Chapter 4: Continuity

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Exercise 4.1. *Suppose f is a real function define on $\mathbb{R}^1$ which satisfies*

$$\lim_{h \to 0} [f(x+h) - f(x-h)] = 0$$

for every $x \in \mathbb{R}^1$. Does this imply that f is continuous?

Proof. $\lim_{h \to 0}[f(x+h) - f(x-h)] = 0$ holds if f is continuous. But the converse of this statement and is not true. For example, define $f : \mathbb{R}^1 \to \mathbb{R}^1$ by

$$f(x) = \begin{cases} 1 & (x = 0), \\ 0 & (x \neq 0). \end{cases}$$

f is not continuous at $x = 0$ but

$$\lim_{h \to 0} [f(x+h) - f(x-h)] = 0$$

for any $x \in \mathbb{R}^1$. (The identity holds for $x \neq 0$ since f is continuous on $\mathbb{R}^1 - \{0\}$. Besides, $\lim_{h \to 0}[f(0+h) - f(0-h)] = \lim_{h \to 0}[0 - 0] = 0$.) $\square$

Exercise 4.2. *If f is a continuous mapping of a metric space X into a metric space Y, prove that $f(\overline{E}) \subseteq \overline{f(E)}$ for every set $E \subseteq X$. ($\overline{E}$ denotes the closure of E.) Show, by an example, that $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$.*

Proof.

(1) Since f is continuous and $\overline{f(E)}$ is closed, $f^{-1}(\overline{f(E)})$ is closed. Hence,

$$\begin{aligned} f^{-1}(\overline{f(E)}) &\supseteq f^{-1}(f(E)) && \text{(Monotonicity of } f^{-1}) \\ &\supseteq E, && \text{(Note in Theorem 4.14)} \\ \overline{E} &\subseteq f^{-1}(\overline{f(E)}), && \text{(Monotonicity of closure)} \\ f(\overline{E}) &\subseteq f(f^{-1}(\overline{f(E)})) && \text{(Monotonicity of } f) \\ &\subseteq \overline{f(E)}. && \text{(Note in Theorem 4.14)} \end{aligned}$$

(2) Let $f : (0, \infty) \to \mathbb{R}$ be a continuous function defined by

$$f(x) = \frac{1}{x}.$$

Consider $E = \mathbb{Z}^+ \subseteq (0, \infty)$. Then $f(E) = \left\{\frac{1}{n} : n \in \mathbb{Z}^+\right\}$, and thus

$$f(\overline{E}) = \left\{\frac{1}{n} : n \in \mathbb{Z}^+\right\}.$$
$$\overline{f(E)} = \left\{\frac{1}{n} : n \in \mathbb{Z}^+\right\} \bigcup \{0\}.$$

$\square$

Supplement (Inverse image).

(1) $E \subseteq f^{-1}[f(E)]$ *for* $E \subseteq X$.

$$\begin{aligned} \forall\, x \in E &\Longrightarrow f(x) \in f(E) \\ &\Longleftrightarrow x \in f^{-1}[f(E)]. \qquad \text{(Definition of the inverse image)} \end{aligned}$$

$\square$

(2) $f[f^{-1}(E)] \subseteq E$ *for* $E \subseteq Y$.

$$\begin{aligned} \forall\, y \in f[f^{-1}(E)] &\Longleftrightarrow \exists\, x \in f^{-1}(E) \text{ such that } y = f(x) \\ &\Longleftrightarrow \exists\, x, f(x) \in E \text{ such that } y = f(x) \\ &\Longrightarrow \exists\, x, y = f(x) \in E. \end{aligned}$$

$\square$

Supplement (Continuity). *Let f be a map from a topological space on X to a topological space on Y. Then, the following statements are equivalent:*

(1) *f is continuous: For each $x \in X$ and every neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$.*

(2) *For every open set O in Y, the inverse image $f^{-1}(O)$ is open in X.*

(3) *For every closed set C in Y, the inverse image $f^{-1}(C)$ is closed in X.*

(4) *$f(A)^\circ \subseteq f(A^\circ)$ for every subset A of X.*

(5) *$f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ$ for every subset B of Y.*

(6) *$f(\overline{A}) \subseteq \overline{f(A)}$ for every subset A of X.*

(7) *$\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for every subset B of Y.*

Exercise 4.3. *Let f be a continuous real function on a metric space X. Let $Z(f)$ (the zero set of f) be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.*

Proof (Corollary to Theorem 4.8). Since f is continuous, $f^{-1}(\{0\}) = Z(f)$ is closed in X for a closed subset $\{0\}$ in $\mathbb{R}^1$. $\square$

Denote the complement of any set E by $\widetilde{E}$.

Proof (Theorem 4.8). Consider the complement of $Z(f)$ in X,

$$\begin{aligned}\widetilde{Z(f)} &= \{x \in X : f(x) \neq 0\}\\ &= f^{-1}((-\infty, 0) \cup (0, \infty)).\end{aligned}$$

Since f is continuous, $f^{-1}((-\infty, 0) \cup (0, \infty)) = \widetilde{Z(f)}$ is open in X for a open subset $(-\infty, 0) \cup (0, \infty)$ in $\mathbb{R}^1$. $\square$

Proof (Definition 2.18(d)). Given any limit point p of $Z(f)$. *Show that $f(p) = 0$ or $p \in Z(f)$.* Since f is continuous, given any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(p)| < \varepsilon$ for all $x \in X$ for which $d_X(x, p) < \delta$. Since p is a limit point of $Z(f)$, for such $\delta > 0$ we have a point $q \neq p$ such that $q \in Z(f)$, or $f(q) = 0$. So $|f(p)| < \varepsilon$ for any $\varepsilon > 0$. $f(p) = 0$. $\square$

Proof (Definition 2.18(f)). Consider the complement of $Z(f)$ in X,

$$\widetilde{Z(f)} = \{x \in X : f(x) \neq 0\} = \{f > 0\} \cup \{f < 0\}$$

where $\{f > 0\} = \{x \in X : f(x) > 0\}$ and $\{f < 0\} = \{x \in X : f(x) < 0\}$. It suffices to show $\{f > 0\}$ is open. ($\{f < 0\}$ is similar.) Given any point p of $\{f > 0\}$ or $f(p) > 0$. *Want to show p is an interior point of $\{f > 0\}$.* Since f is continuous, given any $\varepsilon = \frac{f(p)}{2} > 0$ there exists a $\delta > 0$ such that $|f(x) - f(p)| < \frac{f(p)}{2}$ for all $x \in X$ for which $d_X(x, p) < \delta$. For such x with $d_X(x, p) < \delta$ we have

$$\frac{1}{2}f(p) < f(x) < \frac{3}{2}f(p).$$

That is, $N = \{x : d_X(x, p) < \delta\}$ is a neighborhood p such that $N \subseteq \{f > 0\}$. $\square$

Exercise 4.4. *Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that $f(E)$ is dense in $f(X)$. If $g(p) = f(p)$ for all $p \in E$, prove that $g(p) = f(p)$ for all $p \in X$. (In other words, a continuous mapping is determined by its values on a dense subset of its domain.)*

Proof.

(1) *Show that $f(E)$ is dense in $f(X)$.* It suffices to show that every point $y \in f(X) - f(E)$ is a limit point of $f(E)$. Since $y \in f(X) - f(E)$, there exists a point $x \in X - E$ such that $y = f(x)$. Since E is dense in X, there exists a sequence $\{x_n\}$ in E such that $x_n \to x$ as $n \to \infty$. Let $y_n = f(x_n) \in f(E)$. Take limit and use the continuity of f, $y_n \to y$ as $n \to \infty$, or y is a limit point of $f(E)$.

(2) *Show that $g(p) = f(p)$ for all $p \in X$ if $g(p) = f(p)$ for all $p \in E$.* It suffices to show $g(p) = f(p)$ for all $p \in X - E$. Given any $p \in X - E$, there exists a sequence $\{p_n\}$ in E such that $p_n \to p$ as $n \to \infty$. Notice that $g(p_n) = f(p_n)$ by the assumption. Take limit and use the continuity of f and g, $g(p) = f(p)$ for $p \in X - E$.

