Chapter 7: Sequences and Series of Functions

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Exercise 7.1. *Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.*

Proof (Cauchy criterion). Let $\{f_n\}$ be a uniformly convergent sequence of bounded functions.

(1) Since f_n is bounded, there exists M_n such that $|f_n(x)| \leq M_n$.

(2) Since $\{f_n\}$ converges uniformly, given $1 > 0$ there exists an integer N such that
$$|f_n(x) - f_m(x)| \leq 1 \text{ whenever } n, m \geq N$$
(Theorem 7.8 (Cauchy criterion for uniformly convergence)). Especially,
$$|f_n(x)| \leq |f_n(x) - f_N(x)| + |f_N(x)| \leq 1 + M_N \text{ whenever } n \geq N.$$

(3) Thus, $\{f_n\}$ is uniformly bounded by $M = \max\{M_1, \ldots, M_{N-1}, M_N + 1\}$.

□

Exercise 7.2. *If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n + g_n\}$ converge uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E.*

Proof. Let $f_n \to f$ uniformly and $g_n \to g$ uniformly.

(1) *Show that $\{f_n + g_n\}$ converges uniformly.* Given $\varepsilon > 0$. Since $f_n \to f$ uniformly and $g_n \to g$ uniformly, there exist two integers N_1 and N_2 such that
$$|f_n(x) - f(x)| \leq \frac{\varepsilon}{2} \text{ whenever } n \geq N_1, x \in E$$
$$|g_n(x) - g(x)| \leq \frac{\varepsilon}{2} \text{ whenever } n \geq N_2, x \in E.$$
Take $N = \max\{N_1, N_2\}$, we have
$$\begin{aligned} & |(f_n(x) + g_n(x)) - (f(x) + g(x))| \\ = & |(f_n(x) - f(x)) + (g_n(x) - g(x))| \\ \leq & |f_n(x) - f(x)| + |g_n(x) - g(x)| \\ \leq & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ = & \varepsilon \end{aligned}$$

whenever $n \geq N$, $x \in E$. Hence $f_n + g_n \to f + g$ uniformly on E.

(2) *Show that $\{f_n g_n\}$ converges uniformly if, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions.* Given $\varepsilon > 0$.

(a) By Exercise 7.1, both $\{f_n\}$ and $\{g_n\}$ are uniformly bounded. So there exist M_1 and M_2 such that

$$|f_n(x)| \leq M_1 \text{ and } |g_n(x)| \leq M_2$$

for all n and $x \in E$. Also, $|f(x)| \leq M_1 + 1$ and $|g(x)| \leq M_2 + 1$.

(b) Since $f_n \to f$ uniformly and $g_n \to g$ uniformly, there exist two integers N_1 and N_2 such that

$$|f_n(x) - f(x)| \leq \frac{\varepsilon}{2(M_2+1)} \text{ whenever } n \geq N_1, x \in E$$
$$|g_n(x) - g(x)| \leq \frac{\varepsilon}{2(M_1+1)} \text{ whenever } n \geq N_2, x \in E.$$

(Note that each denominator of $\frac{\varepsilon}{2(M_j+1)}$ $(j = 1, 2)$ is well-defined and positive!) Take $N = \max\{N_1, N_2\}$, we have

$$\begin{aligned}
&|f_n(x)g_n(x) - f(x)g(x)| \\
=&|[f_n(x) - f(x)]g_n(x) + f(x)[g_n(x) - g(x)]| \\
\leq&|f_n(x) - f(x)||g_n(x)| + |f(x)||g_n(x) - g(x)| \\
\leq&\frac{\varepsilon}{2(M_2+1)} \cdot M_2 + (M_1+1) \cdot \frac{\varepsilon}{2(M_1+1)} \\
\leq&\varepsilon
\end{aligned}$$

whenever $n \geq N$, $x \in E$. Hence $f_n g_n \to fg$ uniformly on E.

□

Proof (Cauchy criterion).

(1) *Show that $\{f_n + g_n\}$ converges uniformly.* Given $\varepsilon > 0$. Since $\{f_n\}$ and $\{g_n\}$ converge uniformly, there exist two integers N_1 and N_2 such that

$$|f_n(x) - f_m(x)| \leq \frac{\varepsilon}{2} \text{ whenever } n, m \geq N_1, x \in E$$
$$|g_n(x) - g_m(x)| \leq \frac{\varepsilon}{2} \text{ whenever } n, m \geq N_2, x \in E.$$

Take $N = \max\{N_1, N_2\}$, we have

$$\begin{aligned}
&|(f_n(x) + g_n(x)) - (f_m(x) + g_m(x))| \\
=&|(f_n(x) - f_n(x)) + (g_n(x) - g_m(x))| \\
\leq&|f_n(x) - f_n(x)| + |g_n(x) - g_m(x)| \\
\leq&\frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
=&\varepsilon
\end{aligned}$$

whenever $n, m \geq N$, $x \in E$. Hence $\{f_n + g_n\}$ converges uniformly on E.

(2) *Show that $\{f_n g_n\}$ converges uniformly if, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions.* Given $\varepsilon > 0$.

(a) By Exercise 7.1, both $\{f_n\}$ and $\{g_n\}$ are uniformly bounded. So there exist M_1 and M_2 such that

$$|f_n(x)| \leq M_1 \text{ and } |g_n(x)| \leq M_2$$

for all n and $x \in E$. Also, $|f(x)| \leq M_1 + 1$ and $|g(x)| \leq M_2 + 1$.

