Chapter 4: Determinants

Author: Meng-Gen Tsai
Email: plover@gmail.com

Section 4.1: Determinants of Order 2

Exercise 4.1.1. *Label the following statements as being true or false.*

(a) *The function* $\det : \mathsf{M}_{2\times 2}(F) \to F$ *is a linear transformation.*

(b) *The determinant of a* 2×2 *matrix is a linear function of each row of the matrix when the other row is held fixed.*

(c) *If* $A \in \mathsf{M}_{2\times 2}(F)$ *and* $\det(A) = 0$*, then* A *is invertible.*

(d) *If* u *and* v *are vectors in* $\mathbb{R}^2$ *emanating from the origin, then the area of the parallelogram having* u *and* v *as adjacent side is*

$$\det \begin{pmatrix} u \\ v \end{pmatrix}.$$

(e) *A coordinate system is right-handed if and only if its orientation equals* 1*.*

Proof of (a). False. Example 4.1.1, or take

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathsf{M}_{2\times 2}(F) \text{ and } B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \mathsf{M}_{2\times 2}(F).$$

Then $\det(A + B) = \det(I_2) = 1 \neq 0 = 0 + 0 = \det(A) + \det(B)$. $\square$

Proof of (b). True. Proposition 4.1. $\square$

Proof of (c). False. Proposition 4.2. $\square$

Proof of (d). False. The area should be

$$O \begin{pmatrix} u \\ v \end{pmatrix} \cdot \det \begin{pmatrix} u \\ v \end{pmatrix} = \left| \det \begin{pmatrix} u \\ v \end{pmatrix} \right|.$$

$\square$

Proof of (e). True. See Exercise 4.1.12. $\square$

Exercise 4.1.2. *Compute the determinants of the following elements of* $\mathsf{M}_{2\times 2}(\mathbb{R})$*.*

(a) $\begin{pmatrix} 6 & -3 \\ 2 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} -5 & 2 \\ 6 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 8 & 0 \\ 3 & -1 \end{pmatrix}$

Proof of (a).

$$\det \begin{pmatrix} 6 & -3 \\ 2 & 4 \end{pmatrix} = 6 \cdot 4 - (-3) \cdot 2 = 24 + 6 = 30.$$

□

Proof of (b).

$$\det \begin{pmatrix} -5 & 2 \\ 6 & 1 \end{pmatrix} = (-5) \cdot 1 - 2 \cdot 6 = -5 - 12 = -17.$$

□

Proof of (c).

$$\det \begin{pmatrix} 8 & 0 \\ 3 & -1 \end{pmatrix} = 8 \cdot (-1) - 0 \cdot 3 = -8.$$

□

Exercise 4.1.3. *Compute the determinants of the following elements of* $\mathsf{M}_{2\times 2}(\mathbb{C})$.

(a) $\begin{pmatrix} -1+i & 1-4i \\ 3+2i & 2-3i \end{pmatrix}$

(b) $\begin{pmatrix} 5-2i & 6+4i \\ -3+i & 7i \end{pmatrix}$

(c) $\begin{pmatrix} 2i & 3 \\ 4 & 6i \end{pmatrix}$

Proof of (a).

$$\begin{aligned} \det \begin{pmatrix} -1+i & 1-4i \\ 3+2i & 2-3i \end{pmatrix} &= (-1+i) \cdot (2-3i) - (1-4i) \cdot (3+2i) \\ &= (1+5i) - (11-10i) \\ &= -10 + 15i. \end{aligned}$$

□

Proof of (b).

$$\det\begin{pmatrix} 5-2i & 6+4i \\ -3+i & 7i \end{pmatrix} = (5-2i)\cdot(7i) - (6+4i)\cdot(-3+i)$$
$$= (14+35i) - (-22-6i)$$
$$= 36+41i.$$

□

Proof of (c).

$$\det\begin{pmatrix} 2i & 3 \\ 4 & 6i \end{pmatrix} = (2i)\cdot(6i) - 3\cdot 4 = -12 - 12 = -24.$$

□

Exercise 4.1.4. *For each of the following pairs of vectors u and v in $\mathbb{R}^2$, compute the area of the parallelogram determined by u and v.*

(a) $u = (3,-2)$ *and* $v = (2,5)$

(b) $u = (1,3)$ *and* $v = (-3,1)$

(c) $u = (4,-1)$ *and* $v = (-6,-2)$

(d) $u = (3,4)$ *and* $v = (2,-6)$

Proof of (a).

