

Chapter 9: Functions of Several Variables

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Exercise 9.1. *If S is a nonempty subset of a vector space X , prove (as asserted in Section 9.1) that the span of S is a vector space.*

Denote the span of S by $\text{span}(S)$.

Proof.

- (1) Since $S \neq \emptyset$, there is $\mathbf{z} \in S$. So $1\mathbf{z} = \mathbf{z} \in \text{span}(S) \neq \emptyset$. (In fact, $\text{span}(S) \supseteq S$.)
- (2) If $\mathbf{x}, \mathbf{y} \in \text{span}(S)$, then there exist elements $\mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{y}_1, \dots, \mathbf{y}_n \in S$ and scalars $a_1, \dots, a_m, b_1, \dots, b_n$ such that

$$\mathbf{x} = a_1\mathbf{x}_1 + \dots + a_m\mathbf{x}_m,$$

$$\mathbf{y} = b_1\mathbf{y}_1 + \dots + b_n\mathbf{y}_n.$$

Then

$$\mathbf{x} + \mathbf{y} = a_1\mathbf{x}_1 + \dots + a_m\mathbf{x}_m + b_1\mathbf{y}_1 + \dots + b_n\mathbf{y}_n$$

is a linear combination of the elements of S . For any scalar c ,

$$c\mathbf{x} = (ca_1)\mathbf{x}_1 + \dots + (ca_m)\mathbf{x}_m$$

is again linear combination of the elements of S .

- (3) By (1)(2), $\text{span}(S)$ is a vector space.

□

Note. Any subspace of X that contains S must also contain $\text{span}(S)$.

Exercise 9.2. *Prove (as asserted in Section 9.6) that BA is linear if A and B are linear transformations. Prove also that A^{-1} is linear and invertible if A is invertible.*

Proof. Use the notation in Definitions 9.6.

- (1) *Show that BA is linear if A and B are linear transformations.* Let X, Y, Z be vector spaces, $A \in L(X, Y)$ and $B \in L(Y, Z)$.

(a) Given any $\mathbf{x}_1, \mathbf{x}_2 \in X$.

$$\begin{aligned}
 (BA)(\mathbf{x}_1 + \mathbf{x}_2) &= B(A(\mathbf{x}_1 + \mathbf{x}_2)) \\
 &= B(A\mathbf{x}_1 + A\mathbf{x}_2) \quad (A \text{ is a linear transformation}) \\
 &= B(A\mathbf{x}_1) + B(A\mathbf{x}_2) \quad (B \text{ is a linear transformation}) \\
 &= (BA)\mathbf{x}_1 + (BA)\mathbf{x}_2.
 \end{aligned}$$

(b) For any $\mathbf{x} \in X$ and scalar c ,

$$\begin{aligned}
 (BA)(c\mathbf{x}) &= B(A(c\mathbf{x})) \\
 &= B(cA\mathbf{x}) \quad (A \text{ is a linear transformation}) \\
 &= cB(A\mathbf{x}) \quad (B \text{ is a linear transformation}) \\
 &= c(BA)\mathbf{x}.
 \end{aligned}$$

By (a)(b), $BA \in L(X, Z)$.

(2) Show that A^{-1} is linear if A is invertible.

(a) Given any $\mathbf{y}_1, \mathbf{y}_2 \in X$. Since A is surjective, there exist $\mathbf{x}_1, \mathbf{x}_2 \in X$ such that

$$\begin{aligned}
 \mathbf{y}_1 &= A\mathbf{x}_1 \\
 \mathbf{y}_2 &= A\mathbf{x}_2.
 \end{aligned}$$

So

$$\begin{aligned}
 A^{-1}\mathbf{y}_1 &= A^{-1}(A\mathbf{x}_1) = \mathbf{x}_1 \\
 A^{-1}\mathbf{y}_2 &= A^{-1}(A\mathbf{x}_2) = \mathbf{x}_2
 \end{aligned}$$

(by Definitions 9.4). Hence

$$\begin{aligned}
 A^{-1}(\mathbf{y}_1 + \mathbf{y}_2) &= A^{-1}(A\mathbf{x}_1 + A\mathbf{x}_2) \\
 &= A^{-1}(A(\mathbf{x}_1 + \mathbf{x}_2)) \quad (A \text{ is a linear transformation}) \\
 &= \mathbf{x}_1 + \mathbf{x}_2 \quad (\text{Definitions 9.4}) \\
 &= A^{-1}\mathbf{y}_1 + A^{-1}\mathbf{y}_2.
 \end{aligned}$$

(b) For any $\mathbf{y} \in X$ and scalar c , there is a corresponding $\mathbf{x} \in X$ such that $\mathbf{y} = A\mathbf{x}$ since A is surjective. So $A^{-1}\mathbf{y} = \mathbf{x}$ by Definition 9.4. Hence

$$\begin{aligned}
 A^{-1}(c\mathbf{y}) &= A^{-1}(cA\mathbf{x}) \\
 &= A^{-1}(A(c\mathbf{x})) \quad (A \text{ is a linear transformation}) \\
 &= c\mathbf{x} \quad (\text{Definitions 9.4}) \\
 &= cA^{-1}\mathbf{y}.
 \end{aligned}$$

By (a)(b), $A^{-1} \in L(X)$.

(3) *Show that A^{-1} is invertible if A is invertible.* It suffices to show that A^{-1} is injective and surjective.

(a) *Show that A^{-1} is injective.* Given any $\mathbf{y}_1, \mathbf{y}_2 \in X$. Since A is surjective, there exist $\mathbf{x}_1, \mathbf{x}_2 \in X$ such that

$$\begin{aligned}\mathbf{y}_1 &= A\mathbf{x}_1 \\ \mathbf{y}_2 &= A\mathbf{x}_2.\end{aligned}$$

Suppose $A^{-1}\mathbf{y}_1 = A^{-1}\mathbf{y}_2$. So $A^{-1}(A\mathbf{x}_1) = A^{-1}(A\mathbf{x}_2)$, or $\mathbf{x}_1 = \mathbf{x}_2$, or $\mathbf{y}_1 = A\mathbf{x}_1 = A\mathbf{x}_2 = \mathbf{y}_2$.

(b) *Show that A^{-1} is surjective.* For any $\mathbf{x} \in X$, there exists $A\mathbf{x} \in X$ such that $A^{-1}(A\mathbf{x}) = \mathbf{x}$ by Definitions 9.4.

□

Exercise 9.3. Assume $A \in L(X, Y)$ and $A\mathbf{x} = \mathbf{0}$ only when $\mathbf{x} = \mathbf{0}$. Prove that A is then 1-1.

Proof. Suppose $A\mathbf{x} = A\mathbf{y}$. Since A is a linear transformation, $A(\mathbf{x} - \mathbf{y}) = A\mathbf{x} - A\mathbf{y} = \mathbf{0}$. By assumption, $\mathbf{x} - \mathbf{y} = \mathbf{0}$ or $\mathbf{x} = \mathbf{y}$. □

Exercise 9.4. Prove (as asserted in Section 9.30) that null spaces and ranges of linear transformations are vector spaces.

Proof. Use the notation in Definitions 9.30. Suppose X, Y are vector spaces, and $A \in L(X, Y)$, as in Definition 9.6.

(1) *Show that $\mathcal{N}(A)$ is a vector space in X .*

(a) Note that $\mathbf{0} \in X$. Since $A\mathbf{0} = \mathbf{0}$, $\mathbf{0} \in \mathcal{N}(A) \neq \emptyset$.

(b) Suppose $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{N}(A)$. Then

$$\begin{aligned}A(\mathbf{x}_1 + \mathbf{x}_2) &= A\mathbf{x}_1 + A\mathbf{x}_2 && (A \text{ is a linear transformation}) \\ &= \mathbf{0} + \mathbf{0} && (\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{N}(A)) \\ &= \mathbf{0}.\end{aligned}$$

So $\mathbf{x}_1 + \mathbf{x}_2 \in \mathcal{N}(A)$.

(c) Suppose $\mathbf{x} \in \mathcal{N}(A)$ and c is a scalar. Then

$$\begin{aligned}A(c\mathbf{x}) &= cA\mathbf{x} && (A \text{ is a linear transformation}) \\ &= c\mathbf{0} && (\mathbf{x} \in \mathcal{N}(A)) \\ &= \mathbf{0}.\end{aligned}$$

So $c\mathbf{x} \in \mathcal{N}(A)$.

By (a)(b)(c), $\mathcal{N}(A)$ is a vector space.

(2) Show that $\mathcal{R}(A)$ is a vector space in Y .

(a) Note that $\mathbf{0} \in X$. So $A\mathbf{0} = \mathbf{0} \in \mathcal{R}(A) \neq \emptyset$.

(b) Suppose $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{R}(A)$. Then there exist $\mathbf{x}_1, \mathbf{x}_2 \in X$ such that $A\mathbf{x}_1 = \mathbf{y}_1$ and $A\mathbf{x}_2 = \mathbf{y}_2$. Hence

$$\begin{aligned}\mathbf{y}_1 + \mathbf{y}_2 &= A\mathbf{x}_1 + A\mathbf{x}_2 \\ &= A(\mathbf{x}_1 + \mathbf{x}_2) \quad (A \text{ is a linear transformation}).\end{aligned}$$

So $\mathbf{y}_1 + \mathbf{y}_2 \in \mathcal{R}(A)$.

(c) Suppose $\mathbf{y} \in \mathcal{R}(A)$ and c is a scalar. Then there exists $\mathbf{x} \in X$ such that $A\mathbf{x} = \mathbf{y}$. Hence

$$\begin{aligned}c\mathbf{y} &= cA\mathbf{x} \\ &= A(c\mathbf{x}) \quad (A \text{ is a linear transformation}).\end{aligned}$$

So $c\mathbf{y} \in \mathcal{R}(A)$.

By (a)(b)(c), $\mathcal{R}(A)$ is a vector space.

□

Exercise 9.5. Prove that to every $A \in L(\mathbb{R}^n, \mathbb{R}^1)$ corresponds a unique $\mathbf{y} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$. Prove also that $\|A\| = \|\mathbf{y}\|$. (Hint: Under certain conditions, equality holds in the Schwarz inequality.)

Proof.

(1) Recall that $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the standard basis of \mathbb{R}^n (Definitions 9.1).

Given any $\mathbf{x} \in \mathbb{R}^n$, write $\mathbf{x} = (x_1, \dots, x_n)$ as $\mathbf{x} = \sum x_j \mathbf{e}_j$.

(2) Show that \mathbf{y} exists. Since A is a linear transformation,

$$\begin{aligned}A\mathbf{x} &= A\left(\sum x_j \mathbf{e}_j\right) \\ &= \sum x_j A\mathbf{e}_j \\ &= (x_1, \dots, x_n) \cdot (A\mathbf{e}_1, \dots, A\mathbf{e}_n) \\ &= \mathbf{x} \cdot \sum (A\mathbf{e}_j) \mathbf{e}_j.\end{aligned}$$

Define $\mathbf{y} = \sum (A\mathbf{e}_j) \mathbf{e}_j \in \mathbb{R}^n$ so that $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$.

(3) Show that \mathbf{y} is unique. Suppose there exists some $\mathbf{z} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{x} \cdot \mathbf{z}$. So

$$\begin{aligned}0 &= A\mathbf{x} - A\mathbf{x} \\ &= \mathbf{x} \cdot \mathbf{y} - \mathbf{x} \cdot \mathbf{z} \\ &= \mathbf{x} \cdot (\mathbf{y} - \mathbf{z})\end{aligned}$$

for any $\mathbf{x} \in \mathbb{R}^n$. In particular, take $\mathbf{x} = \mathbf{y} - \mathbf{z} \in \mathbb{R}^n$ to get

$$0 = (\mathbf{y} - \mathbf{z}) \cdot (\mathbf{y} - \mathbf{z}) = |\mathbf{y} - \mathbf{z}|^2$$

or $\mathbf{y} - \mathbf{z} = \mathbf{0}$ or $\mathbf{y} = \mathbf{z}$.