(b) Since $\{f_n\} \to f$ uniformly and $\{g_n\} \to g$ uniformly, there exist two integers N_1 and N_2 such that

$$|f_n(x) - f_m(x)| \leq \frac{\varepsilon}{2(M_2+1)} \text{ whenever } n, m \geq N_1, x \in E$$
$$|g_n(x) - g_m(x)| \leq \frac{\varepsilon}{2(M_1+1)} \text{ whenever } n, m \geq N_2, x \in E.$$

Take $N = \max\{N_1, N_2\}$, we have

$$\begin{aligned}
&|f_n(x)g_n(x) - f_m(x)g_m(x)| \\
=&|[f_n(x) - f_m(x)]g_n(x) + f_m(x)[g_n(x) - g_m(x)]| \\
\leq&|f_n(x) - f_m(x)||g_n(x)| + |f_m(x)||g_n(x) - g_m(x)| \\
\leq&\frac{\varepsilon}{2(M_2+1)} \cdot M_2 + M_1 \cdot \frac{\varepsilon}{2(M_1+1)} \\
\leq&\varepsilon
\end{aligned}$$

whenever $n \geq N$, $x \in E$. Hence $\{f_n g_n\}$ converges uniformly on E.

□

Note. It proved that $f_n g_n \to fg$ in Theorem 7.29.

Exercise 7.3. *Construct sequences $\{f_n\}$, $\{g_n\}$ which converge uniformly on some set E, but such that $\{f_n g_n\}$ does not converge uniformly on E (of course, $\{f_n g_n\}$ must converge on E).*

We provides some examples here.

Proof ($f_n(x) = x + \frac{1}{n}$).

(1) Define $\{f_n(x)\}$ on $E = \mathbb{R}$ by $f_n(x) = x + \frac{1}{n}$ and $f(x) = x$. Clearly, $\{f_n(x)\}$ converges to $f(x)$ pointwise.

(2) *Show that $\{f_n\}$ converges uniformly.* Given $\varepsilon > 0$. There exists an integer $N \geq \frac{1}{\varepsilon}$ such that

$$|f_n(x) - f(x)| = \frac{1}{n} \leq \frac{1}{N} \leq \varepsilon$$

whenever $n \geq N$ and $x \in E$. Hence $\{f_n\} \to f$ uniformly.

(3) *Show that $\{f_n^2\}$ does not converge uniformly.* Clearly, $\{f_n(x)^2\}$ converges to $f(x)^2$ pointwise. Hence

$$\sup_{x\in E} |f_n(x)^2 - f(x)^2| = \sup_{x \in E} \left| \frac{2x}{n} + \frac{1}{n^2} \right| \to \infty$$

as $n \to \infty$ (by considering $x = n^2 \in E$). Hence $\{f_n^2\}$ does not converge uniformly (Theorem 7.9).

□

Proof ($f_n(x) = \frac{1}{x}$, $g_n(x) = \frac{1}{n}$).

(1) Let $E = (0, 1)$. Let $\{f_n(x)\}$ on E be $f_n(x) = \frac{1}{x}$ and $\{g_n(x)\}$ on E be $g_n(x) = \frac{1}{n}$. Clearly, $\{f_n(x)\}$ converges to $f(x) = \frac{1}{x}$ pointwise and $\{g_n(x)\}$ converges to $g(x) = 0$ pointwise.

(2) *Show that $\{f_n\}$ converges uniformly.* Given $\varepsilon > 0$. There exists an integer $N = 1$ such that

$$|f_n(x) - f(x)| = 0 \leq \varepsilon$$

whenever $n \geq N$ and $x \in E$. Hence $\{f_n\} \to f$ uniformly.

(3) *Show that $\{g_n\}$ converges uniformly.* Given $\varepsilon > 0$. There exists an integer $N \geq \frac{1}{\varepsilon}$ such that

$$|g_n(x) - g(x)| = \frac{1}{n} \leq \frac{1}{N} \leq \varepsilon$$

whenever $n \geq N$ and $x \in E$. Hence $\{g_n\} \to g$ uniformly.

(4) *Show that $\{f_n g_n\}$ does not converge uniformly.* Clearly, $\{f_n(x)g_n(x)\}$ converges to $f(x)g(x) = 0$ pointwise. Hence

$$\sup_{x\in E} |f_n(x)g_n(x) - 0| = \sup_{x\in E} \left| \frac{1}{nx} \right| \to \infty$$

as $n \to \infty$ (by considering $x = \frac{1}{n^2} \in E$). Hence $\{f_n g_n\}$ does not converge uniformly (Theorem 7.9).

□

Proof (Exercise 9.2 in Tom M. Apostol, Mathematical Analysis, 2nd edition).

(1) Let $E = [\alpha, \beta] \subseteq \mathbb{R}$ be a bounded interval. Define two sequences $\{f_n\}$ and $\{g_n\}$ on E as follows:

$$f_n(x) = x\left(1 + \frac{1}{n}\right) \text{ if } x \in \mathbb{R},\ n = 1, 2, \cdots,$$

$$g_n(x) = \begin{cases} \frac{1}{n} & \text{if } x = 0 \text{ or if } x \text{ is irrational,} \\ b + \frac{1}{n} & \text{if } x \text{ is rational} \neq 0, \text{ say } x = \frac{a}{b},\ b > 0. \end{cases}$$

Here we assume that $\gcd(a, b) = 1$. Clearly, $f(x) = x$ and

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or if } x \text{ is irrational,} \\ b & \text{if } x \text{ is rational} \neq 0, \text{ say } x = \frac{a}{b},\ b > 0. \end{cases}$$

Let $M = \max\{|\alpha|, |\beta|\} \geq 0$.

(2) *Show that $\{f_n\}$ converges uniformly.* Given $\varepsilon > 0$. There exists an integer $N \geq \frac{M}{\varepsilon}$ such that

$$|f_n(x) - f(x)| = \frac{|x|}{n} \leq \frac{M}{N} \leq \varepsilon$$

whenever $n \geq N$ and $x \in E$. Hence $\{f_n\} \to f$ uniformly.

