Notes on the book: *Atiyah and Macdonald, Introduction to Commutative Algebra*

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Contents

Chapter 1: Rings and Ideals

Exercise 1.1.

Let x be a nilpotent element of A. Show that $1+x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit.

Proof.

(1) Suppose $x^m = 0$ for some odd integer $m \geq 0$. Then

$$1 = 1 + x^m = (1+x)(1 - x + x^2 - \cdots + (-1)^{m-1}x^{m-1}),$$

or $1+x$ is a unit.

(2) If u is any unit and x is any nilpotent, $u + x = u \cdot (1 + u^{-1}x)$ is a product of two units (using that $u^{-1}x$ is nilpotent and applying (1)) and hence a unit again.

□

Proof (Proposition 1.9).

(1) *The nilradical is a subset of the Jacobson radical.*

(a) The nilradical $\mathfrak{N}$ of A is the intersection of all the prime ideals of A by Proposition 1.8.

(b) The Jacobson radical $\mathfrak{J}$ of A is the intersection of all the maximal ideals of A by definition.

(2) By Proposition 1.9, $x \in \mathfrak{J}$ if and only if $1 - xy$ is a unit in A for all $y \in A$. So $1 + x = 1 - (-x) \cdot 1$ is a unit in A since x is a nilpotent and $\mathfrak{J}$ is an ideal.

□

Exercise 1.2.

Let A be a ring and let $A[x]$ be the ring of polynomials in an indeterminate x, with coefficients in A. Let $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$. Prove that

(i) *f is a unit in $A[x]$ if and only if a_0 is a unit in A and $a_1, \ldots, a_n$ are nilpotent. (Hint: If $b_0 + b_1x + \cdots + b_mx^m$ is the inverse of f, prove by induction on r that $a_n^{r+1}b_{m-r} = 0$. Hence show that a_n is nilpotent, and then use Exercise 1.1.)*

(ii) *f is nilpotent if and only if $a_0, a_1, \ldots, a_n$ are nilpotent.*

(iii) *f is a zero-divisor if and only if there exists $a \neq 0$ such that $af = 0$. (Hint: Choose a polynomial $g = b_0 + b_1 x + \cdots + b_m x^m$ of least degree m such that $fg = 0$. Then $a_n b_m = 0$, hence $a_n g = 0$ (because $a_n g$ annihilates f and has degree $< m$). Now show by induction that $a_{n-r} g = 0$ $(0 \leq r \leq n)$.)*

(iv) *f is said to be **primitive** if $(a_0, a_1, \ldots, a_n) = (1)$. Prove that if $f, g \in A[x]$, then fg is primitive if and only if f and g are primitive.*

Proof of (i).

(1) ($\Longleftarrow$) holds by Exercise 1.1.

(2) ($\Longrightarrow$) There exists the inverse g of f, say $g = b_0 + b_1 x + \cdots + b_m x^m$ satisfying $1 = fg$. Clearly, $1 = a_0 b_0$, or a_0 is a unit in A. Also,

$$\begin{aligned}
0 &= a_n b_m, \\
0 &= a_n b_{m-1} + a_{n-1} b_m, \\
0 &= a_n b_{m-2} + a_{n-1} b_{m-1} + a_{n-2} b_m, \\
&\cdots
\end{aligned}$$

A direct computing shows that

$$\begin{aligned}
0 &= a_n^1 b_m, \\
0 &= a_n (a_n b_{m-1} + a_{n-1} b_m) \\
&= a_n^2 b_{m-1} + a_{n-1} a_n b_m \\
&= a_n^2 b_{m-1}, \\
0 &= a_n^2 (a_n b_{m-2} + a_{n-1} b_{m-1} + a_{n-2} b_m) \\
&= a_n^3 b_{m-2} + a_{n-1} a_n^2 b_{m-1} + a_{n-2} a_n^2 b_m \\
&= a_n^3 b_{m-2}, \\
&\cdots
\end{aligned}$$

So we might have $a_n^{r+1} b_{m-r} = 0$ for $r = 0, 1, 2, \ldots, m$.

(3) *Show that $a_n^{r+1} b_{m-r} = 0$ for $r = 0, 1, 2, \ldots, m$ by induction on r.*

(a) As $r = 0$, $a_n b_m = 0$ by comparing the coefficient of $fg = 1$ at x^{n+m}.

(b) For any $r > 0$, comparing the coefficient of $fg = 1$ at x^{n+m-r},

$$0 = a_n b_{m-r} + a_{n-1} b_{m-r+1} + \cdots + a_{n-r} b_m.$$

Multiplying by a_n^r on the both sides,

$$\begin{aligned}
0 &= a_n^{r+1} b_{m-r} + a_{n-1} a_n^r b_{m-r+1} + \cdots + a_{n-r} a_n^r b_m \\
&= a_n^{r+1} b_{m-r}.
\end{aligned}$$

by the induction hypothesis.

(4) a_n *is a nilpotent.* Putting $r = m$ in $a_n^{r+1} b_{m-r} = 0$ and get $a_n^{m+1} b_0 = 0$. Notice that b_0 is a unit, $a_n^{m+1} = 0$, or a_n is a nilpotent.

(5) Consider $f - a_n x^n = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$, a polynomial $\in A[x]$ of degree $n-1$. Note that f is a unit and $a_n x^n$ is a nilpotent. By Exercise 1.1, $f - a_n x^n$ is a unit too. Applying the (2)(3)(4) again, a_{n-1} is a nilpotent as $n - 1 > 0$, that is, applying descending induction on n then yields the desired property.

□

Proof of (ii).

(1) ($\Longleftarrow$) holds since the nilradical of any ring is an ideal.

(2) ($\Longrightarrow$) $f^N = 0$ for some $N > 0$. So $0 = f^N = a_0^n + \cdots + a_n^N x^{nN}$. Compare the coefficient in the lowest term to get $a_0^N = 0$, or a_0 is a nilpotent.

(3) Note that $f - a_0 = a_1 x + \cdots + a_n x^n \in A[x]$ is nilpotent since f and a_0 are nilpotent. $f - a_0$ is a nilpotent too. Continue the same argument in (2), the result is established.

□

Proof of (iii).

(1) ($\Longleftarrow$) holds trivially.

(2) ($\Longrightarrow$) Pick a polynomial $g = b_0 + b_1 x + \cdots + b_m x^m$ of least degree m such that $fg = 0$. Especially, $a_n b_m = 0$.

(3) Consider

$$\begin{aligned} a_n g &= a_n b_0 + \cdots + a_n b_{m-1} x^{m-1} + a_n b_m x^m \\ &= a_n b_0 + \cdots + a_n b_{m-1} x^{m-1} \end{aligned}$$

(since $a_n b_m = 0$). $a_n g$ is a polynomial over A of having degree strictly less than m. Notice that $f \cdot (a_n g) = a_n \cdot (fg) = 0$. By minimality of m, $a_n g = 0$.

(4) Induction on the degree n of f.

(a) As $n = 0$, $f = a_0$. There exists $b_m \neq 0$ such that $b_m f = b_m a_0 = 0$ by (2).

(b) For any zero-divisor f of degree n, there is a polynomial $g = b_0 + b_1 x + \cdots + b_m x^m$ of least degree m such that $fg = 0$. By (2)(3),

$$\begin{aligned} (f - a_n x^n) \cdot g &= fg - a_n x^n g \\ &= 0 - 0 \\ &= 0. \end{aligned}$$

That is, $f - a_n x^n$ is a zero-divisor of degree $n-1$. By the induction hypothesis, there exists $b_m \neq 0$ such that $b_m(f - a_n x^n) = 0$. So $b_m f = b_m(f - a_n x^n) + b_m a_n x^n = 0 + 0 = 0$.

(c) By (a)(b), ($\Longrightarrow$) holds by mathematical induction.

$\square$

Proof of (iv). Note that

(1) $f \notin \mathfrak{m}[x]$ for any maximal ideal $\mathfrak{m}$ of A if and only if f is primitive.

(2) For any maximal ideal $\mathfrak{m}$ of A, $A/\mathfrak{m}$ is a field (or an integral domain).

(3) $A[x]$ is an integral domain if A is an integral domain.

(4) $A[x]/\mathfrak{m}[x] \cong (A/\mathfrak{m})[x]$ as a ring isomorphism.

