

Notes on the book:  
*P.J. Hilton and U. Stammbach, A  
Course in Homological Algebra*

Meng-Gen Tsai  
plover@gmail.com

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# Chapter I: Modules

## §1. Modules

### Exercise 1.1. (Diagram chasing)

Complete the proof of Lemma 1.1. Show moreover that  $\alpha$  is surjective (resp. injective) if  $\alpha'$ ,  $\alpha''$  are surjective (resp. injective).

*Lemma 1.1.* Let  $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$  and  $0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0$  be two short exact sequences. Suppose that in the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A' & \xrightarrow{\mu} & A & \xrightarrow{\varepsilon} & A'' \longrightarrow 0 \\ & & \downarrow \alpha' & & \downarrow \alpha & & \downarrow \alpha'' \\ 0 & \longrightarrow & B' & \xrightarrow{\mu'} & B & \xrightarrow{\varepsilon'} & B'' \longrightarrow 0 \end{array}$$

any two of the three homomorphisms  $\alpha'$ ,  $\alpha$ ,  $\alpha''$  are isomorphisms. Then the third is an isomorphism, too.

*Proof (Diagram chasing).*

(1) Show that  $\alpha$  is surjective if  $\alpha'$ ,  $\alpha''$  are surjective.

- (a) Take any  $b \in B$ , it suffices to find  $a \in A$  such that  $\alpha a = b$ .
- (b) Consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varepsilon} & A'' \\ \downarrow \alpha & & \downarrow \alpha'' \\ B & \xrightarrow{\varepsilon'} & B'' \end{array}$$

$\varepsilon' b \in B''$ . By the surjectivity of  $\alpha''$ ,  $\exists a'' \in A''$  such that  $\alpha'' a'' = \varepsilon' b$ . By the surjectivity of  $\varepsilon$ ,  $\exists \bar{a} \in A$  such that  $\varepsilon \bar{a} = a''$ . Hence

$$\begin{aligned} \varepsilon'(b - \alpha \bar{a}) &= \varepsilon' b - \varepsilon' \alpha \bar{a} \\ &= \varepsilon' b - \alpha'' \varepsilon \bar{a} && \text{(The diagram commutes)} \\ &= \varepsilon' b - \alpha'' a'' \\ &= \varepsilon' b - \varepsilon' b \\ &= 0. \end{aligned}$$

- (c) Consider the short exact sequence

$$0 \longrightarrow B' \xrightarrow{\mu'} B \xrightarrow{\varepsilon'} B'' \longrightarrow 0$$

As  $\varepsilon'(b - \alpha \bar{a}) = 0$ ,  $\exists b' \in B'$  such that  $\mu' b' = b - \alpha \bar{a}$ .

(d) Consider the commutative diagram

$$\begin{array}{ccc} A' & \xrightarrow{\mu} & A \\ \downarrow \alpha' & & \downarrow \alpha \\ B' & \xrightarrow{\mu'} & B \end{array}$$

By the surjectivity of  $\alpha'$ ,  $\exists a' \in A'$  such that  $\alpha'a' = b'$ . Hence

$$\begin{aligned} \alpha(\mu a' + \bar{a}) &= \alpha \mu a' + \alpha \bar{a} \\ &= \mu' \alpha' a' + \alpha \bar{a} && \text{(The diagram commutes)} \\ &= \mu' b' + \alpha \bar{a} \\ &= (b - \alpha \bar{a}) + \alpha \bar{a} \\ &= b. \end{aligned}$$

Therefore, there exists  $a := \mu a' + \bar{a}$  such that  $\alpha a = b$ .

(2) Show that  $\alpha$  is injective if  $\alpha'$ ,  $\alpha''$  are injective.

(a) It suffices to show that  $\ker \alpha = 0$ . Take  $a \in \ker \alpha$ . ( $\alpha(a) = \alpha a = 0$ .)

(b) Consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varepsilon} & A'' \\ \downarrow \alpha & & \downarrow \alpha'' \\ B & \xrightarrow{\varepsilon'} & B'' \end{array}$$

we have  $0 = \varepsilon' \alpha a = \alpha'' \varepsilon a$ . By the injectivity of  $\alpha''$ ,  $\varepsilon a = 0$ .

(c) Consider the short exact sequence

$$0 \longrightarrow A' \xrightarrow{\mu} A \xrightarrow{\varepsilon} A'' \longrightarrow 0$$

As  $\varepsilon a = 0$ ,  $\exists a' \in A'$  such that  $\mu a' = a$ .

