Notes on the book: *Atiyah and Macdonald, Introduction to Commutative Algebra*

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Contents

Chapter 1: Rings and Ideals

Exercise 1.1.

Let x be a nilpotent element of A. Show that $1+x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit.

Proof.

(1) Suppose $x^m = 0$ for some odd integer $m \geq 0$. Then

$$1 = 1 + x^m = (1+x)(1 - x + x^2 - \cdots + (-1)^{m-1}x^{m-1}),$$

or $1+x$ is a unit.

(2) If u is any unit and x is any nilpotent, $u + x = u \cdot (1 + u^{-1}x)$ is a product of two units (using that $u^{-1}x$ is nilpotent and applying (1)) and hence a unit again.

□

Proof (Proposition 1.9).

(1) *The nilradical is a subset of the Jacobson radical.*

  (a) The nilradical $\mathfrak{N}$ of A is the intersection of all the prime ideals of A by Proposition 1.8.

  (b) The Jacobson radical $\mathfrak{J}$ of A is the intersection of all the maximal ideals of A by definition.

(2) By Proposition 1.9, $x \in \mathfrak{J}$ if and only if $1 - xy$ is a unit in A for all $y \in A$. So $1 + x = 1 - (-x) \cdot 1$ is a unit in A since x is a nilpotent and $\mathfrak{J}$ is an ideal.

□

Exercise 1.2.

Let A be a ring and let $A[x]$ be the ring of polynomials in an indeterminate x, with coefficients in A. Let $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$. Prove that

(i) *f is a unit in $A[x]$ if and only if a_0 is a unit in A and $a_1, \ldots, a_n$ are nilpotent. (Hint: If $b_0 + b_1x + \cdots + b_mx^m$ is the inverse of f, prove by induction on r that $a_n^{r+1}b_{m-r} = 0$. Hence show that a_n is nilpotent, and then use Exercise 1.1.)*

(ii) *f is nilpotent if and only if $a_0, a_1, \ldots, a_n$ are nilpotent.*

(iii) *f is a zero-divisor if and only if there exists $a \neq 0$ such that $af = 0$. (Hint: Choose a polynomial $g = b_0 + b_1x + \cdots + b_mx^m$ of least degree m such that $fg = 0$. Then $a_nb_m = 0$, hence $a_ng = 0$ (because a_ng annihilates f and has degree $< m$). Now show by induction that $a_{n-r}g = 0$ $(0 \leq r \leq n)$.)*

(iv) *f is said to be **primitive** if $(a_0, a_1, \ldots, a_n) = (1)$. Prove that if $f, g \in A[x]$, then fg is primitive if and only if f and g are primitive.*

Proof of (i).

(1) $(\Longleftarrow)$ holds by Exercise 1.1.

(2) $(\Longrightarrow)$ There exists the inverse g of f, say $g = b_0 + b_1x + \cdots + b_mx^m$ satisfying $1 = fg$. Clearly, $1 = a_0b_0$, or a_0 is a unit in A. Also,

$$\begin{aligned}
0 &= a_nb_m, \\
0 &= a_nb_{m-1} + a_{n-1}b_m, \\
0 &= a_nb_{m-2} + a_{n-1}b_{m-1} + a_{n-2}b_m, \\
&\cdots
\end{aligned}$$

A direct computing shows that

$$\begin{aligned}
0 &= a_n^1b_m, \\
0 &= a_n(a_nb_{m-1} + a_{n-1}b_m) \\
&= a_n^2b_{m-1} + a_{n-1}a_nb_m \\
&= a_n^2b_{m-1}, \\
0 &= a_n^2(a_nb_{m-2} + a_{n-1}b_{m-1} + a_{n-2}b_m) \\
&= a_n^3b_{m-2} + a_{n-1}a_n^2b_{m-1} + a_{n-2}a_n^2b_m \\
&= a_n^3b_{m-2}, \\
&\cdots
\end{aligned}$$

So we might have $a_n^{r+1}b_{m-r} = 0$ for $r = 0, 1, 2, \ldots, m$.

(3) *Show that $a_n^{r+1}b_{m-r} = 0$ for $r = 0, 1, 2, \ldots, m$ by induction on r.*

(a) As $r = 0$, $a_nb_m = 0$ by comparing the coefficient of $fg = 1$ at x^{n+m}.

(b) For any $r > 0$, comparing the coefficient of $fg = 1$ at x^{n+m-r},

$$0 = a_nb_{m-r} + a_{n-1}b_{m-r+1} + \cdots + a_{n-r}b_m.$$

Multiplying by a_n^r on the both sides,

$$\begin{aligned}
0 &= a_n^{r+1}b_{m-r} + a_{n-1}a_n^rb_{m-r+1} + \cdots + a_{n-r}a_n^rb_m \\
&= a_n^{r+1}b_{m-r}.
\end{aligned}$$

by the induction hypothesis.

(4) a_n *is a nilpotent.* Putting $r = m$ in $a_n^{r+1} b_{m-r} = 0$ and get $a_n^{m+1} b_0 = 0$. Notice that b_0 is a unit, $a_n^{m+1} = 0$, or a_n is a nilpotent.

(5) Consider $f - a_n x^n = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$, a polynomial $\in A[x]$ of degree $n-1$. Note that f is a unit and $a_n x^n$ is a nilpotent. By Exercise 1.1, $f - a_n x^n$ is a unit too. Applying the (2)(3)(4) again, a_{n-1} is a nilpotent as $n - 1 > 0$, that is, applying descending induction on n then yields the desired property.

□

Proof of (ii).

(1) ($\Longleftarrow$) holds since the nilradical of any ring is an ideal.

(2) ($\Longrightarrow$) $f^N = 0$ for some $N > 0$. So $0 = f^N = a_0^n + \cdots + a_n^N x^{nN}$. Compare the coefficient in the lowest term to get $a_0^N = 0$, or a_0 is a nilpotent.

(3) Note that $f - a_0 = a_1 x + \cdots + a_n x^n \in A[x]$ is nilpotent since f and a_0 are nilpotent. $f - a_0$ is a nilpotent too. Continue the same argument in (2), the result is established.

□

Proof of (iii).

(1) ($\Longleftarrow$) holds trivially.

(2) ($\Longrightarrow$) Pick a polynomial $g = b_0 + b_1 x + \cdots + b_m x^m$ of least degree m such that $fg = 0$. Especially, $a_n b_m = 0$.

(3) Consider

$$\begin{aligned} a_n g &= a_n b_0 + \cdots + a_n b_{m-1} x^{m-1} + a_n b_m x^m \\ &= a_n b_0 + \cdots + a_n b_{m-1} x^{m-1} \end{aligned}$$

(since $a_n b_m = 0$). $a_n g$ is a polynomial over A of having degree strictly less than m. Notice that $f \cdot (a_n g) = a_n \cdot (fg) = 0$. By minimality of m, $a_n g = 0$.

(4) Induction on the degree n of f.

(a) As $n = 0$, $f = a_0$. There exists $b_m \neq 0$ such that $b_m f = b_m a_0 = 0$ by (2).

(b) For any zero-divisor f of degree n, there is a polynomial $g = b_0 + b_1 x + \cdots + b_m x^m$ of least degree m such that $fg = 0$. By (2)(3),

$$\begin{aligned} (f - a_n x^n) \cdot g &= fg - a_n x^n g \\ &= 0 - 0 \\ &= 0. \end{aligned}$$

That is, $f - a_n x^n$ is a zero-divisor of degree $n-1$. By the induction hypothesis, there exists $b_m \neq 0$ such that $b_m(f - a_n x^n) = 0$. So $b_m f = b_m(f - a_n x^n) + b_m a_n x^n = 0 + 0 = 0$.

(c) By (a)(b), ($\Longrightarrow$) holds by mathematical induction.

$\square$

Proof of (iv). Note that

(1) $f \notin \mathfrak{m}[x]$ for any maximal ideal $\mathfrak{m}$ of A if and only if f is primitive.

(2) For any maximal ideal $\mathfrak{m}$ of A, $A/\mathfrak{m}$ is a field (or an integral domain).

(3) $A[x]$ is an integral domain if A is an integral domain.

(4) $A[x]/\mathfrak{m}[x] \cong (A/\mathfrak{m})[x]$ as a ring isomorphism.

Hence,

$$\begin{aligned} f, g : \text{ primitive} &\Longleftrightarrow f, g \notin \mathfrak{m}[x] \text{ for any maximal ideal } \mathfrak{m} \\ &\Longleftrightarrow f, g \neq 0 \text{ in } (A/\mathfrak{m})[x] \text{ for any maximal ideal } \mathfrak{m} \\ &\Longleftrightarrow fg \neq 0 \text{ in } (A/\mathfrak{m})[x] \text{ for any maximal ideal } \mathfrak{m} \\ &\Longleftrightarrow fg \notin \mathfrak{m}[x] \text{ for any maximal ideal } \mathfrak{m} \\ &\Longleftrightarrow fg : \text{ primitive.} \end{aligned}$$

$\square$

Exercise 1.3.

Generalize the results of Exercise 1.2 to a polynomial ring $A[x_1, \ldots, x_r]$ in several indeterminates.

Generalization. Let

$$f = \sum_{(i)} a_{(i)} x^{(i)} \in A[x_1, \ldots, x_r]$$

where $\sum_{(i)}$ is the summation over $(i) = (i_1, \ldots, i_r)$ with $i_1 + \cdots + i_r = n$. Then

(i) f is a unit in $A[x_1, \ldots, x_r]$ if and only if $a_{(0)}$ is a unit in A and all other $a_{(i)}$ are nilpotent.

(ii) f is nilpotent if and only if all $a_{(i)}$ are nilpotent.

(iii) f is a zero-divisor if and only if there exists $a \neq 0$ such that $af = 0$.

(iv) If $f, g \in A[x_1, \ldots, x_r]$, then fg is primitive if and only if f and g are primitive.

Proof. Use the mathematical induction to prove (i)(ii)(iii) and apply the same argument in Exercise 1.2 (iv) to prove (iv). □

Exercise 1.4.

In the ring $A[x]$, the Jacobson radical is equal to the nilradical.

Proof.

(1) The nilradical $\mathfrak{N}$ is a subset of the Jacobson radical $\mathfrak{J}$. It suffices to show that $\mathfrak{J} \subseteq \mathfrak{N}$.

(2)

$$\begin{aligned} & f \in \mathfrak{J} \\ \Longleftrightarrow\ & 1 - fy \text{ is a unit in } A[x] \text{ for all } y \in A[x] && \text{(Proposition 1.9)} \\ \Longrightarrow\ & 1 - xf \text{ is a unit in } A[x] && (y = x) \\ \Longrightarrow\ & \text{All coefficients of } f \text{ are nilpotent} && \text{(Exercise 1.2 (i))} \\ \Longrightarrow\ & f \text{ is nilpotent} && \text{(Exercise 1.2 (ii))} \\ \Longrightarrow\ & f \in \mathfrak{N}. \end{aligned}$$

□

Exercise 1.5.

Let A be a ring and let $A[[x]]$ be the ring of formal power series $f = \sum_{n=0}^{\infty} a_n x^n$ with coefficients in A. Show that

(i) *f is a unit in $A[[x]]$ if and only if a_0 is a unit in A.*

(ii) *If f is nilpotent, then a_n is nilpotent for all $n \geq 0$. Is converse true? (See Exercise 7.2.)*

(iii) *f belongs to the Jacobson radical of $A[[x]]$ if and only if a_0 belongs to the Jacobson radical of A.*

(iv) *The contraction of a maximal ideal $\mathfrak{m}$ of $A[[x]]$ is a maximal ideal of A, and $\mathfrak{m}$ is generated by $\mathfrak{m}^c$ and x.*

(v) *Every prime ideal of A is the contraction of a prime ideal of $A[[x]]$.*

Proof of (i).

(1) ($\Longrightarrow$) If $g = \sum_{n=0}^{\infty} b_n x^n$ is an inverse of f, then $fg = 1$ implies that $a_0 b_0 = 1$ so that a_0 is a unit in A.

(2) ($\Longleftarrow$) Our goal is to find $g = \sum_{n=0}^{\infty} b_n x^n$ such that the Cauchy product $fg = \sum_{n=0}^{\infty} c_n x^n$ is equal to $1 \in A[x]$. Here $c_n = \sum_{r=0}^{n} a_r b_{n-r}$. By the assumption we have that $c_0 = 1$ and $c_1 = c_2 = \cdots = 0$. Hence

$$\begin{aligned} b_0 &= a_0^{-1} \\ b_1 &= -a_0^{-1} a_1 b_0 \\ &\cdots \\ b_n &= a_0^{-1} \sum_{r=1}^{n} a_r b_{n-r} \\ &\cdots \end{aligned}$$

by induction.

□

Proof of (ii).

(1) The proof is the same as Exercise 1.2 (ii).

(2) The converse is true if A is Noetherian (by Exercise 7.2).

(3) The converse is not always true. Take

$$A = \mathbb{F}_2[t, t^{-2}, t^{-2^2}, \ldots]/(t)$$

and

$$f(x) = \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} t^{-2^n} x^n \in A[x].$$

Note that A is not Noetherian and all a_n are nilpotent in A. To show f is not nilpotent in $A[x]$, it suffices to show that f^{2^r} is not equal to zero for all positive integers r.

(4) Note that $\mathbb{F}_2$ is a field of characteristic 2. So

$$f^{2^r} = \sum_{n=1}^{\infty} a_n^{2^r} x^n = \sum_{n=1}^{\infty} t^{2^{r-n}} x^n = \sum_{n=r+1}^{\infty} t^{2^{r-n}} x^n \neq 0$$

for all r.

□

Proof of (iii).

$$\begin{aligned}
&f \text{ in the Jacobson radical of } A[[x]] \\
\Longleftrightarrow\ & 1 - fg \in A[[x]] \text{ is unit for all } g = \sum_{n=0}^{\infty} b_n x^n \in A[[x]] && \text{(Proposition 1.9)} \\
\Longleftrightarrow\ & 1 - a_0 b_0 \in A \text{ is unit for all } b_0 \in A && \text{((i))} \\
\Longleftrightarrow\ & a_0 \text{ belongs to the Jacobson radical of } A. && \text{(Proposition 1.9)}
\end{aligned}$$

□

Proof of (iv).

(1) Note that $x = 0 + x$ belongs to the Jacobson radical of $A[[x]]$ since 0 obviously belongs to the Jacobson radical of A (by (iii)).

(2) So $x \in \mathfrak{m}$ or $(x) \subseteq \mathfrak{m}$ for any maximal ideal in $A[[x]]$. So it is clear that $\mathfrak{m} = \mathfrak{m}^c + (x)$.

