Chapter 8: Some Special Functions

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Supplement. Fourier coefficients in Definition 8.9.

(1) Write

$$f(x) = a_0 + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}$$

(as the textbook Rudin, Principles of Mathematical Analysis, Third Edition). Then

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx. \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx, n \in \mathbb{Z}^+. \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx, n \in \mathbb{Z}^+. \end{aligned}$$

(2) One might write in one different form,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

The only difference between the new one and the old one is a_0, so a_0 should be

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx.$$

(3) Again, one might write in one different form,

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

Similarly, a_0 should be

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x)}{\sqrt{2}} dx.$$

(4) Recall $f(x) = \sum_{-N}^{N} c_n e^{inx}$ $(x \in \mathbb{R})$ where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

The relations among a_n, b_n of this textbook and c_n are

$$c_0 = a_0$$
$$c_n = \frac{1}{2}\left(a_n + ib_n\right), n \in \mathbb{Z}^+.$$

(5) In some textbooks (Henryk Iwaniec, Topics in Classical Automorphic Forms), it is convenient to consider periodic functions f of period 1. Define

$$e(n) = e^{2\pi ix} = \cos(2\pi x) + i\sin(2\pi x).$$

Any periodic and piecewise continuous function f has the Fourier series representation

$$f(x) = \sum_{-\infty}^{\infty} a_n e(nx)$$

with coefficients given by

$$a_n = \int_0^1 f(x)e(-nx)dx.$$

Here is one exercise for this representation. *Show that the fractional part of* x, $\{x\} = x - [x]$, *is given by*

$$\{x\} = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{\pi n}.$$

Supplement. Parseval's theorem 8.16.

(1) Given

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

Then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

(2) Given

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

Then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

(3) Given

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)), x \in \mathbb{R}.$$

Then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

Exercise 8.1. *Define*

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

Prove that f has derivatives of all orders at $x = 0$, and that $f^{(n)}(0) = 0$ for $n = 1, 2, 3, \ldots$

$f(x)$ is an example of non-analytic smooth function, that is, infinitely differentiable functions are not necessarily analytic. In this exercise, we will show that Taylor series of f at the origin converges everywhere to the zero function. So the Taylor series does not equal $f(x)$ for $x \neq 0$. Consequently, f is not analytic at $x = 0$.

Proof.

(1) *Show that*

$$\lim_{x \to 0} g(x)e^{-\frac{1}{x^2}} = 0$$

for any rational function $g(x) \in \mathbb{R}(x)$.

(a) Write $g(x) = \frac{p(x)}{q(x)}$ for some $p(x), q(x) \in \mathbb{R}[x]$, $g(x) \neq 0$.

(b) Write $q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0$. $q(x)$ is not identically zero, that is, there exists the unique coefficient of the least power of x in $q(x)$ which is non-zero, say $b_M \neq 0$.

(c) Thus,

$$g(x) = \frac{p(x)/x^M}{q(x)/x^M}.$$

The denominator of $g(x)$ tends to $b_M \neq 0$ as $x \to 0$. By the similar argument in Theorem 8.6(f), we have

$$\frac{p(x)}{x^M} e^{-\frac{1}{x^2}} \to 0 \text{ as } x \to 0.$$

Hence, $\lim_{x \to 0} g(x)e^{-\frac{1}{x^2}} = 0$ for any $g(x) \in \mathbb{R}(x)$.

(2) *Given any real $x \neq 0$, show that*

$$f^{(n)}(x) = g_n(x)e^{-\frac{1}{x^2}}$$

for some rational function $g(x) \in \mathbb{R}(x)$.

(a) Say $g_0(x) = 1 \in \mathbb{R}(x)$.

(b) $\mathbb{R}(x)$ is a field. *Show that $g'(x) \in \mathbb{R}(x)$ for any $g(x) \in \mathbb{R}(x)$.* Write $g(x) = \frac{p(x)}{q(x)}$ for some $p(x), q(x) \in \mathbb{R}[x]$, $q(x) \neq 0$. Thus

$$g'(x) = \frac{p'(x)q(x) - p(x)q'(x)}{q(x)^2}.$$

The numerator of $g'(x)$ is in $\mathbb{R}[x]$ since the differentiation operator on $\mathbb{R}[x]$ is closed in $\mathbb{R}[x]$. Also, the denominator of $g'(x) = q(x)^2 \neq 0$ since $\mathbb{R}[x]$ is an integral domain. Therefore, $g'(x) \in \mathbb{R}(x)$.

(c) Induction on n. For $n = 1$, we have

$$\begin{aligned} f'(x) &= g_0'(x)e^{-\frac{1}{x^2}} + g_0(x) \cdot \left(-\frac{1}{x^2}\right)' e^{-\frac{1}{x^2}} \\ &= \left(g_0'(x) + g_0(x) \cdot \left(-\frac{1}{x^2}\right)'\right) e^{-\frac{1}{x^2}} \\ &= g_1(x)e^{-\frac{1}{x^2}} \end{aligned}$$

where

$$g_1(x) = g_0'(x) + g_0(x) \cdot \left(-\frac{1}{x^2}\right)' \in \mathbb{R}(x).$$

Now assume that the conclusion holds for $n = k$. As $n = k + 1$, similar to the case $n = 1$,

$$f^{(k+1)}(x) = g_{k+1}(x)e^{-\frac{1}{x^2}}$$

where

$$g_{k+1}(x) = g_k'(x) + g_k(x) \cdot \left(-\frac{1}{x^2}\right)' \in \mathbb{R}(x).$$

By induction, the conclusion is true.

