

Chapter 10: Integration of Differential Forms

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Exercise 10.1. ...

Proof.

(1)

(2)

□

Exercise 10.2. For $i = 1, 2, 3, \dots$, let $\varphi_i \in \mathcal{C}(\mathbb{R}^1)$ have support in $(2^{-i}, 2^{1-i})$, such that $\int \varphi_i = 1$. Put

$$f(x, y) = \sum_{i=1}^{\infty} [\varphi_i(x) - \varphi_{i+1}(x)] \varphi_i(y)$$

Then f has compact support in \mathbb{R}^2 , f is continuous except at $(0, 0)$, and

$$\int dy \int f(x, y) dx = 0 \quad \text{but} \quad \int dx \int f(x, y) dy = 1.$$

Observe that f is unbounded in every neighborhood of $(0, 0)$.

Proof.

(1)

(2)

□

Exercise 10.3. ...

Proof.

(1)

(2)

□

Exercise 10.4. For $(x, y) \in \mathbb{R}^2$, define

$$\mathbf{F}(x, y) = (e^x \cos y - 1, e^x \sin y)$$

Prove that $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$, where

$$\mathbf{G}_1(x, y) = (e^x \cos y - 1, y)$$

$$\mathbf{G}_2(u, v) = (u, (1 + u) \tan v)$$

are primitive in some neighborhood of $(0, 0)$. Compute the Jacobians of \mathbf{G}_1 , \mathbf{G}_2 , \mathbf{F} at $(0, 0)$. Define

$$\mathbf{H}_2(x, y) = (x, e^x \sin y)$$

and find

$$\mathbf{H}_1(u, v) = (h(u, v), v)$$

so that $\mathbf{F} = \mathbf{H}_1 \circ \mathbf{H}_2$ is in some neighborhood of $(0, 0)$.

Proof.

(1) By Definition 10.5,

$$\mathbf{G}_1(x, y) = (e^x \cos y - 1)\mathbf{e}_1 + y\mathbf{e}_2,$$

$$\mathbf{G}_2(u, v) = u\mathbf{e}_1 + ((1 + u) \tan v)\mathbf{e}_2$$

are primitive in some neighborhood of $(0, 0)$.

(2) Show that $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$. Given any $(x, y) \in \mathbb{R}^2$, we have

$$\begin{aligned} (\mathbf{G}_2 \circ \mathbf{G}_1)(x, y) &= \mathbf{G}_2(\mathbf{G}_1(x, y)) \\ &= \mathbf{G}_2(e^x \cos y - 1, y) \\ &= (e^x \cos y - 1, (1 + (e^x \cos y - 1)) \tan y) \\ &= (e^x \cos y - 1, e^x \sin y) \\ &= \mathbf{F}(x, y). \end{aligned}$$

(3) Since

$$\begin{aligned} J_{\mathbf{G}_1}(x, y) &= \begin{bmatrix} e^x \cos y & -e^x \sin y \\ 0 & 1 \end{bmatrix} \\ J_{\mathbf{G}_2}(x, y) &= \begin{bmatrix} 1 & 0 \\ \tan y & (1 + x) \sec^2 y \end{bmatrix} \\ J_{\mathbf{F}}(x, y) &= \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}, \end{aligned}$$

$$\begin{aligned}
J_{\mathbf{G}_1}(0,0) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
J_{\mathbf{G}_2}(0,0) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
J_{\mathbf{F}}(0,0) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

(4) Define $h(u, v) = \sqrt{e^{2u} - v^2} - 1$ on

$$B\left((0,0); \frac{1}{64}\right) \subseteq \mathbb{R}^2.$$

$h(u, v)$ is well-defined since $e^{2u} - v^2 > 0$ for all $(u, v) \in B((0,0); \frac{1}{64})$.

(5) Given any $(x, y) \in \mathbb{R}^2$, we have

$$\begin{aligned}
(\mathbf{H}_1 \circ \mathbf{H}_2)(x, y) &= \mathbf{H}_1(\mathbf{H}_2(x, y)) \\
&= \mathbf{H}_1(x, e^x \sin y) \\
&= (\sqrt{e^{2x} - (e^x \sin y)^2} - 1, e^x \sin y) \\
&= (e^x \cos y - 1, e^x \sin y) \\
&= \mathbf{F}(x, y).
\end{aligned}$$

□

Exercise 10.5. *Formulate and prove an analogue of Theorem 10.8, in which K is a compact subset of an arbitrary metric space. (Replace the functions φ_i that occur in the proof of Theorem 10.8 by functions of the type constructed in Exercise 4.22.)*

Proof (Theorem 10.8).

