

## Chapter 10: Integration of Differential Forms

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**Exercise 10.1.** Let  $H$  be a compact convex set in  $\mathbb{R}^k$ , with nonempty interior. Let  $f \in \mathcal{C}(H)$ , put  $f(\mathbf{x}) = 0$  in the complement of  $H$ , and define  $\int_H f$  as in Definition 10.3. Prove that  $\int_H f$  is independent of the order in which the  $k$  integrations are carried out. (Hint: Approximate  $f$  by functions that are continuous on  $\mathbb{R}^k$  and whose supports are in  $H$ , as was done in Example 10.4.)

*Proof.*

(1)

(2)

□

**Exercise 10.2.** For  $i = 1, 2, 3, \dots$ , let  $\varphi_i \in \mathcal{C}(\mathbb{R}^1)$  have support in  $(2^{-i}, 2^{1-i})$ , such that  $\int \varphi_i = 1$ . Put

$$f(x, y) = \sum_{i=1}^{\infty} [\varphi_i(x) - \varphi_{i+1}(x)] \varphi_i(y)$$

Then  $f$  has compact support in  $\mathbb{R}^2$ ,  $f$  is continuous except at  $(0, 0)$ , and

$$\int dy \int f(x, y) dx = 0 \quad \text{but} \quad \int dx \int f(x, y) dy = 1.$$

Observe that  $f$  is unbounded in every neighborhood of  $(0, 0)$ .

*Proof.*

(1) If  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}^m$  are two functions, then

(a)  $\text{supp}(fg) \subseteq \text{supp}(f) \cap \text{supp}(g)$ .

(b)  $\text{supp}(f + g) \subseteq \text{supp}(f) \cup \text{supp}(g)$ .

(2) Note that  $f(x, y)$  is well-defined on  $\mathbb{R}^2$  since only finitely many terms are nonzero for each fixed point  $(x, y) \in \mathbb{R}^2$  (by (1)). Besides,

$$\begin{aligned} & \text{supp}([\varphi_i(x) - \varphi_{i+1}(x)]\varphi_i(y)) \\ & \subseteq \{(x, y) : x \in \text{supp}(\varphi_i) \cup \text{supp}(\varphi_{i+1}), y \in \text{supp}(\varphi_i)\} \\ & \subseteq \{(x, y) : x \in (2^{-i}, 2^{-i+1}) \cup (2^{-i-1}, 2^{-i}), y \in (2^{-i}, 2^{-i+1})\} \\ & \subseteq \{(x, y) : x \in (0, 1), y \in (0, 1)\} \end{aligned}$$

for all  $i = 1, 2, 3, \dots$ . So  $\text{supp}(f) \subseteq (0, 1)^2$ , or  $\text{supp}(f)$  is bounded. As  $\text{supp}(f)$  is closed (by definition),  $\text{supp}(f)$  is compact (Theorem 2.41).

(3) Show that  $f(x, y)$  is not continuous at  $(0, 0)$ .

(a) Note that  $f(0, 0) = 0$  since  $(0, 0) \notin \text{supp}(f) \subseteq (0, 1)^2$ . It suffices to show that there exists a sequence  $\{(t_n, t_n)\}$  in  $\mathbb{R}^2$  such that  $(t_n, t_n) \neq (0, 0)$ ,  $\lim_{n \rightarrow \infty} (t_n, t_n) = (0, 0)$  but  $\lim_{n \rightarrow \infty} f(t_n, t_n)$  does not converge to 0 (Theorem 4.2).

(b) For any  $n = 1, 2, 3, \dots$ ,

$$1 = \int \varphi_n = \int_{2^{-n}}^{2^{-n+1}} \varphi(t) dt \leq 2^{-n} \sup_{t \in [2^{-n}, 2^{-n+1}]} \varphi(t),$$

or  $\sup_{t \in [2^{-n}, 2^{-n+1}]} \varphi(t) \geq 2^n$ . By the continuity of  $\varphi_n$ , there exists  $t_n \in [2^{-n}, 2^{-n+1}]$  such that  $\varphi_n(t_n) \geq 2^n$  (Theorem 4.16).

(c) We construct  $\{(t_n, t_n)\}$  in  $\mathbb{R}^2$  by (b) for all  $n = 1, 2, 3, \dots$ . Clearly,  $(t_n, t_n) \neq (0, 0)$  and  $\lim_{n \rightarrow \infty} (t_n, t_n) = (0, 0)$ . However,

$$f(t_n, t_n) = [\varphi_n(t_n) - \varphi_{n+1}(t_n)]\varphi_n(t_n) = \varphi_n(t_n)^2 \geq 2^{2n}$$

does not converge to 0 as  $n \rightarrow \infty$ .

(4) Show that  $f(x, y)$  is continuous at  $\mathbf{x}_0 = (x_0, y_0) \neq (0, 0)$ . Consider an open neighborhood  $B(\mathbf{x}_0; r)$  of  $\mathbf{x}_0$  with  $r = \frac{\|\mathbf{x}_0\|}{64} > 0$ . Hence,

$$f(x, y)|_{B(\mathbf{x}_0; r)} = \sum_{i=1}^N [\varphi_i(x) - \varphi_{i+1}(x)]\varphi_i(y)$$

is the sum of finitely many terms where  $N = \log_2 \frac{89}{\|\mathbf{x}_0\|} \geq 1$  (since  $[\varphi_i(x) - \varphi_{i+1}(x)]\varphi_i(y) = 0$  on  $B(\mathbf{x}_0; r)$  whenever  $i \geq N$ ). Therefore,  $f(x, y)|_{B(\mathbf{x}_0; r)}$  is continuous by the continuity of  $\varphi_i$ .

(5) Show that  $\int dy \int f(x, y) dx = 0$ . For any fixed  $y$ , there is a positive integer  $N(y)$  such that  $\varphi_{N(y)+1}(y) = \varphi_{N(y)+2}(y) = \dots = 0$  and

$$f(x, y) = \sum_{i=1}^{N(y)} [\varphi_i(x) - \varphi_{i+1}(x)]\varphi_i(y).$$

So

$$\begin{aligned}
\int f(x, y) dx &= \int \sum_{i=1}^{N(y)} [\varphi_i(x) - \varphi_{i+1}(x)] \varphi_i(y) dx \\
&= \sum_{i=1}^{N(y)} \varphi_i(y) \int [\varphi_i(x) - \varphi_{i+1}(x)] dx \\
&= \sum_{i=1}^{N(y)} \varphi_i(y) \left( \int \varphi_i(x) dx - \int \varphi_{i+1}(x) dx \right) \\
&= \sum_{i=1}^{N(y)} \varphi_i(y) (1 - 1) \\
&= 0,
\end{aligned}$$

and thus

$$\int dy \int f(x, y) dx = \int 0 dy = 0.$$

- (6) *Show that  $\int dx \int f(x, y) dy = 0$ . For any fixed  $x$ , there is a positive integer  $N(x)$  such that  $\varphi_{N(x)+1}(x) = \varphi_{N(x)+2}(x) = \dots = 0$  and*

$$f(x, y) = \sum_{i=1}^{N(x)} [\varphi_i(x) - \varphi_{i+1}(x)] \varphi_i(y).$$

So

$$\begin{aligned}
\int f(x, y) dy &= \int \sum_{i=1}^{N(x)} [\varphi_i(x) - \varphi_{i+1}(x)] \varphi_i(y) dy \\
&= \sum_{i=1}^{N(x)} [\varphi_i(x) - \varphi_{i+1}(x)] \int \varphi_i(y) dy \\
&= \sum_{i=1}^{N(x)} [\varphi_i(x) - \varphi_{i+1}(x)] \\
&= \varphi_1(x),
\end{aligned}$$

and thus

$$\int dx \int f(x, y) dy = \int \varphi_1(x) dx = 1.$$

□

**Exercise 10.3.**

- (a) If  $\mathbf{F}$  is as in Theorem 10.7, put  $\mathbf{A} = \mathbf{F}'(\mathbf{0})$ ,  $\mathbf{F}_1(\mathbf{x}) = \mathbf{A}^{-1}\mathbf{F}(\mathbf{x})$ . Then  $\mathbf{F}_1(\mathbf{0}) = \mathbf{I}$ . Show that

$$\mathbf{F}_1(\mathbf{x}) = \mathbf{G}_n \circ \mathbf{G}_{n-1} \circ \cdots \circ \mathbf{G}_1(\mathbf{x})$$

in some neighborhood of  $\mathbf{0}$ , for certain primitive mappings  $\mathbf{G}_1, \dots, \mathbf{G}_n$ . This gives another version of Theorem 10.7:

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}'(\mathbf{0})\mathbf{G}_n \circ \mathbf{G}_{n-1} \circ \cdots \circ \mathbf{G}_1(\mathbf{x}).$$

- (b) Prove that the mapping  $(x, y) \mapsto (y, x)$  of  $\mathbb{R}^2$  onto  $\mathbb{R}^2$  is not the composition of any two primitive mappings, in any neighborhood of the origin. (This shows that the flips  $B_i$  cannot be omitted from the statement of Theorem 10.7.)

*Proof of (a).*

- (1) Suppose  $\mathbf{F}$  is a  $\mathcal{C}'$ -mapping of an open set  $E \subseteq \mathbb{R}^n$  into  $\mathbb{R}^n$ ,  $\mathbf{0} \in E$ ,  $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ , and  $\mathbf{F}'(\mathbf{0})$  is invertible.
- (2) Similar to the proof of Theorem 10.7. Put  $\mathbf{F}_1 = \mathbf{F}$ .
- (3) As  $m = 1$ , there is an open neighborhood  $V_1 \subseteq E$  of  $\mathbf{0}$  such that  $\mathbf{F}_1(\mathbf{0}) = (\mathbf{F}'(\mathbf{0}))^{-1}\mathbf{F}(\mathbf{0}) = \mathbf{0}$ ,  $\mathbf{F}'_1(\mathbf{0}) = \mathbf{I}$  is invertible, and

$$\mathbf{F}_1(\mathbf{x}) = \sum_{i=1}^n \alpha_i(\mathbf{x})\mathbf{e}_i,$$

where  $\alpha_1, \dots, \alpha_n$  are real  $\mathcal{C}'$ -functions in  $V_1$ . Hence

$$\mathbf{F}'_1(\mathbf{0})\mathbf{e}_1 = \sum_{i=1}^n (D_1\alpha_i)(\mathbf{0})\mathbf{e}_i.$$

Note that  $(D_1\alpha_1)(\mathbf{0}) = 1 \neq 0$ , and we might pick  $B_1 = \mathbf{I}$ . Thus we can define

$$\mathbf{G}_1(\mathbf{x}) = \mathbf{x} + [\alpha_1(\mathbf{x}) - x_1]\mathbf{e}_1 \quad (\mathbf{x} \in V_1).$$

Then  $\mathbf{G}_1 \in \mathcal{C}'(V_1)$ ,  $\mathbf{G}_1$  is primitive, and  $\mathbf{G}'_1(\mathbf{0}) = \mathbf{I}$  is invertible.

- (4) Now we make the induction hypothesis for  $1 \leq m \leq n-1$ .
- (5) Since  $\mathbf{G}'_m(\mathbf{0}) = \mathbf{I}$  is invertible, the inverse function theorem shows that there is an open set  $U_m$ , with  $\mathbf{0} \in U_m \subseteq V_m$ , such that  $\mathbf{G}_m$  is an injective mapping of  $U_m$  onto a neighborhood  $V_{m+1}$  of  $\mathbf{0}$ , in which  $\mathbf{G}_m^{-1} \in \mathcal{C}'(V_{m+1})$ . Define  $\mathbf{F}_{m+1}$  by

$$\mathbf{F}_{m+1}(\mathbf{y}) = \mathbf{F}_m \circ \mathbf{G}_m^{-1}(\mathbf{y}) \quad (\mathbf{y} \in V_{m+1}).$$

Then  $\mathbf{F}_{m+1} \in \mathcal{C}'(V_{m+1})$ ,  $\mathbf{F}_m(\mathbf{0}) = \mathbf{0}$ , and  $\mathbf{F}'_{m+1}(\mathbf{0}) = \mathbf{I}$  is invertible by the chain rule and the inverse function theorem. So

$$\mathbf{F}_{m+1}(\mathbf{x}) = P_m \mathbf{x} + \sum_{i=m+1}^n \alpha_i(\mathbf{x}) \mathbf{e}_i,$$

where  $\alpha_1, \dots, \alpha_n$  are real  $\mathcal{C}'$ -functions in  $V_{m+1}$ . Hence

$$\mathbf{F}'_{m+1}(\mathbf{0}) \mathbf{e}_{m+1} = \sum_{i=m+1}^n (D_{m+1} \alpha_i)(\mathbf{0}) \mathbf{e}_i.$$

Note that  $(D_{m+1} \alpha_{m+1})(\mathbf{0}) = 1 \neq 0$ , and we might pick  $B_{m+1} = \mathbf{I}$ . Thus we can define

$$\mathbf{G}_{m+1}(\mathbf{x}) = \mathbf{x} + [\alpha_{m+1}(\mathbf{x}) - x_{m+1}] \mathbf{e}_{m+1} \quad (\mathbf{x} \in V_{m+1}).$$

Then  $\mathbf{G}_{m+1} \in \mathcal{C}'(V_{m+1})$ ,  $\mathbf{G}_{m+1}$  is primitive, and  $\mathbf{G}'_{m+1}(\mathbf{0}) = \mathbf{I}$  is invertible. Our induction hypothesis holds therefore with  $m+1$  in place of  $m$ .

(6) Note that

$$\mathbf{F}_m(\mathbf{x}) = \mathbf{F}_{m+1}(\mathbf{G}_m(\mathbf{x})) \quad (\mathbf{x} \in U_m).$$

If we apply this with  $m = 1, \dots, n-1$ , we successively obtain

$$\mathbf{F}_1 = \mathbf{F}_n \circ \mathbf{G}_{n-1} \circ \dots \circ \mathbf{G}_1$$

in some open neighborhood of  $\mathbf{0}$ . Note that  $\mathbf{F}_n$  is primitive since

$$\mathbf{F}_n(\mathbf{x}) = P_{n-1} \mathbf{x} + \alpha_n(\mathbf{x}) \mathbf{e}_n.$$

This completes the proof.

□

*Proof of (b).*

(1) For  $(x, y) \in \mathbb{R}^2$ , define

$$\mathbf{F}(x, y) = (y, x).$$

(2) (Reductio ad absurdum) If  $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$  for some primitive mappings  $\mathbf{G}_i$  ( $i = 1, 2$ ) in some neighborhood  $V_i$  of the origin,  $\mathbf{G}_i(\mathbf{0}) = \mathbf{0}$  and  $\mathbf{G}'_i$  is invertible, then we may assume that

$$\mathbf{G}_1(x, y) = (x, g_1(x, y)) \quad \text{and} \quad \mathbf{G}_2(x, y) = (g_2(x, y), y).$$

Here the case  $\mathbf{G}_1(x, y) = (g_1(x, y), y)$  and  $\mathbf{G}_2(x, y) = (x, g_2(x, y))$  is similar to the above case. Besides,  $\mathbf{G}_1(x, y) = (x, g_1(x, y))$  and  $\mathbf{G}_2(x, y) = (x, g_2(x, y))$  implies that

$$\mathbf{G}_2 \circ \mathbf{G}_1(x, y) = (x, g_2(x, g_1(x, y))) \neq (y, x) = \mathbf{F}(x, y).$$

Same reason for  $\mathbf{G}_1(x, y) = (g_1(x, y), y)$  and  $\mathbf{G}_2(x, y) = (g_2(x, y), y)$ .

(3) Note that

$$\mathbf{F}'(\mathbf{0}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Since

$$\mathbf{F}'(\mathbf{0}) = \mathbf{G}'_2(\mathbf{G}_1(\mathbf{0}))\mathbf{G}'_1(\mathbf{0}) = \mathbf{G}'_2(\mathbf{0})\mathbf{G}'_1(\mathbf{0}),$$

we have

$$\begin{aligned} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} &= \begin{bmatrix} D_1g_2(0,0) & D_2g_2(0,0) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ D_1g_1(0,0) & D_2g_1(0,0) \end{bmatrix} \\ &= \begin{bmatrix} * & * \\ D_1g_1(0,0) & D_2g_1(0,0) \end{bmatrix}. \end{aligned}$$

So  $D_1g_1(0,0) = 1$  and  $D_2g_1(0,0) = 0$ , and thus  $\mathbf{G}'_1(\mathbf{0}) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$  is not invertible, which is absurd.

□

**Exercise 10.4.** For  $(x, y) \in \mathbb{R}^2$ , define

$$\mathbf{F}(x, y) = (e^x \cos y - 1, e^x \sin y)$$

Prove that  $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$ , where

$$\begin{aligned} \mathbf{G}_1(x, y) &= (e^x \cos y - 1, y) \\ \mathbf{G}_2(u, v) &= (u, (1 + u) \tan v) \end{aligned}$$

are primitive in some neighborhood of  $(0, 0)$ . Compute the Jacobians of  $\mathbf{G}_1$ ,  $\mathbf{G}_2$ ,  $\mathbf{F}$  at  $(0, 0)$ . Define

$$\mathbf{H}_2(x, y) = (x, e^x \sin y)$$

and find

$$\mathbf{H}_1(u, v) = (h(u, v), v)$$

so that  $\mathbf{F} = \mathbf{H}_1 \circ \mathbf{H}_2$  is in some neighborhood of  $(0, 0)$ .

*Proof.*

(1) By Definition 10.5,

$$\begin{aligned} \mathbf{G}_1(x, y) &= (e^x \cos y - 1)\mathbf{e}_1 + y\mathbf{e}_2, \\ \mathbf{G}_2(u, v) &= u\mathbf{e}_1 + ((1 + u) \tan v)\mathbf{e}_2 \end{aligned}$$

are primitive in some neighborhood of  $(0, 0)$ .

(2) Show that  $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$ . Given any  $(x, y) \in \mathbb{R}^2$ , we have

$$\begin{aligned}
(\mathbf{G}_2 \circ \mathbf{G}_1)(x, y) &= \mathbf{G}_2(\mathbf{G}_1(x, y)) \\
&= \mathbf{G}_2(e^x \cos y - 1, y) \\
&= (e^x \cos y - 1, (1 + (e^x \cos y - 1)) \tan y) \\
&= (e^x \cos y - 1, e^x \sin y) \\
&= \mathbf{F}(x, y).
\end{aligned}$$

(3) Since

$$\begin{aligned}
J_{\mathbf{G}_1}(x, y) &= \det \begin{bmatrix} e^x \cos y & -e^x \sin y \\ 0 & 1 \end{bmatrix} = e^x \cos y \\
J_{\mathbf{G}_2}(x, y) &= \det \begin{bmatrix} 1 & 0 \\ \tan y & (1 + x) \sec^2 y \end{bmatrix} = (1 + x) \sec^2 y \\
J_{\mathbf{F}}(x, y) &= \det \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix} = e^{2x},
\end{aligned}$$

$$J_{\mathbf{G}_1}(0, 0) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$

$$J_{\mathbf{G}_2}(0, 0) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$

$$J_{\mathbf{F}}(0, 0) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1.$$

(4) Define  $h(u, v) = \sqrt{e^{2u} - v^2} - 1$  on

$$B\left((0, 0); \frac{1}{64}\right) \subseteq \mathbb{R}^2.$$

$h(u, v)$  is well-defined since  $e^{2u} - v^2 > 0$  for all  $(u, v) \in B\left((0, 0); \frac{1}{64}\right)$ .

(5) Given any  $(x, y) \in \mathbb{R}^2$ , we have

$$\begin{aligned}
(\mathbf{H}_1 \circ \mathbf{H}_2)(x, y) &= \mathbf{H}_1(\mathbf{H}_2(x, y)) \\
&= \mathbf{H}_1(x, e^x \sin y) \\
&= (\sqrt{e^{2x} - (e^x \sin y)^2} - 1, e^x \sin y) \\
&= (e^x \cos y - 1, e^x \sin y) \\
&= \mathbf{F}(x, y).
\end{aligned}$$

□

**Exercise 10.5.** Formulate and prove an analogue of Theorem 10.8, in which  $K$  is a compact subset of an arbitrary metric space. (Replace the functions  $\varphi_i$  that occur in the proof of Theorem 10.8 by functions of the type constructed in Exercise 4.22.)

*Proof (Theorem 10.8).*

- (1) (Partitions of unity.) Suppose  $K$  is a compact subset of a metric space  $X$ , and  $\{V_\alpha\}$  is an open cover of  $K$ . Then there exist functions  $\psi_1, \dots, \psi_s \in \mathcal{C}(X)$  such that

- (a)  $0 \leq \psi_i \leq 1$  for  $1 \leq i \leq s$ .
- (b) each  $\psi_i$  has its support in some  $V_\alpha$ , and
- (c)  $\psi_1(x) + \dots + \psi_s(x) = 1$  for every  $x \in K$ .

- (2) It is trivial that some  $V_\alpha = X$  by taking  $s = 1$  and  $\psi_1(x) = 1 \in \mathcal{C}(X)$ . Now we assume that all  $V_\alpha \subsetneq X$ .

- (3) Associate with each  $x \in K$  an index  $\alpha(x)$  so that  $x \in V_{\alpha(x)}$ . Then there are open balls  $B(x)$  and  $W(x)$ , centered at  $x$ , with

$$x \in B(x) \subseteq \overline{B(x)} \subseteq W(x) \subseteq \overline{W(x)} \subseteq V_{\alpha(x)}$$

(Since  $V_{\alpha(x)}$  is open, there exists  $r > 0$  such that  $B(x; r) \subseteq V_{\alpha(x)}$ . Take  $B(x) = B(x; \frac{r}{89})$  and  $W(x) = B(x; \frac{r}{64})$ .)

- (4) Since  $K$  is compact, there are finitely many points  $x_1, \dots, x_s \in K$  such that

$$K \subseteq B(x_1) \cup \dots \cup B(x_s).$$

Note that

- (a)  $\overline{B(x_i)}$  is a nonempty closed set since  $x_i \in B(x_i) \subseteq \overline{B(x_i)}$ .
- (b)  $X - W(x_i) \supseteq X - V_{\alpha(x_i)}$  is a nonempty closed set by the assumption in (2).
- (c)  $\overline{B(x_i)} \cap (X - W(x_i)) \subseteq W(x_i) \cap (X - W(x_i)) = \emptyset$ .

By Exercise 4.22, there is a function

$$\varphi_i(x) = \frac{\rho_{\overline{B(x_i)}}(x)}{\rho_{\overline{B(x_i)}}(x) + \rho_{X - W(x_i)}(x)} \in \mathcal{C}(X)$$

such that  $\varphi_i(x) = 1$  on  $\overline{B(x_i)}$ ,  $\varphi_i(x) = 0$  outside  $W(x_i)$ , and  $0 \leq \varphi_i(x) \leq 1$  on  $X$  for  $1 \leq i \leq s$ .

- (5) Define  $\psi_1 = \varphi_1 \in \mathcal{C}(X)$  and

$$\psi_{i+1} = (1 - \varphi_1) \cdots (1 - \varphi_i) \varphi_{i+1} \in \mathcal{C}(X)$$



for  $1 \leq i \leq s-1$ . Properties (a) and (b) in (1) are clear. Also,

$$\psi_1(x) + \cdots + \psi_s(x) = 1 - (1 - \varphi_1(x)) \cdots (1 - \varphi_s(x))$$

by the construction of  $\psi_i$ . If  $x \in K$ , then  $x \in B(x_i)$  for some  $i$ , hence  $\varphi_i(x) = 1$ , and the product  $(1 - \varphi_1(x)) \cdots (1 - \varphi_s(x)) = 0$ . This proves property (c) in (1).

□

**Exercise 10.6.** *Strengthen the conclusion of Theorem 10.8 by showing that the functions  $\psi_i$  can be made differentiable, and even infinitely differentiable. (Use Exercise 8.1 in the construction of the auxiliary functions  $\psi_i$ .)*

*Proof (Theorem 10.8).*

- (1) It is trivial that some  $V_\alpha = \mathbb{R}^n$  by taking  $s = 1$  and  $\psi_1(\mathbf{x}) = 1 \in \mathcal{C}^\infty(\mathbb{R}^n)$ . Now we assume that all  $V_\alpha \subsetneq \mathbb{R}^n$ .
- (2) Associate with each  $\mathbf{x} \in K$  an index  $\alpha(x)$  so that  $\mathbf{x} \in V_{\alpha(x)}$ . Then there are open  $n$ -cells  $B(\mathbf{x})$  and  $W(\mathbf{x})$  (Definition 10.1), centered at  $\mathbf{x}$ , with

$$\mathbf{x} \in B(\mathbf{x}) \subseteq \overline{B(\mathbf{x})} \subseteq W(\mathbf{x}) \subseteq \overline{W(\mathbf{x})} \subseteq V_{\alpha(\mathbf{x})}$$

(Since  $V_{\alpha(\mathbf{x})}$  is open, there exists  $r > 0$  such that  $B(\mathbf{x}; r) \subseteq V_{\alpha(\mathbf{x})}$ . Take

$$B(\mathbf{x}) = I\left(\mathbf{x}; \frac{r}{89\sqrt{n}}\right), \quad W(\mathbf{x}) = I\left(\mathbf{x}; \frac{r}{64\sqrt{n}}\right)$$

where  $I(\mathbf{p}; r)$  is the open  $n$ -cell centered at  $\mathbf{p} = (p_1, \dots, p_n)$  defined by

$$I(\mathbf{p}; r) = (p_1 - r, p_1 + r) \times \cdots \times (p_n - r, p_n + r) \subseteq \mathbb{R}^n.$$

- (3) Define

$$f(y) = \begin{cases} e^{-\frac{1}{y^2}} & (y > 0), \\ 0 & (y \leq 0). \end{cases}$$

$f(y) \in \mathcal{C}^\infty(\mathbb{R}^1)$  by applying the similar argument in Exercise 8.1.

