

$\varepsilon(j_1, j_2) = \operatorname{sgn}(j_2 - j_1) = s(j_1, j_2)$.

(2) Inductive step: *Show that for any $s \geq 2$, if $\varepsilon(j_1, \ldots, j_s) = s(j_1, \ldots, j_s)$ holds, then $\varepsilon(j_1, \ldots, j_{s+1}) = s(j_1, \ldots, j_{s+1})$ also holds.*

$$
\begin{aligned}
\varepsilon(j_1, \ldots, j_{s+1}) &= \varepsilon(j_1, \ldots, j_s) \prod_{\substack{1 \leq p \leq s \\ q = s+1}} \operatorname{sgn}(j_q - j_p) \\
&= s(j_1, \ldots, j_s) \prod_{\substack{1 \leq p \leq s \\ q = s+1}} \operatorname{sgn}(j_q - j_p) \\
&= \prod_{1 \leq p < q \leq s} \operatorname{sgn}(j_q - j_p) \prod_{\substack{1 \leq p \leq s \\ q = s+1}} \operatorname{sgn}(j_q - j_p) \\
&= \prod_{1 \leq p < q \leq s+1} \operatorname{sgn}(j_q - j_p) \\
&= s(j_1, \ldots, j_{s+1}).
\end{aligned}
$$

(3) Conclusion: Since both the base case and the inductive step have been proved as true, by mathematical induction the statement holds for every integer $k \geq 2$.

□

Exercise 10.15. *If ω and λ are k- and m-forms, respectively, prove that*

$$\omega \wedge \lambda = (-1)^{km} \lambda \wedge \omega.$$

Proof.

(1) Write

$$\omega = \sum_I b_I(\mathbf{x}) dx_I, \qquad \lambda = \sum_J c_J(\mathbf{x}) dx_J$$

in the stardard presentations, where I and J range over all increasing k-indices and over all increasing m-indices taken from the set $\{1, \ldots, n\}$.

(2) *Show that $dx_I \wedge dx_J = (-1)^{km} dx_J \wedge dx_I$.*

$$
\begin{aligned}
dx_I \wedge dx_J &= dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_J \\
&= (-1)^m dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}} \wedge dx_J \wedge dx_{i_k} \\
&= (-1)^{2m} dx_{i_1} \wedge \cdots \wedge dx_{i_{k-2}} \wedge dx_J \wedge dx_{i_{k-1}} \wedge dx_{i_k} \\
&\cdots \\
&= (-1)^{km} dx_J \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&= (-1)^{km} dx_J \wedge dx_I.
\end{aligned}
$$

(3)

$$\begin{aligned}\omega \wedge \lambda &= \sum_{I,J} b_I(\mathbf{x}) c_J(\mathbf{x}) dx_I \wedge dx_J \\ &= (-1)^{km} \sum_{J,I} c_J(\mathbf{x}) b_I(\mathbf{x}) dx_J \wedge dx_I \\ &= (-1)^{km} \lambda \wedge \omega.\end{aligned}$$

□

Exercise 10.16. *If $k \geq 2$ and $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_k]$ is an oriented affine k-simplex, prove that $\partial^2 \sigma = 0$, directly from the definition of the boundary operator ∂. Deduce from this that $\partial^2 \Psi = 0$ for every chain Ψ. (Hint: For orientation, do it first for $k = 2$, $k = 3$. In general, if $i < j$, let σ_{ij} be the $(k-2)$-simplex obtained by deleting $\mathbf{p}_i$ and $\mathbf{p}_j$ from σ. Show that each σ_{ij} occurs twice in $\partial^2 \sigma$, with opposite sign.)*

Proof (Brute-force).

(1) Write the boundary of the oriented affine k-simplex $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_k]$ as

$$\partial \sigma = \sum_{i=0}^{k} (-1)^i [\mathbf{p}_0, \ldots, \widehat{\mathbf{p}}_i, \ldots, \mathbf{p}_k]$$

where where the oriented $(k-1)$-simplex $[\mathbf{p}_0, \ldots, \widehat{\mathbf{p}}_i, \ldots, \mathbf{p}_k]$ is obtained by deleting σ's i-th vertex (Boundaries 10.29).

(2)

$$\begin{aligned}\partial^2 \sigma =& \partial \left(\sum_i (-1)^i [\mathbf{p}_0, \ldots, \widehat{\mathbf{p}}_i, \ldots, \mathbf{p}_k] \right) \\ =& \sum_i (-1)^i \partial [\mathbf{p}_0, \ldots, \widehat{\mathbf{p}}_i, \ldots, \mathbf{p}_k] \\ =& \sum_{j<i} (-1)^i (-1)^j [\mathbf{p}_0, \ldots, \widehat{\mathbf{p}_j}, \ldots, \widehat{\mathbf{p}}_i, \ldots, \mathbf{p}_k] \\ &+ \sum_{j>i} (-1)^i (-1)^{j-1} [\mathbf{p}_0, \ldots, \widehat{\mathbf{p}}_i, \ldots, \widehat{\mathbf{p}_j}, \ldots, \mathbf{p}_k] \\ =& \sum_{j<i} (-1)^{i+j} [\mathbf{p}_0, \ldots, \widehat{\mathbf{p}_j}, \ldots, \widehat{\mathbf{p}}_i, \ldots, \mathbf{p}_k] \\ &- \sum_{j>i} (-1)^{i+j} [\mathbf{p}_0, \ldots, \widehat{\mathbf{p}}_i, \ldots, \widehat{\mathbf{p}_j}, \ldots, \mathbf{p}_k].\end{aligned}$$

The latter two summations cancel since after switching i and j in the second sum. Therefore $\partial^2 \sigma = 0$.

(3) The boundary of a chain is the linear combination of boundaries of the simplices in the chain. Write $\Psi = \sum_{i=1}^{r} \sigma_i$. where σ_i is an oriented affine simplex. Then

$$\partial^2 \Psi = \partial \left(\partial \sum \sigma_i \right) = \partial \left(\sum \partial \sigma_i \right) = \sum \partial^2 \sigma_i = \sum 0 = 0$$

for any affine chain Ψ.

$\square$

Exercise 10.17. *Put $J^2 = \tau_1 + \tau_2$, where*

$$\tau_1 = [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2], \qquad \tau_2 = -[\mathbf{0}, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1].$$

Explain why it is reasonable to call J^2 the positively oriented unit square in $\mathbb{R}^2$. Show that ∂J^2 is the sum of 4 oriented affine 1-simplexes. Find these. What is $\partial(\tau_1 - \tau_2)$?

Proof.

(1) Note that the unit square $I^2 \in \mathbb{R}^2$ is the union of $\tau_1(Q^2)$ and $\tau_2(Q_2)$, where

$$\begin{aligned}
\tau_1(\mathbf{u}) &= ([\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2])(\mathbf{u}) \\
&= \mathbf{0} + \alpha_1 \mathbf{e}_1 + \alpha_2(\mathbf{e}_1 + \mathbf{e}_2) \\
&= \mathbf{0} + (\alpha_1 + \alpha_2)\mathbf{e}_1 + \alpha_2 \mathbf{e}_2 \\
&= \mathbf{0} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{u}
\end{aligned}$$

and

$$\begin{aligned}
\tau_2(\mathbf{u}) &= (-[\mathbf{0}, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1])(\mathbf{u}) \\
&= ([\mathbf{0}, \mathbf{e}_2 + \mathbf{e}_1, \mathbf{e}_2])(\mathbf{u}) \\
&= \mathbf{0} + \alpha_1(\mathbf{e}_1 + \mathbf{e}_2) + \alpha_2 \mathbf{e}_2 \\
&= \mathbf{0} + \alpha_1 \mathbf{e}_1 + (\alpha_1 + \alpha_2)\mathbf{e}_2 \\
&= \mathbf{0} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{u}
\end{aligned}$$

where $\mathbf{u} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 \in \mathbb{R}^2$ (as in Equation (78)). Both τ_1 and τ_2 have Jacobian $1 > 0$, or positively oriented (Affine simplexes 10.26). So it is reasonable to call J^2 the positively oriented unit square in $\mathbb{R}^2$.

(2)

$$\begin{aligned}
\partial \tau_1 &= [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] - [\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{0}, \mathbf{e}_1], \\
\partial \tau_2 &= [\mathbf{e}_2 + \mathbf{e}_1, \mathbf{e}_2] - [\mathbf{0}, \mathbf{e}_2] + [\mathbf{0}, \mathbf{e}_2 + \mathbf{e}_1] \\
&= [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2] + [\mathbf{e}_2, \mathbf{0}] + [\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2].
\end{aligned}$$

(3) By (2),

$$\partial J^2 = \partial\tau_1 + \partial\tau_2 = [\mathbf{0}, \mathbf{e}_1] + [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2] + [\mathbf{e}_2, \mathbf{0}],$$

which is the positively oriented boundary of I^2.

(4) By (2),

$$\begin{aligned}\partial(\tau_1 - \tau_2) =& \partial\tau_1 - \partial\tau_2 \\ =& [\mathbf{0}, \mathbf{e}_1] + [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{0}] \\ &+ [\mathbf{0}, \mathbf{e}_2] + [\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{0}].\end{aligned}$$

$\square$

Exercise 10.18. *Consider the oriented affine* 3*-simplex*

$$\sigma_1 = [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]$$

in $\mathbb{R}^3$*. Show that* σ_1 *(regarded as a linear transformation) has determinant* 1*. Thus* σ_1 *is positively oriented.*

Let $\sigma_2, \ldots, \sigma_6$ *be five other oriented* 3*-simplexes, obtained as follows: There are five permutations* (i_1, i_2, i_3) *of* $(1, 2, 3)$*, distinct from* $(1, 2, 3)$*. Associate with each* (i_1, i_2, i_3) *the simplex*

$$s(i_1, i_2, i_3)[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}]$$

where s *is the sign that occurs in the definition of the determinant. (This is how* τ_2 *was obtained from* τ_1 *in Exercise 10.17.) Show that* $\sigma_2, \ldots, \sigma_6$ *are positively oriented.*

Put $J^3 = \sigma_1 + \cdots + \sigma_6$*. Then* J^3 *may be called the positively oriented unit cube in* $\mathbb{R}^3$*. Show that* ∂J^3 *is the sum of* 12 *oriented affine* 2*-simplexes. (These* 12 *triangles cover the surface of the unit cube* I^3*.)*

Show that $\mathbf{x} = (x_1, x_2, x_3)$ *is in the range of* σ_1 *if and only if* $0 \leq x_3 \leq x_2 \leq x_1 \leq 1$*.*

Show that the range of $\sigma_1, \ldots, \sigma_6$ *have disjoint interiors, and that their union covers* I^3*. (Compared with Exercise 10.13; note that* $3! = 6$*.)*

Proof.

(1) *Show that* σ_1 *(regarded as a linear transformation) has determinant* 1*.*

Given any $\mathbf{u} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3 \in \mathbb{R}^3$, we have

$$\begin{aligned}\sigma_1(\mathbf{u}) &= ([\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3])(\mathbf{u}) \\ &= \mathbf{0} + \alpha_1\mathbf{e}_1 + \alpha_2(\mathbf{e}_1 + \mathbf{e}_2) + \alpha_3(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) \\ &= \mathbf{0} + (\alpha_1 + \alpha_2 + \alpha_3)\mathbf{e}_1 + (\alpha_2 + \alpha_3)\mathbf{e}_2 + \alpha_3\mathbf{e}_3 \\ &= \mathbf{0} + \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{say } A} \mathbf{u}.\end{aligned}$$

So

$$\det(A) = \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = 1.$$

(2) *Show that $\sigma_2, \ldots, \sigma_6$ are positively oriented.* Define the permutation matrix $P_{(i_1,i_2,i_3)}$ corresponding to a permutation (i_1, i_2, i_3) of $(1, 2, 3)$ by

$$P_{(i_1,i_2,i_3)} = \begin{bmatrix} \mathbf{e}_{i_1} & \mathbf{e}_{i_2} & \mathbf{e}_{i_3} \end{bmatrix}.$$

For example,

$$P_{(2,3,1)} = \begin{bmatrix} \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Note that the sign $s(i_1, i_2, i_3)$ of the permutation (i_1, i_2, i_3) is exactly the same as the determinant of the permutation matrix $P_{(i_1,i_2,i_3)}$. Define a permutation $(j_1, j_2, 3)$ of $(1, 2, 3)$ (for swapping the first and the second coordinates of $\mathbf{u}$) by

$$(j_1, j_2, 3) = \begin{cases} (1, 2, 3) & \text{if } s(i_1, i_2, i_3) = 1, \\ (2, 1, 3) & \text{if } s(i_1, i_2, i_3) = -1. \end{cases}$$

Write

$$\sigma_{(i_1,i_2,i_3)} = s(i_1, i_2, i_3)[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}].$$

(So that $\sigma_1 = \sigma_{(1,2,3)}$.) Hence,

$$\begin{aligned}&\sigma_{(i_1,i_2,i_3)}(\mathbf{u}) \\ =&\mathbf{0} + \alpha_{j_1}\mathbf{e}_{i_1} + \alpha_{j_2}(\mathbf{e}_{i_1} + \mathbf{e}_{i_2}) + \alpha_3(\mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}) \\ =&\mathbf{0} + (\alpha_{j_1} + \alpha_{j_2} + \alpha_3)\mathbf{e}_{i_1} + (\alpha_{j_2} + \alpha_3)\mathbf{e}_{i_2} + \alpha_3\mathbf{e}_{i_3} \\ =&\mathbf{0} + P_{(i_1,i_2,i_3)}AP_{(j_1,j_2,3)}\mathbf{u}\end{aligned}$$

where $\mathbf{u} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3 \in \mathbb{R}^3$. For example,

$$P_{(2,3,1)}AP_{(1,2,3)} = \begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\ 1 & 1 & 1\\ 0 & 1 & 1\end{bmatrix}.$$

So

$$\begin{aligned}\det\bigl(P_{(i_1,i_2,i_3)}AP_{(j_1,j_2,3)}\bigr) &= \det\bigl(P_{(i_1,i_2,i_3)}\bigr)\det(A)\det\bigl(P_{(j_1,j_2,3)}\bigr)\\ &= s(i_1,i_2,i_3)\cdot 1\cdot s(i_1,i_2,i_3)\\ &= 1.\end{aligned}$$

(3) *Show that ∂J^3 is the sum of* 12 *oriented affine* 2*-simplexes.* Note that

$$\begin{aligned}\sum_{(i_1,i_2,i_3)}\sigma_{(i_1,i_2,i_3)} &= \sum_{\substack{(i_1,i_2,i_3)\\ i_1>i_2}}\sigma_{(i_1,i_2,i_3)} + \sum_{\substack{(i_1,i_2,i_3)\\ i_1<i_2}}\sigma_{(i_1,i_2,i_3)}\\ &= \sum_{\substack{(i_1,i_2,i_3)\\ i_1>i_2}} s(i_1,i_2,i_3)[\mathbf{0},\mathbf{e}_{i_1}+\mathbf{e}_{i_2},\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3]\\ &\quad+\sum_{\substack{(i_1,i_2,i_3)\\ i_2>i_1}} -s(i_2,i_1,i_3)[\mathbf{0},\mathbf{e}_{i_2}+\mathbf{e}_{i_1},\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3]\\ &=\mathbf{0}\end{aligned}$$

and

$$\begin{aligned}\sum_{(i_1,i_2,i_3)}\sigma_{(i_1,i_2,i_3)} &= \sum_{\substack{(i_1,i_2,i_3)\\ i_2>i_3}}\sigma_{(i_1,i_2,i_3)} + \sum_{\substack{(i_1,i_2,i_3)\\ i_2<i_3}}\sigma_{(i_1,i_2,i_3)}\\ &= \sum_{\substack{(i_1,i_2,i_3)\\ i_2>i_3}} s(i_1,i_2,i_3)[\mathbf{0},\mathbf{e}_{i_1},\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3]\\ &\quad+\sum_{\substack{(i_1,i_2,i_3)\\ i_3>i_2}} -s(i_1,i_3,i_2)[\mathbf{0},\mathbf{e}_{i_1},\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3]\\ &=\mathbf{0}.\end{aligned}$$

So

$$\begin{aligned}
\partial J^3 &= \sum_{(i_1,i_2,i_3)} \partial\sigma_{(i_1,i_2,i_3)} \\
&= \sum_{(i_1,i_2,i_3)} s(i_1,i_2,i_3)[\mathbf{e}_{i_1}, \mathbf{e}_{i_1}+\mathbf{e}_{i_2}, \mathbf{e}_{i_1}+\mathbf{e}_{i_2}+\mathbf{e}_{i_3}] \\
&\quad - s(i_1,i_2,i_3)[\mathbf{0}, \mathbf{e}_{i_1}+\mathbf{e}_{i_2}, \mathbf{e}_{i_1}+\mathbf{e}_{i_2}+\mathbf{e}_{i_3}] \\
&\quad + s(i_1,i_2,i_3)[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1}+\mathbf{e}_{i_2}+\mathbf{e}_{i_3}] \\
&\quad - s(i_1,i_2,i_3)[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1}+\mathbf{e}_{i_2}] \\
&= \sum_{(i_1,i_2,i_3)} s(i_1,i_2,i_3)[\mathbf{e}_{i_1}, \mathbf{e}_{i_1}+\mathbf{e}_{i_2}, \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3] \\
&\quad \underbrace{- \sum_{(i_1,i_2,i_3)} s(i_1,i_2,i_3)[\mathbf{0}, \mathbf{e}_{i_1}+\mathbf{e}_{i_2}, \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3]}_{=\mathbf{0}} \\
&\quad \underbrace{+ \sum_{(i_1,i_2,i_3)} s(i_1,i_2,i_3)[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3]}_{=\mathbf{0}} \\
&\quad - \sum_{(i_1,i_2,i_3)} s(i_1,i_2,i_3)[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1}+\mathbf{e}_{i_2}].
\end{aligned}$$

Thus,

$$\begin{aligned}
\partial J^3 = &\sum_{(i_1,i_2,i_3)} s(i_1,i_2,i_3)[\mathbf{e}_{i_1}, \mathbf{e}_{i_1}+\mathbf{e}_{i_2}, \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3] \\
&- \sum_{(i_1,i_2,i_3)} s(i_1,i_2,i_3)[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1}+\mathbf{e}_{i_2}]
\end{aligned}$$

is the sum of 12 oriented affine 2-simplexes. (Note that $3! = 6$.)

