

## Chapter 10: Integration of Differential Forms

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**Exercise 10.1.** Let  $H$  be a compact convex set in  $\mathbb{R}^k$ , with nonempty interior. Let  $f \in \mathcal{C}(H)$ , put  $f(\mathbf{x}) = 0$  in the complement of  $H$ , and define  $\int_H f$  as in Definition 10.3. Prove that  $\int_H f$  is independent of the order in which the  $k$  integrations are carried out. (Hint: Approximate  $f$  by functions that are continuous on  $\mathbb{R}^k$  and whose supports are in  $H$ , as was done in Example 10.4.)

*Proof.*

(1)

(2)

□

**Exercise 10.2.** For  $i = 1, 2, 3, \dots$ , let  $\varphi_i \in \mathcal{C}(\mathbb{R}^1)$  have support in  $(2^{-i}, 2^{1-i})$ , such that  $\int \varphi_i = 1$ . Put

$$f(x, y) = \sum_{i=1}^{\infty} [\varphi_i(x) - \varphi_{i+1}(x)] \varphi_i(y)$$

Then  $f$  has compact support in  $\mathbb{R}^2$ ,  $f$  is continuous except at  $(0, 0)$ , and

$$\int dy \int f(x, y) dx = 0 \quad \text{but} \quad \int dx \int f(x, y) dy = 1.$$

Observe that  $f$  is unbounded in every neighborhood of  $(0, 0)$ .

*Proof.*

(1) If  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}^m$  are two functions, then

(a)  $\text{supp}(fg) \subseteq \text{supp}(f) \cap \text{supp}(g)$ .

(b)  $\text{supp}(f + g) \subseteq \text{supp}(f) \cup \text{supp}(g)$ .

(2) Note that  $f(x, y)$  is well-defined on  $\mathbb{R}^2$  since only finitely many terms are nonzero for each fixed point  $(x, y) \in \mathbb{R}^2$  (by (1)). Besides,

$$\begin{aligned} & \text{supp}([\varphi_i(x) - \varphi_{i+1}(x)]\varphi_i(y)) \\ & \subseteq \{(x, y) : x \in \text{supp}(\varphi_i) \cup \text{supp}(\varphi_{i+1}), y \in \text{supp}(\varphi_i)\} \\ & \subseteq \{(x, y) : x \in (2^{-i}, 2^{-i+1}) \cup (2^{-i-1}, 2^{-i}), y \in (2^{-i}, 2^{-i+1})\} \\ & \subseteq \{(x, y) : x \in (0, 1), y \in (0, 1)\} \end{aligned}$$

for all  $i = 1, 2, 3, \dots$ . So  $\text{supp}(f) \subseteq (0, 1)^2$ , or  $\text{supp}(f)$  is bounded. As  $\text{supp}(f)$  is closed (by definition),  $\text{supp}(f)$  is compact (Theorem 2.41).

(3) Show that  $f(x, y)$  is not continuous at  $(0, 0)$ .

(a) Note that  $f(0, 0) = 0$  since  $(0, 0) \notin \text{supp}(f) \subseteq (0, 1)^2$ . It suffices to show that there exists a sequence  $\{(t_n, t_n)\}$  in  $\mathbb{R}^2$  such that  $(t_n, t_n) \neq (0, 0)$ ,  $\lim_{n \rightarrow \infty} (t_n, t_n) = (0, 0)$  but  $\lim_{n \rightarrow \infty} f(t_n, t_n)$  does not converge to 0 (Theorem 4.2).

(b) For any  $n = 1, 2, 3, \dots$ ,

$$1 = \int \varphi_n = \int_{2^{-n}}^{2^{-n+1}} \varphi(t) dt \leq 2^{-n} \sup_{t \in [2^{-n}, 2^{-n+1}]} \varphi(t),$$

or  $\sup_{t \in [2^{-n}, 2^{-n+1}]} \varphi(t) \geq 2^n$ . By the continuity of  $\varphi_n$ , there exists  $t_n \in [2^{-n}, 2^{-n+1}]$  such that  $\varphi_n(t_n) \geq 2^n$  (Theorem 4.16).

(c) We construct  $\{(t_n, t_n)\}$  in  $\mathbb{R}^2$  by (b) for all  $n = 1, 2, 3, \dots$ . Clearly,  $(t_n, t_n) \neq (0, 0)$  and  $\lim_{n \rightarrow \infty} (t_n, t_n) = (0, 0)$ . However,

$$f(t_n, t_n) = [\varphi_n(t_n) - \varphi_{n+1}(t_n)]\varphi_n(t_n) = \varphi_n(t_n)^2 \geq 2^{2n}$$

does not converge to 0 as  $n \rightarrow \infty$ .

(4) Show that  $f(x, y)$  is continuous at  $\mathbf{x}_0 = (x_0, y_0) \neq (0, 0)$ . Consider an open neighborhood  $B(\mathbf{x}_0; r)$  of  $\mathbf{x}_0$  with  $r = \frac{\|\mathbf{x}_0\|}{64} > 0$ . Hence,

$$f(x, y)|_{B(\mathbf{x}_0; r)} = \sum_{i=1}^N [\varphi_i(x) - \varphi_{i+1}(x)]\varphi_i(y)$$

is the sum of finitely many terms where  $N = \log_2 \frac{89}{\|\mathbf{x}_0\|} \geq 1$  (since  $[\varphi_i(x) - \varphi_{i+1}(x)]\varphi_i(y) = 0$  on  $B(\mathbf{x}_0; r)$  whenever  $i \geq N$ ). Therefore,  $f(x, y)|_{B(\mathbf{x}_0; r)}$  is continuous by the continuity of  $\varphi_i$ .

(5) Show that  $\int dy \int f(x, y) dx = 0$ . For any fixed  $y$ , there is a positive integer  $N(y)$  such that  $\varphi_{N(y)+1}(y) = \varphi_{N(y)+2}(y) = \dots = 0$  and

$$f(x, y) = \sum_{i=1}^{N(y)} [\varphi_i(x) - \varphi_{i+1}(x)]\varphi_i(y).$$

So

$$\begin{aligned}
\int f(x, y) dx &= \int \sum_{i=1}^{N(y)} [\varphi_i(x) - \varphi_{i+1}(x)] \varphi_i(y) dx \\
&= \sum_{i=1}^{N(y)} \varphi_i(y) \int [\varphi_i(x) - \varphi_{i+1}(x)] dx \\
&= \sum_{i=1}^{N(y)} \varphi_i(y) \left( \int \varphi_i(x) dx - \int \varphi_{i+1}(x) dx \right) \\
&= \sum_{i=1}^{N(y)} \varphi_i(y) (1 - 1) \\
&= 0,
\end{aligned}$$

and thus

$$\int dy \int f(x, y) dx = \int 0 dy = 0.$$

- (6) Show that  $\int dx \int f(x, y) dy = 0$ . For any fixed  $x$ , there is a positive integer  $N(x)$  such that  $\varphi_{N(x)+1}(x) = \varphi_{N(x)+2}(x) = \dots = 0$  and

$$f(x, y) = \sum_{i=1}^{N(x)} [\varphi_i(x) - \varphi_{i+1}(x)] \varphi_i(y).$$

So

$$\begin{aligned}
\int f(x, y) dy &= \int \sum_{i=1}^{N(x)} [\varphi_i(x) - \varphi_{i+1}(x)] \varphi_i(y) dy \\
&= \sum_{i=1}^{N(x)} [\varphi_i(x) - \varphi_{i+1}(x)] \int \varphi_i(y) dy \\
&= \sum_{i=1}^{N(x)} [\varphi_i(x) - \varphi_{i+1}(x)] \\
&= \varphi_1(x),
\end{aligned}$$

and thus

$$\int dx \int f(x, y) dy = \int \varphi_1(x) dx = 1.$$

□

**Exercise 10.3.**

- (a) If  $\mathbf{F}$  is as in Theorem 10.7, put  $\mathbf{A} = \mathbf{F}'(\mathbf{0})$ ,  $\mathbf{F}_1(\mathbf{x}) = \mathbf{A}^{-1}\mathbf{F}(\mathbf{x})$ . Then  $\mathbf{F}_1(\mathbf{0}) = \mathbf{I}$ . Show that

$$\mathbf{F}_1(\mathbf{x}) = \mathbf{G}_n \circ \mathbf{G}_{n-1} \circ \cdots \circ \mathbf{G}_1(\mathbf{x})$$

in some neighborhood of  $\mathbf{0}$ , for certain primitive mappings  $\mathbf{G}_1, \dots, \mathbf{G}_n$ . This gives another version of Theorem 10.7:

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}'(\mathbf{0})\mathbf{G}_n \circ \mathbf{G}_{n-1} \circ \cdots \circ \mathbf{G}_1(\mathbf{x}).$$

- (b) Prove that the mapping  $(x, y) \mapsto (y, x)$  of  $\mathbb{R}^2$  onto  $\mathbb{R}^2$  is not the composition of any two primitive mappings, in any neighborhood of the origin. (This shows that the flips  $B_i$  cannot be omitted from the statement of Theorem 10.7.)

*Proof of (a).*

- (1) Suppose  $\mathbf{F}$  is a  $\mathcal{C}'$ -mapping of an open set  $E \subseteq \mathbb{R}^n$  into  $\mathbb{R}^n$ ,  $\mathbf{0} \in E$ ,  $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ , and  $\mathbf{F}'(\mathbf{0})$  is invertible.
- (2) Similar to the proof of Theorem 10.7. Put  $\mathbf{F}_1 = \mathbf{F}$ .
- (3) As  $m = 1$ , there is an open neighborhood  $V_1 \subseteq E$  of  $\mathbf{0}$  such that  $\mathbf{F}_1(\mathbf{0}) = (\mathbf{F}'(\mathbf{0}))^{-1}\mathbf{F}(\mathbf{0}) = \mathbf{0}$ ,  $\mathbf{F}'_1(\mathbf{0}) = \mathbf{I}$  is invertible, and

$$\mathbf{F}_1(\mathbf{x}) = \sum_{i=1}^n \alpha_i(\mathbf{x})\mathbf{e}_i,$$

where  $\alpha_1, \dots, \alpha_n$  are real  $\mathcal{C}'$ -functions in  $V_1$ . Hence

$$\mathbf{F}'_1(\mathbf{0})\mathbf{e}_1 = \sum_{i=1}^n (D_1\alpha_i)(\mathbf{0})\mathbf{e}_i.$$

Note that  $(D_1\alpha_1)(\mathbf{0}) = 1 \neq 0$ , and we might pick  $B_1 = \mathbf{I}$ . Thus we can define

$$\mathbf{G}_1(\mathbf{x}) = \mathbf{x} + [\alpha_1(\mathbf{x}) - x_1]\mathbf{e}_1 \quad (\mathbf{x} \in V_1).$$

Then  $\mathbf{G}_1 \in \mathcal{C}'(V_1)$ ,  $\mathbf{G}_1$  is primitive, and  $\mathbf{G}'_1(\mathbf{0}) = \mathbf{I}$  is invertible.

- (4) Now we make the induction hypothesis for  $1 \leq m \leq n-1$ .
- (5) Since  $\mathbf{G}'_m(\mathbf{0}) = \mathbf{I}$  is invertible, the inverse function theorem shows that there is an open set  $U_m$ , with  $\mathbf{0} \in U_m \subseteq V_m$ , such that  $\mathbf{G}_m$  is an injective mapping of  $U_m$  onto a neighborhood  $V_{m+1}$  of  $\mathbf{0}$ , in which  $\mathbf{G}_m^{-1} \in \mathcal{C}'(V_{m+1})$ . Define  $\mathbf{F}_{m+1}$  by

$$\mathbf{F}_{m+1}(\mathbf{y}) = \mathbf{F}_m \circ \mathbf{G}_m^{-1}(\mathbf{y}) \quad (\mathbf{y} \in V_{m+1}).$$

Then  $\mathbf{F}_{m+1} \in \mathcal{C}'(V_{m+1})$ ,  $\mathbf{F}_m(\mathbf{0}) = \mathbf{0}$ , and  $\mathbf{F}'_{m+1}(\mathbf{0}) = \mathbf{I}$  is invertible by the chain rule and the inverse function theorem. So

$$\mathbf{F}_{m+1}(\mathbf{x}) = P_m \mathbf{x} + \sum_{i=m+1}^n \alpha_i(\mathbf{x}) \mathbf{e}_i,$$

where  $\alpha_1, \dots, \alpha_n$  are real  $\mathcal{C}'$ -functions in  $V_{m+1}$ . Hence

$$\mathbf{F}'_{m+1}(\mathbf{0}) \mathbf{e}_{m+1} = \sum_{i=m+1}^n (D_{m+1} \alpha_i)(\mathbf{0}) \mathbf{e}_i.$$

Note that  $(D_{m+1} \alpha_{m+1})(\mathbf{0}) = 1 \neq 0$ , and we might pick  $B_{m+1} = \mathbf{I}$ . Thus we can define

$$\mathbf{G}_{m+1}(\mathbf{x}) = \mathbf{x} + [\alpha_{m+1}(\mathbf{x}) - x_{m+1}] \mathbf{e}_{m+1} \quad (\mathbf{x} \in V_{m+1}).$$

Then  $\mathbf{G}_{m+1} \in \mathcal{C}'(V_{m+1})$ ,  $\mathbf{G}_{m+1}$  is primitive, and  $\mathbf{G}'_{m+1}(\mathbf{0}) = \mathbf{I}$  is invertible. Our induction hypothesis holds therefore with  $m+1$  in place of  $m$ .

(6) Note that

$$\mathbf{F}_m(\mathbf{x}) = \mathbf{F}_{m+1}(\mathbf{G}_m(\mathbf{x})) \quad (\mathbf{x} \in U_m).$$

If we apply this with  $m = 1, \dots, n-1$ , we successively obtain

$$\mathbf{F}_1 = \mathbf{F}_n \circ \mathbf{G}_{n-1} \circ \dots \circ \mathbf{G}_1$$

in some open neighborhood of  $\mathbf{0}$ . Note that  $\mathbf{F}_n$  is primitive since

$$\mathbf{F}_n(\mathbf{x}) = P_{n-1} \mathbf{x} + \alpha_n(\mathbf{x}) \mathbf{e}_n.$$

This completes the proof.

□

*Proof of (b).*

(1) For  $(x, y) \in \mathbb{R}^2$ , define

$$\mathbf{F}(x, y) = (y, x).$$

(2) (Reductio ad absurdum) If  $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$  for some primitive mappings  $\mathbf{G}_i$  ( $i = 1, 2$ ) in some neighborhood  $V_i$  of the origin,  $\mathbf{G}_i(\mathbf{0}) = \mathbf{0}$  and  $\mathbf{G}'_i$  is invertible, then we may assume that

$$\mathbf{G}_1(x, y) = (x, g_1(x, y)) \quad \text{and} \quad \mathbf{G}_2(x, y) = (g_2(x, y), y).$$

Here the case  $\mathbf{G}_1(x, y) = (g_1(x, y), y)$  and  $\mathbf{G}_2(x, y) = (x, g_2(x, y))$  is similar to the above case. Besides,  $\mathbf{G}_1(x, y) = (x, g_1(x, y))$  and  $\mathbf{G}_2(x, y) = (x, g_2(x, y))$  implies that

$$\mathbf{G}_2 \circ \mathbf{G}_1(x, y) = (x, g_2(x, g_1(x, y))) \neq (y, x) = \mathbf{F}(x, y).$$

Same reason for  $\mathbf{G}_1(x, y) = (g_1(x, y), y)$  and  $\mathbf{G}_2(x, y) = (g_2(x, y), y)$ .

(3) Note that

$$\mathbf{F}'(\mathbf{0}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Since

$$\mathbf{F}'(\mathbf{0}) = \mathbf{G}'_2(\mathbf{G}_1(\mathbf{0}))\mathbf{G}'_1(\mathbf{0}) = \mathbf{G}'_2(\mathbf{0})\mathbf{G}'_1(\mathbf{0}),$$

we have

$$\begin{aligned} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} &= \begin{bmatrix} D_1g_2(0,0) & D_2g_2(0,0) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ D_1g_1(0,0) & D_2g_1(0,0) \end{bmatrix} \\ &= \begin{bmatrix} * & * \\ D_1g_1(0,0) & D_2g_1(0,0) \end{bmatrix}. \end{aligned}$$

So  $D_1g_1(0,0) = 1$  and  $D_2g_1(0,0) = 0$ , and thus  $\mathbf{G}'_1(\mathbf{0}) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$  is not invertible, which is absurd.

□

**Exercise 10.4.** For  $(x, y) \in \mathbb{R}^2$ , define

$$\mathbf{F}(x, y) = (e^x \cos y - 1, e^x \sin y)$$

Prove that  $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$ , where

$$\begin{aligned} \mathbf{G}_1(x, y) &= (e^x \cos y - 1, y) \\ \mathbf{G}_2(u, v) &= (u, (1 + u) \tan v) \end{aligned}$$

are primitive in some neighborhood of  $(0, 0)$ . Compute the Jacobians of  $\mathbf{G}_1$ ,  $\mathbf{G}_2$ ,  $\mathbf{F}$  at  $(0, 0)$ . Define

$$\mathbf{H}_2(x, y) = (x, e^x \sin y)$$

and find

$$\mathbf{H}_1(u, v) = (h(u, v), v)$$

so that  $\mathbf{F} = \mathbf{H}_1 \circ \mathbf{H}_2$  is in some neighborhood of  $(0, 0)$ .

*Proof.*

(1) By Definition 10.5,

$$\begin{aligned} \mathbf{G}_1(x, y) &= (e^x \cos y - 1)\mathbf{e}_1 + y\mathbf{e}_2, \\ \mathbf{G}_2(u, v) &= u\mathbf{e}_1 + ((1 + u) \tan v)\mathbf{e}_2 \end{aligned}$$

are primitive in some neighborhood of  $(0, 0)$ .

(2) Show that  $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$ . Given any  $(x, y) \in \mathbb{R}^2$ , we have

$$\begin{aligned}
(\mathbf{G}_2 \circ \mathbf{G}_1)(x, y) &= \mathbf{G}_2(\mathbf{G}_1(x, y)) \\
&= \mathbf{G}_2(e^x \cos y - 1, y) \\
&= (e^x \cos y - 1, (1 + (e^x \cos y - 1)) \tan y) \\
&= (e^x \cos y - 1, e^x \sin y) \\
&= \mathbf{F}(x, y).
\end{aligned}$$

(3) Since

$$\begin{aligned}
J_{\mathbf{G}_1}(x, y) &= \det \begin{bmatrix} e^x \cos y & -e^x \sin y \\ 0 & 1 \end{bmatrix} = e^x \cos y \\
J_{\mathbf{G}_2}(x, y) &= \det \begin{bmatrix} 1 & 0 \\ \tan y & (1 + x) \sec^2 y \end{bmatrix} = (1 + x) \sec^2 y \\
J_{\mathbf{F}}(x, y) &= \det \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix} = e^{2x},
\end{aligned}$$

$$J_{\mathbf{G}_1}(0, 0) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$

$$J_{\mathbf{G}_2}(0, 0) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$

$$J_{\mathbf{F}}(0, 0) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1.$$

(4) Define  $h(u, v) = \sqrt{e^{2u} - v^2} - 1$  on

$$B\left((0, 0); \frac{1}{64}\right) \subseteq \mathbb{R}^2.$$

$h(u, v)$  is well-defined since  $e^{2u} - v^2 > 0$  for all  $(u, v) \in B\left((0, 0); \frac{1}{64}\right)$ .

(5) Given any  $(x, y) \in \mathbb{R}^2$ , we have

$$\begin{aligned}
(\mathbf{H}_1 \circ \mathbf{H}_2)(x, y) &= \mathbf{H}_1(\mathbf{H}_2(x, y)) \\
&= \mathbf{H}_1(x, e^x \sin y) \\
&= (\sqrt{e^{2x} - (e^x \sin y)^2} - 1, e^x \sin y) \\
&= (e^x \cos y - 1, e^x \sin y) \\
&= \mathbf{F}(x, y).
\end{aligned}$$

□

**Exercise 10.5.** Formulate and prove an analogue of Theorem 10.8, in which  $K$  is a compact subset of an arbitrary metric space. (Replace the functions  $\varphi_i$  that occur in the proof of Theorem 10.8 by functions of the type constructed in Exercise 4.22.)

*Proof (Theorem 10.8).*

- (1) (Partitions of unity.) Suppose  $K$  is a compact subset of a metric space  $X$ , and  $\{V_\alpha\}$  is an open cover of  $K$ . Then there exist functions  $\psi_1, \dots, \psi_s \in \mathcal{C}(X)$  such that

- (a)  $0 \leq \psi_i \leq 1$  for  $1 \leq i \leq s$ .
- (b) each  $\psi_i$  has its support in some  $V_\alpha$ , and
- (c)  $\psi_1(x) + \dots + \psi_s(x) = 1$  for every  $x \in K$ .

- (2) It is trivial that some  $V_\alpha = X$  by taking  $s = 1$  and  $\psi_1(x) = 1 \in \mathcal{C}(X)$ . Now we assume that all  $V_\alpha \subsetneq X$ .

- (3) Associate with each  $x \in K$  an index  $\alpha(x)$  so that  $x \in V_{\alpha(x)}$ . Then there are open balls  $B(x)$  and  $W(x)$ , centered at  $x$ , with

$$x \in B(x) \subseteq \overline{B(x)} \subseteq W(x) \subseteq \overline{W(x)} \subseteq V_{\alpha(x)}$$

(Since  $V_{\alpha(x)}$  is open, there exists  $r > 0$  such that  $B(x; r) \subseteq V_{\alpha(x)}$ . Take  $B(x) = B(x; \frac{r}{89})$  and  $W(x) = B(x; \frac{r}{64})$ .)

- (4) Since  $K$  is compact, there are finitely many points  $x_1, \dots, x_s \in K$  such that

$$K \subseteq B(x_1) \cup \dots \cup B(x_s).$$

Note that

- (a)  $\overline{B(x_i)}$  is a nonempty closed set since  $x_i \in B(x_i) \subseteq \overline{B(x_i)}$ .
- (b)  $X - W(x_i) \supseteq X - V_{\alpha(x_i)}$  is a nonempty closed set by the assumption in (2).
- (c)  $\overline{B(x_i)} \cap (X - W(x_i)) \subseteq W(x_i) \cap (X - W(x_i)) = \emptyset$ .

By Exercise 4.22, there is a function

$$\varphi_i(x) = \frac{\rho_{\overline{B(x_i)}}(x)}{\rho_{\overline{B(x_i)}}(x) + \rho_{X - W(x_i)}(x)} \in \mathcal{C}(X)$$

such that  $\varphi_i(x) = 1$  on  $\overline{B(x_i)}$ ,  $\varphi_i(x) = 0$  outside  $W(x_i)$ , and  $0 \leq \varphi_i(x) \leq 1$  on  $X$  for  $1 \leq i \leq s$ .

- (5) Define  $\psi_1 = \varphi_1 \in \mathcal{C}(X)$  and

$$\psi_{i+1} = (1 - \varphi_1) \cdots (1 - \varphi_i) \varphi_{i+1} \in \mathcal{C}(X)$$



for  $1 \leq i \leq s-1$ . Properties (a) and (b) in (1) are clear. Also,

$$\psi_1(x) + \cdots + \psi_s(x) = 1 - (1 - \varphi_1(x)) \cdots (1 - \varphi_s(x))$$

by the construction of  $\psi_i$ . If  $x \in K$ , then  $x \in B(x_i)$  for some  $i$ , hence  $\varphi_i(x) = 1$ , and the product  $(1 - \varphi_1(x)) \cdots (1 - \varphi_s(x)) = 0$ . This proves property (c) in (1).