- (4) Given any  $\mathbf{x} = (x_1, \dots, x_n) \in K$  and construct  $B(\mathbf{x})$  and  $W(\mathbf{x})$  as in (2). Define

$$g_{x_j}(y_j) = \frac{f(y_j)}{f(y_j) + f\left(\frac{r}{64\sqrt{n}} - \frac{r}{89\sqrt{n}} - y_j\right)}$$

for  $1 \leq j \leq n$ .  $g_{x_j}$  is well-defined and  $g_{x_j} \in \mathcal{C}^\infty(\mathbb{R}^1)$ . So

$$g_{x_j}(y_j) = \begin{cases} 0 & \text{if } y_j \leq 0, \\ \text{strictly increasing} & \text{if } 0 \leq y_j \leq \frac{r}{64\sqrt{n}} - \frac{r}{89\sqrt{n}}, \\ 1 & \text{if } y_j \geq \frac{r}{64\sqrt{n}} - \frac{r}{89\sqrt{n}}. \end{cases}$$

Next, define

$$h_{x_j}(y_j) = g_{x_j} \left( y_j - x_j + \frac{r}{64\sqrt{n}} \right) g_{x_j} \left( x_j + \frac{r}{64\sqrt{n}} - y_j \right)$$

for  $1 \leq j \leq n$ .  $h_{x_j} \in \mathcal{C}^\infty(\mathbb{R}^1)$ . So

$$h_{x_j}(y_j) = \begin{cases} 0 & \text{if } y_j \leq x_j - \frac{r}{64\sqrt{n}}, \\ \text{strictly increasing} & \text{if } x_j - \frac{r}{64\sqrt{n}} \leq y_j \leq x_j - \frac{r}{89\sqrt{n}}, \\ 1 & \text{if } x_j - \frac{r}{89\sqrt{n}} \leq y_j \leq x_j + \frac{r}{89\sqrt{n}}, \\ \text{strictly decreasing} & \text{if } x_j + \frac{r}{89\sqrt{n}} \leq y_j \leq x_j + \frac{r}{64\sqrt{n}}, \\ 0 & \text{if } y_j \geq x_j + \frac{r}{64\sqrt{n}}. \end{cases}$$

Finally we define  $\mathbf{h}_{\mathbf{x}} : \mathbb{R}^n \rightarrow \mathbb{R}^1$  by

$$\mathbf{h}_{\mathbf{x}}(\mathbf{y}) = \prod_{j=1}^n h_{x_j}(y_j)$$

where  $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ . Hence,  $\mathbf{h}_{\mathbf{x}} \in \mathcal{C}^\infty(\mathbb{R}^n)$  (Theorem 9.21). Also,  $\mathbf{h}_{\mathbf{x}}(\mathbf{y}) = 1$  on  $B(\mathbf{x})$ ,  $\mathbf{h}_{\mathbf{x}}(\mathbf{y}) = 0$  outside  $W(\mathbf{x})$ , and  $0 \leq \mathbf{h}_{\mathbf{x}}(\mathbf{y}) \leq 1$ .

- (5) Since  $K$  is compact, there are finitely many points  $\mathbf{x}_1, \dots, \mathbf{x}_s \in K$  such that

$$K \subseteq B(\mathbf{x}_1) \cup \dots \cup B(\mathbf{x}_s).$$

Take

$$\varphi_i(\mathbf{x}) = \mathbf{h}_{\mathbf{x}_i}(\mathbf{x}) \in \mathcal{C}^\infty(\mathbb{R}^n)$$

for  $1 \leq i \leq s$ .

- (6) The rest are the same as the proof of Theorem 10.8 or Exercise 10.5.

□

### Exercise 10.7.

- (a) Show that the simplex  $Q^k$  is the smallest convex subset of  $\mathbb{R}^k$  such that contains  $\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_k$ .
- (b) Show that affine mappings take convex sets to convex sets.

*Proof of (a).*

- (1) Show that  $Q^k$  contains  $\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_k$ . Recall

$$Q^k = \{(x_1, \dots, x_k) \in \mathbb{R}^k : x_1 + \dots + x_k \leq 1 \text{ and } x_1, \dots, x_k \geq 0\}$$

(Example 10.14). Hence  $\mathbf{0} = (0, \dots, 0) \in Q^k$  and

$$\mathbf{e}_i = (0, \dots, \underbrace{1}_{i\text{th coordinate}}, \dots, 0) \in Q^k.$$

- (2) Show that  $Q^k$  is a convex subset of  $\mathbb{R}^k$ . Given any  $\mathbf{x} = (x_1, \dots, x_k) \in Q^k$ ,  $\mathbf{y} = (y_1, \dots, y_k) \in Q^k$  and  $0 < \lambda < 1$ . Hence

$$\lambda \mathbf{x} + (1 - \lambda) \mathbf{y} = (\lambda x_1 + (1 - \lambda) y_1, \dots, \lambda x_k + (1 - \lambda) y_k) \in Q^k$$

since each  $\lambda x_i + (1 - \lambda) y_i \geq 0$  and

$$\sum_{i=1}^k (\lambda x_i + (1 - \lambda) y_i) = \lambda \sum_{i=1}^k x_i + (1 - \lambda) \sum_{i=1}^k y_i \leq \lambda + (1 - \lambda) = 1.$$

- (3) Given any convex set  $E \subseteq \mathbb{R}^k$  containing  $\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_k$ . Show that  $E \supseteq Q^k$ .

- (a) Induction on  $k$ . Base case:  $k = 1$ . Given any  $\mathbf{x} = (x_1) \in Q^1$ . We have  $0 \leq x_1 \leq 1$  by the definition of  $Q^1$ . So that  $\mathbf{x} = x_1 \mathbf{e}_1 + (1 - x_1) \mathbf{0} \in E$  since  $\mathbf{0}, \mathbf{e}_1 \in E$  and  $E$  is convex.
- (b) Inductive step: suppose the statement holds for  $k = n$ . Given any  $\mathbf{x} = (x_1, \dots, x_n, x_{n+1}) \in Q^{n+1}$ . If  $x_{n+1} = 1$ , then  $x_1 = \dots = x_n = 0$  by the definition of  $Q^{n+1}$ . So  $\mathbf{x} = \mathbf{e}_{n+1} \in E$  by the assumption of  $E$ . If  $0 \leq x_{n+1} < 1$ , then  $x_1 + \dots + x_n \leq 1 - x_{n+1}$  or

$$\frac{x_1}{1 - x_{n+1}} + \dots + \frac{x_n}{1 - x_{n+1}} \leq 1.$$

So the point

$$\left( \frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}} \right) \in Q^n,$$

or

$$\left( \frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}}, 0 \right), \text{ say } \hat{\mathbf{x}}, \in E$$

by the induction hypothesis. Note that  $\mathbf{e}_{n+1} \in E$ . Hence

$$\mathbf{x} = x_{n+1} \mathbf{e}_{n+1} + (1 - x_{n+1}) \hat{\mathbf{x}} \in E$$

by the convexity of  $E$ .

- (c) Conclusion: Since both the base case and the inductive step have been proved as true, by mathematical induction the statement holds.

□

*Proof of (b).*

- (1) Let  $\mathbf{f}$  be an affine mapping that carries a vector space  $X$  into a vector space  $Y$  such that

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{0}) + A\mathbf{x}$$

for some  $A \in L(X, Y)$ .

- (2) Given any convex subset  $C$  of  $X$ . To show that  $\mathbf{f}(C)$  is convex, it suffices to show that

$$\lambda \mathbf{y}_1 + (1 - \lambda) \mathbf{y}_2 \in \mathbf{f}(C)$$

for any  $\mathbf{y}_1, \mathbf{y}_2 \in \mathbf{f}(C)$  and  $0 < \lambda < 1$ . Write  $\mathbf{y}_1 = \mathbf{f}(\mathbf{x}_1)$ ,  $\mathbf{y}_2 = \mathbf{f}(\mathbf{x}_2)$  for some  $\mathbf{x}_1, \mathbf{x}_2 \in C$ . Note that  $\lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2 \in C$  by the convexity of  $C$ . Hence

$$\begin{aligned} & \mathbf{f}(\lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2) \\ &= \mathbf{f}(\mathbf{0}) + A(\lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2) \\ &= \mathbf{f}(\mathbf{0}) + \lambda A \mathbf{x}_1 + (1 - \lambda) A \mathbf{x}_2 & (A \in L(X, Y)) \\ &= \lambda(\mathbf{f}(\mathbf{0}) + A \mathbf{x}_1) + (1 - \lambda)(\mathbf{f}(\mathbf{0}) + A \mathbf{x}_2) \\ &= \lambda \mathbf{f}(\mathbf{x}_1) + (1 - \lambda) \mathbf{f}(\mathbf{x}_2) \\ &= \lambda \mathbf{y}_1 + (1 - \lambda) \mathbf{y}_2 \in \mathbf{f}(C). \end{aligned}$$

□

**Exercise 10.8.** Let  $H$  be the parallelogram in  $\mathbb{R}^2$  whose vertices are  $(1, 1)$ ,  $(3, 2)$ ,  $(4, 5)$ ,  $(2, 4)$ . Find the affine map  $T$  which sends  $(0, 0)$  to  $(1, 1)$ ,  $(1, 0)$  to  $(3, 2)$ ,  $(1, 1)$  to  $(4, 5)$ ,  $(0, 1)$  to  $(2, 4)$ . Show that  $J_T = 5$ . Use  $T$  to convert the integral

$$\alpha = \int_H e^{x-y} dx dy$$

to an integral over  $I^2$  and thus compute  $\alpha$ .

*Proof.*

- (1) By Affine simplexes 10.26,

$$T(\mathbf{x}) = T(\mathbf{0}) + A\mathbf{x},$$

where  $A \in L(\mathbb{R}^2, \mathbb{R}^2)$ , say  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Note that  $T : \begin{bmatrix} 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . Thus

$$T : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 + ax + by \\ 1 + cx + dy \end{bmatrix}.$$

- (2) By  $T : (1, 0) \mapsto (3, 2)$  and  $T : (0, 1) \mapsto (2, 4)$ , we can solve  $A$  as

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

It is easy to verify such

$$T : \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{x}} \mapsto \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{T(\mathbf{0})} + \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{x}} = \begin{bmatrix} 1 + 2x + y \\ 1 + x + 3y \end{bmatrix}$$

satisfying our requirement.

(3)

$$J_T = \det \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = 5.$$

(4) By Example 10.4 and Theorem 10.9, we have

$$\begin{aligned} \int_H e^{x-y} dx dy &= \int_{I^2} e^{(1+2u+v)-(1+u+3v)} |J_T| du dv \\ &= 5 \int_{I^2} e^{u-2v} du dv \\ &= 5 \left\{ \int_0^1 e^u du \right\} \left\{ \int_0^1 e^{-2v} dv \right\} \quad (\text{Theorem 10.2}) \\ &= \frac{5}{2} (e-1)(1-e^{-2}). \end{aligned}$$

□

**Exercise 10.9.** Define  $(x, y) = T(r, \theta)$  on the rectangle

$$0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi$$

by the equations

$$x = r \cos \theta, \quad y = r \sin \theta.$$

Show that  $T$  maps this rectangle onto the closed disc  $D$  with center at  $(0, 0)$  and radius  $a$ , that  $T$  is one-to-one in the interior of the rectangle, and that  $J_T(r, \theta) = r$ . If  $f \in \mathcal{C}(D)$ , prove the formula for integration in polar coordinates:

$$\int_D f(x, y) dx dy = \int_0^a \int_0^{2\pi} f(T(r, \theta)) r dr d\theta.$$

(Hint: Let  $D_0$  be the interior of  $D$ , minus the interval from  $(0, 0)$  to  $(a, 0)$ . As it stands, Theorem 10.9 applies to continuous functions  $f$  whose support lies in  $D_0$ . To remove this restriction, proceed as in Example 10.4.)

*Proof.* Define  $E = [0, a] \times [0, 2\pi]$ .

(1) Show that  $T$  maps  $E$  onto  $D$ . Given any  $(x, y) \in D$ .

(a) It is equivalent to solve  $(r, \theta)$  from

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned}$$

in terms of  $(x, y)$ . Let  $L$  be the closed interval from  $(0, 0)$  to  $(a, 0)$ , say  $L = \{(r, 0) \in \mathbb{R}^2 : 0 \leq r \leq a\} \subseteq D$ .

- (b) If  $(x, y) \in L$ , say  $(x, y) = (r, 0)$  for some  $r \in [0, a]$ , then there exists  $(r, 0)$  or  $(r, 2\pi)$  such that  $T(r, 0) = T(r, 2\pi) = (r, 0)$ . (Therefore,  $T$  is not one-to-one on  $L$ .)
- (c) If  $(x, y) \in D - L$ , then there is  $r = (x^2 + y^2)^{\frac{1}{2}}$  in  $[0, a]$  (since  $(x, y) \in D = \{(x, y) \in \mathbb{R}^2 : (x^2 + y^2)^{\frac{1}{2}} \leq a\}$ ). Define

$$\theta = \begin{cases} \arccos\left(\frac{x}{r}\right) & \text{if } y \geq 0, \\ 2\pi - \arccos\left(\frac{x}{r}\right) & \text{if } y < 0. \end{cases}$$

It is well-defined since  $r \neq 0$ . Besides,  $\theta \in [0, 2\pi]$  and  $T(r, \theta) = (x, y)$ .

- (2) Show that  $T$  is one-to-one in the interior of the rectangle  $E$ . Suppose  $(r_1, \theta_1), (r_2, \theta_2) \in \text{int}(E) = (0, a) \times (0, 2\pi)$  and  $T(r_1, \theta_1) = T(r_2, \theta_2) = (x, y) \in D$ . Then

$$\begin{aligned} x &= r_1 \cos \theta_1 = r_2 \cos \theta_2, \\ y &= r_1 \sin \theta_1 = r_2 \sin \theta_2. \end{aligned}$$

Note that  $r_1^2 = r_2^2 = x^2 + y^2$  and  $r_1, r_2 > 0$ , we have  $r_1 = r_2$ . Solve  $\cos \theta_1 = \cos \theta_2$  and  $\sin \theta_1 = \sin \theta_2$  to get  $\theta_1 = \theta_2 + 2m\pi$  for all  $m \in \mathbb{Z}$ . Here  $m$  must be zero since  $\theta_1, \theta_2 \in (0, 2\pi)$ . Therefore,  $(r_1, \theta_1) = (r_2, \theta_2)$ .

- (3)  $T(\text{int}(E)) = D_0 = \text{int}(D) - L$  (by (1)(2)).
- (4) Show that  $J_T(r, \theta) = r$ .

$$J_T(r, \theta) = \det \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix} = r.$$

- (5) If  $f \in \mathcal{C}(D)$ , show that

$$\int_D f(x, y) dx dy$$

is well-defined. Similar to Example 10.4.

- (a) Extend  $f$  to a function on  $I^2 = [-a, a]^2$  by setting  $f(x, y) = 0$  off  $D$ , and define

$$\int_D f = \int_{I^2} f.$$

Since  $f$  may be discontinuous on  $I^2$ , the existence of the integral  $\int_{I^2} f$ . We also wish to show that this integral is independent of the order in which the 2 integrations are carried out.

- (b) To do this, suppose  $0 < \delta < 1$ , put

$$\varphi_\delta(t) = \begin{cases} 1 & \text{if } t \leq 1 - \delta, \\ \frac{1-t}{\delta} & \text{if } 1 - \delta \leq t \leq 1, \\ 0 & \text{if } t \geq 1, \end{cases}$$

and define

$$F_\delta(x, y) = \varphi_\delta \left( \frac{\sqrt{x^2 + y^2}}{a} \right) f(x, y).$$

Then  $F_\delta \in \mathcal{C}(I^2)$  (or  $\mathcal{C}(\mathbb{R}^2)$ ).

- (c) For each  $x \in [-a, a]$ , the set of all  $y$  such that  $F_\delta(x, y) \neq f(x, y)$  is contained a union of two segment whose length does not exceed

$$a\sqrt{1^2 - (1 - \delta)^2} = a\sqrt{2\delta - \delta^2} < a\sqrt{2\delta}.$$

Since  $0 \leq \varphi_\delta \leq 1$ , it follows that

$$\left| \int_{-a}^a F_\delta(x, y) dy - \int_{-a}^a f(x, y) dy \right| \leq 2a\sqrt{2\delta} \|f\|$$

where  $\|f\| = \max_{(x, y) \in I^2} |f(x, y)|$ . So the sequence of continuous function

$$\left\{ \int_{-a}^a F_\delta(x, y) dy \right\} \rightarrow \int_{-a}^a f(x, y) dy := g(x)$$

uniformly as  $\delta \rightarrow 0$ . ( $\delta = \frac{1}{n}$  for example.) So  $g(x) \in \mathcal{C}([-a, a])$ , and the further integrations present no problem, that is,  $\int_{I^2} f$  is existed.

- (d) Moreover,

$$\left| \int_{I^2} F_\delta(x, y) dx dy - \int_{I^2} f(x, y) dx dy \right| \leq 4a^2\sqrt{2\delta} \|f\|$$

It is true, regardless of the order in which the 2 single integrations are carried out. Since  $F_\delta \in \mathcal{C}(I^2)$ ,  $\int F_\delta$  is unaffected by any change in this order. Hence the inequality shows that the same is true of  $\int f$ .

- (6) Show that

$$\int_D f(x, y) dx dy = \int_0^a \int_0^{2\pi} f(T(r, \theta)) r dr d\theta.$$

- (a) Note that  $T$  is a one-to-one  $\mathcal{C}'$ -mapping of an open set  $D_0$  such that  $J_T(r, \theta) = r \neq 0$  for all  $(r, \theta) \in \text{int}(E)$ . To apply Theorem 10.9, we will leverage Example 10.4 again.
- (b) Given any  $\min \left\{ \frac{a}{89}, \frac{\pi}{64} \right\} > \delta > 0$ . Define

$$E_\delta = [\delta, a - \delta] \times [\delta, 2\pi - \delta] \subseteq \text{int}(E)$$

and  $D_\delta = T(E_\delta)$ . Similar to Example 10.4, let  $\varphi_\delta$  be a continuous function on  $\mathbb{R}^2$  such that  $\varphi_\delta(x, y) = 1$  on  $D_\delta$  and  $\varphi_\delta(x, y) = 0$  off  $D_{\frac{\delta}{2}}$ . Consider  $f_\delta(x, y) = \varphi_\delta(x, y)f(x, y)$  on  $\mathbb{R}^2$ . By construction,  $f_\delta$  is a continuous function on  $\mathbb{R}^2$  whose support is compact and lies in  $D_{\frac{\delta}{2}} \subseteq D_0$ . Hence, by Theorem 10.9

$$\int_{\mathbb{R}^2} f_\delta = \int_{D_{\frac{\delta}{2}}} f_\delta = \int_{\frac{\delta}{2}}^{a - \frac{\delta}{2}} \int_{\frac{\delta}{2}}^{2\pi - \frac{\delta}{2}} f(T(r, \theta)) r dr d\theta.$$

(c) Since  $f \in \mathcal{C}(D)$  and Exercise 6.7,

$$\lim_{\delta \rightarrow 0} \int_{\frac{\delta}{2}}^{a-\frac{\delta}{2}} \int_{\frac{\delta}{2}}^{2\pi-\frac{\delta}{2}} f(T(r, \theta)) r dr d\theta = \int_0^a \int_0^{2\pi} f(T(r, \theta)) r dr d\theta.$$

Therefore, it suffices to show that

$$\lim_{\delta \rightarrow 0} \int_{\mathbb{R}^2} f_\delta = \int_{\mathbb{R}^2} f.$$

(d) Note that  $\int_{\mathbb{R}^2} f = \int_D f$  (by (5)) and  $\int_{\mathbb{R}^2} f_\delta = \int_{D_{\frac{\delta}{2}}} f_\delta = \int_D f_\delta$  (by (b)). So

$$\left| \int_{\mathbb{R}^2} f - \int_{\mathbb{R}^2} f_\delta \right| = \left| \int_D f - \int_D f_\delta \right|.$$

To estimate the difference between  $\int_D f$  and  $\int_D f_\delta$ , we notice that  $f$  and  $f_\delta$  coincide on  $D_\delta$  by construction. Given any  $(x, y) \in D - D_\delta$ . Fix  $y$ , the set of all  $x$  such that  $f(x, y) \neq f_\delta(x, y)$  is contained a union of two segment whose length does not exceed  $\sqrt{a^2 - (a - \delta)^2 \cos^2 \delta}$ . Similarly as in (5), we have

$$\left| \int_D f - \int_D f_\delta \right| \leq 4a \sqrt{a^2 - (a - \delta)^2 \cos^2 \delta} \|f\|.$$

Hence,  $\lim \int_D f_\delta = \int_D f$ .

□

**Exercise 10.10.** Let  $a \rightarrow \infty$  in Exercise 10.9 and prove that

$$\int_{\mathbb{R}^2} f(x, y) dx dy = \int_0^\infty \int_0^{2\pi} f(T(r, \theta)) r dr d\theta,$$

for continuous functions  $f$  that decrease sufficiently rapidly as  $|x| + |y| \rightarrow \infty$ . (Find a more precise formulation.) Apply this to

$$f(x, y) = \exp(-x^2 - y^2)$$

to derive formula

$$\int_{-\infty}^\infty e^{-s^2} ds = \sqrt{\pi}.$$

*Proof.*

(1) Define

$$D_a = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq a^2\},$$

$$I_a = \{(x, y) \in \mathbb{R}^2 : |x| \leq a, |y| \leq a\},$$



for any  $a > 0$ . Similar to Definition 10.1, define

$$\int_{\mathbb{R}^2} f(x, y) dx dy = \lim_{a \rightarrow \infty} \int_{I_a} f(x, y) dx dy.$$

Besides, since  $f \in \mathcal{C}(\mathbb{R}^2)$ , we have

$$\begin{aligned} \int_0^\infty \int_0^{2\pi} f(T(r, \theta)) r dr d\theta &= \lim_{a \rightarrow \infty} \int_0^a \int_0^{2\pi} f(T(r, \theta)) r dr d\theta \\ &= \lim_{a \rightarrow \infty} \int_{D_a} f(x, y) dx dy \end{aligned}$$

by the definition in Exercise 6.8 and Exercise 10.9. So it suffices to show that both  $\lim_{a \rightarrow \infty} \int_{I_a} f(x, y) dx dy$  and  $\lim_{a \rightarrow \infty} \int_{D_a} f(x, y) dx dy$  exist and they are equal if  $f$  decreases sufficiently rapidly as  $|x| + |y| \rightarrow \infty$ .

- (2) Say that  $f$  decreases sufficiently rapidly as  $|x| + |y| \rightarrow \infty$  if there is a real number  $M > 0$  such that

$$|f(x, y)| \leq \alpha(x^2 + y^2)^{-1-\beta}$$

for some  $\alpha > 0$  and  $\beta > 0$  whenever  $|x| + |y| \geq M$ . Similar to Theorem 3.29, we can give other weaker hypotheses:

$$|f(x, y)| \leq \alpha(x^2 + y^2)^{-1} \log(x^2 + y^2)^{-1-\beta}$$

or

$$|f(x, y)| \leq \alpha(x^2 + y^2)^{-1} \log(x^2 + y^2)^{-1} \log \log(x^2 + y^2)^{-1-\beta}$$

and so on.

- (3) Write

$$\begin{aligned} &\int_0^a \int_0^{2\pi} |f(T(r, \theta))| r dr d\theta \\ &= \int_0^M \int_0^{2\pi} |f(T(r, \theta))| r dr d\theta + \int_M^a \int_0^{2\pi} |f(T(r, \theta))| r dr d\theta. \end{aligned}$$

Note that

$$\int_0^M \int_0^{2\pi} |f(T(r, \theta))| r dr d\theta < +\infty$$

since  $f \in \mathcal{C}(\mathbb{R}^2)$ . Besides,

$$\begin{aligned} \int_M^a \int_0^{2\pi} |f(T(r, \theta))| r dr d\theta &\leq \int_M^a \int_0^{2\pi} \alpha(x^2 + y^2)^{-1-\beta} r dr d\theta \\ &= \int_M^a \int_0^{2\pi} \alpha r^{-1-2\beta} dr d\theta \\ &= \pi \alpha \beta^{-1} (M^{-2\beta} - a^{-2\beta}) \end{aligned}$$

is finite as  $a \rightarrow \infty$ . Hence,  $\int_0^\infty \int_0^{2\pi} |f(T(r, \theta))| r dr d\theta$  converges absolutely, or  $\lim_{a \rightarrow \infty} \int_{D_a} f(x, y) dx dy$  exists.

(4) Note that

$$\int_{D_a} |f(x, y)| dx dy \leq \int_{I_a} |f(x, y)| dx dy \leq \int_{D_{\sqrt{2}a}} |f(x, y)| dx dy.$$

Let  $a \rightarrow \infty$ , we have

$$\lim_{a \rightarrow \infty} \int_{I_a} |f(x, y)| dx dy = \lim_{a \rightarrow \infty} \int_{D_a} |f(x, y)| dx dy$$

also exists.

(5) Show that  $\int_{-\infty}^\infty e^{-s^2} ds = \sqrt{\pi}$ . Take  $f(x, y) = \exp(-x^2 - y^2) \in \mathcal{C}(\mathbb{R}^2)$ . So

$$\lim_{x^2+y^2 \rightarrow \infty} |f(x, y)|(x^2 + y^2)^2 = \lim_{x^2+y^2 \rightarrow \infty} e^{-(x^2+y^2)}(x^2 + y^2)^2 = 0$$

by Theorem 8.6(f). Note that

$$\begin{aligned} \int_0^\infty \int_0^{2\pi} f(T(r, \theta)) r dr d\theta &= \int_0^\infty \int_0^{2\pi} e^{-r^2} r dr d\theta \\ &= 2\pi \int_0^\infty e^{-r^2} r dr \\ &= \pi \left[ -e^{-r^2} \right]_{r=0}^{r=\infty} \\ &= \pi \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^2} f(x, y) dx dy &= \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-x^2-y^2} dx dy \\ &= \left\{ \int_{-\infty}^\infty e^{-x^2} dx \right\} \left\{ \int_{-\infty}^\infty e^{-y^2} dy \right\} \\ &= \left\{ \int_{-\infty}^\infty e^{-x^2} dx \right\}^2. \end{aligned}$$

Hence,

$$\begin{aligned} \int_{\mathbb{R}^2} f(x, y) dx dy &= \int_0^\infty \int_0^{2\pi} f(T(r, \theta)) r dr d\theta \\ \implies \left\{ \int_{-\infty}^\infty e^{-x^2} dx \right\}^2 &= \pi \\ \implies \int_{-\infty}^\infty e^{-x^2} dx &= \sqrt{\pi}. \end{aligned}$$

(Here  $\int_{-\infty}^\infty e^{-x^2} dx$  is always nonnegative.)