Hence,

$$\begin{aligned}
f, g : \text{ primitive} &\Longleftrightarrow f, g \notin \mathfrak{m}[x] \text{ for any maximal ideal } \mathfrak{m} \\
&\Longleftrightarrow f, g \neq 0 \text{ in } (A/\mathfrak{m})[x] \text{ for any maximal ideal } \mathfrak{m} \\
&\Longleftrightarrow fg \neq 0 \text{ in } (A/\mathfrak{m})[x] \text{ for any maximal ideal } \mathfrak{m} \\
&\Longleftrightarrow fg \notin \mathfrak{m}[x] \text{ for any maximal ideal } \mathfrak{m} \\
&\Longleftrightarrow fg : \text{ primitive.}
\end{aligned}$$

$\square$

Exercise 1.3.

Generalize the results of Exercise 1.2 to a polynomial ring $A[x_1, \ldots, x_r]$ in several indeterminates.

Generalization. Let

$$f = \sum_{(i)} a_{(i)} x^{(i)} \in A[x_1, \ldots, x_r]$$

where $\sum_{(i)}$ is the summation over $(i) = (i_1, \ldots, i_r)$ with $i_1 + \cdots + i_r = n$. Then

(i) f is a unit in $A[x_1, \ldots, x_r]$ if and only if $a_{(0)}$ is a unit in A and all other $a_{(i)}$ are nilpotent.

(ii) f is nilpotent if and only if all $a_{(i)}$ are nilpotent.

(iii) f is a zero-divisor if and only if there exists $a \neq 0$ such that $af = 0$.

(iv) If $f, g \in A[x_1, \ldots, x_r]$, then fg is primitive if and only if f and g are primitive.

Proof. Use the mathematical induction to prove (i)(ii)(iii) and apply the same argument in Exercise 1.2 (iv) to prove (iv). $\square$

Exercise 1.4.

In the ring $A[x]$, the Jacobson radical is equal to the nilradical.

Proof.

(1) The nilradical $\mathfrak{N}$ is a subset of the Jacobson radical $\mathfrak{J}$. It suffices to show that $\mathfrak{J} \subseteq \mathfrak{N}$.

(2)

$$
\begin{aligned}
& f \in \mathfrak{J} \\
\Longleftrightarrow\ & 1 - fy \text{ is a unit in } A[x] \text{ for all } y \in A[x] && \text{(Proposition 1.9)} \\
\Longrightarrow\ & 1 - xf \text{ is a unit in } A[x] && (y = x) \\
\Longrightarrow\ & \text{All coefficients of } f \text{ are nilpotent} && \text{(Exercise 1.2 (i))} \\
\Longrightarrow\ & f \text{ is nilpotent} && \text{(Exercise 1.2 (ii))} \\
\Longrightarrow\ & f \in \mathfrak{N}.
\end{aligned}
$$

$\square$

Exercise 1.5.

Let A be a ring and let $A[[x]]$ be the ring of formal power series $f = \sum_{n=0}^{\infty} a_n x^n$ with coefficients in A. Show that

(i) *f is a unit in $A[[x]]$ if and only if a_0 is a unit in A.*

(ii) *If f is nilpotent, then a_n is nilpotent for all $n \geq 0$. Is converse true? (See Exercise 7.2.)*

(iii) *f belongs to the Jacobson radical of $A[[x]]$ if and only if a_0 belongs to the Jacobson radical of A.*

(iv) *The contraction of a maximal ideal $\mathfrak{m}$ of $A[[x]]$ is a maximal ideal of A, and $\mathfrak{m}$ is generated by $\mathfrak{m}^c$ and x.*

(v) *Every prime ideal of A is the contraction of a prime ideal of $A[[x]]$.*

Proof of (i).

(1) ($\Longrightarrow$) If $g = \sum_{n=0}^{\infty} b_n x^n$ is an inverse of f, then $fg = 1$ implies that $a_0 b_0 = 1$ so that a_0 is a unit in A.

(2) ($\Longleftarrow$) Our goal is to find $g = \sum_{n=0}^{\infty} b_n x^n$ such that the Cauchy product $fg = \sum_{n=0}^{\infty} c_n x^n$ is equal to $1 \in A[x]$. Here $c_n = \sum_{r=0}^{n} a_r b_{n-r}$. By the assumption we have that $c_0 = 1$ and $c_1 = c_2 = \cdots = 0$. Hence

$$
\begin{aligned}
b_0 &= a_0^{-1} \\
b_1 &= -a_0^{-1} a_1 b_0 \\
&\cdots \\
b_n &= a_0^{-1} \sum_{r=1}^{n} a_r b_{n-r} \\
&\cdots
\end{aligned}
$$

by induction.

□

Proof of (ii).

(1) The proof is the same as Exercise 1.2 (ii).

(2) The converse is true if A is Noetherian (by Exercise 7.2).

(3) The converse is not always true. Take

$$A = \mathbb{F}_2[t, t^{-2}, t^{-2^2}, \ldots]/(t)$$

and

$$f(x) = \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} t^{-2^n} x^n \in A[x].$$

Note that A is not Noetherian and all a_n are nilpotent in A. To show f is not nilpotent in $A[x]$, it suffices to show that f^{2^r} is not equal to zero for all positive integers r.

(4) Note that $\mathbb{F}_2$ is a field of characteristic 2. So

$$f^{2^r} = \sum_{n=1}^{\infty} a_n^{2^r} x^n = \sum_{n=1}^{\infty} t^{2^{r-n}} x^n = \sum_{n=r+1}^{\infty} t^{2^{r-n}} x^n \neq 0$$

for all r.

□

Proof of (iii).

$$
\begin{aligned}
& f \text{ in the Jacobson radical of } A[[x]] \\
\Longleftrightarrow\ & 1 - fg \in A[[x]] \text{ is unit for all } g = \sum_{n=0}^{\infty} b_n x^n \in A[[x]] && \text{(Proposition 1.9)} \\
\Longleftrightarrow\ & 1 - a_0 b_0 \in A \text{ is unit for all } b_0 \in A && \text{((i))} \\
\Longleftrightarrow\ & a_0 \text{ belongs to the Jacobson radical of } A. && \text{(Proposition 1.9)}
\end{aligned}
$$

□

Proof of (iv).

(1) Note that $x = 0 + x$ belongs to the Jacobson radical of $A[[x]]$ since 0 obviously belongs to the Jacobson radical of A (by (iii)).

(2) So $x \in \mathfrak{m}$ or $(x) \subseteq \mathfrak{m}$ for any maximal ideal in $A[[x]]$. So it is clear that $\mathfrak{m} = \mathfrak{m}^c + (x)$.

(3) Moreover, $\mathfrak{m}^c$ is a maximal ideal since $A/\mathfrak{m}^c \cong A[[x]]/\mathfrak{m}$ is a field.

□

Proof of (v).

(1) Similar to (iv). Suppose $\mathfrak{p}$ is a prime ideal of A. Let $\mathfrak{q} = \mathfrak{p} + (x)$ be an ideal of $A[[x]]$.

(2) $\mathfrak{q}^c = \mathfrak{p}$ clearly. Besides, $\mathfrak{q}^c$ is a prime ideal since

$$A[[x]]/\mathfrak{q}^c \cong A/\mathfrak{p}$$

is an integral domain.

□

Supplement 1.5.1.

(Exercise II.1.2 in the textbook: Jrgen Neukirch, Algebraic Number Theory.) A p-adic integer $a = a_0 + a_1 p + a_2 p^2 + \cdots$ is a unit in the ring $\mathbb{Z}_p$ if and only if $a_0 \neq 0$.

Proof.

(1) ($\Longrightarrow$) If $b = b_0 + b_1 p + b_2 p^2 + \cdots$ is an inverse of a, then $ab = 1$ implies that $a_0 b_0 = 1$ so that a_0 is a unit in $\mathbb{Z}/p\mathbb{Z}$ or $a_0 \neq 0$.

(2) ($\Longleftarrow$) Our goal is to find

$$b = b_0 + b_1 p + b_2 p^2 + \cdots \in \mathbb{Z}_p$$

such that the Cauchy product

$$ab = c_0 + c_1 p + c_2 p^2 + \cdots$$

is equal to $1 \in \mathbb{Z}_p$. Here $c_n = \sum_{\nu=0}^{n} a_\nu b_{n-\nu}$. By the assumption we have that $c_0 = 1$ and $c_1 = c_2 = \cdots = 0$. Hence

$$\begin{aligned} b_0 &= a_0^{-1} \\ b_1 &= -a_0^{-1} a_1 b_0 \\ &\cdots \\ b_n &= a_0^{-1} \sum_{\nu=1}^{n} a_\nu b_{n-\nu} \\ &\cdots \end{aligned}$$

by induction.

$\square$

Exercise 1.6.