(3) Moreover, $\mathfrak{m}^c$ is a maximal ideal since $A/\mathfrak{m}^c \cong A[[x]]/\mathfrak{m}$ is a field.

□

Proof of (v).

(1) Similar to (iv). Suppose $\mathfrak{p}$ is a prime ideal of A. Let $\mathfrak{q} = \mathfrak{p} + (x)$ be an ideal of $A[[x]]$.

(2) $\mathfrak{q}^c = \mathfrak{p}$ clearly. Besides, $\mathfrak{q}^c$ is a prime ideal since

$$A[[x]]/\mathfrak{q}^c \cong A/\mathfrak{p}$$

is an integral domain.

□

Supplement 1.5.1.

(Exercise II.1.2 in the textbook: Jrgen Neukirch, Algebraic Number Theory.) A p-adic integer $a = a_0 + a_1 p + a_2 p^2 + \cdots$ is a unit in the ring $\mathbb{Z}_p$ if and only if $a_0 \neq 0$.

Proof.

(1) ($\Longrightarrow$) If $b = b_0 + b_1 p + b_2 p^2 + \cdots$ is an inverse of a, then $ab = 1$ implies that $a_0 b_0 = 1$ so that a_0 is a unit in $\mathbb{Z}/p\mathbb{Z}$ or $a_0 \neq 0$.

(2) ($\Longleftarrow$) Our goal is to find

$$b = b_0 + b_1 p + b_2 p^2 + \cdots \in \mathbb{Z}_p$$

such that the Cauchy product

$$ab = c_0 + c_1 p + c_2 p^2 + \cdots$$

is equal to $1 \in \mathbb{Z}_p$. Here $c_n = \sum_{\nu=0}^{n} a_\nu b_{n-\nu}$. By the assumption we have that $c_0 = 1$ and $c_1 = c_2 = \cdots = 0$. Hence

$$\begin{aligned} b_0 &= a_0^{-1} \\ b_1 &= -a_0^{-1} a_1 b_0 \\ &\cdots \\ b_n &= a_0^{-1} \sum_{\nu=1}^{n} a_\nu b_{n-\nu} \\ &\cdots \end{aligned}$$

by induction.

$\square$

Exercise 1.6.

A ring A is such that every ideal not contained in the nilradical contains a nonzero idempotent (that is, an element e such that $e^2 = e \neq 0$). Prove that the nilradical and Jacobson radical of A are equal.

Proof.

(1) $\mathfrak{N} \subseteq \mathfrak{J}$ clearly.

(2) Since

$$\begin{aligned} a \notin \mathfrak{N} &\Longrightarrow (a) \not\subseteq \mathfrak{N} \\ &\Longrightarrow \text{there exists a nonzero idempotent } e \in (a) \\ &\Longrightarrow e = ar \text{ for some } r \in A \\ &\Longrightarrow 0 = e - e^2 = e(1-e) = ar(1-ar) \\ &\Longrightarrow 1 - ar \text{ is a zero-divisor, not a unit} \\ &\Longrightarrow a \notin \mathfrak{J}, && \text{(Proposition 1.9)} \end{aligned}$$

we have $\mathfrak{J} \subseteq \mathfrak{N}$.

$\square$

Exercise 1.7.

Let A be a ring in which every element satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal in A is maximal.

Proof. It suffices to show that *for any prime ideal $\mathfrak{p}$ in A, $A/\mathfrak{p}$ is a field.*

(1) Take any $0 \neq \overline{x} \in A/\mathfrak{p}$, which is represented by $x \in A - \mathfrak{p}$. By assumption there exists $n \geq 2$ such that $x^n = x$. So $\overline{x}^n = \overline{x}$ or $\overline{x}(\overline{x}^{n-1} - 1) = 0$.

(2) Since $\mathfrak{p}$ is prime, $A/\mathfrak{p}$ is a integral domain. That is, $\overline{x} = 0$ (impossible) or $\overline{x}^{n-1} - 1 = 0$. Write $\overline{x} \cdot \overline{x}^{n-2} = 1$ in $A/\mathfrak{p}$. So $\overline{x}^{n-2}$ is an inverse of $\overline{x} \neq 0$ in $A/\mathfrak{p}$, which implies that $A/\mathfrak{p}$ is a field (since $\overline{x}$ is arbitrary).

(3) $A/\mathfrak{p}$ is a field if and only if $\mathfrak{p}$ is maximal.

□

Exercise 1.8.

Let A be a ring $\neq 0$. Show that the set of prime ideals of A has minimal elements with respect to inclusion.

Similar to Theorem 1.3.

Proof (Zorn's Lemma).

(1) Let Σ be the set of all prime ideals of A.

(2) Order Σ by $\supseteq$, that is, $\mathfrak{p} \leq \mathfrak{q}$ if $\mathfrak{p} \supseteq \mathfrak{q}$.

(3) Σ is not empty, since every ring $A \neq 0$ has at least one maximal ideal (or prime ideal) (Theorem 1.3).

(4) To apply Zorn's lemma we must show that every chain in Σ has a lower bound in Σ; let then $(\mathfrak{p}_\alpha)$ be a chain of prime ideals in Σ, so that for each pair of indices α, β we have either $\mathfrak{p}_\alpha \subseteq \mathfrak{p}_\beta$ or $\mathfrak{p}_\beta \subseteq \mathfrak{p}_\alpha$. Let $\mathfrak{p} = \bigcap_\alpha \mathfrak{p}_\alpha$.

(5) *Show that $\mathfrak{p}$ is a prime ideal.* Clearly $\mathfrak{p}$ is an ideal. Given any $xy \in \mathfrak{p}$ and $x \notin \mathfrak{p}$. So xy is in all prime ideals $\mathfrak{p}_\alpha$. By assumption $x \notin \mathfrak{p}$, there is some β such that $x \notin \mathfrak{p}_\beta$, or $x \notin \mathfrak{p}_\alpha$ whenever $\alpha \geq \beta$. So $y \in \mathfrak{p}_\alpha$ whenever $\alpha \geq \beta$. Since $y \in \mathfrak{p}_\beta$, $y \in \mathfrak{p}_\gamma$ whenever $\beta \geq \gamma$. Therefore, $y \in \mathfrak{p}_\alpha$ for all α, or $y \in \mathfrak{p}$, or $\mathfrak{p}$ is prime.

□

Exercise 1.9.

Let $\mathfrak{a}$ be an ideal $\neq (1)$ in a ring A. Show that $\mathfrak{a} = r(\mathfrak{a}) \Longleftrightarrow \mathfrak{a}$ is an intersection of prime ideals.

Proof.

(1) ($\Longrightarrow$). By Proposition 1.14, $\mathfrak{a} = r(\mathfrak{a})$ is the intersection of the prime ideals which contain $\mathfrak{a}$.

(2) ($\Longleftarrow$).

$$\begin{aligned}
\mathfrak{a} &= \bigcap\{\mathfrak{p} \in \text{some subset of } \mathrm{Spec}(A)\} \\
&= \bigcap\{\mathfrak{p} \in \text{some subset of } \mathrm{Spec}(A) : \mathfrak{p} \supseteq \mathfrak{a}\} \\
&\supseteq \bigcap\{\mathfrak{p} \in \mathrm{Spec}(A) : \mathfrak{p} \supseteq \mathfrak{a}\} \\
&= r(\mathfrak{a}) \\
&\supseteq \mathfrak{a}.
\end{aligned}$$

□

Exercise 1.10.

Let A be a ring, $\mathfrak{N}$ its nilradical. Show the following are equivalent:

(i) *A has exactly one prime ideal;*

(ii) *every element of A is either a unit or nilpotent;*

(iii) *$A/\mathfrak{N}$ is a field.*

Proof.

$$\begin{aligned}
&A/\mathfrak{N} \text{ is a field} \\
\Longrightarrow\ &\mathfrak{N} \text{ is a maximal ideal} \\
\Longrightarrow\ &\mathfrak{p} = \mathfrak{N} \text{ for every prime ideal } \mathfrak{p} && \text{(Proposition 1.8)} \\
\Longrightarrow\ &A \text{ has exactly one prime ideal } \mathfrak{p} \\
\Longrightarrow\ &\mathfrak{p} = \mathfrak{N} \\
\Longrightarrow\ &A \text{ has exactly one maximal ideal } \mathfrak{p} \\
\Longrightarrow\ &\text{Given any } a \in A,\ a \text{ is a unit or } a \in \mathfrak{p} = \mathfrak{N}. && \text{(Corollary 1.5)} \\
\Longrightarrow\ &A/\mathfrak{N} \text{ is a field.}
\end{aligned}$$

□

Exercise 1.11. (Boolean ring)

*A ring A is **Boolean** if $x^2 = x$ for all $x \in A$. In a Boolean ring A, show that*

(i) $2x = 0$ *for all* $x \in A$*;*

(ii) *every prime ideal* $\mathfrak{p}$ *is maximal, and* $A/\mathfrak{p}$ *is a field with two elements;*

(iii) *every finitely generated ideal in* A *is principal.*

Proof of (i). Note that $2x = x+x = (x+x)^2 = (2x)^2 = 4x^2 = 4x$. So $2x = 0$. $\square$

Proof of (ii). Same as Exercise 1.7 with $n = 2$. $\square$

Proof of (iii).

(1) By induction, it suffices to show that *if* $\mathfrak{a} = (x, y)$ *is an ideal in* A*, then* $\mathfrak{a} = (z)$ *for some* $z \in A$*.*

(2) Take $z = x + y + xy$. $(z) \subseteq \mathfrak{a}$ obviously.

(3) Conversely, note that

$$x = x^2 = x(z - y - xy) = xz - \overbrace{xy - \underbrace{x^2y}_{=xy}}^{=2xy=0} = xz \in (z).$$

Also $y \in (z)$ similarly. So $\mathfrak{a} \subseteq (z)$ and thus $\mathfrak{a} = (z)$ is principal.

$\square$

Exercise 1.12.

A local ring contains no idempotent $\neq 0, 1$.

Proof.

(1) If e is an idempotent $\neq 0, 1$ in a local ring A with the maximal ideal $\mathfrak{m}$, then by definition $0 = e(1 - e)$ shows that both $e \neq 0$ and $1 - e \neq 0$ are not unit.

(2) Thus $e \in \mathfrak{m}$ and $1 - e \in \mathfrak{m}$. So $1 = (1 - e) + e$ is a unit in $\mathfrak{m}$, which is absurd.

$\square$

Construction of an algebraic closure of a field (E. Artin)

Exercise 1.13.

Let K be a field and let Σ be the set of all irreducible monic polynomials f in one indeterminate with coefficients in K. Let A be the polynomial ring over K generated by indeterminates x_f, one for each $f \in \Sigma$. Let $\mathfrak{a}$ be the ideal of A generated by the polynomials $f(x_f)$ for all $f \in \Sigma$. Show that $\mathfrak{a} \neq (1)$.

Let $\mathfrak{m}$ be a maximal ideal of A containing $\mathfrak{a}$ and let $K_1 = A/\mathfrak{m}$. Then K_1 is an extension field of K in which each $f \in \Sigma$ has a root. Repeat the construction with K_1 in place of K, obtaining a field K_2, and so on. Let $L = \bigcup_{n=1}^{\infty} K_n$. Then L is a field in which each $f \in \Sigma$ splits completely into linear factors. Let $\overline{K}$ be the set of all elements of L which are algebraic over K. Then $\overline{K}$ is an algebraic closure of K.

Proof.

(1) *Show that $\mathfrak{a} \neq (1)$.* (Reductio ad absurdum) If $\mathfrak{a} = (1)$, then we can write

$$1 = \sum_{i=1}^{n} g_i(x) f_i(x_{f_i}) \in A$$

where $x = (x_{f_1}, \ldots, x_{f_n}, x_{g_1}, \ldots, x_{g_r})$ is a tuple with finitely many indeterminates. It is possible since it is a finite sum.

(2) Let L be an algebraic extension of K such that each f_i has a root $a_i \in L$ $(i = 1, \ldots, n)$.

(3) Take $x = (a_1, \ldots, a_n, 0, \ldots, 0)$ in the equation $1 = \sum_{i=1}^{n} g_i(x) f_i(x_{f_i})$ to get

$$\begin{aligned} 1 &= \sum_{i=1}^{n} g_i(a_1, \ldots, a_n, 0, \ldots, 0) f_i(a_i) \\ &= \sum_{i=1}^{n} g_i(a_1, \ldots, a_n, 0, \ldots, 0) \cdot 0 \\ &= 0, \end{aligned}$$

which is absurd.

□

Exercise 1.14.

In a ring A, let Σ be the set of all ideals in which every element is a zero-divisor. Show that the set Σ has maximal elements and that every maximal element of Σ is a prime ideal. Hence the set of zero-divisors in A is a union of prime ideals.

Proof.

(1) Suppose $1 \neq 0$.

(2) *Show that the set Σ has maximal elements.* Order Σ by inclusion. Σ is not empty, since $0 \in \Sigma$. To apply Zorn's lemma we must show that every chain in Σ has an upper bound in Σ; let then $(\mathfrak{a}_\alpha)$ be a chain of ideals in Σ, so that for each pair of indices α, β we have either $\mathfrak{a}_\alpha \subseteq \mathfrak{a}_\beta$ or $\mathfrak{a}_\beta \subseteq \mathfrak{a}_\alpha$.

(3) Let $\mathfrak{a} = \bigcup_\alpha \mathfrak{a}_\alpha$. Then $\mathfrak{a}$ is an ideal and every element of $\mathfrak{a}$ is a zero-divisor. Hence $\mathfrak{a} \in \Sigma$, and $\mathfrak{a}$ is an upper bound of the chain. Hence by Zorn's lemma, Σ has maximal elements.

(4) *Show that every maximal element of Σ is a prime ideal.* Let $\mathfrak{p}$ be a maximal element in Σ. Suppose $x, y \notin \mathfrak{p}$. Then there are non-zero-divisors in $\mathfrak{p}+(x)$ and $\mathfrak{p} + (y)$, and their product is an element of $\mathfrak{p} + (xy)$ that is again a non-zero-divisor. So $xy \notin \mathfrak{p}$.

(5) Hence the set of zero-divisors in A is a union of prime ideals (by the construction in (2) and the result of (4)).

$\square$

The prime spectrum of a ring

Lemma 1.15.1.

For any $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$, $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$.

Proof.

(1) If $\mathfrak{p} \supseteq \mathfrak{a}$. We are done.

(2) If $\mathfrak{p} \not\supseteq \mathfrak{a}$, there exists $a \in \mathfrak{a} - \mathfrak{p}$. So for any $b \in \mathfrak{b}$, $b \in \mathfrak{p}$ since $ab \in \mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$ and $\mathfrak{p}$ is a prime ideal, that is, $\mathfrak{p} \supseteq \mathfrak{b}$.

