Chapter 11: The Lebesuge Theory

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Exercise 11.1. *If $f \geq 0$ and $\int_E f d\mu = 0$, prove that $f(x) = 0$ almost everywhere on E. (Hint: Let E_n be the subset of E on which $f(x) > \frac{1}{n}$. Write $A = \bigcup E_n$. Then $\mu(A) = 0$ if and only if $\mu(E_n) = 0$ for every n.)*

Might assume that f is measurable on E.

Proof (Hint).

(1) Define $A = \{x \in E : f(x) > 0\}$. So $f(x) = 0$ almost everywhere on E if and only if $\mu(A) = 0$.

(2) Define
$$E_n = \left\{ x \in E : f(x) > \frac{1}{n} \right\}$$
for $n = 1, 2, 3, \ldots$. Note that $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$ and
$$A = \bigcup_{n=1}^{\infty} E_n.$$
Since μ is a measure,
$$\lim_{n \to \infty} \mu(E_n) = \mu(A)$$
(Theorem 11.3).

(3) (Reductio ad absurdum) If $\mu(A) > 0$, there is an integer N such that $\mu(E_n) \geq \frac{\mu(A)}{2}$ whenever $n \geq N$ (by (2)). In particular, take $n = N$ to get
$$\begin{aligned}
\int_E f d\mu &\geq \int_{E_N} f d\mu && (\mu \text{ is a measure and } E_N \subseteq E) \\
&\geq \frac{1}{N} \cdot \mu(E_N) && (\text{Remarks 11.23(b)}) \\
&\geq \frac{1}{N} \cdot \frac{\mu(A)}{2} \\
&> 0,
\end{aligned}$$
contrary to the assumption that $\int_E f d\mu = 0$.

$\square$

Note. Compare to Exercise 6.2.

Exercise 11.2. *If $\int_A f d\mu = 0$ for every measurable subset A of a measurable set E, then $f(x) = 0$ almost everywhere on E.*

Might assume that f is measurable on E.

Proof.

(1) Define

$$A = \{x \in E : f(x) \geq 0\} \qquad \text{and} \qquad B = \{x \in E : f(x) \leq 0\}.$$

A and B are measurable subsets of a measurable set E since f is measurable.

(2) Apply Exercise 11.1 to the fact that $f \geq 0$ on A (by construction) and $\int_A f d\mu = 0$ (by assumption), we have $f(x) = 0$ almost everywhere on A.

(3) Similarly, apply Exercise 11.1 to the fact that $-f \geq 0$ on B and $\int_B (-f) d\mu = -\int_B f d\mu = 0$, we have $f(x) = 0$ almost everywhere on B.

(4) As $E = A \cup B$, $f(x) = 0$ almost everywhere on E by (2)(3).

□

Exercise 11.3. *If $\{f_n\}$ is a sequence of measurable functions, prove that the set of points x at which $\{f_n(x)\}$ converges is measurable.*

Proof.

(1) It suffices to show that

$$E = \{x : \{f_n(x)\} \text{ is convergent }\} = \{x : \{f_n(x)\} \text{ is Cauchy }\}$$

is measurable (since $\mathbb{R}^1$ is complete).

(2) Write

$$E = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n,m \geq N} \left\{ x : |f_n(x) - f_m(x)| \leq \frac{1}{k} \right\}$$

Since $\{f_n\}$ is a sequence of measurable functions, $x \mapsto |f_n(x) - f_m(x)|$ is measurable (Theorem 11.16 and Theorem 11.18). Hence

$$\left\{ x : |f_n(x) - f_m(x)| \leq \frac{1}{k} \right\}$$

is measurable (Theorem 11.15). Therefore E is measurable.

□

Exercise 11.4. *If $f \in \mathscr{L}(\mu)$ on E and g is bounded and measurable on E, then $fg \in \mathscr{L}(\mu)$ on E.*

Proof (Theorem 11.27).

(1) fg is measurable since both f and g are measurable (Theorem 11.18).

(2) $|g| \leq M$ for some real $M \in \mathbb{R}^1$ by the boundedness of g. Hence

$$|fg| \leq M|f|$$

on E.

(3) To apply Theorem 11.27, it suffices to show that $M|f| \in \mathscr{L}(\mu)$ on E. Theorem 11.26 implies that $|f| \in \mathscr{L}(\mu)$ if $f \in \mathscr{L}(\mu)$. And Remarks 11.23(d) implies that $M|f| \in \mathscr{L}(\mu)$ if $|f| \in \mathscr{L}(\mu)$.

□

Note. It is not true for Riemann integrable functions: *If $f \in \mathscr{R}$ on $[a, b]$ and g is bounded and measurable on $[a, b]$, then fg might be not Riemann integrable.*

Exercise 11.5. *...*

Proof.

(1)

(2)

□

Exercise 11.6. *...*

Proof.

(1)

(2)

□

Exercise 11.7. *...*

Proof.

(1)

(2)

□

Exercise 11.8. ...

Proof.

(1)

(2)

□

Exercise 11.9. ...

Proof.

(1)

(2)

□

Exercise 11.10. ...

Proof.

(1)

(2)

□

Exercise 11.11. ...

Proof.

(1)

(2)

□

Exercise 11.12. ...

Proof.

(1)

(2)

□

Exercise 11.13. ...

Proof.

(1)

(2)

□

Exercise 11.14. ...

Proof.

(1)

(2)

□

Exercise 11.15. ...

Proof.

(1)

(2)

□

Exercise 11.16. ...

Proof.

(1)

(2)

□

Exercise 11.17. ...

Proof.

(1)

(2)

□

Exercise 11.18. ...

Proof.

(1)

(2)

□