(4) *Show that $\|A\| = |\mathbf{y}|$.* By the Schwarz inequality (Theorem 1.37(d)),

$$|A\mathbf{x}| = |\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}| \leq |\mathbf{y}|$$

as $|\mathbf{x}| \leq 1$. Take the sup over all $|\mathbf{x}| \leq 1$ to get

$$\|A\| \leq |\mathbf{y}|.$$

If $\mathbf{y} = \mathbf{0}$, then $\|A\| = |\mathbf{y}| = 0$. If $\mathbf{y} \neq \mathbf{0}$, then the equality holds when $\mathbf{x} = \frac{\mathbf{y}}{|\mathbf{y}|} \in \mathbb{R}^n$. (Here $|\mathbf{x}| = 1$.)

□

Exercise 9.6. *If $f(0,0) = 0$ and*

$$f(x,y) = \frac{xy}{x^2 + y^2} \quad \text{if } (x,y) \neq (0,0),$$

prove that $(D_1f)(x,y)$ and $(D_2f)(x,y)$ exist at every point of \mathbb{R}^2 , although f is not continuous at $(0,0)$.

Proof.

(1) *Show that*

$$(D_1f)(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0), \\ \frac{y(y^2 - x^2)}{(x^2 + y^2)^2} & \text{if } (x,y) \neq (0,0). \end{cases}$$

Write

$$\begin{aligned} (D_1f)(x,y) &= \lim_{t \rightarrow 0} \frac{f((x,y) + t(1,0)) - f(x,y)}{t} \\ &= \lim_{t \rightarrow 0} \frac{f(x+t,y) - f(x,y)}{t}. \end{aligned}$$

If $(x,y) = (0,0)$,

$$(D_1f)(0,0) = \lim_{t \rightarrow 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t \rightarrow 0} \frac{0 - 0}{t} = 0.$$

If $(x, y) \neq (0, 0)$,

$$\begin{aligned}
 (D_1 f)(x, y) &= \lim_{t \rightarrow 0} \frac{f(x+t, y) - f(x, y)}{t} \\
 &= \lim_{t \rightarrow 0} \frac{\frac{(x+t)y}{(x+t)^2 + y^2} - \frac{xy}{x^2 + y^2}}{t} \\
 &= \lim_{t \rightarrow 0} \frac{y(y^2 - x^2) - txy}{((x+t)^2 + y^2)(x^2 + y^2)} \\
 &= \frac{y(y^2 - x^2)}{(x^2 + y^2)^2}.
 \end{aligned}$$

(2) Show that

$$(D_2 f)(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0), \\ \frac{x(x^2 - y^2)}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0). \end{cases}$$

Similar to (1).

(3) Show that f is not continuous at $(0, 0)$. Note that

$$\lim_{n \rightarrow \infty} f\left(\frac{1}{n}, \frac{1}{n}\right) = \lim_{n \rightarrow \infty} \frac{\frac{1}{n} \cdot \frac{1}{n}}{\frac{1}{n^2} + \frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{1}{2} = \frac{1}{2}$$

and

$$\lim_{n \rightarrow \infty} f\left(\frac{1}{n}, 0\right) = \lim_{n \rightarrow \infty} \frac{0}{\frac{1}{n^2} + 0} = \lim_{n \rightarrow \infty} 0 = 0.$$

Hence the limit $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ does not exist.

□

Exercise 9.7. Suppose that f is a real-valued function defined in an open set $E \subseteq \mathbb{R}^n$, and that the partial derivatives $D_1 f, \dots, D_n f$ are bounded in E . Prove that f is continuous in E . (Hint: Proceed as in the proof of Theorem 9.21.)

Proof.

- (1) Since $D_j f$ is bounded in E , there is a real number M_j such that $|D_j f| \leq M_j$ in E . Take $M = \max_{1 \leq j \leq n} M_j$ so that $|D_j f| \leq M$ in E for all $1 \leq j \leq n$.
- (2) Fix $\mathbf{x} \in E$ and $\varepsilon > 0$. Since E is open, there is an open neighborhood

$$B(\mathbf{x}; r) = \{\mathbf{x} + \mathbf{h} \in E : |\mathbf{h}| < r\} \subseteq E$$

with

$$0 < r < \frac{\varepsilon}{n(M+1)}.$$

- (3) Write $\mathbf{h} = \sum h_j \mathbf{e}_j$, $|\mathbf{h}| < r$, put $\mathbf{v}_0 = \mathbf{0}$, and $\mathbf{v}_k = h_1 \mathbf{e}_1 + \cdots + h_k \mathbf{e}_k$ for $1 \leq k \leq n$. Then

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = \sum_{j=1}^n [f(\mathbf{x} + \mathbf{v}_j) - f(\mathbf{x} + \mathbf{v}_{j-1})].$$

Since $|\mathbf{v}_k| < r$ for $1 \leq k \leq n$ and since $B(\mathbf{x}; r)$ is convex, the open interval with end points $\mathbf{x} + \mathbf{v}_{j-1}$ and $\mathbf{x} + \mathbf{v}_j$ lie in $B(\mathbf{x}; r)$. Since $\mathbf{v}_j = \mathbf{v}_{j-1} + h_j \mathbf{e}_j$, the mean value theorem (Theorem 5.10) show that

$$f(\mathbf{x} + \mathbf{v}_j) - f(\mathbf{x} + \mathbf{v}_{j-1}) = h_j (D_j f)(\mathbf{x} + \mathbf{v}_{j-1} + \theta_j h_j \mathbf{e}_j)$$

for some $\theta_j \in (0, 1)$.

- (4) Note that $|h_j| \leq |\mathbf{h}| < r < \frac{\varepsilon}{n(M+1)}$. Hence

$$\begin{aligned} |f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x})| &\leq \sum_{j=1}^n |f(\mathbf{x} + \mathbf{v}_j) - f(\mathbf{x} + \mathbf{v}_{j-1})| \\ &= \sum_{j=1}^n |h_j| |(D_j f)(\mathbf{x} + \mathbf{v}_{j-1} + \theta_j h_j \mathbf{e}_j)| \\ &\leq \sum_{j=1}^n \frac{\varepsilon}{n(M+1)} \cdot M \\ &< \varepsilon \end{aligned}$$

as $|\mathbf{h}| < r < \frac{\varepsilon}{n(M+1)}$. Hence f is continuous at all $\mathbf{x} \in E$.

□

Exercise 9.8. Suppose that f is a differentiable real function in an open set $E \subseteq \mathbb{R}^n$, and that f has a local maximum at a point $\mathbf{x} \in E$. Prove that $f'(\mathbf{x}) = 0$.

Proof (Theorem 5.8).

- (1) Apply Theorem 5.8 to each $D_j f$ for $1 \leq j \leq n$. Since f has a local maximum at a point $\mathbf{x} \in E$, there is an open neighborhood $B(\mathbf{x}; r)$ of \mathbf{x} in E such that

$$f(\mathbf{y}) \leq f(\mathbf{x})$$

for all $\mathbf{y} \in B(\mathbf{x}; r)$. Therefore,

$$f(\mathbf{x} + t\mathbf{e}_j) \leq f(\mathbf{x})$$

for all $|t| < r$ and $1 \leq j \leq n$, or $t \mapsto f(\mathbf{x} + t\mathbf{e}_j)$ has a local maximum at a point $t = 0 \in (-r, r)$.

- (2) Since f is differentiable in E , each partial derivatives $D_j f$ exist (Theorem 9.21). Hence Theorem 5.8 implies that $(D_j f)(\mathbf{x}) = 0$ for all $1 \leq j \leq n$. So

$$f'(\mathbf{x}) = [(D_1 f)(\mathbf{x}) \cdots (D_n f)(\mathbf{x})] = [0 \cdots 0] = 0$$

(as the zero matrix).

□

Exercise 9.9. If \mathbf{f} is a differentiable mapping of a connected open set $E \subseteq \mathbb{R}^n$, and if $\mathbf{f}'(\mathbf{x}) = 0$ for every $\mathbf{x} \in E$, prove that \mathbf{f} is a constant in E .

Proof.

- (1) Show that \mathbf{f} is **locally constant**. Given any $\mathbf{x} \in E$. Since E is open, there exists an open neighborhood $B(\mathbf{x}; r)$ of \mathbf{x} such that $B(\mathbf{x}; r) \subseteq E$ and $r > 0$. Corollary to Theorem 9.19 implies that \mathbf{f} is a constant on $B(\mathbf{x}; r)$, that is, \mathbf{f} is locally constant.
- (2) Show that \mathbf{f} is constant if \mathbf{f} is locally constant in a connected set $E \subseteq \mathbb{R}^n$. Might assume that $E \neq \emptyset$. (Otherwise there is nothing to do.) Take some $\mathbf{x}_0 \in E$.

(a) Let

$$U = \{\mathbf{y} \in E : \mathbf{f}(\mathbf{y}) = \mathbf{f}(\mathbf{x}_0)\}.$$

- (b) U is open since \mathbf{f} is locally constant (by (1)). (Take any $\mathbf{y} \in U$. Since \mathbf{f} is locally constant, there is an open neighborhood $B(\mathbf{y}) \subseteq E$ of \mathbf{y} such that $\mathbf{f}(\mathbf{z}) = \mathbf{f}(\mathbf{y}) = \mathbf{f}(\mathbf{x}_0)$ whenever $\mathbf{z} \in B(\mathbf{y})$. So that $B(\mathbf{y}) \subseteq U$, or U is open.)
- (c) Besides, since \mathbf{f} is continuous (Remarks 9.13(c)), the set U is closed. (The proof is the same as Proof (Definition 2.18(d)) in Exercise 4.3.)
- (d) So U is open and closed. Write $E = U \cup (E - U)$. Here U and $E - U$ are both open and closed. Hence $U \cap \overline{E - U} = U \cap (E - U) = \emptyset$ and $\overline{U} \cap (E - U) = U \cap (E - U) = \emptyset$. Note that $\mathbf{x}_0 \in U \neq \emptyset$. By the connectedness of E , $E - U = \emptyset$, or $E = U$, or \mathbf{f} is constant on E .

Note. The only subsets of a connected set E which are both open and closed are E and \emptyset .

□

Exercise 9.10. If f is a real function defined in a convex open set $E \subseteq \mathbb{R}^n$, such that $(D_1 f)(\mathbf{x}) = 0$ for every $\mathbf{x} \in E$, prove that $f(\mathbf{x})$ depends only on x_2, \dots, x_n . Show that the convexity of E can be replaced by a weaker condition, but that some condition is required. For example, if $n = 2$ and E is shaped like

a horseshoe, the statement may be false.

Proof.

- (1) It suffices to show that

$$f(a, x_2, \dots, x_n) = f(b, x_2, \dots, x_n)$$

whenever $\mathbf{x} = (a, x_2, \dots, x_n) \in E$ and $\mathbf{y} = (b, x_2, \dots, x_n) \in E$ if $(D_1 f)(\mathbf{x}) = 0$ in the convex open set E .

- (2) Might assume that $a < b$. Since $g : t \mapsto f(t, x_2, \dots, x_n)$ is a real continuous function on $[a, b]$ (by the openness of E) and differentiable in (a, b) (by the existence of $D_1 f$),

$$g(b) - g(a) = (b - a)g'(\xi)$$

for some $\xi \in (a, b)$. Note that

$$g'(\xi) = (D_1 f)(\xi, x_2, \dots, x_n) = 0$$

by assumption. $g(b) = g(a)$ or $f(a, x_2, \dots, x_n) = f(b, x_2, \dots, x_n)$.

