

## Chapter 2: Basic Topology

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**Exercise 2.1.** *Prove that the empty set is a subset of every set.*

*Proof.* By Definitions 1.3,

- (1) The set which contains no element will be called the **empty set**,
- (2) If  $A$  and  $B$  are sets, and if every element of  $A$  is an element of  $B$ , we say that  $A$  is a **subset** of  $B$ ,

every element of the empty set (there are none) belongs to every set. That is, the empty set is a subset of every set.  $\square$

**Exercise 2.2.** *A complex number  $z$  is said to be algebraic if there are integers  $a_0, \dots, a_n$ , not all zero, such that*

$$a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n = 0.$$

*Prove that the set of all algebraic numbers is countable. (Hint: For every positive integer  $N$  there are only finitely many equations with*

$$n + |a_0| + |a_1| + \dots + |a_n| = N.$$

Might assume  $a_0 \neq 0$ .

For example, all rational numbers are algebraic since  $p = \frac{\alpha}{\beta}$  (where  $\alpha, \beta \in \mathbb{Z}$ ) is a root of  $\beta z - \alpha = 0$ .

Besides,  $z = \sqrt{2} + \sqrt{3}$  is algebraic since  $z^4 - 10z^2 + 1 = 0$ . In fact,  $z = \pm\sqrt{2} \pm \sqrt{3}$  are also algebraic since  $z^4 - 10z^2 + 1 = (z - \sqrt{2} - \sqrt{3})(z + \sqrt{2} - \sqrt{3})(z - \sqrt{2} + \sqrt{3})(z + \sqrt{2} + \sqrt{3})$ .

**Lemma.** *The set of all polynomials over  $\mathbb{Z}$  is countable implies that the set of algebraic numbers is countable.*

*Proof of Lemma.* By definition, we write the set of algebraic numbers as

$$S = \bigcup_{f(x) \in \mathbb{Z}[x]} \{z \in \mathbb{C} : f(z) = 0\}.$$

Since each polynomial of degree  $n$  has at most  $n$  roots,  $\{z \in \mathbb{C} : f(z) = 0\}$  is finite for each given  $f(x) \in \mathbb{Z}[x]$ . So  $S$  is a countable union (by assumption) of finite sets, and hence at most countable.  $S$  is infinite since every integer  $\alpha$  is a root of  $f(z) = z - \alpha$ . So  $S$  is countable.  $\square$

Thus, it suffices to show that *the set of all polynomials over  $\mathbb{Z}$  is countable*.

*Proof (Hint).* For every positive integer  $N$  there are only finitely many equations with  $n + |a_0| + |a_1| + \cdots + |a_n| = N$ . Write

$$P_N = \{f(x) \in \mathbb{Z}[x] : n + |a_0| + |a_1| + \cdots + |a_n| = N\}$$

where  $f(x) = a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n$  with  $a_0 \neq 0$ , and

$$P = \bigcup_{N=1}^{\infty} P_N.$$

$P$  is the set of all polynomials over  $\mathbb{Z}$ .

Each  $P_N$  is finite for given  $N$  (since the equation  $n + |a_0| + |a_1| + \cdots + |a_n| = N$  has finitely many solutions  $(n, a_0, a_1, \dots, a_n) \in \mathbb{Z}^{n+2}$ ). So  $P$  is a countable union of finite sets, and hence at most countable.  $P$  is infinite since  $\mathbb{Z}$  is a subring of  $\mathbb{Z}[x]$ . So  $P$  is countable.  $\square$

*Proof (Theorem 2.13).*

- (1)  $\mathbb{Z}^N$  is countable for any integer  $N > 0$ . Theorem 2.13.
- (2) The set of all polynomials over  $\mathbb{Z}$  is countable. Let

$$P_n = \{f \in \mathbb{Z}[x] : \deg f = n\},$$

and

$$P = \bigcup_{n=1}^{\infty} P_n = \mathbb{Z}[x].$$

*Claim:*  $P_n$  is countable. Define a 1-1 map  $\varphi_n : P_n \rightarrow \mathbb{Z}^{n+1}$  by

$$\varphi_n(a_0z^n + a_1z^{n-1} + \cdots + a_n) = (a_0, a_1, \dots, a_{n-1}, a_n).$$

By (1) and Theorem 2.8,  $P_n$  is countable. ( $P_n$  is infinite since  $a_n \in \mathbb{Z}$ .) Now  $P$  is a countable union of countable sets, and hence countable by Theorem 2.12.

$\square$

*Proof (Unique factorization theorem).*

- (1) *The set of prime numbers is countable.* Write all primes in the ascending order as  $p_1, p_2, \dots, p_n, \dots$  where  $p_1 = 2, p_2 = 3, \dots, p_{10001} = 104743, \dots$  (See ProjectEuler 7: 10001st prime. Use sieve of Eratosthenes to get  $p_{10001}$ .)
- (2) *The set of all polynomials over  $\mathbb{Z}$  is countable.* Let

$$P_n = \{f \in \mathbb{Z}[x] : \deg f = n\},$$

and

$$P = \bigcup_{n=1}^{\infty} P_n = \mathbb{Z}[x].$$

*Claim:*  $P_n$  is countable. Define a map  $\varphi_n : P_n \rightarrow \mathbb{Z}^+$  by

$$\varphi_n(a_0 z^n + a_1 z^{n-1} + \dots + a_n) = p_1^{\psi(a_0)} p_2^{\psi(a_1)} \dots p_{n+1}^{\psi(a_n)},$$

where  $\psi$  is a 1-1 correspondence from  $\mathbb{Z}$  to  $\mathbb{Z}^+$  (Example 2.5). By the unique factorization theorem,  $\varphi_n$  is 1-1. So  $P_n$  is countable by Theorem 2.8. ( $P_n$  is infinite since  $a_n \in \mathbb{Z}$ .) Now  $P$  is a countable union of countable sets, and hence countable by Theorem 2.12.

□

**Exercise 2.3.** *Prove that there exist real numbers which are not algebraic.*

*Proof (Exercise 2.2).* If all real numbers were algebraic, then  $\mathbb{R}$  is countable by Exercise 2.2, contrary to the fact that  $\mathbb{R}$  is uncountable (Corollary to Theorem 2.43). □

*Proof (Liouville, 1844).*

- (1) **Lemma.** *If  $\xi$  is a real algebraic number of degree  $n > 1$ , then there is a constant  $A > 0$  (depending on  $\xi$ ) such that*

$$\left| \xi - \frac{h}{k} \right| \geq \frac{A}{k^n}$$

*for all rational numbers  $\frac{h}{k}$ .*

- (a) If  $\left| \xi - \frac{h}{k} \right| \geq 1$ , pick  $A = 1 > 0$ .
- (b) If  $\left| \xi - \frac{h}{k} \right| < 1$ , let  $f(x) = a_0 + a_1 x + \dots + a_n x^n$  be an irreducible polynomial of degree  $n > 1$  over  $\mathbb{Z}$  such that  $f(\xi) = 0$ . By the mean value theorem,

$$f(\xi) - f\left(\frac{h}{k}\right) = \left(\xi - \frac{h}{k}\right) f'(c)$$

for some  $c \in \left(\xi - \frac{h}{k}, \xi + \frac{h}{k}\right) \subseteq (\xi - 1, \xi + 1)$ . Notice that

- (i)  $f(\xi) = 0$  by definition.
- (ii)  $f\left(\frac{h}{k}\right) \neq 0$  since  $\frac{h}{k}$  cannot be a root of  $f(x)$ . Otherwise  $f$  is of degree 1, contrary to the assumption of  $f$ .
- (iii)  $\left|f\left(\frac{h}{k}\right)\right| \geq \frac{1}{k^n}$  since

$$\begin{aligned} f\left(\frac{h}{k}\right) &= a_0 + a_1\left(\frac{h}{k}\right) + \cdots + a_n\left(\frac{h}{k}\right)^n \neq 0, \\ k^n f\left(\frac{h}{k}\right) &= a_0 k^n + h k^{n-1} a_1 + \cdots + h^n a_n \neq 0, \\ k^n \left|f\left(\frac{h}{k}\right)\right| &\geq 1. \end{aligned}$$

- (iv)  $|f'(c)| \leq \sup_{x \in [\xi-1, \xi+1]} |f'(x)|$  since  $c \in [\xi-1, \xi+1]$  and  $f'(x)$  is continuous or bounded on a compact set  $[\xi-1, \xi+1]$ .

