Chapter 11: The Lebesuge Theory

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Exercise 11.1. *If $f \geq 0$ and $\int_E f d\mu = 0$, prove that $f(x) = 0$ almost everywhere on E. (Hint: Let E_n be the subset of E on which $f(x) > \frac{1}{n}$. Write $A = \bigcup E_n$. Then $\mu(A) = 0$ if and only if $\mu(E_n) = 0$ for every n.)*

Might assume that f is measurable on E.

Proof (Hint).

(1) Define $A = \{x \in E : f(x) > 0\}$. So $f(x) = 0$ almost everywhere on E if and only if $\mu(A) = 0$.

(2) Define
$$E_n = \left\{ x \in E : f(x) > \frac{1}{n} \right\}$$
for $n = 1, 2, 3, \ldots$. Note that $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$ and
$$A = \bigcup_{n=1}^{\infty} E_n.$$
Since μ is a measure,
$$\lim_{n \to \infty} \mu(E_n) = \mu(A)$$
(Theorem 11.3).

(3) (Reductio ad absurdum) If $\mu(A) > 0$, there is an integer N such that $\mu(E_n) \geq \frac{\mu(A)}{2}$ whenever $n \geq N$ (by (2)). In particular, take $n = N$ to get
$$\begin{aligned}
\int_E f d\mu &\geq \int_{E_N} f d\mu && (\mu \text{ is a measure and } E_N \subseteq E) \\
&\geq \frac{1}{N} \cdot \mu(E_N) && (\text{Remarks 11.23(b)}) \\
&\geq \frac{1}{N} \cdot \frac{\mu(A)}{2} \\
&> 0,
\end{aligned}$$
contrary to the assumption that $\int_E f d\mu = 0$.

$\square$

Note. Compare to Exercise 6.2.

Exercise 11.2. *If $\int_A f d\mu = 0$ for every measurable subset A of a measurable set E, then $f(x) = 0$ almost everywhere on E.*

Might assume that f is measurable on E.

Proof.

(1) Define

$$A = \{x \in E : f(x) \geq 0\} \qquad \text{and} \qquad B = \{x \in E : f(x) \leq 0\}.$$

A and B are measurable subsets of a measurable set E since f is measurable.

(2) Apply Exercise 11.1 to the fact that $f \geq 0$ on A (by construction) and $\int_A f d\mu = 0$ (by assumption), we have $f(x) = 0$ almost everywhere on A.

(3) Similarly, apply Exercise 11.1 to the fact that $-f \geq 0$ on B and $\int_B (-f) d\mu = -\int_B f d\mu = 0$, we have $f(x) = 0$ almost everywhere on B.

(4) As $E = A \cup B$, $f(x) = 0$ almost everywhere on E by (2)(3).

□

Exercise 11.3. *If $\{f_n\}$ is a sequence of measurable functions, prove that the set of points x at which $\{f_n(x)\}$ converges is measurable.*

Proof.

(1) It suffices to show that

$$E = \{x : \{f_n(x)\} \text{ is convergent } \} = \{x : \{f_n(x)\} \text{ is Cauchy } \}$$

is measurable (since $\mathbb{R}^1$ is complete).

(2) Write

$$E = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n,m \geq N} \left\{ x : |f_n(x) - f_m(x)| \leq \frac{1}{k} \right\}$$

Since $\{f_n\}$ is a sequence of measurable functions, $x \mapsto |f_n(x) - f_m(x)|$ is measurable (Theorem 11.16 and Theorem 11.18). Hence

$$\left\{ x : |f_n(x) - f_m(x)| \leq \frac{1}{k} \right\}$$

is measurable (Theorem 11.15). Therefore E is measurable.

$\square$

Exercise 11.4. *If $f \in \mathscr{L}(\mu)$ on E and g is bounded and measurable on E, then $fg \in \mathscr{L}(\mu)$ on E.*

Proof (Theorem 11.27).

(1) fg is measurable since both f and g are measurable (Theorem 11.18).

(2) $|g| \leq M$ for some real $M \in \mathbb{R}^1$ by the boundedness of g. Hence

$$|fg| \leq M|f|$$

on E.

(3) To apply Theorem 11.27, it suffices to show that $M|f| \in \mathscr{L}(\mu)$ on E. Theorem 11.26 implies that $|f| \in \mathscr{L}(\mu)$ if $f \in \mathscr{L}(\mu)$. And Remarks 11.23(d) implies that $M|f| \in \mathscr{L}(\mu)$ if $|f| \in \mathscr{L}(\mu)$.

$\square$

Note (Riemann integral). *If $f \in \mathscr{R}$ on $[a, b]$ and g is bounded and measurable on $[a, b]$, then fg might be not Riemann integrable.*

Exercise 11.5. *Put*

$$g(x) = \begin{cases} 0 & (0 \leq x \leq \frac{1}{2}), \\ 1 & (\frac{1}{2} < x \leq 1), \end{cases}$$

and

$$\begin{aligned} f_{2k}(x) &= g(x) && (0 \leq x \leq 1), \\ f_{2k+1}(x) &= g(1-x) && (0 \leq x \leq 1). \end{aligned}$$

Show that

$$\liminf_{n \to \infty} f_n(x) = 0 \qquad (0 \leq x \leq 1),$$

but

$$\int_0^1 f_n(x) dx = \frac{1}{2}.$$

(Compare with the Fatou's theorem.)

Proof.

