Chapter 5: Differentiation

Exercise 5.1. *Let f be defined for all real x, and suppose that*

$$|f(x) - f(y)| \leq (x-y)^2$$

for all real x and y. Prove that f is a constant.

Proof.

$$\left|\frac{f(x)-f(y)}{x-y}\right| \leq |x-y|$$

for $x \neq y$. Given any $y \in \mathbb{R}$, $|\frac{f(x)-f(y)}{x-y}| \to 0$ as $x \to y$, or $|f'(y)| = 0$. (Or using ϵ-δ argument. Fix $y \in \mathbb{R}$. Given any $\epsilon > 0$, there exists $\delta = \epsilon > 0$ such that

$$\left|\frac{f(x)-f(y)}{x-y} - 0\right| \leq |x-y| < \delta = \epsilon$$

whenever $|x-y| < \delta$. That is, $|f'(y)| = 0$.) So $f'(y) = 0$ for any $y \in \mathbb{R}$. By Theorem 5.11 (b), f is a contstant. $\square$

Exercise 5.4. *If*

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$

where $C_0, ..., C_n$ are real contstants, prove that the equation

$$C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n = 0$$

has at least one real root between 0 and 1.

Proof. Let

$$g(x) = C_0 x + \frac{C_1}{2}x^2 + \cdots + \frac{C_{n-1}}{n}x^n + \frac{C_n}{n+1}x^{n+1} \in \mathbb{R}[x].$$

Then $g(0) = g(1) = 0$, and $g'(x) = C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n$. By the mean value theorem (Theorem 5.10), there exists a point $\xi \in (0,1)$ at which

$$g(1) - g(0) = g'(\xi)(1-0),$$

or $g'(\xi) = 0$. That is, there exists a real root $x = \xi$ between 0 and 1 at which $C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n = 0$. $\square$