

Chapter 2: Modules

*Author: Meng-Gen Tsai
Email: plover@gmail.com*

Exercise 2.1 Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$ if m, n are coprime.

It suffices to show that

$$(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$$

where d is the greatest common divisor of m and n .

Outlines.

- (1) Define $\tilde{\varphi}$ by

$$\begin{array}{ccc} \tilde{\varphi}: & (\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z}) & \longrightarrow \mathbb{Z}/d\mathbb{Z} \\ & \Downarrow & \Downarrow \\ & (x + m\mathbb{Z}, y + n\mathbb{Z}) & \longmapsto xy + d\mathbb{Z}. \end{array}$$

$\tilde{\varphi}$ is well-defined and \mathbb{Z} -bilinear.

- (2) By the universal property, $\tilde{\varphi}$ factors through a \mathbb{Z} -linear map

$$\varphi: (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \rightarrow \mathbb{Z}/d\mathbb{Z}$$

(such that $\varphi(x \otimes y) = \tilde{\varphi}(x, y)$).

- (3) To show that φ is isomorphic, might find the inverse map $\psi: \mathbb{Z}/d\mathbb{Z} \rightarrow (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$ of φ . Define ψ by

$$\begin{array}{ccc} \psi: & \mathbb{Z}/d\mathbb{Z} & \longrightarrow (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \\ & \Downarrow & \Downarrow \\ & z + d\mathbb{Z} & \longmapsto (z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}). \end{array}$$

ψ is well-defined and \mathbb{Z} -linear.

- (4) $\psi \circ \varphi = \text{id}$.

- (5) $\varphi \circ \psi = \text{id}$.

Proof of (1).

(a) $\tilde{\varphi}$ is well-defined. Say $x' = x + am$ for some $a \in \mathbb{Z}$ and $y' = y + bn$ for some $b \in \mathbb{Z}$. Then $x'y' - xy = yam + xbn + abmn \in \mathbb{Z}/d\mathbb{Z}$. That is, $\tilde{\varphi}$ is independent of coset representative.

(b) $\tilde{\varphi}$ is \mathbb{Z} -bilinear.

(i) For any $\lambda \in \mathbb{Z}$, $\tilde{\varphi}(\lambda x, y) = \tilde{\varphi}(x, \lambda y) = \lambda \tilde{\varphi}(x, y)$. In fact,

$$\begin{aligned}\tilde{\varphi}(\lambda(x + m\mathbb{Z}), y + n\mathbb{Z}) &= \tilde{\varphi}(\lambda x + m\mathbb{Z}, y + n\mathbb{Z}) = \lambda xy + d\mathbb{Z}, \\ \tilde{\varphi}(x + m\mathbb{Z}, \lambda(y + n\mathbb{Z})) &= \tilde{\varphi}(x + m\mathbb{Z}, \lambda y + n\mathbb{Z}) = \lambda xy + d\mathbb{Z}, \\ \tilde{\varphi}(x_1 + m\mathbb{Z}, y + n\mathbb{Z}) &= \lambda(xy + d\mathbb{Z}) = \lambda xy + d\mathbb{Z}.\end{aligned}$$

(ii) $\tilde{\varphi}(x_1 + x_2, y) = \tilde{\varphi}(x_1, y) + \tilde{\varphi}(x_2, y)$. In fact,

$$\begin{aligned}\tilde{\varphi}((x_1 + x_2) + m\mathbb{Z}, y + n\mathbb{Z}) &= (x_1 + x_2)y + d\mathbb{Z}, \\ \tilde{\varphi}(x_1 + m\mathbb{Z}, y + n\mathbb{Z}) + \tilde{\varphi}(x_2 + m\mathbb{Z}, y + n\mathbb{Z}) &= (x_1y + d\mathbb{Z}) + (x_2y + d\mathbb{Z}) \\ &= (x_1 + x_2)y + d\mathbb{Z}.\end{aligned}$$

(iii) $\tilde{\varphi}(x, y_1 + y_2) = \tilde{\varphi}(x, y_1) + \tilde{\varphi}(x, y_2)$. Similar to (ii).

□

Proof of (3).

(a) ψ is well-defined. Say $z' = z + cd$ for some $c \in \mathbb{Z}$. Note that $d = \alpha m + \beta n$ for some $\alpha, \beta \in \mathbb{Z}$. Thus

$$\begin{aligned}\psi(z' + d\mathbb{Z}) &= \psi(z + cd + d\mathbb{Z}) \\ &= \psi(z + c(\alpha m + \beta n) + d\mathbb{Z}) \\ &= (z + c(\alpha m + \beta n) + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\ &= (z + c\beta n + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\ &= (z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) + (c\beta n + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\ &= \psi(z + d\mathbb{Z}) + (1 + m\mathbb{Z}) \otimes (c\beta n + n\mathbb{Z}) \\ &= \psi(z + d\mathbb{Z}).\end{aligned}$$

(b) ψ is \mathbb{Z} -linear.

(i) For any $\lambda \in \mathbb{Z}$, $\psi(\lambda z) = \lambda \psi(z)$. In fact,

$$\begin{aligned}\psi(\lambda(z + d\mathbb{Z})) &= \psi(\lambda z + d\mathbb{Z}) = (\lambda z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}), \\ \lambda \psi(z + d\mathbb{Z}) &= \lambda((z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z})) = (\lambda z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}).\end{aligned}$$

(ii) $\psi(z_1 + z_2) = \psi(z_1) + \psi(z_2)$.

$$\begin{aligned}\psi((z_1 + z_2) + d\mathbb{Z}) &= (z_1 + z_2 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}), \\ \psi(z_1 + d\mathbb{Z}) + \psi(z_2 + d\mathbb{Z}) &= (z_1 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) + (z_2 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\ &= (z_1 + z_2 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}).\end{aligned}$$

□

Proof of (4). For any $(x + m\mathbb{Z}) \otimes (y + n\mathbb{Z}) \in (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$,

$$\begin{aligned}\psi(\varphi((x + m\mathbb{Z}) \otimes (y + n\mathbb{Z}))) &= \psi(xy + d\mathbb{Z}) \\ &= (xy + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\ &= (x + m\mathbb{Z}) \otimes (y + n\mathbb{Z}).\end{aligned}$$

□

Proof of (5). For any $z + d\mathbb{Z} \in \mathbb{Z}/d\mathbb{Z}$,

$$\begin{aligned}\varphi(\psi(z + d\mathbb{Z})) &= \varphi((z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z})) \\ &= z + d\mathbb{Z}.\end{aligned}$$

□