

Notes on the book:  
*Apostol, Modular Functions and  
Dirichlet Series in Number Theory,  
2nd edition*

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## Chapter 1: Elliptic functions

### Exercise 1.1.

Given two pairs of complex numbers  $(\omega_1, \omega_2)$  and  $(\omega'_1, \omega'_2)$  with nonreal ratios  $\omega_1/\omega_2$  and  $\omega'_1/\omega'_2$ . Prove that they generate the same set of periods if, and only if, there is a  $2 \times 2$  matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  with integer entries and determinant  $\pm 1$  such that

$$\begin{pmatrix} \omega'_2 \\ \omega'_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix}.$$

*Proof.*

- (1) ( $\implies$ ) Suppose  $(\omega_1, \omega_2)$  and  $(\omega'_1, \omega'_2)$  generate the same set of periods.

In particular, there is a  $2 \times 2$  matrix  $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{M}_{2 \times 2}(\mathbb{Z})$  (resp.

$A' := \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathbf{M}_{2 \times 2}(\mathbb{Z})$ ) such that

$$\begin{pmatrix} \omega'_2 \\ \omega'_1 \end{pmatrix} = A \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix}, \quad \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} = A' \begin{pmatrix} \omega'_2 \\ \omega'_1 \end{pmatrix}.$$

Hence it suffices to show  $\det(A) = \pm 1$ .

- (2) Note that

$$\begin{pmatrix} \omega'_2 \\ \omega'_1 \end{pmatrix} = AA' \begin{pmatrix} \omega'_2 \\ \omega'_1 \end{pmatrix}.$$

Hence

$$AA' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Take the determinant on the both sides to get

$$\det(A) \det(A') = 1.$$

Since  $\det(\mathbf{M}_{2 \times 2}(\mathbb{Z})) \subseteq \mathbb{Z}$ ,  $\det(A) = \pm 1$ .

- (3) ( $\impliedby$ )  $\Omega(\omega'_1, \omega'_2) \subseteq \Omega(\omega_1, \omega_2)$  is obvious. Note that

$$\begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} = \underbrace{\frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}_{\in \mathbf{M}_{2 \times 2}(\mathbb{Z})} \begin{pmatrix} \omega'_2 \\ \omega'_1 \end{pmatrix}.$$

Thus  $\Omega(\omega_1, \omega_2) \subseteq \Omega(\omega'_1, \omega'_2)$ . Therefore  $\Omega(\omega_1, \omega_2) = \Omega(\omega'_1, \omega'_2)$ .

□

**Supplement 1.1.1.**

(Exercise I.1.1 in the textbook: *Jürgen Neukirch, Algebraic Number Theory*.)  
 $\alpha \in \mathbb{Z}[i]$  is a unit if and only if  $N(\alpha) = 1$ .

*Proof.*

- (1) ( $\implies$ ) Since  $\alpha$  is a unit, there is  $\beta \in \mathbb{Z}[i]$  such that  $\alpha\beta = 1$ . So  $N(\alpha\beta) = N(1)$ , or  $N(\alpha)N(\beta) = 1$ . Since the image of  $N$  is nonnegative integers,  $N(\alpha) = 1$ .
- (2) ( $\impliedby$ )  $N(\alpha) = \alpha\bar{\alpha}$ , or  $1 = \alpha\bar{\alpha}$  since  $N(\alpha) = 1$ . That is,  $\bar{\alpha} \in \mathbb{Z}[i]$  is the inverse of  $\alpha \in \mathbb{Z}[i]$ . (Or we solve the equation  $N(\alpha) = a^2 + b^2 = 1$ , and show that all four solutions ( $\pm 1$  and  $\pm i$ ) are units.)
- (3) Conclusion: a unit  $\alpha = a + bi$  of  $\mathbb{Z}[i]$  is satisfying the equation  $N(\alpha) = a^2 + b^2 = 1$  by (1)(2). That is, the only unit of  $\mathbb{Z}[i]$  are  $\pm 1$  and  $\pm i$ .