- (3) (2) shows that the convexity of E can be replaced by a weaker condition that $E \subseteq \mathbb{R}^n$ is convex in the first coordinate, say E is open and

$$\lambda \mathbf{x} + (1 - \lambda) \mathbf{y} = (\lambda a + (1 - \lambda)b, x_2, \dots, x_n) \in E$$

whenever $\mathbf{x} = (a, x_2, \dots, x_n) \in E$, $\mathbf{y} = (b, x_2, \dots, x_n) \in E$, and $0 < \lambda < 1$.

- (4) Show that the convexity of E or some weaker condition is required. Define $f(x, y) = \operatorname{sgn}(x)$ on $E = \{(x, y) \in \mathbb{R}^2 : x \neq 0\}$. E is open and $(D_1 f)(x, y) = 0$ in E . Note that $f(1989, 0) = 1$ and $f(-64, 0) = -1$, and thus $f(x, y)$ does not depend only on $y = 0$.

□

Exercise 9.11. If f and g are differentiable real functions in \mathbb{R}^n , prove that

$$\nabla(fg) = f\nabla g + g\nabla f$$

and that

$$\nabla\left(\frac{1}{f}\right) = -\frac{1}{f^2}\nabla f$$

whenever $f \neq 0$.

Proof. Recall Example 9.18:

$$(\nabla(f))(\mathbf{x}) = \sum_{i=1}^n (D_i f)(\mathbf{x}) \mathbf{e}_i.$$

(1) Show that $\nabla(fg) = f\nabla g + g\nabla f$. For any $\mathbf{x} \in \mathbb{R}^n$,

$$\begin{aligned}
(\nabla(fg))(\mathbf{x}) &= \sum_{i=1}^n (D_i(fg))(\mathbf{x}) \mathbf{e}_i \\
&= \sum_{i=1}^n (g(D_i f) + f(D_i g))(\mathbf{x}) \mathbf{e}_i && \text{(Theorem 5.3(b))} \\
&= \sum_{i=1}^n [g(\mathbf{x})(D_i f)(\mathbf{x}) + f(\mathbf{x})(D_i g)(\mathbf{x})] \mathbf{e}_i \\
&= g(\mathbf{x}) \sum_{i=1}^n (D_i f)(\mathbf{x}) \mathbf{e}_i + f(\mathbf{x}) \sum_{i=1}^n (D_i g)(\mathbf{x}) \mathbf{e}_i \\
&= g(\mathbf{x})(\nabla f)(\mathbf{x}) + f(\mathbf{x})(\nabla g)(\mathbf{x}) \\
&= (f\nabla g + g\nabla f)(\mathbf{x}).
\end{aligned}$$

(2) Show that

$$\nabla \left(\frac{1}{f} \right) = -\frac{1}{f^2} \nabla f$$

whenever $f \neq 0$. Note that $\nabla(1) = 0$ since

$$\nabla(1)(\mathbf{x}) = \sum (D_i 1)(\mathbf{x}) \mathbf{e}_i = \sum (0)(\mathbf{x}) \mathbf{e}_i = \sum 0 \mathbf{e}_i = 0.$$

Hence as $f \neq 0$, we have

$$\begin{aligned}
0 &= \nabla(1) \\
&= \nabla \left(f \frac{1}{f} \right) && (f \neq 0) \\
&= f \nabla \left(\frac{1}{f} \right) + \frac{1}{f} \nabla f && ((1)),
\end{aligned}$$

$$\text{or } \nabla \left(\frac{1}{f} \right) = -\frac{1}{f^2} \nabla f.$$

□

Exercise 9.12. Fix two real numbers a and b , $0 < a < b$. Define a mapping $\mathbf{f} = (f_1, f_2, f_3)$ of \mathbb{R}^2 into \mathbb{R}^3 by

$$\begin{aligned}
f_1(s, t) &= (b + a \cos s) \cos t \\
f_2(s, t) &= (b + a \cos s) \sin t \\
f_3(s, t) &= a \sin s.
\end{aligned}$$

Describe the range K of \mathbf{f} . (It is a certain compact subset of \mathbb{R}^3 .)

- (a) Show that there are exactly 4 points $\mathbf{p} \in K$ such that

$$(\nabla f_1)(\mathbf{f}^{-1}(\mathbf{p})) = \mathbf{0}.$$

Find these points.

- (b) Determine the set of all $\mathbf{q} \in K$ such that

$$(\nabla f_3)(\mathbf{f}^{-1}(\mathbf{q})) = \mathbf{0}.$$

- (c) Show that one of the point \mathbf{p} found in part (a) corresponds to a local maximum of f_1 , one corresponds to a local minimum, and that the other two are neither (they are so-called “saddle points”). Which of the points \mathbf{q} found in part (b) corresponds to maxima or minima?
- (d) Let λ be an irrational real number, and define $\mathbf{g}(t) = \mathbf{f}(t, \lambda t)$. Prove that \mathbf{g} is a one-to-one mapping of \mathbb{R}^1 onto a dense subset of K . Prove that

$$|\mathbf{g}'(t)|^2 = a^2 + \lambda^2(b + a \cos t)^2.$$

Proof.

- (1) K is a torus, where
- (a) s, t are angles which make a full circle (so that their values start and end at the same point).
 - (b) b is the distance from the center of the tube to the center of the torus.
 - (c) a is the radius of the tube.
- (2) Show that K is compact. Since \sin and \cos are periodic (with period 2π), $K = \mathbf{f}([0, 2\pi]^2)$ is compact by the compactness of $[0, 2\pi]^2$ and the continuity of \mathbf{f} (Theorem 4.14).

□

Proof of (a).

- (1)

$$\begin{aligned} (\nabla f_1)(\mathbf{x}) &= (D_1 f_1)(\mathbf{x})\mathbf{e}_1 + (D_2 f_1)(\mathbf{x})\mathbf{e}_2 \\ &= ((D_1 f_1)(s, t), (D_2 f_1)(s, t)) \\ &= (-a \sin s \cos t, -(b + a \cos t) \sin t) \end{aligned}$$

So $(\nabla f_1)(\mathbf{x}) = \mathbf{0}$ if and only if

$$\begin{aligned} 0 &= -a \sin s \cos t, \\ 0 &= -(b + a \cos t) \sin t. \end{aligned}$$

- (2) Note that $b + a \cos t > 0$ for any $b > a > 0$ and $t \in \mathbb{R}^1$. Hence $(\nabla f_1)(\mathbf{x}) = \mathbf{0}$ if and only if $\sin t = \sin s = 0$. Therefore, $\mathbf{p} = (\pm(b \pm a), 0, 0)$, or there are exactly 4 points $\mathbf{p} = (b + a, 0, 0)$, $(b - a, 0, 0)$, $(-b - a, 0, 0)$, or $(-b + a, 0, 0) \in K$.

□

Proof of (b).

(1)

$$\begin{aligned} (\nabla f_3)(\mathbf{x}) &= (D_1 f_3)(\mathbf{x})\mathbf{e}_1 + (D_2 f_3)(\mathbf{x})\mathbf{e}_2 \\ &= ((D_1 f_3)(s, t), (D_2 f_3)(s, t)) \\ &= (a \cos s, 0) \end{aligned}$$

So $(\nabla f_1)(\mathbf{x}) = \mathbf{0}$ if and only if $\cos s = 0$ (since $a > 0$).

- (2) Therefore, $\mathbf{q} = (b \cos t, b \sin t, \pm a)$.

□

Proof of (c).

- (1) Since $-1 \leq \cos s \leq 1$ and $-1 \leq \cos t \leq 1$, $-b - a \leq f_1(s, t) \leq b + a$.

- (a) $(b + a, 0, 0)$ corresponds to a local maximum of f_1 .
- (b) $(-b - a, 0, 0)$ corresponds to a local minimum of f_1 .
- (c) $(b - a, 0, 0)$ and $(-b + a, 0, 0)$ are saddle points by considering any open neighborhood of (s, t) at which $\cos s = \pm 1$ and $\cos t = \mp 1$.

- (2) Since $-1 \leq \sin s \leq 1$, $-a \leq f_3(s, t) \leq a$.

- (a) $(b \cos t, b \sin t, a)$ corresponds to a local maximum of f_3 .
- (b) $(b \cos t, b \sin t, -a)$ corresponds to a local minimum of f_3 .

□

Proof of (d).

(1)

$$\mathbf{g}(t) = \mathbf{f}(t, \lambda t) = ((b + a \cos t) \cos(\lambda t), (b + a \cos t) \sin(\lambda t), a \sin t).$$

- (2) Show that \mathbf{g} is a one-to-one mapping of \mathbb{R}^1 . It suffices to show that $\mathbf{g}(t) = \mathbf{g}(s)$ implies $t = s$.

(a) By $\mathbf{g}(t) = \mathbf{g}(s)$,

$$(b + a \cos t) \cos(\lambda t) = (b + a \cos s) \cos(\lambda s), \quad (\text{I})$$

$$(b + a \cos t) \sin(\lambda t) = (b + a \cos s) \sin(\lambda s), \quad (\text{II})$$

$$a \sin t = a \sin s. \quad (\text{III})$$

(I) and (II) imply that $\cos t = \cos s$ (since $b > a > 0$). (III) implies that $\sin t = \sin s$. Hence

$$t = s + 2n\pi$$

for some integer n .

(b) Again, (I) and (II) imply that

$$\cos(\lambda t) = \cos(\lambda s) \quad \text{and} \quad \sin(\lambda t) = \sin(\lambda s).$$

Hence

$$\lambda t = \lambda s + 2m\pi$$

for some integer m . By assumption that $t = s + 2n\pi$, we have $m = n\lambda$. Since λ is irrational, $m = n = 0$. Therefore $t = s$ holds.

(3) Show that $\mathbf{g}(\mathbb{R}^1)$ is dense in K . Note that $\mathbf{f}([0, 2\pi]^2) = K$. Use the notations $\{x\}$ in Exercise 4.16. It suffices to show that the set

$$\left\{ \left(2\pi \left\{ \frac{t}{2\pi} \right\}, 2\pi \left\{ \frac{\lambda t}{2\pi} \right\} \right) : t \in \mathbb{R}^1 \right\}$$

is dense in $[0, 2\pi]^2$ (Exercise 4.4), or to show that

$$\{(\{t\}, \{\lambda t\}) : t \in \mathbb{R}^1\}$$

is dense in $[0, 1]^2$, which is the conclusion of Exercise 4.25(b).

(4) Show that $|\mathbf{g}'(t)|^2 = a^2 + \lambda^2(b + a \cos t)^2$. By

$$\begin{aligned} \mathbf{g}'(t) = & \begin{pmatrix} -a \sin t \cos(\lambda t) - \lambda(b + a \cos t) \sin(\lambda t), \\ -a \sin t \sin(\lambda t) + \lambda(b + a \cos t) \cos(\lambda t), \\ a \cos t \end{pmatrix}, \end{aligned}$$

$$\begin{aligned}
|\mathbf{g}'(t)|^2 &= \mathbf{g}'(t) \cdot \mathbf{g}'(t) \\
&= (-a \sin t \cos(\lambda t) - \lambda(b + a \cos t) \sin(\lambda t))^2 \\
&\quad + (-a \sin t \sin(\lambda t) + \lambda(b + a \cos t) \cos(\lambda t))^2 + (a \cos t)^2 \\
&= \underbrace{a^2 \sin^2 t \cos^2(\lambda t) + a^2 \cos^2 t}_{=a^2} \\
&\quad + \underbrace{\lambda^2(b + a \cos t)^2 \sin^2(\lambda t) + \lambda^2(b + a \cos t)^2 \cos^2(\lambda t)}_{=\lambda^2(b+a \cos t)^2} \\
&\quad + 2a\lambda \sin t \cos(\lambda t) \lambda(b + a \cos t) \sin(\lambda t) \\
&\quad - 2a\lambda \sin t \sin(\lambda t) \lambda(b + a \cos t) \cos(\lambda t) \\
&= a^2 + \lambda^2(b + a \cos t)^2.
\end{aligned}$$

□

Exercise 9.13. Suppose \mathbf{f} is a differentiable mapping of \mathbb{R}^1 into \mathbb{R}^3 such that $|\mathbf{f}(t)| = 1$ for every t . Prove that $\mathbf{f}'(t) \cdot \mathbf{f}(t) = 0$. Interpret this result geometrically.