□

Exercise 4.5. *If f is a real continuous function defined on a closed set $E \subseteq \mathbb{R}^1$, prove that there exist continuous real function g on $\mathbb{R}^1$ such that $g(x) = f(x)$ for all $x \in E$. (Such functions g are called **continuous extensions** of f from E to $\mathbb{R}^1$.) Show that the result becomes false if the word "closed" is omitted. Extend the result to vector valued functions. (Hint: Let the graph of g be a straight line on each of the segments which constitute the complement of E (compare Exercise 2.29). The result remains true if $\mathbb{R}^1$ is replaced by any metric space, but the proof is not so simple.)*

Proof.

(1) Every open set in $\mathbb{R}^1$ is the union of an at most countable collection of disjoint segments (Exercise 2.29).

(2) We need to construct a continuous real function on the complement of E. By (1), write $\widetilde{E} = \bigcup_{i \in \mathscr{C}} (a_i, b_i)$ where $\mathscr{C}$ is at most countable and $a_i < b_i$. (a_i, b_i could be $\pm\infty$.) Define $g(x)$ by

$$g(x) = \begin{cases} f(x) & (x \in E), \\ f(a_i) + \frac{f(b_i) - f(a_i)}{b_i - a_i}(x - a_i) & (x \in (a_i, b_i) : \text{finite interval}), \\ f(a_i) & (x \in (a_i, b_i) : a_i : \text{finite}, b_i = +\infty), \\ f(b_i) & (x \in (a_i, b_i) : a_i = -\infty, b_i : \text{finite}), \\ 0 & (x \in (a_i, b_i) : a_i = -\infty, b_i = +\infty). \end{cases}$$

Show that g is continuous in $\mathbb{R}^1$, or show that $g(x)$ is continuous at $x = p$ for any point $p \in \mathbb{R}^1$.

(a) Given a point $p \in \widetilde{E}$. There is an open interval $I = (a_i, b_i)$ such that $p \in I$. Since the graph of g in an open interval I is a straight line, g is continuous at $x = p$.

(b) Given an isolated point $p \in E$. There are two open intervals $I = (a_i, b_i)$ and $J = (a_j, b_j)$ such that $b_i = p = a_j$. So $\lim_{x \to p^-} g(x) = \lim_{x \to p^+} g(x) = f(p)$ by the construction of g, which says g is continuous at $x = p$.

(c) Given a limit point $p \in E$. So that $g(p) = f(p)$. Given $\varepsilon > 0$. Consider $\lim_{x \to p^+} g(x)$ first. (The case $\lim_{x \to p^-} g(x)$ is similar.)

(i) For such $\varepsilon > 0$, there is a $\delta' > 0$ such that

$$f(p) - \varepsilon < f(x) < f(p) + \varepsilon$$

whenever

$$x \in E \text{ and } p < x < \delta'.$$

Since p is a limit point of E, there is a point $q \neq p$ such that $|q - p| < \delta'$. Might assume that $q > p$, and then retake $\delta = \min\{\delta', q - p\} > 0$. (If no such q, $\lim_{x \to p^+} g(x) = f(p)$ trivially.)

(ii) For any x such that $p < x < q$, consider $x \in E$ or else $x \in \widetilde{E}$. As $x \in E$, nothing to do by (i).

(iii) As $x \in \widetilde{E}$, there exists an open interval $I = (a_i, b_i)$ such that $x \in I \subseteq (p, q)$. Therefore,

$$f(a_i) \leq g(x) \leq f(b_i) \text{ or } f(a_i) \geq g(x) \geq f(b_i).$$

By (i),

$$\begin{aligned} f(p) - \varepsilon < f(a_i) < f(p) + \varepsilon \text{ and} \\ f(p) - \varepsilon < f(b_i) < f(p) + \varepsilon, \\ f(p) - \varepsilon < f(a_i) \leq g(x) \leq f(b_i) < f(p) + \varepsilon \text{ or} \\ f(p) - \varepsilon < f(b_i) \leq g(x) \leq f(a_i) < f(p) + \varepsilon. \end{aligned}$$

Hence, given $\varepsilon > 0$ there is a $\delta > 0$ such that $|g(x) - g(p)| < \varepsilon$ whenever $p < x < \delta$ (and $x \in \mathbb{R}^1$), or $\lim_{x \to p^+} g(x) = g(p)$.

(3) Consider $f(x) = \log(x)$ in $(0, \infty)$. Since $\lim_{x \to 0} f(x) = -\infty$, we cannot find any real continuous function g defined on $x = 0$.

(4) For a vector-valued function $\mathbf{f} = (f_1, \ldots, f_k)$, with each f_i is continuous on a closed set $E \subseteq \mathbb{R}^1$, extend f_i to a continuous function g_i on $\mathbb{R}^1$ as (2). Put $\mathbf{g} = (g_1, \ldots, g_k)$. Clearly $\mathbf{g}$ is an extension of $\mathbf{f}$. Besides, $\mathbf{g}$ is continuous in $\mathbb{R}^1$ by Theorem 4.10.

$\square$

Supplement (Tietze's Extension Theorem). *If X is a normal topological space and $f : A \to \mathbb{R}$ is a continuous map from a closed subset A of X into the real numbers carrying the standard topology, then there exists a continuous map $g : X \to \mathbb{R}$ with $g(a) = f(a)$ for all $a \in A$.*

Exercise 4.6. *If f is defined on E, the graph of f is the set of points $(x, f(x))$, for $x \in E$. In particular, if E is a set of real numbers, and f is real-valued, the graph of f is a subset of the plain. Suppose E is compact, and prove that that f is continuous on E if and only if its graph is compact.*

Proof. Let $G = \{(x, f(x)) : x \in E\}$ be the graph of f.

(1) $(\Longrightarrow)$ Let $\mathbf{f} : E \to G$ defined by

$$\mathbf{f}(x) = (x, f(x)).$$

$\mathbf{f}(E) = G$ exactly. Since f and x are continuous in E, $\mathbf{f}$ is continuous (Theorem 4.10). As E is compact, $\mathbf{f}(E)$ is compact (Theorem 4.14).

(2) $(\Longleftarrow)$ Let $\pi : G \to E$ be a projection map defined by

$$\pi(x, f(x)) = x.$$

Notice that $\pi \circ \mathbf{f} = \mathrm{id}_E$ and $\mathbf{f} \circ \pi = \mathrm{id}_G$. Besides, π is a continuous one-to-one mapping of a compact set G onto E. Then the inverse mapping $\pi^{-1} = \mathbf{f}$ is a continuous mapping of E onto G (Theorem 4.17). So f is continuous (Theorem 4.10).

□

Exercise 4.7. *If $E \subseteq X$ and if f is a function defined on X, the* ***restriction*** *of f to E is the function g whose domain of definition is E, such that $g(p) = f(p)$ for $p \in E$. Define f and g on $\mathbb{R}^2$ by:*

$$f(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0) \\ \frac{xy^2}{x^2+y^4} & \text{if } (x, y) \neq (0, 0), \end{cases}$$

$$g(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0) \\ \frac{xy^2}{x^2+y^6} & \text{if } (x, y) \neq (0, 0), \end{cases}$$

Prove that f is bounded on $\mathbb{R}^2$, that g is unbounded in every neighborhood of $(0, 0)$, and that f is not continuous at $(0, 0)$; nevertheless, the restrictions of both f and g to every straight line in $\mathbb{R}^2$ are continuous!

Proof.

(1) *Show that f is bounded on $\mathbb{R}^2$.*

$$\begin{aligned}(|x|-|y^2|)^2 \geq 0 &\iff |x|^2 - 2|x||y^2| + |y^2|^2 \geq 0\\ &\iff |x|^2 + |y^2|^2 \geq 2|x||y^2|\\ &\iff |x^2+y^4| \geq 2|xy^2|\\ &\implies \frac{1}{2} \geq \left|\frac{xy^2}{x^2+y^2}\right| \text{ whenever } (x,y)\neq(0,0)\\ &\implies |f(x,y)| \leq \frac{1}{2} \text{ whenever } (x,y)\neq(0,0).\end{aligned}$$

Note that $f(0,0) = 0 \leq \frac{1}{2}$. Hence f is bounded by $\frac{1}{2}$ on $\mathbb{R}^2$.