□

**Exercise 1.2.**

Let  $S(0)$  denote the sum of the zeros of an elliptic function  $f$  in a period parallelogram, and let  $S(\infty)$  denote the sum of the poles in the same parallelogram. Prove that  $S(0) - S(\infty)$  is a period of  $f$ . (Hint: Integrate  $z \frac{f'(z)}{f(z)}$ .)

*Proof.*

- (1) Similar to Theorem 1.8, The integral

$$\frac{1}{2\pi i} \int_C z \frac{f'(z)}{f(z)}$$

taken around the boundary  $C$  of a cell (no zeros or poles on its boundary) counts the difference between the sum of the zeros and the sum of the poles inside the cell, that is,

$$S(0) - S(\infty) = \frac{1}{2\pi i} \int_C z \frac{f'(z)}{f(z)}.$$

(The proof is similar to the proof of the argument principle.)

- (2) Let  $C_1$  be the path from 0 to  $\omega_1$ ,  $C_2$  be the path from  $\omega_1$  to  $\omega_1 + \omega_2$ ,  $C_3$

be the path from  $\omega_1 + \omega_2$  to  $\omega_2$ , and  $C_4$  be the path from  $\omega_2$  to 0. Hence

$$\begin{aligned}
& \frac{1}{2\pi i} \int_{C_1} z \frac{f'(z)}{f(z)} + \frac{1}{2\pi i} \int_{C_3} z \frac{f'(z)}{f(z)} \\
&= \frac{1}{2\pi i} \int_{C_1} z \frac{f'(z)}{f(z)} + \frac{1}{2\pi i} \int_{-C_1} (z + \omega_2) \frac{f'(z + \omega_2)}{f(z + \omega_2)} \\
&= \frac{1}{2\pi i} \int_{C_1} z \frac{f'(z)}{f(z)} - \frac{1}{2\pi i} \int_{C_1} (z + \omega_2) \frac{f'(z)}{f(z)} \\
&= -\omega_2 \frac{1}{2\pi i} \int_{C_1} \frac{f'(z)}{f(z)}
\end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{2\pi i} \int_{C_2} z \frac{f'(z)}{f(z)} + \frac{1}{2\pi i} \int_{C_4} z \frac{f'(z)}{f(z)} \\
&= \frac{1}{2\pi i} \int_{-C_4} (z + \omega_1) \frac{f'(z + \omega_1)}{f(z + \omega_1)} + \frac{1}{2\pi i} \int_{C_4} z \frac{f'(z)}{f(z)} \\
&= -\frac{1}{2\pi i} \int_{C_4} (z + \omega_1) \frac{f'(z)}{f(z)} + \frac{1}{2\pi i} \int_{C_4} z \frac{f'(z)}{f(z)} \\
&= -\omega_1 \frac{1}{2\pi i} \int_{C_4} \frac{f'(z)}{f(z)}
\end{aligned}$$

Therefore

$$S(0) - S(\infty) = -\omega_1 \frac{1}{2\pi i} \int_{C_4} \frac{f'(z)}{f(z)} - \omega_2 \frac{1}{2\pi i} \int_{C_1} \frac{f'(z)}{f(z)}.$$

So it suffices to show that  $\frac{1}{2\pi i} \int_{C_1} \frac{f'(z)}{f(z)} \in \mathbb{Z}$ . (Other cases are similar.)

(3) By choosing one branch of log, we have

$$\begin{aligned}
\frac{1}{2\pi i} \int_{C_1} \frac{f'(z)}{f(z)} &= \frac{1}{2\pi i} \log \frac{f(\omega_1)}{f(0)} \\
&= \frac{1}{2\pi i} \log(1) & (f(\omega_1) = f(0)) \\
&= \frac{1}{2\pi i} (2\pi i m) \text{ for some } m \in \mathbb{Z} \\
&= m \in \mathbb{Z}.
\end{aligned}$$

□

### Exercise 1.5.

*Prove that every elliptic function  $f$  can be expressed in the form*

$$f(z) = R_1[\wp(z)] + \wp'(z)R_2[\wp(z)]$$

where  $R_1$  and  $R_2$  are rational functions and  $\wp$  has the same set of periods as  $f$ .

