Chapter 8: Some Special Functions

Exercise 8.1. *Define*

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

Prove that f has derivatives of all orders at $x = 0$, and that $f^{(n)}(0) = 0$ for $n = 1, 2, 3, ...$

$f(x)$ is an example of non-analytic smooth function, that is, infinitely differentiable functions are not necessarily analytic. In this exercise, we will show that Taylor series of f at the origin converges everywhere to the zero function. So the Taylor series does not equal $f(x)$ for $x \neq 0$. Consequently, f is not analytic at $x = 0$.

Claim 1.

$$\lim_{x \to 0} g(x) e^{-\frac{1}{x^2}} = 0$$

for any rational function $g(x) \in \mathbb{R}(x)$.

Proof. Write $g(x) = \frac{p(x)}{q(x)}$ for some $p(x), q(x) \in \mathbb{R}[x]$. Write $q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0$. $q(x)$ is not identically zero, that is, there exists the unique coefficient of the least power of x in $q(x)$ which is non-zero, say $b_M \neq 0$. Now write $g(x)$ as $g(x) = \frac{p(x)/x^M}{q(x)/x^M}$. The denominator of $g(x)$ tends to $b_M \neq 0$ as $x \to 0$. By the similar argument of Theorem 8.6(f) ($\lim_{x \to \infty} x^n e^{-x} = 0$ for any $n \in \mathbb{Z}$),

$$\frac{p(x)}{x^M} e^{-\frac{1}{x^2}} \to 0 \text{ as } x \to 0.$$

Hence, $\lim_{x \to 0} g(x) e^{-\frac{1}{x^2}} = 0$ for any $g(x) \in \mathbb{R}(x)$. $\square$

Claim 2. *Given any real $x \neq 0$*

$$f^{(n)}(x) = g_n(x) e^{-\frac{1}{x^2}}$$

for some rational function $g(x) \in \mathbb{R}(x)$.

Proof. Say $g_0(x) = 1 \in \mathbb{R}(x)$. Notice that $\mathbb{R}(x)$ is a field and $g'(x) \in \mathbb{R}(x)$ for any $g(x) \in \mathbb{R}(x)$. (Write $g(x) = \frac{p(x)}{q(x)}$ for some $p(x), q(x) \in \mathbb{R}[x]$. Notice that $p'(x) \in \mathbb{R}[x]$ for any $p(x) \in \mathbb{R}[x]$.) Now we prove by mathematical induction.

For $n = 1$, we have

$$\begin{aligned}
f'(x) &= g_0'(x)e^{-\frac{1}{x^2}} + g_0(x) \cdot \left(-\frac{1}{x^2}\right)' e^{-\frac{1}{x^2}} \\
&= \left(g_0'(x) + g_0(x) \cdot \left(-\frac{1}{x^2}\right)'\right) e^{-\frac{1}{x^2}} \\
&= g_1(x)e^{-\frac{1}{x^2}}
\end{aligned}$$

where $g_1(x) = g_0'(x) + g_0(x) \cdot (-\frac{1}{x^2})' \in \mathbb{R}(x)$. Now assume $n = k$ holds. For $n = k+1$, similar to $n = 1$, $f^{(k+1)}(x) = g_{k+1}(x)e^{-\frac{1}{x^2}}$ where $g_{k+1}(x) = g_k'(x) + g_k(x) \cdot (-\frac{1}{x^2})' \in \mathbb{R}(x)$. $\square$

Proof of Exercise 8.1. Prove by mathematical induction. For $n = 1$,

$$f'(0) = \lim_{t \to 0} \frac{e^{-\frac{1}{t^2}} - 0}{t} = 0.$$

(Use Claim 1.) Now assume $n = k$ holds. For $n = k+1$,

$$f^{(k+1)}(0) = \lim_{t \to 0} \frac{f^{(k)}(t) - f^{(k)}(0)}{t} = \lim_{t \to 0} \frac{g_k(t)e^{-\frac{1}{t^2}} - 0}{t} = 0.$$

(Use Claim 1 and 2.) Thus, $f^{(n)}(0) = 0$ for $n \in \mathbb{Z}^+$. $\square$

Exercise 8.6. *Suppose $f(x)f(y) = f(x+y)$ for all real x and y.*
(a) Assuming that f is differentiable and not zero, prove that

$$f(x) = e^{cx}$$

where c is a constant.
(b) Prove the same thing, assuming only that f is continuous.

(b) implies (a). We prove (b) directly.

Proof of (b). Since $f(x)$ is not zero, there exists $x_0 \in \mathbb{R}$ such that $f(x_0) \neq 0$. So $f(0)f(x_0) = f(x_0)$, or $f(0) = 1$ by cancelling $f(x_0) \neq 0$.