(2) *Show that g is unbounded in every neighborhood of $\mathbb{R}^2$.* Consider a sequence $\{\mathbf{p}_n\}_{n\geq 1} \subseteq \mathbb{R}^2$

$$\mathbf{p}_n = (x_n, y_n) = \left(\frac{1}{n^3}, \frac{1}{n}\right)$$

such that $\mathbf{p}_n \neq \mathbf{0}$ and $\lim \mathbf{p}_n = \mathbf{0}$. Thus,

$$\lim_{n\to\infty} g(\mathbf{p}_n) = \lim_{n\to\infty} \frac{x_n y_n^2}{x_n^2 + y_n^6} = \lim_{n\to\infty} \frac{(\frac{1}{n^3})(\frac{1}{n})^2}{(\frac{1}{n^3})^2 + (\frac{1}{n})^6} = \lim_{n\to\infty} \frac{n}{2} = \infty.$$

Hence g is unbounded in every neighborhood of $\mathbb{R}^2$.

(3) *Show that f is not continuous at $(0,0)$.* Consider a sequence $\{\mathbf{p}_n\}_{n\geq 1} \subseteq \mathbb{R}^2$

$$\mathbf{p}_n = (x_n, y_n) = \left(\frac{1}{n^2}, \frac{1}{n}\right)$$

such that $\mathbf{p}_n \neq \mathbf{0}$ and $\lim \mathbf{p}_n = \mathbf{0}$. Thus,

$$\lim_{n\to\infty} f(\mathbf{p}_n) = \lim_{n\to\infty} \frac{x_n y_n^2}{x_n^2 + y_n^4} = \lim_{n\to\infty} \frac{(\frac{1}{n^2})(\frac{1}{n})^2}{(\frac{1}{n^2})^2 + (\frac{1}{n})^4} = \frac{1}{2}.$$

So, $\lim f(\mathbf{p}_n) = \frac{1}{2} \neq 0$. By Theorem 4.6, f is not continuous at $(0,0)$.

(4) *The restrictions of f to every straight line in $\mathbb{R}^2$ is continuous.*

(a) *The line $L_\infty = \{(0,y) : y \in \mathbb{R}\}$.* Hence $f|_{L_\infty}(x,y) = 0$ for all $(x,y) \in L_\infty$ (including $(0,0) \in L_\infty$). Therefore $f|_{L_\infty}$ is continuous.

(b) *The line $L_\alpha = \{(x,\alpha x) : x \in \mathbb{R}\}$ for some $\alpha \in \mathbb{R}$.* $f|_{L_\alpha}(x,y)$ is continuous on $L_\alpha - \{(0,0)\}$.

$$f|_{L_\alpha}(x,y) = f|_{L_\alpha}(x,\alpha x) = \begin{cases} 0 & \text{if } (x,y) = (0,0)\\ \frac{\alpha^2 x}{1+\alpha^4 x^2} & \text{if } (x,y) \neq (0,0). \end{cases}$$

So

$$\lim_{(x,y)\to(0,0)} f|_{L_\alpha}(x,y) = \lim_{x\to 0} \frac{\alpha^2 x}{1+\alpha^4 x^2} = 0 = f(0,0),$$

or $f|_{L_\alpha}(x,y)$ is continuous at $(0,0)$. Therefore, $f|_{L_\alpha}(x,y)$ is continuous on L_α.

(c) *The line L not passing* $(0,0)$. It is clear since $f(x,y)$ is continuous on $\mathbb{R}^2 - \{(0,0)\}$.

(5) *The restrictions of g to every straight line in $\mathbb{R}^2$ is continuous.* Similar to (4).

(a) *The line $L_\infty = \{(0,y) : y \in \mathbb{R}\}$.* Hence $g|_{L_\infty}(x,y) = 0$ for all $(x,y) \in L_\infty$ (including $(0,0) \in L_\infty$). Therefore $g|_{L_\infty}$ is continuous.

(b) *The line $L_\alpha = \{(x,\alpha x) : x \in \mathbb{R}\}$ for some $\alpha \in \mathbb{R}$.* $g|_{L_\alpha}(x,y)$ is continuous on $L_\alpha - \{(0,0)\}$.

$$g|_{L_\alpha}(x,y) = g|_{L_\alpha}(x,\alpha x) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \frac{\alpha^2 x}{1+\alpha^6 x^4} & \text{if } (x,y) \neq (0,0). \end{cases}$$

So

$$\lim_{(x,y)\to(0,0)} g|_{L_\alpha}(x,y) = \lim_{x\to 0} \frac{\alpha^2 x}{1+\alpha^6 x^4} = 0 = g(0,0),$$

or $g|_{L_\alpha}(x,y)$ is continuous at $(0,0)$. Therefore, $g|_{L_\alpha}(x,y)$ is continuous on L_α.

(c) *The line L not passing* $(0,0)$. It is clear since $g(x,y)$ is continuous on $\mathbb{R}^2 - \{(0,0)\}$.

□

Exercise 4.8. *Let f be a real uniformly continuous function on the bounded set E in $\mathbb{R}$. Prove that f is bounded on E. Show that the conclusion is false if boundedness of E is omitted from the hypothesis.*

The conclusion is false if boundedness of E is omitted from the hypothesis. For example, $f(x) = x$ on $\mathbb{R}$ is uniformly continuous on $\mathbb{R}$ but $f(\mathbb{R}) = \mathbb{R}$ is unbounded.

Proof (Brute-force).

(1) Since $f : E \to \mathbb{R}$ is uniformly continuous, given any $\varepsilon > 0$, there is $\delta > 0$ such that $\left|f(x) - f(y)\right| < \varepsilon$ whenever $|x - y| < \delta$. In particular, pick $\varepsilon = 1$.

(2) By the boundedness of E, there is $M > 0$ such that $|x| < M$ for all $x \in E$.

(3) For such $\delta > 0$, we construct a covering of $E \subseteq \mathbb{R}$. Construct a special collection $\mathscr{C}$ of intervals

$$I_a = \left[\frac{\delta}{2}a, \frac{\delta}{2}(a+1)\right]$$

where $a \in \mathbb{Z}$ satisfying

$$|a| < \frac{2M}{\delta} + 1.$$

By construction, $\mathscr{C}$ is a finite covering of E.

(4) For every interval I_a of the collection $\mathscr{C}$, pick a point $x_a \in E \bigcap I_a$ if possible. This process will terminate eventually since $\mathscr{C}$ is a finite. Collect these representative points as $\mathscr{D} = \{x_a\}$. Notice that $\mathscr{D}$ is finite again.

(5) Now for any point $x \in E$, x lies in some I_a containing x_a. Both x and x_a are in the same interval and their distance satisfies

$$|x - x_a| \leq \frac{\delta}{2} < \delta$$

and thus by (1)

$$\big|f(x) - f(x_a)\big| < 1, \text{ or } \big|f(x)\big| < 1 + \big|f(x_a)\big|.$$

(6) Let

$$M = 1 + \max_{x_{\mathbf{a}} \in \mathscr{D}} \big|f(x_a)\big|.$$

So given any $x \in E$, $\big|f(x)\big| < M$.

□

Proof (Heine-Borel Theorem). Heine-Borel theorem provides the finiteness property to construct the boundedness property of f.

(1) *Let E be a bounded subset of a metric space X. Show that the closure of E in X is also bounded in X.* E is bounded if $E \subseteq B_X(a; r)$ for some $r > 0$ and some $a \in X$. (The ball $B_X(a; r)$ is defined to the set of all $x \in X$ such that $d_X(x, a) < r$.) Take the closure on the both sides,

$$\overline{E} \subseteq \overline{B_X(a; r)} = \{x \in X : d_X(x, a) \leq r\} \subseteq B_X(a; 2r),$$

or $\overline{E}$ is bounded.

(2) Since $f : E \to \mathbb{R}$ is uniformly continuous, given any $\varepsilon > 0$, there is $\delta > 0$ such that $\big|f(x) - f(y)\big| < \varepsilon$ whenever $|x - y| < \delta$. In particular, pick $\varepsilon = 1$.

