Chapter 2: Some Basic Notions of Set Theory

Author: Meng-Gen Tsai
Email: plover@gmail.com

Exercise 2.6. *Let $f : S \to T$ be a function. If A and B are arbitrary subsets of S, prove that*

$$f(A \cup B) = f(A) \cap f(B) \text{ and } f(A \cap B) \subseteq f(A) \cup f(B).$$

Generalize to arbitrary unions and intersections.

Generalization. Let $f : S \to T$ be a function. If $\mathscr{F}$ is an arbitrary collection of sets, then

$$f\left(\bigcup_{A \in \mathscr{F}} A\right) = \bigcap_{A \in \mathscr{F}} f(A) \text{ and } f\left(\bigcap_{A \in \mathscr{F}} A\right) \subseteq \bigcup_{A \in \mathscr{F}} f(A).$$

Note. $f(A \cap B)$ might not be equal to $f(A) \cup f(B)$. For example, let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$. Then for any nonempty disjoint subsets A and B, we have $\varnothing = f(A \cap B) \not\supseteq f(A) \cup f(B) = \{0\}$.

Proof.

(1)

$$\begin{aligned}
\forall\, y \in f\left(\bigcup_{A \in \mathscr{F}} A\right) &\Longleftrightarrow \exists\, x \in \bigcup_{A \in \mathscr{F}} A \text{ such that } f(x) = y \\
&\Longleftrightarrow \exists\, x \in A \text{ for some } A \in \mathscr{F} \text{ such that } f(x) = y \\
&\Longleftrightarrow \exists\, A \in \mathscr{F} \text{ such that } y \in f(A) \\
&\Longleftrightarrow \forall\, y \in \bigcap_{A \in \mathscr{F}} f(A)
\end{aligned}$$

(2)

$$
\begin{aligned}
\forall\, y \in f\left(\bigcap_{A\in\mathscr{F}} A\right) &\Longleftrightarrow \exists\, x \in \bigcap_{A\in\mathscr{F}} A \text{ such that } f(x) = y \\
&\Longleftrightarrow \exists\, x \text{ in all } A \in \mathscr{F} \text{ such that } f(x) = y \\
&\qquad (x \text{ not depending on } A) \\
&\Longrightarrow \forall\, A \in \mathscr{F}, \exists\, x \in A \text{ such that } f(x) = y \\
&\qquad (x \text{ depending on } A) \\
&\Longleftrightarrow \forall\, A \in \mathscr{F}, y \in f(A) \\
&\Longleftrightarrow \forall\, y \in \bigcup_{A\in\mathscr{F}} f(A).
\end{aligned}
$$

□

Exercise 2.7. *Let $f : S \to T$ be a function. If $Y \subseteq T$, we denote by $f^{-1}(Y)$ the largest subset of S which f maps into Y. That is,*

$$f^{-1}(Y) = \{x : x \in S \text{ and } f(x) \in Y\}.$$

The set $f^{-1}(Y)$ is called the inverse image of Y under f. Prove the following for arbitrary subsets X of S and Y of T.

(a) $X \subseteq f^{-1}[f(X)]$.

(b) $f[f^{-1}(Y)] \subseteq Y$.

(c) $f^{-1}(Y_1 \cup Y_2) = f^{-1}(Y_1) \cup f^{-1}(Y_2)$.

(d) $f^{-1}(Y_1 \cap Y_2) = f^{-1}(Y_1) \cap f^{-1}(Y_2)$.

(e) $f^{-1}(T - Y) = S - f^{-1}(Y)$.

(f) *Generalize (c) and (d) to arbitrary unions and intersections.*

Proof of (a).

$$
\begin{aligned}
\forall\, x \in X &\Longrightarrow f(x) \in f(X) \\
&\Longleftrightarrow x \in f^{-1}[f(X)]. \qquad \text{(Definition of the inverse image)}
\end{aligned}
$$

□

Proof of (b).

$$
\begin{aligned}
\forall\, y \in f[f^{-1}(Y)] &\Longleftrightarrow \exists\, x \in f^{-1}(Y) \text{ such that } y = f(x) \\
&\Longleftrightarrow \exists\, x, f(x) \in Y \text{ such that } y = f(x) \\
&\Longrightarrow \exists\, x, y = f(x) \in Y.
\end{aligned}
$$

□

Proof of (c). For an arbitrary collection $\mathscr{F}$ of subsets Y of T, show that

$$f^{-1}\left(\bigcup_{Y\in\mathscr{F}} Y\right) = \bigcup_{Y\in\mathscr{F}} f^{-1}(Y).$$

$$\begin{aligned}
\forall\, x \in f^{-1}\left(\bigcup_{Y\in\mathscr{F}} Y\right) &\Longleftrightarrow f(x) \in \bigcup_{Y\in\mathscr{F}} Y \\
&\Longleftrightarrow f(x) \in Y \text{ for some } Y \in \mathscr{F} \\
&\Longleftrightarrow x \in f^{-1}(Y) \text{ for some } Y \in \mathscr{F} \\
&\Longleftrightarrow x \in \bigcup_{Y\in\mathscr{F}} f^{-1}(Y).
\end{aligned}$$

□

Proof of (d). Similar to (c). *For an arbitrary collection $\mathscr{F}$ of subsets Y of T, show that*

$$f^{-1}\left(\bigcap_{Y\in\mathscr{F}} Y\right) = \bigcap_{Y\in\mathscr{F}} f^{-1}(Y).$$

$$\begin{aligned}
\forall\, x \in f^{-1}\left(\bigcap_{Y\in\mathscr{F}} Y\right) &\Longleftrightarrow f(x) \in \bigcap_{Y\in\mathscr{F}} Y \\
&\Longleftrightarrow f(x) \in Y \text{ for all } Y \in \mathscr{F} \\
&\Longleftrightarrow x \in f^{-1}(Y) \text{ for all } Y \in \mathscr{F} \\
&\Longleftrightarrow x \in \bigcap_{Y\in\mathscr{F}} f^{-1}(Y).
\end{aligned}$$