A ring A is such that every ideal not contained in the nilradical contains a nonzero idempotent (that is, an element e such that $e^2 = e \neq 0$). Prove that the nilradical and Jacobson radical of A are equal.

Proof.

(1) $\mathfrak{N} \subseteq \mathfrak{J}$ clearly.

(2) Since

$$\begin{aligned} a \notin \mathfrak{N} &\Longrightarrow (a) \not\subseteq \mathfrak{N} \\ &\Longrightarrow \text{there exists a nonzero idempotent } e \in (a) \\ &\Longrightarrow e = ar \text{ for some } r \in A \\ &\Longrightarrow 0 = e - e^2 = e(1-e) = ar(1-ar) \\ &\Longrightarrow 1 - ar \text{ is a zero-divisor, not a unit} \\ &\Longrightarrow a \notin \mathfrak{J}, && \text{(Proposition 1.9)} \end{aligned}$$

we have $\mathfrak{J} \subseteq \mathfrak{N}$.

$\square$

Exercise 1.7.

Let A be a ring in which every element satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal in A is maximal.

Proof. It suffices to show that *for any prime ideal $\mathfrak{p}$ in A, $A/\mathfrak{p}$ is a field.*

(1) Take any $0 \neq \overline{x} \in A/\mathfrak{p}$, which is represented by $x \in A - \mathfrak{p}$. By assumption there exists $n \geq 2$ such that $x^n = x$. So $\overline{x}^n = \overline{x}$ or $\overline{x}(\overline{x}^{n-1} - 1) = 0$.

(2) Since $\mathfrak{p}$ is prime, $A/\mathfrak{p}$ is a integral domain. That is, $\overline{x} = 0$ (impossible) or $\overline{x}^{n-1} - 1 = 0$. Write $\overline{x} \cdot \overline{x}^{n-2} = 1$ in $A/\mathfrak{p}$. So $\overline{x}^{n-2}$ is an inverse of $\overline{x} \neq 0$ in $A/\mathfrak{p}$, which implies that $A/\mathfrak{p}$ is a field (since $\overline{x}$ is arbitrary).

(3) $A/\mathfrak{p}$ is a field if and only if $\mathfrak{p}$ is maximal.

□

Exercise 1.8.

Let A be a ring $\neq 0$. Show that the set of prime ideals of A has minimal elements with respect to inclusion.

Similar to Theorem 1.3.

Proof (Zorn's Lemma).

(1) Let Σ be the set of all prime ideals of A.

(2) Order Σ by $\supseteq$, that is, $\mathfrak{p} \leq \mathfrak{q}$ if $\mathfrak{p} \supseteq \mathfrak{q}$.

(3) Σ is not empty, since every ring $A \neq 0$ has at least one maximal ideal (or prime ideal) (Theorem 1.3).

(4) To apply Zorn's lemma we must show that every chain in Σ has a lower bound in Σ; let then $(\mathfrak{p}_\alpha)$ be a chain of prime ideals in Σ, so that for each pair of indices α, β we have either $\mathfrak{p}_\alpha \subseteq \mathfrak{p}_\beta$ or $\mathfrak{p}_\beta \subseteq \mathfrak{p}_\alpha$. Let $\mathfrak{p} = \bigcap_\alpha \mathfrak{p}_\alpha$.

(5) *Show that $\mathfrak{p}$ is a prime ideal.* Clearly $\mathfrak{p}$ is an ideal. Given any $xy \in \mathfrak{p}$ and $x \notin \mathfrak{p}$. So xy is in all prime ideals $\mathfrak{p}_\alpha$. By assumption $x \notin \mathfrak{p}$, there is some β such that $x \notin \mathfrak{p}_\beta$, or $x \notin \mathfrak{p}_\alpha$ whenever $\alpha \geq \beta$. So $y \in \mathfrak{p}_\alpha$ whenever $\alpha \geq \beta$. Since $y \in \mathfrak{p}_\beta$, $y \in \mathfrak{p}_\gamma$ whenever $\beta \geq \gamma$. Therefore, $y \in \mathfrak{p}_\alpha$ for all α, or $y \in \mathfrak{p}$, or $\mathfrak{p}$ is prime.

□

Exercise 1.9.

Let $\mathfrak{a}$ be an ideal $\neq (1)$ in a ring A. Show that $\mathfrak{a} = r(\mathfrak{a}) \Longleftrightarrow \mathfrak{a}$ is an intersection of prime ideals.

Proof.

(1) ($\Longrightarrow$). By Proposition 1.14, $\mathfrak{a} = r(\mathfrak{a})$ is the intersection of the prime ideals which contain $\mathfrak{a}$.

(2) ($\Longleftarrow$).

$$\begin{aligned}
\mathfrak{a} &= \bigcap\{\mathfrak{p} \in \text{some subset of } \mathrm{Spec}(A)\} \\
&= \bigcap\{\mathfrak{p} \in \text{some subset of } \mathrm{Spec}(A) : \mathfrak{p} \supseteq \mathfrak{a}\} \\
&\supseteq \bigcap\{\mathfrak{p} \in \mathrm{Spec}(A) : \mathfrak{p} \supseteq \mathfrak{a}\} \\
&= r(\mathfrak{a}) \\
&\supseteq \mathfrak{a}.
\end{aligned}$$

□

Exercise 1.10.

Let A be a ring, $\mathfrak{N}$ its nilradical. Show the following are equivalent:

(i) *A has exactly one prime ideal;*

(ii) *every element of A is either a unit or nilpotent;*

(iii) *$A/\mathfrak{N}$ is a field.*

Proof.

$$\begin{aligned}
& A/\mathfrak{N} \text{ is a field} \\
\Longrightarrow\ & \mathfrak{N} \text{ is a maximal ideal} \\
\Longrightarrow\ & \mathfrak{p} = \mathfrak{N} \text{ for every prime ideal } \mathfrak{p} && \text{(Proposition 1.8)} \\
\Longrightarrow\ & A \text{ has exactly one prime ideal } \mathfrak{p} \\
\Longrightarrow\ & \mathfrak{p} = \mathfrak{N} \\
\Longrightarrow\ & A \text{ has exactly one maximal ideal } \mathfrak{p} \\
\Longrightarrow\ & \text{Given any } a \in A,\ a \text{ is a unit or } a \in \mathfrak{p} = \mathfrak{N}. && \text{(Corollary 1.5)} \\
\Longrightarrow\ & A/\mathfrak{N} \text{ is a field.}
\end{aligned}$$

□

Exercise 1.11. (Boolean ring)

A ring A is ***Boolean*** *if $x^2 = x$ for all $x \in A$. In a Boolean ring A, show that*

(i) $2x = 0$ *for all $x \in A$;*

(ii) *every prime ideal $\mathfrak{p}$ is maximal, and $A/\mathfrak{p}$ is a field with two elements;*

(iii) *every finitely generated ideal in A is principal.*

Proof of (i). Note that $2x = x+x = (x+x)^2 = (2x)^2 = 4x^2 = 4x$. So $2x = 0$. □

Proof of (ii). Same as Exercise 1.7 with $n = 2$. □

Proof of (iii).

(1) By induction, it suffices to show that *if $\mathfrak{a} = (x, y)$ is an ideal in A, then $\mathfrak{a} = (z)$ for some $z \in A$.*

(2) Take $z = x + y + xy$. $(z) \subseteq \mathfrak{a}$ obviously.

(3) Conversely, note that

$$x = x^2 = x(z - y - xy) = xz - \overbrace{xy - \underbrace{x^2y}_{=xy}}^{=2xy=0} = xz \in (z).$$

Also $y \in (z)$ similarly. So $\mathfrak{a} \subseteq (z)$ and thus $\mathfrak{a} = (z)$ is principal.

□

Exercise 1.12.

A local ring contains no idempotent $\neq 0, 1$.

Proof.

(1) If e is an idempotent $\neq 0, 1$ in a local ring A with the maximal ideal $\mathfrak{m}$, then by definition $0 = e(1 - e)$ shows that both $e \neq 0$ and $1 - e \neq 0$ are not unit.

(2) Thus $e \in \mathfrak{m}$ and $1 - e \in \mathfrak{m}$. So $1 = (1 - e) + e$ is a unit in $\mathfrak{m}$, which is absurd.

□

Construction of an algebraic closure of a field (E. Artin)

Exercise 1.13.

