

Chapter 1: Curves

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Section 1-1: Introduction

Classical differential geometry: the study of local properties of curves and surfaces.

Global differential geometry: the study of the influence of the local properties on the behavior of the entire curve and surface.

No exercises.

Section 1-2: Parametrized Curves

Exercise 1-2.1. Find a parametrized curve $\alpha(t)$ whose trace is the circle $x^2 + y^2 = 1$ such that $\alpha(t)$ runs clockwise around the circle with $\alpha(0) = (0, 1)$.

Proof. $\alpha(t) = (\sin t, \cos t)$, $t \in \mathbb{R}$. \square

Exercise 1-2.2. Let $\alpha(t)$ be a parametrized curve which does not pass through the origin. If $\alpha(t_0)$ is the point of the trace of α closest to the origin and $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

Proof. Let $f(t) = |\alpha(t)|^2 = \alpha(t) \cdot \alpha(t)$. $f(t)$ is differentiable and $f(t)$ has a local minimum at a point $t = t_0 \in I$. So $f'(t_0) = 0$. [Theorem 5.8 in *W. Rudin, Principles of Mathematical Analysis*, 3rd edition.] Since

$$f'(t) = 2\alpha(t) \cdot \alpha'(t),$$

$f'(t_0) = 2\alpha(t_0) \cdot \alpha'(t_0) = 0$, or $\alpha(t_0) \cdot \alpha'(t_0) = 0$. Since $\alpha(t_0) \neq 0$ and $\alpha'(t_0) \neq 0$, $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$. \square

Exercise 1-2.3. A parametrized curve $\alpha(t)$ has a property that its second derivative $\alpha''(t)$ is identically zero. What can be said about α ?

$\alpha(t)$ is a straight line.

Proof. Since $\alpha''(t)$ is identically zero, $\alpha'(t) = a$ is a constant. [Theorem 5.11 in *W. Rudin, Principles of Mathematical Analysis*, 3rd edition.] Define

$f(t) = \alpha(t) - at$ (on I). Since $f'(t) = \alpha'(t) - a = 0$, $f(t) = \alpha(t) - at = b$ is a constant again. Therefore, $\alpha(t) = at + b$, which is a straight line (on I). \square

Exercise 1-2.4. Let $\alpha : I \rightarrow \mathbb{R}^3$ be a parametrized curve and let $v \in \mathbb{R}^3$ be a fixed vector. Assume that $\alpha'(t)$ is orthogonal to v for all $t \in I$ and that $\alpha(0)$ is orthogonal to v . Prove that $\alpha(t)$ is orthogonal to v for all $t \in I$.

Need to assume that $\alpha(t) \neq 0$ for all $t \in I$.

Proof. Given any $t \neq 0 \in I$. (Nothing to do at $t = 0$.) Define $f : I \rightarrow \mathbb{R}$ by $f(t) = \alpha(t) \cdot v$. By the mean value theorem, there exists a point ξ between 0 and t such that

$$f(t) - f(0) = f'(\xi)(t - 0),$$

where $f'(t) = \alpha'(t) \cdot v + \alpha(t) \cdot v' = \alpha'(t) \cdot v$. Note that $f(0) = 0$ since $\alpha(0)$ is orthogonal to v , and $f'(\xi) = 0$ since $\alpha'(\xi)$ is orthogonal to v . So the identity is reduced to

$$f(t) = 0,$$

or $\alpha(t) \cdot v = 0$, or $\alpha(t)$ is orthogonal to v . \square

Exercise 1-2.5. Let $\alpha : I \rightarrow \mathbb{R}^3$ be a parametrized curve, with $\alpha'(t) \neq 0$ for all $t \in I$. Show that $|\alpha(t)|$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.

The same trick in Exercise 1-2.2.

Proof. It is equivalent to show that $|\alpha(t)|^2$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$. Let

$$f(t) = |\alpha(t)|^2 = \alpha(t) \cdot \alpha(t).$$

Notice that $\alpha'(t) \neq 0$, and thus

$$\begin{aligned} & |\alpha(t)| \text{ is a nonzero constant} \\ \iff & f(t) = |\alpha(t)|^2 \text{ is a nonzero constant} \\ \iff & f'(t) = 0 \text{ and } f(t) \text{ is a nonzero constant} \\ \iff & \alpha(t) \cdot \alpha'(t) = 0 \text{ and } \alpha(t) \text{ is a nonzero constant} \\ \iff & \alpha(t) \text{ is orthogonal to } \alpha'(t) \text{ for all } t \in I. \end{aligned}$$

\square

Section 1-3: Regular Curves; Arc Length

Exercise 1-3.1. Show that the tangent lines to the regular parametrized curve $\alpha(t) = (3t, 3t^2, 2t^3)$ make a constant angle with the line $y = 0, z = x$.