*Proof.*

$$\begin{aligned} f(z) &= \underbrace{\frac{f(z) + f(-z)}{2}}_{\text{even}} + \wp'(z) \underbrace{\frac{f(z) - f(-z)}{2\wp'(z)}}_{\text{even}} \\ &= R_1[\wp(z)] + \wp'(z)R_2[\wp(z)] \text{ for some rational functions } R_1, R_2 \end{aligned}$$

(by Exercise 1.4).  $\square$

### Exercise 1.7.

The discriminant of the polynomial  $f(x) = 4(x - x_1)(x - x_2)(x - x_3)$  is the product  $16\{(x_2 - x_1)(x_3 - x_2)(x_3 - x_1)\}^2$ . Prove that the discriminant of  $f(x) = 4x^3 - ax - b$  is  $a^3 - 27b^2$ .

*Proof.*

(1) Since

$$f'(x) = 4(x - x_2)(x - x_3) + 4(x - x_1)(x - x_3) + 4(x - x_1)(x - x_2),$$

we have

$$\begin{aligned} f'(x_1) &= 4(x_1 - x_2)(x_1 - x_3), \\ f'(x_2) &= 4(x_2 - x_1)(x_2 - x_3), \\ f'(x_3) &= 4(x_3 - x_1)(x_3 - x_2). \end{aligned}$$

Hence

$$f'(x_1)f'(x_2)f'(x_3) = -4\text{disc}(f)$$

where  $\text{disc}(f)$  be the discriminant of  $f(x)$ .

(2) As  $f(x) = 4x^3 - ax - b$ , we have  $f'(x) = 12x^2 - a$ . So

$$f'(x_1)f'(x_2)f'(x_3) = (12x_1^2 - a)(12x_2^2 - a)(12x_3^2 - a).$$

Note that

$$\begin{aligned} x_1x_2x_3 &= \frac{b}{4}, \\ x_1x_2 + x_2x_3 + x_3x_1 &= -\frac{a}{4}, \\ x_1 + x_2 + x_3 &= 0, \end{aligned}$$

we have

$$\begin{aligned}
x_1^2 x_2^2 x_3^2 &= \frac{b^2}{4^2}, \\
x_1^2 x_2^2 + x_2^2 x_3^2 + x_3^2 x_1^2 &= (x_1 x_2 + x_2 x_3 + x_3 x_1)^2 - 2x_1 x_2 x_3 (x_1 + x_2 + x_3) \\
&= \frac{a^2}{4^2}, \\
x_1^2 + x_2^2 + x_3^2 &= (x_1 + x_2 + x_3)^2 - 2(x_1 x_2 + x_2 x_3 + x_3 x_1) \\
&= \frac{a}{2}.
\end{aligned}$$

(3) Hence

$$\begin{aligned}
f'(x_1)f'(x_2)f'(x_3) &= (12x_1^2 - a)(12x_2^2 - a)(12x_3^2 - a) \\
&= 12^3(x_1^2 x_2^2 x_3^2) - 12^2 a(x_1^2 x_2^2 + x_2^2 x_3^2 + x_3^2 x_1^2) \\
&\quad + 12a^2(x_1^2 + x_2^2 + x_3^2) - a^3 \\
&= 12^3 \cdot \frac{b^2}{4^2} - 12^2 a \cdot \frac{a^2}{4^2} + 12a^2 \cdot \frac{a}{2} - a^3 \\
&= -4(a^3 - 27b^2).
\end{aligned}$$

Therefore

$$\text{disc}(4x^3 - ax - b) = a^3 - 27b^2.$$

□

### Exercise 1.11.

If  $k \geq 2$  and  $\tau \in H$  prove that the Eisenstein series

$$G_{2k}(\tau) = \sum_{(m,n) \neq (0,0)} (m + n\tau)^{-2k}$$

has the Fourier expansion

$$G_{2k}(\tau) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) e^{2\pi i n \tau}.$$

*Proof.*

(1) Let  $q = e^{2\pi i \tau}$ . Similar to Lemma 1.3 on page 19, we have

$$(2k-1)! \sum_{m=-\infty}^{+\infty} \frac{1}{(\tau + m)^{2k}} = (2\pi i)^{2k} \sum_{r=1}^{\infty} r^{2k-1} q^r.$$