□

**Exercise 10.11.** Define  $(u, v) = T(s, t)$  on the strip

$$0 < s < \infty, \quad 0 < t < 1$$

by setting  $u = s - st$ ,  $v = st$ . Show that  $T$  is a 1-1 mapping of the strip onto the positive quadrant  $Q$  in  $\mathbb{R}^2$ . Show that  $J_T(s, t) = s$ . For  $x > 0$ ,  $y > 0$ , integrate

$$u^{x-1}e^{-u}v^{y-1}e^{-v}$$

over  $Q$ , use Theorem 10.9 to convert the integral to one over the strip, and derive

$$\int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

in this way. (For this application, Theorem 10.9 has to be extended so as to cover certain improper integrals. Provide this extension.)

*Proof.*

(1)

(2)

□

**Exercise 10.12.** Let  $I^k$  be the set of all  $\mathbf{u} = (u_1, \dots, u_k) \in \mathbb{R}^k$  with  $0 \leq u_i \leq 1$  for all  $i$ ; let  $Q^k$  be the set of all  $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}^k$  with  $x_i \geq 0$ ,  $\sum x_i \leq 1$ . ( $I^k$  is the unit cube;  $Q^k$  is the standard simplex in  $\mathbb{R}^k$ .) Define  $\mathbf{x} = T(\mathbf{u})$  by

$$\begin{aligned} x_1 &= u_1 \\ x_2 &= (1 - u_1)u_2 \\ &\dots \\ x_k &= (1 - u_1) \cdots (1 - u_{k-1})u_k. \end{aligned}$$

Show that

$$\sum_{i=1}^k x_i = 1 - \prod_{i=1}^k (1 - u_i).$$

Show that  $T$  maps  $I^k$  onto  $Q^k$ , that  $T$  is 1-1 in the interior of  $I^k$ , and that its inverse  $S$  is defined in the interior of  $Q^k$  by  $u_1 = x_1$  and

$$u_i = \frac{x_i}{1 - x_1 - \cdots - x_{i-1}}$$

for  $i = 2, \dots, k$ . Show that

$$J_T(\mathbf{u}) = (1 - u_1)^{k-1}(1 - u_2)^{k-2} \cdots (1 - u_{k-1}),$$

and

$$J_S(\mathbf{x}) = [(1 - x_1)(1 - x_1 - x_2) \cdots (1 - x_1 - \cdots - x_{k-1})]^{-1}.$$

*Proof.*

(1) *Show that*

$$\sum_{i=1}^m x_i = 1 - \prod_{i=1}^m (1 - u_i)$$

for all  $1 \leq m \leq k$ . Induction on  $m$ . Base case:  $x_1 = 1 - (1 - u_1)$ . Inductive step: Suppose the case  $m = h$  is true. Consider the case  $m = h + 1$ :

$$\begin{aligned} \sum_{i=1}^{h+1} x_i &= \left( \sum_{i=1}^h x_i \right) + x_{h+1} \\ &= 1 - \prod_{i=1}^h (1 - u_i) + x_{h+1} && \text{(Induction hypothesis)} \\ &= 1 - \prod_{i=1}^h (1 - u_i) + u_{h+1} \prod_{i=1}^h (1 - u_i) && \text{(Definition of } x_{h+1}) \\ &= 1 - (1 - u_{h+1}) \prod_{i=1}^h (1 - u_i) \\ &= 1 - \prod_{i=1}^{h+1} (1 - u_i). \end{aligned}$$

Since both the base case and the inductive step have been proved as true, by mathematical induction the statement is established.

(2) *Show that  $T$  maps  $I^k$  onto  $Q^k$ .* Given any  $\mathbf{x} = (x_1, \dots, x_k) \in Q^k$ . It is equivalent to solve  $\mathbf{u} = (u_1, \dots, u_k)$  from

$$\begin{aligned} x_1 &= u_1 \\ x_2 &= (1 - u_1)u_2 \\ &\dots \\ x_k &= (1 - u_1) \cdots (1 - u_{k-1})u_k \end{aligned}$$

in terms of  $\mathbf{x} = (x_1, \dots, x_k)$ . It is clear that  $u_1 = x_1$  and

$$u_i = \begin{cases} x_i(1 - x_1 - \cdots - x_{i-1})^{-1} & \text{if } x_1 + \cdots + x_{i-1} \neq 1, \\ 0 & \text{if } x_1 + \cdots + x_{i-1} = 1. \end{cases}$$

for  $i = 2, \dots, k$ . (If  $x_1 + \cdots + x_{i-1} \neq 1$ , by (1) we have

$$\prod_{j=1}^{i-1} (1 - u_j) = 1 - \sum_{j=1}^{i-1} x_j \neq 0$$

and thus

$$u_i = x_i \left\{ \prod_{j=1}^{i-1} (1 - u_j) \right\}^{-1} = x_i (1 - x_1 - \cdots - x_{i-1})^{-1}.$$

If  $x_1 + \cdots + x_{i-1} = 1$ , then  $x_i = \cdots = x_k = 0$ . We may take  $u_i = 0$  to set the expression  $x_i = (1 - u_1) \cdots (1 - u_{i-1})u_i$  to zero.) Note that the solution  $\mathbf{u} \in I^k$  is well-defined by construction, or  $T(I^k) = Q^k$ .

- (3) Show that  $T$  is 1-1 in the interior of  $I^k$ . Suppose  $T(\mathbf{u}) = T(\mathbf{v}) = \mathbf{x}$  with  $\mathbf{u}, \mathbf{v} \in \text{int}(I^k)$ . Then we consider the following equation:

$$\begin{aligned} x_1 &= u_1 = v_1 \\ x_2 &= (1 - u_1)u_2 = (1 - v_1)v_2 \\ &\dots \\ x_k &= (1 - u_1) \cdots (1 - u_{k-1})u_k = (1 - v_1) \cdots (1 - v_{k-1})v_k. \end{aligned}$$

By (1),

$$\mathbf{x} \in \text{int}(Q^k) = \left\{ (x_1, \dots, x_k) \in \mathbb{R}^k : x_i > 0, \sum x_i < 1 \right\}.$$

Hence,

$$\begin{aligned} u_1 &= v_1 = x_1 \\ u_2 &= v_1 = x_2(1 - x_1)^{-1} \\ &\dots \\ u_k &= v_k = x_k(1 - x_1 - \cdots - x_{k-1})^{-1}. \end{aligned}$$

Here all  $(1 - x_1)^{-1}, \dots, (1 - x_1 - \cdots - x_i)^{-1}$  are well-defined since  $\mathbf{x} \in \text{int}(Q^k)$ . Therefore,  $T$  is injective on  $\text{int}(I^k)$ .

- (4) By (2)(3),  $T$  maps  $\text{int}(I^k)$  onto  $\text{int}(Q^k)$ . That is, given any  $\mathbf{x} = (x_1, \dots, x_k) \in \text{int}(Q^k)$ , we can pick

$$\begin{aligned} u_1 &= x_1 \\ u_i &= x_i(1 - x_1 - \cdots - x_{i-1})^{-1} \quad (i = 2, \dots, k) \end{aligned}$$

such that  $\mathbf{u} \in \text{int}(I^k)$  and  $T(\mathbf{u}) = \mathbf{x}$ .

- (5) Note that  $T(\mathbf{u}) = (u_1, (1 - u_1)u_2, \dots, (1 - u_1) \cdots (1 - u_{k-1})u_k)$  on  $\text{int}(I^k)$ . So

$$T'(\mathbf{u}) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ * & (1 - u_1) & 0 & \cdots & 0 \\ * & * & \prod_{i=1}^2 (1 - u_i) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & \prod_{i=1}^{k-1} (1 - u_i) \end{bmatrix}$$

is a lower triangular matrix. Hence,

$$\begin{aligned} J_T(\mathbf{u}) &= \det T'(\mathbf{u}) \\ &= 1 \cdot (1 - u_1) \cdot \prod_{i=1}^2 (1 - u_i) \cdots \prod_{i=1}^{k-1} (1 - u_i) \\ &= \prod_{i=1}^{k-1} (1 - u_i)^{k-i}. \end{aligned}$$

(6) Similar to (5).  $S(\mathbf{x}) = (x_1, x_2(1 - x_1)^{-1}, \dots, x_k(1 - x_1 - \dots - x_{k-1})^{-1})$  on  $\text{int}(Q^k)$ . So

$$S'(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ * & (1 - x_1)^{-1} & 0 & \cdots & 0 \\ * & * & (1 - x_1 - x_2)^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & (1 - x_1 - \cdots - x_{k-1})^{-1} \end{bmatrix}$$

is a lower triangular matrix. Hence,

$$\begin{aligned} J_S(\mathbf{x}) &= \det S'(\mathbf{x}) \\ &= 1 \cdot (1 - x_1)^{-1} \cdot (1 - x_1 - x_2)^{-1} \cdots (1 - x_1 - \cdots - x_{k-1})^{-1} \\ &= [(1 - x_1)(1 - x_1 - x_2) \cdots (1 - x_1 - \cdots - x_{k-1})]^{-1}. \end{aligned}$$

□

**Exercise 10.13.** Let  $r_1, \dots, r_k$  be nonnegative integers, and prove that

$$\int_{Q^k} x_1^{r_1} \cdots x_k^{r_k} d\mathbf{x} = \frac{r_1! \cdots r_k!}{(k + r_1 + \cdots + r_k)!}$$

(Hint: Use Exercise 10.12, Theorems 10.9 and 8.20.) Note that the special case  $r_1 = \cdots = r_k = 0$  shows that the volume of  $Q^k$  is  $\frac{1}{k!}$ .

*Proof.*

(1) Define  $T : I^k$  onto  $Q^k$  as in Exercise 10.12, and  $f : Q^k \rightarrow \mathbb{R}^1$  by

$$f(\mathbf{x}) = f(x_1, \dots, x_k) = x_1^{r_1} \cdots x_k^{r_k} = \prod_{i=1}^k x_i^{r_i}.$$

(2) By Exercise 10.12, Example 10.4 and Theorems 10.9, we have

$$\begin{aligned}
\int_{Q^k} x_1^{r_1} \cdots x_k^{r_k} d\mathbf{x} &= \int_{Q^k} f(\mathbf{x}) d\mathbf{x} \\
&= \int_{I^k} f(T(\mathbf{u})) |J_T(\mathbf{u})| d\mathbf{u} \\
&= \int_{I^k} \prod_{i=1}^k \left( u_i \prod_{j=1}^{i-1} (1 - u_j) \right)^{r_i} \prod_{i=1}^k (1 - u_i)^{k-i} d\mathbf{u} \\
&= \int_{I^k} \prod_{i=1}^k u_i^{r_i} (1 - u_i)^{k-i+\sum_{j=i+1}^k r_j} d\mathbf{u} \\
&= \prod_{i=1}^k \int_0^1 u_i^{r_i} (1 - u_i)^{k-i+\sum_{j=i+1}^k r_j} du_i && \text{(Theorem 10.2)} \\
&= \prod_{i=1}^k \frac{r_i! \left( k - i + \sum_{j=i+1}^k r_j \right)!}{\left( k - i + 1 + \sum_{j=i}^k r_j \right)!} && \text{(Theorem 8.20)} \\
&= \frac{r_1! \cdots r_k!}{(k + r_1 + \cdots + r_k)!}.
\end{aligned}$$

□

**Exercise 10.14 (Levi-Civita symbol).** Prove  $\varepsilon(j_1, \dots, j_k) = s(j_1, \dots, j_k)$ , where

$$s(j_1, \dots, j_k) = \prod_{p < q} \operatorname{sgn}(j_q - j_p).$$

It is usually to define the Levi-Civita symbol by

$$\varepsilon(j_1, \dots, j_k) = \begin{cases} 1 & \text{if } (j_1, \dots, j_k) \text{ is an even permutation of } J, \\ -1 & \text{if } (j_1, \dots, j_k) \text{ is an odd permutation of } J, \\ 0 & \text{otherwise} \end{cases}$$

(Basic  $k$ -forms 10.14). Thus, it is the sign of the permutation in the case of a permutation, and zero otherwise. So  $\varepsilon(j_1, \dots, j_k)$  is equivalent to an explicit expression  $s(j_1, \dots, j_k) = \prod_{p < q} \operatorname{sgn}(j_q - j_p)$ .

*Proof.*

(1) Induction on  $k$ . Base case: Show that  $\varepsilon(j_1, j_2) = s(j_1, j_2)$ . Since

$$\varepsilon(j_1, j_2) = \begin{cases} 1 & \text{if } j_1 < j_2 \\ -1 & \text{if } j_1 > j_2, \end{cases}$$

$$\varepsilon(j_1, j_2) = \operatorname{sgn}(j_2 - j_1) = s(j_1, j_2).$$

- (2) Inductive step: *Show that for any  $s \geq 2$ , if  $\varepsilon(j_1, \dots, j_s) = s(j_1, \dots, j_s)$  holds, then  $\varepsilon(j_1, \dots, j_{s+1}) = s(j_1, \dots, j_{s+1})$  also holds.*

$$\begin{aligned} \varepsilon(j_1, \dots, j_{s+1}) &= \varepsilon(j_1, \dots, j_s) \prod_{\substack{1 \leq p \leq s \\ q=s+1}} \operatorname{sgn}(j_q - j_p) \\ &= s(j_1, \dots, j_s) \prod_{\substack{1 \leq p \leq s \\ q=s+1}} \operatorname{sgn}(j_q - j_p) \\ &= \prod_{1 \leq p < q \leq s} \operatorname{sgn}(j_q - j_p) \prod_{\substack{1 \leq p \leq s \\ q=s+1}} \operatorname{sgn}(j_q - j_p) \\ &= \prod_{1 \leq p < q \leq s+1} \operatorname{sgn}(j_q - j_p) \\ &= s(j_1, \dots, j_{s+1}). \end{aligned}$$

- (3) Conclusion: Since both the base case and the inductive step have been proved as true, by mathematical induction the statement holds for every integer  $k \geq 2$ .

□

**Exercise 10.15.** *If  $\omega$  and  $\lambda$  are  $k$ - and  $m$ -forms, respectively, prove that*

$$\omega \wedge \lambda = (-1)^{km} \lambda \wedge \omega.$$

*Proof.*

- (1) Write

$$\omega = \sum_I b_I(\mathbf{x}) dx_I, \quad \lambda = \sum_J c_J(\mathbf{x}) dx_J$$

in the standard presentations, where  $I$  and  $J$  range over all increasing  $k$ -indices and over all increasing  $m$ -indices taken from the set  $\{1, \dots, n\}$ .

- (2) Show that  $dx_I \wedge dx_J = (-1)^{km} dx_J \wedge dx_I$ .

$$\begin{aligned} dx_I \wedge dx_J &= dx_{i_1} \wedge \dots \wedge dx_{i_k} \wedge dx_J \\ &= (-1)^m dx_{i_1} \wedge \dots \wedge dx_{i_{k-1}} \wedge dx_J \wedge dx_{i_k} \\ &= (-1)^{2m} dx_{i_1} \wedge \dots \wedge dx_{i_{k-2}} \wedge dx_J \wedge dx_{i_{k-1}} \wedge dx_{i_k} \\ &\dots \\ &= (-1)^{km} dx_J \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k} \\ &= (-1)^{km} dx_J \wedge dx_I. \end{aligned}$$



(3)

$$\begin{aligned}
\omega \wedge \lambda &= \sum_{I,J} b_I(\mathbf{x}) c_J(\mathbf{x}) dx_I \wedge dx_J \\
&= (-1)^{km} \sum_{J,I} c_J(\mathbf{x}) b_I(\mathbf{x}) dx_J \wedge dx_I \\
&= (-1)^{km} \lambda \wedge \omega.
\end{aligned}$$

□

**Exercise 10.16.** If  $k \geq 2$  and  $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_k]$  is an oriented affine  $k$ -simplex, prove that  $\partial^2 \sigma = 0$ , directly from the definition of the boundary operator  $\partial$ . Deduce from this that  $\partial^2 \Psi = 0$  for every chain  $\Psi$ . (Hint: For orientation, do it first for  $k = 2$ ,  $k = 3$ . In general, if  $i < j$ , let  $\sigma_{ij}$  be the  $(k-2)$ -simplex obtained by deleting  $\mathbf{p}_i$  and  $\mathbf{p}_j$  from  $\sigma$ . Show that each  $\sigma_{ij}$  occurs twice in  $\partial^2 \sigma$ , with opposite sign.)

*Proof (Brute-force).*

- (1) Write the boundary of the oriented affine  $k$ -simplex  $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_k]$  as

$$\partial \sigma = \sum_{i=0}^k (-1)^i [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k]$$

where where the oriented  $(k-1)$ -simplex  $[\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k]$  is obtained by deleting  $\sigma$ 's  $i$ -th vertex (Boundaries 10.29).

- (2)

$$\begin{aligned}
\partial^2 \sigma &= \partial \left( \sum_i (-1)^i [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k] \right) \\
&= \sum_i (-1)^i \partial [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k] \\
&= \sum_{j < i} (-1)^i (-1)^j [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_j, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k] \\
&\quad + \sum_{j > i} (-1)^i (-1)^{j-1} [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \widehat{\mathbf{p}}_j, \dots, \mathbf{p}_k] \\
&= \sum_{j < i} (-1)^{i+j} [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_j, \dots, \widehat{\mathbf{p}}_i, \dots, \mathbf{p}_k] \\
&\quad - \sum_{j > i} (-1)^{i+j} [\mathbf{p}_0, \dots, \widehat{\mathbf{p}}_i, \dots, \widehat{\mathbf{p}}_j, \dots, \mathbf{p}_k].
\end{aligned}$$

The latter two summations cancel since after switching  $i$  and  $j$  in the second sum. Therefore  $\partial^2 \sigma = 0$ .

- (3) The boundary of a chain is the linear combination of boundaries of the simplices in the chain. Write  $\Psi = \sum_{i=1}^r \sigma_i$ , where  $\sigma_i$  is an oriented affine simplex. Then

$$\partial^2 \Psi = \partial \left( \partial \sum \sigma_i \right) = \partial \left( \sum \partial \sigma_i \right) = \sum \partial^2 \sigma_i = \sum 0 = 0$$

for any affine chain  $\Psi$ .

□

**Exercise 10.17.** Put  $J^2 = \tau_1 + \tau_2$ , where

$$\tau_1 = [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2], \quad \tau_2 = -[\mathbf{0}, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1].$$

Explain why it is reasonable to call  $J^2$  the positively oriented unit square in  $\mathbb{R}^2$ . Show that  $\partial J^2$  is the sum of 4 oriented affine 1-simplexes. Find these. What is  $\partial(\tau_1 - \tau_2)$ ?

*Proof.*

- (1) Note that the unit square  $I^2 \in \mathbb{R}^2$  is the union of  $\tau_1(Q^2)$  and  $\tau_2(Q^2)$ , where

$$\begin{aligned} \tau_1(\mathbf{u}) &= ([\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2])(\mathbf{u}) \\ &= \mathbf{0} + \alpha_1 \mathbf{e}_1 + \alpha_2 (\mathbf{e}_1 + \mathbf{e}_2) \\ &= \mathbf{0} + (\alpha_1 + \alpha_2) \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 \\ &= \mathbf{0} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{u} \end{aligned}$$

and

$$\begin{aligned} \tau_2(\mathbf{u}) &= (-[\mathbf{0}, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1])(\mathbf{u}) \\ &= ([\mathbf{0}, \mathbf{e}_2 + \mathbf{e}_1, \mathbf{e}_2])(\mathbf{u}) \\ &= \mathbf{0} + \alpha_1 (\mathbf{e}_1 + \mathbf{e}_2) + \alpha_2 \mathbf{e}_2 \\ &= \mathbf{0} + \alpha_1 \mathbf{e}_1 + (\alpha_1 + \alpha_2) \mathbf{e}_2 \\ &= \mathbf{0} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{u} \end{aligned}$$

where  $\mathbf{u} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 \in \mathbb{R}^2$  (as in Equation (78)). Both  $\tau_1$  and  $\tau_2$  have Jacobian  $1 > 0$ , or positively oriented (Affine simplexes 10.26). So it is reasonable to call  $J^2$  the positively oriented unit square in  $\mathbb{R}^2$ .

- (2)

$$\begin{aligned} \partial \tau_1 &= [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] - [\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{0}, \mathbf{e}_1], \\ \partial \tau_2 &= [\mathbf{e}_2 + \mathbf{e}_1, \mathbf{e}_2] - [\mathbf{0}, \mathbf{e}_2] + [\mathbf{0}, \mathbf{e}_2 + \mathbf{e}_1] \\ &= [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2] + [\mathbf{e}_2, \mathbf{0}] + [\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2]. \end{aligned}$$

(3) By (2),

$$\partial J^2 = \partial\tau_1 + \partial\tau_2 = [\mathbf{0}, \mathbf{e}_1] + [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2] + [\mathbf{e}_2, \mathbf{0}],$$

which is the positively oriented boundary of  $I^2$ .

(4) By (2),

$$\begin{aligned} \partial(\tau_1 - \tau_2) &= \partial\tau_1 - \partial\tau_2 \\ &= [\mathbf{0}, \mathbf{e}_1] + [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{0}] \\ &\quad + [\mathbf{0}, \mathbf{e}_2] + [\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{0}]. \end{aligned}$$

□

**Exercise 10.18.** Consider the oriented affine 3-simplex

$$\sigma_1 = [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]$$

in  $\mathbb{R}^3$ . Show that  $\sigma_1$  (regarded as a linear transformation) has determinant 1. Thus  $\sigma_1$  is positively oriented.

Let  $\sigma_2, \dots, \sigma_6$  be five other oriented 3-simplexes, obtained as follows: There are five permutations  $(i_1, i_2, i_3)$  of  $(1, 2, 3)$ , distinct from  $(1, 2, 3)$ . Associate with each  $(i_1, i_2, i_3)$  the simplex

$$s(i_1, i_2, i_3)[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}]$$

where  $s$  is the sign that occurs in the definition of the determinant. (This is how  $\tau_2$  was obtained from  $\tau_1$  in Exercise 10.17.) Show that  $\sigma_2, \dots, \sigma_6$  are positively oriented.

Put  $J^3 = \sigma_1 + \dots + \sigma_6$ . Then  $J^3$  may be called the positively oriented unit cube in  $\mathbb{R}^3$ . Show that  $\partial J^3$  is the sum of 12 oriented affine 2-simplexes. (These 12 triangles cover the surface of the unit cube  $I^3$ .)

Show that  $\mathbf{x} = (x_1, x_2, x_3)$  is in the range of  $\sigma_1$  if and only if  $0 \leq x_3 \leq x_2 \leq x_1 \leq 1$ .

Show that the range of  $\sigma_1, \dots, \sigma_6$  have disjoint interiors, and that their union covers  $I^3$ . (Compared with Exercise 10.13; note that  $3! = 6$ .)

*Proof.*

(1) Show that  $\sigma_1$  (regarded as a linear transformation) has determinant 1.

Given any  $\mathbf{u} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3 \in \mathbb{R}^3$ , we have

$$\begin{aligned}\sigma_1(\mathbf{u}) &= ([\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3])(\mathbf{u}) \\ &= \mathbf{0} + \alpha_1 \mathbf{e}_1 + \alpha_2 (\mathbf{e}_1 + \mathbf{e}_2) + \alpha_3 (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) \\ &= \mathbf{0} + (\alpha_1 + \alpha_2 + \alpha_3) \mathbf{e}_1 + (\alpha_2 + \alpha_3) \mathbf{e}_2 + \alpha_3 \mathbf{e}_3 \\ &= \mathbf{0} + \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{say } A} \mathbf{u}.\end{aligned}$$

So

$$\det(A) = \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = 1.$$

- (2) *Show that  $\sigma_2, \dots, \sigma_6$  are positively oriented.* Define the permutation matrix  $P_{(i_1, i_2, i_3)}$  corresponding to a permutation  $(i_1, i_2, i_3)$  of  $(1, 2, 3)$  by

$$P_{(i_1, i_2, i_3)} = [\mathbf{e}_{i_1} \quad \mathbf{e}_{i_2} \quad \mathbf{e}_{i_3}].$$

For example,

$$P_{(2, 3, 1)} = [\mathbf{e}_2 \quad \mathbf{e}_3 \quad \mathbf{e}_1] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Note that the sign  $s(i_1, i_2, i_3)$  of the permutation  $(i_1, i_2, i_3)$  is exactly the same as the determinant of the permutation matrix  $P_{(i_1, i_2, i_3)}$ . Define a permutation  $(j_1, j_2, 3)$  of  $(1, 2, 3)$  (for swapping the first and the second coordinates of  $\mathbf{u}$ ) by

$$(j_1, j_2, 3) = \begin{cases} (1, 2, 3) & \text{if } s(i_1, i_2, i_3) = 1, \\ (2, 1, 3) & \text{if } s(i_1, i_2, i_3) = -1. \end{cases}$$

Write

$$\sigma_{(i_1, i_2, i_3)} = s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}].$$

(So that  $\sigma_1 = \sigma_{(1, 2, 3)}$ .) Hence,

$$\begin{aligned}\sigma_{(i_1, i_2, i_3)}(\mathbf{u}) &= \mathbf{0} + \alpha_{j_1} \mathbf{e}_{i_1} + \alpha_{j_2} (\mathbf{e}_{i_1} + \mathbf{e}_{i_2}) + \alpha_3 (\mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}) \\ &= \mathbf{0} + (\alpha_{j_1} + \alpha_{j_2} + \alpha_3) \mathbf{e}_{i_1} + (\alpha_{j_2} + \alpha_3) \mathbf{e}_{i_2} + \alpha_3 \mathbf{e}_{i_3} \\ &= \mathbf{0} + P_{(i_1, i_2, i_3)} A P_{(j_1, j_2, 3)} \mathbf{u}\end{aligned}$$

where  $\mathbf{u} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3 \in \mathbb{R}^3$ . For example,

$$P_{(2,3,1)}AP_{(1,2,3)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

So

$$\begin{aligned} \det(P_{(i_1, i_2, i_3)}AP_{(j_1, j_2, 3)}) &= \det(P_{(i_1, i_2, i_3)}) \det(A) \det(P_{(j_1, j_2, 3)}) \\ &= s(i_1, i_2, i_3) \cdot 1 \cdot s(i_1, i_2, i_3) \\ &= 1. \end{aligned}$$

(3) Show that  $\partial J^3$  is the sum of 12 oriented affine 2-simplexes. Note that

$$\begin{aligned} \sum_{(i_1, i_2, i_3)} \sigma_{(i_1, i_2, i_3)} &= \sum_{\substack{(i_1, i_2, i_3) \\ i_1 > i_2}} \sigma_{(i_1, i_2, i_3)} + \sum_{\substack{(i_1, i_2, i_3) \\ i_1 < i_2}} \sigma_{(i_1, i_2, i_3)} \\ &= \sum_{\substack{(i_1, i_2, i_3) \\ i_1 > i_2}} s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\ &\quad + \sum_{\substack{(i_1, i_2, i_3) \\ i_2 > i_1}} -s(i_2, i_1, i_3) [\mathbf{0}, \mathbf{e}_{i_2} + \mathbf{e}_{i_1}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\ &= \mathbf{0} \end{aligned}$$

and

$$\begin{aligned} \sum_{(i_1, i_2, i_3)} \sigma_{(i_1, i_2, i_3)} &= \sum_{\substack{(i_1, i_2, i_3) \\ i_2 > i_3}} \sigma_{(i_1, i_2, i_3)} + \sum_{\substack{(i_1, i_2, i_3) \\ i_2 < i_3}} \sigma_{(i_1, i_2, i_3)} \\ &= \sum_{\substack{(i_1, i_2, i_3) \\ i_2 > i_3}} s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\ &\quad + \sum_{\substack{(i_1, i_2, i_3) \\ i_3 > i_2}} -s(i_1, i_3, i_2) [\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\ &= \mathbf{0}. \end{aligned}$$

So

$$\begin{aligned}
\partial J^3 &= \sum_{(i_1, i_2, i_3)} \partial \sigma_{(i_1, i_2, i_3)} \\
&= \sum_{(i_1, i_2, i_3)} s(i_1, i_2, i_3) [\mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}] \\
&\quad - s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}] \\
&\quad + s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}] \\
&\quad - s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}] \\
&= \sum_{(i_1, i_2, i_3)} s(i_1, i_2, i_3) [\mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\
&\quad - \underbrace{\sum_{(i_1, i_2, i_3)} s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]}_{=0} \\
&\quad + \underbrace{\sum_{(i_1, i_2, i_3)} s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]}_{=0} \\
&\quad - \sum_{(i_1, i_2, i_3)} s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}].
\end{aligned}$$

Thus,

$$\begin{aligned}
\partial J^3 &= \sum_{(i_1, i_2, i_3)} s(i_1, i_2, i_3) [\mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\
&\quad - \sum_{(i_1, i_2, i_3)} s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}]
\end{aligned}$$

is the sum of 12 oriented affine 2-simplexes. (Note that  $3! = 6$ .)

