

## Chapter 9: Functions of Several Variables

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**Exercise 9.1.** If  $S$  is a nonempty subset of a vector space  $X$ , prove (as asserted in Section 9.1) that the span of  $S$  is a vector space.

Denote the span of  $S$  by  $\text{span}(S)$ .

*Proof.*

- (1) Since  $S \neq \emptyset$ , there is  $\mathbf{z} \in S$ . So  $1\mathbf{z} = \mathbf{z} \in \text{span}(S) \neq \emptyset$ . (In fact,  $\text{span}(S) \supseteq S$ .)
- (2) If  $\mathbf{x}, \mathbf{y} \in \text{span}(S)$ , then there exist elements  $\mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{y}_1, \dots, \mathbf{y}_n \in S$  and scalars  $a_1, \dots, a_m, b_1, \dots, b_n$  such that

$$\mathbf{x} = a_1\mathbf{x}_1 + \cdots + a_m\mathbf{x}_m,$$

$$\mathbf{y} = b_1\mathbf{y}_1 + \cdots + b_n\mathbf{y}_n.$$

Then

$$\mathbf{x} + \mathbf{y} = a_1\mathbf{x}_1 + \cdots + a_m\mathbf{x}_m + b_1\mathbf{y}_1 + \cdots + b_n\mathbf{y}_n$$

is a linear combination of the elements of  $S$ . For any scalar  $c$ ,

$$c\mathbf{x} = (ca_1)\mathbf{x}_1 + \cdots + (ca_m)\mathbf{x}_m$$

is again linear combination of the elements of  $S$ .

- (3) By (1)(2),  $\text{span}(S)$  is a vector space.

□

*Note.* Any subspace of  $X$  that contains  $S$  must also contain  $\text{span}(S)$ .

**Exercise 9.2.** Prove (as asserted in Section 9.6) that  $BA$  is linear if  $A$  and  $B$  are linear transformations. Prove also that  $A^{-1}$  is linear and invertible if  $A$  is invertible.

*Proof.* Use the notation in Definitions 9.6.

- (1) Show that  $BA$  is linear if  $A$  and  $B$  are linear transformations. Let  $X, Y, Z$  be vector spaces,  $A \in L(X, Y)$  and  $B \in L(Y, Z)$ .

(a) Given any  $\mathbf{x}_1, \mathbf{x}_2 \in X$ .

$$\begin{aligned}
(BA)(\mathbf{x}_1 + \mathbf{x}_2) &= B(A(\mathbf{x}_1 + \mathbf{x}_2)) \\
&= B(A\mathbf{x}_1 + A\mathbf{x}_2) && (A \text{ is a linear transformation}) \\
&= B(A\mathbf{x}_1) + B(A\mathbf{x}_2) && (B \text{ is a linear transformation}) \\
&= (BA)\mathbf{x}_1 + (BA)\mathbf{x}_2.
\end{aligned}$$

(b) For any  $\mathbf{x} \in X$  and scalar  $c$ ,

$$\begin{aligned}
(BA)(c\mathbf{x}) &= B(A(c\mathbf{x})) \\
&= B(cA\mathbf{x}) && (A \text{ is a linear transformation}) \\
&= cB(A\mathbf{x}) && (B \text{ is a linear transformation}) \\
&= c(BA)\mathbf{x}.
\end{aligned}$$

By (a)(b),  $BA \in L(X, Z)$ .

(2) Show that  $A^{-1}$  is linear if  $A$  is invertible.

(a) Given any  $\mathbf{y}_1, \mathbf{y}_2 \in X$ . Since  $A$  is surjective, there exist  $\mathbf{x}_1, \mathbf{x}_2 \in X$  such that

$$\begin{aligned}
\mathbf{y}_1 &= A\mathbf{x}_1 \\
\mathbf{y}_2 &= A\mathbf{x}_2.
\end{aligned}$$

So

$$\begin{aligned}
A^{-1}\mathbf{y}_1 &= A^{-1}(A\mathbf{x}_1) = \mathbf{x}_1 \\
A^{-1}\mathbf{y}_2 &= A^{-1}(A\mathbf{x}_2) = \mathbf{x}_2
\end{aligned}$$

