

Chapter 7: Sequences and Series of Functions

Author: Meng-Gen Tsai

Email: plover@gmail.com

Exercise 7.1. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

Proof (Cauchy criterion). Let $\{f_n\}$ be a uniformly convergent sequence of bounded functions.

- (1) Since f_n is bounded, there exists M_n such that $|f_n(x)| \leq M_n$.
- (2) Since $\{f_n\}$ converges uniformly, given $1 > 0$ there exists an integer N such that

$$|f_n(x) - f_m(x)| \leq 1 \text{ whenever } n, m \geq N$$

(Theorem 7.8 (Cauchy criterion for uniform convergence)). Especially,

$$|f_n(x)| \leq |f_n(x) - f_N(x)| + |f_N(x)| \leq 1 + M_N \text{ whenever } n \geq N.$$

- (3) Thus, $\{f_n\}$ is uniformly bounded by $M = \max\{M_1, \dots, M_{N-1}, M_N + 1\}$.

□

Exercise 7.2. If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E , prove that $\{f_n + g_n\}$ converge uniformly on E . If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E .

Proof. Let $f_n \rightarrow f$ uniformly and $g_n \rightarrow g$ uniformly.

- (1) Show that $\{f_n + g_n\}$ converges uniformly. Given $\varepsilon > 0$. Since $f_n \rightarrow f$ uniformly and $g_n \rightarrow g$ uniformly, there exist two integers N_1 and N_2 such that

$$|f_n(x) - f(x)| \leq \frac{\varepsilon}{2} \text{ whenever } n \geq N_1, x \in E$$

$$|g_n(x) - g(x)| \leq \frac{\varepsilon}{2} \text{ whenever } n \geq N_2, x \in E.$$

Take $N = \max\{N_1, N_2\}$, we have

$$\begin{aligned} & |(f_n(x) + g_n(x)) - (f(x) + g(x))| \\ &= |(f_n(x) - f(x)) + (g_n(x) - g(x))| \\ &\leq |f_n(x) - f(x)| + |g_n(x) - g(x)| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

whenever $n \geq N$, $x \in E$. Hence $f_n + g_n \rightarrow f + g$ uniformly on E .

- (2) Show that $\{f_n g_n\}$ converges uniformly if, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions. Given $\varepsilon > 0$.

- (a) By Exercise 7.1, both $\{f_n\}$ and $\{g_n\}$ are uniformly bounded. So there exist M_1 and M_2 such that

$$|f_n(x)| \leq M_1 \text{ and } |g_n(x)| \leq M_2$$

for all n and $x \in E$. Also, $|f(x)| \leq M_1 + 1$ and $|g(x)| \leq M_2 + 1$.

- (b) Since $f_n \rightarrow f$ uniformly and $g_n \rightarrow g$ uniformly, there exist two integers N_1 and N_2 such that

$$|f_n(x) - f(x)| \leq \frac{\varepsilon}{2(M_2 + 1)} \text{ whenever } n \geq N_1, x \in E$$

$$|g_n(x) - g(x)| \leq \frac{\varepsilon}{2(M_1 + 1)} \text{ whenever } n \geq N_2, x \in E.$$

(Note that each denominator of $\frac{\varepsilon}{2(M_j + 1)}$ ($j = 1, 2$) is well-defined and positive!) Take $N = \max\{N_1, N_2\}$, we have

$$\begin{aligned} & |f_n(x)g_n(x) - f(x)g(x)| \\ &= |[f_n(x) - f(x)]g_n(x) + f(x)[g_n(x) - g(x)]| \\ &\leq |f_n(x) - f(x)||g_n(x)| + |f(x)||g_n(x) - g(x)| \\ &\leq \frac{\varepsilon}{2(M_2 + 1)} \cdot M_2 + (M_1 + 1) \cdot \frac{\varepsilon}{2(M_1 + 1)} \\ &\leq \varepsilon \end{aligned}$$

whenever $n \geq N$, $x \in E$. Hence $f_n g_n \rightarrow fg$ uniformly on E .

□

Proof (Cauchy criterion).