Proof.

- (1) Write $\mathbf{f} = (f_1, f_2, f_3)$ as a vector-valued function. By Remarks 5.16, \mathbf{f} is differentiable if and only if each f_1, f_2, f_3 is differentiable. So $\mathbf{f}' = (f'_1, f'_2, f'_3)$. Hence

$$\begin{aligned}
|\mathbf{f}(t)| &= 1 \text{ for every } t \\
\iff \mathbf{f}(t) \cdot \mathbf{f}(t) &= 1 \\
\iff f_1(t)^2 + f_2(t)^2 + f_3(t)^2 &= 1 \\
\implies 2f_1(t)f'_1(t) + 2f_2(t)f'_2(t) + 2f_3(t)f'_3(t) &= 0 \\
\iff f_1(t)f'_1(t) + f_2(t)f'_2(t) + f_3(t)f'_3(t) &= 0 \\
\iff (f_1(t), f_2(t), f_3(t)) \cdot (f'_1(t), f'_2(t), f'_3(t)) &= 0 \\
\iff \mathbf{f}(t) \cdot \mathbf{f}'(t) = \mathbf{f}'(t) \cdot \mathbf{f}(t) &= 0.
\end{aligned}$$

- (2) The vector $\mathbf{f}'(t)$ is called the **tangent vector** (or **velocity vector**) of \mathbf{f} at t . Geometrically, given any mapping \mathbf{f} lying on the sphere S^2 , its tangent vector at t is lying on the tangent plane of S^2 at t .

□

Exercise 9.14. Define $f(0, 0) = 0$ and

$$f(x, y) = \frac{x^3}{x^2 + y^2} \quad \text{if } (x, y) \neq (0, 0).$$

- (a) Prove that D_1f and D_2f are bounded functions in \mathbb{R}^2 . (Hence f is continuous.)
- (b) Let \mathbf{u} be any unit vector in \mathbb{R}^2 . Show that the directional derivative $(D_{\mathbf{u}}f)(0,0)$ exists, and that its absolute value is at most 1.
- (c) Let γ be a differentiable mapping of \mathbb{R}^1 into \mathbb{R}^2 (in other words, γ is a differentiable curve in \mathbb{R}^2), with $\gamma(t) = (0,0)$ and $\gamma'(t) \neq (0,0)$ for any $t \in \mathbb{R}^1$. Put $g(t) = f(\gamma(t))$ and prove that g is differentiable for every $t \in \mathbb{R}^1$. If $\gamma \in \mathcal{C}'$, prove that $g \in \mathcal{C}'$.
- (d) In spite of this, prove that f is not differentiable at $(0,0)$.

Proof of (a).

- (1) Show that

$$(D_1f)(x,y) = \begin{cases} 1 & \text{if } (x,y) = (0,0), \\ \frac{x^2(x^2+3y^2)}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0). \end{cases}$$

If $(x,y) = (0,0)$,

$$(D_1f)(0,0) = \lim_{t \rightarrow 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t \rightarrow 0} \frac{t-0}{t} = 1.$$

If $(x,y) \neq (0,0)$,

$$\begin{aligned} (D_1f)(x,y) &= \lim_{t \rightarrow 0} \frac{f(x+t,y) - f(x,y)}{t} \\ &= \lim_{t \rightarrow 0} \frac{\frac{(x+t)^3}{(x+t)^2+y^2} - \frac{x^3}{x^2+y^2}}{t} \\ &= \lim_{t \rightarrow 0} \frac{x^2(x^2+3y^2) + tx(2x^2+3y^2) + t^2(x^2+y^2)}{((x+t)^2+y^2)(x^2+y^2)} \\ &= \frac{x^2(x^2+3y^2)}{(x^2+y^2)^2}. \end{aligned}$$

(Or differentiate directly.)

- (2) Show that $(D_1f)(x,y)$ is bounded. It suffices to show that $(D_1f)(x,y)$ is bounded if $(x,y) \neq (0,0)$. Write $x = r \cos \theta$ and $y = r \sin \theta$ in the polar coordinates. (Here $r > 0$.) Hence

$$(D_1f)(x,y) = \frac{x^2(x^2+3y^2)}{(x^2+y^2)^2} = \cos^2 \theta (\cos^2 \theta + 3 \sin^2 \theta)$$

is bounded by $1 \cdot (1+3) = 4$.

(3) Show that

$$(D_2f)(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0), \\ \frac{-2x^3y}{(x^2+y^2)^2} & \text{if } (x, y) \neq (0, 0). \end{cases}$$

If $(x, y) = (0, 0)$,

$$(D_2f)(0, 0) = \lim_{t \rightarrow 0} \frac{f(0, t) - f(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{0 - 0}{t} = 0.$$

If $(x, y) \neq (0, 0)$,

$$\begin{aligned} (D_2f)(x, y) &= \lim_{t \rightarrow 0} \frac{f(x, y+t) - f(x, y)}{t} \\ &= \lim_{t \rightarrow 0} \frac{\frac{x^3}{x^2+(y+t)^2} - \frac{x^3}{x^2+y^2}}{t} \\ &= \lim_{t \rightarrow 0} \frac{-2x^3y - tx^3}{(x^2 + (y+t)^2)(x^2 + y^2)} \\ &= \frac{-2x^3y}{(x^2 + y^2)^2}. \end{aligned}$$

(Or differentiate directly.)

(4) Show that $(D_2f)(x, y)$ is bounded. Similar to (2).

(5) Show that f is continuous. Apply Exercise 9.7 to (2)(4).

□

Proof of (b).

(1) Write $\mathbf{u} = (u_1, u_2)$. The formula

$$(D_{\mathbf{u}}f)(0, 0) = (D_1f)(0, 0)u_1 + (D_2f)(0, 0)u_2 = u_1$$

might be false since we don't know if f is differentiable or not. Actually, we will show that $(D_{\mathbf{u}}f)(0, 0) = u_1^3 \neq u_1$.

(2)

$$\begin{aligned} (D_{\mathbf{u}}f)(0, 0) &= \lim_{t \rightarrow 0} \frac{f(tu_1, tu_2) - f(0, 0)}{t} \\ &= \lim_{t \rightarrow 0} \frac{\frac{t^3 u_1^3}{t^2 u_1^2 + t^2 u_2^2} - 0}{t} \\ &= \lim_{t \rightarrow 0} u_1^3 \quad (|\mathbf{u}| = 1) \\ &= u_1^3. \end{aligned}$$

Also $|(D_{\mathbf{u}}f)(0, 0)| = |u_1|^3 \leq 1$ since $|\mathbf{u}| = 1$.

□

Proof of (c).

(1) Given any $t \in \mathbb{R}^1$.

$$g'(t) = \lim_{x \rightarrow t} \frac{g(x) - g(t)}{x - t} = \lim_{x \rightarrow t} \frac{f(\gamma(x)) - f(\gamma(t))}{x - t}.$$

Write $\gamma(t) = (\gamma_1(t), \gamma_2(t))$.

(2) Suppose that $\gamma(t) \neq (0, 0)$. Since γ is differentiable, γ is continuous. So there exists an open neighborhood $B(t) \subseteq \mathbb{R}^1$ of t such that $\gamma(x) \neq (0, 0)$ whenever $x \in B(t)$. Hence

$$\begin{aligned} g'(t) &= \lim_{x \rightarrow t} \frac{\frac{\gamma_1(x)^3}{\gamma_1(x)^2 + \gamma_2(x)^2} - \frac{\gamma_1(t)^3}{\gamma_1(t)^2 + \gamma_2(t)^2}}{x - t} \\ &= \frac{d}{dt} \left(\frac{\gamma_1(t)^3}{\gamma_1(t)^2 + \gamma_2(t)^2} \right) \\ &= \frac{3\gamma_1(t)^2 \gamma_1'(t)}{\gamma_1(t)^2 + \gamma_2(t)^2} - \frac{\gamma_1(t)^3 (2\gamma_1(t) \gamma_1'(t) + 2\gamma_2(t) \gamma_2'(t))}{(\gamma_1(t)^2 + \gamma_2(t)^2)^2}. \end{aligned}$$

exists since γ_1 and γ_2 are differentiable.

(3) Suppose that $\gamma(t) = (0, 0)$ and thus $\gamma'(t) \neq (0, 0)$. So

$$g'(t) = \lim_{x \rightarrow t} \frac{f(\gamma(x))}{x - t}$$

Note that $\gamma(x) \neq (0, 0)$ in some open neighborhood of t since

$$\lim_{\substack{x \rightarrow t \\ \gamma(x) = (0, 0)}} \frac{\gamma(x) - \gamma(t)}{x - t} = (0, 0),$$

contrary to the assumption that $\gamma'(t) \neq (0, 0)$. Note that $\gamma_1(t) = \gamma_2(t) = 0$. So

$$\begin{aligned} g'(t) &= \lim_{x \rightarrow t} \frac{f(\gamma(x))}{x - t} \\ &= \lim_{x \rightarrow t} \frac{\gamma_1(x)^3}{\gamma_1(x)^2 + \gamma_2(x)^2} \cdot \frac{1}{x - t} \\ &= \lim_{x \rightarrow t} \frac{(\gamma_1(x) - \gamma_1(t))^3}{(\gamma_1(x) - \gamma_1(t))^2 + (\gamma_2(x) - \gamma_2(t))^2} \cdot \frac{1}{x - t} \\ &= \lim_{x \rightarrow t} \frac{\left(\frac{\gamma_1(x) - \gamma_1(t)}{x - t} \right)^3}{\left(\frac{\gamma_1(x) - \gamma_1(t)}{x - t} \right)^2 + \left(\frac{\gamma_2(x) - \gamma_2(t)}{x - t} \right)^2} \\ &= \frac{\gamma_1'(t)^3}{\gamma_1'(t)^2 + \gamma_2'(t)^2} \end{aligned}$$

since $\gamma'(t) \neq (0, 0)$.