(3) For such $\delta > 0$, we construct an open covering of $\overline{E} \subseteq \mathbb{R}$. Pick a collection $\mathscr{C}$ of open balls $B(a; \delta) \subseteq \mathbb{R}$ where a runs over all elements of E. $\mathscr{C}$ covers $\overline{E}$ (by the definition of accumulation points). Since $\overline{E}$ is closed and bounded (by applying (1) on the boundedness of E), $\overline{E}$ is compact (Heine-Borel theorem). That is, there is a finite subcollection $\mathscr{C}'$ of $\mathscr{C}$ also covers $\overline{E}$, say

$$\mathscr{C}' = \big\{B(a_1; \delta)), B(a_2; \delta), \ldots, B(a_m; \delta)\big\}.$$

(4) Given any $x \in E \subseteq \overline{E}$, there is some $a_i \in E$ $(1 \leq i \leq m)$ such that $x \in B(a_i; \delta)$. In such ball, $|x - a_i| < \delta$. By (2), $\big|f(x) - f(a_i)\big| < 1$, or $\big|f(x)\big| < 1 + \big|f(a_i)\big|$. Almost done. Notice that a_i depends on x, and thus we might use finiteness of $\{a_1, a_2, \dots, a_m\}$ to remove dependence of a_i.

(5) Let

$$M = 1 + \max_{1 \leq i \leq m} \big|f(a_i)\big|.$$

So given any $x \in E$, $\big|f(x)\big| < M$.

$\square$

Supplement. Exercise about considering the closure. (Problem 3.5 in H. L. Royden, Real Analysis, 3rd Edition.) *Let $A = \mathbb{Q} \cap [0, 1]$, and let $\{I_n\}$ be a finite collection of open intervals covering A. Then $\sum l(I_n) \geq 1$.*

Proof.

$$\begin{aligned} 1 = m^*[0,1] = m^*\overline{A} \leq m^*\left(\overline{\bigcup I_n}\right) &= m^*\left(\bigcup \overline{I_n}\right) \\ &\leq \sum m^*(\overline{I_n}) = \sum l(\overline{I_n}) = \sum l(I_n). \end{aligned}$$

$\square$

Exercise 4.9. *Show that the requirement in the definition of uniformly continuity can be rephrased as follows, in terms of diameters of sets: To every $\varepsilon > 0$ there exists a $\delta > 0$ such that* $\mathrm{diam} f(E) < \varepsilon$ *for all $E \subseteq X$ with* $\mathrm{diam} E < \delta$.

Proof.

(1) ($\Longrightarrow$) Given $\varepsilon > 0$. By Definition 4.18, there exists a $\delta > 0$ such that

$$d(f(p), f(q)) < \frac{\varepsilon}{64}$$

for all p and q in X for which $d(p, q) < \delta$. Let E be any subset of X satisfying $\mathrm{diam} E < \delta$. Then for any $p, q \in E$,

$$d(p, q) \leq \mathrm{diam} E < \delta.$$

So that

$$d(f(p), f(q)) < \frac{\varepsilon}{64},$$

or $\frac{\varepsilon}{64}$ is an upper bound of $S = \{d(f(p), f(q)) : p, q \in E\}$. Hence

$$\mathrm{diam} f(E) = \sup S \leq \frac{\varepsilon}{64} < \varepsilon.$$

(Here we pick "$\frac{\varepsilon}{64}$" instead of ε since we want to get "$\mathrm{diam} f(E) < \varepsilon$" instead of $\mathrm{diam} f(E) \leq \varepsilon$.)

(2) ($\Longleftarrow$) Easy. Given $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mathrm{diam} f(E) < \varepsilon$ for all $E \subseteq X$ with $\mathrm{diam} E < \delta$. In particular, for any $p, q \in X$ with $d(p, q) < \delta$, we can take $E = \{p, q\} \subseteq X$ and its diameter

$$\mathrm{diam} E = d(p, q) < \delta.$$

So that

$$d(f(p), f(q)) = \mathrm{diam} f(E) < \varepsilon,$$

or Definition 4.18 holds.

□

Exercise 4.10. *Complete the details of the following alternative proof of Theorem 4.19 (Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X): If f is not uniformly continuous, then for some $\varepsilon > 0$ there are sequences $\{p_n\}$, $\{q_n\}$ in X such that $d_X(p_n, q_n) \to 0$ but $d_Y(f(p_n), f(q_n)) > \varepsilon$. Use Theorem 2.37 to obtain a contradiction.*

Proof.

(1) (Reductio ad absurdum) If f were not uniformly continuous, then for some $\varepsilon > 0$ there are sequences $\{p_n\}$, $\{q_n\}$ in X such that $d_X(p_n, q_n) \to 0$ but $d_Y(f(p_n), f(q_n)) > \varepsilon$.

(2) By Theorem 2.37, there is a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ such that $\{p_{n_k}\}$ converges to $p \in X$. Similar argument to $\{q_n\}$, we have a subsequence $\{q_{n'_k}\}$ of $\{q_n\}$ converging to $q \in X$.

(3) Since

$$d_X(p, q) \leq d_X(p, p_{n_k}) + d_X(p_{n_k}, q_{n'_k}) + d_X(q_{n'_k}, q) \to 0$$

(by assumption and (2)) and $d_X(p, q)$ is a constant, $d_X(p, q) = 0$ or $p = q$.

(4) Since f is continuous,

$$\lim_{k\to\infty} f(p_{n_k}) = f(p) = f(q) = \lim_{k\to\infty} f(q_{n'_k})$$

or $d_Y(f(p_{n_k}), f(q_{n'_k})) \to 0$, contrary to the assumption.

□

Exercise 4.11. *Suppose f is a uniformly continuous mapping of a metric space X into a metric space Y and prove that $\{f(x_n)\}$ is a Cauchy sequence in Y for every Cauchy sequence $\{x_n\}$ in X. Use this result to give an alternative proof* *of the theorem stated in Exercise 4.13.*

An alternative proof of Exercise 4.13 will be in Exercise 4.13 itself.

Proof (Definition 4.18). Given any Cauchy sequence $\{x_n\}$ in X.

(1) Given any $\varepsilon > 0$. Since f is uniformly continuous, there exists a $\delta > 0$ such that
$$d_Y(f(p), f(q)) < \varepsilon$$
for all p and q in X for which $d_X(p, q) < \delta$.

(2) Since $\{x_n\}$ is Cauchy in X, for such $\delta > 0$ there is an integer N such that
$$d_X(x_n, x_m) < \delta$$
whenever $n, m \geq N$.

(3) By (1)(2),
$$d_Y(f(x_n), f(x_m)) < \varepsilon$$
whenever $n, m \geq N$. Hence $\{f(x_n)\}$ is Cauchy in Y.

□

Proof (Exercise 4.9). Given any Cauchy sequence $\{x_n\}$ in X.

(1) Given any $\varepsilon > 0$. Since f is uniformly continuous, there exists a $\delta > 0$ such that $\text{diam} f(E) < \varepsilon$ for all $E \subseteq X$ with $\text{diam} E < \delta$.

(2) Since $\{x_n\}$ is Cauchy in X, for such $\delta > 0$ there is an integer N such that
$$d_X(x_n, x_m) < \frac{\delta}{64}$$
whenever $n, m \geq N$.

(3) Consider $E = \{x_N, x_{N+1}, \ldots\}$. By (2), $\text{diam} E \leq \frac{\delta}{64} < \delta$. By (1),
$$d_Y(f(x_n), f(x_m)) \leq \text{diam} f(E) < \varepsilon$$
whenever $n, m \geq N$. Hence $\{f(x_n)\}$ is Cauchy in Y.

□

Exercise 4.12. *A uniformly continuous function of a uniformly continuous function is uniformly continuous. State this more precisely and prove it.*

Statement (similar to Theorem 4.7): *suppose X, Y, Z are metric space, $E \subseteq X$, f maps E into Y, g maps the range of f, $f(E)$, into Z, and h is the mapping of E into Z defined by*

$$h(x) = g(f(x)) \qquad (x \in E).$$

If f is uniformly continuous on E and g is uniformly continuous on $f(E)$, then h is uniformly continuous on E.

Proof.

(1) Given $\varepsilon > 0$. Since g is uniformly continuous on $f(E)$, there exists $\eta > 0$ such that

$$d_Z(g(f(p)), g(f(q))) < \varepsilon \quad \text{if} \quad d_Y(f(p), f(q)) < \eta \quad \text{and} \quad f(p), f(q) \in f(E).$$

(2) Since f is uniformly continuous on E, there exists $\delta > 0$ such that

$$d_Y(f(p), f(q)) < \eta \quad \text{if} \quad d_X(p, q) < \delta \quad \text{and} \quad p, q \in E.$$

(3) By (1)(2),

$$d_Z(h(p), h(q)) = d_Z(g(f(p)), g(f(q))) < \varepsilon$$

if $d_X(p, q) < \delta$ and $p, q \in E$. Hence h is uniformly continuous on E.