(4) *Show that* $\mathbf{x} = (x_1, x_2, x_3)$ *is in the range of* σ_1 *if and only if* $0 \leq x_3 \leq x_2 \leq x_1 \leq 1$.

(a) By (1), $\mathbf{x}$ is in the range of σ_1 if and only if $\mathbf{x} = A\mathbf{u}$ for $\mathbf{u} = (u_1, u_2, u_3) \in Q^3$, or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1+u_2+u_3 \\ u_2+u_3 \\ u_3 \end{bmatrix}.$$

(b) Since $\mathbf{u} = (u_1, u_2, u_3) \in Q^3$, $u_1 + u_2 + u_3 \leq 1$ and $u_1, u_2, u_3 \geq 0$. Hence $0 \leq u_3 \leq u_2 + u_3 \leq u_1 + u_2 + u_3 \leq 1$ or $0 \leq x_3 \leq x_2 \leq x_1 \leq 1$.

(c) Conversely, if $0 \leq x_3 \leq x_2 \leq x_1 \leq 1$, we define

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_3 \end{bmatrix}.$$

Clearly, $\mathbf{v} \in Q^3$.

(5) *Show that the range of $\sigma_1, \dots, \sigma_6$ have disjoint interiors, and that their union covers I^3.* Similar to (4). By (2), $\mathbf{x} = P_{(i_1,i_2,i_3)} A P_{(j_1,j_2,3)} \mathbf{u}$, or $P_{(i_1,i_2,i_3)^{-1}} \mathbf{x} = A P_{(j_1,j_2,3)} \mathbf{u}$, or

$$\begin{bmatrix} x_{i_1} \\ x_{i_2} \\ x_{i_3} \end{bmatrix} = \begin{bmatrix} u_1 + u_2 + u_3 \\ u_{j_2} + u_3 \\ u_3 \end{bmatrix}.$$

In any case, we always have $0 \leq u_3 \leq u_{j_2} + u_3 \leq u_1 + u_2 + u_3 \leq 1$. Hence $\mathbf{x} = (x_1, x_2, x_3)$ is in the range of $\sigma_{(i_1,i_2,i_3)}$ if and only if

$$0 \leq x_{i_3} \leq x_{i_2} \leq x_{i_1} \leq 1.$$

The interior of $\sigma_{(i_1,i_2,i_3)}$ is

$$\{\mathbf{x} \in \mathbb{R}^3 : 0 < x_{i_3} < x_{i_2} < x_{i_1} < 1\},$$

and thus the range of $\sigma_1, \dots, \sigma_6$ have disjoint interiors. Also, any $\mathbf{x} \in I^3$ has the relation

$$0 \leq x_{i_3} \leq x_{i_2} \leq x_{i_1} \leq 1$$

for some permutation (i_1, i_2, i_3) of $(1, 2, 3)$. Hence

$$I^3 = \bigcup_{(i_1,i_2,i_3)} \sigma_{(i_1,i_2,i_3)}(Q^3) = \bigcup_{i=1}^{6} \sigma_i(Q^3).$$

$\square$

Exercise 10.19. *Let J^2 and J^3 be as in Exercise 10.17 and Exercise 10.18. Define*

$$\begin{aligned} B_{01}(u,v) &= (0,u,v), & B_{11}(u,v) &= (1,u,v), \\ B_{02}(u,v) &= (u,0,v), & B_{12}(u,v) &= (u,1,v), \\ B_{03}(u,v) &= (u,v,0), & B_{13}(u,v) &= (u,v,1). \end{aligned}$$

These are affine, and map $\mathbb{R}^2$ into $\mathbb{R}^3$. Put $\beta_{ri} = B_{ri}(J^2)$, for $r = 0, 1$, $i = 1, 2, 3$. Each β_{ri} is an affine-oriented 2-chain. (See Section 10.30.) Verify that

$$\partial J^3 = \sum_{i=1}^{3} (-1)^i (\beta_{0i} - \beta_{1i}),$$

in agreement with Exercise 10.18.)

Proof.

(1) A direct calculation shows that

$$
\begin{aligned}
B_{01}(\tau_1) - B_{11}(\tau_1) =& [\mathbf{0}, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3] - [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\
B_{02}(\tau_1) - B_{12}(\tau_1) =& [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_3] - [\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\
B_{03}(\tau_1) - B_{13}(\tau_1) =& [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] - [\mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\
B_{01}(\tau_2) - B_{11}(\tau_2) =& - [\mathbf{0}, \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3] + [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\
B_{02}(\tau_2) - B_{12}(\tau_2) =& - [\mathbf{0}, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_3] + [\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\
B_{03}(\tau_2) - B_{13}(\tau_2) =& - [\mathbf{0}, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3].
\end{aligned}
$$

(2) To express the formula in (1) clearly, we define

$$
\omega_{(i_1,i_2,i_3)} = [\mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] - [\mathbf{0}, \mathbf{e}_{i_2}, \mathbf{e}_{i_2} + \mathbf{e}_{i_3}],
$$

and thus

$$
\begin{aligned}
-(B_{01}(\tau_1) - B_{11}(\tau_1)) &= s(1,2,3)\omega_{(1,2,3)} \\
B_{02}(\tau_1) - B_{12}(\tau_1) &= s(2,1,3)\omega_{(2,1,3)} \\
-(B_{03}(\tau_1) - B_{13}(\tau_1)) &= s(3,1,2)\omega_{(3,1,2)} \\
-(B_{01}(\tau_2) - B_{11}(\tau_2)) &= s(1,3,2)\omega_{(1,3,2)} \\
B_{02}(\tau_2) - B_{12}(\tau_2) &= s(2,3,1)\omega_{(2,3,1)} \\
-(B_{03}(\tau_2) - B_{13}(\tau_2)) &= s(3,2,1)\omega_{(3,2,1)}.
\end{aligned}
$$

(3) Note that

$$
\begin{aligned}
\beta_{0i} - \beta_{1i} &= B_{0i}(J^2) - B_{1i}(J^2) \\
&= B_{0i}(\tau_1 + \tau_2) - B_{1i}(\tau_1 + \tau_2) \\
&= B_{0i}(\tau_1) + B_{0i}(\tau_2) - B_{1i}(\tau_1) - B_{1i}(\tau_2) \\
&= (B_{0i}(\tau_1) - B_{1i}(\tau_1)) + (B_{0i}(\tau_2) - B_{1i}(\tau_2)).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\sum_{i=1}^{3}(-1)^i(\beta_{0i} - \beta_{1i}) \\
=& \sum_{i=1}^{3}(-1)^i(B_{0i}(\tau_1) - B_{1i}(\tau_1)) + \sum_{i=1}^{3}(-1)^i(B_{0i}(\tau_2) - B_{1i}(\tau_2)) \\
=& \sum_{(i_1,i_2,i_3)} s(i_1,i_2,i_3)\omega_{(i_1,i_2,i_3)} \\
=& \sum_{(i_1,i_2,i_3)} s(i_1,i_2,i_3)[\mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\
&- \sum_{(i_1,i_2,i_3)} s(i_1,i_2,i_3)[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}] \\
=& \partial J^3.
\end{aligned}
$$

$\square$

Exercise 10.20. *State conditions under which the formula*

$$\int_{\Phi} f d\omega = \int_{\partial\Phi} f\omega - \int_{\Phi} (df) \wedge \omega$$

is valid, and show that it generalizes the formula for integration by parts. (Hint: $d(f\omega) = (df) \wedge \omega + f d\omega$.)

Proof.

(1) *If*

(a) *Φ is a k-chain of class $\mathscr{C}''$ in an open set $V \subseteq \mathbb{R}^m$,*

(b) *ω is a $(k-1)$-form of class $\mathscr{C}'$ in V,*

(c) *f is a 0-form of class $\mathscr{C}'$ in V,*

then

$$\int_{\Phi} f d\omega = \int_{\partial\Phi} f\omega - \int_{\Phi} (df) \wedge \omega$$

(2) Theorem 10.20(a) implies that

$$d(f\omega) = (df) \wedge \omega + f d\omega.$$

(3) The Stokes' theorem (Theorem 10.33) shows that

$$\int_{\Phi} d(f\omega) = \int_{\partial\Phi} f\omega.$$

Hence

$$\int_{\Phi} f d\omega = \int_{\Phi} d(f\omega) - \int_{\Phi} (df) \wedge \omega = \int_{\partial\Phi} f\omega - \int_{\Phi} (df) \wedge \omega.$$

(4) Define $\Phi : Q^1 = [0,1] \to [a,b]$ by

$$\Phi(\alpha) = a + \alpha(b-a).$$

Φ is a 1-simplex of class $\mathscr{C}''$ in an open set $V \supseteq [a,b]$. Also,

$$\partial\Phi = [b] - [a].$$

Let $\omega = g$ be a 0-form of class $\mathscr{C}'(V)$.

(5) Note that

$$\int_\Phi f d\omega = \int_\Phi f dg = \int_0^1 f(\Phi(t))g'(\Phi(t))\Phi'(t)dt = \int_a^b f(u)g'(u)du,$$
$$\int_{\partial\Phi} f\omega = \int_{[b]} fg + \int_{-[a]} fg = f(b)g(b) + (-1)f(a)f(a),$$
$$\int_\Phi (df)\wedge\omega = \int_\Phi (df)g = \int_0^1 f'(\Phi(t))g(\Phi(t))\Phi'(t)dt = \int_a^b f'(u)g(u)du.$$

Hence

$$\int_a^b f(u)g'(u)du = f(b)g(b) - f(a)f(a) - \int_a^b f'(u)g(u)du,$$

which is the same as the integration by parts (Theorem 6.22).

□

Exercise 10.21. *As in Example 10.36, consider the* 1*-form*

$$\eta = \frac{xdy - ydx}{x^2 + y^2}$$

in $\mathbb{R}^2 - \{\mathbf{0}\}$.

(a) *Carry out the computation that leads to*

$$\int_\gamma \eta = 2\pi \neq 0,$$

and prove that $d\eta = 0$.

(b) *Let* $\gamma(t) = (r\cos t, r\sin t)$, *for some* $r > 0$, *and let* Γ *be a* $\mathscr{C}''$*-curve in* $\mathbb{R}^2 - \{\mathbf{0}\}$, *with parameter interval* $[0, 2\pi]$, *with* $\Gamma(0) = \Gamma(2\pi)$, *such that the intervals* $[\gamma(t), \Gamma(t)]$ *do not contain* $\mathbf{0}$ *for any* $t \in [0, 2\pi]$. *Prove that*

$$\int_\Gamma \eta = 2\pi.$$

(Hint: For $0 \leq t \leq 2\pi$, $0 \leq u \leq 1$, *define*

$$\Phi(t, u) = (1 - u)\Gamma(t) + u\gamma(t).$$

Then Φ *is a* 2*-surface in* $\mathbb{R}^2 - \{\mathbf{0}\}$ *whose parameter domain is the indicated rectangle. Because of cancellations (as in Example 10.32),*

$$\partial\Phi = \Gamma - \gamma.$$

Use Stokes' theorem to deduce that

$$\int_\Gamma \eta = \int_\gamma \eta$$

because $d\eta = 0$.*)*

(c) *Take* $\Gamma(t) = (a\cos t, b\sin t)$ *where* $a > 0$, $b > 0$ *are fixed. Use part (b) to show that*

$$\int_0^{2\pi} \frac{ab}{a^2\cos^2 t + b^2\sin^2 t} dt = 2\pi.$$

(d) *Show that*

$$\eta = d\left(\arctan\frac{y}{x}\right)$$

in any convex open set in which $x \neq 0$, *and that*

$$\eta = d\left(-\arctan\frac{x}{y}\right)$$

in any convex open set in which $y \neq 0$. *Explain why this justifies the notation* $\eta = d\theta$, *in spite of the fact that* η *is not exact in* $\mathbb{R}^2 - \{0\}$.

(e) *Show that (b) can be derived from (d).*

(f) *If* Γ *is any closed* $\mathscr{C}'$*-curve in* $\mathbb{R}^2 - \{\mathbf{0}\}$, *prove that*

$$\frac{1}{2\pi}\int_\Gamma \eta = \mathrm{Ind}(\Gamma).$$

(See Exercise 8.23 for the definition of the index of a curve.)

Proof of (a).

(1)

$$\begin{aligned}
\int_\gamma \eta &= \int_0^{2\pi} \frac{(r\cos t)d(r\sin t) - (r\sin t)d(r\cos t)}{(r\cos t)^2 + (r\sin t)^2} \\
&= \int_0^{2\pi} \frac{(r\cos t)(r\cos t) - (r\sin t)(-r\sin t)}{(r\cos t)^2 + (r\sin t)^2} dt \\
&= \int_0^{2\pi} dt \\
&= 2\pi.
\end{aligned}$$

(2)

$$\begin{aligned}
d\eta =& d\left(\frac{xdy - ydx}{x^2+y^2}\right) \\
=& d\left(\frac{x}{x^2+y^2}\right) \wedge dy - d\left(\frac{y}{x^2+y^2}\right) \wedge dx && (d^2 = 0) \\
=& D_1\left(\frac{x}{x^2+y^2}\right) dx \wedge dy && (dy \wedge dy = 0) \\
& - D_2\left(\frac{y}{x^2+y^2}\right) dy \wedge dx && (dx \wedge dx = 0) \\
=& \left(\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}\right) dx \wedge dy \\
& + \left(\frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}\right) dx \wedge dy \\
=& 0
\end{aligned}$$

□

Note.

1. η is closed and locally exact, that is, $\eta = dt$ on $\mathbb{R}^2 - L$ where L is any line passing through $\mathbf{0}$. η is not exact since $\int_\gamma \eta = 2\pi \neq 0$. (See Exercise 10.22(g).)
2. *(Poincaré's Lemma for 1-form.) Let $\omega = \sum a_i dx_i$ be defined in an open set $U \subseteq \mathbb{R}^n$. Then $d\omega = 0$ if and only if for each $p \in U$ there is a neighborhood $V \subseteq U$ of p and a differentiable function $f : V \to \mathbb{R}^1$ with $df = \omega$ (i.e., ω is locally exact).*

Proof of (b).

1. For $0 \leq t \leq 2\pi$, $0 \leq u \leq 1$, define
$$\Phi(t,u) = (1-u)\Gamma(t) + u\gamma(t).$$
Then Φ is a 2-surface in $\mathbb{R}^2 - \{\mathbf{0}\}$ whose parameter domain $D = \{(t,u) : 0 \leq t \leq 2\pi, 0 \leq u \leq 1\}$ is the indicated rectangle.
2. Similar to Example 10.32,
$$\partial\Phi = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$$
where
$$\begin{aligned}
\gamma_1(t) &= \Phi(t,0) = \Gamma(t), \\
\gamma_2(u) &= \Phi(2\pi,u) = (1-u)\Gamma(2\pi) + u\gamma(2\pi), \\
\gamma_3(t) &= \Phi(2\pi - t, 1) = \gamma(2\pi - t), \\
\gamma_4(u) &= \Phi(0, 1-u) = u\Gamma(0) + (1-u)\gamma(0).
\end{aligned}$$

Because of cancellations (as in Example 10.32), $\gamma(0) = \gamma(2\pi)$ and $\Gamma(0) = \Gamma(2\pi)$, $\gamma_4 = -\gamma_2$ and $\gamma_3 = -\gamma$. Hence,

$$\partial\Phi = \Gamma - \gamma.$$

(3) The Stokes' theorem (Theorem 10.33) implies that

$$\int_\Phi d\eta = \int_{\partial\Phi} \eta = \int_{\Gamma-\gamma} \eta = \int_\Gamma \eta - \int_\gamma \eta.$$

Hence,

$$\int_\Gamma \eta = \int_\gamma \eta$$

(since $d\eta = 0$ by (a)).

□

Proof of (c).

(1) Γ satisfies all conditions described in (b). So

$$\int_\Gamma \eta = 2\pi.$$

(2) A direct calculation shows that

$$\begin{aligned}
2\pi = \int_\Gamma \eta = \int_\Gamma \frac{xdy - ydx}{x^2+y^2} \\
&= \int_0^{2\pi} \frac{a\cos(t)d(b\sin(t)) - b\sin(t)d(a\cos(t))}{(a\cos(t))^2 + (b\sin(t))^2} \\
&= \int_0^{2\pi} \frac{ab(\cos^2 t + \sin^2 t)}{a^2\cos^2 t + b^2\sin^2 t} \\
&= \int_0^{2\pi} \frac{ab}{a^2\cos^2 t + b^2\sin^2 t}.
\end{aligned}$$

□

Proof of (d).

(1) In any convex open set in which $x \neq 0$, we have

$$\begin{aligned}
d\left(\arctan\frac{y}{x}\right) &= \left(D_1 \arctan\frac{y}{x}\right) dx + \left(D_2 \arctan\frac{y}{x}\right) dy \\
&= -\frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy \\
&= \eta.
\end{aligned}$$

(2) In any convex open set in which $y \neq 0$, we have

$$\begin{aligned} d\left(-\arctan\frac{x}{y}\right) &= \left(D_1\left(-\arctan\frac{x}{y}\right)\right)dx + \left(D_2\left(-\arctan\frac{x}{y}\right)\right)dy \\ &= -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy \\ &= \eta. \end{aligned}$$

(3) By (1)(2), η is locally exact. Note that $\theta_1 = \arctan\frac{y}{x}$ and $\theta_2 = -\arctan\frac{x}{y}$ cannot be patched together to defined a global 0-form θ on $\mathbb{R}^2 - \{\mathbf{0}\}$.

$\square$

Proof of (e).

