

## Chapter 10: Integration of Differential Forms

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### Exercise 10.1. ...

*Proof.*

(1)

(2)

□

**Exercise 10.2.** For  $i = 1, 2, 3, \dots$ , let  $\varphi_i \in \mathcal{C}(\mathbb{R}^1)$  have support in  $(2^{-i}, 2^{1-i})$ , such that  $\int \varphi_i = 1$ . Put

$$f(x, y) = \sum_{i=1}^{\infty} [\varphi_i(x) - \varphi_{i+1}(x)] \varphi_i(y)$$

Then  $f$  has compact support in  $\mathbb{R}^2$ ,  $f$  is continuous except at  $(0, 0)$ , and

$$\int dy \int f(x, y) dx = 0 \quad \text{but} \quad \int dx \int f(x, y) dy = 1.$$

Observe that  $f$  is unbounded in every neighborhood of  $(0, 0)$ .

*Proof.*

(1)

(2)

□

### Exercise 10.3. ...

*Proof.*

(1)

(2)

□

**Exercise 10.4.** For  $(x, y) \in \mathbb{R}^2$ , define

$$\mathbf{F}(x, y) = (e^x \cos y - 1, e^x \sin y)$$

Prove that  $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$ , where

$$\mathbf{G}_1(x, y) = (e^x \cos y - 1, y)$$

$$\mathbf{G}_2(u, v) = (u, (1 + u) \tan v)$$

are primitive in some neighborhood of  $(0, 0)$ . Compute the Jacobians of  $\mathbf{G}_1$ ,  $\mathbf{G}_2$ ,  $\mathbf{F}$  at  $(0, 0)$ . Define

$$\mathbf{H}_2(x, y) = (x, e^x \sin y)$$

and find

$$\mathbf{H}_1(u, v) = (h(u, v), v)$$

so that  $\mathbf{F} = \mathbf{H}_1 \circ \mathbf{H}_2$  is in some neighborhood of  $(0, 0)$ .

*Proof.*

(1) By Definition 10.5,

$$\mathbf{G}_1(x, y) = (e^x \cos y - 1)\mathbf{e}_1 + y\mathbf{e}_2,$$

$$\mathbf{G}_2(u, v) = u\mathbf{e}_1 + ((1 + u) \tan v)\mathbf{e}_2$$

are primitive in some neighborhood of  $(0, 0)$ .

(2) Show that  $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$ . Given any  $(x, y) \in \mathbb{R}^2$ , we have

$$\begin{aligned} (\mathbf{G}_2 \circ \mathbf{G}_1)(x, y) &= \mathbf{G}_2(\mathbf{G}_1(x, y)) \\ &= \mathbf{G}_2(e^x \cos y - 1, y) \\ &= (e^x \cos y - 1, (1 + (e^x \cos y - 1)) \tan y) \\ &= (e^x \cos y - 1, e^x \sin y) \\ &= \mathbf{F}(x, y). \end{aligned}$$

(3) Since

$$\begin{aligned} J_{\mathbf{G}_1}(x, y) &= \begin{bmatrix} e^x \cos y & -e^x \sin y \\ 0 & 1 \end{bmatrix} \\ J_{\mathbf{G}_2}(x, y) &= \begin{bmatrix} 1 & 0 \\ \tan y & (1 + x) \sec^2 y \end{bmatrix} \\ J_{\mathbf{F}}(x, y) &= \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}, \end{aligned}$$

$$\begin{aligned}
J_{\mathbf{G}_1}(0,0) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
J_{\mathbf{G}_2}(0,0) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
J_{\mathbf{F}}(0,0) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

(4) Define  $h(u, v) = \sqrt{e^{2u} - v^2} - 1$  on

$$B\left((0,0); \frac{1}{64}\right) \subseteq \mathbb{R}^2.$$

$h(u, v)$  is well-defined since  $e^{2u} - v^2 > 0$  for all  $(u, v) \in B\left((0,0); \frac{1}{64}\right)$ .

(5) Given any  $(x, y) \in \mathbb{R}^2$ , we have

$$\begin{aligned}
(\mathbf{H}_1 \circ \mathbf{H}_2)(x, y) &= \mathbf{H}_1(\mathbf{H}_2(x, y)) \\
&= \mathbf{H}_1(x, e^x \sin y) \\
&= (\sqrt{e^{2x} - (e^x \sin y)^2} - 1, e^x \sin y) \\
&= (e^x \cos y - 1, e^x \sin y) \\
&= \mathbf{F}(x, y).
\end{aligned}$$

□

#### Exercise 10.5. ...

*Proof.*

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#### Exercise 10.6. ...

*Proof.*

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□

**Exercise 10.7. ...**

*Proof.*

(1)

(2)

□

**Exercise 10.8.** Let  $H$  be the parallelogram in  $\mathbb{R}^2$  whose vertices are  $(1, 1)$ ,  $(3, 2)$ ,  $(4, 5)$ ,  $(2, 4)$ . Find the affine map  $T$  which sends  $(0, 0)$  to  $(1, 1)$ ,  $(1, 0)$  to  $(3, 2)$ ,  $(1, 1)$  to  $(4, 5)$ ,  $(0, 1)$  to  $(2, 4)$ . Show that  $J_T = 5$ . Use  $T$  to convert the integral

$$\alpha = \int_H e^{x-y} dx dy$$

to an integral over  $I^2$  and thus compute  $\alpha$ .

