Chapter 10: Integration of Differential Forms

Author: Meng-Gen Tsai
Email: plover@gmail.com

Exercise 10.1. *...*

Proof.

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□

Exercise 10.2. *For $i = 1, 2, 3, \ldots$, let $\varphi_i \in \mathscr{C}(\mathbb{R}^1)$ have support in $(2^{-i}, 2^{1-i})$, such that $\int \varphi_i = 1$. Put*

$$f(x, y) = \sum_{i=1}^{\infty} [\varphi_i(x) - \varphi_{i+1}(x)]\varphi_i(y)$$

Then f has compact support in $\mathbb{R}^2$, f is continuous except at $(0, 0)$, and

$$\int dy \int f(x, y)dx = 0 \qquad \text{but} \qquad \int dx \int f(x, y)dy = 1.$$

Observe that f is unbounded in every neighborhood of $(0, 0)$.

Proof.

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Exercise 10.3. *...*

Proof.

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$\square$

Exercise 10.4. *For* $(x, y) \in \mathbb{R}^2$, *define*

$$\mathbf{F}(x, y) = (e^x \cos y - 1, e^x \sin y)$$

Prove that $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$, *where*

$$\begin{aligned} \mathbf{G}_1(x, y) &= (e^x \cos y - 1, y) \\ \mathbf{G}_2(u, v) &= (u, (1 + u) \tan v) \end{aligned}$$

are primitive in some neighborhood of $(0, 0)$. *Compute the Jacobians of* $\mathbf{G}_1$, $\mathbf{G}_2$, $\mathbf{F}$ *at* $(0, 0)$. *Define*

$$\mathbf{H}_2(x, y) = (x, e^x \sin y)$$

and find

$$\mathbf{H}_1(u, v) = (h(u, v), v)$$

so that $\mathbf{F} = \mathbf{H}_1 \circ \mathbf{H}_2$ *is in some neighborhood of* $(0, 0)$.

Proof.

(1) By Definition 10.5,

$$\begin{aligned} \mathbf{G}_1(x, y) &= (e^x \cos y - 1)\mathbf{e}_1 + y\mathbf{e}_2, \\ \mathbf{G}_2(u, v) &= u\mathbf{e}_1 + ((1 + u) \tan v)\mathbf{e}_2 \end{aligned}$$

are primitive in some neighborhood of $(0, 0)$.

(2) *Show that* $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$. Given any $(x, y) \in \mathbb{R}^2$, we have

$$\begin{aligned} (\mathbf{G}_2 \circ \mathbf{G}_1)(x, y) &= \mathbf{G}_2(\mathbf{G}_1(x, y)) \\ &= \mathbf{G}_2(e^x \cos y - 1, y) \\ &= (e^x \cos y - 1, (1 + (e^x \cos y - 1)) \tan y) \\ &= (e^x \cos y - 1, e^x \sin y) \\ &= \mathbf{F}(x, y). \end{aligned}$$

(3) Since

$$\begin{aligned} J_{\mathbf{G}_1}(x, y) &= \begin{bmatrix} e^x \cos y & -e^x \sin y \\ 0 & 1 \end{bmatrix} \\ J_{\mathbf{G}_2}(x, y) &= \begin{bmatrix} 1 & 0 \\ \tan y & (1 + x) \sec^2 y \end{bmatrix} \\ J_{\mathbf{F}}(x, y) &= \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}, \end{aligned}$$

$$
\begin{aligned}
J_{\mathbf{G}_1}(0,0) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
J_{\mathbf{G}_2}(0,0) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
J_{\mathbf{F}}(0,0) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}
$$

(4) Define $h(u,v) = \sqrt{e^{2u} - v^2} - 1$ on

$$
B\left((0,0); \frac{1}{64}\right) \subseteq \mathbb{R}^2.
$$

$h(u,v)$ is well-defined since $e^{2u} - v^2 > 0$ for all $(u,v) \in B\left((0,0); \frac{1}{64}\right)$.

(5) Given any $(x,y) \in \mathbb{R}^2$, we have

$$
\begin{aligned}
(\mathbf{H}_1 \circ \mathbf{H}_2)(x,y) &= \mathbf{H}_1(\mathbf{H}_2(x,y)) \\
&= \mathbf{H}_1(x, e^x \sin y) \\
&= (\sqrt{e^{2x} - (e^x \sin y)^2} - 1, e^x \sin y) \\
&= (e^x \cos y - 1, e^x \sin y) \\
&= \mathbf{F}(x,y).
\end{aligned}
$$

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Exercise 10.5. ...

Proof.

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Exercise 10.6. ...

Proof.

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Exercise 10.7. ...

Proof.

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Exercise 10.8. ...

Proof.

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Exercise 10.9. ...

Proof.

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Exercise 10.10. ...

Proof.

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Exercise 10.11. ...

Proof.

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Exercise 10.12. ...

Proof.

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Exercise 10.13. ...

Proof.

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Exercise 10.14. ...

Proof.

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Exercise 10.15. *If ω and λ are k- and m-forms, respectively, prove that*

$$\omega \wedge \lambda = (-1)^{km} \lambda \wedge \omega.$$

Proof.

(1) Write
$$\omega = \sum_I b_I(\mathbf{x}) dx_I, \qquad \lambda = \sum_J c_J(\mathbf{x}) dx_J$$
in the stardard presentations, where I and J range over all increasing k-indices and over all increasing m-indices taken from the set $\{1, \ldots, n\}$.

(2) *Show that* $dx_I \wedge dx_J = (-1)^{km} dx_J \wedge dx_I$.

$$\begin{aligned}
dx_I \wedge dx_J &= dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_J \\
&= (-1)^m dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}} \wedge dx_J \wedge dx_{i_k} \\
&= (-1)^{2m} dx_{i_1} \wedge \cdots \wedge dx_{i_{k-2}} \wedge dx_J \wedge dx_{i_{k-1}} \wedge dx_{i_k} \\
&\cdots \\
&= (-1)^{km} dx_J \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&= (-1)^{km} dx_J \wedge dx_I.
\end{aligned}$$

(3)

$$\begin{aligned}
\omega \wedge \lambda &= \sum_{I,J} b_I(\mathbf{x}) c_J(\mathbf{x}) dx_I \wedge dx_J \\
&= (-1)^{km} \sum_{J,I} c_J(\mathbf{x}) b_I(\mathbf{x}) dx_J \wedge dx_I \\
&= (-1)^{km} \lambda \wedge \omega.
\end{aligned}$$

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Exercise 10.16. ...

Proof.

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Exercise 10.17. ...

Proof.

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Exercise 10.18. ...

Proof.

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Exercise 10.19. ...

Proof.

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Exercise 10.20. ...

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Exercise 10.21. ...

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Exercise 10.22. ...

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Exercise 10.23. ...

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Exercise 10.24. ...

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Exercise 10.25. ...

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Exercise 10.26. ...

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Exercise 10.27. ...

Proof.

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Exercise 10.28. ...

Proof.

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Exercise 10.29. ...

Proof.

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Exercise 10.30. ...

Proof.

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Exercise 10.31. ...

Proof.

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Exercise 10.32. ...

Proof.

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