Chapter 5: Differentiation

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Exercise 5.1. *Let f be defined for all real x, and suppose that*

$$|f(x) - f(y)| \leq (x - y)^2$$

for all real x and y. Prove that f is a constant.

Proof.

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq |x - y|$$

for $x \neq y$. Given any $y \in \mathbb{R}$, $|\frac{f(x)-f(y)}{x-y}| \to 0$ as $x \to y$, or $|f'(y)| = 0$. (Or using ε-δ argument. Fix $y \in \mathbb{R}$. Given any $\varepsilon > 0$, there exists $\delta = \varepsilon > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} - 0 \right| \leq |x - y| < \delta = \varepsilon$$

whenever $|x - y| < \delta$. That is, $|f'(y)| = 0$.) So $f'(y) = 0$ for any $y \in \mathbb{R}$. By Theorem 5.11 (b), f is a constant. $\square$

Exercise 5.4. *If*

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$

where $C_0, ..., C_n$ are real constants, prove that the equation

$$C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n = 0$$

has at least one real root between 0 and 1.

Proof. Let

$$g(x) = C_0 x + \frac{C_1}{2} x^2 + \cdots + \frac{C_{n-1}}{n} x^n + \frac{C_n}{n+1} x^{n+1} \in \mathbb{R}[x].$$

Then $g(0) = g(1) = 0$, and $g'(x) = C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n$. By the mean value theorem (Theorem 5.10), there exists a point $\xi \in (0, 1)$ at which

$$g(1) - g(0) = g'(\xi)(1 - 0),$$

or $g'(\xi) = 0$. That is, there exists a real root $x = \xi$ between 0 and 1 at which $C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n = 0$. $\square$

Exercise 5.14. *Let f be a differentiable real function defined in (a, b). Prove that f is convex if and only if f' is monotonically increasing. Assume next $f''(x)$ exists for every $x \in (a, b)$, and prove that f is convex if and only if $f''(x) \geq 0$ for all $x \in (a, b)$.*

Proof.

(1) *Show that f' is monotonically increasing if f is convex.*

   (a) Since f is convex, by definition (Exercise 4.23)

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

   whenever $a < x < b$, $a < y < b$, $0 < \lambda < 1$.

   (b) As $x \neq y$, we have

$$\begin{aligned} f(y) - f(x) &\geq \frac{f(x + \lambda(y - x)) - f(x)}{\lambda} \\ &= \frac{f(x + \lambda(y - x)) - f(x)}{\lambda(y - x)} \cdot (y - x) \end{aligned}$$

   and let $\lambda \to 0$ to get

$$f(y) - f(x) \geq f'(x)(y - x)$$

   (since $f'(x)$ exists). Similarly, we have

$$f(x) - f(y) \geq f'(y)(x - y).$$

   (c) Given any $y > x$, we have

$$f'(y)(y - x) \geq f(y) - f(x) \geq f'(x)(y - x).$$

   Hence $f'(y) \geq f'(x)$ whenever $y > x$, or f' is monotonically increasing.

(2) *Show that f is convex if f' is monotonically increasing.* Given any $y > x$ and any $0 < \lambda < 1$.

   (a) By Theorem 5.10 (mean value theorem), there is a point $x < \xi < y$ such that

$$f(y) - f(x) = f'(\xi)(y - x).$$

   Since f' is monotonically increasing,

$$f'(y)(y - x) \geq f(y) - f(x) \geq f'(x)(y - x).$$

(b) Write $z = \lambda x + (1-\lambda)y$. Hence

$$\begin{aligned} f(y) - f(z) &\geq f'(z)(y - z), \\ f(z) - f(x) &\leq f'(z)(z - x), \end{aligned}$$

or

$$\begin{aligned} f(y) &\geq f(z) + f'(z)(y - z), \\ f(x) &\geq f(z) + f'(z)(x - z), \end{aligned}$$

or

$$\begin{aligned} \lambda f(x) + (1-\lambda)f(y) \geq& \lambda[f(z) + f'(z)(x - z)] \\ &+ (1-\lambda)[f(z) + f'(z)(y - z)] \\ =& f(z) \\ =& f(\lambda x + (1-\lambda)y). \end{aligned}$$

Hence f is convex.

(3) *Show that $f''(x) \geq 0$ if f is convex and f'' exists.* By (1), f' is monotonically increasing since f is convex. Given any $x \neq y$, we have

$$\frac{f'(y) - f'(x)}{y - x} \geq 0.$$

Let $y \to x$, we have $f''(x) \geq 0$ if f'' exists.

(4) *Show that f is convex if f'' exists and $f''(x) \geq 0$.* By Theorem 5.11(a), f' is monotonically increasing. By (2), f is convex.

$\square$