By (1)(2), $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$. $\square$

Exercise 1.15.

Let A be a ring and let X be the set of all prime ideals of A. For each subset E of A, let $V(E)$ denote the set of all prime ideals of A which contain E. Prove that

(i) *if $\mathfrak{a}$ is the ideal generated by E, then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$.*

(ii) $V(0) = X$, $V(1) = \varnothing$.

(iii) *if $(E_i)_{i \in I}$ is any family of subsets of A, then*

$$V\left(\bigcup_{i \in I} E_i\right) = \bigcap_{i \in I} V(E_i).$$

(iv) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ *for any ideals $\mathfrak{a}$, $\mathfrak{b}$ of A.*

*The results show that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. The resulting topology is called the **Zariski topology**. The topological space X is called the **prime spectrum** of A, and is written* $\mathrm{Spec}(A)$.

Note that if $E_1 \subseteq E_2$, then $V(E_1) \supseteq V(E_2)$.

Proof of (i).

(1) *Show that $V(E) = V(\mathfrak{a})$.*

  (a) *Show that $V(E) \subseteq V(\mathfrak{a})$.* Given any $\mathfrak{p} \in V(E)$, $\mathfrak{p} \supseteq E$. For any $a \in \mathfrak{a}$, since $\mathfrak{a}$ is generated by E, we can write a as a finite sum $a = \sum \alpha\beta$ where $\alpha \in A$ and $\beta \in E$. Since $E \subseteq \mathfrak{p}$, all $\beta \in \mathfrak{p}$. Since $\mathfrak{p}$ is an ideal, $a = \sum \alpha\beta \in \mathfrak{p}$. That is, $\mathfrak{p} \supseteq \mathfrak{a}$, or $\mathfrak{p} \in V(\mathfrak{a})$.

  (b) $V(E) \supseteq V(\mathfrak{a})$ since $\mathfrak{a} \supseteq E$.

(2) *Show that $V(\mathfrak{a}) = V(r(\mathfrak{a}))$.*

  (a) *Show that $V(\mathfrak{a}) \subseteq V(r(\mathfrak{a}))$.* Given any $\mathfrak{p} \in V(\mathfrak{a})$,

$$\begin{aligned}
\mathfrak{p} \in V(\mathfrak{a}) &\Longrightarrow \mathfrak{p} \supseteq \mathfrak{a} \\
&\Longrightarrow \mathfrak{p} \supseteq \text{the intersection of the primes ideals } \mathfrak{p} \supseteq \mathfrak{a} \\
&\Longrightarrow \mathfrak{p} \supseteq r(\mathfrak{a}) \text{ (by Proposition 1.14)} \\
&\Longrightarrow \mathfrak{p} \in V(r(\mathfrak{a})).
\end{aligned}$$

  (b) $V(\mathfrak{a}) \supseteq V(r(\mathfrak{a}))$ since $r(\mathfrak{a}) \supseteq \mathfrak{a}$.

$\square$

Proof of (ii).

(1) $V(1) = \varnothing$ since no prime ideal contains 1 by definition.

(2) $V(0) = X$ since 0 is in every ideal (especially in every prime ideal).

$\square$

Proof of (iii).

$$\begin{aligned}
\mathfrak{p} \in V\left(\bigcup_{i \in I} E_i\right) &\Longleftrightarrow \mathfrak{p} \supseteq \bigcup_{i \in I} E_i \\
&\Longleftrightarrow \mathfrak{p} \supseteq E_i \text{ for all } i \in I \\
&\Longleftrightarrow \mathfrak{p} \in V(E_i) \text{ for all } i \in I \\
&\Longleftrightarrow \mathfrak{p} \in \bigcap_{i \in I} V(E_i).
\end{aligned}$$

$\square$

Proof of (iv).

(1) *Show that* $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$.

  (a) $V(\mathfrak{a} \cap \mathfrak{b}) \subseteq V(\mathfrak{a}\mathfrak{b})$ since $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b}$.

  (b) *Show that* $V(\mathfrak{a} \cap \mathfrak{b}) \supseteq V(\mathfrak{a}\mathfrak{b})$. Given any $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$, $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$. By Lemma 15.1.1, $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$. Notice that $\mathfrak{a} \supseteq \mathfrak{a} \cap \mathfrak{b}$ and $\mathfrak{b} \supseteq \mathfrak{a} \cap \mathfrak{b}$. In any case, $\mathfrak{p} \supseteq \mathfrak{a} \cap \mathfrak{b}$, $\mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b})$.

(2) *Show that* $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.

  (a) *Show that* $V(\mathfrak{a}\mathfrak{b}) \subseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$. Given any $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$, $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$. By Lemma 15.1.1, $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$, $\mathfrak{p} \in V(\mathfrak{a})$ or $\mathfrak{p} \in V(\mathfrak{b})$, $\mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$.

  (b) *Show that* $V(\mathfrak{a}\mathfrak{b}) \supseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$. Given any $\mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$, $\mathfrak{p} \in V(\mathfrak{a})$ or $\mathfrak{p} \in V(\mathfrak{b})$, $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$. Notice that $\mathfrak{a} \supseteq \mathfrak{a}\mathfrak{b}$ and $\mathfrak{b} \supseteq \mathfrak{a}\mathfrak{b}$. In any cases, $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$, or $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$.

$\square$

Exercise 1.16.

Draw pictures of $\mathrm{Spec}(\mathbb{Z})$, $\mathrm{Spec}(\mathbb{R})$, $\mathrm{Spec}(\mathbb{C}[x])$, $\mathrm{Spec}(\mathbb{R}[x])$, $\mathrm{Spec}(\mathbb{Z}[x])$.

Proof.

(1) *Show that* $\mathrm{Spec}(\mathbb{Z}) = \{(0)\} \cup \{(p) : p \textit{ is a rational prime}\}$. Note that $\mathbb{Z}$ is a PID. So all non-trivial prime ideals are of the form (π) where π are irreducible.

(2) *Show that* $\mathrm{Spec}(\mathbb{R}) = \{(0)\}$. Note that $\mathbb{R}$ is a field.

(3) *Show that* $\mathrm{Spec}(\mathbb{C}[x]) = \{(0)\} \cup \{(x-z) : z \in \mathbb{C}\}$. Note that $\mathbb{C}[x]$ is a PID and $\mathbb{C}$ is algebraically closed. Hence all non-trivial prime ideals are of the form $(x-z)$ where $z \in \mathbb{C}$.

(4) *Show that* $\mathrm{Spec}(\mathbb{R}[x])$ *are*

   (i) (0).

   (ii) $\{(x-r) : r \in \mathbb{R}\}$.

   (iii) $\{(x-z)(x-\overline{z}) : z \in \mathbb{C}, \mathrm{Im}(z) > 0\}$.

   Here is the proof.

   (a) Note that $\mathbb{R}[x]$ is a PID and all non-trivial prime ideals are of the form (f) where f are irreducible. Might assume f is monic. By the fundamental theorem of algebra, f has a root $z \in \mathbb{C}$.

   (b) The case $r := z \in \mathbb{R}$. $x - r$ is a factor of f. Hence $f = x - r$.

   (c) The case $z \in \mathbb{C} \setminus \mathbb{R}$. Since the conjugate of f is also in $\mathbb{R}[x]$, $\overline{z}$ is also a root of f. So $(x-z)(x-\overline{z}) \in \mathbb{R}[x]$ is an irreducible factor of f. Hence $f = (x-z)(x-\overline{z})$ by the irreducibility of f.

(5) *Show that* $\mathrm{Spec}(\mathbb{Z}[x])$ *are*

   (i) (0).

   (ii) (p) where p are rational primes.

   (iii) (f) where $f \in \mathbb{Z}[x]$ are irreducible.

   (iv) (p, f) where p are rational primes and $f \in \mathbb{Z}[x]$ are irreducible when viewed in $\mathbb{F}_p[x]$.

   Before giving a proof, it is worth taking a look at the book: *David Mumford, The red book of varieties and schemes.*

   (a) Let $\phi : \mathbb{Z} \to \mathbb{Z}[x]$ be the natural inclusion map. Hence $\phi^* : \mathrm{Spec}(\mathbb{Z}[x]) \to \mathrm{Spec}(\mathbb{Z})$ is continuous (Exercise 1.21). Suppose $\mathfrak{P} \in \mathrm{Spec}(\mathbb{Z}[x])$, then $\phi^*(\mathfrak{P}) = (0)$ or (p) where p is a rational prime.

   (b) The case $\phi^*(\mathfrak{P}) = (0)$. A non-trivial prime ideal $\mathfrak{P}$ must be generated by a set of nonconstant polynomials which, since $\mathfrak{P}$ is prime, may be assumed to be irreducible in $\mathbb{Z}[x]$. Note that $\mathbb{Z}[x]$ is not a PID.

   (c) By Gauss' lemma, these polynomials are also irreducible in $\mathbb{Q}[x]$. Since $\mathbb{Q}[x]$ is a Euclidean domain, if there are at least two distinct irreducible polynomials f, g generating $\mathfrak{P}$, then $1 = af + bg$ for some $a, b \in \mathbb{Q}[x]$. Clearing all denominators to get that $n = \widetilde{a}f + \widetilde{b}g$ for some $\widetilde{a}, \widetilde{b} \in \mathbb{Z}[x]$ and some $n \in \mathbb{Z} \setminus \{0\}$, contrary to $\phi^*(\mathfrak{P}) = (0)$. Therefore, $\mathfrak{P} = (f)$ for one irreducible polynomial $f \in \mathbb{Z}[x]$.

(d) The case $\phi^*(\mathfrak{P}) = (p)$ where p is a rational prime. Note that

$$\begin{aligned}\mathbb{Z}[x]/\mathfrak{P} &\cong (\mathbb{Z}[x]/p\mathbb{Z}[x])/(\mathfrak{P}/p\mathbb{Z}[x]) \\ &\cong (\underbrace{\mathbb{Z}/p\mathbb{Z}}_{:=\mathbb{F}_p})[x]/(\mathfrak{P}/p\mathbb{Z}[x])\end{aligned}$$

is an integral domain (since $\mathfrak{P}$ is prime). So $\mathfrak{P}/p\mathbb{Z}[x]$ is a prime ideal in $\mathbb{F}_p[x]$. Note that $\mathbb{F}_p[x]$ is a PID and all non-trivial prime ideals are of the form (f) where f are irreducible.

(e) As $\mathfrak{P}/p\mathbb{Z}[x] = (0)$, $\mathfrak{P} = p\mathbb{Z}[x] = (p) \in \mathbb{Z}[x]$.

(f) As $\mathfrak{P}/p\mathbb{Z}[x] = (f)$ where $f \in \mathbb{Z}[x]$ is irreducible when viewed in $\mathbb{F}_p[x]$, $\mathfrak{P} = (p, f)$.

$\square$

Exercise 1.17.

For each $f \in A$, let X_f denote the complement of $V(f)$ in $X = \operatorname{Spec}(A)$. The sets X_f are open. Show that they form a basis of open sets for the Zariski topology, and that

(i) $X_f \cap X_g = X_{fg}$.

(ii) $X_f = \varnothing \iff f$ *is nilpotent.*

(iii) $X_f = X \iff f$ *is a unit.*

(iv) $X_f = X_g \iff r((f)) = r((g))$.

(v) *X is quasi-compact (compact), that is, every open covering of X has a finite subcovering.*

(vi) *More generally, each X_f is quasi-compact.*

(vii) *An open subset of X is quasi-compact if and only if it is a finite union of sets X_f.*

The sets X_f are called basic open sets of $X = \operatorname{Spec}(A)$.

(Hint: To prove (v), remark that it is enough to consider a covering of X by basic open sets $X_{f_i} (i \in I)$. Show that the f_i generate the unit ideal and hence that there is an equation of the form

$$1 = \sum_{i \in J} g_i f_i \quad (g_i \in A)$$

where J is some finite subset of I. Then the $X_{f_i} (i \in J)$ cover X.)

Proof of basis. It is equivalent to Exercise 1.15 (iii). Given any open set O in X. Write $O = X - V(\mathfrak{a})$ for some ideal $\mathfrak{a}$ of A. Since

$$V(\mathfrak{a}) = V\left(\bigcup_{f\in\mathfrak{a}}(f)\right) = \bigcap_{f\in\mathfrak{a}} V(f),$$

we have

$$O = X - V(\mathfrak{a}) = X - \bigcap_{f\in\mathfrak{a}} V(f) = \bigcup_{f\in\mathfrak{a}}(X - V(f)) = \bigcup_{f\in\mathfrak{a}} X_f,$$

or any open set is a union of basic open sets. □

Proof of (i). $X_f \cap X_g = X_{fg} \Longleftrightarrow V(f) \cup V(g) = V(fg)$ holds by Exercise 1.15 (iv). □

Proof of (ii).

$$\begin{aligned}
X_f = \varnothing &\Longleftrightarrow V(f) = X \\
&\Longleftrightarrow f \in \mathfrak{p} \text{ for all prime ideal } \mathfrak{p} \text{ of } A \\
&\Longleftrightarrow f \in \mathfrak{N}, \text{the nilradical of } A \text{ (Proposition 1.8)} \\
&\Longleftrightarrow f \text{ is nilpotent (Proposition 1.7)}
\end{aligned}$$

□

Proof of (ii)(Using (iv)).

$$\begin{aligned}
X_f = \varnothing &\Longleftrightarrow X_f = X_0 && \text{(Exercise 15(ii))} \\
&\Longleftrightarrow r(f) = r(0) && \text{((iv))} \\
&\Longleftrightarrow f \in r(f) = r(0) \\
&\Longleftrightarrow f^m = 0 \text{ for some } m > 0 \\
&\Longleftrightarrow f \text{ is nilpotent}
\end{aligned}$$

□

Proof of (iii).

$$\begin{aligned}
X_f = X &\Longleftrightarrow V(f) = \varnothing \\
&\Longleftrightarrow f \notin \mathfrak{p} \text{ for all prime ideal } \mathfrak{p} \text{ of } A \\
&\Longleftrightarrow f \text{ is unit (Corollary 1.5)}
\end{aligned}$$

□

Proof of (iii)(Using (iv)).

$$\begin{aligned}
X_f = X &\iff X_f = X_1 && \text{(Exercise 15(ii))}\\
&\iff r(f) = r(1) && \text{((iv))}\\
&\iff f \in r(f) = r(1)\\
&\iff f^m = 1 \text{ for some } m > 0\\
&\iff f \text{ is unit}
\end{aligned}$$

□

Proof of (iv).