$\square$

Exercise 4.13. *Let E be a dense subset of a metric space X, and let f be a uniformly continuous real function defined on E. Prove that f has a continuous extension from E to X (see Exercise 4.5 for terminology). (Uniqueness follows from Exercise 4.4.) (Hint: For each $p \in X$ and each positive integer n, let $V_n(p)$ be the set of all $q \in E$ with $d(p, q) < \frac{1}{n}$. Use Exercise 4.9 to show that the intersection of the closures of the sets $f(V_1(p)), f(V_2(p)), \ldots$, consists of a single point, say $g(p)$, of $\mathbb{R}^1$. Prove that the function g so define on X is the desired extension of f.) Could the range space $\mathbb{R}^1$ be replaced by $\mathbb{R}^k$? By any compact metric space? By any complete metric space? By any metric space?*

Proof (Hint). We prove the case that the range metric space is complete.

(1) Given any $p \in X$. We will extend f on $x = p$. For any positive integer n, let $V_n(p)$ be the set of all $q \in E$ with $d(p, q) < \frac{1}{n}$.

(2) *Show that $\overline{f(V_n(p))} \supseteq \overline{f(V_{n+1}(p))}$.* By construction, $V_n(p) \supseteq V_{n+1}(p)$. Thus $f(V_n(p)) \supseteq f(V_{n+1}(p))$ and

$$\overline{f(V_n(p))} \supseteq \overline{f(V_{n+1}(p))}.$$

(3) *Show that* $\lim_{n\to\infty} \operatorname{diam}\overline{f(V_n(p))} = 0$.

(a) Since E is dense in X, $V_n(p) \neq \varnothing$ and thus

$$f(V_n(p)) \neq \varnothing.$$

Especially,

$$\overline{f(V_n(p))} \supseteq f(V_n(p)) \neq \varnothing.$$

Hence $\mathrm{diam} V_n(p)$ and $\mathrm{diam} f(V_n(p))$ are well-defined.

(b) By the definition of $V_n(p)$ or $0 \leq \mathrm{diam} V_n(p) \leq \frac{2}{n}$,

$$\lim_{n \to \infty} \mathrm{diam} V_n(p) = 0.$$

(c) By the uniformly continuity of f (Exercise 4.9),

$$\lim_{n \to \infty} \mathrm{diam} f(V_n(p)) = 0.$$

(d) Since $\mathrm{diam} \overline{f(V_n(p))} = \mathrm{diam} f(V_n(p))$ (Theorem 3.10(a)),

$$\lim_{n \to \infty} \mathrm{diam} \overline{f(V_n(p))} = 0.$$

(4) *Show that there is an integer N such that $\overline{f(V_n(p))}$ is closed and bounded whenever $n \geq N$.*

(a) (Closeness.) Each $\overline{f(V_n(p))}$ is closed.

(b) (Boundedness.) Since $\lim_{n \to \infty} \mathrm{diam} \overline{f(V_n(p))} = 0$ by (3), there is an integer N such that

$$\mathrm{diam} \overline{f(V_n(p))} \leq \frac{1}{89}$$

whenever $n \geq N$. By the definition of diameters of $\overline{f(V_n(p))}$, each $\overline{f(V_n(p))}$ is bounded by $\frac{1}{64}$ whenever $n \geq N$.

(c) *Note.* If we apply Exercise 4.8 instead, we need extra efforts to generalize Exercise 4.8 to different range spaces for answering the following questions.

(5) By (2)(3)(4) and Exercise 3.21,

$$\bigcap_{n=N}^{\infty} \overline{f(V_n(p))}$$

or

$$\bigcap_{n=1}^{\infty} \overline{f(V_n(p))}$$

consists of exactly one point, say $g(p)$. This point $g(p)$ is an extension of f at $x = p$. Clearly, $g(p) = f(p)$ if $p \in E$.

(6) *Define*

$$g(p) = \begin{cases} \bigcap_{n=1}^{\infty} \overline{f(V_n(p))} = f(p) & (p \in E), \\ \bigcap_{n=1}^{\infty} \overline{f(V_n(p))} & (p \notin E). \end{cases}$$

Show that g is uniformly continuous.

(a) Given any $\varepsilon > 0$. Since f is uniformly continuous on E, there exists $\delta > 0$ such that

$$d(f(p), f(q)) < \frac{\varepsilon}{3} < \varepsilon$$

whenever $d(p, q) < \delta$ and $p, q \in E$. We will show that such δ also holds for g. Now given any $p, q \in X$ with $d(p, q) < \delta$.

(b) Since $\mathrm{diam} f(V_n(p)) = \mathrm{diam} \overline{f(V_n(p))}$ and $\lim_{n\to\infty} \mathrm{diam} \overline{f(V_n(p))} = 0$ (whether $p \in E$ or not), there is an integer N_1 such that

$$\mathrm{diam} f(V_n(p)) < \frac{\varepsilon}{3}$$

whenever $n \geq N_1$. Similarly, there is an integer N_2 such that

$$\mathrm{diam} f(V_n(q)) < \frac{\varepsilon}{3}$$

whenever $n \geq N_2$.

(c) Take an integer N_3 satisfying

$$N_3 > \frac{4}{\delta - d(p, q)} > 0.$$

For any $p' \in V_n(p) \neq \varnothing$ and $q' \in V_n(q) \neq \varnothing$ as $n \geq N_3$, we have

$$\begin{aligned} d(p', q') &\leq d(p', p) + d(p, q) + d(q, q') \\ &\leq \frac{2}{n} + d(p, q) + \frac{2}{n} \\ &\leq \frac{2}{N_3} + d(p, q) + \frac{2}{N_3} \\ &< \frac{2(\delta - d(p, q))}{4} + d(p, q) + \frac{2(\delta - d(p, q))}{4} \\ &= \delta. \end{aligned}$$

(d) Take $N = \max\{N_1, N_2, N_3\}$. For any $p' \in V_N(p)$ and $q' \in V_N(p)$, we have

$$\begin{aligned} d(g(p), f(p')) &\leq \mathrm{diam} f(V_N(p)) < \frac{\varepsilon}{3}, \\ d(f(p'), f(q')) &< \frac{\varepsilon}{3}, \\ d(f(q'), g(q)) &\leq \mathrm{diam} f(V_N(q)) < \frac{\varepsilon}{3}. \end{aligned}$$

Hence

$$\begin{aligned} d(g(p), g(q)) &\leq d(g(p), f(p')) + d(f(p'), f(q')) + d(f(q'), g(q)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

(7) *Show that the range space $\mathbb{R}^1$ cannot be replaced by any metric space.*

(a) Take $X = \mathbb{R}$ and $Y = \mathbb{Q}$ with the Euclidean metric. Let $E = \mathbb{Q}$ be a dense subset of $X = \mathbb{R}$. Define $f : E \to Y$ by

$$f(x) = x.$$

(b) f is uniformly continuous on E.

(c) (Reductio ad absurdum) If f were having a continuous extension g on X, then

$$\lim_{n\to\infty} g(p_n) = g(p)$$

for any sequence $\{p_n\}$ in X such that $p_n \neq p$ and $\lim_{n\to\infty} p_n = p$.

(d) In particular, for some rational sequence $\{p_n\}$ in $E = \mathbb{Q}$ converging to $\sqrt{2} \in X$, we have

$$\lim_{n\to\infty} g(p_n) = \lim_{n\to\infty} f(p_n) = \lim_{n\to\infty} p_n = \sqrt{2} = g(p) \in \mathbb{Q},$$

which is absurd.

□

Proof (Exercise 4.11). We prove the case that the range metric space is complete.

(1) Given any $p \in X$. We will extend f on $x = p$. Since E is dense in X, there exists a sequence $\{p_n\}$ in E converging to p (whether $p \in E$ or not).

(2) Since E is dense in X, there exists a sequence $\{p_n\}$ in E converging to p (whether $p \in E$ or not). Hence $\{p_n\}$ is Cauchy in E (Theorem 3.11(a)). Since f is uniformly continuous, $\{f(p_n)\}$ is Cauchy. Since the range space is complete, $\{f(p_n)\}$ converges to a point, say $g(p)$. This point $g(p)$ is an extension of f at $x = p$. Clearly, $g(p) = f(p)$ if $p \in E$.