Proof. $\alpha'(t) = (3, 6t, 6t^2)$. The line $y = 0, z = x$ is $\beta(t) = (1, 0, 1)$. The cosine of the angle θ between these two curves is

$$\begin{aligned}\cos \theta &= \frac{(3, 6t, 6t^2) \cdot (1, 0, 1)}{|(3, 6t, 6t^2)| |(1, 0, 1)|} \\ &= \frac{3 + 6t^2}{\sqrt{3^2 + (6t)^2 + (6t^2)^2} \sqrt{2}} \\ &= \frac{3 + 6t^2}{\sqrt{9 + 36t^2 + 36t^4} \sqrt{2}} \\ &= \frac{3 + 6t^2}{\sqrt{(3 + 6t^2)^2} \sqrt{2}} \\ &= \frac{1}{\sqrt{2}}.\end{aligned}$$

(Notice $3 + 6t^2 > 0$ for all $t \in \mathbb{R}$.) That is, the angle between α' and β is a constant ($= \pi/4$). \square

Exercise 1-3.2. A circular disk of radius 1 in the plane xy rolls without slipping along the x axis. The figure described by a point of the circumference of the disk is called a **cycloid** (Figure 1-7 in Mantreda P. do Carmo, *Differential Geometry of Curves and Surfaces*).

- Obtain a parametrized curve $\alpha : \mathbb{R} \rightarrow \mathbb{R}^2$ the trace of which is the cycloid and determine its singular points.
- Compute the arc length of the cycloid corresponding to a complete rotation of the disk.

Proof of (a).

- Since

$$\begin{cases} x = t - \sin t \\ y = 1 - \cos t, \end{cases}$$

we define $\alpha(t) = (t - \sin t, 1 - \cos t)$.

- $\alpha'(t) = (1 - \cos t, \sin t)$. $\alpha'(t) = 0$ if and only if $t = 2n\pi$ where $n \in \mathbb{Z}$. That is, all singular points are $\alpha(2n\pi) = (2n\pi, 0)$ where $n \in \mathbb{Z}$.

\square

Proof of (b). The arc length of the cycloid corresponding to a complete rotation of the disk is

$$\begin{aligned}
 \int_0^{2\pi} |\alpha'(t)| dt &= \int_0^{2\pi} \sqrt{(1 - \cos t)^2 + (\sin t)^2} dt \\
 &= \int_0^{2\pi} \sqrt{2} \sqrt{1 - \cos t} dt \\
 &= \int_0^{2\pi} 2 \sin \frac{t}{2} dt \\
 &= \left[-4 \cos \frac{t}{2} \right]_{t=0}^{t=2\pi} \\
 &= 8.
 \end{aligned}$$

□

Supplement. The cycloid is not an algebraic curve.

Exercise 1-3.4. Let $\alpha : (0, \pi) \rightarrow \mathbb{R}^2$ be given by

$$\alpha(t) = \left(\sin t, \cos t + \log \tan \frac{t}{2} \right),$$

where t is the angle that the y axis makes with the vector $\alpha(t)$. The trace of α is called the **tractrix**. (Figure 1-9 in Mantredo P. do Carmo, *Differential Geometry of Curves and Surfaces*). Show that

- (a) α is a differentiable parametrized curve, regular except at $t = \frac{\pi}{2}$.
- (b) The length of the segment of the tangent of the tractrix between the point of tangency and the y axis is constantly equal to 1.