Let K be a field and let Σ be the set of all irreducible monic polynomials f in one indeterminate with coefficients in K. Let A be the polynomial ring over K generated by indeterminates x_f, one for each $f \in \Sigma$. Let $\mathfrak{a}$ be the ideal of A generated by the polynomials $f(x_f)$ for all $f \in \Sigma$. Show that $\mathfrak{a} \neq (1)$.

Let $\mathfrak{m}$ be a maximal ideal of A containing $\mathfrak{a}$ and let $K_1 = A/\mathfrak{m}$. Then K_1 is an extension field of K in which each $f \in \Sigma$ has a root. Repeat the construction with K_1 in place of K, obtaining a field K_2, and so on. Let $L = \bigcup_{n=1}^{\infty} K_n$. Then L is a field in which each $f \in \Sigma$ splits completely into linear factors. Let $\overline{K}$ be the set of all elements of L which are algebraic over K. Then $\overline{K}$ is an algebraic closure of K.

Proof.

(1) *Show that $\mathfrak{a} \neq (1)$.* (Reductio ad absurdum) If $\mathfrak{a} = (1)$, then we can write

$$1 = \sum_{i=1}^{n} g_i(x) f_i(x_{f_i}) \in A$$

where $x = (x_{f_1}, \ldots, x_{f_n}, x_{g_1}, \ldots, x_{g_r})$ is a tuple with finitely many indeterminates. It is possible since it is a finite sum.

(2) Let L be an algebraic extension of K such that each f_i has a root $a_i \in L$ $(i = 1, \ldots, n)$.

(3) Take $x = (a_1, \ldots, a_n, 0, \ldots, 0)$ in the equation $1 = \sum_{i=1}^{n} g_i(x) f_i(x_{f_i})$ to get

$$\begin{aligned} 1 &= \sum_{i=1}^{n} g_i(a_1, \ldots, a_n, 0, \ldots, 0) f_i(a_i) \\ &= \sum_{i=1}^{n} g_i(a_1, \ldots, a_n, 0, \ldots, 0) \cdot 0 \\ &= 0, \end{aligned}$$

which is absurd.

□

Exercise 1.14.

In a ring A, let Σ be the set of all ideals in which every element is a zero-divisor. Show that the set Σ has maximal elements and that every maximal element of Σ is a prime ideal. Hence the set of zero-divisors in A is a union of prime ideals.

Proof.

(1) Suppose $1 \neq 0$.

(2) *Show that the set Σ has maximal elements.* Order Σ by inclusion. Σ is not empty, since $0 \in \Sigma$. To apply Zorn's lemma we must show that every chain in Σ has an upper bound in Σ; let then $(\mathfrak{a}_\alpha)$ be a chain of ideals in Σ, so that for each pair of indices α, β we have either $\mathfrak{a}_\alpha \subseteq \mathfrak{a}_\beta$ or $\mathfrak{a}_\beta \subseteq \mathfrak{a}_\alpha$.

(3) Let $\mathfrak{a} = \bigcup_\alpha \mathfrak{a}_\alpha$. Then $\mathfrak{a}$ is an ideal and every element of $\mathfrak{a}$ is a zero-divisor. Hence $\mathfrak{a} \in \Sigma$, and $\mathfrak{a}$ is an upper bound of the chain. Hence by Zorn's lemma, Σ has maximal elements.

(4) *Show that every maximal element of Σ is a prime ideal.* Let $\mathfrak{p}$ be a maximal element in Σ. Suppose $x, y \notin \mathfrak{p}$. Then there are non-zero-divisors in $\mathfrak{p}+(x)$ and $\mathfrak{p}+(y)$, and their product is an element of $\mathfrak{p}+(xy)$ that is again a non-zero-divisor. So $xy \notin \mathfrak{p}$.

(5) Hence the set of zero-divisors in A is a union of prime ideals (by the construction in (2) and the result of (4)).

□

The prime spectrum of a ring

Exercise 1.15.

Let A be a ring and let X be the set of all prime ideals of A. For each subset E of A, let $V(E)$ denote the set of all prime ideals of A which contain E. Prove that

(i) *if $\mathfrak{a}$ is the ideal generated by E, then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$.*

(ii) $V(0) = X$, $V(1) = \varnothing$.

(iii) *if $(E_i)_{i \in I}$ is any family of subsets of A, then*

$$V\left(\bigcup_{i \in I} E_i\right) = \bigcap_{i \in I} V(E_i).$$

(iv) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ *for any ideals* $\mathfrak{a}$, $\mathfrak{b}$ *of* A.

The results show that the sets $V(E)$ *satisfy the axioms for closed sets in a topological space. The resulting topology is called the* ***Zariski topology****. The topological space* X *is called the* ***prime spectrum*** *of* A*, and is written* $\operatorname{Spec}(A)$.

Note that if $E_1 \subseteq E_2$, then $V(E_1) \supseteq V(E_2)$.

Proof of (i).

(1) *Show that* $V(E) = V(\mathfrak{a})$.

  (a) *Show that* $V(E) \subseteq V(\mathfrak{a})$. Given any $\mathfrak{p} \in V(E)$, $\mathfrak{p} \supseteq E$. For any $a \in \mathfrak{a}$, since $\mathfrak{a}$ is generated by E, we can write a as a finite sum $a = \sum \alpha\beta$ where $\alpha \in A$ and $\beta \in E$. Since $E \subseteq \mathfrak{p}$, all $\beta \in \mathfrak{p}$. Since $\mathfrak{p}$ is an ideal, $a = \sum \alpha\beta \in \mathfrak{p}$. That is, $\mathfrak{p} \supseteq \mathfrak{a}$, or $\mathfrak{p} \in V(\mathfrak{a})$.

  (b) $V(E) \supseteq V(\mathfrak{a})$ since $\mathfrak{a} \supseteq E$.

(2) *Show that* $V(\mathfrak{a}) = V(r(\mathfrak{a}))$.

  (a) *Show that* $V(\mathfrak{a}) \subseteq V(r(\mathfrak{a}))$. Given any $\mathfrak{p} \in V(\mathfrak{a})$,

$$\begin{aligned}
\mathfrak{p} \in V(\mathfrak{a}) &\Longrightarrow \mathfrak{p} \supseteq \mathfrak{a} \\
&\Longrightarrow \mathfrak{p} \supseteq \text{the intersection of the primes ideals } \mathfrak{p} \supseteq \mathfrak{a} \\
&\Longrightarrow \mathfrak{p} \supseteq r(\mathfrak{a}) \text{ (by Proposition 1.14)} \\
&\Longrightarrow \mathfrak{p} \in V(r(\mathfrak{a})).
\end{aligned}$$

  (b) $V(\mathfrak{a}) \supseteq V(r(\mathfrak{a}))$ since $r(\mathfrak{a}) \supseteq \mathfrak{a}$.

□

Proof of (ii).

(1) $V(1) = \varnothing$ since no prime ideal contains 1 by definition.

(2) $V(0) = X$ since 0 is in every ideal (especially in every prime ideal).

□

Proof of (iii).

$$\begin{aligned}
\mathfrak{p} \in V\left(\bigcup_{i \in I} E_i\right) &\Longleftrightarrow \mathfrak{p} \supseteq \bigcup_{i \in I} E_i \\
&\Longleftrightarrow \mathfrak{p} \supseteq E_i \text{ for all } i \in I \\
&\Longleftrightarrow \mathfrak{p} \in V(E_i) \text{ for all } i \in I \\
&\Longleftrightarrow \mathfrak{p} \in \bigcap_{i \in I} V(E_i).
\end{aligned}$$

□

Lemma. *For any* $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$, $\mathfrak{p} \supseteq \mathfrak{a}$ *or* $\mathfrak{p} \supseteq \mathfrak{b}$.

Proof of Lemma.

(1) If $\mathfrak{p} \supseteq \mathfrak{a}$. We are done.

(2) If $\mathfrak{p} \not\supseteq \mathfrak{a}$, there exists $a \in \mathfrak{a} - \mathfrak{p}$. So for any $b \in \mathfrak{b}$, $b \in \mathfrak{p}$ since $ab \in \mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$ and $\mathfrak{p}$ is a prime ideal, that is, $\mathfrak{p} \supseteq \mathfrak{b}$.

By (1)(2), $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$. □

Proof of (iv).

(1) *Show that* $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$.

   (a) $V(\mathfrak{a} \cap \mathfrak{b}) \subseteq V(\mathfrak{a}\mathfrak{b})$ since $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b}$.