- (4) Show that  $\mathbf{x} = (x_1, x_2, x_3)$  is in the range of  $\sigma_1$  if and only if  $0 \leq x_3 \leq x_2 \leq x_1 \leq 1$ .

- (a) By (1),  $\mathbf{x}$  is in the range of  $\sigma_1$  if and only if  $\mathbf{x} = A\mathbf{u}$  for  $\mathbf{u} = (u_1, u_2, u_3) \in Q^3$ , or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 + u_2 + u_3 \\ u_2 + u_3 \\ u_3 \end{bmatrix}.$$

- (b) Since  $\mathbf{u} = (u_1, u_2, u_3) \in Q^3$ ,  $u_1 + u_2 + u_3 \leq 1$  and  $u_1, u_2, u_3 \geq 0$ .

Hence  $0 \leq u_3 \leq u_2 + u_3 \leq u_1 + u_2 + u_3 \leq 1$  or  $0 \leq x_3 \leq x_2 \leq x_1 \leq 1$ .

- (c) Conversely, if  $0 \leq x_3 \leq x_2 \leq x_1 \leq 1$ , we define

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_3 \end{bmatrix}.$$

Clearly,  $\mathbf{v} \in Q^3$ .

- (5) Show that the range of  $\sigma_1, \dots, \sigma_6$  have disjoint interiors, and that their union covers  $I^3$ . Similar to (4). By (2),  $\mathbf{x} = P_{(i_1, i_2, i_3)} AP_{(j_1, j_2, 3)} \mathbf{u}$ , or  $P_{(i_1, i_2, i_3)}^{-1} \mathbf{x} = AP_{(j_1, j_2, 3)} \mathbf{u}$ , or

$$\begin{bmatrix} x_{i_1} \\ x_{i_2} \\ x_{i_3} \end{bmatrix} = \begin{bmatrix} u_1 + u_2 + u_3 \\ u_{j_2} + u_3 \\ u_3 \end{bmatrix}.$$

In any case, we always have  $0 \leq u_3 \leq u_{j_2} + u_3 \leq u_1 + u_2 + u_3 \leq 1$ . Hence  $\mathbf{x} = (x_1, x_2, x_3)$  is in the range of  $\sigma_{(i_1, i_2, i_3)}$  if and only if

$$0 \leq x_{i_3} \leq x_{i_2} \leq x_{i_1} \leq 1.$$

The interior of  $\sigma_{(i_1, i_2, i_3)}$  is

$$\{\mathbf{x} \in \mathbb{R}^3 : 0 < x_{i_3} < x_{i_2} < x_{i_1} < 1\},$$

and thus the range of  $\sigma_1, \dots, \sigma_6$  have disjoint interiors. Also, any  $\mathbf{x} \in I^3$  has the relation

$$0 \leq x_{i_3} \leq x_{i_2} \leq x_{i_1} \leq 1$$

for some permutation  $(i_1, i_2, i_3)$  of  $(1, 2, 3)$ . Hence

$$I^3 = \bigcup_{(i_1, i_2, i_3)} \sigma_{(i_1, i_2, i_3)}(Q^3) = \bigcup_{i=1}^6 \sigma_i(Q^3).$$

□

**Exercise 10.19.** Let  $J^2$  and  $J^3$  be as in Exercise 10.17 and Exercise 10.18. Define

$$\begin{aligned} B_{01}(u, v) &= (0, u, v), & B_{11}(u, v) &= (1, u, v), \\ B_{02}(u, v) &= (u, 0, v), & B_{12}(u, v) &= (u, 1, v), \\ B_{03}(u, v) &= (u, v, 0), & B_{13}(u, v) &= (u, v, 1). \end{aligned}$$

These are affine, and map  $\mathbb{R}^2$  into  $\mathbb{R}^3$ . Put  $\beta_{ri} = B_{ri}(J^2)$ , for  $r = 0, 1$ ,  $i = 1, 2, 3$ . Each  $\beta_{ri}$  is an affine-oriented 2-chain. (See Section 10.30.) Verify that

$$\partial J^3 = \sum_{i=1}^3 (-1)^i (\beta_{0i} - \beta_{1i}),$$

in agreement with Exercise 10.18.)

*Proof.*

(1) A direct calculation shows that

$$\begin{aligned}
B_{01}(\tau_1) - B_{11}(\tau_1) &= [\mathbf{0}, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3] - [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\
B_{02}(\tau_1) - B_{12}(\tau_1) &= [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_3] - [\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\
B_{03}(\tau_1) - B_{13}(\tau_1) &= [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] - [\mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\
B_{01}(\tau_2) - B_{11}(\tau_2) &= -[\mathbf{0}, \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3] + [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\
B_{02}(\tau_2) - B_{12}(\tau_2) &= -[\mathbf{0}, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_3] + [\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\
B_{03}(\tau_2) - B_{13}(\tau_2) &= -[\mathbf{0}, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3].
\end{aligned}$$

(2) To express the formula in (1) clearly, we define

$$\omega_{(i_1, i_2, i_3)} = [\mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] - [\mathbf{0}, \mathbf{e}_{i_2}, \mathbf{e}_{i_2} + \mathbf{e}_{i_3}],$$

and thus

$$\begin{aligned}
-(B_{01}(\tau_1) - B_{11}(\tau_1)) &= s(1, 2, 3)\omega_{(1, 2, 3)} \\
B_{02}(\tau_1) - B_{12}(\tau_1) &= s(2, 1, 3)\omega_{(2, 1, 3)} \\
-(B_{03}(\tau_1) - B_{13}(\tau_1)) &= s(3, 1, 2)\omega_{(3, 1, 2)} \\
-(B_{01}(\tau_2) - B_{11}(\tau_2)) &= s(1, 3, 2)\omega_{(1, 3, 2)} \\
B_{02}(\tau_2) - B_{12}(\tau_2) &= s(2, 3, 1)\omega_{(2, 3, 1)} \\
-(B_{03}(\tau_2) - B_{13}(\tau_2)) &= s(3, 2, 1)\omega_{(3, 2, 1)}.
\end{aligned}$$

(3) Note that

$$\begin{aligned}
\beta_{0i} - \beta_{1i} &= B_{0i}(J^2) - B_{1i}(J^2) \\
&= B_{0i}(\tau_1 + \tau_2) - B_{1i}(\tau_1 + \tau_2) \\
&= B_{0i}(\tau_1) + B_{0i}(\tau_2) - B_{1i}(\tau_1) - B_{1i}(\tau_2) \\
&= (B_{0i}(\tau_1) - B_{1i}(\tau_1)) + (B_{0i}(\tau_2) - B_{1i}(\tau_2)).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{i=1}^3 (-1)^i (\beta_{0i} - \beta_{1i}) \\
&= \sum_{i=1}^3 (-1)^i (B_{0i}(\tau_1) - B_{1i}(\tau_1)) + \sum_{i=1}^3 (-1)^i (B_{0i}(\tau_2) - B_{1i}(\tau_2)) \\
&= \sum_{(i_1, i_2, i_3)} s(i_1, i_2, i_3) \omega_{(i_1, i_2, i_3)} \\
&= \sum_{(i_1, i_2, i_3)} s(i_1, i_2, i_3) [\mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\
&\quad - \sum_{(i_1, i_2, i_3)} s(i_1, i_2, i_3) [\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}] \\
&= \partial J^3.
\end{aligned}$$



□

**Exercise 10.20.** *State conditions under which the formula*

$$\int_{\Phi} f d\omega = \int_{\partial\Phi} f\omega - \int_{\Phi} (df) \wedge \omega$$

*is valid, and show that it generalizes the formula for integration by parts. (Hint:  $d(f\omega) = (df) \wedge \omega + f d\omega$ .)*

*Proof.*

(1) *If*

- (a)  $\Phi$  is a  $k$ -chain of class  $\mathcal{C}''$  in an open set  $V \subseteq \mathbb{R}^m$ ,
- (b)  $\omega$  is a  $(k-1)$ -form of class  $\mathcal{C}'$  in  $V$ ,
- (c)  $f$  is a 0-form of class  $\mathcal{C}'$  in  $V$ ,

*then*

$$\int_{\Phi} f d\omega = \int_{\partial\Phi} f\omega - \int_{\Phi} (df) \wedge \omega$$

(2) Theorem 10.20(a) implies that

$$d(f\omega) = (df) \wedge \omega + f d\omega.$$

(3) The Stokes' theorem (Theorem 10.33) shows that

$$\int_{\Phi} d(f\omega) = \int_{\partial\Phi} f\omega.$$

Hence

$$\int_{\Phi} f d\omega = \int_{\Phi} d(f\omega) - \int_{\Phi} (df) \wedge \omega = \int_{\partial\Phi} f\omega - \int_{\Phi} (df) \wedge \omega.$$

(4) Define  $\Phi : Q^1 = [0, 1] \rightarrow [a, b]$  by

$$\Phi(\alpha) = a + \alpha(b - a).$$

$\Phi$  is a 1-simplex of class  $\mathcal{C}''$  in an open set  $V \supseteq [a, b]$ . Also,

$$\partial\Phi = [b] - [a].$$

Let  $\omega = g$  be a 0-form of class  $\mathcal{C}'(V)$ .

(5) Note that

$$\begin{aligned}\int_{\Phi} f d\omega &= \int_{\Phi} f dg = \int_0^1 f(\Phi(t))g'(\Phi(t))\Phi'(t)dt = \int_a^b f(u)g'(u)du, \\ \int_{\partial\Phi} f\omega &= \int_{[b]} fg + \int_{-[a]} fg = f(b)g(b) + (-1)f(a)f(a), \\ \int_{\Phi} (df) \wedge \omega &= \int_{\Phi} (df)g = \int_0^1 f'(\Phi(t))g(\Phi(t))\Phi'(t)dt = \int_a^b f'(u)g(u)du.\end{aligned}$$

Hence

$$\int_a^b f(u)g'(u)du = f(b)g(b) - f(a)f(a) - \int_a^b f'(u)g(u)du,$$

which is the same as the integration by parts (Theorem 6.22).

□

**Exercise 10.21.** *As in Example 10.36, consider the 1-form*

$$\eta = \frac{xdy - ydx}{x^2 + y^2}$$

*in  $\mathbb{R}^2 - \{\mathbf{0}\}$ .*

(a) *Carry out the computation that leads to*

$$\int_{\gamma} \eta = 2\pi \neq 0,$$

*and prove that  $d\eta = 0$ .*

(b) *Let  $\gamma(t) = (r \cos t, r \sin t)$ , for some  $r > 0$ , and let  $\Gamma$  be a  $\mathcal{C}''$ -curve in  $\mathbb{R}^2 - \{\mathbf{0}\}$ , with parameter interval  $[0, 2\pi]$ , with  $\Gamma(0) = \Gamma(2\pi)$ , such that the intervals  $[\gamma(t), \Gamma(t)]$  do not contain  $\mathbf{0}$  for any  $t \in [0, 2\pi]$ . Prove that*

$$\int_{\Gamma} \eta = 2\pi.$$

*(Hint: For  $0 \leq t \leq 2\pi$ ,  $0 \leq u \leq 1$ , define*

$$\Phi(t, u) = (1 - u)\Gamma(t) + u\gamma(t).$$

*Then  $\Phi$  is a 2-surface in  $\mathbb{R}^2 - \{\mathbf{0}\}$  whose parameter domain is the indicated rectangle. Because of cancellations (as in Example 10.32),*

$$\partial\Phi = \Gamma - \gamma.$$

*Use Stokes' theorem to deduce that*

$$\int_{\Gamma} \eta = \int_{\gamma} \eta$$

*because  $d\eta = 0$ .)*

- (c) Take  $\Gamma(t) = (a \cos t, b \sin t)$  where  $a > 0$ ,  $b > 0$  are fixed. Use part (b) to show that

$$\int_0^{2\pi} \frac{ab}{a^2 \cos^2 t + b^2 \sin^2 t} dt = 2\pi.$$

- (d) Show that

$$\eta = d \left( \arctan \frac{y}{x} \right)$$

in any convex open set in which  $x \neq 0$ , and that

$$\eta = d \left( -\arctan \frac{x}{y} \right)$$

in any convex open set in which  $y \neq 0$ . Explain why this justifies the notation  $\eta = d\theta$ , in spite of the fact that  $\eta$  is not exact in  $\mathbb{R}^2 - \{0\}$ .

- (e) Show that (b) can be derived from (d).

- (f) If  $\Gamma$  is any closed  $\mathcal{C}'$ -curve in  $\mathbb{R}^2 - \{0\}$ , prove that

$$\frac{1}{2\pi} \int_{\Gamma} \eta = \text{Ind}(\Gamma).$$

(See Exercise 8.23 for the definition of the index of a curve.)

*Proof of (a).*

(1)

$$\begin{aligned} \int_{\gamma} \eta &= \int_0^{2\pi} \frac{(r \cos t)d(r \sin t) - (r \sin t)d(r \cos t)}{(r \cos t)^2 + (r \sin t)^2} \\ &= \int_0^{2\pi} \frac{(r \cos t)(r \cos t) - (r \sin t)(-r \sin t)}{(r \cos t)^2 + (r \sin t)^2} dt \\ &= \int_0^{2\pi} dt \\ &= 2\pi. \end{aligned}$$

(2)

$$\begin{aligned}
d\eta &= d\left(\frac{xdy - ydx}{x^2 + y^2}\right) \\
&= d\left(\frac{x}{x^2 + y^2}\right) \wedge dy - d\left(\frac{y}{x^2 + y^2}\right) \wedge dx \quad (d^2 = 0) \\
&= D_1\left(\frac{x}{x^2 + y^2}\right) dx \wedge dy \quad (dy \wedge dy = 0) \\
&\quad - D_2\left(\frac{y}{x^2 + y^2}\right) dy \wedge dx \quad (dx \wedge dx = 0) \\
&= \left(\frac{1}{x^2 + y^2} - \frac{2x^2}{(x^2 + y^2)^2}\right) dx \wedge dy \\
&\quad + \left(\frac{1}{x^2 + y^2} - \frac{2y^2}{(x^2 + y^2)^2}\right) dx \wedge dy \\
&= 0
\end{aligned}$$

□

*Note.*

- (1)  $\eta$  is closed and locally exact, that is,  $\eta = dt$  on  $\mathbb{R}^2 - L$  where  $L$  is any line passing through  $\mathbf{0}$ .  $\eta$  is not exact since  $\int_\gamma \eta = 2\pi \neq 0$ . (See Exercise 10.22(g).)
- (2) (*Poincaré's Lemma for 1-form.*) Let  $\omega = \sum a_i dx_i$  be defined in an open set  $U \subseteq \mathbb{R}^n$ . Then  $d\omega = 0$  if and only if for each  $p \in U$  there is a neighborhood  $V \subseteq U$  of  $p$  and a differentiable function  $f : V \rightarrow \mathbb{R}^1$  with  $df = \omega$  (i.e.,  $\omega$  is locally exact).

*Proof of (b).*

- (1) For  $0 \leq t \leq 2\pi$ ,  $0 \leq u \leq 1$ , define

$$\Phi(t, u) = (1 - u)\Gamma(t) + u\gamma(t).$$

Then  $\Phi$  is a 2-surface in  $\mathbb{R}^2 - \{\mathbf{0}\}$  whose parameter domain  $D = \{(t, u) : 0 \leq t \leq 2\pi, 0 \leq u \leq 1\}$  is the indicated rectangle.

- (2) Similar to Example 10.32,

$$\partial\Phi = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$$

where

$$\begin{aligned}
\gamma_1(t) &= \Phi(t, 0) = \Gamma(t), \\
\gamma_2(u) &= \Phi(2\pi, u) = (1 - u)\Gamma(2\pi) + u\gamma(2\pi), \\
\gamma_3(t) &= \Phi(2\pi - t, 1) = \gamma(2\pi - t), \\
\gamma_4(u) &= \Phi(0, 1 - u) = u\Gamma(0) + (1 - u)\gamma(0).
\end{aligned}$$

Because of cancellations (as in Example 10.32),  $\gamma(0) = \gamma(2\pi)$  and  $\Gamma(0) = \Gamma(2\pi)$ ,  $\gamma_4 = -\gamma_2$  and  $\gamma_3 = -\gamma_1$ . Hence,

$$\partial\Phi = \Gamma - \gamma.$$

(3) The Stokes' theorem (Theorem 10.33) implies that

$$\int_{\Phi} d\eta = \int_{\partial\Phi} \eta = \int_{\Gamma-\gamma} \eta = \int_{\Gamma} \eta - \int_{\gamma} \eta.$$

Hence,

$$\int_{\Gamma} \eta = \int_{\gamma} \eta$$

(since  $d\eta = 0$  by (a)).

□

*Proof of (c).*

(1)  $\Gamma$  satisfies all conditions described in (b). So

$$\int_{\Gamma} \eta = 2\pi.$$

(2) A direct calculation shows that

$$\begin{aligned} 2\pi &= \int_{\Gamma} \eta = \int_{\Gamma} \frac{xdy - ydx}{x^2 + y^2} \\ &= \int_0^{2\pi} \frac{a \cos(t)d(b \sin(t)) - b \sin(t)d(a \cos(t))}{(a \cos(t))^2 + (b \sin(t))^2} \\ &= \int_0^{2\pi} \frac{ab(\cos^2 t + \sin^2 t)}{a^2 \cos^2 t + b^2 \sin^2 t} \\ &= \int_0^{2\pi} \frac{ab}{a^2 \cos^2 t + b^2 \sin^2 t}. \end{aligned}$$

□

*Proof of (d).*

(1) In any convex open set in which  $x \neq 0$ , we have

$$\begin{aligned} d\left(\arctan \frac{y}{x}\right) &= \left(D_1 \arctan \frac{y}{x}\right) dx + \left(D_2 \arctan \frac{y}{x}\right) dy \\ &= -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy \\ &= \eta. \end{aligned}$$

(2) In any convex open set in which  $y \neq 0$ , we have

$$\begin{aligned} d\left(-\arctan \frac{x}{y}\right) &= \left(D_1\left(-\arctan \frac{x}{y}\right)\right) dx + \left(D_2\left(-\arctan \frac{x}{y}\right)\right) dy \\ &= -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy \\ &= \eta. \end{aligned}$$

(3) By (1)(2),  $\eta$  is locally exact. Note that  $\theta_1 = \arctan \frac{y}{x}$  and  $\theta_2 = -\arctan \frac{x}{y}$  cannot be patched together to defined a global 0-form  $\theta$  on  $\mathbb{R}^2 - \{\mathbf{0}\}$ .

□

*Proof of (e).*

(1) Partition  $[0, 2\pi]$  into five subintervals

$$I_i = \left[ \frac{(2i-3)\pi}{4}, \frac{(2i-1)\pi}{4} \right] \cap [0, 2\pi].$$

for  $i = 1, 2, 3, 4, 5$ . Hence

$$\begin{aligned} \int_{\gamma} \eta &= \sum_{i=1}^5 \int_{\gamma(I_i)} \eta \\ &= \sum_{i=1,3,5} \int_{\gamma(I_i)} d\left(\arctan \frac{y}{x}\right) + \sum_{i=2,4} \int_{\gamma(I_i)} d\left(-\arctan \frac{x}{y}\right). \end{aligned}$$

(2) The Stokes' theorem (Theorem 10.33) implies that

$$\begin{aligned} \int_{\gamma(I_1)} d\left(\arctan \frac{y}{x}\right) &= \int_{\partial\gamma(I_1)} \arctan \frac{y}{x} \\ &= \left[ \arctan \frac{r \cos t}{r \sin t} \right]_{t=0}^{t=\frac{\pi}{4}} \\ &= [\arctan(\tan(t))]_{t=0}^{t=\frac{\pi}{4}} \\ &= \frac{\pi}{4}, \end{aligned}$$

and

$$\begin{aligned} \int_{\gamma(I_2)} d\left(-\arctan \frac{x}{y}\right) &= \int_{\partial\gamma(I_2)} -\arctan \frac{x}{y} \\ &= \left[ \arctan \frac{r \sin t}{r \cos t} \right]_{t=\frac{\pi}{4}}^{t=\frac{3\pi}{4}} \\ &= [\arctan(\cot(t))]_{t=\frac{\pi}{4}}^{t=\frac{3\pi}{4}} \\ &= \frac{\pi}{2}. \end{aligned}$$

Similarly,

$$\begin{aligned}\int_{\gamma(I_3)} d\left(\arctan \frac{y}{x}\right) &= \frac{\pi}{2} \\ \int_{\gamma(I_4)} d\left(-\arctan \frac{x}{y}\right) &= \frac{\pi}{2} \\ \int_{\gamma(I_5)} d\left(\arctan \frac{y}{x}\right) &= \frac{\pi}{4}.\end{aligned}$$

(3) Therefore,

$$\int_{\gamma} \eta = \left(\frac{\pi}{4} + \frac{\pi}{2} + \frac{\pi}{4}\right) + \left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 2\pi.$$

□

*Proof of (f).*

(1) Regard  $\Gamma(t)$  as a plane curve  $(\Gamma_1(t), \Gamma_2(t))$  over  $\mathbb{R}^2$  or  $\Gamma_1(t) + i\Gamma_2(t)$  over  $\mathbb{C}^1$ . Note that

$$\begin{aligned}\frac{\Gamma'(t)}{\Gamma(t)} &= \frac{\Gamma_1'(t) + i\Gamma_2'(t)}{\Gamma_1(t) + i\Gamma_2(t)} \\ &= \frac{\Gamma_1'(t)\Gamma_1(t) + \Gamma_2'(t)\Gamma_2(t)}{\Gamma_1(t)^2 + \Gamma_2(t)^2} + i\frac{\Gamma_1(t)\Gamma_2'(t) - \Gamma_2(t)\Gamma_1'(t)}{\Gamma_1(t)^2 + \Gamma_2(t)^2}.\end{aligned}$$

So

$$\operatorname{Im}\left(\frac{\Gamma'(t)}{\Gamma(t)}\right) = \frac{\Gamma_1(t)\Gamma_2'(t) - \Gamma_2(t)\Gamma_1'(t)}{\Gamma_1(t)^2 + \Gamma_2(t)^2}.$$

(2) By Exercise 8.23,

$$\operatorname{Ind}(\Gamma) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\Gamma'(t)}{\Gamma(t)} dt$$

is always an integer. That is,

$$\begin{aligned}\operatorname{Ind}(\Gamma) &= \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Im}\left(\frac{\Gamma'(t)}{\Gamma(t)}\right) dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{\Gamma_1(t)\Gamma_2'(t) - \Gamma_2(t)\Gamma_1'(t)}{\Gamma_1(t)^2 + \Gamma_2(t)^2} dt \\ &= \frac{1}{2\pi} \int_{\Gamma} \frac{xdy - ydx}{x^2 + y^2} \\ &= \frac{1}{2\pi} \int_{\Gamma} \eta.\end{aligned}$$

(Note that  $\operatorname{Ind}(\Gamma) = 1$  if  $\Gamma$  is defined as in (c). Hence the integral in (c) is equal to  $2\pi\operatorname{Ind}(\Gamma) = 2\pi$ .)

□

**Exercise 10.22.** As in Example 10.37, define  $\zeta$  in  $\mathbb{R}^3 - \{\mathbf{0}\}$  by

$$\zeta = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{r^3}$$

where  $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$ , let  $D$  be the rectangle given by  $0 \leq u \leq \pi$ ,  $0 \leq v \leq 2\pi$ , and let  $\Sigma$  be the 2-surface in  $\mathbb{R}^3$ , with parameter domain  $D$ , given by

$$x = \sin u \cos v, \quad y = \sin u \sin v, \quad z = \cos u.$$

- (a) Prove that  $d\zeta = 0$  in  $\mathbb{R}^3 - \{\mathbf{0}\}$ .
- (b) Let  $S$  denote the restriction of  $\Sigma$  to a parameter domain  $E \subseteq D$ . Prove that

$$\int_S \zeta = \int_E \sin u \, du \, dv = A(S),$$

where  $A$  denotes area, as in Section 10.46. Note that this contains

$$\int_{\Sigma} \zeta = \int_D \sin u \, du \, dv = 4\pi \neq 0$$

as a special case.