□

**Exercise 10.6.** *Strengthen the conclusion of Theorem 10.8 by showing that the functions  $\psi_i$  can be made differentiable, and even infinitely differentiable. (Use Exercise 8.1 in the construction of the auxiliary functions  $\psi_i$ .)*

*Proof (Theorem 10.8).*

- (1) It is trivial that some  $V_\alpha = \mathbb{R}^n$  by taking  $s = 1$  and  $\psi_1(\mathbf{x}) = 1 \in \mathcal{C}^\infty(\mathbb{R}^n)$ . Now we assume that all  $V_\alpha \subsetneq \mathbb{R}^n$ .
- (2) Associate with each  $\mathbf{x} \in K$  an index  $\alpha(x)$  so that  $\mathbf{x} \in V_{\alpha(x)}$ . Then there are open  $n$ -cells  $B(\mathbf{x})$  and  $W(\mathbf{x})$  (Definition 10.1), centered at  $\mathbf{x}$ , with

$$\mathbf{x} \in B(\mathbf{x}) \subseteq \overline{B(\mathbf{x})} \subseteq W(\mathbf{x}) \subseteq \overline{W(\mathbf{x})} \subseteq V_{\alpha(\mathbf{x})}$$

(Since  $V_{\alpha(\mathbf{x})}$  is open, there exists  $r > 0$  such that  $B(\mathbf{x}; r) \subseteq V_{\alpha(\mathbf{x})}$ . Take

$$B(\mathbf{x}) = I\left(\mathbf{x}; \frac{r}{89\sqrt{n}}\right), \quad W(\mathbf{x}) = I\left(\mathbf{x}; \frac{r}{64\sqrt{n}}\right)$$

where  $I(\mathbf{p}; r)$  is the open  $n$ -cell centered at  $\mathbf{p} = (p_1, \dots, p_n)$  defined by

$$I(\mathbf{p}; r) = (p_1 - r, p_1 + r) \times \cdots \times (p_n - r, p_n + r) \subseteq \mathbb{R}^n.$$

- (3) Define

$$f(y) = \begin{cases} e^{-\frac{1}{y^2}} & (y > 0), \\ 0 & (y \leq 0). \end{cases}$$

$f(y) \in \mathcal{C}^\infty(\mathbb{R}^1)$  by applying the similar argument in Exercise 8.1.

- (4) Given any  $\mathbf{x} = (x_1, \dots, x_n) \in K$  and construct  $B(\mathbf{x})$  and  $W(\mathbf{x})$  as in (2). Define

$$g_{x_j}(y_j) = \frac{f(y_j)}{f(y_j) + f\left(\frac{r}{64\sqrt{n}} - \frac{r}{89\sqrt{n}} - y_j\right)}$$

for  $1 \leq j \leq n$ .  $g_{x_j}$  is well-defined and  $g_{x_j} \in \mathcal{C}^\infty(\mathbb{R}^1)$ . So

$$g_{x_j}(y_j) = \begin{cases} 0 & \text{if } y_j \leq 0, \\ \text{strictly increasing} & \text{if } 0 \leq y_j \leq \frac{r}{64\sqrt{n}} - \frac{r}{89\sqrt{n}}, \\ 1 & \text{if } y_j \geq \frac{r}{64\sqrt{n}} - \frac{r}{89\sqrt{n}}. \end{cases}$$

Next, define

$$h_{x_j}(y_j) = g_{x_j} \left( y_j - x_j + \frac{r}{64\sqrt{n}} \right) g_{x_j} \left( x_j + \frac{r}{64\sqrt{n}} - y_j \right)$$

for  $1 \leq j \leq n$ .  $h_{x_j} \in \mathcal{C}^\infty(\mathbb{R}^1)$ . So

$$h_{x_j}(y_j) = \begin{cases} 0 & \text{if } y_j \leq x_j - \frac{r}{64\sqrt{n}}, \\ \text{strictly increasing} & \text{if } x_j - \frac{r}{64\sqrt{n}} \leq y_j \leq x_j - \frac{r}{89\sqrt{n}}, \\ 1 & \text{if } x_j - \frac{r}{89\sqrt{n}} \leq y_j \leq x_j + \frac{r}{89\sqrt{n}}, \\ \text{strictly decreasing} & \text{if } x_j + \frac{r}{89\sqrt{n}} \leq y_j \leq x_j + \frac{r}{64\sqrt{n}}, \\ 0 & \text{if } y_j \geq x_j + \frac{r}{64\sqrt{n}}. \end{cases}$$

Finally we define  $\mathbf{h}_{\mathbf{x}} : \mathbb{R}^n \rightarrow \mathbb{R}^1$  by

$$\mathbf{h}_{\mathbf{x}}(\mathbf{y}) = \prod_{j=1}^n h_{x_j}(y_j)$$

where  $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ . Hence,  $\mathbf{h}_{\mathbf{x}} \in \mathcal{C}^\infty(\mathbb{R}^n)$  (Theorem 9.21). Also,  $\mathbf{h}_{\mathbf{x}}(\mathbf{y}) = 1$  on  $B(\mathbf{x})$ ,  $\mathbf{h}_{\mathbf{x}}(\mathbf{y}) = 0$  outside  $W(\mathbf{x})$ , and  $0 \leq \mathbf{h}_{\mathbf{x}}(\mathbf{y}) \leq 1$ .

- (5) Since  $K$  is compact, there are finitely many points  $\mathbf{x}_1, \dots, \mathbf{x}_s \in K$  such that

$$K \subseteq B(\mathbf{x}_1) \cup \dots \cup B(\mathbf{x}_s).$$

Take

$$\varphi_i(\mathbf{x}) = \mathbf{h}_{\mathbf{x}_i}(\mathbf{x}) \in \mathcal{C}^\infty(\mathbb{R}^n)$$

for  $1 \leq i \leq s$ .

- (6) The rest are the same as the proof of Theorem 10.8 or Exercise 10.5.

□

### Exercise 10.7.

- (a) Show that the simplex  $Q^k$  is the smallest convex subset of  $\mathbb{R}^k$  such that contains  $\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_k$ .
- (b) Show that affine mappings take convex sets to convex sets.

*Proof of (a).*

- (1) Show that  $Q^k$  contains  $\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_k$ . Recall

$$Q^k = \{(x_1, \dots, x_k) \in \mathbb{R}^k : x_1 + \dots + x_k \leq 1 \text{ and } x_1, \dots, x_k \geq 0\}$$

(Example 10.14). Hence  $\mathbf{0} = (0, \dots, 0) \in Q^k$  and

$$\mathbf{e}_i = (0, \dots, \underbrace{1}_{i\text{th coordinate}}, \dots, 0) \in Q^k.$$

- (2) Show that  $Q^k$  is a convex subset of  $\mathbb{R}^k$ . Given any  $\mathbf{x} = (x_1, \dots, x_k) \in Q^k$ ,  $\mathbf{y} = (y_1, \dots, y_k) \in Q^k$  and  $0 < \lambda < 1$ . Hence

$$\lambda \mathbf{x} + (1 - \lambda) \mathbf{y} = (\lambda x_1 + (1 - \lambda) y_1, \dots, \lambda x_k + (1 - \lambda) y_k) \in Q^k$$

since each  $\lambda x_i + (1 - \lambda) y_i \geq 0$  and

$$\sum_{i=1}^k (\lambda x_i + (1 - \lambda) y_i) = \lambda \sum_{i=1}^k x_i + (1 - \lambda) \sum_{i=1}^k y_i \leq \lambda + (1 - \lambda) = 1.$$

- (3) Given any convex set  $E \subseteq \mathbb{R}^k$  containing  $\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_k$ . Show that  $E \supseteq Q^k$ .

- (a) Induction on  $k$ . Base case:  $k = 1$ . Given any  $\mathbf{x} = (x_1) \in Q^1$ . We have  $0 \leq x_1 \leq 1$  by the definition of  $Q^1$ . So that  $\mathbf{x} = x_1 \mathbf{e}_1 + (1 - x_1) \mathbf{0} \in E$  since  $\mathbf{0}, \mathbf{e}_1 \in E$  and  $E$  is convex.
- (b) Inductive step: suppose the statement holds for  $k = n$ . Given any  $\mathbf{x} = (x_1, \dots, x_n, x_{n+1}) \in Q^{n+1}$ . If  $x_{n+1} = 1$ , then  $x_1 = \dots = x_n = 0$  by the definition of  $Q^{n+1}$ . So  $\mathbf{x} = \mathbf{e}_{n+1} \in E$  by the assumption of  $E$ . If  $0 \leq x_{n+1} < 1$ , then  $x_1 + \dots + x_n \leq 1 - x_{n+1}$  or

$$\frac{x_1}{1 - x_{n+1}} + \dots + \frac{x_n}{1 - x_{n+1}} \leq 1.$$

So the point

$$\left( \frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}} \right) \in Q^n,$$

or

$$\left( \frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}}, 0 \right), \text{ say } \hat{\mathbf{x}}, \in E$$

by the induction hypothesis. Note that  $\mathbf{e}_{n+1} \in E$ . Hence

$$\mathbf{x} = x_{n+1} \mathbf{e}_{n+1} + (1 - x_{n+1}) \hat{\mathbf{x}} \in E$$

by the convexity of  $E$ .

- (c) Conclusion: Since both the base case and the inductive step have been proved as true, by mathematical induction the statement holds.

□

*Proof of (b).*

- (1) Let  $\mathbf{f}$  be an affine mapping that carries a vector space  $X$  into a vector space  $Y$  such that

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{0}) + A\mathbf{x}$$

for some  $A \in L(X, Y)$ .

- (2) Given any convex subset  $C$  of  $X$ . To show that  $\mathbf{f}(C)$  is convex, it suffices to show that

$$\lambda \mathbf{y}_1 + (1 - \lambda) \mathbf{y}_2 \in \mathbf{f}(C)$$

for any  $\mathbf{y}_1, \mathbf{y}_2 \in \mathbf{f}(C)$  and  $0 < \lambda < 1$ . Write  $\mathbf{y}_1 = \mathbf{f}(\mathbf{x}_1)$ ,  $\mathbf{y}_2 = \mathbf{f}(\mathbf{x}_2)$  for some  $\mathbf{x}_1, \mathbf{x}_2 \in C$ . Note that  $\lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2 \in C$  by the convexity of  $C$ . Hence

$$\begin{aligned} & \mathbf{f}(\lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2) \\ &= \mathbf{f}(\mathbf{0}) + A(\lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2) \\ &= \mathbf{f}(\mathbf{0}) + \lambda A \mathbf{x}_1 + (1 - \lambda) A \mathbf{x}_2 & (A \in L(X, Y)) \\ &= \lambda(\mathbf{f}(\mathbf{0}) + A \mathbf{x}_1) + (1 - \lambda)(\mathbf{f}(\mathbf{0}) + A \mathbf{x}_2) \\ &= \lambda \mathbf{f}(\mathbf{x}_1) + (1 - \lambda) \mathbf{f}(\mathbf{x}_2) \\ &= \lambda \mathbf{y}_1 + (1 - \lambda) \mathbf{y}_2 \in \mathbf{f}(C). \end{aligned}$$

□

**Exercise 10.8.** Let  $H$  be the parallelogram in  $\mathbb{R}^2$  whose vertices are  $(1, 1)$ ,  $(3, 2)$ ,  $(4, 5)$ ,  $(2, 4)$ . Find the affine map  $T$  which sends  $(0, 0)$  to  $(1, 1)$ ,  $(1, 0)$  to  $(3, 2)$ ,  $(1, 1)$  to  $(4, 5)$ ,  $(0, 1)$  to  $(2, 4)$ . Show that  $J_T = 5$ . Use  $T$  to convert the integral

$$\alpha = \int_H e^{x-y} dx dy$$

to an integral over  $I^2$  and thus compute  $\alpha$ .

*Proof.*

- (1) By Affine simplexes 10.26,

$$T(\mathbf{x}) = T(\mathbf{0}) + A\mathbf{x},$$

where  $A \in L(\mathbb{R}^2, \mathbb{R}^2)$ , say  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Note that  $T : \begin{bmatrix} 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . Thus

$$T : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 + ax + by \\ 1 + cx + dy \end{bmatrix}.$$

- (2) By  $T : (1, 0) \mapsto (3, 2)$  and  $T : (0, 1) \mapsto (2, 4)$ , we can solve  $A$  as

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

It is easy to verify such

$$T : \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{x}} \mapsto \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{T(\mathbf{0})} + \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{x}} = \begin{bmatrix} 1 + 2x + y \\ 1 + x + 3y \end{bmatrix}$$

satisfying our requirement.

(3)

$$J_T = \det \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = 5.$$

(4) By Example 10.4 and Theorem 10.9, we have

$$\begin{aligned} \int_H e^{x-y} dx dy &= \int_{I^2} e^{(1+2u+v)-(1+u+3v)} |J_T| du dv \\ &= 5 \int_{I^2} e^{u-2v} du dv \\ &= 5 \left\{ \int_0^1 e^u du \right\} \left\{ \int_0^1 e^{-2v} dv \right\} \quad (\text{Theorem 10.2}) \\ &= \frac{5}{2} (e-1)(1-e^{-2}). \end{aligned}$$

□

**Exercise 10.9.** Define  $(x, y) = T(r, \theta)$  on the rectangle

$$0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi$$

by the equations

$$x = r \cos \theta, \quad y = r \sin \theta.$$

Show that  $T$  maps this rectangle onto the closed disc  $D$  with center at  $(0, 0)$  and radius  $a$ , that  $T$  is one-to-one in the interior of the rectangle, and that  $J_T(r, \theta) = r$ . If  $f \in \mathcal{C}(D)$ , prove the formula for integration in polar coordinates:

$$\int_D f(x, y) dx dy = \int_0^a \int_0^{2\pi} f(T(r, \theta)) r dr d\theta.$$

(Hint: Let  $D_0$  be the interior of  $D$ , minus the interval from  $(0, 0)$  to  $(0, a)$ . As it stands, Theorem 10.9 applies to continuous functions  $f$  whose support lies in  $D_0$ . To remove this restriction, proceed as in Example 10.4.)

*Proof.*

(1)

(2)

□

**Exercise 10.10.** Let  $a \rightarrow \infty$  in Exercise 10.9 and prove that

$$\int_{\mathbb{R}^2} f(x, y) dx dy = \int_0^\infty \int_0^{2\pi} f(T(r, \theta)) r dr d\theta,$$

for continuous functions  $f$  that decrease sufficiently rapidly as  $|x| + |y| \rightarrow \infty$ .  
(Find a more precise formulation.) Apply this to

$$f(x, y) = \exp(-x^2 - y^2)$$

to derive formula

$$\int_{-\infty}^{\infty} e^{-s^2} ds = \sqrt{\pi}.$$

*Proof.*

(1)

(2)

□

**Exercise 10.11.** Define  $(u, v) = T(s, t)$  on the strip

$$0 < s < \infty, \quad 0 < t < 1$$

by setting  $u = s - st$ ,  $v = st$ . Show that  $T$  is a 1-1 mapping of the strip onto the positive quadrant  $Q$  in  $\mathbb{R}^2$ . Show that  $J_T(s, t) = s$ . For  $x > 0$ ,  $y > 0$ , integrate

$$u^{x-1} e^{-u} v^{y-1} e^{-v}$$

over  $Q$ , use Theorem 10.9 to convert the integral to one over the strip, and derive

$$\int_0^1 t^{x-1} (1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

in this way. (For this application, Theorem 10.9 has to be extended so as to cover certain improper integrals. Provide this extension.)

*Proof.*

(1)

(2)

□

**Exercise 10.12.** Let  $I^k$  be the set of all  $\mathbf{u} = (u_1, \dots, u_k) \in \mathbb{R}^k$  with  $0 \leq u_i \leq 1$  for all  $i$ ; let  $Q^k$  be the set of all  $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}^k$  with  $x_i \geq 0$ ,  $\sum x_i \leq 1$ . ( $I^k$  is the unit cube;  $Q^k$  is the standard simplex in  $\mathbb{R}^k$ .) Define  $\mathbf{x} = T(\mathbf{u})$  by

$$x_1 = u_1$$

$$x_2 = (1 - u_1)u_2$$

$$\dots$$

$$x_k = (1 - u_1) \cdots (1 - u_{k-1})u_k.$$

Show that

$$\sum_{i=1}^k x_i = 1 - \prod_{i=1}^k (1 - u_i).$$

Show that  $T$  maps  $I^k$  onto  $Q^k$ , that  $T$  is 1-1 in the interior of  $I^k$ , and that its inverse  $S$  is defined in the interior of  $Q^k$  by  $u_1 = x_1$  and

$$u_i = \frac{x_i}{1 - x_1 - \cdots - x_{i-1}}$$

for  $i = 2, \dots, k$ . Show that

$$J_T(\mathbf{u}) = (1 - u_1)^{k-1} (1 - u_2)^{k-2} \cdots (1 - u_{k-1}),$$

and

$$J_S(\mathbf{x}) = [(1 - x_1)(1 - x_1 - x_2) \cdots (1 - x_1 - \cdots - x_{k-1})]^{-1}.$$

*Proof.*

(1) Show that

$$\sum_{i=1}^m x_i = 1 - \prod_{i=1}^m (1 - u_i)$$

for all  $1 \leq m \leq k$ . Induction on  $m$ . Base case:  $x_1 = 1 - (1 - u_1)$ . Inductive step: Suppose the case  $m = h$  is true. Consider the case  $m = h + 1$ :

$$\begin{aligned} \sum_{i=1}^{h+1} x_i &= \left( \sum_{i=1}^h x_i \right) + x_{h+1} \\ &= 1 - \prod_{i=1}^h (1 - u_i) + x_{h+1} && \text{(Induction hypothesis)} \\ &= 1 - \prod_{i=1}^h (1 - u_i) + u_{h+1} \prod_{i=1}^h (1 - u_i) && \text{(Definition of } x_{h+1}) \\ &= 1 - (1 - u_{h+1}) \prod_{i=1}^h (1 - u_i) \\ &= 1 - \prod_{i=1}^{h+1} (1 - u_i). \end{aligned}$$

Since both the base case and the inductive step have been proved as true, by mathematical induction the statement is established.

- (2) Show that  $T$  maps  $I^k$  onto  $Q^k$ . Given any  $\mathbf{x} = (x_1, \dots, x_k) \in Q^k$ . It is equivalent to solve  $\mathbf{u} = (u_1, \dots, u_k)$  from

$$\begin{aligned} x_1 &= u_1 \\ x_2 &= (1 - u_1)u_2 \\ &\dots \\ x_k &= (1 - u_1) \cdots (1 - u_{k-1})u_k \end{aligned}$$

in terms of  $\mathbf{x} = (x_1, \dots, x_k)$ . It is clear that  $u_1 = x_1$  and

$$u_i = \begin{cases} x_i(1 - x_1 - \cdots - x_{i-1})^{-1} & \text{if } x_1 + \cdots + x_{i-1} \neq 1, \\ 0 & \text{if } x_1 + \cdots + x_{i-1} = 1. \end{cases}$$

for  $i = 2, \dots, k$ . (If  $x_1 + \cdots + x_{i-1} \neq 1$ , by (1) we have

$$\prod_{j=1}^{i-1} (1 - u_j) = 1 - \sum_{j=1}^{i-1} x_j \neq 0$$

and thus

$$u_i = x_i \left\{ \prod_{j=1}^{i-1} (1 - u_j) \right\}^{-1} = x_i (1 - x_1 - \cdots - x_{i-1})^{-1}.$$

If  $x_1 + \cdots + x_{i-1} = 1$ , then  $x_i = \cdots = x_k = 0$ . We may take  $u_i = 0$  to set the expression  $x_i = (1 - u_1) \cdots (1 - u_{i-1})u_i$  to zero.) Note that the solution  $\mathbf{u} \in I^k$  is well-defined by construction, or  $T(I^k) = Q^k$ .

- (3) Show that  $T$  is 1-1 in the interior of  $I^k$ . Suppose  $T(\mathbf{u}) = T(\mathbf{v}) = \mathbf{x}$  with  $\mathbf{u}, \mathbf{v} \in \text{int}(I^k)$ . Then we consider the following equation:

$$\begin{aligned} x_1 &= u_1 = v_1 \\ x_2 &= (1 - u_1)u_2 = (1 - v_1)v_2 \\ &\dots \\ x_k &= (1 - u_1) \cdots (1 - u_{k-1})u_k = (1 - v_1) \cdots (1 - v_{k-1})v_k. \end{aligned}$$

By (1),

$$\mathbf{x} \in \text{int}(Q^k) = \left\{ (x_1, \dots, x_k) \in \mathbb{R}^k : x_i > 0, \sum x_i < 1 \right\}.$$

Hence,

$$\begin{aligned} u_1 &= v_1 = x_1 \\ u_2 &= v_1 = x_2(1 - x_1)^{-1} \\ &\dots \\ u_k &= v_k = x_k(1 - x_1 - \cdots - x_{k-1})^{-1}. \end{aligned}$$

Here all  $(1 - x_1)^{-1}, \dots, (1 - x_1 - \cdots - x_i)^{-1}$  are well-defined since  $\mathbf{x} \in \text{int}(Q^k)$ . Therefore,  $T$  is injective on  $\text{int}(I^k)$ .



- (4) By (2)(3),  $T$  maps  $\text{int}(I^k)$  onto  $\text{int}(Q^k)$ . That is, given any  $\mathbf{x} = (x_1, \dots, x_k) \in \text{int}(Q^k)$ , we can pick

$$\begin{aligned} u_1 &= x_1 \\ u_i &= x_i(1 - x_1 - \dots - x_{i-1})^{-1} \quad (i = 2, \dots, k) \end{aligned}$$

such that  $\mathbf{u} \in \text{int}(I^k)$  and  $T(\mathbf{u}) = \mathbf{x}$ .

- (5) Note that  $T(\mathbf{u}) = (u_1, (1 - u_1)u_2, \dots, (1 - u_1) \cdots (1 - u_{k-1})u_k)$  on  $\text{int}(I^k)$ . So

$$T'(\mathbf{u}) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ * & (1 - u_1) & 0 & \cdots & 0 \\ * & * & \prod_{i=1}^2 (1 - u_i) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & \prod_{i=1}^{k-1} (1 - u_i) \end{bmatrix}$$

is a lower triangular matrix. Hence,

$$\begin{aligned} J_T(\mathbf{u}) &= \det T'(\mathbf{u}) \\ &= 1 \cdot (1 - u_1) \cdot \prod_{i=1}^2 (1 - u_i) \cdots \prod_{i=1}^{k-1} (1 - u_i) \\ &= \prod_{i=1}^{k-1} (1 - u_i)^{k-i}. \end{aligned}$$

- (6) Similar to (5).  $S(\mathbf{x}) = (x_1, x_2(1 - x_1)^{-1}, \dots, x_k(1 - x_1 - \dots - x_{k-1})^{-1})$  on  $\text{int}(Q^k)$ . So

$$S'(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ * & (1 - x_1)^{-1} & 0 & \cdots & 0 \\ * & * & (1 - x_1 - x_2)^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & (1 - x_1 - \dots - x_{k-1})^{-1} \end{bmatrix}$$

is a lower triangular matrix. Hence,

$$\begin{aligned} J_S(\mathbf{x}) &= \det S'(\mathbf{x}) \\ &= 1 \cdot (1 - x_1)^{-1} \cdot (1 - x_1 - x_2)^{-1} \cdots (1 - x_1 - \dots - x_{k-1})^{-1} \\ &= [(1 - x_1)(1 - x_1 - x_2) \cdots (1 - x_1 - \dots - x_{k-1})]^{-1}. \end{aligned}$$

□

**Exercise 10.13.** Let  $r_1, \dots, r_k$  be nonnegative integers, and prove that

$$\int_{Q^k} x_1^{r_1} \cdots x_k^{r_k} d\mathbf{x} = \frac{r_1! \cdots r_k!}{(k + r_1 + \cdots + r_k)!}$$

(Hint: Use Exercise 10.12, Theorems 10.9 and 8.20.) Note that the special case  $r_1 = \cdots = r_k = 0$  shows that the volume of  $Q^k$  is  $\frac{1}{k!}$ .