(1) *Show that* $\liminf_{n \to \infty} f_n(x) = 0$. Note that

$$g(1-x) = \begin{cases} 1 & (0 \leq x < \frac{1}{2}), \\ 0 & (\frac{1}{2} < x \leq 1). \end{cases}$$

Since $f_n(x) \geq 0$ by definition, $\liminf_{n\to\infty} f_n(x) \geq 0$. Since $f_{2k}(0) = f_{2k+1}(1) = 0$ for all positive integers k, $\liminf_{n\to\infty} f_n(x) \leq 0$. Therefore the result is established.

(2) *Show that* $\int_0^1 f_n(x)dx = \frac{1}{2}$. Since

$$\int_0^1 f_{2k}(x)dx = \int_0^1 g(x)dx = \frac{1}{2},$$
$$\int_0^1 f_{2k+1}(x)dx = \int_0^1 g(1-x)dx = \frac{1}{2},$$

in any case $\int_0^1 f_n(x)dx = \frac{1}{2}$ for all positive integers n.

(3) This example shows that we may have the strict inequality in the Fatou's theorem.

□

Supplement (Similar exercise). *Consider the sequence $\{f_n\}$ defined by $f_n(x) = 1$ if $n \leq x < n+1$, with $f_n(x) = 0$ otherwise. Show that we may have the strict inequality in the Fatou's theorem.*

Exercise 11.6. *Let*

$$f_n(x) = \begin{cases} \frac{1}{n} & (|x| \leq n), \\ 0 & (|x| > n). \end{cases}$$

Then $f_n(x) \to 0$ uniformly on $\mathbb{R}^1$, but

$$\int_{-\infty}^{\infty} f_n(x)dx = 2 \qquad (n = 1, 2, 3, \ldots).$$

(We write $\int_{-\infty}^{\infty}$ in place of $\int_{\mathbb{R}^1}$.) Thus uniform convergence does not imply dominated convergence in the sense of Theorem 11.32. However, on sets of finite measure, uniformly convergent sequences of bounded functions do satisfy Theorem 11.32.

Proof.

(1) *Show that $f_n(x) \to 0$ uniformly on $\mathbb{R}^1$.* Given any $\varepsilon > 0$, there is an integer $N > \frac{1}{\varepsilon}$ such that

$$|f_n(x) - 0| \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

whenever $n \geq N$ and $x \in \mathbb{R}^1$. Hence $f_n(x) \to 0$ uniformly.

(2) *Show that* $\int_{-\infty}^{\infty} f_n(x)dx = 2$.

$$\int_{-\infty}^{\infty} f_n(x)dx = \int_{-n}^{n} \frac{1}{n}dx = 2.$$

(3) By (1)(2),

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx \neq \int_{-\infty}^{\infty} \lim_{n\to\infty} f_n(x)dx$$

suggests that the Lebesgue's dominated convergence theorem (Theorem 11.32) does not hold in this case. In fact, if there were $g \in \mathscr{L}$ such that $|f_n(x)| \leq g(x)$, then

$$\begin{aligned}
\int_{-\infty}^{\infty} g(x)dx &\geq \int_{0}^{\infty} g(x)dx && \text{(Theorem 11.24)} \\
&= \sum_{n=1}^{\infty} \int_{n-1}^{n} g(x)dx && \text{(Theorem 11.24)} \\
&\geq \sum_{n=1}^{\infty} \int_{n-1}^{n} |f_n(x)|dx \\
&= \sum_{n=1}^{\infty} \int_{n-1}^{n} \frac{1}{n}dx \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \\
&= \infty,
\end{aligned}$$

which is absurd.

(4) *Show that on sets of finite measure, uniformly convergent sequences of bounded functions* $\{f_n\}$ *do satisfy Theorem 11.32.*

(a) Since $\{f_n\}$ is uniformly convergent, $\{f_n\}$ is uniformly bounded (Exercise 7.1), or there exists a real number M such that

$$|f_n(x)| \leq M$$

for all positive integer n and $x \in E$.

(b) Define $g(x) = M$ on E. It is clear that

$$\int_E g(x)dx = M\mu(E) < +\infty.$$

Now we can apply the Lebesgue's dominated convergence theorem (Theorem 11.32) to get

$$\lim_{n\to\infty} \int_E f_n d\mu = \int_E \lim_{n\to\infty} f_n d\mu.$$

□

Exercise 11.7. *Find a necessary and sufficient condition that $f \in \mathscr{R}(\alpha)$ on $[a, b]$. (Hint: Consider Example 11.6(b) and Theorem 11.33.)*

Proof.

(1) Defines the regular measure μ by

$$\begin{aligned}
\mu([a, b)) &= \alpha(b-) - \alpha(a-) \\
\mu([a, b]) &= \alpha(b+) - \alpha(a-) \\
\mu((a, b]) &= \alpha(b+) - \alpha(a+) \\
\mu((a, b)) &= \alpha(b-) - \alpha(a+)
\end{aligned}$$

where $\alpha : \mathbb{R}^1 \to \mathbb{R}^1$ might be defined by

$$\alpha(x) = \begin{cases} \alpha(a) & \text{if } x < a, \\ \alpha(x) & \text{if } a \leq x \leq b, \\ \alpha(b) & \text{if } x > b. \end{cases}$$

(Example 11.6(b)).