- (1) *(Partitions of unity.) Suppose K is a compact subset of a metric space X , and $\{V_\alpha\}$ is an open cover of K . Then there exist functions $\psi_1, \dots, \psi_s \in \mathcal{C}(X)$ such that*
 - (a) $0 \leq \psi_i \leq 1$ for $1 \leq i \leq s$.
 - (b) *each ψ_i has its support in some V_α , and*
 - (c) $\psi_1(x) + \dots + \psi_s(x) = 1$ for every $x \in K$.
- (2) It is trivial that some $V_\alpha = X$ by taking $s = 1$ and $\psi_1(x) = 1 \in \mathcal{C}(X)$. Now we assume that all $V_\alpha \subsetneq X$.
- (3) Associate with each $x \in K$ an index $\alpha(x)$ so that $x \in V_{\alpha(x)}$. Then there are open balls $B(x)$ and $W(x)$, centered at x , with

$$x \in B(x) \subseteq \overline{B(x)} \subseteq W(x) \subseteq \overline{W(x)} \subseteq V_{\alpha(x)}$$

(Since $V_{\alpha(x)}$ is open, there exists $r > 0$ such that $B(x; r) \subseteq V_{\alpha(x)}$. Take $B(x) = B(x; \frac{r}{89})$ and $W(x) = B(x; \frac{r}{64})$.)

- (4) Since K is compact, there are finitely many points $x_1, \dots, x_s \in K$ such that

$$K \subseteq B(x_1) \cup \dots \cup B(x_s).$$

Note that

- (a) $\overline{B(x_i)}$ is a nonempty closed set since $x_i \in B(x_i) \subseteq \overline{B(x_i)}$.
- (b) $X - W(x_i) \supseteq X - V_{\alpha(x_i)}$ is a nonempty closed set by the assumption in (2).
- (c) $\overline{B(x_i)} \cap (X - W(x_i)) \subseteq W(x_i) \cap (X - W(x_i)) = \emptyset$.

By Exercise 4.22, there is a function

$$\varphi_i(x) = \frac{\rho_{\overline{B(x_i)}}(x)}{\rho_{\overline{B(x_i)}}(x) + \rho_{X - W(x_i)}(x)} \in \mathcal{C}(X)$$

such that $\varphi_i(x) = 1$ on $\overline{B(x_i)}$, $\varphi_i(x) = 0$ outside $W(x_i)$, and $0 \leq \varphi_i(x) \leq 1$ on X for $1 \leq i \leq s$.

- (5) Define $\psi_1 = \varphi_1 \in \mathcal{C}(X)$ and

$$\psi_{i+1} = (1 - \varphi_1) \cdots (1 - \varphi_i) \varphi_{i+1} \in \mathcal{C}(X)$$

for $1 \leq i \leq s - 1$. Properties (a) and (b) in (1) are clear. Also,

$$\psi_1(x) + \dots + \psi_s(x) = 1 - (1 - \varphi_1(x)) \cdots (1 - \varphi_s(x))$$

by the construction of ψ_i . If $x \in K$, then $x \in B(x_i)$ for some i , hence $\varphi_i(x) = 1$, and the product $(1 - \varphi_1(x)) \cdots (1 - \varphi_s(x)) = 0$. This proves property (c) in (1).

□

Exercise 10.6. Strengthen the conclusion of Theorem 10.8 by showing that the functions ψ_i can be made differentiable, and even infinitely differentiable. (Use Exercise 8.1 in the construction of the auxiliary functions ψ_i .)

Proof (Theorem 10.8).

- (1) It is trivial that some $V_\alpha = \mathbb{R}^n$ by taking $s = 1$ and $\psi_1(\mathbf{x}) = 1 \in \mathcal{C}^\infty(\mathbb{R}^n)$. Now we assume that all $V_\alpha \subsetneq \mathbb{R}^n$.
- (2) Associate with each $\mathbf{x} \in K$ an index $\alpha(x)$ so that $\mathbf{x} \in V_{\alpha(x)}$. Then there are open n -cells $B(\mathbf{x})$ and $W(\mathbf{x})$ (Definition 10.1), centered at \mathbf{x} , with

$$\mathbf{x} \in B(\mathbf{x}) \subseteq \overline{B(\mathbf{x})} \subseteq W(\mathbf{x}) \subseteq \overline{W(\mathbf{x})} \subseteq V_{\alpha(\mathbf{x})}$$

(Since $V_{\alpha(\mathbf{x})}$ is open, there exists $r > 0$ such that $B(\mathbf{x}; r) \subseteq V_{\alpha(\mathbf{x})}$. Take

$$B(\mathbf{x}) = I\left(\mathbf{x}; \frac{r}{89\sqrt{n}}\right), \quad W(\mathbf{x}) = I\left(\mathbf{x}; \frac{r}{64\sqrt{n}}\right)$$

where $I(\mathbf{p}; r)$ is the open n -cell centered at $\mathbf{p} = (p_1, \dots, p_n)$ defined by

$$I(\mathbf{p}; r) = (p_1 - r, p_1 + r) \times \cdots \times (p_n - r, p_n + r) \subseteq \mathbb{R}^n.$$

(3) Define

$$f(y) = \begin{cases} e^{-\frac{1}{y^2}} & (y > 0), \\ 0 & (y \leq 0). \end{cases}$$

$f(y) \in \mathcal{C}^\infty(\mathbb{R}^1)$ by applying the similar argument in Exercise 8.1.