*Proof.*

(1) Define  $T : I^k$  onto  $Q^k$  as in Exercise 10.12, and  $f : Q^k \rightarrow \mathbb{R}^1$  by

$$f(\mathbf{x}) = f(x_1, \dots, x_k) = x_1^{r_1} \cdots x_k^{r_k} = \prod_{i=1}^k x_i^{r_i}.$$

(2) By Exercise 10.12, Example 10.4 and Theorems 10.9, we have

$$\begin{aligned} \int_{Q^k} x_1^{r_1} \cdots x_k^{r_k} d\mathbf{x} &= \int_{Q^k} f(\mathbf{x}) d\mathbf{x} \\ &= \int_{I^k} f(T(\mathbf{u})) |J_T(\mathbf{u})| d\mathbf{u} \\ &= \int_{I^k} \prod_{i=1}^k \left( u_i \prod_{j=1}^{i-1} (1 - u_j) \right)^{r_i} \prod_{i=1}^k (1 - u_i)^{k-i} d\mathbf{u} \\ &= \int_{I^k} \prod_{i=1}^k u_i^{r_i} (1 - u_i)^{k-i+\sum_{j=i+1}^k r_j} d\mathbf{u} \\ &= \prod_{i=1}^k \int_0^1 u_i^{r_i} (1 - u_i)^{k-i+\sum_{j=i+1}^k r_j} du_i && \text{(Theorem 10.2)} \\ &= \prod_{i=1}^k \frac{r_i! \left( k - i + \sum_{j=i+1}^k r_j \right)!}{\left( k - i + 1 + \sum_{j=i}^k r_j \right)!} && \text{(Theorem 8.20)} \\ &= \frac{r_1! \cdots r_k!}{(k + r_1 + \cdots + r_k)!}. \end{aligned}$$

□

**Exercise 10.14 (Levi-Civita symbol).** Prove  $\varepsilon(j_1, \dots, j_k) = s(j_1, \dots, j_k)$ , where

$$s(j_1, \dots, j_k) = \prod_{p < q} \text{sgn}(j_q - j_p).$$

It is usually to define the Levi-Civita symbol by

$$\varepsilon(j_1, \dots, j_k) = \begin{cases} 1 & \text{if } (j_1, \dots, j_k) \text{ is an even permutation of } J, \\ -1 & \text{if } (j_1, \dots, j_k) \text{ is an odd permutation of } J, \\ 0 & \text{otherwise} \end{cases}$$

(Basic  $k$ -forms 10.14). Thus, it is the sign of the permutation in the case of a permutation, and zero otherwise. So  $\varepsilon(j_1, \dots, j_k)$  is equivalent to an explicit expression  $s(j_1, \dots, j_k) = \prod_{p < q} \text{sgn}(j_q - j_p)$ .

*Proof.*

- (1) Induction on  $k$ . Base case: Show that  $\varepsilon(j_1, j_2) = s(j_1, j_2)$ . Since

$$\varepsilon(j_1, j_2) = \begin{cases} 1 & \text{if } j_1 < j_2 \\ -1 & \text{if } j_1 > j_2, \end{cases}$$

$$\varepsilon(j_1, j_2) = \text{sgn}(j_2 - j_1) = s(j_1, j_2).$$

- (2) Inductive step: Show that for any  $s \geq 2$ , if  $\varepsilon(j_1, \dots, j_s) = s(j_1, \dots, j_s)$  holds, then  $\varepsilon(j_1, \dots, j_{s+1}) = s(j_1, \dots, j_{s+1})$  also holds.

$$\begin{aligned} \varepsilon(j_1, \dots, j_{s+1}) &= \varepsilon(j_1, \dots, j_s) \prod_{\substack{1 \leq p \leq s \\ q=s+1}} \text{sgn}(j_q - j_p) \\ &= s(j_1, \dots, j_s) \prod_{\substack{1 \leq p \leq s \\ q=s+1}} \text{sgn}(j_q - j_p) \\ &= \prod_{1 \leq p < q \leq s} \text{sgn}(j_q - j_p) \prod_{\substack{1 \leq p \leq s \\ q=s+1}} \text{sgn}(j_q - j_p) \\ &= \prod_{1 \leq p < q \leq s+1} \text{sgn}(j_q - j_p) \\ &= s(j_1, \dots, j_{s+1}). \end{aligned}$$

- (3) Conclusion: Since both the base case and the inductive step have been proved as true, by mathematical induction the statement holds for every integer  $k \geq 2$ .

□

**Exercise 10.15.** If  $\omega$  and  $\lambda$  are  $k$ - and  $m$ -forms, respectively, prove that

$$\omega \wedge \lambda = (-1)^{km} \lambda \wedge \omega.$$

*Proof.*

(1) Write

$$\omega = \sum_I b_I(\mathbf{x}) dx_I, \quad \lambda = \sum_J c_J(\mathbf{x}) dx_J$$

in the standard presentations, where  $I$  and  $J$  range over all increasing  $k$ -indices and over all increasing  $m$ -indices taken from the set  $\{1, \dots, n\}$ .

(2) Show that  $dx_I \wedge dx_J = (-1)^{km} dx_J \wedge dx_I$ .

$$\begin{aligned} dx_I \wedge dx_J &= dx_{i_1} \wedge \dots \wedge dx_{i_k} \wedge dx_J \\ &= (-1)^m dx_{i_1} \wedge \dots \wedge dx_{i_{k-1}} \wedge dx_J \wedge dx_{i_k} \\ &= (-1)^{2m} dx_{i_1} \wedge \dots \wedge dx_{i_{k-2}} \wedge dx_J \wedge dx_{i_{k-1}} \wedge dx_{i_k} \\ &\dots \\ &= (-1)^{km} dx_J \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k} \\ &= (-1)^{km} dx_J \wedge dx_I. \end{aligned}$$

(3)

$$\begin{aligned} \omega \wedge \lambda &= \sum_{I,J} b_I(\mathbf{x}) c_J(\mathbf{x}) dx_I \wedge dx_J \\ &= (-1)^{km} \sum_{J,I} c_J(\mathbf{x}) b_I(\mathbf{x}) dx_J \wedge dx_I \\ &= (-1)^{km} \lambda \wedge \omega. \end{aligned}$$

□

**Exercise 10.16.** If  $k \geq 2$  and  $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_k]$  is an oriented affine  $k$ -simplex, prove that  $\partial^2 \sigma = 0$ , directly from the definition of the boundary operator  $\partial$ . Deduce from this that  $\partial^2 \Psi = 0$  for every chain  $\Psi$ . (Hint: For orientation, do it first for  $k = 2$ ,  $k = 3$ . In general, if  $i < j$ , let  $\sigma_{ij}$  be the  $(k-2)$ -simplex obtained by deleting  $\mathbf{p}_i$  and  $\mathbf{p}_j$  from  $\sigma$ . Show that each  $\sigma_{ij}$  occurs twice in  $\partial^2 \sigma$ , with opposite sign.)

*Proof (Brute-force).*

(1) Write the boundary of the oriented affine  $k$ -simplex  $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_k]$  as

$$\partial \sigma = \sum_{i=0}^k (-1)^i [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k]$$

where where the oriented  $(k-1)$ -simplex  $[\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k]$  is obtained by deleting  $\sigma$ 's  $i$ -th vertex (Boundaries 10.29).

(2)

$$\begin{aligned}
\partial^2 \sigma &= \partial \left( \sum_i (-1)^i [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k] \right) \\
&= \sum_i (-1)^i \partial [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k] \\
&= \sum_{j < i} (-1)^i (-1)^j [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_j, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k] \\
&\quad + \sum_{j > i} (-1)^i (-1)^{j-1} [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \widehat{\mathbf{p}}_j, \dots, \mathbf{p}_k] \\
&= \sum_{j < i} (-1)^{i+j} [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_j, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k] \\
&\quad - \sum_{j > i} (-1)^{i+j} [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \widehat{\mathbf{p}}_j, \dots, \mathbf{p}_k].
\end{aligned}$$

The latter two summations cancel since after switching  $i$  and  $j$  in the second sum. Therefore  $\partial^2 \sigma = 0$ .

- (3) The boundary of a chain is the linear combination of boundaries of the simplices in the chain. Write  $\Psi = \sum_{i=1}^r \sigma_i$ , where  $\sigma_i$  is an oriented affine simplex. Then

$$\partial^2 \Psi = \partial \left( \partial \sum \sigma_i \right) = \partial \left( \sum \partial \sigma_i \right) = \sum \partial^2 \sigma_i = \sum 0 = 0$$

for any affine chain  $\Psi$ .

□

**Exercise 10.17.** Put  $J^2 = \tau_1 + \tau_2$ , where

$$\tau_1 = [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2], \quad \tau_2 = -[\mathbf{0}, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1].$$

Explain why it is reasonable to call  $J^2$  the positively oriented unit square in  $\mathbb{R}^2$ . Show that  $\partial J^2$  is the sum of 4 oriented affine 1-simplexes. Find these. What is  $\partial(\tau_1 - \tau_2)$ ?

*Proof.*

- (1) Note that the unit square  $I^2 \in \mathbb{R}^2$  is the union of  $\tau_1(Q^2)$  and  $\tau_2(Q^2)$ , where

$$\begin{aligned}
\tau_1(\mathbf{u}) &= ([\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2])(\mathbf{u}) \\
&= \mathbf{0} + \alpha_1 \mathbf{e}_1 + \alpha_2 (\mathbf{e}_1 + \mathbf{e}_2) \\
&= \mathbf{0} + (\alpha_1 + \alpha_2) \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 \\
&= \mathbf{0} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{u}
\end{aligned}$$

and

$$\begin{aligned}
\tau_2(\mathbf{u}) &= (-[\mathbf{0}, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1])(\mathbf{u}) \\
&= ([\mathbf{0}, \mathbf{e}_2 + \mathbf{e}_1, \mathbf{e}_2])(\mathbf{u}) \\
&= \mathbf{0} + \alpha_1(\mathbf{e}_1 + \mathbf{e}_2) + \alpha_2\mathbf{e}_2 \\
&= \mathbf{0} + \alpha_1\mathbf{e}_1 + (\alpha_1 + \alpha_2)\mathbf{e}_2 \\
&= \mathbf{0} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{u}
\end{aligned}$$

where  $\mathbf{u} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 \in \mathbb{R}^2$  (as in Equation (78)). Both  $\tau_1$  and  $\tau_2$  have Jacobian  $1 > 0$ , or positively oriented (Affine simplexes 10.26). So it is reasonable to call  $J^2$  the positively oriented unit square in  $\mathbb{R}^2$ .

(2)

$$\begin{aligned}
\partial\tau_1 &= [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] - [\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{0}, \mathbf{e}_1], \\
\partial\tau_2 &= [\mathbf{e}_2 + \mathbf{e}_1, \mathbf{e}_2] - [\mathbf{0}, \mathbf{e}_2] + [\mathbf{0}, \mathbf{e}_2 + \mathbf{e}_1] \\
&= [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2] + [\mathbf{e}_2, \mathbf{0}] + [\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2].
\end{aligned}$$

(3) By (2),

$$\partial J^2 = \partial\tau_1 + \partial\tau_2 = [\mathbf{0}, \mathbf{e}_1] + [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2] + [\mathbf{e}_2, \mathbf{0}],$$

which is the positively oriented boundary of  $I^2$ .

(4) By (2),

$$\begin{aligned}
\partial(\tau_1 - \tau_2) &= \partial\tau_1 - \partial\tau_2 \\
&= [\mathbf{0}, \mathbf{e}_1] + [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{0}] \\
&\quad + [\mathbf{0}, \mathbf{e}_2] + [\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{0}].
\end{aligned}$$

□

**Exercise 10.18.** Consider the oriented affine 3-simplex

$$\sigma_1 = [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]$$

in  $\mathbb{R}^3$ . Show that  $\sigma_1$  (regarded as a linear transformation) has determinant 1. Thus  $\sigma_1$  is positively oriented.

Let  $\sigma_2, \dots, \sigma_6$  be five other oriented 3-simplexes, obtained as follows: There are five permutations  $(i_1, i_2, i_3)$  of  $(1, 2, 3)$ , distinct from  $(1, 2, 3)$ . Associate with each  $(i_1, i_2, i_3)$  the simplex

$$s(i_1, i_2, i_3)[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}]$$

where  $s$  is the sign that occurs in the definition of the determinant. (This is how  $\tau_2$  was obtained from  $\tau_1$  in Exercise 10.17.) Show that  $\sigma_2, \dots, \sigma_6$  are positively oriented.

Put  $J^3 = \sigma_1 + \dots + \sigma_6$ . Then  $J^3$  may be called the positively oriented unit cube in  $\mathbb{R}^3$ . Show that  $\partial J^3$  is the sum of 12 oriented affine 2-simplexes. (These 12 triangles cover the surface of the unit cube  $I^3$ .)

Show that  $\mathbf{x} = (x_1, x_2, x_3)$  is in the range of  $\sigma_1$  if and only if  $0 \leq x_3 \leq x_2 \leq x_1 \leq 1$ .

Show that the range of  $\sigma_1, \dots, \sigma_6$  have disjoint interiors, and that their union covers  $I^3$ . (Compared with Exercise 10.13; note that  $3! = 6$ .)

*Proof.*

- (1) Show that  $\sigma_1$  (regarded as a linear transformation) has determinant 1. Given any  $\mathbf{u} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3 \in \mathbb{R}^3$ , we have

$$\begin{aligned} \sigma_1(\mathbf{u}) &= ([\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3])(\mathbf{u}) \\ &= \mathbf{0} + \alpha_1 \mathbf{e}_1 + \alpha_2(\mathbf{e}_1 + \mathbf{e}_2) + \alpha_3(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) \\ &= \mathbf{0} + (\alpha_1 + \alpha_2 + \alpha_3)\mathbf{e}_1 + (\alpha_2 + \alpha_3)\mathbf{e}_2 + \alpha_3 \mathbf{e}_3 \\ &= \mathbf{0} + \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{say } A} \mathbf{u}. \end{aligned}$$

So

$$\det(A) = \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = 1.$$

- (2) Show that  $\sigma_2, \dots, \sigma_6$  are positively oriented. Define the permutation matrix  $P_{(i_1, i_2, i_3)}$  corresponding to a permutation  $(i_1, i_2, i_3)$  of  $(1, 2, 3)$  by

$$P_{(i_1, i_2, i_3)} = [\mathbf{e}_{i_1} \quad \mathbf{e}_{i_2} \quad \mathbf{e}_{i_3}].$$

For example,

$$P_{(2,3,1)} = [\mathbf{e}_2 \quad \mathbf{e}_3 \quad \mathbf{e}_1] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Note that the sign  $s(i_1, i_2, i_3)$  of the permutation  $(i_1, i_2, i_3)$  is exactly the same as the determinant of the permutation matrix  $P_{(i_1, i_2, i_3)}$ . Define a

permutation  $(j_1, j_2, 3)$  of  $(1, 2, 3)$  (for swapping the first and the second coordinates of  $\mathbf{u}$ ) by

$$(j_1, j_2, 3) = \begin{cases} (1, 2, 3) & \text{if } s(i_1, i_2, i_3) = 1, \\ (2, 1, 3) & \text{if } s(i_1, i_2, i_3) = -1. \end{cases}$$

Write

$$\sigma_{(i_1, i_2, i_3)} = s(i_1, i_2, i_3)[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}].$$

(So that  $\sigma_1 = \sigma_{(1,2,3)}$ .) Hence,

$$\begin{aligned} & \sigma_{(i_1, i_2, i_3)}(\mathbf{u}) \\ &= \mathbf{0} + \alpha_{j_1} \mathbf{e}_{i_1} + \alpha_{j_2}(\mathbf{e}_{i_1} + \mathbf{e}_{i_2}) + \alpha_3(\mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}) \\ &= \mathbf{0} + (\alpha_{j_1} + \alpha_{j_2} + \alpha_3) \mathbf{e}_{i_1} + (\alpha_{j_2} + \alpha_3) \mathbf{e}_{i_2} + \alpha_3 \mathbf{e}_{i_3} \\ &= \mathbf{0} + P_{(i_1, i_2, i_3)} A P_{(j_1, j_2, 3)} \mathbf{u} \end{aligned}$$

where  $\mathbf{u} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3 \in \mathbb{R}^3$ . For example,

$$P_{(2,3,1)} A P_{(1,2,3)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

So

$$\begin{aligned} \det(P_{(i_1, i_2, i_3)} A P_{(j_1, j_2, 3)}) &= \det(P_{(i_1, i_2, i_3)}) \det(A) \det(P_{(j_1, j_2, 3)}) \\ &= s(i_1, i_2, i_3) \cdot 1 \cdot s(i_1, i_2, i_3) \\ &= 1. \end{aligned}$$

(3) Show that  $\partial J^3$  is the sum of 12 oriented affine 2-simplexes. Note that

$$\begin{aligned} \sum_{(i_1, i_2, i_3)} \sigma_{(i_1, i_2, i_3)} &= \sum_{\substack{(i_1, i_2, i_3) \\ i_1 > i_2}} \sigma_{(i_1, i_2, i_3)} + \sum_{\substack{(i_1, i_2, i_3) \\ i_1 < i_2}} \sigma_{(i_1, i_2, i_3)} \\ &= \sum_{\substack{(i_1, i_2, i_3) \\ i_1 > i_2}} s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\ &\quad + \sum_{\substack{(i_1, i_2, i_3) \\ i_2 > i_1}} -s(i_2, i_1, i_3) [\mathbf{0}, \mathbf{e}_{i_2} + \mathbf{e}_{i_1}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\ &= \mathbf{0} \end{aligned}$$



and

$$\begin{aligned}
\sum_{(i_1, i_2, i_3)} \sigma_{(i_1, i_2, i_3)} &= \sum_{\substack{(i_1, i_2, i_3) \\ i_2 > i_3}} \sigma_{(i_1, i_2, i_3)} + \sum_{\substack{(i_1, i_2, i_3) \\ i_2 < i_3}} \sigma_{(i_1, i_2, i_3)} \\
&= \sum_{\substack{(i_1, i_2, i_3) \\ i_2 > i_3}} s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\
&\quad + \sum_{\substack{(i_1, i_2, i_3) \\ i_3 > i_2}} -s(i_1, i_3, i_2) [\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\
&= \mathbf{0}.
\end{aligned}$$

So

$$\begin{aligned}
\partial J^3 &= \sum_{(i_1, i_2, i_3)} \partial \sigma_{(i_1, i_2, i_3)} \\
&= \sum_{(i_1, i_2, i_3)} s(i_1, i_2, i_3) [\mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}] \\
&\quad - s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}] \\
&\quad + s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}] \\
&\quad - s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}] \\
&= \sum_{(i_1, i_2, i_3)} s(i_1, i_2, i_3) [\mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\
&\quad - \underbrace{\sum_{(i_1, i_2, i_3)} s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]}_{=0} \\
&\quad + \underbrace{\sum_{(i_1, i_2, i_3)} s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]}_{=0} \\
&\quad - \sum_{(i_1, i_2, i_3)} s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}].
\end{aligned}$$

Thus,

$$\begin{aligned}
\partial J^3 &= \sum_{(i_1, i_2, i_3)} s(i_1, i_2, i_3) [\mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\
&\quad - \sum_{(i_1, i_2, i_3)} s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}]
\end{aligned}$$

is the sum of 12 oriented affine 2-simplexes. (Note that  $3! = 6$ .)

- (4) Show that  $\mathbf{x} = (x_1, x_2, x_3)$  is in the range of  $\sigma_1$  if and only if  $0 \leq x_3 \leq x_2 \leq x_1 \leq 1$ .

- (a) By (1),  $\mathbf{x}$  is in the range of  $\sigma_1$  if and only if  $\mathbf{x} = A\mathbf{u}$  for  $\mathbf{u} = (u_1, u_2, u_3) \in Q^3$ , or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 + u_2 + u_3 \\ u_2 + u_3 \\ u_3 \end{bmatrix}.$$

- (b) Since  $\mathbf{u} = (u_1, u_2, u_3) \in Q^3$ ,  $u_1 + u_2 + u_3 \leq 1$  and  $u_1, u_2, u_3 \geq 0$ . Hence  $0 \leq u_3 \leq u_2 + u_3 \leq u_1 + u_2 + u_3 \leq 1$  or  $0 \leq x_3 \leq x_2 \leq x_1 \leq 1$ .  
(c) Conversely, if  $0 \leq x_3 \leq x_2 \leq x_1 \leq 1$ , we define

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_3 \end{bmatrix}.$$

Clearly,  $\mathbf{v} \in Q^3$ .

- (5) Show that the range of  $\sigma_1, \dots, \sigma_6$  have disjoint interiors, and that their union covers  $I^3$ . Similar to (4). By (2),  $\mathbf{x} = P_{(i_1, i_2, i_3)} A P_{(j_1, j_2, 3)} \mathbf{u}$ , or  $P_{(i_1, i_2, i_3)}^{-1} \mathbf{x} = A P_{(j_1, j_2, 3)} \mathbf{u}$ , or

$$\begin{bmatrix} x_{i_1} \\ x_{i_2} \\ x_{i_3} \end{bmatrix} = \begin{bmatrix} u_1 + u_2 + u_3 \\ u_{j_2} + u_3 \\ u_3 \end{bmatrix}.$$

In any case, we always have  $0 \leq u_3 \leq u_{j_2} + u_3 \leq u_1 + u_2 + u_3 \leq 1$ . Hence  $\mathbf{x} = (x_1, x_2, x_3)$  is in the range of  $\sigma_{(i_1, i_2, i_3)}$  if and only if

$$0 \leq x_{i_3} \leq x_{i_2} \leq x_{i_1} \leq 1.$$

The interior of  $\sigma_{(i_1, i_2, i_3)}$  is

$$\{\mathbf{x} \in \mathbb{R}^3 : 0 < x_{i_3} < x_{i_2} < x_{i_1} < 1\},$$

and thus the range of  $\sigma_1, \dots, \sigma_6$  have disjoint interiors. Also, any  $\mathbf{x} \in I^3$  has the relation

$$0 \leq x_{i_3} \leq x_{i_2} \leq x_{i_1} \leq 1$$

for some permutation  $(i_1, i_2, i_3)$  of  $(1, 2, 3)$ . Hence

$$I^3 = \bigcup_{(i_1, i_2, i_3)} \sigma_{(i_1, i_2, i_3)}(Q^3) = \bigcup_{i=1}^6 \sigma_i(Q^3).$$

□

**Exercise 10.19.** Let  $J^2$  and  $J^3$  be as in Exercise 10.17 and Exercise 10.18. Define

$$\begin{aligned} B_{01}(u, v) &= (0, u, v), & B_{11}(u, v) &= (1, u, v), \\ B_{02}(u, v) &= (u, 0, v), & B_{12}(u, v) &= (u, 1, v), \\ B_{03}(u, v) &= (u, v, 0), & B_{13}(u, v) &= (u, v, 1). \end{aligned}$$

These are affine, and map  $\mathbb{R}^2$  into  $\mathbb{R}^3$ . Put  $\beta_{ri} = B_{ri}(J^2)$ , for  $r = 0, 1$ ,  $i = 1, 2, 3$ . Each  $\beta_{ri}$  is an affine-oriented 2-chain. (See Section 10.30.) Verify that

$$\partial J^3 = \sum_{i=1}^3 (-1)^i (\beta_{0i} - \beta_{1i}),$$

in agreement with Exercise 10.18.)