(3) *Show that $g(p)$ is well-defined.* If $\{p'_n\}$ is another sequence in E converging to p, we construct a new sequence $\{p''_n\}$ based on $\{p_n\}$ and $\{p'_n\}$ by

$$p''_n = \begin{cases} p_{\frac{n+1}{2}} & (n \equiv 1 \pmod 2), \\ p'_{\frac{n}{2}} & (n \equiv 0 \pmod 2). \end{cases}$$

Clearly $\{p''_n\}$ also converges to p. So $\{f(p''_n)\}$ converges to a single point. Note that $\{f(p_n)\}$ and $\{f(p'_n)\}$ are two subsequences of $\{f(p''_n)\}$, and thus both subsequences converge to the same point.

(4) *Define*

$$g(p) = \begin{cases} f(p) & (p \in E), \\ \lim_{n\to\infty} f(p_n) & (p \notin E) \end{cases}$$

where $\{p_n\}$ *is any sequence in* E *converging to* p*. Show that* g *is uniformly continuous.*

(a) Given any $\varepsilon > 0$. Since f is uniformly continuous on E, there exists $\delta > 0$ such that

$$d(f(p), f(q)) < \frac{\varepsilon}{3} < \varepsilon$$

whenever $d(p, q) < \delta$ and $p, q \in E$. We will show that such δ also holds for g. Now given any $p, q \in X$ with $d(p, q) < \delta$.

(b) By (2), there exists a sequence $\{p_n\}$ in E such that $\lim p_n = p$. Take an integer N_1 such that

$$d(p_n, p) < \frac{\delta - d(p, q)}{2}$$

whenever $n \geq N_1$. Similarly, there exists a sequence $\{q_n\}$ in E such that $\lim q_n = q$. Take an integer N_2 such that

$$d(q_n, q) < \frac{\delta - d(p, q)}{2}$$

whenever $n \geq N_2$. Therefore,

$$\begin{aligned} d(p_n, q_n) &\leq d(p_n, p) + d(p, q) + d(q, q_n) \\ &< \frac{\delta - d(p, q)}{2} + d(p, q) + \frac{\delta - d(p, q)}{2} \\ &= \delta. \end{aligned}$$

whenever $n \geq N_1$ and $n \geq N_2$.

(c) Since $\lim f(p_n) = g(p)$, there is an integer N_3 such that

$$d(f(p_n), g(p)) < \frac{\varepsilon}{3}$$

whenever $n \geq N_3$. Similarly, since $\lim f(q_n) = g(q)$, there is an integer N_4 such that

$$d(f(q_n), g(q)) < \frac{\varepsilon}{3}$$

whenever $n \geq N_4$.

(d) Take $N = \max\{N_1, N_2, N_3, N_4\}$, we have

$$\begin{aligned} d(g(p), f(p_N)) &< \frac{\varepsilon}{3}, \\ d(f(p_N), f(q_N)) &< \frac{\varepsilon}{3}, \\ d(f(q_N), g(q)) &< \frac{\varepsilon}{3}. \end{aligned}$$

Hence

$$\begin{aligned} d(g(p), g(q)) &\leq d(g(p), f(p_N)) + d(f(p_N), f(q_N)) + d(f(q_N), g(q)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

□

Exercise 4.14 (Brouwer's fixed-point theorem). *Let $I = [0, 1]$ be the closed unit interval. Suppose f is continuous mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$.*

Proof (Theorem 4.23). Let $g(x) = f(x) - x$ in I.

(1) $g(0) = 0$. Take $x = 0$.

(2) $g(1) = 0$. Take $x = 1$.

(3) Suppose $g(0) \neq 0$ ($f(0) \neq 0$) and $g(1) \neq 0$ ($f(1) \neq 1$). Since $f : I \to I$, $f(0) > 0$ and $f(1) < 1$. That is, $g(0) > 0$ and $g(1) < 0$. Applying the intermediate value theorem (Theorem 4.23), there is a point in $\xi \in (0, 1)$ such that $g(\xi) = 0$. That is, $f(\xi) = \xi$ for some $\xi \in (0, 1)$.

In any case, the conclusion holds. □

Supplement. Brouwer's fixed-point theorem.

(1) In the $\mathbb{R}^1$, see Exercise 4.14 itself.

(2) In the $\mathbb{R}^2$, see Exercise 8.29.

(3) In the $\mathbb{R}^n$, every continuous function from a closed ball of a Euclidean space $\mathbb{R}^n$ into itself has a fixed point (without proof).

(4) In a Banach space, Schauder fixed-point theorem.

Exercise 4.15. *Call a mapping of X into Y **open** if $f(V)$ is an open set in Y whenever V is an open set in X. Prove that every continuous open mapping of $\mathbb{R}^1$ into $\mathbb{R}^1$ is monotonic.*

In fact, f is strictly monotonic.

Proof.

(1) (Reductio ad absurdum) If f were not strictly monotonic, then there exist $a < c < b \in \mathbb{R}^1$ such that

$$f(a) \leq f(c) \geq f(b)$$

or

$$f(a) \geq f(c) \leq f(b).$$

(2) In any case, f is a real continuous function on a compact set $[a, b]$. By Theorem 4.16, there exists $p, q \in [a, b]$ such that

$$M = \sup_{x \in [a,b]} f(x) = f(p),$$
$$m = \inf_{x \in [a,b]} f(x) = f(q).$$

(3) As $f(a) \leq f(c) \geq f(b)$, we consider where f reaches its maximum value M (by (2)).

   (a) $f(a) = M$ or $f(b) = M$. Since $f(a) \leq f(c) \geq f(b)$, by the maximality of M, $f(c) = M$ or $M \in f((a, b))$.

   (b) $f(a) < M$ and $f(b) < M$. Hence $M \in f((a, b))$ clearly.

   In any case, $M \in f((a, b))$. Note that $f((a, b))$ is open since f is an open mapping and (a, b) is open.

   Since M is in an open set $f((a, b))$, there exists an open neighborhood $B(M; r) \subseteq f((a, b))$ where $r > 0$. Hence

$$M + \frac{r}{64} \in B(M; r) \subseteq f((a, b)),$$

   contrary to the maximality of M.

(4) As $f(a) \geq f(c) \leq f(b)$, we consider where f reaches its minimum value m (by (2)). Similar to (3), we can reach a contradiction again.

(5) By (3)(4), (1) is absurd, and thus f is strictly monotonic.

□

Exercise 4.16. *Let $[x]$ denote the largest integer contained in x, this is, $[x]$ is a integer such that $x - 1 < [x] \leq x$; and let $(x) = x - [x]$ denote the fractional part of x. What discontinuities do the function $[x]$ and (x) have?*

Proof.

(1) *The function $[x]$ only has discontinuities at $x \in \mathbb{Z}$.*

(a) For any $p \notin \mathbb{Z}$, there is an integer n such that $n < p < n+1$. Given any $\varepsilon > 0$, there is a $\delta = \min\{p-n, (n+1)-p\} > 0$ such that $\big|[x]-[p]\big| < \varepsilon$ whenever $|x-p| < \delta$. In fact, $|x-p| < \delta$ is equivalent to $n < x < n+1$ and therefore $\big|[x]-[p]\big| = |n-n| = 0 < \varepsilon$.

(b) For any $p \in \mathbb{Z}$, $\lim_{x \to p^+} [x] = p$ and $\lim_{x \to p^-} [x] = p-1$.

(2) *The function (x) only has discontinuities at $x \in \mathbb{Z}$.*

(a) Since $[x]$ is continuous on $\mathbb{R} - \mathbb{Z}$ and x is continuous on $\mathbb{R}$, especially on $\mathbb{R} - \mathbb{Z}$, $(x) = x - [x]$ is continuous on $\mathbb{R} - \mathbb{Z}$.

(b) For any $p \in \mathbb{Z}$, $\lim_{x \to p^+} (x) = 0$ and $\lim_{x \to p^-} (x) = 1$.