(3) *Show that $\{g_n\}$ converges uniformly.* Given $\varepsilon > 0$. There exists an integer $N \geq \frac{1}{\varepsilon}$ such that

$$|g_n(x) - g(x)| = \frac{1}{n} \leq \frac{1}{N} \leq \varepsilon$$

whenever $n \geq N$ and $x \in E$. Hence $\{g_n\} \to g$ uniformly.

(4) *Show that $\{f_n g_n\}$ does not converge uniformly.*

(a) Clearly, $\{f_n(x)g_n(x)\}$ converges to $f(x)g(x)$ pointwise where

$$f(x)g(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or if } x \text{ is irrational,} \\ a & \text{if } x = \frac{a}{b} \text{ is rational} \neq 0,\ b > 0. \end{cases}$$

(b) Note that

$$f_n(x)g_n(x) = \begin{cases} \frac{x}{n}\left(1 + \frac{1}{n}\right) & \text{if } x = 0 \text{ or if } x \text{ is irrational,} \\ \left(a + \frac{x}{n}\right)\left(1 + \frac{1}{n}\right) & \text{if } x = \frac{a}{b} \text{ is rational} \neq 0,\ b > 0. \end{cases}$$

Therefore,

$$f_n(x)g_n(x) - f(x)g(x) = \begin{cases} \frac{x}{n}\left(1 + \frac{1}{n}\right) & \text{if } x = 0 \text{ or if } x \text{ is irrational,} \\ \frac{x}{n}\left(1 + b + \frac{1}{n}\right) & \text{if } x = \frac{a}{b} \text{ is rational} \neq 0,\ b > 0. \end{cases}$$

(c) Hence

$$\begin{aligned} \sup_{x \in E} |f_n(x)g_n(x) - f(x)g(x)| &\geq \sup_{x \in E \cap \mathbb{Q}} |f_n(x)g_n(x) - f(x)g(x)| \\ &= \sup_{x \in E \cap \mathbb{Q}} |a| \left(\frac{1}{n} + \frac{1}{bn} + \frac{1}{bn^2}\right) \\ &\geq \sup_{x \in E \cap \mathbb{Q}} |a| \left(\frac{1}{n}\right) \\ &= \sup_{x \in E \cap \mathbb{Q}} \frac{|a|}{n}. \end{aligned}$$

(d) *Given any irrational number $\gamma \in E$, there exists a sequence*

$$\left\{ r_m = \frac{a_m}{b_m} \right\}$$

of nonzero rational numbers in E such that $\lim r_m = \gamma$. Show that $\{a_m\}$ is unbounded. If it is true, we can find $x_n = r_{m_n} = \frac{a_{m_n}}{b_{m_n}}$ such that $|a_{m_n}| \geq n^2$ and

$$\sup_{x \in E} |f_n(x)g_n(x) - f(x)g(x)| \geq \sup_{x \in E \cap \mathbb{Q}} \frac{|a|}{n} \geq \frac{n^2}{n} = n \to \infty$$

as $n \to \infty$.

(e) (Reductio ad absurdum) If $\{a_m\}$ were bounded, then there exists a **constant** subsequence of $\{a_{m_k}\}$ such that $\lim a_{m_k} = a \in \mathbb{Z}$. Since $\lim_{m \to \infty} r_m = \gamma$, $\lim_{k \to \infty} r_{m_k} = \gamma$ or

$$\lim_{k \to \infty} b_{m_k} = \lim_{k \to \infty} \frac{a_{m_k}}{r_{m_k}} = \frac{a}{\gamma}$$

(it is well-defined since r_{m_k} and γ cannot be zero). Since all b_{m_k} are positive integers, the limit $\lim b_{m_k} = b$ is a positive integer too, or $b = \frac{a}{\gamma} \in \mathbb{Z}^+$, or $\gamma = \frac{a}{b} \in \mathbb{Z}$, which is absurd.

Therefore, $\{f_n g_n\}$ does not converge uniformly.

□

Exercise 7.4. *Consider*

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}.$$

For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous whenever the series converges? Is f bounded?

Proof. Clearly, $f(x)$ is defined on $\mathbb{R} - \{-1, -\frac{1}{4}, -\frac{1}{9}, \ldots\}$.

(1)

PLACEHOLDER

Exercise 7.5. *Let*

$$f_n(x) = \begin{cases} 0 & (x < \frac{1}{n+1}), \\ \sin^2 \frac{\pi}{x} & (\frac{1}{n+1} \leq x \leq \frac{1}{n}), \\ 0 & (\frac{1}{n} < x). \end{cases}$$

Show that $\{f_n\}$ converges to a continuous function, but not uniformly. Use the series $\sum f_n$ to show that absolute convergence, even for all x, does not imply uniform convergence.

Proof.

(1) *Show that $\lim_{n\to\infty} f_n(x) = 0$. Hence $\{f_n\}$ converges to a continuous function 0 pointwise.* Clearly, $f_n(x) = 0$ for all $x \not\in (0,1)$. Next, for any fixed $x \in (0,1)$, there exists an integer $N > \frac{1}{x}$ such that

$$x > \frac{1}{N} \geq \frac{1}{n}$$

whenever $n \geq N$. Hence $f_n(x) = 0$ whenever $n \geq N$.

(2) *Show that $f_n \to f = 0$ not uniformly.* Let

$$x_n = \frac{1}{n + \frac{1}{2}} \to 0$$

for all $n = 1, 2, 3, \ldots$. Thus, $f_m(x_n) = \delta_{mn}$, where δ_{mn} is Kronecker delta.