(by Definitions 9.4). Hence

$$\begin{aligned}
A^{-1}(\mathbf{y}_1 + \mathbf{y}_2) &= A^{-1}(A\mathbf{x}_1 + A\mathbf{x}_2) \\
&= A^{-1}(A(\mathbf{x}_1 + \mathbf{x}_2)) && (A \text{ is a linear transformation}) \\
&= \mathbf{x}_1 + \mathbf{x}_2 && (\text{Definitions 9.4}) \\
&= A^{-1}\mathbf{y}_1 + A^{-1}\mathbf{y}_2.
\end{aligned}$$

(b) For any  $\mathbf{y} \in X$  and scalar  $c$ , there is a corresponding  $\mathbf{x} \in X$  such that  $\mathbf{y} = A\mathbf{x}$  since  $A$  is surjective. So  $A^{-1}\mathbf{y} = \mathbf{x}$  by Definition 9.4. Hence

$$\begin{aligned}
A^{-1}(c\mathbf{y}) &= A^{-1}(cA\mathbf{x}) \\
&= A^{-1}(A(c\mathbf{x})) && (A \text{ is a linear transformation}) \\
&= c\mathbf{x} && (\text{Definitions 9.4}) \\
&= cA^{-1}\mathbf{y}.
\end{aligned}$$

By (a)(b),  $A^{-1} \in L(X)$ .

(3) *Show that  $A^{-1}$  is invertible if  $A$  is invertible.* It suffices to show that  $A^{-1}$  is injective and surjective.

(a) *Show that  $A^{-1}$  is injective.* Given any  $\mathbf{y}_1, \mathbf{y}_2 \in X$ . Since  $A$  is surjective, there exist  $\mathbf{x}_1, \mathbf{x}_2 \in X$  such that

$$\begin{aligned}\mathbf{y}_1 &= A\mathbf{x}_1 \\ \mathbf{y}_2 &= A\mathbf{x}_2.\end{aligned}$$

Suppose  $A^{-1}\mathbf{y}_1 = A^{-1}\mathbf{y}_2$ . So  $A^{-1}(A\mathbf{x}_1) = A^{-1}(A\mathbf{x}_2)$ , or  $\mathbf{x}_1 = \mathbf{x}_2$ , or  $\mathbf{y}_1 = A\mathbf{x}_1 = A\mathbf{x}_2 = \mathbf{y}_2$ .

(b) *Show that  $A^{-1}$  is surjective.* For any  $\mathbf{x} \in X$ , there exists  $A\mathbf{x} \in X$  such that  $A^{-1}(A\mathbf{x}) = \mathbf{x}$  by Definitions 9.4.

□

**Exercise 9.3.** Assume  $A \in L(X, Y)$  and  $A\mathbf{x} = \mathbf{0}$  only when  $\mathbf{x} = \mathbf{0}$ . Prove that  $A$  is then 1-1.

*Proof.* Suppose  $A\mathbf{x} = A\mathbf{y}$ . Since  $A$  is a linear transformation,  $A(\mathbf{x} - \mathbf{y}) = A\mathbf{x} - A\mathbf{y} = \mathbf{0}$ . By assumption,  $\mathbf{x} - \mathbf{y} = \mathbf{0}$  or  $\mathbf{x} = \mathbf{y}$ . □

**Exercise 9.4.** Prove (as asserted in Section 9.30) that null spaces and ranges of linear transformations are vector spaces.

*Proof.* Use the notation in Definitions 9.30. Suppose  $X, Y$  are vector spaces, and  $A \in L(X, Y)$ , as in Definition 9.6.

(1) *Show that  $\mathcal{N}(A)$  is a vector space in  $X$ .*

(a) Note that  $\mathbf{0} \in X$ . Since  $A\mathbf{0} = \mathbf{0}$ ,  $\mathbf{0} \in \mathcal{N}(A) \neq \emptyset$ .

(b) Suppose  $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{N}(A)$ . Then

$$\begin{aligned}A(\mathbf{x}_1 + \mathbf{x}_2) &= A\mathbf{x}_1 + A\mathbf{x}_2 && (A \text{ is a linear transformation}) \\ &= \mathbf{0} + \mathbf{0} && (\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{N}(A)) \\ &= \mathbf{0}.\end{aligned}$$

So  $\mathbf{x}_1 + \mathbf{x}_2 \in \mathcal{N}(A)$ .