Proof of (a).

$$\begin{aligned}
 \alpha'(t) &= \left(\cos t, -\sin t + \frac{1}{\tan \frac{t}{2}} \frac{1}{\cos^2 \frac{t}{2}} \frac{1}{2} \right) \\
 &= \left(\cos t, -\sin t + \frac{1}{2 \sin \frac{t}{2} \cos \frac{t}{2}} \right) \\
 &= \left(\cos t, \frac{\cos^2 t}{\sin t} \right)
 \end{aligned}$$

exists. And $\alpha'(t) = 0$ if and only if $t = \frac{\pi}{2}$. That is, there is a unique singular point at $t = \frac{\pi}{2}$. □

Proof of (b). The tangent line of the tractrix through the regular point t is parametrized by $\beta : \mathbb{R} \rightarrow \mathbb{R}^2$ which is defined by

$$\begin{aligned}\beta(u) &= u\alpha'(t) + \alpha(t) \\ &= \left(u \cos t + \sin t, u \frac{\cos^2 t}{\sin t} + \cos t + \log \tan \frac{t}{2} \right).\end{aligned}$$

By construction, this tangent line $\beta(u)$ meets the tractrix at $u = 0$, and meets the y -axis when $u \cos t + \sin t = 0$ or $u = -\tan t$. So the length of the segment is

$$\begin{aligned}|\beta(0) - \beta(-\tan t)| &= \sqrt{(-\tan t \cos t)^2 + \left(-\tan t \frac{\cos^2 t}{\sin t}\right)^2} \\ &= \sqrt{(\sin t)^2 + (\cos t)^2} \\ &= 1.\end{aligned}$$

□

Exercise 1-3.8. Let $\alpha : I \rightarrow \mathbb{R}^3$ be a differentiable curve and let $[a, b] \subseteq I$ be a closed interval. For every partition

$$a = t_0 < t_1 < \cdots < t_n = b$$

of $[a, b]$, consider the sum

$$\sum_{i=1}^n |\alpha(t_i) - \alpha(t_{i-1})| = l(\alpha, P),$$

where P stands for the given partition. The norm $|P|$ of a partition P is defined as

$$|P| = \max(t_i - t_{i-1}), i = 1, \dots, n.$$

Geometrically, $l(\alpha, P)$ is the length of a polygon inscribed in $\alpha([a, b])$ with vertices in $\alpha(t_i)$ (see Figure 1-3 in Mantredo P. do Carmo, *Differential Geometry of Curves and Surfaces*). The point of the exercise is to show that the arc length of $\alpha([a, b])$ is, in some sense, a limit of lengths of inscribed polygons. Prove that given $\varepsilon > 0$ there exists $\delta > 0$ such that if $|P| < \delta$ then

$$\left| \int_a^b |\alpha'(t)| dt - l(\alpha, P) \right| < \varepsilon.$$

Assume that $\alpha'(t)$ is continuous.

Proof. Given $\varepsilon > 0$.

- (1) Since $\alpha'(t)$ is continuous on a compact set $[a, b]$, $\alpha'(t)$ is uniformly continuous, that is, there exists $\delta > 0$ such that

$$|\alpha'(s) - \alpha'(t)| < \frac{\varepsilon}{2(b-a)} \text{ whenever } |s - t| < \delta.$$

- (2) Let $P = \{a = t_0, t_1, \dots, t_n = b\}$ be a partition of $[a, b]$, with $\Delta t_i = t_i - t_{i-1} < \delta$ for all $i = 1, \dots, n$. If $t_{i-1} \leq t \leq t_i$, it follows that

$$|\alpha'(t_i)| - \frac{\varepsilon}{2(b-a)} \leq |\alpha'(t)| \leq |\alpha'(t_i)| + \frac{\varepsilon}{2(b-a)}.$$

Hence,

$$\begin{aligned} & \int_{t_{i-1}}^{t_i} |\alpha'(t)| dt \\ & \geq |\alpha'(t_i)| \Delta t_i - \frac{\varepsilon}{2(b-a)} \Delta t_i \\ & = \left| \int_{t_{i-1}}^{t_i} [\alpha'(t) + \alpha'(t_i) - \alpha'(t)] dt \right| - \frac{\varepsilon}{2(b-a)} \Delta t_i \\ & \geq \left| \int_{t_{i-1}}^{t_i} \alpha'(t) dt \right| - \left| \int_{t_{i-1}}^{t_i} [\alpha'(t_i) - \alpha'(t)] dt \right| - \frac{\varepsilon}{2(b-a)} \Delta t_i \\ & \geq |\alpha(t_i) - \alpha(t_{i-1})| - \frac{\varepsilon}{b-a} \Delta t_i \end{aligned}$$