(1) Partition $[0, 2\pi]$ into five subintervals

$$I_i = \left[\frac{(2i-3)\pi}{4}, \frac{(2i-1)\pi}{4}\right] \cap [0, 2\pi].$$

for $i = 1, 2, 3, 4, 5$. Hence

$$\begin{aligned} \int_\gamma \eta &= \sum_{i=1}^{5} \int_{\gamma(I_i)} \eta \\ &= \sum_{i=1,3,5} \int_{\gamma(I_i)} d\left(\arctan\frac{y}{x}\right) + \sum_{i=2,4} \int_{\gamma(I_i)} d\left(-\arctan\frac{x}{y}\right). \end{aligned}$$

(2) The Stokes' theorem (Theorem 10.33) implies that

$$\begin{aligned} \int_{\gamma(I_1)} d\left(\arctan\frac{y}{x}\right) &= \int_{\partial\gamma(I_1)} \arctan\frac{y}{x} \\ &= \left[\arctan\frac{r\cos t}{r\sin t}\right]_{t=0}^{t=\frac{\pi}{4}} \\ &= [\arctan(\tan(t))]_{t=0}^{t=\frac{\pi}{4}} \\ &= \frac{\pi}{4}, \end{aligned}$$

and

$$\begin{aligned} \int_{\gamma(I_2)} d\left(-\arctan\frac{x}{y}\right) &= \int_{\partial\gamma(I_2)} -\arctan\frac{x}{y} \\ &= \left[\arctan\frac{r\sin t}{r\cos t}\right]_{t=\frac{\pi}{4}}^{t=\frac{3\pi}{4}} \\ &= [\arctan(\cot(t))]_{t=\frac{\pi}{4}}^{t=\frac{3\pi}{4}} \\ &= \frac{\pi}{2}. \end{aligned}$$

Similarly,

$$\int_{\gamma(I_3)} d\left(\arctan\frac{y}{x}\right) = \frac{\pi}{2}$$
$$\int_{\gamma(I_4)} d\left(-\arctan\frac{x}{y}\right) = \frac{\pi}{2}$$
$$\int_{\gamma(I_5)} d\left(\arctan\frac{y}{x}\right) = \frac{\pi}{4}.$$

(3) Therefore,

$$\int_\gamma \eta = \left(\frac{\pi}{4}+\frac{\pi}{2}+\frac{\pi}{4}\right)+\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = 2\pi.$$

□

Proof of (f).

(1) Regard $\Gamma(t)$ as a plane curve $(\Gamma_1(t), \Gamma_2(t))$ over $\mathbb{R}^2$ or $\Gamma_1(t) + i\Gamma_2(t)$ over $\mathbb{C}^1$. Note that

$$\begin{aligned}\frac{\Gamma'(t)}{\Gamma(t)} &= \frac{\Gamma_1'(t)+i\Gamma_2'(t)}{\Gamma_1(t)+i\Gamma_2(t)} \\ &= \frac{\Gamma_1'(t)\Gamma_1'(t)+\Gamma_2'(t)\Gamma_2'(t)}{\Gamma_1(t)^2+\Gamma_2(t)^2} + i\frac{\Gamma_1(t)\Gamma_2'(t)-\Gamma_2(t)\Gamma_1'(t)}{\Gamma_1(t)^2+\Gamma_2(t)^2}.\end{aligned}$$

So

$$\operatorname{Im}\left(\frac{\Gamma'(t)}{\Gamma(t)}\right) = \frac{\Gamma_1(t)\Gamma_2'(t)-\Gamma_2(t)\Gamma_1'(t)}{\Gamma_1(t)^2+\Gamma_2(t)^2}.$$

(2) By Exercise 8.23,

$$\operatorname{Ind}(\Gamma) = \frac{1}{2\pi i}\int_0^{2\pi}\frac{\Gamma'(t)}{\Gamma(t)}dt$$

is always an integer. That is,

$$\begin{aligned}\operatorname{Ind}(\Gamma) &= \frac{1}{2\pi}\int_0^{2\pi}\operatorname{Im}\left(\frac{\Gamma'(t)}{\Gamma(t)}\right)dt \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{\Gamma_1(t)\Gamma_2'(t)-\Gamma_2(t)\Gamma_1'(t)}{\Gamma_1(t)^2+\Gamma_2(t)^2}dt \\ &= \frac{1}{2\pi}\int_\Gamma\frac{xdy-ydx}{x^2+y^2} \\ &= \frac{1}{2\pi}\int_\Gamma\eta.\end{aligned}$$

(Note that $\operatorname{Ind}(\Gamma) = 1$ if Γ is defined as in (c). Hence the integral in (c) is equal to $2\pi\operatorname{Ind}(\Gamma) = 2\pi$.)

□

Exercise 10.22. *As in Example 10.37, define ζ in $\mathbb{R}^3 - \{\mathbf{0}\}$ by*

$$\zeta = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{r^3}$$

where $r = (x^2+y^2+z^2)^{\frac{1}{2}}$, let D be the rectangle given by $0 \le u \le \pi$, $0 \le v \le 2\pi$, and let Σ be the 2-surface in $\mathbb{R}^3$, with parameter domain D, given by

$$x = \sin u \cos v, \qquad y = \sin u \sin v, \qquad z = \cos u.$$

(a) *Prove that $d\zeta = 0$ in $\mathbb{R}^3 - \{\mathbf{0}\}$.*

(b) *Let S denote the restriction of Σ to a parameter domain $E \subseteq D$. Prove that*

$$\int_S \zeta = \int_E \sin u \, du \, dv = A(S),$$

where A denotes area, as in Section 10.46. Note that this contains

$$\int_\Sigma \zeta = \int_D \sin u \, du \, dv = 4\pi \neq 0$$

as a special case.

(c) *Suppose g, h_1, h_2, h_3, are $\mathscr{C}''$-functions on $[0,1]$, $g > 0$. Let $(x,y,z) = \Phi(s,t)$ define a 2-surface Φ, with parameter domain I^2, by*

$$x = g(t)h_1(s), \qquad y = g(t)h_2(s), \qquad z = g(t)h_3(s).$$

Prove that

$$\int_\Phi \zeta = 0,$$

directly from Equation (35) in Chapter 10. Note the shape of the range of Φ: For fixed s, $\Phi(s,t)$ runs over an interval on a line through $\mathbf{0}$. The range of Φ thus lies in a "cone" with vertex at the origin.

(d) *Let E be a closed rectangle in D, with edges parallel to those of D. Suppose $f \in \mathscr{C}''(D)$, $f > 0$. Let Ω be the 2-surface with parameter domain E, defined by*

$$\Omega(u,v) = f(u,v)\Sigma(u,v).$$

Define S as in (b) and prove that

$$\int_\Omega \zeta = \int_S \zeta = A(S).$$

(Since S is the "radical projection" of Ω into the unit sphere, this result makes it reasonable to call $\int_\Omega \zeta$ the "solid angle" subtended by the range of Ω at the origin.) (Hint: Consider the 3-surface Ψ given by

$$\Psi(t,u,v) = [1-t+tf(u,v)]\Sigma(u,v),$$

where $(u,v) \in E$, $0 \leq t \leq 1$. For fixed v, the mapping $(t,u) \mapsto \Psi(t,u,v)$ is a 2-surface Φ to which (c) can be applied to show that $\int_\Phi \zeta = 0$. The same thing holds when u is fixed. By (a) and Stokes' theorem,

$$\int_{\partial\Psi} \zeta = \int_\Psi d\zeta = 0.)$$

(e) *Put $\lambda = -\frac{z}{r}\eta$, where*

$$\eta = \frac{xdy - ydx}{x^2+y^2},$$

as in Exercise 10.21. Then λ is a 1-form in the open set $V \subseteq \mathbb{R}^3$ in which $x^2+y^2 > 0$. Show that ζ is exact in V by showing that

$$\zeta = d\lambda.$$

(f) *Derive (d) from (e), without using (c). (Hint: To begin with, assume $0 < u < \pi$ on E. By (e),*

$$\int_\Omega \zeta = \int_{\partial\Omega} \lambda \qquad \text{and} \qquad \int_S \zeta = \int_{\partial S} \lambda.$$

Show that the two integrals of λ are equal, by using part (d) of Exercise 10.21, and by noting that $\frac{z}{r}$ is the same at $\Sigma(u,v)$ as at $\Omega(u,v)$.)

(g) *Is ζ exact in the complement of every line through the origin?*

Proof of (a).

(1) Note that ζ is well-defined on $\mathbb{R}^3 - \{\mathbf{0}\}$. Hence,

$$\begin{aligned}
d\zeta &= d\left(\frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{r^3}\right) \\
&= d\left(\frac{x}{r^3}\right) \wedge dy \wedge dz + d\left(\frac{y}{r^3}\right) \wedge dz \wedge dx + d\left(\frac{z}{r^3}\right) \wedge dx \wedge dy \\
&= D_1\left(\frac{x}{r^3}\right) dx \wedge dy \wedge dz + D_2\left(\frac{y}{r^3}\right) dy \wedge dz \wedge dx + D_3\left(\frac{z}{r^3}\right) dz \wedge dx \wedge dy \\
&= \frac{r^3 - 3rx^2}{r^6} dx \wedge dy \wedge dz + \frac{r^3 - 3ry^2}{r^6} dy \wedge dz \wedge dx + \frac{r^3 - 3rz^2}{r^6} dz \wedge dx \wedge dy \\
&= \left(\frac{r^3 - 3rx^2}{r^6} + \frac{r^3 - 3ry^2}{r^6} + \frac{r^3 - 3rz^2}{r^6}\right) dx \wedge dy \wedge dz \\
&= 0 dx \wedge dy \wedge dz \\
&= 0
\end{aligned}$$

in $\mathbb{R}^3 - \{\mathbf{0}\}$.

(2) Or write
$$\mathbf{F} = \frac{x}{r^3}\mathbf{e}_1 + \frac{y}{r^3}\mathbf{e}_2 + \frac{z}{r^3}\mathbf{e}_3$$
as in Vector fields 10.42. So
$$\omega_{\mathbf{F}} = \zeta$$
and
$$d\omega_{\mathbf{F}} = (\nabla \cdot \mathbf{F}) dx \wedge dy \wedge dz$$
as in the proof of the divergence theorem (Theorem 10.51). Note that the divergence of $\mathbf{F}$ is zero.

$\square$

Proof of (b).

(1) By Area elements in $\mathbb{R}^3$ 10.46,
$$\begin{aligned}\mathbf{N}(u,v) &= \frac{\partial(y,z)}{\partial(u,v)}\mathbf{e}_1 + \frac{\partial(z,x)}{\partial(u,v)}\mathbf{e}_2 + \frac{\partial(x,y)}{\partial(u,v)}\mathbf{e}_3 \\ &= (\sin^2 u \cos v)\mathbf{e}_1 + (\sin^2 u \sin v)\mathbf{e}_2 + (\sin u \cos u)\mathbf{e}_3.\end{aligned}$$
Here $|\mathbf{N}(u,v)| = \sin u \geq 0$ (by noting that $u \in [0,\pi]$), and
$$\mathbf{n}(u,v) = \frac{\mathbf{N}(u,v)}{|\mathbf{N}(u,v)|} = (\sin u \cos v, \sin u \sin v, \cos u).$$

(2) Note that $\zeta = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$ on $S \subseteq \Sigma$. Hence, by Integrals of 2-forms in $\mathbb{R}^3$ 10.49,
$$\begin{aligned}\int_S \zeta &= \int_S xdy \wedge dz + ydz \wedge dx + zdx \wedge dy \\ &= \int_E (\sin u \cos v, \sin u \sin v, \cos u) \cdot \mathbf{N}(u,v)\, du\, dv \\ &= \int_E \mathbf{n}(u,v) \cdot \mathbf{n}(u,v) |\mathbf{N}(u,v)|\, du\, dv \\ &= \int_E |\mathbf{N}(u,v)|\, du\, dv \\ &= A(S).\end{aligned}$$

(3) In particular,

$$\begin{aligned}
\int_\Sigma \zeta &= \int_D \sin u \, du \, dv \\
&= \int_0^\pi \int_0^{2\pi} \sin u \, du \, dv \\
&= \left(\int_0^\pi \sin u \, du \right) \left(\int_0^{2\pi} dv \right) \\
&= 2 \cdot 2\pi \\
&= 4\pi.
\end{aligned}$$

□

Proof of (c).

(1) Similar to (b).

$$\begin{aligned}
\mathbf{N}(s,t) &= \frac{\partial(y,z)}{\partial(s,t)}\mathbf{e}_1 + \frac{\partial(z,x)}{\partial(s,t)}\mathbf{e}_2 + \frac{\partial(x,y)}{\partial(s,t)}\mathbf{e}_3 \\
&= g(t)g'(t)[(h_1(s), h_2(s), h_3(s)) \times (h_1'(s), h_2'(s), h_3'(s))] \\
&= g(t)g'(t)[\mathbf{h}(s) \times \mathbf{h}'(s)],
\end{aligned}$$

where $\mathbf{h}(s) = (h_1(s), h_2(s), h_3(s))$ and $\mathbf{h}'(s) = (h_1'(s), h_2'(s), h_3'(s))$. (Here "$\times$" is the cross product in $\mathbb{R}^3$.)

(2) Assume ζ is well-defined, i.e., $\mathbf{h}(s) \neq \mathbf{0}$ for all $s \in [0,1]$. By Integrals of 2-forms in $\mathbb{R}^3$ 10.49,

$$\begin{aligned}
\int_\Phi \zeta &= \int_\Phi \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{r^3} \\
&= \int_{I^2} \frac{g(t)}{g(t)^3 |\mathbf{h}(s)|^3} \mathbf{h}(s) \cdot \mathbf{N}(s,t) \, ds \, dt \\
&= \int_{I^2} \frac{g(t)}{g(t)^3 |\mathbf{h}(s)|^3} \mathbf{h}(s) \cdot g(t)g'(t)[\mathbf{h}(s) \times \mathbf{h}'(s)] \, ds \, dt \\
&= \int_{I^2} \frac{g'(t)}{g(t) |\mathbf{h}(s)|^3} \mathbf{h}(s) \cdot [\mathbf{h}(s) \times \mathbf{h}'(s)] \, ds \, dt \\
&= 0
\end{aligned}$$

(since $\mathbf{h}(s) \cdot [\mathbf{h}(s) \times \mathbf{h}'(s)] = 0$.)

(3) Note that Σ in spherical coordinate system cannot be parameterized as $(x,y,z) = g(t)\mathbf{h}(s)$, and thus $\int_S \zeta$ could be nonzero as shown in (b).

□

Proof of (d) (Hint).

(1) Consider the 3-surface Ψ given by

$$\Psi(t,u,v) = [1 - t + tf(u,v)]\Sigma(u,v),$$

where $(u,v) \in E$, $0 \leq t \leq 1$. Write

$$E = [a_1, b_1] \times [a_2, b_2] \subseteq D = [0,\pi] \times [0, 2\pi].$$

Note that $\Psi(t,u,v) \subseteq \mathbb{R}^3 - \{\mathbf{0}\}$. So the boundary of Ψ is

$$\begin{aligned}
\partial\Psi &= \Psi(0,u,v) - \Psi(1,u,v) \\
&\quad + \Psi(t,a_1,v) - \Psi(t,b_1,v) \\
&\quad + \Psi(t,u,a_2) - \Psi(t,u,b_2) \\
&= S(u,v) - \Omega(u,v) \\
&\quad + \Psi|_{u=a_1}(t,v) - \Psi|_{u=b_1}(t,v) \\
&\quad + \Psi|_{v=a_2}(t,u) - \Psi|_{v=b_2}(t,u),
\end{aligned}$$

where $\Psi|_{u=u_0}(t,v) = \Psi(t,u_0,v)$ and $\Psi|_{v=v_0}(t,u) = \Psi(t,u,v_0)$.

(2) *Show that*

$$\int_{\Psi|_{v=v_0}} \zeta = 0$$

for any fixed $v = v_0 \in [a_2, b_2]$. Note that ζ is well-defined on $\Psi|_{v=v_0}$. Write $\Psi|_{v=v_0}(t,u) = (x,y,z) = (x(t,u), y(t,u), z(t,u))$. By definition of Ψ, we have

$$\begin{aligned}
x &= g(t,u)\sin u \cos v_0 \\
y &= g(t,u)\sin u \sin v_0 \\
z &= g(t,u)\cos u,
\end{aligned}$$

where $g(t,u) = 1 - t + tf(u,v_0)$. Similar to (c),

$$\begin{aligned}
\mathbf{N}(t,u) &= \frac{\partial(y,z)}{\partial(t,u)}\mathbf{e}_1 + \frac{\partial(z,x)}{\partial(t,u)}\mathbf{e}_2 + \frac{\partial(x,y)}{\partial(t,u)}\mathbf{e}_3 \\
&= g(t,u)D_1 g(t,u)(-\sin v_0, \cos v_0, 0).
\end{aligned}$$

Note that

$$(x(t,u), y(t,u), z(t,u)) \cdot \mathbf{N}(t,u) = 0.$$

So

$$\begin{aligned}
\int_{\Psi|_{v=v_0}} \zeta &= \int_{\Psi|_{v=v_0}} r^{-3}(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy) \\
&= \int_{[0,1]\times[a_1,b_1]} r^{-3}(x(t,u), y(t,u), z(t,u)) \cdot \mathbf{N}(t,u)\, dt\, du \\
&= \int_{[0,1]\times[a_1,b_1]} 0\, dt\, du \\
&= 0.
\end{aligned}$$

(3) *Show that*

$$\int_{\Psi|_{u=u_0}} \zeta = 0$$

for any fixed $u = u_0 \in [a_1, b_1]$. Similar to (2).

$$\begin{aligned}\mathbf{N}(t,v) &= \frac{\partial(y,z)}{\partial(t,v)}\mathbf{e}_1 + \frac{\partial(z,x)}{\partial(t,v)}\mathbf{e}_2 + \frac{\partial(x,y)}{\partial(t,v)}\mathbf{e}_3 \\ &= \sin u_0 g(t,v) D_1 g(t,v)(-\cos u_0 \cos v, -\cos u_0 \sin v, \sin u_0).\end{aligned}$$

where $g(t,v) = 1 - t + tf(u_0, v)$. So $(x(t,v), y(t,v), z(t,v)) \cdot \mathbf{N}(t,v) = 0$ and thus $\int_{\Psi|_{u=u_0}} \zeta = 0$.