(2) *Suppose f is bounded on $[a, b]$. Then $f \in \mathscr{R}(\alpha)$ on $[a, b]$ if and only if f and α satisfy both properties (I) and (II) below.*

(I) *f and α cannot be both left-discontinuous, or both right-discontinuous at same point.*

(II) *f is continuous almost everywhere with respect to μ on $[a, b] - D_\alpha$ where D_α is the set of discontinuities of α on $[a, b]$.*

(3) Similar to Theorem 11.33. By Definition 6.2 and Theorem 6.4 there is a sequence $\{P_k\}$ of partitions of $[a, b]$, such that P_{k+1} is a refinement of P_k, such that the distance between adjacent points of P_k is less than $\frac{1}{k}$, and such that

$$\lim_{k\to\infty} L(P_k, f, \alpha) = \mathscr{R} \underline{\int} f d\alpha, \qquad \lim_{k\to\infty} L(P_k, f, \alpha) = \mathscr{R} \overline{\int} f d\alpha.$$

(In this proof, all integrals are taken over $[a, b]$.)

(4) If $P_k = \{x_0, x_1, \ldots, x_n\}$, with $x_0 = a$, $x_n = b$, define

$$L_k(a) = U_k(a) = f(a);$$

put $U_k(x) = M_i$ and $L_k(x) = m_i$ for $x_{i-1} < x \leq x_i$, $1 \leq i \leq n$, using the notation introduced in Definition 6.1. Then

$$L(P_k, f, \alpha) = \int L_k d\mu, \qquad U(P_k, f, \alpha) = \int U_k d\mu$$

and

$$L_1(x) \leq L_2(x) \leq \cdots \leq f(x) \leq \cdots \leq U_2(x) \leq U_1(x)$$

for all $x \in [a, b]$, since P_{k+1} refines P_k.

(5) So there exist

$$L(x) = \lim_{k \to \infty} L_k(x), \qquad U(x) = \lim_{k \to \infty} U_k(x).$$

Observe that L and U are bounded μ-measurable function on $[a, b]$, that

$$L(x) \leq f(x) \leq U(x) \qquad (a \leq x \leq b),$$

and that

$$\int L d\mu = \lim \int L_k d\mu = \lim L(P_k, f, \alpha) = \mathscr{R} \underline{\int} f d\alpha,$$
$$\int U d\mu = \lim \int U_k d\mu = \lim U(P_k, f, \alpha) = \mathscr{R} \overline{\int} f d\alpha$$

by the Lebesgue's monotone convergence theorem (Theorem 11.28).

(6) So $f \in \mathscr{R}(\alpha)$ on $[a, b]$ if and only if $\int L d\mu = \int U d\mu$ if and only if

(III) $L(x) = U(x)$ almost everywhere with respect to μ

by Exercise 11.1 and the fact that $U(x) - L(x) \geq 0$.

(7) *Show that $f \in \mathscr{R}(\alpha)$ on $[a, b]$ implies the property (I).* It is independent of the Lebesgue theory.

(a) Suppose both f and α are discontinuous from the right at $x = c$; that is, suppose that there exists an $\varepsilon > 0$ such that for every $\delta > 0$ there are values of $x, y \in (c, c + \delta) \subseteq [a, b]$ for which

$$|f(x) - f(c)| \geq \sqrt{\varepsilon}, \qquad |\alpha(y) - \alpha(c)| \geq \sqrt{\varepsilon}.$$

(b) Let $P = \{x_0 < \cdots < x_n\}$ be any partition of $[a, b]$ containing c, say $c = x_{i-1}$ for some $i = 1, \ldots, n$. Then

$$\begin{aligned} U(P, f, \alpha) - L(P, f, \alpha) &= \sum_{j=1}^{n} (M_j - m_j)(\alpha(x_j) - \alpha(x_{j-1})) \\ &\geq (M_i - m_i)(\alpha(x_i) - \alpha(x_{i-1})). \end{aligned}$$

Take $\delta = x_i - x_{i-1}$. $x_i = x_{i-1} + \delta = c + \delta$. Then

$$\alpha(x_i) - \alpha(x_{i-1}) = \alpha(c + \delta) - \alpha(c) \geq \alpha(y) - \alpha(c) \geq \sqrt{\varepsilon}$$

(by the monotonicity of α). Besides,

$$M_i - m_i \geq |f(x) - f(c)| \geq \sqrt{\varepsilon}.$$

Hence,

$$U(P, f, \alpha) - L(P, f, \alpha) \geq \varepsilon.$$

Therefore, Theorem 6.6 implies that $f \notin \mathscr{R}(\alpha)$ on $[a, b]$.

(c) The argument is similar if c is a common discontinuity from the left.

(8) *Show that (III) implies (II).*

(a) *Show that f is continuous at $x \in [a, b] - D_\alpha$ if $U(x) = L(x)$ and $x \notin \bigcup_{k=1}^{\infty} P_k$.* (Reductio ad absurdum) If f were not continuous at x, then there exists an $\varepsilon > 0$ such that there is a sequence $\{y_m\}$ in $[a, b]$ such that $|y_m - x| < \frac{1}{m}$ but

$$|f(y_m) - f(x)| > \varepsilon$$

for $m = 1, 2, 3, \ldots$ (Theorem 4.2).

(b) Given any P_k. Since $x \notin P_k$, $x \in (x_{i-1}, x_i)$ for some i. Since (x_{i-1}, x_i) is open, there exists an integer N_k such that $y_m \in (x_{i-1}, x_i)$ whenever $m \geq N_k$. Hence,

$$U_k(x) - L_k(x) = M_i - m_i \geq |f(y_m) - f(x)| > \varepsilon.$$

Take the limit to get

$$U(x) - L(x) \geq \varepsilon,$$

which is absurd. Therefore, the statement in (a) is proved.