(a) *(Definition 7.7.)* (Reductio ad absurdum) If $\{f_n\}$ were convergent uniformly, then given $\varepsilon = \frac{1}{64} > 0$, there exists an integer N such that $n \geq N$ implies

$$|f_n(x) - f(x)| \leq \frac{1}{64}$$

for all real x. However,

$$|f_N(x_N) - f(x_N)| = 1 > \frac{1}{64},$$

which is absurd.

(b) *(Theorem 7.8)* (Reductio ad absurdum) If $\{f_n\}$ were convergent uniformly, then given $\varepsilon = \frac{1}{64} > 0$, there exists an integer N such that $n, m \geq N$ implies

$$|f_n(x) - f_m(x)| \leq \frac{1}{64}$$

for all real x. However,

$$|f_N(x_N) - f_{N+1}(x_N)| = 1 > \frac{1}{64},$$

which is absurd.

(c) *(Theorem 7.9)* Since

$$M_n = \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| \geq |f_n(x_n) - f(x_n)| = 1,$$

$f_n \to f$ not uniformly.

(d) *(Exercise 7.9.)* Since each f_n is continuous and

$$\lim_{n\to\infty} f_n(x_n) = \lim_{n\to\infty} 1 = 1 \neq 0 = f(0),$$

$f_n \to f = 0$ not uniformly.

(3) *Show that $\sum f_n$ converges absolutely.* Write $F_n = \sum_{k=1}^n f_k$ and $F = \sum f_n$. Clearly,

$$F(x) = \begin{cases} 0 & (x \leq 0), \\ \sin^2 \frac{\pi}{x} & (0 < x \leq 1), \\ 0 & (x \geq 1). \end{cases}$$

Note that $f_n \geq 0$ for each n. Hence $\sum f_n$ converges absolutely.

(4) *Show that $\sum f_n$ does not converge uniformly.* Similar to (2). Let

$$x_n = \frac{1}{n + \frac{1}{2}} \to 0$$

for all $n = 1, 2, 3, \ldots$. Thus

$$F_m(x_n) = \begin{cases} 1 & (m \geq n), \\ 0 & (m < n). \end{cases}$$

(a) *(Definition 7.7.)* (Reductio ad absurdum) If $\{F_n\}$ were convergent uniformly, then given $\varepsilon = \frac{1}{64} > 0$, there exists an integer N such that $n \geq N$ implies

$$|F_n(x) - F(x)| \leq \frac{1}{64}$$

for all real x. However,

$$|F_N(x_{N+1}) - F(x_{N+1})| = 1 > \frac{1}{64},$$

which is absurd.

(b) *(Theorem 7.8)* (Reductio ad absurdum) If $\{F_n\}$ were convergent uniformly, then given $\varepsilon = \frac{1}{64} > 0$, there exists an integer N such that $n, m \geq N$ implies

$$|F_n(x) - F_m(x)| \leq \frac{1}{64}$$

for all real x. However,

$$|F_N(x_{N+1}) - F_{N+1}(x_{N+1})| = 1 > \frac{1}{64},$$

which is absurd.

(c) *(Theorem 7.9)* Since

$$M_n = \sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \geq |F_n(x_{n+1}) - F(x_{n+1})| = 1,$$

$F_n \to F$ not uniformly.

(d) *(Exercise 7.9.)* Since each F_n is continuous and

$$\lim_{n\to\infty} F_n(x_{n+1}) = \lim_{n\to\infty} 0 \neq 1 = F(x_{n+1}),$$

$F_n \to F$ not uniformly.

(e) *(Theorem 7.12.)* (Reductio ad absurdum) If $\{F_n\}$ were converging to F uniformly, then F were continuous since each F_n is continuous by Theorem 7.12. However, F is not continuous at $x = 0$.

□

Exercise 7.6. *Prove that the series*

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

Proof (Dirichlet's test). Given any bounded interval $E = [\alpha, \beta] \subseteq \mathbb{R}$. Write $f_n(x) = (-1)^n$ on E and $g_n(x) = \frac{x^2+n}{n^2}$ on E.

(1) The partial sums $F_n(x)$ of $\sum f_n(x)$ form a uniformly bounded sequence.

(2) $g_1(x) \geq g_2(x) \geq \cdots$ since

$$g_{n+1}(x) = \frac{x^2}{(n+1)^2} + \frac{1}{n+1} < \frac{x^2}{n^2} + \frac{1}{n} = g_n(x).$$

(3) Write $M = \max\{|\alpha|, |\beta|\}$. Since

$$|g_n(x)| = \frac{x^2}{n^2} + \frac{1}{n} \leq \frac{M^2}{n^2} + \frac{1}{n} \to \infty$$

as $n \to \infty$, $\lim_{n\to\infty} g_n(x) = 0$. By Dirichlet's test (Exercise 7.11), $\sum_{n=1}^{\infty} f_n(x) g_n(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}$ converges.

(4)

$$\begin{aligned}\sum |f_n(x)| &= \sum \frac{x^2+n}{n^2} \\ &\geq \sum \frac{n}{n^2} \\ &= \sum \frac{1}{n} \to \log n + \gamma\end{aligned}$$

(Exercise 8.9). Hence $\sum (-1)^n \frac{x^2+n}{n^2}$ does not converge absolutely for any value of x.

$\square$

Exercise 7.7. *For $n = 1, 2, 3, \ldots$, x real, put*

$$f_n(x) = \frac{x}{1 + nx^2}.$$

Show that $\{f_n\}$ converges uniformly to a function f, and that the equation

$$f'(x) = \lim_{n \to \infty} f_n'(x)$$

is correct if $x \neq 0$, but false if $x = 0$.

$f_n(x)$ is defined on $\mathbb{R}$.

Proof.