(4) So

$$\begin{aligned}0 &= \int_\Psi d\zeta && (d\zeta = 0 \text{ on } \mathbb{R}^3 - \{\mathbf{0}\}) \\ &= \int_{\partial\Psi} \zeta && (\text{Theorem 10.33}) \\ &= \int_S \zeta - \int_\Omega \zeta \\ &\quad + \underbrace{\int_{\Psi|_{u=a_1}} \zeta - \int_{\Psi|_{u=b_1}} \zeta}_{\text{all are zero by (2)}} \\ &\quad + \underbrace{\int_{\Psi|_{v=a_2}} \zeta - \int_{\Psi|_{v=b_2}} \zeta}_{\text{all are zero by (3)}} && ((1)) \\ &= \int_S \zeta - \int_\Omega \zeta.\end{aligned}$$

Hence

$$\int_\Omega \zeta = \underbrace{\int_S \zeta = A(S)}_{\text{by (b)}}.$$

□

Proof of (e).

(1) Note that

$$d\left(-\frac{z}{r}\right) = \frac{xz}{r^3}dx + \frac{yz}{r^3}dy - \frac{r^2 - z^2}{r^3}dz = \frac{xz}{r^3}dx + \frac{yz}{r^3}dy - \frac{x^2 + y^2}{r^3}dz$$

since $r^2 = x^2 + y^2 + z^2$.

(2)

$$
\begin{aligned}
d\lambda &= d\left(-\frac{z}{r}\eta\right) \\
&= \underbrace{d\left(-\frac{z}{r}\right)}_{\text{apply (1)}} \wedge \eta + (-1)^1 \left(-\frac{z}{r}\right) \wedge \underbrace{d\eta}_{=0} \\
&= \left(\frac{xz}{r^3}dx + \frac{yz}{r^3}dy - \frac{x^2+y^2}{r^3}dz\right) \wedge \left(\frac{-ydx + xdy}{x^2+y^2}\right) \\
&= \left(\frac{x(x^2+y^2)}{r^3(x^2+y^2)}\right) dy \wedge dz + \left(\frac{y(x^2+y^2)}{r^3(x^2+y^2)}\right) dz \wedge dx + \left(\frac{x^2z + y^2z}{r^3(x^2+y^2)}\right) dx \wedge dy \\
&= \left(\frac{x}{r^3}\right) dy \wedge dz + \left(\frac{y}{r^3}\right) dz \wedge dx + \left(\frac{z}{r^3}\right) dx \wedge dy \\
&= \zeta.
\end{aligned}
$$

□

Proof of (f).

(1) To ensure that η is well-defined on E, we might assume $x^2+y^2 = \sin^2 u \neq 0$ or $0 < u < \pi$ on E. It is fine since $\int_\Omega \zeta$ and $\int_S \zeta$ is well-defined on any closed rectangle in D and we can apply the argument in Exercise 6.7 to remove the additional restriction.

(2) By the Stokes' theorem (Theorem 10.33) and (e),

$$\int_\Omega \zeta = \int_{\partial\Omega} \lambda \qquad \text{and} \qquad \int_S \zeta = \int_{\partial S} \lambda.$$

So it suffices to show that

$$\int_{\partial\Omega} \lambda = \int_{\partial S} \lambda.$$

Note that $\lambda = -\frac{z}{r}\eta$, and thus it suffices to show that $\frac{z}{r}\Big|_{\partial\Omega} = \frac{z}{r}\Big|_{\partial\Sigma}$ and $\eta|_{\partial\Omega} = \eta|_{\partial S}$.

(3) *Show that* $\frac{z}{r}\Big|_{\partial\Omega} = \frac{z}{r}\Big|_{\partial\Sigma}$. For any $(x_\Omega, y_\Omega, z_\Omega) \in \partial\Omega$,

$$(x_\Omega, y_\Omega, z_\Omega) = f(u,v)(x_\Sigma, y_\Sigma, z_\Sigma)$$

where $(x_\Sigma, y_\Sigma, z_\Sigma) \in \partial S$. So

$$
\begin{aligned}
\frac{z}{r}\Big|_{\partial\Omega} &= \frac{z_\Omega}{(x_\Omega^2 + y_\Omega^2 + z_\Omega^2)^{\frac{1}{2}}} \\
&= \frac{f(u,v) z_\Sigma}{f(u,v)(x_\Sigma^2 + y_\Sigma^2 + z_\Sigma^2)^{\frac{1}{2}}} \\
&= \frac{z_\Sigma}{(x_\Sigma^2 + y_\Sigma^2 + z_\Sigma^2)^{\frac{1}{2}}} \\
&= \frac{z}{r}\Big|_{\partial S}.
\end{aligned}
$$

(Note that $f > 0$.)

(4) *Show that $\eta|_{\partial\Omega} = \eta|_{\partial S}$.* Similar to (3). If $x_\Omega \neq 0$ (or $x_\Sigma \neq 0$), then by Exercise 10.21(d)

$$\begin{aligned}
\eta|_{\partial\Omega} &= d\left(\arctan \frac{y_\Omega}{x_\Omega}\right) \\
&= d\left(\arctan \frac{f(u,v)y_\Sigma}{f(u,v)x_\Sigma}\right) \\
&= d\left(\arctan \frac{y_\Sigma}{x_\Sigma}\right) \\
&= \eta|_{\partial S}.
\end{aligned}$$

Similarly, $\eta|_{\partial\Omega} = \eta|_{\partial S}$ is also true if $y_\Omega \neq 0$. Note that $(x_\Omega, y_\Omega) \neq (0,0)$ by assumption. Therefore the result is established.

□

Proof of (g).

(1) Yes. Given any line L passing through $\mathbf{0}$, say

$$(r\sin u\cos v, r\sin u\sin v, r\cos u) \in L \qquad (r \in \mathbb{R}^1),$$

for some $u \in [0,\pi]$ and $v \in [0,2\pi]$. We will show that ζ is exact in $U = \mathbb{R}^3 - L$.

(2) Linear algebra says that all rotation matrices $T \in SO(3)$ can be obtained from

$$\begin{aligned}
R_x(u) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos u & -\sin u \\ 0 & \sin u & \cos u \end{bmatrix} \\
R_y(v) &= \begin{bmatrix} \cos v & 0 & -\sin v \\ 0 & 1 & 0 \\ \sin v & 0 & \cos v \end{bmatrix} \\
R_z(w) &= \begin{bmatrix} \cos w & -\sin w & 0 \\ \sin w & \cos w & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

using matrix multiplication, say $T = R_x(u)R_y(v)R_z(w)$. For example, the rotation

$$T = R_y\left(u - \frac{\pi}{2}\right) R_z(-v)$$

maps L to the z-axis (by showing that $T(r\sin u\cos v, r\sin u\sin v, r\cos u) = (0,0,r)$). By Theorem 10.22 it suffices to show that ζ is invariant under $R_x(u)$, $R_x(v)$ and $R_z(w)$. By the symmetricity of ζ, it suffices to show that ζ is invariant under $T = R_x(u)$.

(3) *Show that ζ is invariant under $T = R_x(u)$.* By

$$T : (x, y, z) \mapsto (x, y\cos u - z\sin u, y\sin u + z\cos u),$$

we have

$$\begin{aligned} r &\mapsto r \\ dx &\mapsto dx \\ dy &\mapsto \cos u dy - \sin u dz \\ dz &\mapsto \sin u dy + \cos u dz. \end{aligned}$$

So

$$\begin{aligned} dy \wedge dz &\mapsto (\cos u dy - \sin u dz) \wedge (\sin u dy + \cos u dz) \\ &= dy \wedge dz, \\ dz \wedge dx &\mapsto (\sin u dy + \cos u dz) \wedge dx \\ &= -\sin u dx \wedge dy + \cos u dz \wedge dx, \\ dx \wedge dy &\mapsto dx \wedge (\sin u dy + \cos u dz) \\ &= \cos u dx \wedge dy + \sin u dz \wedge dx. \end{aligned}$$

Thus

$$\begin{aligned} \zeta \mapsto r^{-3}\{ & x dy \wedge dz \\ & + (y\cos u - z\sin u)(-\sin u dx \wedge dy + \cos u dz \wedge dx) \\ & + (y\sin u + z\cos u)(\cos u dx \wedge dy + \sin u dz \wedge dx)\} \\ = r^{-3}\{ & x dy \wedge dz \\ & + [\cos u(y\cos u - z\sin u) + \sin u(y\sin u + z\cos u)]dz \wedge dx \\ & + [-\sin u(y\cos u - z\sin u) + \cos u(y\sin u + z\cos u)]dx \wedge dy\} \\ = r^{-3}\{ & x dy \wedge dz + y dz \wedge dx + z dx \wedge dy\} \\ = \zeta. & \end{aligned}$$

(4) Let $V = \mathbb{R}^3 - z$-axis. Since $\zeta_T = \zeta$ (by (3)) is well-defined in V, $\zeta_T = \zeta = d\lambda$ by (e). Here λ is in V, not necessary in U (if $L \neq z$-axis). Luckily, we can use T^{-1} to pullback λ in U. Thus

$$\zeta = (\zeta_T)_{T^{-1}} = (d\lambda)_{T^{-1}} = d(\lambda_{T^{-1}})$$

by Theorems 10.22 and 10.23. That is, ζ is exact in $U = \mathbb{R}^3 - L$. (Or ζ is locally exact in $\mathbb{R}^3 - \{\mathbf{0}\}$.)

□

Exercise 10.23. *Fix n. Define $r_k = (x_1^2 + \cdots + x_k^2)^{\frac{1}{2}}$ for $1 \le k \le n$, let E_k be the set of all $\mathbf{x} \in \mathbb{R}^n$ at which $r_k > 0$, and let ω_k be the $(k-1)$-form defined in E_k by*

$$\omega_k = (r_k)^{-k} \sum_{i=1}^{k} (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k$$

Note that $\omega_2 = \eta$, $\omega_3 = \zeta$ in the terminology of Exercise 10.21 and Exercise 10.22. Note also that

$$E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n = \mathbb{R}^n.$$

(a) *Prove that $d\omega_k = 0$ in E_k.*

(b) *For $k = 2, \ldots, n$, prove that ω_k is exact in E_{k-1}, by showing that*

$$\omega_k = d(f_k \omega_{k-1}) = df_k \wedge \omega_{k-1}$$

where $f_k(\mathbf{x}) = (-1)^k g_k\left(\frac{x_k}{r_k}\right)$ where

$$g_k(t) = \int_{-1}^{t} (1 - s^2)^{\frac{k-3}{2}} ds \qquad (-1 < t < 1).$$

(Hint: f_k satisfies the differential equations

$$\mathbf{x} \cdot (\nabla f_k)(\mathbf{x}) = 0$$

and

$$(D_k f_k)(\mathbf{x}) = \frac{(-1)^k (r_{k-1})^{k-1}}{(r_k)^k}.)$$

(c) *Is ω_n exact in E_n?*

(d) *Note that (b) is a generalization of part (e) of Exercise 10.22. Try to extend some of the other assertions of Exercise 10.21 and Exercise 10.22 to ω_n, for arbitrary n.*

Proof of (a).

(1) Note that

$$D_i r_k = \frac{1}{2r_k} \cdot (2x_i) = \frac{x_i}{r_k}.$$

(2)

$$
\begin{aligned}
d\omega_k &= \sum_{i=1}^{k} d\left((-1)^{i-1}(r_k)^{-k} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k\right) \\
&= \sum_{i=1}^{k} D_i \left((-1)^{i-1}(r_k)^{-k} x_i\right) dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k \\
&= \sum_{i=1}^{k} (-1)^{i-1} \left((r_k)^{-k} \cdot 1 + \underbrace{(-k)(r_k)^{-k-1} \frac{x_i}{r_k}}_{\text{chain rule}} \cdot x_i \right) \underbrace{(-1)^{i-1} dx_1 \wedge \cdots \wedge dx_k}_{\text{anticommutative relation}} \\
&= (r_k)^{-k-2} \underbrace{\sum_{i=1}^{k} \left((r_k)^2 - k x_i^2\right)}_{=0} dx_1 \wedge \cdots \wedge dx_k \\
&= 0.
\end{aligned}
$$

□

Proof of (b).

(1) Note that

$$
D_i \left(\frac{x_k}{r_k}\right) = \frac{\delta_{ik}(r_k)^2 - x_i x_k}{(r_k)^3}
$$

where δ_{ik} is the Kronecker delta. So

$$
\begin{aligned}
(D_i f_k)(\mathbf{x}) &= D_i \left((-1)^k g_k \left(\frac{x_k}{r_k}\right)\right) \\
&= D_i \left((-1)^k \int_{-1}^{\frac{x_k}{r_k}} (1 - s^2)^{\frac{k-3}{2}} ds\right) \\
&= (-1)^k D_i \left(\frac{x_k}{r_k}\right) \left(1 - \left(\frac{x_k}{r_k}\right)^2\right)^{\frac{k-3}{2}} \\
&= (-1)^k \frac{\delta_{ik}(r_k)^2 - x_i x_k}{(r_k)^3} \frac{(r_{k-1})^{k-3}}{(r_k)^{k-3}} \\
&= (-1)^k \frac{(r_{k-1})^{k-3}}{(r_k)^k} (\delta_{ik}(r_k)^2 - x_i x_k).
\end{aligned}
$$

In particular,

$$
(D_k f_k)(\mathbf{x}) = (-1)^k \frac{(r_{k-1})^{k-3}}{(r_k)^k} ((r_k)^2 - (x_k)^2) = (-1)^k \frac{(r_{k-1})^{k-1}}{(r_k)^k}
$$

(since $(r_k)^2 - (x_k)^2 = (r_{k-1})^2$).

(2) Since

$$\sum_i x_i(\delta_{ik}(r_k)^2 - x_i x_k) = (r_k)^2 \underbrace{\sum_i x_i \delta_{ik}}_{=x_k} - x_k \underbrace{\sum_i x_i^2}_{=(r_k)^2} = 0,$$

we have

$$\begin{aligned}
\mathbf{x} \cdot (\nabla f_k)(\mathbf{x}) &= \sum_i x_i (D_i f_k)(\mathbf{x}) \\
&= \sum_i x_i (-1)^k \frac{(r_{k-1})^{k-3}}{(r_k)^k} (\delta_{ik}(r_k)^2 - x_i x_k) \\
&= (-1)^k \frac{(r_{k-1})^{k-3}}{(r_k)^k} \sum_i x_i (\delta_{ik}(r_k)^2 - x_i x_k) \\
&= 0.
\end{aligned}$$

(3) On $E_{k-1} \subsetneq E_k$, we write

$$\begin{aligned}
& d(f_k \omega_{k-1}) \\
&= (df_k) \wedge \omega_{k-1} + (-1)^0 f_k \wedge \underbrace{(d\omega_{k-1})}_{=0} \\
&= (df_k) \wedge \omega_{k-1} \\
&= \left\{ \sum_{i=1}^{k} D_i f_k(\mathbf{x}) dx_i \right\} \wedge \\
&\quad \left\{ \frac{1}{(r_{k-1})^{k-1}} \sum_{j=1}^{k-1} (-1)^{j-1} x_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{k-1} \right\} \\
&= \frac{1}{(r_{k-1})^{k-1}} \sum_{\substack{1 \le i \le k \\ 1 \le j \le k-1}} (-1)^{j-1} x_j D_i f_k(\mathbf{x}) dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{k-1} \\
&= \frac{1}{(r_{k-1})^{k-1}} \sum_{j=1}^{k-1} (-1)^{j-1} x_j D_j f_k(\mathbf{x}) dx_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{k-1} \\
&\quad + \frac{1}{(r_{k-1})^{k-1}} \sum_{j=1}^{k-1} (-1)^{j-1} x_j D_k f_k(\mathbf{x}) dx_k \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{k-1}.
\end{aligned}$$

(4) By (2),

$$\begin{aligned}
&\frac{1}{(r_{k-1})^{k-1}} \sum_{j=1}^{k-1} (-1)^{j-1} x_j D_j f_k(\mathbf{x}) dx_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{k-1} \\
&= \frac{1}{(r_{k-1})^{k-1}} \sum_{j=1}^{k-1} x_j D_j f_k(\mathbf{x}) dx_1 \wedge \cdots \wedge dx_{k-1} \\
&= \frac{1}{(r_{k-1})^{k-1}} (-D_k f_k(x) x_k) dx_1 \wedge \cdots \wedge dx_{k-1} \\
&= \frac{-D_k f_k(\mathbf{x})}{(r_{k-1})^{k-1}} x_k dx_1 \wedge \cdots \wedge dx_{k-1} \wedge \widehat{dx_k} \\
&= (r_k)^{-k} (-1)^{k-1} x_k dx_1 \wedge \cdots \wedge dx_{k-1} \wedge \widehat{dx_k} && ((1)).
\end{aligned}$$

Also,

$$\begin{aligned}
&\frac{1}{(r_{k-1})^{k-1}} \sum_{j=1}^{k-1} (-1)^{j-1} x_j D_k f_k(\mathbf{x}) dx_k \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{k-1} \\
&= \frac{(-1)^k D_k f_k(\mathbf{x})}{(r_{k-1})^{k-1}} \sum_{j=1}^{k-1} (-1)^{j-1} x_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_k \\
&= (r_k)^{-k} \sum_{j=1}^{k-1} (-1)^{j-1} x_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_k && ((1)).
\end{aligned}$$

(5) Hence,

$$\begin{aligned}
&d(f_k \omega_{k-1}) \\
&= (r_k)^{-k} (-1)^{k-1} x_k dx_1 \wedge \cdots \wedge dx_{k-1} \wedge \widehat{dx_k} \\
&\quad + (r_k)^{-k} \sum_{j=1}^{k-1} (-1)^{j-1} x_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_k \\
&= (r_k)^{-k} \sum_{j=1}^{k} (-1)^{j-1} x_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_k \\
&= \omega_k.
\end{aligned}$$

□

Proof of (c).