□

Proof of (e).

$$\begin{aligned}
\forall\, x \in f^{-1}(T-Y) &\Longleftrightarrow f(x) \in T-Y \\
&\Longleftrightarrow f(x) \notin Y \\
&\Longleftrightarrow x \notin f^{-1}(Y) \\
&\Longleftrightarrow x \in S - f^{-1}(Y).
\end{aligned}$$

□

Proof of (f). Proved in (c)(d). □

Exercise 2.15. *A real number is called algebraic if it is a root of an algebraic equation $f(x) = 0$, where $a_0 + a_1x + \cdots + a_nx^n = 0$ is a polynomial with integer coefficients. Prove that the set of all polynomials with integer coefficients is countable and deduce that the set of algebraic numbers is also countable.*

Might assume $a_n \neq 0$.

For example, all rational numbers are algebraic since $p = \frac{\alpha}{\beta}$ (where $\alpha, \beta \in \mathbb{Z}$) is a root of $\beta x - \alpha = 0$.

Besides, $x = \sqrt{2}+\sqrt{3}$ is algebraic since $x^4-10x^2+1 = 0$. In fact, $x = \pm\sqrt{2}+\pm\sqrt{3}$ are also algebraic since $x^4 - 10x^2 + 1 = (x - \sqrt{2} - \sqrt{3})(x + \sqrt{2} - \sqrt{3})(x - \sqrt{2} + \sqrt{3})(x + \sqrt{2} + \sqrt{3})$.

Note. *Countable set* in the sense of Tom M. Apostol is equivalent to *at most countable set* in the sense of Walter Rudin.

Lemma. *The set of all polynomials over $\mathbb{Z}$ is countable implies that the set of algebraic numbers is countable.*

Proof of Lemma. By definition, we write the set of algebraic numbers as

$$S = \bigcup_{f(x)\in\mathbb{Z}[x]} \{\alpha \in \mathbb{R} : f(\alpha) = 0\}.$$

Since each polynomial of degree n has at most n roots, $\{\alpha \in \mathbb{R} : f(\alpha) = 0\}$ is finite (or countable) for each given $f(x) \in \mathbb{Z}[x]$. So S is a countable union (by assumption) of countable sets, and hence countable by Theorem 2.27. $\square$

Now we show that *the set of all polynomials over $\mathbb{Z}$ is countable.*

Proof (Walter Rudin). For every positive integer N there are only finitely many equations with $n + |a_0| + |a_1| + \cdots + |a_n| = N$. Write

$$P_N = \{f(x) \in \mathbb{Z}[x] : n + |a_0| + |a_1| + \cdots + |a_n| = N\}$$

where $f(x) = a_0 + a_1x + \cdots + a_nx^n$ with $a_n \neq 0$, and

$$P = \bigcup_{N=1}^{\infty} P_N.$$

P is the set of all polynomials over $\mathbb{Z}$.

Each P_N is finite (or countable) for given N (since the equation $n + |a_0| + |a_1| + \cdots + |a_n| = N$ has finitely many solutions $(n, a_0, a_1, ..., a_n) \in \mathbb{Z}^{n+2}$). So P is a countable union of countable sets, and hence countable by Theorem 2.27. $\square$

Proof (Theorem 2.18).

(1) $\mathbb{Z}^N$ *is countable for any integer* $N > 0$. Induction on N and apply the same argument of Theorem 2.18.

(2) *The set of all polynomials over* $\mathbb{Z}$ *is countable.* Let

$$P_n = \{f \in \mathbb{Z}[x] : \deg f = n\},$$

and

$$P = \bigcup_{n=1}^{\infty} P_n = \mathbb{Z}[x].$$

Claim: P_n *is countable.* Define a one-to-one map $\varphi_n : P_n \to \mathbb{Z}^{n+1}$ by

$$\varphi_n(a_0 + a_1 x + \cdots + a_n x^n) = (a_0, a_1, ..., a_n).$$

By (1) and Theorem 2.16, P_n is countable. Now P is a countable union of countable sets, and hence countable by Theorem 2.27.

□

Proof (Unique factorization theorem).

(1) *The set of prime numbers is countable.* Write all primes in the ascending order as $p_1, p_2, ..., p_n, ...$ where $p_1 = 2, p_2 = 3, ..., p_{10001} = 104743, ...$ (See ProjectEuler 7: 10001st prime. Use sieve of Eratosthenes to get p_{10001}.)

(2) *The set of all polynomials over* $\mathbb{Z}$ *is countable.* Let

$$P_n = \{f \in \mathbb{Z}[x] : \deg f = n\},$$

and

$$P = \bigcup_{n=1}^{\infty} P_n = \mathbb{Z}[x].$$

Claim: P_n *is countable.* Define a map $\varphi_n : P_n \to \mathbb{Z}^+$ by

$$\varphi_n(a_0 + a_1 x + \cdots + a_n x^n) = p_1^{\psi(a_0)} p_2^{\psi(a_1)} \cdots p_{n+1}^{\psi(a_n)},$$

where ψ is a one-to-one correspondence from $\mathbb{Z}$ to $\mathbb{Z}^+$. By the unique factorization theorem, φ_n is one-to-one. So P_n is countable by Theorem 2.16. Now P is a countable union of countable sets, and hence countable by Theorem 2.27.

□