Chapter 8: Some Special Functions

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Supplement. Fourier coefficients in Definition 8.9.

(1) Write

$$f(x) = a_0 + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}$$

(as the textbook Rudin, Principles of Mathematical Analysis, Third Edition). Then

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx.$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx, n \in \mathbb{Z}^+.$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx, n \in \mathbb{Z}^+.$$

(2) One might write in one different form,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

The only difference between the new one and the old one is a_0, so a_0 should be

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx.$$

(3) Again, one might write in one different form,

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

Similarly, a_0 should be

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x)}{\sqrt{2}} dx.$$

(4) Recall $f(x) = \sum_{-N}^{N} c_n e^{inx}$ $(x \in \mathbb{R})$ where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

The relations among a_n, b_n of this textbook and c_n are

$$c_0 = a_0$$

$$c_n = \frac{1}{2}(a_n + ib_n), n \in \mathbb{Z}^+.$$

(5) In some textbooks (Henryk Iwaniec, Topics in Classical Automorphic Forms), it is convenient to consider periodic functions f of period 1. Define

$$e(n) = e^{2\pi ix} = \cos(2\pi x) + i\sin(2\pi x).$$

Any periodic and piecewise continuous function f has the Fourier series representation

$$f(x) = \sum_{-\infty}^{\infty} a_n e(nx)$$

with coefficients given by

$$a_n = \int_0^1 f(x)e(-nx)dx.$$

Here is one exercise for this representation. *Show that the fractional part of x, $\{x\} = x - [x]$, is given by*

$$\{x\} = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{\pi n}.$$

Supplement. Parseval's theorem 8.16.

(1) Given

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

Then

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

(2) Given

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

Then

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

(3) Given
$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$
Then
$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

Exercise 8.1. *Define*
$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$
Prove that f has derivatives of all orders at $x = 0$, and that $f^{(n)}(0) = 0$ for $n = 1, 2, 3, \ldots$

$f(x)$ is an example of non-analytic smooth function, that is, infinitely differentiable functions are not necessarily analytic. In this exercise, we will show that Taylor series of f at the origin converges everywhere to the zero function. So the Taylor series does not equal $f(x)$ for $x \neq 0$. Consequently, f is not analytic at $x = 0$.

Proof.

(1) *Show that*
$$\lim_{x \to 0} g(x) e^{-\frac{1}{x^2}} = 0$$
for any rational function $g(x) \in \mathbb{R}(x)$.

(a) Write $g(x) = \frac{p(x)}{q(x)}$ for some $p(x), q(x) \in \mathbb{R}[x]$, $g(x) \neq 0$.

(b) Write $q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0$. $q(x)$ is not identically zero, that is, there exists the unique coefficient of the least power of x in $q(x)$ which is non-zero, say $b_M \neq 0$.

(c) Thus,
$$g(x) = \frac{p(x)/x^M}{q(x)/x^M}.$$
The denominator of $g(x)$ tends to $b_M \neq 0$ as $x \to 0$. By the similar argument in Theorem 8.6(f), we have
$$\frac{p(x)}{x^M} e^{-\frac{1}{x^2}} \to 0 \text{ as } x \to 0.$$
Hence, $\lim_{x \to 0} g(x) e^{-\frac{1}{x^2}} = 0$ for any $g(x) \in \mathbb{R}(x)$.

(2) *Given any real $x \neq 0$, show that*
$$f^{(n)}(x) = g_n(x) e^{-\frac{1}{x^2}}$$
for some rational function $g(x) \in \mathbb{R}(x)$.

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(a) Say $g_0(x) = 1 \in \mathbb{R}(x)$.

(b) $\mathbb{R}(x)$ is a field. *Show that $g'(x) \in \mathbb{R}(x)$ for any $g(x) \in \mathbb{R}(x)$.* Write $g(x) = \frac{p(x)}{q(x)}$ for some $p(x), q(x) \in \mathbb{R}[x]$, $q(x) \neq 0$. Thus

$$g'(x) = \frac{p'(x)q(x) - p(x)q'(x)}{q(x)^2}.$$

The numerator of $g'(x)$ is in $\mathbb{R}[x]$ since the differentiation operator on $\mathbb{R}[x]$ is closed in $\mathbb{R}[x]$. Also, the denominator of $g'(x) = q(x)^2 \neq 0$ since $\mathbb{R}[x]$ is an integral domain. Therefore, $g'(x) \in \mathbb{R}(x)$.