*Proof.*

(1) A direct calculation shows that

$$\begin{aligned} B_{01}(\tau_1) - B_{11}(\tau_1) &= [\mathbf{0}, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3] - [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\ B_{02}(\tau_1) - B_{12}(\tau_1) &= [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_3] - [\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\ B_{03}(\tau_1) - B_{13}(\tau_1) &= [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] - [\mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\ B_{01}(\tau_2) - B_{11}(\tau_2) &= -[\mathbf{0}, \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3] + [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\ B_{02}(\tau_2) - B_{12}(\tau_2) &= -[\mathbf{0}, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_3] + [\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\ B_{03}(\tau_2) - B_{13}(\tau_2) &= -[\mathbf{0}, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]. \end{aligned}$$

(2) To express the formula in (1) clearly, we define

$$\omega_{(i_1, i_2, i_3)} = [\mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] - [\mathbf{0}, \mathbf{e}_{i_2}, \mathbf{e}_{i_2} + \mathbf{e}_{i_3}],$$

and thus

$$\begin{aligned} -(B_{01}(\tau_1) - B_{11}(\tau_1)) &= s(1, 2, 3)\omega_{(1, 2, 3)} \\ B_{02}(\tau_1) - B_{12}(\tau_1) &= s(2, 1, 3)\omega_{(2, 1, 3)} \\ -(B_{03}(\tau_1) - B_{13}(\tau_1)) &= s(3, 1, 2)\omega_{(3, 1, 2)} \\ -(B_{01}(\tau_2) - B_{11}(\tau_2)) &= s(1, 3, 2)\omega_{(1, 3, 2)} \\ B_{02}(\tau_2) - B_{12}(\tau_2) &= s(2, 3, 1)\omega_{(2, 3, 1)} \\ -(B_{03}(\tau_2) - B_{13}(\tau_2)) &= s(3, 2, 1)\omega_{(3, 2, 1)}. \end{aligned}$$

(3) Note that

$$\begin{aligned} \beta_{0i} - \beta_{1i} &= B_{0i}(J^2) - B_{1i}(J^2) \\ &= B_{0i}(\tau_1 + \tau_2) - B_{1i}(\tau_1 + \tau_2) \\ &= B_{0i}(\tau_1) + B_{0i}(\tau_2) - B_{1i}(\tau_1) - B_{1i}(\tau_2) \\ &= (B_{0i}(\tau_1) - B_{1i}(\tau_1)) + (B_{0i}(\tau_2) - B_{1i}(\tau_2)). \end{aligned}$$

Thus,

$$\begin{aligned}
& \sum_{i=1}^3 (-1)^i (\beta_{0i} - \beta_{1i}) \\
&= \sum_{i=1}^3 (-1)^i (B_{0i}(\tau_1) - B_{1i}(\tau_1)) + \sum_{i=1}^3 (-1)^i (B_{0i}(\tau_2) - B_{1i}(\tau_2)) \\
&= \sum_{(i_1, i_2, i_3)} s(i_1, i_2, i_3) \omega_{(i_1, i_2, i_3)} \\
&= \sum_{(i_1, i_2, i_3)} s(i_1, i_2, i_3) [\mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\
&\quad - \sum_{(i_1, i_2, i_3)} s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}] \\
&= \partial J^3.
\end{aligned}$$

□

**Exercise 10.20.** *State conditions under which the formula*

$$\int_{\Phi} f d\omega = \int_{\partial\Phi} f\omega - \int_{\Phi} (df) \wedge \omega$$

*is valid, and show that it generalizes the formula for integration by parts. (Hint:  $d(f\omega) = (df) \wedge \omega + f d\omega$ .)*

*Proof.*

(1) *If*

- (a)  $\Phi$  is a  $k$ -chain of class  $\mathcal{C}''$  in an open set  $V \subseteq \mathbb{R}^m$ ,
- (b)  $\omega$  is a  $(k-1)$ -form of class  $\mathcal{C}'$  in  $V$ ,
- (c)  $f$  is a 0-form of class  $\mathcal{C}'$  in  $V$ ,

*then*

$$\int_{\Phi} f d\omega = \int_{\partial\Phi} f\omega - \int_{\Phi} (df) \wedge \omega$$

(2) Theorem 10.20(a) implies that

$$d(f\omega) = (df) \wedge \omega + f d\omega.$$

(3) The Stokes' theorem (Theorem 10.33) shows that

$$\int_{\Phi} d(f\omega) = \int_{\partial\Phi} f\omega.$$

Hence

$$\int_{\Phi} f d\omega = \int_{\Phi} d(f\omega) - \int_{\Phi} (df) \wedge \omega = \int_{\partial\Phi} f\omega - \int_{\Phi} (df) \wedge \omega.$$

(4) Define  $\Phi : Q^1 = [0, 1] \rightarrow [a, b]$  by

$$\Phi(\alpha) = a + \alpha(b - a).$$

$\Phi$  is a 1-simplex of class  $\mathcal{C}''$  in an open set  $V \supseteq [a, b]$ . Also,

$$\partial\Phi = [b] - [a].$$

Let  $\omega = g$  be a 0-form of class  $\mathcal{C}'(V)$ .

(5) Note that

$$\begin{aligned} \int_{\Phi} f d\omega &= \int_{\Phi} f dg = \int_0^1 f(\Phi(t))g'(\Phi(t))\Phi'(t)dt = \int_a^b f(u)g'(u)du, \\ \int_{\partial\Phi} f\omega &= \int_{[b]} fg + \int_{-[a]} fg = f(b)g(b) + (-1)f(a)f(a), \\ \int_{\Phi} (df) \wedge \omega &= \int_{\Phi} (df)g = \int_0^1 f'(\Phi(t))g(\Phi(t))\Phi'(t)dt = \int_a^b f'(u)g(u)du. \end{aligned}$$

Hence

$$\int_a^b f(u)g'(u)du = f(b)g(b) - f(a)f(a) - \int_a^b f'(u)g(u)du,$$

which is the same as the integration by parts (Theorem 6.22).

□

**Exercise 10.21.** *As in Example 10.36, consider the 1-form*

$$\eta = \frac{xdy - ydx}{x^2 + y^2}$$

*in  $\mathbb{R}^2 - \{\mathbf{0}\}$ .*

(a) *Carry out the computation that leads to*

$$\int_{\gamma} \eta = 2\pi \neq 0,$$

*and prove that  $d\eta = 0$ .*

- (b) Let  $\gamma(t) = (r \cos t, r \sin t)$ , for some  $r > 0$ , and let  $\Gamma$  be a  $\mathcal{C}''$ -curve in  $\mathbb{R}^2 - \{\mathbf{0}\}$ , with parameter interval  $[0, 2\pi]$ , with  $\Gamma(0) = \Gamma(2\pi)$ , such that the intervals  $[\gamma(t), \Gamma(t)]$  do not contain  $\mathbf{0}$  for any  $t \in [0, 2\pi]$ . Prove that

$$\int_{\Gamma} \eta = 2\pi.$$

(Hint: For  $0 \leq t \leq 2\pi$ ,  $0 \leq u \leq 1$ , define

$$\Phi(t, u) = (1 - u)\Gamma(t) + u\gamma(t).$$

Then  $\Phi$  is a 2-surface in  $\mathbb{R}^2 - \{\mathbf{0}\}$  whose parameter domain is the indicated rectangle. Because of cancellations (as in Example 10.32),

$$\partial\Phi = \Gamma - \gamma.$$

Use Stokes' theorem to deduce that

$$\int_{\Gamma} \eta = \int_{\gamma} \eta$$

because  $d\eta = 0$ .)

- (c) Take  $\Gamma(t) = (a \cos t, b \sin t)$  where  $a > 0$ ,  $b > 0$  are fixed. Use part (b) to show that

$$\int_0^{2\pi} \frac{ab}{a^2 \cos^2 t + b^2 \sin^2 t} dt = 2\pi.$$

- (d) Show that

$$\eta = d\left(\arctan \frac{y}{x}\right)$$

in any convex open set in which  $x \neq 0$ , and that

$$\eta = d\left(-\arctan \frac{x}{y}\right)$$

in any convex open set in which  $y \neq 0$ . Explain why this justifies the notation  $\eta = d\theta$ , in spite of the fact that  $\eta$  is not exact in  $\mathbb{R}^2 - \{\mathbf{0}\}$ .

- (e) Show that (b) can be derived from (d).

- (f) If  $\Gamma$  is any closed  $\mathcal{C}'$ -curve in  $\mathbb{R}^2 - \{\mathbf{0}\}$ , prove that

$$\frac{1}{2\pi} \int_{\Gamma} \eta = \text{Ind}(\Gamma).$$

(See Exercise 8.23 for the definition of the index of a curve.)

*Proof of (a).*

(1)

$$\begin{aligned}
\int_{\gamma} \eta &= \int_0^{2\pi} \frac{(r \cos t)d(r \sin t) - (r \sin t)d(r \cos t)}{(r \cos t)^2 + (r \sin t)^2} \\
&= \int_0^{2\pi} \frac{(r \cos t)(r \cos t) - (r \sin t)(-r \sin t)}{(r \cos t)^2 + (r \sin t)^2} dt \\
&= \int_0^{2\pi} dt \\
&= 2\pi.
\end{aligned}$$

(2)

$$\begin{aligned}
d\eta &= d\left(\frac{xdy - ydx}{x^2 + y^2}\right) \\
&= d\left(\frac{x}{x^2 + y^2}\right) \wedge dy - d\left(\frac{y}{x^2 + y^2}\right) \wedge dx \quad (d^2 = 0) \\
&= D_1\left(\frac{x}{x^2 + y^2}\right) dx \wedge dy \quad (dy \wedge dy = 0) \\
&\quad - D_2\left(\frac{y}{x^2 + y^2}\right) dy \wedge dx \quad (dx \wedge dx = 0) \\
&= \left(\frac{1}{x^2 + y^2} - \frac{2x^2}{(x^2 + y^2)^2}\right) dx \wedge dy \\
&\quad + \left(\frac{1}{x^2 + y^2} - \frac{2y^2}{(x^2 + y^2)^2}\right) dx \wedge dy \\
&= 0
\end{aligned}$$

□

*Note.*

- (1)  $\eta$  is closed and locally exact, that is,  $\eta = dt$  on  $\mathbb{R}^2 - L$  where  $L$  is any line passing through  $\mathbf{0}$ .  $\eta$  is not exact since  $\int_{\gamma} \eta = 2\pi \neq 0$ . (See Exercise 10.22(g).)
- (2) (*Poincaré's Lemma for 1-form.*) Let  $\omega = \sum a_i dx_i$  be defined in an open set  $U \subseteq \mathbb{R}^n$ . Then  $d\omega = 0$  if and only if for each  $p \in U$  there is a neighborhood  $V \subseteq U$  of  $p$  and a differentiable function  $f : V \rightarrow \mathbb{R}^1$  with  $df = \omega$  (i.e.,  $\omega$  is locally exact).

*Proof of (b).*

- (1) For  $0 \leq t \leq 2\pi$ ,  $0 \leq u \leq 1$ , define

$$\Phi(t, u) = (1 - u)\Gamma(t) + u\gamma(t).$$

Then  $\Phi$  is a 2-surface in  $\mathbb{R}^2 - \{\mathbf{0}\}$  whose parameter domain  $D = \{(t, u) : 0 \leq t \leq 2\pi, 0 \leq u \leq 1\}$  is the indicated rectangle.

(2) Similar to Example 10.32,

$$\partial\Phi = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$$

where

$$\begin{aligned}\gamma_1(t) &= \Phi(t, 0) = \Gamma(t), \\ \gamma_2(u) &= \Phi(2\pi, u) = (1 - u)\Gamma(2\pi) + u\gamma(2\pi), \\ \gamma_3(t) &= \Phi(2\pi - t, 1) = \gamma(2\pi - t), \\ \gamma_4(u) &= \Phi(0, 1 - u) = u\Gamma(0) + (1 - u)\gamma(0).\end{aligned}$$

Because of cancellations (as in Example 10.32),  $\gamma(0) = \gamma(2\pi)$  and  $\Gamma(0) = \Gamma(2\pi)$ ,  $\gamma_4 = -\gamma_2$  and  $\gamma_3 = -\gamma_1$ . Hence,

$$\partial\Phi = \Gamma - \gamma.$$

(3) The Stokes' theorem (Theorem 10.33) implies that

$$\int_{\Phi} d\eta = \int_{\partial\Phi} \eta = \int_{\Gamma - \gamma} \eta = \int_{\Gamma} \eta - \int_{\gamma} \eta.$$

Hence,

$$\int_{\Gamma} \eta = \int_{\gamma} \eta$$

(since  $d\eta = 0$  by (a)).

□

*Proof of (c).*

(1)  $\Gamma$  satisfies all conditions described in (b). So

$$\int_{\Gamma} \eta = 2\pi.$$

(2) A direct calculation shows that

$$\begin{aligned}2\pi &= \int_{\Gamma} \eta = \int_{\Gamma} \frac{xdy - ydx}{x^2 + y^2} \\ &= \int_0^{2\pi} \frac{a \cos(t) d(b \sin(t)) - b \sin(t) d(a \cos(t))}{(a \cos(t))^2 + (b \sin(t))^2} \\ &= \int_0^{2\pi} \frac{ab(\cos^2 t + \sin^2 t)}{a^2 \cos^2 t + b^2 \sin^2 t} \\ &= \int_0^{2\pi} \frac{ab}{a^2 \cos^2 t + b^2 \sin^2 t}.\end{aligned}$$



□

*Proof of (d).*

(1) In any convex open set in which  $x \neq 0$ , we have

$$\begin{aligned} d\left(\arctan \frac{y}{x}\right) &= \left(D_1 \arctan \frac{y}{x}\right) dx + \left(D_2 \arctan \frac{y}{x}\right) dy \\ &= -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy \\ &= \eta. \end{aligned}$$

(2) In any convex open set in which  $y \neq 0$ , we have

$$\begin{aligned} d\left(-\arctan \frac{x}{y}\right) &= \left(D_1 \left(-\arctan \frac{x}{y}\right)\right) dx + \left(D_2 \left(-\arctan \frac{x}{y}\right)\right) dy \\ &= -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy \\ &= \eta. \end{aligned}$$

(3) By (1)(2),  $\eta$  is locally exact. Note that  $\theta_1 = \arctan \frac{y}{x}$  and  $\theta_2 = -\arctan \frac{x}{y}$  cannot be patched together to defined a global 0-form  $\theta$  on  $\mathbb{R}^2 - \{\mathbf{0}\}$ .

□

*Proof of (e).*

(1) Partition  $[0, 2\pi]$  into five subintervals

$$I_i = \left[ \frac{(2i-3)\pi}{4}, \frac{(2i-1)\pi}{4} \right] \cap [0, 2\pi].$$

for  $i = 1, 2, 3, 4, 5$ . Hence

$$\begin{aligned} \int_{\gamma} \eta &= \sum_{i=1}^5 \int_{\gamma(I_i)} \eta \\ &= \sum_{i=1,3,5} \int_{\gamma(I_i)} d\left(\arctan \frac{y}{x}\right) + \sum_{i=2,4} \int_{\gamma(I_i)} d\left(-\arctan \frac{x}{y}\right). \end{aligned}$$

(2) The Stokes' theorem (Theorem 10.33) implies that

$$\begin{aligned} \int_{\gamma(I_1)} d\left(\arctan \frac{y}{x}\right) &= \int_{\partial\gamma(I_1)} \arctan \frac{y}{x} \\ &= \left[ \arctan \frac{r \cos t}{r \sin t} \right]_{t=0}^{t=\frac{\pi}{4}} \\ &= [\arctan(\tan(t))]_{t=0}^{t=\frac{\pi}{4}} \\ &= \frac{\pi}{4}, \end{aligned}$$

and

$$\begin{aligned}
\int_{\gamma(I_2)} d\left(-\arctan \frac{x}{y}\right) &= \int_{\partial\gamma(I_2)} -\arctan \frac{x}{y} \\
&= \left[ \arctan \frac{r \sin t}{r \cos t} \right]_{t=\frac{\pi}{4}}^{t=\frac{3\pi}{4}} \\
&= [\arctan(\cot(t))]_{t=\frac{\pi}{4}}^{t=\frac{3\pi}{4}} \\
&= \frac{\pi}{2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_{\gamma(I_3)} d\left(\arctan \frac{y}{x}\right) &= \frac{\pi}{2} \\
\int_{\gamma(I_4)} d\left(-\arctan \frac{x}{y}\right) &= \frac{\pi}{2} \\
\int_{\gamma(I_5)} d\left(\arctan \frac{y}{x}\right) &= \frac{\pi}{4}.
\end{aligned}$$

(3) Therefore,

$$\int_{\gamma} \eta = \left(\frac{\pi}{4} + \frac{\pi}{2} + \frac{\pi}{4}\right) + \left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 2\pi.$$

□

*Proof of (f).*

(1) Regard  $\Gamma(t)$  as a plane curve  $(\Gamma_1(t), \Gamma_2(t))$  over  $\mathbb{R}^2$  or  $\Gamma_1(t) + i\Gamma_2(t)$  over  $\mathbb{C}^1$ . Note that

$$\begin{aligned}
\frac{\Gamma'(t)}{\Gamma(t)} &= \frac{\Gamma'_1(t) + i\Gamma'_2(t)}{\Gamma_1(t) + i\Gamma_2(t)} \\
&= \frac{\Gamma'_1(t)\Gamma_1(t) + \Gamma'_2(t)\Gamma_2(t)}{\Gamma_1(t)^2 + \Gamma_2(t)^2} + i \frac{\Gamma_1(t)\Gamma'_2(t) - \Gamma_2(t)\Gamma'_1(t)}{\Gamma_1(t)^2 + \Gamma_2(t)^2}.
\end{aligned}$$

So

$$\operatorname{Im} \left( \frac{\Gamma'(t)}{\Gamma(t)} \right) = \frac{\Gamma_1(t)\Gamma'_2(t) - \Gamma_2(t)\Gamma'_1(t)}{\Gamma_1(t)^2 + \Gamma_2(t)^2}.$$

(2) By Exercise 8.23,

$$\operatorname{Ind}(\Gamma) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\Gamma'(t)}{\Gamma(t)} dt$$

is always an integer. That is,

$$\begin{aligned}
 \text{Ind}(\Gamma) &= \frac{1}{2\pi} \int_0^{2\pi} \text{Im} \left( \frac{\Gamma'(t)}{\Gamma(t)} \right) dt \\
 &= \frac{1}{2\pi} \int_0^{2\pi} \frac{\Gamma_1(t)\Gamma_2'(t) - \Gamma_2(t)\Gamma_1'(t)}{\Gamma_1(t)^2 + \Gamma_2(t)^2} dt \\
 &= \frac{1}{2\pi} \int_{\Gamma} \frac{xdy - ydx}{x^2 + y^2} \\
 &= \frac{1}{2\pi} \int_{\Gamma} \eta.
 \end{aligned}$$

(Note that  $\text{Ind}(\Gamma) = 1$  if  $\Gamma$  is defined as in (c). Hence the integral in (c) is equal to  $2\pi\text{Ind}(\Gamma) = 2\pi$ .)

□

**Exercise 10.22.** As in Example 10.37, define  $\zeta$  in  $\mathbb{R}^3 - \{\mathbf{0}\}$  by

$$\zeta = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{r^3}$$

where  $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$ , let  $D$  be the rectangle given by  $0 \leq u \leq \pi$ ,  $0 \leq v \leq 2\pi$ , and let  $\Sigma$  be the 2-surface in  $\mathbb{R}^3$ , with parameter domain  $D$ , given by

$$x = \sin u \cos v, \quad y = \sin u \sin v, \quad z = \cos u.$$

- (a) Prove that  $d\zeta = 0$  in  $\mathbb{R}^3 - \{\mathbf{0}\}$ .
- (b) Let  $S$  denote the restriction of  $\Sigma$  to a parameter domain  $E \subseteq D$ . Prove that

$$\int_S \zeta = \int_E \sin u \, du \, dv = A(S),$$

where  $A$  denotes area, as in Section 10.46. Note that this contains

$$\int_{\Sigma} \zeta = \int_D \sin u \, du \, dv = 4\pi \neq 0$$

as a special case.

- (c) Suppose  $g, h_1, h_2, h_3$ , are  $\mathcal{C}''$ -functions on  $[0, 1]$ ,  $g > 0$ . Let  $(x, y, z) = \Phi(s, t)$  define a 2-surface  $\Phi$ , with parameter domain  $I^2$ , by

$$x = g(t)h_1(s), \quad y = g(t)h_2(s), \quad z = g(t)h_3(s).$$

Prove that

$$\int_{\Phi} \zeta = 0,$$

directly from Equation (35) in Chapter 10. Note the shape of the range of  $\Phi$ : For fixed  $s$ ,  $\Phi(s, t)$  runs over an interval on a line through  $\mathbf{0}$ . The range of  $\Phi$  thus lies in a “cone” with vertex at the origin.

- (d) Let  $E$  be a closed rectangle in  $D$ , with edges parallel to those of  $D$ . Suppose  $f \in \mathcal{C}''(D)$ ,  $f > 0$ . Let  $\Omega$  be the 2-surface with parameter domain  $E$ , defined by

$$\Omega(u, v) = f(u, v)\Sigma(u, v).$$

Define  $S$  as in (b) and prove that

$$\int_{\Omega} \zeta = \int_S \zeta = A(S).$$

(Since  $S$  is the “radical projection” of  $\Omega$  into the unit sphere, this result makes it reasonable to call  $\int_{\Omega} \zeta$  the “solid angle” subtended by the range of  $\Omega$  at the origin.) (Hint: Consider the 3-surface  $\Psi$  given by

$$\Psi(t, u, v) = [1 - t + tf(u, v)]\Sigma(u, v),$$

where  $(u, v) \in E$ ,  $0 \leq t \leq 1$ . For fixed  $v$ , the mapping  $(t, u) \mapsto \Psi(t, u, v)$  is a 2-surface  $\Phi$  to which (c) can be applied to show that  $\int_{\Phi} \zeta = 0$ . The same thing holds when  $u$  is fixed. By (a) and Stokes’ theorem,

$$\int_{\partial\Psi} \zeta = \int_{\Psi} d\zeta = 0.)$$

- (e) Put  $\lambda = -\frac{z}{r}\eta$ , where

$$\eta = \frac{xdy - ydx}{x^2 + y^2},$$

as in Exercise 10.21. Then  $\lambda$  is a 1-form in the open set  $V \subseteq \mathbb{R}^3$  in which  $x^2 + y^2 > 0$ . Show that  $\zeta$  is exact in  $V$  by showing that

$$\zeta = d\lambda.$$

- (f) Derive (d) from (e), without using (c). (Hint: To begin with, assume  $0 < u < \pi$  on  $E$ . By (e),

$$\int_{\Omega} \zeta = \int_{\partial\Omega} \lambda \quad \text{and} \quad \int_S \zeta = \int_{\partial S} \lambda.$$

Show that the two integrals of  $\lambda$  are equal, by using part (d) of Exercise 10.21, and by noting that  $\frac{z}{r}$  is the same at  $\Sigma(u, v)$  as at  $\Omega(u, v)$ .)