(4) Given any $\mathbf{x} = (x_1, \dots, x_n) \in K$ and construct $B(\mathbf{x})$ and $W(\mathbf{x})$ as in (2). Define

$$g_{x_j}(y_j) = \frac{f(y_j)}{f(y_j) + f\left(\frac{r}{64\sqrt{n}} - \frac{r}{89\sqrt{n}} - y_j\right)}$$

for $1 \leq j \leq n$. g_{x_j} is well-defined and $g_{x_j} \in \mathcal{C}^\infty(\mathbb{R}^1)$. So

$$g_{x_j}(y_j) = \begin{cases} 0 & \text{if } y_j \leq 0, \\ \text{strictly increasing} & \text{if } 0 \leq y_j \leq \frac{r}{64\sqrt{n}} - \frac{r}{89\sqrt{n}}, \\ 1 & \text{if } y_j \geq \frac{r}{64\sqrt{n}} - \frac{r}{89\sqrt{n}}. \end{cases}$$

Next, define

$$h_{x_j}(y_j) = g_{x_j}\left(y_j - x_j + \frac{r}{64\sqrt{n}}\right) g_{x_j}\left(x_j + \frac{r}{64\sqrt{n}} - y_j\right)$$

for $1 \leq j \leq n$. $h_{x_j} \in \mathcal{C}^\infty(\mathbb{R}^1)$. So

$$h_{x_j}(y_j) = \begin{cases} 0 & \text{if } y_j \leq x_j - \frac{r}{64\sqrt{n}}, \\ \text{strictly increasing} & \text{if } x_j - \frac{r}{64\sqrt{n}} \leq y_j \leq x_j - \frac{r}{89\sqrt{n}}, \\ 1 & \text{if } x_j - \frac{r}{89\sqrt{n}} \leq y_j \leq x_j + \frac{r}{89\sqrt{n}}, \\ \text{strictly decreasing} & \text{if } x_j + \frac{r}{89\sqrt{n}} \leq y_j \leq x_j + \frac{r}{64\sqrt{n}}, \\ 0 & \text{if } y_j \geq x_j + \frac{r}{64\sqrt{n}}. \end{cases}$$

Finally we define $\mathbf{h}_{\mathbf{x}} : \mathbb{R}^n \rightarrow \mathbb{R}^1$ by

$$\mathbf{h}_{\mathbf{x}}(\mathbf{y}) = \prod_{j=1}^n h_{x_j}(y_j)$$

where $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$. Hence, $\mathbf{h}_{\mathbf{x}} \in \mathcal{C}^\infty(\mathbb{R}^n)$ (Theorem 9.21). Also, $\mathbf{h}_{\mathbf{x}}(\mathbf{y}) = 1$ on $B(\mathbf{x})$, $\mathbf{h}_{\mathbf{x}}(\mathbf{y}) = 0$ outside $W(\mathbf{x})$, and $0 \leq \mathbf{h}_{\mathbf{x}}(\mathbf{y}) \leq 1$.

- (5) Since K is compact, there are finitely many points $\mathbf{x}_1, \dots, \mathbf{x}_s \in K$ such that

$$K \subseteq B(\mathbf{x}_1) \cup \dots \cup B(\mathbf{x}_s).$$

Take

$$\varphi_i(\mathbf{x}) = \mathbf{h}_{\mathbf{x}_i}(\mathbf{x}) \in \mathcal{C}^\infty(\mathbb{R}^n)$$

for $1 \leq i \leq s$.

- (6) The rest are the same as the proof of Theorem 10.8 or Exercise 10.5.

□

Exercise 10.7.

- (a) Show that the simplex Q^k is the smallest convex subset of \mathbb{R}^k such that contains $\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_k$.
- (b) Show that affine mappings take convex sets to convex sets.

Proof of (a).

- (1) Show that Q^k contains $\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_k$. Recall

$$Q^k = \{(x_1, \dots, x_k) \in \mathbb{R}^k : x_1 + \dots + x_k \leq 1 \text{ and } x_1, \dots, x_k \geq 0\}$$

(Example 10.14). Hence $\mathbf{0} = (0, \dots, 0) \in Q^k$ and

$$\mathbf{e}_i = (0, \dots, \underbrace{1}_{i\text{th coordinate}}, \dots, 0) \in Q^k.$$

- (2) Show that Q^k is a convex subset of \mathbb{R}^k . Given any $\mathbf{x} = (x_1, \dots, x_k) \in Q^k$, $\mathbf{y} = (y_1, \dots, y_k) \in Q^k$ and $0 < \lambda < 1$. Hence

$$\lambda \mathbf{x} + (1 - \lambda) \mathbf{y} = (\lambda x_1 + (1 - \lambda) y_1, \dots, \lambda x_k + (1 - \lambda) y_k) \in Q^k$$

since each $\lambda x_i + (1 - \lambda) y_i \geq 0$ and

$$\sum_{i=1}^k (\lambda x_i + (1 - \lambda) y_i) = \lambda \sum_{i=1}^k x_i + (1 - \lambda) \sum_{i=1}^k y_i \leq \lambda + (1 - \lambda) = 1.$$

- (3) Given any convex set $E \subseteq \mathbb{R}^k$ containing $\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_k$. Show that $E \supseteq Q^k$.