and

$$\begin{aligned} & \int_{t_{i-1}}^{t_i} |\alpha'(t)| dt \\ & \leq |\alpha'(t_i)| \Delta t_i + \frac{\varepsilon}{2(b-a)} \Delta t_i \\ & = \left| \int_{t_{i-1}}^{t_i} [\alpha'(t) + \alpha'(t_i) - \alpha'(t)] dt \right| + \frac{\varepsilon}{2(b-a)} \Delta t_i \\ & \leq \left| \int_{t_{i-1}}^{t_i} \alpha'(t) dt \right| + \left| \int_{t_{i-1}}^{t_i} [\alpha'(t_i) - \alpha'(t)] dt \right| + \frac{\varepsilon}{2(b-a)} \Delta t_i \\ & \leq |\alpha(t_i) - \alpha(t_{i-1})| + \frac{\varepsilon}{b-a} \Delta t_i. \end{aligned}$$

- (3) If we add these inequalities, we obtain

$$l(\alpha, P) - \varepsilon \leq \int_a^b |\alpha'(t)| dt \leq l(\alpha, P) + \varepsilon.$$

□

Exercise 1-3.9.

- (a) Let $\alpha : I \rightarrow \mathbb{R}^3$ be a curve of class C^0 (compare Exercise 1-3.7). Use the approximation by polygons described in Exercise 1-3.8 to give a reasonable definition of arc length of α .
- (b) (A Nonrectifiable Curve.) The following example shows that, with any reasonable definition, the arc length of a C^0 curve in a closed interval may be unbounded. Let $\alpha : [0, 1] \rightarrow \mathbb{R}^2$ be given as $\alpha(t) = (t, t \sin(\frac{\pi}{t}))$ if $t \neq 0$, and $\alpha(0) = (0, 0)$. Show, geometrically, that the arc length of the portion of the curve corresponding to $\frac{1}{n+1} \leq t \leq \frac{1}{n}$ is at least $\frac{2}{n+\frac{1}{2}}$. Use this to show that the length of curve in the interval $\frac{1}{N} \leq t \leq 1$ is greater than $2 \sum_{n=1}^{N-1} \frac{1}{n+1}$, and thus it tends to infinity as $N \rightarrow \infty$.

Proof of (a). Define

$$l(\alpha) = \sup\{l(\alpha, P) : P \text{ is a partition of } [a, b]\}.$$

□

Note. (Theorem 6.17 in Tom. M. Apostol, *Mathematical Analysis*, 2nd edition.). α is rectifiable if and only if α is of bounded variation on $[a, b]$.

Proof of (b).

- (1) Consider a partition $P = \left\{ \frac{1}{n+1}, \frac{1}{n+\frac{1}{2}}, \frac{1}{n} \right\}$ of $\left[\frac{1}{n+1}, \frac{1}{n} \right]$. So that $\alpha(\frac{1}{n+1}) = \alpha(\frac{1}{n}) = 0$ and $\alpha(\frac{1}{n+\frac{1}{2}}) = \pm 1$.

- (2) Thus,

$$\begin{aligned} & \text{The arc length of the portion of } \alpha \text{ over } \left[\frac{1}{n+1}, \frac{1}{n} \right] \\ & \geq \text{The sum of each length of the individual chords} \\ & = \sqrt{\left(\frac{1}{n+\frac{1}{2}} - \frac{1}{n+1} \right)^2 + \left(\frac{1}{n+\frac{1}{2}} \right)^2} + \sqrt{\left(\frac{1}{n} - \frac{1}{n+\frac{1}{2}} \right)^2 + \left(\frac{1}{n+\frac{1}{2}} \right)^2} \\ & \geq \frac{2}{n+\frac{1}{2}}. \end{aligned}$$

(3) So

$$\begin{aligned}
& \text{The arc length of } \alpha \text{ over } \left[\frac{1}{N}, 1 \right] \\
&= \sum_{n=1}^{N-1} \left\{ \text{The arc length of } \alpha \text{ over } \left[\frac{1}{n+1}, \frac{1}{n} \right] \right\} \\
&\geq \sum_{n=1}^{N-1} \frac{2}{n + \frac{1}{2}} \\
&> 2 \sum_{n=1}^{N-1} \frac{1}{n+1}.
\end{aligned}$$

It tends to infinity as $N \rightarrow \infty$, or α is nonrectifiable.

