Chapter 2: Modules

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Exercise 2.1. *Show that* $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$ *if* m, n *are coprime.*

It suffices to show that

$$(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$$

where d is the greatest common divisor of m and n.

Outlines.

(1) Define $\widetilde{\varphi}$ by

$$\begin{array}{cccc} \widetilde{\varphi}: & (\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z}) & \longrightarrow & \mathbb{Z}/d\mathbb{Z} \\ & \Cup & & \Cup \\ & (x + m\mathbb{Z}, y + n\mathbb{Z}) & \longmapsto & xy + d\mathbb{Z}. \end{array}$$

$\widetilde{\varphi}$ is well-defined and $\mathbb{Z}$-bilinear.

(2) By the universal property, $\widetilde{\varphi}$ factors through a $\mathbb{Z}$-linear map

$$\varphi : (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/d\mathbb{Z}$$

(such that $\varphi(x \otimes y) = \widetilde{\varphi}(x, y)$).

(3) To show that φ is isomorphic, might find the inverse map $\psi : \mathbb{Z}/d\mathbb{Z} \to (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$ of φ. Define ψ by

$$\begin{array}{cccc} \psi: & \mathbb{Z}/d\mathbb{Z} & \longrightarrow & (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \\ & \Cup & & \Cup \\ & z + d\mathbb{Z} & \longmapsto & (z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}). \end{array}$$

ψ is well-defined and $\mathbb{Z}$-linear.

(4) $\psi \circ \varphi = \mathrm{id}$.

(5) $\varphi \circ \psi = \mathrm{id}$.

Proof of (1).

(a) $\widetilde{\varphi}$ *is well-defined.* Say $x' = x + am$ for some $a \in \mathbb{Z}$ and $y' = y + bn$ for some $b \in \mathbb{Z}$. Then $x'y' - xy = yam + xbn + abmn \in \mathbb{Z}/d\mathbb{Z}$. That is, $\widetilde{\varphi}$ is independent of coset representative.

(b) $\widetilde{\varphi}$ *is $\mathbb{Z}$-bilinear.*

(i) *For any $\lambda \in \mathbb{Z}$, $\widetilde{\varphi}(\lambda x, y) = \widetilde{\varphi}(x, \lambda y) = \lambda\widetilde{\varphi}(x, y)$.* In fact,

$$\begin{aligned}
\widetilde{\varphi}(\lambda(x + m\mathbb{Z}), y + n\mathbb{Z}) &= \widetilde{\varphi}(\lambda x + m\mathbb{Z}, y + n\mathbb{Z}) = \lambda xy + d\mathbb{Z}, \\
\widetilde{\varphi}(x + m\mathbb{Z}, \lambda(y + n\mathbb{Z})) &= \widetilde{\varphi}(x + m\mathbb{Z}, \lambda y + n\mathbb{Z}) = \lambda xy + d\mathbb{Z}, \\
\widetilde{\varphi}(x_1 + m\mathbb{Z}, y + n\mathbb{Z}) &= \lambda(xy + d\mathbb{Z}) = \lambda xy + d\mathbb{Z}.
\end{aligned}$$

(ii) $\widetilde{\varphi}(x_1 + x_2, y) = \widetilde{\varphi}(x_1, y) + \widetilde{\varphi}(x_2, y)$. In fact,

$$\begin{aligned}
\widetilde{\varphi}((x_1 + x_2) + m\mathbb{Z}, y + n\mathbb{Z}) &= (x_1 + x_2)y + d\mathbb{Z}, \\
\widetilde{\varphi}(x_1 + m\mathbb{Z}, y + n\mathbb{Z}) + \widetilde{\varphi}(x_2 + m\mathbb{Z}, y + n\mathbb{Z}) &= (x_1 y + d\mathbb{Z}) + (x_2 y + d\mathbb{Z}) \\
&= (x_1 + x_2)y + d\mathbb{Z}.
\end{aligned}$$

(iii) $\widetilde{\varphi}(x, y_1 + y_2) = \widetilde{\varphi}(x, y_1) + \widetilde{\varphi}(x, y_2)$. Similar to (ii).

□

Proof of (3).