(1) ω_n is not exact in E_n (though it is locally exact).

(2) Let

$$\begin{aligned}
\mathbb{S}^{n-1} &= \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\} \\
\mathbb{B}^n &= \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leq 1\}.
\end{aligned}$$

It suffices to show that

$$\int_{\mathbb{S}^{n-1}} \omega_n = \frac{n\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} \neq 0.$$

Therefore, ω_n is not exact in E_n.

(3) Define

$$\omega = \frac{1}{n}\sum_{i=1}^{n}(-1)^{i-1}x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

on $\mathbb{S}^{n-1}$. Note that

$$\omega = \frac{1}{n}\omega_n$$

on $\mathbb{S}^{n-1}$ (and that's why we pick $\mathbb{S}^{n-1}$). The Stokes' theorem (Theorem 10.33) implies that

$$\int_{\mathbb{S}^{n-1}} \frac{1}{n}\omega_n = \int_{\partial\mathbb{B}^n} \omega = \int_{\mathbb{B}^n} d\omega = \int_{\mathbb{B}^n} dx_1 \wedge \cdots \wedge dx_n = \mathrm{vol}(\mathbb{B}^n),$$

where $\mathrm{vol}(\mathbb{B}^n)$ is the volume of $\mathbb{B}^n$. Thus *it suffices to show that*

$$\mathrm{vol}(\mathbb{B}^n) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} \neq 0.$$

There are many proofs for this. We give a direct integration in spherical coordinates.

(4) Similar to Exercise 10.9. The spherical coordinate system has a radial coordinate r and angular coordinates $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_{n-1})$, where the domain of each $\varphi_1, \ldots, \varphi_{n-2}$ is $[0, \pi]$ and the domain of φ_{n-1} is $[0, 2\pi]$. That is,

$$\begin{aligned}
x_1 &= \cos\varphi_1 \\
x_2 &= \sin\varphi_1 \cos\varphi_2 \\
x_3 &= \sin\varphi_1 \sin\varphi_2 \cos\varphi_3 \\
&\cdots \\
x_{n-1} &= \sin\varphi_1 \cdots \sin\varphi_{n-2} \cos\varphi_{n-1} \\
x_n &= \sin\varphi_1 \cdots \sin\varphi_{n-2} \sin\varphi_{n-1}.
\end{aligned}$$

(It is different from Exercise 10.22.) The spherical volume element is

$$r^{n-1}\sin^{n-2}\varphi_1 \sin^{n-3}\varphi_2 \cdots \sin\varphi_{n-2} dr\, d\boldsymbol{\varphi}.$$

Thus by Some consequences 8.21,

$$\begin{aligned}
\mathrm{vol}(\mathbb{B}^n) &= \int_{\mathbb{B}^n} d\mathbf{x} \\
&= \int_0^1 \int_0^\pi \cdots \int_0^{2\pi} r^{n-1} \sin^{n-2}\varphi_1 \cdots \sin\varphi_{n-2} dr\, d\boldsymbol{\varphi} \\
&= \left(\int_0^1 r^{n-1} dr\right)\left(\int_0^\pi \sin^{n-2}\varphi_1 d\varphi_1\right)\cdots\left(\int_0^{2\pi} d\varphi_{n-1}\right) \\
&= \frac{1}{n}\cdot\frac{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2})}\cdot\frac{\Gamma(\frac{n-2}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n-1}{2})}\cdots\frac{\Gamma(1)\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})}\cdot 2\pi \\
&= \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}.
\end{aligned}$$

(Use the similar argument in (d)(ii) to get the spherical volume element.)

(5) Note that we can apply the spherical coordinate system to $\int_{\mathbb{S}^{n-1}} \omega_n$ directly (without the Stokes' theorem). The area element is

$$\sin^{n-2}\varphi_1 \sin^{n-3}\varphi_2 \cdots \sin\varphi_{n-2} d\boldsymbol{\varphi}.$$

A long calculation shows that

$$\begin{aligned}
&\int_{\mathbb{S}^{n-1}} \omega_n \\
&= \int_0^\pi \cdots \int_0^{2\pi} \sin^{n-2}\varphi_1 \sin^{n-3}\varphi_2 \cdots \sin\varphi_{n-2} d\boldsymbol{\varphi} \\
&= \left(\int_0^\pi \sin^{n-2}\varphi_1 d\varphi_1\right)\cdots\left(\int_0^{2\pi} d\varphi_{n-1}\right) \\
&= \frac{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2})}\cdot\frac{\Gamma(\frac{n-2}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n-1}{2})}\cdots\frac{\Gamma(1)\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})}\cdot 2\pi \\
&= \frac{n\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}.
\end{aligned}$$

(See (d)(ii) for more details.)

$\square$

Outline of (d).

(i) One generalization of Exercise 10.21(a) and 10.22(a). See Exercise 10.23(a).

(ii) One generalization of Exercise 10.22(b). *Let* $\Sigma = \mathbb{S}^{n-1}$ *be the* $(n-1)$*-surface in* $\mathbb{R}^n$*, with parameter domain* $D = [0,\pi]^{n-2}\times[0,2\pi]$*, given* *by*

$$\begin{aligned} x_1 &= \cos\varphi_1 \\ x_2 &= \sin\varphi_1\cos\varphi_2 \\ x_3 &= \sin\varphi_1\sin\varphi_2\cos\varphi_3 \\ &\cdots \\ x_{n-1} &= \sin\varphi_1\cdots\sin\varphi_{n-2}\cos\varphi_{n-1} \\ x_n &= \sin\varphi_1\cdots\sin\varphi_{n-2}\sin\varphi_{n-1}. \end{aligned}$$

Let S denote the restriction of Σ to a parameter domain $E \subseteq D$. Prove that

$$\begin{aligned} \int_S \omega_n &= \int_E \sin^{n-2}\varphi_1 \sin^{n-3}\varphi_2\cdots\sin\varphi_{n-2}d\boldsymbol{\varphi} \\ &= A(S), \end{aligned}$$

where A denotes surface area.

(iii) One generalization of Exercise 10.22(c). *Suppose $g \in \mathscr{C}''([0,1])$, $\mathbf{h} = (h_1,\ldots,h_n) \in \mathscr{C}''([0,1]^{n-2})$, and $g > 0$. Write $\mathbf{x} = (x_1,\ldots,x_n)$ and $\mathbf{s} = (s_1,\ldots,s_{n-2})$. Let*

$$\mathbf{x} = \Phi(\mathbf{s}, t)$$

define a $(n-1)$-surface Φ, with parameter domain $[0,1]^{n-1}$, by

$$\mathbf{x} = g(t)\mathbf{h}(\mathbf{s}).$$

Prove that

$$\int_\Phi \omega_n = 0.$$

(iv) One generalization of Exercise 10.21(b) and 10.22(d). *Let E be a closed cell in D, with edges parallel to those of D. Suppose $f \in \mathscr{C}''(D)$, $f > 0$. Let Ω be the $(n-1)$-surface with parameter domain E, defined by*

$$\Omega(\boldsymbol{\varphi}) = f(\boldsymbol{\varphi})\Sigma(\boldsymbol{\varphi}).$$

Define S as in (ii) and prove that

$$\int_\Omega \omega_n = \int_S \omega_n = A(S).$$

(v) One generalization of Exercise 10.21(d) and 10.22(e). See Exercise 10.23(b).

(vi) One generalization of Examples 10.36 and 10.37. See Exercise 10.23(c).

(vii) One generalization of Exercise 10.21(e) and 10.22(f). *Derive (iv) from Exercise 10.23(b), without using (iii).*

(viii) One generalization of Exercise 10.21(f). $\pi_{n-1}(\mathbb{S}^{n-1}) = \mathbb{Z}$ (without proof).

(ix) One generalization of Exercise 10.22(g). *Show that ω_n is exact in the complement of every line L passing through the origin.*

Proof of (d)(ii).

(1) On $S \subseteq \mathbb{S}^{n-1}$, we have

$$\begin{aligned}
\int_S \omega_n &= \int_S \sum_{i=1}^{n} (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\
&= \int_S \sum_{i=1}^{n} (-1)^{i-1} x_i(\boldsymbol{\varphi}) \frac{\partial(x_1, \ldots, \widehat{x_i}, \ldots, x_n)}{\partial(\varphi_1, \ldots, \varphi_{n-1})} d\varphi_1 \wedge \cdots \wedge d\varphi_{n-1} \\
&= \int_S \sum_{i=1}^{n} (-1)^{i-1} x_i(\boldsymbol{\varphi}) \det \begin{bmatrix} \frac{\partial x_1}{\partial \varphi_1} & \cdots & \frac{\partial x_1}{\partial \varphi_{n-1}} \\ \vdots & \ddots & \vdots \\ \widehat{\frac{\partial x_i}{\partial \varphi_1}} & \cdots & \widehat{\frac{\partial x_i}{\partial \varphi_{n-1}}} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial \varphi_1} & \cdots & \frac{\partial x_n}{\partial \varphi_{n-1}} \end{bmatrix} d\varphi_1 \wedge \cdots \wedge d\varphi_{n-1} \\
&= \int_S \det \underbrace{\begin{bmatrix} x_1 & \frac{\partial x_1}{\partial \varphi_1} & \cdots & \frac{\partial x_1}{\partial \varphi_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ x_i & \frac{\partial x_i}{\partial \varphi_1} & \cdots & \frac{\partial x_i}{\partial \varphi_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ x_n & \frac{\partial x_n}{\partial \varphi_1} & \cdots & \frac{\partial x_n}{\partial \varphi_{n-1}} \end{bmatrix}}_{\text{say } A_n} d\varphi_1 \wedge \cdots \wedge d\varphi_{n-1}.
\end{aligned}$$

Hence, it suffices to show that

$$\det(A_n) = \sin^{n-2} \varphi_1 \sin^{n-3} \varphi_2 \cdots \sin \varphi_{n-2}.$$

(2) *Show that* $\det(A_n) = \sin^{n-2} \varphi_1 \sin^{n-3} \varphi_2 \cdots \sin \varphi_{n-2}$. Induction on n.

(a) When $n = 3$, a straightforward computation shows that the determinant is

$$\begin{aligned}
&\det(A_3) \\
&= \det \begin{bmatrix} \cos \varphi_1 & -\sin \varphi_1 & 0 \\ \sin \varphi_1 \cos \varphi_2 & \cos \varphi_1 \cos \varphi_2 & -\sin \varphi_1 \sin \varphi_2 \\ \sin \varphi_1 \sin \varphi_2 & \cos \varphi_1 \sin \varphi_2 & \sin \varphi_1 \cos \varphi_2 \end{bmatrix} \\
&= \sin \varphi_1.
\end{aligned}$$

(b) When $n = 4$,

$$\begin{aligned}
&\det(A_4)\\
&= \det \begin{bmatrix} \cos\varphi_1 & -\sin\varphi_1 & 0 & 0 \\ \sin\varphi_1\cos\varphi_2 & \cos\varphi_1\cos\varphi_2 & -\sin\varphi_1\sin\varphi_2 & 0 \\ \sin\varphi_1\sin\varphi_2\cos\varphi_3 & \cos\varphi_1\sin\varphi_2\cos\varphi_3 & \sin\varphi_1\cos\varphi_2\cos\varphi_3 & -\sin\varphi_1\sin\varphi_2\sin\varphi_3 \\ \sin\varphi_1\sin\varphi_2\sin\varphi_3 & \cos\varphi_1\sin\varphi_2\sin\varphi_3 & \sin\varphi_1\cos\varphi_2\sin\varphi_3 & \sin\varphi_1\sin\varphi_2\cos\varphi_3 \end{bmatrix}.
\end{aligned}$$

Expand along the last column to get

$$\begin{aligned}
&\det(A_4)\\
&= (-1)^{3+4}(-\sin\varphi_1\sin\varphi_2\sin\varphi_3)\\
&\quad \det \begin{bmatrix} \cos\varphi_1 & -\sin\varphi_1 & 0 \\ \sin\varphi_1\cos\varphi_2 & \cos\varphi_1\cos\varphi_2 & -\sin\varphi_1\sin\varphi_2 \\ \sin\varphi_1\sin\varphi_2\sin\varphi_3 & \cos\varphi_1\sin\varphi_2\sin\varphi_3 & \sin\varphi_1\cos\varphi_2\sin\varphi_3 \end{bmatrix}\\
&\quad + (-1)^{4+4}(\sin\varphi_1\sin\varphi_2\cos\varphi_3)\\
&\quad \det \begin{bmatrix} \cos\varphi_1 & -\sin\varphi_1 & 0 \\ \sin\varphi_1\cos\varphi_2 & \cos\varphi_1\cos\varphi_2 & -\sin\varphi_1\sin\varphi_2 \\ \sin\varphi_1\sin\varphi_2\cos\varphi_3 & \cos\varphi_1\sin\varphi_2\cos\varphi_3 & \sin\varphi_1\cos\varphi_2\cos\varphi_3 \end{bmatrix}\\
&= (\sin\varphi_1\sin\varphi_2\sin^2\varphi_3)\det(A_3) + (\sin\varphi_1\sin\varphi_2\cos^2\varphi_3)\det(A_3)\\
&= \sin\varphi_1\sin\varphi_2\det(A_3)\\
&= \sin^2\varphi_1\sin\varphi_2.
\end{aligned}$$

(c) Now for large n, as (b) we expand along the last column to get

$$\begin{aligned}
&\det(A_n)\\
&= (-1)^{(n-1)+n}(-\sin\varphi_1\cdots\sin\varphi_{n-2}\sin\varphi_{n-1})(\sin\varphi_{n-1}\det(A_{n-1}))\\
&\quad + (-1)^{n+n}(\sin\varphi_1\cdots\sin\varphi_{n-2}\cos\varphi_{n-1})(\cos\varphi_{n-1}\det(A_{n-1}))\\
&= (\sin\varphi_1\cdots\sin\varphi_{n-2})\det(A_{n-1})\\
&= (\sin\varphi_1\cdots\sin\varphi_{n-2})(\sin^{n-3}\varphi_1\sin^{n-4}\varphi_2\cdots\sin\varphi_{n-3})\\
&= \sin^{n-2}\varphi_1\sin^{n-3}\varphi_2\cdots\sin^2\varphi_{n-3}\sin\varphi_{n-2}.
\end{aligned}$$

(3) (Area elements in $\mathbb{R}^3$ 10.46.) Given any $\mathbf{x} = (x_1, \ldots, x_n) \in S$. Define the vector $\mathbf{N}(\boldsymbol{\varphi})$ by

$$\mathbf{N}(\boldsymbol{\varphi}) = \sum_{i=1}^{n} \frac{\partial(x_1, \ldots, \widehat{x_i}, \ldots, x_n)}{\partial(\varphi_1, \ldots, \varphi_{n-1})}\mathbf{e}_i.$$

So the area of S is defined by

$$A(S) = \int_E |\mathbf{N}(\boldsymbol{\varphi})| d\boldsymbol{\varphi}.$$

(4) By the similar proof in (2),

$$\begin{aligned}
&\frac{\partial(x_1, \dots, \widehat{x_i}, \dots, x_n)}{\partial(\varphi_1, \dots, \varphi_{n-1})} \\
&= (-1)^{i-1} \sin^{n-2}\varphi_1 \sin^{n-3}\varphi_2 \cdots \sin^2\varphi_{n-3} \sin\varphi_{n-2} x_i
\end{aligned}$$

if $i = 1, \dots, n$. Since $\sum x_i^2 = 1$ on S,

$$|\mathbf{N}(\boldsymbol{\varphi})| = \sin^{n-2}\varphi_1 \sin^{n-3}\varphi_2 \cdots \sin^2\varphi_{n-3} \sin\varphi_{n-2}.$$

Thus,

$$\begin{aligned}
A(S) &= \int_E |\mathbf{N}(\boldsymbol{\varphi})| d\boldsymbol{\varphi} \\
&= \int_E \sin^{n-2}\varphi_1 \sin^{n-3}\varphi_2 \cdots \sin^2\varphi_{n-3} \sin\varphi_{n-2} d\boldsymbol{\varphi}.
\end{aligned}$$

(5) Note that we can apply (3) on (2) to get the same conclusion.

$$\begin{aligned}
\int_S \omega_n &= \int_S \sum_{i=1}^n (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\
&= \int_S \sum_{i=1}^n (-1)^{i-1} x_i \frac{\partial(x_1, \dots, \widehat{x_i}, \dots, x_n)}{\partial(\varphi_1, \dots, \varphi_{n-1})} d\varphi_1 \wedge \cdots \wedge d\varphi_{n-1} \\
&= \int_E \sum_{i=1}^n (-1)^{i-1} x_i \\
&\qquad (-1)^{i-1} \sin^{n-2}\varphi_1 \sin^{n-3}\varphi_2 \cdots \sin^2\varphi_{n-3} \sin\varphi_{n-2} x_i d\boldsymbol{\varphi} \\
&= \int_E \sin^{n-2}\varphi_1 \sin^{n-3}\varphi_2 \cdots \sin^2\varphi_{n-3} \sin\varphi_{n-2} d\boldsymbol{\varphi}.
\end{aligned}$$

□

Proof of (d)(iii).