(c) Induction on n. For $n = 1$, we have

$$f'(x) = g_0'(x)e^{-\frac{1}{x^2}} + g_0(x) \cdot \left(-\frac{1}{x^2}\right)' e^{-\frac{1}{x^2}}$$

$$= \left(g_0'(x) + g_0(x) \cdot \left(-\frac{1}{x^2}\right)'\right) e^{-\frac{1}{x^2}}$$

$$= g_1(x)e^{-\frac{1}{x^2}}$$

where

$$g_1(x) = g_0'(x) + g_0(x) \cdot \left(-\frac{1}{x^2}\right)' \in \mathbb{R}(x).$$

Now assume that the conclusion holds for $n = k$. As $n = k + 1$, similar to the case $n = 1$,

$$f^{(k+1)}(x) = g_{k+1}(x)e^{-\frac{1}{x^2}}$$

where

$$g_{k+1}(x) = g_k'(x) + g_k(x) \cdot \left(-\frac{1}{x^2}\right)' \in \mathbb{R}(x).$$

By induction, the conclusion is true.

(3) Induction on n. For $n = 1$, by (1) we have

$$f'(0) = \lim_{t \to 0} \frac{e^{-\frac{1}{t^2}} - 0}{t} = 0.$$

Now assume that the statement holds for $n = k$. As $n = k + 1$, by (1)(2) we have

$$f^{(k+1)}(0) = \lim_{t \to 0} \frac{f^{(k)}(t) - f^{(k)}(0)}{t} = \lim_{t \to 0} \frac{g_k(t)e^{-\frac{1}{t^2}} - 0}{t} = 0.$$

Thus, $f^{(n)}(0) = 0$ for $n \in \mathbb{Z}^+$.

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□

Exercise 8.2. *Let a_{ij} be the number in the ith row and jth column of the array*

$$
\begin{array}{ccccc}
-1 & 0 & 0 & 0 & \cdots \\
\frac{1}{2} & -1 & 0 & 0 & \cdots \\
\frac{1}{4} & \frac{1}{2} & -1 & 0 & \cdots \\
\frac{1}{8} & \frac{1}{4} & \frac{1}{2} & -1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

so that

$$
a_{ij} = \begin{cases}
0 & (i < j), \\
-1 & (i = j), \\
2^{j-i} & (i > j).
\end{cases}
$$

Prove that

$$
\sum_i \sum_j a_{ij} = -2, \quad \sum_j \sum_i a_{ij} = 0.
$$

Also see Theorem 8.3.

Proof (Brute-force).

$$
\begin{aligned}
\sum_i \sum_j a_{ij} &= \sum_{i=1}^{\infty} \left(\sum_{j=i} a_{ij} + \sum_{j<i} a_{ij} \right) \\
&= \sum_{i=1}^{\infty} \left(-1 + \sum_{j=1}^{i-1} 2^{j-i} \right) \\
&= \sum_{i=1}^{\infty} (-1 + (1 - 2^{1-i})) \\
&= \sum_{i=1}^{\infty} -2^{1-i} \\
&= -2.
\end{aligned}
$$

$$\sum_j \sum_i a_{ij} = \sum_{j=1}^{\infty} \left(\sum_{i=j} a_{ij} + \sum_{i>j} a_{ij} \right)$$

$$= \sum_{j=1}^{\infty} \left(-1 + \sum_{i=j+1}^{\infty} 2^{j-i} \right)$$

$$= \sum_{j=1}^{\infty} (-1 + 1)$$

$$= \sum_{j=1}^{\infty} 0$$

$$= 0.$$

$\square$

Exercise 8.3.

Exercise 8.4.

Exercise 8.5.

Exercise 8.6. *Suppose* $f(x)f(y) = f(x+y)$ *for all real* x *and* y.

(a) *Assuming that* f *is differentiable and not zero, prove that*

$$f(x) = e^{cx}$$

where c *is a constant.*

(b) *Prove the same thing, assuming only that* f *is continuous.*

Part (b) implies part (a). We prove part (b) directly.

Proof of (b).