*Proof.*

(1) By Affine simplexes 10.26,

$$T(\mathbf{x}) = T(\mathbf{0}) + A\mathbf{x},$$

where  $A \in L(\mathbb{R}^2, \mathbb{R}^2)$ , say  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Note that  $T : \begin{bmatrix} 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . Thus

$$T : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 + ax + by \\ 1 + cx + dy \end{bmatrix}.$$

(2) By  $T : (1, 0) \mapsto (3, 2)$  and  $T : (0, 1) \mapsto (2, 4)$ , we can solve  $A$  as

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

It is easy to verify such

$$T : \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{x}} \mapsto \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{T(\mathbf{0})} + \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{x}} = \begin{bmatrix} 1 + 2x + y \\ 1 + x + 3y \end{bmatrix}$$

satisfying our requirement.

(3)

$$J_T = \det \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = 5.$$

(4)

$$\begin{aligned}\int_H e^{x-y} dx dy &= \int_{[0,1]^2} e^{(1+2u+v)-(1+u+3v)} |J_T| du dv \\ &= 5 \int_{[0,1]^2} e^{u-2v} du dv \\ &= 5 \left\{ \int_0^1 e^u du \right\} \left\{ \int_0^1 e^{-2v} dv \right\} \quad (\text{Theorem 10.2}) \\ &= \frac{5}{2} (e-1)(1-e^{-2}).\end{aligned}$$

□

**Exercise 10.9. ...**

*Proof.*

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(2)

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**Exercise 10.10. ...**

*Proof.*

(1)

(2)

□

**Exercise 10.11. ...**

*Proof.*

(1)

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□

**Exercise 10.12. ...**

*Proof.*

(1)

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**Exercise 10.13.** ...

*Proof.*

(1)

(2)

□

**Exercise 10.14 (Levi-Civita symbol).** *Prove  $\varepsilon(j_1, \dots, j_k) = s(j_1, \dots, j_k)$ , where*

$$s(j_1, \dots, j_k) = \prod_{p < q} \text{sgn}(j_q - j_p).$$

It is usually to define the Levi-Civita symbol by

$$\varepsilon(j_1, \dots, j_k) = \begin{cases} 1 & \text{if } (j_1, \dots, j_k) \text{ is an even permutation of } J, \\ -1 & \text{if } (j_1, \dots, j_k) \text{ is an odd permutation of } J, \\ 0 & \text{otherwise} \end{cases}$$

(Basic  $k$ -forms 10.14). Thus, it is the sign of the permutation in the case of a permutation, and zero otherwise. So  $\varepsilon(j_1, \dots, j_k)$  is equivalent to an explicit expression  $s(j_1, \dots, j_k) = \prod_{p < q} \text{sgn}(j_q - j_p)$ .

*Proof.*

(1) Induction on  $k$ . Base case: *Show that  $\varepsilon(j_1, j_2) = s(j_1, j_2)$ .* Since

$$\varepsilon(j_1, j_2) = \begin{cases} 1 & \text{if } j_1 < j_2 \\ -1 & \text{if } j_1 > j_2, \end{cases}$$

$$\varepsilon(j_1, j_2) = \text{sgn}(j_2 - j_1) = s(j_1, j_2).$$

(2) Inductive step: *Show that for any  $s \geq 2$ , if  $\varepsilon(j_1, \dots, j_s) = s(j_1, \dots, j_s)$*

holds, then  $\varepsilon(j_1, \dots, j_{s+1}) = s(j_1, \dots, j_{s+1})$  also holds.

$$\begin{aligned}
\varepsilon(j_1, \dots, j_{s+1}) &= \varepsilon(j_1, \dots, j_s) \prod_{\substack{1 \leq p \leq s \\ q=s+1}} \operatorname{sgn}(j_q - j_p) \\
&= s(j_1, \dots, j_s) \prod_{\substack{1 \leq p \leq s \\ q=s+1}} \operatorname{sgn}(j_q - j_p) \\
&= \prod_{1 \leq p < q \leq s} \operatorname{sgn}(j_q - j_p) \prod_{\substack{1 \leq p \leq s \\ q=s+1}} \operatorname{sgn}(j_q - j_p) \\
&= \prod_{1 \leq p < q \leq s+1} \operatorname{sgn}(j_q - j_p) \\
&= s(j_1, \dots, j_{s+1}).
\end{aligned}$$

- (3) Conclusion: Since both the base case and the inductive step have been proved as true, by mathematical induction the statement holds for every integer  $k \geq 2$ .

□

**Exercise 10.15.** If  $\omega$  and  $\lambda$  are  $k$ - and  $m$ -forms, respectively, prove that

$$\omega \wedge \lambda = (-1)^{km} \lambda \wedge \omega.$$

*Proof.*

- (1) Write

$$\omega = \sum_I b_I(\mathbf{x}) dx_I, \quad \lambda = \sum_J c_J(\mathbf{x}) dx_J$$

in the standard presentations, where  $I$  and  $J$  range over all increasing  $k$ -indices and over all increasing  $m$ -indices taken from the set  $\{1, \dots, n\}$ .