- (a) Induction on k . Base case: $k = 1$. Given any $\mathbf{x} = (x_1) \in Q^1$. We have $0 \leq x_1 \leq 1$ by the definition of Q^1 . So that $\mathbf{x} = x_1 \mathbf{e}_1 + (1 - x_1) \mathbf{0} \in E$ since $\mathbf{0}, \mathbf{e}_1 \in E$ and E is convex.

- (b) Inductive step: suppose the statement holds for $k = n$. Given any $\mathbf{x} = (x_1, \dots, x_n, x_{n+1}) \in Q^{n+1}$. If $x_{n+1} = 1$, then $x_1 = \dots = x_n = 0$ by the definition of Q^{n+1} . So $\mathbf{x} = \mathbf{e}_{n+1} \in E$ by the assumption of E . If $0 \leq x_{n+1} < 1$, then $x_1 + \dots + x_n \leq 1 - x_{n+1}$ or

$$\frac{x_1}{1 - x_{n+1}} + \dots + \frac{x_n}{1 - x_{n+1}} \leq 1.$$

So the point

$$\left(\frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}} \right) \in Q^n,$$

or

$$\left(\frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}}, 0 \right), \text{ say } \hat{\mathbf{x}}, \in E$$

by the induction hypothesis. Note that $\mathbf{e}_{n+1} \in E$. Hence

$$\mathbf{x} = x_{n+1}\mathbf{e}_{n+1} + (1 - x_{n+1})\hat{\mathbf{x}} \in E$$

by the convexity of E .

- (c) Conclusion: Since both the base case and the inductive step have been proved as true, by mathematical induction the statement holds.

□

Proof of (b).

- (1) Let \mathbf{f} be an affine mapping that carries a vector space X into a vector space Y such that

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{0}) + A\mathbf{x}$$

for some $A \in L(X, Y)$.

- (2) Given any convex subset C of X . To show that $\mathbf{f}(C)$ is convex, it suffices to show that

$$\lambda \mathbf{y}_1 + (1 - \lambda) \mathbf{y}_2 \in \mathbf{f}(C)$$

for any $\mathbf{y}_1, \mathbf{y}_2 \in \mathbf{f}(C)$ and $0 < \lambda < 1$. Write $\mathbf{y}_1 = \mathbf{f}(\mathbf{x}_1)$, $\mathbf{y}_2 = \mathbf{f}(\mathbf{x}_2)$ for some $\mathbf{x}_1, \mathbf{x}_2 \in C$. Note that $\lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2 \in C$ by the convexity of C . Hence

$$\begin{aligned} & \mathbf{f}(\lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2) \\ &= \mathbf{f}(\mathbf{0}) + A(\lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2) \\ &= \mathbf{f}(\mathbf{0}) + \lambda A\mathbf{x}_1 + (1 - \lambda)A\mathbf{x}_2 & (A \in L(X, Y)) \\ &= \lambda(\mathbf{f}(\mathbf{0}) + A\mathbf{x}_1) + (1 - \lambda)(\mathbf{f}(\mathbf{0}) + A\mathbf{x}_2) \\ &= \lambda \mathbf{f}(\mathbf{x}_1) + (1 - \lambda) \mathbf{f}(\mathbf{x}_2) \\ &= \lambda \mathbf{y}_1 + (1 - \lambda) \mathbf{y}_2 \in \mathbf{f}(C). \end{aligned}$$

□

Exercise 10.8. Let H be the parallelogram in \mathbb{R}^2 whose vertices are $(1, 1)$, $(3, 2)$, $(4, 5)$, $(2, 4)$. Find the affine map T which sends $(0, 0)$ to $(1, 1)$, $(1, 0)$ to $(3, 2)$, $(1, 1)$ to $(4, 5)$, $(0, 1)$ to $(2, 4)$. Show that $J_T = 5$. Use T to convert the integral

$$\alpha = \int_H e^{x-y} dx dy$$

to an integral over I^2 and thus compute α .

Proof.

(1) By Affine simplexes 10.26,

$$T(\mathbf{x}) = T(\mathbf{0}) + A\mathbf{x},$$

where $A \in L(\mathbb{R}^2, \mathbb{R}^2)$, say $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Note that $T : \begin{bmatrix} 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus

$$T : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 + ax + by \\ 1 + cx + dy \end{bmatrix}.$$

(2) By $T : (1, 0) \mapsto (3, 2)$ and $T : (0, 1) \mapsto (2, 4)$, we can solve A as

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

It is easy to verify such

$$T : \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{x}} \mapsto \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{T(\mathbf{0})} + \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{x}} = \begin{bmatrix} 1 + 2x + y \\ 1 + x + 3y \end{bmatrix}$$

satisfying our requirement.