(1) Similar to Exercise 10.22(c). Assume that ω_n is well-defined, i.e., $\mathbf{h}(\mathbf{s}) \neq 0$

for all $\mathbf{s} \in [0,1]^{n-2}$.

$$\frac{\partial(x_1,\ldots,\widehat{x_i},\ldots,x_n)}{\partial(s_1,\ldots,s_{n-2},t)} = \det\begin{bmatrix} \frac{\partial x_1}{\partial s_1} & \cdots & \frac{\partial x_1}{\partial s_{n-2}} & \frac{\partial x_1}{\partial t} \\ \vdots & \ddots & \vdots & \vdots \\ \widehat{\frac{\partial x_i}{\partial s_1}} & \cdots & \widehat{\frac{\partial x_i}{\partial s_{n-2}}} & \widehat{\frac{\partial x_i}{\partial t}} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial x_1}{\partial s_1} & \cdots & \frac{\partial x_n}{\partial s_{n-2}} & \frac{\partial x_n}{\partial t} \end{bmatrix}$$

$$= \det\begin{bmatrix} g\frac{\partial h_1}{\partial s_1} & \cdots & g\frac{\partial h_1}{\partial s_{n-2}} & g'h_1 \\ \vdots & \ddots & \vdots & \vdots \\ \widehat{g\frac{\partial h_i}{\partial s_1}} & \cdots & \widehat{g\frac{\partial h_i}{\partial s_{n-2}}} & \widehat{g'h_i} \\ \vdots & \ddots & \vdots & \vdots \\ g\frac{\partial h_n}{\partial s_1} & \cdots & g\frac{\partial h_n}{\partial s_{n-2}} & g'h_n \end{bmatrix}$$

$$= g^{n-2}g' \det\underbrace{\begin{bmatrix} \frac{\partial h_1}{\partial s_1} & \cdots & \frac{\partial h_1}{\partial s_{n-2}} & h_1 \\ \vdots & \ddots & \vdots & \vdots \\ \widehat{\frac{\partial h_i}{\partial s_1}} & \cdots & \widehat{\frac{\partial h_i}{\partial s_{n-2}}} & \widehat{h_i} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial h_n}{\partial s_1} & \cdots & \frac{\partial h_n}{\partial s_{n-2}} & h_n \end{bmatrix}}_{\text{say } A}.$$

(2) So

$$\int_\Phi \omega_n = \int_{[0,1]^{n-1}} \frac{1}{g(t)^n|\mathbf{h}(\mathbf{s})|^n}\sum_{i=1}^n(-1)^{i-1}g(t)h_i g(t)^{n-2}g'(t)\det(A)\,d\mathbf{s}\,dt$$

$$= \int_{[0,1]^{n-1}} \frac{g'(t)}{g(t)|\mathbf{h}(\mathbf{s})|^n}\sum_{i=1}^n(-1)^{i-1}h_i \det\begin{bmatrix} \frac{\partial h_1}{\partial s_1} & \cdots & \frac{\partial h_1}{\partial s_{n-2}} & h_1 \\ \vdots & \ddots & \vdots & \vdots \\ \widehat{\frac{\partial h_i}{\partial s_1}} & \cdots & \widehat{\frac{\partial h_i}{\partial s_{n-2}}} & \widehat{h_i} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial h_n}{\partial s_1} & \cdots & \frac{\partial h_n}{\partial s_{n-2}} & h_n \end{bmatrix} d\mathbf{s}\,dt$$

$$= \int_{[0,1]^{n-1}} \frac{g'(t)}{g(t)|\mathbf{h}(\mathbf{s})|^n}\det\underbrace{\begin{bmatrix} h_1 & \frac{\partial h_1}{\partial s_1} & \cdots & \frac{\partial h_1}{\partial s_{n-2}} & h_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h_n & \frac{\partial h_n}{\partial s_1} & \cdots & \frac{\partial h_n}{\partial s_{n-2}} & h_n \end{bmatrix}}_{\text{say } B} d\mathbf{s}\,dt.$$

Since the first column is the same as the last column in B, $\det(B) = 0$ (Theorem 9.34(d)). Therefore, $\int_\Phi \omega_n = \int_{[0,1]^{n-1}} 0\,d\mathbf{s}\,dt = 0$.

□

Proof of (d)(iv).

(1) Consider the n-surface Ψ given by

$$\Psi(t, \boldsymbol{\varphi}) = [1 - t + tf(\boldsymbol{\varphi})]\Sigma(\boldsymbol{\varphi}),$$

where $\boldsymbol{\varphi} \in E \subseteq D$, $0 \leq t \leq 1$.

(2) Write

$$E = [a_1, b_1] \times \cdots \times [a_{n-1}, b_{n-1}] \subseteq D.$$

Note that $\Psi(t, \boldsymbol{\varphi}) \subseteq \mathbb{R}^n - \{\mathbf{0}\}$. So the boundary of Ψ is

$$\partial\Psi = \Psi(0, \boldsymbol{\varphi}) - \Psi(1, \boldsymbol{\varphi}) + \sum_{i=1}^{n-1}(\Psi|_{\varphi_i = a_i} - \Psi|_{\varphi_i = b_i}),$$

where $\Psi|_{\varphi_i=\theta} : [a_1, b_1] \times \cdots \times \widehat{[a_i, b_i]} \times \cdots \times [a_{n-1}, b_{n-1}] \to \Omega$ is a mapping defined by

$$\begin{aligned}
\Psi|_{\varphi_i=\theta}(t, \varphi_1, \ldots, \widehat{\varphi}_i, \ldots, \varphi_{n-1}) &= \Psi(t, \varphi_1, \ldots, \varphi_{i-1}, \theta, \varphi_{i+1}, \ldots, \varphi_{n-1}) \\
&= \Psi(t, \boldsymbol{\varphi} + (\theta - \varphi_i)\mathbf{e}_i).
\end{aligned}$$

(3) *Show that*

$$\int_{\Psi|_{\varphi_1=\theta}} \omega_n = 0$$

for any fixed $\varphi_1 = \theta \in [a_1, b_1]$. Note that ω_n is well-defined on $\Psi|_{\varphi_1=\theta}$. Write

$$\Psi|_{\varphi_1=\theta}(t, \widehat{\varphi_1}, \varphi_2, \ldots, \varphi_{n-1}) = \mathbf{x}(t, \widehat{\varphi_1}, \varphi_2, \ldots, \varphi_{n-1}).$$

By definition of Ψ, we have

$$\begin{aligned}
x_1 &= g(t, \boldsymbol{\varphi} + (\theta - \varphi_1)\mathbf{e}_1)\cos\theta \\
x_2 &= g(t, \boldsymbol{\varphi} + (\theta - \varphi_1)\mathbf{e}_1)\sin\theta\cos\varphi_2 \\
&\cdots \\
x_{n-1} &= g(t, \boldsymbol{\varphi} + (\theta - \varphi_1)\mathbf{e}_1)\sin\theta\cdots\sin\varphi_{n-2}\cos\varphi_{n-1} \\
x_n &= g(t, \boldsymbol{\varphi} + (\theta - \varphi_1)\mathbf{e}_1)\sin\theta\cdots\sin\varphi_{n-2}\sin\varphi_{n-1},
\end{aligned}$$

where $g(t, \boldsymbol{\varphi} + (\theta - \varphi_1)\mathbf{e}_1) = 1 - t + tf(\boldsymbol{\varphi} + (\theta - \varphi_1)\mathbf{e}_1)$.

(4) Note that $r_n = g > 0$. Since

$$\frac{\partial x_i}{\partial t} = \frac{\partial g}{\partial t} g^{-1} x_i,$$

$$\frac{\partial(x_1,\ldots,\widehat{x_i},\ldots,x_n)}{\partial(t,\widehat{\varphi_1},\varphi_2,\ldots,\varphi_{n-1})} = \det\begin{bmatrix} \frac{\partial x_1}{\partial t} & \frac{\partial x_1}{\partial \varphi_2} & \cdots & \frac{\partial x_1}{\partial \varphi_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{\frac{\partial x_i}{\partial t}} & \widehat{\frac{\partial x_i}{\partial \varphi_2}} & \cdots & \widehat{\frac{\partial x_i}{\partial \varphi_{n-1}}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial t} & \frac{\partial x_n}{\partial \varphi_2} & \cdots & \frac{\partial x_n}{\partial \varphi_{n-1}} \end{bmatrix}$$

$$= \det\begin{bmatrix} \frac{\partial g}{\partial t}g^{-1}x_1 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{\frac{\partial g}{\partial t}g^{-1}x_i} & \widehat{*} & \cdots & \widehat{*} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g}{\partial t}g^{-1}x_n & * & \cdots & * \end{bmatrix}$$

$$= \frac{\partial g}{\partial t}g^{-1}\det\underbrace{\begin{bmatrix} x_1 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{x_i} & \widehat{*} & \cdots & \widehat{*} \\ \vdots & \vdots & \ddots & \vdots \\ x_n & * & \cdots & * \end{bmatrix}}_{\text{say } A}.$$

So

$$\int_{\Psi|_{\varphi_1=\theta}} \omega_n = \int_E g^{-n}\sum_{i=1}^{n}(-1)^{i-1}x_i\frac{\partial g}{\partial t}g^{-1}\det(A)\,dt\,d\varphi_2\cdots d\varphi_{n-1}$$

$$= \int_E \frac{\partial g}{\partial t}g^{-n-1}\sum_{i=1}^{n}(-1)^{i-1}x_i\det\begin{bmatrix} x_1 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{x_i} & \widehat{*} & \cdots & \widehat{*} \\ \vdots & \vdots & \ddots & \vdots \\ x_n & * & \cdots & * \end{bmatrix}\,dt\,d\varphi_2\cdots d\varphi_{n-1}$$

$$= \int_E \frac{\partial g}{\partial t}g^{-n-1}\det\underbrace{\begin{bmatrix} x_1 & x_1 & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_i & x_i & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n & x_n & * & \cdots & * \end{bmatrix}}_{\text{say } B}\,dt\,d\varphi_2\cdots d\varphi_{n-1}.$$

Since the first column is the same as the second column in B, $\det(B) = 0$ (Theorem 9.34(d)). Therefore, $\int_{\Psi|_{\varphi_1=\theta}} \omega_n = 0$.

(5) $\int_{\Psi|_{\varphi_i=\theta}} \omega_n = 0$ is also true for all $i = 1,\ldots,n-1$ by the same argument

in (3)(4). Hence,

$$\begin{aligned}
0 &= \int_{\Psi} d\omega_n \\
&= \int_{\partial\Psi} \omega_n \\
&= \int_S \omega_n - \int_{\Omega} \omega_n + \sum_{i=1}^{n-1} \left(\int_{\Psi|_{\varphi_i = a_i}} \omega_n - \int_{\Psi|_{\varphi_i = b_i}} \omega_n \right) \\
&= \int_S \omega_n - \int_{\Omega} \omega_n
\end{aligned}$$

by (a) and the Stokes' theorem (Theorem 10.33), or

$$\int_{\Omega} \omega_n = \underbrace{\int_S \omega_n = A(S)}_{\text{by (d)(ii)}}.$$

□

Proof of (d)(vii). Similar to Exercise 10.22(f).

(1) To ensure that ω_n is well-defined on $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n = \mathbb{R}^n - \{\mathbf{0}\}$, we might assume $0 < \varphi_1 < \pi$. It is fine since $\int_{\Omega} \omega_n$ and $\int_S \omega_n$ is well-defined on any closed rectangle in D and we can apply the argument in Exercise 6.7 to remove the additional restriction.

(2) By the Stokes' theorem (Theorem 10.33) and (b),

$$\int_{\Omega} \omega_n = \int_{\partial\Omega} f_n \omega_{n-1} \qquad \text{and} \qquad \int_S \omega_n = \int_{\partial S} f_n \omega_{n-1}.$$

So it suffices to show that

$$\int_{\partial\Omega} f_n \omega_{n-1} = \int_{\partial S} f_n \omega_{n-1}.$$

So it suffices to show that $f_n|_{\partial\Omega} = f_n|_{\partial S}$ and $\omega_{n-1}|_{\partial\Omega} = \omega_{n-1}|_{\partial S}$.

(3) *Show that* $f_n|_{\partial\Omega} = f_n|_{\partial S}$. For any $\mathbf{x}_{\Omega} \in \partial\Omega$,

$$\mathbf{x}_{\Omega} = f(\boldsymbol{\varphi})\mathbf{x}_{\Sigma}.$$

So

$$\begin{aligned}
f_n(\mathbf{x}_\Omega) &= (-1)^n g_n \left(\frac{(x_n)_\Omega}{((x_1)_\Omega^2 + \cdots + (x_n)_\Omega^2)^{\frac{1}{2}}} \right) \\
&= (-1)^n g_n \left(\frac{f(\boldsymbol{\varphi})(x_n)_\Sigma}{f(\boldsymbol{\varphi})((x_1)_\Sigma^2 + \cdots + (x_n)_\Sigma^2)^{\frac{1}{2}}} \right) \\
&= (-1)^n g_n \left(\frac{(x_n)_\Sigma}{((x_1)_\Sigma^2 + \cdots + (x_n)_\Sigma^2)^{\frac{1}{2}}} \right) \\
&= f_n(\mathbf{x}_\Sigma).
\end{aligned}$$

(Note that $f > 0$.)

(4) *Show that* $\omega_{n-1}|_{\partial\Omega} = \omega_{n-1}|_{\partial S}$. Induction on n. When $n = 2$ or $n = 3$, it is proved in Exercise 10.22(f). Now for large $n-1$, (3) is also true for $n-1$. Hence,

$$\omega_{n-1}|_{\partial\Omega} = d(f_{n-1}\omega_{n-2})|_{\partial\Omega} = d(f_{n-1}\omega_{n-2})|_{\partial S} = \omega_{n-1}|_{\partial S}\,.$$

By induction, the result is established.

□

Proof of (d)(ix). Similar to Exercise 10.22(g).

(1) Given any line L passing through $\mathbf{0}$, say

$$(r\cos\varphi_1, \cdots, \sin\varphi_1 \cdots \sin\varphi_{n-2}\sin\varphi_{n-1}) \in L \subseteq \mathbb{R}^n$$

where $r \in \mathbb{R}^1$ for some $\boldsymbol{\varphi} \in [0,\pi]^{n-2} \times [0, 2\pi]$. We will show that ω_n is exact in $U = \mathbb{R}^n - L$.

(2) Linear algebra says that all rotation matrices $T \in SO(n)$ can be obtained from

$$R_i(u) = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & \text{\huge 0} & \\ & & 1 & & & & \\ & & & R(u) & & & \\ & & & & 1 & & \\ & \text{\huge 0} & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

using matrix multiplication. Here

$$R(u) = \begin{bmatrix} \cos u & -\sin u \\ \sin u & \cos u \end{bmatrix}$$

is a 2-by-2 rotation matrix at the ith row and ith column. For example, the rotation

$$T = R_1(-\varphi_1)R_2(-\varphi_2)\cdots R_{n-2}(-\varphi_{n-2})R_{n-1}(-\varphi_{n-1})$$

maps L to the x_n-axis. Similar to Exercise 10.22(g), it suffices to show that ω_n is invariant under $T = R_1(u)$.

(3) *Show that ω_n is invariant under $T = R_1(u)$.* By

$$T : \mathbf{x} \mapsto (x_1 \cos u - x_2 \sin u, x_1 \sin u + x_2 \cos u, x_3, \ldots, x_n),$$

we have

$$\begin{aligned}
r_n &\mapsto r_n \\
dx_1 &\mapsto \cos u dx_1 - \sin u dx_2 \\
dx_2 &\mapsto \sin u dx_1 + \cos u dx_2 \\
dx_3 &\mapsto dx_3 \\
&\cdots \\
dx_n &\mapsto dx_n.
\end{aligned}$$

So $dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ maps to

$$\begin{cases}
\cos u\, \widehat{dx_1} \wedge \cdots \wedge dx_n + \sin u\, dx_1 \wedge \widehat{dx_2} \wedge \cdots \wedge dx_n & \text{if } i = 1 \\
-\sin u\, \widehat{dx_1} \wedge \cdots \wedge dx_n + \cos u\, dx_1 \wedge \widehat{dx_2} \wedge \cdots \wedge dx_n & \text{if } i = 2 \\
dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n & \text{otherwise.}
\end{cases}$$

Thus

$$\begin{aligned}
\omega_n \mapsto\ & (r_n)^{-n}(x_1 \cos u - x_2 \sin u) \\
& \qquad \Big(\cos u\, \widehat{dx_1} \wedge \cdots \wedge dx_n + \sin u\, dx_1 \wedge \widehat{dx_2} \wedge \cdots \wedge dx_n\Big) \\
& + (r_n)^{-n}(x_1 \sin u + x_2 \cos u) \\
& \qquad \Big(-\sin u\, \widehat{dx_1} \wedge \cdots \wedge dx_n + \cos u\, dx_1 \wedge \widehat{dx_2} \wedge \cdots \wedge dx_n\Big) \\
& + (r_n)^{-n} \sum_{i=3}^{n} (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\
=\ & (r_n)^{-n} x_1 \widehat{dx_1} \wedge \cdots \wedge dx_n \\
& - (r_n)^{-n} x_2\, dx_1 \wedge \widehat{dx_2} \wedge \cdots \wedge dx_n \\
& + (r_n)^{-n} \sum_{i=3}^{n} (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\
=\ & (r_n)^{-n} \sum_{i=1}^{n} (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\
=\ & \omega_n.
\end{aligned}$$

(4) Similar to Exercise 10.22(g), ω_n is exact in $\mathbb{R}^n - L$. (Or ω_n is locally exact in $\mathbb{R}^n - \{\mathbf{0}\}$.)

$\square$

Exercise 10.24. *Let $\omega = \sum a_i(\mathbf{x})dx_i$ be a 1-form of class $\mathscr{C}''$ in a convex open set $E \subseteq \mathbb{R}^n$. Assume $d\omega = 0$ and prove that ω is exact in E, by completing the following outline:*

Fix $\mathbf{p} \in E$. Define

$$f(\mathbf{x}) = \int_{[\mathbf{p},\mathbf{x}]} \omega \qquad (\mathbf{x} \in E).$$

Apply Stokes' theorem to affine-oriented 2-simplexs $[\mathbf{p},\mathbf{x},\mathbf{y}]$ in E. Deduce that

$$f(\mathbf{y}) - f(\mathbf{x}) = \sum_{i=1}^{n} (y_i - x_i) \int_0^1 a_i((1-t)\mathbf{x} + t\mathbf{y})dt$$

for $\mathbf{x} \in E$, $\mathbf{y} \in E$. Hence $(D_i f)(\mathbf{x}) = a_i(\mathbf{x})$.