(c) Now it suffices to show that both sets

$$E_1 = \{x \in [a, b] - D_\alpha : L(x) \neq U(x)\}$$
$$E_2 = \left\{x \in [a, b] - D_\alpha : x \in \bigcup_{k=1}^{\infty} P_k\right\}$$

are μ-measure zero. E_1 is μ-measure zero by (III). E_2 is μ-measure zero since E_2 is countable and defined on $[a, b] - D_\alpha$.

Therefore, (II) is established.

(9) *Show that (I)(II) implies (III).* Use the notation in (8).

(a) It suffices to show that

$$\begin{aligned} &\{x \in [a, b] : L(x) \neq U(x)\} \\ =&\underbrace{\{x \in [a, b] - D_\alpha : L(x) \neq U(x)\}}_{=E_1} \bigcup \underbrace{\{x \in D_\alpha : L(x) \neq U(x)\}}_{=E_3} \end{aligned}$$

is μ-measure zero.

(b) Note that E_2 is μ-measure zero. Hence (II) and (8)(a) imply that E_1 is μ-measure zero. So it suffices to show that E_3 is μ-measure zero. In fact, we will show that $E_3 = \varnothing$, or $L(x) = U(x)$ whenever $x \in D_\alpha$.

(c) Write
$$D_\alpha = \{y_1, \ldots, y_m, \ldots\}$$
since D_α is at most countable (Theorem 4.30). (Set $y_m = y_{m+1} = \cdots$ if D_α is finite.) Define a refinement of P_k by
$$P_k \bigcup \{y_1, \ldots, y_k\}$$
and use the same symbol P_k to denote this refinement.

(d) Given any $x \in D_\alpha$. Suppose α is discontinuous from the right at x. By the construction in (c), there is an integer N_1 such that $x = x_{i-1}$ is in some subinterval $[x_{i-1}, x_i]$ of P_k whenever $k \geq N_1$.

(e) Note that f is continuous from the right at x by (I). Given an $\varepsilon > 0$, there exists a $\delta > 0$ such that
$$|f(y) - f(x)| \leq \frac{\varepsilon}{2}$$
whenever $y \in [x, x + \delta)$. So
$$|f(y) - f(z)| \leq |f(y) - f(x)| + |f(x) - f(z)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$
whenever $y, z \in [x, x + \delta)$.

(f) Take an integer $N_2 > \frac{1}{\delta}$ such that for any any subinterval $[x_{i-1}, x_i]$ of P_k we have $[x_{i-1}, x_i] \subseteq [x, x + \delta)$ whenever $k \geq N_2$.

(g) Take $N = \max\{N_1, N_2\}$. As $k \geq N$, $[x_{i-1}, x_i] \subseteq [x, x + \delta)$ and
$$U_k(x) - L_k(x) = M_i - m_i = \sup_{y,z \in [x_{i-1}, x_i]} |f(y) - f(z)| \leq \varepsilon.$$
Take the limit to get
$$U(x) - L(x) \leq \varepsilon.$$
Since ε is arbitrary, $L(x) = U(x)$.

(h) The argument is similar if α is discontinuous from the left at x.

□

Exercise 11.8. *If $f \in \mathscr{R}$ on $[a, b]$ and if $F(x) = \int_a^x f(t)dt$, prove that $F'(x) = f(x)$ almost everywhere on $[a, b]$.*

Proof.

(1) Theorem 6.20 implies that $F'(x_0) = f(x_0)$ if f is continuous at $x_0 \in [a, b]$.

(2) Since $f \in \mathscr{R}$ on $[a, b]$, f is bounded on $[a, b]$. Theorem 11.33 implies that f is continuous almost everywhere on $[a, b]$.

By (1)(2), $F'(x) = f(x)$ almost everywhere on $[a, b]$. $\Box$

Exercise 11.9. *Prove that the function F given by*

$$F(x) = \int_a^x f dt \qquad (a \leq x \leq b)$$

(where $f \in \mathscr{L}$ on $[a, b]$) is continuous on $[a, b]$.

Proof.

(1) *Let $f \in \mathscr{L}$ on E. Show that given any $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$\int_A f d\mu < \varepsilon$$

whenever $A \subseteq E$ with $\mu(A) < \delta$.

(a) Define $f_n(x) = \min\{f(x), n\}$ on E for $n = 1, 2, 3, \ldots$. Then $\{f_n\}$ is a sequence of measurable functions such that

$$0 \leq f_1(x) \leq f_2(x) \leq \cdots.$$

Also, $f_n \to f$. Then by the Lebesuge's monotone convergenece theorem (Theorem 11.28),

$$\lim_{n \to \infty} \int_E f_n d\mu = \int_E f d\mu.$$

(b) For such $\varepsilon > 0$, there is an integer $N \geq 1$ such that

$$\int_E (f - f_N) d\mu < \frac{\varepsilon}{2}.$$

Choose $\delta > 0$ so that $\delta < \frac{\varepsilon}{2N}$. If $\mu(A) < \delta$, we have

$$\begin{aligned}
\int_A f d\mu &= \int_A (f - f_N) d\mu + \int_A f_N d\mu \\
&\leq \int_E (f - f_N) d\mu + N\mu(A) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

(2) Apply (1) to f^+ and f^- on $E = [a, b]$. Given any $\varepsilon > 0$, there is a common $\delta > 0$ such that

$$\left| \int_x^y f^+ dt \right| < \frac{\varepsilon}{2} \qquad \text{and} \qquad \left| \int_x^y f^- dt \right| < \frac{\varepsilon}{2}$$

whenever $|y - x| < \delta$. So

$$|F(y) - F(x)| \leq \left| \int_x^y f^+ dt \right| + \left| \int_x^y f^- dt \right| < \varepsilon$$

whenever $|y - x| < \delta$. Hence F is uniformly continuous. (In fact, F is absolutely continuous by the same argument.)