By (i)-(iv),

$$\begin{aligned} \left|f(\xi) - f\left(\frac{h}{k}\right)\right| &= \left|\left(\xi - \frac{h}{k}\right) f'(c)\right|, \\ \frac{1}{k^n} &\leq \left|f\left(\frac{h}{k}\right)\right| = \left|\xi - \frac{h}{k}\right| |f'(c)| \leq \left|\xi - \frac{h}{k}\right| \cdot \sup_{x \in [\xi-1, \xi+1]} |f'(x)|. \end{aligned}$$

Pick  $A = (1 + \sup_{x \in [\xi-1, \xi+1]} |f'(x)|)^{-1} > 0$ .

By (a)(b), we arrange  $A = \min(1, (1 + \sup_{x \in [\xi-1, \xi+1]} |f'(x)|)^{-1}) > 0$  to fit the inequality.

- (2)  $\xi = \sum_{n=0}^{\infty} 10^{-n!}$  is transcendental.

- (a) Let  $k_j = 10^{j!}$ ,  $h_j = 10^{j!} \sum_{n=0}^j 10^{-n!}$ . Then

$$\left|\xi - \frac{h_j}{k_j}\right| = \sum_{n=j+1}^{\infty} 10^{-n!} < \sum_{n=(j+1)!}^{\infty} 10^{-n} = \frac{A_j}{k_j^j}$$

where  $A_j = \frac{10}{9} \cdot 10^{-j!}$ .

- (b) If  $\xi$  were a real algebraic number of degree  $d > 1$ , then by Lemma and (a),

$$\frac{A}{k_j^d} < \left|\xi - \frac{h_j}{k_j}\right| < \frac{A_j}{k_j^j} < \frac{A_j}{k_j^d}$$

for some  $A > 0$  and  $j \geq d$ , or  $0 < A < A_j$ . Since  $j$  is arbitrary,  $A_j \rightarrow 0$  as  $j \rightarrow \infty$ , contrary to  $A > 0$ .

- (c) If  $\xi$  were a real algebraic number of degree  $d = 1$ ,  $\xi = \frac{h}{k}$  is a rational number. So

$$\left|\xi - \frac{h_j}{k_j}\right| = \left|\frac{h}{k} - \frac{h_j}{k_j}\right| = \left|\frac{hk_j - kh_j}{kk_j}\right| \geq \left|\frac{1}{kk_j}\right| = \frac{|k|^{-1}}{k_j}$$

for all  $j$ . (It is impossible that  $hk_j - kh_j = 0$  or  $\frac{h}{k} = \frac{h_j}{k_j}$  since  $|\frac{h}{k} - \frac{h_j}{k_j}| = \sum_{n=j+1}^{\infty} 10^{-n!} > 0$  for all  $j$ .) Again by (a),

$$\frac{|k|^{-1}}{k_j} \leq \left| \xi - \frac{h_j}{k_j} \right| < \frac{A_j}{k_j^j} < \frac{A_j}{k_j},$$

or  $0 < |k|^{-1} < A_j$ . (Similar to (b).) Since  $j$  is arbitrary,  $A_j \rightarrow 0$  as  $j \rightarrow \infty$ , contrary to  $|k|^{-1} > 0$ .

□

**Exercise 2.4.** *Is the set of all irrational real numbers countable?*

*Proof (Reductio ad absurdum).* If  $\mathbb{R} - \mathbb{Q}$  were countable, then  $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} - \mathbb{Q})$  is countable (Theorem 2.12), contrary to the fact that  $\mathbb{R}$  is uncountable (Corollary to Theorem 2.43). □

**Exercise 2.5.** *Construct a bounded set of real numbers with exactly three limit points.*

*Proof (Exercise 2.12).* Let

$$K_p = \{p\} \cup \left\{ p + \frac{1}{n} : n \in \mathbb{Z}^+ \right\} \subseteq \mathbb{R}^1$$

be a compact set of  $\mathbb{R}^1$  with exactly one limit point  $p \in \mathbb{R}^1$  (Exercise 2.12). Then

$$K_{1989} \cup K_6 \cup K_4$$

is a compact set of  $\mathbb{R}^1$  with exactly three limit points  $1989, 6, 4 \in \mathbb{R}^1$ . □

**Exercise 2.6.** *Let  $E'$  be the set of all limit points of a set  $E$ . Prove that  $E'$  is closed. Prove that  $E$  and  $\overline{E}$  have the same limit points. (Recall that  $\overline{E} = E \cup E'$ .) Do  $E$  and  $E'$  always have the same limit points?*

*Proof.*

(1) *Show that  $E'$  is closed.*

(a) *Use Definition 2.18 (d).*

(i) It suffices to show every limit point of  $E'$  is a limit point of  $E$ . Given a limit point  $p$  of  $E'$ , so that every open neighborhood  $U$  of  $p$  contains a point  $q_0 \neq p$  such that  $q_0 \in E'$ .

- (ii) Since  $q_0$  is a limit point of  $E$ , there is an open neighborhood  $V$  of  $q_0$  contains a point  $q \neq q_0$  such that  $q \in E$ , where

$$V = U \cap B\left(q_0; \frac{1}{2}d_E(p, q_0)\right) \subseteq U$$

( $B(x; r)$  is the open ball with center at  $x$  and radius  $r$ ).

- (iii) By the construction of  $V$ , for such open neighborhood  $U$  of  $p$ , there is  $q \neq p$  and  $q \in V \subseteq U$  and  $q \in E$ . That is,  $p$  is a limit point of  $E$ .

(b) Use Definition 2.18 (e).

- (i) To show  $E'$  is closed or  $X - E'$  is open, it suffices to show every point of  $X - E'$  is an interior point of  $X - E'$ .  
(ii) Given a point  $p \in X - E'$ , or  $p$  is not a limit point of  $E$ . There is an open neighborhood  $U$  of  $p$  contains no point  $q \neq p$  such that  $q \in E$ .  
(iii) To show  $U$  is an open neighborhood of  $p$  such that  $U \subseteq X - E'$ , it suffices to no point  $q \neq p$  such that  $q \in E'$ . If there were a limit point  $q$  of  $E$  such that  $q \neq p$  and  $q \in U$ , then

$$V = U \cap B\left(q; \frac{1}{2}d_E(p, q)\right) \subseteq U$$

is an open neighborhood of  $q$  contains no point of  $E$ , contrary to the assumption  $q \in E'$ . So  $U \subseteq X - E'$  is an open neighborhood of  $p \in X - E'$ .