(1) Since

$$|f_n(x)| = \left| \frac{x}{1 + nx^2} \right| \leq \frac{|x|}{\sqrt{n}|x|} = \frac{1}{\sqrt{n}} \to \infty$$

as $n \to \infty$, $f_n \to 0$ uniformly (Theorem 7.9).

(2) Clearly, $f'(x) = 0$. Since

$$f_n'(x) = \frac{1 - nx^2}{(1 + nx^2)^2},$$

$$\lim_{n \to \infty} f_n'(x) = \begin{cases} 1 & (x = 0), \\ 0 & (x \neq 0). \end{cases}$$

So that the equation

$$f'(x) = \lim_{n \to \infty} f_n'(x)$$

is correct if $x \neq 0$, but false if $x = 0$.

$\square$

Note. $f_n'(x)$ does not converge uniformly by considering

$$\lim_{n \to \infty} f_n' \left(\frac{1}{n} \right) = \lim_{n \to \infty} \frac{1 - \frac{1}{n}}{(1 + \frac{1}{n})^2} = 1.$$

Exercise 7.8. *If*

$$I(x) = \begin{cases} 0 & (x \leq 0), \\ 1 & (x > 0), \end{cases}$$

if $\{x_n\}$ *is a sequence of distinct points of* (a, b)*, and if* $\sum |c_n|$ *converges, prove that the series*

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n) \qquad (a \leq x \leq b)$$

converges uniformly, and that f *is continuous for every* $x \neq x_n$*.*

Proof.

(1) Define $f_n(x) = c_n I(x - x_n)$ on (a, b). So

$$|f_n(x)| = |c_n||I(x - x_n)| \leq |c_n| \qquad (x \in (a, b), n = 1, 2, 3, \ldots).$$

Since $\sum |c_n|$ converges, $f = \sum f_n$ converges uniformly (Theorem 7.10).

(2) Given any $p \in (a, b)$ with $p \neq x_n$ for all $n = 1, 2, 3, \ldots$. So each $I(x - x_n)$ is continuous at $x = p$, and thus each partial sum $\sum_{n=1}^{N} f_n(x)$ is continuous.

(3) By Theorem 7.11

$$\begin{aligned} \lim_{x \to p} f(x) &= \lim_{x \to p} \sum_{n=1}^{\infty} f_n(x) \\ &= \lim_{N \to \infty} \left(\lim_{x \to p} \sum_{n=1}^{N} f_n(x) \right) \\ &= \lim_{N \to \infty} \sum_{n=1}^{N} f_n(p) \\ &= \sum_{n=1}^{\infty} f_n(p) \\ &= f(p). \end{aligned}$$

$f(x)$ is continuous at $x = p$ too.

□

Exercise 7.9. *Let* $\{f_n\}$ *be a sequence of continuous functions which converges uniformly to a function* f *on a set* E*. Prove that*

$$\lim_{n \to \infty} f_n(x_n) = f(x)$$

for every sequence of points $x_n \in E$ *such that* $x_n \to x$*, and* $x \in E$*. Is the converse of this true?*

Proof.

(1) Given any $x \in E$ and any $\varepsilon > 0$. Since each f_n is continuous and $f_n \to f$ uniformly, f is continuous (Theorem 7.12). Hence as $x_n \to x$, there exists an integer N_1 such that

$$|f(x_n) - f(x)| \leq \frac{\varepsilon}{2} \text{ whenever } n \geq N_1$$

(Theorem 4.2). Also, $f_n \to f$ uniformly implies that there exists an integer N_2 such that

$$|f_n(x_n) - f(x_n)| \leq \frac{\varepsilon}{2} \text{ whenever } n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$ be an integer. Then

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $n \geq N$. Therefore, $\lim_{n\to\infty} f_n(x_n) = f(x)$.

(2) *Show that the converse is false.* Let $E = (0,1)$ and $f_n = \frac{1}{nx}$ on E. Given any $x \in E$. First,

$$f(x) = \lim_{n\to\infty} f_n = \lim_{n\to\infty} \frac{1}{nx} = 0$$

Next, for each sequence of points $x_n \in E$ such that $x_n \to x$ (note that each $x_n \neq 0$ and $x \neq 0$), we have

$$\lim_{n\to\infty} f_n(x_n) = \lim_{n\to\infty} \frac{1}{nx_n} = \lim_{n\to\infty} \frac{1}{n} \lim_{n\to\infty} \frac{1}{x_n} = 0 \cdot \frac{1}{x} = 0.$$

Hence $\lim_{n\to\infty} f_n(x_n) = f(x) = 0$. However, $\{f_n\}$ does not converge uniformly. (See *Proof ($f_n(x) = \frac{1}{x}$, $g_n(x) = \frac{1}{n}$)* in Exercise 7.3.)

□

Exercise 7.10. *Letting (x) denote the fractional part of the real number x (see Exercise 4.16 for the definition), consider the function*

$$f(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2} \qquad (x \in \mathbb{R}).$$

Find all discontinuities of f, and show that they form a countable dense set. Show that f is nevertheless Riemann-integrable on every bounded interval.

Proof. Let $f_n(x) = \frac{(nx)}{n^2}$ on $\mathbb{R}$, $F_n(x) = \sum_{k=1}^{n} f_k(x)$ on $\mathbb{R}$.

(1) Since

$$|f_n(x)| = \left| \frac{(nx)}{n^2} \right| \leq \frac{1}{n^2}$$

for all $x \in \mathbb{R}$ and $n = 1, 2, 3, \ldots$ and $\sum \frac{1}{n^2}$ converges (to $\frac{\pi^2}{6}$), $F_n = \sum f_k$ converges uniformly to f on $\mathbb{R}$ (Theorem 7.10).