(d) Consider the commutative diagram

$$\begin{array}{ccc} A' & \xrightarrow{\mu} & A \\ \downarrow \alpha' & & \downarrow \alpha \\ B' & \xrightarrow{\mu'} & B \end{array}$$

$0 = \alpha a = \alpha \mu a' = \mu' \alpha' a'$ . By the injectivity of  $\mu' \alpha'$ ,  $a' = 0$ .  
Therefore,  $a = \mu a' = 0$ .

(3) Suppose  $\alpha$  is surjective. Show that  $\alpha''$  is surjective.

(a) Take any  $b'' \in B''$ , it suffices to find  $a'' \in A''$  such that  $\alpha'' a'' = b''$ .

(b) Consider the commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{\varepsilon} & A'' \\
\downarrow \alpha & & \downarrow \alpha'' \\
B & \xrightarrow{\varepsilon'} & B''
\end{array}$$

By the surjectivity of  $\varepsilon'$ ,  $\exists b \in B$  such that  $\varepsilon'b = b''$ . By the surjectivity of  $\alpha$ ,  $\exists a \in A$  such that  $\alpha a = b$ . Take  $a'' := \varepsilon a \in A''$ . Hence

$$\begin{aligned}
\alpha'' a'' &= \alpha'' \varepsilon a \\
&= \varepsilon' \alpha a && \text{(The diagram commutes)} \\
&= \varepsilon' b \\
&= b''.
\end{aligned}$$

(4) Suppose  $\alpha'$  is surjective and  $\alpha$  is injective. Show that  $\alpha''$  is injective.

- (a) It suffices to show that  $\ker \alpha'' = 0$ . Take  $a'' \in \ker \alpha''$ . ( $\alpha''(a'') = \alpha'' a'' = 0$ .)
- (b) Consider the commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{\varepsilon} & A'' \\
\downarrow \alpha & & \downarrow \alpha'' \\
B & \xrightarrow{\varepsilon'} & B''
\end{array}$$

By the surjectivity of  $\varepsilon$ ,  $\exists a \in A$  such that  $\varepsilon a = a''$ . So

$$\begin{aligned}
0 &= \alpha'' a'' \\
&= \alpha'' \varepsilon a \\
&= \varepsilon' \alpha a. && \text{(The diagram commutes)}
\end{aligned}$$

(c) Consider the short exact sequence

$$0 \longrightarrow B' \xrightarrow{\mu'} B \xrightarrow{\varepsilon'} B'' \longrightarrow 0$$

As  $\varepsilon'(\alpha a) = 0$ ,  $\exists b' \in B'$  such that  $\mu' b' = \alpha a$ .

(d) Consider the commutative diagram

$$\begin{array}{ccc}
A' & \xrightarrow{\mu} & A \\
\downarrow \alpha' & & \downarrow \alpha \\
B' & \xrightarrow{\mu'} & B
\end{array}$$

By surjectivity of  $\alpha'$ ,  $\exists a' \in A'$  such that  $\alpha' a' = b'$ . So

$$\begin{aligned}
\alpha a &= \mu' b' \\
&= \mu' \alpha' a' \\
&= \alpha \mu a'. && \text{(The diagram commutes)}
\end{aligned}$$

By the injectivity of  $\alpha$ ,  $a = \mu a'$ . Hence

$$a'' = \varepsilon a = \varepsilon \mu a' = 0.$$

Therefore  $\ker \alpha'' = 0$ .

(5) By (3)(4),  $\alpha''$  is an isomorphism if both  $\alpha'$  and  $\alpha$  are isomorphisms.

(6) Suppose  $\alpha$  is surjective and  $\alpha''$  is injective. Show that  $\alpha'$  is surjective.

(a) Take any  $b' \in B'$ , it suffices to find  $a' \in A'$  such that  $\alpha' a' = b'$ . Let  $b := \mu' b' \in B$  and note that  $\varepsilon' b = 0$  by the exactness of

$$0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0.$$

(b) Consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varepsilon} & A'' \\ \downarrow \alpha & & \downarrow \alpha'' \\ B & \xrightarrow{\varepsilon'} & B'' \end{array}$$

By the surjectivity of  $\alpha$ ,  $\exists a \in A$  such that  $\alpha a = b$ . So

$$\begin{aligned} 0 &= \varepsilon' b \\ &= \varepsilon' \alpha a \\ &= \alpha'' \varepsilon a. \end{aligned} \quad (\text{The diagram commutes})$$

By the injectivity of  $\alpha''$ ,  $\varepsilon a = 0$ .