(4) By (2)(3), $g'(t)$ exists and

$$g'(t) = \begin{cases} \frac{3\gamma_1(t)^2\gamma_1'(t)}{\gamma_1(t)^2 + \gamma_2(t)^2} - \frac{\gamma_1(t)^3(2\gamma_1(t)\gamma_1'(t) + 2\gamma_2(t)\gamma_2'(t))}{(\gamma_1(t)^2 + \gamma_2(t)^2)^2} & \text{if } \gamma(t) \neq (0, 0), \\ \frac{\gamma_1'(t)^3}{\gamma_1'(t)^2 + \gamma_2'(t)^2} & \text{if } \gamma(t) = (0, 0). \end{cases}$$

(5) Now suppose $\gamma \in \mathcal{C}'$. To show $g' \in \mathcal{C}'$, it suffices to show that

$$\lim_{x \rightarrow t} g'(x) = g'(t)$$

if $\gamma(t) = (0, 0)$ since $g'(t)$ is always continuous if $\gamma(t) \neq (0, 0)$. Here all $\gamma_1, \gamma_2, \gamma_1', \gamma_2'$ are continuous and $\gamma_1(t)^2 + \gamma_2(t)^2 \neq 0$ by assumption. So

$$\begin{aligned} & \lim_{x \rightarrow t} \frac{3\gamma_1(x)^2\gamma_1'(x)}{\gamma_1(x)^2 + \gamma_2(x)^2} \\ &= \lim_{x \rightarrow t} \frac{3 \left(\frac{\gamma_1(x) - \gamma_1(t)}{x - t} \right)^2 \gamma_1'(x)}{\left(\frac{\gamma_1(x) - \gamma_1(t)}{x - t} \right)^2 + \left(\frac{\gamma_2(x) - \gamma_2(t)}{x - t} \right)^2} \\ &= \frac{3\gamma_1'(t)^2 \cdot \gamma_1'(t)}{\gamma_1'(t)^2 + \gamma_2'(t)^2} \\ &= \frac{3\gamma_1'(t)^3}{\gamma_1'(t)^2 + \gamma_2'(t)^2} \end{aligned}$$

and similarly

$$\begin{aligned} & \lim_{x \rightarrow t} \frac{\gamma_1(t)^3(2\gamma_1(t)\gamma_1'(t) + 2\gamma_2(t)\gamma_2'(t))}{(\gamma_1(t)^2 + \gamma_2(t)^2)^2} \\ &= \lim_{x \rightarrow t} \frac{\left(\frac{\gamma_1(x) - \gamma_1(t)}{x - t} \right)^3 \left(2\frac{\gamma_1(x) - \gamma_1(t)}{x - t} \gamma_1'(t) + 2\frac{\gamma_2(x) - \gamma_2(t)}{x - t} \gamma_2'(t) \right)}{\left(\left(\frac{\gamma_1(x) - \gamma_1(t)}{x - t} \right)^2 + \left(\frac{\gamma_2(x) - \gamma_2(t)}{x - t} \right)^2 \right)^2} \\ &= \frac{\gamma_1'(t)^3 \cdot (2\gamma_1'(t)\gamma_1'(t) + 2\gamma_2'(t)\gamma_2'(t))}{(\gamma_1'(t)^2 + \gamma_2'(t)^2)^2} \\ &= \frac{2\gamma_1'(t)^3}{\gamma_1'(t)^2 + \gamma_2'(t)^2}. \end{aligned}$$

Hence

$$\lim_{x \rightarrow t} g'(x) = \frac{3\gamma_1'(t)^3}{\gamma_1'(t)^2 + \gamma_2'(t)^2} - \frac{2\gamma_1'(t)^3}{\gamma_1'(t)^2 + \gamma_2'(t)^2} = g'(t).$$

□

Proof of (d). (Reductio ad absurdum) If f were differentiable, then

$$(D_{\mathbf{u}}f)(0, 0) = (D_1f)(0, 0)u_1 + (D_2f)(0, 0)u_2 = u_1$$

(Formula (40) in Chapter 9), contrary to (b) if we take $\mathbf{u} = \left(\frac{1}{64}, \frac{\sqrt{4095}}{64}\right)$. \square

Exercise 9.15. Define $f(0, 0) = 0$, and put

$$f(x, y) = x^2 + y^2 - 2x^2y - \frac{4x^6y^2}{(x^4 + y^2)^2}$$

if $(x, y) \neq (0, 0)$.

(a) Prove, for all $(x, y) \in \mathbb{R}^2$, that

$$4x^4y^2 \leq (x^4 + y^2)^2.$$

Conclude that f is continuous.

(b) For $0 \leq \theta \leq 2\pi$, $-\infty < t < \infty$, define

$$g_\theta(t) = f(t \cos \theta, t \sin \theta).$$

Show that $g_\theta(0) = 0$, $g'_\theta(0) = 0$, $g''_\theta(0) = 2$. Each g_θ has therefore a strict local minimum at $t = 0$. In other words, the restriction of f to each line through $(0, 0)$ has a strict local minimum at $(0, 0)$.

(c) Show that $(0, 0)$ is nevertheless not a local minimum for f , since $f(x, x^2) = -x^4$.

Proof of (a).

(1) Since $t^2 \geq 0$ for all $t \in \mathbb{R}^1$,

$$(x^4 + y^2)^2 - 4x^4y^2 = (x^4 - y^2)^2 \geq 0.$$

Hence $4x^4y^2 \leq (x^4 + y^2)^2$.

(2) $f(x, y)$ is continuous at $(x, y) \neq (0, 0)$. Besides,

$$\begin{aligned} |f(x, y)| &= \left| x^2 + y^2 - 2x^2y - \frac{4x^6y^2}{(x^4 + y^2)^2} \right| \\ &\leq |x^2| + |y^2| + |2x^2y| + |x^2| \left| \frac{4x^4y^2}{(x^4 + y^2)^2} \right| \\ &\leq |x^2| + |y^2| + |2x^2y| + |x^2|. \end{aligned}$$

Hence $|x^2| + |y^2| + |2x^2y| + |x^2| \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$, or

$$\lim_{(x, y) \rightarrow (0, 0)} |f(x, y)| = 0 = f(0, 0),$$

or $\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = f(0, 0)$, or $f(x, y)$ is continuous at $(0, 0)$.

□

Proof of (b).

(1)

$$g_\theta(t) = \begin{cases} t^2 - 2t^3 \cos^2 \theta \sin \theta - \frac{4t^4 \cos^6 \theta \sin^2 \theta}{(t^2 \cos^4 \theta + \sin^2 \theta)^2} & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

(Note that $\frac{4t^4 \cos^6 \theta \sin^2 \theta}{(t^2 \cos^4 \theta + \sin^2 \theta)^2}$ is undefined as $t = 0$ and $\sin \theta = 0$.)

(2) $g_\theta(0) = 0$ by definition.

(3) Show that $g'_\theta(0) = 0$ for any $\theta \in [0, 2\pi]$. If $\sin \theta \neq 0$ ($\theta \neq 0, \pi, 2\pi$), then

$$\begin{aligned} g'_\theta(0) &= \lim_{t \rightarrow 0} \frac{t^2 - 2t^3 \cos^2 \theta \sin \theta - \frac{4t^4 \cos^6 \theta \sin^2 \theta}{(t^2 \cos^4 \theta + \sin^2 \theta)^2} - 0}{t} \\ &= \lim_{t \rightarrow 0} \left(t - 2t^2 \cos^2 \theta \sin \theta - \frac{4t^3 \cos^6 \theta \sin^2 \theta}{(t^2 \cos^4 \theta + \sin^2 \theta)^2} \right) \\ &= 0. \end{aligned}$$

If $\sin \theta = 0$, then

$$g'_\theta(0) = \lim_{t \rightarrow 0} \frac{t^2 - 0}{t} = \lim_{t \rightarrow 0} t = 0.$$

(4) Combine (3) and a direct calculation for the case $t \neq 0$, we have

$$g'_\theta(t) = \begin{cases} 2t - 6t^2 \cos^2 \theta \sin \theta - \frac{16t^3 \cos^6 \theta \sin^4 \theta}{(t^2 \cos^4 \theta + \sin^2 \theta)^3} & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

(5) Show that $g''_\theta(0) = 2$ for any $\theta \in [0, 2\pi]$. If $\sin \theta \neq 0$ ($\theta \neq 0, \pi, 2\pi$), then

$$\begin{aligned} g''_\theta(0) &= \lim_{t \rightarrow 0} \frac{2t - 6t^2 \cos^2 \theta \sin \theta - \frac{16t^3 \cos^6 \theta \sin^4 \theta}{(t^2 \cos^4 \theta + \sin^2 \theta)^3} - 0}{t} \\ &= \lim_{t \rightarrow 0} \left(t - 6t \cos^2 \theta \sin \theta - \frac{16t^2 \cos^6 \theta \sin^4 \theta}{(t^2 \cos^4 \theta + \sin^2 \theta)^3} \right) \\ &= 2. \end{aligned}$$

If $\sin \theta = 0$, then

$$g''_\theta(0) = \lim_{t \rightarrow 0} \frac{2t - 0}{t} = \lim_{t \rightarrow 0} 2 = 2.$$

(6) Since $g''_\theta(0) > 0$ and $g'_\theta(0) = 0$, g_θ has a strict local minimum at $t = 0$. As θ is fixed, f is restricted to some line through $(0, 0)$. Hence, such restriction of f has a strict local minimum at $t = 0$.

□

Proof of (c). Since $f(x, x^2) = -x^4 \leq 0 = f(0, 0)$ in any open neighborhood of $(0, 0)$, $f(0, 0) = 0$ cannot be a local minimum for f . □

Exercise 9.16. Show that the continuity of \mathbf{f}' at the point \mathbf{a} is needed in the inverse function theorem, even in the case $n = 1$: If

$$f(t) = t + 2t^2 \sin \frac{1}{t}$$

for $t \neq 0$, and $f(0) = 0$, then $f'(0) = 1$, f' is bounded in $(-1, 1)$, but f is not one-to-one in any neighborhood of 0.

Proof.

(1) Show that

$$f'(t) = \begin{cases} 1 + 4t \sin \frac{1}{t} - 2 \cos \frac{1}{t} & \text{if } t \neq 0, \\ 1 & \text{if } t = 0. \end{cases}$$

It suffices to show that $f'(0) = 1$. In fact,

$$f'(0) = \lim_{t \rightarrow 0} \frac{t + 2t^2 \sin \frac{1}{t} - 0}{t - 0} = \lim_{t \rightarrow 0} \left(1 + 2t \sin \frac{1}{t} \right) = 1$$

(since $\sin \frac{1}{t}$ is bounded and $2t \rightarrow 0$ as $t \rightarrow 0$).

Note. $f'(t)$ is not continuous at $t = 0$.

(2) Show that f' is bounded in $(-1, 1)$.

$$|f'(t)| \leq 1 + 4|t| \left| \sin \frac{1}{t} \right| + 2 \left| \cos \frac{1}{t} \right| \leq 1 + 4 + 2 = 7$$

if $t \neq 0$. Hence f' is bounded by 7 in $(-1, 1)$.

(3) Show that f is not one-to-one in any neighborhood of 0. Take

$$x_n = \frac{1}{2n\pi} \quad \text{and} \quad y_n = \frac{1}{2n\pi + \pi}$$

for $n = 1, 2, 3, \dots$. So that

$$f'(x_n) = -1 < 0 \quad \text{and} \quad f'(y_n) = 3 > 0.$$

Since $f'(t)$ is continuous if $t \neq 0$, there exists $\xi_n \in (y_n, x_n)$ such that $f'(\xi_n) = 0$ (Theorem 4.23). Then Theorem 5.11 implies that f has a local maximum at ξ_n , that is, f is not one-to-one in the interval $[y_n, x_n]$ (by applying Theorem 4.23 again). Since $x_n \rightarrow 0$ and $y_n \rightarrow 0$ as $n \rightarrow \infty$, f is not one-to-one in any neighborhood of 0.

□

Exercise 9.17. Let $\mathbf{f} = (f_1, f_2)$ be the mapping of \mathbb{R}^2 into \mathbb{R}^2 given by

$$f_1(x, y) = e^x \cos y, \quad f_2(x, y) = e^x \sin y.$$

- (a) What is the range of \mathbf{f} ?
- (b) Show that the Jacobian of \mathbf{f} is not zero at any point of \mathbb{R}^2 . Thus every point of \mathbb{R}^2 has a neighborhood in which \mathbf{f} is one-to-one. Nevertheless, \mathbf{f} is not one-to-one on \mathbb{R}^2 .
- (c) Put $\mathbf{a} = (0, \frac{\pi}{3})$, $\mathbf{b} = \mathbf{f}(\mathbf{a})$, let \mathbf{g} be the continuous inverse of \mathbf{f} , defined in a neighborhood of \mathbf{b} such that $\mathbf{g}(\mathbf{b}) = \mathbf{a}$. Find an explicit formula for \mathbf{g} , compute $\mathbf{f}'(\mathbf{a})$ and $\mathbf{g}'(\mathbf{b})$, and verify the formula

$$\mathbf{g}'(\mathbf{y}) = \{\mathbf{f}'(\mathbf{g}(\mathbf{y}))\}^{-1}.$$

- (d) What are the images under \mathbf{f} of lines parallel to the coordinate axes?