(c) Suppose  $\mathbf{x} \in \mathcal{N}(A)$  and  $c$  is a scalar. Then

$$\begin{aligned}A(c\mathbf{x}) &= cA\mathbf{x} && (A \text{ is a linear transformation}) \\ &= c\mathbf{0} && (\mathbf{x} \in \mathcal{N}(A)) \\ &= \mathbf{0}.\end{aligned}$$

So  $c\mathbf{x} \in \mathcal{N}(A)$ .

By (a)(b)(c),  $\mathcal{N}(A)$  is a vector space.

(2) Show that  $\mathcal{R}(A)$  is a vector space in  $Y$ .

(a) Note that  $\mathbf{0} \in X$ . So  $A\mathbf{0} = \mathbf{0} \in \mathcal{R}(A) \neq \emptyset$ .

(b) Suppose  $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{R}(A)$ . Then there exist  $\mathbf{x}_1, \mathbf{x}_2 \in X$  such that  $A\mathbf{x}_1 = \mathbf{y}_1$  and  $A\mathbf{x}_2 = \mathbf{y}_2$ . Hence

$$\begin{aligned}\mathbf{y}_1 + \mathbf{y}_2 &= A\mathbf{x}_1 + A\mathbf{x}_2 \\ &= A(\mathbf{x}_1 + \mathbf{x}_2) \quad (A \text{ is a linear transformation}).\end{aligned}$$

So  $\mathbf{y}_1 + \mathbf{y}_2 \in \mathcal{R}(A)$ .

(c) Suppose  $\mathbf{y} \in \mathcal{R}(A)$  and  $c$  is a scalar. Then there exists  $\mathbf{x} \in X$  such that  $A\mathbf{x} = \mathbf{y}$ . Hence

$$\begin{aligned}c\mathbf{y} &= cA\mathbf{x} \\ &= A(c\mathbf{x}) \quad (A \text{ is a linear transformation}).\end{aligned}$$

So  $c\mathbf{y} \in \mathcal{R}(A)$ .

By (a)(b)(c),  $\mathcal{R}(A)$  is a vector space.

□

**Exercise 9.5.** Prove that to every  $A \in L(\mathbb{R}^n, \mathbb{R}^1)$  corresponds a unique  $\mathbf{y} \in \mathbb{R}^n$  such that  $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$ . Prove also that  $\|A\| = \|\mathbf{y}\|$ . (Hint: Under certain conditions, equality holds in the Schwarz inequality.)

*Proof.*

(1) Recall that  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  is the standard basis of  $\mathbb{R}^n$  (Definitions 9.1).

Given any  $\mathbf{x} \in \mathbb{R}^n$ , write  $\mathbf{x} = (x_1, \dots, x_n)$  as  $\mathbf{x} = \sum x_j \mathbf{e}_j$ .

(2) Show that  $\mathbf{y}$  exists. Since  $A$  is a linear transformation,

$$\begin{aligned}A\mathbf{x} &= A\left(\sum x_j \mathbf{e}_j\right) \\ &= \sum x_j A\mathbf{e}_j \\ &= (x_1, \dots, x_n) \cdot (A\mathbf{e}_1, \dots, A\mathbf{e}_n) \\ &= \mathbf{x} \cdot \sum (A\mathbf{e}_j) \mathbf{e}_j.\end{aligned}$$

Define  $\mathbf{y} = \sum (A\mathbf{e}_j) \mathbf{e}_j \in \mathbb{R}^n$  so that  $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$ .

(3) Show that  $\mathbf{y}$  is unique. Suppose there exists some  $\mathbf{z} \in \mathbb{R}^n$  such that  $A\mathbf{x} = \mathbf{x} \cdot \mathbf{z}$ . So

$$\begin{aligned}0 &= A\mathbf{x} - A\mathbf{x} \\ &= \mathbf{x} \cdot \mathbf{y} - \mathbf{x} \cdot \mathbf{z} \\ &= \mathbf{x} \cdot (\mathbf{y} - \mathbf{z})\end{aligned}$$

for any  $\mathbf{x} \in \mathbb{R}^n$ . In particular, take  $\mathbf{x} = \mathbf{y} - \mathbf{z} \in \mathbb{R}^n$  to get

$$0 = (\mathbf{y} - \mathbf{z}) \cdot (\mathbf{y} - \mathbf{z}) = |\mathbf{y} - \mathbf{z}|^2$$

or  $\mathbf{y} - \mathbf{z} = \mathbf{0}$  or  $\mathbf{y} = \mathbf{z}$ .

