Chapter 2: Basic Topology

Author: Meng-Gen Tsai
Email: plover@gmail.com

Exercise 2.1. *Prove that the empty set is a subset of every set.*

Proof. By Definitions 1.3,

(1) The set which contains no element will be called the **empty set**,

(2) If A and B are sets, and if every element of A is an element of B, we say that A is a **subset** of B,

every element of the empty set (there are none) belongs to every set. That is, the empty set is a subset of every set. □

Exercise 2.2. *A complex number z is said to be algebraic if there are integers $a_0, ..., a_n$, not all zero, such that*

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic numbers is countable. (Hint: For every positive integer N there are only finitely many equations with

$$n + |a_0| + |a_1| + \cdots + |a_n| = N.$$

Might assume $a_0 \neq 0$.

For example, all rational numbers are algebraic since $p = \frac{\alpha}{\beta}$ (where $\alpha, \beta \in \mathbb{Z}$) is a root of $\beta z - \alpha = 0$.

Besides, $z = \sqrt{2} + \sqrt{3}$ is algebraic since $z^4 - 10z^2 + 1 = 0$. In fact, $z = \pm\sqrt{2} + \pm\sqrt{3}$ are also algebraic since $z^4 - 10z^2 + 1 = (z - \sqrt{2} - \sqrt{3})(z + \sqrt{2} - \sqrt{3})(z - \sqrt{2} + \sqrt{3})(z + \sqrt{2} + \sqrt{3})$.

Lemma. *The set of all polynomials over $\mathbb{Z}$ is countable implies that the set of algebraic numbers is countable.*

Proof of Lemma. By definition, we write the set of algebraic numbers as

$$S = \bigcup_{f(x) \in \mathbb{Z}[x]} \{z \in \mathbb{C} : f(z) = 0\}.$$

Since each polynomial of degree n has at most n roots, $\{z \in \mathbb{C} : f(z) = 0\}$ is finite for each given $f(x) \in \mathbb{Z}[x]$. So S is a countable union (by assumption) of finite sets, and hence at most countable. S is infinite since every integer α is a root of $f(z) = z - \alpha$. So S is countable. $\square$

Thus, it suffices to show that *the set of all polynomials over $\mathbb{Z}$ is countable.*

Proof (Hint). For every positive integer N there are only finitely many equations with $n + |a_0| + |a_1| + \cdots + |a_n| = N$. Write

$$P_N = \{f(x) \in \mathbb{Z}[x] : n + |a_0| + |a_1| + \cdots + |a_n| = N\}$$

where $f(x) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$ with $a_0 \neq 0$, and

$$P = \bigcup_{N=1}^{\infty} P_N.$$

P is the set of all polynomials over $\mathbb{Z}$.

Each P_N is finite for given N (since the equation $n + |a_0| + |a_1| + \cdots + |a_n| = N$ has finitely many solutions $(n, a_0, a_1, ..., a_n) \in \mathbb{Z}^{n+2}$). So P is a countable union of finite sets, and hence at most countable. P is infinite since $\mathbb{Z}$ is a subring of $\mathbb{Z}[x]$. So P is countable. $\square$

Proof (Theorem 2.13).

(1) *$\mathbb{Z}^N$ is countable for any integer $N > 0$.* Theorem 2.13.

(2) *The set of all polynomials over $\mathbb{Z}$ is countable.* Let

$$P_n = \{f \in \mathbb{Z}[x] : \deg f = n\},$$

and

$$P = \bigcup_{n=1}^{\infty} P_n = \mathbb{Z}[x].$$

Claim: P_n is countable. Define a 1-1 map $\varphi_n : P_n \to \mathbb{Z}^{n+1}$ by

$$\varphi_n(a_0 z^n + a_1 z^{n-1} + \cdots + a_n) = (a_0, a_1, ..., a_{n-1}, a_n).$$

By (1) and Theorem 2.8, P_n is countable. (P_n is infinite since $a_n \in \mathbb{Z}$.) Now P is a countable union of countable sets, and hence countable by Theorem 2.12.

$\square$

Proof (Unique factorization theorem).

(1) *The set of prime numbers is countable.* Write all primes in the ascending order as $p_1, p_2, ..., p_n, ...$ where $p_1 = 2, p_2 = 3, ..., p_{10001} = 104743, ...$ (See ProjectEuler 7: 10001st prime. Use sieve of Eratosthenes to get p_{10001}.)