(1) Since $f(x)$ is not zero, there exists $x_0 \in \mathbb{R}$ such that $f(x_0) \neq 0$. So $f(0)f(x_0) = f(x_0)$, or $f(0) = 1$ by cancelling $f(x_0) \neq 0$.

(2) Next, $f(\frac{n}{m}) = f(\frac{1}{m})^n$ for $m \in \mathbb{Z}$, $n \in \mathbb{Z}^+$. Since f is continuous at $x = 0$, f is positive in the neighborhood of $x = 0$. That is, there exists $N \in \mathbb{Z}^+$ such that $f(\frac{1}{m}) > 0$ whenever $|m| \geq N$. So, $f(\frac{n}{m}) = f(\frac{1}{m})^n > 0$. (Since $f(\frac{n}{m}) = f(\frac{kn}{km})$ for any $k \in \mathbb{Z}^+$, we can rescale m to km such that $|km| \geq N$.) That is, f is positive on $\mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and f is continuous on $\mathbb{R}$, f is positive on $\mathbb{R}$.

(3) Now let $c = \log f(1)$ (which is well-defined since $f > 0$). We write $f(1)$ in the two ways. Firstly, $f(1) = f(\frac{n}{n}) = f(\frac{1}{n})^n$ where $n \in \mathbb{Z}^+$. Secondly, $f(1) = e^c = (e^{\frac{c}{n}})^n$. Since the positive n-th root is unique (Theorem 1.21),

$f(\frac{1}{n}) = e^{\frac{c}{n}}$ for $n \in \mathbb{Z}^+$. By $f(x)f(-x) = f(0) = 1$ or $f(-x) = \frac{1}{f(x)}$, $f(-\frac{1}{n}) = \frac{1}{e^{\frac{c}{n}}} = e^{-\frac{c}{n}}$ for $n \in \mathbb{Z}^+$. Therefore,

$$f\left(\frac{1}{m}\right) = e^{\frac{c}{m}} \text{ where } m \in \mathbb{Z}.$$

(4) By using $f(\frac{n}{m}) = f(\frac{1}{m})^n$ for $m \in \mathbb{Z}$, $n \in \mathbb{Z}^+$ again, $f(\frac{n}{m}) = e^{c\frac{n}{m}}$ where $m \in \mathbb{Z}, n \in \mathbb{Z}^+$, or

$$f(x) = e^{cx} \text{ where } x \in \mathbb{Q}.$$

Since $g(x) = f(x) - e^{cx}$ vanishes on a dense set of $\mathbb{Q}$ and g is continuous on $\mathbb{R}$, g vanishes on $\mathbb{R}$. Therefore, $f(x) = e^{cx}$ for $x \in \mathbb{R}$.

$\square$

Supplement. *Proof of (a).*

(1) Since $f(x)$ is not zero, there exists $x_0 \in \mathbb{R}$ such that $f(x_0) \neq 0$. So $f(0)f(x_0) = f(x_0)$, or $f(0) = 1$ by cancelling $f(x_0) \neq 0$.

(2) Since f is differentiable, for any $x \in \mathbb{R}$,

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x)f(h) - f(x)}{h} \\
&= f(x) \lim_{h \to 0} \frac{f(h) - 1}{h} \\
&= f(x) \lim_{h \to 0} \frac{f(h) - f(0)}{h} \\
&= f(x) f'(0).
\end{aligned}$$

Let $c = f'(0)$ be a constant. Then $f'(x) = cf(x)$. So $f(x) = e^{cx}$ for $x \in \mathbb{R}$. (To see this, let $g(x) = \frac{f(x)}{e^{cx}}$ be well-defined on $\mathbb{R}$. $g(0) = 1$. $g'(x) = 0$ since $f'(x) = cf(x)$. So $g(x)$ is a constant, or $g(x) = 1$ since $g(0) = 1$. Therefore, $f(x) = e^{cx}$ on $\mathbb{R}$.)

$\square$

Supplement. Cauchy's functional equation.

(1) *(Cauchy's functional equation.) Suppose $f(x) + f(y) = f(x+y)$ for all real x and y. Assuming that f is continuous, prove that $f(x) = cx$ where c is a constant.*

Notice that we cannot let $g(x) = \log f(x)$ and apply Cauchy's functional equation on $g(x)$ to prove Exercise 8.6 since $f(x)$ is not necessary positive

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and thus $g(x) = \log f(x)$ might be meaningless. However, this wrong approach gives you some useful ideas such as you need to prove that $f(x)$ is positive first, and $f(x)$ should be equal to e^{cx} where $c = g(1) = \log f(1)$.