Next, $f(\frac{n}{m}) = f(\frac{1}{m})^n$ for $m \in \mathbb{Z}$, $n \in \mathbb{Z}^+$. Since f is continuous at $x = 0$, f is positive in the neighborhood of $x = 0$. That is, there exists $N \in \mathbb{Z}^+$ such that $f(\frac{1}{m}) > 0$ whenever $|m| \geq N$. So, $f(\frac{n}{m}) = f(\frac{1}{m})^n > 0$. (Since $f(\frac{n}{m}) = f(\frac{kn}{km})$ for any $k \in \mathbb{Z}^+$, we can rescale m to km such that $|km| \geq N$.) That is, f is positive on $\mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and f is continuous on $\mathbb{R}$, f is positive on $\mathbb{R}$.

Now let $c = \log f(1)$ (which is well-defined since $f > 0$). We write $f(1)$ in the two ways. Firstly, $f(1) = f(\frac{n}{n}) = f(\frac{1}{n})^n$ where $n \in \mathbb{Z}^+$. Secondly, $f(1) = e^c = (e^{\frac{c}{n}})^n$. Since the positive n-th root is unique (Theorem 1.21), $f(\frac{1}{n}) = e^{\frac{c}{n}}$ for

$n \in \mathbb{Z}^+$. By $f(x)f(-x) = f(0) = 1$ or $f(-x) = \frac{1}{f(x)}$, $f(-\frac{1}{n}) = \frac{1}{e^{\frac{c}{n}}} = e^{-\frac{c}{n}}$ for $n \in \mathbb{Z}^+$. Therefore,

$$f\left(\frac{1}{m}\right) = e^{\frac{c}{m}} \text{ where } m \in \mathbb{Z}.$$

By using $f(\frac{n}{m}) = f(\frac{1}{m})^n$ for $m \in \mathbb{Z}$, $n \in \mathbb{Z}^+$ again, $f(\frac{n}{m}) = e^{c\frac{n}{m}}$ where $m \in \mathbb{Z}, n \in \mathbb{Z}^+$, or

$$f(x) = e^{cx} \text{ where } x \in \mathbb{Q}.$$

Since $g(x) = f(x) - e^{cx}$ vanishes on a dense set of $\mathbb{Q}$ and g is continuous on $\mathbb{R}$, g vanishes on $\mathbb{R}$. Therefore, $f(x) = e^{cx}$ for $x \in \mathbb{R}$. $\square$

Supplement. Proof of (a).
Proof of (a). Since $f(x)$ is not zero, there exists $x_0 \in \mathbb{R}$ such that $f(x_0) \neq 0$. So $f(0)f(x_0) = f(x_0)$, or $f(0) = 1$ by cancelling $f(x_0) \neq 0$.

Since f is differentiable, for any $x \in \mathbb{R}$,

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x)f(h) - f(x)}{h} \\ &= f(x) \lim_{h \to 0} \frac{f(h) - 1}{h} \\ &= f(x) \lim_{h \to 0} \frac{f(h) - f(0)}{h} \\ &= f(x)f'(0). \end{aligned}$$

Let $c = f'(0)$ be a constant. Then $f'(x) = cf(x)$. So $f(x) = e^{cx}$ for $x \in \mathbb{R}$. (To see this, let $g(x) = \frac{f(x)}{e^{cx}}$ be well-defined on $\mathbb{R}$. $g(0) = 1$. $g'(x) = 0$ since $f'(x) = cf(x)$. So $g(x)$ is a constant, or $g(x) = 1$ since $g(0) = 1$. Therefore, $f(x) = e^{cx}$ on $\mathbb{R}$.) $\square$

Supplement. Cauchy's functional equation.

(1) *(Cauchy's functional equation.) Suppose $f(x) + f(y) = f(x+y)$ for all real x and y. Assuming that f is continuous, prove that $f(x) = cx$ where c is a constant.*

 Notice that we cannot let $g(x) = \log f(x)$ and apply Cauchy's functional equation on $g(x)$ to prove Exercise 8.6 since $f(x)$ is not necessary positive and thus $g(x) = \log f(x)$ might be meaningless. However, this wrong approach gives you some useful ideas such as you need to prove that $f(x)$ is positive first, and $f(x)$ should be equal to e^{cx} where $c = g(1) = \log f(1)$.

(2) *Suppose $f(xy) = f(x) + f(y)$ for all positive real x and y. Assuming that f is continuous, prove that $f(x) = c \log x$ where c is a constant.*

(3) *Suppose* $f(xy) = f(x)f(y)$ *for all positive real* x *and* y. *Assuming that* f *is continuous and positive, prove that* $f(x) = x^c$ *where* c *is a constant.*

(4) *Suppose* $f(x + y) = f(x) + f(y) + xy$ *for all real* x *and* y. *Assuming that* f *is continuous, prove that* $f(x) = \frac{1}{2}x^2 + cx$ *where* c *is a constant.*

(5) *(USA 2002.) Suppose* $f(x^2 - y^2) = xf(x) - yf(y)$ *for all real* x *and* y. *Assuming that* f *is continuous, prove that* $f(x) = cx$ *where* c *is a constant.*