(3) Induction on n. For $n = 1$, by (1) we have

$$f'(0) = \lim_{t \to 0} \frac{e^{-\frac{1}{t^2}} - 0}{t} = 0.$$

Now assume that the statement holds for $n = k$. As $n = k + 1$, by (1)(2) we have

$$f^{(k+1)}(0) = \lim_{t \to 0} \frac{f^{(k)}(t) - f^{(k)}(0)}{t} = \lim_{t \to 0} \frac{g_k(t)e^{-\frac{1}{t^2}} - 0}{t} = 0.$$

Thus, $f^{(n)}(0) = 0$ for $n \in \mathbb{Z}^+$.

□

Exercise 8.2. *Let a_{ij} be the number in the ith row and jth column of the array*

$$\begin{array}{ccccc} -1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & -1 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{2} & -1 & 0 & \cdots \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

so that

$$a_{ij} = \begin{cases} 0 & (i < j), \\ -1 & (i = j), \\ 2^{j-i} & (i > j). \end{cases}$$

Prove that

$$\sum_i \sum_j a_{ij} = -2, \quad \sum_j \sum_i a_{ij} = 0.$$

Also see Theorem 8.3.

Proof (Brute-force).

$$\begin{aligned} \sum_i \sum_j a_{ij} &= \sum_{i=1}^{\infty} \left(\sum_{j=i} a_{ij} + \sum_{j<i} a_{ij} \right) \\ &= \sum_{i=1}^{\infty} \left(-1 + \sum_{j=1}^{i-1} 2^{j-i} \right) \\ &= \sum_{i=1}^{\infty} (-1 + (1 - 2^{1-i})) \\ &= \sum_{i=1}^{\infty} -2^{1-i} \\ &= -2. \end{aligned}$$

$$
\begin{aligned}
\sum_j \sum_i a_{ij} &= \sum_{j=1}^{\infty} \left(\sum_{i=j} a_{ij} + \sum_{i>j} a_{ij} \right) \\
&= \sum_{j=1}^{\infty} \left(-1 + \sum_{i=j+1}^{\infty} 2^{j-i} \right) \\
&= \sum_{j=1}^{\infty} (-1+1) \\
&= \sum_{j=1}^{\infty} 0 \\
&= 0.
\end{aligned}
$$

□

Exercise 8.3. PLACEHOLDER.

Exercise 8.4. PLACEHOLDER.

Exercise 8.5. PLACEHOLDER.

Exercise 8.6. *Suppose $f(x)f(y) = f(x+y)$ for all real x and y.*

(a) *Assuming that f is differentiable and not zero, prove that*

$$f(x) = e^{cx}$$

where c is a constant.

(b) *Prove the same thing, assuming only that f is continuous.*

Part (b) implies part (a). We prove part (b) directly.

Proof of (b).

(1) Since $f(x)$ is not zero, there exists $x_0 \in \mathbb{R}$ such that $f(x_0) \neq 0$. So $f(0)f(x_0) = f(x_0)$, or $f(0) = 1$ by cancelling $f(x_0) \neq 0$.

(2) Next, $f(\frac{n}{m}) = f(\frac{1}{m})^n$ for $m \in \mathbb{Z}$, $n \in \mathbb{Z}^+$. Since f is continuous at $x = 0$, f is positive in the neighborhood of $x = 0$. That is, there exists $N \in \mathbb{Z}^+$ such that $f(\frac{1}{m}) > 0$ whenever $|m| \geq N$. So, $f(\frac{n}{m}) = f(\frac{1}{m})^n > 0$. (Since $f(\frac{n}{m}) = f(\frac{kn}{km})$ for any $k \in \mathbb{Z}^+$, we can rescale m to km such that

$|km| \geq N$.) That is, f is positive on $\mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and f is continuous on $\mathbb{R}$, f is positive on $\mathbb{R}$.

(3) Now let $c = \log f(1)$ (which is well-defined since $f > 0$). We write $f(1)$ in the two ways. Firstly, $f(1) = f(\frac{n}{n}) = f(\frac{1}{n})^n$ where $n \in \mathbb{Z}^+$. Secondly, $f(1) = e^c = (e^{\frac{c}{n}})^n$. Since the positive n-th root is unique (Theorem 1.21), $f(\frac{1}{n}) = e^{\frac{c}{n}}$ for $n \in \mathbb{Z}^+$. By $f(x)f(-x) = f(0) = 1$ or $f(-x) = \frac{1}{f(x)}$, $f(-\frac{1}{n}) = \frac{1}{e^{\frac{c}{n}}} = e^{-\frac{c}{n}}$ for $n \in \mathbb{Z}^+$. Therefore,

$$f\left(\frac{1}{m}\right) = e^{\frac{c}{m}} \text{ where } m \in \mathbb{Z}.$$

(4) By using $f(\frac{n}{m}) = f(\frac{1}{m})^n$ for $m \in \mathbb{Z}$, $n \in \mathbb{Z}^+$ again, $f(\frac{n}{m}) = e^{c\frac{n}{m}}$ where $m \in \mathbb{Z}, n \in \mathbb{Z}^+$, or

$$f(x) = e^{cx} \text{ where } x \in \mathbb{Q}.$$

Since $g(x) = f(x) - e^{cx}$ vanishes on a dense set of $\mathbb{Q}$ and g is continuous on $\mathbb{R}$, g vanishes on $\mathbb{R}$. Therefore, $f(x) = e^{cx}$ for $x \in \mathbb{R}$.