(2) *Suppose $f(xy) = f(x) + f(y)$ for all positive real x and y. Assuming that f is continuous, prove that $f(x) = c \log x$ where c is a constant.*

(3) *Suppose $f(xy) = f(x)f(y)$ for all positive real x and y. Assuming that f is continuous and positive, prove that $f(x) = x^c$ where c is a constant.*

(4) *Suppose $f(x + y) = f(x) + f(y) + xy$ for all real x and y. Assuming that f is continuous, prove that $f(x) = \frac{1}{2}x^2 + cx$ where c is a constant.*

(5) *(USA 2002.) Suppose $f(x^2 - y^2) = xf(x) - yf(y)$ for all real x and y. Assuming that f is continuous, prove that $f(x) = cx$ where c is a constant.*

Supplement. *Show that the only automorphism of $\mathbb{Q}$ is the identity.*

Proof. Given any $\sigma \in \mathrm{Aut}(\mathbb{Q})$.

(1) *Show that $\sigma(1) = 1$.* Since $1^2 = 1$, $\sigma(1)\sigma(1) = \sigma(1)$. $\sigma(1) = 0$ or 1. There are only two possible cases.

  (a) Assume that $\sigma(1) = 0$. So

  $$\sigma(a) = \sigma(a \cdot 1) = \sigma(a) \cdot \sigma(1) = \sigma(a) \cdot 0 = 0$$

  for any $a \in \mathbb{Q}$. That is, $\sigma = 0 \in \mathrm{Aut}(\mathbb{Q})$, which is absurd.

  (b) Therefore, $\sigma(1) = 1$.

(2) *Show that $\sigma(n) = n$ for all $n \in \mathbb{Z}^+$.* Write $n = 1 + 1 + \cdots + 1$ (n times 1). Applying the additivity of σ, we have

  $$\sigma(n) = \sigma(1) + \sigma(1) + \cdots + \sigma(1) = 1 + 1 + \cdots + 1 = n.$$

  (Might use induction on n to eliminate $\cdots$ symbols.)

(3) *Show that $\sigma(n) = n$ for all $n \in \mathbb{Z}$.* By the additivity of σ, $\sigma(-n) = -\sigma(n) = -n$ for $n \geq 0$. The result is established.

For any $a = \frac{n}{m} \in \mathbb{Q}$ ($m, n \in \mathbb{Z}$, $n \neq 0$), applying the multiplication of σ on $am = n$, that is, $\sigma(a)\sigma(m) = \sigma(n)$. By (3), we have $\sigma(a)m = n$, or

$$\sigma(a) = \frac{m}{n} = a$$

provided $n \neq 0$, or σ is the identity. $\square$

Exercise 8.8. *For $n = 0, 1, 2, \ldots$, and x real, prove that*

$$|\sin(nx)| \leq n|\sin x|.$$

Note that this inequality may be false for other values of n. For instance,

$$\left|\sin\left(\frac{1}{2}\pi\right)\right| > \frac{1}{2}|\sin \pi|.$$

Proof. Induction on n.

(1) Note that
$$\sin(a + b) = \sin a \cos b + \cos a \sin b$$
for any $a, b \in \mathbb{R}$.

(2) $n = 0, 1$ are clearly true.

(3) Assume the induction hypothesis that for the single case $n = k$ holds, meaning
$$|\sin(kx)| \leq k|\sin x|$$
is true. It follows that

$$
\begin{aligned}
|\sin((k+1)x)| &= |\sin(kx)\cos x + \cos(kx)\sin x| && (1) \\
&\leq |\sin(kx)||\cos x| + |\cos(kx)||\sin x| && \text{(Triangle inequality)} \\
&\leq |\sin(kx)| + |\sin x| && (|\cos(\cdot)| \leq 1) \\
&\leq k|\sin x| + |\sin x| && \text{(Induction hypothesis)} \\
&\leq (k+1)|\sin x|.
\end{aligned}
$$

$\square$

Exercise 8.10. *Prove that $\sum \frac{1}{p}$ diverges; the sum extends over all primes.*

There are many proofs of this result. We provide some of them.

Proof (Due to hint). Given N.

