Chapter 10: Integration of Differential Forms

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Exercise 10.1. *...*

Proof.

(1)

(2)

□

Exercise 10.2. *For $i = 1, 2, 3, \ldots$, let $\varphi_i \in \mathscr{C}(\mathbb{R}^1)$ have support in $(2^{-i}, 2^{1-i})$, such that $\int \varphi_i = 1$. Put*

$$f(x, y) = \sum_{i=1}^{\infty} [\varphi_i(x) - \varphi_{i+1}(x)]\varphi_i(y)$$

Then f has compact support in $\mathbb{R}^2$, f is continuous except at $(0, 0)$, and

$$\int dy \int f(x, y) dx = 0 \qquad \textit{but} \qquad \int dx \int f(x, y) dy = 1.$$

Observe that f is unbounded in every neighborhood of $(0, 0)$.

Proof.

(1)

(2)

□

Exercise 10.3. *...*

Proof.

(1)

(2)

$\square$

Exercise 10.4. *For* $(x, y) \in \mathbb{R}^2$, *define*

$$\mathbf{F}(x, y) = (e^x \cos y - 1, e^x \sin y)$$

Prove that $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$, *where*

$$\begin{aligned}\mathbf{G}_1(x, y) &= (e^x \cos y - 1, y)\\ \mathbf{G}_2(u, v) &= (u, (1+u)\tan v)\end{aligned}$$

are primitive in some neighborhood of $(0, 0)$. *Compute the Jacobians of* $\mathbf{G}_1$, $\mathbf{G}_2$, $\mathbf{F}$ *at* $(0, 0)$. *Define*

$$\mathbf{H}_2(x, y) = (x, e^x \sin y)$$

and find

$$\mathbf{H}_1(u, v) = (h(u, v), v)$$

so that $\mathbf{F} = \mathbf{H}_1 \circ \mathbf{H}_2$ *is in some neighborhood of* $(0, 0)$.

Proof.

(1) By Definition 10.5,

$$\begin{aligned}\mathbf{G}_1(x, y) &= (e^x \cos y - 1)\mathbf{e}_1 + y\mathbf{e}_2,\\ \mathbf{G}_2(u, v) &= u\mathbf{e}_1 + ((1+u)\tan v)\mathbf{e}_2\end{aligned}$$

are primitive in some neighborhood of $(0, 0)$.

(2) *Show that* $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$. Given any $(x, y) \in \mathbb{R}^2$, we have

$$\begin{aligned}(\mathbf{G}_2 \circ \mathbf{G}_1)(x, y) &= \mathbf{G}_2(\mathbf{G}_1(x, y))\\ &= \mathbf{G}_2(e^x \cos y - 1, y)\\ &= (e^x \cos y - 1, (1 + (e^x \cos y - 1))\tan y)\\ &= (e^x \cos y - 1, e^x \sin y)\\ &= \mathbf{F}(x, y).\end{aligned}$$

(3) Since

$$\begin{aligned}J_{\mathbf{G}_1}(x, y) &= \begin{bmatrix} e^x \cos y & -e^x \sin y \\ 0 & 1 \end{bmatrix}\\ J_{\mathbf{G}_2}(x, y) &= \begin{bmatrix} 1 & 0 \\ \tan y & (1+x)\sec^2 y \end{bmatrix}\\ J_{\mathbf{F}}(x, y) &= \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix},\end{aligned}$$

$$J_{\mathbf{G}_1}(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$J_{\mathbf{G}_2}(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$J_{\mathbf{F}}(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(4) Define $h(u,v) = \sqrt{e^{2u} - v^2} - 1$ on

$$B\left((0,0); \frac{1}{64}\right) \subseteq \mathbb{R}^2.$$

$h(u,v)$ is well-defined since $e^{2u} - v^2 > 0$ for all $(u,v) \in B\left((0,0); \frac{1}{64}\right)$.

(5) Given any $(x,y) \in \mathbb{R}^2$, we have

$$\begin{aligned}
(\mathbf{H}_1 \circ \mathbf{H}_2)(x,y) &= \mathbf{H}_1(\mathbf{H}_2(x,y)) \\
&= \mathbf{H}_1(x, e^x \sin y) \\
&= (\sqrt{e^{2x} - (e^x \sin y)^2} - 1, e^x \sin y) \\
&= (e^x \cos y - 1, e^x \sin y) \\
&= \mathbf{F}(x,y).
\end{aligned}$$

□

Exercise 10.5. ...

Proof.

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(2)

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Exercise 10.6. ...

Proof.

(1)

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□

Exercise 10.7. ...

Proof.

(1)

(2)

$\square$

Exercise 10.8. *Let H be the parallelogram in $\mathbb{R}^2$ whose vertices are $(1,1)$, $(3,2)$, $(4,5)$, $(2,4)$. Find the affine map T which sends $(0,0)$ to $(1,1)$, $(1,0)$ to $(3,2)$, $(1,1)$ to $(4,5)$, $(0,1)$ to $(2,4)$. Show that $J_T = 5$. Use T to convert the integral*

$$\alpha = \int_H e^{x-y} dxdy$$

to an integral over I^2 and thus compute α.