   (b) *Show that* $V(\mathfrak{a} \cap \mathfrak{b}) \supseteq V(\mathfrak{a}\mathfrak{b})$. Given any $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$, $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$. By Lemma, $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$. Notice that $\mathfrak{a} \supseteq \mathfrak{a} \cap \mathfrak{b}$ and $\mathfrak{b} \supseteq \mathfrak{a} \cap \mathfrak{b}$. In any case, $\mathfrak{p} \supseteq \mathfrak{a} \cap \mathfrak{b}$, $\mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b})$.

(2) *Show that* $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.

   (a) *Show that* $V(\mathfrak{a}\mathfrak{b}) \subseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$. Given any $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$, $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$. By Lemma, $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$, $\mathfrak{p} \in V(\mathfrak{a})$ or $\mathfrak{p} \in V(\mathfrak{b})$, $\mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$.

   (b) *Show that* $V(\mathfrak{a}\mathfrak{b}) \supseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$. Given any $\mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$, $\mathfrak{p} \in V(\mathfrak{a})$ or $\mathfrak{p} \in V(\mathfrak{b})$, $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$. Notice that $\mathfrak{a} \supseteq \mathfrak{a}\mathfrak{b}$ and $\mathfrak{b} \supseteq \mathfrak{a}\mathfrak{b}$. In any cases, $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$, or $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$.

□

Exercise 1.17.

For each $f \in A$, *let* X_f *denote the complement of* $V(f)$ *in* $X = \operatorname{Spec}(A)$. *The sets* X_f *are open. Show that they form a basis of open sets for the Zariski topology, and that*

(i) $X_f \cap X_g = X_{fg}$.

(ii) $X_f = \varnothing \Longleftrightarrow f$ *is nilpotent.*

(iii) $X_f = X \Longleftrightarrow f$ *is a unit.*

(iv) $X_f = X_g \Longleftrightarrow r((f)) = r((g))$.

(v) X *is quasi-compact (compact), that is, every open covering of* X *has a finite subcovering.*

(vi) *More generally, each X_f is quasi-compact.*

(vii) *An open subset of X is quasi-compact if and only if it is a finite union of sets X_f.*

The sets X_f are called basic open sets of $X = \operatorname{Spec}(A)$.

(Hint: To prove (v), remark that it is enough to consider a covering of X by basic open sets $X_{f_i} (i \in I)$. Show that the f_i generate the unit ideal and hence that there is an equation of the form

$$1 = \sum_{i \in J} g_i f_i \quad (g_i \in A)$$

where J is some finite subset of I. Then the $X_{f_i} (i \in J)$ cover X.)

Proof of basis. It is equivalent to Exercise 1.15 (iii). Given any open set O in X. Write $O = X - V(\mathfrak{a})$ for some ideal $\mathfrak{a}$ of A. Since

$$V(\mathfrak{a}) = V\left(\bigcup_{f \in \mathfrak{a}} (f)\right) = \bigcap_{f \in \mathfrak{a}} V(f),$$

we have

$$O = X - V(\mathfrak{a}) = X - \bigcap_{f \in \mathfrak{a}} V(f) = \bigcup_{f \in \mathfrak{a}} (X - V(f)) = \bigcup_{f \in \mathfrak{a}} X_f,$$

or any open set is a union of basic open sets. $\square$

Proof of (i). $X_f \cap X_g = X_{fg} \Longleftrightarrow V(f) \cup V(g) = V(fg)$ holds by Exercise 1.15 (iv). $\square$

Proof of (ii).

$$\begin{aligned} X_f = \varnothing &\Longleftrightarrow V(f) = X \\ &\Longleftrightarrow f \in \mathfrak{p} \text{ for all prime ideal } \mathfrak{p} \text{ of } A \\ &\Longleftrightarrow f \in \mathfrak{N}, \text{the nilradical of } A \text{ (Proposition 1.8)} \\ &\Longleftrightarrow f \text{ is nilpotent (Proposition 1.7)} \end{aligned}$$

$\square$

Proof of (ii)(Using (iv)).

$$\begin{aligned} X_f = \varnothing &\Longleftrightarrow X_f = X_0 && \text{(Exercise 15(ii))} \\ &\Longleftrightarrow r(f) = r(0) && \text{((iv))} \\ &\Longleftrightarrow f \in r(f) = r(0) \\ &\Longleftrightarrow f^m = 0 \text{ for some } m > 0 \\ &\Longleftrightarrow f \text{ is nilpotent} \end{aligned}$$

□

Proof of (iii).

$$\begin{aligned} X_f = X &\iff V(f) = \varnothing \\ &\iff f \notin \mathfrak{p} \text{ for all prime ideal } \mathfrak{p} \text{ of } A \\ &\iff f \text{ is unit (Corollary 1.5)} \end{aligned}$$

□

Proof of (iii)(Using (iv)).

$$\begin{aligned} X_f = X &\iff X_f = X_1 && \text{(Exercise 15(ii))} \\ &\iff r(f) = r(1) && \text{((iv))} \\ &\iff f \in r(f) = r(1) \\ &\iff f^m = 1 \text{ for some } m > 0 \\ &\iff f \text{ is unit} \end{aligned}$$

□

Proof of (iv).

(1) *Show that $X_f \subseteq X_g \iff r((f)) \subseteq r((g))$.* Actually,

$$\begin{aligned} X_f \subseteq X_g &\Longrightarrow V(f) \supseteq V(g) \\ &\Longrightarrow \{\mathfrak{p} \in \mathrm{Spec}(A) : \mathfrak{p} \supseteq (f)\} \supseteq \{\mathfrak{p} \in \mathrm{Spec}(A) : \mathfrak{p} \supseteq (g)\} \\ &\Longrightarrow \bigcap_{(f)\subseteq\mathfrak{p}\in\mathrm{Spec}(A)} \mathfrak{p} \subseteq \bigcap_{(g)\subseteq\mathfrak{p}\in\mathrm{Spec}(A)} \mathfrak{p} \\ &\overset{1.14}{\Longrightarrow} r(f) \subseteq r(g) \\ &\Longrightarrow V(r(f)) \supseteq V(r(g)) \\ &\Longrightarrow V(f) \supseteq V(g) \\ &\Longrightarrow X_f \subseteq X_g. \end{aligned}$$

(2) By (1),

$$\begin{aligned} X_f \subseteq X_g &\iff r((f)) \subseteq r((g)), \\ X_f \supseteq X_g &\iff r((f)) \supseteq r((g)). \end{aligned}$$

Hence,

$$X_f = X_g \iff r((f)) = r((g)).$$

□

Proof of (v). Notice that it is enough to consider a covering of X by basic open sets $X_{f_i} (i \in I)$.

(1) Since X is covered by $X_{f_i} (i \in I)$,

$$\begin{aligned} X = \bigcup_{i \in I} X_{f_i} &\Longrightarrow X - V(1) = \bigcup_{i \in I} (X - V(f_i)) \\ &\Longrightarrow V(1) = \bigcap_{i \in I} V(f_i) \\ &\Longrightarrow V(1) = V\left(\sum_{i \in I} f_i\right) \\ &\Longrightarrow r(1) = r\left(\sum_{i \in I} f_i\right). \end{aligned}$$

Hence, $1 \in r(1) = r\left(\sum_{i \in I} f_i\right)$ can be expressed as

$$1 = 1^m = \sum_{j \in J} g_j f_j$$

where J is a finite subset of I and $g_j \in A$. That is, $(1) = \sum_{j \in J} f_j$.

(2) Hence, $V(1) = V\left(\sum_{j \in J} f_j\right)$. Therefore, X is covered by finite subcovering $\{X_{f_j}\} (j \in J)$.

□

Proof of (v)(Using (vi)). Since $X = X_1$, X is quasi-compact by (vi). □

Proof of (vi). Notice that it is enough to consider a covering of X_f by basic open sets $X_{f_i} (i \in I)$.

(1) Since X_f is covered by $X_{f_i} (i \in I)$,

$$\begin{aligned} X_f = \bigcup_{i \in I} X_{f_i} &\Longrightarrow X - V(f) = \bigcup_{i \in I} (X - V(f_i)) \\ &\Longrightarrow V(f) = \bigcap_{i \in I} V(f_i) \\ &\Longrightarrow V(f) = V\left(\sum_{i \in I} f_i\right) \\ &\Longrightarrow r(f) = r\left(\sum_{i \in I} f_i\right). \end{aligned}$$

Hence, $f \in r(f) = r\left(\sum_{i \in I} f_i\right)$ can be expressed as

$$f^m = \sum_{j \in J} g_j f_j$$

where J is a finite subset of I and $g_j \in A$. That is, $f^m \in \sum_{j \in J} f_j$.

(2) *Show that* $V\left(\sum_{j\in J} f_j\right) = V(f)$.