- (1) Show that $\{f_n + g_n\}$ converges uniformly. Given $\varepsilon > 0$. Since $\{f_n\}$ and $\{g_n\}$ converge uniformly, there exist two integers N_1 and N_2 such that

$$|f_n(x) - f_m(x)| \leq \frac{\varepsilon}{2} \text{ whenever } n, m \geq N_1, x \in E$$

$$|g_n(x) - g_m(x)| \leq \frac{\varepsilon}{2} \text{ whenever } n, m \geq N_2, x \in E.$$

Take $N = \max\{N_1, N_2\}$, we have

$$\begin{aligned} & |(f_n(x) + g_n(x)) - (f_m(x) + g_m(x))| \\ &= |(f_n(x) - f_m(x)) + (g_n(x) - g_m(x))| \\ &\leq |f_n(x) - f_m(x)| + |g_n(x) - g_m(x)| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

whenever $n, m \geq N$, $x \in E$. Hence $\{f_n + g_n\}$ converges uniformly on E .

- (2) Show that $\{f_n g_n\}$ converges uniformly if, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions. Given $\varepsilon > 0$.

- (a) By Exercise 7.1, both $\{f_n\}$ and $\{g_n\}$ are uniformly bounded. So there exist M_1 and M_2 such that

$$|f_n(x)| \leq M_1 \text{ and } |g_n(x)| \leq M_2$$

for all n and $x \in E$. Also, $|f(x)| \leq M_1 + 1$ and $|g(x)| \leq M_2 + 1$.

- (b) Since $\{f_n\} \rightarrow f$ uniformly and $\{g_n\} \rightarrow g$ uniformly, there exist two integers N_1 and N_2 such that

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq \frac{\varepsilon}{2(M_2 + 1)} \text{ whenever } n, m \geq N_1, x \in E \\ |g_n(x) - g_m(x)| &\leq \frac{\varepsilon}{2(M_1 + 1)} \text{ whenever } n, m \geq N_2, x \in E. \end{aligned}$$

Take $N = \max\{N_1, N_2\}$, we have

$$\begin{aligned} &|f_n(x)g_n(x) - f_m(x)g_m(x)| \\ &= |[f_n(x) - f_m(x)]g_n(x) + f_m(x)[g_n(x) - g_m(x)]| \\ &\leq |f_n(x) - f_m(x)||g_n(x)| + |f_m(x)||g_n(x) - g_m(x)| \\ &\leq \frac{\varepsilon}{2(M_2 + 1)} \cdot M_2 + M_1 \cdot \frac{\varepsilon}{2(M_1 + 1)} \\ &\leq \varepsilon \end{aligned}$$

whenever $n \geq N$, $x \in E$. Hence $\{f_n g_n\}$ converges uniformly on E .

□

Exercise 7.3. Construct sequences $\{f_n\}$, $\{g_n\}$ which converge uniformly on some set E , but such that $\{f_n g_n\}$ does not converge uniformly on E (of course, $\{f_n g_n\}$ must converge on E).

We provides some examples here.

Proof ($f_n(x) = x + \frac{1}{n}$).

- (1) Define $\{f_n(x)\}$ on $E = \mathbb{R}$ by $f_n(x) = x + \frac{1}{n}$ and $f(x) = x$. Clearly, $\{f_n(x)\}$ converges to $f(x)$ pointwise.
- (2) Show that $\{f_n\}$ converges uniformly. Given $\varepsilon > 0$. There exists an integer $N \geq \frac{1}{\varepsilon}$ such that

$$|f_n(x) - f(x)| = \frac{1}{n} \leq \frac{1}{N} \leq \varepsilon$$

whenever $n \geq N$ and $x \in E$. Hence $\{f_n\} \rightarrow f$ uniformly.

- (3) Show that $\{f_n^2\}$ does not converge uniformly. Clearly, $\{f_n(x)^2\}$ converges to $f(x)^2$ pointwise. Hence

$$\sup_{x \in E} |f_n(x)^2 - f(x)^2| = \sup_{x \in E} \left| \frac{2x}{n} + \frac{1}{n^2} \right| \rightarrow \infty$$

as $n \rightarrow \infty$ (by considering $x = n^2 \in E$). Hence $\{f_n^2\}$ does not converge uniformly (Theorem 7.9).

□

Proof ($f_n(x) = \frac{1}{x}$, $g_n(x) = \frac{1}{n}$).