□

Exercise 4.17.
PLACEHOLDER

Exercise 4.18 (Thomae's function). *Every rational x can be written in the form $x = \frac{m}{n}$, where $n > 0$, and m and n are integers without any common divisors. When $x = 0$, we take $n = 1$. Consider the function f defined on $\mathbb{R}^1$ by*

$$f(x) = \begin{cases} 0 & (x \text{ irrational}), \\ \frac{1}{n} & (x = \frac{m}{n}). \end{cases}$$

Prove that f is continuous at every irrational point, and that f has a simple discontinuity at every rational point.

Proof.

(1) *Show that f has period 1.*

(a) As x is irrational, $x+1$ is irrational too. Hence $f(x+1) = 0 = f(x)$.

(b) As $x = \frac{m}{n} \in \mathbb{Q}$, where $n > 0$, and m and n are integers without any common divisors.

$$x + 1 = \frac{m+n}{n} \in \mathbb{Q},$$

where $n > 0$, and $\gcd(m+n, n) = \gcd(m, n) = 1$. Hence $f(x+1) = \frac{1}{n} = f(x)$.

In any case, $f(x+1) = f(x)$ for any $x \in \mathbb{R}$.

(2) *Show that $f(p+) = f(p-) = 0$ for any $p \in \mathbb{R}$.*

(a) By (1), we might assume $p \in [0, 1]$. For the edge cases $f(0-)$ and $f(1+)$, note that $f(0-) = f(1-)$ and $f(1+) = f(0+)$.

(b) Now given any point $p \in [0,1]$. Given $\varepsilon > 0$, it suffices to find $\delta > 0$ such that

$$\left|f(x) - 0\right| < \varepsilon \quad \text{whenever} \quad p < x < p + \delta.$$

(The case $f(p-) = 0$ is similar.)

(c) For such $\varepsilon > 0$, there is an integer $N \geq s$ such that

$$\frac{1}{N} < \varepsilon.$$

Hence

$$E = \left\{ \frac{m}{n} \in [0,1] : n \leq N \text{ where } \frac{m}{n} \text{ is defined as this exercise} \right\} - \{p\}$$

is finite since m and n are satisfying the relation $0 \leq m \leq n \leq N$ and there are only finitely many pairs of (m, n).

(d) Now we can take

$$\delta = \min\{|q - p| : q \in E\} > 0.$$

As $p < x < p + \delta$, $\left|f(x) - 0\right| = 0 < \varepsilon$ if x is irrational;

$$\left|f(x) - 0\right| < \frac{1}{N} < \varepsilon$$

if x is rational. (Note that $x \notin E$ by the construction of δ and thus the denominator of x is $> N$.) In any case, $\left|f(x) - 0\right| < \varepsilon$.

(3) Since $f(x) = 0$ if $x \in \mathbb{R} - \mathbb{Q}$ and $f(x) \neq 0$ if $x \in \mathbb{Q}$, f is continuous at every irrational point and f has a simple discontinuity at every rational point (by (2)).

□

Note.

(1) *f is nowhere differentiable.* (Hint: Use Lemma in Proof (Liouville, 1844) of Exercise 2.3.)

(2) *$f \in \mathscr{R}$ on any bounded interval $[a,b]$ and $\int_a^b f dx = 0$.* To prove this, use the similar argument in Exercise 6.6.

Exercise 4.19. *Suppose f is a real function with domain $\mathbb{R}^1$ which has the intermediate value property: If $f(a) < c < f(b)$, then $f(x) = c$ for some x between a and b. Suppose also, for every rational r, that the set of all x with $f(x) = r$ is closed. Prove that f is continuous. (Hint: If $x_n \to x_0$ but*

$f(x_n) > r > f(x_0)$ for some r and all n, then $f(t_n) = r$ for some t_n between x_0 and x_n; thus $t_n \to x_0$. Find a contradiction. (N. J. Fine, Amer. Math. Monthly, vol. 73, 1966, p.782.))

Proof.

(1) (Reductio ad absurdum) If f were not continuous at $x = x_0$, then there exists a sequence $\{x_n\}$ in $\mathbb{R}^1$ such that $\{f(x_n)\}$ in $\mathbb{R}^1$ not converging to $f(x_0)$. So there exists $\varepsilon_0 > 0$ and a subsequence $\{x_{n'}\}$ of $\{x_n\}$ such that

$$\left| f(x_{n'}) - f(x_0) \right| \geq \varepsilon_0$$

for all n'. Hence

$$f(x_{n'}) \geq f(x_0) + \varepsilon_0 \quad \text{or} \quad f(x_{n'}) \leq f(x_0) - \varepsilon_0$$

for all n'.

(2) Since there are infinitely many $\{x_{n'}\}$, there are only two possible cases:

(a) There is a subsequence $\{x_{n''}\}$ of $\{x_{n'}\}$ such that $f(x_{n''}) \geq f(x_0) + \varepsilon_0$ for all n''.

(b) There is a subsequence $\{x_{n''}\}$ of $\{x_{n'}\}$ such that $f(x_{n''}) \leq f(x_0) - \varepsilon_0$ for all n''.

In any case, we may assume that there exists $\varepsilon_0 > 0$ and a sequence $\{x_n\}$ such that

$$f(x_n) \geq f(x_0) + \varepsilon_0$$

for all n. (The case that $f(x_n) \leq f(x_0) - \varepsilon_0$ is similar.)

(3) Pick any rational number $r \in (f(x_0), f(x_0) + \varepsilon_0)$ since $\mathbb{Q}$ is dense in $\mathbb{R}$. (In fact, $\mathbb{Q}$ can be replaced by any dense subset of $\mathbb{R}$ in this exercise.) Since

$$f(x_n) \geq f(x_0) + \varepsilon_0 > r > f(x_0),$$

by the intermediate value property, there exists some t_n between x_0 and x_n such that $f(t_n) = r$.

(4) Let $E = \{t : f(t) = r\}$ be a closed set (since $r \in \mathbb{Q}$ and the assumption). Hence $t_n \in E$ for all n (by (3)).

(5) $\lim t_n = x_0$ since $\lim x_n = x_0$. Hence x_0 is a limit point of E. (Note that $x_0 \notin E$ since $f(x_0) < r$ (by (3)).) Since E is closed, $x_0 \in E$ or $f(x_0) = r$, which is absurd.

□

Supplement. *Suppose f is a differentiable real function with domain $\mathbb{R}^1$. Suppose for every rational r, that the set of all x with $f'(x) = r$ is closed.*

Prove that f' is continuous. Theorem 5.12 (Darboux's theorem) and Exercise 4.19 itself implies all.

Exercise 4.20.
PLACEHOLDER

Exercise 4.21.
PLACEHOLDER

Exercise 4.22.
PLACEHOLDER

Exercise 4.23. *A real-valued function f defined in (a, b) is said to be* ***convex*** *if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

whenever $a < x < b$, $a < y < b$, $0 < \lambda < 1$. Prove that every convex function is continuous. Prove that every increasing convex function of a convex function is convex. (For example, if f is convex, so is e^f.)

If f is convex in (a, b) and if $a < s < t < u < b$, show that

$$\frac{f(t)-f(s)}{t-s} \leq \frac{f(u)-f(s)}{u-s} \leq \frac{f(u)-f(t)}{u-t}.$$

Proof.

(1) *Show that $\frac{f(t)-f(s)}{t-s} \leq \frac{f(u)-f(s)}{u-s} \leq \frac{f(u)-f(t)}{u-t}$.* Since

$$\begin{aligned} t &= \frac{t-s}{u-s}u + \left(1 - \frac{t-s}{u-s}\right)s \\ &= \left(1 - \frac{u-t}{u-s}\right)u + \frac{u-t}{u-s}s \end{aligned}$$

and $0 < \frac{t-s}{u-s}, \frac{u-t}{u-s} < 1$, by the convexity of f we have

$$\begin{aligned} f(t) &\leq \frac{t-s}{u-s}f(u) + \left(1 - \frac{t-s}{u-s}\right)f(s), \\ f(t) &\leq \left(1 - \frac{u-t}{u-s}\right)f(u) + \frac{u-t}{u-s}f(s). \end{aligned}$$

It is equivalent to

$$\frac{f(t)-f(s)}{t-s} \leq \frac{f(u)-f(s)}{u-s} \leq \frac{f(u)-f(t)}{u-t}.$$

$\square$

(2) *If x, y, x', y' are points of (a,b) with $x \leq x' < y'$ and $x < y \leq y'$, then the chord over (x', y') has larger slope than the chord over (x, y); that is,*

$$\frac{f(y)-f(x)}{y-x} \leq \frac{f(y')-f(x')}{y'-x'}.$$

It is a corollary to (1).