(2) Similar to Theorem 1.18, we have

$$\begin{aligned}
G_{2k}(\tau) &= \sum_{(m,n) \neq (0,0)} (m+n\tau)^{-2k} \\
&= \sum_{\substack{m=-\infty \\ m \neq 0 (n=0)}}^{+\infty} m^{-2k} + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{+\infty} ((m+n\tau)^{-2k} + (m-n\tau)^{-2k}) \\
&= 2\zeta(2k) + 2 \sum_{n=1}^{\infty} \sum_{m=-\infty}^{+\infty} (m+n\tau)^{-2k} \\
&= 2\zeta(2k) + 2 \sum_{n=1}^{\infty} \frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{r=1}^{\infty} r^{2k-1} q^{nr} \\
&= 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \underbrace{\sum_{d|n} d^{2k-1}}_{=\sigma_{2k-1}(n)} q^n.
\end{aligned}$$

In the last double sum we collect together those terms for which  $nr$  is constant.

□

### Exercise 1.12.

Refer to Exercise 1.11. If  $\tau \in H$  prove that

$$G_{2k}\left(-\frac{1}{\tau}\right) = \tau^{2k} G_{2k}(\tau)$$

and deduce that

$$\begin{aligned}
G_{2k}\left(\frac{i}{2}\right) &= (-4)^k G_{2k}(2i) && \text{for all } k \geq 2, \\
G_{2k}(i) &= 0 && \text{if } k \text{ is odd,} \\
G_{2k}(e^{\frac{2\pi i}{3}}) &= 0 && \text{if } k \not\equiv 0 \pmod{3}.
\end{aligned}$$

*Proof.*

(1)

$$\begin{aligned}
G_{2k}\left(-\frac{1}{\tau}\right) &= \sum_{(m,n) \neq (0,0)} \left(m - \frac{n}{\tau}\right)^{-2k} \\
&= \tau^{2k} \sum_{(m,n) \neq (0,0)} (\tau m - n)^{-2k} \\
&= \tau^{2k} G_{2k}(\tau).
\end{aligned}$$

(2) Let  $\tau = 2i$ . We have  $G_{2k}\left(\frac{i}{2}\right) = (-4)^k G_{2k}(2i)$ .

(3) Let  $\tau = i$ . We have  $G_{2k}(i) = (-1)^k G_{2k}(i)$ . Hence  $G_{2k}(i) = 0$  if  $k$  is odd.

(4) Let  $\tau = e^{\frac{\pi i}{3}}$ . We have  $G_{2k}(e^{\frac{2\pi i}{3}}) = e^{\frac{2k\pi i}{3}} G_{2k}(e^{\frac{\pi i}{3}})$ . Since

$$e^{\frac{2\pi i}{3}} = -1 + e^{\frac{\pi i}{3}}$$

and each Eisenstein series is a periodic function of  $\tau$  of period 1, we have  $G_{2k}(e^{\frac{2\pi i}{3}}) = G_{2k}(e^{\frac{\pi i}{3}})$ . So  $G_{2k}(e^{\frac{2\pi i}{3}}) = e^{\frac{2k\pi i}{3}} G_{2k}(e^{\frac{2\pi i}{3}})$ . Therefore  $G_{2k}(e^{\frac{2\pi i}{3}}) = 0$  if  $k \not\equiv 0 \pmod{3}$ .

□

### Exercise 1.13.

Ramanujan's tau function  $\tau(n)$  is defined by the Fourier expansion

$$\Delta(\tau) = (2\pi)^{12} \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n \tau},$$

derived in Theorem 1.19. Prove that

$$\tau(n) = 8000\{(\sigma_3 \circ \sigma_3) \circ \sigma_3\}(n) - 147(\sigma_5 \circ \sigma_5)(n),$$

where  $f \circ g$  denotes the Cauchy product of two sequences,

$$(f \circ g)(n) = \sum_{k=0}^n f(k)g(n-k),$$

and  $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$  for  $n \geq 1$ , with  $\sigma_3(0) = \frac{1}{240}$ ,  $\sigma_5(0) = -\frac{1}{504}$ . (Hint: Theorem 1.18.)

