Chapter 11: The Lebesuge Theory

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Exercise 11.1. *If $f \geq 0$ and $\int_E f d\mu = 0$, prove that $f(x) = 0$ almost everywhere on E. (Hint: Let E_n be the subset of E on which $f(x) > \frac{1}{n}$. Write $A = \bigcup E_n$. Then $\mu(A) = 0$ if and only if $\mu(E_n) = 0$ for every n.)*

Proof (Hint).

(1) Define $A = \{x \in E : f(x) > 0\}$. So $f(x) = 0$ almost everywhere on E if and only if $\mu(A) = 0$.

(2) Define
$$E_n = \left\{x \in E : f(x) > \frac{1}{n}\right\}$$
for $n = 1, 2, 3, \ldots$. Note that $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$ and
$$A = \bigcup_{n=1}^{\infty} E_n.$$
Since μ is a measure,
$$\lim_{n \to \infty} \mu(E_n) = \mu(A)$$
(Theorem 11.3).

(3) (Reductio ad absurdum) If $\mu(A) > 0$, there is an integer N such that $\mu(E_n) \geq \frac{\mu(A)}{2}$ whenever $n \geq N$ (by (2)). In particular, take $n = N$ to get
$$\begin{aligned}
\int_E f d\mu &\geq \int_{E_N} f d\mu && (\mu \text{ is a measure and } E_N \subseteq E) \\
&\geq \frac{1}{N} \cdot \mu(E_N) && (\text{Remarks 11.23(b)}) \\
&\geq \frac{1}{N} \cdot \frac{\mu(A)}{2} \\
&> 0,
\end{aligned}$$
contrary to the assumption that $\int_E f d\mu = 0$.

$\square$

Note. Compare to Exercise 6.2.