(3) *Show that f is continuous.* Let $[c,d] \subseteq (a,b)$. Then by (2),

$$\frac{f(c)-f(a)}{c-a} \leq \frac{f(y)-f(x)}{y-x} \leq \frac{f(b)-f(d)}{b-d}$$

for x, y in $[c,d]$. Thus $|f(y)-f(x)| \leq M|y-x|$ in $[c,d]$ (where $M = \max\left(|\frac{f(c)-f(a)}{c-a}|, |\frac{f(b)-f(d)}{b-d}|\right)$), and so f is absolutely continuous on each closed subinterval of (a,b). Especially, f is continuous.

(4) *Let f be a convex function, g be an increasing convex function, and $h = g \circ f$. Show that h is convex.*

$$\begin{aligned}
f(\lambda x + (1-\lambda)y) &\leq \lambda f(x) + (1-\lambda) f(y), && \text{(Convexity of } f\text{)} \\
g(f(\lambda x + (1-\lambda)y)) &\leq g(\lambda f(x) + (1-\lambda) f(y)) && \text{(Increasing of } g\text{)} \\
&\leq \lambda g(f(x)) + (1-\lambda) g(f(y)), && \text{(Convexity of } g\text{)} \\
h(\lambda x + (1-\lambda)y) &\leq \lambda h(x) + (1-\lambda) h(y). &&
\end{aligned}$$

$\square$

Exercise 4.24. *Assume that f is a continuous real function defined in (a,b) such that*

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$$

for all $x, y \in (a,b)$. Prove that f is convex.

Proof.

(1) *Show that*

$$f\left(\frac{x_1 + \cdots + x_n}{n}\right) \leq \frac{f(x_1) + \cdots + f(x_n)}{n}$$

whenever $a < x_i < b$ $(1 \le i \le n)$. Apply Cauchy induction and use the same argument in proving the AM-GM inequality. As $n = 1, 2$, the inequality holds by assumption. Suppose $n = 2^k$ $(k \ge 1)$ the inequality holds. As $n = 2^{k+1}$,

$$\begin{aligned}
& f\left(\frac{x_1 + \cdots + x_{2^{k+1}}}{2^{k+1}}\right) \\
=& f\left(\frac{1}{2}\left(\frac{x_1 + \cdots + x_{2^k}}{2^k} + \frac{x_{2^k+1} + \cdots + x_{2^{k+1}}}{2^k}\right)\right) \\
\le& \frac{1}{2}\left(f\left(\frac{x_1 + \cdots + x_{2^k}}{2^k}\right) + f\left(\frac{x_{2^k+1} + \cdots + x_{2^{k+1}}}{2^k}\right)\right) \\
\le& \frac{1}{2}\left(\frac{f(x_1) + \cdots + f(x_{2^k})}{2^k} + \frac{f(x_{2^k+1}) + \cdots + f(x_{2^{k+1}})}{2^k}\right) \\
=& \frac{f(x_1) + \cdots + f(x_{2^k}) + f(x_{2^k+1}) + \cdots + f(x_{2^{k+1}})}{2^{k+1}} \\
=& \frac{f(x_1) + \cdots + f(x_{2^{k+1}})}{2^{k+1}}.
\end{aligned}$$

As n is not a power of 2, then it is certainly less than some natural power of 2, say $n < 2^m$ for some m. Let

$$x_{n+1} = \cdots = x_{2^m} = \frac{x_1 + \cdots + x_n}{n} = \alpha.$$

Then by the induction hypothesis,

$$\begin{aligned}
f(\alpha) &= f\left(\frac{x_1 + \cdots + x_n + \alpha + \cdots + \alpha}{2^m}\right) \\
&\le \frac{f(x_1) + \cdots + f(x_n) + f(\alpha) + \cdots + f(\alpha)}{2^m} \\
&\le \frac{f(x_1) + \cdots + f(x_n) + (2^m - n)f(\alpha)}{2^m}, \\
2^m f(\alpha) &\le f(x_1) + \cdots + f(x_n) + (2^m - n)f(\alpha), \\
nf(\alpha) &\le f(x_1) + \cdots + f(x_n),
\end{aligned}$$

or $f\left(\frac{1}{n}(x_1 + \cdots + x_n)\right) \le \frac{1}{n}(f(x_1) + \cdots f(x_n))$.

(2) Hence,

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$$

for any rational λ in $(0,1)$. (Given any positive integers $p < q$, put $n = q$, $x_1 = \cdots = x_p = x$ and $x_{p+1} = \cdots = x_n = y$ in (1).)

(3) Given any real $\lambda \in (0,1)$, there is a sequence of rational numbers $\{r_n\} \subseteq (0,1)$ such that $r_n \to \lambda$. By (2),

$$f(r_n x + (1-r_n)y) \le r_n f(x) + (1-r_n)f(y)$$

for any rational r_n in $(0, 1)$. Taking limit on the both sides and using the continuity of f, we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

□

Proof (Reductio ad absurdum). If f were not convex, then there is a subinterval $[c, d] \subseteq (a, b)$ such that

$$\frac{f(d) - f(c)}{d - c} < \frac{f(x_0) - f(c)}{x_0 - c}$$

for some $x_0 \in [c, d]$. Let

$$g(x) = f(x) - f(c) - \frac{f(d) - f(c)}{d - c}(x - c)$$

for $x \in [c, d]$. Therefore,

(1) $g(x)$ is continuous and midpoint convex.

(2) $g(c) = g(d) = 0$.

(3) Let $M = \sup\{g(x) : x \in [c, d]\}$. $\infty > M > 0$ due to the continuity of g and the existence of x_0. And let $\xi = \inf\{x \in [c, d] : g(x) = M\}$. By the continuity of g, $g(\xi) = M$. $\xi \in (c, d)$ by (2).

(4) Since (c, d) is open, there is $h > 0$ such that $(\xi - h, \xi + h) \subseteq (c, d)$. By the minimality of ξ and M, $g(\xi - h) < g(\xi)$ and $g(\xi + h) \leq g(\xi)$.

Therefore,

$$\begin{aligned}
g(\xi - h) + g(\xi + h) &< 2g(\xi), \\
\frac{g(\xi - h) + g(\xi + h)}{2} &< g(h) \\
&= g\left(\frac{(\xi - h) + (\xi + h)}{2}\right),
\end{aligned}$$

contrary to the midpoint convexity of g. □

The result becomes false if "continuity of f" is omitted.

Exercise 4.25. *If $A \subset \mathbb{R}^k$ and $B \subset \mathbb{R}^k$, define $A + B$ to be the set of all sums $\mathbf{x} + \mathbf{y}$ with $\mathbf{x} \in A$, $\mathbf{y} \in B$.*

(a) *If K is compact and C is closed in $\mathbb{R}^k$, prove that $K + C$ is closed. (Hint: Take $\mathbf{z} \notin K + C$, put $F = \mathbf{z} - C$, the set of all $\mathbf{z} - \mathbf{y}$ with $\mathbf{y} \in C$. Then K and F are disjoint. Choose δ as in Exercise 4.21. Show that the open ball with center $\mathbf{z}$ and radius δ does not intersect $K + C$.)*

(b) *Let α be an irrational real number. Let C_1 be the set of all integers, let C_2 be the set of all $n\alpha$ with $n \in C_1$. Show that C_1 and C_2 are closed subsets of $\mathbb{R}^1$ whose sum $C_1 + C_2$ is not closed, by showing that $C_1 + C_2$ is a countable dense subset of $\mathbb{R}^1$.*

Proof. TODO.

Exercise 4.26. *Suppose X, Y, Z are metric spaces, and Y is compact. Let f map X into Y, let g be a continuous one-to-one mapping of Y into Z, and put $h(x) = g(f(x))$ for $x \in X$.*

Prove that f is uniformly continuous if h is uniformly continuous. (Hint: g^{-1} has compact domain $g(Y)$, and $f(x) = g^{-1}(h(x))$.)

Prove also that f is continuous if h is continuous.

Show (by modifying Example 4.21, or by finding a different example) that the compactness of Y cannot be omitted from the hypotheses, even when X and Z are compact.

Proof. TODO.