(c) Consider the short exact sequence

$$0 \longrightarrow A' \xrightarrow{\mu} A \xrightarrow{\varepsilon} A'' \longrightarrow 0$$

As  $\varepsilon a = 0$ ,  $\exists a' \in A'$  such that  $\mu a' = a$ .

(d) Consider the commutative diagram

$$\begin{array}{ccc} A' & \xrightarrow{\mu} & A \\ \downarrow \alpha' & & \downarrow \alpha \\ B' & \xrightarrow{\mu'} & B \end{array}$$

Note that

$$\begin{aligned} \mu'(\alpha' a') &= \mu' \alpha' a' \\ &= \alpha \mu a' & (\text{The diagram commutes}) &= \alpha a \\ &= b \\ &= \mu' b'. \end{aligned}$$

By the injectivity of  $\mu'$ ,  $b' = \alpha' a'$  for some  $a' \in A'$ .

(7) Suppose  $\alpha$  is injective. Show that  $\alpha'$  is injective.

(a) It suffices to show that  $\ker \alpha' = 0$ . Take  $a' \in \ker \alpha'$ . ( $\alpha'(a') = \alpha'a' = 0$ .)

(b) Consider the commutative diagram

$$\begin{array}{ccc} A' & \xrightarrow{\mu} & A \\ \downarrow \alpha' & & \downarrow \alpha \\ B' & \xrightarrow{\mu'} & B \end{array}$$

Note that

$$\begin{aligned} 0 &= \mu'0 \\ &= \mu'\alpha'a' \\ &= \alpha\mu a'. \end{aligned} \quad (\text{The diagram commutes})$$

The injectivity of  $\alpha\mu$  shows that  $a' = 0$ .

(8) By (6)(7),  $\alpha'$  is an isomorphism if both  $\alpha$  and  $\alpha''$  are isomorphisms.

□

### Exercise 1.2. (Five lemma)

Show that, given a commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 & \longrightarrow & \cdots \\ & & \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & \downarrow \varphi_4 & & \downarrow \varphi_5 & & \\ \cdots & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 & \longrightarrow & \cdots \end{array}$$

with exact rows, in which  $\varphi_1, \varphi_2, \varphi_4, \varphi_5$  are isomorphisms, then  $\varphi_3$  is also an isomorphism. Can we weaken the hypotheses in a reasonable way?

One reasonable hypotheses:

- (a) If  $\varphi_1$  is surjective and  $\varphi_2, \varphi_4$  is injective, then  $\varphi_3$  is injective.
- (b) If  $\varphi_5$  is injective and  $\varphi_2, \varphi_4$  is surjective, then  $\varphi_3$  is surjective.

*Proof of (a).*

(1) Write

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & A_5 & \longrightarrow & \cdots \\ & & \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & \downarrow \varphi_4 & & \downarrow \varphi_5 & & \\ \cdots & \longrightarrow & B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 & \xrightarrow{\beta_4} & B_5 & \longrightarrow & \cdots \end{array}$$

Take  $a \in \ker(\varphi_3)$  and then we need to show  $a = 0$ .

(2) The commutative diagram

$$\begin{array}{ccc} A_3 & \xrightarrow{\alpha_3} & A_4 \\ \downarrow \varphi_3 & & \downarrow \varphi_4 \\ B_3 & \xrightarrow{\beta_3} & B_4 \end{array}$$

implies that  $0 = \beta_3 0 = \beta_3 \varphi_3 a = \varphi_4 \alpha_3 a$ . The injectivity of  $\varphi_4$  implies that  $\alpha_3 a = 0$ .

(3) The exact sequence

$$\cdots \longrightarrow A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} A_4 \longrightarrow \cdots$$

shows that  $a \in \ker(\alpha_3) = \text{im}(\alpha_2)$ . So there exists  $a_2 \in A_2$  such that  $\alpha_2 a_2 = a$ .

(4) The commutative diagram

$$\begin{array}{ccc} A_2 & \xrightarrow{\alpha_2} & A_3 \\ \downarrow \varphi_2 & & \downarrow \varphi_3 \\ B_2 & \xrightarrow{\beta_2} & B_3 \end{array}$$

implies that  $0 = \varphi_3 a = \varphi_3 \alpha_2 a_2 = \beta_2 \varphi_2 a_2$ .