(3)

$$J_T = \det \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = 5.$$

(4)

$$\begin{aligned} \int_H e^{x-y} dx dy &= \int_{[0,1]^2} e^{(1+2u+v)-(1+u+3v)} |J_T| du dv \\ &= 5 \int_{[0,1]^2} e^{u-2v} du dv \\ &= 5 \left\{ \int_0^1 e^u du \right\} \left\{ \int_0^1 e^{-2v} dv \right\} \quad (\text{Theorem 10.2}) \\ &= \frac{5}{2} (e - 1) (1 - e^{-2}). \end{aligned}$$

□

Exercise 10.9. ...

Proof.

(1)

(2)

□

Exercise 10.10. ...

Proof.

(1)

(2)

□

Exercise 10.11. ...

Proof.

(1)

(2)

□

Exercise 10.12. ...

Proof.

(1)

(2)

□

Exercise 10.13. ...

Proof.

(1)

(2)

□

Exercise 10.14 (Levi-Civita symbol). Prove $\varepsilon(j_1, \dots, j_k) = s(j_1, \dots, j_k)$, where

$$s(j_1, \dots, j_k) = \prod_{p < q} \text{sgn}(j_q - j_p).$$

It is usually to define the Levi-Civita symbol by

$$\varepsilon(j_1, \dots, j_k) = \begin{cases} 1 & \text{if } (j_1, \dots, j_k) \text{ is an even permutation of } J, \\ -1 & \text{if } (j_1, \dots, j_k) \text{ is an odd permutation of } J, \\ 0 & \text{otherwise} \end{cases}$$

(Basic k -forms 10.14). Thus, it is the sign of the permutation in the case of a permutation, and zero otherwise. So $\varepsilon(j_1, \dots, j_k)$ is equivalent to an explicit expression $s(j_1, \dots, j_k) = \prod_{p < q} \text{sgn}(j_q - j_p)$.

Proof.

(1) Induction on k . Base case: Show that $\varepsilon(j_1, j_2) = s(j_1, j_2)$. Since

$$\varepsilon(j_1, j_2) = \begin{cases} 1 & \text{if } j_1 < j_2 \\ -1 & \text{if } j_1 > j_2, \end{cases}$$

$$\varepsilon(j_1, j_2) = \text{sgn}(j_2 - j_1) = s(j_1, j_2).$$

(2) Inductive step: Show that for any $s \geq 2$, if $\varepsilon(j_1, \dots, j_s) = s(j_1, \dots, j_s)$ holds, then $\varepsilon(j_1, \dots, j_{s+1}) = s(j_1, \dots, j_{s+1})$ also holds.

$$\begin{aligned} \varepsilon(j_1, \dots, j_{s+1}) &= \varepsilon(j_1, \dots, j_s) \prod_{\substack{1 \leq p \leq s \\ q=s+1}} \text{sgn}(j_q - j_p) \\ &= s(j_1, \dots, j_s) \prod_{\substack{1 \leq p \leq s \\ q=s+1}} \text{sgn}(j_q - j_p) \\ &= \prod_{1 \leq p < q \leq s} \text{sgn}(j_q - j_p) \prod_{\substack{1 \leq p \leq s \\ q=s+1}} \text{sgn}(j_q - j_p) \\ &= \prod_{1 \leq p < q \leq s+1} \text{sgn}(j_q - j_p) \\ &= s(j_1, \dots, j_{s+1}). \end{aligned}$$

- (3) Conclusion: Since both the base case and the inductive step have been proved as true, by mathematical induction the statement holds for every integer $k \geq 2$.

□

Exercise 10.15. If ω and λ are k - and m -forms, respectively, prove that

$$\omega \wedge \lambda = (-1)^{km} \lambda \wedge \omega.$$

Proof.

- (1) Write

$$\omega = \sum_I b_I(\mathbf{x}) dx_I, \quad \lambda = \sum_J c_J(\mathbf{x}) dx_J$$

in the standard presentations, where I and J range over all increasing k -indices and over all increasing m -indices taken from the set $\{1, \dots, n\}$.

- (2) Show that $dx_I \wedge dx_J = (-1)^{km} dx_J \wedge dx_I$.

$$\begin{aligned} dx_I \wedge dx_J &= dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_J \\ &= (-1)^m dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}} \wedge dx_J \wedge dx_{i_k} \\ &= (-1)^{2m} dx_{i_1} \wedge \cdots \wedge dx_{i_{k-2}} \wedge dx_J \wedge dx_{i_{k-1}} \wedge dx_{i_k} \\ &\quad \dots \\ &= (-1)^{km} dx_J \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= (-1)^{km} dx_J \wedge dx_I. \end{aligned}$$

