Chapter 3: Operators

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Problem 3.6. *Prove that* $\hat{A} + \hat{B} = \hat{B} + \hat{A}$*.*

Two operators $\hat{A}$ and $\hat{B}$ are said to be equal if $\hat{A}f = \hat{B}f$ for all functions f.

Proof.

$$(\hat{A} + \hat{B})f = \hat{A}f + \hat{B}f = \hat{B}f + \hat{A}f = (\hat{B} + \hat{A})f$$

holds for any function f. By definition, $\hat{A} + \hat{B} = \hat{B} + \hat{A}$. □

Problem 3.7. *Let* $\hat{D} = d/dx$*. Verify that* $(\hat{D} + x)(\hat{D} - x) = \hat{D}^2 - x^2 - 1$*.*

Proof.

$$\begin{aligned}
((\hat{D} + x)(\hat{D} - x))f &= (\hat{D} + x)((\hat{D} - x)f) \\
&= (\hat{D} + x)(f' - xf) \\
&= (f' - xf)' + x(f' - xf) \\
&= (f'' - f - xf') + (xf' - x^2 f) \\
&= f'' - f - x^2 f \\
&= (\hat{D}^2 - x^2 - 1)f
\end{aligned}$$

holds for any function f. By definition, $(\hat{D} + x)(\hat{D} - x) = \hat{D}^2 - x^2 - 1$. □

Problem 3.27. *Evaluate the commutators*

(a) $[\hat{x}, \hat{p}_x]$;

(b) $[\hat{x}, \hat{p}_x^2]$;

(c) $[\hat{x}, \hat{p}_y]$;

(d) $[\hat{x}, \hat{V}(x, y, z)]$;

(e) $[\hat{x}, \hat{H}]$*, where the Hamiltonian operator is*

$$\hat{H} = -\frac{\hbar}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + V(x, y, z);$$

(f) $[\hat{x}\hat{y}\hat{z}, \hat{p}_x^2]$.

Proof of (a).

$$\begin{aligned}
[\hat{x}, \hat{p}_x]f &= (\hat{x}\hat{p}_x - \hat{p}_x\hat{x})f \\
&= (\hat{x}\hat{p}_x)f - (\hat{p}_x\hat{x})f \\
&= (\hat{x})\left(\frac{\hbar}{i}\frac{\partial f}{\partial x}\right) - (\hat{p}_x)(xf) \\
&= x\frac{\hbar}{i}\frac{\partial f}{\partial x} - \frac{\hbar}{i}\left(f + x\frac{\partial f}{\partial x}\right) \\
&= -\frac{\hbar}{i}f
\end{aligned}$$

holds for any function f. By definition, $[\hat{x}, \hat{p}_x] = -\frac{\hbar}{i}$. □

Proof of (b).

$$\begin{aligned}
[\hat{x}, \hat{p}_x^2]f &= (\hat{x}\hat{p}_x^2 - \hat{p}_x^2\hat{x})f \\
&= (\hat{x}\hat{p}_x^2)f - (\hat{p}_x^2\hat{x})f \\
&= (\hat{x}\hat{p}_x)\left(\frac{\hbar}{i}\frac{\partial f}{\partial x}\right) - (\hat{p}_x^2)(xf) \\
&= (\hat{x})\left(\frac{\hbar}{i}\frac{\hbar}{i}\frac{\partial^2 f}{\partial x^2}\right) - (\hat{p}_x)\frac{\hbar}{i}\left(f + x\frac{\partial f}{\partial x}\right) \\
&= x\left(\frac{\hbar}{i}\frac{\hbar}{i}\frac{\partial^2 f}{\partial x^2}\right) - \frac{\hbar}{i}\frac{\hbar}{i}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial x} + x\frac{\partial^2 f}{\partial x^2}\right) \\
&= -\frac{\hbar}{i}\frac{\hbar}{i}\cdot 2\frac{\partial f}{\partial x} \\
&= \left(2\hbar\frac{\partial}{\partial x}\right)f
\end{aligned}$$

holds for any function f. By definition, $[\hat{x}, \hat{p}_x^2] = 2\hbar\frac{\partial}{\partial x}$. □

Proof of (c).