(4) *Show that  $\|A\| = |\mathbf{y}|$ .* By the Schwarz inequality (Theorem 1.37(d)),

$$|A\mathbf{x}| = |\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}| \leq |\mathbf{y}|$$

as  $|\mathbf{x}| \leq 1$ . Take the sup over all  $|\mathbf{x}| \leq 1$  to get

$$\|A\| \leq |\mathbf{y}|.$$

If  $\mathbf{y} = \mathbf{0}$ , then  $\|A\| = |\mathbf{y}| = 0$ . If  $\mathbf{y} \neq \mathbf{0}$ , then the equality holds when  $\mathbf{x} = \frac{\mathbf{y}}{|\mathbf{y}|} \in \mathbb{R}^n$ . (Here  $|\mathbf{x}| = 1$ .)

□

**Exercise 9.6.** *If  $f(0,0) = 0$  and*

$$f(x,y) = \frac{xy}{x^2 + y^2} \quad \text{if } (x,y) \neq (0,0),$$

*prove that  $(D_1f)(x,y)$  and  $(D_2f)(x,y)$  exist at every point of  $\mathbb{R}^2$ , although  $f$  is not continuous at  $(0,0)$ .*

*Proof.*

(1) *Show that*

$$(D_1f)(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0), \\ \frac{y(y^2 - x^2)}{(x^2 + y^2)^2} & \text{if } (x,y) \neq (0,0). \end{cases}$$

Write

$$\begin{aligned} (D_1f)(x,y) &= \lim_{t \rightarrow 0} \frac{f((x,y) + t(1,0)) - f(x,y)}{t} \\ &= \lim_{t \rightarrow 0} \frac{f(x+t,y) - f(x,y)}{t}. \end{aligned}$$

If  $(x,y) = (0,0)$ ,

$$(D_1f)(0,0) = \lim_{t \rightarrow 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t \rightarrow 0} \frac{0 - 0}{t} = 0.$$

If  $(x, y) \neq (0, 0)$ ,

$$\begin{aligned}
 (D_1 f)(x, y) &= \lim_{t \rightarrow 0} \frac{f(x+t, y) - f(x, y)}{t} \\
 &= \lim_{t \rightarrow 0} \frac{\frac{(x+t)y}{(x+t)^2 + y^2} - \frac{xy}{x^2 + y^2}}{t} \\
 &= \lim_{t \rightarrow 0} \frac{y(y^2 - x^2) - txy}{((x+t)^2 + y^2)(x^2 + y^2)} \\
 &= \frac{y(y^2 - x^2)}{(x^2 + y^2)^2}.
 \end{aligned}$$

(2) Show that

$$(D_2 f)(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0), \\ \frac{x(x^2 - y^2)}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0). \end{cases}$$

Similar to (1).

(3) Show that  $f$  is not continuous at  $(0, 0)$ . Note that

$$\lim_{n \rightarrow \infty} f\left(\frac{1}{n}, \frac{1}{n}\right) = \lim_{n \rightarrow \infty} \frac{\frac{1}{n} \cdot \frac{1}{n}}{\frac{1}{n^2} + \frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{1}{2} = \frac{1}{2}$$

and

$$\lim_{n \rightarrow \infty} f\left(\frac{1}{n}, 0\right) = \lim_{n \rightarrow \infty} \frac{0}{\frac{1}{n^2} + 0} = \lim_{n \rightarrow \infty} 0 = 0.$$

Hence the limit  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  does not exist.

□

**Exercise 9.7.** Suppose that  $f$  is a real-valued function defined in an open set  $E \subseteq \mathbb{R}^n$ , and that the partial derivatives  $D_1 f, \dots, D_n f$  are bounded in  $E$ . Prove that  $f$  is continuous in  $E$ . (Hint: Proceed as in the proof of Theorem 9.21.)