(1) *Show that* $X_f \subseteq X_g \iff r((f)) \subseteq r((g))$. Actually,

$$\begin{aligned}
X_f \subseteq X_g &\Longrightarrow V(f) \supseteq V(g)\\
&\Longrightarrow \{\mathfrak{p} \in \mathrm{Spec}(A) : \mathfrak{p} \supseteq (f)\} \supseteq \{\mathfrak{p} \in \mathrm{Spec}(A) : \mathfrak{p} \supseteq (g)\}\\
&\Longrightarrow \bigcap_{(f)\subseteq\mathfrak{p}\in\mathrm{Spec}(A)} \mathfrak{p} \subseteq \bigcap_{(g)\subseteq\mathfrak{p}\in\mathrm{Spec}(A)} \mathfrak{p}\\
&\overset{1.14}{\Longrightarrow} r(f) \subseteq r(g)\\
&\Longrightarrow V(r(f)) \supseteq V(r(g))\\
&\Longrightarrow V(f) \supseteq V(g)\\
&\Longrightarrow X_f \subseteq X_g.
\end{aligned}$$

(2) By (1),

$$\begin{aligned}
X_f \subseteq X_g &\iff r((f)) \subseteq r((g)),\\
X_f \supseteq X_g &\iff r((f)) \supseteq r((g)).
\end{aligned}$$

Hence,

$$X_f = X_g \iff r((f)) = r((g)).$$

□

Proof of (v). Notice that it is enough to consider a covering of X by basic open sets $X_{f_i} (i \in I)$.

(1) Since X is covered by $X_{f_i}(i \in I)$,

$$\begin{aligned} X = \bigcup_{i \in I} X_{f_i} &\Longrightarrow X - V(1) = \bigcup_{i \in I}(X - V(f_i)) \\ &\Longrightarrow V(1) = \bigcap_{i \in I} V(f_i) \\ &\Longrightarrow V(1) = V\left(\sum_{i \in I} f_i\right) \\ &\Longrightarrow r(1) = r\left(\sum_{i \in I} f_i\right). \end{aligned}$$

Hence, $1 \in r(1) = r\left(\sum_{i \in I} f_i\right)$ can be expressed as

$$1 = 1^m = \sum_{j \in J} g_j f_j$$

where J is a finite subset of I and $g_j \in A$. That is, $(1) = \sum_{j \in J} f_j$.

(2) Hence, $V(1) = V\left(\sum_{j \in J} f_j\right)$. Therefore, X is covered by finite subcovering $\{X_{f_j}\}(j \in J)$.

□

Proof of (v)(Using (vi)). Since $X = X_1$, X is quasi-compact by (vi). □

Proof of (vi). Notice that it is enough to consider a covering of X_f by basic open sets $X_{f_i}(i \in I)$.

(1) Since X_f is covered by $X_{f_i}(i \in I)$,

$$\begin{aligned} X_f = \bigcup_{i \in I} X_{f_i} &\Longrightarrow X - V(f) = \bigcup_{i \in I}(X - V(f_i)) \\ &\Longrightarrow V(f) = \bigcap_{i \in I} V(f_i) \\ &\Longrightarrow V(f) = V\left(\sum_{i \in I} f_i\right) \\ &\Longrightarrow r(f) = r\left(\sum_{i \in I} f_i\right). \end{aligned}$$

Hence, $f \in r(f) = r\left(\sum_{i \in I} f_i\right)$ can be expressed as

$$f^m = \sum_{j \in J} g_j f_j$$

where J is a finite subset of I and $g_j \in A$. That is, $f^m \in \sum_{j \in J} f_j$.

(2) *Show that* $V\left(\sum_{j\in J} f_j\right) = V(f)$.

 (a) ($\subseteq$) For any prime ideal $\mathfrak{p} \supseteq \sum_{j\in J} f_j$, $f^m \in \mathfrak{p}$ or $f \in \mathfrak{p}$ (since $\mathfrak{p}$ is prime). So $\mathfrak{p} \supseteq (f)$, or $V\left(\sum_{j\in J} f_j\right) \subseteq V(f)$.

 (b) ($\supseteq$)

$$\sum_{j\in J} f_j \subseteq \sum_{i\in I} f_i \Longrightarrow V\left(\sum_{j\in J} f_j\right) \supseteq V\left(\sum_{i\in I} f_i\right) = V(f).$$

(3) Therefore, X_f is covered by finite subcovering $\{X_{f_j}\}(j \in J)$.

□

Proof of (vi)(Using (v)). Exercise 3.21 (i) shows that X_f is the spectrum of A_f. By (v), X_f is quasi-compact. □

Proof of (vii).

(1) ($\Longrightarrow$) Given an open subset O. Since X_f form a basis of open sets,

$$O = \bigcup_{f\in\mathfrak{a}} X_f \text{ for some ideal } \mathfrak{a} \text{ of } A$$

 Especially, $\{X_f\}_{f\in\mathfrak{a}}$ is an open covering of O. Since O is quasi-compact, there exists a finite subcovering $\{X_f\}_{f\in J}$ of O, where J is a finite subset of $\mathfrak{a}$ (as a set). That is, $O = \bigcup_{f\in J} X_f$ is a finite union of sets X_f.

(2) ($\Longleftarrow$) Since X_f is quasi-compact, any finite union of quasi-compact sets is quasi-compact again.

□

Exercise 1.18.

For psychological reasons it is sometimes convenient to denote a prime ideal of A by a letter such as x or y when thinking of it as a point of $X = \operatorname{Spec}(A)$. When thinking of x as a prime ideal of A, we denote it by $\mathfrak{p}_x$ (logically, of course, it is the same thing). Show that

(i) *The set $\{x\}$ is closed (we say that x is a "closed point") in* $\operatorname{Spec}(A)$ *if and only if $\mathfrak{p}_x$ is maximal;*

(ii) $\overline{\{x\}} = V(\mathfrak{p}_x)$*;*

(iii) $y \in \overline{\{x\}}$ *if and only if* $\mathfrak{p}_x \subseteq \mathfrak{p}_y$*;*

(iv) *X is a T_0-space (this means that if x, y are distinct points of X, then either there is a neighborhood of x which does not contain y, or else there is a neighborhood of y which does not contain x).*

Proof of (i).

$$\{x\} = \overline{\{x\}} \overset{\text{(ii)}}{\Longleftrightarrow} \{x\} = V(\mathfrak{p}_x) \Longleftrightarrow \mathfrak{p}_x \text{ is maximal.}$$

□

Proof of (ii). Since $\overline{\{x\}}$ is the intersection of all closed sets containing x and Exercise 1.15 (iii), we have

$$\overline{\{x\}} = \bigcap_{\mathfrak{a} \subseteq \mathfrak{p}_x} V(\mathfrak{a}) = V\left(\sum_{\mathfrak{a} \subseteq \mathfrak{p}_x} \mathfrak{a}\right) = V(\mathfrak{p}_x).$$

□

Proof of (iii).

$$y \in \overline{\{x\}} \overset{\text{(ii)}}{\Longleftrightarrow} y \in V(\mathfrak{p}_x) \Longleftrightarrow \mathfrak{p}_y \supseteq \mathfrak{p}_x.$$

□

Proof of (iv).

(1) Suppose x and y are two points in X such that $y \in \overline{\{x\}}$ and $x \in \overline{\{y\}}$. Note that $x = y$ implies that X is a T_0-space. So it suffices to show that $x = y$.

(2) By (iii), $\mathfrak{p}_y \supseteq \mathfrak{p}_x$ and $\mathfrak{p}_x \supseteq \mathfrak{p}_y$. So $\mathfrak{p}_x = \mathfrak{p}_y$ or $x = y$.

□

Exercise 1.19.

A topological space X is said to be irreducible if $X \neq \varnothing$ and if every pair of non-empty open sets in X intersect, or equivalently if every non-empty open set is dense in X. Show that Spec(A) *is irreducible if and only if the nilradical of A is a prime ideal.*

Proof. Use the notations in Proposition 1.7 and Exercise 1.17.

$$\begin{aligned}
&\text{Spec}(A) \text{ is irreducible} \\
\Longleftrightarrow\ & X_f \cap X_g \neq \varnothing \text{ for nonempty } X_f, X_g \in \text{Spec}(A) \\
\Longleftrightarrow\ & X_{fg} \neq \varnothing \text{ for nonempty } X_f, X_g \in \text{Spec}(A) && \text{(Exercise 1.17 (i))} \\
\Longleftrightarrow\ & fg \notin \mathfrak{N} \text{ for } f, g \notin \mathfrak{N} && \text{(Exercise 1.17 (ii))} \\
\Longleftrightarrow\ & \mathfrak{N} \text{ is prime.}
\end{aligned}$$

□

Exercise 1.20.

Let X be a topological space.

(i) *If Y is an irreducible subspace of X, then the closure $\overline{Y}$ of Y in X is irreducible.*

(ii) *Every irreducible subspace of X is contained in a maximal irreducible subspace.*

(iii) *The maximal irreducible subspaces of X are closed and cover X. They are called the irreducible components of X. What are the irreducible components of a Hausdorff space?*

(iv) *If A is a ring and $X = \mathrm{Spec}(A)$, then the irreducible components of X are the closed sets $V(\mathfrak{p})$, where $\mathfrak{p}$ is a minimal prime ideal of A (Exercise 1.8).*

Proof of (i).

(1) *Y is irreducible if and only if Y cannot be represented as the union of two proper closed subspaces.*

$$\begin{aligned}
&\forall \text{ nonempty open sets } U_1 \text{ and } U_2, U_1 \cap U_2 \neq \varnothing \\
\Longleftrightarrow\ &\forall \text{ nonempty open sets } U_1 \text{ and } U_2, X - (U_1 \cap U_2) \neq X \\
\Longleftrightarrow\ &\forall \text{ nonempty open sets } U_1 \text{ and } U_2, (X - U_1) \cup (X - U_2) \neq X \\
\Longleftrightarrow\ &\forall \text{ proper closed sets } Y_1 \text{ and } Y_2, Y_1 \cup Y_2 \neq X \\
\Longleftrightarrow\ &\nexists \text{ proper closed sets } Y_1 \text{ and } Y_2, Y_1 \cup Y_2 = X.
\end{aligned}$$

(2) If $\overline{Y}$ were reducible, there are two closed set Y_1 and Y_2 such that

$$\overline{Y} \subseteq Y_1 \cup Y_2, \qquad \overline{Y} \not\subseteq Y_i (i = 1, 2).$$

(a) $Y \subseteq \overline{Y} \subseteq Y_1 \cup Y_2$.

(b) $Y \not\subseteq Y_i (i = 1, 2)$. If not, $Y \subseteq Y_i$ for some i. Take closure to get $\overline{Y} \subseteq \overline{Y_i} = Y_i$ (since Y_i is closed), contrary to the assumption.

By (a)(b), Y is reducible, which is absurd.

□

Proof of (ii).

(1) This is a standard application of Zorn's lemma.

(2) Suppose Y is an irreducible subspace of X. Let Σ be the set of all irreducible subspaces of X containing Y. Order Σ by inclusion. Σ is not empty, since $Y \in \Sigma$. To apply Zorn's lemma we must show that every chain in Σ has an upper bound in Σ; let then (Y_α) be a chain in Σ. Let $Z = \bigcup_\alpha Y_\alpha$. $Z \supseteq Y$ clearly.

(3) *Show that Z is irreducible.* Given two non-empty open sets U and V contained in $Z = \bigcup_\alpha Y_\alpha$. Then $U \cap Y_\alpha \neq \varnothing$ and $V \cap Y_\beta \neq \varnothing$ for some α, β. Since (Y_α) is a chain, we might have $V \cap Y_\alpha \supseteq V \cap Y_\beta \neq \varnothing$ if $\beta \leq \alpha$. (The case $\alpha \leq \beta$ is similar.) So $U \cap V \cap Z \supseteq U \cap V \cap Y_\alpha \neq \varnothing$ since Z contains an irreducible subspace Y_α in X.

(4) Hence $Z \in \Sigma$, and Z is an upper bound of the chain (Y_α). Hence by Zorn's lemma Σ has a maximal element.

□

Proof of (iii).

(1) *Show that the maximal irreducible subspaces of X are closed.* Suppose Y is a maximal irreducible subspaces of X. So $\overline{Y}$ of Y in X is irreducible (by part (i)). The maximality of Y implies that $Y = \overline{Y}$.

(2) *Show that the maximal irreducible subspaces of X cover X.* Note that each element $P \in X$ forms an irreducible subset $\{P\}$ and thus $\{P\}$ is contained in one irreducible component (by (ii)).

(3) One point subsets are the irreducible components of a Hausdorff space.

□

Proof of (iv).

(1) Suppose Y is an irreducible components of X. *Show that $Y = V(\mathfrak{p})$ where $\mathfrak{p}$ is a prime ideal.* Similar to the proof of Exercise 1.19.

(2) *Show that $\mathfrak{p}$ is a minimal prime ideal of A.* Suppose $\mathfrak{q} \subseteq \mathfrak{p}$. Then $V(\mathfrak{q}) \supseteq V(\mathfrak{p})$. By the maximality of $Y = V(\mathfrak{p})$, $V(\mathfrak{q}) = V(\mathfrak{p})$ or $r(\mathfrak{q}) = r(\mathfrak{p})$ or $\mathfrak{q} = \mathfrak{p}$. Hence $\mathfrak{p}$ is a minimal prime ideal of A.

□

Exercise 1.21.

Let $\phi : A \to B$ be a ring homomorphism. Let $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$. If $\mathfrak{q} \in Y$, then $\phi^{-1}(\mathfrak{q})$ is a prime ideal of A, i.e., a point of X. Hence ϕ induces a mapping $\phi^* : Y \to X$. Show that

(i) *If* $f \in A$ *then* $\phi^{*-1}(X_f) = Y_{\phi(f)}$, *and hence that* ϕ^* *is continuous.*

(ii) *If* $\mathfrak{a}$ *is an ideal of* A, *then* $\phi^{*-1}(V(\mathfrak{a})) = V(\mathfrak{a}^e)$.

(iii) *If* $\mathfrak{b}$ *is an ideal of* B, *then* $\overline{\phi^*(V(\mathfrak{b}))} = V(\mathfrak{b}^c)$.

(iv) *If* ϕ *is surjective, then* ϕ^* *is a homeomorphism of* Y *onto the closed subset* $V(\ker(\phi))$ *of* X. *(In particular,* $\operatorname{Spec}(A)$ *and* $\operatorname{Spec}(A/\mathfrak{N})$ *(where* $\mathfrak{N}$ *is the nilradical of* A*) are naturally homeomorphic.)*

(v) *If* ϕ *is injective, then* $\phi^*(Y)$ *is dense in* X. *More precisely,* $\phi^*(Y)$ *is dense in* X *if and only if* $\ker(\phi) \subseteq \mathfrak{N}$.