*Proof.*



(1) Let  $q = e^{2\pi i\tau}$ . Write

$$g_2(\tau) = \frac{4\pi^4}{3} \left\{ 1 + 240 \sum_{k=1}^{\infty} \sigma_3(k) q^k \right\} = \frac{4\pi^4}{3} \left\{ 240 \sum_{k=0}^{\infty} \sigma_3(k) q^k \right\},$$

$$g_3(\tau) = \frac{8\pi^6}{27} \left\{ 1 - 504 \sum_{k=1}^{\infty} \sigma_5(k) q^k \right\} = \frac{8\pi^6}{27} \left\{ -504 \sum_{k=0}^{\infty} \sigma_5(k) q^k \right\}$$

(Theorem 1.18).

(2) Similar to the proof of Theorem 1.19,

$$\begin{aligned} \Delta(\tau) &= g_2(\tau)^3 - 27g_3(\tau)^2 \\ &= \frac{64\pi^{12}}{27} \left\{ \left( 240 \sum_{k=0}^{\infty} \sigma_3(k) q^k \right)^3 - \left( -504 \sum_{k=0}^{\infty} \sigma_5(k) q^k \right)^2 \right\} \\ &= (2\pi)^{12} \left\{ 8000 \left( \sum_{k=0}^{\infty} \sigma_3(k) q^k \right)^3 - 147 \left( \sum_{k=0}^{\infty} \sigma_5(k) q^k \right)^2 \right\} \\ &= (2\pi)^{12} \sum_{n=0}^{\infty} \{ 8000 \{ (\sigma_3 \circ \sigma_3) \circ \sigma_3 \}(n) - 147 (\sigma_5 \circ \sigma_5)(n) \} q^n \\ &= (2\pi)^{12} \sum_{n=1}^{\infty} \{ 8000 \{ (\sigma_3 \circ \sigma_3) \circ \sigma_3 \}(n) - 147 (\sigma_5 \circ \sigma_5)(n) \} q^n. \end{aligned}$$

(Here  $8000 \{ (\sigma_3 \circ \sigma_3) \circ \sigma_3 \}(0) - 147 (\sigma_5 \circ \sigma_5)(0) = 0$ .)

(3) Therefore

$$\tau(n) = 8000 \{ (\sigma_3 \circ \sigma_3) \circ \sigma_3 \}(n) - 147 (\sigma_5 \circ \sigma_5)(n)$$

for  $n \geq 1$ .

□

#### Exercise 1.14. (Lambert series)

A series of the form  $\sum_{n=1}^{\infty} f(n) \frac{x^n}{1-x^n}$  is called a **Lambert series**. Assuming absolute convergence, prove that

$$\sum_{n=1}^{\infty} f(n) \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} F(n) x^n,$$

where

$$F(n) = \sum_{d|n} f(d).$$

Apply this result to obtain the following formulas, valid for  $|x| < 1$ .

(a)

$$\sum_{n=1}^{\infty} \frac{\mu(n)x^n}{1-x^n} = x.$$

(b)

$$\sum_{n=1}^{\infty} \frac{\varphi(n)x^n}{1-x^n} = \frac{x}{(1-x)^2}.$$

(c)

$$\sum_{n=1}^{\infty} \frac{n^{\alpha}x^n}{1-x^n} = \sum_{n=1}^{\infty} \sigma_{\alpha}(n)x^n.$$

(d)

$$\sum_{n=1}^{\infty} \frac{\lambda(n)x^n}{1-x^n} = \sum_{n=1}^{\infty} x^{n^2}.$$

(e) Use the result in (c) to express  $g_2(\tau)$  and  $g_3(\tau)$  in terms of Lambert series in  $x = e^{2\pi i\tau}$ .

*Note.* In (a),  $\mu(n)$  is the Möbius function; In (b),  $\varphi(n)$  is Euler's totient; and in (d),  $\lambda(n)$  is Liouville's function.