- (g) Is  $\zeta$  exact in the complement of every line through the origin?

*Proof of (a).*

(1) Note that  $\zeta$  is well-defined on  $\mathbb{R}^3 - \{\mathbf{0}\}$ . Hence,

$$\begin{aligned}
d\zeta &= d\left(\frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{r^3}\right) \\
&= d\left(\frac{x}{r^3}\right) \wedge dy \wedge dz + d\left(\frac{y}{r^3}\right) \wedge dz \wedge dx + d\left(\frac{z}{r^3}\right) \wedge dx \wedge dy \\
&= D_1\left(\frac{x}{r^3}\right) dx \wedge dy \wedge dz + D_2\left(\frac{y}{r^3}\right) dy \wedge dz \wedge dx + D_3\left(\frac{z}{r^3}\right) dz \wedge dx \wedge dy \\
&= \frac{r^3 - 3rx^2}{r^6} dx \wedge dy \wedge dz + \frac{r^3 - 3ry^2}{r^6} dy \wedge dz \wedge dx + \frac{r^3 - 3rz^2}{r^6} dz \wedge dx \wedge dy \\
&= \left(\frac{r^3 - 3rx^2}{r^6} + \frac{r^3 - 3ry^2}{r^6} + \frac{r^3 - 3rz^2}{r^6}\right) dx \wedge dy \wedge dz \\
&= 0 dx \wedge dy \wedge dz \\
&= 0
\end{aligned}$$

in  $\mathbb{R}^3 - \{\mathbf{0}\}$ .

(2) Or write

$$\mathbf{F} = \frac{x}{r^3} \mathbf{e}_1 + \frac{y}{r^3} \mathbf{e}_2 + \frac{z}{r^3} \mathbf{e}_3$$

as in Vector fields 10.42. So

$$\omega_{\mathbf{F}} = \zeta$$

and

$$d\omega_{\mathbf{F}} = (\nabla \cdot \mathbf{F}) dx \wedge dy \wedge dz$$

as in the proof of the divergence theorem (Theorem 10.51). Note that the divergence of  $\mathbf{F}$  is zero.

□

*Proof of (b).*

(1) By Area elements in  $\mathbb{R}^3$  10.46,

$$\begin{aligned}
\mathbf{N}(u, v) &= \frac{\partial(y, z)}{\partial(u, v)} \mathbf{e}_1 + \frac{\partial(z, x)}{\partial(u, v)} \mathbf{e}_2 + \frac{\partial(x, y)}{\partial(u, v)} \mathbf{e}_3 \\
&= (\sin^2 u \cos v) \mathbf{e}_1 + (\sin^2 u \sin v) \mathbf{e}_2 + (\sin u \cos u) \mathbf{e}_3.
\end{aligned}$$

Here  $|\mathbf{N}(u, v)| = \sin u \geq 0$  (by noting that  $u \in [0, \pi]$ ), and

$$\mathbf{n}(u, v) = \frac{\mathbf{N}(u, v)}{|\mathbf{N}(u, v)|} = (\sin u \cos v, \sin u \sin v, \cos u).$$

(2) Note that  $\zeta = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$  on  $S \subseteq \Sigma$ . Hence, by Integrals

of 2-forms in  $\mathbb{R}^3$  10.49,

$$\begin{aligned}
\int_S \zeta &= \int_S x dy \wedge dz + y dz \wedge dx + z dx \wedge dy \\
&= \int_E (\sin u \cos v, \sin u \sin v, \cos u) \cdot \mathbf{N}(u, v) \, du \, dv \\
&= \int_E \mathbf{n}(u, v) \cdot \mathbf{n}(u, v) |\mathbf{N}(u, v)| \, du \, dv \\
&= \int_E |\mathbf{N}(u, v)| \, du \, dv \\
&= A(S).
\end{aligned}$$

(3) In particular,

$$\begin{aligned}
\int_\Sigma \zeta &= \int_D \sin u \, du \, dv \\
&= \int_0^\pi \int_0^{2\pi} \sin u \, du \, dv \\
&= \left( \int_0^\pi \sin u \, du \right) \left( \int_0^{2\pi} dv \right) \\
&= 2 \cdot 2\pi \\
&= 4\pi.
\end{aligned}$$

□

*Proof of (c).*

(1) Similar to (b).

$$\begin{aligned}
\mathbf{N}(s, t) &= \frac{\partial(y, z)}{\partial(s, t)} \mathbf{e}_1 + \frac{\partial(z, x)}{\partial(s, t)} \mathbf{e}_2 + \frac{\partial(x, y)}{\partial(s, t)} \mathbf{e}_3 \\
&= g(t)g'(t)[(h_1(s), h_2(s), h_3(s)) \times (h'_1(s), h'_2(s), h'_3(s))] \\
&= g(t)g'(t)[\mathbf{h}(s) \times \mathbf{h}'(s)],
\end{aligned}$$

where  $\mathbf{h}(s) = (h_1(s), h_2(s), h_3(s))$  and  $\mathbf{h}'(s) = (h'_1(s), h'_2(s), h'_3(s))$ . (Here “ $\times$ ” is the cross product in  $\mathbb{R}^3$ .)

(2) Assume  $\zeta$  is well-defined, i.e.,  $\mathbf{h}(s) \neq \mathbf{0}$  for all  $s \in [0, 1]$ . By Integrals of

2-forms in  $\mathbb{R}^3$  10.49,

$$\begin{aligned}
\int_{\Phi} \zeta &= \int_{\Phi} \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{r^3} \\
&= \int_{I^2} \frac{g(t)}{g(t)^3 |\mathbf{h}(s)|^3} \mathbf{h}(s) \cdot \mathbf{N}(s, t) \, ds \, dt \\
&= \int_{I^2} \frac{g(t)}{g(t)^3 |\mathbf{h}(s)|^3} \mathbf{h}(s) \cdot g(t)g'(t)[\mathbf{h}(s) \times \mathbf{h}'(s)] \, ds \, dt \\
&= \int_{I^2} \frac{g'(t)}{g(t)|\mathbf{h}(s)|^3} \mathbf{h}(s) \cdot [\mathbf{h}(s) \times \mathbf{h}'(s)] \, ds \, dt \\
&= 0
\end{aligned}$$

(since  $\mathbf{h}(s) \cdot [\mathbf{h}(s) \times \mathbf{h}'(s)] = 0$ .)

- (3) Note that  $\Sigma$  in spherical coordinate system cannot be parameterized as  $(x, y, z) = g(t)\mathbf{h}(s)$ , and thus  $\int_S \zeta$  could be nonzero as shown in (b).

□

*Proof of (d) (Hint).*

- (1) Consider the 3-surface  $\Psi$  given by

$$\Psi(t, u, v) = [1 - t + tf(u, v)]\Sigma(u, v),$$

where  $(u, v) \in E$ ,  $0 \leq t \leq 1$ . Write

$$E = [a_1, b_1] \times [a_2, b_2] \subseteq D = [0, \pi] \times [0, 2\pi].$$

Note that  $\Psi(t, u, v) \subseteq \mathbb{R}^3 - \{\mathbf{0}\}$ . So the boundary of  $\Psi$  is

$$\begin{aligned}
\partial\Psi &= \Psi(0, u, v) - \Psi(1, u, v) \\
&\quad + \Psi(t, a_1, v) - \Psi(t, b_1, v) \\
&\quad + \Psi(t, u, a_2) - \Psi(t, u, b_2) \\
&= S(u, v) - \Omega(u, v) \\
&\quad + \Psi|_{u=a_1}(t, v) - \Psi|_{u=b_1}(t, v) \\
&\quad + \Psi|_{v=a_2}(t, u) - \Psi|_{v=b_2}(t, u),
\end{aligned}$$

where  $\Psi|_{u=u_0}(t, v) = \Psi(t, u_0, v)$  and  $\Psi|_{v=v_0}(t, u) = \Psi(t, u, v_0)$ .

- (2) *Show that*

$$\int_{\Psi|_{v=v_0}} \zeta = 0$$

for any fixed  $v = v_0 \in [a_2, b_2]$ . Note that  $\zeta$  is well-defined on  $\Psi|_{v=v_0}$ . Write  $\Psi|_{v=v_0}(t, u) = (x, y, z) = (x(t, u), y(t, u), z(t, u))$ . By definition of

$\Psi$ , we have

$$\begin{aligned}x &= g(t, u) \sin u \cos v_0 \\y &= g(t, u) \sin u \sin v_0 \\z &= g(t, u) \cos u,\end{aligned}$$

where  $g(t, u) = 1 - t + tf(u, v_0)$ . Similar to (c),

$$\begin{aligned}\mathbf{N}(t, u) &= \frac{\partial(y, z)}{\partial(t, u)} \mathbf{e}_1 + \frac{\partial(z, x)}{\partial(t, u)} \mathbf{e}_2 + \frac{\partial(x, y)}{\partial(t, u)} \mathbf{e}_3 \\&= g(t, u) D_1 g(t, u) (-\sin v_0, \cos v_0, 0).\end{aligned}$$

Note that

$$(x(t, u), y(t, u), z(t, u)) \cdot \mathbf{N}(t, u) = 0.$$

So

$$\begin{aligned}\int_{\Psi|_{v=v_0}} \zeta &= \int_{\Psi|_{v=v_0}} r^{-3} (x dy \wedge dz + y dz \wedge dx + z dx \wedge dy) \\&= \int_{[0,1] \times [a_1, b_1]} r^{-3} (x(t, u), y(t, u), z(t, u)) \cdot \mathbf{N}(t, u) dt du \\&= \int_{[0,1] \times [a_1, b_1]} 0 dt du \\&= 0.\end{aligned}$$

(3) *Show that*

$$\int_{\Psi|_{u=u_0}} \zeta = 0$$

for any fixed  $u = u_0 \in [a_1, b_1]$ . Similar to (2).

$$\begin{aligned}\mathbf{N}(t, v) &= \frac{\partial(y, z)}{\partial(t, v)} \mathbf{e}_1 + \frac{\partial(z, x)}{\partial(t, v)} \mathbf{e}_2 + \frac{\partial(x, y)}{\partial(t, v)} \mathbf{e}_3 \\&= \sin u_0 g(t, v) D_1 g(t, v) (-\cos u_0 \cos v, -\cos u_0 \sin v, \sin u_0).\end{aligned}$$

where  $g(t, v) = 1 - t + tf(u_0, v)$ . So  $(x(t, v), y(t, v), z(t, v)) \cdot \mathbf{N}(t, v) = 0$  and thus  $\int_{\Psi|_{u=u_0}} \zeta = 0$ .



(4) So

$$\begin{aligned}
0 &= \int_{\Psi} d\zeta && (d\zeta = 0 \text{ on } \mathbb{R}^3 - \{\mathbf{0}\}) \\
&= \int_{\partial\Psi} \zeta && (\text{Theorem 10.33}) \\
&= \int_S \zeta - \int_{\Omega} \zeta \\
&\quad + \underbrace{\int_{\Psi|u=a_1} \zeta - \int_{\Psi|u=b_1} \zeta}_{\text{all are zero by (2)}} \\
&\quad + \underbrace{\int_{\Psi|v=a_2} \zeta - \int_{\Psi|v=b_2} \zeta}_{\text{all are zero by (3)}} && ((1)) \\
&= \int_S \zeta - \int_{\Omega} \zeta.
\end{aligned}$$

Hence

$$\int_{\Omega} \zeta = \underbrace{\int_S \zeta}_{\text{by (b)}} = A(S).$$

□

*Proof of (e).*

(1) Note that

$$\begin{aligned}
d\left(-\frac{z}{r}\right) &= \frac{xz}{r^3}dx + \frac{yz}{r^3}dy - \frac{r^2 - z^2}{r^3}dz = \frac{xz}{r^3}dx + \frac{yz}{r^3}dy - \frac{x^2 + y^2}{r^3}dz \\
&\text{since } r^2 = x^2 + y^2 + z^2.
\end{aligned}$$

(2)

$$\begin{aligned}
d\lambda &= d\left(-\frac{z}{r}\eta\right) \\
&= \underbrace{d\left(-\frac{z}{r}\right)}_{\text{apply (1)}} \wedge \eta + (-1)^1 \left(-\frac{z}{r}\right) \wedge \underbrace{d\eta}_{=0} \\
&= \left(\frac{xz}{r^3}dx + \frac{yz}{r^3}dy - \frac{x^2 + y^2}{r^3}dz\right) \wedge \left(\frac{-ydx + xdy}{x^2 + y^2}\right) \\
&= \left(\frac{x(x^2 + y^2)}{r^3(x^2 + y^2)}\right) dy \wedge dz + \left(\frac{y(x^2 + y^2)}{r^3(x^2 + y^2)}\right) dz \wedge dx + \left(\frac{x^2z + y^2z}{r^3(x^2 + y^2)}\right) dx \wedge dy \\
&= \left(\frac{x}{r^3}\right) dy \wedge dz + \left(\frac{y}{r^3}\right) dz \wedge dx + \left(\frac{z}{r^3}\right) dx \wedge dy \\
&= \zeta.
\end{aligned}$$

□

*Proof of (f).*

- (1) To ensure that  $\eta$  is well-defined on  $E$ , we might assume  $x^2 + y^2 = \sin^2 u \neq 0$  or  $0 < u < \pi$  on  $E$ . It is fine since  $\int_{\Omega} \zeta$  and  $\int_S \zeta$  is well-defined on any closed rectangle in  $D$  and we can apply the argument in Exercise 6.7 to remove the additional restriction.

- (2) By the Stokes' theorem (Theorem 10.33) and (e),

$$\int_{\Omega} \zeta = \int_{\partial\Omega} \lambda \quad \text{and} \quad \int_S \zeta = \int_{\partial S} \lambda.$$

So it suffices to show that

$$\int_{\partial\Omega} \lambda = \int_{\partial S} \lambda.$$

Note that  $\lambda = -\frac{z}{r}\eta$ , and thus it suffices to show that  $\frac{z}{r}|_{\partial\Omega} = \frac{z}{r}|_{\partial S}$  and  $\eta|_{\partial\Omega} = \eta|_{\partial S}$ .

- (3) Show that  $\frac{z}{r}|_{\partial\Omega} = \frac{z}{r}|_{\partial S}$ . For any  $(x_{\Omega}, y_{\Omega}, z_{\Omega}) \in \partial\Omega$ ,

$$(x_{\Omega}, y_{\Omega}, z_{\Omega}) = f(u, v)(x_{\Sigma}, y_{\Sigma}, z_{\Sigma})$$

where  $(x_{\Sigma}, y_{\Sigma}, z_{\Sigma}) \in \partial S$ . So

$$\begin{aligned} \frac{z}{r}|_{\partial\Omega} &= \frac{z_{\Omega}}{(x_{\Omega}^2 + y_{\Omega}^2 + z_{\Omega}^2)^{\frac{1}{2}}} \\ &= \frac{f(u, v)z_{\Sigma}}{f(u, v)(x_{\Sigma}^2 + y_{\Sigma}^2 + z_{\Sigma}^2)^{\frac{1}{2}}} \\ &= \frac{z_{\Sigma}}{(x_{\Sigma}^2 + y_{\Sigma}^2 + z_{\Sigma}^2)^{\frac{1}{2}}} \\ &= \frac{z}{r}|_{\partial S}. \end{aligned}$$

(Note that  $f > 0$ .)

- (4) Show that  $\eta|_{\partial\Omega} = \eta|_{\partial S}$ . Similar to (3). If  $x_{\Omega} \neq 0$  (or  $x_{\Sigma} \neq 0$ ), then by Exercise 10.21(d)

$$\begin{aligned} \eta|_{\partial\Omega} &= d\left(\arctan \frac{y_{\Omega}}{x_{\Omega}}\right) \\ &= d\left(\arctan \frac{f(u, v)y_{\Sigma}}{f(u, v)x_{\Sigma}}\right) \\ &= d\left(\arctan \frac{y_{\Sigma}}{x_{\Sigma}}\right) \\ &= \eta|_{\partial S}. \end{aligned}$$

Similarly,  $\eta|_{\partial\Omega} = \eta|_{\partial S}$  is also true if  $y_{\Omega} \neq 0$ . Note that  $(x_{\Omega}, y_{\Omega}) \neq (0, 0)$  by assumption. Therefore the result is established.

□

*Proof of (g).*

- (1) Yes. Given any line  $L$  passing through  $\mathbf{0}$ , say

$$(r \sin u \cos v, r \sin u \sin v, r \cos u) \in L \quad (r \in \mathbb{R}^1),$$

for some  $u \in [0, \pi]$  and  $v \in [0, 2\pi]$ . We will show that  $\zeta$  is exact in  $U = \mathbb{R}^3 - L$ .

- (2) Linear algebra says that all rotation matrices  $T \in SO(3)$  can be obtained from

$$\begin{aligned} R_x(u) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos u & -\sin u \\ 0 & \sin u & \cos u \end{bmatrix} \\ R_y(v) &= \begin{bmatrix} \cos v & 0 & -\sin v \\ 0 & 1 & 0 \\ \sin v & 0 & \cos v \end{bmatrix} \\ R_z(w) &= \begin{bmatrix} \cos w & -\sin w & 0 \\ \sin w & \cos w & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

using matrix multiplication, say  $T = R_x(u)R_y(v)R_z(w)$ . For example, the rotation

$$T = R_y\left(u - \frac{\pi}{2}\right) R_z(-v)$$

maps  $L$  to the  $z$ -axis (by showing that  $T(r \sin u \cos v, r \sin u \sin v, r \cos u) = (0, 0, r)$ ). By Theorem 10.22 it suffices to show that  $\zeta$  is invariant under  $R_x(u)$ ,  $R_y(v)$  and  $R_z(w)$ . By the symmetricity of  $\zeta$ , it suffices to show that  $\zeta$  is invariant under  $T = R_x(u)$ .

- (3) Show that  $\zeta$  is invariant under  $T = R_x(u)$ . By

$$T : (x, y, z) \mapsto (x, y \cos u - z \sin u, y \sin u + z \cos u),$$

we have

$$\begin{aligned} r &\mapsto r \\ dx &\mapsto dx \\ dy &\mapsto \cos u dy - \sin u dz \\ dz &\mapsto \sin u dy + \cos u dz. \end{aligned}$$

So

$$\begin{aligned}
dy \wedge dz &\mapsto (\cos u dy - \sin u dz) \wedge (\sin u dy + \cos u dz) \\
&= dy \wedge dz, \\
dz \wedge dx &\mapsto (\sin u dy + \cos u dz) \wedge dx \\
&= -\sin u dx \wedge dy + \cos u dz \wedge dx, \\
dx \wedge dy &\mapsto dx \wedge (\sin u dy + \cos u dz) \\
&= \cos u dx \wedge dy + \sin u dz \wedge dx.
\end{aligned}$$

Thus

$$\begin{aligned}
\zeta &\mapsto r^{-3} \{ x dy \wedge dz \\
&\quad + (y \cos u - z \sin u)(-\sin u dx \wedge dy + \cos u dz \wedge dx) \\
&\quad + (y \sin u + z \cos u)(\cos u dx \wedge dy + \sin u dz \wedge dx) \} \\
&= r^{-3} \{ x dy \wedge dz \\
&\quad + [\cos u(y \cos u - z \sin u) + \sin u(y \sin u + z \cos u)] dz \wedge dx \\
&\quad + [-\sin u(y \cos u - z \sin u) + \cos u(y \sin u + z \cos u)] dx \wedge dy \} \\
&= r^{-3} \{ x dy \wedge dz + y dz \wedge dx + z dx \wedge dy \} \\
&= \zeta.
\end{aligned}$$

- (4) Let  $V = \mathbb{R}^3 - z\text{-axis}$ . Since  $\zeta_T = \zeta$  (by (3)) is well-defined in  $V$ ,  $\zeta_T = \zeta = d\lambda$  by (e). Here  $\lambda$  is in  $V$ , not necessary in  $U$  (if  $L \neq z\text{-axis}$ ). Luckily, we can use  $T^{-1}$  to pullback  $\lambda$  in  $U$ . Thus

$$\zeta = (\zeta_T)_{T^{-1}} = (d\lambda)_{T^{-1}} = d(\lambda_{T^{-1}})$$

by Theorems 10.22 and 10.23. That is,  $\zeta$  is exact in  $U = \mathbb{R}^3 - L$ . (Or  $\zeta$  is locally exact in  $\mathbb{R}^3 - \{\mathbf{0}\}$ .)

□

**Exercise 10.23.** Fix  $n$ . Define  $r_k = (x_1^2 + \cdots + x_k^2)^{\frac{1}{2}}$  for  $1 \leq k \leq n$ , let  $E_k$  be the set of all  $\mathbf{x} \in \mathbb{R}^n$  at which  $r_k > 0$ , and let  $\omega_k$  be the  $(k-1)$ -form defined in  $E_k$  by

$$\omega_k = (r_k)^{-k} \sum_{i=1}^k (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k$$

Note that  $\omega_2 = \eta$ ,  $\omega_3 = \zeta$  in the terminology of Exercise 10.21 and Exercise 10.22. Note also that

$$E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n = \mathbb{R}^n.$$

- (a) Prove that  $d\omega_k = 0$  in  $E_k$ .

(b) For  $k = 2, \dots, n$ , prove that  $\omega_k$  is exact in  $E_{k-1}$ , by showing that

$$\omega_k = d(f_k \omega_{k-1}) = df_k \wedge \omega_{k-1}$$

where  $f_k(\mathbf{x}) = (-1)^k g_k\left(\frac{x_k}{r_k}\right)$  where

$$g_k(t) = \int_{-1}^t (1-s^2)^{\frac{k-3}{2}} ds \quad (-1 < t < 1).$$

(Hint:  $f_k$  satisfies the differential equations

$$\mathbf{x} \cdot (\nabla f_k)(\mathbf{x}) = 0$$

and

$$(D_k f_k)(\mathbf{x}) = \frac{(-1)^k (r_{k-1})^{k-1}}{(r_k)^k}.)$$

(c) Is  $\omega_n$  exact in  $E_n$ ?

(d) Note that (b) is a generalization of part (e) of Exercise 10.22. Try to extend some of the other assertions of Exercise 10.21 and Exercise 10.22 to  $\omega_n$ , for arbitrary  $n$ .