- (2) Show that  $E' = \overline{E}'$ . It suffices to show  $E' \supseteq \overline{E}'$ . ( $E' \subseteq \overline{E}'$  holds trivially since  $E \subseteq \overline{E}$ ). Given a limit point  $p$  of  $\overline{E} = E \cup E'$ .

- (a)  $p$  is a limit point of  $E$ . Nothing to do.  
(b)  $p$  is a limit point of  $E'$ . Since  $p$  is a limit point of  $E'$  and  $E'$  is a closed set,  $p \in E'$ , or  $p$  is a limit point of  $E$ .

In any case,  $E' \supseteq \overline{E}'$ .

- (3)  $E$  and  $E'$  might not have the same limit points. Let

$$E = \left\{ \frac{1}{n} : n \in \mathbb{Z}^+ \right\} \subseteq \mathbb{R}^1.$$

Then  $E' = \{0\}$  and thus  $(E')' = \emptyset$ .

□

**Exercise 2.7.** Let  $A_1, A_2, A_3, \dots$  be subsets of a metric space.

- (a) If  $B_n = \bigcup_{i=1}^n A_i$ , prove that  $\overline{B_n} = \bigcup_{i=1}^n \overline{A_i}$ , for  $n = 1, 2, 3, \dots$   
(b) If  $B = \bigcup_{i=1}^{\infty} A_i$ , prove that  $\overline{B} \supseteq \bigcup_{i=1}^{\infty} \overline{A_i}$ .

Show, by an example, that this inclusion can be proper.

*Proof of (a).*

- (1) Show that  $\overline{B_n} \subseteq \bigcup_{i=1}^n \overline{A_i}$ . Since  $A_i \subseteq \overline{A_i}$  for any  $i$ , we have

$$B_n = \bigcup_{i=1}^n A_i \subseteq \bigcup_{i=1}^n \overline{A_i}.$$

Since  $\bigcup_{i=1}^n \overline{A_i}$  is a union of finitely many closed set  $\overline{A_i}$ ,  $\bigcup_{i=1}^n \overline{A_i}$  is closed (Theorem 2.24(d)). By Theorem 2.27(c),  $\overline{B_n} \subseteq \bigcup_{i=1}^n \overline{A_i}$ .

- (2) Show that  $\overline{B_n} \supseteq \bigcup_{i=1}^n \overline{A_i}$ . Same argument in the proof of (b).

□

*Proof of (b).* Since  $\bigcup_{j=1}^{\infty} A_j \supseteq A_i$  for any  $i$ , by the monotonicity of closure, we have  $\overline{\bigcup_{j=1}^{\infty} A_j} \supseteq \overline{A_i}$  for any  $i$ , or  $\overline{B} \supseteq \bigcup_{i=1}^{\infty} \overline{A_i}$ . □

*Proof of proper inclusion in (b).* Let

$$A_n = \left( \frac{1}{n}, \infty \right) \subseteq \mathbb{R}^1$$

for any  $n \in \mathbb{Z}^+$ . Then

$$\begin{aligned} \bigcup_{n=1}^{\infty} A_n = (0, \infty) &\implies \overline{\bigcup_{n=1}^{\infty} A_n} = \overline{(0, \infty)} = [0, \infty), \\ \overline{A_n} = \left[ \frac{1}{n}, \infty \right) &\implies \bigcup_{n=1}^{\infty} \overline{A_n} = \bigcup_{n=1}^{\infty} \left[ \frac{1}{n}, \infty \right) = (0, \infty). \end{aligned}$$

□

**Exercise 2.8.** Is every point of every open set  $E \subseteq \mathbb{R}^2$  a limit point of  $E$ ? Answer the same question for closed sets in  $\mathbb{R}^2$ .

It is not true for all metric spaces  $X$ . The (discrete) metric in Exercise 2.10 implies no limit point exists in  $X$ .

*Proof.*

- (1) Show that for every open set  $E \subseteq \mathbb{R}^k$ ,  $E \subseteq E'$ . Given any point  $\mathbf{p} \in E$ , we shall show  $\mathbf{p}$  is a limit point of  $E$ .

- (a) Since  $E$  is open, there is an open neighborhood  $B(\mathbf{p}; r_0) \subseteq E$  for some  $r_0 > 0$ .
- (b) In particular, given any  $s \in \mathbb{R}$  such that  $0 < s < r_0$ , we can find

$$\mathbf{q} \in B(\mathbf{p}; s) \subseteq B(\mathbf{p}; r_0) \subseteq E$$

such that  $\mathbf{q} \neq \mathbf{p}$ . Explicitly, write

$$\mathbf{p} = (p_1, \dots, p_k)$$

and choose

$$\mathbf{q} = \left(p_1 + \frac{s}{89}, p_2, \dots, p_k\right) \neq \mathbf{p}$$

(since  $s > 0$ ). Clearly,  $\mathbf{q}$  is well-defined in  $\mathbb{R}^k$  and  $|\mathbf{q} - \mathbf{p}| = \frac{s}{89} < s$  or  $\mathbf{q} \in B(\mathbf{p}; s)$ .

- (c) Now given every open neighborhood  $B(\mathbf{p}, r)$  of  $\mathbf{p}$ . We can choose  $s \in \mathbb{R}$  such that  $0 < s < \min\{r_0, r\} \leq r_0$ . (might pick  $s = \frac{1}{64} \min\{r_0, r\}$ .) By (b), there exists  $\mathbf{q} \neq \mathbf{p}$  such that

$$\mathbf{q} \in B(\mathbf{p}; s) \subseteq B(\mathbf{p}; r) \subseteq E.$$

- (2) Give an example of a closed set  $E \subseteq \mathbb{R}^k$  such that  $E \not\subseteq E'$ . Pick  $E = \{\mathbf{0}\}$ . So  $E' = \emptyset$  and thus  $E \not\subseteq E'$ .

□

**Exercise 2.9.** Let  $E^\circ$  denote the set of all interior points of a set  $E$ . [See Definition 2.18(e);  $E^\circ$  is called the interior of  $E$ .]

- (a) Prove that  $E^\circ$  is always open.
- (b) Prove that  $E$  is open if and only if  $E^\circ = E$ .
- (c) If  $G$  is contained in  $E$  and  $G$  is open, prove that  $G$  is contained in  $E^\circ$ .
- (d) Prove that the complement of  $E^\circ$  is the closure of the complement of  $E$ .
- (e) Do  $E$  and  $\overline{E}$  always have the same interiors?
- (f) Do  $E$  and  $E^\circ$  always have the same closures?

Similar to Theorem 2.27.

*Proof of (a).* It is equivalent to show that  $E^\circ \subseteq (E^\circ)^\circ$ .

- (1) Given any point  $x \in E^\circ$ , there is  $r > 0$  such that  $B(x; r) \subseteq E$ .



- (2) It suffices to show that  $B(x; \frac{2}{r}) \subseteq E^\circ$ . Given any point  $y \in B(x; \frac{2}{r})$ , we will show that there is an open neighborhood  $B(y; \frac{2}{r})$  of  $y$  such that  $B(y; \frac{2}{r}) \subseteq E$ .
- (3) Given any point  $z \in B(y; \frac{2}{r})$ , we have

$$d(z, x) \leq d(z, y) + d(y, x) < \frac{2}{r} + \frac{2}{r} = r,$$

or  $z \in B(x; r) \subseteq E$ . Therefore,  $B(y; \frac{2}{r}) \subseteq E$ , or  $y \in E^\circ$ , or  $B(x; \frac{2}{r}) \subseteq E^\circ$ , or  $x \in (E^\circ)^\circ$ , or  $E^\circ \subseteq (E^\circ)^\circ$ .