- (2) Show that  $dx_I \wedge dx_J = (-1)^{km} dx_J \wedge dx_I$ .

$$\begin{aligned}
dx_I \wedge dx_J &= dx_{i_1} \wedge \dots \wedge dx_{i_k} \wedge dx_J \\
&= (-1)^m dx_{i_1} \wedge \dots \wedge dx_{i_{k-1}} \wedge dx_J \wedge dx_{i_k} \\
&= (-1)^{2m} dx_{i_1} \wedge \dots \wedge dx_{i_{k-2}} \wedge dx_J \wedge dx_{i_{k-1}} \wedge dx_{i_k} \\
&\dots \\
&= (-1)^{km} dx_J \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k} \\
&= (-1)^{km} dx_J \wedge dx_I.
\end{aligned}$$

(3)

$$\begin{aligned}
\omega \wedge \lambda &= \sum_{I,J} b_I(\mathbf{x}) c_J(\mathbf{x}) dx_I \wedge dx_J \\
&= (-1)^{km} \sum_{J,I} c_J(\mathbf{x}) b_I(\mathbf{x}) dx_J \wedge dx_I \\
&= (-1)^{km} \lambda \wedge \omega.
\end{aligned}$$

□

**Exercise 10.16.** If  $k \geq 2$  and  $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_k]$  is an oriented affine  $k$ -simplex, prove that  $\partial^2 \sigma = 0$ , directly from the definition of the boundary operator  $\partial$ . Deduce from this that  $\partial^2 \Psi = 0$  for every chain  $\Psi$ . (Hint: For orientation, do it first for  $k = 2$ ,  $k = 3$ . In general, if  $i < j$ , let  $\sigma_{ij}$  be the  $(k-2)$ -simplex obtained by deleting  $\mathbf{p}_i$  and  $\mathbf{p}_j$  from  $\sigma$ . Show that each  $\sigma_{ij}$  occurs twice in  $\partial^2 \sigma$ , with opposite sign.)

*Proof (Brute-force).*

- (1) Write the boundary of the oriented affine  $k$ -simplex  $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_k]$  as

$$\partial \sigma = \sum_{i=0}^k (-1)^i [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k]$$

where where the oriented  $(k-1)$ -simplex  $[\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k]$  is obtained by deleting  $\sigma$ 's  $i$ -th vertex (Boundaries 10.29).

- (2)

$$\begin{aligned}
\partial^2 \sigma &= \partial \left( \sum_i (-1)^i [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k] \right) \\
&= \sum_i (-1)^i \partial [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k] \\
&= \sum_{j < i} (-1)^i (-1)^j [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_j, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k] \\
&\quad + \sum_{j > i} (-1)^i (-1)^{j-1} [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \widehat{\mathbf{p}}_j, \dots, \mathbf{p}_k] \\
&= \sum_{j < i} (-1)^{i+j} [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_j, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k] \\
&\quad - \sum_{j > i} (-1)^{i+j} [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \widehat{\mathbf{p}}_j, \dots, \mathbf{p}_k].
\end{aligned}$$

The latter two summations cancel since after switching  $i$  and  $j$  in the second sum. Therefore  $\partial^2 \sigma = 0$ .



- (3) The boundary of a chain is the linear combination of boundaries of the simplices in the chain. Write  $\Psi = \sum_{i=1}^r \sigma_i$ , where  $\sigma_i$  is an oriented affine simplex. Then

$$\partial^2 \Psi = \partial \left( \partial \sum \sigma_i \right) = \partial \left( \sum \partial \sigma_i \right) = \sum \partial^2 \sigma_i = \sum 0 = 0$$

for any affine chain  $\Psi$ .

□

**Exercise 10.17. ...**

*Proof.*

(1)

(2)

□

**Exercise 10.18. ...**

*Proof.*

(1)

(2)

□

**Exercise 10.19. ...**

*Proof.*

(1)

(2)

□

**Exercise 10.20. ...**

*Proof.*

(1)

(2)

□

**Exercise 10.21. ...**

*Proof.*

(1)

(2)

□

**Exercise 10.22. ...**

*Proof.*

(1)

(2)

□

**Exercise 10.23. ...**

*Proof.*

(1)

(2)

□

**Exercise 10.24. ...**

*Proof.*

(1)

(2)

□

**Exercise 10.25. ...**

*Proof.*

(1)

(2)

□

**Exercise 10.26. ...**

*Proof.*

(1)

(2)

□

**Exercise 10.27. ...**

*Proof.*

(1)

(2)

□

**Exercise 10.28. ...**

*Proof.*

(1)

(2)

□

**Exercise 10.29. ...**

*Proof.*

(1)

(2)

□

**Exercise 10.30. ...**

*Proof.*

(1)

(2)

□

**Exercise 10.31. ...**

*Proof.*

(1)

(2)

□

**Exercise 10.32. ...**

*Proof.*

(1)

(2)

□