Proof.

(1) By Affine simplexes 10.26,

$$T(\mathbf{x}) = T(\mathbf{0}) + A\mathbf{x},$$

where $A \in L(\mathbb{R}^2, \mathbb{R}^2)$, say $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Note that $T : \begin{bmatrix} 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus

$$T : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 + ax + by \\ 1 + cx + dy \end{bmatrix}.$$

(2) By $T : (1,0) \mapsto (3,2)$ and $T : (0,1) \mapsto (2,4)$, we can solve A as

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

It is easy to verify such

$$T : \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{x}} \mapsto \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{T(\mathbf{0})} + \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{x}} = \begin{bmatrix} 1 + 2x + y \\ 1 + x + 3y \end{bmatrix}$$

satisfying our requirement.

(3)

$$J_T = \det \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = 5.$$

(4)

$$\begin{aligned}
\int_H e^{x-y} dxdy &= \int_{[0,1]^2} e^{(1+2u+v)-(1+u+3v)} |J_T| dudv \\
&= 5 \int_{[0,1]^2} e^{u-2v} dudv \\
&= 5 \left\{ \int_0^1 e^u du \right\} \left\{ \int_0^1 e^{-2v} dv \right\} && \text{(Theorem 10.2)} \\
&= \frac{5}{2}(e-1)(1-e^{-2}).
\end{aligned}$$

□

Exercise 10.9. ...

Proof.

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Exercise 10.10. ...

Proof.

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Exercise 10.11. ...

Proof.

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Exercise 10.12. ...

Proof.

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Exercise 10.13. *...*

Proof.

(1)

(2)

□

Exercise 10.14 (Levi-Civita symbol). *Prove* $\varepsilon(j_1, \ldots, j_k) = s(j_1, \ldots, j_k)$, *where*

$$s(j_1, \ldots, j_k) = \prod_{p<q} \operatorname{sgn}(j_q - j_p).$$

It is usually to define the Levi-Civita symbol by

$$\varepsilon(j_1, \ldots, j_k) = \begin{cases} 1 & \text{if } (j_1, \cdots, j_k) \text{ is an even permutation of } J, \\ -1 & \text{if } (j_1, \cdots, j_k) \text{ is an odd permutation of } J, \\ 0 & \text{otherwise} \end{cases}$$

(Basic k-forms 10.14). Thus, it is the sign of the permutation in the case of a permutation, and zero otherwise. So $\varepsilon(j_1, \ldots, j_k)$ is equivalent to an explicit expression $s(j_1, \ldots, j_k) = \prod_{p<q} \operatorname{sgn}(j_q - j_p)$.

Proof.

(1) Induction on k. Base case: *Show that* $\varepsilon(j_1, j_2) = s(j_1, j_2)$. Since

$$\varepsilon(j_1, j_2) = \begin{cases} 1 & \text{if } j_1 < j_2 \\ -1 & \text{if } j_1 > j_2, \end{cases}$$

$\varepsilon(j_1, j_2) = \operatorname{sgn}(j_2 - j_1) = s(j_1, j_2)$.

(2) Inductive step: *Show that for any* $s \geq 2$, *if* $\varepsilon(j_1, \ldots, j_s) = s(j_1, \ldots, j_s)$

holds, then $\varepsilon(j_1, \ldots, j_{s+1}) = s(j_1, \ldots, j_{s+1})$ *also holds.*

$$\begin{aligned}
\varepsilon(j_1, \ldots, j_{s+1}) &= \varepsilon(j_1, \ldots, j_s) \prod_{\substack{1 \le p \le s \\ q = s+1}} \operatorname{sgn}(j_q - j_p) \\
&= s(j_1, \ldots, j_s) \prod_{\substack{1 \le p \le s \\ q = s+1}} \operatorname{sgn}(j_q - j_p) \\
&= \prod_{1 \le p < q \le s} \operatorname{sgn}(j_q - j_p) \prod_{\substack{1 \le p \le s \\ q = s+1}} \operatorname{sgn}(j_q - j_p) \\
&= \prod_{1 \le p < q \le s+1} \operatorname{sgn}(j_q - j_p) \\
&= s(j_1, \ldots, j_{s+1}).
\end{aligned}$$

(3) Conclusion: Since both the base case and the inductive step have been proved as true, by mathematical induction the statement holds for every integer $k \geq 2$.

□

Exercise 10.15. *If* ω *and* λ *are* k*- and* m*-forms, respectively, prove that*

$$\omega \wedge \lambda = (-1)^{km} \lambda \wedge \omega.$$

Proof.

(1) Write

$$\omega = \sum_I b_I(\mathbf{x}) dx_I, \qquad \lambda = \sum_J c_J(\mathbf{x}) dx_J$$

in the stardard presentations, where I and J range over all increasing k-indices and over all increasing m-indices taken from the set $\{1, \ldots, n\}$.