□

Supplement. *Proof of (a).*

(1) Since $f(x)$ is not zero, there exists $x_0 \in \mathbb{R}$ such that $f(x_0) \neq 0$. So $f(0)f(x_0) = f(x_0)$, or $f(0) = 1$ by cancelling $f(x_0) \neq 0$.

(2) Since f is differentiable, for any $x \in \mathbb{R}$,

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x)f(h) - f(x)}{h} \\ &= f(x) \lim_{h \to 0} \frac{f(h) - 1}{h} \\ &= f(x) \lim_{h \to 0} \frac{f(h) - f(0)}{h} \\ &= f(x)f'(0). \end{aligned}$$

Let $c = f'(0)$ be a constant. Then $f'(x) = cf(x)$. So $f(x) = e^{cx}$ for $x \in \mathbb{R}$. (To see this, let $g(x) = \frac{f(x)}{e^{cx}}$ be well-defined on $\mathbb{R}$. $g(0) = 1$. $g'(x) = 0$ since $f'(x) = cf(x)$. So $g(x)$ is a constant, or $g(x) = 1$ since $g(0) = 1$. Therefore, $f(x) = e^{cx}$ on $\mathbb{R}$.)

□

Supplement. Cauchy's functional equation.

(1) *(Cauchy's functional equation.) Suppose $f(x)+f(y)=f(x+y)$ for all real x and y. Assuming that f is continuous, prove that $f(x)=cx$ where c is a constant.*

Notice that we cannot let $g(x)=\log f(x)$ and apply Cauchy's functional equation on $g(x)$ to prove Exercise 8.6 since $f(x)$ is not necessary positive and thus $g(x)=\log f(x)$ might be meaningless. However, this wrong approach gives you some useful ideas such as you need to prove that $f(x)$ is positive first, and $f(x)$ should be equal to e^{cx} where $c=g(1)=\log f(1)$.

(2) *Suppose $f(xy)=f(x)+f(y)$ for all positive real x and y. Assuming that f is continuous, prove that $f(x)=c\log x$ where c is a constant.*

(3) *Suppose $f(xy)=f(x)f(y)$ for all positive real x and y. Assuming that f is continuous and positive, prove that $f(x)=x^c$ where c is a constant.*

(4) *Suppose $f(x+y)=f(x)+f(y)+xy$ for all real x and y. Assuming that f is continuous, prove that $f(x)=\frac{1}{2}x^2+cx$ where c is a constant.*

(5) *(USA 2002.) Suppose $f(x^2-y^2)=xf(x)-yf(y)$ for all real x and y. Assuming that f is continuous, prove that $f(x)=cx$ where c is a constant.*

Supplement. *Show that the only automorphism of $\mathbb{Q}$ is the identity.*

Proof. Given any $\sigma\in\mathrm{Aut}(\mathbb{Q})$.

(1) *Show that $\sigma(1)=1$.* Since $1^2=1$, $\sigma(1)\sigma(1)=\sigma(1)$. $\sigma(1)=0$ or 1. There are only two possible cases.

   (a) Assume that $\sigma(1)=0$. So

$$\sigma(a)=\sigma(a\cdot 1)=\sigma(a)\cdot\sigma(1)=\sigma(a)\cdot 0=0$$

   for any $a\in\mathbb{Q}$. That is, $\sigma=0\in\mathrm{Aut}(\mathbb{Q})$, which is absurd.

   (b) Therefore, $\sigma(1)=1$.

(2) *Show that $\sigma(n)=n$ for all $n\in\mathbb{Z}^+$.* Write $n=1+1+\cdots+1$ (n times 1). Applying the additivity of σ, we have

$$\sigma(n)=\sigma(1)+\sigma(1)+\cdots+\sigma(1)=1+1+\cdots+1=n.$$

(Might use induction on n to eliminate $\cdots$ symbols.)

(3) *Show that $\sigma(n)=n$ for all $n\in\mathbb{Z}$.* By the additivity of σ, $\sigma(-n)=-\sigma(n)=-n$ for $n\geq 0$. The result is established.

For any $a = \frac{n}{m} \in \mathbb{Q}$ ($m, n \in \mathbb{Z}$, $n \neq 0$), applying the multiplication of σ on $am = n$, that is, $\sigma(a)\sigma(m) = \sigma(n)$. By (3), we have $\sigma(a)m = n$, or

$$\sigma(a) = \frac{m}{n} = a$$

provided $n \neq 0$, or σ is the identity. $\square$

Exercise 8.7. PLACEHOLDER.