(2) *The set of all polynomials over $\mathbb{Z}$ is countable.* Let

$$P_n = \{f \in \mathbb{Z}[x] : \deg f = n\},$$

and

$$P = \bigcup_{n=1}^{\infty} P_n = \mathbb{Z}[x].$$

Claim: P_n is countable. Define a map $\varphi_n : P_n \to \mathbb{Z}^+$ by

$$\varphi_n(a_0 z^n + a_1 z^{n-1} + \cdots + a_n) = p_1^{\psi(a_0)} p_2^{\psi(a_1)} \cdots p_{n+1}^{\psi(a_n)},$$

where ψ is a 1-1 correspondence from $\mathbb{Z}$ to $\mathbb{Z}^+$ (Example 2.5). By the unique factorization theorem, φ_n is 1-1. So P_n is countable by Theorem 2.8. (P_n is infinite since $a_n \in \mathbb{Z}$.) Now P is a countable union of countable sets, and hence countable by Theorem 2.12.

$\square$

Exercise 2.3. *Prove that there exist real numbers which are not algebraic.*

Proof (Exercise 2.2). If all real numbers were algebraic, then $\mathbb{R}$ is countable by Exercise 2.2, contrary to the fact that $\mathbb{R}$ is uncountable (Corollary to Theorem 2.43). $\square$

Proof (Liouville, 1844).

(1) **Lemma.** *If ξ is a real algebraic number of degree $n > 1$, then there is a constant $A > 0$ (depending on ξ) such that*

$$\left| \xi - \frac{h}{k} \right| \geq \frac{A}{k^n}$$

for all rational numbers $\frac{h}{k}$.

(a) If $\left|\xi - \frac{h}{k}\right| \geq 1$, pick $A = 1 > 0$.

(b) If $\left|\xi - \frac{h}{k}\right| < 1$, let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ be an irreducible polynomial of degree $n > 1$ over $\mathbb{Z}$ such that $f(\xi) = 0$. By the mean value theorem,

$$f(\xi) - f\left(\frac{h}{k}\right) = \left(\xi - \frac{h}{k}\right) f'(c)$$

for some $c \in \left(\xi - \frac{h}{k}, \xi + \frac{h}{k}\right) \subseteq (\xi - 1, \xi + 1)$. Notice that

(i) $f(\xi) = 0$ by definition.

(ii) $f\left(\frac{h}{k}\right) \neq 0$ since $\frac{h}{k}$ cannot be a root of $f(x)$. Otherwise f is of degree 1, contrary to the assumption of f.

(iii) $\left|f\left(\frac{h}{k}\right)\right| \geq \frac{1}{k^n}$ since

$$\begin{aligned} f\left(\frac{h}{k}\right) &= a_0 + a_1\left(\frac{h}{k}\right) + \cdots + a_n\left(\frac{h}{k}\right)^n \neq 0, \\ k^n f\left(\frac{h}{k}\right) &= a_0 k^n + hk^{n-1}a_1 + \cdots + h^n a_n \neq 0, \\ k^n\left|f\left(\frac{h}{k}\right)\right| &\geq 1. \end{aligned}$$

(iv) $|f'(c)| \leq \sup_{x\in[\xi-1,\xi+1]} |f'(x)|$ since $c \in [\xi-1, \xi+1]$ and $f'(x)$ is continuous or bounded on a compact set $[\xi-1, \xi+1]$.

By (i)-(iv),

$$\begin{aligned} \left|f(\xi) - f\left(\frac{h}{k}\right)\right| &= \left|\left(\xi - \frac{h}{k}\right) f'(c)\right|, \\ \frac{1}{k^n} \leq \left|f\left(\frac{h}{k}\right)\right| &= \left|\xi - \frac{h}{k}\right| |f'(c)| \leq \left|\xi - \frac{h}{k}\right| \cdot \sup_{x\in[\xi-1,\xi+1]} |f'(x)|. \end{aligned}$$

Pick $A = (1 + \sup_{x\in[\xi-1,\xi+1]} |f'(x)|)^{-1} > 0$.

By (a)(b), we arrange $A = \min(1, (1+\sup_{x\in[\xi-1,\xi+1]} |f'(x)|)^{-1}) > 0$ to fit the inequality.

(2) $\xi = \sum_{n=0}^{\infty} 10^{-n!}$ *is transcendental.*

(a) Let $k_j = 10^{j!}$, $h_j = 10^{j!}\sum_{n=0}^{j} 10^{-n!}$. Then

$$\left|\xi - \frac{h_j}{k_j}\right| = \sum_{n=j+1}^{\infty} 10^{-n!} < \sum_{n=(j+1)!}^{\infty} 10^{-n} = \frac{A_j}{k_j^j}$$

where $A_j = \frac{10}{9} \cdot 10^{-j!}$.