□

Exercise 1-3.10. (Straight Lines as Shortest.) Let $\alpha : I \rightarrow \mathbb{R}^3$ be a parametrized curve. Let $[a, b] \subseteq I$ and set $\alpha(a) = p$, $\alpha(b) = q$.

(a) Show that, for any constant vector v , $|v| = 1$,

$$(q - p) \cdot v = \int_a^b \alpha'(t) \cdot v dt \leq \int_a^b |\alpha'(t)| dt.$$

(b) Set

$$v = \frac{q - p}{|q - p|}$$

and show that

$$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)| dt;$$

that is, the curve of shortest length from $\alpha(a)$ to $\alpha(b)$ is the straight line joining these points.

Assume $p \neq q$ (otherwise $v = \frac{q-p}{|q-p|}$ is meaningless).

Proof of (a). Let $f(t) = \alpha(t) \cdot v$ defined on I . By the fundamental theorem of calculus,

$$\int_a^b f'(t) dt = f(b) - f(a).$$

Since $f'(t) = \alpha'(t) \cdot v$,

$$(\alpha(b) - \alpha(a)) \cdot v = \int_a^b \alpha'(t) \cdot v dt.$$

Therefore,

$$\begin{aligned}
(q-p) \cdot v &= \int_a^b \alpha'(t) \cdot v dt \\
&\leq \int_a^b |\alpha'(t) \cdot v| dt \\
&\leq \int_a^b |\alpha'(t)| |v| dt \\
&= \int_a^b |\alpha'(t)| dt.
\end{aligned}$$

□

Proof of (b). $|v| = \frac{|q-p|}{|q-p|} = 1$. So,

$$\begin{aligned}
(q-p) \cdot \frac{q-p}{|q-p|} &\leq \int_a^b |\alpha'(t)| dt, \\
|q-p| &\leq \int_a^b |\alpha'(t)| dt.
\end{aligned}$$

□

Section 1-4: The Vector Product in \mathbb{R}^3

Exercise 1-4.1. Check whether the following bases are positive:

- (a) The basis $\{(1, 3), (4, 2)\}$ in \mathbb{R}^2 .
- (b) The basis $\{(1, 3, 5), (2, 3, 7), (4, 8, 3)\}$ in \mathbb{R}^3 .

Proof of (a). Write $u = (1, 3)$ and $v = (4, 2)$. Then

$$\det(u, v) = \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = -10 < 0.$$

Thus $\{u, v\}$ is negative w.r.t. the natural order basis $\{e_1 = (1, 0), e_2 = (0, 1)\}$. □

Proof of (b). Write $u = (1, 3, 5)$, $v = (2, 3, 7)$, $w = (4, 8, 3)$. Then

$$\det(u, v, w) = \begin{vmatrix} 1 & 3 & 5 \\ 2 & 3 & 7 \\ 4 & 8 & 3 \end{vmatrix} = 39 > 0.$$

Thus $\{u, v, w\}$ is positive w.r.t. the natural order basis $\{e_1, e_2, e_3\}$. □

Exercise 1-4.2. A plane P contained in \mathbb{R}^3 is given by the equation $ax + by + cz + d = 0$. Show that the vector $v = (a, b, c)$ is perpendicular to the plane and that $|d|/\sqrt{a^2 + b^2 + c^2}$ measures the distance from the plane to the origin $(0, 0, 0)$.

Say v is a normal vector of E .

In general, the distance from the plane E to any point $(x_0, y_0, z_0) \in \mathbb{R}^3$ is

$$\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Proof.

- (1) To show $v = (a, b, c)$ is perpendicular to the plane, it suffices to show that $v \cdot u = 0$ for any vector u lying on the plane E . Write $u = \overrightarrow{PQ}$ where $P = (x_1, y_1, z_1) \in E$ and $Q = (x_2, y_2, z_2) \in E$. Hence $u = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

$$\begin{aligned} v \cdot u &= (a, b, c) \cdot (x_2 - x_1, y_2 - y_1, z_2 - z_1) \\ &= a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) \\ &= (ax_2 + by_2 + cz_2) - (ax_1 + by_1 + cz_1) \\ &= (-d) - (-d) \\ &= 0. \end{aligned}$$

- (2) Pick any point $(x_1, y_1, z_1) \in E$. The distance from the plane E to the point (x_0, y_0, z_0) is

$$\begin{aligned} &\left| (x_1 - x_0, y_1 - y_0, z_1 - z_0) \cdot \frac{v}{|v|} \right| \\ &= \left| (x_1 - x_0, y_1 - y_0, z_1 - z_0) \cdot \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}} \right| \\ &= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|-d - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}. \end{aligned}$$

□

Exercise 1-4.3. Determine the angle of intersection of the two planes $5x + 3y + 2z - 4 = 0$ and $3x + 4y - 7z = 0$.