Exercise 8.8. *For $n = 0, 1, 2, \ldots$, and x real, prove that*

$$|\sin(nx)| \leq n|\sin x|.$$

Note that this inequality may be false for other values of n. For instance,

$$\left|\sin\left(\frac{1}{2}\pi\right)\right| > \frac{1}{2}|\sin \pi|.$$

Proof. Induction on n.

(1) Note that
$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$
for any $a, b \in \mathbb{R}$.

(2) $n = 0, 1$ are clearly true.

(3) Assume the induction hypothesis that for the single case $n = k$ holds, meaning
$$|\sin(kx)| \leq k|\sin x|$$
is true. It follows that

$$\begin{aligned}
|\sin((k+1)x)| &= |\sin(kx)\cos x + \cos(kx)\sin x| && ((1)) \\
&\leq |\sin(kx)||\cos x| + |\cos(kx)||\sin x| && \text{(Triangle inequality)} \\
&\leq |\sin(kx)| + |\sin x| && (|\cos(\cdot)| \leq 1) \\
&\leq k|\sin x| + |\sin x| && \text{(Induction hypothesis)} \\
&\leq (k+1)|\sin x|.
\end{aligned}$$

$\square$

Exercise 8.9 (The Euler-Mascheroni constant).

(a) *Put* $s_N = 1 + \frac{1}{2} + \cdots + \frac{1}{N}$. *Prove that*

$$\lim_{N \to \infty} (s_N - \log N)$$

exists. (The limit, often denoted by γ*, is called Euler's constant. Its numerical value is* $0.5772\ldots$*. It is not known whether* γ *is rational or not.)*

(b) *Roughly how large must* m *be so that* $N = 10^m$ *satisfies* $s_N > 100$*?*

Proof of (a) (Theorem 3.14).

(1) Note that

$$\begin{aligned}
&\frac{1}{1+\frac{1}{n}} \leq \frac{1}{x} \leq 1 \text{ for } x \in \left[1, 1+\frac{1}{n}\right] \\
\Longrightarrow &\int_1^{1+\frac{1}{n}} \frac{dx}{1+\frac{1}{n}} \leq \int_1^{1+\frac{1}{n}} \frac{dx}{x} \leq \int_1^{1+\frac{1}{n}} dx && \text{(Theorem 6.12(b))} \\
\Longrightarrow &\frac{1}{n+1} \leq \int_1^{1+\frac{1}{n}} \frac{dx}{x} \leq \frac{1}{n} \\
\Longrightarrow &\frac{1}{n+1} \leq \log\left(1+\frac{1}{n}\right) \leq \frac{1}{n}. && \text{(Equation (39) on page 180)}
\end{aligned}$$

(2) Define

$$\gamma_n = s_n - \log n.$$

It suffices to show that $\{\gamma_n\}$ is monotonic and bounded (Theorem 3.14).

(3) *Show that* $\{\gamma_n\}$ *is decreasing.*

$$\begin{aligned}
\gamma_{n+1} - \gamma_n &= (s_{n+1} - \log(n+1)) - (s_n - \log n) \\
&= (s_{n+1} - s_n) - (\log(n+1) - \log n) \\
&= \frac{1}{n+1} - \log\left(\frac{n+1}{n}\right) \\
&= \frac{1}{n+1} - \log\left(1+\frac{1}{n}\right) \\
&\leq 0. && ((1))
\end{aligned}$$

Note. $\gamma_n \leq \cdots \leq \gamma_1 = 1$ for all $n = 1, 2, 3, \ldots$.

(4) *Show that* $\gamma_n \geq 0$ *for all* $n = 1, 2, 3, \ldots$. Since

$$\begin{aligned}
\log n &= \sum_{k=1}^{n-1} (\log(k+1) - \log k) \\
&= \sum_{k=1}^{n-1} \log \frac{k+1}{k} \\
&= \sum_{k=1}^{n-1} \log \left(1 + \frac{1}{k}\right) \\
&\leq \sum_{k=1}^{n-1} \frac{1}{k} \qquad \text{((1))} \\
&= s_{n-1},
\end{aligned}$$

we have

$$\gamma_n = s_n - \log n \geq s_n - s_{n-1} = \frac{1}{n} > 0.$$

By (3)(4), $\{\gamma_n\}$ converges to $\lim_{N \to \infty}(s_N - \log N) = \gamma$. $\square$

Supplement.

(1) This proof is based on **integral test** (Theorem 8.23) in the textbook: *Tom. M. Apostol, Mathematical Analysis, 2nd edition.*

(2) *Show that if* $f \geq 0$ *on* $[0, \infty)$ *and* f *is monotonically decreasing, and if*

$$c_n = \sum_{k=1}^{n} f(k) - \int_1^n f(x)dx,$$

then $\lim_{n \to \infty} c_n$ *exists.* (Exercise 10 of Section 5.2 in the textbook: *R Creighton Buck, Advanced Calculus, 3rd edition.* See page 235.) If this exercise is true, we can get the existence of γ by taking $f(x) = \frac{1}{x}$.