$$\left|\det\begin{pmatrix} 3 & -2 \\ 2 & 5 \end{pmatrix}\right| = |19| = 19.$$

□

Proof of (b).

$$\left|\det\begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}\right| = |10| = 10.$$

□

Proof of (c).

$$\left|\det\begin{pmatrix} 4 & -1 \\ -6 & -2 \end{pmatrix}\right| = |-14| = 14.$$

□

Proof of (d).

$$\left|\det\begin{pmatrix}3 & 4\\ 2 & -6\end{pmatrix}\right| = |-26| = 26.$$

$\square$

Exercise 4.1.5. *Prove that if B is the matrix obtained by interchanging the rows of a 2×2 matrix A, then $\det(B) = -\det(A)$.*

Proof. Write

$$A = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathsf{M}_{2\times 2}(F).$$

Then

$$B = \begin{pmatrix}c & d\\ a & b\end{pmatrix} \in \mathsf{M}_{2\times 2}(F).$$

Then $\det(B) = cb - ad = -(ad - bc) = -\det(A)$. $\square$

Exercise 4.1.6. *Prove that if the two columns of $A \in \mathsf{M}_{2\times 2}(F)$ are identical, then $\det(A) = 0$.*

Proof. By assumption, write

$$A = \begin{pmatrix}a & a\\ c & c\end{pmatrix} \in \mathsf{M}_{2\times 2}(F).$$

Then $\det(A) = ac - ac = 0$. $\square$

Exercise 4.1.7. *Prove that for any $A \in \mathsf{M}_{2\times 2}(F)$, $\det(A^t) = \det(A)$.*

Proof. Write

$$A = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathsf{M}_{2\times 2}(F),$$

then

$$A^t = \begin{pmatrix}a & c\\ b & d\end{pmatrix} \in \mathsf{M}_{2\times 2}(F).$$

So $\det(A) = ad - bc = ad - cb = \det(A^t)$. $\square$

Exercise 4.1.8. *Prove that if $A \in \mathsf{M}_{2\times 2}(F)$ is upper triangular, then $\det(A)$ equals the product of the diagonal entries of A.*

Proof. Write
$$A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathsf{M}_{2\times 2}(F)$$
since A is upper triangular. Then $\det(A) = ad$, which is equal to the product of the diagonal entries, a and d, of A. □

Exercise 4.1.9. *Prove that for any $A, B \in \mathsf{M}_{2\times 2}(F)$ we have $\det(AB) = \det(A)\cdot\det(B)$.*

Proof. Write
$$\begin{aligned} A &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{M}_{2\times 2}(F), \\ B &= \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in \mathsf{M}_{2\times 2}(F). \end{aligned}$$
Then
$$AB = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} \in \mathsf{M}_{2\times 2}(F).$$
A direct calculation shows
$$\begin{aligned} \det(AB) &= (ae+bg)(cf+dh) - (af+bh)(ce+dg) \\ &= (acef + adeh + bcfg + bdgh) - (acef + adfg + bceh + bdgh) \\ &= adeh + bcfg - adfg - bceh \\ &= (ad-bc)(eh-fg) \\ &= \det(A)\det(B). \end{aligned}$$
□

Exercise 4.1.10. *The **classical adjoint** of a 2×2 matrix $A \in \mathsf{M}_{2\times 2}(F)$ is the matrix*
$$C = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}.$$
Prove

(a) $CA = AC = [\det(A)]I$.

(b) $\det(C) = \det(A)$.

(c) The classical adjoint of A^t is C^t.

(d) If A is invertible, then $A^{-1} = [\det(A)]^{-1}C$.

Note that

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Proof of (a).

$$\begin{aligned} CA &= \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \\ &= \begin{pmatrix} A_{22}A_{11} - A_{12}A_{21} & A_{22}A_{12} - A_{12}A_{22} \\ -A_{21}A_{11} + A_{11}A_{21} & -A_{21}A_{12} + A_{11}A_{22} \end{pmatrix} \\ &= \begin{pmatrix} \det(A) & 0 \\ 0 & \det(A) \end{pmatrix} \\ &= [\det(A)]I. \end{aligned}$$

$$\begin{aligned} AC &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \\ &= \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & -A_{11}A_{12} + A_{12}A_{11} \\ A_{21}A_{22} - A_{22}A_{21} & -A_{21}A_{12} + A_{22}A_{11} \end{pmatrix} \\ &= \begin{pmatrix} \det(A) & 0 \\ 0 & \det(A) \end{pmatrix} \\ &= [\det(A)]I. \end{aligned}$$