□

*Proof of (b).*

- (1) ( $\implies$ )(Definition 2.18) Since  $E$  is open, every point of  $E$  is an interior point of  $E$ . Hence  $E \subseteq E^\circ$ . Note that  $E^\circ \subseteq E$  is trivial, and thus  $E^\circ = E$ .
- (2) ( $\Leftarrow$ )(a) By (a),  $E = E^\circ$  is always open.
- (3) ( $\Leftarrow$ )(Definition 2.18) Every point of  $E$  is an interior point of  $E$  since  $E = E^\circ$ . Hence  $E$  is open by Definition 2.18(f).

□

*Proof of (c).*  $G \subseteq E$  implies  $G^\circ \subseteq E^\circ$ .  $G = G^\circ$  since  $G$  is open ((b)). Hence  $G = G^\circ \subseteq E^\circ$ , that is,  $E^\circ$  is the largest open set contained in  $E$ . (Similarly,  $\overline{E}$  is the smallest closed set containing  $E$ .) □

*Proof of (d).* Show that  $X - E^\circ = \overline{X - E}$  and  $(X - E)^\circ = X - \overline{E}$ .

- (1) (Theorem 2.27 and (c))

$$\begin{aligned}
X - E^\circ &= X - \bigcup_{\text{Open } V \subseteq E} V \\
&= \bigcap_{\text{Open } V \subseteq E} (X - V) \\
&= \bigcap_{\text{Closed } W \supseteq X - E} W \\
&= \overline{X - E}. \\
X - \overline{E} &= X - \bigcap_{\text{Closed } W \supseteq E} W \\
&= \bigcup_{\text{Closed } W \supseteq E} (X - W) \\
&= \bigcup_{\text{Open } V \subseteq X - E} V \\
&= (X - E)^\circ.
\end{aligned}$$

(2) (Brute-force)

$$\begin{aligned}
x \in E^\circ &\iff \exists r > 0 \text{ such that } B(x; r) \subseteq E \\
&\iff \exists r > 0 \text{ such that } B(x; r) \cap (X - E) = \emptyset \\
&\iff x \notin \overline{X - E} \\
&\iff x \in X - \overline{X - E}. \\
x \in (X - E)^\circ &\iff \exists r > 0 \text{ such that } B(x; r) \subseteq (X - E) \\
&\iff \exists r > 0 \text{ such that } B(x; r) \cap E = \emptyset \\
&\iff x \notin \overline{E} \\
&\iff x \in X - \overline{E}.
\end{aligned}$$

Note that  $X - E^\circ = \overline{X - E}$  is equivalent to  $(X - E)^\circ = X - \overline{E}$  by mapping  $E \mapsto X - E$ .  $\square$

*Proof of (e).* No.

- (1) Let  $X = \mathbb{R}^1$  equipped with the Euclidean metric, and  $E = \mathbb{Q} \subseteq X$ .
- (2)  $E^\circ = \emptyset$  since  $\tilde{\mathbb{Q}}$  is dense in  $\mathbb{R}$ .
- (3)  $(\overline{E})^\circ = (\mathbb{R}^1)^\circ = \mathbb{R}^1$  since  $\mathbb{Q}$  is dense in  $\mathbb{R}$  and  $\mathbb{R}^1$  is open.

$\square$

*Proof of (f).* No.

- (1) Let  $X = \mathbb{R}^1$  equipped with the Euclidean metric, and  $E = \mathbb{Q} \subseteq X$ .
- (2)  $\overline{E} = \mathbb{R}^1$  since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ .
- (3)  $\overline{E^\circ} = \overline{\emptyset} = \emptyset$  since  $\tilde{\mathbb{Q}}$  is dense in  $\mathbb{R}$ .

$\square$

**Exercise 2.10.** Let  $X$  be an infinite set. For  $p \in X$  and  $q \in X$ , define

$$d(p, q) = \begin{cases} 1 & (\text{if } p \neq q) \\ 0 & (\text{if } p = q). \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

(The statement holds even if  $X$  is finite.) We called  $d$  the discrete metric, and the corresponding topology on  $X$  induces the discrete topology. Conversely, if  $X$  has the discrete topology,  $X$  is always metrizable by the discrete metric.

*Proof.*

- (1)  $d(p, q)$  is a metric.
- (a)  $d(p, q) > 0$  if  $p \neq q$ ;  $d(p, p) = 0$ . Trivial.
  - (b)  $d(p, q) = d(q, p)$ . Trivial.
  - (c)  $d(p, q) \leq d(p, r) + d(r, q)$  for any  $r \in X$ . If  $p = q$ , nothing to do. If  $p \neq q$ ,  $r \neq p$  or  $r \neq q$  for any  $r \in X$ . (Assume not true,  $r = p$  and  $r = q$  implies that  $p = q$  which is a contradiction.) In any case  $d(p, r) + d(r, q) \geq 1 = d(p, q)$ .
- (2) *Every subset is open.* Let  $E$  be any subset of  $X$ . Then every point  $p \in E$  is an interior point of  $E$ . In fact, we can pick one open neighborhood  $U = B(p; \frac{1}{2})$  of  $p$  containing only one point  $p \in E$  or  $U = \{p\}$ , and such open neighborhood  $U$  is a subset of  $E$ . So every subset of  $X$  is open.
- (3) *Every subset is closed.* Since every subset is open, every subset is closed by Theorem 2.23.

**Supplement.** Might use Definition 2.18 (d) to prove directly since there are no limit points in  $X$  if we consider one open neighborhood  $U = B(p; \frac{1}{2})$  of  $p$ . Therefore, every subset is closed. Again we apply Theorem 2.23 to get that every subset is open without using Definition 2.18 (f).

- (4) *A subset is compact iff it is finite.*
- (a) *Any finite subset is compact.* Say  $E = \{p_1, p_2, \dots, p_k\}$ , and  $\{G_\alpha\}$  be an open covering of  $E$ . From  $\{G_\alpha\}$  we pick  $G_{\alpha_1}$  containing  $p_1$ ,  $G_{\alpha_2}$  containing  $p_2$ , ..., and  $G_{\alpha_k}$  containing  $p_k$ . This process can be done in the finitely many steps. Therefore,

$$\{G_{\alpha_1}, G_{\alpha_2}, \dots, G_{\alpha_k}\}$$

is a finite subcovering of  $\{G_\alpha\}$  covering  $E$ .

- (b) *Any infinite subset is not compact.* Take a collection

$$\mathcal{G} = \{G_p = \{p\}\}$$

of open subsets where  $p$  runs all points in  $E$ . Clearly,  $\{G_p\}$  is an open covering. Assume

$$\mathcal{G}' = \{G_{p_1}, G_{p_2}, \dots, G_{p_k}\}$$

is any finite subcovering of  $\mathcal{G}$ . Since  $E$  is infinite, there exist a point  $p \in E$  such that  $p \neq p_1, p \neq p_2, \dots, p \neq p_k$ . Therefore,  $\mathcal{G}'$  does not cover  $p$ , or  $\mathcal{G}$  does not contain any finite subcovering  $\mathcal{G}'$ .

□

Notice that every subset is bounded. Therefore, every subset is closed and bounded, but only finite subset is compact, i.e., Heine-Borel theorem is not true in the infinite discrete topology.