(1) *Show that*
$$\sum_{n \leq N} \frac{1}{n} \leq \prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1}.$$

By the unique factorization theorem on $n \leq N$,

$$\sum_{n \leq N} \frac{1}{n} \leq \prod_{p \leq N} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1}.$$

(2) By (1) and the fact that $\sum \frac{1}{n}$ diverges, there are infinitely many primes.

(3) *Show that*

$$\prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1} \leq \exp\left(\sum_{p \leq N} \frac{2}{p}\right).$$

By applying the inequality $(1 - x)^{-1} < e^{2x}$ where $x \in (0, \frac{1}{2}]$ on any prime p,

$$\left(1 - \frac{1}{p}\right)^{-1} < \exp\left(\frac{2}{p}\right).$$

Now multiplying the inequality over all primes $p \leq N$ and noticing that $\exp(x) \cdot \exp(y) = \exp(x + y)$, we have

$$\prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1} \leq \exp\left(\sum_{p \leq N} \frac{2}{p}\right).$$

(4) By (1)(3),

$$\sum_{n \leq N} \frac{1}{n} \leq \exp\left(\sum_{p \leq N} \frac{2}{p}\right).$$

Since $\sum_{n \leq N} \frac{1}{n}$ diverges, the result holds.

□

Proof (Due to Kenneth Ireland and Michael Rosen). The proof in Kenneth Ireland and Michael Rosen, A Classical Introduction to Modern Number Theory, Second Edition (Theorem 3 in Chapter 2) does not use the inequality $(1-x)^{-1} < e^{2x}$ $(x \in (0, \frac{1}{2}])$ directly. Instead, the authors take the logarithm on $(1 - p^{-1})^{-1}$ and estimate it. (So the length of proof is longer than the proof due to hint.) That is,

$$
\begin{aligned}
-\log\left(1 - p^{-1}\right) &= \sum_{n=1}^{\infty} \frac{p^{-n}}{n} \\
&= \frac{1}{p} + \sum_{n=2}^{\infty} \frac{p^{-n}}{n} \\
&< \frac{1}{p} + \sum_{n=2}^{\infty} p^{-n} \\
&= \frac{1}{p} + \frac{p^{-2}}{1 - p^{-1}} \\
&< \frac{1}{p} + 2 \cdot \frac{1}{p^2}.
\end{aligned}
$$

Now we sum over all primes $p \leq N$,

$$\log \left(\prod_{p \leq N} \left(1 - \frac{1}{p} \right)^{-1} \right) < \sum_{p \leq N} \frac{1}{p} + 2 \sum_{p \leq N} \frac{1}{p^2}.$$

So

$$\log \sum_{n \leq N} \frac{1}{n} < \sum_{p \leq N} \frac{1}{p} + 2 \sum_{p \leq N} \frac{1}{p^2}.$$

Notice that $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{p^2}$ converges (since $\sum \frac{1}{n^2}$ converges). Therefore, $\sum \frac{1}{p}$ diverges. $\square$

Proof (Due to I. Niven). It is an exercise in Kenneth Ireland and Michael Rosen, A Classical Introduction to Modern Number Theory, Second Edition. See Exercise 27 in Chapter 2.

(1) *Show that $\sum' \frac{1}{n}$, the sum being over square free integers, diverges.* For any positive integers n, we can write $n = a^2 b$ where $a \in \mathbb{Z}^+$ and b is a square free integer. Given N,

$$\sum_{n \leq N} \frac{1}{n} \leq \left(\sum_{a=1}^{\infty} \frac{1}{a^2} \right) \left(\sum_{b \leq N}' \frac{1}{b} \right).$$

Notice that $\sum_{a=1}^{\infty} \frac{1}{a^2}$ converges. Since $\sum_{n \leq N} \frac{1}{n} \to \infty$ as $N \to \infty$, $\sum_{b \leq N}' \frac{1}{b} \to \infty$ as $N \to \infty$.

(2) *Show that*

$$\prod_{p \leq N} (1 + \frac{1}{p}) \to \infty \text{ as } N \to \infty.$$

By the unique factorization theorem on $n \leq N$,

$$\prod_{p \leq N} \left(1 + \frac{1}{p} \right) \geq \sum_{n \leq N}' \frac{1}{n}.$$

Since $\sum_{n \leq N}' \frac{1}{n} \to \infty$ as $N \to \infty$ by (1), the conclusion is established.

