

Notes on the book:
*Apostol, Introduction to Analytic
Number Theory*

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Chapter 1: The Fundamental Theorem of Arithmetic

Exercise 1.30.

If $n > 1$ prove that the sum

$$\sum_{k=1}^n \frac{1}{k}$$

is not an integer.

Proof.

(1) (Reductio ad absurdum) Suppose

$$H := \sum_{k=1}^n \frac{1}{k}$$

were an integer.

(2) Let s be the largest integer such that $2^s \leq n$. So the integer number

$$\begin{aligned} 2^{s-1}H &= \sum_{k=1}^n \frac{2^{s-1}}{k} \\ &= 2^{s-1} + 2^{s-2} + \frac{2^{s-1}}{3} + 2^{s-3} + \frac{2^{s-1}}{5} + \frac{2^{s-2}}{3} + \cdots + \frac{1}{2} + \cdots. \end{aligned}$$

has only one term of even denominators (as $n > 1$) if we write all terms in irreducible fractions. That is,

$$2^{s-1}H = \frac{1}{2} + \frac{c}{d} \in \mathbb{Z}$$

where $\frac{c}{d}$ is an irreducible fraction with odd d . Hence it suffices to show that $2 \nmid d$ to get a contradiction.

(3) By

$$\frac{1}{2} + \frac{c}{d} = \frac{d+2c}{2d} \in \mathbb{Z}$$

we have $d+2c = 2dd'$ for some $d' \in \mathbb{Z}$. Note that 2 is a prime. So $2 \mid (d+2c)$ or $2 \mid d$, which is absurd.

□