**Exercise 2.11.** For  $x \in \mathbb{R}^1$  and  $y \in \mathbb{R}^1$ , define

$$\begin{aligned} d_1(x, y) &= (x - y)^2, \\ d_2(x, y) &= \sqrt{|x - y|}, \\ d_3(x, y) &= |x^2 - y^2|, \\ d_4(x, y) &= |x - 2y|, \\ d_5(x, y) &= \frac{|x - y|}{1 + |x - y|}. \end{aligned}$$

Determine, for each of these, whether it is a metric or not.

*Proof.*

- (1)  $d = d_1$  is not a metric. (Reductio ad absurdum) If  $d$  were a metric, then

$$d(0, 2) > d(0, 1) + d(1, 2),$$

contrary to Definition 2.15(c) that  $d(p, q) \leq d(p, r) + d(r, q)$ .

- (2)  $d = d_2$  is a metric. It suffices to show that  $d'(x, y) = \sqrt{d(x, y)}$  is a metric if  $d(x, y)$  is a metric. For any  $p, q, r \in \mathbb{R}^1$ ,

(a)  $d'(p, q) = \sqrt{d(p, q)} > 0$  if  $p \neq q$ ;  $d'(p, p) = \sqrt{d(p, p)} = 0$ .

(b)  $d'(p, q) = \sqrt{d(p, q)} = \sqrt{d(q, p)} = d'(q, p)$ .

(c)

$$\begin{aligned} \sqrt{d(p, r) + d(r, q)} &\leq \sqrt{d(p, r)} + \sqrt{d(r, q)} \\ \iff (\sqrt{d(p, r)} + \sqrt{d(r, q)})^2 &\leq (\sqrt{d(p, r)} + \sqrt{d(r, q)})^2 \\ \iff d(p, r) + d(r, q) &\leq d(p, r) + d(r, q) + 2\sqrt{d(p, r)}\sqrt{d(r, q)} \\ \iff 0 &\leq 2\sqrt{d(p, r)}\sqrt{d(r, q)}. \end{aligned}$$

(d)

$$\begin{aligned} d'(p, q) &= \sqrt{d(p, q)} \\ &\leq \sqrt{d(p, r) + d(r, q)} && \text{(Triangle inequality)} \\ &\leq \sqrt{d(p, r)} + \sqrt{d(r, q)} && ((c)) \\ &= d'(p, r) + d'(r, q). \end{aligned}$$

By Definition 2.15,  $d'$  is a metric.

(3)  $d = d_3$  is not a metric. (Reductio ad absurdum) If  $d$  were a metric, then

$$d(1, -1) = 0,$$

contrary to Definition 2.15(a):  $d(p, q) > 0$  if  $p \neq q$ ;  $d(p, p) = 0$ .

(4)  $d = d_4$  is not a metric. (Reductio ad absurdum) If  $d$  were a metric, then

$$d(1, 1) = 1,$$

contrary to Definition 2.15(a):  $d(p, q) > 0$  if  $p \neq q$ ;  $d(p, p) = 0$ .

(5)  $d = d_5$  is a metric. It suffices to show that  $d'(x, y) = \frac{d(x, y)}{1+d(x, y)}$  is a metric if  $d(x, y)$  is a metric. For any  $p, q, r \in \mathbb{R}^1$ ,

$$(a) \quad d'(p, q) = \frac{d(p, q)}{1+d(p, q)} > 0 \text{ if } p \neq q; \quad d'(p, p) = \frac{d(p, p)}{1+d(p, p)} = 0.$$

$$(b) \quad d'(p, q) = \frac{d(p, q)}{1+d(p, q)} = \frac{d(q, p)}{1+d(q, p)} = d'(q, p).$$

(c) Write  $x = d(p, q)$ ,  $y = d(p, r)$  and  $z = d(r, q)$ . So  $x, y, z \geq 0$  and

$$\begin{aligned} x &\leq y + z \\ \iff x + x(y + z) &\leq y + z + x(y + z) \\ \iff x(1 + y + z) &\leq (1 + x)(y + z) \\ \iff \frac{x}{1 + x} &\leq \frac{y + z}{1 + y + z}. \end{aligned}$$

(d)

$$\begin{aligned} d'(p, q) &= \frac{d(p, q)}{1 + d(p, q)} \\ &\leq \frac{d(p, r) + d(r, q)}{1 + d(p, r) + d(r, q)} && ((c)) \\ &= \frac{d(p, r)}{1 + d(p, r) + d(r, q)} + \frac{d(r, q)}{1 + d(p, r) + d(r, q)} \\ &= \frac{d(p, r)}{1 + d(p, r)} + \frac{d(r, q)}{1 + d(r, q)} \\ &= d'(p, r) + d'(r, q). \end{aligned}$$

(e) Or we can show  $d'(p, q) \leq d'(p, r) + d'(r, q)$  by

$$\begin{aligned} \frac{x}{1 + x} &\leq \frac{y}{1 + y} + \frac{z}{1 + z} \\ \iff x(1 + y)(1 + z) &\leq y(1 + z)(1 + x) + z(1 + x)(1 + y) \\ \iff x + xy + xz + xyz &\leq (y + xy + yz + xyz) + (z + xz + yz + xyz) \\ \iff x &\leq y + z + 2yz + xyz \\ \iff x &\leq y + z && (d \text{ is nonnegative}) \end{aligned}$$

By Definition 2.15,  $d'$  is a metric.

□

**Exercise 2.12.** Let  $K \subseteq \mathbb{R}^1$  consist of 0 and the numbers  $\frac{1}{n}$ , for  $n = 1, 2, 3, \dots$ . Prove that  $K$  is compact directly from the definition (without using the Heine-Borel theorem).

*Proof.* Let  $\{G_\alpha\}$  be an open covering of  $K$ . There is an open set  $G_0 \in \{G_\alpha\}$  containing 0. So there exists an open neighborhood  $U = B(0; r)$  of 0 such that  $U \subseteq G_0$ . So  $U$  contains all points  $q = \frac{1}{n}$  of  $K$  whenever  $n > \frac{1}{r}$ . To construct a finite subcovering of  $\{G_\alpha\}$ , we need to pick finitely many open sets from  $\{G_\alpha\}$  to cover the remaining points  $q = \frac{1}{n}$  where  $n = 1, 2, \dots, [\frac{1}{r}]$ , say  $G_1$  contains  $q = \frac{1}{1}$ ,  $G_2$  contains  $q = \frac{1}{2}$ , ...,  $G_{[\frac{1}{r}]}$  contains  $q = \frac{1}{[\frac{1}{r}]}$ . (Might be duplicated.)

Hence,

$$\left\{G_0, G_1, G_2, \dots, G_{[\frac{1}{r}]}\right\}$$

is a finite subcovering of  $\{G_\alpha\}$  covering  $K$ . □

*Proof (Heine-Borel theorem).*

(1)  $K$  is closed. In fact, the only limit point of  $K$  is 0, which is in  $K$ .

- (a)  $p = 0$  is a limit point. Given  $r > 0$ . There always exists  $n \in \mathbb{Z}^+$  such that  $r > \frac{1}{n}$ . So any open neighborhood  $B(0; r)$  of  $p = 0$  contains at least one point  $q = \frac{1}{n} \neq 0$  in  $K$ .
- (b)  $p < 0$  is not a limit point. Pick an open neighborhood  $B(p; r)$  of  $p$  where  $r = |p| > 0$ . Then  $B(p; r) \cap K = \emptyset$ .
- (c)  $p > 0$  is not a limit point. There always exists  $m \in \mathbb{Z}^+$  such that  $p > \frac{1}{m}$  whenever  $n \geq m$ . Pick an open neighborhood  $B(p; r)$  of  $p$  where  $r = p - \frac{1}{m} > 0$ . Then  $B(p; r)$  does not have all points  $q = \frac{1}{n} \in K$  whenever  $n \geq m$ . By Theorem 2.20,  $p$  cannot be a limit point of  $K$ .