- (c) Suppose  $g, h_1, h_2, h_3$ , are  $\mathcal{C}''$ -functions on  $[0, 1]$ ,  $g > 0$ . Let  $(x, y, z) = \Phi(s, t)$  define a 2-surface  $\Phi$ , with parameter domain  $I^2$ , by

$$x = g(t)h_1(s), \quad y = g(t)h_2(s), \quad z = g(t)h_3(s).$$

Prove that

$$\int_{\Phi} \zeta = 0,$$

directly from Equation (35) in Chapter 10. Note the shape of the range of  $\Phi$ : For fixed  $s$ ,  $\Phi(s, t)$  runs over an interval on a line through  $\mathbf{0}$ . The range of  $\Phi$  thus lies in a “cone” with vertex at the origin.

- (d) Let  $E$  be a closed rectangle in  $D$ , with edges parallel to those of  $D$ . Suppose  $f \in \mathcal{C}''(D)$ ,  $f > 0$ . Let  $\Omega$  be the 2-surface with parameter domain  $E$ , defined by

$$\Omega(u, v) = f(u, v)\Sigma(u, v).$$

Define  $S$  as in (b) and prove that

$$\int_{\Omega} \zeta = \int_S \zeta = A(S).$$



(Since  $S$  is the “radical projection” of  $\Omega$  into the unit sphere, this result makes it reasonable to call  $\int_{\Omega} \zeta$  the “solid angle” subtended by the range of  $\Omega$  at the origin.) (Hint: Consider the 3-surface  $\Psi$  given by

$$\Psi(t, u, v) = [1 - t + tf(u, v)]\Sigma(u, v),$$

where  $(u, v) \in E$ ,  $0 \leq t \leq 1$ . For fixed  $v$ , the mapping  $(t, u) \mapsto \Psi(t, u, v)$  is a 2-surface  $\Phi$  to which (c) can be applied to show that  $\int_{\Phi} \zeta = 0$ . The same thing holds when  $u$  is fixed. By (a) and Stokes’ theorem,

$$\int_{\partial\Psi} \zeta = \int_{\Psi} d\zeta = 0.)$$

(e) Put  $\lambda = -\frac{z}{r}\eta$ , where

$$\eta = \frac{xdy - ydx}{x^2 + y^2},$$

as in Exercise 10.21. Then  $\lambda$  is a 1-form in the open set  $V \subseteq \mathbb{R}^3$  in which  $x^2 + y^2 > 0$ . Show that  $\zeta$  is exact in  $V$  by showing that

$$\zeta = d\lambda.$$

(f) Derive (d) from (e), without using (c). (Hint: To begin with, assume  $0 < u < \pi$  on  $E$ . By (e),

$$\int_{\Omega} \zeta = \int_{\partial\Omega} \lambda \quad \text{and} \quad \int_S \zeta = \int_{\partial S} \lambda.$$

Show that the two integrals of  $\lambda$  are equal, by using part (d) of Exercise 10.21, and by noting that  $\frac{z}{r}$  is the same at  $\Sigma(u, v)$  as at  $\Omega(u, v)$ .)

(g) Is  $\zeta$  exact in the complement of every line through the origin?

*Proof of (a).*

(1) Note that  $\zeta$  is well-defined on  $\mathbb{R}^3 - \{\mathbf{0}\}$ . Hence,

$$\begin{aligned} d\zeta &= d\left(\frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{r^3}\right) \\ &= d\left(\frac{x}{r^3}\right) \wedge dy \wedge dz + d\left(\frac{y}{r^3}\right) \wedge dz \wedge dx + d\left(\frac{z}{r^3}\right) \wedge dx \wedge dy \\ &= D_1\left(\frac{x}{r^3}\right) dx \wedge dy \wedge dz + D_2\left(\frac{y}{r^3}\right) dy \wedge dz \wedge dx + D_3\left(\frac{z}{r^3}\right) dz \wedge dx \wedge dy \\ &= \frac{r^3 - 3rx^2}{r^6} dx \wedge dy \wedge dz + \frac{r^3 - 3ry^2}{r^6} dy \wedge dz \wedge dx + \frac{r^3 - 3rz^2}{r^6} dz \wedge dx \wedge dy \\ &= \left(\frac{r^3 - 3rx^2}{r^6} + \frac{r^3 - 3ry^2}{r^6} + \frac{r^3 - 3rz^2}{r^6}\right) dx \wedge dy \wedge dz \\ &= 0dx \wedge dy \wedge dz \\ &= 0 \end{aligned}$$

in  $\mathbb{R}^3 - \{\mathbf{0}\}$ .

(2) Or write

$$\mathbf{F} = \frac{x}{r^3} \mathbf{e}_1 + \frac{y}{r^3} \mathbf{e}_2 + \frac{z}{r^3} \mathbf{e}_3$$

as in Vector fields 10.42. So

$$\omega_{\mathbf{F}} = \zeta$$

and

$$d\omega_{\mathbf{F}} = (\nabla \cdot \mathbf{F}) dx \wedge dy \wedge dz$$

as in the proof of the divergence theorem (Theorem 10.51). Note that the divergence of  $\mathbf{F}$  is zero.

□

*Proof of (b).*

(1) By Area elements in  $\mathbb{R}^3$  10.46,

$$\begin{aligned} \mathbf{N}(u, v) &= \frac{\partial(y, z)}{\partial(u, v)} \mathbf{e}_1 + \frac{\partial(z, x)}{\partial(u, v)} \mathbf{e}_2 + \frac{\partial(x, y)}{\partial(u, v)} \mathbf{e}_3 \\ &= (\sin^2 u \cos v) \mathbf{e}_1 + (\sin^2 u \sin v) \mathbf{e}_2 + (\sin u \cos u) \mathbf{e}_3. \end{aligned}$$

Here  $|\mathbf{N}(u, v)| = \sin u \geq 0$  (by noting that  $u \in [0, \pi]$ ), and

$$\mathbf{n}(u, v) = \frac{\mathbf{N}(u, v)}{|\mathbf{N}(u, v)|} = (\sin u \cos v, \sin u \sin v, \cos u).$$

(2) Note that  $\zeta = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$  on  $S \subseteq \Sigma$ . Hence, by Integrals of 2-forms in  $\mathbb{R}^3$  10.49,

$$\begin{aligned} \int_S \zeta &= \int_S xdy \wedge dz + ydz \wedge dx + zdx \wedge dy \\ &= \int_E (\sin u \cos v, \sin u \sin v, \cos u) \cdot \mathbf{N}(u, v) du dv \\ &= \int_E \mathbf{n}(u, v) \cdot \mathbf{n}(u, v) |\mathbf{N}(u, v)| du dv \\ &= \int_E |\mathbf{N}(u, v)| du dv \\ &= A(S). \end{aligned}$$

(3) In particular,

$$\begin{aligned}
\int_{\Sigma} \zeta &= \int_D \sin u \, du \, dv \\
&= \int_0^\pi \int_0^{2\pi} \sin u \, du \, dv \\
&= \left( \int_0^\pi \sin u \, du \right) \left( \int_0^{2\pi} dv \right) \\
&= 2 \cdot 2\pi \\
&= 4\pi.
\end{aligned}$$

□

*Proof of (c).*

(1) Similar to (b).

$$\begin{aligned}
\mathbf{N}(s, t) &= \frac{\partial(y, z)}{\partial(s, t)} \mathbf{e}_1 + \frac{\partial(z, x)}{\partial(s, t)} \mathbf{e}_2 + \frac{\partial(x, y)}{\partial(s, t)} \mathbf{e}_3 \\
&= g(t)g'(t)[(h_1(s), h_2(s), h_3(s)) \times (h'_1(s), h'_2(s), h'_3(s))] \\
&= g(t)g'(t)[\mathbf{h}(s) \times \mathbf{h}'(s)],
\end{aligned}$$

where  $\mathbf{h}(s) = (h_1(s), h_2(s), h_3(s))$  and  $\mathbf{h}'(s) = (h'_1(s), h'_2(s), h'_3(s))$ . (Here “ $\times$ ” is the cross product in  $\mathbb{R}^3$ .)

(2) Assume  $\zeta$  is well-defined, i.e.,  $\mathbf{h}(s) \neq \mathbf{0}$  for all  $s \in [0, 1]$ . By Integrals of 2-forms in  $\mathbb{R}^3$  10.49,

$$\begin{aligned}
\int_{\Phi} \zeta &= \int_{\Phi} \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{r^3} \\
&= \int_{I^2} \frac{g(t)}{g(t)^3 |\mathbf{h}(s)|^3} \mathbf{h}(s) \cdot \mathbf{N}(s, t) \, ds \, dt \\
&= \int_{I^2} \frac{g(t)}{g(t)^3 |\mathbf{h}(s)|^3} \mathbf{h}(s) \cdot g(t)g'(t)[\mathbf{h}(s) \times \mathbf{h}'(s)] \, ds \, dt \\
&= \int_{I^2} \frac{g'(t)}{g(t) |\mathbf{h}(s)|^3} \mathbf{h}(s) \cdot [\mathbf{h}(s) \times \mathbf{h}'(s)] \, ds \, dt \\
&= 0
\end{aligned}$$

(since  $\mathbf{h}(s) \cdot [\mathbf{h}(s) \times \mathbf{h}'(s)] = 0$ .)

(3) Note that  $\Sigma$  in spherical coordinate system cannot be parameterized as  $(x, y, z) = g(t)\mathbf{h}(s)$ , and thus  $\int_S \zeta$  could be nonzero as shown in (b).

□

*Proof of (d) (Hint).*

(1) Consider the 3-surface  $\Psi$  given by

$$\Psi(t, u, v) = [1 - t + tf(u, v)]\Sigma(u, v),$$

where  $(u, v) \in E$ ,  $0 \leq t \leq 1$ . Write

$$E = [a_1, b_1] \times [a_2, b_2] \subseteq D = [0, \pi] \times [0, 2\pi].$$

Note that  $\Psi(t, u, v) \subseteq \mathbb{R}^3 - \{\mathbf{0}\}$ . So the boundary of  $\Psi$  is

$$\begin{aligned} \partial\Psi &= \Psi(0, u, v) - \Psi(1, u, v) \\ &\quad + \Psi(t, a_1, v) - \Psi(t, b_1, v) \\ &\quad + \Psi(t, u, a_2) - \Psi(t, u, b_2) \\ &= S(u, v) - \Omega(u, v) \\ &\quad + \Psi|_{u=a_1}(t, v) - \Psi|_{u=b_1}(t, v) \\ &\quad + \Psi|_{v=a_2}(t, u) - \Psi|_{v=b_2}(t, u), \end{aligned}$$

where  $\Psi|_{u=u_0}(t, v) = \Psi(t, u_0, v)$  and  $\Psi|_{v=v_0}(t, u) = \Psi(t, u, v_0)$ .

(2) Show that

$$\int_{\Psi|_{v=v_0}} \zeta = 0$$

for any fixed  $v = v_0 \in [a_2, b_2]$ . Note that  $\zeta$  is well-defined on  $\Psi|_{v=v_0}$ . Write  $\Psi|_{v=v_0}(t, u) = (x, y, z) = (x(t, u), y(t, u), z(t, u))$ . By definition of  $\Psi$ , we have

$$\begin{aligned} x &= g(t, u) \sin u \cos v_0 \\ y &= g(t, u) \sin u \sin v_0 \\ z &= g(t, u) \cos u, \end{aligned}$$

where  $g(t, u) = 1 - t + tf(u, v_0)$ . Similar to (c),

$$\begin{aligned} \mathbf{N}(t, u) &= \frac{\partial(y, z)}{\partial(t, u)} \mathbf{e}_1 + \frac{\partial(z, x)}{\partial(t, u)} \mathbf{e}_2 + \frac{\partial(x, y)}{\partial(t, u)} \mathbf{e}_3 \\ &= g(t, u) D_1 g(t, u) (-\sin v_0, \cos v_0, 0). \end{aligned}$$

Note that

$$(x(t, u), y(t, u), z(t, u)) \cdot \mathbf{N}(t, u) = 0.$$

So

$$\begin{aligned} \int_{\Psi|_{v=v_0}} \zeta &= \int_{\Psi|_{v=v_0}} r^{-3} (x dy \wedge dz + y dz \wedge dx + z dx \wedge dy) \\ &= \int_{[0,1] \times [a_1, b_1]} r^{-3} (x(t, u), y(t, u), z(t, u)) \cdot \mathbf{N}(t, u) dt du \\ &= \int_{[0,1] \times [a_1, b_1]} 0 dt du \\ &= 0. \end{aligned}$$

(3) Show that

$$\int_{\Psi|u=u_0} \zeta = 0$$

for any fixed  $u = u_0 \in [a_1, b_1]$ . Similar to (2).

$$\begin{aligned} \mathbf{N}(t, v) &= \frac{\partial(y, z)}{\partial(t, v)} \mathbf{e}_1 + \frac{\partial(z, x)}{\partial(t, v)} \mathbf{e}_2 + \frac{\partial(x, y)}{\partial(t, v)} \mathbf{e}_3 \\ &= \sin u_0 g(t, v) D_1 g(t, v) (-\cos u_0 \cos v, -\cos u_0 \sin v, \sin u_0). \end{aligned}$$

where  $g(t, v) = 1 - t + tf(u_0, v)$ . So  $(x(t, v), y(t, v), z(t, v)) \cdot \mathbf{N}(t, v) = 0$  and thus  $\int_{\Psi|u=u_0} \zeta = 0$ .

(4) So

$$\begin{aligned} 0 &= \int_{\Psi} d\zeta && (d\zeta = 0 \text{ on } \mathbb{R}^3 - \{\mathbf{0}\}) \\ &= \int_{\partial\Psi} \zeta && (\text{Theorem 10.33}) \\ &= \int_S \zeta - \int_{\Omega} \zeta \\ &\quad + \underbrace{\int_{\Psi|u=a_1} \zeta - \int_{\Psi|u=b_1} \zeta}_{\text{all are zero by (2)}} \\ &\quad + \underbrace{\int_{\Psi|v=a_2} \zeta - \int_{\Psi|v=b_2} \zeta}_{\text{all are zero by (3)}} && ((1)) \\ &= \int_S \zeta - \int_{\Omega} \zeta. \end{aligned}$$

Hence

$$\int_{\Omega} \zeta = \underbrace{\int_S \zeta}_{\text{by (b)}} = A(S).$$

□

*Proof of (e).*

(1) Note that

$$d\left(-\frac{z}{r}\right) = \frac{xz}{r^3} dx + \frac{yz}{r^3} dy - \frac{r^2 - z^2}{r^3} dz = \frac{xz}{r^3} dx + \frac{yz}{r^3} dy - \frac{x^2 + y^2}{r^3} dz$$

since  $r^2 = x^2 + y^2 + z^2$ .

(2)

$$\begin{aligned}
d\lambda &= d\left(-\frac{z}{r}\eta\right) \\
&= \underbrace{d\left(-\frac{z}{r}\right)}_{\text{apply (1)}} \wedge \eta + (-1)^1 \left(-\frac{z}{r}\right) \wedge \underbrace{d\eta}_{=0} \\
&= \left(\frac{xz}{r^3}dx + \frac{yz}{r^3}dy - \frac{x^2+y^2}{r^3}dz\right) \wedge \left(\frac{-ydx + xdy}{x^2+y^2}\right) \\
&= \left(\frac{x(x^2+y^2)}{r^3(x^2+y^2)}\right) dy \wedge dz + \left(\frac{y(x^2+y^2)}{r^3(x^2+y^2)}\right) dz \wedge dx + \left(\frac{x^2z+y^2z}{r^3(x^2+y^2)}\right) dx \wedge dy \\
&= \left(\frac{x}{r^3}\right) dy \wedge dz + \left(\frac{y}{r^3}\right) dz \wedge dx + \left(\frac{z}{r^3}\right) dx \wedge dy \\
&= \zeta.
\end{aligned}$$

□

*Proof of (f).*

- (1) To ensure that  $\eta$  is well-defined on  $E$ , we might assume  $x^2+y^2 = \sin^2 u \neq 0$  or  $0 < u < \pi$  on  $E$ . It is fine since  $\int_{\Omega} \zeta$  and  $\int_S \zeta$  is well-defined on any closed rectangle in  $D$  and we can apply the argument in Exercise 6.7 to remove the additional restriction.
- (2) By the Stokes' theorem (Theorem 10.33) and (e),

$$\int_{\Omega} \zeta = \int_{\partial\Omega} \lambda \quad \text{and} \quad \int_S \zeta = \int_{\partial S} \lambda.$$

So it suffices to show that

$$\int_{\partial\Omega} \lambda = \int_{\partial S} \lambda.$$

Note that  $\lambda = -\frac{z}{r}\eta$ , and thus it suffices to show that  $\frac{z}{r}|_{\partial\Omega} = \frac{z}{r}|_{\partial S}$  and  $\eta|_{\partial\Omega} = \eta|_{\partial S}$ .

- (3) Show that  $\frac{z}{r}|_{\partial\Omega} = \frac{z}{r}|_{\partial S}$ . For any  $(x_{\Omega}, y_{\Omega}, z_{\Omega}) \in \partial\Omega$ ,  
 $(x_{\Omega}, y_{\Omega}, z_{\Omega}) = f(u, v)(x_{\Sigma}, y_{\Sigma}, z_{\Sigma})$

where  $(x_{\Sigma}, y_{\Sigma}, z_{\Sigma}) \in \partial S$ . So

$$\begin{aligned}
\frac{z}{r}|_{\partial\Omega} &= \frac{z_{\Omega}}{(x_{\Omega}^2 + y_{\Omega}^2 + z_{\Omega}^2)^{\frac{1}{2}}} \\
&= \frac{f(u, v)z_{\Sigma}}{f(u, v)(x_{\Sigma}^2 + y_{\Sigma}^2 + z_{\Sigma}^2)^{\frac{1}{2}}} \\
&= \frac{z_{\Sigma}}{(x_{\Sigma}^2 + y_{\Sigma}^2 + z_{\Sigma}^2)^{\frac{1}{2}}} \\
&= \frac{z}{r}|_{\partial S}.
\end{aligned}$$

(Note that  $f > 0$ .)

- (4) Show that  $\eta|_{\partial\Omega} = \eta|_{\partial S}$ . Similar to (3). If  $x_\Omega \neq 0$  (or  $x_\Sigma \neq 0$ ), then by Exercise 10.21(d)

$$\begin{aligned}\eta|_{\partial\Omega} &= d\left(\arctan \frac{y_\Omega}{x_\Omega}\right) \\ &= d\left(\arctan \frac{f(u,v)y_\Sigma}{f(u,v)x_\Sigma}\right) \\ &= d\left(\arctan \frac{y_\Sigma}{x_\Sigma}\right) \\ &= \eta|_{\partial S}.\end{aligned}$$

Similarly,  $\eta|_{\partial\Omega} = \eta|_{\partial S}$  is also true if  $y_\Omega \neq 0$ . Note that  $(x_\Omega, y_\Omega) \neq (0, 0)$  by assumption. Therefore the result is established.

□

*Proof of (g).*

- (1) Yes. Given any line  $L$  passing through  $\mathbf{0}$ , say

$$(r \sin u \cos v, r \sin u \sin v, r \cos u) \in L \quad (r \in \mathbb{R}^1),$$

for some  $u \in [0, \pi]$  and  $v \in [0, 2\pi]$ . We will show that  $\zeta$  is exact in  $U = \mathbb{R}^3 - L$ .

- (2) Linear algebra says that all rotation matrices  $T \in SO(3)$  can be obtained from

$$\begin{aligned}R_x(u) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos u & -\sin u \\ 0 & \sin u & \cos u \end{bmatrix} \\ R_y(v) &= \begin{bmatrix} \cos v & 0 & -\sin v \\ 0 & 1 & 0 \\ \sin v & 0 & \cos v \end{bmatrix} \\ R_z(w) &= \begin{bmatrix} \cos w & -\sin w & 0 \\ \sin w & \cos w & 0 \\ 0 & 0 & 1 \end{bmatrix}\end{aligned}$$

using matrix multiplication, say  $T = R_x(u)R_y(v)R_z(w)$ . For example, the rotation

$$T = R_y\left(u - \frac{\pi}{2}\right)R_z(-v)$$

maps  $L$  to the  $z$ -axis (by showing that  $T(r \sin u \cos v, r \sin u \sin v, r \cos u) = (0, 0, r)$ ). By Theorem 10.22 it suffices to show that  $\zeta$  is invariant under  $R_x(u)$ ,  $R_y(v)$  and  $R_z(w)$ . By the symmetricity of  $\zeta$ , it suffices to show that  $\zeta$  is invariant under  $T = R_x(u)$ .

(3) Show that  $\zeta$  is invariant under  $T = R_x(u)$ . By

$$T : (x, y, z) \mapsto (x, y \cos u - z \sin u, y \sin u + z \cos u),$$

we have

$$\begin{aligned} r &\mapsto r \\ dx &\mapsto dx \\ dy &\mapsto \cos u dy - \sin u dz \\ dz &\mapsto \sin u dy + \cos u dz. \end{aligned}$$

So

$$\begin{aligned} dy \wedge dz &\mapsto (\cos u dy - \sin u dz) \wedge (\sin u dy + \cos u dz) \\ &= dy \wedge dz, \\ dz \wedge dx &\mapsto (\sin u dy + \cos u dz) \wedge dx \\ &= -\sin u dx \wedge dy + \cos u dz \wedge dx, \\ dx \wedge dy &\mapsto dx \wedge (\sin u dy + \cos u dz) \\ &= \cos u dx \wedge dy + \sin u dz \wedge dx. \end{aligned}$$

Thus

$$\begin{aligned} \zeta &\mapsto r^{-3} \{ xdy \wedge dz \\ &\quad + (y \cos u - z \sin u)(-\sin u dx \wedge dy + \cos u dz \wedge dx) \\ &\quad + (y \sin u + z \cos u)(\cos u dx \wedge dy + \sin u dz \wedge dx) \} \\ &= r^{-3} \{ xdy \wedge dz \\ &\quad + [\cos u(y \cos u - z \sin u) + \sin u(y \sin u + z \cos u)] dz \wedge dx \\ &\quad + [-\sin u(y \cos u - z \sin u) + \cos u(y \sin u + z \cos u)] dx \wedge dy \} \\ &= r^{-3} \{ xdy \wedge dz + ydz \wedge dx + zdx \wedge dy \} \\ &= \zeta. \end{aligned}$$

(4) Let  $V = \mathbb{R}^3 - z\text{-axis}$ . Since  $\zeta_T = \zeta$  (by (3)) is well-defined in  $V$ ,  $\zeta_T = \zeta = d\lambda$  by (e). Here  $\lambda$  is in  $V$ , not necessary in  $U$  (if  $L \neq z\text{-axis}$ ). Luckily, we can use  $T^{-1}$  to pullback  $\lambda$  in  $U$ . Thus

$$\zeta = (\zeta_T)_{T^{-1}} = (d\lambda)_{T^{-1}} = d(\lambda_{T^{-1}})$$

by Theorems 10.22 and 10.23. That is,  $\zeta$  is exact in  $U = \mathbb{R}^3 - L$ . (Or  $\zeta$  is locally exact in  $\mathbb{R}^3 - \{\mathbf{0}\}$ .)

□



**Exercise 10.23.** Fix  $n$ . Define  $r_k = (x_1^2 + \cdots + x_k^2)^{\frac{1}{2}}$  for  $1 \leq k \leq n$ , let  $E_k$  be the set of all  $\mathbf{x} \in \mathbb{R}^n$  at which  $r_k > 0$ , and let  $\omega_k$  be the  $(k-1)$ -form defined in  $E_k$  by

$$\omega_k = (r_k)^{-k} \sum_{i=1}^k (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k$$

Note that  $\omega_2 = \eta$ ,  $\omega_3 = \zeta$  in the terminology of Exercise 10.21 and Exercise 10.22. Note also that

$$E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n = \mathbb{R}^n.$$

- (a) Prove that  $d\omega_k = 0$  in  $E_k$ .  
(b) For  $k = 2, \dots, n$ , prove that  $\omega_k$  is exact in  $E_{k-1}$ , by showing that

$$\omega_k = d(f_k \omega_{k-1}) = df_k \wedge \omega_{k-1}$$

where  $f_k(\mathbf{x}) = (-1)^k g_k\left(\frac{x_k}{r_k}\right)$  where

$$g_k(t) = \int_{-1}^t (1-s^2)^{\frac{k-3}{2}} ds \quad (-1 < t < 1).$$

(Hint:  $f_k$  satisfies the differential equations

$$\mathbf{x} \cdot (\nabla f_k)(\mathbf{x}) = 0$$

and

$$(D_k f_k)(\mathbf{x}) = \frac{(-1)^k (r_{k-1})^{k-1}}{(r_k)^k}.)$$

- (c) Is  $\omega_n$  exact in  $E_n$ ?  
(d) Note that (b) is a generalization of part (e) of Exercise 10.22. Try to extend some of the other assertions of Exercise 10.21 and Exercise 10.22 to  $\omega_n$ , for arbitrary  $n$ .

*Proof of (a).*

- (1) Note that

$$D_i r_k = \frac{1}{2r_k} \cdot (2x_i) = \frac{x_i}{r_k}.$$

(2)

$$\begin{aligned}
d\omega_k &= \sum_{i=1}^k d \left( (-1)^{i-1} (r_k)^{-k} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k \right) \\
&= \sum_{i=1}^k D_i \left( (-1)^{i-1} (r_k)^{-k} x_i \right) dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k \\
&= \sum_{i=1}^k (-1)^{i-1} \left( (r_k)^{-k} \cdot 1 + \underbrace{(-k)(r_k)^{-k-1} \frac{x_i}{r_k}}_{\text{chain rule}} \cdot x_i \right) \underbrace{(-1)^{i-1} dx_1 \wedge \cdots \wedge dx_k}_{\text{anticommutative relation}} \\
&= (r_k)^{-k-2} \underbrace{\sum_{i=1}^k ((r_k)^2 - kx_i^2)}_{=0} dx_1 \wedge \cdots \wedge dx_k \\
&= 0.
\end{aligned}$$

□

*Proof of (b).*

(1) Note that

$$D_i \left( \frac{x_k}{r_k} \right) = \frac{\delta_{ik}(r_k)^2 - x_i x_k}{(r_k)^3}$$

where  $\delta_{ik}$  is the Kronecker delta. So

$$\begin{aligned}
(D_i f_k)(\mathbf{x}) &= D_i \left( (-1)^k g_k \left( \frac{x_k}{r_k} \right) \right) \\
&= D_i \left( (-1)^k \int_{-1}^{\frac{x_k}{r_k}} (1-s^2)^{\frac{k-3}{2}} ds \right) \\
&= (-1)^k D_i \left( \frac{x_k}{r_k} \right) \left( 1 - \left( \frac{x_k}{r_k} \right)^2 \right)^{\frac{k-3}{2}} \\
&= (-1)^k \frac{\delta_{ik}(r_k)^2 - x_i x_k}{(r_k)^3} \frac{(r_{k-1})^{k-3}}{(r_k)^{k-3}} \\
&= (-1)^k \frac{(r_{k-1})^{k-3}}{(r_k)^k} (\delta_{ik}(r_k)^2 - x_i x_k).
\end{aligned}$$

In particular,

$$(D_k f_k)(\mathbf{x}) = (-1)^k \frac{(r_{k-1})^{k-3}}{(r_k)^k} ((r_k)^2 - (x_k)^2) = (-1)^k \frac{(r_{k-1})^{k-1}}{(r_k)^k}$$

(since  $(r_k)^2 - (x_k)^2 = (r_{k-1})^2$ ).