*Proof.*

- (1) Since  $D_j f$  is bounded in  $E$ , there is a real number  $M_j$  such that  $|D_j f| \leq M_j$  in  $E$ . Take  $M = \max_{1 \leq j \leq n} M_j$  so that  $|D_j f| \leq M$  in  $E$  for all  $1 \leq j \leq n$ .
- (2) Fix  $\mathbf{x} \in E$  and  $\varepsilon > 0$ . Since  $E$  is open, there is an open neighborhood

$$B(\mathbf{x}; r) = \{\mathbf{x} + \mathbf{h} \in E : |\mathbf{h}| < r\} \subseteq E$$

with

$$0 < r < \frac{\varepsilon}{n(M+1)}.$$

- (3) Write  $\mathbf{h} = \sum h_j \mathbf{e}_j$ ,  $|\mathbf{h}| < r$ , put  $\mathbf{v}_0 = \mathbf{0}$ , and  $\mathbf{v}_k = h_1 \mathbf{e}_1 + \cdots + h_k \mathbf{e}_k$  for  $1 \leq k \leq n$ . Then

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = \sum_{j=1}^n [f(\mathbf{x} + \mathbf{v}_j) - f(\mathbf{x} + \mathbf{v}_{j-1})].$$

Since  $|\mathbf{v}_k| < r$  for  $1 \leq k \leq n$  and since  $B(\mathbf{x}; r)$  is convex, the open interval with end points  $\mathbf{x} + \mathbf{v}_{j-1}$  and  $\mathbf{x} + \mathbf{v}_j$  lie in  $B(\mathbf{x}; r)$ . Since  $\mathbf{v}_j = \mathbf{v}_{j-1} + h_j \mathbf{e}_j$ , the mean value theorem (Theorem 5.10) show that

$$f(\mathbf{x} + \mathbf{v}_j) - f(\mathbf{x} + \mathbf{v}_{j-1}) = h_j (D_j f)(\mathbf{x} + \mathbf{v}_{j-1} + \theta_j h_j \mathbf{e}_j)$$

for some  $\theta_j \in (0, 1)$ .

- (4) Note that  $|h_j| \leq |\mathbf{h}| < r < \frac{\varepsilon}{n(M+1)}$ . Hence

$$\begin{aligned} |f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x})| &\leq \sum_{j=1}^n |f(\mathbf{x} + \mathbf{v}_j) - f(\mathbf{x} + \mathbf{v}_{j-1})| \\ &= \sum_{j=1}^n |h_j| |(D_j f)(\mathbf{x} + \mathbf{v}_{j-1} + \theta_j h_j \mathbf{e}_j)| \\ &\leq \sum_{j=1}^n \frac{\varepsilon}{n(M+1)} \cdot M \\ &< \varepsilon \end{aligned}$$

as  $|\mathbf{h}| < r < \frac{\varepsilon}{n(M+1)}$ . Hence  $f$  is continuous at all  $\mathbf{x} \in E$ .

□

**Exercise 9.8.** Suppose that  $f$  is a differentiable real function in an open set  $E \subseteq \mathbb{R}^n$ , and that  $f$  has a local maximum at a point  $\mathbf{x} \in E$ . Prove that  $f'(\mathbf{x}) = 0$ .

*Proof (Theorem 5.8).*

- (1) Apply Theorem 5.8 to each  $D_j f$  for  $1 \leq j \leq n$ . Since  $f$  has a local maximum at a point  $\mathbf{x} \in E$ , there is an open neighborhood  $B(\mathbf{x}; r)$  of  $\mathbf{x}$  in  $E$  such that

$$f(\mathbf{y}) \leq f(\mathbf{x})$$

for all  $\mathbf{y} \in B(\mathbf{x}; r)$ . Therefore,

$$f(\mathbf{x} + t\mathbf{e}_j) \leq f(\mathbf{x})$$

for all  $|t| < r$  and  $1 \leq j \leq n$ , or  $t \mapsto f(\mathbf{x} + t\mathbf{e}_j)$  has a local maximum at a point  $t = 0 \in (-r, r)$ .

- (2) Since  $f$  is differentiable in  $E$ , each partial derivatives  $D_j f$  exist (Theorem 9.21). Hence Theorem 5.8 implies that  $(D_j f)(\mathbf{x}) = 0$  for all  $1 \leq j \leq n$ . So

$$f'(\mathbf{x}) = [(D_1 f)(\mathbf{x}) \cdots (D_n f)(\mathbf{x})] = [0 \cdots 0] = 0$$

(as the zero matrix).

□

**Exercise 9.9.** If  $\mathbf{f}$  is a differentiable mapping of a connected open set  $E \subseteq \mathbb{R}^n$ , and if  $\mathbf{f}'(\mathbf{x}) = 0$  for every  $\mathbf{x} \in E$ , prove that  $\mathbf{f}$  is a constant in  $E$ .