*Proof.* Similar to the proof of Exercise 1.11.

$$\begin{aligned} \sum_{n=1}^{\infty} f(n) \frac{x^n}{1-x^n} &= \sum_{n=1}^{\infty} f(n) \sum_{r=1}^{\infty} x^{rn} \\ &= \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} f(n) x^{rn} \\ &= \sum_{n=1}^{\infty} \underbrace{\left( \sum_{d|n} f(d) \right)}_{=F(n)} x^n. \end{aligned}$$

□

*Proof of (a).* Theorem 2.1 in the textbook: *T. M. Apostol, Introduction to Analytic Number Theory* shows that

$$F(n) := \sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

Hence

$$\sum_{n=1}^{\infty} \mu(n) \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} F(n) x^n = x.$$

□

*Proof of (b).* Theorem 2.2 in the textbook: *T. M. Apostol, Introduction to Analytic Number Theory* shows that  $F(n) := \sum_{d|n} \varphi(d) = n$ . Hence

$$\sum_{n=1}^{\infty} \varphi(n) \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} F(n) x^n = \sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}.$$

□

*Proof of (c).* Since

$$F(n) := \sum_{d|n} d^\alpha = \sigma_\alpha(n),$$

we have

$$\sum_{n=1}^{\infty} n^\alpha \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} F(n) x^n = \sum_{n=1}^{\infty} \sigma_\alpha(n) x^n.$$

□

*Proof of (d).* Theorem 2.19 in the textbook: *T. M. Apostol, Introduction to Analytic Number Theory* shows that

$$F(n) := \sum_{d|n} \lambda(d) = \begin{cases} 1 & \text{if } n \text{ is a square,} \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\sum_{n=1}^{\infty} \lambda(n) \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} F(n) x^n = \sum_{n=1}^{\infty} x^{n^2}.$$

□

*Proof of (e).*

(1) Let  $q = x = e^{2\pi i \tau}$ .

$$\begin{aligned} g_2(\tau) &= \frac{4\pi^4}{3} \left\{ 1 + 240 \sum_{k=1}^{\infty} \sigma_3(k) q^k \right\} && \text{(Theorem 1.18)} \\ &= \frac{4\pi^4}{3} \left\{ 1 + 240 \sum_{k=1}^{\infty} \frac{k^3 q^k}{1-q^k} \right\} && ((c)). \end{aligned}$$

(2) Similarly,

$$\begin{aligned} g_3(\tau) &= \frac{8\pi^6}{27} \left\{ 1 - 504 \sum_{k=1}^{\infty} \sigma_5(k) q^k \right\} && \text{(Theorem 1.18)} \\ &= \frac{8\pi^6}{27} \left\{ 1 - 504 \sum_{k=1}^{\infty} \frac{k^5 q^k}{1-q^k} \right\} && ((c)). \end{aligned}$$

□

**Exercise 1.15.**

Let

$$G(x) = \sum_{n=1}^{\infty} \frac{n^5 x^n}{1 - x^n},$$

and let

$$F(x) = \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^5 x^n}{1 + x^n}.$$

(a) Prove that  $F(x) = G(x) - 34G(x^2) + 64(x^4)$ .

(b) Prove that

$$\sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^5}{1 + e^{n\pi}} = \frac{31}{504}.$$

(c) Use Theorem 12.17 in the textbook: T. M. Apostol, *Introduction to Analytic Number Theory* to prove the more general result

$$\sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^{4k+1}}{1 + e^{n\pi}} = \frac{2^{4k+1} - 1}{8k + 4} B_{4k+2}.$$

*Proof of (a).*

(1) Consider the general case. Let

$$G(x) = \sum_{n=1}^{\infty} \frac{n^{4k+1} x^n}{1 - x^n},$$

and let

$$F(x) = \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^{4k+1} x^n}{1 + x^n}.$$

Show that  $F(x) = G(x) - (2^{4k+1} + 2)G(x^2) + 2^{4k+2}G(x^4)$ .

(2) The identity

$$\sum_{n=1}^{\infty} \frac{x^n}{1 + x^n} = \sum_{n=1}^{\infty} \frac{x^n}{1 - x^n} - 2 \sum_{n=1}^{\infty} \frac{x^{2n}}{1 - x^{2n}}$$

is always true. Hence  $H(x) := \sum_{n=1}^{\infty} \frac{n^{4k+1} x^n}{1 + x^n} = G(x) - 2G(x^2)$ .