(2)  $K$  is bounded. There is a real number  $M = 2$  and a point  $q = 0 \in \mathbb{R}^1$  such that  $|p - q| = |p| < 2$  for all  $p \in K$ .

By Heine-Borel theorem,  $K$  is compact in  $\mathbb{R}^1$ . □

**Exercise 2.13.** Construct a compact set of real numbers whose limit points form a countable set.

*Proof (Exercise 2.12).* Let  $K(p; r) \subseteq \mathbb{R}^1$  be

$$K(p; r) = \left\{p + \frac{r}{n} : n = 2, 3, \dots\right\} \cup \{p\}$$

and

$$K = \left( \bigcup_{i=0}^{\infty} K(2^{-i}; 2^{-i}) \right) \cup \{0\}.$$

- (1) *The set of limit points of  $K$  is  $K' = \{2^{-i} : i = 0, 1, 2, \dots\} \cup \{0\}$ , which is (infinitely) countable.*
  - (a) The unique limit point of  $K(2^{-i}; 2^{-i})$  is  $2^{-i}$  for each  $i = 0, 1, 2, \dots$  (Exercise 2.12).
  - (b) 0 is a limit point of  $K$ .
  - (c) No other limit points of  $K$ . Similar to the argument of the proof of Exercise 2.12.
- (2)  $K$  is closed. All limit points are in  $K$ .
- (3)  $K$  is bounded. There is a real number  $M = 2$  and a point  $q = 0 \in \mathbb{R}^1$  such that  $|p - q| = |p| < 2$  for all  $p \in K$ .

By Heine-Borel theorem,  $K$  is compact in  $\mathbb{R}^1$ , and has infinitely countable limit points.  $\square$

**Exercise 2.14.** *Give an example of an open cover of the segment  $(0, 1)$  which has no finite subcover.*

*Proof.* In  $\mathbb{R}^1$ , take a collection

$$\mathcal{G} = \left\{ G_n = \left( \frac{1}{n}, 1 \right) \right\}$$

of open subsets where  $n \in \mathbb{Z}^+$ .

- (1)  $\mathcal{G}$  is an open covering of  $(0, 1) \subseteq \mathbb{R}^1$ . Actually, given  $x \in (0, 1)$ , there exists an positive integer  $n$  such that  $x > \frac{1}{n}$ . That is,  $x \in (\frac{1}{n}, 1) = G_n$ .
- (2) *There is no finite subcovering of  $\mathcal{G}$ .* Assume

$$\mathcal{G}' = \{G_{n_1}, G_{n_2}, \dots, G_{n_k}\}$$

is any finite subcovering of  $\mathcal{G}$  where  $n_1 < n_2 < \dots < n_k$ . Take  $x \in \left(0, \frac{1}{n_k}\right) \neq \emptyset$ ,  $x = \frac{1}{2n_k}$  for example. Then  $x \notin G_{n_1}$ ,  $x \notin G_{n_2}$ , ...,  $x \notin G_{n_k}$ , which contradicts that  $\mathcal{G}'$  is a finite subcovering of  $\mathcal{G}$  covering  $(0, 1)$ .

$\square$

**Exercise 2.15.** *Show that Theorem 2.36 and its Corollary become false (in  $\mathbb{R}^1$ , for example) if the word “compact” is replaced by “closed” or by “bounded.”*

*Recall:*

- (1) Theorem 2.36: If  $\{K_\alpha\}$  is a collection of compact subsets of a metric space  $X$  such that the intersection of every finite subcollection of  $\{K_\alpha\}$  is nonempty, then  $\bigcap K_\alpha$  is nonempty.
- (2) Corollary: If  $\{K_n\}$  is a sequence of nonempty compact sets such that  $K_n$  contains  $K_{n+1}$  ( $n = 1, 2, 3, \dots$ ), then  $\bigcap K_n$  is not empty.

*Proof.* Let  $X = \mathbb{R}^1$  with the usual Euclidean metric.

- (1) For the closeness, let  $K_n = [n, \infty) \subseteq X$ .
- (2) For the boundedness, let  $K_n = (0, \frac{1}{n}) \subseteq X$ .

In any case,  $K_1 \supseteq K_2 \supseteq \dots$  and  $\bigcap K_n = \emptyset$ .  $\square$

**Exercise 2.16.** Regard  $\mathbb{Q}$ , the set of all rational numbers, as a metric space, with  $d(p, q) = |p - q|$ . Let  $E$  be the set of all  $p \in \mathbb{Q}$  such that  $2 < p^2 < 3$ . Show that  $E$  is closed and bounded in  $\mathbb{Q}$ , but that  $E$  is not compact. Is  $E$  open in  $\mathbb{Q}$ ?

**Lemma.** Assume  $S \subseteq T \subseteq M$ . Then  $S$  is compact in  $(M, d)$  if, and only if,  $S$  is compact in the metric subspace  $(T, d)$ .

*Proof of Lemma.*

- (1) ( $\implies$ ) Let  $\mathcal{F}$  be an open covering of  $S$  in  $(T, d)$ , say  $S \subseteq \bigcup_{A \in \mathcal{F}} A$  where  $A$  is open in  $T$ . Then  $A = B \cap T$  for some open set  $B$  in  $M$  (Theorem 3.33). Let  $\mathcal{G}$  be the collection of  $B$ . Then

$$S \subseteq \bigcup_{A \in \mathcal{F}} A = \bigcup_{B \in \mathcal{G}} (B \cap T) \subseteq \bigcup_{B \in \mathcal{G}} B,$$

or  $\mathcal{G}$  be an open covering of  $S$  in  $(M, d)$ . Since  $S$  is compact in  $(M, d)$ ,  $\mathcal{G}$  contains a finite subcovering, say

$$S \subseteq B_1 \cap \dots \cap B_p.$$

So

$$S \cap T \subseteq (B_1 \cap T) \cap \dots \cap (B_p \cap T),$$

or

$$S \subseteq A_1 \cap \dots \cap A_p$$

(since  $S \subseteq T$  or  $S \cap T = S$ ). So there is a finite subcovering of  $\mathcal{F}$  covering  $S$ , or  $S$  is compact in  $(T, d)$ .

- (2) ( $\impliedby$ ) Let  $\mathcal{G}$  be an open covering of  $S$  in  $(M, d)$ , say  $S \subseteq \bigcup_{B \in \mathcal{G}} B$  where  $B$  is open in  $M$ . Then  $A = B \cap T$  is open in  $T$ . Let  $\mathcal{F}$  be the collection of  $A$ . Then

$$S \cap T \subseteq \bigcup_{B \in \mathcal{G}} (B \cap T) = \bigcup_{A \in \mathcal{F}} A,$$



or  $\mathcal{F}$  be an open covering of  $S \cap T = S$  in  $(T, d)$ . Since  $S$  is compact in  $(T, d)$ ,  $\mathcal{F}$  contains a finite subcovering, say

$$S \subseteq A_1 \cap \cdots \cap A_p.$$

Clearly,  $S \subseteq B_1 \cap \cdots \cap B_p$  since  $A = B \cap T \subseteq B$ . So there is a finite subcovering of  $\mathcal{G}$  covering  $S$ , or  $S$  is compact in  $(M, d)$ .

□

*Proof.* Write  $E_0 = (\sqrt{2}, \sqrt{3}) \cup (-\sqrt{3}, -\sqrt{2})$ , and  $E = E_0 \cap \mathbb{Q}$ .