(vi) *Let* $\psi : B \to C$ *be another ring homomorphism. Then* $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

(vii) *Let* A *be an integral domain with just one nonzero prime ideal* $\mathfrak{p}$, *and let* K *be the field of fractions of* A. *Let* $B = (A/\mathfrak{p}) \times K$. *Define* $\phi : A \to B$ *by* $\phi(x) = (\overline{x}, x)$, *where* $\overline{x}$ *is the image of* x *in* $A/\mathfrak{p}$. *Show that* ϕ^* *is bijective but not a homeomorphism.*

Proof of (i). Since

$$\begin{aligned}
&\mathfrak{q} \in Y_{\phi(f)} = Y - V(\phi(f)) \\
\Longleftrightarrow\ &\mathfrak{q} \notin V(\phi(f)) = \{\text{all prime ideals in } B \text{ containing } \phi(f)\} \\
\Longleftrightarrow\ &\phi(f) \notin \mathfrak{q} \\
\Longleftrightarrow\ &f \notin \phi^{-1}(\mathfrak{q}) \\
\Longleftrightarrow\ &\phi^{-1}(\mathfrak{q}) \notin V(f) = \{\text{all prime ideals in } A \text{ containing } f\} \\
\Longleftrightarrow\ &\phi^*(\mathfrak{q}) = \phi^{-1}(\mathfrak{q}) \in X_f,
\end{aligned}$$

ϕ^* is continuous. $\square$

Proof of (ii).

(1) *Use the same notation of Proposition 1.17. Show that*

$$\mathfrak{b}^c \supseteq \mathfrak{a} \Longleftrightarrow \mathfrak{b} \supseteq \mathfrak{a}^e.$$

Suppose $\mathfrak{b}^c \supseteq \mathfrak{a}$, then $\mathfrak{b}^{ce} \supseteq \mathfrak{a}^e$. Proposition 1.17 (i) suggests that $\mathfrak{b} \supseteq \mathfrak{b}^{ce} \supseteq \mathfrak{a}^e$. The converse is similar.

(2) So

$$\begin{aligned}
&\mathfrak{q} \in \phi^{*-1}(V(\mathfrak{a})) \\
\Longleftrightarrow\ &\phi^*(\mathfrak{q}) \in V(\mathfrak{a}) = \{\text{all prime ideals containing } \mathfrak{a}\} \\
\Longleftrightarrow\ &\phi^*(\mathfrak{q}) \supseteq \mathfrak{a} \\
\Longleftrightarrow\ &\mathfrak{q}^c \supseteq \mathfrak{a} \\
\Longleftrightarrow\ &\mathfrak{q} \supseteq \mathfrak{a}^e && ((1)) \\
\Longleftrightarrow\ &\mathfrak{q} \in V(\mathfrak{a}^e) = \{\text{all prime ideals containing } \mathfrak{a}^e\}.
\end{aligned}$$

□

Proof of (iii).

(1) Might assume that $\mathfrak{b} = r(\mathfrak{b})$ is radical by Exercise 1.15 (i).

(2) *Show that* $\overline{\phi^*(V(\mathfrak{b}))} \supseteq V(\mathfrak{b}^c)$. Write $\overline{\phi^*(V(\mathfrak{b}))} = V(\mathfrak{a})$ for some radical ideal $\mathfrak{a}$ in A since $\overline{\phi^*(V(\mathfrak{b}))}$ is closed. So

$$\begin{aligned} & V(\mathfrak{a}^e) = \phi^{*-1}(V(\mathfrak{a})) = \phi^{*-1}(\overline{\phi^*(V(\mathfrak{b}))}) \supseteq V(\mathfrak{b}) \qquad \text{((ii))} \\ \Longrightarrow\ & r(\mathfrak{a}^e) \subseteq r(\mathfrak{b}) \\ \Longrightarrow\ & r(\mathfrak{a})^e \subseteq r(\mathfrak{a}^e) \subseteq r(\mathfrak{b}) \\ \Longrightarrow\ & \mathfrak{a}^e \subseteq \mathfrak{b} \\ \Longrightarrow\ & \mathfrak{a} \subseteq \mathfrak{b}^c \\ \Longrightarrow\ & V(\mathfrak{a}) \supseteq V(\mathfrak{b}^c). \end{aligned}$$

(3) *Show that* $\overline{\phi^*(V(\mathfrak{b}))} \subseteq V(\mathfrak{b}^c)$. It suffices to show that $\phi^*(V(\mathfrak{b})) \subseteq V(\mathfrak{b}^c)$ since $V(\mathfrak{b}^c)$ is closed. Suppose $\mathfrak{p} \in \phi^*(V(\mathfrak{b}))$. Then there is $\mathfrak{q} \in V(\mathfrak{b})$ such that

$$\mathfrak{p} = \phi^*(\mathfrak{q}) = \mathfrak{q}^c \supseteq \mathfrak{b}^c.$$

So $\mathfrak{p} \in V(\mathfrak{b}^c)$.

□

Proof of (iv). Note that $A/\ker\phi \cong B$ since ϕ is surjective. The correspondence theorem shows that $\phi^* : Y \to V(\ker\phi)$ is bijective. As the continuity of ϕ^* is given by (i), ϕ^* is a homeomorphism of Y onto $V(\ker(\phi)) \subseteq X$. □

Proof of (v).

(1) It suffices to *show that* $\phi^*(Y)$ *is dense in* X *if and only if* $\ker(\phi) \subseteq \mathfrak{N}$.

(2)

$$\begin{aligned} & \phi^*(Y) \text{ is dense in } X \\ \Longleftrightarrow\ & X = \overline{\phi^*(Y)} = \overline{\phi^*(V(0))} = V(0^c) = V(\ker\phi) \\ \Longleftrightarrow\ & \ker\phi \text{ is contained in every prime ideal of } A \\ \Longleftrightarrow\ & \ker\phi \subseteq \mathfrak{N}. \end{aligned}$$

□

Proof of (vi).

$$(\psi\circ\phi)^*(\mathfrak{p}) = (\psi\circ\phi)^{-1}(\mathfrak{p}) = \phi^{-1}(\psi^{-1}(\mathfrak{p})) = \phi^*(\psi^*(\mathfrak{p})) = (\phi^*\circ\psi^*)(\mathfrak{p})$$

for every prime ideal $\mathfrak{p}$ in $\mathrm{Spec}(C)$. □

Proof of (vii).

(1) *Show that ϕ^* is bijective.* Note that

$$\begin{aligned} X &= \operatorname{Spec}(A) = \{(0), \mathfrak{p}\} \\ Y &= \operatorname{Spec}(B) = \{A/\mathfrak{p} \times (0), (0) \times K\} \end{aligned}$$

and thus

$$\begin{aligned} \phi^*(A/\mathfrak{p} \times (0)) &= (0) \\ \phi^*((0) \times K) &= (\mathfrak{p}). \end{aligned}$$

Hence ϕ^* is a bijective.

(2) *Show that ϕ^* is not a homeomorphism.* Note that $\overline{\{(0)\}} = X$ (Exercise 1.18 (iii)) and Y is equipped with the discrete topology since each prime ideal of B is maximal (Exercise 1.18 (i)). So ϕ^* cannot be a homeomorphism.

□

Exercise 1.22.

Let $A = \prod_{i=1}^{n} A_i$ be a direct product of rings A_i. Show that $\operatorname{Spec}(A)$ *is the disjoint union of open (and closed) subspaces X_i, where X_i is canonically homeomorphic with* $\operatorname{Spec}(A_i)$*.*

Conversely, let A be any ring. Show that the following statements are equivalent:

(i) $X = \operatorname{Spec}(A)$ *is disconnected.*

(ii) $A \cong A_1 \times A_2$ *where neither of the rings A_1, A_2 is the zero ring.*

(iii) *A contains an idempotent $\neq 0, 1$ In particular, the spectrum of a local ring is always connected (Exercise 1.12).*

Proof.

(1) *Show that* $\operatorname{Spec}(A)$ *is the union of closed subspaces X_i, where $X_i \cong$* $\operatorname{Spec}(A_i)$. Let $\phi_i : A \to A_i$ be the projection map. So

$$\ker \phi_i = A_1 \times \cdots \times A_{i-1} \times 0 \times A_{i+1} \times \cdots \times A_n.$$

So

$$\operatorname{Spec}(A) = V(0) = V\left(\bigcap_{i=1}^{n} \ker \phi_i\right) = \bigcup_{i=1}^{n} V(\ker \phi_i)$$

where $X_i := V(\ker \phi_i) \cong \operatorname{Spec}(A_i)$ (Exercise 1.21).

(2) *Show that $V(\ker \phi_i)$ and $V(\ker \phi_j)$ are disjoint if $i \neq j$.*

$$V(\ker \phi_i) \cap V(\ker \phi_j) = V(\ker \phi_i + \ker \phi_j) = V(A) = V(1) = \varnothing.$$

(3) *Show that $V(\ker \phi_i)$ is open.* $\mathrm{Spec}(A) = \bigcup_{j=1}^n V(\ker \phi_j)$ and $V(\ker \phi_i) \cap V(\ker \phi_j) = \varnothing$ (if $i \neq j$) implies that $\mathrm{Spec}(A) \setminus V(\ker \phi_i) = \bigcup_{j \neq i} V(\ker \phi_j)$ is closed. Thus $V(\ker \phi_i)$ is open.

(4) ((ii) $\Longrightarrow$ (i)) See (1)(2)(3).

(5) ((i) $\Longrightarrow$ (iii)) Write X as a disjoint union of two nonempty closed sets $V(\mathfrak{a}), V(\mathfrak{b})$ where $\mathfrak{a}, \mathfrak{b}$ are radical ideals in A (Exercise 1.15). Since

$$\begin{aligned} V(0) = X = V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) \\ V(1) = \varnothing = V(\mathfrak{a}) \cap V(\mathfrak{b}) = V(\mathfrak{a} + \mathfrak{b}), \end{aligned}$$

there exist $a \in \mathfrak{a}$, $b \in \mathfrak{b}$ such that $a + b = 1$ and $(ab)^n = 0$ for one positive integer n. So $ab = 0$ since $\mathfrak{a}\mathfrak{b}$ is radical. (Note that $\mathfrak{a} + \mathfrak{b} = 1$ and Exercise 1.13 on page 9.) So

$$a^2 = a(1 - b) = a - ab = a$$

is an idempotent. Also $a \neq 0, 1$ since $V(\mathfrak{a}), V(\mathfrak{b})$ are proper subsets of X.

(6) ((iii) $\Longrightarrow$ (ii)) Take an idempotent $e \neq 0, 1$ in A. Two ideals (e) and $(1-e)$ are proper and coprime. So $(e) \cap (1 - e) = (e)(1 - e) = (0)$ (Proposition 1.10 (i)). Proposition 1.10 (ii) and (iii) imply that the ring homomorphism

$$A \to A/(e) \times A/(1 - e)$$

is an isomorphism. Also $A/(e), A/(1 - e) \neq 0$ since $e \neq 0, 1$.

$\square$

Exercise 1.23.

Let A be a Boolean ring (Exercise 1.11), and let $X = \mathrm{Spec}(A)$.

(i) *For each $f \in A$, the set X_f (Exercise 1.17) is both open and closed in X.*

(ii) *Let $f_1, \ldots, f_n \in A$. Show that $X_{f_1} \cup \cdots \cup X_{f_n} = X_f$ for some $f \in A$.*

(iii) *The sets X_f are the only open subsets of X which are both open and closed.*

(iv) *X is a compact Hausdorff space.*

Proof of (i).

(1) *Show that X is the disjoint union of subspaces X_f and X_{1-f}.* Note that every element in a Boolean ring is an idempotent. Hence

$$\begin{aligned}
&X_f \cap X_{1-f} = X_{f(1-f)} = X_0 = \varnothing \\
&X_f \cup X_{1-f} = X \setminus (V(f) \cap V(1-f)) = X \setminus \underbrace{V(f + (1-f))}_{=V(1)=\varnothing} = X.
\end{aligned}$$

(2) Hence $X_f = X \setminus X_{1-f}$ is both open and closed.

$\square$

Proof of (ii). Similar to (i),

$$\begin{aligned}
X_{f_1} \cup \cdots \cup X_{f_n} &= X \setminus (V(f_1) \cap \cdots \cap V(f_n)) \\
&= X \setminus V(f_1, \ldots, f_n) \\
&= X \setminus V(f) && \text{(Exercise 1.11 (iii))} \\
&= X_f
\end{aligned}$$

for some $f \in A$. $\square$

Proof of (iii).

(1) Suppose Y is both open and closed in X.

(2) Since Y is closed and X is quasi-compact (Exercise 1.17 (vi)), Y is quasi-compact.

(3) Since Y is open, Y is a finite union of sets X_{f_i} for $i = 1, \ldots, n$ (Exercise 1.17 (vii)). Hence $Y = X_f$ for some $f \in A$ (by (ii)).

$\square$

Proof of (iv).

(1) The compactness of X is followed by Exercise 1.17 (v).

(2) *Show that X is Hausdorff.* Exercise 1.18 shows that X is a T_0-space. This means that if x, y are distinct points of X, we might assume that there is a neighborhood U of x which does not contain y.

(3) Write $U = X_f$ for some $f \in A$ (by Exercise 1.17 and (ii)). As $x \in X_f$, $y \in X \setminus X_f = X_{1-f}$ and $X_f \cap X_{1-f} = \varnothing$ by (i). Hence X is Hausdorff.

$\square$

Exercise 1.24. (Boolean lattice)

Let L be a lattice, in which the sup and inf of two elements a, b are denoted by $a \vee b$ and $a \wedge b$ respectively. L is a ***Boolean lattice*** *(or* ***Boolean algebra****) if*

(i) *L has a least element and a greatest element (denoted by 0, 1 respectively);*

(ii) *Each of $\vee$, $\wedge$ is distributive over the other;*

(iii) *Each $a \in L$ has a unique "complement" $a' \in L$ such that $a \vee a' = 1$ and $a \wedge a' = 0$.*

(For example, the set of all subsets of a set, ordered by inclusion, is a Boolean lattice.)

Let L be a Boolean lattice. Define addition and multiplication in L by the rules

$$a + b = (a \wedge b') \vee (a' \wedge b), \qquad ab = a \wedge b.$$

Verify that in this way L becomes a Boolean ring, say $A(L)$.