(a) Note that

$$f(n+1) \leq \int_n^{n+1} f(x)dx \leq f(n).$$

(b) *Show that* $\{c_n\}$ *is decreasing.*

$$c_{n+1} - c_n = f(n+1) - \int_n^{n+1} f(x)dx \leq 0.$$

(c) *Show that* $c_n \geq 0$. Since $f(k) \geq \int_k^{k+1} f(x)dx$,

$$\begin{aligned}
\sum_{k=1}^{n} f(k) &\geq \sum_{k=1}^{n} \int_k^{k+1} f(x)dx \\
&= \int_1^{n+1} f(x)dx \\
&\geq \int_1^{n} f(x)dx. && (f \geq 0)
\end{aligned}$$

So that $c_n = \sum_{k=1}^{n} f(k) - \int_1^n f(x)dx \geq 0$.

(d) By (b)(c), $\{c_n\}$ converges (Theorem 3.14).

$\square$

Proof of (a) (Limit comparison test). Inspired by this paper: *Philippe Flajolet and Ilan Vardi, Zeta Function Expansions of Classical Constants.*

(1) Rewrite

$$\gamma_n + \log n - \log(n+1) = \sum_{k=1}^{n} \left(\frac{1}{k} - \log\left(1 + \frac{1}{k}\right) \right)$$

(similar to the argument in (a)(4)(Theorem 3.14)).

(2) *Show that*

$$\lim_{k \to \infty} \frac{\frac{1}{k} - \log\left(1 + \frac{1}{k}\right)}{\frac{1}{k^2}} = \frac{1}{2}.$$

In fact,

$$\begin{aligned}
&\lim_{k \to \infty} \frac{\frac{1}{k} - \log\left(1 + \frac{1}{k}\right)}{\frac{1}{k^2}} \\
=&\lim_{x \to 0} \frac{x - \log(1+x)}{x^2} && (\text{Put } x = \frac{1}{k}) \\
=&\lim_{x \to 0} \frac{1 - \frac{1}{1+x}}{2x} && (\text{L'Hospital's rule}) \\
=&\lim_{x \to 0} \frac{1}{2(x+1)} \\
=&\frac{1}{2}.
\end{aligned}$$

(3) By limit comparison test or comparison test, $\sum \left(\frac{1}{k} - \log\left(1 + \frac{1}{k}\right)\right)$ converges since $\sum \frac{1}{k^2}$ converges. Also,

$$\lim_{n \to \infty} \log n - \log(n+1) = 0.$$

Therefore, $\lim_{n \to \infty} \gamma_n$ exists.

□

Note. This proof is based on **limit comparison test** (Theorem 8.21) in this textbook: *Tom. M. Apostol, Mathematical Analysis, 2nd edition. It is easy to prove by the original comparison test.*

Proof of (a) (Uniformly convergence of $\sum \frac{x}{n(x+n)}$*).* (One example to Exercise 7 of Section 6.2 in the textbook: *R Creighton Buck, Advanced Calculus, 3rd edition*. See pages 270 to 271.)

(1) Let
$$f_n(x) = \frac{x}{n(x+n)} = \frac{1}{n} - \frac{1}{x+n}$$
defined on $E = [0,1]$.

(2) Note that
$$|f_n(x)| \leq \frac{1}{n^2}$$
for all $x \in [0,1]$. Since $\sum \frac{1}{n^2}$ converges, $\sum f_n$ converges uniformly on $[0,1]$ (Theorem 7.10).

(3) Corollary to Theorem 7.16 implies that
$$\begin{aligned}
\int_0^1 \sum_{n=1}^{\infty} \frac{x}{n(x+n)} dx &= \sum_{n=1}^{\infty} \int_0^1 \frac{x}{n(x+n)} dx \\
&= \sum_{n=1}^{\infty} \int_0^1 \left(\frac{1}{n} - \frac{1}{x+n} \right) dx \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \log \frac{n+1}{n} \right) \\
&= \lim_{N \to \infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \log(N+1) \right) \\
&= \lim_{N \to \infty} (s_N - \log(N+1))
\end{aligned}$$
exists. Since $\lim_{N \to \infty}(\log(N+1) - \log N) = 0$,
$$\begin{aligned}
\gamma &= \lim_{N \to \infty} (s_N - \log N) \\
&= \lim_{N \to \infty} (s_N - \log(N+1)) + \lim_{N \to \infty} (\log(N+1) - \log N)
\end{aligned}$$
exists.

□

Proof of (a) (Existence of $\int_1^{\infty} \frac{\{x\}}{x^2} dx$*).*

(1) Define $\{x\} = x - [x]$ where $[x]$ is the greatest integer $\leq x$ (Exercise 6.16). *Show that*

$$\int_1^\infty \frac{\{x\}}{x^2} dx < \infty.$$

Use the similar argument in Exercise 6.16(b). Since $\frac{\{x\}}{x^2} \leq \frac{1}{x^2}$ on $[1, \infty)$ and $\int_1^\infty \frac{1}{x^2} dx = 1$ exists, the result is established (Theorem 6.12(b)).