*Proof.*

- (1) Show that  $\mathbf{f}$  is **locally constant**. Given any  $\mathbf{x} \in E$ . Since  $E$  is open, there exists an open neighborhood  $B(\mathbf{x}; r)$  of  $\mathbf{x}$  such that  $B(\mathbf{x}; r) \subseteq E$  and  $r > 0$ . Corollary to Theorem 9.19 implies that  $\mathbf{f}$  is a constant on  $B(\mathbf{x}; r)$ , that is,  $\mathbf{f}$  is locally constant.
- (2) Show that  $\mathbf{f}$  is constant if  $\mathbf{f}$  is locally constant in a connected set  $E \subseteq \mathbb{R}^n$ . Might assume that  $E \neq \emptyset$ . (Otherwise there is nothing to do.) Take some  $\mathbf{x}_0 \in E$ .

(a) Let

$$U = \{\mathbf{y} \in E : \mathbf{f}(\mathbf{y}) = \mathbf{f}(\mathbf{x}_0)\}.$$

- (b)  $U$  is open since  $\mathbf{f}$  is locally constant (by (1)). (Take any  $\mathbf{y} \in U$ . Since  $\mathbf{f}$  is locally constant, there is an open neighborhood  $B(\mathbf{y}) \subseteq E$  of  $\mathbf{y}$  such that  $\mathbf{f}(\mathbf{z}) = \mathbf{f}(\mathbf{y}) = \mathbf{f}(\mathbf{x}_0)$  whenever  $\mathbf{z} \in B(\mathbf{y})$ . So that  $B(\mathbf{y}) \subseteq U$ , or  $U$  is open.)
- (c) Besides, since  $\mathbf{f}$  is continuous (Remarks 9.13(c)), the set  $U$  is closed. (The proof is the same as Proof (Definition 2.18(d)) in Exercise 4.3.)
- (d) So  $U$  is open and closed. Write  $E = U \cup (E - U)$ . Here  $U$  and  $E - U$  are both open and closed. Hence  $U \cap \overline{E - U} = U \cap (E - U) = \emptyset$  and  $\overline{U} \cap (E - U) = U \cap (E - U) = \emptyset$ . Note that  $\mathbf{x}_0 \in U \neq \emptyset$ . By the connectedness of  $E$ ,  $E - U = \emptyset$ , or  $E = U$ , or  $\mathbf{f}$  is constant on  $E$ .

*Note.* The only subsets of a connected set  $E$  which are both open and closed are  $E$  and  $\emptyset$ .

□

**Exercise 9.10.** ...

*Proof.*

- (1)



(2)

□

**Exercise 9.11.** If  $f$  and  $g$  are differentiable real functions in  $\mathbb{R}^n$ , prove that

$$\nabla(fg) = f\nabla g + g\nabla f$$

and that

$$\nabla\left(\frac{1}{f}\right) = -\frac{1}{f^2}\nabla f$$

whenever  $f \neq 0$ .

*Proof.* Recall Example 9.18:

$$(\nabla(f))(\mathbf{x}) = \sum_{i=1}^n (D_i f)(\mathbf{x}) \mathbf{e}_i.$$

(1) Show that  $\nabla(fg) = f\nabla g + g\nabla f$ . For any  $\mathbf{x} \in \mathbb{R}^n$ ,

$$\begin{aligned} (\nabla(fg))(\mathbf{x}) &= \sum_{i=1}^n (D_i(fg))(\mathbf{x}) \mathbf{e}_i \\ &= \sum_{i=1}^n (g(D_i f) + f(D_i g))(\mathbf{x}) \mathbf{e}_i && \text{(Theorem 5.3(b))} \\ &= \sum_{i=1}^n [g(\mathbf{x})(D_i f)(\mathbf{x}) + f(\mathbf{x})(D_i g)(\mathbf{x})] \mathbf{e}_i \\ &= g(\mathbf{x}) \sum_{i=1}^n (D_i f)(\mathbf{x}) \mathbf{e}_i + f(\mathbf{x}) \sum_{i=1}^n (D_i g)(\mathbf{x}) \mathbf{e}_i \\ &= g(\mathbf{x})(\nabla f)(\mathbf{x}) + f(\mathbf{x})(\nabla g)(\mathbf{x}) \\ &= (f\nabla g + g\nabla f)(\mathbf{x}). \end{aligned}$$