Conversely, starting from a Boolean ring A, define an ordering on A as follows: $a \leq b$ means that $a = ab$. Show that, with respect to this ordering, A is a Boolean lattice. In this way we obtain a one-to-one correspondence between (isomorphism classes of) Boolean rings and (isomorphism classes of) Boolean lattices.

Proof.

(1) Some properties about $\vee$ and $\wedge$:

(a) (Commutativity) *Show that*

$$a \vee b = b \vee a, \qquad a \wedge b = b \wedge a.$$

Say $z_1 := a \vee b$ and $z_2 := b \vee a$. By the definition of the sup,

$$\begin{aligned} &z_1 \geq a, b \text{ such that for all other } w_1 \geq a, b \text{ we have } w_1 \geq z_1 \\ &z_2 \geq b, a \text{ such that for all other } w_2 \geq b, a \text{ we have } w_2 \geq z_2. \end{aligned}$$

So $z_1 \geq z_2$ and $z_2 \geq z_1$ and thus $z_1 = z_2$. Hence $a \vee b = b \vee a$. Similarly, $a \wedge b = b \wedge a$.

(b) (Associativity) *Show that*

$$\begin{aligned} (a \vee b) \vee c = a \vee b \vee c = a \vee (b \vee c), \\ (a \wedge b) \wedge c = a \wedge b \wedge c = a \wedge (b \wedge c). \end{aligned}$$

Say $z_1 := (a \wedge b) \wedge c$, $z_2 := a \wedge b \wedge c$, and $z_3 := a \wedge (b \wedge c)$. By the definition of inf, z_1 is a unique greatest element such that $z_1 \leq a \wedge b, c$. So $z_1 \leq a, b, c$ or $z_1 \leq z_2$. Besides, $z_2 \leq a, b, c$ implies that $z_2 \leq a, b \wedge c$. So $z_2 \leq z_3$. Hence $z_1 \leq z_2 \leq z_3$. Similarly, $z_3 \leq z_2 \leq z_1$. So $z_1 = z_2 = z_3$. Similarly, $(a \vee b) \vee c = a \vee b \vee c = a \vee (b \vee c)$

(c) (De Morgan's laws) *Show that*

$$(a \vee b)' = a' \wedge b', \qquad (a \wedge b)' = a' \vee b'.$$

Since

$$\begin{aligned}
(a \vee b) \vee (a' \wedge b') &= (a \vee b \vee a') \wedge (a \vee b \vee b') \\
&= (a \vee a' \vee b) \wedge (a \vee b \vee b') \\
&= (1 \vee b) \wedge (a \vee 1) \\
&= 1 \wedge 1 \\
&= 1.
\end{aligned}$$

and

$$\begin{aligned}
(a \vee b) \wedge (a' \wedge b') &= (a \wedge a' \wedge b') \vee (b \wedge a' \wedge b') \\
&= (a \wedge a' \wedge b') \vee (a' \wedge b \wedge b') \\
&= (0 \wedge b') \vee (a' \wedge 0) \\
&= 0 \vee 0 \\
&= 0,
\end{aligned}$$

The complement of $a \vee b$ is $a' \wedge b'$. Similarly, $(a \wedge b)' = a' \vee b'$.

(2) *Show that $A(L)$ is an abelian group under addition.*

(a) (Commutativity) *Show that $a + b = b + a$.* By (1)(a),

$$\begin{aligned}
a + b &= (a \wedge b') \vee (a' \wedge b) \\
&= (a' \wedge b) \vee (a \wedge b') \\
&= (b \wedge a') \vee (b' \wedge a) \\
&= b + a.
\end{aligned}$$

(b) (Associativity) *Show that $(a + b) + c = a + (b + c)$.* By (1)(a)(b),

$$\begin{aligned}
&(a + b) + c \\
&= ((a + b) \wedge c') \vee ((a + b)' \wedge c) \\
&= (((a \wedge b') \vee (a' \wedge b)) \wedge c') \\
&\quad \vee (((a \wedge b') \vee (a' \wedge b))' \wedge c) \\
&= (a \wedge b' \wedge c') \vee (a' \wedge b \wedge c') \\
&\quad \vee ((a' \vee b) \wedge (a \vee b') \wedge c) && ((\text{ii}),(1)(\text{c})) \\
&= (a \wedge b' \wedge c') \vee (a' \wedge b \wedge c') \\
&\quad \vee (((a' \wedge a) \vee (a' \wedge b') \vee (b \wedge a) \vee (b \wedge b')) \wedge c) && ((\text{ii})) \\
&= (a \wedge b' \wedge c') \vee (a' \wedge b \wedge c') \\
&\quad \vee ((a' \wedge b') \vee (a \wedge b)) \wedge c) && ((\text{iii}),(1)(\text{a})) \\
&= (a \wedge b' \wedge c') \vee (a' \wedge b \wedge c') \\
&\quad \vee (a' \wedge b' \wedge c) \vee (a \wedge b \wedge c) && ((\text{ii}))
\end{aligned}$$

and

$$\begin{aligned}
&a + (b + c) \\
&= (b + c) + a && ((a)) \\
&= (c \wedge b' \wedge a') \vee (c' \wedge b \wedge a') \vee (c' \wedge b' \wedge a) \vee (c \wedge b \wedge a) \\
&= (a' \wedge b' \wedge c) \vee (a' \wedge b \wedge c') \vee (a \wedge b' \wedge c') \vee (a \wedge b \wedge c) && ((1)(a)) \\
&= (a \wedge b' \wedge c') \vee (a' \wedge b \wedge c') \vee (a' \wedge b' \wedge c) \vee (a \wedge b \wedge c). && ((1)(a))
\end{aligned}$$

Thus $(a + b) + c = a + (b + c)$.

(c) (Identity) *Show that* $a + 0 = 0 + a = a$. The complement of 0 in L is $0' = 1$ and vice versa ((iii)). Hence

$$\begin{aligned}
a + 0 &= (a \wedge 0') \vee (a' \wedge 0) \\
&= (a \wedge 1) \vee (a' \wedge 0) \\
&= a \vee 0 \\
&= a.
\end{aligned}$$

Note that $A(L)$ is commutative under addition.

(d) (Invertibility) *Show that* $a + a = 0$, *that is,* a *itself is the additive inverse of* a.

$$a + a = (a \wedge a') \vee (a' \wedge a) = 0 \vee 0 = 0.$$

(3) *Show that* $A(L)$ *is commutative under multiplication.* It is (1)(a).

(4) *Show that* $A(L)$ *is a monoid under multiplication.*

(a) (Associativity) *Show that* $(ab)c = a(bc)$. It is (1)(b).

(b) (Identity) *Show that* $a1 = 1a = a$.

$$a1 = a \wedge 1 = a, \qquad 1a = 1 \wedge a = a.$$

(5) *Show that multiplication is distributive with respect to addition in* $A(L)$.

(a) (Left distributivity) *Show that* $a(b + c) = ab + ac$. Note that

$$\begin{aligned}
a(b + c) &= a \wedge (b + c) \\
&= a \wedge ((b \wedge c') \vee (b' \wedge c)) \\
&= (a \wedge b \wedge c') \vee (a \wedge b' \wedge c) && ((ii))
\end{aligned}$$

and

$$\begin{aligned}
ab + ac &= (a \wedge b) + (a \wedge c) \\
&= ((a \wedge b) \wedge (a \wedge c)') \vee ((a \wedge b)' \wedge (a \wedge c)) \\
&= ((a \wedge b) \wedge (a' \vee c')) \vee ((a' \vee b') \wedge (a \wedge c)) && ((1)(c)) \\
&= ((a \wedge b \wedge a') \vee (a \wedge b \wedge c')) \\
&\quad \vee ((a' \wedge a \wedge c) \vee (b' \wedge a \wedge c)) && ((ii)) \\
&= ((a \wedge a' \wedge b) \vee (a \wedge b \wedge c')) \\
&\quad \vee ((a' \wedge a \wedge c) \vee (a \wedge b' \wedge c)) && ((1)(a)) \\
&= 0 \vee (a \wedge b \wedge c') \vee 0 \vee (a \wedge b' \wedge c) && ((iii)) \\
&= (a \wedge b \wedge c') \vee (a \wedge b' \wedge c). && ((i))
\end{aligned}$$

(b) (Right distributivity) The left distributivity implies the right distributivity by (1)(a).

(6) (2)-(5) show that $A(L)$ is a commutative ring. Also $a^2 = a \wedge a = a$ implies that $A(L)$ is a Boolean ring.

(7) Conversely, starting from a Boolean ring A, define an ordering on A as follows: $a \leq b$ means that $a = ab$. The ordering is well-defined (since A is a Boolean ring).

(8) *Define $a \vee b = a + b + ab$ and $a \wedge b = ab$. Show that $a \vee b$ is the sup and $a \wedge b$ is the inf.* Similar to the proof of Exercise 1.11 (iii),

$$a(a \vee b) = a(a + b + ab) = a^2 + ab + a^2 b = a + ab + ab = a.$$

So $a \leq a \vee b$. Similarly, $b \leq a \vee b$. So $a \vee b$ is an upper bound of a and b. To show $a \vee b$ is the least upper bound, it suffices to show that all other $z \geq a, b$ we have $z \geq a \vee b$. In fact,

$$(a \vee b)z = (a + b + ab)z = az + bz + abz = a + b + ab = a \vee b.$$

Hence $a \vee b$ is the sup. Similarly, $a \wedge b$ is the inf. Therefore we define a lattice $L(A)$ on a Boolean ring A.

(9) *Show that $L(A)$ is a Boolean lattice.* $0 \in A$ is a least element, 1 is a greatest element, each of $\vee$ and $\wedge$ is distributive over the other, and $a' = 1 - a$ is the unique complement of a.

(10) It is easy to see that $A(L(A)) = A$ and $L(A(L)) = L$ (up to isomorphism). Hence there is a one-to-one correspondence between Boolean rings and Boolean lattices.

$\square$

Exercise 1.25. (Stone's theorem)

From the last two exercises deduce Stone's theorem, that every Boolean lattice is isomorphic to the lattice of open-and-closed subsets of some compact Hausdorff topological space.

Proof.

(1) Suppose L is a Boolean lattice and $A = A(L)$ is the corresponding Boolean ring (Exercise 1.24). Observe that $X = \mathrm{Spec}(A)$ is a compact Hausdorff space (Exercise 1.23).

(2) Define a map
$$\alpha : L \to \mathcal{P}(X)$$
by $\alpha(f) = X_f$ where $\mathcal{P}(X)$ is the power set of a set X. View $\mathcal{P}(X)$ as a Boolean lattice, ordered by inclusion.

(3) The image of α is the collection of all open-and-closed sets in X (Exercise 1.23). Note that $\mathrm{im}(\alpha)$ is a Boolean lattice (Exercises 1.17 and 1.23).

(4) *Show that $\alpha : L \to \mathrm{im}(\alpha)$ is injective.* Suppose $X_f = X_g$. Exercise 1.17 shows that $r((f)) = r((g))$. In particular, $f \in r((g))$. So $f = g^n$ for some $n \geq 1$. Hence $f = g^n = g^{n-1} = \cdots = g$ since A is a Boolean ring.

(5) Since
$$\begin{aligned} f \leq g &\Longleftrightarrow f = fg \\ &\Longleftrightarrow X_f = X_{fg} && \text{(Injectivity of } \alpha\text{)} \\ &\Longleftrightarrow X_f = X_f \cap X_g \\ &\Longleftrightarrow X_f \subseteq X_g, \end{aligned}$$
$\alpha : L \to \mathrm{im}(\alpha)$ preserves the ordering. Hence α is an isomorphism between two Boolean lattices.

□

Exercise 1.26. (Maximal spectrum)

Let A be a ring. The subspace of $\mathrm{Spec}(A)$ *consisting of the maximal ideals of A, with the induced topology, is called the **maximal spectrum** of A and is denoted by* $\mathrm{Max}(A)$*. For arbitrary commutative rings it does not have the nice functorial properties of* $\mathrm{Spec}(A)$ *(see Exercise 1.21), because the inverse image of a maximal ideal under a ring homomorphism need not be maximal.*

Let X be a compact Hausdorff space and let $C(X)$ denote the ring of all real-valued continuous functions on X (add and multiply functions by adding and multiplying *their values). For each $x \in X$, let $\mathfrak{m}_x$ be the set of all $f \in C(X)$ such that $f(x) = 0$. The ideal $\mathfrak{m}_x$ is maximal, because it is the kernel of the (surjective) homomorphism $C(X) \to \mathbb{R}$ which takes f to $f(x)$. If $\widetilde{X}$ denotes* $\mathrm{Max}(C(X))$, *we have therefore defined a mapping $\mu : X \to \widetilde{X}$, namely $x \mapsto \mathfrak{m}_x$.*

We shall show that μ is a homeomorphism of X onto $\widetilde{X}$.

(i) *Let $\mathfrak{m}$ be any maximal ideal of $C(X)$, and let $V = V(\mathfrak{m})$ be the set of common zeros of the functions in $\mathfrak{m}$: that is,*

$$V = \{x \in X : f(x) = 0 \textit{ for all } f \in \mathfrak{m}\}.$$

Suppose that V is empty. Then for each $x \in X$ there exists $f_x \in \mathfrak{m}$ such that $f_x(x) \neq 0$. Since f_x is continuous, there is an open neighborhood U_x of x in X on which f_x does not vanish. By compactness a finite number of the neighborhoods, say $U_{x_1}, \ldots, U_{x_n}$, cover X. Let

$$f = f_{x_1}^2 + \cdots + f_{x_n}^2.$$

Then f does not vanish at any point of X, hence is a unit in $C(X)$. But this contradicts $f \in \mathfrak{m}$, hence V is not empty. Let x be a point of V. Then $\mathfrak{m} \subseteq \mathfrak{m}_x$, hence $\mathfrak{m} = \mathfrak{m}_x$ because $\mathfrak{m}$ is maximal. Hence μ is surjective.

(ii) *By Urysohn's lemma (this is the only non-trivial fact required in the argument) the continuous functions separate the points of X. Hence $x \neq y \Longrightarrow \mathfrak{m}_x \neq \mathfrak{m}_y$, and therefore μ is injective.*

(iii) *Let $f \in C(X)$; let*

$$U_f = \{x \in X : f(x) \neq 0\}$$

and let

$$\widetilde{U}_f = \{\mathfrak{m} \in \widetilde{X} : f \notin \mathfrak{m}\}.$$

Show that $\mu(U_f) = \widetilde{U}_f$. The open sets U_f (resp. $\widetilde{U}_f$) form a basis of the topology of X (resp. $\widetilde{X}$) and therefore μ is a homeomorphism. Thus X can be reconstructed from the ring of functions $C(X)$.