(2) *Show that*

$$\int_1^N \frac{[x]}{x^2} dx = s_N - 1.$$

Use the similar argument in Exercise 6.16(a),

$$\begin{aligned}
\int_1^N \frac{[x]}{x^2} dx &= \sum_{k=1}^{N-1} \int_k^{k+1} \frac{[x]}{x^2} dx \\
&= \sum_{k=1}^{N-1} \int_k^{k+1} \frac{k}{x^2} dx \\
&= \sum_{k=1}^{N-1} \int_k^{k+1} \frac{k}{x^2} dx \\
&= \sum_{k=1}^{N-1} \frac{1}{k+1} \\
&= \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} \\
&= s_N - 1.
\end{aligned}$$

(3) *Show that*

$$\int_1^N \frac{\{x\}}{x^2} dx = \log N - s_N + 1 = 1 - \gamma_N.$$

In fact,

$$\begin{aligned}
\int_1^N \frac{\{x\}}{x^2} dx &= \int_1^N \frac{x - [x]}{x^2} dx \\
&= \int_1^N \frac{1}{x} dx - \int_1^N \frac{[x]}{x^2} dx \\
&= \log N - (s_N - 1) \\
&= \log N - s_N + 1 \\
&= 1 - \gamma_N.
\end{aligned}$$

(4) Since
$$\lim_{N\to\infty}\int_1^N \frac{\{x\}}{x^2}dx = \int_1^\infty \frac{\{x\}}{x^2}dx$$
exists (by (1)), $\gamma = \lim \gamma_N$ exists.

$\square$

Proof of (b). By $s_n - \log n > 0$ in (a)(4)(Theorem 3.14), it suffices to choose $N = 10^m$ such that $s_N \geq \log(N+1) > 100$, or
$$m > \frac{\log(\exp(100)-1)}{\log 10},$$
or choose m satisfying
$$m > \frac{100}{\log 10} > \frac{\log(\exp(100)-1)}{\log 10},$$
or $m = 44$. $\square$

Note. The exact value of N is
$$15092688622113788323693563264538101449859497 \approx 1.509 \times 10^{43}.$$

Exercise 8.10. *Prove that $\sum \frac{1}{p}$ diverges; the sum extends over all primes.*

There are many proofs of this result. We provide some of them.

Proof (Due to hint). Given N.

(1) *Show that*
$$\sum_{n\leq N}\frac{1}{n} \leq \prod_{p\leq N}\left(1-\frac{1}{p}\right)^{-1}.$$

By the unique factorization theorem on $n \leq N$,
$$\sum_{n\leq N}\frac{1}{n} \leq \prod_{p\leq N}\left(1+\frac{1}{p}+\frac{1}{p^2}+\cdots\right) = \prod_{p\leq N}\left(1-\frac{1}{p}\right)^{-1}.$$

(2) By (1) and the fact that $\sum \frac{1}{n}$ diverges, there are infinitely many primes.

(3) *Show that*
$$\prod_{p\leq N}\left(1-\frac{1}{p}\right)^{-1} \leq \exp\left(\sum_{p\leq N}\frac{2}{p}\right).$$

By applying the inequality $(1-x)^{-1} < e^{2x}$ where $x \in (0, \frac{1}{2}]$ on any prime p,

$$\left(1 - \frac{1}{p}\right)^{-1} < \exp\left(\frac{2}{p}\right).$$

Now multiplying the inequality over all primes $p \leq N$ and noticing that $\exp(x) \cdot \exp(y) = \exp(x+y)$, we have

$$\prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1} \leq \exp\left(\sum_{p \leq N} \frac{2}{p}\right).$$

(4) By (1)(3),

$$\sum_{n \leq N} \frac{1}{n} \leq \exp\left(\sum_{p \leq N} \frac{2}{p}\right).$$

Since $\sum_{n \leq N} \frac{1}{n}$ diverges, the result holds.

□

Proof (Due to Kenneth Ireland and Michael Rosen). The proof in Kenneth Ireland and Michael Rosen, A Classical Introduction to Modern Number Theory, Second Edition (Theorem 3 in Chapter 2) does not use the inequality $(1-x)^{-1} < e^{2x}$ ($x \in (0, \frac{1}{2}]$) directly. Instead, the authors take the logarithm on $(1-p^{-1})^{-1}$ and estimate it. (So the length of proof is longer than the proof due to hint.) That is,

$$\begin{aligned}
-\log\left(1 - p^{-1}\right) &= \sum_{n=1}^{\infty} \frac{p^{-n}}{n} \\
&= \frac{1}{p} + \sum_{n=2}^{\infty} \frac{p^{-n}}{n} \\
&< \frac{1}{p} + \sum_{n=2}^{\infty} p^{-n} \\
&= \frac{1}{p} + \frac{p^{-2}}{1 - p^{-1}} \\
&< \frac{1}{p} + 2 \cdot \frac{1}{p^2}.
\end{aligned}$$

Now we sum over all primes $p \leq N$,

$$\log\left(\prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1}\right) < \sum_{p \leq N} \frac{1}{p} + 2 \sum_{p \leq N} \frac{1}{p^2}.$$

So

$$\log \sum_{n \leq N} \frac{1}{n} < \sum_{p \leq N} \frac{1}{p} + 2 \sum_{p \leq N} \frac{1}{p^2}.$$

Notice that $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{p^2}$ converges (since $\sum \frac{1}{n^2}$ converges). Therefore, $\sum \frac{1}{p}$ diverges. $\square$

Proof (Due to I. Niven). It is an exercise in Kenneth Ireland and Michael Rosen, A Classical Introduction to Modern Number Theory, Second Edition. See Exercise 27 in Chapter 2.