- (3)

$$\begin{aligned} \omega \wedge \lambda &= \sum_{I,J} b_I(\mathbf{x}) c_J(\mathbf{x}) dx_I \wedge dx_J \\ &= (-1)^{km} \sum_{J,I} c_J(\mathbf{x}) b_I(\mathbf{x}) dx_J \wedge dx_I \\ &= (-1)^{km} \lambda \wedge \omega. \end{aligned}$$

□

Exercise 10.16. If $k \geq 2$ and $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_k]$ is an oriented affine k -simplex, prove that $\partial^2 \sigma = 0$, directly from the definition of the boundary operator ∂ . Deduce from this that $\partial^2 \Psi = 0$ for every chain Ψ . (Hint: For orientation, do it first for $k = 2$, $k = 3$. In general, if $i < j$, let σ_{ij} be the $(k - 2)$ -simplex obtained by deleting \mathbf{p}_i and \mathbf{p}_j from σ . Show that each σ_{ij} occurs twice in $\partial^2 \sigma$, with opposite sign.)

Proof (Brute-force).

- (1) Write the boundary of the oriented affine k -simplex $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_k]$ as

$$\partial\sigma = \sum_{i=0}^k (-1)^i [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k]$$

where where the oriented $(k-1)$ -simplex $[\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k]$ is obtained by deleting σ 's i -th vertex (Boundaries 10.29).

- (2)

$$\begin{aligned} \partial^2\sigma &= \partial \left(\sum_i (-1)^i [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k] \right) \\ &= \sum_i (-1)^i \partial [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k] \\ &= \sum_{j < i} (-1)^i (-1)^j [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_j, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k] \\ &\quad + \sum_{j > i} (-1)^i (-1)^{j-1} [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \widehat{\mathbf{p}}_j, \dots, \mathbf{p}_k] \\ &= \sum_{j < i} (-1)^{i+j} [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_j, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k] \\ &\quad - \sum_{j > i} (-1)^{i+j} [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \widehat{\mathbf{p}}_j, \dots, \mathbf{p}_k]. \end{aligned}$$

The latter two summations cancel since after switching i and j in the second sum. Therefore $\partial^2\sigma = 0$.

- (3) The boundary of a chain is the linear combination of boundaries of the simplices in the chain. Write $\Psi = \sum_{i=1}^r \sigma_i$. where σ_i is an oriented affine simplex. Then

$$\partial^2\Psi = \partial \left(\partial \sum \sigma_i \right) = \partial \left(\sum \partial\sigma_i \right) = \sum \partial^2\sigma_i = \sum 0 = 0$$

for any affine chain Ψ .

□

Exercise 10.17. Put $J^2 = \tau_1 + \tau_2$, where

$$\tau_1 = [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2], \quad \tau_2 = -[\mathbf{0}, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1].$$

Explain why it is reasonable to call J^2 the positively oriented unit square in \mathbb{R}^2 . Show that ∂J^2 is the sum of 4 oriented affine 1-simplexes. Find these. What is $\partial(\tau_1 - \tau_2)$?

Proof.

- (1) Note that the unit square $I^2 \in \mathbb{R}^2$ is the union of $\tau_1(Q^2)$ and $\tau_2(Q^2)$, where

$$\begin{aligned}\tau_1(\mathbf{u}) &= ([\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2])(\mathbf{u}) \\ &= \mathbf{0} + \alpha_1 \mathbf{e}_1 + \alpha_2 (\mathbf{e}_1 + \mathbf{e}_2) \\ &= \mathbf{0} + (\alpha_1 + \alpha_2) \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 \\ &= \mathbf{0} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{u}\end{aligned}$$

and

$$\begin{aligned}\tau_2(\mathbf{u}) &= (-[\mathbf{0}, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1])(\mathbf{u}) \\ &= ([\mathbf{0}, \mathbf{e}_2 + \mathbf{e}_1, \mathbf{e}_2])(\mathbf{u}) \\ &= \mathbf{0} + \alpha_1 (\mathbf{e}_1 + \mathbf{e}_2) + \alpha_2 \mathbf{e}_2 \\ &= \mathbf{0} + \alpha_1 \mathbf{e}_1 + (\alpha_1 + \alpha_2) \mathbf{e}_2 \\ &= \mathbf{0} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{u}\end{aligned}$$

where $\mathbf{u} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 \in \mathbb{R}^2$ (as in Equation (78)). Both τ_1 and τ_2 have Jacobian $1 > 0$, or positively oriented (Affine simplexes 10.26). So it is reasonable to call J^2 the positively oriented unit square in \mathbb{R}^2 .

- (2)

$$\begin{aligned}\partial\tau_1 &= [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] - [\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{0}, \mathbf{e}_1], \\ \partial\tau_2 &= [\mathbf{e}_2 + \mathbf{e}_1, \mathbf{e}_2] - [\mathbf{0}, \mathbf{e}_2] + [\mathbf{0}, \mathbf{e}_2 + \mathbf{e}_1] \\ &= [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2] + [\mathbf{e}_2, \mathbf{0}] + [\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2].\end{aligned}$$

- (3) By (2),

$$\partial J^2 = \partial\tau_1 + \partial\tau_2 = [\mathbf{0}, \mathbf{e}_1] + [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2] + [\mathbf{e}_2, \mathbf{0}],$$

which is the positively oriented boundary of I^2 .