(b) If ξ were a real algebraic number of degree $d > 1$, then by Lemma and (a),

$$\frac{A}{k_j^d} < \left|\xi - \frac{h_j}{k_j}\right| < \frac{A_j}{k_j^j} < \frac{A_j}{k_j^d}$$

for some $A > 0$ and $j \geq d$, or $0 < A < A_j$. Since j is arbitrary, $A_j \to 0$ as $j \to \infty$, contrary to $A > 0$.

(c) If ξ were a real algebraic number of degree $d = 1$, $\xi = \frac{h}{k}$ is a rational number. So

$$\left|\xi - \frac{h_j}{k_j}\right| = \left|\frac{h}{k} - \frac{h_j}{k_j}\right| = \left|\frac{hk_j - kh_j}{kk_j}\right| \geq \left|\frac{1}{kk_j}\right| = \frac{|k|^{-1}}{k_j}$$

for all j. (It is impossible that $hk_j - kh_j = 0$ or $\frac{h}{k} = \frac{h_i}{k_j}$ since $|\frac{h}{k} - \frac{h_j}{k_j}| = \sum_{n=j+1}^{\infty} 10^{-n!} > 0$ for all j.) Again by (a),

$$\frac{|k|^{-1}}{k_j} \leq \left| \xi - \frac{h_j}{k_j} \right| < \frac{A_j}{k_j^j} < \frac{A_j}{k_j},$$

or $0 < |k|^{-1} < A_j$. (Similar to (b).) Since j is arbitrary, $A_j \to 0$ as $j \to \infty$, contrary to $|k|^{-1} > 0$.

□

Exercise 2.10. *Let X be an infinite set. For $p \in X$ and $q \in X$, define*

$$d(p,q) = \begin{cases} 1 & (\text{if } p \neq q) \\ 0 & (\text{if } p = q). \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

(The statement holds even if X is finite.) We called d the discrete metric, and the corresponding topology on X induces the discrete topology. Conversely, if X has the discrete topology, X is always metrizable by the discrete metric.

Proof.

(1) *$d(p,q)$ is a metric.*

   (a) *$d(p,q) > 0$ if $p \neq q$; $d(p,p) = 0$.* Trivial.

   (b) *$d(p,q) = d(q,p)$.* Trivial.

   (c) *$d(p,q) \leq d(p,r) + d(r,q)$ for any $r \in X$.* If $p = q$, nothing to do. If $p \neq q$, $r \neq p$ or $r \neq q$ for any $r \in X$. (Assume not true, $r = p$ and $r = q$ implies that $p = q$ which is a contradiction.) In any cases $d(p,r) + d(r,q) \geq 1 = d(p,q)$.

(2) *Every subset is open.* Let E be any subset of X. Then every point $p \in E$ is an interior point of E. In fact, we can pick one neighborhood $N_{\frac{1}{2}}(p)$ of p containing only one point $p \in E$ or $N_{\frac{1}{2}}(p) = \{p\}$, and such neighborhood $N_{\frac{1}{2}}(p)$ is a subset of E. So every subset of X is open.

(3) *Every subset is closed.* Since every subset is open, every subset is closed by Theorem 2.23.

   Supplement. Might use Definition 2.18 (d) to prove directly since there are no limit points in X if we consider one neighborhood $N_{\frac{1}{2}}(p)$ of p. Therefore, every subset is closed. Again we apply Theorem 2.23 to get that every subset is open without using Definition 2.18 (f).

(4) *A subset is compact iff it is finite.*

   (a) *Any finite subset is compact.* Say $E = \{p_1, p_2, ..., p_k\}$, and $\{G_\alpha\}$ be an open covering of E. From $\{G_\alpha\}$ we pick G_{α_1} containing p_1, G_{α_2} containing p_2, ..., and G_{α_k} containing p_k. This process can be done in the finitely many steps. Therefore,

$$\{G_{\alpha_1}, G_{\alpha_2}, ..., G_{\alpha_k}\}$$

   is a finite subcovering of $\{G_\alpha\}$ covering E.

