Chapter 3: Lebesgue Measure

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Section 3.1: Introduction

Problem 3.1. *If A and B are two sets in $\mathfrak{M}$ with $A \subseteq B$, then $mA \leq mB$. This property is called monotonicity.*

Proof. Write

$$B = B \cap X = B \cap (A \cup \widetilde{A}) = (B \cap A) \cup (B \cap \widetilde{A}) = A \cup (B - A).$$

Here $B \cap A = A$ comes from $A \subseteq B$ (Problem 1.9). Notice that A and $B - A$ are disjoint. Since m is a countably additive measure (m is nonnegative) on a σ-algebra $\mathfrak{M}$,

$$mB = mA + m(B - A) \geq mA.$$

$\square$

Problem 3.2. *Let $\langle E_n \rangle$ be any sequence of sets in $\mathfrak{M}$. Then $m(\bigcup E_n) \leq \sum mE_n$. (Hint: Use Proposition 1.2) This property of a measure is called countable subadditivity.*

As the argument in Problem 3.1.

Proof. Since $\langle E_n \rangle$ is a sequence of sets in σ-algebra $\mathfrak{M}$, by Proposition 1.2 and its proof, there is a sequence $\langle F_n \rangle$ of sets in σ-algebra $\mathfrak{M}$ such that all F_n are pairwise disjoint, $F_n \subseteq E_n$, and

$$\bigcup E_n = \bigcup F_n.$$

Since m is a countably additive measure on a σ-algebra $\mathfrak{M}$,

$$m\left(\bigcup E_n\right) = m\left(\bigcup F_n\right) = \sum mF_n \geq \sum mE_n.$$

The last inequality holds by applying Exercise 3.1 on $F_n \subseteq E_n$ for any n. $\square$

Problem 3.3. *If there is a set A in $\mathfrak{M}$ such that $mA < \infty$, then $m\varnothing = 0$.*

Proof. For such A, write $A = A \cup \varnothing$. A and $\varnothing$ are disjoint. Since m is a countably additive measure on a σ-algebra $\mathfrak{M}$,

$$mA = mA + m\varnothing.$$

Since $mA < \infty$, we can cancel out mA on the both sides to get $m\varnothing = 0$. $\square$

Section 3.2: Outer Measure

Problem 3.5. *Let A be the set of rational numbers between 0 and 1, and let $\{I_n\}$ be a finite collection of open intervals covering A. Then $\sum l(I_n) \geq 1$.*

Proof. Look at the closure of A.

$$\begin{aligned}
1 &= m^*[0,1] && \text{(Proposition 3.1)} \\
&= m^*\overline{A} && \text{(A is dense in $[0,1]$)} \\
&\leq m^*\left(\overline{\bigcup I_n}\right) && \text{(Proposition 2.10)} \\
&= m^*\left(\bigcup \overline{I_n}\right) && \text{(Proposition 2.10)} \\
&\leq \sum m^*(\overline{I_n}) && \text{(Proposition 3.2)} \\
&= \sum l(\overline{I_n}) && \text{(Proposition 3.1)} \\
&= \sum l(I_n) && \text{(definition of length)}
\end{aligned}$$

$\square$

Problem 3.6. *Prove Proposition 5: Given any set A and any $\epsilon > 0$, there is an open set O such that $A \subseteq O$ and $m^*O \leq m^*A + \epsilon$. There is a $G \in G_\delta$ such that $m^*G = m^*A$.*

Proof.

(1) *Use the definition of the outer measure.* By the definition of m^*, for such $\epsilon > 0$ there exists a countable collection $\{I_n\}$ of open intervals that covers A and
$$m^*A + \epsilon \geq \sum l(I_n).$$

(2) *Construct an open set O.* Let $O = \bigcup I_n \supseteq A$ which is the union of any collection of open sets I_n. By Proposition 2.7, O is open.

(3) *Show that $m^*O \leq m^*A + \epsilon$.* By Proposition 3.2 and 3.1,
$$m^*O = m^*\left(\bigcup I_n\right) \leq \sum m^*I_n = \sum l(I_n) \leq m^*A + \epsilon.$$

Therefore, given any set A and any $\epsilon > 0$, there is an open set O such that $A \subseteq O$ and $m^*O \leq m^*A + \epsilon$.

(4) *Construct $G \in G_\delta$ in a natural way.* Given any $n \in \mathbb{N}$, there exists an open set O_n such that $O_n \supseteq A$ and $m^*O_n \leq m^*A + \frac{1}{n}$. Let

$$G = \bigcap_{n=1}^{\infty} O_n \in G_\delta.$$

(5) *Show that $m^*G = m^*A$.*

(a) Since $A \subseteq O_n$ for any $n \in \mathbb{N}$, $A \subseteq \bigcap_{n=1}^{\infty} O_n = G$. Thus $m^*A \leq m^*G$.

(b) Since $O_n \supseteq \bigcap_{n=1}^{\infty} O_n = G$ for any $n \in \mathbb{N}$,

$$m^*A + \frac{1}{n} \geq m^*O_n \geq m^*G$$

for any $n \in \mathbb{N}$. Since $n \in \mathbb{N}$ is arbitrary, $m^*A \geq m^*G$.

By (a)(b), $m^*A = m^*G$.

$\square$

Problem 3.7. *Prove that m^* is translation invariant.*

Proof. Given $E \in \mathfrak{M}$ and $y \in \mathbb{R}$.

(1) $m^*(E+y) \leq m^*E$. Let $\{I_n\}$ of open intervals that cover E. Then $\{I_n+y\}$ of open intervals that cover $E+y$. Notice that the definition of m^* and $l(I_n+y) = l(I_n)$, then

$$m^*(E+y) \leq \sum l(I_n+y) = \sum l(I_n).$$

Take the infimum of all such sum $\sum l(I_n)$, $m^*(E+y) \leq m^*E$.

(2) $m^*(E) \leq m^*(E+y)$. Similar to (1).

By (1)(2), $m^*(E+y) = m^*E$, that is, m^* is translation invariant. $\square$

Problem 3.8. *Prove that if $m^*A = 0$, then $m^*(A \cup B) = m^*B$.*

Proof.

(1) $m^*(A \cup B) \geq m^*B$ since $A \cup B \supseteq B$ and the definition of m^*. (Any covering of $A \cup B$ by open intervals is also a covering of B so that the latter infimum is taken over a larger collection than the former.)

(2) $m^*(A \cup B) \leq m^*B$. By Proposition 3.2,

$$m^*(A \cup B) \leq m^*A + m^*B = 0 + m^*B = m^*B.$$

By (1)(2), $m^*(A \cup B) = m^*B$. $\square$