*Proof of (a).*

(1) Note that

$$D_i r_k = \frac{1}{2r_k} \cdot (2x_i) = \frac{x_i}{r_k}.$$

(2)

$$\begin{aligned} d\omega_k &= \sum_{i=1}^k d \left( (-1)^{i-1} (r_k)^{-k} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k \right) \\ &= \sum_{i=1}^k D_i \left( (-1)^{i-1} (r_k)^{-k} x_i \right) dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k \\ &= \sum_{i=1}^k (-1)^{i-1} \left( (r_k)^{-k} \cdot 1 + \underbrace{(-k)(r_k)^{-k-1} \frac{x_i}{r_k}}_{\text{chain rule}} \cdot x_i \right) \underbrace{(-1)^{i-1} dx_1 \wedge \cdots \wedge dx_k}_{\text{anticommutative relation}} \\ &= (r_k)^{-k-2} \underbrace{\sum_{i=1}^k ((r_k)^2 - kx_i^2)}_{=0} dx_1 \wedge \cdots \wedge dx_k \\ &= 0. \end{aligned}$$

□

*Proof of (b).*

(1) Note that

$$D_i \left( \frac{x_k}{r_k} \right) = \frac{\delta_{ik}(r_k)^2 - x_i x_k}{(r_k)^3}$$

where  $\delta_{ik}$  is the Kronecker delta. So

$$\begin{aligned} (D_i f_k)(\mathbf{x}) &= D_i \left( (-1)^k g_k \left( \frac{x_k}{r_k} \right) \right) \\ &= D_i \left( (-1)^k \int_{-1}^{\frac{x_k}{r_k}} (1-s^2)^{\frac{k-3}{2}} ds \right) \\ &= (-1)^k D_i \left( \frac{x_k}{r_k} \right) \left( 1 - \left( \frac{x_k}{r_k} \right)^2 \right)^{\frac{k-3}{2}} \\ &= (-1)^k \frac{\delta_{ik}(r_k)^2 - x_i x_k}{(r_k)^3} \frac{(r_{k-1})^{k-3}}{(r_k)^{k-3}} \\ &= (-1)^k \frac{(r_{k-1})^{k-3}}{(r_k)^k} (\delta_{ik}(r_k)^2 - x_i x_k). \end{aligned}$$

In particular,

$$(D_k f_k)(\mathbf{x}) = (-1)^k \frac{(r_{k-1})^{k-3}}{(r_k)^k} ((r_k)^2 - (x_k)^2) = (-1)^k \frac{(r_{k-1})^{k-1}}{(r_k)^k}$$

(since  $(r_k)^2 - (x_k)^2 = (r_{k-1})^2$ ).

(2) Since

$$\sum_i x_i (\delta_{ik}(r_k)^2 - x_i x_k) = (r_k)^2 \underbrace{\sum_i x_i \delta_{ik}}_{=x_k} - x_k \underbrace{\sum_i x_i^2}_{=(r_k)^2} = 0,$$

we have

$$\begin{aligned} \mathbf{x} \cdot (\nabla f_k)(\mathbf{x}) &= \sum_i x_i (D_i f_k)(\mathbf{x}) \\ &= \sum_i x_i (-1)^k \frac{(r_{k-1})^{k-3}}{(r_k)^k} (\delta_{ik}(r_k)^2 - x_i x_k) \\ &= (-1)^k \frac{(r_{k-1})^{k-3}}{(r_k)^k} \sum_i x_i (\delta_{ik}(r_k)^2 - x_i x_k) \\ &= 0. \end{aligned}$$

(3) On  $E_{k-1} \subsetneq E_k$ , we write

$$\begin{aligned}
& d(f_k \omega_{k-1}) \\
&= (df_k) \wedge \omega_{k-1} + (-1)^0 f_k \wedge \underbrace{(d\omega_{k-1})}_{=0} \\
&= (df_k) \wedge \omega_{k-1} \\
&= \left\{ \sum_{i=1}^k D_i f_k(\mathbf{x}) dx_i \right\} \wedge \\
&\quad \left\{ \frac{1}{(r_{k-1})^{k-1}} \sum_{j=1}^{k-1} (-1)^{j-1} x_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{k-1} \right\} \\
&= \frac{1}{(r_{k-1})^{k-1}} \sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq k-1}} (-1)^{j-1} x_j D_i f_k(\mathbf{x}) dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{k-1} \\
&= \frac{1}{(r_{k-1})^{k-1}} \sum_{j=1}^{k-1} (-1)^{j-1} x_j D_j f_k(\mathbf{x}) dx_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{k-1} \\
&\quad + \frac{1}{(r_{k-1})^{k-1}} \sum_{j=1}^{k-1} (-1)^{j-1} x_j D_k f_k(\mathbf{x}) dx_k \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{k-1}.
\end{aligned}$$

(4) By (2),

$$\begin{aligned}
& \frac{1}{(r_{k-1})^{k-1}} \sum_{j=1}^{k-1} (-1)^{j-1} x_j D_j f_k(\mathbf{x}) dx_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{k-1} \\
&= \frac{1}{(r_{k-1})^{k-1}} \sum_{j=1}^{k-1} x_j D_j f_k(\mathbf{x}) dx_1 \wedge \cdots \wedge dx_{k-1} \\
&= \frac{1}{(r_{k-1})^{k-1}} (-D_k f_k(\mathbf{x}) x_k) dx_1 \wedge \cdots \wedge dx_{k-1} \\
&= \frac{-D_k f_k(\mathbf{x})}{(r_{k-1})^{k-1}} x_k dx_1 \wedge \cdots \wedge dx_{k-1} \wedge \widehat{dx_k} \\
&= (r_k)^{-k} (-1)^{k-1} x_k dx_1 \wedge \cdots \wedge dx_{k-1} \wedge \widehat{dx_k} \quad ((1)).
\end{aligned}$$

Also,

$$\begin{aligned}
& \frac{1}{(r_{k-1})^{k-1}} \sum_{j=1}^{k-1} (-1)^{j-1} x_j D_k f_k(\mathbf{x}) dx_k \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{k-1} \\
&= \frac{(-1)^k D_k f_k(\mathbf{x})}{(r_{k-1})^{k-1}} \sum_{j=1}^{k-1} (-1)^{j-1} x_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_k \\
&= (r_k)^{-k} \sum_{j=1}^{k-1} (-1)^{j-1} x_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_k \quad ((1)).
\end{aligned}$$

(5) Hence,

$$\begin{aligned}
& d(f_k \omega_{k-1}) \\
&= (r_k)^{-k} (-1)^{k-1} x_k dx_1 \wedge \cdots \wedge dx_{k-1} \wedge \widehat{dx_k} \\
&\quad + (r_k)^{-k} \sum_{j=1}^{k-1} (-1)^{j-1} x_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_k \\
&= (r_k)^{-k} \sum_{j=1}^k (-1)^{j-1} x_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_k \\
&= \omega_k.
\end{aligned}$$

□

*Proof of (c).*

(1)  $\omega_n$  is not exact in  $E_n$  (though it is locally exact).

(2) Let

$$\begin{aligned}
\mathbb{S}^{n-1} &= \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\} \\
\mathbb{B}^n &= \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leq 1\}.
\end{aligned}$$

*It suffices to show that*

$$\int_{\mathbb{S}^{n-1}} \omega_n = \frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)} \neq 0.$$

Therefore,  $\omega_n$  is not exact in  $E_n$ .

(3) Define

$$\omega = \frac{1}{n} \sum_{i=1}^n (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

on  $\mathbb{S}^{n-1}$ . Note that

$$\omega = \frac{1}{n} \omega_n$$



on  $\mathbb{S}^{n-1}$  (and that's why we pick  $\mathbb{S}^{n-1}$ ). The Stokes' theorem (Theorem 10.33) implies that

$$\int_{\mathbb{S}^{n-1}} \frac{1}{n} \omega_n = \int_{\partial \mathbb{B}^n} \omega = \int_{\mathbb{B}^n} d\omega = \int_{\mathbb{B}^n} dx_1 \wedge \cdots \wedge dx_n = \text{vol}(\mathbb{B}^n),$$

where  $\text{vol}(\mathbb{B}^n)$  is the volume of  $\mathbb{B}^n$ . Thus *it suffices to show that*

$$\text{vol}(\mathbb{B}^n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)} \neq 0.$$

There are many proofs for this. We give a direct integration in spherical coordinates.

- (4) Similar to Exercise 10.9. The spherical coordinate system has a radial coordinate  $r$  and angular coordinates  $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_{n-1})$ , where the domain of each  $\varphi_1, \dots, \varphi_{n-2}$  is  $[0, \pi]$  and the domain of  $\varphi_{n-1}$  is  $[0, 2\pi]$ . That is,

$$\begin{aligned} x_1 &= \cos \varphi_1 \\ x_2 &= \sin \varphi_1 \cos \varphi_2 \\ x_3 &= \sin \varphi_1 \sin \varphi_2 \cos \varphi_3 \\ &\vdots \\ x_{n-1} &= \sin \varphi_1 \cdots \sin \varphi_{n-2} \cos \varphi_{n-1} \\ x_n &= \sin \varphi_1 \cdots \sin \varphi_{n-2} \sin \varphi_{n-1}. \end{aligned}$$

(It is different from Exercise 10.22.) The spherical volume element is

$$r^{n-1} \sin(\varphi_1)^{n-2} \sin(\varphi_2)^{n-3} \cdots \sin(\varphi_{n-2}) dr d\boldsymbol{\varphi}.$$

Thus by Some consequences 8.21,

$$\begin{aligned} \text{vol}(\mathbb{B}^n) &= \int_{\mathbb{B}^n} d\mathbf{x} \\ &= \int_0^1 \int_0^\pi \cdots \int_0^{2\pi} r^{n-1} \sin(\varphi_1)^{n-2} \cdots \sin(\varphi_{n-2}) dr d\boldsymbol{\varphi} \\ &= \left( \int_0^1 r^{n-1} dr \right) \left( \int_0^\pi \sin(\varphi_1)^{n-2} d\varphi_1 \right) \cdots \left( \int_0^{2\pi} d\varphi_{n-1} \right) \\ &= \frac{1}{n} \cdot \frac{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2})} \cdot \frac{\Gamma(\frac{n-2}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n-1}{2})} \cdots \frac{\Gamma(1)\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})} \cdot 2\pi \\ &= \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)}. \end{aligned}$$

- (5) Note that we can apply the spherical coordinate system to  $\int_{\mathbb{S}^{n-1}} \omega_n$  directly (without the Stokes' theorem). The area element of the 2-sphere, is given by

$$\sin(\varphi_1)^{n-2} \sin(\varphi_2)^{n-3} \cdots \sin(\varphi_{n-2}) d\boldsymbol{\varphi}.$$

A tedious calculation (in the proof of (d)) shows that

$$\begin{aligned}
& \int_{\mathbb{S}^{n-1}} \omega_n \\
&= \int_0^\pi \cdots \int_0^{2\pi} \sin(\varphi_1)^{n-2} \sin(\varphi_2)^{n-3} \cdots \sin(\varphi_{n-2}) d\boldsymbol{\varphi} \\
&= \left( \int_0^\pi \sin(\varphi_1)^{n-2} d\varphi_1 \right) \cdots \left( \int_0^{2\pi} d\varphi_{n-1} \right) \\
&= \frac{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2})} \cdot \frac{\Gamma(\frac{n-2}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n-1}{2})} \cdots \frac{\Gamma(1)\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})} \cdot 2\pi \\
&= \frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)}.
\end{aligned}$$

□

*Outline of (d).*

- (i) One generalization of Exercise 10.21(a) and 10.22(a). See Exercise 10.23(a).
- (ii) One generalization of Exercise 10.22(b). Let  $\Sigma = \mathbb{S}^{n-1}$  be the  $(n-1)$ -surface in  $\mathbb{R}^n$ , with parameter domain  $D = [0, \pi]^{n-2} \times [0, 2\pi]$ , given by

$$\begin{aligned}
x_1 &= \cos \varphi_1 \\
x_2 &= \sin \varphi_1 \cos \varphi_2 \\
x_3 &= \sin \varphi_1 \sin \varphi_2 \cos \varphi_3 \\
&\vdots \\
x_{n-1} &= \sin \varphi_1 \cdots \sin \varphi_{n-2} \cos \varphi_{n-1} \\
x_n &= \sin \varphi_1 \cdots \sin \varphi_{n-2} \sin \varphi_{n-1}.
\end{aligned}$$

Let  $S$  denote the restriction of  $\Sigma$  to a parameter domain  $E \subseteq D$ . Prove that

$$\begin{aligned}
\int_S \omega_n &= \int_E \sin(\varphi_1)^{n-2} \sin(\varphi_2)^{n-3} \cdots \sin(\varphi_{n-2}) d\boldsymbol{\varphi} \\
&= A(S),
\end{aligned}$$

where  $A$  denotes surface area.

- (iii) One generalization of Exercise 10.22(c). Suppose  $g \in \mathcal{C}''([0, 1])$ ,  $\mathbf{h} = (h_1, \dots, h_n) \in \mathcal{C}''([0, 1]^{n-2})$ , and  $g > 0$ . Write  $\mathbf{x} = (x_1, \dots, x_n)$  and  $\mathbf{s} = (s_1, \dots, s_{n-2})$ . Let

$$\mathbf{x} = \Phi(\mathbf{s}, t)$$

define a  $(n-1)$ -surface  $\Phi$ , with parameter domain  $[0, 1]^{n-1}$ , by

$$\mathbf{x} = g(t)\mathbf{h}(\mathbf{s}).$$

Prove that

$$\int_{\Phi} \omega_n = 0.$$

- (iv) One generalization of Exercise 10.21(b) and 10.22(d). *Let  $E$  be a closed cell in  $D$ , with edges parallel to those of  $D$ . Suppose  $f \in \mathcal{C}''(D)$ ,  $f > 0$ . Let  $\Omega$  be the  $(n-1)$ -surface with parameter domain  $E$ , defined by*

$$\Omega(\varphi) = f(\varphi)\Sigma(\varphi).$$

Define  $S$  as in (ii) and prove that

$$\int_{\Omega} \omega_n = \int_S \omega_n = A(S).$$

- (v) One generalization of Exercise 10.21(d) and 10.22(e). See Exercise 10.23(b).  
(vi) One generalization of Examples 10.36 and 10.37. See Exercise 10.23(c).  
(vii) One generalization of Exercise 10.21(e) and 10.22(f). *Derive (iv) from Exercise 10.23(b), without using (iii).*  
(viii) One generalization of Exercise 10.21(f).  $\pi_{n-1}(\mathbb{S}^{n-1}) = \mathbb{Z}$  (without proof).  
(ix) One generalization of Exercise 10.22(g). *Show that  $\omega_n$  is exact in the complement of every line  $L$  passing through the origin.*

*Proof of (d)(ii).*

- (1) On  $S \subseteq \mathbb{S}^{n-1}$ , we have

$$\begin{aligned} \int_S \omega_n &= \int_S \sum_{i=1}^n (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\ &= \int_S \sum_{i=1}^n (-1)^{i-1} x_i(\varphi) \frac{\partial(x_1, \dots, \widehat{x_i}, \dots, x_n)}{\partial(\varphi_1, \dots, \varphi_{n-1})} d\varphi_1 \wedge \cdots \wedge d\varphi_{n-1}. \end{aligned}$$

Since  $\sum_{i=1}^n x_i(\varphi)^2 = 1$ , it suffices to show that

$$\frac{\partial(x_1, \dots, \widehat{x_i}, \dots, x_n)}{\partial(\varphi_1, \dots, \varphi_{n-1})} = (-1)^{i-1} \sin^{n-2} \varphi_1 \sin^{n-3} \varphi_2 \cdots \sin \varphi_{n-2} x_i.$$

- (2) Similar to Exercise 10.22(f), we might assume that  $\varphi_1 \in (0, \pi), \dots, \varphi_{n-2} \in (0, \pi)$  and  $\varphi_{n-1} \in (0, \pi) \cup (\pi, 2\pi)$ . To remove this additional restriction,

apply the argument in Exercise 6.7. Note that

$$\frac{\partial(x_1, \dots, \widehat{x_i}, \dots, x_n)}{\partial(\varphi_1, \dots, \varphi_{n-1})} = \det \underbrace{\begin{bmatrix} \frac{\partial x_1}{\partial \varphi_1} & \frac{\partial x_1}{\partial \varphi_2} & \dots & \frac{\partial x_1}{\partial \varphi_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{\frac{\partial x_i}{\partial \varphi_1}} & \widehat{\frac{\partial x_i}{\partial \varphi_2}} & \dots & \widehat{\frac{\partial x_i}{\partial \varphi_{n-1}}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial \varphi_1} & \frac{\partial x_n}{\partial \varphi_2} & \dots & \frac{\partial x_n}{\partial \varphi_{n-1}} \end{bmatrix}}_{\text{say } A_i}$$

where

$$\frac{\partial x_i}{\partial \varphi_j} = \begin{cases} 0 & \text{if } i < j, \\ -\tan(\varphi_i)x_i & \text{if } i = j, \\ \cot(\varphi_j)x_i & \text{if } i > j, \end{cases}$$

with  $\cos \varphi_n := 1$  or  $\varphi_n := 0$  by convention.

(3) Show that

$$\det(A_i) = (-1)^{i-1} \sin^{n-2} \varphi_1 \sin^{n-3} \varphi_2 \sin \varphi_{n-2} x_i$$

by induction.

□

*Proof of (d)(iii).*

(1) Similar to Exercise 10.22(c). Assume that  $\omega_n$  is well-defined, i.e.,  $\mathbf{h}(\mathbf{s}) \neq 0$

for all  $\mathbf{s} \in [0, 1]^{n-2}$ .

$$\begin{aligned}
\frac{\partial(x_1, \dots, \widehat{x_i}, \dots, x_n)}{\partial(s_1, \dots, s_{n-2}, t)} &= \det \begin{bmatrix} \frac{\partial x_1}{\partial s_1} & \dots & \frac{\partial x_1}{\partial s_{n-2}} & \frac{\partial x_1}{\partial t} \\ \vdots & \ddots & \vdots & \vdots \\ \widehat{\frac{\partial x_i}{\partial s_1}} & \dots & \widehat{\frac{\partial x_i}{\partial s_{n-2}}} & \widehat{\frac{\partial x_i}{\partial t}} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial x_n}{\partial s_1} & \dots & \frac{\partial x_n}{\partial s_{n-2}} & \frac{\partial x_n}{\partial t} \end{bmatrix} \\
&= \det \begin{bmatrix} g \frac{\partial h_1}{\partial s_1} & \dots & g \frac{\partial h_1}{\partial s_{n-2}} & g' h_1 \\ \vdots & \ddots & \vdots & \vdots \\ g \widehat{\frac{\partial h_i}{\partial s_1}} & \dots & g \widehat{\frac{\partial h_i}{\partial s_{n-2}}} & g' h_i \\ \vdots & \ddots & \vdots & \vdots \\ g \frac{\partial h_n}{\partial s_1} & \dots & g \frac{\partial h_n}{\partial s_{n-2}} & g' h_n \end{bmatrix} \\
&= g^{n-2} g' \det \underbrace{\begin{bmatrix} \frac{\partial h_1}{\partial s_1} & \dots & \frac{\partial h_1}{\partial s_{n-2}} & h_1 \\ \vdots & \ddots & \vdots & \vdots \\ \widehat{\frac{\partial h_i}{\partial s_1}} & \dots & \widehat{\frac{\partial h_i}{\partial s_{n-2}}} & \widehat{h_i} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial h_n}{\partial s_1} & \dots & \frac{\partial h_n}{\partial s_{n-2}} & h_n \end{bmatrix}}_{\text{say } A}.
\end{aligned}$$

(2) So

$$\begin{aligned}
\int_{\Phi} \omega_n &= \int_{[0,1]^{n-1}} \frac{1}{g(t)^n |\mathbf{h}(\mathbf{s})|^n} \sum_{i=1}^n (-1)^{i-1} g(t) h_i g(t)^{n-2} g'(t) \det(A) \, ds \, dt \\
&= \int_{[0,1]^{n-1}} \frac{g'(t)}{g(t) |\mathbf{h}(\mathbf{s})|^n} \sum_{i=1}^n (-1)^{i-1} h_i \det \begin{bmatrix} \frac{\partial h_1}{\partial s_1} & \dots & \frac{\partial h_1}{\partial s_{n-2}} & h_1 \\ \vdots & \ddots & \vdots & \vdots \\ \widehat{\frac{\partial h_i}{\partial s_1}} & \dots & \widehat{\frac{\partial h_i}{\partial s_{n-2}}} & \widehat{h_i} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial h_n}{\partial s_1} & \dots & \frac{\partial h_n}{\partial s_{n-2}} & h_n \end{bmatrix} \, ds \, dt \\
&= \int_{[0,1]^{n-1}} \frac{g'(t)}{g(t) |\mathbf{h}(\mathbf{s})|^n} \det \underbrace{\begin{bmatrix} h_1 & \frac{\partial h_1}{\partial s_1} & \dots & \frac{\partial h_1}{\partial s_{n-2}} & h_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h_n & \frac{\partial h_n}{\partial s_1} & \dots & \frac{\partial h_n}{\partial s_{n-2}} & h_n \end{bmatrix}}_{\text{say } B} \, ds \, dt.
\end{aligned}$$

Since the first column is the same as the last column in  $B$ ,  $\det(B) = 0$  (Theorem 9.34(d)). Therefore,  $\int_{\Phi} \omega_n = \int_{[0,1]^{n-1}} 0 \, ds \, dt = 0$ .

□

*Proof of (d)(iv).*

- (1) Consider the  $n$ -surface  $\Psi$  given by

$$\Psi(t, \boldsymbol{\varphi}) = [1 - t + tf(\boldsymbol{\varphi})]\Sigma(\boldsymbol{\varphi}),$$

where  $\boldsymbol{\varphi} \in E \subseteq D$ ,  $0 \leq t \leq 1$ .

- (2) Write

$$E = [a_1, b_1] \times \cdots \times [a_{n-1}, b_{n-1}] \subseteq D.$$

Note that  $\Psi(t, \boldsymbol{\varphi}) \subseteq \mathbb{R}^n - \{\mathbf{0}\}$ . So the boundary of  $\Psi$  is

$$\partial\Psi = \Psi(0, \boldsymbol{\varphi}) - \Psi(1, \boldsymbol{\varphi}) + \sum_{i=1}^{n-1} (\Psi|_{\varphi_i=a_i} - \Psi|_{\varphi_i=b_i}),$$

where  $\Psi|_{\varphi_i=\theta} : [a_1, b_1] \times \cdots \times \widehat{[a_i, b_i]} \times \cdots \times [a_{n-1}, b_{n-1}] \rightarrow \Omega$  is a mapping defined by

$$\begin{aligned} \Psi|_{\varphi_i=\theta}(t, \varphi_1, \dots, \widehat{\varphi_i}, \dots, \varphi_{n-1}) &= \Psi(t, \varphi_1, \dots, \varphi_{i-1}, \theta, \varphi_{i+1}, \dots, \varphi_{n-1}) \\ &= \Psi(t, \boldsymbol{\varphi} + (\theta - \varphi_i)\mathbf{e}_i). \end{aligned}$$

- (3) Show that

$$\int_{\Psi|_{\varphi_1=\theta}} \omega_n = 0$$

for any fixed  $\varphi_1 = \theta \in [a_1, b_1]$ . Note that  $\omega_n$  is well-defined on  $\Psi|_{\varphi_1=\theta}$ . Write

$$\Psi|_{\varphi_1=\theta}(t, \widehat{\varphi_1}, \varphi_2, \dots, \varphi_{n-1}) = \mathbf{x}(t, \widehat{\varphi_1}, \varphi_2, \dots, \varphi_{n-1}).$$

By definition of  $\Psi$ , we have

$$\begin{aligned} x_1 &= g(t, \boldsymbol{\varphi} + (\theta - \varphi_1)\mathbf{e}_1) \cos \theta \\ x_2 &= g(t, \boldsymbol{\varphi} + (\theta - \varphi_1)\mathbf{e}_1) \sin \theta \cos \varphi_2 \\ &\dots \\ x_{n-1} &= g(t, \boldsymbol{\varphi} + (\theta - \varphi_1)\mathbf{e}_1) \sin \theta \cdots \sin \varphi_{n-2} \cos \varphi_{n-1} \\ x_n &= g(t, \boldsymbol{\varphi} + (\theta - \varphi_1)\mathbf{e}_1) \sin \theta \cdots \sin \varphi_{n-2} \sin \varphi_{n-1}, \end{aligned}$$

where  $g(t, \boldsymbol{\varphi} + (\theta - \varphi_1)\mathbf{e}_1) = 1 - t + tf(\boldsymbol{\varphi} + (\theta - \varphi_1)\mathbf{e}_1)$ .