□

Proof of (b).

$$\begin{aligned} \det(C) &= A_{22}A_{11} - (-A_{12})(-A_{21}) \\ &= A_{11}A_{22} - A_{12}A_{21} \\ &= \det(A). \end{aligned}$$

□

Proof of (c).

$$A^t = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix}.$$

The classical adjoint of A^t is

$$\begin{pmatrix} A_{22} & -A_{21} \\ -A_{12} & A_{11} \end{pmatrix} = C^t.$$

□

Proof of (d). Proposition 4.2. □

Exercise 4.1.11. *Let $\delta : \mathsf{M}_{2\times 2}(F) \to F$ be a function with the following three properties.*

(i) *δ is a linear function of each row of the matrix when the other row is held fixed.*

(ii) *If the two rows of $A \in \mathsf{M}_{2\times 2}(F)$ are identical, then $\delta(A) = 0$.*

(iii) *If I is the 2×2 identity matrix, then $\delta(I) = 1$.*

Prove that $\delta(A) = \det(A)$ for all $A \in \mathsf{M}_{2\times 2}(F)$. (This result is generalized in Section 4.5.)

Proof. Write

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

(1) *If u, v are elements of F^2 and k is a scalar, then*

$$\delta \begin{pmatrix} u \\ v + ku \end{pmatrix} = \delta \begin{pmatrix} u + kv \\ v \end{pmatrix} = \delta \begin{pmatrix} u \\ v \end{pmatrix}.$$

In fact,

$$\begin{aligned} \delta \begin{pmatrix} u \\ v + ku \end{pmatrix} &= \delta \begin{pmatrix} u \\ v \end{pmatrix} + \delta \begin{pmatrix} u \\ ku \end{pmatrix} && \text{(Property (i))} \\ &= \delta \begin{pmatrix} u \\ v \end{pmatrix} + k\delta \begin{pmatrix} u \\ u \end{pmatrix} && \text{(Property (i))} \\ &= \delta \begin{pmatrix} u \\ v \end{pmatrix}. && \text{(Property (ii))} \end{aligned}$$

Similarly, $\delta \begin{pmatrix} u + kv \\ v \end{pmatrix} = \delta \begin{pmatrix} u \\ v \end{pmatrix}$.

(2) *If u, v are elements of F^2, then*

$$\delta \begin{pmatrix} u \\ v \end{pmatrix} = -\delta \begin{pmatrix} v \\ u \end{pmatrix}.$$

In fact,

$$\begin{aligned} 0 &= \delta \begin{pmatrix} u + v \\ u + v \end{pmatrix} && \text{(Property (ii))} \\ &= \delta \begin{pmatrix} u + v \\ u \end{pmatrix} + \delta \begin{pmatrix} u + v \\ v \end{pmatrix} && \text{(Property (i))} \\ &= \delta \begin{pmatrix} v \\ u \end{pmatrix} + \delta \begin{pmatrix} u \\ v \end{pmatrix}. && \text{((1))} \end{aligned}$$

(3) *If v is an element of F^2, then*

$$\delta\begin{pmatrix}0\\v\end{pmatrix}=0.$$

In fact,

$$\begin{aligned}\delta\begin{pmatrix}0\\v\end{pmatrix}&=\delta\begin{pmatrix}0+0\\v\end{pmatrix}\\&=\delta\begin{pmatrix}0\\v\end{pmatrix}+\delta\begin{pmatrix}0\\v\end{pmatrix}. &&\text{(Property (i))}\end{aligned}$$

In particular, $\delta\begin{pmatrix}0\\v\end{pmatrix}=0=\det\begin{pmatrix}0\\v\end{pmatrix}$.