Proof of (a).

- (1) The range of \mathbf{f} is $\mathbb{R}^2 - \{(0, 0)\}$.
- (2) If $(a, b) \neq (0, 0)$, then $\mathbf{f} : (\log \sqrt{a^2 + b^2}, \text{atan2}(b, a)) \mapsto (a, b)$ where

$$\text{atan2}(b, a) = \begin{cases} \arctan\left(\frac{b}{a}\right) & \text{if } a > 0, \\ \arctan\left(\frac{b}{a}\right) + \pi & \text{if } a < 0 \text{ and } b \geq 0, \\ \arctan\left(\frac{b}{a}\right) - \pi & \text{if } a < 0 \text{ and } b < 0, \\ \frac{\pi}{2} & \text{if } a = 0 \text{ and } b > 0, \\ -\frac{\pi}{2} & \text{if } a = 0 \text{ and } b < 0. \end{cases}$$

(Or apply Theorem 8.7(d).)

- (3) If $(a, b) = (0, 0)$, then for any $(x, y) \in \mathbb{R}^2$ we have $f_1(x, y)^2 + f_2(x, y)^2 = e^{2x} \neq 0$. So that there is no (x, y) such that $\mathbf{f} : (x, y) \mapsto (0, 0)$.

□

Proof of (b).

- (1)

$$[\mathbf{f}'(x, y)] = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}.$$

So \mathbf{f}' is continuous and

$$J_{\mathbf{f}}(x, y) = \det \mathbf{f}'(x, y) = e^{2x} \neq 0.$$

- (2) Since $J_{\mathbf{f}}(x, y) \neq 0$, $\mathbf{f}'(x, y)$ is invertible (Theorem 9.36). So the inverse function theorem (Theorem 9.24) implies that there exists an open neighborhood $B(x, y)$ of (x, y) such that \mathbf{f} is injective on $B(x, y)$.

- (3) Note that

$$\mathbf{f}(0, 0) = \mathbf{f}(0, 2\pi) = (1, 0).$$

So that \mathbf{f} is not injective on the whole \mathbb{R}^2 . (Injectivity of \mathbf{f} is a local property.)

□

Proof of (c).

- (1) If $\mathbf{a} = (0, \frac{\pi}{3})$, then $\mathbf{b} = \mathbf{f}(\mathbf{a}) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$.

- (2) Similar to (2) in the proof of (a), define $\mathbf{g} : E \rightarrow \mathbb{R}^2$ by

$$\mathbf{g}(x, y) = \left(\log \sqrt{x^2 + y^2}, \arctan \left(\frac{y}{x} \right) \right).$$

where E is some open neighborhood of the point $\mathbf{b} \in \mathbb{R}^2$ described in (b). So \mathbf{g} is a continuous inverse of \mathbf{f} .

- (3) Since

$$[\mathbf{f}'(x, y)] = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix},$$

$$[\mathbf{f}'(\mathbf{a})] = [\mathbf{f}'(0, \frac{\pi}{3})] = \begin{bmatrix} e^0 \cos \frac{\pi}{3} & -e^0 \sin \frac{\pi}{3} \\ e^0 \sin \frac{\pi}{3} & e^0 \cos \frac{\pi}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

- (4) Since

$$[\mathbf{g}'(x, y)] = \begin{bmatrix} \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix},$$

$$[\mathbf{g}'(\mathbf{b})] = \left[\mathbf{g}' \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right) \right] = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

Here we can see $[\mathbf{f}'(\mathbf{a})][\mathbf{g}'(\mathbf{b})] = [\mathbf{g}'(\mathbf{b})][\mathbf{f}'(\mathbf{a})] = 1$.

- (5)

$$\begin{aligned} [\mathbf{g}'(\mathbf{y})] &= [\mathbf{g}'(\mathbf{f}(\mathbf{x}))] \\ &= [\mathbf{g}'(e^x \cos y, e^x \sin y)] \\ &= \begin{bmatrix} \frac{e^x \cos y}{e^{2x}} & \frac{e^x \sin y}{e^{2x}} \\ \frac{-e^x \sin y}{e^{2x}} & \frac{e^x \cos y}{e^{2x}} \end{bmatrix} \\ &= \begin{bmatrix} e^{-x} \cos y & e^{-x} \sin y \\ -e^{-x} \sin y & e^{-x} \cos y \end{bmatrix}, \end{aligned}$$

and

$$[\mathbf{f}'(\mathbf{g}(\mathbf{y}))] = [\mathbf{f}'(\mathbf{x})] = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}.$$

Note that

$$\begin{bmatrix} e^{-x} \cos y & e^{-x} \sin y \\ -e^{-x} \sin y & e^{-x} \cos y \end{bmatrix} \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix} = 1.$$

Therefore

$$\mathbf{g}'(\mathbf{y}) = \{\mathbf{f}'(\mathbf{g}(\mathbf{y}))\}^{-1}$$

on $\mathbf{g}(E)$.

□

Proof of (d).

- (1) The case $L_r = \{(x, y) \in \mathbb{R}^2 : x = r\}$ parallel to y -axis where $r \in \mathbb{R}^1$ is constant. The image under \mathbf{f} is

$$\begin{aligned} \mathbf{f}(L_r) &= \{(e^r \cos y, e^r \sin y) \in \mathbb{R}^2 : y \in \mathbb{R}^1\} \\ &= \{(s, t) \in \mathbb{R}^2 : s^2 + t^2 = (e^r)^2\}, \end{aligned}$$

a circle which is centered at the origin $(0, 0) \in \mathbb{R}^2$ with radius $e^r > 0$.

- (2) The case $L_\theta = \{(x, y) \in \mathbb{R}^2 : y = \theta\}$ parallel to x -axis where $\theta \in \mathbb{R}^1$ is constant. The image under \mathbf{f} is

$$\begin{aligned} \mathbf{f}(L_\theta) &= \{(e^x \cos \theta, e^x \sin \theta) \in \mathbb{R}^2 : x \in \mathbb{R}^1\} \\ &= \{(y \cos \theta, y \sin \theta) \in \mathbb{R}^2 : y > 0\}, \end{aligned}$$

which is a ray from the origin $(0, 0)$ (not included) to the infinity passing through a point $(\cos \theta, \sin \theta)$ in the unit circle.

□

Exercise 9.18. ...

Proof.

(1)

(2)

□

Exercise 9.19. Show that the system of equations

$$\begin{aligned} 3x + y - z + u^2 &= 0 \\ x - y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0 \end{aligned}$$

can be solved for x, y, u in terms of z ; for x, z, u in terms of y ; for y, z, u in terms of x ; but not for x, y, z in terms of u .

Proof (Brute-force).

(1) Denote

$$3x + y - z + u^2 = 0 \quad (\text{I})$$

$$x - y + 2z + u = 0 \quad (\text{II})$$

$$2x + 2y - 3z + 2u = 0 \quad (\text{III})$$

So (I) - 3(II) implies that

$$4y + u(u - 3) = 7z, \quad (\text{IV})$$

and (III) - 2(II) implies that

$$4y = 7z. \quad (\text{V})$$

By (IV)(V), we have $u(u - 3) = 0$. Hence $u = 0$ or $u = 3$ in any case.

(2) Show that (I)(II)(III) can be solved for x, y, u in terms of z . (V) implies that $y = \frac{7z}{4}$. Hence

$$(x, y, u) = \left(-\frac{z}{4}, \frac{7z}{4}, 0 \right), \left(-\frac{z}{4} - 3, \frac{7z}{4}, 3 \right).$$

(3) Show that (I)(II)(III) can be solved for x, z, u in terms of y .

$$(x, z, u) = \left(-\frac{y}{7}, \frac{4y}{7}, 0 \right), \left(-\frac{y}{7} - 3, \frac{4y}{7}, 3 \right).$$

(4) Show that (I)(II)(III) can be solved for y, z, u in terms of x .

$$(y, z, u) = (-7x, -4x, 0), (-7x - 21, -4x - 12, 3).$$

(5) Show that (I)(II)(III) can not be solved for x, y, z in terms of u . Actually,

$$(x, y, z) = (-t - u, 7t, 4t)$$

for all $t \in \mathbb{R}^1$.

□

Proof (The implicit function theorem).

- (1) Define \mathbf{f} be a \mathcal{C}' -mapping of \mathbb{R}^{3+1} into \mathbb{R}^3 by

$$\mathbf{f}(x, y, z, u) = (3x + y - z + u^2, x - y + 2z + u, 2x + 2y - 3z + 2u).$$

Note that $\mathbf{f}(0, 0, 0, 0) = \mathbf{0}$ and $\mathbf{f}(-3, 0, 0, 3) = \mathbf{0}$.

- (2) Since

$$[\mathbf{f}'(x, y, z, u)] = \begin{bmatrix} 3 & 1 & -1 & 2u \\ 1 & -1 & 2 & 1 \\ 2 & 2 & -3 & 2 \end{bmatrix},$$

\mathbf{f}' is continuous,

$$[\mathbf{f}'(0, 0, 0, 0)] = \begin{bmatrix} 3 & 1 & -1 & 0 \\ 1 & -1 & 2 & 1 \\ 2 & 2 & -3 & 2 \end{bmatrix},$$

and

$$[\mathbf{f}'(-3, 0, 0, 3)] = \begin{bmatrix} 3 & 1 & -1 & 6 \\ 1 & -1 & 2 & 1 \\ 2 & 2 & -3 & 2 \end{bmatrix}.$$

- (3) The submatrix

$$[\mathbf{f}'(0, 0, 0, 0)]_x = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & -3 & 2 \end{bmatrix}$$

is invertible since its determinant is $3 \neq 0$. By the implicit function theorem (Theorem 9.28), the system can be solved for y, z, u in terms of x . Similar arguments to $[\mathbf{f}'(0, 0, 0, 0)]_y$, $[\mathbf{f}'(0, 0, 0, 0)]_z$, $[\mathbf{f}'(-3, 0, 0, 3)]_x$, $[\mathbf{f}'(-3, 0, 0, 3)]_y$, and $[\mathbf{f}'(-3, 0, 0, 3)]_z$.

- (4) Note that $[\mathbf{f}'(0, 0, 0, 0)]_u$ and $[\mathbf{f}'(-3, 0, 0, 3)]_u$ are not invertible, we cannot apply the implicit function theorem (Theorem 9.28). We need to show by brute-force in this case.

□

Exercise 9.20. Take $n = m = 1$ in the implicit function theorem, and interpret the theorem (as well as its proof) graphically.

Implicit function theorem (for $n = m = 1$). Let $f(x, y)$ be a \mathcal{C}' -mapping of an open set $E \subseteq \mathbb{R}^2$ into \mathbb{R} , such that $f(a, b) = 0$ for some point $(a, b) \in E$. Assume that

$$D_1 f(a, b) \neq 0.$$

Then there exist open sets $U \subseteq E$ and $W \subseteq \mathbb{R}^1$, with $(a, b) \in U$ and $b \in W$, having the following property:

To every $y \in W$ corresponds a unique x such that

$$(x, y) \in U \quad \text{and} \quad f(x, y) = 0.$$

If this x is defined to be $g(y)$, then g is a \mathcal{C}' -mapping of W into \mathbb{R}^1 ,
 $g(b) = a$,

$$f(g(y), y) = 0 \quad (y \in W),$$

and

$$g'(b) = -\frac{D_2 f(a, b)}{D_1 f(a, b)}.$$

Proof.