(1) *Show that* $\sum' \frac{1}{n}$*, the sum being over square free integers, diverges.* For any positive integers n, we can write $n = a^2 b$ where $a \in \mathbb{Z}^+$ and b is a square free integer. Given N,

$$\sum_{n \leq N} \frac{1}{n} \leq \left(\sum_{a=1}^{\infty} \frac{1}{a^2} \right) \left({\sum_{b \leq N}}' \frac{1}{b} \right).$$

Notice that $\sum_{a=1}^{\infty} \frac{1}{a^2}$ converges. Since $\sum_{n \leq N} \frac{1}{n} \to \infty$ as $N \to \infty$, $\sum'_{b \leq N} \frac{1}{b} \to \infty$ as $N \to \infty$.

(2) *Show that*

$$\prod_{p \leq N} (1 + \frac{1}{p}) \to \infty \text{ as } N \to \infty.$$

By the unique factorization theorem on $n \leq N$,

$$\prod_{p \leq N} \left(1 + \frac{1}{p} \right) \geq {\sum_{n \leq N}}' \frac{1}{n}.$$

Since ${\sum_{n \leq N}}' \frac{1}{n} \to \infty$ as $N \to \infty$ by (1), the conclusion is established.

(3) By applying the inequality $e^x > 1 + x$ on any prime p,

$$\exp \left(\frac{1}{p} \right) > 1 + \frac{1}{p}.$$

Now multiplying the inequality over all primes $p \leq N$ and noticing that $\exp(x) \cdot \exp(y) = \exp(x + y)$, we have

$$\exp \left(\sum_{p \leq N} \frac{1}{p} \right) > \prod_{p \leq N} \left(1 + \frac{1}{p} \right).$$

By (2), $\exp \left(\sum_{p \leq N} \frac{1}{p} \right) \to \infty$ as $N \to \infty$, or $\sum_{p \leq N} \frac{1}{p} \to \infty$ as $N \to \infty$.

$\square$

Exercise 8.11. PLACEHOLDER.

Exercise 8.12. *Suppose* $0 < \delta < \pi$*,*

$$f(x) = \begin{cases} 1 & \text{if } |x| \leq \delta, \\ 0 & \text{if } \delta < |x| \leq \pi, \end{cases}$$

and $f(x + 2\pi) = f(x)$ *for all* x*.*

(a) *Compute the Fourier coefficients of* f*.*

(b) *Compute that*

$$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2} \qquad (0 < \delta < \pi).$$

(c) *Deduce from Parseval's theorem that*

$$\sum_{n=1}^{\infty} \frac{(\sin(n\delta))^2}{n^2\delta} = \frac{\pi - \delta}{2}.$$

(d) *Let* $\delta \to 0$ *and prove that*

$$\int_0^{\infty} \left(\frac{\sin x}{x} \right)^2 dx = \frac{\pi}{2}.$$

(e) *Put* $\delta = \frac{\pi}{2}$ *in (c). What do you get?*

It is a centered square pulse around $x = 0$ with shift δ. Besides, $f(x)$ is an even function.

Proof of (a).

$$\begin{aligned} c_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \\ &= \frac{1}{2\pi} \int_{-\delta}^{\delta} dx \\ &= \frac{\delta}{\pi}. \end{aligned}$$

For $0 \neq n \in \mathbb{Z}$,

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \\ &= \frac{1}{2\pi} \int_{-\delta}^{\delta} e^{-inx} dx \\ &= \frac{1}{2\pi} \cdot \frac{2\sin(n\delta)}{n} \\ &= \frac{\sin(n\delta)}{n\pi}. \end{aligned}$$

□

Supplement. Find a_n and b_n of this textbook.
By (a), $a_0 = \frac{\delta}{\pi}$, $a_n = \frac{2\sin(n\delta)}{n\pi}$, $b_n = 0$ for $n \in \mathbb{Z}^+$. Surely, we can compute a_n and b_n $(n > 0)$ directly. Since $f(x)$ is an even function, $b_n = 0$. And

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \\ &= \frac{2}{\pi} \int_{0}^{\delta} \cos(nx) dx \\ &= \frac{2\sin(n\delta)}{n\pi}. \end{aligned}$$

Proof of (b). Given $x = 0$, there are constants $\delta' = \delta > 0$ and $M = 1 < \infty$ such that

$$|f(0+t) - f(0)| \leq M|t|$$

for all $t \in (-\delta', \delta')$. By Theorem 8.14,

$$\sum_{-\infty}^{\infty} c_n = f(0).$$

Notice that $c_{-n} = c_n$ for $n \in \mathbb{Z}^+$, so

$$\begin{aligned} \frac{\delta}{\pi} + 2\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n\pi} &= 1 \\ \sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} &= \frac{\pi - \delta}{2}. \end{aligned}$$