(2) Show that

$$\nabla\left(\frac{1}{f}\right) = -\frac{1}{f^2}\nabla f$$

whenever  $f \neq 0$ . Note that  $\nabla(1) = 0$  since

$$\nabla(1)(\mathbf{x}) = \sum (D_i 1)(\mathbf{x}) \mathbf{e}_i = \sum (0)(\mathbf{x}) \mathbf{e}_i = \sum 0 \mathbf{e}_i = 0.$$

Hence as  $f \neq 0$ , we have

$$\begin{aligned} 0 &= \nabla(1) \\ &= \nabla\left(f \frac{1}{f}\right) && (f \neq 0) \\ &= f\nabla\left(\frac{1}{f}\right) + \frac{1}{f}\nabla f && ((1)), \end{aligned}$$

$$\text{or } \nabla \left( \frac{1}{f} \right) = -\frac{1}{f^2} \nabla f.$$

□

**Exercise 9.12.** ...

*Proof.*

(1)

(2)

□

**Exercise 9.13.** Suppose  $\mathbf{f}$  is a differentiable mapping of  $\mathbb{R}^1$  into  $\mathbb{R}^3$  such that  $|\mathbf{f}(t)| = 1$  for every  $t$ . Prove that  $\mathbf{f}'(t) \cdot \mathbf{f}(t) = 0$ . Interpret this result geometrically.

*Proof.*

- (1) Write  $\mathbf{f} = (f_1, f_2, f_3)$  as a vector-valued function. By Remarks 5.16,  $\mathbf{f}$  is differentiable if and only if each  $f_1, f_2, f_3$  is differentiable. So  $\mathbf{f}' = (f'_1, f'_2, f'_3)'$ . Hence

$$\begin{aligned} |\mathbf{f}(t)| &= 1 \text{ for every } t \\ \iff \mathbf{f}(t) \cdot \mathbf{f}(t) &= 1 \\ \iff f_1(t)^2 + f_2(t)^2 + f_3(t)^2 &= 1 \\ \implies 2f_1(t)f'_1(t) + 2f_2(t)f'_2(t) + 2f_3(t)f'_3(t) &= 0 \\ \iff f_1(t)f'_1(t) + f_2(t)f'_2(t) + f_3(t)f'_3(t) &= 0 \\ \iff (f_1(t), f_2(t), f_3(t)) \cdot (f'_1(t), f'_2(t), f'_3(t)) &= 0 \\ \iff \mathbf{f}(t) \cdot \mathbf{f}'(t) = \mathbf{f}'(t) \cdot \mathbf{f}(t) &= 0. \end{aligned}$$

- (2) The vector  $\mathbf{f}'(t)$  is called the **tangent vector** (or **velocity vector**) of  $\mathbf{f}$  at  $t$ . Geometrically, given any mapping  $\mathbf{f}$  lying on the sphere  $S^2$ , its tangent vector at  $t$  is lying on the tangent plane of  $S^2$  at  $t$ .

□

**Exercise 9.14.** Define  $f(0, 0) = 0$  and

$$f(x, y) = \frac{x^3}{x^2 + y^2} \quad \text{if } (x, y) \neq (0, 0).$$

- (a) Prove that  $D_1f$  and  $D_2f$  are bounded functions in  $\mathbb{R}^2$ . (Hence  $f$  is continuous.)
- (b) Let  $\mathbf{u}$  be any unit vector in  $\mathbb{R}^2$ . Show that the directional derivative  $(D_{\mathbf{u}}f)(0,0)$  exists, and that its absolute value is at most 1.
- (c)
- (d) In spite of this, prove that  $f$  is not differentiable at  $(0,0)$ .

*Proof of (a).*

- (1) Show that

$$(D_1f)(x, y) = \begin{cases} 1 & \text{if } (x, y) = (0, 0), \\ \frac{x^2(x^2+3y^2)}{(x^2+y^2)^2} & \text{if } (x, y) \neq (0, 0). \end{cases}$$

If  $(x, y) = (0, 0)$ ,

$$(D_1f)(0, 0) = \lim_{t \rightarrow 0} \frac{f(t, 0) - f(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{t - 0}{t} = 1.$$

If  $(x, y) \neq (0, 0)$ ,

$$\begin{aligned} (D_1f)(x, y) &= \lim_{t \rightarrow 0} \frac{f(x+t, y) - f(x, y)}{t} \\ &= \lim_{t \rightarrow 0} \frac{\frac{(x+t)^3}{(x+t)^2+y^2} - \frac{x^3}{x^2+y^2}}{t} \\ &= \lim_{t \rightarrow 0} \frac{x^2(x^2+3y^2) + tx(2x^2+3y^2) + t^2(x^2+y^2)}{((x+t)^2+y^2)(x^2+y^2)} \\ &= \frac{x^2(x^2+3y^2)}{(x^2+y^2)^2}. \end{aligned}$$

(Or differentiate directly.)

