Chapter 2: Four Important Linear PDE

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Problem 2.2. *Prove that Laplace's equation $\Delta u = 0$ is rotation invariant; that is, if O is an orthogonal $n \times n$ matrix and we define*

$$v(x) := u(Ox) \ \ (x \in \mathbb{R}^n),$$

then $\Delta v = 0$.

Proof.

(1) Let $O = [O_{ij}]$. O is orthogonal if $OO^t = O^tO = I$, or

$$\sum_{i=1}^n O_{pi}O_{qi} = \delta_{pq}$$

where δ_{pq} is the Kronecker delta.

(2) Let $y = Ox$. So that

$$\begin{aligned}
D_i v(x) &= \sum_{p=1}^n D_p u(y) O_{pi}, \\
D_{ij} v(x) &= \sum_{q=1}^n \sum_{p=1}^n D_{pq} u(y) O_{pi} O_{qj}, \\
\Delta v(x) &= \sum_{i=1}^n D_{ii} v(x) \\
&= \sum_{i=1}^n \sum_{q=1}^n \sum_{p=1}^n D_{pq} u(y) O_{pi} O_{qi} \\
&= \sum_{q=1}^n \sum_{p=1}^n D_{pq} u(y) \left(\sum_{i=1}^n O_{pi} O_{qi} \right) \\
&= \sum_{q=1}^n \sum_{p=1}^n D_{pq} \delta_{pq} \\
&= \sum_{q=1}^n D_{qq} u(y) \\
&= \Delta u(y).
\end{aligned}$$

(3) As $\Delta u(y) = 0$, $\Delta v(x) = 0$.

□