- (1)  $E$  is a subset of  $\mathbb{Q}$ .
- (2) *Show that  $E$  is bounded in  $\mathbb{Q}$ .* Since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , there is  $p \in \mathbb{Q}$  such that  $\sqrt{2} < p < \sqrt{3}$ , or  $p \in E$ . Let  $r = p + \sqrt{3} > 0$ . Therefore,  $E \subseteq B(p; r)$  for some  $r > 0$  and  $p \in E$ , or  $E$  is bounded.
- (3) *Show that  $E$  is closed in  $\mathbb{Q}$ .* It suffices to show its complement is open in  $\mathbb{Q}$ . Given any

$$p \in \tilde{E} = ((-\infty, -\sqrt{3}] \cup [-\sqrt{2}, \sqrt{2}] \cup [\sqrt{3}, \infty)) \cap \mathbb{Q}.$$

$$p \leq -\sqrt{3} \text{ or } -\sqrt{2} \leq p \leq \sqrt{2} \text{ or } p \geq \sqrt{3}.$$

- (a)  $p \leq -\sqrt{3}$ .  $p \neq -\sqrt{3}$  since  $p \in \mathbb{Q}$  and  $-\sqrt{3}$  is irrational. So  $p < -\sqrt{3}$  and thus there exists  $q \in \mathbb{Q}$  such that  $p < q < -\sqrt{3}$  since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ . Let  $r = \max\{-\sqrt{3} - q, q - p\} > 0$ . The ball  $B(q; r)$  is contained in  $\tilde{E}$ .
- (b)  $-\sqrt{2} \leq p \leq \sqrt{2}$ . Similar to (a).
- (c)  $p \geq \sqrt{3}$ . Similar to (a).

By (a)(b),  $\tilde{E}$  is open in  $\mathbb{Q}$ , or  $E$  is closed in  $\mathbb{Q}$ .

- (4) *Show that  $E$  is not compact in  $\mathbb{Q}$ .* (Reductio ad absurdum) If  $E_0$  were compact in the metric space  $\mathbb{Q}$ ,  $E_0$  is compact in the metric space  $\mathbb{R}$  (Lemma), which is absurd.
- (5) *Show that  $E$  is open.* Similar to (3).

□

**Exercise 2.17.** Let  $E$  be the set of all  $x \in [0, 1]$  whose decimal expansion contains only the digits 4 and 7. Is  $E$  countable? Is  $E$  dense in  $[0, 1]$ ? Is  $E$  compact? Is  $E$  perfect?

*Proof.*

- (1) *Show that  $E$  is uncountable.*

□

**Exercise 2.18.** *Is there a nonempty perfect set in  $\mathbb{R}^1$  which contains no rational number?*

**Exercise 2.19.**

- (a) *If  $A$  and  $B$  are disjoint closed sets in some metric space  $X$ , prove that they are separated.*
- (b) *Prove the same for disjoint open sets.*
- (c) *Fix  $p \in X$ ,  $\delta > 0$ , define  $A$  to be the set of all  $q \in X$  for which  $d(p, q) < \delta$ , define  $B$  similarly, with  $>$  in place of  $<$ . Prove that  $A$  and  $B$  are separated.*
- (d) *Prove that every connected metric space with at least two points is uncountable. Hint: Use (c).*

**Exercise 2.20.** *Are closures and interiors of connected sets always connected? (Look at subsets of  $\mathbb{R}^2$ .)*

*Proof.*

- (1) *Interiors of connected sets are not always connected.* Let  $X = \mathbb{R}^2$  with the usual Euclidean metric be a metric space. Take

$$E = B(89; 1) \cup B(64; 1) \cup \{(x, 0) \in \mathbb{R}^2 : 64 \leq x \leq 89\}.$$

$E$  is connected and

$$E^\circ = B(89; 1) \cup B(64; 1)$$

is disconnected.

- (2) *Closures of connected sets are always connected. It suffices to show that  $E$  is disconnected if  $\overline{E}$  is disconnected.*

- (a) Write  $\overline{E} = A \cup B$  as a union of two nonempty separated sets. Here  $A \neq \emptyset$ ,  $B \neq \emptyset$ ,  $A \cap \overline{B} = \emptyset$  and  $\overline{A} \cap B = \emptyset$ .

- (b) Write

$$E = (A \cap E) \cup (B \cap E)$$

and we will show that  $E$  is disconnected.

- (c) *Show that  $A \cap E$  and  $B \cap E$  are separated.* In fact,

$$(A \cap E) \cap \overline{B \cap E} \subseteq A \cap \overline{B} = \emptyset,$$

$$\overline{A \cap E} \cap (B \cap E) \subseteq \overline{A} \cap B = \emptyset.$$

- (d) Show that  $A \cap E$  and  $B \cap E$  are nonempty. (Reductio ad absurdum)  
If  $A \cap E = \emptyset$ , then

$$E = (A \cap E) \cup (B \cap E) = B \cap E \implies E \subseteq B.$$

So

$$\begin{aligned} A &= (A \cup B) \cap A & (A \subseteq A \cup B) \\ &= \overline{E} \cap A \\ &\subseteq \overline{B} \cap A & (E \subseteq B) \\ &= \emptyset \end{aligned}$$

which contradicts  $A \neq \emptyset$  in (a). Therefore,  $A \cap E \neq \emptyset$ . Similarly,  $B \cap E \neq \emptyset$ .

Hence,  $E$  is disconnected if  $\overline{E}$  is disconnected, or closures of connected sets are always connected

□

**Exercise 2.21.** Let  $A$  and  $B$  be separated subsets of some  $\mathbb{R}^k$ , suppose  $\mathbf{a} \in A$ ,  $\mathbf{b} \in B$ , and define

$$\mathbf{p}(t) = (1-t)\mathbf{a} + t\mathbf{b}$$

for  $t \in \mathbb{R}^1$ . Put  $A_0 = \mathbf{p}^{-1}(A)$ ,  $B_0 = \mathbf{p}^{-1}(B)$ . [Thus  $t \in A_0$  if and only if  $\mathbf{p}(t) \in A$ .]

- Prove that  $A_0$  and  $B_0$  are separated subsets of  $\mathbb{R}^1$ .
- Prove that there exists  $t_0 \in (0, 1)$  such that  $\mathbf{p}(t_0) \notin A \cup B$ .
- Prove that every convex subset of  $\mathbb{R}^k$  is connected.

**Exercise 2.22.** A metric space is called separable if it contains a countable dense subset. Show that  $\mathbb{R}^k$  is separable. (Hint: Consider the set of points which have only rational coordinates.)

*Proof.* Let  $E$  be the set of points which have only rational coordinates.

- Show that  $E$  is countable.  $\mathbb{Q}$  is countable and thus  $E = \mathbb{Q}^k$  is countable (Theorem 2.13).
- Show that  $E$  is dense. Given any  $\mathbf{p} = (p_1, \dots, p_k) \in \mathbb{R}^k$ . We want to show that  $\mathbf{p}$  is a limit point of  $E$ .
  - Given any open neighborhood  $B(\mathbf{p}; r)$  of  $\mathbf{p}$ ,  $r > 0$ .