(3) Note that

$$\begin{aligned}
H(x) &= \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^{4k+1}x^n}{1+x^n} + \sum_{\substack{n=1 \\ (n \text{ even})}}^{\infty} \frac{n^{4k+1}x^n}{1+x^n} \\
&= F(x) + \sum_{n=1}^{\infty} \frac{(2n)^{4k+1}x^{2n}}{1+x^{2n}} \\
&= F(x) + 2^{4k+1} \sum_{n=1}^{\infty} \frac{n^{4k+1}x^{2n}}{1+x^{2n}} \\
&= F(x) + 2^{4k+1}H(x^2).
\end{aligned}$$

Hence

$$\begin{aligned}
F(x) &= H(x) - 2^{4k+1}H(x^2) \\
&= [G(x) - 2G(x^2)] - 2^{4k+1}[G(x^2) - 2G(x^4)] \\
&= G(x) - (2^{4k+1} + 2)G(x^2) + 2^{4k+2}G(x^4).
\end{aligned}$$

□

*Proof of (b).* Take  $k = 1$  in part (c), we have

$$\sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^5}{1+e^{n\pi}} = \frac{31}{12} \cdot \frac{1}{42} = \frac{31}{504}.$$

□

*Proof of (c).*

(1) Let  $q = e^{2\pi i\tau}$ . So

$$\begin{aligned}
G_{4k+2}(\tau) &= 2\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!} \sum_{n=1}^{\infty} \sigma_{4k+1}(n)q^n && \text{(Exercise 1.11)} \\
&= 2\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!} G(q) && \text{(Exercise 1.14(c))}
\end{aligned}$$

Hence

$$\begin{aligned}
& G_{4k+2}(\tau) - (2^{4k+1} + 2)G_{4k+2}(2\tau) + 2^{4k+2}G_{4k+2}(4\tau) \\
&= \left[ 2\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!} G(q) \right] \\
&\quad - (2^{4k+1} + 2) \left[ 2\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!} G(q^2) \right] \\
&\quad + 2^{4k+2} \left[ 2\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!} G(q^4) \right] \\
&= (1 - (2^{4k+1} + 2) + 2^{4k+2}) \cdot 2\zeta(4k+2) \\
&\quad + \frac{2(2\pi i)^{4k+2}}{(4k+1)!} [G(q) - (2^{4k+1} + 2)G(q^2) + 2^{4k+2}G(q^4)] \\
&= (2^{4k+2} - 2)\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!} F(q).
\end{aligned}$$

(2) By taking  $\tau = \frac{i}{2}$ , we have

$$F(q) = F(e^{-\pi}) = \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^{4k+1}}{1 + e^{n\pi}}$$

and

$$\begin{aligned}
& G_{4k+2}(\tau) - (2^{4k+1} + 2)G_{4k+2}(2\tau) + 2^{4k+2}G_{4k+2}(4\tau) \\
&= G_{4k+2}\left(\frac{i}{2}\right) - (2^{4k+1} + 2)G_{4k+2}(i) + 2^{4k+2}G_{4k+2}(2i) \\
&= (-4)^{2k+1}G_{4k+2}(2i) - (2^{4k+1} + 2) \cdot 0 + 2^{4k+2}G_{4k+2}(2i) \\
&= 0.
\end{aligned}$$

(Exercise 1.12). Hence

$$0 = (2^{4k+2} - 2)\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!} \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^{4k+1}}{1 + e^{n\pi}}.$$

(3) Theorem 12.17 in the textbook: *T. M. Apostol, Introduction to Analytic Number Theory* shows that

$$\zeta(4k+2) = (-1)^{2k+1+1} \frac{(2\pi)^{4k+2} B_{4k+2}}{2(4k+2)!} = \frac{(2\pi)^{4k+2} B_{4k+2}}{2(4k+2)!}.$$

Hence

$$\sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^{4k+1}}{1 + e^{n\pi}} = \frac{2^{4k+1} - 1}{8k+4} B_{4k+2}.$$

□