Proof.

(1) *Show that the inverse image of a maximal ideal under a ring homomorphism need not be maximal.* Let $\phi : \mathbb{Z}[x] \to \mathbb{R}[x]$ be a natural inclusion map. The ideal $\mathfrak{P} = (x)$ in $\mathbb{R}[x]$ is maximal. But $\phi^{-1}(\mathfrak{P}) = (x)$ in $\mathbb{Z}[x]$ is not maximal since $(x) \subsetneq (x, 2)$ in $\mathbb{Z}[x]$.

(2) *Show that $\mu(U_f) = \widetilde{U}_f$.*

$$\begin{aligned} x \in U_f &\Longleftrightarrow x \in X \text{ such that } f(x) \neq 0 \\ &\Longleftrightarrow x \in X \text{ such that } f \notin \mathfrak{m}_x \\ &\Longleftrightarrow \mathfrak{m}_x \in \widetilde{X} \text{ such that } f \notin \mathfrak{m}_x \\ &\Longleftrightarrow \mu(x) = \mathfrak{m}_x \in \widetilde{U}_f. \end{aligned}$$

(3) *Show that U_f form a basis of the topology of X.* Let U be open in X. For any $x \in U$, it suffices to find $f \in C(X)$ such that $x \in U_f \subseteq U$. Note that one-point set $\{x\}$ is closed (since X is Hausdorff). By Urysohn's lemma, there is $f \in C(X)$ such that $f = 1$ on $\{x\}$ and $f = 0$ on $X \setminus U$.

(4) *Show that $\widetilde{U}_f$ form a basis of the topology of $\widetilde{X}$.* Let $\widetilde{U} = \widetilde{W} \cap \widetilde{X}$ be open in $\widetilde{X}$ where $\widetilde{W}$ is open in $\operatorname{Spec}(C(X))$ (w.r.t. the induced topology). For any $\mathfrak{m} \in \widetilde{U} = \widetilde{W} \cap \widetilde{X} \subseteq \widetilde{W}$, Exercise 1.17 shows that

$$\mathfrak{m} \in \operatorname{Spec}(C(X))_f \subseteq \widetilde{W}$$

for some $f \in C(X)$. So

$$\mathfrak{m} \in \underbrace{\operatorname{Spec}(C(X))_f \cap \widetilde{X}}_{=\widetilde{U}_f} \subseteq \underbrace{\widetilde{W} \cap \widetilde{X}}_{=\widetilde{U}}.$$

□

Affine algebraic varieties

Exercise 1.27. (Hilbert's Nullstellensatz)

Let k be an algebraically closed field and let

$$f_\alpha(t_1, \ldots, t_n) = 0$$

*be a set of polynomial equations in n variables with coefficients in k. The set X of all points $x = (x_1, \ldots, x_n) \in k^n$ which satisfy these equations is an **affine algebraic variety**.*

*Consider the set of all polynomials $g \in k[t_1, \ldots, t_n]$ with the property that $g(x) = 0$ for all $x \in X$. This set is an ideal $I(X)$ in the polynomial ring, and is called the **ideal of the variety** X. The quotient ring*

$$P(X) = k[t_1, \ldots, t_n]/I(X)$$

is the ring of polynomial functions on X, because two polynomials g, h define the same polynomial function on X if and only if $g - h$ vanishes at every point of X, that is, if and only if $g - h \in I(X)$.

*Let ξ_i be the image of t_i in $P(X)$. The ξ_i ($1 \leq i \leq n$) are the **coordinate functions** on X: if $x \in X$, then $\xi_i(x)$ is the ith coordinate of x. $P(X)$ is generated as a k-algebra by the coordinate functions, and is called the **coordinate ring** (or affine algebra) of X.*

As in Exercise 1.26, for each $x \in X$ let $\mathfrak{m}_x$ be the ideal of all $f \in P(X)$ such that $f(x) = 0$; it is a maximal ideal of $P(X)$. Hence, if $\widetilde{X} = \mathrm{Max}(P(X))$, we have defined a mapping $\mu : X \to \widetilde{X}$, namely $x \mapsto \mathfrak{m}_x$. It is easy to show that μ is injective: if $x \neq y$, we must have $x_i \neq y_i$ for some i $(1 \leq i \leq n)$, and hence $\xi_i - x_i$ is in $\mathfrak{m}_x$ but not in $\mathfrak{m}_y$, so that $\mathfrak{m}_x \neq \mathfrak{m}_y$. What is less obvious (but still true) is that μ is surjective. This is one form of Hilbert's Nullstellensatz (see Chapter 7).

Proof.

(1) *Show that μ is surjective.* If $\mathfrak{m}$ is a maximal ideal of $P(X)$, then $B := P(X)/\mathfrak{m}$ is a finitely generated k-algebra. Note that B is also a field, Corollary 5.24 implies that B is a finite algebraic extension of k.

(2) In fact, $B \cong k$ since $k = \overline{k}$. Let x_i be the image of ξ_i in k for each i. So $\xi_i - x_i = 0 \in k \cong B$ or $\xi_i - x_i \in \mathfrak{m}$. So

$$\mathfrak{m} \subseteq (\xi_1 - x_1, \dots, \xi_n - x_n) = \mathfrak{m}_x.$$

Hence $\mathfrak{m} = \mathfrak{m}_x$ by the maximality of $\mathfrak{m}$.

□

Exercise 1.28.

Let $f_1, \dots, f_m$ be elements of $k[t_1, \dots, t_n]$. They determine a ***polynomial mapping*** *$\phi : k^n \to k^m$: if $x \in k^n$, the coordinates of $\phi(x)$ are $f_1(x), \dots, f_m(x)$.*

Let X, Y be affine algebraic varieties in k^n, k^m respectively. A mapping $\phi : X \to Y$ is said to be ***regular*** *if ϕ is the restriction to X of a polynomial mapping from k^n to k^m.*

If η is a polynomial function on Y, then $\eta \circ \phi$ is a polynomial function on X. Hence ϕ induces a k-algebra homomorphism $P(Y) \to P(X)$, namely $\eta \mapsto \eta \circ \phi$. Show that in this way we obtain a one-to-one correspondence between the regular mappings $X \to Y$ and the k-algebra homomorphisms $P(Y) \to P(X)$.

Proof.

(1) Let $P(X) = k[t_1, \dots, t_n]/I(X)$ and $P(Y) = k[s_1, \dots, s_m]/I(Y)$. Let η_j be the image of s_j in $P(Y)$. Suppose ϕ induces a k-algebra homomorphism $P(Y) \to P(X)$ by $\widetilde{\phi} : \eta \mapsto \eta \circ \phi$.

(2) *Show that the correspondence is injective.* Suppose $\widetilde{\alpha} = \widetilde{\beta}$ for some regular mappings $\alpha = (\alpha_1, \dots, \alpha_m)$ and $\beta = (\beta_1, \dots, \beta_m)$. Hence

$$\alpha_j = \eta_j \circ \alpha = \widetilde{\alpha}(\eta_j) = \widetilde{\beta}(\eta_j) = \eta_j \circ \beta = \beta_j$$

for $1 \leq j \leq m$. Hence $\alpha_j = \beta_j$ on X and thus $\alpha = \beta$ on X.

(3) *Show that the correspondence is surjective.* Suppose $\Psi : P(Y) \to P(X)$ is a k-algebra homomorphism. Say $\psi_j + I(X) := \Psi(\eta_j) \in P(X)$ for some $\psi_j \in k[t_1, \ldots, t_n]$ (where $1 \leq j \leq m$).

(4) Define $\psi : X \to k^m$ by

$$\psi(P) = (\psi_1(P), \ldots, \psi_m(P))$$

where $P = (t_1, \ldots, t_n) \in X$. ψ is well-defined (since ψ is independent of the choice of ψ_j). To show ψ is regular, it suffices to show that the image of ψ is contained in Y. It is guaranteed by $\Psi(0) = 0$. Lastly note that $\widetilde{\psi} = \Psi$.

$\square$

Chapter 2: Modules

Exercise 2.1.

Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$ *if* m, n *are coprime.*

It suffices to show that

$$(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$$

where d is the greatest common divisor of m and n.

Outlines.

(1) Define $\widetilde{\varphi}$ by

$$\begin{array}{rccc} \widetilde{\varphi}: & (\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z}) & \longrightarrow & \mathbb{Z}/d\mathbb{Z} \\ & \Psi & & \Psi \\ & (x+m\mathbb{Z}, y+n\mathbb{Z}) & \longmapsto & xy + d\mathbb{Z}. \end{array}$$

$\widetilde{\varphi}$ is well-defined and $\mathbb{Z}$-bilinear.

(2) By the universal property, $\widetilde{\varphi}$ factors through a $\mathbb{Z}$-bilinear map

$$\varphi : (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/d\mathbb{Z}$$

(such that $\varphi(x \otimes y) = \widetilde{\varphi}(x, y)$).

(3) To show that φ is isomorphic, might find the inverse map $\psi : \mathbb{Z}/d\mathbb{Z} \to (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$ of φ. Define ψ by

$$\begin{array}{rccc} \psi: & \mathbb{Z}/d\mathbb{Z} & \longrightarrow & (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \\ & \Psi & & \Psi \\ & z + d\mathbb{Z} & \longmapsto & (z+m\mathbb{Z}) \otimes (1+n\mathbb{Z}). \end{array}$$

ψ is well-defined and $\mathbb{Z}$-linear.

(4) $\psi \circ \varphi = \mathrm{id}$.

(5) $\varphi \circ \psi = \mathrm{id}$.

Proof of (1).

(a) $\widetilde{\varphi}$ *is well-defined.* Say $x' = x + am$ for some $a \in \mathbb{Z}$ and $y' = y + bn$ for some $b \in \mathbb{Z}$. Then $x'y' - xy = yam + xbn + abmn \in \mathbb{Z}/d\mathbb{Z}$. That is, $\widetilde{\varphi}$ is independent of coset representative.

(b) *$\widetilde{\varphi}$ is $\mathbb{Z}$-bilinear.*

(i) *For any $\lambda \in \mathbb{Z}$, $\widetilde{\varphi}(\lambda x, y) = \widetilde{\varphi}(x, \lambda y) = \lambda\widetilde{\varphi}(x, y)$.* In fact,

$$\begin{aligned}
\widetilde{\varphi}(\lambda(x + m\mathbb{Z}), y + n\mathbb{Z}) &= \widetilde{\varphi}(\lambda x + m\mathbb{Z}, y + n\mathbb{Z}) = \lambda xy + d\mathbb{Z}, \\
\widetilde{\varphi}(x + m\mathbb{Z}, \lambda(y + n\mathbb{Z})) &= \widetilde{\varphi}(x + m\mathbb{Z}, \lambda y + n\mathbb{Z}) = \lambda xy + d\mathbb{Z}, \\
\widetilde{\varphi}(x_1 + m\mathbb{Z}, y + n\mathbb{Z}) &= \lambda(xy + d\mathbb{Z}) = \lambda xy + d\mathbb{Z}.
\end{aligned}$$

(ii) $\widetilde{\varphi}(x_1 + x_2, y) = \widetilde{\varphi}(x_1, y) + \widetilde{\varphi}(x_2, y)$. In fact,

$$\begin{aligned}
\widetilde{\varphi}((x_1 + x_2) + m\mathbb{Z}, y + n\mathbb{Z}) &= (x_1 + x_2)y + d\mathbb{Z}, \\
\widetilde{\varphi}(x_1 + m\mathbb{Z}, y + n\mathbb{Z}) + \widetilde{\varphi}(x_2 + m\mathbb{Z}, y + n\mathbb{Z}) &= (x_1 y + d\mathbb{Z}) + (x_2 y + d\mathbb{Z}) \\
&= (x_1 + x_2)y + d\mathbb{Z}.
\end{aligned}$$

(iii) $\widetilde{\varphi}(x, y_1 + y_2) = \widetilde{\varphi}(x, y_1) + \widetilde{\varphi}(x, y_2)$. Similar to (ii).

□

Proof of (3).

(a) *ψ is well-defined.* Say $z' = z + cd$ for some $c \in \mathbb{Z}$. Note that $d = \alpha m + \beta n$ for some $\alpha, \beta \in \mathbb{Z}$. Thus

$$\begin{aligned}
\psi(z' + d\mathbb{Z}) &= \psi(z + cd + d\mathbb{Z}) \\
&= \psi(z + c(\alpha m + \beta n) + d\mathbb{Z}) \\
&= (z + c(\alpha m + \beta n) + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\
&= (z + c\beta n + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\
&= (z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) + (c\beta n + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\
&= \psi(z + d\mathbb{Z}) + (1 + m\mathbb{Z}) \otimes (c\beta n + n\mathbb{Z}) \\
&= \psi(z + d\mathbb{Z}).
\end{aligned}$$

(b) *ψ is $\mathbb{Z}$-linear.*

(i) *For any $\lambda \in \mathbb{Z}$, $\psi(\lambda z) = \lambda\psi(z)$.* In fact,

$$\begin{aligned}
\psi(\lambda(z + d\mathbb{Z})) &= \psi(\lambda z + d\mathbb{Z}) = (\lambda z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}), \\
\lambda\psi(z + d\mathbb{Z}) &= \lambda((z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z})) = (\lambda z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}).
\end{aligned}$$

(ii) $\psi(z_1 + z_2) = \psi(z_1) + \psi(z_2)$.

$$\begin{aligned}
\psi((z_1 + z_2) + d\mathbb{Z}) &= (z_1 + z_2 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}), \\
\psi(z_1 + d\mathbb{Z}) + \psi(z_2 + d\mathbb{Z}) &= (z_1 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) + (z_2 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\
&= (z_1 + z_2 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}).
\end{aligned}$$

$\square$

Proof of (4). For any $(x+m\mathbb{Z})\otimes(y+n\mathbb{Z})\in(\mathbb{Z}/m\mathbb{Z})\otimes_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z})$,

$$\begin{aligned}\psi(\varphi((x+m\mathbb{Z})\otimes(y+n\mathbb{Z}))) &= \psi(xy+d\mathbb{Z})\\ &= (xy+m\mathbb{Z})\otimes(1+n\mathbb{Z})\\ &= (x+m\mathbb{Z})\otimes(y+n\mathbb{Z}).\end{aligned}$$

$\square$

Proof of (5). For any $z+d\mathbb{Z}\in\mathbb{Z}/d\mathbb{Z}$,

$$\begin{aligned}\varphi(\psi(z+d\mathbb{Z}) &= \varphi((z+m\mathbb{Z})\otimes(1+n\mathbb{Z}))\\ &= z+d\mathbb{Z}.\end{aligned}$$

$\square$

Exercise 2.2.