(3) By applying the inequality $e^x > 1 + x$ on any prime p,

$$\exp \left(\frac{1}{p} \right) > 1 + \frac{1}{p}.$$

Now multiplying the inequality over all primes $p \leq N$ and noticing that $\exp(x) \cdot \exp(y) = \exp(x + y)$, we have

$$\exp \left(\sum_{p \leq N} \frac{1}{p} \right) > \prod_{p \leq N} \left(1 + \frac{1}{p} \right).$$

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By (2), $\exp\left(\sum_{p\leq N}\frac{1}{p}\right) \to \infty$ as $N \to \infty$, or $\sum_{p\leq N}\frac{1}{p} \to \infty$ as $N \to \infty$.

$\square$

Exercise 8.12. *Suppose $0 < \delta < \pi$,*

$$f(x) = \begin{cases} 1 & \text{if } |x| \leq \delta, \\ 0 & \text{if } \delta < |x| \leq \pi, \end{cases}$$

and $f(x + 2\pi) = f(x)$ for all x.

(a) *Compute the Fourier coefficients of f.*

(b) *Compute that*

$$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2} \qquad (0 < \delta < \pi).$$

(c) *Deduce from Parseval's theorem that*

$$\sum_{n=1}^{\infty} \frac{(\sin(n\delta))^2}{n^2\delta} = \frac{\pi - \delta}{2}.$$

(d) *Let $\delta \to 0$ and prove that*

$$\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}.$$

(e) *Put $\delta = \frac{\pi}{2}$ in (c). What do you get?*

It is a centered square pulse around $x = 0$ with shift δ. Besides, $f(x)$ is an even function.

Proof of (a).

$$\begin{aligned}
c_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx \\
&= \frac{1}{2\pi} \int_{-\delta}^{\delta} dx \\
&= \frac{\delta}{\pi}.
\end{aligned}$$

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For $0 \neq n \in \mathbb{Z}$,

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

$$= \frac{1}{2\pi} \int_{-\delta}^{\delta} e^{-inx} dx$$

$$= \frac{1}{2\pi} \cdot \frac{2\sin(n\delta)}{n}$$

$$= \frac{\sin(n\delta)}{n\pi}.$$

□

Supplement. Find a_n and b_n of this textbook.

By (a), $a_0 = \frac{\delta}{\pi}$, $a_n = \frac{2\sin(n\delta)}{n\pi}$, $b_n = 0$ for $n \in \mathbb{Z}^+$. Surely, we can compute a_n and b_n $(n > 0)$ directly. Since $f(x)$ is an even function, $b_n = 0$. And

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

$$= \frac{2}{\pi} \int_0^{\delta} \cos(nx) dx$$

$$= \frac{2\sin(n\delta)}{n\pi}.$$

Proof of (b). Given $x = 0$, there are constants $\delta' = \delta > 0$ and $M = 1 < \infty$ such that

$$|f(0 + t) - f(0)| \leq M|t|$$

for all $t \in (-\delta', \delta')$. By Theorem 8.14,

$$\sum_{-\infty}^{\infty} c_n = f(0).$$

Notice that $c_{-n} = c_n$ for $n \in \mathbb{Z}^+$, so

$$\frac{\delta}{\pi} + 2 \sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n\pi} = 1$$

$$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2}.$$

□

We can also use the expression a_n and b_n to prove the same thing. Besides, taking $\delta = 1$ yields

$$\sum_{n=1}^{\infty} \frac{\sin n}{n} = \frac{\pi - 1}{2}.$$

Proof of (c). Since $f(x)$ is a Riemann-integrable function with period 2π, by Parseval's theorem

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx = \sum_{-\infty}^{\infty}|c_n|^2.$$

So

$$\frac{\delta}{\pi} = \frac{\delta^2}{\pi^2} + 2\sum_{n=1}^{\infty}\frac{(\sin(n\delta))^2}{n^2\pi^2},$$

or

$$\sum_{n=1}^{\infty}\frac{(\sin(n\delta))^2}{n^2\delta} = \frac{\pi-\delta}{2}.$$

□

Notices that

$$\sum_{n=1}^{\infty}\frac{(\sin n)^2}{n^2} = \frac{\pi-1}{2}$$

as $\delta = 1$.