(5) The exact sequence

$$\cdots \longrightarrow B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \longrightarrow \cdots$$

shows that  $\varphi_2 a_2 \in \ker(\beta_2) = \text{im}(\beta_1)$ . So there exists  $b_1 \in B_1$  such that  $\varphi_2 a_2 = \beta_1 b_1$ .

(6) Consider the commutative diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\alpha_1} & A_2 \\ \downarrow \varphi_1 & & \downarrow \varphi_2 \\ B_1 & \xrightarrow{\beta_1} & B_2 \end{array}$$

The surjectivity of  $\varphi_i$  implies that  $\exists a_1 \in A_1$  such that  $\varphi_1 a_1 = b_1$ . Hence the commutative diagram implies that  $\varphi_2(\alpha_1 a_1) = \varphi_2 \alpha_1 a_1 = \beta_1 \varphi_1 a_1 = \beta_1 b_1 = \varphi_2 a_2$ . The injectivity of  $\varphi_2$  implies that  $\alpha_1 a_1 = a_2$ .

(7) The exact sequence

$$\cdots \longrightarrow A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \longrightarrow \cdots$$

shows that  $a = \alpha_2 a_2 = \alpha_2 \alpha_1 a_1 = 0$ . Therefore  $\varphi_3$  is injective.

□

*Proof of (b).*

- (1) Take any  $b \in B_3$ , it suffices to find  $a \in A$  such that  $\varphi_3 a = b$ .
- (2) Let  $b_4 := \beta_3 b \in B_4$ . The exact sequence

$$\cdots \longrightarrow B_3 \xrightarrow{\beta_3} B_4 \xrightarrow{\beta_4} B_5 \longrightarrow \cdots$$

shows that  $\beta_4 b_4 = \beta_4(\beta_3 b) = 0$ .

- (3) Look at the commutative diagram

$$\begin{array}{ccc} A_4 & \xrightarrow{\alpha_4} & A_5 \\ \downarrow \varphi_4 & & \downarrow \varphi_5 \\ B_4 & \xrightarrow{\beta_4} & B_5 \end{array}$$

By the surjectivity of  $\varphi_4$ ,  $\exists a_4 \in A_4$  such that  $\varphi_4 a_4 = b_4$ . So the commutative diagram says that  $0 = \beta_4 b_4 = \beta_4 \varphi_4 a_4 = \varphi_5 \alpha_4 a_4$ . By the injectivity of  $\varphi_5$ ,  $\alpha_4 a_4 = 0$ .

- (4) The exact sequence

$$\cdots \longrightarrow A_3 \xrightarrow{\alpha_3} A_4 \xrightarrow{\alpha_4} A_5 \longrightarrow \cdots$$

shows that  $a_4 \in \ker(\alpha_4) = \text{im}(\alpha_3)$ . So there exists  $a_3 \in A_3$  such that  $\alpha_3 a_3 = a_4$ .

- (5) Let  $\bar{b} = b - \varphi_3 a_3 \in B_3$ . The commutative diagram

$$\begin{array}{ccc} A_3 & \xrightarrow{\alpha_3} & A_4 \\ \downarrow \varphi_3 & & \downarrow \varphi_4 \\ B_3 & \xrightarrow{\beta_3} & B_4 \end{array}$$

implies that  $\beta_3 \bar{b} = \beta_3 b - \beta_3 \varphi_3 a_3 = \beta_3 b - \varphi_4 \alpha_3 a_3 = \beta_3 b - \varphi_4 a_4 = \beta_3 b - b_4 = \beta_3 b - \beta_3 b = 0$ . So  $\bar{b} \in \ker(\beta_3)$ .

- (6) The exact sequence

$$\cdots \longrightarrow B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} B_4 \longrightarrow \cdots$$



shows that  $\bar{b} \in \ker(\beta_3) = \text{im}(\beta_2)$ . Hence  $\exists b_2 \in B_2$  such that  $\bar{b} = \beta_2 b_2$ .

(7) Look at the commutative diagram

$$\begin{array}{ccc} A_2 & \xrightarrow{\alpha_2} & A_3 \\ \downarrow \varphi_2 & & \downarrow \varphi_3 \\ B_2 & \xrightarrow{\beta_2} & B_3 \end{array}$$

The surjectivity of  $\varphi_2$  implies that  $\exists a_2 \in A_2$  such that  $b_2 = \varphi_2 a_2$ . Let  $a := \alpha_2 a_2 + a_3$ . Hence

$$\begin{aligned} \varphi_3(a) &= \varphi_3 \alpha_2 a_2 + \varphi_3 a_3 \\ &= \beta_2 \varphi_2 a_2 + \varphi_3 a_3 && \text{(The diagram commutes)} \\ &= \beta_2 b_2 + \varphi_3 a_3 \\ &= \bar{b} + \varphi_3 a_3 \\ &= (b - \varphi_3 a_3) + \varphi_3 a_3 \\ &= b. \end{aligned}$$

□

#### Exercise 1.4.

Show that the abelian group  $A$  admits the structure of a  $\mathbb{Z}/(m)$ -module if and only if  $mA = 0$ .