- (4) Note that  $r_n = g > 0$ . Since

$$\frac{\partial x_i}{\partial t} = \frac{\partial g}{\partial t} g^{-1} x_i,$$

$$\begin{aligned}
\frac{\partial(x_1, \dots, \widehat{x_i}, \dots, x_n)}{\partial(t, \widehat{\varphi_1}, \varphi_2, \dots, \varphi_{n-1})} &= \det \begin{bmatrix} \frac{\partial x_1}{\partial t} & \frac{\partial x_1}{\partial \varphi_2} & \cdots & \frac{\partial x_1}{\partial \varphi_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{\frac{\partial x_i}{\partial t}} & \widehat{\frac{\partial x_i}{\partial \varphi_2}} & \cdots & \widehat{\frac{\partial x_i}{\partial \varphi_{n-1}}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial t} & \frac{\partial x_n}{\partial \varphi_2} & \cdots & \frac{\partial x_n}{\partial \varphi_{n-1}} \end{bmatrix} \\
&= \det \begin{bmatrix} \frac{\partial g}{\partial t} g^{-1} x_1 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{\frac{\partial g}{\partial t} g^{-1} x_i} & \widehat{*} & \cdots & \widehat{*} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g}{\partial t} g^{-1} x_n & * & \cdots & * \end{bmatrix} \\
&= \frac{\partial g}{\partial t} g^{-1} \det \underbrace{\begin{bmatrix} x_1 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{x_i} & \widehat{*} & \cdots & \widehat{*} \\ \vdots & \vdots & \ddots & \vdots \\ x_n & * & \cdots & * \end{bmatrix}}_{\text{say } A}.
\end{aligned}$$

So

$$\begin{aligned}
\int_{\Psi|_{\varphi_1=\theta}} \omega_n &= \int_E g^{-n} \sum_{i=1}^n (-1)^{i-1} x_i \frac{\partial g}{\partial t} g^{-1} \det(A) dt d\varphi_2 \cdots d\varphi_{n-1} \\
&= \int_E \frac{\partial g}{\partial t} g^{-n-1} \sum_{i=1}^n (-1)^{i-1} x_i \det \begin{bmatrix} x_1 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{x_i} & \widehat{*} & \cdots & \widehat{*} \\ \vdots & \vdots & \ddots & \vdots \\ x_n & * & \cdots & * \end{bmatrix} dt d\varphi_2 \cdots d\varphi_{n-1} \\
&= \int_E \frac{\partial g}{\partial t} g^{-n-1} \det \underbrace{\begin{bmatrix} x_1 & x_1 & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_i & x_i & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n & x_n & * & \cdots & * \end{bmatrix}}_{\text{say } B} dt d\varphi_2 \cdots d\varphi_{n-1}.
\end{aligned}$$

Since the first column is the same as the second column in  $B$ ,  $\det(B) = 0$  (Theorem 9.34(d)). Therefore,  $\int_{\Psi|_{\varphi_1=\theta}} \omega_n = 0$ .

(5)  $\int_{\Psi|_{\varphi_i=\theta}} \omega_n = 0$  is also true for all  $i = 1, \dots, n-1$  by the same argument

in (3)(4). Hence,

$$\begin{aligned}
0 &= \int_{\Psi} d\omega_n \\
&= \int_{\partial\Psi} \omega_n \\
&= \int_S \omega_n - \int_{\Omega} \omega_n + \sum_{i=1}^{n-1} \left( \int_{\Psi|_{\varphi_i=a_i}} \omega_n - \int_{\Psi|_{\varphi_i=b_i}} \omega_n \right) \\
&= \int_S \omega_n - \int_{\Omega} \omega_n
\end{aligned}$$

by (a) and the Stokes' theorem (Theorem 10.33), or

$$\int_{\Omega} \omega_n = \underbrace{\int_S \omega_n}_{\text{by (d)(ii)}} = A(S).$$

□

*Proof of (d)(vii).*

(1)

(2)

□

*Proof of (d)(ix).* Similar to Exercise 10.22(g).

(1) Given any line  $L$  passing through  $\mathbf{0}$ , say

$$(r \cos \varphi_1, \dots, \sin \varphi_1 \cdots \sin \varphi_{n-2} \sin \varphi_{n-1}) \in L \subseteq \mathbb{R}^n$$

where  $r \in \mathbb{R}^1$  for some  $\varphi \in [0, \pi]^{n-2} \times [0, 2\pi]$ . We will show that  $\omega_n$  is exact in  $U = \mathbb{R}^n - L$ .

(2) Linear algebra says that all rotation matrices  $T \in SO(n)$  can be obtained from

$$R_i(u) = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & 0 \\ & & 1 & & & \\ & & & R(u) & & \\ & & & & 1 & \\ 0 & & & & & \ddots \\ & & & & & & 1 \end{bmatrix}$$



using matrix multiplication. Here

$$R(u) = \begin{bmatrix} \cos u & -\sin u \\ \sin u & \cos u \end{bmatrix}$$

is a 2-by-2 rotation matrix at the  $i$ th row and  $i$ th column. For example, the rotation

$$T = R_1(-\varphi_1)R_2(-\varphi_2)\cdots R_{n-2}(-\varphi_{n-2})R_{n-1}(-\varphi_{n-1})$$

maps  $L$  to the  $x_n$ -axis. Similar to Exercise 10.22(g), it suffices to show that  $\omega_n$  is invariant under  $T = R_1(u)$ .

(3) Show that  $\omega_n$  is invariant under  $T = R_1(u)$ . By

$$T : \mathbf{x} \mapsto (x_1 \cos u - x_2 \sin u, x_1 \sin u + x_2 \cos u, x_3, \dots, x_n),$$

we have

$$\begin{aligned} r_n &\mapsto r_n \\ dx_1 &\mapsto \cos u dx_1 - \sin u dx_2 \\ dx_2 &\mapsto \sin u dx_1 + \cos u dx_2 \\ dx_3 &\mapsto dx_3 \\ &\dots \\ dx_n &\mapsto dx_n. \end{aligned}$$

So  $dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$  maps to

$$\begin{cases} \cos u \widehat{dx_1} \wedge \cdots \wedge dx_n + \sin u dx_1 \wedge \widehat{dx_2} \wedge \cdots \wedge dx_n & \text{if } i = 1 \\ -\sin u \widehat{dx_1} \wedge \cdots \wedge dx_n + \cos u dx_1 \wedge \widehat{dx_2} \wedge \cdots \wedge dx_n & \text{if } i = 2 \\ dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n & \text{otherwise.} \end{cases}$$

Thus

$$\begin{aligned}
\omega_n &\mapsto (r_n)^{-n}(x_1 \cos u - x_2 \sin u) \\
&\quad \left( \cos u \widehat{dx_1} \wedge \cdots \wedge dx_n + \sin u dx_1 \wedge \widehat{dx_2} \wedge \cdots \wedge dx_n \right) \\
&\quad + (r_n)^{-n}(x_1 \sin u + x_2 \cos u) \\
&\quad \left( -\sin u \widehat{dx_1} \wedge \cdots \wedge dx_n + \cos u dx_1 \wedge \widehat{dx_2} \wedge \cdots \wedge dx_n \right) \\
&\quad + (r_n)^{-n} \sum_{i=3}^n (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\
&= (r_n)^{-n} x_1 \widehat{dx_1} \wedge \cdots \wedge dx_n \\
&\quad - (r_n)^{-n} x_2 dx_1 \wedge \widehat{dx_2} \wedge \cdots \wedge dx_n \\
&\quad + (r_n)^{-n} \sum_{i=3}^n (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\
&= (r_n)^{-n} \sum_{i=1}^n (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\
&= \omega_n.
\end{aligned}$$

- (4) Similar to Exercise 10.22(g),  $\omega_n$  is exact in  $\mathbb{R}^n - L$ . (Or  $\omega_n$  is locally exact in  $\mathbb{R}^n - \{\mathbf{0}\}$ .)

□

**Exercise 10.24.** Let  $\omega = \sum a_i(\mathbf{x})dx_i$  be a 1-form of class  $\mathcal{C}''$  in a convex open set  $E \subseteq \mathbb{R}^n$ . Assume  $d\omega = 0$  and prove that  $\omega$  is exact in  $E$ , by completing the following outline:

Fix  $\mathbf{p} \in E$ . Define

$$f(\mathbf{x}) = \int_{[\mathbf{p}, \mathbf{x}]} \omega \quad (\mathbf{x} \in E).$$

Apply Stokes' theorem to affine-oriented 2-simplices  $[\mathbf{p}, \mathbf{x}, \mathbf{y}]$  in  $E$ . Deduce that

$$f(\mathbf{y}) - f(\mathbf{x}) = \sum_{i=1}^n (y_i - x_i) \int_0^1 a_i((1-t)\mathbf{x} + t\mathbf{y}) dt$$

for  $\mathbf{x} \in E$ ,  $\mathbf{y} \in E$ . Hence  $(D_i f)(\mathbf{x}) = a_i(\mathbf{x})$ .

*Proof.*

- (1) Fix  $\mathbf{p} \in E$ . Define

$$f(\mathbf{x}) = \int_{[\mathbf{p}, \mathbf{x}]} \omega \quad (\mathbf{x} \in E).$$

- (2) Given any  $\mathbf{x} \in E$ ,  $\mathbf{y} \in E$ , and  $\mathbf{x} \neq \mathbf{y}$ . The affine-oriented 2-simplex  $\Psi = [\mathbf{p}, \mathbf{x}, \mathbf{y}]$  is in  $E$  by the convexity of  $E$ . (If  $E$  is open but not convex, we can show that  $\omega = df$  **locally** as the note in Exercise 10.21(a). That is why we say that  $\omega$  is locally exact. The proof is exactly the same.)

- (3) Note that

$$\partial\Psi = \partial[\mathbf{p}, \mathbf{x}, \mathbf{y}] = [\mathbf{x}, \mathbf{y}] - [\mathbf{p}, \mathbf{y}] + [\mathbf{p}, \mathbf{x}].$$

The Stokes' theorem (Theorem 10.33) implies that

$$\begin{aligned} \int_{\Psi} d\omega &= \int_{\partial\Psi} \omega \iff \int_{\Psi} 0 = \int_{[\mathbf{x}, \mathbf{y}]} \omega - \int_{[\mathbf{p}, \mathbf{y}]} \omega + \int_{[\mathbf{p}, \mathbf{x}]} \omega \\ &\iff 0 = \int_{[\mathbf{x}, \mathbf{y}]} \omega - f(\mathbf{y}) + f(\mathbf{x}) \\ &\iff f(\mathbf{y}) - f(\mathbf{x}) = \int_{[\mathbf{x}, \mathbf{y}]} \omega. \end{aligned}$$

- (4) Define  $\gamma : [0, 1] \rightarrow E$  by

$$\begin{aligned} \gamma(t) &= \mathbf{x} + t(\mathbf{y} - \mathbf{x}) \\ &= \sum_{i=1}^n x_i + t(y_i - x_i) \end{aligned}$$

(where  $\mathbf{x} = (x_1, \dots, x_n)$  and  $\mathbf{y} = (y_1, \dots, y_n)$ ). Hence  $[0, 1]$  is the parameter domain of  $[\mathbf{x}, \mathbf{y}]$  with respect to  $\gamma$ . So

$$\begin{aligned} \int_{[\mathbf{x}, \mathbf{y}]} \omega &= \int_0^1 \sum_{i=1}^n a_i(\gamma(t)) \frac{\partial(x_i + t(y_i - x_i))}{\partial t} dt \\ &= \int_0^1 \sum_{i=1}^n a_i(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))(y_i - x_i) dt \\ &= \sum_{i=1}^n (y_i - x_i) \int_0^1 a_i(\mathbf{x} + t(\mathbf{y} - \mathbf{x})) dt. \end{aligned}$$

Thus,

$$f(\mathbf{y}) - f(\mathbf{x}) = \sum_{i=1}^n (y_i - x_i) \int_0^1 a_i(\mathbf{x} + t(\mathbf{y} - \mathbf{x})) dt.$$

- (5) Note that

$$\begin{aligned} f(\mathbf{x} + h\mathbf{e}_j) - f(\mathbf{x}) &= \sum_{i=1}^n ((x_i + h\delta_{ij}) - x_i) \int_0^1 a_i(\mathbf{x} + t((\mathbf{x} + h\mathbf{e}_j) - \mathbf{x})) dt \\ &= \sum_{i=1}^n h\delta_{ij} \int_0^1 a_i(\mathbf{x} + t h\mathbf{e}_j) dt \\ &= h \int_0^1 a_j(\mathbf{x} + t h\mathbf{e}_j) dt. \end{aligned}$$

(Here  $\delta_{ij}$  is the Kronecker delta.) So

$$\begin{aligned}
(D_j f)(\mathbf{x}) &= \lim_{h \rightarrow 0} \frac{f(\mathbf{x} + h\mathbf{e}_j) - f(\mathbf{x})}{h} \\
&= \lim_{h \rightarrow 0} \int_0^1 a_j(\mathbf{x} + th\mathbf{e}_j) dt \\
&= \int_0^1 a_j(\mathbf{x}) dt \quad (a_j \in \mathcal{C}'') \\
&= a_j(\mathbf{x}).
\end{aligned}$$

Thus,

$$df = \sum_{j=1}^n (D_j f)(\mathbf{x}) dx_j = \sum_{j=1}^n a_j(\mathbf{x}) dx_j = \omega,$$

or  $\omega$  is exact in  $E$ .

□

**Exercise 10.25.** Assume  $\omega$  is a 1-form in an open set  $E \subseteq \mathbb{R}^n$  such that

$$\int_{\gamma} \omega = 0$$

for every closed curve  $\gamma$  in  $E$ , of class  $\mathcal{C}'$ . Prove that  $\omega$  is exact in  $E$ , by imitating part of the argument sketched in Exercise 10.24.

*Proof.*

(1) Assume that  $E$  is a **connected** open subset of  $\mathbb{R}^n$ . Show that  $\omega$  is exact in  $E$  if  $\int_{\gamma} \omega = 0$  for every closed curve  $\gamma$  in  $E$ , of class  $\mathcal{C}'$ .

(2) Fix  $\mathbf{p} \in E$ . Define

$$f(\mathbf{x}) = \int_{[\mathbf{p}, \mathbf{x}]} \omega \quad (\mathbf{x} \in E).$$

It is well-defined since  $E$  is connected and  $\int_{\gamma} \omega = 0$  for every closed curve  $\gamma$  in  $E$ .

(3) Given any  $\mathbf{x} \in E$ ,  $\mathbf{y} \in E$ , and  $\mathbf{x} \neq \mathbf{y}$ . Let

$$\gamma = [\mathbf{x}, \mathbf{y}] - [\mathbf{p}, \mathbf{y}] + [\mathbf{p}, \mathbf{x}]$$

be a closed curve in  $E$ . Hence,

$$\begin{aligned}
0 &= \int_{\gamma} \omega && \text{(Assumption)} \\
&= \int_{[\mathbf{x}, \mathbf{y}]} \omega - \int_{[\mathbf{p}, \mathbf{y}]} \omega + \int_{[\mathbf{p}, \mathbf{x}]} \omega \\
&= \int_{[\mathbf{x}, \mathbf{y}]} \omega - f(\mathbf{y}) + f(\mathbf{x}).
\end{aligned}$$

So

$$f(\mathbf{y}) - f(\mathbf{x}) = \int_{[\mathbf{x}, \mathbf{y}]} \omega$$

- (4) Similar to (4)(5) in the proof of Exercise 10.24, we have  $df = \omega$ . So the statement in (1) is proved. In general, we can define each  $f_\alpha$  on each connected component  $E_\alpha$  (which is open) of  $E$  such that  $df_\alpha = \omega$  on  $E_\alpha$ . Take

$$f|_{E_\alpha} = f_\alpha$$

on  $E$ . Hence,  $df = \omega$  on the whole  $E$ .

□

**Exercise 10.26.** Assume  $\omega$  is a 1-form in  $\mathbb{R}^3 - \{\mathbf{0}\}$ , of class  $\mathcal{C}'$  and  $d\omega = 0$ . Prove that  $\omega$  is exact in  $\mathbb{R}^3 - \{\mathbf{0}\}$ . (Hint: Every closed continuously differentiable curve in  $\mathbb{R}^3 - \{\mathbf{0}\}$  is the boundary of a 2-surface in  $\mathbb{R}^3 - \{\mathbf{0}\}$ . Apply Stokes' theorem and Exercise 10.25.)

*Proof.*

- (1) Let  $E = \mathbb{R}^3 - \{\mathbf{0}\}$ . By Exercise 10.25, it suffices to show that

$$\int_\gamma \omega = 0$$

for every closed curve  $\gamma$  in  $E$ , of class  $\mathcal{C}'$ .

- (2) Intuitively, every closed continuously differentiable curve in  $\mathbb{R}^3 - \{\mathbf{0}\}$  is the boundary of a 2-surface in  $\mathbb{R}^3 - \{\mathbf{0}\}$ . So there is some 2-surface  $\Psi$  such that  $\partial\Psi = \gamma$ . The Stokes' theorem (Theorem 10.33) implies that

$$\int_\gamma \omega = \int_{\partial\Psi} \omega = \int_\Psi d\omega = \int_\Psi 0 = 0.$$

□

**Exercise 10.27.** Let  $E$  be an open 3-cell in  $\mathbb{R}^3$ , with edges parallel to the coordinate axes. Suppose  $(a, b, c) \in E$ ,  $f_i \in \mathcal{C}'(E)$  for  $i = 1, 2, 3$ ,

$$\omega = f_1 dy \wedge dz + f_2 dz \wedge dx + f_3 dx \wedge dy,$$

and assume that  $d\omega = 0$  in  $E$ . Define

$$\lambda = g_1 dx + g_2 dy$$

where

$$\begin{aligned} g_1(x, y, z) &= \int_c^z f_2(x, y, s)ds - \int_b^y f_3(x, t, c)dt \\ g_2(x, y, z) &= - \int_c^z f_1(x, y, s)ds, \end{aligned}$$

for  $(x, y, z) \in E$ . Prove that  $d\lambda = \omega$  in  $E$ . Evaluate these integrals when  $\omega = \zeta$  and thus find the form  $\lambda$  that occurs in part (e) of Exercise 10.22.

*Proof.*

(1) Let  $\mathbf{F} = f_1\mathbf{e}_1 + f_2\mathbf{e}_2 + f_3\mathbf{e}_3$  as in Vector fields 10.42. Then

$$d\omega = (\nabla \cdot \mathbf{F})dx \wedge dy \wedge dz.$$

As  $d\omega = 0$  by assumption,  $\nabla \cdot \mathbf{F} = D_1f_1 + D_2f_2 + D_3f_3 = 0$ .

(2) As

$$\begin{aligned} d\lambda &= d(g_1dx + g_2dy) \\ &= (D_1g_1dx + D_2g_1dy + D_3g_1dz) \wedge dx \\ &\quad + (D_1g_2dx + D_2g_2dy + D_3g_2dz) \wedge dy \\ &= (-D_3g_2)dy \wedge dz + (D_3g_1)dz \wedge dx + (D_1g_2 - D_2g_1)dx \wedge dy, \end{aligned}$$

it suffices to show that

$$\begin{aligned} f_1 &= -D_3g_2, \\ f_2 &= D_3g_1, \\ f_3 &= D_1g_2 - D_2g_1 \end{aligned}$$

on  $E$ .

(3) Theorem 6.20 implies that

$$-D_3g_2 = D_3 \int_c^z f_1(x, y, s)ds = f_1(x, y, z)$$

and

$$D_3g_1 = D_3 \int_c^z f_2(x, y, s)ds - D_3 \int_b^y f_3(x, t, c)dt = f_2(x, y, z).$$

Also,

$$\begin{aligned}
& D_1 g_2 - D_2 g_1 \\
&= D_1 \left( - \int_c^z f_1(x, y, s) ds \right) \\
&\quad - D_2 \left( \int_c^z f_2(x, y, s) ds - \int_b^y f_3(x, t, c) dt \right) \\
&= - \int_c^z D_1 f_1(x, y, s) ds \quad (f_1 \in \mathcal{C}') \\
&\quad - \int_c^z D_2 f_2(x, y, s) ds + f_3(x, y, c) \quad (f_2 \in \mathcal{C}', \text{ Theorem 6.20}) \\
&= \int_c^z D_3 f_3(x, y, s) ds + f_3(x, y, c) \quad ((1)) \\
&= f_3(x, y, z) \quad (\text{Theorem 6.21}).
\end{aligned}$$

Therefore,  $d\lambda = \omega$  in  $E$ .

(4) When  $\omega = \zeta = r^{-3}(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy)$ , we get

$$\begin{aligned}
f_1(x, y, z) &= x(x^2 + y^2 + z^2)^{-\frac{3}{2}}, \\
f_2(x, y, z) &= y(x^2 + y^2 + z^2)^{-\frac{3}{2}}, \\
f_3(x, y, z) &= z(x^2 + y^2 + z^2)^{-\frac{3}{2}}.
\end{aligned}$$

So,

$$\begin{aligned}
\int_c^z f_2(x, y, s) ds &= \left[ ys(x^2 + y^2)^{-1}(x^2 + y^2 + s^2)^{-\frac{1}{2}} \right]_{s=c}^{s=z}, \\
\int_b^y f_3(x, t, c) dt &= \left[ ct(x^2 + c^2)^{-1}(x^2 + t^2 + c^2)^{-\frac{1}{2}} \right]_{t=b}^{t=y}, \\
\int_c^z f_1(x, y, s) ds &= \left[ xs(x^2 + y^2)^{-1}(x^2 + y^2 + s^2)^{-\frac{1}{2}} \right]_{s=c}^{s=z}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\lambda &= g_1 dx + g_2 dy \\
&= \left[ ys(x^2 + y^2)^{-1}(x^2 + y^2 + s^2)^{-\frac{1}{2}} \right]_{s=c}^{s=z} dx \\
&\quad - \left[ ct(x^2 + c^2)^{-1}(x^2 + t^2 + c^2)^{-\frac{1}{2}} \right]_{t=b}^{t=y} dx \\
&\quad + \left[ xs(x^2 + y^2)^{-1}(x^2 + y^2 + s^2)^{-\frac{1}{2}} \right]_{s=c}^{s=z} dy \\
&= - \left[ zr^{-1} - c(x^2 + y^2 + c^2)^{-\frac{1}{2}} \right] \eta \quad (\text{Definition of } \eta) \\
&\quad - c(x^2 + c^2)^{-1} \left[ y(x^2 + y^2 + c^2)^{-\frac{1}{2}} - b(x^2 + b^2 + c^2)^{-\frac{1}{2}} \right] dx.
\end{aligned}$$

As we pick  $(a, b, c) = (a, 0, 0) \in \mathbb{R}^3 - \{\mathbf{0}\}$  (or  $a \neq 0$ ), we have  $\lambda = -zr^{-1}\eta$  such that  $d\lambda = \omega = \zeta$ , which is the same as part (e) in Exercise 10.22.