Let A be a ring, $\mathfrak{a}$ an ideal, M an A-module. Show that $(A/\mathfrak{a})\otimes_A M$ is isomorphic to $M/\mathfrak{a}M$. (Hint: Tensor the exact sequence $0\to\mathfrak{a}\to A\to A/\mathfrak{a}\to 0$ with M.

Proof (Hint). There is a natural exact sequence E:

$$E : 0\to\mathfrak{a}\xrightarrow{i} A\xrightarrow{\pi} A/\mathfrak{a}\to 0$$

where i is the inclusion map (and π is the projection map). Tensor E with M:

$$E' : \mathfrak{a}\otimes_A M\xrightarrow{i\otimes 1} A\otimes_A M\xrightarrow{\pi\otimes 1}(A/\mathfrak{a})\otimes_A M\to 0$$

is exact, or

$$(A/\mathfrak{a})\otimes_A M\cong A\otimes_A M/\mathrm{im}(i\otimes 1).$$

By Proposition 2.14, There is an unique isomorphism $A\otimes_A M\to M$ defined by $a\otimes x\mapsto ax$. This isomorphism sends $\mathrm{im}(i\otimes 1)$ to $\mathfrak{a}M$. Therefore,

$$(A/\mathfrak{a})\otimes_A M\cong M/\mathfrak{a}M.$$

$\square$

Proof (Brute-force).

(1) Define $\widetilde{\varphi}$ by

$$\begin{array}{rccc} \widetilde{\varphi}: & A/\mathfrak{a} \times M & \longrightarrow & M/\mathfrak{a}M \\ & \cup & & \cup \\ & (a+\mathfrak{a}, x) & \longmapsto & ax + \mathfrak{a}M. \end{array}$$

$\widetilde{\varphi}$ is well-defined and A-bilinear.

(2) By the universal property, $\widetilde{\varphi}$ factors through a A-bilinear map

$$\varphi : A/\mathfrak{a} \otimes_A M \to M/\mathfrak{a}M$$

(such that $\varphi(a \otimes x) = \widetilde{\varphi}(a, x)$).

(3) To show that φ is isomorphic, might find the inverse map $\psi : M/\mathfrak{a}M \to A/\mathfrak{a} \otimes_A M$ of φ. Define ψ by

$$\begin{array}{rccc} \psi: & M/\mathfrak{a}M & \longrightarrow & A/\mathfrak{a} \otimes_A M \\ & \cup & & \cup \\ & x + \mathfrak{a}M & \longmapsto & (1+\mathfrak{a}) \otimes x. \end{array}$$

ψ is well-defined and A-linear.

(4) $\psi \circ \varphi = \mathrm{id}$.

(5) $\varphi \circ \psi = \mathrm{id}$.

□

Exercise 2.3.

Let A be a local ring, M and N finitely generated A-modules. Prove that if $M \otimes_A N = 0$, then $M = 0$ or $N = 0$. (Hint: Let $\mathfrak{m}$ be the maximal ideal, $k = A/\mathfrak{m}$ the residue field. Let $M_k = k \otimes_A M \cong M/\mathfrak{m}M$ by Exercise 2.2. By Nakayama's lemma, $M_k = 0 \Longrightarrow M = 0$. But $M \otimes_A N = 0 \Longrightarrow (M \otimes_A N)_k = 0 \Longrightarrow M_k \otimes_k N_k = 0 \Longrightarrow M_k = 0$ or $N_k = 0$ since M_k, N_k are vector spaces over a field.)

The conclusion might be false if A is not local. For example, Exercise 2.1.

Proof (Hint). Let $\mathfrak{m}$ be the maximal ideal, $k = A/\mathfrak{m}$ the residue field. Let $M_k = k \otimes_A M$.

(1) *(Base extension) Show that* $(M \otimes_A N)_k = M_k \otimes_k N_k$. In fact, by Proposition 2.14

$$\begin{aligned}
(M \otimes_A N)_k &= k \otimes_A (M \otimes_A N) \\
&= (k \otimes_A M) \otimes_A N \\
&= M_k \otimes_A N \\
&= (M_k \otimes_k k) \otimes_A N \\
&= M_k \otimes_k (k \otimes_A N) \\
&= M_k \otimes_k N_k.
\end{aligned}$$

(2)

$$\begin{aligned}
M \otimes_A N = 0 &\Longrightarrow (M \otimes_A N)_k = 0 \\
&\Longrightarrow M_k \otimes_k N_k = 0 && ((1)) \\
&\Longrightarrow M_k = 0 \text{ or } N_k = 0 && (M_k, N_k\text{: vector spaces}) \\
&\Longrightarrow M/\mathfrak{m}M = 0 \text{ or } M/\mathfrak{m}M = 0 && (\text{Exercise 2.2}) \\
&\Longrightarrow M = 0 \text{ or } N = 0. && (\text{Nakayama's lemma})
\end{aligned}$$

□

Exercise 2.4.

Let M_i $(i \in I)$ be any family of A-modules, and let M be their direct sum. Prove that M is flat $\Longleftrightarrow$ each M_i is flat.

Proof. Given any A-module homomorphism $f : N' \to N$.

(1) Similar to Proposition 2.14 (iii), we have two isomorphisms

(a)

$$\varphi : \bigoplus_{i \in I} (N' \otimes M_i) \cong N' \otimes_A \bigoplus_{i \in I} M_i$$

defined by

$$\varphi((x \otimes m_i)_{i \in I}) = x \otimes (m_i)_{i \in I}$$

where $x \in N'$, $m_i \in M_i$ $(i \in I)$.

(b)

$$\psi : N \otimes_A \bigoplus_{i \in I} M_i \cong \bigoplus_{i \in I} (N \otimes M_i)$$

defined by

$$\psi(y \otimes (m_i)_{i \in I}) = (y \otimes m_i)_{i \in I}$$

where $y \in N$, $m_i \in M_i$ $(i \in I)$.

(2) $f : N' \to N$ induces an A-module homomorphism

$$f \otimes \mathrm{id}_M : N' \otimes_A M \to N \otimes_A M.$$

(3) $\psi \circ f \otimes \mathrm{id}_M \circ \varphi$ defines an A-module homomorphism

$$\psi \circ f \otimes \mathrm{id}_M \circ \varphi : \bigoplus_{i \in I} (N' \otimes M_i) \to \bigoplus_{i \in I} (N \otimes M_i)$$

which sends $(x \otimes m_i)_{i \in I}$ to $(f(x) \otimes m_i)_{i \in I}$. That is,

$$\psi \circ f \otimes \mathrm{id}_M \circ \varphi = \bigoplus_{i \in I} f \otimes \mathrm{id}_{M_i}.$$

(4) *Show that M is flat if and only if each M_i is flat.* Suppose f is injective.

$$\begin{aligned}
& M_i \text{ is flat } \forall\, i \in I \\
\Longleftrightarrow\ & f \otimes \mathrm{id}_{M_i} \text{ is injective } \forall\, i \in I \\
\Longleftrightarrow\ & \bigoplus_{i \in I} f \otimes \mathrm{id}_{M_i} \text{ is injective} && \text{(Injectivity)} \\
\Longleftrightarrow\ & \psi \circ f \otimes \mathrm{id}_M \circ \varphi \text{ is injective} && \text{((3))} \\
\Longleftrightarrow\ & f \otimes \mathrm{id}_M \text{ is injective} && (\varphi \text{ and } \psi \text{ are isomorphic}) \\
\Longleftrightarrow\ & M \text{ is flat.}
\end{aligned}$$

□

Exercise 2.5.

Let $A[x]$ be the ring of polynomials in one indeterminate over a ring A. Prove that $A[x]$ is a flat A-algebra. (Hint: Use Exercise 2.4.)

Proof (Hint).

(1) A is a flat A-module by Proposition 2.14 (iv).

(2) As an A-module,

$$A[x] \cong \bigoplus_{n \in \mathbb{Z}^+} Ax^n \cong \bigoplus_{n \in \mathbb{Z}^+} A$$

(since $Ax^n \cong A$).

(3) By Exercise 2.4, $A[x] \cong \bigoplus_{n \in \mathbb{Z}^+} A$ is flat.

□

Exercise 2.8.

(i) *If M and N are flat A-modules, then so is $M \otimes_A N$.*

(ii) *If B is a flat A-algebra and N is a flat B-module, then N is flat as A-module.*

Proof of (i). Given any exact sequence of A-modules $0 \to N_1 \to N_2 \to N_3 \to 0$. Since M is flat,

$$0 \to N_1 \otimes_A M \to N_2 \otimes_A M \to N_3 \otimes_A M \to 0$$

is exact. Since N is flat,

$$0 \to (N_1 \otimes_A M) \otimes_A N \to (N_2 \otimes_A M) \otimes_A N \to (N_3 \otimes_A M) \otimes_A N \to 0$$

is exact. By Proposition 2.14 (ii),

$$0 \to N_1 \otimes_A (M \otimes_A N) \to N_2 \otimes_A (M \otimes_A N) \to N_3 \otimes_A (M \otimes_A N) \to 0$$

is exact, or $M \otimes_A N$ is flat. $\square$

Proof of (ii). Given any exact sequence of A-modules $0 \to N_1 \to N_2 \to N_3 \to 0$. Since B is a flat A-algebra (A-module),

$$0 \to N_1 \otimes_A B \to N_2 \otimes_A B \to N_3 \otimes_A B \to 0$$

is exact. Since N is a flat B-module,

$$0 \to (N_1 \otimes_A B) \otimes_B N \to (N_2 \otimes_A B) \otimes_B N \to (N_3 \otimes_A B) \otimes_B N \to 0$$

is exact. By Exercise 2.15 on page 27,

$$0 \to N_1 \otimes_A (B \otimes_B N) \to N_2 \otimes_A (B \otimes_B N) \to N_3 \otimes_A (B \otimes_B N) \to 0$$

is exact. By Proposition 2.14 (iv),

$$0 \to N_1 \otimes_A N \to N_2 \otimes_A N \to N_3 \otimes_A N \to 0$$

is exact, or N is flat. $\square$

Exercise 2.9.

Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. If M' and M'' are finitely generated, then so is M.

Proof.

(1) Write

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0.$$

Also write

$$\begin{aligned} &x_1, \ldots, x_n \text{ as generators of } M', \\ &z_1, \ldots, z_m \text{ as generators of } M'' \end{aligned}$$

(since M' and M'' are finitely generated).

(2) Since the map $g : M \to M''$ is surjective, there exists $y_j \in M$ such that $g(y_j) = z_j$ for $j = 1, \ldots, m$.

(3) *Show that M is generated by*

$$f(x_1), \ldots, f(x_n), y_1, \ldots, y_m.$$

Given any $y \in M$.

$$\begin{aligned} y \in M &\Longrightarrow g(y) \in M'' \\ &\Longrightarrow g(y) = \sum_{j=1}^{m} s_j z_j \text{ where } s_j \in A \\ &\Longrightarrow g(y) = \sum_{j=1}^{m} s_j g(y_j) \\ &\Longrightarrow g(y) = g\left(\sum_{j=1}^{m} s_j y_j\right) \\ &\Longrightarrow y - \sum_{j=1}^{m} s_j y_j \in \ker(g) = \operatorname{im}(f) \\ &\Longrightarrow \exists\, x \in M' \text{ such that } f(x) = y - \sum_{j=1}^{m} s_j y_j \end{aligned}$$

Write $x = \sum_{i=1}^{n} r_i x_i$ where $r_i \in A$. So,

$$\begin{aligned} y \in M &\Longrightarrow f\left(\sum_{i=1}^{n} r_i x_i\right) = y - \sum_{j=1}^{m} s_j y_j \\ &\Longrightarrow \sum_{i=1}^{n} r_i f(x_i) = y - \sum_{j=1}^{m} s_j y_j \\ &\Longrightarrow y = \sum_{i=1}^{n} r_i f(x_i) + \sum_{j=1}^{m} s_j y_j. \end{aligned}$$

Hence, every $y \in M$ is a linear combination of $f(x_1), \ldots, f(x_n), y_1, \ldots, y_m$, or M is finitely generated (by $f(x_1), \ldots, f(x_n), y_1, \ldots, y_m$).

$\square$

Direct limits

Exercise 2.14.

A partially ordered set I is said to be a ***directed*** *set if for each pair i, j in I there exists $k \in I$ such that $i \leq k$ and $j \leq k$.*

Let A be a ring, let I be a directed set and let $(M_i)_{i \in I}$ be a family of A-modules indexed by I. For each pair i, j in I such that $i \leq j$, let $\mu_{ij} : M_i \to M_j$ be an A-homomorphism, and suppose that the following axioms are satisfied:

(1) *μ_{ii} is the identity mapping of M_i, for all $i \in I$;*

(2) *$\mu_{ik} = \mu_{jk} \circ \mu_{ij}$ whenever $i \leq j \leq k$.*

Then the modules M_i and homomorphisms μ_{ij} are said to form a direct system $\mathbf{M} = (M_i, \mu_{ij})$ over the directed set I.

We shall construct an A-module M called the ***direct limit*** *of the direct system $\mathbf{M}$. Let C be the direct sum of the M_i, and identify each module M_i with its canonical image in C. Let D be the submodule of C generated by all elements of the form $x_i - \mu_{ij}(x_i)$ where $i \leq j$ and $x_i \in M_i$. Let $M = C/D$, let $\mu : C \to M$ be the projection and let μ_i be the restriction of μ to M_i.*

The module M or more correctly the pair consisting of M and the family of homomorphisms $\mu_i : M_i \to M$, is called the ***direct limit*** *of the direct system $\mathbf{M}$, and is written $\varinjlim M_i$. From the construction it is clear that $\mu_i = \mu_j \circ \mu_{ij}$ whenever $i \leq j$.*

Proof. Show that $\mu_i = \mu_j \circ \mu_{ij}$ whenever $i \leq j$. For any $x_i \in M_i$, we have

$$\begin{aligned} \mu_i(x_i) - (\mu_j \circ \mu_{ij})(x) &= \mu_i(x_i) - \mu_j(\mu_{ij}(x)) \\ &= \mu(x_i) - \mu(\mu_{ij}(x)) \\ &= \mu(x_i - \mu_{ij}(x)) \in \ker(\mu). \end{aligned}$$

$\square$

Note. A concrete example. The set $p^{-\infty}\mathbb{Z}$ of rational numbers whose denominators are powers of a rational prime p is a colimit $\varinjlim p^{-i}\mathbb{Z}$. More precisely, $p^{-\infty}\mathbb{Z}$ is the colimit of the diagram

$$\mathbb{Z} \to p^{-1}\mathbb{Z} \to p^{-2}\mathbb{Z} \to \cdots .$$