□

Note. Compare to Theorem 6.20.

Exercise 11.10. *If $\mu(X) < +\infty$ and $f \in \mathscr{L}^2(\mu)$ on X, prove that $f \in \mathscr{L}$ on X. If*

$$\mu(X) = +\infty,$$

this is false. For instance, if

$$f(x) = \frac{1}{1 + |x|},$$

then $f^2 \in \mathscr{L}$ on $\mathbb{R}^1$, but $f \notin \mathscr{L}$ on $\mathbb{R}^1$.

Proof.

(1) Since $\mu(X) < +\infty$, $1 \in \mathscr{L}^2(\mu)$ on X. By Theorem 11.35, $f \in \mathscr{L}(\mu)$, and

$$\int_X |f| d\mu \leq \|f\| \|1\|.$$

(2) *Show that $f^2 \in \mathscr{L}$ on $\mathbb{R}^1$.* To apply Theorem 11.33, we might restrict the measure space $X = \mathbb{R}^1$ to some interval $[a, b]$. Then apply the Lebesgue's monotone convergence theorem (Theorem 11.28) to get the conclusion.

(a) Write

$$f(x)^2 = \left(\frac{1}{1 + |x|} \right)^2 = \frac{1}{1 + 2|x| + x^2} \leq \frac{1}{1 + x^2}.$$

By Theorem 11.27, *it suffices to show that $\frac{1}{1+x^2} \in \mathscr{L}$ on $\mathbb{R}^1$.*

(b) Consider the sequence $\{f_n\}$ defined by

$$f_n(x) = \frac{1}{1 + x^2} \chi_{[-n,n]}(x).$$

(Here $\chi_{[-n,n]} = K_{[-n,n]}$ is the characteristic function of $[-n, n]$ defined in Definition 11.19.) By construction,

$$0 \leq f_1(x) \leq f_2(x) \leq \cdots \qquad (x \in \mathbb{R}^1)$$

and

$$f_n(x) \to \frac{1}{1+x^2} \qquad (x \in \mathbb{R}^1).$$

(c) Hence

$$\begin{aligned}
\int_{\mathbb{R}^1} \frac{1}{1+x^2} dx &= \lim_{n\to\infty} \int_{\mathbb{R}^1} f_n(x) dx && \text{(Theorem 11.28)} \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^1} \frac{1}{1+x^2} \chi_{[-n,n]}(x) dx \\
&= \lim_{n\to\infty} \int_{-n}^{n} \frac{1}{1+x^2} dx \\
&= \lim_{n\to\infty} \mathscr{R} \int_{-n}^{n} \frac{1}{1+x^2} dx && \text{(Theorem 11.33)} \\
&= \lim_{n\to\infty} 2\arctan(n) \\
&= \pi < \infty.
\end{aligned}$$

(4) *Show that $f \notin \mathscr{L}$ on $\mathbb{R}^1$.*

(a) Consider the sequence $\{f_n\}$ defined by

$$f_n(x) = f(x)\chi_{[-n,n]}(x) = \frac{1}{1+|x|}\chi_{[-n,n]}(x).$$

By construction,

$$0 \leq f_1(x) \leq f_2(x) \leq \cdots \qquad (x \in \mathbb{R}^1)$$

and

$$f_n(x) \to f(x) \qquad (x \in \mathbb{R}^1).$$

(b) Hence

$$\begin{aligned}
\int_{\mathbb{R}^1} f(x) dx &= \lim_{n\to\infty} \int_{\mathbb{R}^1} f_n(x) dx && \text{(Theorem 11.28)} \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^1} \frac{1}{1+|x|} \chi_{[-n,n]}(x) dx \\
&= \lim_{n\to\infty} \int_{-n}^{n} \frac{1}{1+|x|} dx \\
&= \lim_{n\to\infty} \mathscr{R} \int_{-n}^{n} \frac{1}{1+|x|} dx && \text{(Theorem 11.33)} \\
&= \lim_{n\to\infty} 2\log(n+1) \\
&= \infty,
\end{aligned}$$

or $f \notin \mathscr{L}$ on $\mathbb{R}^1$.

□

Note. Compare to Exercise 6.5.

Exercise 11.11. *If $f, g \in \mathscr{L}(\mu)$ on X, defined the distance between f and g by*

$$\int_X |f-g|d\mu.$$

Prove that $\mathscr{L}(\mu)$ is a complete metric space.

Proof.

(1) Define

$$\|f-g\|_1 = \int_X |f-g|d\mu.$$

Thus $\|f-g\|_1 = 0$ if and only if $f = g$ almost everywhere on X (Exercise 11.1). As in Remark 11.37, we identify two functions to be equivalent if they are equal almost everywhere.

(2) *Show that $\mathscr{L}(\mu)$ is a metric space.*

(a) By definition, $\|f-g\|_1 \geq 0$. Besides, $\|f-g\|_1 = 0$ if and only if $f = g$ almost everywhere by (1).

(b) $\|f-g\|_1 = \|g-f\|_1$ since $|f(x)-g(x)| = |g(x)-f(x)|$ for all $x \in X$.

(c) Since $|f(x)-g(x)| \leq |f(x)-h(x)| + |h(x)-g(x)|$ for all $x \in X$, Remarks 11.23(c) and Theorem 11.29 imply that

$$\|f-g\|_1 \leq \|f-h\|_1 + \|h-g\|_1.$$

(3) *Show that $\mathscr{L}(\mu)$ is complete.* Similar to the proof of Theorem 11.42.