(4) To show $\delta(A)=\det(A)$, we consider three possible cases about the first row: $A_{11}\neq 0$, $A_{12}\neq 0$, or $A_{11}=A_{12}=0$. The case $A_{11}=A_{12}=0$ is proved in (3). We prove the rest two cases in (5) and (6). Write

$$u=(A_{11},A_{12})\text{ and }v=(A_{21},A_{22}).$$

(5) *Show that $\delta(A)=\det(A)$ if $A_{11}\neq 0$.* So

$$\begin{aligned}\delta(A)&=\delta\begin{pmatrix}u\\v\end{pmatrix}\\&=\delta\begin{pmatrix}u\\v-\frac{A_{21}}{A_{11}}u\end{pmatrix} &&\text{((1))}\\&=\delta\begin{pmatrix}A_{11}&A_{12}\\0&A_{22}-\frac{A_{12}A_{21}}{A_{11}}\end{pmatrix}\\&=\left(A_{22}-\frac{A_{12}A_{21}}{A_{11}}\right)\delta\begin{pmatrix}A_{11}&A_{12}\\0&1\end{pmatrix} &&\text{(Property (i))}\\&=\left(A_{22}-\frac{A_{12}A_{21}}{A_{11}}\right)\delta\begin{pmatrix}A_{11}&0\\0&1\end{pmatrix} &&\text{((1))}\\&=A_{11}\left(A_{22}-\frac{A_{12}A_{21}}{A_{11}}\right)\delta\begin{pmatrix}1&0\\0&1\end{pmatrix} &&\text{(Property (i))}\\&=\det(A)\delta(I)\\&=\det(A). &&\text{(Property (iii))}\end{aligned}$$

(6) *Show that* $\delta(A) = \det(A)$ *if* $A_{12} \neq 0$. So

$$\begin{aligned}
\delta(A) &= \delta \begin{pmatrix} u \\ v \end{pmatrix} \\
&= \delta \begin{pmatrix} u \\ v - \frac{A_{22}}{A_{12}} u \end{pmatrix} && ((1)) \\
&= \delta \begin{pmatrix} A_{11} & A_{12} \\ A_{21} - \frac{A_{22}A_{11}}{A_{12}} & 0 \end{pmatrix} \\
&= \left(A_{21} - \frac{A_{22}A_{11}}{A_{12}} \right) \delta \begin{pmatrix} A_{11} & A_{12} \\ 1 & 0 \end{pmatrix} && \text{(Property (i))} \\
&= \left(A_{21} - \frac{A_{22}A_{11}}{A_{12}} \right) \delta \begin{pmatrix} 0 & A_{12} \\ 1 & 0 \end{pmatrix} && ((1)) \\
&= A_{12} \left(A_{21} - \frac{A_{22}A_{11}}{A_{12}} \right) \delta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} && \text{(Property (i))} \\
&= -A_{12} \left(A_{21} - \frac{A_{22}A_{11}}{A_{12}} \right) \delta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} && ((2)) \\
&= \det(A)\delta(I) \\
&= \det(A). && \text{(Property (iii))}
\end{aligned}$$

□

Exercise 4.1.12. *Let* $\{u, v\}$ *be an ordered basis for* $\mathbb{R}^2$. *Prove that*

$$O \begin{pmatrix} u \\ v \end{pmatrix} = 1$$

if and only if $\{u, v\}$ *forms a right-handed coordinate system. (Hint: Recall the definition of a rotation given in Example 2.1.2.)*

If $\beta = \{u, v\}$ is an ordered basis for $\mathbb{R}^2$, define the orientation of β as

$$O \begin{pmatrix} u \\ v \end{pmatrix} = \frac{\det \begin{pmatrix} u \\ v \end{pmatrix}}{\left| \det \begin{pmatrix} u \\ v \end{pmatrix} \right|}.$$

A coordinate system $\{u, v\}$ is called right-handed if u can be rotated in a counterclockwise direction through an angle θ $(0 < \theta < \pi)$ to coincide with v.

Example 2.1.2. For any angle θ, define $\mathsf{T}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\mathsf{T}_\theta(a_1, a_2) = (a_1 \cos\theta - a_2 \sin\theta, a_1 \sin\theta + a_2 \cos\theta).$$

T_θ is called the rotation by θ.

Proof.

(1) By Example 2.1.2, for any coordinate system $\{u, v\}$, there is $0 < \theta < 2\pi$ and $\alpha > 0$ such that $v = \alpha \mathsf{T}_\theta(u)$. Write $u = (u_1, u_2) \in \mathbb{R}^2, v = (v_1, v_2) \in \mathbb{R}^2$.