$$\begin{aligned}
[\hat{x}, \hat{p}_y]f &= (\hat{x}\hat{p}_y - \hat{p}_y\hat{x})f \\
&= (\hat{x}\hat{p}_y)f - (\hat{p}_y\hat{x})f \\
&= (\hat{x})\left(\frac{\hbar}{i}\frac{\partial f}{\partial y}\right) - (\hat{p}_y)(xf) \\
&= x\frac{\hbar}{i}\frac{\partial f}{\partial x} - \frac{\hbar}{i}\cdot x\frac{\partial f}{\partial y} \\
&= 0
\end{aligned}$$

holds for any function f. By definition, $[\hat{x}, \hat{p}_y] = 0$. $\square$

Proof of (d).

$$\begin{aligned}
[\hat{x}, \hat{V}(x,y,z)]f &= (\hat{x}\hat{V}(x,y,z) - \hat{V}(x,y,z)\hat{x})f \\
&= (\hat{x}\hat{V}(x,y,z))f - (\hat{V}(x,y,z)\hat{x})f \\
&= \hat{x}(V(x,y,z)f) - \hat{V}(x,y,z)(xf) \\
&= xV(x,y,z)f - V(x,y,z)xf \\
&= 0
\end{aligned}$$

holds for any function f. By definition, $[\hat{x}, \hat{V}(x,y,z)] = 0$. $\square$

Proof of (e).

(1) Given any function f,

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}(fx) &= \frac{\partial}{\partial x}\left(x\frac{\partial f}{\partial x} + f\right) \\
&= \frac{\partial f}{\partial x} + x\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial x} \\
&= 2\frac{\partial f}{\partial x} + x\frac{\partial^2 f}{\partial x^2}, \\
\frac{\partial^2}{\partial y^2}(fx) &= x\frac{\partial^2 f}{\partial y^2}, \\
\frac{\partial^2}{\partial z^2}(fx) &= x\frac{\partial^2 f}{\partial z^2}.
\end{aligned}$$

(2)

$$\begin{aligned}
(\hat{H}\hat{x})f &= (\hat{H})(xf) \\
&= -\frac{\hbar}{2m}\left(\frac{\partial^2}{\partial x^2}(xf) + \frac{\partial^2}{\partial y^2}(xf) + \frac{\partial^2}{\partial z^2}(xf)\right) + V(x,y,z)xf \\
&= -\frac{\hbar}{2m}\left(2\frac{\partial f}{\partial x} + x\frac{\partial^2 f}{\partial x^2} + x\frac{\partial^2 f}{\partial y^2} + x\frac{\partial^2 f}{\partial z^2}\right) + V(x,y,z)xf \\
&= -\frac{\hbar}{m}\frac{\partial f}{\partial x} + (\hat{x}\hat{H})f
\end{aligned}$$

(3)

$$\begin{aligned}
[\hat{x}, \hat{H}]f &= (\hat{x}\hat{H} - \hat{H}\hat{x})f \\
&= (\hat{x}\hat{H})f - (\hat{H}\hat{x})f \\
&= \frac{\hbar}{m}\frac{\partial f}{\partial x}
\end{aligned}$$

holds for any function f. By definition, $[\hat{x}, \hat{H}] = \frac{\hbar}{m}\frac{\partial}{\partial x}$.

□

Proof of (f). Similar to (b), $[\hat{x}\hat{y}\hat{z}, \hat{p}_x^2] = 2\hbar yz\frac{\partial}{\partial x}$. □

Problem 3.33. *Prove the multiple-integral identity*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x)G(y)H(z)dxdydz = \int_{-\infty}^{\infty} F(x)dx \int_{-\infty}^{\infty} G(y)dy \int_{-\infty}^{\infty} H(z)dz.$$

Proof. Write $\int = \int_{-\infty}^{\infty}$.

$$\begin{aligned}
&\int\int\int F(x)G(y)H(z)dxdydz \\
&= \int\int\left(\int F(x) \underbrace{G(y)H(z)}_{\text{constant w.r.t } x} dx\right) dydz \\
&= \int\int G(y)H(z) \underbrace{\int F(x)dx}_{\text{constant w.r.t } y \text{ and } z} dydz \\
&= \int F(x)dx \int\int G(y)H(z)dydz \\
&= \int F(x)dx \int G(y)dy \int H(z)dz. && \text{(Similar arguments)}
\end{aligned}$$

□