(a) *Let $\{f_n\}$ be a Cauchy sequence in $\mathscr{L}(\mu)$, show that there exists a function $f \in \mathscr{L}(\mu)$ such that $\{f_n\}$ converges to $f \in \mathscr{L}(\mu)$.*

(b) Since $\{f_n\}$ is a Cauchy sequence, we can find a sequence $\{n_k\}$, $k = 1, 2, 3, \ldots$, such that

$$\left\|f_{n_k} - f_{n_{k+1}}\right\|_1 = \int_X \left|f_{n_k} - f_{n_{k+1}}\right| d\mu < \frac{1}{2^k} \qquad (k = 1, 2, 3, \ldots).$$

Hence

$$\sum_{k=1}^{\infty} \int_X \left|f_{n_k} - f_{n_{k+1}}\right| d\mu \leq \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 < +\infty.$$

(c) By Theorem 11.30, we may interchange the summation and integration to get

$$\int_X \sum_{k=1}^{\infty} \left| f_{n_k} - f_{n_{k+1}} \right| d\mu < +\infty,$$

or

$$\sum_{k=1}^{\infty} \left| f_{n_k}(x) - f_{n_{k+1}}(x) \right| = \sum_{k=1}^{\infty} \left| f_{n_{k+1}}(x) - f_{n_k}(x) \right| < +\infty$$

almost everywhere on X.

(d) Since the kth partial sum of the series

$$\sum_{k=1}^{\infty} (f_{n_{k+1}}(x) - f_{n_k}(x))$$

which converges almost everywhere on X (Theorem 3.45), is

$$f_{n_{k+1}}(x) - f_{n_1}(x),$$

we see that the equation

$$f(x) = \lim_{k\to\infty} f_{n_k}(x)$$

defines $f(x)$ for almost all $x \in X$, and it does not matter how we define $f(x)$ at the remaining points of X.

(e) We shall now show that this function f has the desired properties. Let $\varepsilon > 0$ be given, and choose N such that

$$\|f_n - f_m\|_1 \leq \varepsilon$$

whenever $n, m \geq N$. If $n_k > N$, Fatou's theorem shows that

$$\|f - f_{n_k}\|_1 \leq \liminf_{i\to\infty} \|f_{n_i} - f_{n_k}\|_1 \leq \varepsilon.$$

Thus $f - f_{n_k} \in \mathscr{L}(\mu)$, and since $f = (f - f_{n_k}) + f_{n_k} \in \mathscr{L}(\mu)$, we see that $f \in \mathscr{L}(\mu)$. Also, since ε is arbitrary,

$$\lim_{k\to\infty} \|f - f_{n_k}\|_1 = 0.$$

(f) Finally, the inequality

$$\|f - f_n\|_1 \leq \|f - f_{n_k}\|_1 + \|f_{n_k} - f_n\|_1$$

shows that $\{f_n\}$ converges to $f \in \mathscr{L}(\mu)$; for if we take n and n_k large enough, each of the two terms can be made arbitrary small.

□

Exercise 11.12. *Suppose*

(a) $|f(x,y)| \leq 1$ *if* $0 \leq x \leq 1$, $0 \leq y \leq 1$.

(b) *for fixed* x, $f(x,y)$ *is a continuous function of* y.

(c) *for fixed* y, $f(x,y)$ *is a continuous function of* x.

Put

$$g(x) = \int_0^1 f(x,y)dy \qquad (0 \leq x \leq 1).$$

Is g *continuous?*

Proof.

(1) *Show that* g *is continuous.*

(2) Let $\{x_n\}$ be a sequence in $[0,1]$ such that $x_n \neq x$ and $\lim x_n = x$. It suffices to show that

$$\begin{aligned} \lim_{n\to\infty} g(x_n) &= \lim_{n\to\infty} \int_0^1 f(x_n,y)dy \\ &= \int_0^1 \lim_{n\to\infty} f(x_n,y)dy \\ &= \int_0^1 f(x,y)dy \\ &= g(x) \end{aligned}$$

(Theorem 4.2). Since $\lim_{n\to\infty} f(x_n,y) = f(x,y)$ for any fixed y (by (c)), it suffices to show that

$$\lim_{n\to\infty} \int_0^1 f(x_n,y)dy = \int_0^1 \lim_{n\to\infty} f(x_n,y)dy.$$

(3) Define $\{f_n\}$ by $f_n(y) = f(x_n,y)$. $f_n(y)$ is a continuous function of y for every fixed n (by (b)). Thus $f_n(y)$ is measurable (Example 11.14). Besides, $|f_n(y)| \leq 1$ and $1 \in \mathscr{L}$ on $[0,1]$ (by (a)). The Lebesgue's dominated convergence theorem (Theorem 11.32) implies that

$$\lim_{n\to\infty} \int_0^1 f(x_n,y)dy = \int_0^1 \lim_{n\to\infty} f(x_n,y)dy.$$

□

Supplement (Similar exercise). *Suppose*

(a) $|f(x,y)| \leq g(y)$ *if* $0 \leq x \leq 1$, $0 \leq y \leq 1$, *where* $g \in \mathscr{L}$ *on* $[0,1]$.

(b) *for fixed* x, $f(x,y)$ *is a measurable function of* y.

(c) *for fixed* y, $f(x,y)$ *is a continuous function of* x.

Show that

$$h(x) = \int_0^1 f(x,y)dy \qquad (0 \leq x \leq 1).$$

is continuous.