   (a) ($\subseteq$) For any prime ideal $\mathfrak{p} \supseteq \sum_{j\in J} f_j$, $f^m \in \mathfrak{p}$ or $f \in \mathfrak{p}$ (since $\mathfrak{p}$ is prime). So $\mathfrak{p} \supseteq (f)$, or $V\left(\sum_{j\in J} f_j\right) \subseteq V(f)$.

   (b) ($\supseteq$)
$$\sum_{j\in J} f_j \subseteq \sum_{i\in I} f_i \Longrightarrow V\left(\sum_{j\in J} f_j\right) \supseteq V\left(\sum_{i\in I} f_i\right) = V(f).$$

(3) Therefore, X_f is covered by finite subcovering $\{X_{f_j}\}(j \in J)$.

$\square$

Proof of (vi)(Using (v)). Exercise 3.21 (i) shows that X_f is the spectrum of A_f. By (v), X_f is quasi-compact. $\square$

Proof of (vii).

(1) ($\Longrightarrow$) Given an open subset O. Since X_f form a basis of open sets,
$$O = \bigcup_{f\in\mathfrak{a}} X_f \text{ for some ideal } \mathfrak{a} \text{ of } A$$
Especially, $\{X_f\}_{f\in\mathfrak{a}}$ is an open covering of O. Since O is quasi-compact, there exists a finite subcovering $\{X_f\}_{f\in J}$ of O, where J is a finite subset of $\mathfrak{a}$ (as a set). That is, $O = \bigcup_{f\in J} X_f$ is a finite union of sets X_f.

(2) ($\Longleftarrow$) Since X_f is quasi-compact, any finite union of quasi-compact sets is quasi-compact again.

$\square$

Exercise 1.19.

A topological space X is said to be irreducible if $X \neq \varnothing$ and if every pair of non-empty open sets in X intersect, or equivalently if every non-empty open set is dense in X. Show that $\mathrm{Spec}(A)$ *is irreducible if and only if the nilradical of A is a prime ideal.*

Proof. Use the notations in Proposition 1.7 and Exercise 1.17.

$$\begin{aligned}
&\mathrm{Spec}(A) \text{ is irreducible} \\
\Longleftrightarrow\ & X_f \cap X_g \neq \varnothing \text{ for nonempty } X_f, X_g \in \mathrm{Spec}(A) \\
\Longleftrightarrow\ & X_{fg} \neq \varnothing \text{ for nonempty } X_f, X_g \in \mathrm{Spec}(A) && \text{(Exercise 1.17 (i))} \\
\Longleftrightarrow\ & fg \notin \mathfrak{N} \text{ for } f, g \notin \mathfrak{N} && \text{(Exercise 1.17 (ii))} \\
\Longleftrightarrow\ & \mathfrak{N} \text{ is prime.}
\end{aligned}$$

□

Exercise 1.20.

Let X be a topological space.

(i) *If Y is an irreducible subspace of X, then the closure $\overline{Y}$ of Y in X is irreducible.*

(ii) *Every irreducible subspace of X is contained in a maximal irreducible subspace.*

(iii) *The maximal irreducible subspaces of X are closed and cover X. They are called the irreducible components of X. What are the irreducible components of a Hausdorff space?*

Proof of (i).

(1) *Y is irreducible if and only if Y cannot be represented as the union of two proper closed subspaces.*

$$\begin{aligned}
&\forall \text{ nonempty open sets } U_1 \text{ and } U_2, U_1 \cap U_2 \neq \varnothing \\
\Longleftrightarrow\ &\forall \text{ nonempty open sets } U_1 \text{ and } U_2, X - (U_1 \cap U_2) \neq X \\
\Longleftrightarrow\ &\forall \text{ nonempty open sets } U_1 \text{ and } U_2, (X - U_1) \cup (X - U_2) \neq X \\
\Longleftrightarrow\ &\forall \text{ proper closed sets } Y_1 \text{ and } Y_2, Y_1 \cup Y_2 \neq X \\
\Longleftrightarrow\ &\nexists \text{ proper closed sets } Y_1 \text{ and } Y_2, Y_1 \cup Y_2 = X.
\end{aligned}$$

(2) If $\overline{Y}$ were reducible, there are two closed set Y_1 and Y_2 such that

$$\overline{Y} \subseteq Y_1 \cup Y_2, \qquad \overline{Y} \not\subseteq Y_i (i = 1, 2).$$

(a) $Y \subseteq \overline{Y} \subseteq Y_1 \cup Y_2$.

(b) $Y \not\subseteq Y_i (i = 1, 2)$. If not, $Y \subseteq Y_i$ for some i. Take closure to get $\overline{Y} \subseteq \overline{Y_i} = Y_i$ (since Y_i is closed), contrary to the assumption.

By (a)(b), Y is reducible, which is absurd.

□

Proof of (ii).

(1) This is a standard application of Zorn's lemma.

(2) Suppose Y is an irreducible subspace of X. Let Σ be the set of all irreducible subspaces of X containing Y. Order Σ by inclusion. Σ is not empty, since $Y \in \Sigma$. To apply Zorn's lemma we must show that every chain in Σ has an upper bound in Σ; let then (Y_α) be a chain in Σ. Let $Z = \bigcup_\alpha Y_\alpha$. $Z \supseteq Y$ clearly.

(3) *Show that Z is irreducible.* Given two non-empty open sets U and V contained in $Z = \bigcup_\alpha Y_\alpha$. Then $U \cap Y_\alpha \neq \varnothing$ and $V \cap Y_\beta \neq \varnothing$ for some α, β. Since (Y_α) is a chain, we might have $V \cap Y_\alpha \supseteq V \cap Y_\beta \neq \varnothing$ if $\beta \leq \alpha$. (The case $\alpha \leq \beta$ is similar.) So $U \cap V \cap Z \supseteq U \cap V \cap Y_\alpha \neq \varnothing$ since Z contains an irreducible subspace Y_α in X.

(4) Hence $Z \in \Sigma$, and Z is an upper bound of the chain (Y_α). Hence by Zorn's lemma Σ has a maximal element.

□

Proof of (iii).

(1) *Show that the maximal irreducible subspaces of X are closed.* Suppose Y is a maximal irreducible subspaces of X. So $\overline{Y}$ of Y in X is irreducible (by part (i)). The maximality of Y implies that $Y = \overline{Y}$.

(2) *Show that the maximal irreducible subspaces of X cover X.* Note that each element $P \in X$ forms an irreducible subset $\{P\}$ and thus $\{P\}$ is contained in one irreducible component (by (ii)).

(3) One point subsets are the irreducible components of a Hausdorff space.

□

Chapter 2: Modules

Exercise 2.1.

Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$ *if* m, n *are coprime.*

It suffices to show that

$$(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$$

where d is the greatest common divisor of m and n.

Outlines.

(1) Define $\widetilde{\varphi}$ by

$$\begin{array}{rccc} \widetilde{\varphi}: & (\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z}) & \longrightarrow & \mathbb{Z}/d\mathbb{Z} \\ & \cup & & \cup \\ & (x + m\mathbb{Z}, y + n\mathbb{Z}) & \longmapsto & xy + d\mathbb{Z}. \end{array}$$

$\widetilde{\varphi}$ is well-defined and $\mathbb{Z}$-bilinear.

(2) By the universal property, $\widetilde{\varphi}$ factors through a $\mathbb{Z}$-bilinear map

$$\varphi : (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/d\mathbb{Z}$$

(such that $\varphi(x \otimes y) = \widetilde{\varphi}(x, y)$).

(3) To show that φ is isomorphic, might find the inverse map $\psi : \mathbb{Z}/d\mathbb{Z} \to (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$ of φ. Define ψ by

$$\begin{array}{rccc} \psi: & \mathbb{Z}/d\mathbb{Z} & \longrightarrow & (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \\ & \cup & & \cup \\ & z + d\mathbb{Z} & \longmapsto & (z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}). \end{array}$$

ψ is well-defined and $\mathbb{Z}$-linear.

(4) $\psi \circ \varphi = \mathrm{id}$.

(5) $\varphi \circ \psi = \mathrm{id}$.

Proof of (1).

(a) $\widetilde{\varphi}$ *is well-defined.* Say $x' = x + am$ for some $a \in \mathbb{Z}$ and $y' = y + bn$ for some $b \in \mathbb{Z}$. Then $x'y' - xy = yam + xbn + abmn \in \mathbb{Z}/d\mathbb{Z}$. That is, $\widetilde{\varphi}$ is independent of coset representative.