- (4) By (2),

$$\begin{aligned}\partial(\tau_1 - \tau_2) &= \partial\tau_1 - \partial\tau_2 \\ &= [\mathbf{0}, \mathbf{e}_1] + [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{0}] \\ &\quad + [\mathbf{0}, \mathbf{e}_2] + [\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{0}].\end{aligned}$$

□

Exercise 10.18. Consider the oriented affine 3-simplex

$$\sigma_1 = [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]$$

in \mathbb{R}^3 . Show that σ_1 (regarded as a linear transformation) has determinant 1. Thus σ_1 is positively oriented.

Let $\sigma_2, \dots, \sigma_6$ be five other oriented 3-simplexes, obtained as follows: There are five permutations (i_1, i_2, i_3) of $(1, 2, 3)$, distinct from $(1, 2, 3)$. Associate with each (i_1, i_2, i_3) the simplex

$$s(i_1, i_2, i_3)[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}]$$

where s is the sign that occurs in the definition of the determinant. (This is how τ_2 was obtained from τ_1 in Exercise 10.17.) Show that $\sigma_2, \dots, \sigma_6$ are positively oriented.

Put $J^3 = \sigma_1 + \dots + \sigma_6$. Then J^3 may be called the positively oriented unit cube in \mathbb{R}^3 . Show that ∂J^3 is the sum of 12 oriented affine 2-simplexes. (These 12 triangles cover the surface of the unit cube I^3 .)

Show that $\mathbf{x} = (x_1, x_2, x_3)$ is in the range of σ_1 if and only if $0 \leq x_3 \leq x_2 \leq x_1 \leq 1$.

Show that the range of $\sigma_1, \dots, \sigma_6$ have disjoint interiors, and that their union covers I^3 . (Compared with Exercise 10.13; note that $3! = 6$.)

Proof.

- (1) Show that σ_1 (regarded as a linear transformation) has determinant 1. Given any $\mathbf{u} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3 \in \mathbb{R}^3$, we have

$$\begin{aligned} \sigma_1(\mathbf{u}) &= ([\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3])(\mathbf{u}) \\ &= \mathbf{0} + \alpha_1 \mathbf{e}_1 + \alpha_2(\mathbf{e}_1 + \mathbf{e}_2) + \alpha_3(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) \\ &= \mathbf{0} + (\alpha_1 + \alpha_2 + \alpha_3)\mathbf{e}_1 + (\alpha_2 + \alpha_3)\mathbf{e}_2 + \alpha_3 \mathbf{e}_3 \\ &= \mathbf{0} + \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{say } A} \mathbf{u}. \end{aligned}$$

So

$$\det(A) = \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = 1.$$

- (2) Show that $\sigma_2, \dots, \sigma_6$ are positively oriented. Define the permutation matrix $P_{(i_1, i_2, i_3)}$ corresponding to a permutation (i_1, i_2, i_3) of $(1, 2, 3)$ by

$$P_{(i_1, i_2, i_3)} = [\mathbf{e}_{i_1} \quad \mathbf{e}_{i_2} \quad \mathbf{e}_{i_3}].$$

For example,

$$P_{(2,3,1)} = [\mathbf{e}_2 \quad \mathbf{e}_3 \quad \mathbf{e}_1] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Note that the sign $s(i_1, i_2, i_3)$ of the permutation (i_1, i_2, i_3) is exactly the same as the determinant of the permutation matrix $P_{(i_1, i_2, i_3)}$. Hence,

$$\begin{aligned} & (s(i_1, i_2, i_3)[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}])(\mathbf{u}) \\ &= s(i_1, i_2, i_3)(\mathbf{0} + \alpha_1 \mathbf{e}_{i_1} + \alpha_2(\mathbf{e}_{i_1} + \mathbf{e}_{i_2}) + \alpha_3(\mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3})) \\ &= \mathbf{0} + s(i_1, i_2, i_3)(\alpha_1 \mathbf{e}_{i_1} + \alpha_2(\mathbf{e}_{i_1} + \mathbf{e}_{i_2}) + \alpha_3(\mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3})) \\ &= \mathbf{0} + s(i_1, i_2, i_3)((\alpha_1 + \alpha_2 + \alpha_3)\mathbf{e}_{i_1} + (\alpha_2 + \alpha_3)\mathbf{e}_{i_2} + \alpha_3 \mathbf{e}_{i_3}) \\ &= \mathbf{0} + s(i_1, i_2, i_3)P_{(i_1, i_2, i_3)}A\mathbf{u} \end{aligned}$$

where $\mathbf{u} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3 \in \mathbb{R}^3$. So

$$\begin{aligned} \det(s(i_1, i_2, i_3)P_{(i_1, i_2, i_3)}A) &= s(i_1, i_2, i_3)^3 \det(P_{(i_1, i_2, i_3)}) \det(A) \\ &= s(i_1, i_2, i_3)^3 \cdot s(i_1, i_2, i_3) \cdot 1 \\ &= 1. \end{aligned}$$

(3)

□

Exercise 10.19. ...