- (1) Let $E = (0, 1)$. Let $\{f_n(x)\}$ on E be $f_n(x) = \frac{1}{x}$ and $\{g_n(x)\}$ on E be $g_n(x) = \frac{1}{n}$. Clearly, $\{f_n(x)\}$ converges to $f(x) = \frac{1}{x}$ pointwise and $\{g_n(x)\}$ converges to $g(x) = 0$ pointwise.
- (2) Show that $\{f_n\}$ converges uniformly. Given $\varepsilon > 0$. There exists an integer $N = 1$ such that

$$|f_n(x) - f(x)| = 0 \leq \varepsilon$$

whenever $n \geq N$ and $x \in E$. Hence $\{f_n\} \rightarrow f$ uniformly.

- (3) Show that $\{g_n\}$ converges uniformly. Given $\varepsilon > 0$. There exists an integer $N \geq \frac{1}{\varepsilon}$ such that

$$|g_n(x) - g(x)| = \frac{1}{n} \leq \frac{1}{N} \leq \varepsilon$$

whenever $n \geq N$ and $x \in E$. Hence $\{g_n\} \rightarrow g$ uniformly.

- (4) Show that $\{f_n g_n\}$ does not converge uniformly. Clearly, $\{f_n(x)g_n(x)\}$ converges to $f(x)g(x) = 0$ pointwise. Hence

$$\sup_{x \in E} |f_n(x)g_n(x) - 0| = \sup_{x \in E} \left| \frac{1}{nx} \right| \rightarrow \infty$$

as $n \rightarrow \infty$ (by considering $x = \frac{1}{n^2} \in E$). Hence $\{f_n g_n\}$ does not converge uniformly (Theorem 7.9).

□

Proof (Exercise 9.2 in Tom M. Apostol, *Mathematical Analysis*, 2nd edition).

- (1) Let $E = [\alpha, \beta] \subseteq \mathbb{R}$ be a bounded interval. Define two sequences $\{f_n\}$ and $\{g_n\}$ on E as follows:

$$f_n(x) = x \left(1 + \frac{1}{n} \right) \text{ if } x \in \mathbb{R}, n = 1, 2, \dots,$$

$$g_n(x) = \begin{cases} \frac{1}{n} & \text{if } x = 0 \text{ or if } x \text{ is irrational,} \\ b + \frac{1}{n} & \text{if } x \text{ is rational } \neq 0, \text{ say } x = \frac{a}{b}, b > 0. \end{cases}$$

Here we assume that $\gcd(a, b) = 1$. Clearly, $f(x) = x$ and

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or if } x \text{ is irrational,} \\ b & \text{if } x \text{ is rational } \neq 0, \text{ say } x = \frac{a}{b}, b > 0. \end{cases}$$

Let $M = \max\{|\alpha|, |\beta|\} \geq 0$.

- (2) *Show that $\{f_n\}$ converges uniformly.* Given $\varepsilon > 0$. There exists an integer $N \geq \frac{M}{\varepsilon}$ such that

$$|f_n(x) - f(x)| = \frac{|x|}{n} \leq \frac{M}{N} \leq \varepsilon$$

whenever $n \geq N$ and $x \in E$. Hence $\{f_n\} \rightarrow f$ uniformly.

- (3) *Show that $\{g_n\}$ converges uniformly.* Given $\varepsilon > 0$. There exists an integer $N \geq \frac{1}{\varepsilon}$ such that

$$|g_n(x) - g(x)| = \frac{1}{n} \leq \frac{1}{N} \leq \varepsilon$$

whenever $n \geq N$ and $x \in E$. Hence $\{g_n\} \rightarrow g$ uniformly.

- (4) *Show that $\{f_n g_n\}$ does not converge uniformly.*

(a) Clearly, $\{f_n(x)g_n(x)\}$ converges to $f(x)g(x)$ pointwise where

$$f(x)g(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or if } x \text{ is irrational,} \\ a & \text{if } x = \frac{a}{b} \text{ is rational } \neq 0, b > 0. \end{cases}$$

(b) Note that

$$f_n(x)g_n(x) = \begin{cases} \frac{x}{n} \left(1 + \frac{1}{n}\right) & \text{if } x = 0 \text{ or if } x \text{ is irrational,} \\ \left(a + \frac{x}{n}\right) \left(1 + \frac{1}{n}\right) & \text{if } x = \frac{a}{b} \text{ is rational } \neq 0, b > 0. \end{cases}$$

Therefore,

$$f_n(x)g_n(x) - f(x)g(x) = \begin{cases} \frac{x}{n} \left(1 + \frac{1}{n}\right) & \text{if } x = 0 \text{ or if } x \text{ is irrational,} \\ \frac{x}{n} \left(1 + b + \frac{1}{n}\right) & \text{if } x = \frac{a}{b} \text{ is rational } \neq 0, b > 0. \end{cases}$$