Exercise 11.13. *Consider the functions*

$$f_n(x) = \sin(nx) \qquad (n = 1, 2, 3, \ldots, -\pi \leq x \leq \pi)$$

as points of $\mathscr{L}^2$. *Prove that the set of these points is closed and bounded, but not compact.*

Proof. Define $E = \{f_n\}$ as a set in $\mathscr{L}^2$.

(1) *Show that* E *is bounded.* Note that

$$\|f_n\|_2 = \left(\int_{-\pi}^{\pi} \sin(nx)^2 dx \right)^{\frac{1}{2}} = \sqrt{\pi}$$

for all n (Definition 8.10). So E is bounded by $\sqrt{\pi}$.

(2) *Show that* E *is closed.* It suffices to show that E has no limit points.

$$\begin{aligned}
\|f_n - f_m\|_2 &= \left(\int_{-\pi}^{\pi} (\sin(nx) - \sin(mx))^2 dx \right)^{\frac{1}{2}} \\
&= \left(\int_{-\pi}^{\pi} \sin(nx)^2 - 2\sin(nx)\sin(mx) + \sin(mx)^2 dx \right)^{\frac{1}{2}} \\
&= (\pi + 0 + \pi)^{\frac{1}{2}} \\
&= \sqrt{2\pi}
\end{aligned}$$

for all $n \neq m$ (Definition 8.10). So all points of E are isolated.

(3) *Show that* E *is not compact.*

(a) Take a collection

$$\mathscr{G} = \{G_n = B(f_n; 1)\}$$

of open subsets $(n = 1, 2, 3, \ldots)$.

(b) $\mathscr{G}$ is an open covering of $E \subseteq \mathscr{L}^2$ since $f_n \in G_n$ for each $n = 1, 2, 3, \ldots$.

(c) *Show that there is no finite subcoverings of $\mathscr{G}$.* (Reductio ad absurdum) If

$$\mathscr{G}' = \{G_{n_1}, G_{n_2}, \dots, G_{n_k}\}$$

were a finite subcovering of $\mathscr{G}$ with $n_1 < n_2 < \cdots < n_k$, then f_{n_k+1} is not in any open sets from $\mathscr{G}'$ (by (2)), which is absurd.

□

Exercise 11.14. *Prove that a complex function f is measurable if and only if $f^{-1}(V)$ is measurable for every open set V in the plane.*

Proof.

(1)

(2)

□

Exercise 11.15. *Let $\mathscr{R}$ be the ring of all elementary subsets of $(0, 1]$. If $0 < a \leq b \leq 1$, define*

$$\phi([a, b]) = \phi([a, b)) = \phi((a, b]) = \phi((a, b)) = b - a,$$

but define

$$\phi((0, b)) = \phi((0, b]) = 1 + b$$

if $0 < b \leq 1$. Show that this gives an additive set function ϕ on $\mathscr{R}$, which is not regular and which cannot be extended to a countably additive set function on a σ-ring.

Proof.

(1) Define $\phi : \mathscr{R} \to \mathbb{R} \cup \{\pm\infty\}$ by

$$\phi(A) = \sum_{i=1}^{n} \phi(I_i)$$

where A is a finite number of disjoint intervals $I_1, \dots, I_n$ (Definition 11.4).

(2) *Show that ϕ is an additive set function.* Given any two elementary sets $A, B \in \mathscr{R}$ with $A \bigcap B = \varnothing$. By Definition 11.4,

$$A = \bigcup_{i=1}^{n} I_i, \qquad B = \bigcup_{j=1}^{m} J_j$$

where $I_i \bigcap J_j = \varnothing$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$ (since $A \bigcap B = \varnothing$). Hence,

$$\begin{aligned}\phi\left(A \bigcup B\right) &= \phi\left(\left\{\bigcup_{i=1}^{n} I_i\right\} \bigcup \left\{\bigcup_{j=1}^{m} J_j\right\}\right) \\ &= \sum_{i=1}^{n} \phi(I_i) + \sum_{j=1}^{m} \phi(J_j) \\ &= \phi(A) + \phi(B).\end{aligned}$$

(3) *Show that ϕ is not countably additive.* Write

$$(0, 1] = \bigcup_{i=1}^{\infty} A_i$$

where $A_i = \left(2^{-i}, 2^{-i+1}\right]$. Note that $A_i \bigcap A_j = \varnothing$ if $i \neq j$. So

$$\sum_{i=1}^{\infty} \phi(A_i) = \sum_{i=1}^{\infty} 2^{-i} = 1 \neq 2 = \phi((0, 1]).$$

(4) *Show that ϕ is not regular.*

(a) *Given any closed set $F \in \mathscr{R}$. Show that $\phi(F) \leq 1$.* Write $F = \bigcup_{i=1}^{n} I_i$ where each I_i are disjoint by Definition 11.4. Here every I_i is never of the form $(0, b]$ or $(0, b)$ where $b > 0$. (Otherwise 0 is a limit point of F, or $0 \in \overline{F} = F$, which is absurd.) Hence $\phi(F) = \sum \phi(I_i) \leq 1$.

(b) Take $A = \left(0, \frac{1}{2}\right] \in \mathscr{R}$ and $\varepsilon = \frac{1}{64} > 0$. Then for every closed set $F \in \mathscr{R}$, we have

$$\phi(A) = \frac{3}{2} > 1 + \frac{1}{64} \geq \phi(F) + \varepsilon.$$

That is, ϕ cannot be regular (Definition 11.5).