(2) *Show that* $dx_I \wedge dx_J = (-1)^{km} dx_J \wedge dx_I$.

$$\begin{aligned}
dx_I \wedge dx_J &= dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_J \\
&= (-1)^m dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}} \wedge dx_J \wedge dx_{i_k} \\
&= (-1)^{2m} dx_{i_1} \wedge \cdots \wedge dx_{i_{k-2}} \wedge dx_J \wedge dx_{i_{k-1}} \wedge dx_{i_k} \\
&\cdots \\
&= (-1)^{km} dx_J \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&= (-1)^{km} dx_J \wedge dx_I.
\end{aligned}$$

(3)

$$\begin{aligned}\omega \wedge \lambda &= \sum_{I,J} b_I(\mathbf{x}) c_J(\mathbf{x}) dx_I \wedge dx_J \\ &= (-1)^{km} \sum_{J,I} c_J(\mathbf{x}) b_I(\mathbf{x}) dx_J \wedge dx_I \\ &= (-1)^{km} \lambda \wedge \omega.\end{aligned}$$

□

Exercise 10.16. *If $k \geq 2$ and $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_k]$ is an oriented affine k-simplex, prove that $\partial^2\sigma = 0$, directly from the definition of the boundary operator ∂. Deduce from this that $\partial^2\Psi = 0$ for every chain Ψ. (Hint: For orientation, do it first for $k = 2$, $k = 3$. In general, if $i < j$, let σ_{ij} be the $(k-2)$-simplex obtained by deleting $\mathbf{p}_i$ and $\mathbf{p}_j$ from σ. Show that each σ_{ij} occurs twice in $\partial^2\sigma$, with opposite sign.)*

Proof (Brute-force).

(1) Write the boundary of the oriented affine k-simplex $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_k]$ as

$$\partial\sigma = \sum_{i=0}^{k} (-1)^i [\mathbf{p}_0, \ldots, \widehat{\mathbf{p}}_i, \ldots, \mathbf{p}_k]$$

where where the oriented $(k-1)$-simplex $[\mathbf{p}_0, \ldots, \widehat{\mathbf{p}}_i, \ldots, \mathbf{p}_k]$ is obtained by deleting σ's i-th vertex (Boundaries 10.29).

(2)

$$\begin{aligned}\partial^2\sigma =& \partial\left(\sum_i (-1)^i [\mathbf{p}_0, \ldots, \widehat{\mathbf{p}}_i, \ldots, \mathbf{p}_k]\right) \\ =& \sum_i (-1)^i \partial[\mathbf{p}_0, \ldots, \widehat{\mathbf{p}}_i, \ldots, \mathbf{p}_k] \\ =& \sum_{j<i} (-1)^i (-1)^j [\mathbf{p}_0, \ldots, \widehat{\mathbf{p}_j}, \ldots, \widehat{\mathbf{p}}_i, \ldots, \mathbf{p}_k] \\ &+ \sum_{j>i} (-1)^i (-1)^{j-1} [\mathbf{p}_0, \ldots, \widehat{\mathbf{p}}_i, \ldots, \widehat{\mathbf{p}_j}, \ldots, \mathbf{p}_k] \\ =& \sum_{j<i} (-1)^{i+j} [\mathbf{p}_0, \ldots, \widehat{\mathbf{p}_j}, \ldots, \widehat{\mathbf{p}}_i, \ldots, \mathbf{p}_k] \\ &- \sum_{j>i} (-1)^{i+j} [\mathbf{p}_0, \ldots, \widehat{\mathbf{p}}_i, \ldots, \widehat{\mathbf{p}_j}, \ldots, \mathbf{p}_k].\end{aligned}$$

The latter two summations cancel since after switching i and j in the second sum. Therefore $\partial^2\sigma = 0$.

(3) The boundary of a chain is the linear combination of boundaries of the simplices in the chain. Write $\Psi = \sum_{i=1}^{r} \sigma_i$. where σ_i is an oriented affine simplex. Then

$$\partial^2 \Psi = \partial\left(\partial \sum \sigma_i\right) = \partial\left(\sum \partial \sigma_i\right) = \sum \partial^2 \sigma_i = \sum 0 = 0$$

for any affine chain Ψ.

□

Exercise 10.17. ...

Proof.

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(2)

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Exercise 10.18. ...

Proof.

(1)

(2)

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Exercise 10.19. ...

Proof.

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Exercise 10.20. ...

Proof.

(1)

(2)

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Exercise 10.21. ...

Proof.

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(2)

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Exercise 10.22. ...

Proof.

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Exercise 10.23. ...

Proof.

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Exercise 10.24. ...

Proof.

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Exercise 10.25. ...

Proof.

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Exercise 10.26. ...

Proof.

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Exercise 10.27. ...

Proof.

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Exercise 10.28. ...

Proof.

(1)

(2)

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Exercise 10.29. ...

Proof.

(1)

(2)

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Exercise 10.30. ...

Proof.

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(2)

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Exercise 10.31. ...

Proof.

(1)

(2)

□

Exercise 10.32. ...

Proof.

(1)

(2)

□