Proof.

(1) Fix $\mathbf{p} \in E$. Define

$$f(\mathbf{x}) = \int_{[\mathbf{p},\mathbf{x}]} \omega \qquad (\mathbf{x} \in E).$$

(2) Given any $\mathbf{x} \in E$, $\mathbf{y} \in E$, and $\mathbf{x} \neq \mathbf{y}$. The affine-oriented 2-simplexs $\Psi = [\mathbf{p},\mathbf{x},\mathbf{y}]$ is in E by the convexity of E. (If E is open but not convex, we can show that $\omega = df$ ***locally*** as the note in Exercise 10.21(a). That is why we say that ω is locally exact. The proof is exactly the same.)

(3) Note that

$$\partial\Psi = \partial[\mathbf{p},\mathbf{x},\mathbf{y}] = [\mathbf{x},\mathbf{y}] - [\mathbf{p},\mathbf{y}] + [\mathbf{p},\mathbf{x}].$$

The Stokes' theorem (Theorem 10.33) implies that

$$\begin{aligned}
\int_\Psi d\omega = \int_{\partial\Psi} \omega \Longleftrightarrow \int_\Psi 0 &= \int_{[\mathbf{x},\mathbf{y}]} \omega - \int_{[\mathbf{p},\mathbf{y}]} \omega + \int_{[\mathbf{p},\mathbf{x}]} \omega \\
\Longleftrightarrow 0 &= \int_{[\mathbf{x},\mathbf{y}]} \omega - f(\mathbf{y}) + f(\mathbf{x}) \\
\Longleftrightarrow f(\mathbf{y}) - f(\mathbf{x}) &= \int_{[\mathbf{x},\mathbf{y}]} \omega.
\end{aligned}$$

(4) Define $\gamma : [0,1] \to E$ by

$$\begin{aligned}
\gamma(t) &= \mathbf{x} + t(\mathbf{y} - \mathbf{x}) \\
&= \sum_{i=1}^{n} x_i + t(y_i - x_i)
\end{aligned}$$

(where $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$). Hence $[0, 1]$ is the parameter domain of $[\mathbf{x}, \mathbf{y}]$ with respect to γ. So

$$\begin{aligned}
\int_{[\mathbf{x},\mathbf{y}]} \omega &= \int_0^1 \sum_{i=1}^n a_i(\gamma(t)) \frac{\partial(x_i + t(y_i - x_i))}{\partial t} dt \\
&= \int_0^1 \sum_{i=1}^n a_i(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))(y_i - x_i) dt \\
&= \sum_{i=1}^n (y_i - x_i) \int_0^1 a_i(\mathbf{x} + t(\mathbf{y} - \mathbf{x})) dt.
\end{aligned}$$

Thus,

$$f(\mathbf{y}) - f(\mathbf{x}) = \sum_{i=1}^n (y_i - x_i) \int_0^1 a_i(\mathbf{x} + t(\mathbf{y} - \mathbf{x})) dt.$$

(5) Note that

$$\begin{aligned}
f(\mathbf{x} + h\mathbf{e}_j) - f(\mathbf{x}) &= \sum_{i=1}^n ((x_i + h\delta_{ij}) - x_i) \int_0^1 a_i(\mathbf{x} + t((\mathbf{x} + h\mathbf{e}_j) - \mathbf{x})) dt \\
&= \sum_{i=1}^n h\delta_{ij} \int_0^1 a_i(\mathbf{x} + th\mathbf{e}_j) dt \\
&= h \int_0^1 a_j(\mathbf{x} + th\mathbf{e}_j) dt.
\end{aligned}$$

(Here δ_{ij} is the Kronecker delta.) So

$$\begin{aligned}
(D_j f)(\mathbf{x}) &= \lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{e}_j) - f(\mathbf{x})}{h} \\
&= \lim_{h \to 0} \int_0^1 a_j(\mathbf{x} + th\mathbf{e}_j) dt \\
&= \int_0^1 a_j(\mathbf{x}) dt && (a_j \in \mathscr{C}'') \\
&= a_j(\mathbf{x}).
\end{aligned}$$

Thus,

$$df = \sum_{j=1}^n (D_j f)(\mathbf{x}) dx_j = \sum_{j=1}^n a_j(\mathbf{x}) dx_j = \omega,$$

or ω is exact in E.

□

Exercise 10.25. *Assume ω is a 1-form in an open set $E \subseteq \mathbb{R}^n$ such that*

$$\int_\gamma \omega = 0$$

for every closed curve γ in E, of class $\mathscr{C}'$. Prove that ω is exact in E, by imitating part of the argument sketched in Exercise 10.24.

Proof.

(1) *Assume that E is a* ***connected*** *open subset of $\mathbb{R}^n$. Show that ω is exact in E if $\int_\gamma \omega = 0$ for every closed curve γ in E, of class $\mathscr{C}'$.*

(2) Fix $\mathbf{p} \in E$. Define

$$f(\mathbf{x}) = \int_{[\mathbf{p},\mathbf{x}]} \omega \qquad (\mathbf{x} \in E).$$

It is well-defined since E is connected and $\int_\gamma \omega = 0$ for every closed curve γ in E.

(3) Given any $\mathbf{x} \in E$, $\mathbf{y} \in E$, and $\mathbf{x} \neq \mathbf{y}$. Let

$$\gamma = [\mathbf{x},\mathbf{y}] - [\mathbf{p},\mathbf{y}] + [\mathbf{p},\mathbf{x}]$$

be a closed curve in E. Hence,

$$\begin{aligned} 0 &= \int_\gamma \omega && \text{(Assumption)} \\ &= \int_{[\mathbf{x},\mathbf{y}]} \omega - \int_{[\mathbf{p},\mathbf{y}]} \omega + \int_{[\mathbf{p},\mathbf{x}]} \omega \\ &= \int_{[\mathbf{x},\mathbf{y}]} \omega - f(\mathbf{y}) + f(\mathbf{x}). \end{aligned}$$

So

$$f(\mathbf{y}) - f(\mathbf{x}) = \int_{[\mathbf{x},\mathbf{y}]} \omega$$

(4) Similar to (4)(5) in the proof of Exercise 10.24, we have $df = \omega$. So the statement in (1) is proved. In general, we can define each f_α on each connected component E_α (which is open) of E such that $df_\alpha = \omega$ on E_α. Take

$$f|_{E_\alpha} = f_\alpha$$

on E. Hence, $df = \omega$ on the whole E.

$\square$

Exercise 10.26. *Assume ω is a 1-form in $\mathbb{R}^3 - \{\mathbf{0}\}$, of class $\mathscr{C}'$ and $d\omega = 0$. Prove that ω is exact in $\mathbb{R}^3 - \{\mathbf{0}\}$. (Hint: Every closed continuously differentiable curve in $\mathbb{R}^3 - \{\mathbf{0}\}$ is the boundary of a 2-surface in $\mathbb{R}^3 - \{\mathbf{0}\}$. Apply Stokes' theorem and Exercise 10.25.)*

Proof.

(1) Let $E = \mathbb{R}^3 - \{\mathbf{0}\}$. By Exercise 10.25, it suffices to show that

$$\int_\gamma \omega = 0$$

for every closed curve γ in E, of class $\mathscr{C}'$.

(2) Intuitively, every closed continuously differentiable curve in $\mathbb{R}^3 - \{\mathbf{0}\}$ is the boundary of a 2-surface in $\mathbb{R}^3 - \{\mathbf{0}\}$. So there is some 2-surface Ψ such that $\partial\Psi = \gamma$. The Stokes' theorem (Theorem 10.33) implies that

$$\int_\gamma \omega = \int_{\partial\Psi} \omega = \int_\Psi d\omega = \int_\Psi 0 = 0.$$

□

Exercise 10.27. *Let E be an open 3-cell in $\mathbb{R}^3$, with edges parallel to the coordinate axes. Suppose $(a, b, c) \in E$, $f_i \in \mathscr{C}'(E)$ for $i = 1, 2, 3$,*

$$\omega = f_1 dy \wedge dz + f_2 dz \wedge dx + f_3 dx \wedge dy,$$

and assume that $d\omega = 0$ in E. Define

$$\lambda = g_1 dx + g_2 dy$$

where

$$\begin{aligned} g_1(x, y, z) &= \int_c^z f_2(x, y, s) ds - \int_b^y f_3(x, t, c) dt \\ g_2(x, y, z) &= -\int_c^z f_1(x, y, s) ds, \end{aligned}$$

for $(x, y, z) \in E$. Prove that $d\lambda = \omega$ in E. Evaluate these integrals when $\omega = \zeta$ and thus find the form λ that occurs in part (e) of Exercise 10.22.

Proof.

(1) Let $\mathbf{F} = f_1\mathbf{e}_1 + f_2\mathbf{e}_2 + f_3\mathbf{e}_3$ as in Vector fields 10.42. Then

$$d\omega = (\nabla \cdot F) dx \wedge dy \wedge dz.$$

As $d\omega = 0$ by assumption, $\nabla \cdot F = D_1 f_1 + D_2 f_2 + D_3 f_3 = 0$.

(2) As

$$\begin{aligned} d\lambda &= d(g_1 dx + g_2 dy) \\ &= (D_1 g_1 dx + D_2 g_1 dy + D_3 g_1 dz) \wedge dx \\ &\quad + (D_1 g_2 dx + D_2 g_2 dy + D_3 g_2 dz) \wedge dy \\ &= (-D_3 g_2) dy \wedge dz + (D_3 g_1) dz \wedge dx + (D_1 g_2 - D_2 g_1) dx \wedge dy, \end{aligned}$$

it suffices to show that

$$\begin{aligned} f_1 &= -D_3 g_2, \\ f_2 &= D_3 g_1, \\ f_3 &= D_1 g_2 - D_2 g_1 \end{aligned}$$

on E.

(3) Theorem 6.20 implies that

$$-D_3 g_2 = D_3 \int_c^z f_1(x,y,s)ds = f_1(x,y,z)$$

and

$$D_3 g_1 = D_3 \int_c^z f_2(x,y,s)ds - D_3 \int_b^y f_3(x,t,c)dt = f_2(x,y,z).$$

Also,

$$\begin{aligned} & D_1 g_2 - D_2 g_1 \\ =& D_1 \left(-\int_c^z f_1(x,y,s)ds \right) \\ & - D_2 \left(\int_c^z f_2(x,y,s)ds - \int_b^y f_3(x,t,c)dt \right) \\ =& -\int_c^z D_1 f_1(x,y,s)ds & (f_1 \in \mathscr{C}') \\ & - \int_c^z D_2 f_2(x,y,s)ds + f_3(x,y,c) & (f_2 \in \mathscr{C}', \text{ Theorem 6.20}) \\ =& \int_c^z D_3 f_3(x,y,s)ds + f_3(x,y,c) & ((1)) \\ =& f_3(x,y,z) & (\text{Theorem 6.21}). \end{aligned}$$

Therefore, $d\lambda = \omega$ in E.

(4) When $\omega = \zeta = r^{-3}(x dy \wedge dz + y dz \wedge dx + z dx \wedge dy)$, we get

$$\begin{aligned} f_1(x,y,z) &= x(x^2+y^2+z^2)^{-\frac{3}{2}}, \\ f_2(x,y,z) &= y(x^2+y^2+z^2)^{-\frac{3}{2}}, \\ f_3(x,y,z) &= z(x^2+y^2+z^2)^{-\frac{3}{2}}. \end{aligned}$$

So,

$$\begin{aligned}
\int_c^z f_2(x,y,s)ds &= \left[ys(x^2+y^2)^{-1}(x^2+y^2+s^2)^{-\frac{1}{2}}\right]_{s=c}^{s=z}, \\
\int_b^y f_3(x,t,c)dt &= \left[ct(x^2+c^2)^{-1}(x^2+t^2+c^2)^{-\frac{1}{2}}\right]_{t=b}^{t=y}, \\
\int_c^z f_1(x,y,s)ds &= \left[xs(x^2+y^2)^{-1}(x^2+y^2+s^2)^{-\frac{1}{2}}\right]_{s=c}^{s=z}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\lambda &= g_1dx + g_2dy \\
&= \left[ys(x^2+y^2)^{-1}(x^2+y^2+s^2)^{-\frac{1}{2}}\right]_{s=c}^{s=z} dx \\
&\quad - \left[ct(x^2+c^2)^{-1}(x^2+t^2+c^2)^{-\frac{1}{2}}\right]_{t=b}^{t=y} dx \\
&\quad + \left[xs(x^2+y^2)^{-1}(x^2+y^2+s^2)^{-\frac{1}{2}}\right]_{s=c}^{s=z} dy \\
&= -\left[zr^{-1} - c(x^2+y^2+c^2)^{-\frac{1}{2}}\right]\eta && \text{(Definition of } \eta) \\
&\quad - c(x^2+c^2)^{-1}\left[y(x^2+y^2+c^2)^{-\frac{1}{2}} - b(x^2+b^2+c^2)^{-\frac{1}{2}}\right] dx.
\end{aligned}$$

As we pick $(a,b,c) = (a,0,0) \in \mathbb{R}^3 - \{\mathbf{0}\}$ (or $a \neq 0$), we have $\lambda = -zr^{-1}\eta$ such that $d\lambda = \omega = \zeta$, which is the same as part (e) in Exercise 10.22.

$\square$

Exercise 10.28. *Fix $b > a > 0$, define*

$$\Phi(r,\theta) = (r\cos\theta, r\sin\theta)$$

for $a \leq r \leq b$, $0 \leq \theta \leq 2\pi$. (The range of Φ is an annulus in $\mathbb{R}^2$.) Put $\omega = x^3dy$, and compute both

$$\int_\Phi d\omega \qquad \text{and} \qquad \int_{\partial\Phi} \omega$$

to verify that they are equal.

Proof.

(1) Note that
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r.$$

So

$$\begin{aligned}
\int_{\Phi} d\omega &= \int_{\Phi} 3x^2 dx \wedge dy && (dy \wedge dy = 0) \\
&= \int_{[a,b]\times[0,2\pi]} 3(r\cos\theta)^2 \frac{\partial(x,y)}{\partial(r,\theta)} dr d\theta \\
&= \int_a^b \int_0^{2\pi} 3r^3(\cos\theta)^2 dr d\theta \\
&= \frac{3\pi}{4}(b^4 - a^4).
\end{aligned}$$

(2) Similar to Exercise 10.21(b), write

$$\partial\Phi = \Gamma - \gamma,$$

where $\Gamma(t) = (b\cos t, b\sin t)$ on $[0, 2\pi]$ and $\gamma(t) = (a\cos t, a\sin t)$ on $[0, 2\pi]$. Hence

$$\begin{aligned}
\int_{\partial\Phi} \omega &= \int_{\Gamma} \omega - \int_{\gamma} \omega \\
&= \int_{\Gamma} x^3 dy - \int_{\gamma} x^3 dy \\
&= \int_{[0,2\pi]} (b\cos\theta)^3 \frac{\partial y}{\partial\theta} d\theta - \int_{[0,2\pi]} (a\cos\theta)^3 \frac{\partial y}{\partial\theta} d\theta \\
&= \int_0^{2\pi} b^4(\cos\theta)^4 d\theta - \int_0^{2\pi} a^4(\cos\theta)^4 d\theta \\
&= \frac{3\pi}{4}(b^4 - a^4).
\end{aligned}$$

(3)

$$\int_{\Phi} d\omega = \int_{\partial\Phi} \omega = \frac{3\pi}{4}(b^4 - a^4).$$

□

Exercise 10.29. *Prove the existence of a function α with the properties needed in the proof of Theorem 10.38, and prove that the resulting function F is of class $\mathscr{C}'$. (Both assertions become trivial if E is an open cell or an open ball, since α can then be taken to be a constant. Refer to Theorem 9.42.)*

Proof.

(1)

(2)

□

Exercise 10.30. *If* $\mathbf{N}$ *is the vector given by*

$$\mathbf{N} = (\alpha_2\beta_3 - \alpha_3\beta_2)\mathbf{e}_1 + (\alpha_3\beta_1 - \alpha_1\beta_3)\mathbf{e}_2 + (\alpha_1\beta_2 - \alpha_2\beta_1)\mathbf{e}_3$$

(Equation (135)), prove that

$$\det \begin{bmatrix} \alpha_1 & \beta_1 & \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_2 & \beta_2 & \alpha_3\beta_1 - \alpha_1\beta_3 \\ \alpha_3 & \beta_3 & \alpha_1\beta_2 - \alpha_2\beta_1 \end{bmatrix} = |\mathbf{N}|^2$$

Also, verify

$$\mathbf{N} \cdot (T\mathbf{e}_1) = \mathbf{N} \cdot (T\mathbf{e}_2)$$

(Equation (137)).

Proof.