*Proof.*

(1) ( $\implies$ ) It suffices to show that  $ma = 0$  for all  $a \in A$ . Let  $\Lambda = \mathbb{Z}/(m)$ .

$$\begin{aligned} ma &= \underbrace{a + \cdots + a}_{m \text{ times}} \\ &= \underbrace{1_\Lambda a + \cdots + 1_\Lambda a}_{m \text{ times}} && \text{(Axiom M3)} \\ &= \underbrace{(1_\Lambda + \cdots + 1_\Lambda)}_{m \text{ times}} a && \text{(Axiom M1)} \\ &= 0_\Lambda a && (\text{char}(\Lambda) = m) \\ &= 0. && \text{(Axiom M1)} \end{aligned}$$

(2) ( $\impliedby$ ) Write  $\bar{\lambda} \in \Lambda := \mathbb{Z}/(m)$  where  $\lambda \in \mathbb{Z}$  and  $\bar{\lambda}$  is the residue class of  $\lambda$  in  $\Lambda$ . Define  $\omega : \Lambda \rightarrow \text{End}(A, A)$  by

$$\omega(\bar{\lambda})(a) = \lambda a$$

for all  $a \in A$  and  $\bar{\lambda} \in \Lambda$ .  $\omega$  is well-defined since  $mA = 0$ . Note that all four module axioms hold for  $A$  (as a  $\Lambda$ -module).

□

## §2. The Group of Homomorphisms

### Exercise 2.1.

Show that in the setting of Theorem 2.1  $\varepsilon_* = \text{Hom}(A, \varepsilon)$  is not, in general, surjective even if  $\varepsilon$  is. (Hint: Take  $\Lambda = \mathbb{Z}$ ,  $A = \mathbb{Z}/(n)$ , the integers mod  $n$ , and the short exact sequence  $\mathbb{Z} \xrightarrow{\mu} \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z}/(n)$  where  $\mu$  is multiplication by  $n$ .)

*Theorem 2.1.* Let  $B' \xrightarrow{\mu} B \xrightarrow{\varepsilon} B''$  be an exact sequence of  $\Lambda$ -modules. For every  $\Lambda$ -module  $A$  the induced sequence

$$0 \longrightarrow \text{Hom}_{\Lambda}(A, B') \xrightarrow{\mu_*} \text{Hom}_{\Lambda}(A, B) \xrightarrow{\varepsilon_*} \text{Hom}_{\Lambda}(A, B'')$$

is exact.

*Proof.*

(1) Consider

$$\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/(m), \mathbb{Z}) \xrightarrow{\varepsilon_*} \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/(m), \mathbb{Z}/(m)).$$

Note that  $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/(m), \mathbb{Z}/(m))$  is not trivial. So to prove that  $\varepsilon_*$  is not surjective, it suffices to show that  $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/(m), \mathbb{Z}) = 0$ .

(2) Show that  $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/(m), \mathbb{Z}) = 0$ . Suppose  $\alpha \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/(m), \mathbb{Z})$ . Given any  $a \in \mathbb{Z}/(m)$ . So  $ma = 0$  by the Lagrange's theorem in group theory. So

$$0 = \alpha(0) = \alpha(ma) = m\alpha(a) \in \mathbb{Z}.$$

So  $\alpha(a) = 0 \in \mathbb{Z}$ . Hence  $\alpha$  is a zero map.

□

**Exercise 2.2.**

*Prove Theorem 2.2.*

*Theorem 2.1.* Let  $A' \xrightarrow{\mu} A \xrightarrow{\varepsilon} A''$  be an exact sequence of  $\Lambda$ -modules. For every  $\Lambda$ -module  $B$  the induced sequence

$$0 \longrightarrow \operatorname{Hom}_{\Lambda}(A'', B) \xrightarrow{\varepsilon^*} \operatorname{Hom}_{\Lambda}(A, B) \xrightarrow{\mu^*} \operatorname{Hom}_{\Lambda}(A', B)$$

is exact.

*Proof of Theorem 2.2.*

(1)

□