(2) Calculate $\det \begin{pmatrix} u \\ v \end{pmatrix}$.

$$\begin{aligned}
\det \begin{pmatrix} u \\ v \end{pmatrix} &= \det \begin{pmatrix} u \\ \alpha \mathsf{T}_\theta(u) \end{pmatrix} \\
&= \alpha \det \begin{pmatrix} u \\ \mathsf{T}_\theta(u) \end{pmatrix} \\
&= \alpha \det \begin{pmatrix} u_1 & u_2 \\ u_1 \cos\theta - u_2 \sin\theta & u_1 \sin\theta + u_2 \cos\theta \end{pmatrix} \\
&= \alpha(u_1^2 + u_2^2) \sin\theta.
\end{aligned}$$

(3)

$$\begin{aligned}
O \begin{pmatrix} u \\ v \end{pmatrix} = 1 &\Longleftrightarrow \det \begin{pmatrix} u \\ v \end{pmatrix} = \alpha(u_1^2 + u_2^2) \sin\theta > 0 \\
&\Longleftrightarrow \sin\theta > 0 \\
&\Longleftrightarrow 0 < \theta < \pi \\
&\Longleftrightarrow \{u, v\} \text{ is a right-handed coordinate system.}
\end{aligned}$$

□

Section 4.2: Determinants of Order n

Exercise 4.2.2. *Find the value of k that satisfies the following equation.*

$$\det \begin{pmatrix} 3a_1 & 3a_2 & 3a_3 \\ 3b_1 & 3b_2 & 3b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{pmatrix} = k \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

Proof (Exercise 4.2.25). By Exercise 4.2.25, $\det(3A) = 3^3 \det(A)$ for any $A \in \mathsf{M}_{3\times 3}(F)$, or $k = 3^3 = 27$. □

Exercise 4.2.26. *Let $A \in \mathsf{M}_{n\times n}(F)$. Under what conditions is* $\det(-A) = \det(A)$*?*

Proof (Exercise 4.2.25). By Exercise 4.2.25, $\det(-A) = (-1)^n \det(A)$ for any $A \in \mathsf{M}_{n\times n}(F)$. That is, n is even if and only if $\det(-A) = \det(A)$. $\square$

Section 4.3: Properties of Determinants

Exercise 4.3.9. *A matrix $M \in \mathsf{M}_{n\times n}(\mathbb{C})$ is called nilpotent if, for some positive integer k, $M^k = O$, where O is the $n \times n$ zero matrix. Prove that if M is nilpotent, then* $\det(M) = 0$.

Proof. Given any nilpotent matrix $M \in \mathsf{M}_{n\times n}(\mathbb{C})$ such that $M^k = O$ for some $k \in \mathbb{Z}^+$.

$$\begin{aligned} M^k = O &\Longrightarrow \det\left(M^k\right) = \det(O) \\ &\Longleftrightarrow \det(M)^k = 0 && \text{(Theorem 4.7)} \\ &\Longleftrightarrow \det(M) = 0. \end{aligned}$$

$\square$

Exercise 4.3.11. *A matrix $Q \in \mathsf{M}_{n\times n}(\mathbb{R})$ is called orthogonal if $QQ^t = I$. Prove that if Q is orthogonal, then* $\det(Q) = \pm 1$.

Proof. By the orthogonality of Q, $QQ^t = I$. So

$$\begin{aligned} QQ^t = I &\Longrightarrow \det\left(QQ^t\right) = \det(I) \\ &\Longleftrightarrow \det(Q)\det\left(Q^t\right) = \det(I) && \text{(Theorem 4.7)} \\ &\Longleftrightarrow \det(Q)\det(Q) = \det(I) && \text{(Theorem 4.8)} \\ &\Longleftrightarrow \det(Q)^2 = 1 && \text{(Example 4.2.4)} \\ &\Longleftrightarrow \det(Q) = \pm 1. \end{aligned}$$

$\square$

Exercise 4.3.14. *Prove that if $A, B \in \mathsf{M}_{n\times n}(F)$ are similar, then* $\det(A) = \det(B)$.

Proof. Since A, B are similar, there exists an invertible matrix Q such that

$B = Q^{-1}AQ$. So

$$\begin{aligned}
\det(B) &= \det\bigl(Q^{-1}AQ\bigr) \\
&= \det\bigl(Q^{-1}\bigr)\det(A)\det(Q) && \text{(Theorem 4.7)} \\
&= \det(Q)\det\bigl(Q^{-1}\bigr)\det(A) && \text{(F is field)} \\
&= \det\bigl(QQ^{-1}\bigr)\det(A) && \text{(Theorem 4.7)} \\
&= \det(I)\det(A) \\
&= 1 \cdot \det(A) && \text{(Example 4.2.4)} \\
&= \det(A).
\end{aligned}$$

□