(2) Since

$$\sum_i x_i (\delta_{ik} (r_k)^2 - x_i x_k) = (r_k)^2 \underbrace{\sum_i x_i \delta_{ik}}_{=x_k} - x_k \underbrace{\sum_i x_i^2}_{=(r_k)^2} = 0,$$

we have

$$\begin{aligned} \mathbf{x} \cdot (\nabla f_k)(\mathbf{x}) &= \sum_i x_i (D_i f_k)(\mathbf{x}) \\ &= \sum_i x_i (-1)^k \frac{(r_{k-1})^{k-3}}{(r_k)^k} (\delta_{ik} (r_k)^2 - x_i x_k) \\ &= (-1)^k \frac{(r_{k-1})^{k-3}}{(r_k)^k} \sum_i x_i (\delta_{ik} (r_k)^2 - x_i x_k) \\ &= 0. \end{aligned}$$

(3) On  $E_{k-1} \subsetneq E_k$ , we write

$$\begin{aligned} & d(f_k \omega_{k-1}) \\ &= (df_k) \wedge \omega_{k-1} + (-1)^0 f_k \wedge \underbrace{(d\omega_{k-1})}_{=0} \\ &= (df_k) \wedge \omega_{k-1} \\ &= \left\{ \sum_{i=1}^k D_i f_k(\mathbf{x}) dx_i \right\} \wedge \\ & \quad \left\{ \frac{1}{(r_{k-1})^{k-1}} \sum_{j=1}^{k-1} (-1)^{j-1} x_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{k-1} \right\} \\ &= \frac{1}{(r_{k-1})^{k-1}} \sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq k-1}} (-1)^{j-1} x_j D_i f_k(\mathbf{x}) dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{k-1} \\ &= \frac{1}{(r_{k-1})^{k-1}} \sum_{j=1}^{k-1} (-1)^{j-1} x_j D_j f_k(\mathbf{x}) dx_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{k-1} \\ & \quad + \frac{1}{(r_{k-1})^{k-1}} \sum_{j=1}^{k-1} (-1)^{j-1} x_j D_k f_k(\mathbf{x}) dx_k \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{k-1}. \end{aligned}$$

(4) By (2),

$$\begin{aligned}
& \frac{1}{(r_{k-1})^{k-1}} \sum_{j=1}^{k-1} (-1)^{j-1} x_j D_j f_k(\mathbf{x}) dx_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{k-1} \\
&= \frac{1}{(r_{k-1})^{k-1}} \sum_{j=1}^{k-1} x_j D_j f_k(\mathbf{x}) dx_1 \wedge \cdots \wedge dx_{k-1} \\
&= \frac{1}{(r_{k-1})^{k-1}} (-D_k f_k(x) x_k) dx_1 \wedge \cdots \wedge dx_{k-1} \\
&= \frac{-D_k f_k(\mathbf{x})}{(r_{k-1})^{k-1}} x_k dx_1 \wedge \cdots \wedge dx_{k-1} \wedge \widehat{dx_k} \\
&= (r_k)^{-k} (-1)^{k-1} x_k dx_1 \wedge \cdots \wedge dx_{k-1} \wedge \widehat{dx_k} \tag{(1)}.
\end{aligned}$$

Also,

$$\begin{aligned}
& \frac{1}{(r_{k-1})^{k-1}} \sum_{j=1}^{k-1} (-1)^{j-1} x_j D_k f_k(\mathbf{x}) dx_k \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{k-1} \\
&= \frac{(-1)^k D_k f_k(\mathbf{x})}{(r_{k-1})^{k-1}} \sum_{j=1}^{k-1} (-1)^{j-1} x_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_k \\
&= (r_k)^{-k} \sum_{j=1}^{k-1} (-1)^{j-1} x_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_k \tag{(1)}.
\end{aligned}$$

(5) Hence,

$$\begin{aligned}
& d(f_k \omega_{k-1}) \\
&= (r_k)^{-k} (-1)^{k-1} x_k dx_1 \wedge \cdots \wedge dx_{k-1} \wedge \widehat{dx_k} \\
&\quad + (r_k)^{-k} \sum_{j=1}^{k-1} (-1)^{j-1} x_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_k \\
&= (r_k)^{-k} \sum_{j=1}^k (-1)^{j-1} x_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_k \\
&= \omega_k.
\end{aligned}$$

□

*Proof of (c).*

(1)  $\omega_n$  is not exact in  $E_n$  (though it is locally exact).

(2) Let

$$\begin{aligned}
\mathbb{S}^{n-1} &= \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\} \\
\mathbb{B}^n &= \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leq 1\}.
\end{aligned}$$

It suffices to show that

$$\int_{\mathbb{S}^{n-1}} \omega_n = \frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)} \neq 0.$$

Therefore,  $\omega_n$  is not exact in  $E_n$ .

(3) Define

$$\omega = \frac{1}{n} \sum_{i=1}^n (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

on  $\mathbb{S}^{n-1}$ . Note that

$$\omega = \frac{1}{n} \omega_n$$

on  $\mathbb{S}^{n-1}$  (and that's why we pick  $\mathbb{S}^{n-1}$ ). The Stokes' theorem (Theorem 10.33) implies that

$$\int_{\mathbb{S}^{n-1}} \frac{1}{n} \omega_n = \int_{\partial \mathbb{B}^n} \omega = \int_{\mathbb{B}^n} d\omega = \int_{\mathbb{B}^n} dx_1 \wedge \cdots \wedge dx_n = \text{vol}(\mathbb{B}^n),$$

where  $\text{vol}(\mathbb{B}^n)$  is the volume of  $\mathbb{B}^n$ . Thus *it suffices to show that*

$$\text{vol}(\mathbb{B}^n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)} \neq 0.$$

There are many proofs for this. We give a direct integration in spherical coordinates.

(4) Similar to Exercise 10.9. The spherical coordinate system has a radial coordinate  $r$  and angular coordinates  $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_{n-1})$ , where the domain of each  $\varphi_1, \dots, \varphi_{n-2}$  is  $[0, \pi]$  and the domain of  $\varphi_{n-1}$  is  $[0, 2\pi]$ . That is,

$$\begin{aligned} x_1 &= \cos \varphi_1 \\ x_2 &= \sin \varphi_1 \cos \varphi_2 \\ x_3 &= \sin \varphi_1 \sin \varphi_2 \cos \varphi_3 \\ &\dots \\ x_{n-1} &= \sin \varphi_1 \cdots \sin \varphi_{n-2} \cos \varphi_{n-1} \\ x_n &= \sin \varphi_1 \cdots \sin \varphi_{n-2} \sin \varphi_{n-1}. \end{aligned}$$

(It is different from Exercise 10.22.) The spherical volume element is

$$r^{n-1} \sin^{n-2} \varphi_1 \sin^{n-3} \varphi_2 \cdots \sin \varphi_{n-2} dr d\boldsymbol{\varphi}.$$

Thus by Some consequences 8.21,

$$\begin{aligned}
\text{vol}(\mathbb{B}^n) &= \int_{\mathbb{B}^n} d\mathbf{x} \\
&= \int_0^1 \int_0^\pi \cdots \int_0^{2\pi} r^{n-1} \sin^{n-2} \varphi_1 \cdots \sin \varphi_{n-2} dr d\boldsymbol{\varphi} \\
&= \left( \int_0^1 r^{n-1} dr \right) \left( \int_0^\pi \sin^{n-2} \varphi_1 d\varphi_1 \right) \cdots \left( \int_0^{2\pi} d\varphi_{n-1} \right) \\
&= \frac{1}{n} \cdot \frac{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2})} \cdot \frac{\Gamma(\frac{n-2}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n-1}{2})} \cdots \frac{\Gamma(1)\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})} \cdot 2\pi \\
&= \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)}.
\end{aligned}$$

(Use the similar argument in (d)(ii) to get the spherical volume element.)

- (5) Note that we can apply the spherical coordinate system to  $\int_{\mathbb{S}^{n-1}} \omega_n$  directly (without the Stokes' theorem). The area element is

$$\sin^{n-2} \varphi_1 \sin^{n-3} \varphi_2 \cdots \sin \varphi_{n-2} d\boldsymbol{\varphi}.$$

A long calculation shows that

$$\begin{aligned}
&\int_{\mathbb{S}^{n-1}} \omega_n \\
&= \int_0^\pi \cdots \int_0^{2\pi} \sin^{n-2} \varphi_1 \sin^{n-3} \varphi_2 \cdots \sin \varphi_{n-2} d\boldsymbol{\varphi} \\
&= \left( \int_0^\pi \sin^{n-2} \varphi_1 d\varphi_1 \right) \cdots \left( \int_0^{2\pi} d\varphi_{n-1} \right) \\
&= \frac{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2})} \cdot \frac{\Gamma(\frac{n-2}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n-1}{2})} \cdots \frac{\Gamma(1)\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})} \cdot 2\pi \\
&= \frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)}.
\end{aligned}$$

(See (d)(ii) for more details.)

□

*Outline of (d).*

- (i) One generalization of Exercise 10.21(a) and 10.22(a). See Exercise 10.23(a).
- (ii) One generalization of Exercise 10.22(b). Let  $\Sigma = \mathbb{S}^{n-1}$  be the  $(n-1)$ -surface in  $\mathbb{R}^n$ , with parameter domain  $D = [0, \pi]^{n-2} \times [0, 2\pi]$ , given

by

$$\begin{aligned}
x_1 &= \cos \varphi_1 \\
x_2 &= \sin \varphi_1 \cos \varphi_2 \\
x_3 &= \sin \varphi_1 \sin \varphi_2 \cos \varphi_3 \\
&\dots \\
x_{n-1} &= \sin \varphi_1 \cdots \sin \varphi_{n-2} \cos \varphi_{n-1} \\
x_n &= \sin \varphi_1 \cdots \sin \varphi_{n-2} \sin \varphi_{n-1}.
\end{aligned}$$

Let  $S$  denote the restriction of  $\Sigma$  to a parameter domain  $E \subseteq D$ . Prove that

$$\begin{aligned}
\int_S \omega_n &= \int_E \sin^{n-2} \varphi_1 \sin^{n-3} \varphi_2 \cdots \sin \varphi_{n-2} d\boldsymbol{\varphi} \\
&= A(S),
\end{aligned}$$

where  $A$  denotes surface area.

- (iii) One generalization of Exercise 10.22(c). Suppose  $g \in \mathcal{C}''([0, 1])$ ,  $\mathbf{h} = (h_1, \dots, h_n) \in \mathcal{C}''([0, 1]^{n-2})$ , and  $g > 0$ . Write  $\mathbf{x} = (x_1, \dots, x_n)$  and  $\mathbf{s} = (s_1, \dots, s_{n-2})$ . Let

$$\mathbf{x} = \Phi(\mathbf{s}, t)$$

define a  $(n-1)$ -surface  $\Phi$ , with parameter domain  $[0, 1]^{n-1}$ , by

$$\mathbf{x} = g(t)\mathbf{h}(\mathbf{s}).$$

Prove that

$$\int_{\Phi} \omega_n = 0.$$

- (iv) One generalization of Exercise 10.21(b) and 10.22(d). Let  $E$  be a closed cell in  $D$ , with edges parallel to those of  $D$ . Suppose  $f \in \mathcal{C}''(D)$ ,  $f > 0$ . Let  $\Omega$  be the  $(n-1)$ -surface with parameter domain  $E$ , defined by

$$\Omega(\boldsymbol{\varphi}) = f(\boldsymbol{\varphi})\Sigma(\boldsymbol{\varphi}).$$

Define  $S$  as in (ii) and prove that

$$\int_{\Omega} \omega_n = \int_S \omega_n = A(S).$$

- (v) One generalization of Exercise 10.21(d) and 10.22(e). See Exercise 10.23(b).  
(vi) One generalization of Examples 10.36 and 10.37. See Exercise 10.23(c).  
(vii) One generalization of Exercise 10.21(e) and 10.22(f). Derive (iv) from Exercise 10.23(b), without using (iii).

- (viii) One generalization of Exercise 10.21(f).  $\pi_{n-1}(\mathbb{S}^{n-1}) = \mathbb{Z}$  (without proof).
- (ix) One generalization of Exercise 10.22(g). *Show that  $\omega_n$  is exact in the complement of every line  $L$  passing through the origin.*

*Proof of (d)(ii).*

- (1) On  $S \subseteq \mathbb{S}^{n-1}$ , we have

$$\begin{aligned}
\int_S \omega_n &= \int_S \sum_{i=1}^n (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\
&= \int_S \sum_{i=1}^n (-1)^{i-1} x_i(\varphi) \frac{\partial(x_1, \dots, \widehat{x_i}, \dots, x_n)}{\partial(\varphi_1, \dots, \varphi_{n-1})} d\varphi_1 \wedge \cdots \wedge d\varphi_{n-1} \\
&= \int_S \sum_{i=1}^n (-1)^{i-1} x_i(\varphi) \det \begin{bmatrix} \frac{\partial x_1}{\partial \varphi_1} & \cdots & \frac{\partial x_1}{\partial \varphi_{n-1}} \\ \vdots & \ddots & \vdots \\ \widehat{\frac{\partial x_i}{\partial \varphi_1}} & \cdots & \widehat{\frac{\partial x_i}{\partial \varphi_{n-1}}} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial \varphi_1} & \cdots & \frac{\partial x_n}{\partial \varphi_{n-1}} \end{bmatrix} d\varphi_1 \wedge \cdots \wedge d\varphi_{n-1} \\
&= \int_S \det \underbrace{\begin{bmatrix} x_1 & \frac{\partial x_1}{\partial \varphi_1} & \cdots & \frac{\partial x_1}{\partial \varphi_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ x_i & \frac{\partial x_i}{\partial \varphi_1} & \cdots & \frac{\partial x_i}{\partial \varphi_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ x_n & \frac{\partial x_n}{\partial \varphi_1} & \cdots & \frac{\partial x_n}{\partial \varphi_{n-1}} \end{bmatrix}}_{\text{say } A_n} d\varphi_1 \wedge \cdots \wedge d\varphi_{n-1}.
\end{aligned}$$

Hence, it suffices to show that

$$\det(A_n) = \sin^{n-2} \varphi_1 \sin^{n-3} \varphi_2 \cdots \sin \varphi_{n-2}.$$

- (2) *Show that  $\det(A_n) = \sin^{n-2} \varphi_1 \sin^{n-3} \varphi_2 \cdots \sin \varphi_{n-2}$ . Induction on  $n$ .*

- (a) When  $n = 3$ , a straightforward computation shows that the determinant is

$$\begin{aligned}
&\det(A_3) \\
&= \det \begin{bmatrix} \cos \varphi_1 & -\sin \varphi_1 & 0 \\ \sin \varphi_1 \cos \varphi_2 & \cos \varphi_1 \cos \varphi_2 & -\sin \varphi_1 \sin \varphi_2 \\ \sin \varphi_1 \sin \varphi_2 & \cos \varphi_1 \sin \varphi_2 & \sin \varphi_1 \cos \varphi_2 \end{bmatrix} \\
&= \sin \varphi_1.
\end{aligned}$$



(b) When  $n = 4$ ,

$$\begin{aligned} & \det(A_4) \\ &= \det \begin{bmatrix} \cos \varphi_1 & -\sin \varphi_1 & 0 & 0 \\ \sin \varphi_1 \cos \varphi_2 & \cos \varphi_1 \cos \varphi_2 & -\sin \varphi_1 \sin \varphi_2 & 0 \\ \sin \varphi_1 \sin \varphi_2 \cos \varphi_3 & \cos \varphi_1 \sin \varphi_2 \cos \varphi_3 & \sin \varphi_1 \cos \varphi_2 \cos \varphi_3 & -\sin \varphi_1 \sin \varphi_2 \sin \varphi_3 \\ \sin \varphi_1 \sin \varphi_2 \sin \varphi_3 & \cos \varphi_1 \sin \varphi_2 \sin \varphi_3 & \sin \varphi_1 \cos \varphi_2 \sin \varphi_3 & \sin \varphi_1 \sin \varphi_2 \cos \varphi_3 \end{bmatrix}. \end{aligned}$$

Expand along the last column to get

$$\begin{aligned} & \det(A_4) \\ &= (-1)^{3+4}(-\sin \varphi_1 \sin \varphi_2 \sin \varphi_3) \\ & \quad \det \begin{bmatrix} \cos \varphi_1 & -\sin \varphi_1 & 0 \\ \sin \varphi_1 \cos \varphi_2 & \cos \varphi_1 \cos \varphi_2 & -\sin \varphi_1 \sin \varphi_2 \\ \sin \varphi_1 \sin \varphi_2 \sin \varphi_3 & \cos \varphi_1 \sin \varphi_2 \sin \varphi_3 & \sin \varphi_1 \cos \varphi_2 \sin \varphi_3 \end{bmatrix} \\ & \quad + (-1)^{4+4}(\sin \varphi_1 \sin \varphi_2 \cos \varphi_3) \\ & \quad \det \begin{bmatrix} \cos \varphi_1 & -\sin \varphi_1 & 0 \\ \sin \varphi_1 \cos \varphi_2 & \cos \varphi_1 \cos \varphi_2 & -\sin \varphi_1 \sin \varphi_2 \\ \sin \varphi_1 \sin \varphi_2 \cos \varphi_3 & \cos \varphi_1 \sin \varphi_2 \cos \varphi_3 & \sin \varphi_1 \cos \varphi_2 \cos \varphi_3 \end{bmatrix} \\ &= (\sin \varphi_1 \sin \varphi_2 \sin^2 \varphi_3) \det(A_3) + (\sin \varphi_1 \sin \varphi_2 \cos^2 \varphi_3) \det(A_3) \\ &= \sin \varphi_1 \sin \varphi_2 \det(A_3) \\ &= \sin^2 \varphi_1 \sin \varphi_2. \end{aligned}$$

(c) Now for large  $n$ , as (b) we expand along the last column to get

$$\begin{aligned} & \det(A_n) \\ &= (-1)^{(n-1)+n}(-\sin \varphi_1 \cdots \sin \varphi_{n-2} \sin \varphi_{n-1})(\sin \varphi_{n-1} \det(A_{n-1})) \\ & \quad + (-1)^{n+n}(\sin \varphi_1 \cdots \sin \varphi_{n-2} \cos \varphi_{n-1})(\cos \varphi_{n-1} \det(A_{n-1})) \\ &= (\sin \varphi_1 \cdots \sin \varphi_{n-2}) \det(A_{n-1}) \\ &= (\sin \varphi_1 \cdots \sin \varphi_{n-2})(\sin^{n-3} \varphi_1 \sin^{n-4} \varphi_2 \cdots \sin \varphi_{n-3}) \\ &= \sin^{n-2} \varphi_1 \sin^{n-3} \varphi_2 \cdots \sin^2 \varphi_{n-3} \sin \varphi_{n-2}. \end{aligned}$$

(3) (Area elements in  $\mathbb{R}^3$  10.46.) Given any  $\mathbf{x} = (x_1, \dots, x_n) \in S$ . Define the vector  $\mathbf{N}(\boldsymbol{\varphi})$  by

$$\mathbf{N}(\boldsymbol{\varphi}) = \sum_{i=1}^n \frac{\partial(x_1, \dots, \widehat{x_i}, \dots, x_n)}{\partial(\varphi_1, \dots, \varphi_{n-1})} \mathbf{e}_i.$$

So the area of  $S$  is defined by

$$A(S) = \int_E |\mathbf{N}(\boldsymbol{\varphi})| d\boldsymbol{\varphi}.$$

(4) By the similar proof in (2),

$$\begin{aligned} & \frac{\partial(x_1, \dots, \widehat{x_i}, \dots, x_n)}{\partial(\varphi_1, \dots, \varphi_{n-1})} \\ &= (-1)^{i-1} \sin^{n-2} \varphi_1 \sin^{n-3} \varphi_2 \cdots \sin^2 \varphi_{n-3} \sin \varphi_{n-2} x_i \end{aligned}$$

if  $i = 1, \dots, n$ . Since  $\sum x_i^2 = 1$  on  $S$ ,

$$|\mathbf{N}(\varphi)| = \sin^{n-2} \varphi_1 \sin^{n-3} \varphi_2 \cdots \sin^2 \varphi_{n-3} \sin \varphi_{n-2}.$$

Thus,

$$\begin{aligned} A(S) &= \int_E |\mathbf{N}(\varphi)| d\varphi \\ &= \int_E \sin^{n-2} \varphi_1 \sin^{n-3} \varphi_2 \cdots \sin^2 \varphi_{n-3} \sin \varphi_{n-2} d\varphi. \end{aligned}$$

(5) Note that we can apply (3) on (2) to get the same conclusion.

$$\begin{aligned} \int_S \omega_n &= \int_S \sum_{i=1}^n (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\ &= \int_S \sum_{i=1}^n (-1)^{i-1} x_i \frac{\partial(x_1, \dots, \widehat{x_i}, \dots, x_n)}{\partial(\varphi_1, \dots, \varphi_{n-1})} d\varphi_1 \wedge \cdots \wedge d\varphi_{n-1} \\ &= \int_E \sum_{i=1}^n (-1)^{i-1} x_i \\ &\quad (-1)^{i-1} \sin^{n-2} \varphi_1 \sin^{n-3} \varphi_2 \cdots \sin^2 \varphi_{n-3} \sin \varphi_{n-2} x_i d\varphi \\ &= \int_E \sin^{n-2} \varphi_1 \sin^{n-3} \varphi_2 \cdots \sin^2 \varphi_{n-3} \sin \varphi_{n-2} d\varphi. \end{aligned}$$

□

*Proof of (d)(iii).*

(1) Similar to Exercise 10.22(c). Assume that  $\omega_n$  is well-defined, i.e.,  $\mathbf{h}(\mathbf{s}) \neq 0$

for all  $\mathbf{s} \in [0, 1]^{n-2}$ .

$$\begin{aligned}
\frac{\partial(x_1, \dots, \widehat{x_i}, \dots, x_n)}{\partial(s_1, \dots, s_{n-2}, t)} &= \det \begin{bmatrix} \frac{\partial x_1}{\partial s_1} & \dots & \frac{\partial x_1}{\partial s_{n-2}} & \frac{\partial x_1}{\partial t} \\ \vdots & \ddots & \vdots & \vdots \\ \widehat{\frac{\partial x_i}{\partial s_1}} & \dots & \widehat{\frac{\partial x_i}{\partial s_{n-2}}} & \widehat{\frac{\partial x_i}{\partial t}} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial x_n}{\partial s_1} & \dots & \frac{\partial x_n}{\partial s_{n-2}} & \frac{\partial x_n}{\partial t} \end{bmatrix} \\
&= \det \begin{bmatrix} g \frac{\partial h_1}{\partial s_1} & \dots & g \frac{\partial h_1}{\partial s_{n-2}} & g' h_1 \\ \vdots & \ddots & \vdots & \vdots \\ g \widehat{\frac{\partial h_i}{\partial s_1}} & \dots & g \widehat{\frac{\partial h_i}{\partial s_{n-2}}} & g' h_i \\ \vdots & \ddots & \vdots & \vdots \\ g \frac{\partial h_n}{\partial s_1} & \dots & g \frac{\partial h_n}{\partial s_{n-2}} & g' h_n \end{bmatrix} \\
&= g^{n-2} g' \det \underbrace{\begin{bmatrix} \frac{\partial h_1}{\partial s_1} & \dots & \frac{\partial h_1}{\partial s_{n-2}} & h_1 \\ \vdots & \ddots & \vdots & \vdots \\ \widehat{\frac{\partial h_i}{\partial s_1}} & \dots & \widehat{\frac{\partial h_i}{\partial s_{n-2}}} & \widehat{h_i} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial h_n}{\partial s_1} & \dots & \frac{\partial h_n}{\partial s_{n-2}} & h_n \end{bmatrix}}_{\text{say } A}.
\end{aligned}$$

(2) So

$$\begin{aligned}
\int_{\Phi} \omega_n &= \int_{[0,1]^{n-1}} \frac{1}{g(t)^n |\mathbf{h}(\mathbf{s})|^n} \sum_{i=1}^n (-1)^{i-1} g(t) h_i g(t)^{n-2} g'(t) \det(A) \, ds \, dt \\
&= \int_{[0,1]^{n-1}} \frac{g'(t)}{g(t) |\mathbf{h}(\mathbf{s})|^n} \sum_{i=1}^n (-1)^{i-1} h_i \det \begin{bmatrix} \frac{\partial h_1}{\partial s_1} & \dots & \frac{\partial h_1}{\partial s_{n-2}} & h_1 \\ \vdots & \ddots & \vdots & \vdots \\ \widehat{\frac{\partial h_i}{\partial s_1}} & \dots & \widehat{\frac{\partial h_i}{\partial s_{n-2}}} & \widehat{h_i} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial h_n}{\partial s_1} & \dots & \frac{\partial h_n}{\partial s_{n-2}} & h_n \end{bmatrix} \, ds \, dt \\
&= \int_{[0,1]^{n-1}} \frac{g'(t)}{g(t) |\mathbf{h}(\mathbf{s})|^n} \det \underbrace{\begin{bmatrix} h_1 & \frac{\partial h_1}{\partial s_1} & \dots & \frac{\partial h_1}{\partial s_{n-2}} & h_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h_n & \frac{\partial h_n}{\partial s_1} & \dots & \frac{\partial h_n}{\partial s_{n-2}} & h_n \end{bmatrix}}_{\text{say } B} \, ds \, dt.
\end{aligned}$$

Since the first column is the same as the last column in  $B$ ,  $\det(B) = 0$  (Theorem 9.34(d)). Therefore,  $\int_{\Phi} \omega_n = \int_{[0,1]^{n-1}} 0 \, ds \, dt = 0$ .