$\square$

Exercise 11.16. *Suppose $\{n_k\}$ is an increasing sequence of positive integers and E is the set of all $x \in (-\pi, \pi)$ at which $\{\sin(n_k x)\}$ converges. Prove that $m(E) = 0$. (Hint: For every $A \subseteq E$,*

$$\int_A \sin(n_k x) dx \to 0,$$

and

$$2 \int_A (\sin(n_k x))^2 dx = \int_A (1 - \cos(2 n_k x)) dx \to m(A)$$

as $k \to \infty$.)

Proof (Hint).

(1) Define $\{f_k\}$ by $f_k(x) = \sin(n_k x)$ on $[-\pi, \pi]$ for $k = 1, 2, 3, \ldots$. $\{f_k\}$ is a sequence of measurable functions on $[-\pi, \pi]$ since each $f_k : x \to \sin(n_k x)$ is continuous (Example 11.14). By Exercise 11.3, E is measurable. Given any measurable subset A of E, $\{f_k\}$ is a sequence of measurable functions on A and $f(x) = \lim_{k\to\infty} f_k(x)$ is well-defined by the definition of $A \subseteq E$.

(2) Apply the Bessel inequality (Theorem 8.12 and Definition 11.39) to the function $\chi_A \in \mathscr{L}^2$ on $[-\pi, \pi]$, we have

$$c_{-n} = \int_{[-\pi,\pi]} \chi_A e^{inx} dx \to 0$$

as $n \to \infty$. Hence

$$\lim_{k\to\infty} \int_A \sin(n_k x) dx = 0$$

for any measurable subset A of E.

(3) *Show that $f(x) = 0$ almost everywhere on E.* Note that

$$|f_k(x)| = |\sin(n_k x)| \leq 1$$

on A and

$$\int_A dx = m(A) \leq m([-\pi, \pi]) = 2\pi < \infty.$$

By (2) and the Lebesgue's dominated convergence theorem (Theorem 11.32),

$$\int_A f dx = \lim_{k\to\infty} \int_A f_k dx = \lim_{k\to\infty} \int_A \sin(n_k x) dx = 0$$

for any measurable subset A of E. By Exercise 11.2, the conclusion holds.

(4) Apply (1)(2)(3) to the sequence of measurable functions $\{f_k^2\}$ on $[-\pi, \pi]$, we have

$$\begin{aligned}
0 &= 2\int_A f^2 dx && (f^2(x) = 0 \text{ a.e. on } A) \\
&= \lim_{k\to\infty} 2\int_A f_k^2 dx \\
&= \lim_{k\to\infty} 2\int_A \sin(n_k x)^2 dx \\
&= \lim_{k\to\infty} \int_A (1 - \cos(2n_k x)) dx \\
&= m(A) - \lim_{k\to\infty} \int_A \cos(2n_k x) dx \\
&= m(A)
\end{aligned}$$

for any measurable subset A of E. In particular, take $A = E$ to get $m(E) = 0$.

$\square$

Exercise 11.17. *Suppose $E \subseteq (-\pi, \pi)$, $m(E) > 0$, $\delta > 0$. Use the Bessel inequality to prove that there are at most finitely many integers n such that $\sin(nx) \geq \delta$ for all $x \in E$.*

Proof.

(1) (Reductio ad absurdum) If there were infinitely many integers n such that $\sin(nx) \geq \delta$ for all $x \in E$, then there exists an increasing sequence of positive integers $\{n_k\}$ such that $\sin(n_k x) \geq \delta$ for all $x \in E$ and n_k.

(2) Since E is measurable, we apply the Bessel inequality (Theorem 8.12 and Definition 11.39) to the function $\chi_E \in \mathscr{L}^2$ on $[-\pi, \pi]$:

$$c_{-n} = \int_{[-\pi,\pi]} \chi_E e^{inx} dx \to 0$$

as $n \to \infty$. Hence

$$\lim_{k\to\infty} \int_E \sin(n_k x) dx = 0.$$

(3) Note that for all n_k, we have

$$\int_E \sin(n_k x) dx \geq m(E)\delta.$$

Here $m(E)\delta > 0$ is a constant, contrary to $\lim_{k\to\infty} \int_E \sin(n_k x) dx = 0$.

$\square$

Exercise 11.18. *Suppose $f \in \mathscr{L}^2(\mu)$, $g \in \mathscr{L}^2(\mu)$. Prove that*

$$\left| \int f\overline{g} d\mu \right|^2 = \int |f|^2 d\mu \int |g|^2 d\mu$$

if and only if $f(x) = 0$ almost everywhere or there is a constant c such that $g(x) = cf(x)$ almost everywhere. (Compare Theorem 11.35.)

Proof.

(1) Since $g \in \mathscr{L}^2(\mu)$, $\overline{g} \in \mathscr{L}^2(\mu)$. Theorem 11.35 implies that $f\overline{g} \in \mathscr{L}^2(\mu)$, and

$$\left\{ \int |f\overline{g}| d\mu \right\}^2 \leq \int |f|^2 d\mu \int |g|^2 d\mu.$$

With Theorem 11.26, we have

$$\left|\int f\overline{g}d\mu\right|^2 \leq \left\{\int |f\overline{g}|d\mu\right\}^2 \leq \int |f|^2 d\mu \int |g|^2 d\mu.$$

Thus

$$\left|\int f\overline{g}d\mu\right|^2 = \int |f|^2 d\mu \int |g|^2 d\mu$$

if and only if

$$\left\{\int |f\overline{g}|d\mu\right\}^2 = \int |f|^2 d\mu \int |g|^2 d\mu \text{ and } \left|\int f\overline{g}d\mu\right| = \int |f\overline{g}|d\mu.$$

(2)

□

Note. Compare Exercise 1.15 and Exercise 6.10.