□

**Exercise 10.28.** Fix  $b > a > 0$ , define

$$\Phi(r, \theta) = (r \cos \theta, r \sin \theta)$$

for  $a \leq r \leq b$ ,  $0 \leq \theta \leq 2\pi$ . (The range of  $\Phi$  is an annulus in  $\mathbb{R}^2$ .) Put  $\omega = x^3 dy$ , and compute both

$$\int_{\Phi} d\omega \quad \text{and} \quad \int_{\partial\Phi} \omega$$

to verify that they are equal.

*Proof.*

(1) Note that

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \det \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix} = r.$$

So

$$\begin{aligned} \int_{\Phi} d\omega &= \int_{\Phi} 3x^2 dx \wedge dy & (dy \wedge dy = 0) \\ &= \int_{[a, b] \times [0, 2\pi]} 3(r \cos \theta)^2 \frac{\partial(x, y)}{\partial(r, \theta)} dr d\theta \\ &= \int_a^b \int_0^{2\pi} 3r^3 (\cos \theta)^2 dr d\theta \\ &= \frac{3\pi}{4} (b^4 - a^4). \end{aligned}$$

(2) Similar to Exercise 10.21(b), write

$$\partial\Phi = \Gamma - \gamma,$$

where  $\Gamma(t) = (b \cos t, b \sin t)$  on  $[0, 2\pi]$  and  $\gamma(t) = (a \cos t, a \sin t)$  on  $[0, 2\pi]$ .

Hence

$$\begin{aligned} \int_{\partial\Phi} \omega &= \int_{\Gamma} \omega - \int_{\gamma} \omega \\ &= \int_{\Gamma} x^3 dy - \int_{\gamma} x^3 dy \\ &= \int_{[0, 2\pi]} (b \cos \theta)^3 \frac{\partial y}{\partial \theta} d\theta - \int_{[0, 2\pi]} (a \cos \theta)^3 \frac{\partial y}{\partial \theta} d\theta \\ &= \int_0^{2\pi} b^4 (\cos \theta)^4 d\theta - \int_0^{2\pi} a^4 (\cos \theta)^4 d\theta \\ &= \frac{3\pi}{4} (b^4 - a^4). \end{aligned}$$



(3)

$$\int_{\Phi} d\omega = \int_{\partial\Phi} \omega = \frac{3\pi}{4}(b^4 - a^4).$$

□

**Exercise 10.29.** *Prove the existence of a function  $\alpha$  with the properties needed in the proof of Theorem 10.38, and prove that the resulting function  $F$  is of class  $\mathcal{C}'$ . (Both assertions become trivial if  $E$  is an open cell or an open ball, since  $\alpha$  can then be taken to be a constant. Refer to Theorem 9.42.)*

*Proof.*

(1)

(2)

□

**Exercise 10.30.** *If  $\mathbf{N}$  is the vector given by*

$$\mathbf{N} = (\alpha_2\beta_3 - \alpha_3\beta_2)\mathbf{e}_1 + (\alpha_3\beta_1 - \alpha_1\beta_3)\mathbf{e}_2 + (\alpha_1\beta_2 - \alpha_2\beta_1)\mathbf{e}_3$$

*(Equation (135)), prove that*

$$\det \begin{bmatrix} \alpha_1 & \beta_1 & \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_2 & \beta_2 & \alpha_3\beta_1 - \alpha_1\beta_3 \\ \alpha_3 & \beta_3 & \alpha_1\beta_2 - \alpha_2\beta_1 \end{bmatrix} = |\mathbf{N}|^2$$

*Also, verify*

$$\mathbf{N} \cdot (T\mathbf{e}_1) = \mathbf{N} \cdot (T\mathbf{e}_2)$$

*(Equation (137)).*

*Proof.*

(1) By Laplace's expansion along the third column,

$$\begin{aligned}
& \det \begin{bmatrix} \alpha_1 & \beta_1 & \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_2 & \beta_2 & \alpha_3\beta_1 - \alpha_1\beta_3 \\ \alpha_3 & \beta_3 & \alpha_1\beta_2 - \alpha_2\beta_1 \end{bmatrix} \\
&= (-1)^{1+3}(\alpha_2\beta_3 - \alpha_3\beta_2) \det \begin{bmatrix} \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} \\
&\quad + (-1)^{2+3}(\alpha_3\beta_1 - \alpha_1\beta_3) \det \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_3 & \beta_3 \end{bmatrix} \\
&\quad + (-1)^{3+3}(\alpha_1\beta_2 - \alpha_2\beta_1) \det \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix} \\
&= (\alpha_2\beta_3 - \alpha_3\beta_2)^2 + (\alpha_3\beta_1 - \alpha_1\beta_3)^2 + (\alpha_1\beta_2 - \alpha_2\beta_1)^2 \\
&= |\mathbf{N}|^2.
\end{aligned}$$

(2)

$$\begin{aligned}
\mathbf{N} \cdot (T\mathbf{e}_1) &= (\alpha_2\beta_3 - \alpha_3\beta_2, \alpha_3\beta_1 - \alpha_1\beta_3, \alpha_1\beta_2 - \alpha_2\beta_1) \cdot (\alpha_1, \alpha_2, \alpha_3) \\
&= (\alpha_2\beta_3 - \alpha_3\beta_2)\alpha_1 + (\alpha_3\beta_1 - \alpha_1\beta_3)\alpha_2 + (\alpha_1\beta_2 - \alpha_2\beta_1)\alpha_3 \\
&= (\alpha_3\alpha_2 - \alpha_2\alpha_3)\beta_1 + (\alpha_1\alpha_3 - \alpha_3\alpha_1)\beta_2 + (\alpha_2\alpha_1 - \alpha_1\alpha_2)\beta_3 \\
&= 0.
\end{aligned}$$

(3)

$$\begin{aligned}
\mathbf{N} \cdot (T\mathbf{e}_2) &= (\alpha_2\beta_3 - \alpha_3\beta_2, \alpha_3\beta_1 - \alpha_1\beta_3, \alpha_1\beta_2 - \alpha_2\beta_1) \cdot (\beta_1, \beta_2, \beta_3) \\
&= (\alpha_2\beta_3 - \alpha_3\beta_2)\beta_1 + (\alpha_3\beta_1 - \alpha_1\beta_3)\beta_2 + (\alpha_1\beta_2 - \alpha_2\beta_1)\beta_3 \\
&= (\beta_2\beta_3 - \beta_3\beta_2)\alpha_1 + (\beta_3\beta_1 - \beta_1\beta_3)\alpha_2 + (\beta_1\beta_2 - \beta_2\beta_1)\alpha_3 \\
&= 0.
\end{aligned}$$

□

**Exercise 10.31.** Let  $E \subseteq \mathbb{R}^3$  be open, suppose  $g \in \mathcal{C}''(E)$ ,  $h \in \mathcal{C}'''(E)$ , and consider the vector field

$$\mathbf{F} = g\nabla h$$

(a) Prove that

$$\nabla \cdot \mathbf{F} = g\nabla^2 h + (\nabla g) \cdot (\nabla h)$$

where  $\nabla^2 h = \nabla \cdot (\nabla h) = \sum \frac{\partial^2 h}{\partial x_i^2}$  is the so-called “Laplacian” of  $h$ .

(b) If  $\Omega$  is a closed subset of  $E$  with positively oriented boundary  $\partial\Omega$  (as in Theorem 10.51), prove that

$$\int_{\Omega} [g\nabla^2 h + (\nabla g) \cdot (\nabla h)] dV = \int_{\partial\Omega} g \frac{\partial h}{\partial n} dA$$

where (as is customary) we have written  $\frac{\partial h}{\partial n}$  in place of  $(\nabla h) \cdot \mathbf{n}$ . (Thus  $\frac{\partial h}{\partial n}$  is the directional derivative of  $h$  in the direction of the outward normal to  $\partial\Omega$ , the so-called **normal derivative** of  $h$ .) Interchange  $g$  and  $h$ , subtract the resulting formula from the first one, to obtain

$$\int_{\Omega} (g\nabla^2 h - h\nabla^2 g) dV = \int_{\partial\Omega} \left( g \frac{\partial h}{\partial n} - h \frac{\partial g}{\partial n} \right) dA.$$

These two formulas are usually called **Green's identities**.

- (c) Assume that  $h$  is **harmonic** in  $E$ ; this means that  $\nabla^2 h = 0$ . Take  $g = 1$  and conclude that

$$\int_{\partial\Omega} \frac{\partial h}{\partial n} dA = 0.$$

Take  $g = h$ , and conclude that  $h = 0$  in  $\Omega$  if  $h = 0$  on  $\partial\Omega$ .

- (d) Show that Green's identities are also valid in  $\mathbb{R}^2$ .

*Proof of (a).*

- (1) Since

$$\mathbf{F} = g\nabla h = g \left( \sum (D_i h) \mathbf{e}_i \right) = \sum g(D_i h) \mathbf{e}_i,$$

we have

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \nabla \cdot \left( \sum g(D_i h) \mathbf{e}_i \right) \\ &= \sum D_i (g(D_i h)) \\ &= \sum \{ (D_i g)(D_i h) + g D_i (D_i h) \} \\ &= \sum (D_i g)(D_i h) + g \sum D_i (D_i h). \end{aligned}$$

- (2) Also,

$$\begin{aligned} g\nabla^2 h + (\nabla g) \cdot (\nabla h) &= g\nabla \cdot (\nabla h) + (\nabla g) \cdot (\nabla h) \\ &= g\nabla \cdot \left( \sum (D_i h) \mathbf{e}_i \right) + \left( \sum (D_i g) \mathbf{e}_i \right) \cdot \left( \sum (D_i h) \mathbf{e}_i \right) \\ &= g \sum D_i (D_i h) + \sum (D_i g)(D_i h). \end{aligned}$$

- (3) By (1)(2), the result is established.

□

*Proof of (b).*

(1) The divergence theorem (Theorem 10.51) implies that

$$\begin{aligned} \int_{\Omega} (\nabla \cdot \mathbf{F}) dV &= \int_{\partial\Omega} (\mathbf{F} \cdot \mathbf{n}) dA \\ \implies \int_{\Omega} [g \nabla^2 h + (\nabla g) \cdot (\nabla h)] dV &= \int_{\partial\Omega} g \underbrace{\nabla h \cdot \mathbf{n}}_{=\frac{\partial h}{\partial n}} dA. \end{aligned}$$

(2) Green's identities are a set of three identities in vector calculus relating the bulk with the boundary of a region on which differential operators act. (*Green's third identity.*) Assume that  $h$  is harmonic in  $E$ . If  $G(\mathbf{x}, \mathbf{x}_0)$  is the Green's function, then

$$h(\mathbf{x}_0) = \int_{\partial\Omega} \left[ h(\mathbf{x}) \frac{\partial G(\mathbf{x}, \mathbf{x}_0)}{\partial n} - G(\mathbf{x}, \mathbf{x}_0) \frac{\partial h(\mathbf{x})}{\partial n} \right] dA.$$

For example, in  $\mathbb{R}^3$

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi \|\mathbf{x} - \mathbf{x}_0\|}.$$

□

*Proof of (c).* Assume  $\nabla^2 h = 0$ .

(1) Take  $g = 1$  in

$$\int_{\Omega} [g \nabla^2 h + (\nabla g) \cdot (\nabla h)] dV = \int_{\partial\Omega} g \frac{\partial h}{\partial n} dA$$

to get the conclusion. (Here  $\nabla g = \mathbf{0}$  as  $g = 1$ .)

(2) Assume  $h = 0$  on  $\partial\Omega$ . Take  $g = h$  in

$$\int_{\Omega} [g \nabla^2 h + (\nabla g) \cdot (\nabla h)] dV = \int_{\partial\Omega} g \frac{\partial h}{\partial n} dA$$

to get

$$\int_{\Omega} |\nabla h|^2 dV = \int_{\partial\Omega} h \frac{\partial h}{\partial n} dA = 0$$

(since  $h = 0$  on  $\partial\Omega$ ). Since  $h \in \mathcal{C}'(\Omega)$ , Exercise 6.2 implies that  $|\nabla h|^2 = 0$  on  $\Omega$ . So  $D_1 h = D_2 h = D_3 h = 0$  on  $\Omega$ . Since  $h \in \mathcal{C}'(\Omega)$ , Theorem 9.21 implies that  $h = 0$  on  $\Omega$ , or  $h$  is locally constant in  $\Omega$  (Exercise 9.9). Note that  $h = 0$  globally on  $\partial\Omega$ , and thus  $h = 0$  globally on  $\Omega$ .

□

*Proof of (d).*

- (1) (The divergence theorem in  $\mathbb{R}^2$ .) If  $\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2$  is a vector field of class  $\mathcal{C}'$  in an open set  $E \subseteq \mathbb{R}^2$ , and if  $\Omega$  is a closed subset of  $E$  with positively oriented boundary  $\partial\Omega$  then

$$\int_{\Omega} (\nabla \cdot \mathbf{F}) dA = \int_{\partial\Omega} (\mathbf{F} \cdot \mathbf{n}) ds.$$

Define a 1-form by

$$\omega_{\mathbf{F}} = F_1 dy - F_2 dx.$$

So

$$d\omega_{\mathbf{F}} = (\nabla \cdot \mathbf{F}) dx \wedge dy = (\nabla \cdot \mathbf{F}) dA.$$

Hence the Stokes' theorem (Theorem 10.33) implies that

$$\int_{\Omega} (\nabla \cdot \mathbf{F}) dA = \int_{\Omega} d\omega_{\mathbf{F}} = \int_{\partial\Omega} \omega_{\mathbf{F}} = \int_{\partial\Omega} (\mathbf{F} \cdot \mathbf{n}) ds.$$

- (2) Note that

$$\nabla \cdot \mathbf{F} = g \nabla^2 h + (\nabla g) \cdot (\nabla h)$$

is also true in  $\mathbb{R}^2$ . Similar to (b), two Green's identities are also true in  $\mathbb{R}^2$ . (In  $\mathbb{R}^1$ , the Green's first identity is the integration by parts (Theorem 6.22).)

□

**Exercise 10.32 (Möbius band).** Fix  $\delta$ ,  $0 < \delta < 1$ . Let  $D$  be the set of all  $(\theta, t) \in \mathbb{R}^2$  such that  $0 \leq \theta \leq \pi$ ,  $-\delta \leq t \leq \delta$ . Let  $\Phi$  be the 2-surface in  $\mathbb{R}^3$ , with parameter domain  $D$ , given by

$$\begin{aligned} x &= (1 - t \sin \theta) \cos(2\theta) \\ y &= (1 - t \sin \theta) \sin(2\theta) \\ z &= t \cos \theta \end{aligned}$$

where  $(x, y, z) = \Phi(\theta, t)$ . Note that  $\Phi(\pi, t) = \Phi(0, -t)$ , and that  $\Phi$  is one-to-one on the rest of  $D$ .

The range  $M = \Phi(D)$  of  $\Phi$  is known as a **Möbius band**. It is the simplest example of a nonorientable surface.

Prove the various assertions made in the following description: Put  $\mathbf{p}_1 = (0, -\delta)$ ,  $\mathbf{p}_2 = (\pi, -\delta)$ ,  $\mathbf{p}_3 = (\pi, \delta)$ ,  $\mathbf{p}_4 = (0, \delta)$ ,  $\mathbf{p}_5 = \mathbf{p}_1$ . Put  $\gamma_i = [\mathbf{p}_i, \mathbf{p}_{i+1}]$ ,  $i = 1, \dots, 4$ , and put  $\Gamma_i = \Phi \circ \gamma_i$ . Then

$$\partial\Phi = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4.$$

Put  $\mathbf{a} = (1, 0, -\delta)$ ,  $\mathbf{b} = (1, 0, \delta)$ . Then

$$\Phi(\mathbf{p}_1) = \Phi(\mathbf{p}_3) = \mathbf{a}, \quad \Phi(\mathbf{p}_2) = \Phi(\mathbf{p}_4) = \mathbf{b},$$

and  $\partial\Phi$  can be described as follows.

- (1)  $\Gamma_1$  spirals up from  $\mathbf{a}$  to  $\mathbf{b}$ ; its projection into the  $(x, y)$ -plane has winding number  $+1$  around the origin. (See Exercise 8.23.)
- (2)  $\Gamma_2 = [\mathbf{b}, \mathbf{a}]$ .
- (3)  $\Gamma_3$  spirals up from  $\mathbf{a}$  to  $\mathbf{b}$ ; its projection into the  $(x, y)$ -plane has winding number  $-1$  around the origin.
- (4)  $\Gamma_4 = [\mathbf{b}, \mathbf{a}]$ .

Thus  $\partial\Phi = \Gamma_1 + \Gamma_3 + 2\Gamma_2$ .

If we go from  $\mathbf{a}$  to  $\mathbf{b}$  along  $\Gamma_1$  and continue along the “edge” of  $M$  until we return to  $\mathbf{a}$ , the curve traced out is

$$\Gamma = \Gamma_1 - \Gamma_3,$$

which may also be represented on the parameter interval  $[0, 2\pi]$  by the equations

$$\begin{aligned} x &= (1 + \delta \sin \theta) \cos(2\theta) \\ y &= (1 + \delta \sin \theta) \sin(2\theta) \\ z &= -\delta \cos \theta. \end{aligned}$$

It should be emphasized that  $\Gamma \neq \partial\Phi$ : Let  $\eta = \frac{xdy - ydx}{x^2 + y^2}$  be the 1-form discussed in Exercise 10.21 and Exercise 10.22. Since  $d\eta = 0$ , Stokes’ theorem shows that

$$\int_{\partial\Phi} \eta = 0.$$

But although  $\Gamma$  is the “geometric” boundary of  $M$ , we have

$$\int_{\Gamma} \eta = 4\pi.$$

In order to avoid this possible source of confusion, Stokes’ formula (Theorem 10.50) is frequently stated only for orientable surfaces  $\Phi$ .

*Proof.*

- (1) Show that  $\partial\Phi = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$ .

$$\begin{aligned} \partial\Phi &= \Phi \circ (\partial D) \\ &= \Phi \circ (\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4) \\ &= \Phi \circ \gamma_1 + \Phi \circ \gamma_2 + \Phi \circ \gamma_3 + \Phi \circ \gamma_4 \\ &= \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4. \end{aligned}$$

- (2) It is trivial that  $\Phi(\mathbf{p}_1) = \Phi(\mathbf{p}_3) = \mathbf{a} = (1, 0, -\delta)$  and  $\Phi(\mathbf{p}_2) = \Phi(\mathbf{p}_4) = \mathbf{b} = (1, 0, \delta)$  by the definition of  $\Phi$ .

- (3) Show that  $\Gamma_1$  spirals up from  $\mathbf{a}$  to  $\mathbf{b}$ ; its projection into the  $(x, y)$ -plane has winding number  $+1$  around the origin. By definition,  $\Gamma_1 = \Phi \circ \gamma_1 = \Phi([\mathbf{p}_1, \mathbf{p}_2])$ . That is,  $\Gamma_1$  spirals up from  $\Phi(\mathbf{p}_1) = \mathbf{a}$  to  $\Phi(\mathbf{p}_2) = \mathbf{b}$ . Besides, the projection  $P_{\Gamma_1}$  of  $\Gamma_1$  into the  $(x, y)$ -plane ( $z = 0$ ) can be parameterized as

$$\begin{aligned} x &= \left(1 + \delta \sin \frac{t}{2}\right) \cos t \\ y &= \left(1 + \delta \sin \frac{t}{2}\right) \sin t \end{aligned}$$

for  $0 \leq t \leq 2\pi$ . Note that  $P_{\Gamma_1}$  satisfies the condition in Exercise 10.21(b). Hence  $\int_{P_{\Gamma_1}} \eta = 2\pi$ . (Here  $\eta$  is well-defined.) Apply Exercise 10.21(f) to get

$$\text{Ind}(P_{\Gamma_1}) = \frac{1}{2\pi} \int_{P_{\Gamma_1}} \eta = \frac{1}{2\pi} \cdot 2\pi = 1.$$

- (4) Show that  $\Gamma_2 = [\mathbf{b}, \mathbf{a}]$ . By definition,  $\Gamma_2 = \Phi \circ \gamma_2 = \Phi([\mathbf{p}_2, \mathbf{p}_3])$  is  $[\mathbf{b}, \mathbf{a}]$  exactly.
- (5) Show that  $\Gamma_3$  spirals up from  $\mathbf{a}$  to  $\mathbf{b}$ ; its projection into the  $(x, y)$ -plane has winding number  $-1$  around the origin. Similar to (3),  $\Gamma_3$  spirals up from  $\Phi(\mathbf{p}_3) = \mathbf{a}$  to  $\Phi(\mathbf{p}_4) = \mathbf{b}$ . Now we consider  $-\Gamma_3$  instead of  $\Gamma_3$ . The projection  $P_{-\Gamma_3}$  of  $-\Gamma_3$  into the  $(x, y)$ -plane ( $z = 0$ ) can be parameterized as

$$\begin{aligned} x &= \left(1 - \delta \sin \frac{t}{2}\right) \cos t \\ y &= \left(1 - \delta \sin \frac{t}{2}\right) \sin t \end{aligned}$$

for  $0 \leq t \leq 2\pi$ . Similar to (3),  $\text{Ind}(P_{-\Gamma_3}) = 1$ . Therefore,

$$\text{Ind}(\Gamma_3) = -\text{Ind}(-\Gamma_3) = -\text{Ind}(P_{-\Gamma_3}) = -1.$$

- (6) Show that  $\Gamma_4 = [\mathbf{b}, \mathbf{a}]$ . Similar to (4).
- (7) Show that  $\Gamma = \Gamma_1 - \Gamma_3$  is the trace of from  $\mathbf{a}$  to  $\mathbf{b}$  along  $\Gamma_1$  and continue along the “edge” of  $M$  until we return to  $\mathbf{a}$ . By definition,  $\Gamma$  can be parameterized as

$$\begin{aligned} x &= (1 + \delta \sin t) \cos(2t) \\ y &= (1 + \delta \sin t) \sin(2t) \\ z &= -\delta \cos t \end{aligned}$$

for  $t \in [0, 2\pi]$ . Thus,  $\Gamma$  is  $\Gamma_1$  if  $t \in [0, \pi]$  and  $\Gamma$  is  $-\Gamma_3$  if  $t \in [\pi, 2\pi]$  by (3)(5). So  $\Gamma = \Gamma_1 - \Gamma_3$ .

- (8) *Show that  $\int_{\partial\Phi} \eta = 0$ .* Note that  $\eta$  is well-defined since  $M$  does not intersect the  $z$ -axis. So the Stokes' theorem (Theorem 10.33) and  $d\eta = 0$  on  $M$  implies that

$$\int_{\partial\Phi} \eta = \int_{\Phi} d\eta = 0.$$

- (9) *Show that  $\int_{\Gamma} \eta = 4\pi$ .*

$$\begin{aligned} \int_{\Gamma} \eta &= \int_{\Gamma} \frac{xdy - ydx}{x^2 + y^2} \\ &= \int_0^{2\pi} \frac{x(t)y'(t) - y(t)x'(t)}{x(t)^2 + y(t)^2} dt \\ &= \int_0^{2\pi} 2 dt \\ &= 4\pi. \end{aligned} \tag{7}$$

(So the winding number of  $\Gamma$  around of  $\mathbf{0}$  is 2.)

- (10) By (8)(9),  $\Gamma \neq \partial\Phi$ .

□