(a) ψ *is well-defined.* Say $z' = z + cd$ for some $c \in \mathbb{Z}$. Note that $d = \alpha m + \beta n$ for some $\alpha, \beta \in \mathbb{Z}$. Thus

$$\begin{aligned}
\psi(z' + d\mathbb{Z}) &= \psi(z + cd + d\mathbb{Z}) \\
&= \psi(z + c(\alpha m + \beta n) + d\mathbb{Z}) \\
&= (z + c(\alpha m + \beta n) + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\
&= (z + c\beta n + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\
&= (z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) + (c\beta n + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\
&= \psi(z + d\mathbb{Z}) + (1 + m\mathbb{Z}) \otimes (c\beta n + n\mathbb{Z}) \\
&= \psi(z + d\mathbb{Z}).
\end{aligned}$$

(b) ψ *is $\mathbb{Z}$-linear.*

(i) *For any $\lambda \in \mathbb{Z}$, $\psi(\lambda z) = \lambda\psi(z)$.* In fact,

$$\begin{aligned}
\psi(\lambda(z + d\mathbb{Z})) &= \psi(\lambda z + d\mathbb{Z}) = (\lambda z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}), \\
\lambda\psi(z + d\mathbb{Z}) &= \lambda((z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z})) = (\lambda z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}).
\end{aligned}$$

(ii) $\psi(z_1 + z_2) = \psi(z_1) + \psi(z_2)$.

$$\begin{aligned}
\psi((z_1 + z_2) + d\mathbb{Z}) &= (z_1 + z_2 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}), \\
\psi(z_1 + d\mathbb{Z}) + \psi(z_2 + d\mathbb{Z}) &= (z_1 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) + (z_2 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\
&= (z_1 + z_2 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}).
\end{aligned}$$

$\square$

Proof of (4). For any $(x + m\mathbb{Z}) \otimes (y + n\mathbb{Z}) \in (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$,

$$\begin{aligned}\psi(\varphi((x + m\mathbb{Z}) \otimes (y + n\mathbb{Z}))) &= \psi(xy + d\mathbb{Z}) \\ &= (xy + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\ &= (x + m\mathbb{Z}) \otimes (y + n\mathbb{Z}).\end{aligned}$$

$\square$

Proof of (5). For any $z + d\mathbb{Z} \in \mathbb{Z}/d\mathbb{Z}$,

$$\begin{aligned}\varphi(\psi(z + d\mathbb{Z}) &= \varphi((z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z})) \\ &= z + d\mathbb{Z}.\end{aligned}$$

$\square$

Exercise 2.3. *Let A be a local ring, M and N finitely generated A-modules. Prove that if $M \otimes_A N = 0$, then $M = 0$ or $N = 0$. (Hint: Let $\mathfrak{m}$ be the maximal ideal, $k = A/\mathfrak{m}$ the residue field. Let $M_k = k \otimes_A M \cong M/\mathfrak{m}M$ by Exercise 2.2. By Nakayama's lemma, $M_k = 0 \Longrightarrow M = 0$. But $M \otimes_A N = 0 \Longrightarrow (M \otimes_A N)_k = 0 \Longrightarrow M_k \otimes_k N_k = 0 \Longrightarrow M_k = 0$ or $N_k = 0$ since M_k, N_k are vector spaces over a field.)*

Proof (Hint). Let $\mathfrak{m}$ be the maximal ideal, $k = A/\mathfrak{m}$ the residue field. Let $M_k = k \otimes_A M$.

(1) *(Base extension) Show that $(M \otimes_A N)_k = M_k \otimes_k N_k$.* In fact, by Proposition 2.14

$$\begin{aligned}(M \otimes_A N)_k &= k \otimes_A (M \otimes_A N) \\ &= (k \otimes_A M) \otimes_A N \\ &= M_k \otimes_A N \\ &= (M_k \otimes_k k) \otimes_A N \\ &= M_k \otimes_k (k \otimes_A N) \\ &= M_k \otimes_k N_k.\end{aligned}$$

(2)

$$\begin{aligned}M \otimes_A N = 0 &\Longrightarrow (M \otimes_A N)_k = 0 \\ &\Longrightarrow M_k \otimes_k N_k = 0 && ((1)) \\ &\Longrightarrow M_k = 0 \text{ or } N_k = 0 && (M_k, N_k\text{: vector spaces}) \\ &\Longrightarrow M/\mathfrak{m}M = 0 \text{ or } M/\mathfrak{m}M = 0 && (\text{Exercise 2.2}) \\ &\Longrightarrow M = 0 \text{ or } N = 0. && (\text{Nakayama's lemma})\end{aligned}$$

$\square$