(2) Note that (x) is continuous on $\mathbb{R} - \mathbb{Z}$ and not continuous on $\mathbb{Z}$ (Exercise 4.16). Now we define $E_n = \{x \in \mathbb{R} : nx \in \mathbb{Z}\}$. So $E_1 = \mathbb{Z}$, and

$$\bigcup_{n=1}^{\infty} E_n = \mathbb{Q}.$$

So f_n is continuous on $\mathbb{R} - E_n$ and not continuous on E_n. So $F_n = \sum f_k$ is continuous on $\mathbb{R} - \bigcup_{k=1}^{n} E_k \supseteq \mathbb{R}$ - $\mathbb{Q}$.

(3) *Show that $f(x)$ is continuous on $\mathbb{R}$ - $\mathbb{Q}$.* Since $\{F_n\}$ is a sequence of continuous functions on $\mathbb{R}$ - $\mathbb{Q}$ (by (2)) and $F_n \to f$ uniformly (by (1)), f is continuous on $\mathbb{R}$ - $\mathbb{Q}$ (Theorem 7.12).

(4) *Show that $f(x)$ is not continuous on $\mathbb{Q}$, which is a countable dense set of $\mathbb{R}$.*

(a) (Reductio ad absurdum) If there were $p = \frac{a}{b} \in \mathbb{Q}$ with $a, b \in \mathbb{Z}$, $(a, b) = 1$ and $b > 0$ such that $f(x)$ is continuous at $x = p$, then

$$\lim_{x \to p^-} f(x) = \lim_{x \to p^+} f(x).$$

(b) As $b \mid n$, say $n = bq$ for some $q \in \mathbb{Z}^+$, we have

$$\begin{aligned} \lim_{x \to p^-} f_n(x) &= \lim_{x \to p^-} \frac{1}{b^2 q^2} = \frac{1}{b^2 q^2}, \\ \lim_{x \to p^+} f_n(x) &= \lim_{x \to p^+} \frac{0}{b^2 q^2} = 0. \end{aligned}$$

As $b \nmid n$,

$$\lim_{x \to p^-} f_n(x) = \lim_{x \to p^+} f_n(x) = f_n(p).$$

Thus,

$$\lim_{x \to p^-} F_n(x) - \lim_{x \to p^+} F_n(x) = \frac{1}{b^2} \sum_{q=1}^{[\frac{n}{b}]} \frac{1}{q^2}.$$

(c) Since $F_n \to f$ uniformly, given $\varepsilon = \frac{64}{1989 b^2} > 0$, there exists an integer N' such that

$$\left| \sum_{n=m}^{\infty} f_n(x) \right| = \sum_{n=m}^{\infty} f_n(x) \leq \frac{64}{1989 b^2}$$

whenever $m \geq N'$.

(d) Take $N = \max\{N', b\}$.

$$
\begin{aligned}
&\left| \underbrace{\lim_{x \to p^-} f(x)}_{\text{exists}} - \underbrace{\lim_{x \to p^+} f(x)}_{\text{exists}} \right| \\
=&\left| \underbrace{\lim_{x \to p^-} F_N(x)}_{\text{exists}} - \underbrace{\lim_{x \to p^+} F_N(x)}_{\text{exists}} + \underbrace{\lim_{x \to p^-} \sum_{n=N+1}^{\infty} f_n(x)}_{\text{exists}} - \underbrace{\lim_{x \to p^+} \sum_{n=N+1}^{\infty} f_n(x)}_{\text{exists}} \right| \\
\geq&\left| \lim_{x \to p^-} F_N(x) - \lim_{x \to p^+} F_N(x) \right| - \left| \lim_{x \to p^-} \sum_{n=N+1}^{\infty} f_n(x) \right| - \left| \lim_{x \to p^+} \sum_{n=N+1}^{\infty} f_n(x) \right| \\
\geq&\frac{1}{b^2} \sum_{q=1}^{[\frac{n}{b}]} \frac{1}{q^2} - \frac{64}{1989b^2} - \frac{64}{1989b^2} \\
\geq&\frac{1}{q^2} - \frac{64}{1989b^2} - \frac{64}{1989b^2} \\
=&\frac{1861}{1989b^2} \\
>&0,
\end{aligned}
$$

which is absurd.

(4) *Show that f is nevertheless Riemann-integrable on every bounded interval.* Since each $f_n \in \mathscr{R}$ on every bounded interval, $F_n \in \mathscr{R}$ on every bounded interval. Since $F_n \to f$ uniformly, $f \in \mathscr{R}$ on every bounded interval by Theorem 7.16.

□

Exercise 7.11 (Dirichlet's test). *Suppose $\{f_n\}$, $\{g_n\}$ are defined on E, and*

(a) *$\sum f_n(x)$ has uniformly bounded partial sums;*

(b) *$g_n(x) \to 0$ uniformly on E;*

(b) *$g_1(x) \geq g_2(x) \geq g_3(x) \geq \cdots$ for every $x \in E$.*

Prove that $\sum f_n(x) g_n(x)$ converges uniformly on E. (Hint: Compare with Theorem 3.42.)

Theorem 3.42 (Dirichlet's test). Suppose

(a) the partial sums A_n of $\sum a_n$ form a bounded sequence;

(b) $b_0 \geq b_1 \geq b_2 \geq \cdots$;

(c) $\lim_{n\to\infty} b_n = 0$.

Then $\sum a_n b_n$ converges.