Proof.

- (1) The angle of intersection of the two planes is equal to a angle between two normal vectors of planes.
- (2) Let
 - (a) the angle of intersection of the two planes be θ .
 - (b) the normal vector of $5x + 3y + 2z - 4 = 0$ be $n_1 = (5, 3, 2)$.
 - (c) the normal vector of $3x + 4y - 7z = 0$ be $n_2 = (3, 4, -7)$.
- (3) Hence,

$$\cos \theta = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{13}{2\sqrt{703}}.$$

$$\theta = \cos^{-1} \left(\frac{13}{2\sqrt{703}} \right).$$

□

Exercise 1-4.13. Let $u(t) = (u_1(t), u_2(t), u_3(t))$ and $v(t) = (v_1(t), v_2(t), v_3(t))$ be differentiable maps from the interval (a, b) into \mathbb{R}^3 . If the derivatives $u'(t)$ and $v'(t)$ satisfy the conditions

$$u'(t) = au(t) + bv(t), v'(t) = cu(t) - av(t),$$

where a , b , and c are constants, show that $u(t) \wedge v(t)$ is a constant vector.

Proof. Since

$$\begin{aligned} \frac{d}{dt}(u(t) \wedge v(t)) &= u'(t) \wedge v(t) + u(t) \wedge v'(t) \\ &= (au(t) + bv(t)) \wedge v(t) + u(t) \wedge (cu(t) - av(t)) \\ &= au(t) \wedge v(t) + u(t) \wedge (-av(t)) \\ &= a(u(t) \wedge v(t)) + (-a)(u(t) \wedge v(t)) \\ &= (0, 0, 0), \end{aligned}$$

$u(t) \wedge v(t)$ is a constant vector. □

Section 1-5: The Local Theory of Curves Parametrized by Arc Length

Exercise 1-5.2. Show that the torsion τ of α is given by

$$\tau(s) = -\frac{\alpha'(s) \wedge \alpha''(s) \cdot \alpha'''(s)}{|\kappa(s)|^2}.$$

Proof.

- (1) Take inner product $n(s)$ to the definition of torsion $\tau(s)n(s) = b'(s)$, we have

$$\tau(s) = b'(s) \cdot n(s).$$

Since $b'(s) = t(s) \wedge n'(s)$, we have to compute $n'(s)$ first.

- (2) Compute $n'(s)$.

$$n'(s) = \frac{d}{ds} \left(\frac{\alpha''(s)}{\kappa(s)} \right) = \frac{\alpha'''(s)}{\kappa(s)} - \frac{\alpha''(s)\kappa'(s)}{\kappa(s)^2}.$$

- (3) By (1)(2),

$$\begin{aligned} \tau(s) &= b'(s) \cdot n(s) \\ &= (t(s) \wedge n'(s)) \cdot n(s) \\ &= \left(\alpha'(s) \wedge \left(\frac{\alpha'''(s)}{\kappa(s)} - \frac{\alpha''(s)\kappa'(s)}{\kappa(s)^2} \right) \right) \cdot \frac{\alpha''(s)}{\kappa(s)} \\ &= \left(\alpha'(s) \wedge \frac{\alpha'''(s)}{\kappa(s)} \right) \cdot \frac{\alpha''(s)}{\kappa(s)} \\ &= \frac{\alpha'(s) \wedge \alpha'''(s) \cdot \alpha''(s)}{|\kappa(s)|^2}, \end{aligned}$$

or

$$\tau(s) = \frac{\alpha'(s) \wedge \alpha'''(s) \cdot \alpha''(s)}{|\kappa(s)|^2}.$$

□

Section 1-6: The Local Canonical Form

Section 1-7: Global Properties of Plane Curves