□

*Proof of (d)(iv).*

- (1) Consider the  $n$ -surface  $\Psi$  given by

$$\Psi(t, \boldsymbol{\varphi}) = [1 - t + tf(\boldsymbol{\varphi})]\Sigma(\boldsymbol{\varphi}),$$

where  $\boldsymbol{\varphi} \in E \subseteq D$ ,  $0 \leq t \leq 1$ .

- (2) Write

$$E = [a_1, b_1] \times \cdots \times [a_{n-1}, b_{n-1}] \subseteq D.$$

Note that  $\Psi(t, \boldsymbol{\varphi}) \subseteq \mathbb{R}^n - \{\mathbf{0}\}$ . So the boundary of  $\Psi$  is

$$\partial\Psi = \Psi(0, \boldsymbol{\varphi}) - \Psi(1, \boldsymbol{\varphi}) + \sum_{i=1}^{n-1} (\Psi|_{\varphi_i=a_i} - \Psi|_{\varphi_i=b_i}),$$

where  $\Psi|_{\varphi_i=\theta} : [a_1, b_1] \times \cdots \times \widehat{[a_i, b_i]} \times \cdots \times [a_{n-1}, b_{n-1}] \rightarrow \Omega$  is a mapping defined by

$$\begin{aligned} \Psi|_{\varphi_i=\theta}(t, \varphi_1, \dots, \widehat{\varphi_i}, \dots, \varphi_{n-1}) &= \Psi(t, \varphi_1, \dots, \varphi_{i-1}, \theta, \varphi_{i+1}, \dots, \varphi_{n-1}) \\ &= \Psi(t, \boldsymbol{\varphi} + (\theta - \varphi_i)\mathbf{e}_i). \end{aligned}$$

- (3) Show that

$$\int_{\Psi|_{\varphi_1=\theta}} \omega_n = 0$$

for any fixed  $\varphi_1 = \theta \in [a_1, b_1]$ . Note that  $\omega_n$  is well-defined on  $\Psi|_{\varphi_1=\theta}$ . Write

$$\Psi|_{\varphi_1=\theta}(t, \widehat{\varphi_1}, \varphi_2, \dots, \varphi_{n-1}) = \mathbf{x}(t, \widehat{\varphi_1}, \varphi_2, \dots, \varphi_{n-1}).$$

By definition of  $\Psi$ , we have

$$\begin{aligned} x_1 &= g(t, \boldsymbol{\varphi} + (\theta - \varphi_1)\mathbf{e}_1) \cos \theta \\ x_2 &= g(t, \boldsymbol{\varphi} + (\theta - \varphi_1)\mathbf{e}_1) \sin \theta \cos \varphi_2 \\ &\dots \\ x_{n-1} &= g(t, \boldsymbol{\varphi} + (\theta - \varphi_1)\mathbf{e}_1) \sin \theta \cdots \sin \varphi_{n-2} \cos \varphi_{n-1} \\ x_n &= g(t, \boldsymbol{\varphi} + (\theta - \varphi_1)\mathbf{e}_1) \sin \theta \cdots \sin \varphi_{n-2} \sin \varphi_{n-1}, \end{aligned}$$

where  $g(t, \boldsymbol{\varphi} + (\theta - \varphi_1)\mathbf{e}_1) = 1 - t + tf(\boldsymbol{\varphi} + (\theta - \varphi_1)\mathbf{e}_1)$ .

- (4) Note that  $r_n = g > 0$ . Since

$$\frac{\partial x_i}{\partial t} = \frac{\partial g}{\partial t} g^{-1} x_i,$$

$$\begin{aligned}
\frac{\partial(x_1, \dots, \widehat{x_i}, \dots, x_n)}{\partial(t, \widehat{\varphi_1}, \varphi_2, \dots, \varphi_{n-1})} &= \det \begin{bmatrix} \frac{\partial x_1}{\partial t} & \frac{\partial x_1}{\partial \varphi_2} & \cdots & \frac{\partial x_1}{\partial \varphi_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{\frac{\partial x_i}{\partial t}} & \widehat{\frac{\partial x_i}{\partial \varphi_2}} & \cdots & \widehat{\frac{\partial x_i}{\partial \varphi_{n-1}}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial t} & \frac{\partial x_n}{\partial \varphi_2} & \cdots & \frac{\partial x_n}{\partial \varphi_{n-1}} \end{bmatrix} \\
&= \det \begin{bmatrix} \frac{\partial g}{\partial t} g^{-1} x_1 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{\frac{\partial g}{\partial t} g^{-1} x_i} & \widehat{*} & \cdots & \widehat{*} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g}{\partial t} g^{-1} x_n & * & \cdots & * \end{bmatrix} \\
&= \frac{\partial g}{\partial t} g^{-1} \det \underbrace{\begin{bmatrix} x_1 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{x_i} & \widehat{*} & \cdots & \widehat{*} \\ \vdots & \vdots & \ddots & \vdots \\ x_n & * & \cdots & * \end{bmatrix}}_{\text{say } A}.
\end{aligned}$$

So

$$\begin{aligned}
\int_{\Psi|_{\varphi_1=\theta}} \omega_n &= \int_E g^{-n} \sum_{i=1}^n (-1)^{i-1} x_i \frac{\partial g}{\partial t} g^{-1} \det(A) dt d\varphi_2 \cdots d\varphi_{n-1} \\
&= \int_E \frac{\partial g}{\partial t} g^{-n-1} \sum_{i=1}^n (-1)^{i-1} x_i \det \begin{bmatrix} x_1 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{x_i} & \widehat{*} & \cdots & \widehat{*} \\ \vdots & \vdots & \ddots & \vdots \\ x_n & * & \cdots & * \end{bmatrix} dt d\varphi_2 \cdots d\varphi_{n-1} \\
&= \int_E \frac{\partial g}{\partial t} g^{-n-1} \det \underbrace{\begin{bmatrix} x_1 & x_1 & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_i & x_i & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n & x_n & * & \cdots & * \end{bmatrix}}_{\text{say } B} dt d\varphi_2 \cdots d\varphi_{n-1}.
\end{aligned}$$

Since the first column is the same as the second column in  $B$ ,  $\det(B) = 0$  (Theorem 9.34(d)). Therefore,  $\int_{\Psi|_{\varphi_1=\theta}} \omega_n = 0$ .

(5)  $\int_{\Psi|_{\varphi_i=\theta}} \omega_n = 0$  is also true for all  $i = 1, \dots, n-1$  by the same argument

in (3)(4). Hence,

$$\begin{aligned}
0 &= \int_{\Psi} d\omega_n \\
&= \int_{\partial\Psi} \omega_n \\
&= \int_S \omega_n - \int_{\Omega} \omega_n + \sum_{i=1}^{n-1} \left( \int_{\Psi|_{\varphi_i=a_i}} \omega_n - \int_{\Psi|_{\varphi_i=b_i}} \omega_n \right) \\
&= \int_S \omega_n - \int_{\Omega} \omega_n
\end{aligned}$$

by (a) and the Stokes' theorem (Theorem 10.33), or

$$\int_{\Omega} \omega_n = \underbrace{\int_S \omega_n}_{\text{by (d)(ii)}} = A(S).$$

□

*Proof of (d)(vii).* Similar to Exercise 10.22(f).

- (1) To ensure that  $\omega_n$  is well-defined on  $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n = \mathbb{R}^n - \{\mathbf{0}\}$ , we might assume  $0 < \varphi_1 < \pi$ . It is fine since  $\int_{\Omega} \omega_n$  and  $\int_S \omega_n$  is well-defined on any closed rectangle in  $D$  and we can apply the argument in Exercise 6.7 to remove the additional restriction.
- (2) By the Stokes' theorem (Theorem 10.33) and (b),

$$\int_{\Omega} \omega_n = \int_{\partial\Omega} f_n \omega_{n-1} \quad \text{and} \quad \int_S \omega_n = \int_{\partial S} f_n \omega_{n-1}.$$

So it suffices to show that

$$\int_{\partial\Omega} f_n \omega_{n-1} = \int_{\partial S} f_n \omega_{n-1}.$$

So it suffices to show that  $f_n|_{\partial\Omega} = f_n|_{\partial S}$  and  $\omega_{n-1}|_{\partial\Omega} = \omega_{n-1}|_{\partial S}$ .

- (3) Show that  $f_n|_{\partial\Omega} = f_n|_{\partial S}$ . For any  $\mathbf{x}_{\Omega} \in \partial\Omega$ ,

$$\mathbf{x}_{\Omega} = f(\varphi)\mathbf{x}_{\Sigma}.$$

So

$$\begin{aligned}
f_n(\mathbf{x}_\Omega) &= (-1)^n g_n \left( \frac{(x_n)_\Omega}{((x_1)_\Omega^2 + \cdots + (x_n)_\Omega^2)^{\frac{1}{2}}} \right) \\
&= (-1)^n g_n \left( \frac{f(\boldsymbol{\varphi})(x_n)_\Sigma}{f(\boldsymbol{\varphi})((x_1)_\Sigma^2 + \cdots + (x_n)_\Sigma^2)^{\frac{1}{2}}} \right) \\
&= (-1)^n g_n \left( \frac{(x_n)_\Sigma}{((x_1)_\Sigma^2 + \cdots + (x_n)_\Sigma^2)^{\frac{1}{2}}} \right) \\
&= f_n(\mathbf{x}_\Sigma).
\end{aligned}$$

(Note that  $f > 0$ .)

- (4) *Show that  $\omega_{n-1}|_{\partial\Omega} = \omega_{n-1}|_{\partial S}$ .* Induction on  $n$ . When  $n = 2$  or  $n = 3$ , it is proved in Exercise 10.22(f). Now for large  $n - 1$ , (3) is also true for  $n - 1$ . Hence,

$$\omega_{n-1}|_{\partial\Omega} = d(f_{n-1}\omega_{n-2})|_{\partial\Omega} = d(f_{n-1}\omega_{n-2})|_{\partial S} = \omega_{n-1}|_{\partial S}.$$

By induction, the result is established.

□

*Proof of (d)(ix).* Similar to Exercise 10.22(g).

- (1) Given any line  $L$  passing through  $\mathbf{0}$ , say

$$(r \cos \varphi_1, \dots, \sin \varphi_1 \cdots \sin \varphi_{n-2} \sin \varphi_{n-1}) \in L \subseteq \mathbb{R}^n$$

where  $r \in \mathbb{R}^1$  for some  $\boldsymbol{\varphi} \in [0, \pi]^{n-2} \times [0, 2\pi]$ . We will show that  $\omega_n$  is exact in  $U = \mathbb{R}^n - L$ .

- (2) Linear algebra says that all rotation matrices  $T \in SO(n)$  can be obtained from

$$R_i(u) = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & 0 \\ & & 1 & & & \\ & & & R(u) & & \\ & & & & 1 & \\ 0 & & & & & \ddots \\ & & & & & & 1 \end{bmatrix}$$

using matrix multiplication. Here

$$R(u) = \begin{bmatrix} \cos u & -\sin u \\ \sin u & \cos u \end{bmatrix}$$

is a 2-by-2 rotation matrix at the  $i$ th row and  $i$ th column. For example, the rotation

$$T = R_1(-\varphi_1)R_2(-\varphi_2) \cdots R_{n-2}(-\varphi_{n-2})R_{n-1}(-\varphi_{n-1})$$

maps  $L$  to the  $x_n$ -axis. Similar to Exercise 10.22(g), it suffices to show that  $\omega_n$  is invariant under  $T = R_1(u)$ .

(3) Show that  $\omega_n$  is invariant under  $T = R_1(u)$ . By

$$T : \mathbf{x} \mapsto (x_1 \cos u - x_2 \sin u, x_1 \sin u + x_2 \cos u, x_3, \dots, x_n),$$

we have

$$\begin{aligned} r_n &\mapsto r_n \\ dx_1 &\mapsto \cos u dx_1 - \sin u dx_2 \\ dx_2 &\mapsto \sin u dx_1 + \cos u dx_2 \\ dx_3 &\mapsto dx_3 \\ &\dots \\ dx_n &\mapsto dx_n. \end{aligned}$$

So  $dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$  maps to

$$\begin{cases} \cos u \widehat{dx_1} \wedge \cdots \wedge dx_n + \sin u dx_1 \wedge \widehat{dx_2} \wedge \cdots \wedge dx_n & \text{if } i = 1 \\ -\sin u \widehat{dx_1} \wedge \cdots \wedge dx_n + \cos u dx_1 \wedge \widehat{dx_2} \wedge \cdots \wedge dx_n & \text{if } i = 2 \\ dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n & \text{otherwise.} \end{cases}$$

Thus

$$\begin{aligned} \omega_n &\mapsto (r_n)^{-n} (x_1 \cos u - x_2 \sin u) \\ &\quad \left( \cos u \widehat{dx_1} \wedge \cdots \wedge dx_n + \sin u dx_1 \wedge \widehat{dx_2} \wedge \cdots \wedge dx_n \right) \\ &\quad + (r_n)^{-n} (x_1 \sin u + x_2 \cos u) \\ &\quad \left( -\sin u \widehat{dx_1} \wedge \cdots \wedge dx_n + \cos u dx_1 \wedge \widehat{dx_2} \wedge \cdots \wedge dx_n \right) \\ &\quad + (r_n)^{-n} \sum_{i=3}^n (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\ &= (r_n)^{-n} x_1 \widehat{dx_1} \wedge \cdots \wedge dx_n \\ &\quad - (r_n)^{-n} x_2 dx_1 \wedge \widehat{dx_2} \wedge \cdots \wedge dx_n \\ &\quad + (r_n)^{-n} \sum_{i=3}^n (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\ &= (r_n)^{-n} \sum_{i=1}^n (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\ &= \omega_n. \end{aligned}$$



- (4) Similar to Exercise 10.22(g),  $\omega_n$  is exact in  $\mathbb{R}^n - L$ . (Or  $\omega_n$  is locally exact in  $\mathbb{R}^n - \{\mathbf{0}\}$ .)

□

**Exercise 10.24.** Let  $\omega = \sum a_i(\mathbf{x})dx_i$  be a 1-form of class  $\mathcal{C}''$  in a convex open set  $E \subseteq \mathbb{R}^n$ . Assume  $d\omega = 0$  and prove that  $\omega$  is exact in  $E$ , by completing the following outline:

Fix  $\mathbf{p} \in E$ . Define

$$f(\mathbf{x}) = \int_{[\mathbf{p}, \mathbf{x}]} \omega \quad (\mathbf{x} \in E).$$

Apply Stokes' theorem to affine-oriented 2-simplexes  $[\mathbf{p}, \mathbf{x}, \mathbf{y}]$  in  $E$ . Deduce that

$$f(\mathbf{y}) - f(\mathbf{x}) = \sum_{i=1}^n (y_i - x_i) \int_0^1 a_i((1-t)\mathbf{x} + t\mathbf{y}) dt$$

for  $\mathbf{x} \in E, \mathbf{y} \in E$ . Hence  $(D_i f)(\mathbf{x}) = a_i(\mathbf{x})$ .

*Proof.*

- (1) Fix  $\mathbf{p} \in E$ . Define

$$f(\mathbf{x}) = \int_{[\mathbf{p}, \mathbf{x}]} \omega \quad (\mathbf{x} \in E).$$

- (2) Given any  $\mathbf{x} \in E, \mathbf{y} \in E$ , and  $\mathbf{x} \neq \mathbf{y}$ . The affine-oriented 2-simplex  $\Psi = [\mathbf{p}, \mathbf{x}, \mathbf{y}]$  is in  $E$  by the convexity of  $E$ . (If  $E$  is open but not convex, we can show that  $\omega = df$  **locally** as the note in Exercise 10.21(a). That is why we say that  $\omega$  is locally exact. The proof is exactly the same.)

- (3) Note that

$$\partial\Psi = \partial[\mathbf{p}, \mathbf{x}, \mathbf{y}] = [\mathbf{x}, \mathbf{y}] - [\mathbf{p}, \mathbf{y}] + [\mathbf{p}, \mathbf{x}].$$

The Stokes' theorem (Theorem 10.33) implies that

$$\begin{aligned} \int_{\Psi} d\omega &= \int_{\partial\Psi} \omega \iff \int_{\Psi} 0 = \int_{[\mathbf{x}, \mathbf{y}]} \omega - \int_{[\mathbf{p}, \mathbf{y}]} \omega + \int_{[\mathbf{p}, \mathbf{x}]} \omega \\ &\iff 0 = \int_{[\mathbf{x}, \mathbf{y}]} \omega - f(\mathbf{y}) + f(\mathbf{x}) \\ &\iff f(\mathbf{y}) - f(\mathbf{x}) = \int_{[\mathbf{x}, \mathbf{y}]} \omega. \end{aligned}$$

- (4) Define  $\gamma : [0, 1] \rightarrow E$  by

$$\begin{aligned} \gamma(t) &= \mathbf{x} + t(\mathbf{y} - \mathbf{x}) \\ &= \sum_{i=1}^n x_i + t(y_i - x_i) \end{aligned}$$

(where  $\mathbf{x} = (x_1, \dots, x_n)$  and  $\mathbf{y} = (y_1, \dots, y_n)$ ). Hence  $[0, 1]$  is the parameter domain of  $[\mathbf{x}, \mathbf{y}]$  with respect to  $\gamma$ . So

$$\begin{aligned}\int_{[\mathbf{x}, \mathbf{y}]} \omega &= \int_0^1 \sum_{i=1}^n a_i(\gamma(t)) \frac{\partial(x_i + t(y_i - x_i))}{\partial t} dt \\ &= \int_0^1 \sum_{i=1}^n a_i(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))(y_i - x_i) dt \\ &= \sum_{i=1}^n (y_i - x_i) \int_0^1 a_i(\mathbf{x} + t(\mathbf{y} - \mathbf{x})) dt.\end{aligned}$$

Thus,

$$f(\mathbf{y}) - f(\mathbf{x}) = \sum_{i=1}^n (y_i - x_i) \int_0^1 a_i(\mathbf{x} + t(\mathbf{y} - \mathbf{x})) dt.$$

(5) Note that

$$\begin{aligned}f(\mathbf{x} + h\mathbf{e}_j) - f(\mathbf{x}) &= \sum_{i=1}^n ((x_i + h\delta_{ij}) - x_i) \int_0^1 a_i(\mathbf{x} + t((\mathbf{x} + h\mathbf{e}_j) - \mathbf{x})) dt \\ &= \sum_{i=1}^n h\delta_{ij} \int_0^1 a_i(\mathbf{x} + t h\mathbf{e}_j) dt \\ &= h \int_0^1 a_j(\mathbf{x} + t h\mathbf{e}_j) dt.\end{aligned}$$

(Here  $\delta_{ij}$  is the Kronecker delta.) So

$$\begin{aligned}(D_j f)(\mathbf{x}) &= \lim_{h \rightarrow 0} \frac{f(\mathbf{x} + h\mathbf{e}_j) - f(\mathbf{x})}{h} \\ &= \lim_{h \rightarrow 0} \int_0^1 a_j(\mathbf{x} + t h\mathbf{e}_j) dt \\ &= \int_0^1 a_j(\mathbf{x}) dt \quad (a_j \in \mathcal{C}'') \\ &= a_j(\mathbf{x}).\end{aligned}$$

Thus,

$$df = \sum_{j=1}^n (D_j f)(\mathbf{x}) dx_j = \sum_{j=1}^n a_j(\mathbf{x}) dx_j = \omega,$$

or  $\omega$  is exact in  $E$ .

□

**Exercise 10.25.** Assume  $\omega$  is a 1-form in an open set  $E \subseteq \mathbb{R}^n$  such that

$$\int_{\gamma} \omega = 0$$

for every closed curve  $\gamma$  in  $E$ , of class  $\mathcal{C}'$ . Prove that  $\omega$  is exact in  $E$ , by imitating part of the argument sketched in Exercise 10.24.

*Proof.*

- (1) Assume that  $E$  is a **connected** open subset of  $\mathbb{R}^n$ . Show that  $\omega$  is exact in  $E$  if  $\int_{\gamma} \omega = 0$  for every closed curve  $\gamma$  in  $E$ , of class  $\mathcal{C}'$ .

- (2) Fix  $\mathbf{p} \in E$ . Define

$$f(\mathbf{x}) = \int_{[\mathbf{p}, \mathbf{x}]} \omega \quad (\mathbf{x} \in E).$$

It is well-defined since  $E$  is connected and  $\int_{\gamma} \omega = 0$  for every closed curve  $\gamma$  in  $E$ .

- (3) Given any  $\mathbf{x} \in E$ ,  $\mathbf{y} \in E$ , and  $\mathbf{x} \neq \mathbf{y}$ . Let

$$\gamma = [\mathbf{x}, \mathbf{y}] - [\mathbf{p}, \mathbf{y}] + [\mathbf{p}, \mathbf{x}]$$

be a closed curve in  $E$ . Hence,

$$\begin{aligned} 0 &= \int_{\gamma} \omega && \text{(Assumption)} \\ &= \int_{[\mathbf{x}, \mathbf{y}]} \omega - \int_{[\mathbf{p}, \mathbf{y}]} \omega + \int_{[\mathbf{p}, \mathbf{x}]} \omega \\ &= \int_{[\mathbf{x}, \mathbf{y}]} \omega - f(\mathbf{y}) + f(\mathbf{x}). \end{aligned}$$

So

$$f(\mathbf{y}) - f(\mathbf{x}) = \int_{[\mathbf{x}, \mathbf{y}]} \omega$$

- (4) Similar to (4)(5) in the proof of Exercise 10.24, we have  $df = \omega$ . So the statement in (1) is proved. In general, we can define each  $f_{\alpha}$  on each connected component  $E_{\alpha}$  (which is open) of  $E$  such that  $df_{\alpha} = \omega$  on  $E_{\alpha}$ . Take

$$f|_{E_{\alpha}} = f_{\alpha}$$

on  $E$ . Hence,  $df = \omega$  on the whole  $E$ .

□

**Exercise 10.26.** Assume  $\omega$  is a 1-form in  $\mathbb{R}^3 - \{\mathbf{0}\}$ , of class  $\mathcal{C}'$  and  $d\omega = 0$ . Prove that  $\omega$  is exact in  $\mathbb{R}^3 - \{\mathbf{0}\}$ . (Hint: Every closed continuously differentiable curve in  $\mathbb{R}^3 - \{\mathbf{0}\}$  is the boundary of a 2-surface in  $\mathbb{R}^3 - \{\mathbf{0}\}$ . Apply Stokes' theorem and Exercise 10.25.)

*Proof.*

- (1) Let  $E = \mathbb{R}^3 - \{\mathbf{0}\}$ . By Exercise 10.25, it suffices to show that

$$\int_{\gamma} \omega = 0$$

for every closed curve  $\gamma$  in  $E$ , of class  $\mathcal{C}'$ .

- (2) Intuitively, every closed continuously differentiable curve in  $\mathbb{R}^3 - \{\mathbf{0}\}$  is the boundary of a 2-surface in  $\mathbb{R}^3 - \{\mathbf{0}\}$ . So there is some 2-surface  $\Psi$  such that  $\partial\Psi = \gamma$ . The Stokes' theorem (Theorem 10.33) implies that

$$\int_{\gamma} \omega = \int_{\partial\Psi} \omega = \int_{\Psi} d\omega = \int_{\Psi} 0 = 0.$$

□

**Exercise 10.27.** Let  $E$  be an open 3-cell in  $\mathbb{R}^3$ , with edges parallel to the coordinate axes. Suppose  $(a, b, c) \in E$ ,  $f_i \in \mathcal{C}'(E)$  for  $i = 1, 2, 3$ ,

$$\omega = f_1 dy \wedge dz + f_2 dz \wedge dx + f_3 dx \wedge dy,$$

and assume that  $d\omega = 0$  in  $E$ . Define

$$\lambda = g_1 dx + g_2 dy$$

where

$$\begin{aligned} g_1(x, y, z) &= \int_c^z f_2(x, y, s) ds - \int_b^y f_3(x, t, c) dt \\ g_2(x, y, z) &= - \int_c^z f_1(x, y, s) ds, \end{aligned}$$

for  $(x, y, z) \in E$ . Prove that  $d\lambda = \omega$  in  $E$ . Evaluate these integrals when  $\omega = \zeta$  and thus find the form  $\lambda$  that occurs in part (e) of Exercise 10.22.

*Proof.*

- (1) Let  $\mathbf{F} = f_1 \mathbf{e}_1 + f_2 \mathbf{e}_2 + f_3 \mathbf{e}_3$  as in Vector fields 10.42. Then

$$d\omega = (\nabla \cdot \mathbf{F}) dx \wedge dy \wedge dz.$$

As  $d\omega = 0$  by assumption,  $\nabla \cdot \mathbf{F} = D_1 f_1 + D_2 f_2 + D_3 f_3 = 0$ .

(2) As

$$\begin{aligned}
d\lambda &= d(g_1 dx + g_2 dy) \\
&= (D_1 g_1 dx + D_2 g_1 dy + D_3 g_1 dz) \wedge dx \\
&\quad + (D_1 g_2 dx + D_2 g_2 dy + D_3 g_2 dz) \wedge dy \\
&= (-D_3 g_2) dy \wedge dz + (D_3 g_1) dz \wedge dx + (D_1 g_2 - D_2 g_1) dx \wedge dy,
\end{aligned}$$

it suffices to show that

$$\begin{aligned}
f_1 &= -D_3 g_2, \\
f_2 &= D_3 g_1, \\
f_3 &= D_1 g_2 - D_2 g_1
\end{aligned}$$

on  $E$ .

(3) Theorem 6.20 implies that

$$-D_3 g_2 = D_3 \int_c^z f_1(x, y, s) ds = f_1(x, y, z)$$

and

$$D_3 g_1 = D_3 \int_c^z f_2(x, y, s) ds - D_3 \int_b^y f_3(x, t, c) dt = f_2(x, y, z).$$

Also,

$$\begin{aligned}
&D_1 g_2 - D_2 g_1 \\
&= D_1 \left( - \int_c^z f_1(x, y, s) ds \right) \\
&\quad - D_2 \left( \int_c^z f_2(x, y, s) ds - \int_b^y f_3(x, t, c) dt \right) \\
&= - \int_c^z D_1 f_1(x, y, s) ds \quad (f_1 \in \mathcal{C}') \\
&\quad - \int_c^z D_2 f_2(x, y, s) ds + f_3(x, y, c) \quad (f_2 \in \mathcal{C}', \text{ Theorem 6.20}) \\
&= \int_c^z D_3 f_3(x, y, s) ds + f_3(x, y, c) \quad ((1)) \\
&= f_3(x, y, z) \quad (\text{Theorem 6.21}).
\end{aligned}$$

Therefore,  $d\lambda = \omega$  in  $E$ .