(b) *$\widetilde{\varphi}$ is $\mathbb{Z}$-bilinear.*

(i) *For any $\lambda \in \mathbb{Z}$, $\widetilde{\varphi}(\lambda x, y) = \widetilde{\varphi}(x, \lambda y) = \lambda\widetilde{\varphi}(x, y)$.* In fact,

$$\begin{aligned}
\widetilde{\varphi}(\lambda(x + m\mathbb{Z}), y + n\mathbb{Z}) &= \widetilde{\varphi}(\lambda x + m\mathbb{Z}, y + n\mathbb{Z}) = \lambda xy + d\mathbb{Z}, \\
\widetilde{\varphi}(x + m\mathbb{Z}, \lambda(y + n\mathbb{Z})) &= \widetilde{\varphi}(x + m\mathbb{Z}, \lambda y + n\mathbb{Z}) = \lambda xy + d\mathbb{Z}, \\
\widetilde{\varphi}(x_1 + m\mathbb{Z}, y + n\mathbb{Z}) &= \lambda(xy + d\mathbb{Z}) = \lambda xy + d\mathbb{Z}.
\end{aligned}$$

(ii) $\widetilde{\varphi}(x_1 + x_2, y) = \widetilde{\varphi}(x_1, y) + \widetilde{\varphi}(x_2, y)$. In fact,

$$\begin{aligned}
\widetilde{\varphi}((x_1 + x_2) + m\mathbb{Z}, y + n\mathbb{Z}) &= (x_1 + x_2)y + d\mathbb{Z}, \\
\widetilde{\varphi}(x_1 + m\mathbb{Z}, y + n\mathbb{Z}) + \widetilde{\varphi}(x_2 + m\mathbb{Z}, y + n\mathbb{Z}) &= (x_1 y + d\mathbb{Z}) + (x_2 y + d\mathbb{Z}) \\
&= (x_1 + x_2)y + d\mathbb{Z}.
\end{aligned}$$

(iii) $\widetilde{\varphi}(x, y_1 + y_2) = \widetilde{\varphi}(x, y_1) + \widetilde{\varphi}(x, y_2)$. Similar to (ii).

□

Proof of (3).

(a) *ψ is well-defined.* Say $z' = z + cd$ for some $c \in \mathbb{Z}$. Note that $d = \alpha m + \beta n$ for some $\alpha, \beta \in \mathbb{Z}$. Thus

$$\begin{aligned}
\psi(z' + d\mathbb{Z}) &= \psi(z + cd + d\mathbb{Z}) \\
&= \psi(z + c(\alpha m + \beta n) + d\mathbb{Z}) \\
&= (z + c(\alpha m + \beta n) + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\
&= (z + c\beta n + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\
&= (z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) + (c\beta n + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\
&= \psi(z + d\mathbb{Z}) + (1 + m\mathbb{Z}) \otimes (c\beta n + n\mathbb{Z}) \\
&= \psi(z + d\mathbb{Z}).
\end{aligned}$$

(b) *ψ is $\mathbb{Z}$-linear.*

(i) *For any $\lambda \in \mathbb{Z}$, $\psi(\lambda z) = \lambda\psi(z)$.* In fact,

$$\begin{aligned}
\psi(\lambda(z + d\mathbb{Z})) &= \psi(\lambda z + d\mathbb{Z}) = (\lambda z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}), \\
\lambda\psi(z + d\mathbb{Z}) &= \lambda((z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z})) = (\lambda z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}).
\end{aligned}$$

(ii) $\psi(z_1 + z_2) = \psi(z_1) + \psi(z_2)$.

$$\begin{aligned}
\psi((z_1 + z_2) + d\mathbb{Z}) &= (z_1 + z_2 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}), \\
\psi(z_1 + d\mathbb{Z}) + \psi(z_2 + d\mathbb{Z}) &= (z_1 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) + (z_2 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\
&= (z_1 + z_2 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}).
\end{aligned}$$

□

Proof of (4). For any $(x+m\mathbb{Z})\otimes(y+n\mathbb{Z})\in(\mathbb{Z}/m\mathbb{Z})\otimes_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z})$,

$$\begin{aligned}\psi(\varphi((x+m\mathbb{Z})\otimes(y+n\mathbb{Z}))) &= \psi(xy+d\mathbb{Z})\\ &= (xy+m\mathbb{Z})\otimes(1+n\mathbb{Z})\\ &= (x+m\mathbb{Z})\otimes(y+n\mathbb{Z}).\end{aligned}$$

□

Proof of (5). For any $z+d\mathbb{Z}\in\mathbb{Z}/d\mathbb{Z}$,

$$\begin{aligned}\varphi(\psi(z+d\mathbb{Z}) &= \varphi((z+m\mathbb{Z})\otimes(1+n\mathbb{Z}))\\ &= z+d\mathbb{Z}.\end{aligned}$$

□

Exercise 2.2.

Let A be a ring, $\mathfrak{a}$ an ideal, M an A-module. Show that $(A/\mathfrak{a})\otimes_A M$ is isomorphic to $M/\mathfrak{a}M$. (Hint: Tensor the exact sequence $0\to\mathfrak{a}\to A\to A/\mathfrak{a}\to 0$ with M.

Proof (Hint). There is a natural exact sequence E:

$$E: 0\to\mathfrak{a}\xrightarrow{i}A\xrightarrow{\pi}A/\mathfrak{a}\to 0$$

where i is the inclusion map (and π is the projection map). Tensor E with M:

$$E': \mathfrak{a}\otimes_A M\xrightarrow{i\otimes 1}A\otimes_A M\xrightarrow{\pi\otimes 1}(A/\mathfrak{a})\otimes_A M\to 0$$

is exact, or

$$(A/\mathfrak{a})\otimes_A M\cong A\otimes_A M/\mathrm{im}(i\otimes 1).$$

By Proposition 2.14, There is an unique isomorphism $A\otimes_A M\to M$ defined by $a\otimes x\mapsto ax$. This isomorphism sends $\mathrm{im}(i\otimes 1)$ to $\mathfrak{a}M$. Therefore,

$$(A/\mathfrak{a})\otimes_A M\cong M/\mathfrak{a}M.$$

□

Proof (Brute-force).

(1) Define $\widetilde{\varphi}$ by

$$\begin{array}{rccc} \widetilde{\varphi}: & A/\mathfrak{a} \times M & \longrightarrow & M/\mathfrak{a}M \\ & \cup & & \cup \\ & (a+\mathfrak{a}, x) & \longmapsto & ax + \mathfrak{a}M. \end{array}$$

$\widetilde{\varphi}$ is well-defined and A-bilinear.

(2) By the universal property, $\widetilde{\varphi}$ factors through a A-bilinear map

$$\varphi : A/\mathfrak{a} \otimes_A M \to M/\mathfrak{a}M$$

(such that $\varphi(a \otimes x) = \widetilde{\varphi}(a, x)$).

(3) To show that φ is isomorphic, might find the inverse map $\psi : M/\mathfrak{a}M \to A/\mathfrak{a} \otimes_A M$ of φ. Define ψ by

$$\begin{array}{rccc} \psi: & M/\mathfrak{a}M & \longrightarrow & A/\mathfrak{a} \otimes_A M \\ & \cup & & \cup \\ & x + \mathfrak{a}M & \longmapsto & (1+\mathfrak{a}) \otimes x. \end{array}$$

ψ is well-defined and A-linear.

(4) $\psi \circ \varphi = \mathrm{id}$.

(5) $\varphi \circ \psi = \mathrm{id}$.

$\square$

Exercise 2.3.

Let A be a local ring, M and N finitely generated A-modules. Prove that if $M \otimes_A N = 0$, then $M = 0$ or $N = 0$. (Hint: Let $\mathfrak{m}$ be the maximal ideal, $k = A/\mathfrak{m}$ the residue field. Let $M_k = k \otimes_A M \cong M/\mathfrak{m}M$ by Exercise 2.2. By Nakayama's lemma, $M_k = 0 \Longrightarrow M = 0$. But $M \otimes_A N = 0 \Longrightarrow (M \otimes_A N)_k = 0 \Longrightarrow M_k \otimes_k N_k = 0 \Longrightarrow M_k = 0$ or $N_k = 0$ since M_k, N_k are vector spaces over a field.)

The conclusion might be false if A is not local. For example, Exercise 2.1.

Proof (Hint). Let $\mathfrak{m}$ be the maximal ideal, $k = A/\mathfrak{m}$ the residue field. Let $M_k = k \otimes_A M$.