- (1) In the notations of Exercise 4.6, define the graph of f by the set

$$S = \{(x, y) \in E : f(x, y) = 0\}.$$

- (2) Consider the graph S . As $D_1 f(a, b) \neq 0$ and $f(x, y) \in \mathcal{C}'$, there are an open neighborhood $U \subseteq E$ of (a, b) and an open neighborhood W of b such that $x \mapsto f(x, y)$ is strictly monotonic whenever $y \in W$. “Graphically” by the monotony of $f(x, y)$, for any fixed y there is a unique x such that $f(x, y) = 0$.

- (3) “Graphically” the tangent line passing through (a, b) is

$$D_1 f(a, b)(x - a) + D_2 f(a, b)(y - b) = 0.$$

Thus $g'(b) = -\frac{D_2 f(a, b)}{D_1 f(a, b)}$ if $D_1 f(a, b) \neq 0$.

□

Exercise 9.21. Define f in \mathbb{R}^2 by

$$f(x, y) = 2x^3 - 3x^2 + 2y^3 + 3y^2.$$

- (a) Find the four points in \mathbb{R}^2 at which the gradient of f is zero. Show that f has exactly one local maximum and one local minimum in \mathbb{R}^1 .
- (b) Let S be the set of all $(x, y) \in \mathbb{R}^2$ at which $f(x, y) = 0$. Find those points of S that have no neighborhoods in which the equation $f(x, y) = 0$ can be solved for y in terms of x (or for x in terms of y). Describe S as precisely as you can.

Proof of (a).

- (1)

$$(\nabla f)(x, y) = ((D_1 f)(x, y), (D_2 f)(x, y)) = (6x(x - 1), 6y(y + 1)).$$

So $(\nabla f)(x, y) = 0$ if and only if $(x, y) = (0, 0), (0, -1), (1, 0), (1, -1)$.

- (2) $x \mapsto 2x^3 - 3x^2$ have one local maximum at $x = 0$ and one local minimum at $x = 1$. $y \mapsto 2y^3 + 3y^2$ have one local maximum at $y = -1$ and one local minimum at $y = 0$.
- (3) Hence $f : (x, y) \mapsto to(2x^3 - 3x^2) + (2y^3 + 3y^2)$ have one local maximum at $(x, y) = (0, -1)$ and one local minimum at $(x, y) = (1, 0)$. Other two points $(0, 0)$ and $(1, -1)$ are saddle points.

□

Proof of (b).

- (1) By definition,

$$\begin{aligned} S &= \{f(x, y) = 0\} \\ &= \{(x + y)(2x^2 - 2xy - 3x + 2y^2 + 3y) = 0\} \\ &= \{x + y = 0\} \cup \{2x^2 - 2xy - 3x + 2y^2 + 3y = 0\}, \end{aligned}$$

which is a union of a line $L = \{x + y = 0\}$ and an ellipse $E = \{2x^2 - 2xy - 3x + 2y^2 + 3y = 0\}$. The intersection of $L \cap E$ is $\{(0, 0), (1, -1)\}$, and it suggested that $f(x, y) = 0$ cannot be solved for y in terms of x (or for x in terms of y) on $L \cap E = \{(0, 0), (1, -1)\}$.

- (2) By (1) in the proof of (a) and the implicit function theorem (Theorem 9.28), $f(x, y) = 0$ can be solved for y in terms of x (or for x in terms of y) whenever $(D_2f)(x, y) \neq 0$ (or $(D_1f)(x, y) \neq 0$).
- (3) Show that $f(x, y) = 0$ cannot be solved for y in terms of x if $(D_2f)(x, y) = 0$. $(D_2f)(x, y) = 0$ if and only if

$$(x, y) \in T = \left\{ (0, 0), \left(\frac{3}{2}, 0\right), (1, -1), \left(-\frac{1}{2}, -1\right) \right\}.$$

Solve y to get

$$\begin{aligned} y &= -x \\ y &= \frac{1}{4} \left(2x - 3 + \sqrt{-3(2x + 1)(2x - 3)} \right) \\ y &= \frac{1}{4} \left(2x - 3 - \sqrt{-3(2x + 1)(2x - 3)} \right) \end{aligned}$$

In any case, y can not be uniquely determined by x for any $(x, y) \in T$. (“Graphically” we can see the set S to get the conclusion. Explicitly, we can take the limit to each expression (as $(s, t) \rightarrow (x, y) \in T$), and observe that not all limits are equal.)

- (4) Show that $f(x, y) = 0$ cannot be solved for x in terms of y if $(D_1f)(x, y) = 0$. $(D_1f)(x, y) = 0$ if and only if

$$(x, y) \in T = \left\{ (0, 0), \left(0, -\frac{3}{2}\right), (1, -1), \left(1, \frac{1}{2}\right) \right\}.$$

Similar to (3), x can not be uniquely determined by y for any $(x, y) \in T$.

□

Supplement (Second-derivative test for extrema).

- (1) (Theorem 13.11 in Tom M. Apostol, *Mathematical Analysis*, 2nd edition).
Let f be a real-valued function with continuous second-order partial derivatives at a stationary point $\mathbf{a} \in \mathbb{R}^2$. Let

$$A = (D_{11}f)(\mathbf{a}), \quad B = (D_{12}f)(\mathbf{a}), \quad C = (D_{22}f)(\mathbf{a}),$$

and let

$$\Delta = \det \begin{bmatrix} A & B \\ B & C \end{bmatrix} = AC - B^2.$$

Then we have:

- (a) If $\Delta > 0$ and $A > 0$, f has a local minimum at \mathbf{a} .
 - (b) If $\Delta > 0$ and $A < 0$, f has a local maximum at \mathbf{a} .
 - (c) If $\Delta < 0$, f has a saddle point at \mathbf{a} .
- (2) We can give another proof of (a) by the second-derivative test for extrema.

Exercise 9.22. Given a similar discussion for

$$f(x, y) = 2x^3 + 6xy^2 - 3x^2 + 3y^2.$$

Outline.

- (a) Find the two points in \mathbb{R}^2 at which the gradient of f is zero. Show that f has one saddle point and one local minimum in \mathbb{R}^2 .
- (b) Let S be the set of all $(x, y) \in \mathbb{R}^2$ at which $f(x, y) = 0$. Find those points of S that have no neighborhoods in which the equation $f(x, y) = 0$ can be solved for y in terms of x (or for x in terms of y). Describe S as precisely as you can.

Proof of (a).

(1)

$$(\nabla f)(x, y) = ((D_1f)(x, y), (D_2f)(x, y)) = (6(x^2 + y^2 - x), 6y(2x + 1)).$$

So $(\nabla f)(x, y) = 0$ if and only if $(x, y) = (0, 0)$ or $(1, 0)$.

- (2) Show that f has one saddle point at $(x, y) = (0, 0)$. Since $f(x, x) = 8x^3$, $f(x, x) \leq 0 = f(0, 0)$ if $x < 0$ and $f(x, x) \geq 0 = f(0, 0)$ if $x > 0$. Hence (x, y) is not a local maximum or a local minimum for f .

- (3) Show that f has one local minimum at $(x, y) = (1, 0)$. Write

$$f(x, y) = 2x^3 - 3x^2 + (6x + 3)y^2.$$

Note that $2x^3 - 3x^2 \geq -1$ and $(6x + 3)y^2 \geq 0$ in some open neighborhood $B((1, 0); \frac{1}{64})$ of $(1, 0)$. Therefore f has one local minimum at $(x, y) = (1, 0)$.

□

Proof of (b).

- (1) S is a folium of Descartes with a double point at the origin and asymptote $x + \frac{1}{2} = 0$.
whenever $(D_2f)(x, y) \neq 0$ (or $(D_1f)(x, y) \neq 0$).
- (3) Show that $f(x, y) = 0$ cannot be solved for y in terms of x if $(D_2f)(x, y) = 0$. $(D_2f)(x, y) = 0$ if and only if

$$(x, y) \in T = \left\{ (0, 0), \left(\frac{3}{2}, 0 \right) \right\}.$$

Solve y to get

$$y = \sqrt{\frac{-x^2(2x-3)}{3(2x+1)}}$$

$$y = -\sqrt{\frac{-x^2(2x-3)}{3(2x+1)}}$$

In any case, y can not be uniquely determined by x for any $(x, y) \in T$. (“Graphically” we can see the set S to get the conclusion. Explicitly, we can take the limit to each expression (as $(s, t) \rightarrow (x, y) \in T$), and observe that two limits are different.)

- (4) Show that $f(x, y) = 0$ cannot be solved for x in terms of y if $(D_1f)(x, y) = 0$. $(D_1f)(x, y) = 0$ if and only if

$$(x, y) \in T = \left\{ (0, 0), \pm \sqrt{-\frac{3}{4} + \sqrt{\frac{3}{4}}} \right\}.$$

Similar to (3), x can not be uniquely determined by y for any $(x, y) \in T$. That is,

$$x = g(y)$$

$$= \frac{1 - 4y^2}{2} \left\{ 2\sqrt{16y^6 + 24y^4 - 3y^2 - 12y^2 + 1} \right\}^{-\frac{1}{3}}$$

$$+ \left\{ 2\sqrt{16y^6 + 24y^4 - 3y^2 - 12y^2 + 1} \right\}^{\frac{1}{3}} + 1.$$

So as $y \neq 0$, $x = g(y) = g(-y)$. The expression $x = g(y)$ is not unique.

□

Exercise 9.23. Define f in \mathbb{R}^3 by

$$f(x, y_1, y_2) = x^2 y_1 + e^x + y_2.$$

Show that $f(0, 1, -1) = 0$, $(D_1 f)(0, 1, -1) \neq 0$, and that there exists therefore a differentiable function g in some neighborhood of $(1, -1)$ in \mathbb{R}^2 , such that $g(1, -1) = 0$ and

$$f(g(y_1, y_2), y_1, y_2) = 0.$$

Find $(D_1 g)(1, -1)$ and $(D_2 g)(1, -1)$.

Proof.

(1) Note that $f(0, 1, -1) = 0$. Since

$$\begin{aligned} [\nabla f((x, y_1, y_2))]_{(x, y_1, y_2) = (0, 1, -1)} &= [(2xy_1 + e^x, x^2, 1)]_{(x, y_1, y_2) = (0, 1, -1)} \\ &= (1, 0, 1), \end{aligned}$$

$A_x = (1)$ and $A_y = (0, 1)$. By the implicit function theorem (Theorem 9.28), there exists a \mathcal{C}^1 function in some open neighborhood of $(1, -1)$ such that $g(1, -1) = 0$ and $f(g(y_1, y_2), y_1, y_2) = 0$.

(2) Besides, $g'(1, -1) = -(A_x)^{-1} A_y = (0, -1)$ implies that $(D_1 g)(1, -1) = 0$ and $(D_2 g)(1, -1) = -1$.

□

Exercise 9.24. For $(x, y) \neq (0, 0)$, define $\mathbf{f} = (f_1, f_2)$ by

$$f_1(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \quad f_2(x, y) = \frac{xy}{x^2 + y^2}.$$

Compute the rank of $\mathbf{f}'(x, y)$, and find the range of \mathbf{f} .

Proof.

(1)

$$[\mathbf{f}'(x, y)] = \begin{bmatrix} \frac{4xy^2}{(x^2+y^2)^2} & \frac{4x^2y}{(x^2+y^2)^2} \\ \frac{-y(x^2-y^2)}{(x^2+y^2)^2} & \frac{x(x^2-y^2)}{(x^2+y^2)^2} \end{bmatrix}.$$

(2) Show that $\text{rank}([\mathbf{f}'(x, y)]) \neq 2$. It is equivalent to show that $\det[\mathbf{f}'(x, y)] = 0$. Actually,

$$\det[\mathbf{f}'(x, y)] = \frac{4xy^2}{(x^2+y^2)^2} \cdot \frac{x(x^2-y^2)}{(x^2+y^2)^2} - \frac{4x^2y}{(x^2+y^2)^2} \cdot \frac{-y(x^2-y^2)}{(x^2+y^2)^2} = 0.$$

(3) Show that $\text{rank}([\mathbf{f}'(x, y)]) \neq 0$.

$$\begin{aligned} [\mathbf{f}'(x, y)] \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} \frac{4xy^2}{(x^2+y^2)^2} & \frac{4x^2y}{(x^2+y^2)^2} \\ \frac{-y(x^2-y^2)}{(x^2+y^2)^2} & \frac{x(x^2-y^2)}{(x^2+y^2)^2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{4xy^2}{(x^2+y^2)^2} \\ \frac{-y(x^2-y^2)}{(x^2+y^2)^2} \end{bmatrix} \\ &\neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

for all $(x, y) \in \mathbb{R}^2 - \{(0, 0)\}$.