□

We can also use the expression a_n and b_n to prove the same thing. Besides, taking $\delta = 1$ yields

$$\sum_{n=1}^{\infty} \frac{\sin n}{n} = \frac{\pi - 1}{2}.$$

Proof of (c). Since $f(x)$ is a Riemann-integrable function with period 2π, by Parseval's theorem

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx = \sum_{-\infty}^{\infty}|c_n|^2.$$

So

$$\frac{\delta}{\pi} = \frac{\delta^2}{\pi^2} + 2\sum_{n=1}^{\infty}\frac{(\sin(n\delta))^2}{n^2\pi^2},$$

or

$$\sum_{n=1}^{\infty}\frac{(\sin(n\delta))^2}{n^2\delta} = \frac{\pi-\delta}{2}.$$

□

Notices that

$$\sum_{n=1}^{\infty}\frac{(\sin n)^2}{n^2} = \frac{\pi-1}{2}$$

as $\delta = 1$.

Proof of (d). Given $\varepsilon > 0$. By Exercise 6.8,

$$\int_0^{\infty}\left(\frac{\sin x}{x}\right)^2 dx$$

exists. So there exists $b > 0$ such that

$$\left|\int_0^{b}\left(\frac{\sin x}{x}\right)^2 dx - \int_0^{\infty}\left(\frac{\sin x}{x}\right)^2 dx\right| < \frac{\varepsilon}{4}$$

By Supplement in Chapter 6, there exists $\delta > 0$ such that for any partition $P_m = \{0, \frac{b}{m}, \frac{2b}{m}, \dots, \frac{(m-1)b}{m}, b\}$ of $[0,b]$ with $\|P\| = \frac{b}{m} < \delta$, or $m > \frac{b}{\delta}$, we have

$$\left|\sum_{n=1}^{m}\frac{(\sin(n\frac{b}{m}))^2}{(n\frac{b}{m})^2}\cdot\frac{b}{m} - \int_0^{b}\left(\frac{\sin x}{x}\right)^2 dx\right| < \frac{\varepsilon}{4},$$

$$\left|\sum_{n=1}^{m}\frac{(\sin(n\frac{b}{m}))^2}{n^2\frac{b}{m}} - \int_0^{b}\left(\frac{\sin x}{x}\right)^2 dx\right| < \frac{\varepsilon}{4}.$$

For simplicity we resize δ to $\delta < \pi$ to make $0 < \frac{b}{m} < \delta < \pi$. Besides, since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, there exists $N > 0$ such that

$$\left|\sum_{n=1}^{\infty}\frac{(\sin(n\frac{b}{m}))^2}{n^2\frac{b}{m}} - \sum_{n=1}^{m}\frac{(\sin(n\frac{b}{m}))^2}{n^2\frac{b}{m}}\right| < \frac{\varepsilon}{4}$$

whenever $m \geq N$. By (c),

$$\left| \frac{\pi - \frac{b}{m}}{2} - \sum_{n=1}^{m} \frac{(\sin(n\frac{b}{m}))^2}{n^2 \frac{b}{m}} \right| < \frac{\varepsilon}{4}$$

whenever $m \geq N$. Last, it is easy to get

$$\left| \frac{\pi}{2} - \frac{\pi - \frac{b}{m}}{2} \right| < \frac{\varepsilon}{4}$$

whenever $m > \frac{2b}{\varepsilon}$. Now we have

$$\left| \frac{\pi}{2} - \int_0^\infty \left(\frac{\sin x}{x} \right)^2 dx \right| < \varepsilon$$

whenever $m > \max(\frac{b}{\delta}, N, \frac{2b}{\varepsilon})$. Since ε is arbitrary, $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$. $\square$

Proof of (e).

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

Write

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^2} &= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \\
&= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2},
\end{aligned}$$

so

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{6}.$$

$\square$

Exercise 8.13. *Put $f(x) = x$ if $0 \leq x < 2\pi$, and apply Parseval's theorem to conclude that*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi}{6}.$$

Proof.

$$\begin{aligned}
c_0 &= \frac{1}{2\pi} \int_0^{2\pi} x dx \\
&= \pi,
\end{aligned}$$

For $n \neq 0$,

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_0^{2\pi} x e^{-inx} dx \\ &= \frac{1}{2\pi} \left(\left[-\frac{1}{in} x e^{-inx} \right]_{x=0}^{x=2\pi} - \int_0^{2\pi} -\frac{1}{in} e^{-inx} dx \right) \\ &= \frac{i}{n}. \end{aligned}$$

Since $f(x)$ is a Riemann-integrable function with period 2π, by Parseval's theorem

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{-\infty}^{\infty} |c_n|^2.$$

So

$$\frac{1}{2\pi} \cdot \frac{(2\pi)^3}{3} = \pi^2 + 2 \sum_{n=1}^{\infty} \frac{1}{n^2},$$

or

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

□

Supplement. *Put $f(x) = x^n$ if $n \in \mathbb{Z}^+$ and $0 \leq x < 2\pi$. Might get*

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!}.$$

Exercise 8.14-8.31. PLACEHOLDER.