(1) By Laplace's expansion along the third column,

$$\begin{aligned} &\det \begin{bmatrix} \alpha_1 & \beta_1 & \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_2 & \beta_2 & \alpha_3\beta_1 - \alpha_1\beta_3 \\ \alpha_3 & \beta_3 & \alpha_1\beta_2 - \alpha_2\beta_1 \end{bmatrix} \\ =&(-1)^{1+3}(\alpha_2\beta_3 - \alpha_3\beta_2) \det \begin{bmatrix} \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} \\ &+ (-1)^{2+3}(\alpha_3\beta_1 - \alpha_1\beta_3) \det \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_3 & \beta_3 \end{bmatrix} \\ &+ (-1)^{3+3}(\alpha_1\beta_2 - \alpha_2\beta_1) \det \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix} \\ =&(\alpha_2\beta_3 - \alpha_3\beta_2)^2 + (\alpha_3\beta_1 - \alpha_1\beta_3)^2 + (\alpha_1\beta_2 - \alpha_2\beta_1)^2 \\ =&|\mathbf{N}|^2. \end{aligned}$$

(2)

$$\begin{aligned} \mathbf{N} \cdot (T\mathbf{e}_1) &= (\alpha_2\beta_3 - \alpha_3\beta_2, \alpha_3\beta_1 - \alpha_1\beta_3, \alpha_1\beta_2 - \alpha_2\beta_1) \cdot (\alpha_1, \alpha_2, \alpha_3) \\ &= (\alpha_2\beta_3 - \alpha_3\beta_2)\alpha_1 + (\alpha_3\beta_1 - \alpha_1\beta_3)\alpha_2 + (\alpha_1\beta_2 - \alpha_2\beta_1))\alpha_3 \\ &= (\alpha_3\alpha_2 - \alpha_2\alpha_3)\beta_1 + (\alpha_1\alpha_3 - \alpha_3\alpha_1)\beta_2 + (\alpha_2\alpha_1 - \alpha_1\alpha_2)\beta_3 \\ &= 0. \end{aligned}$$

(3)

$$\begin{aligned} \mathbf{N} \cdot (T\mathbf{e}_2) &= (\alpha_2\beta_3 - \alpha_3\beta_2, \alpha_3\beta_1 - \alpha_1\beta_3, \alpha_1\beta_2 - \alpha_2\beta_1) \cdot (\beta_1, \beta_2, \beta_3) \\ &= (\alpha_2\beta_3 - \alpha_3\beta_2)\beta_1 + (\alpha_3\beta_1 - \alpha_1\beta_3)\beta_2 + (\alpha_1\beta_2 - \alpha_2\beta_1))\beta_3 \\ &= (\beta_2\beta_3 - \beta_3\beta_2)\alpha_1 + (\beta_3\beta_1 - \beta_1\beta_3)\alpha_2 + (\beta_1\beta_2 - \beta_2\beta_1)\alpha_3 \\ &= 0. \end{aligned}$$

□

Exercise 10.31. *Let $E \subseteq \mathbb{R}^3$ be open, suppose $g \in \mathscr{C}''(E)$, $h \in \mathscr{C}''(E)$, and consider the vector field*
$$\mathbf{F} = g\nabla h$$

(a) *Prove that*
$$\nabla \cdot \mathbf{F} = g\nabla^2 h + (\nabla g) \cdot (\nabla h)$$
where $\nabla^2 h = \nabla \cdot (\nabla h) = \sum \frac{\partial^2 h}{\partial x_i^2}$ is the so-called "Laplacian" of h.

(b) *If Ω is a closed subset of E with positively oriented boundary $\partial\Omega$ (as in Theorem 10.51), prove that*
$$\int_\Omega [g\nabla^2 h + (\nabla g) \cdot (\nabla h)] dV = \int_{\partial\Omega} g\frac{\partial h}{\partial n} dA$$
*where (as is customary) we have written $\frac{\partial h}{\partial n}$ in place of $(\nabla h) \cdot \mathbf{n}$. (Thus $\frac{\partial h}{\partial n}$ is the directional derivative of h in the direction of the outward normal to $\partial\Omega$, the so-called **normal derivative** of h.) Interchange g and h, substract the resulting formula from the first one, to obtain*
$$\int_\Omega (g\nabla^2 h - h\nabla^2 g) dV = \int_{\partial\Omega} \left(g\frac{\partial h}{\partial n} - h\frac{\partial g}{\partial n} \right) dA.$$
*These two formulas are usually called **Green's identities**.*

(c) *Assume that h is **harmonic** in E; this means that $\nabla^2 h = 0$. Take $g = 1$ and conclude that*
$$\int_{\partial\Omega} \frac{\partial h}{\partial n} dA = 0.$$
Take $g = h$, and conclude that $h = 0$ in Ω if $h = 0$ on $\partial\Omega$.

(d) *Show that Green's identities are also valid in $\mathbb{R}^2$.*

Proof of (a).

(1) Since
$$\mathbf{F} = g\nabla h = g\left(\sum (D_i h)\mathbf{e}_i \right) = \sum g(D_i h)\mathbf{e}_i,$$
we have
$$\begin{aligned} \nabla \cdot \mathbf{F} &= \nabla \cdot \left(\sum g(D_i h)\mathbf{e}_i \right) \\ &= \sum D_i(g(D_i h)) \\ &= \sum \{(D_i g)(D_i h) + g D_i(D_i h)\} \\ &= \sum (D_i g)(D_i h) + g \sum D_i(D_i h). \end{aligned}$$

(2) Also,

$$\begin{aligned} g\nabla^2 h + (\nabla g) \cdot (\nabla h) &= g\nabla \cdot (\nabla h) + (\nabla g) \cdot (\nabla h) \\ &= g\nabla \cdot \left(\sum (D_i h)\mathbf{e}_i\right) + \left(\sum (D_i g)\mathbf{e}_i\right) \cdot \left(\sum (D_i h)\mathbf{e}_i\right) \\ &= g\sum D_i(D_i h) + \sum (D_i g)(D_i h). \end{aligned}$$

(3) By (1)(2), the result is established.

□

Proof of (b).

(1) The divergence theorem (Theorem 10.51) implies that

$$\begin{aligned} &\int_\Omega (\nabla \cdot \mathbf{F}) dV = \int_{\partial\Omega} (\mathbf{F} \cdot \mathbf{n}) dA \\ \Longrightarrow &\int_\Omega [g\nabla^2 h + (\nabla g) \cdot (\nabla h)] dV = \int_{\partial\Omega} g \underbrace{\nabla h \cdot \mathbf{n}}_{=\frac{\partial h}{\partial n}} dA. \end{aligned}$$

(2) Green's identities are a set of three identities in vector calculus relating the bulk with the boundary of a region on which differential operators act. *(Green's third identity.) Assume that h is harmonic in E. If $G(\mathbf{x}, \mathbf{x}_0)$ is the Green's function, then*

$$h(\mathbf{x}_0) = \int_{\partial\Omega} \left[h(\mathbf{x}) \frac{\partial G(\mathbf{x}, \mathbf{x}_0)}{\partial n} - G(\mathbf{x}, \mathbf{x}_0) \frac{\partial h(\mathbf{x})}{\partial n} \right] dA.$$

For example, in $\mathbb{R}^3$

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi \|\mathbf{x} - \mathbf{x}_0\|}.$$

□

Proof of (c). Assume $\nabla^2 h = 0$.

(1) Take $g = 1$ in

$$\int_\Omega [g\nabla^2 h + (\nabla g) \cdot (\nabla h)] dV = \int_{\partial\Omega} g \frac{\partial h}{\partial n} dA$$

to get the conclusion. (Here $\nabla g = \mathbf{0}$ as $g = 1$.)

(2) Assume $h = 0$ on $\partial\Omega$. Take $g = h$ in

$$\int_\Omega [g\nabla^2 h + (\nabla g) \cdot (\nabla h)] dV = \int_{\partial\Omega} g \frac{\partial h}{\partial n} dA$$

to get

$$\int_\Omega |\nabla h|^2 dV = \int_{\partial\Omega} h\frac{\partial h}{\partial n} dA = 0$$

(since $h = 0$ on $\partial\Omega$). Since $h \in \mathscr{C}'(\Omega)$, Exercise 6.2 implies that $|\nabla h|^2 = 0$ on Ω. So $D_1 h = D_2 h = D_3 h = 0$ on Ω. Since $h \in \mathscr{C}'(\Omega)$, Theorem 9.21 implies that $h = 0$ on Ω, or h is locally constant in Ω (Exercise 9.9). Note that $h = 0$ globally on $\partial\Omega$, and thus $h = 0$ globally on Ω.

□

Proof of (d).

(1) *(The divergence theorem in $\mathbb{R}^2$.) If $\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2$ is a vector field of class $\mathscr{C}'$ in an open set $E \subseteq \mathbb{R}^2$, and if Ω is a closed subset of E with positively oriented boundary $\partial\Omega$ then*

$$\int_\Omega (\nabla \cdot \mathbf{F}) dA = \int_{\partial\Omega} (\mathbf{F} \cdot \mathbf{n}) ds.$$

Define a 1-form by

$$\omega_{\mathbf{F}} = F_1 dy - F_2 dx.$$

So

$$d\omega_{\mathbf{F}} = (\nabla \cdot \mathbf{F}) dx \wedge dy = (\nabla \cdot \mathbf{F}) dA.$$

Hence the Stokes' theorem (Theorem 10.33) implies that

$$\int_\Omega (\nabla \cdot \mathbf{F}) dA = \int_\Omega d\omega_{\mathbf{F}} = \int_{\partial\Omega} \omega_{\mathbf{F}} = \int_{\partial\Omega} (\mathbf{F} \cdot \mathbf{n}) ds.$$

(2) Note that

$$\nabla \cdot \mathbf{F} = g\nabla^2 h + (\nabla g) \cdot (\nabla h)$$

is also true in $\mathbb{R}^2$. Similar to (b), two Green's identities are also true in $\mathbb{R}^2$. (In $\mathbb{R}^1$, the Green's first identity is the integration by parts (Theorem 6.22).)

□

Exercise 10.32 (Möbius band). *Fix δ, $0 < \delta < 1$. Let D be the set of all $(\theta, t) \in \mathbb{R}^2$ such that $0 \leq \theta \leq \pi$, $-\delta \leq t \leq \delta$. Let Φ be the 2-surface in $\mathbb{R}^3$, with parameter domain D, given by*

$$\begin{aligned} x &= (1 - t\sin\theta)\cos(2\theta) \\ y &= (1 - t\sin\theta)\sin(2\theta) \\ z &= t\cos\theta \end{aligned}$$

where $(x, y, z) = \Phi(\theta, t)$*. Note that* $\Phi(\pi, t) = \Phi(0, -t)$*, and that* Φ *is one-to-one on the rest of* D*.*

The range $M = \Phi(D)$ *of* Φ *is known as a* ***Möbius band****. It is the simplest example of a nonorientable surface.*

Prove the various assertions made in the following description: Put $\mathbf{p}_1 = (0, -\delta)$*,* $\mathbf{p}_2 = (\pi, -\delta)$*,* $\mathbf{p}_3 = (\pi, \delta)$*,* $\mathbf{p}_4 = (0, \delta)$*,* $\mathbf{p}_5 = \mathbf{p}_1$*. Put* $\gamma_i = [\mathbf{p}_i, \mathbf{p}_{i+1}]$*,* $i = 1, \ldots, 4$*, and put* $\Gamma_i = \Phi \circ \gamma_i$*. Then*

$$\partial\Phi = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4.$$

Put $\mathbf{a} = (1, 0, -\delta)$*,* $\mathbf{b} = (1, 0, \delta)$*. Then*

$$\Phi(\mathbf{p}_1) = \Phi(\mathbf{p}_3) = \mathbf{a}, \qquad \Phi(\mathbf{p}_2) = \Phi(\mathbf{p}_4) = \mathbf{b},$$

and $\partial\Phi$ *can be described as follows.*

(1) Γ_1 *spirals up from* $\mathbf{a}$ *to* $\mathbf{b}$*; its projection into the* (x, y)*-plane has winding number* $+1$ *around the origin. (See Exercise 8.23.)*

(2) $\Gamma_2 = [\mathbf{b}, \mathbf{a}]$.

(3) Γ_3 *spirals up from* $\mathbf{a}$ *to* $\mathbf{b}$*; its projection into the* (x, y)*-plane has winding number* -1 *around the origin.*

(4) $\Gamma_4 = [\mathbf{b}, \mathbf{a}]$.

Thus $\partial\Phi = \Gamma_1 + \Gamma_3 + 2\Gamma_2$.

If we go from $\mathbf{a}$ *to* $\mathbf{b}$ *along* Γ_1 *and continue along the "edge" of* M *until we return to* $\mathbf{a}$*, the curve traced out is*

$$\Gamma = \Gamma_1 - \Gamma_3,$$

which may also be represented on the parameter interval $[0, 2\pi]$ *by the equations*

$$\begin{aligned} x &= (1 + \delta \sin\theta)\cos(2\theta) \\ y &= (1 + \delta \sin\theta)\sin(2\theta) \\ z &= -\delta \cos\theta. \end{aligned}$$

It should be emphasized that $\Gamma \neq \partial\Phi$*: Let* $\eta = \frac{x dy - y dx}{x^2 + y^2}$ *be the* 1*-form discussed in Exercise 10.21 and Exercise 10.22. Since* $d\eta = 0$*, Stokes' theorem shows that*

$$\int_{\partial\Phi} \eta = 0.$$

But although Γ *is the "geometric" boundary of* M*, we have*

$$\int_{\Gamma} \eta = 4\pi.$$

In order to avoid this possible source of confusion, Stokes' formula (Theorem 10.50) is frequently stated only for orientable surfaces Φ.

Proof.

(1) *Show that* $\partial\Phi = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$.

$$\begin{aligned}\partial\Phi &= \Phi \circ (\partial D)\\ &= \Phi \circ (\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4)\\ &= \Phi \circ \gamma_1 + \Phi \circ \gamma_2 + \Phi \circ \gamma_3 + \Phi \circ \gamma_4\\ &= \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4.\end{aligned}$$

(2) It is trivial that $\Phi(\mathbf{p}_1) = \Phi(\mathbf{p}_3) = \mathbf{a} = (1, 0, -\delta)$ and $\Phi(\mathbf{p}_2) = \Phi(\mathbf{p}_4) = \mathbf{b} = (1, 0, \delta)$ by the definition of Φ.

(3) *Show that* Γ_1 *spirals up from* $\mathbf{a}$ *to* $\mathbf{b}$*; its projection into the* (x, y)*-plane has winding number* $+1$ *around the origin.* By definition, $\Gamma_1 = \Phi \circ \gamma_1 = \Phi([\mathbf{p}_1, \mathbf{p}_2])$. That is, Γ_1 spirals up from $\Phi(\mathbf{p}_1) = \mathbf{a}$ to $\Phi(\mathbf{p}_2) = \mathbf{b}$. Besides, the projection P_{Γ_1} of Γ_1 into the (x, y)-plane ($z = 0$) can be parameterized as

$$\begin{aligned}x &= \left(1 + \delta \sin \frac{t}{2}\right) \cos t\\ y &= \left(1 + \delta \sin \frac{t}{2}\right) \sin t\end{aligned}$$

for $0 \leq t \leq 2\pi$. Note that P_{Γ_1} satisfies the condition in Exercise 10.21(b). Hence $\int_{P_{\Gamma_1}} \eta = 2\pi$. (Here η is well-defined.) Apply Exercise 10.21(f) to get

$$\mathrm{Ind}(P_{\Gamma_1}) = \frac{1}{2\pi} \int_{P_{\Gamma_1}} \eta = \frac{1}{2\pi} \cdot 2\pi = 1.$$

(4) *Show that* $\Gamma_2 = [\mathbf{b}, \mathbf{a}]$. By definition, $\Gamma_2 = \Phi \circ \gamma_2 = \Phi([\mathbf{p}_2, \mathbf{p}_3])$ is $[\mathbf{b}, \mathbf{a}]$ exactly.

(5) *Show that* Γ_3 *spirals up from* $\mathbf{a}$ *to* $\mathbf{b}$*; its projection into the* (x, y)*-plane has winding number* -1 *around the origin.* Similar to (3), Γ_3 spirals up from $\Phi(\mathbf{p}_3) = \mathbf{a}$ to $\Phi(\mathbf{p}_4) = \mathbf{b}$. Now we consider $-\Gamma_3$ instead of Γ_3. The projection $P_{-\Gamma_3}$ of $-\Gamma_3$ into the (x, y)-plane ($z = 0$) can be parameterized as

$$\begin{aligned}x &= \left(1 - \delta \sin \frac{t}{2}\right) \cos t\\ y &= \left(1 - \delta \sin \frac{t}{2}\right) \sin t\end{aligned}$$

for $0 \leq t \leq 2\pi$. Similar to (3), $\mathrm{Ind}(P_{-\Gamma_3}) = 1$. Therefore,

$$\mathrm{Ind}(P_{\Gamma_3}) = -\mathrm{Ind}(-P_{\Gamma_3}) = -\mathrm{Ind}(P_{-\Gamma_3}) = -1.$$

(6) *Show that* $\Gamma_4 = [\mathbf{b}, \mathbf{a}]$. Similar to (4).

(7) *Show that* $\Gamma = \Gamma_1 - \Gamma_3$ *is the trace of from* $\mathbf{a}$ *to* $\mathbf{b}$ *along* Γ_1 *and continue along the "edge" of* M *until we return to* $\mathbf{a}$. By definition, Γ can be parameterized as

$$\begin{aligned} x &= (1 + \delta \sin t) \cos(2t) \\ y &= (1 + \delta \sin t) \sin(2t) \\ z &= -\delta \cos t \end{aligned}$$

for $t \in [0, 2\pi]$. Thus, Γ is Γ_1 if $t \in [0, \pi]$ and Γ is $-\Gamma_3$ if $t \in [\pi, 2\pi]$ by (3)(5). So $\Gamma = \Gamma_1 - \Gamma_3$.

(8) *Show that* $\int_{\partial\Phi} \eta = 0$. Note that η is well-defined since M does not intersect the z-axis. So the Stokes' theorem (Theorem 10.33) and $d\eta = 0$ on M implies that

$$\int_{\partial\Phi} \eta = \int_{\Phi} d\eta = 0.$$

(9) *Show that* $\int_{\Gamma} \eta = 4\pi$.

$$\begin{aligned} \int_{\Gamma} \eta &= \int_{\Gamma} \frac{xdy - ydx}{x^2 + y^2} \\ &= \int_0^{2\pi} \frac{x(t)y'(t) - y(t)x'(t)}{x(t)^2 + y(t)^2} dt \qquad ((7)) \\ &= \int_0^{2\pi} 2 \, dt \\ &= 4\pi. \end{aligned}$$

(So the winding number of Γ around of $\mathbf{0}$ is 2.)

(10) By (8)(9), $\Gamma \neq \partial\Phi$.

$\square$