(4) When  $\omega = \zeta = r^{-3}(x dy \wedge dz + y dz \wedge dx + z dx \wedge dy)$ , we get

$$\begin{aligned}
f_1(x, y, z) &= x(x^2 + y^2 + z^2)^{-\frac{3}{2}}, \\
f_2(x, y, z) &= y(x^2 + y^2 + z^2)^{-\frac{3}{2}}, \\
f_3(x, y, z) &= z(x^2 + y^2 + z^2)^{-\frac{3}{2}}.
\end{aligned}$$

So,

$$\begin{aligned}\int_c^z f_2(x, y, s) ds &= \left[ ys(x^2 + y^2)^{-1}(x^2 + y^2 + s^2)^{-\frac{1}{2}} \right]_{s=c}^{s=z}, \\ \int_b^y f_3(x, t, c) dt &= \left[ ct(x^2 + c^2)^{-1}(x^2 + t^2 + c^2)^{-\frac{1}{2}} \right]_{t=b}^{t=y}, \\ \int_c^z f_1(x, y, s) ds &= \left[ xs(x^2 + y^2)^{-1}(x^2 + y^2 + s^2)^{-\frac{1}{2}} \right]_{s=c}^{s=z}.\end{aligned}$$

Hence,

$$\begin{aligned}\lambda &= g_1 dx + g_2 dy \\ &= \left[ ys(x^2 + y^2)^{-1}(x^2 + y^2 + s^2)^{-\frac{1}{2}} \right]_{s=c}^{s=z} dx \\ &\quad - \left[ ct(x^2 + c^2)^{-1}(x^2 + t^2 + c^2)^{-\frac{1}{2}} \right]_{t=b}^{t=y} dx \\ &\quad + \left[ xs(x^2 + y^2)^{-1}(x^2 + y^2 + s^2)^{-\frac{1}{2}} \right]_{s=c}^{s=z} dy \\ &= - \left[ zr^{-1} - c(x^2 + y^2 + c^2)^{-\frac{1}{2}} \right] \eta \quad (\text{Definition of } \eta) \\ &\quad - c(x^2 + c^2)^{-1} \left[ y(x^2 + y^2 + c^2)^{-\frac{1}{2}} - b(x^2 + b^2 + c^2)^{-\frac{1}{2}} \right] dx.\end{aligned}$$

As we pick  $(a, b, c) = (a, 0, 0) \in \mathbb{R}^3 - \{\mathbf{0}\}$  (or  $a \neq 0$ ), we have  $\lambda = -zr^{-1}\eta$  such that  $d\lambda = \omega = \zeta$ , which is the same as part (e) in Exercise 10.22.

□

**Exercise 10.28.** Fix  $b > a > 0$ , define

$$\Phi(r, \theta) = (r \cos \theta, r \sin \theta)$$

for  $a \leq r \leq b$ ,  $0 \leq \theta \leq 2\pi$ . (The range of  $\Phi$  is an annulus in  $\mathbb{R}^2$ .) Put  $\omega = x^3 dy$ , and compute both

$$\int_{\Phi} d\omega \quad \text{and} \quad \int_{\partial\Phi} \omega$$

to verify that they are equal.

*Proof.*

(1) Note that

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \det \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix} = r.$$

So

$$\begin{aligned}
\int_{\Phi} d\omega &= \int_{\Phi} 3x^2 dx \wedge dy & (dy \wedge dy = 0) \\
&= \int_{[a,b] \times [0,2\pi]} 3(r \cos \theta)^2 \frac{\partial(x,y)}{\partial(r,\theta)} dr d\theta \\
&= \int_a^b \int_0^{2\pi} 3r^3 (\cos \theta)^2 dr d\theta \\
&= \frac{3\pi}{4} (b^4 - a^4).
\end{aligned}$$

(2) Similar to Exercise 10.21(b), write

$$\partial\Phi = \Gamma - \gamma,$$

where  $\Gamma(t) = (b \cos t, b \sin t)$  on  $[0, 2\pi]$  and  $\gamma(t) = (a \cos t, a \sin t)$  on  $[0, 2\pi]$ .  
Hence

$$\begin{aligned}
\int_{\partial\Phi} \omega &= \int_{\Gamma} \omega - \int_{\gamma} \omega \\
&= \int_{\Gamma} x^3 dy - \int_{\gamma} x^3 dy \\
&= \int_{[0,2\pi]} (b \cos \theta)^3 \frac{\partial y}{\partial \theta} d\theta - \int_{[0,2\pi]} (a \cos \theta)^3 \frac{\partial y}{\partial \theta} d\theta \\
&= \int_0^{2\pi} b^4 (\cos \theta)^4 d\theta - \int_0^{2\pi} a^4 (\cos \theta)^4 d\theta \\
&= \frac{3\pi}{4} (b^4 - a^4).
\end{aligned}$$

(3)

$$\int_{\Phi} d\omega = \int_{\partial\Phi} \omega = \frac{3\pi}{4} (b^4 - a^4).$$

□

**Exercise 10.29.** *Prove the existence of a function  $\alpha$  with the properties needed in the proof of Theorem 10.38, and prove that the resulting function  $F$  is of class  $\mathcal{C}^1$ . (Both assertions become trivial if  $E$  is an open cell or an open ball, since  $\alpha$  can then be taken to be a constant. Refer to Theorem 9.42.)*

*Proof.*

(1)

(2)

□

**Exercise 10.30.** If  $\mathbf{N}$  is the vector given by

$$\mathbf{N} = (\alpha_2\beta_3 - \alpha_3\beta_2)\mathbf{e}_1 + (\alpha_3\beta_1 - \alpha_1\beta_3)\mathbf{e}_2 + (\alpha_1\beta_2 - \alpha_2\beta_1)\mathbf{e}_3$$

(Equation (135)), prove that

$$\det \begin{bmatrix} \alpha_1 & \beta_1 & \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_2 & \beta_2 & \alpha_3\beta_1 - \alpha_1\beta_3 \\ \alpha_3 & \beta_3 & \alpha_1\beta_2 - \alpha_2\beta_1 \end{bmatrix} = |\mathbf{N}|^2$$

Also, verify

$$\mathbf{N} \cdot (T\mathbf{e}_1) = \mathbf{N} \cdot (T\mathbf{e}_2)$$

(Equation (137)).

*Proof.*

(1) By Laplace's expansion along the third column,

$$\begin{aligned} & \det \begin{bmatrix} \alpha_1 & \beta_1 & \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_2 & \beta_2 & \alpha_3\beta_1 - \alpha_1\beta_3 \\ \alpha_3 & \beta_3 & \alpha_1\beta_2 - \alpha_2\beta_1 \end{bmatrix} \\ &= (-1)^{1+3}(\alpha_2\beta_3 - \alpha_3\beta_2) \det \begin{bmatrix} \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} \\ & \quad + (-1)^{2+3}(\alpha_3\beta_1 - \alpha_1\beta_3) \det \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_3 & \beta_3 \end{bmatrix} \\ & \quad + (-1)^{3+3}(\alpha_1\beta_2 - \alpha_2\beta_1) \det \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix} \\ &= (\alpha_2\beta_3 - \alpha_3\beta_2)^2 + (\alpha_3\beta_1 - \alpha_1\beta_3)^2 + (\alpha_1\beta_2 - \alpha_2\beta_1)^2 \\ &= |\mathbf{N}|^2. \end{aligned}$$

(2)

$$\begin{aligned} \mathbf{N} \cdot (T\mathbf{e}_1) &= (\alpha_2\beta_3 - \alpha_3\beta_2, \alpha_3\beta_1 - \alpha_1\beta_3, \alpha_1\beta_2 - \alpha_2\beta_1) \cdot (\alpha_1, \alpha_2, \alpha_3) \\ &= (\alpha_2\beta_3 - \alpha_3\beta_2)\alpha_1 + (\alpha_3\beta_1 - \alpha_1\beta_3)\alpha_2 + (\alpha_1\beta_2 - \alpha_2\beta_1)\alpha_3 \\ &= (\alpha_3\alpha_2 - \alpha_2\alpha_3)\beta_1 + (\alpha_1\alpha_3 - \alpha_3\alpha_1)\beta_2 + (\alpha_2\alpha_1 - \alpha_1\alpha_2)\beta_3 \\ &= 0. \end{aligned}$$

(3)

$$\begin{aligned} \mathbf{N} \cdot (T\mathbf{e}_2) &= (\alpha_2\beta_3 - \alpha_3\beta_2, \alpha_3\beta_1 - \alpha_1\beta_3, \alpha_1\beta_2 - \alpha_2\beta_1) \cdot (\beta_1, \beta_2, \beta_3) \\ &= (\alpha_2\beta_3 - \alpha_3\beta_2)\beta_1 + (\alpha_3\beta_1 - \alpha_1\beta_3)\beta_2 + (\alpha_1\beta_2 - \alpha_2\beta_1)\beta_3 \\ &= (\beta_2\beta_3 - \beta_3\beta_2)\alpha_1 + (\beta_3\beta_1 - \beta_1\beta_3)\alpha_2 + (\beta_1\beta_2 - \beta_2\beta_1)\alpha_3 \\ &= 0. \end{aligned}$$



□

**Exercise 10.31.** Let  $E \subseteq \mathbb{R}^3$  be open, suppose  $g \in \mathcal{C}''(E)$ ,  $h \in \mathcal{C}''(E)$ , and consider the vector field

$$\mathbf{F} = g\nabla h$$

(a) Prove that

$$\nabla \cdot \mathbf{F} = g\nabla^2 h + (\nabla g) \cdot (\nabla h)$$

where  $\nabla^2 h = \nabla \cdot (\nabla h) = \sum \frac{\partial^2 h}{\partial x_i^2}$  is the so-called “Laplacian” of  $h$ .

(b) If  $\Omega$  is a closed subset of  $E$  with positively oriented boundary  $\partial\Omega$  (as in Theorem 10.51), prove that

$$\int_{\Omega} [g\nabla^2 h + (\nabla g) \cdot (\nabla h)] dV = \int_{\partial\Omega} g \frac{\partial h}{\partial n} dA$$

where (as is customary) we have written  $\frac{\partial h}{\partial n}$  in place of  $(\nabla h) \cdot \mathbf{n}$ . (Thus  $\frac{\partial h}{\partial n}$  is the directional derivative of  $h$  in the direction of the outward normal to  $\partial\Omega$ , the so-called **normal derivative** of  $h$ .) Interchange  $g$  and  $h$ , subtract the resulting formula from the first one, to obtain

$$\int_{\Omega} (g\nabla^2 h - h\nabla^2 g) dV = \int_{\partial\Omega} \left( g \frac{\partial h}{\partial n} - h \frac{\partial g}{\partial n} \right) dA.$$

These two formulas are usually called **Green’s identities**.

(c) Assume that  $h$  is **harmonic** in  $E$ ; this means that  $\nabla^2 h = 0$ . Take  $g = 1$  and conclude that

$$\int_{\partial\Omega} \frac{\partial h}{\partial n} dA = 0.$$

Take  $g = h$ , and conclude that  $h = 0$  in  $\Omega$  if  $h = 0$  on  $\partial\Omega$ .

(d) Show that Green’s identities are also valid in  $\mathbb{R}^2$ .

*Proof of (a).*

(1) Since

$$\mathbf{F} = g\nabla h = g \left( \sum (D_i h) \mathbf{e}_i \right) = \sum g(D_i h) \mathbf{e}_i,$$

we have

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \nabla \cdot \left( \sum g(D_i h) \mathbf{e}_i \right) \\ &= \sum D_i (g(D_i h)) \\ &= \sum \{ (D_i g)(D_i h) + g D_i (D_i h) \} \\ &= \sum (D_i g)(D_i h) + g \sum D_i (D_i h). \end{aligned}$$

(2) Also,

$$\begin{aligned}
g\nabla^2 h + (\nabla g) \cdot (\nabla h) &= g\nabla \cdot (\nabla h) + (\nabla g) \cdot (\nabla h) \\
&= g\nabla \cdot \left( \sum (D_i h) \mathbf{e}_i \right) + \left( \sum (D_i g) \mathbf{e}_i \right) \cdot \left( \sum (D_i h) \mathbf{e}_i \right) \\
&= g \sum D_i (D_i h) + \sum (D_i g) (D_i h).
\end{aligned}$$

(3) By (1)(2), the result is established.

□

*Proof of (b).*

(1) The divergence theorem (Theorem 10.51) implies that

$$\begin{aligned}
\int_{\Omega} (\nabla \cdot \mathbf{F}) dV &= \int_{\partial\Omega} (\mathbf{F} \cdot \mathbf{n}) dA \\
\Rightarrow \int_{\Omega} [g\nabla^2 h + (\nabla g) \cdot (\nabla h)] dV &= \int_{\partial\Omega} g \underbrace{\nabla h \cdot \mathbf{n}}_{=\frac{\partial h}{\partial n}} dA.
\end{aligned}$$

(2) Green's identities are a set of three identities in vector calculus relating the bulk with the boundary of a region on which differential operators act. (*Green's third identity.*) Assume that  $h$  is harmonic in  $E$ . If  $G(\mathbf{x}, \mathbf{x}_0)$  is the Green's function, then

$$h(\mathbf{x}_0) = \int_{\partial\Omega} \left[ h(\mathbf{x}) \frac{\partial G(\mathbf{x}, \mathbf{x}_0)}{\partial n} - G(\mathbf{x}, \mathbf{x}_0) \frac{\partial h(\mathbf{x})}{\partial n} \right] dA.$$

For example, in  $\mathbb{R}^3$

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi \|\mathbf{x} - \mathbf{x}_0\|}.$$

□

*Proof of (c).* Assume  $\nabla^2 h = 0$ .

(1) Take  $g = 1$  in

$$\int_{\Omega} [g\nabla^2 h + (\nabla g) \cdot (\nabla h)] dV = \int_{\partial\Omega} g \frac{\partial h}{\partial n} dA$$

to get the conclusion. (Here  $\nabla g = \mathbf{0}$  as  $g = 1$ .)

(2) Assume  $h = 0$  on  $\partial\Omega$ . Take  $g = h$  in

$$\int_{\Omega} [g\nabla^2 h + (\nabla g) \cdot (\nabla h)] dV = \int_{\partial\Omega} g \frac{\partial h}{\partial n} dA$$

to get

$$\int_{\Omega} |\nabla h|^2 dV = \int_{\partial\Omega} h \frac{\partial h}{\partial n} dA = 0$$

(since  $h = 0$  on  $\partial\Omega$ ). Since  $h \in \mathcal{C}'(\Omega)$ , Exercise 6.2 implies that  $|\nabla h|^2 = 0$  on  $\Omega$ . So  $D_1 h = D_2 h = D_3 h = 0$  on  $\Omega$ . Since  $h \in \mathcal{C}'(\Omega)$ , Theorem 9.21 implies that  $h = 0$  on  $\Omega$ , or  $h$  is locally constant in  $\Omega$  (Exercise 9.9). Note that  $h = 0$  globally on  $\partial\Omega$ , and thus  $h = 0$  globally on  $\Omega$ .

□

*Proof of (d).*

- (1) *(The divergence theorem in  $\mathbb{R}^2$ .) If  $\mathbf{F} = F_1 \mathbf{e}_1 + F_2 \mathbf{e}_2$  is a vector field of class  $\mathcal{C}'$  in an open set  $E \subseteq \mathbb{R}^2$ , and if  $\Omega$  is a closed subset of  $E$  with positively oriented boundary  $\partial\Omega$  then*

$$\int_{\Omega} (\nabla \cdot \mathbf{F}) dA = \int_{\partial\Omega} (\mathbf{F} \cdot \mathbf{n}) ds.$$

Define a 1-form by

$$\omega_{\mathbf{F}} = F_1 dy - F_2 dx.$$

So

$$d\omega_{\mathbf{F}} = (\nabla \cdot \mathbf{F}) dx \wedge dy = (\nabla \cdot \mathbf{F}) dA.$$

Hence the Stokes' theorem (Theorem 10.33) implies that

$$\int_{\Omega} (\nabla \cdot \mathbf{F}) dA = \int_{\Omega} d\omega_{\mathbf{F}} = \int_{\partial\Omega} \omega_{\mathbf{F}} = \int_{\partial\Omega} (\mathbf{F} \cdot \mathbf{n}) ds.$$

- (2) Note that

$$\nabla \cdot \mathbf{F} = g \nabla^2 h + (\nabla g) \cdot (\nabla h)$$

is also true in  $\mathbb{R}^2$ . Similar to (b), two Green's identities are also true in  $\mathbb{R}^2$ . (In  $\mathbb{R}^1$ , the Green's first identity is the integration by parts (Theorem 6.22).)

□

**Exercise 10.32 (Möbius band).** Fix  $\delta$ ,  $0 < \delta < 1$ . Let  $D$  be the set of all  $(\theta, t) \in \mathbb{R}^2$  such that  $0 \leq \theta \leq \pi$ ,  $-\delta \leq t \leq \delta$ . Let  $\Phi$  be the 2-surface in  $\mathbb{R}^3$ , with parameter domain  $D$ , given by

$$\begin{aligned} x &= (1 - t \sin \theta) \cos(2\theta) \\ y &= (1 - t \sin \theta) \sin(2\theta) \\ z &= t \cos \theta \end{aligned}$$

where  $(x, y, z) = \Phi(\theta, t)$ . Note that  $\Phi(\pi, t) = \Phi(0, -t)$ , and that  $\Phi$  is one-to-one on the rest of  $D$ .

The range  $M = \Phi(D)$  of  $\Phi$  is known as a **Möbius band**. It is the simplest example of a nonorientable surface.

Prove the various assertions made in the following description: Put  $\mathbf{p}_1 = (0, -\delta)$ ,  $\mathbf{p}_2 = (\pi, -\delta)$ ,  $\mathbf{p}_3 = (\pi, \delta)$ ,  $\mathbf{p}_4 = (0, \delta)$ ,  $\mathbf{p}_5 = \mathbf{p}_1$ . Put  $\gamma_i = [\mathbf{p}_i, \mathbf{p}_{i+1}]$ ,  $i = 1, \dots, 4$ , and put  $\Gamma_i = \Phi \circ \gamma_i$ . Then

$$\partial\Phi = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4.$$

Put  $\mathbf{a} = (1, 0, -\delta)$ ,  $\mathbf{b} = (1, 0, \delta)$ . Then

$$\Phi(\mathbf{p}_1) = \Phi(\mathbf{p}_3) = \mathbf{a}, \quad \Phi(\mathbf{p}_2) = \Phi(\mathbf{p}_4) = \mathbf{b},$$

and  $\partial\Phi$  can be described as follows.

- (1)  $\Gamma_1$  spirals up from  $\mathbf{a}$  to  $\mathbf{b}$ ; its projection into the  $(x, y)$ -plane has winding number  $+1$  around the origin. (See Exercise 8.23.)
- (2)  $\Gamma_2 = [\mathbf{b}, \mathbf{a}]$ .
- (3)  $\Gamma_3$  spirals up from  $\mathbf{a}$  to  $\mathbf{b}$ ; its projection into the  $(x, y)$ -plane has winding number  $-1$  around the origin.
- (4)  $\Gamma_4 = [\mathbf{b}, \mathbf{a}]$ .

Thus  $\partial\Phi = \Gamma_1 + \Gamma_3 + 2\Gamma_2$ .

If we go from  $\mathbf{a}$  to  $\mathbf{b}$  along  $\Gamma_1$  and continue along the “edge” of  $M$  until we return to  $\mathbf{a}$ , the curve traced out is

$$\Gamma = \Gamma_1 - \Gamma_3,$$

which may also be represented on the parameter interval  $[0, 2\pi]$  by the equations

$$\begin{aligned} x &= (1 + \delta \sin \theta) \cos(2\theta) \\ y &= (1 + \delta \sin \theta) \sin(2\theta) \\ z &= -\delta \cos \theta. \end{aligned}$$

It should be emphasized that  $\Gamma \neq \partial\Phi$ : Let  $\eta = \frac{xdy - ydx}{x^2 + y^2}$  be the 1-form discussed in Exercise 10.21 and Exercise 10.22. Since  $d\eta = 0$ , Stokes' theorem shows that

$$\int_{\partial\Phi} \eta = 0.$$

But although  $\Gamma$  is the “geometric” boundary of  $M$ , we have

$$\int_{\Gamma} \eta = 4\pi.$$

In order to avoid this possible source of confusion, Stokes' formula (Theorem 10.50) is frequently stated only for orientable surfaces  $\Phi$ .

*Proof.*

- (1) Show that  $\partial\Phi = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$ .

$$\begin{aligned}\partial\Phi &= \Phi \circ (\partial D) \\ &= \Phi \circ (\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4) \\ &= \Phi \circ \gamma_1 + \Phi \circ \gamma_2 + \Phi \circ \gamma_3 + \Phi \circ \gamma_4 \\ &= \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4.\end{aligned}$$

- (2) It is trivial that  $\Phi(\mathbf{p}_1) = \Phi(\mathbf{p}_3) = \mathbf{a} = (1, 0, -\delta)$  and  $\Phi(\mathbf{p}_2) = \Phi(\mathbf{p}_4) = \mathbf{b} = (1, 0, \delta)$  by the definition of  $\Phi$ .
- (3) Show that  $\Gamma_1$  spirals up from  $\mathbf{a}$  to  $\mathbf{b}$ ; its projection into the  $(x, y)$ -plane has winding number  $+1$  around the origin. By definition,  $\Gamma_1 = \Phi \circ \gamma_1 = \Phi([ \mathbf{p}_1, \mathbf{p}_2 ])$ . That is,  $\Gamma_1$  spirals up from  $\Phi(\mathbf{p}_1) = \mathbf{a}$  to  $\Phi(\mathbf{p}_2) = \mathbf{b}$ . Besides, the projection  $P_{\Gamma_1}$  of  $\Gamma_1$  into the  $(x, y)$ -plane ( $z = 0$ ) can be parameterized as

$$\begin{aligned}x &= \left(1 + \delta \sin \frac{t}{2}\right) \cos t \\ y &= \left(1 + \delta \sin \frac{t}{2}\right) \sin t\end{aligned}$$

for  $0 \leq t \leq 2\pi$ . Note that  $P_{\Gamma_1}$  satisfies the condition in Exercise 10.21(b). Hence  $\int_{P_{\Gamma_1}} \eta = 2\pi$ . (Here  $\eta$  is well-defined.) Apply Exercise 10.21(f) to get

$$\text{Ind}(P_{\Gamma_1}) = \frac{1}{2\pi} \int_{P_{\Gamma_1}} \eta = \frac{1}{2\pi} \cdot 2\pi = 1.$$

- (4) Show that  $\Gamma_2 = [\mathbf{b}, \mathbf{a}]$ . By definition,  $\Gamma_2 = \Phi \circ \gamma_2 = \Phi([ \mathbf{p}_2, \mathbf{p}_3 ])$  is  $[\mathbf{b}, \mathbf{a}]$  exactly.
- (5) Show that  $\Gamma_3$  spirals up from  $\mathbf{a}$  to  $\mathbf{b}$ ; its projection into the  $(x, y)$ -plane has winding number  $-1$  around the origin. Similar to (3),  $\Gamma_3$  spirals up from  $\Phi(\mathbf{p}_3) = \mathbf{a}$  to  $\Phi(\mathbf{p}_4) = \mathbf{b}$ . Now we consider  $-\Gamma_3$  instead of  $\Gamma_3$ . The projection  $P_{-\Gamma_3}$  of  $-\Gamma_3$  into the  $(x, y)$ -plane ( $z = 0$ ) can be parameterized as

$$\begin{aligned}x &= \left(1 - \delta \sin \frac{t}{2}\right) \cos t \\ y &= \left(1 - \delta \sin \frac{t}{2}\right) \sin t\end{aligned}$$

for  $0 \leq t \leq 2\pi$ . Similar to (3),  $\text{Ind}(P_{-\Gamma_3}) = 1$ . Therefore,

$$\text{Ind}(\Gamma_3) = -\text{Ind}(-\Gamma_3) = -\text{Ind}(P_{-\Gamma_3}) = -1.$$

- (6) Show that  $\Gamma_4 = [\mathbf{b}, \mathbf{a}]$ . Similar to (4).
- (7) Show that  $\Gamma = \Gamma_1 - \Gamma_3$  is the trace of from  $\mathbf{a}$  to  $\mathbf{b}$  along  $\Gamma_1$  and continue along the “edge” of  $M$  until we return to  $\mathbf{a}$ . By definition,  $\Gamma$  can be parameterized as

$$\begin{aligned}x &= (1 + \delta \sin t) \cos(2t) \\y &= (1 + \delta \sin t) \sin(2t) \\z &= -\delta \cos t\end{aligned}$$

for  $t \in [0, 2\pi]$ . Thus,  $\Gamma$  is  $\Gamma_1$  if  $t \in [0, \pi]$  and  $\Gamma$  is  $-\Gamma_3$  if  $t \in [\pi, 2\pi]$  by (3)(5). So  $\Gamma = \Gamma_1 - \Gamma_3$ .

- (8) Show that  $\int_{\partial\Phi} \eta = 0$ . Note that  $\eta$  is well-defined since  $M$  does not intersect the  $z$ -axis. So the Stokes’ theorem (Theorem 10.33) and  $d\eta = 0$  on  $M$  implies that

$$\int_{\partial\Phi} \eta = \int_{\Phi} d\eta = 0.$$

- (9) Show that  $\int_{\Gamma} \eta = 4\pi$ .

$$\begin{aligned}\int_{\Gamma} \eta &= \int_{\Gamma} \frac{xdy - ydx}{x^2 + y^2} \\&= \int_0^{2\pi} \frac{x(t)y'(t) - y(t)x'(t)}{x(t)^2 + y(t)^2} dt \\&= \int_0^{2\pi} 2 dt \\&= 4\pi.\end{aligned} \tag{7}$$

(So the winding number of  $\Gamma$  around of  $\mathbf{0}$  is 2.)

- (10) By (8)(9),  $\Gamma \neq \partial\Phi$ .

□