(1) *(Base extension) Show that* $(M \otimes_A N)_k = M_k \otimes_k N_k$. In fact, by Proposition 2.14

$$\begin{aligned}
(M \otimes_A N)_k &= k \otimes_A (M \otimes_A N) \\
&= (k \otimes_A M) \otimes_A N \\
&= M_k \otimes_A N \\
&= (M_k \otimes_k k) \otimes_A N \\
&= M_k \otimes_k (k \otimes_A N) \\
&= M_k \otimes_k N_k.
\end{aligned}$$

(2)

$$\begin{aligned}
M \otimes_A N = 0 &\Longrightarrow (M \otimes_A N)_k = 0 \\
&\Longrightarrow M_k \otimes_k N_k = 0 && ((1)) \\
&\Longrightarrow M_k = 0 \text{ or } N_k = 0 && (M_k, N_k\text{: vector spaces}) \\
&\Longrightarrow M/\mathfrak{m}M = 0 \text{ or } M/\mathfrak{m}M = 0 && (\text{Exercise 2.2}) \\
&\Longrightarrow M = 0 \text{ or } N = 0. && (\text{Nakayama's lemma})
\end{aligned}$$

□

Exercise 2.4.

Let M_i $(i \in I)$ be any family of A-modules, and let M be their direct sum. Prove that M is flat $\Leftrightarrow$ each M_i is flat.

Proof. Given any A-module homomorphism $f : N' \to N$.

(1) Similar to Proposition 2.14(iii), we have two isomorphisms

(a)
$$\varphi : \bigoplus_{i \in I} (N' \otimes M_i) \cong N' \otimes_A \bigoplus_{i \in I} M_i$$
defined by
$$\varphi((x \otimes m_i)_{i \in I}) = x \otimes (m_i)_{i \in I}$$
where $x \in N'$, $m_i \in M_i$ $(i \in I)$.

(b)
$$\psi : N \otimes_A \bigoplus_{i \in I} M_i \cong \bigoplus_{i \in I} (N \otimes M_i)$$
defined by
$$\psi(y \otimes (m_i)_{i \in I}) = (y \otimes m_i)_{i \in I}$$
where $y \in N$, $m_i \in M_i$ $(i \in I)$.

(2) $f : N' \to N$ induces an A-module homomorphism

$$f \otimes \mathrm{id}_M : N' \otimes_A M \to N \otimes_A M.$$

(3) $\psi \circ f \otimes \mathrm{id}_M \circ \varphi$ defines an A-module homomorphism

$$\psi \circ f \otimes \mathrm{id}_M \circ \varphi : \bigoplus_{i \in I} (N' \otimes M_i) \to \bigoplus_{i \in I} (N \otimes M_i)$$

which sends $(x \otimes m_i)_{i \in I}$ to $(f(x) \otimes m_i)_{i \in I}$. That is,

$$\psi \circ f \otimes \mathrm{id}_M \circ \varphi = \bigoplus_{i \in I} f \otimes \mathrm{id}_{M_i}$$

.

(4) *Show that M is flat if and only if each M_i is flat.* Suppose f is injective.

$$\begin{aligned}
& M_i \text{ is flat } \forall\, i \in I \\
\Longleftrightarrow & f \otimes \mathrm{id}_{M_i} \text{ is injective } \forall\, i \in I \\
\Longleftrightarrow & \bigoplus_{i \in I} f \otimes \mathrm{id}_{M_i} \text{ is injective} && \text{(Injectivity)} \\
\Longleftrightarrow & \psi \circ f \otimes \mathrm{id}_M \circ \varphi \text{ is injective} && \text{((3))} \\
\Longleftrightarrow & f \otimes \mathrm{id}_M \text{ is injective} && (\varphi,\ \psi \text{ are isomorphic}) \\
\Longleftrightarrow & M \text{ is flat.}
\end{aligned}$$

□

Exercise 2.5.

Let $A[x]$ be the ring of polynomials in one indeterminate over a ring A. Prove that $A[x]$ is a flat A-algebra. (Hint: Use Exercise 2.4.)

Proof (Hint).

(1) A is a flat A-module by Proposition 2.14(iv).

(2) As an A-module,

$$A[x] \cong \bigoplus_{n \in \mathbb{Z}^+} Ax^n \cong \bigoplus_{n \in \mathbb{Z}^+} A$$

(since $Ax^n \cong A$).

(3) By Exercise 2.4, $A[x] \cong \bigoplus_{n \in \mathbb{Z}^+} A$ is flat.

□

Exercise 2.8.

(i) *If M and N are flat A-modules, then so is $M \otimes_A N$.*

(ii) *If B is a flat A-algebra and N is a flat B-module, then N is flat as A-module.*

Proof of (i). Given any exact sequence of A-modules $0 \to N_1 \to N_2 \to N_3 \to 0$. Since M is flat,

$$0 \to N_1 \otimes_A M \to N_2 \otimes_A M \to N_3 \otimes_A M \to 0$$

is exact. Since N is flat,

$$0 \to (N_1 \otimes_A M) \otimes_A N \to (N_2 \otimes_A M) \otimes_A N \to (N_3 \otimes_A M) \otimes_A N \to 0$$

is exact. By Proposition 2.14 (ii),

$$0 \to N_1 \otimes_A (M \otimes_A N) \to N_2 \otimes_A (M \otimes_A N) \to N_3 \otimes_A (M \otimes_A N) \to 0$$

is exact, or $M \otimes_A N$ is flat. □

Proof of (ii). Given any exact sequence of A-modules $0 \to N_1 \to N_2 \to N_3 \to 0$. Since B is a flat A-algebra (A-module),

$$0 \to N_1 \otimes_A B \to N_2 \otimes_A B \to N_3 \otimes_A B \to 0$$

is exact. Since N is a flat B-module,

$$0 \to (N_1 \otimes_A B) \otimes_B N \to (N_2 \otimes_A B) \otimes_B N \to (N_3 \otimes_A B) \otimes_B N \to 0$$

is exact. By "Exercise 2.15" on page 27,

$$0 \to N_1 \otimes_A (B \otimes_B N) \to N_2 \otimes_A (B \otimes_B N) \to N_3 \otimes_A (B \otimes_B N) \to 0$$

is exact. By Proposition 2.14 (iv),

$$0 \to N_1 \otimes_A N \to N_2 \otimes_A N \to N_3 \otimes_A N \to 0$$

is exact, or N is flat. □

Exercise 2.9.

Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. If M' and M'' are finitely generated, then so is M.

Proof.

(1) Write
$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0.$$
Also write
$$\begin{aligned} &x_1, \ldots, x_n \text{ as generators of } M', \\ &z_1, \ldots, z_m \text{ as generators of } M'' \end{aligned}$$
(since M' and M'' are finitely generated).

(2) Since the map $g : M \to M''$ is surjective, there exists $y_j \in M$ such that $g(y_j) = z_j$ for $j = 1, \ldots, m$.

(3) *Show that M is generated by*
$$f(x_1), \ldots, f(x_n), y_1, \ldots, y_m.$$
Given any $y \in M$.
$$\begin{aligned} y \in M &\Longrightarrow g(y) \in M'' \\ &\Longrightarrow g(y) = \sum_{j=1}^{m} s_j z_j \text{ where } s_j \in A \\ &\Longrightarrow g(y) = \sum_{j=1}^{m} s_j g(y_j) \\ &\Longrightarrow g(y) = g\left(\sum_{j=1}^{m} s_j y_j\right) \\ &\Longrightarrow y - \sum_{j=1}^{m} s_j y_j \in \ker(g) = \operatorname{im}(f) \\ &\Longrightarrow \exists\, x \in M' \text{ such that } f(x) = y - \sum_{j=1}^{m} s_j y_j \end{aligned}$$
Write $x = \sum_{i=1}^{n} r_i x_i$ where $r_i \in A$. So,
$$\begin{aligned} y \in M &\Longrightarrow f\left(\sum_{i=1}^{n} r_i x_i\right) = y - \sum_{j=1}^{m} s_j y_j \\ &\Longrightarrow \sum_{i=1}^{n} r_i f(x_i) = y - \sum_{j=1}^{m} s_j y_j \\ &\Longrightarrow y = \sum_{i=1}^{n} r_i f(x_i) + \sum_{j=1}^{m} s_j y_j. \end{aligned}$$
Hence, every $y \in M$ is a linear combination of $f(x_1), \ldots, f(x_n), y_1, \ldots, y_m$, or M is finitely generated (by $f(x_1), \ldots, f(x_n), y_1, \ldots, y_m$).

□