Proof.

(1)

(2)

□

Exercise 10.20. ...

Proof.

(1)

(2)

□

Exercise 10.21. ...

Proof.

(1)

(2)

□

Exercise 10.22. ...

Proof.

(1)

(2)

□

Exercise 10.23. ...

Proof.

(1)

(2)

□

Exercise 10.24. ...

Proof.

(1)

(2)

□

Exercise 10.25. ...

Proof.

(1)

(2)

□

Exercise 10.26. ...

Proof.

(1)

(2)

□

Exercise 10.27. ...

Proof.

(1)

(2)

□

Exercise 10.28. ...

Proof.

(1)

(2)

□

Exercise 10.29. ...

Proof.

(1)

(2)

□

Exercise 10.30. *If \mathbf{N} is the vector given by*

$$\mathbf{N} = (\alpha_2\beta_3 - \alpha_3\beta_2)\mathbf{e}_1 + (\alpha_3\beta_1 - \alpha_1\beta_3)\mathbf{e}_2 + (\alpha_1\beta_2 - \alpha_2\beta_1)\mathbf{e}_3$$

(Equation (135)), prove that

$$\det \begin{bmatrix} \alpha_1 & \beta_1 & \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_2 & \beta_2 & \alpha_3\beta_1 - \alpha_1\beta_3 \\ \alpha_3 & \beta_3 & \alpha_1\beta_2 - \alpha_2\beta_1 \end{bmatrix} = |\mathbf{N}|^2$$

Also, verify

$$\mathbf{N} \cdot (T\mathbf{e}_1) = \mathbf{N} \cdot (T\mathbf{e}_2)$$

(Equation (137)).

Proof.

(1) By Laplace's expansion along the third column,

$$\begin{aligned} & \det \begin{bmatrix} \alpha_1 & \beta_1 & \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_2 & \beta_2 & \alpha_3\beta_1 - \alpha_1\beta_3 \\ \alpha_3 & \beta_3 & \alpha_1\beta_2 - \alpha_2\beta_1 \end{bmatrix} \\ &= (-1)^{1+3}(\alpha_2\beta_3 - \alpha_3\beta_2) \det \begin{bmatrix} \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} \\ & \quad + (-1)^{2+3}(\alpha_3\beta_1 - \alpha_1\beta_3) \det \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_3 & \beta_3 \end{bmatrix} \\ & \quad + (-1)^{3+3}(\alpha_1\beta_2 - \alpha_2\beta_1) \det \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix} \\ &= (\alpha_2\beta_3 - \alpha_3\beta_2)^2 + (\alpha_3\beta_1 - \alpha_1\beta_3)^2 + (\alpha_1\beta_2 - \alpha_2\beta_1)^2 \\ &= |\mathbf{N}|^2. \end{aligned}$$

(2)

$$\begin{aligned} \mathbf{N} \cdot (T\mathbf{e}_1) &= (\alpha_2\beta_3 - \alpha_3\beta_2, \alpha_3\beta_1 - \alpha_1\beta_3, \alpha_1\beta_2 - \alpha_2\beta_1) \cdot (\alpha_1, \alpha_2, \alpha_3) \\ &= (\alpha_2\beta_3 - \alpha_3\beta_2)\alpha_1 + (\alpha_3\beta_1 - \alpha_1\beta_3)\alpha_2 + (\alpha_1\beta_2 - \alpha_2\beta_1)\alpha_3 \\ &= (\alpha_3\alpha_2 - \alpha_2\alpha_3)\beta_1 + (\alpha_1\alpha_3 - \alpha_3\alpha_1)\beta_2 + (\alpha_2\alpha_1 - \alpha_1\alpha_2)\beta_3 \\ &= 0. \end{aligned}$$

(3)

$$\begin{aligned} \mathbf{N} \cdot (T\mathbf{e}_2) &= (\alpha_2\beta_3 - \alpha_3\beta_2, \alpha_3\beta_1 - \alpha_1\beta_3, \alpha_1\beta_2 - \alpha_2\beta_1) \cdot (\beta_1, \beta_2, \beta_3) \\ &= (\alpha_2\beta_3 - \alpha_3\beta_2)\beta_1 + (\alpha_3\beta_1 - \alpha_1\beta_3)\beta_2 + (\alpha_1\beta_2 - \alpha_2\beta_1)\beta_3 \\ &= (\beta_2\beta_3 - \beta_3\beta_2)\alpha_1 + (\beta_3\beta_1 - \beta_1\beta_3)\alpha_2 + (\beta_1\beta_2 - \beta_2\beta_1)\alpha_3 \\ &= 0. \end{aligned}$$

□

Exercise 10.31. ...

Proof.

(1)

(2)

□

Exercise 10.32. ...

Proof.

(1)

(2)

□