Proof (Theorem 3.42). Let $F_n(x) = \sum_{k=1}^{n} f_k(x)$. Choose M such that $|F_n(x)| \leq M$ for all n, all $x \in E$. Given $\varepsilon > 0$, there is an integer N such that $g_N(x) \leq \frac{\varepsilon}{2(M+1)}$ for all $x \in E$. For $N \leq p \leq q$, we have

$$\begin{aligned}
&\left|\sum_{n=p}^{q} f_n(x)g_n(x)\right| \\
=&\left|\sum_{n=p}^{q-1} F_n(x)(g_n(x) - g_{n+1}(x)) + F_q(x)g_q(x) - F_{p-1}(x)g_p(x)\right| \\
\leq& M\left|\sum_{n=p}^{q-1}(g_n(x) - g_{n+1}(x)) + g_q(x) + g_p(x)\right| \\
=&2Mg_p(x) \\
\leq&2Mg_N(x) \\
\leq&\varepsilon
\end{aligned}$$

for all $x \in E$. Uniformly convergence now follows from the Cauchy criterion (Theorem 7.8). Note that the first inequality in the above chain depends of course on the fact that $g_n(x) - g_{n+1}(x) \geq 0$. □

Exercise 7.12. PLACEHOLDER
Exercise 7.13. PLACEHOLDER
Exercise 7.14. PLACEHOLDER
Exercise 7.15.
PLACEHOLDER

Exercise 7.16. *Suppose $\{f_n\}$ is an equicontinuous sequence of functions on a compact set K, and $\{f_n\}$ converges pointwise on K. Prove that $\{f_n\}$ converges uniformly on K.*

(Assume that $\{f_n\}$ is a sequence of complex-valued functions.)

Proof. Given any $\varepsilon > 0$.

(1) Since $\{f_n\}$ is equicontinuous, there is $\delta > 0$ such that

$$|f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$$

whenever $x, y \in K$, $|x - y| < \delta$, $n = 1, 2, 3, \ldots$ (where d is the metric function).

(2) (Similar to Proof (Heine-Borel Theorem) in Exercise 4.8.) For such $\delta > 0$, we construct an open covering of K. Pick a collection $\mathscr{C}$ of open balls $B(a;\delta) \subseteq K$ where a runs over all elements of K. Since $\mathscr{C}$ is an open covering of a compact set K, there is a finite subcollection $\mathscr{C}'$ of $\mathscr{C}$ also covers K, say

$$\mathscr{C}' = \{B(a_1;\delta)), B(a_2;\delta), ..., B(a_m;\delta)\}.$$

(3) Since f_n converges pointwise on K, for each i there is an integer N_i such that

$$|f_n(a_i) - f_m(a_i)| < \frac{\varepsilon}{3}$$

whenever $n, m \geq N_i$.

(4) Now given any $x \in K$, by (2) there exists a_j $(1 \leq j \leq m)$ such that $x \in B(a_j;\delta)$. Take $N = \max\{N_1, \ldots, N_m\}$. Hence

$$\begin{aligned}|f_n(x) - f_m(x)| &\leq |f_n(x) - f_n(a_j)| + |f_n(a_j) - f_m(a_j)| + |f_m(a_j) - f_m(x)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon.\end{aligned}$$

whenever $n, m \geq N$. Hence $\{f_n\}$ converges uniformly (Theorem 7.8).

□

Exercise 7.17. PLACEHOLDER
Exercise 7.18. PLACEHOLDER
Exercise 7.19.
PLACEHOLDER

Exercise 7.20. *If f is continuous on $[0, 1]$ and if*

$$\int_0^1 f(x)x^n dx = 0 \qquad (n = 0, 1, 2, \ldots),$$

prove that $f(x) = 0$ on $[0, 1]$. (Hint: The integral of the product of f with any polynomial is zero. Use the Weierstrass theorem to show that $\int_0^1 f^2(x)dx = 0$.)

Proof.

(1) Since $\int_0^1 f(x)x^n dx = 0$ for all $n = 0, 1, 2, \ldots$,

$$\int_0^1 f(x)P(x)dx = 0 \text{ for all } P(x) \in \mathbb{R}[x].$$

(2) By Theorem 7.26 (Stone-Weierstrass Theorem), there exists a sequence of $P_n(x) \in \mathbb{R}[x]$ such that

$$P_n(x) \to f(x)$$

uniformly on $[0,1]$. Since $f(x)$ is continuous on the compact set $[0,1]$, $f(x)$ is bounded on $[0,1]$. Hence

$$f(x)P_n(x) \to f^2(x)$$

uniformly on $[0,1]$.

(3) Since each $f(x)P_n(x)$ is continuous, $f(x)P_n(x) \in \mathscr{R}$ on $[0,1]$ (Theorem 6.8). By Theorem 7.16,

$$\int_0^1 f^2(x)dx = \lim_{n\to\infty} \int_0^1 f(x)P_n(x)dx = \lim_{n\to\infty} 0 = 0.$$

(4) Since $f^2(x)$ is continuous, $f^2(x) = 0$ or $f(x) = 0$ by (3) and Exercise 6.2.

□

Exercise 7.21.
PLACEHOLDER

Exercise 7.22. *Assume $f \in \mathscr{R}(\alpha)$ on $[a,b]$, and prove that there are polynomials P_n such that*

$$\lim_{n\to\infty} \int_a^b |f - P_n|^2 d\alpha = 0.$$

(Compare with Exercise 6.12.)

Notation. For $u \in \mathscr{R}(\alpha)$ on $[a,b]$, define

$$\|u\|_2 = \left\{ \int_a^b |u|^2 d\alpha \right\}^{\frac{1}{2}}.$$

Proof. Given any $\varepsilon = \frac{1}{n} > 0$ $(n = 1, 2, 3, \ldots)$.