(c) Hence

$$\begin{aligned} \sup_{x \in E} |f_n(x)g_n(x) - f(x)g(x)| &\geq \sup_{x \in E \cap \mathbb{Q}} |f_n(x)g_n(x) - f(x)g(x)| \\ &= \sup_{x \in E \cap \mathbb{Q}} |a| \left(\frac{1}{n} + \frac{1}{bn} + \frac{1}{bn^2} \right) \\ &\geq \sup_{x \in E \cap \mathbb{Q}} |a| \left(\frac{1}{n} \right) \\ &= \sup_{x \in E \cap \mathbb{Q}} \frac{|a|}{n}. \end{aligned}$$

(d) Given any irrational number $\gamma \in E$, there exists a sequence

$$\left\{ r_m = \frac{a_m}{b_m} \right\}$$

of nonzero rational numbers in E such that $\lim r_m = \gamma$. Show that $\{a_m\}$ is unbounded. If it is true, we can find $x_n = r_{m_n} = \frac{a_{m_n}}{b_{m_n}}$ such that $|a_{m_n}| \geq n^2$ and

$$\sup_{x \in E} |f_n(x)g_n(x) - f(x)g(x)| \geq \sup_{x \in E \cap \mathbb{Q}} \frac{|a|}{n} \geq \frac{n^2}{n} = n \rightarrow \infty$$

as $n \rightarrow \infty$.

(e) (Reductio ad absurdum) If $\{a_m\}$ were bounded, then there exists a **constant** subsequence of $\{a_{m_k}\}$ such that $\lim a_{m_k} = a \in \mathbb{Z}$. Since $\lim_{m \rightarrow \infty} r_m = \gamma$, $\lim_{k \rightarrow \infty} r_{m_k} = \gamma$ or

$$\lim_{k \rightarrow \infty} b_{m_k} = \lim_{k \rightarrow \infty} \frac{a_{m_k}}{r_{m_k}} = \frac{a}{\gamma}$$

(it is well-defined since r_{m_k} and γ cannot be zero). Since all b_{m_k} are positive integers, the limit $\lim b_{m_k} = b$ is a positive integer too, or $b = \frac{a}{\gamma} \in \mathbb{Z}^+$, or $\gamma = \frac{a}{b} \in \mathbb{Z}$, which is absurd.

Therefore, $\{f_n g_n\}$ does not converge uniformly.

□

Dirichlet's test for convergence of a series

See Theorem 3.42. Suppose

- (a) the partial sums A_n of $\sum a_n$ form a bounded sequence;
- (b) $b_0 \geq b_1 \geq b_2 \geq \cdots$;
- (c) $\lim_{n \rightarrow \infty} b_n = 0$.

Then $\sum a_n b_n$ converges.

Dirichlet's test for uniformly convergence of a function series. Suppose

- (a) the partial sums $F_n(x)$ of $\sum f_n(x)$ form a uniformly bounded sequence;
- (b) $g_1(x) \geq g_2(x) \geq \cdots$;
- (c) $\lim_{n \rightarrow \infty} g_n(x) = 0$.

Then $\sum f_n(x)g_n(x)$ converges.

Proof (Theorem 3.42). Choose M such that $|F_n(x)| \leq M$ for all n . Given $\varepsilon > 0$, there is an integer N such that $g_N(x) \leq \frac{\varepsilon}{2(M+1)}$. For $N \leq p \leq q$, we have

$$\begin{aligned}
& \left| \sum_{n=p}^q f_n(x)g_n(x) \right| \\
&= \left| \sum_{n=p}^{q-1} F_n(x)(g_n(x) - g_{n+1}(x)) + F_q(x)g_q(x) - F_{p-1}(x)g_p(x) \right| \\
&\leq M \left| \sum_{n=p}^{q-1} (g_n(x) - g_{n+1}(x)) + g_q(x) + g_p(x) \right| \\
&= 2Mg_p(x) \\
&\leq 2Mg_N(x) \\
&\leq \varepsilon.
\end{aligned}$$

Uniform convergence now follows from the Cauchy criterion (Theorem 7.8). Note that the first inequality in the above chain depends of course on the fact that $g_n(x) - g_{n+1}(x) \geq 0$. \square

Exercise 7.6. Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x .