Proof of (d). Given $\varepsilon > 0$. By Exercise 6.8,

$$\int_0^{\infty}\left(\frac{\sin x}{x}\right)^2 dx$$

exists. So there exists $b > 0$ such that

$$\left|\int_0^b\left(\frac{\sin x}{x}\right)^2 dx - \int_0^{\infty}\left(\frac{\sin x}{x}\right)^2 dx\right| < \frac{\varepsilon}{4}$$

By Supplement in Chapter 6, there exists $\delta > 0$ such that for any partition $P_m = \{0, \frac{b}{m}, \frac{2b}{m}, \ldots, \frac{(m-1)b}{m}, b\}$ of $[0,b]$ with $\|P\| = \frac{b}{m} < \delta$, or $m > \frac{b}{\delta}$, we have

$$\left|\sum_{n=1}^{m}\frac{(\sin(n\frac{b}{m}))^2}{(n\frac{b}{m})^2}\cdot\frac{b}{m} - \int_0^b\left(\frac{\sin x}{x}\right)^2 dx\right| < \frac{\varepsilon}{4},$$

$$\left|\sum_{n=1}^{m}\frac{(\sin(n\frac{b}{m}))^2}{n^2\frac{b}{m}} - \int_0^b\left(\frac{\sin x}{x}\right)^2 dx\right| < \frac{\varepsilon}{4}.$$

For simplicity we resize δ to $\delta < \pi$ to make $0 < \frac{b}{m} < \delta < \pi$. Besides, since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, there exists $N > 0$ such that

$$\left|\sum_{n=1}^{\infty}\frac{(\sin(n\frac{b}{m}))^2}{n^2\frac{b}{m}} - \sum_{n=1}^{m}\frac{(\sin(n\frac{b}{m}))^2}{n^2\frac{b}{m}}\right| < \frac{\varepsilon}{4}$$

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whenever $m \geq N$. By (c),

$$\left| \frac{\pi - \frac{b}{m}}{2} - \sum_{n=1}^{m} \frac{(\sin(n\frac{b}{m}))^2}{n^2 \frac{b}{m}} \right| < \frac{\varepsilon}{4}$$

whenever $m \geq N$. Last, it is easy to get

$$\left| \frac{\pi}{2} - \frac{\pi - \frac{b}{m}}{2} \right| < \frac{\varepsilon}{4}$$

whenever $m > \frac{2b}{\varepsilon}$. Now we have

$$\left| \frac{\pi}{2} - \int_0^{\infty} \left(\frac{\sin x}{x} \right)^2 dx \right| < \varepsilon$$

whenever $m > \max(\frac{b}{\delta}, N, \frac{2b}{\varepsilon})$. Since ε is arbitrary, $\int_0^{\infty} \left(\frac{\sin x}{x} \right)^2 dx = \frac{\pi}{2}$. $\square$

Proof of (e).

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

Write

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2}$$

$$= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2},$$

so

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{6}.$$

$\square$

Exercise 8.13. *Put $f(x) = x$ if $0 \leq x < 2\pi$, and apply Parseval's theorem to conclude that*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi}{6}.$$

Proof.

$$c_0 = \frac{1}{2\pi} \int_0^{2\pi} x \, dx$$

$$= \pi,$$

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For $n \neq 0$,

$$\begin{aligned}
c_n &= \frac{1}{2\pi} \int_0^{2\pi} x e^{-inx} dx \\
&= \frac{1}{2\pi} \left(\left[-\frac{1}{in} x e^{-inx} \right]_{x=0}^{x=2\pi} - \int_0^{2\pi} -\frac{1}{in} e^{-inx} dx \right) \\
&= \frac{i}{n}.
\end{aligned}$$

Since $f(x)$ is a Riemann-integrable function with period 2π, by Parseval's theorem

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{-\infty}^{\infty} |c_n|^2.$$

So

$$\frac{1}{2\pi} \cdot \frac{(2\pi)^3}{3} = \pi^2 + 2 \sum_{n=1}^{\infty} \frac{1}{n^2},$$

or

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

$\square$

Supplement. *Put $f(x) = x^k$ if $k \in \mathbb{Z}^+$ and $0 \leq x < 2\pi$. Might show that*

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = r_k \pi^{2k}, r_k \in \mathbb{Q}.$$