Chapter 3: Operators

Author: Meng-Gen Tsai
Email: plover@gmail.com

Problem 3.6. *Prove that* $\hat{A} + \hat{B} = \hat{B} + \hat{A}$.

Two operators $\hat{A}$ and $\hat{B}$ are said to be equal if $\hat{A}f = \hat{B}f$ for all functions f.

Proof.

$$(\hat{A} + \hat{B})f = \hat{A}f + \hat{B}f = \hat{B}f + \hat{A}f = (\hat{B} + \hat{A})f$$

holds for any function f. By definition, $\hat{A} + \hat{B} = \hat{B} + \hat{A}$. □

Problem 3.7. *Let* $\hat{D} = d/dx$. *Verify that* $(\hat{D} + x)(\hat{D} - x) = \hat{D}^2 - x^2 - 1$.

Proof.

$$\begin{aligned}
((\hat{D} + x)(\hat{D} - x))f &= (\hat{D} + x)((\hat{D} - x)f) \\
&= (\hat{D} + x)(f' - xf) \\
&= (f' - xf)' + x(f' - xf) \\
&= (f'' - f - xf') + (xf' - x^2 f) \\
&= f'' - f - x^2 f \\
&= (\hat{D}^2 - x^2 - 1)f
\end{aligned}$$

holds for any function f. By definition, $(\hat{D} + x)(\hat{D} - x) = \hat{D}^2 - x^2 - 1$. □

Problem 3.27. *Evaluate the commutators*

(a) $[\hat{x}, \hat{p}_x]$;

(b) $[\hat{x}, \hat{p}_x^2]$;

(c) $[\hat{x}, \hat{p}_y]$;

(d) $[\hat{x}, \hat{V}(x, y, z)]$;

(e) $[\hat{x}, \hat{H}]$, *where the Hamiltonian operator is*

$$\hat{H} = -\frac{\hbar}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + V(x, y, z);$$

(f) $[\hat{x}\hat{y}\hat{z}, \hat{p}_x^2]$.

Proof of (a).

$$\begin{aligned}
[\hat{x}, \hat{p}_x]f &= (\hat{x}\hat{p}_x - \hat{p}_x\hat{x})f \\
&= (\hat{x}\hat{p}_x)f - (\hat{p}_x\hat{x})f \\
&= (\hat{x})\left(\frac{\hbar}{i}\frac{\partial f}{\partial x}\right) - (\hat{p}_x)(xf) \\
&= x\frac{\hbar}{i}\frac{\partial f}{\partial x} - \frac{\hbar}{i}\left(f + x\frac{\partial f}{\partial x}\right) \\
&= -\frac{\hbar}{i}f
\end{aligned}$$

holds for any function f. By definition, $[\hat{x}, \hat{p}_x] = -\frac{\hbar}{i}$. □