Proof (Dirichlet's test). Given any bounded interval $E = [\alpha, \beta] \subseteq \mathbb{R}$. Write $f_n(x) = (-1)^n$ on E and $g_n(x) = \frac{x^2 + n}{n^2}$ on E .

(1) The partial sums $F_n(x)$ of $\sum f_n(x)$ form a uniformly bounded sequence.

(2) $g_1(x) \geq g_2(x) \geq \dots$ since

$$g_{n+1}(x) = \frac{x^2}{(n+1)^2} + \frac{1}{n+1} < \frac{x^2}{n^2} + \frac{1}{n} = g_n(x).$$

(3) Write $M = \max\{|\alpha|, |\beta|\}$. Since

$$|g_n(x)| = \frac{x^2}{n^2} + \frac{1}{n} \leq \frac{M^2}{n^2} + \frac{1}{n} \rightarrow 0$$

as $n \rightarrow \infty$, $\lim_{n \rightarrow \infty} g_n(x) = 0$.

By Dirichlet's test, $\sum_{n=1}^{\infty} f_n(x)g_n(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}$ converges. \square

Exercise 7.7. For $n = 1, 2, 3, \dots$, x real, put

$$f_n(x) = \frac{x}{1 + nx^2}.$$

Show that $\{f_n\}$ converges uniformly to a function f , and that the equation

$$f'(x) = \lim_{n \rightarrow \infty} f'_n(x)$$

is correct if $x \neq 0$, but false if $x = 0$.

$f_n(x)$ is defined on \mathbb{R} .

Proof.

(1) Since

$$|f_n(x)| = \left| \frac{x}{1 + nx^2} \right| \leq \frac{|x|}{\sqrt{n}|x|} = \frac{1}{\sqrt{n}} \rightarrow 0$$

as $n \rightarrow \infty$, $f_n \rightarrow 0$ uniformly (Theorem 7.9).

(2) Clearly, $f'(x) = 0$. Since

$$f'_n(x) = \frac{1 - nx^2}{(1 + nx^2)^2},$$

$$\lim_{n \rightarrow \infty} f'_n(x) = \begin{cases} 1 & (x = 0), \\ 0 & (x \neq 0). \end{cases}$$

So that the equation

$$f'(x) = \lim_{n \rightarrow \infty} f'_n(x)$$

is correct if $x \neq 0$, but false if $x = 0$.

\square

Note. $f'_n(x)$ does not converge uniformly by considering

$$\lim_{n \rightarrow \infty} f'_n\left(\frac{1}{n}\right) = \lim_{n \rightarrow \infty} \frac{1 - \frac{1}{n}}{(1 + \frac{1}{n})^2} = 1.$$

Exercise 7.8. If

$$I(x) = \begin{cases} 0 & (x \leq 0), \\ 1 & (x > 0), \end{cases}$$

if $\{x_n\}$ is a sequence of distinct points of (a, b) , and if $\sum |c_n|$ converges, prove that the series

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n) \quad (a \leq x \leq b)$$

converges uniformly, and that f is continuous for every $x \neq x_n$.

Proof.

(1) Define $f_n(x) = c_n I(x - x_n)$ on (a, b) . So

$$|f_n(x)| = |c_n| |I(x - x_n)| \leq |c_n| \quad (x \in (a, b), n = 1, 2, 3, \dots).$$

Since $\sum |c_n|$ converges, $f = \sum f_n$ converges uniformly (Theorem 7.10).

(2) Given any $p \in (a, b)$ with $p \neq x_n$ for all $n = 1, 2, 3, \dots$. So each $I(x - x_n)$ is continuous at $x = p$, and thus each partial sum $\sum_{n=1}^N f_n(x)$ is continuous.

(3) By Theorem 7.11

$$\begin{aligned} \lim_{x \rightarrow p} f(x) &= \lim_{x \rightarrow p} \sum_{n=1}^{\infty} f_n(x) \\ &= \lim_{N \rightarrow \infty} \left(\lim_{x \rightarrow p} \sum_{n=1}^N f_n(x) \right) \\ &= \lim_{N \rightarrow \infty} \sum_{n=1}^N f_n(p) \\ &= \sum_{n=1}^{\infty} f_n(p) \\ &= f(p). \end{aligned}$$

$f(x)$ is continuous at $x = p$ too.

□