   (b) *Any infinite subset is not compact.* Take a collection

$$\mathscr{G} = \{G_p = \{p\}\}$$

   of open subsets where p runs all points in E. Clearly, $\{G_p\}$ is an open covering. Assume

$$\mathscr{G}' = \{G_{p_1}, G_{p_2}, ..., G_{p_k}\}$$

   is any finite subcovering of $\mathscr{G}$. Since E is infinite, there exist a point $p \in E$ such that $p \neq p_1$, $p \neq p_2$, ..., $p \neq p_k$. Therefore, $\mathscr{G}'$ does not cover p, or $\mathscr{G}$ does not contains any finite subcovering $\mathscr{G}'$.

□

Notice that every subset is bounded. Therefore, every subset is closed and bounded, but only finite subset is compact, i.e., Heine-Borel theorem is not true in the infinite discrete topology.

Exercise 2.12. *Let $K \subseteq \mathbb{R}^1$ consist of 0 and the numbers $\frac{1}{n}$, for $n = 1, 2, 3, ...$. Prove that K is compact directly from the definition (without using the Heine-Borel theorem).*

Proof. Let $\{G_\alpha\}$ be an open covering of K. There is an open set $G_0 \in \{G_\alpha\}$ containing 0. So there exists a neighborhood $N_r(0)$ of 0 such that $N_r(0) \subseteq G_0$. So $N_r(0)$ contains all points $q = \frac{1}{n}$ of K whenever $n > \frac{1}{r}$. To construct a finite subcovering of $\{G_\alpha\}$, we need to pick finitely many open sets from $\{G_\alpha\}$ to cover the remaining points $q = \frac{1}{n}$ where $n = 1, 2, ..., \left[\frac{1}{r}\right]$, say G_1 contains $q = \frac{1}{1}$, G_2 contains $q = \frac{1}{2}$, ..., $G_{\left[\frac{1}{r}\right]}$ contains $q = \frac{1}{\left[\frac{1}{r}\right]}$. (Might be duplicated.) Hence,

$$\left\{G_0, G_1, G_2, ..., G_{\left[\frac{1}{r}\right]}\right\}$$

is a finite subcovering of $\{G_\alpha\}$ covering K. □

Proof (Heine-Borel theorem).

(1) K is closed. In fact, the only limit point of K is 0, which is in K.

(a) $p = 0$ *is a limit point.* Given $r > 0$. There always exists $n \in \mathbb{Z}^+$ such that $r > \frac{1}{n}$. So any neighborhood $N_r(0)$ of $p = 0$ contains at least one point $q = \frac{1}{n} \neq 0$ in K.

(b) $p < 0$ *is not a limit point.* Pick a neighborhood $N_r(p)$ of p where $r = |p| > 0$. Then $N_r(p) \cap K = \varnothing$.

(c) $p > 0$ *is not a limit point.* There always exists $m \in \mathbb{Z}^+$ such that $p > \frac{1}{n}$ whenever $n \geq m$. Pick a neighborhood $N_r(p)$ of p where $r = p - \frac{1}{m} > 0$. Then $N_r(p)$ does not have all points $q = \frac{1}{n} \in K$ whenever $n \geq m$. By Theorem 2.20, p cannot be a limit point of K.

(2) K is bounded. There is a real number $M = 2$ and a point $q = 0 \in \mathbb{R}^1$ such that $|p - q| = |p| < 2$ for all $p \in K$.

By Heine-Borel theorem, K is compact in $\mathbb{R}^1$. $\square$

Exercise 2.14. *Give an example of an open cover of the segment* $(0, 1)$ *which has no finite subcover.*

Proof. In $\mathbb{R}^1$, take a collection

$$\mathscr{G} = \left\{ G_n = \left(\frac{1}{n}, 1 \right) \right\}$$

of open subsets where $n \in \mathbb{Z}^+$.

(1) $\mathscr{G}$ *is an open covering of* $(0, 1) \subseteq \mathbb{R}^1$. Actually, given $x \in (0, 1)$, there exists an positive integer n such that $x > \frac{1}{n}$. That is, $x \in \left(\frac{1}{n}, 1\right) = G_n$.

(2) *There is no finite subcovering of* $\mathscr{G}$. Assume

$$\mathscr{G}' = \{G_{n_1}, G_{n_2}, ..., G_{n_k}\}$$

is any finite subcovering of $\mathscr{G}$ where $n_1 < n_2 < ... < n_k$. Take $x \in \left(0, \frac{1}{n_k}\right) \neq \varnothing$, $x = \frac{1}{2n_k}$ for example. Then $x \notin G_{n_1}$, $x \notin G_{n_1}$, ..., $x \notin G_{n_k}$, which contradicts that $\mathscr{G}'$ is a finite subcovering of $\mathscr{G}$ covering $(0, 1)$.

$\square$