(1) By Exercise 6.12, there exists a continuous function g_n on $[a,b]$ such that

$$\|f - g_n\|_2 < \frac{1}{n}.$$

(2) By Theorem 7.26 (Stone-Weierstrass Theorem), there is a polynomial P_n such that

$$|g_n(x) - P_n(x)| < \frac{1}{n}$$

for all $x \in [a,b]$. Thus

$$\|g_n - P_n\|_2 \leq \left\{ \int_a^b \left(\frac{1}{n}\right)^2 d\alpha \right\}^{\frac{1}{2}} = \frac{(\alpha(b) - \alpha(a))^{\frac{1}{2}}}{n}.$$

(3) By Exercise 6.11,

$$\|f - P_n\|_2 \leq \|f - g_n\|_2 + \|g_n - P_n\|_2 \leq \frac{1 + (\alpha(b) - \alpha(a))^{\frac{1}{2}}}{n},$$

or

$$0 \leq \int_a^b |f - P_n|^2 d\alpha \leq \frac{[1 + (\alpha(b) - \alpha(a))^{\frac{1}{2}}]^2}{n^2}.$$

As $n \to \infty$, $\int_a^b |f - P_n|^2 d\alpha \to 0$.

□

Exercise 7.23. *Put $P_0 = 0$, and define, for $n = 0, 1, 2, \ldots$,*

$$P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n^2(x)}{2}.$$

Prove that

$$\lim_{n \to \infty} P_n(x) = |x|,$$

uniformly on $[-1, 1]$. (This makes it possible to prove the Stone-Weierstrass theorem without first proving Theorem 7.26.) (Hint: Use the identity

$$|x| - P_{n+1} = [|x| - P_n(x)] \left[1 - \frac{|x| + P_n(x)}{2}\right]$$

to prove that $0 \leq P_n(x) \leq P_{n+1}(x) \leq |x|$ if $|x| \leq 1$, and that

$$|x| - P_n(x) \leq |x| \left(1 - \frac{|x|}{2}\right)^n < \frac{2}{n+1}$$

if $|x| \leq 1$.)

Proof (Hint).

(1)

$$\begin{aligned} |x| - P_{n+1}(x) &= |x| - P_n(x) - \frac{|x|^2 - P_n^2(x)}{2} \\ &= |x| - P_n(x) - \frac{(|x| + P_n(x))(|x| - P_n(x))}{2} \\ &= [|x| - P_n(x)] \left[1 - \frac{|x| + P_n(x)}{2}\right]. \end{aligned}$$

(2) *Show that* $0 \le P_n(x) \le |x|$ *if* $|x| \le 1$. Induction on n.

(a) If $n = 0$, then $P_n(x) = P_0(x) = 0$ and thus $0 \le P_0(x) \le |x|$.

(b) Assume the induction hypothesis that for the single case $n = k$ holds, and thus $0 \le P_k(x) \le |x|$ if $|x| \le 1$. So

$$0 \le |x| - P_k(x) \le |x|,$$
$$0 \le 1 - |x| \le 1 - \frac{|x| + P_k(x)}{2} \le 1 - \frac{|x|}{2} \le 1$$

if $|x| \le 1$. Hence

$$0 \le [|x| - P_k(x)] \left[1 - \frac{|x| + P_k(x)}{2}\right] \le |x|.$$

By (1),

$$0 \le |x| - P_{k+1}(x) \le |x|$$

or $0 \le P_{k+1}(x) \le |x|$ if $|x| \le 1$

(c) Since both the base case in (a) and the inductive step in (b) have been proved as true, by mathematical induction the result holds.

(3) *Show that* $0 \le P_n(x) \le P_{n+1}(x) \le |x|$ *if* $|x| \le 1$. By (2), it suffices to show that $P_n(x) \le P_{n+1}(x)$. By (1)(2), we have

$$\begin{aligned} |x| - P_{n+1}(x) &= [|x| - P_n(x)] \left[1 - \frac{|x| + P_n(x)}{2}\right] \\ &\le |x| - P_n(x) \end{aligned}$$

or $P_n(x) \le P_{n+1}(x)$.

(4) *Define* $f_n(t) = t(1-t)^n$ *on* $\left[0, \frac{1}{2}\right]$ *for* $n = 1, 2, 3, \ldots$. *Show that* $f_n(t) \le \frac{1}{n+1}$. Since

$$f_n'(t) = (1-t)^{n-1}(1 - (n+1)t)$$

$f_n'(t) = 0$ on $\left[0, \frac{1}{2}\right]$ if and only if $t = \frac{1}{n+1}$. By Theorem 5.11, $f_n(t)$ reaches its maximum at $t = \frac{1}{n+1}$. Hence

$$f_n(t) \le f_n\left(\frac{1}{n+1}\right) = \frac{1}{n+1}\left(\frac{n}{n+1}\right)^n < \frac{1}{n+1}.$$

(5) *Show that*

$$|x| - P_n(x) \le |x|\left(1 - \frac{|x|}{2}\right)^n < \frac{2}{n+1}$$

if $|x| \leq 1$. Note that

$$|x| - P_n(x) \leq [|x| - P_0(x)] \prod_{k=0}^{n-1} \left[1 - \frac{|x| + P_k(x)}{2}\right] \tag{(1)}$$

$$\leq |x| \prod_{k=0}^{n-1} \left[1 - \frac{|x|}{2}\right] \tag{(2)}$$

$$\leq |x| \left[1 - \frac{|x|}{2}\right]^n$$

$$< \frac{2}{n+1} \qquad \left(\text{Put } t = \frac{|x|}{2} \text{ in (4)}\right).$$

(6) (5) implies that

$$\lim_{n \to \infty} P_n(x) = |x|$$

and

$$\lim_{n \to \infty} \sup_{x \in [-1,1]} |P_n(x) - |x|| = 0.$$

By Theorem 7.9, $P_n(x) \to |x|$ uniformly on $[-1, 1]$.

□

Exercise 7.24. PLACEHOLDER
Exercise 7.25. PLACEHOLDER
Exercise 7.26. PLACEHOLDER