- (2) Show that  $(D_1f)(x, y)$  is bounded. It suffices to show that  $(D_1f)(x, y)$  is bounded if  $(x, y) \neq (0, 0)$ . Write  $x = r \cos \theta$  and  $y = r \sin \theta$  in the polar coordinates. (Here  $r > 0$ .) Hence

$$(D_1f)(x, y) = \frac{x^2(x^2+3y^2)}{(x^2+y^2)^2} = \cos^2 \theta (\cos^2 \theta + 3 \sin^2 \theta)$$

is bounded by  $1 \cdot (1 + 3) = 4$ .

- (3) Show that

$$(D_2f)(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0), \\ \frac{-2x^3y}{(x^2+y^2)^2} & \text{if } (x, y) \neq (0, 0). \end{cases}$$

If  $(x, y) = (0, 0)$ ,

$$(D_2 f)(0, 0) = \lim_{t \rightarrow 0} \frac{f(0, t) - f(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{0 - 0}{t} = 0.$$

If  $(x, y) \neq (0, 0)$ ,

$$\begin{aligned} (D_2 f)(x, y) &= \lim_{t \rightarrow 0} \frac{f(x, y+t) - f(x, y)}{t} \\ &= \lim_{t \rightarrow 0} \frac{\frac{x^3}{x^2 + (y+t)^2} - \frac{x^3}{x^2 + y^2}}{t} \\ &= \lim_{t \rightarrow 0} \frac{-2x^3 y - tx^3}{(x^2 + (y+t)^2)(x^2 + y^2)} \\ &= \frac{-2x^3 y}{(x^2 + y^2)^2}. \end{aligned}$$

(Or differentiate directly.)

(4) Show that  $(D_2 f)(x, y)$  is bounded. Similar to (2).

(5) Show that  $f$  is continuous. Apply Exercise 9.7 to (2)(4).

□

*Proof of (b).*

(1) Write  $\mathbf{u} = (u_1, u_2)$ . The formula

$$(D_{\mathbf{u}} f)(0, 0) = (D_1 f)(0, 0)u_1 + (D_2 f)(0, 0)u_2 = u_1$$

might be false since we don't know if  $f$  is differentiable or not. Actually, we will show that  $(D_{\mathbf{u}} f)(0, 0) = u_1^3 \neq u_1$ .

(2)

$$\begin{aligned} (D_{\mathbf{u}} f)(0, 0) &= \lim_{t \rightarrow 0} \frac{f(tu_1, tu_2) - f(0, 0)}{t} \\ &= \lim_{t \rightarrow 0} \frac{\frac{t^3 u_1^3}{t^2 u_1^2 + t^2 u_2^2} - 0}{t} \\ &= \lim_{t \rightarrow 0} u_1^3 \quad (|\mathbf{u}| = 1) \\ &= u_1^3. \end{aligned}$$

Also  $|(D_{\mathbf{u}} f)(0, 0)| = |u_1|^3 \leq 1$  since  $|\mathbf{u}| = 1$ .

□

*Proof of (d).* (Reductio ad absurdum) If  $f$  were differentiable, then

$$(D_{\mathbf{u}}f)(0,0) = (D_1f)(0,0)u_1 + (D_2f)(0,0)u_2 = u_1$$

(Formula (40) in Chapter 9), contrary to (b).  $\square$

**Exercise 9.15. ...**

*Proof.*

(1)

(2)

$\square$

**Exercise 9.16. ...**

*Proof.*

(1)

(2)

$\square$

**Exercise 9.17. ...**

*Proof.*

(1)

(2)

$\square$

**Exercise 9.18. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.19. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.20. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.21. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.22. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.23. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.24. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.25. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.26. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.27. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.28. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.29. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.30. ...**

*Proof.*

(1)

(2)

□

**Exercise 9.31. ...**

*Proof.*

(1)

(2)

□