(4) Since the rank of \mathbf{f}' is the dimension of the subspace $\mathcal{R}(\mathbf{f}')$ in \mathbb{R}^2 , $\text{rank}([\mathbf{f}'(x, y)]) = 0, 1, 2$. By (2)(3), $\text{rank}([\mathbf{f}'(x, y)]) = 1$.

(5) Show that the range of \mathbf{f} is an ellipse

$$E = \{(s, t) \in \mathbb{R}^2 : s^2 + 4t^2 = 1\}.$$

(a) Clearly, $(f_1(x, y), f_2(x, y)) \in E$.

(b) Conversely, for any $(s, t) \in E$ write

$$s = \cos \theta \quad \text{and} \quad t = \frac{1}{2} \sin \theta$$

for some unique $\theta \in [0, 2\pi)$ (Theorem 8.7(d)). By the tangent half-angle formula,

$$s = \cos \theta = \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}} \quad \text{and} \quad t = \frac{1}{2} \sin \theta = \frac{\tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}.$$

Thus, there exists a point $(1, \tan \frac{\theta}{2}) \in \mathbb{R}^2$ such that

$$f : \left(1, \tan \frac{\theta}{2}\right) \mapsto (s, t) \in E.$$

(c) Or we can do a linear projection from a given point $P = (1, 0)$, say for any $\lambda \in \mathbb{R}^1$ we define a line through P with slope $-\lambda$ meeting E in a further point

$$Q_\lambda = \left(\frac{\lambda^2 - 1}{\lambda^2 + 1}, \frac{\lambda}{\lambda^2 + 1} \right).$$

Might define $Q_\infty = P$. Graphically and informally,

$$\{Q_\lambda : \lambda \in \mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}\} = E.$$

Therefore, $f(1, 0) = P$ and $f(\lambda, 1) \in E - \{P\}$.

By (a)(b), the range of \mathbf{f} is exactly the same as an ellipse E .

□

Exercise 9.25. ...

Proof.

(1)

(2)

□

Exercise 9.26. *Show that the existence (and even the continuity) of $D_{12}f$ does not imply the existence of D_1f . For example, let $f(x, y) = g(x)$, where g is nowhere differentiable.*

Proof.

(1) Consider the function g defined on \mathbb{R}^1 by

$$g(x) = \begin{cases} 0 & (x \text{ irrational}), \\ \frac{1}{n} & (x = \frac{m}{n}). \end{cases}$$

$g(x)$ is nowhere differentiable by (1) in the note of Exercise 4.18. Define $f(x, y) = g(x)$ on \mathbb{R}^2 .

(2) $(D_1f)(x, y) = g'(x)$ does not exist on \mathbb{R}^2 . However, $(D_{12}f)(x, y) = (D_10)(x, y) = 0$ is continuous on \mathbb{R}^2 .

□

Note. Some nowhere differentiable functions.

(1) Exercise 4.18.

(2) Theorem 7.18.

(3) (Weierstrass functions.)

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

where $0 < a < 1$, b is a positive odd integer, and $ab > 1 + \frac{3}{2}\pi$.

(4)

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(n^2 \pi x).$$

(And so on.)

Exercise 9.27. Put $f(0,0) = 0$, and

$$f(x,y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$$

if $(x,y) \neq (0,0)$. Prove that

- (a) f, D_1f, D_2f are continuous in \mathbb{R}^2 .
- (b) $D_{12}f$ and $D_{21}f$ exist at every point of \mathbb{R}^2 , and are continuous except at $(0,0)$.
- (c) $(D_{12}f)(0,0) = 1$, and $(D_{21}f)(0,0) = -1$.

Proof of (a).

- (1) Show that f is continuous in \mathbb{R}^2 .

- (a) Clearly, $f(x,y)$ is continuous if $(x,y) \neq (0,0)$. So it suffices to show that

$$\lim_{(x,y) \rightarrow (0,0)} f(x,y) = f(0,0) = 0.$$

- (b) Write $x = r \cos \theta$ and $y = r \sin \theta$ in the polar coordinates. (Here $r > 0$.) Hence

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} f(x,y) &= \lim_{(x,y) \rightarrow (0,0)} \frac{xy(x^2 - y^2)}{x^2 + y^2} \\ &= \lim_{r \rightarrow 0} r^2 \cos \theta \sin \theta (\cos^2 \theta - \sin^2 \theta) \\ &= 0 \end{aligned}$$

since $\cos \theta \sin \theta (\cos^2 \theta - \sin^2 \theta)$ is bounded by 2.

- (2) Show that D_1f is continuous in \mathbb{R}^2 .

- (a) $(x,y) \neq (0,0)$ implies that

$$(D_1f)(x,y) = \frac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2}.$$

Besides,

$$\begin{aligned}(D_1f)(0,0) &= \lim_{x \rightarrow 0} \frac{f(x,0) - f(0,0)}{x - 0} \\ &= \lim_{x \rightarrow 0} \frac{0}{x} \\ &= 0.\end{aligned}$$

In summary,

$$(D_1f)(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \frac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2} & \text{if } (x,y) \neq (0,0). \end{cases}$$

- (b) Clearly, $(D_1f)(x,y)$ is continuous if $(x,y) \neq (0,0)$. So it suffices to show that

$$\lim_{(x,y) \rightarrow (0,0)} (D_1f)(x,y) = (D_1f)(0,0) = 0.$$

- (c) Similar to (1)(b). Write $x = r \cos \theta$ and $y = r \sin \theta$ in the polar coordinates. (Here $r > 0$.) Hence

$$\begin{aligned}\lim_{(x,y) \rightarrow (0,0)} (D_1f)(x,y) &= \lim_{(x,y) \rightarrow (0,0)} \frac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2} \\ &= \lim_{r \rightarrow 0} r(\cos^4 \theta \sin \theta + 4 \cos^2 \theta \sin^3 \theta - \sin^5 \theta) \\ &= 0\end{aligned}$$

since $\cos^4 \theta \sin \theta + 4 \cos^2 \theta \sin^3 \theta - \sin^5 \theta$ is bounded by 6.

- (3) Similar to (2). Show that D_2f is continuous in \mathbb{R}^2 .

- (a) $(x,y) \neq (0,0)$ implies that

$$(D_2f)(x,y) = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2 + y^2)^2}.$$

Besides,

$$\begin{aligned}(D_2f)(0,0) &= \lim_{y \rightarrow 0} \frac{f(0,y) - f(0,0)}{y - 0} \\ &= \lim_{y \rightarrow 0} \frac{0}{y} \\ &= 0.\end{aligned}$$

In summary,

$$(D_2f)(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \frac{x^5 - 4x^3y^2 - xy^4}{(x^2 + y^2)^2} & \text{if } (x,y) \neq (0,0). \end{cases}$$

- (b) Clearly, $(D_2f)(x, y)$ is continuous if $(x, y) \neq (0, 0)$. So it suffices to show that

$$\lim_{(x,y) \rightarrow (0,0)} (D_2f)(x, y) = (D_2f)(0, 0) = 0.$$

- (c) Similar to (1)(b). Write $x = r \cos \theta$ and $y = r \sin \theta$ in the polar coordinates. (Here $r > 0$.) Hence

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} (D_2f)(x, y) &= \lim_{(x,y) \rightarrow (0,0)} \frac{x^5 - 4x^3y^2 - xy^4}{(x^2 + y^2)^2} \\ &= \lim_{r \rightarrow 0} r(\cos^5 \theta - 4 \cos^3 \theta \sin^2 \theta - \cos \theta \sin^4 \theta) \\ &= 0 \end{aligned}$$

since $\cos^5 \theta - 4 \cos^3 \theta \sin^2 \theta - \cos \theta \sin^4 \theta$ is bounded by 6.

□

Proof of (b).

- (1) Show that $D_{12}f$ exists at every point of \mathbb{R}^2 .

- (a) $(x, y) \neq (0, 0)$ implies that

$$(D_{12}f)(x, y) = (D_1 D_2 f)(x, y) = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2 + y^2)^3}.$$

- (b) Besides,

$$\begin{aligned} (D_{12}f)(0, 0) &= \lim_{x \rightarrow 0} \frac{(D_2f)(x, 0) - (D_2f)(0, 0)}{x - 0} \\ &= \lim_{x \rightarrow 0} \frac{x}{x} \\ &= 1. \end{aligned}$$

In summary,

$$(D_{12}f)(x, y) = \begin{cases} 1 & \text{if } (x, y) = (0, 0) \\ \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2 + y^2)^3} & \text{if } (x, y) \neq (0, 0). \end{cases}$$

- (2) Show that $D_{12}f$ is continuous except at $(0, 0)$.

- (a) Clearly, $(D_{12}f)(x, y)$ is continuous if $(x, y) \neq (0, 0)$. So it suffices to show that

$$\lim_{(x,y) \rightarrow (0,0)} (D_{12}f)(x, y)$$

does not exist.

(b) Take

$$\mathbf{p}_n = \left(\frac{1}{n}, 0\right) \quad \text{and} \quad \mathbf{q}_n = \left(0, \frac{1}{n}\right)$$

for $n = 1, 2, 3, \dots$. So $\lim \mathbf{p}_n = \lim \mathbf{q}_n = \mathbf{0}$,

$$\lim(D_{12}f)(\mathbf{p}_n) = 1 \quad \text{and} \quad \lim(D_{12}f)(\mathbf{q}_n) = -1.$$

Hence $\lim_{(x,y) \rightarrow (0,0)} (D_{12}f)(x, y)$ does not exist.

(3) Show that $D_{21}f$ exists at every point of \mathbb{R}^2 . Similar to (1).

(a) $(x, y) \neq (0, 0)$ implies that

$$(D_{21}f)(x, y) = (D_2 D_1 f)(x, y) = \frac{x^6 + 9x^4 y^2 - 9x^2 y^4 - y^6}{(x^2 + y^2)^3},$$

which is the same as $(D_{12}f)(x, y)$.

(b) Besides,

$$\begin{aligned} (D_{21}f)(0, 0) &= \lim_{y \rightarrow 0} \frac{(D_1 f)(0, y) - (D_1 f)(0, 0)}{y - 0} \\ &= \lim_{y \rightarrow 0} \frac{-y}{y} \\ &= -1. \end{aligned}$$

In summary,

$$(D_{21}f)(x, y) = \begin{cases} -1 & \text{if } (x, y) = (0, 0) \\ \frac{x^6 + 9x^4 y^2 - 9x^2 y^4 - y^6}{(x^2 + y^2)^3} & \text{if } (x, y) \neq (0, 0). \end{cases}$$

(4) Show that $D_{21}f$ is continuous except at $(0, 0)$. Exactly the same as (2) since $(D_{21}f)(x, y) = (D_{12}f)(x, y)$ if $(x, y) \neq (0, 0)$.

□

Proof of (c). See (2)(4) in the proof of (b). □

Exercise 9.28. ...

Proof.

(1)

(2)

□

Exercise 9.29. ...

Proof.

(1)

(2)

□

Exercise 9.30. ...

Proof.

(1)

(2)

□

Exercise 9.31. ...

Proof.

(1)

(2)

□