- (b) Since  $\mathbb{Q}$  is dense in  $\mathbb{R}$  (Theorem 1.20), every coordinate of  $\mathbf{p}$  is a limit point of  $\mathbb{Q}$ . In particular, for every  $i = 1, 2, \dots, k$ , the open neighborhood  $B\left(p_i, \frac{r}{\sqrt{k}}\right)$  of  $p_i$  contains a point  $q_i \neq p_i$  and  $q_i \in \mathbb{Q}$ .
- (c) Collect all  $q_i$  in (b) and define  $\mathbf{q} = (q_1, \dots, q_k) \in \mathbb{Q}^k = E$ . By construction  $\mathbf{q} \neq \mathbf{p}$  and

$$\begin{aligned} |\mathbf{p} - \mathbf{q}| &= \sqrt{(p_1 - q_1)^2 + \dots + (p_k - q_k)^2} \\ &< \sqrt{\left(\frac{r}{\sqrt{k}}\right)^2 + \dots + \left(\frac{r}{\sqrt{k}}\right)^2} \\ &= \sqrt{k \cdot \frac{r^2}{k}} \\ &= r \end{aligned}$$

or  $\mathbf{q} \in B(\mathbf{p}; r)$ .

By (a)(b)(c),  $E$  is dense in  $\mathbb{R}^k$ .

By (1)(2),  $\mathbb{R}^k$  is separable.  $\square$

**Exercise 2.23.** A collection  $\{V_\alpha\}$  of open subsets of  $X$  is said to be a base for  $X$  if the following is true: For every  $x \in X$  and every open set  $G \subseteq X$  such that  $x \in G$ , we have  $x \in V_\alpha \subseteq G$  for some  $\alpha$ . In other words, every open set in  $X$  is the union of a subcollection of  $\{V_\alpha\}$ .

*Prove that every separable metric space has a countable base. (Hint: Take all neighborhoods with rational radius and center in some countable dense subset of  $X$ .)*

*Note.*  $\mathbb{R}^k$  has a countable base (Exercise 2.22).

*Proof (Hint).* Let  $X$  be a separable metric space, and  $E$  be a countable dense subset of  $X$ . Let  $\mathcal{B}$  be a collection of all neighborhoods with rational radius and center in  $E$ .

- (1)  $\mathcal{B}$  is countable (Theorem 2.12).
- (2)  $\mathcal{B}$  is a base for  $X$ . Similar to Exercise 2.9(a). Given any  $p \in X$  and every open set  $G \subseteq X$  such that  $p \in G$ . Since  $p$  is in an open set  $G$ , there exists an open neighborhood  $B(p; r)$  of  $p$  such that  $B(p; r) \subseteq G$ .
- (3) Let  $r_0$  be rational such that  $0 < r_0 < \frac{r}{2}$  (Theorem 1.20(b)). Since  $E$  is dense in  $X$ , there is  $q \in E$  such that  $d_X(p, q) < r_0$ . For such  $r_0 \in \mathbb{Q}$  we pick an open neighborhood  $B(q; r_0)$  of  $q$ . Clearly,  $B(q; r_0) \in \mathcal{B}$ .
- (4)  $p \in B(q; r_0)$  since  $d_X(p, q) < r_0$ .

- (5) Show that  $B(q; r_0) \subseteq B(p; r) \subseteq G$ . For any  $z \in B(q; r_0)$ ,  $d_X(z, p) \leq d_X(z, q) + d_X(q, p) < r_0 + r_0 < \frac{r}{2} + \frac{r}{2} = r$ . That is,  $z \in B(p; r)$ .

By (3)(4)(5), (2) is established. By (1)(2),  $\mathcal{B}$  is a countable base for  $X$ .  $\square$

**Supplement.**

- (1) In topology, a second-countable space, also called a completely separable space, is a topological space whose topology has a countable base.
- (2) Every second-countable space is separable.
- (3) The reverse implication of (2) does not hold in general. However, for metric spaces the properties of being second-countable and separable are equivalent.
- (4) Show that every second-countable metric space  $X$  is separable.

- (a) Let  $\mathcal{B} = \{B_n : n \in \mathbb{Z}^+\}$  be a countable base of  $X$ .
- (b) For every  $B_n \in \mathcal{B}$ , pick any point  $p_n$  of  $B_n$  and collect them as

$$E = \{p_n : p_n \in B_n \text{ for } n \in \mathbb{Z}^+\}.$$

- (c)  $E$  is countable.
- (d) Show that  $E$  is dense. Given any  $x \in X$ . For any open neighborhood  $B(x)$  of  $x$ ,  $B(x)$  is a union of subcollection of  $\mathcal{B}$ . That is, there is always a point in  $E$  by the construction of  $E$ .

$\square$

**Exercise 2.24.** Let  $X$  be a metric space in which every infinite subset has a limit point. Prove that  $X$  is separable.

(Hint: Fix  $\delta > 0$ , and pick  $x_1 \in X$ . Having chosen  $x_1, \dots, x_j \in X$ , choose  $x_{j+1}$ , if possible, so that  $d(x_i, x_{j+1}) \geq \delta$  for  $i = 1, \dots, j$ . Show that this process must stop after finite number of steps, and that  $X$  can therefore be covered by finite many neighborhoods of radius  $\delta$ . Take  $\delta = \frac{1}{n}$  ( $n = 1, 2, 3, \dots$ ) and consider the centers of the corresponding neighborhoods.)

Note. The reverse implication does not hold (Exercise 2.10).

Proof (Hint).

- (1) Fix  $\delta > 0$ , and pick  $x_1 \in X$ . Show that every limit point compact metric space  $X$  is totally bounded.
  - (a) Having chosen  $x_1, \dots, x_j \in X$ , choose  $x_{j+1}$ , if possible, so that  $d(x_i, x_{j+1}) \geq \delta$  for  $i = 1, \dots, j$ . Let  $E_\delta$  be the set of these  $x_i$ .

(b) *Show that this process must stop after finite number of steps, and that  $X$  can therefore be covered by finite many neighborhoods of radius  $\delta$ . (Reductio ad absurdum)*

- (i) If not,  $E_\delta$  is an infinite subset of  $X$ . By assumption there is a limit point of  $E_\delta$ , say  $p \in X$ .
- (ii) In particular, an open neighborhood  $B(p; \frac{\delta}{64})$  of  $p$  contains a point  $x_n \in E_\delta$  with  $p \neq x_n$ .
- (iii) The neighborhood  $B(p; \frac{\delta}{64})$  contains no other point  $x_m \in E_\delta$  with  $m \neq n$ . If so,

$$d_X(x_n, x_m) \leq d_X(x_n, p) + d_X(p, x_m) < \frac{\delta}{64} + \frac{\delta}{64} < \delta,$$

contrary to the construction of  $E_\delta$ .

- (iv) Note that  $p \notin E_\delta$  as a corollary to (iii).
- (v) So another open neighborhood  $B(p; r)$  of  $p$  with  $r = d_X(p, x_n) > 0$  contains no points  $x_m \in E_\delta$  with  $p \neq x_m$ , contrary to the assumption that  $p$  is a limit point of  $E_\delta$ .

(2) *Show that every totally bounded metric space  $X$  is separable. Take  $\delta = \frac{1}{n}$  ( $n = 1, 2, 3, \dots$ ) in (1), and union all  $E_{\frac{1}{n}}$  as*

$$E = \bigcup_{n=1}^{\infty} E_{\frac{1}{n}} \subseteq X.$$

*Show that  $E$  is a countable dense subset of  $X$ .*

- (a) *Show that  $E$  is countable.* Since  $E$  is the countable union of finite set  $E_{\frac{1}{n}}$ ,  $E$  is countable (Theorem 2.12).
- (b) *Show that  $E$  is dense in  $X$ .* Given any  $p \in X$ . It suffices to show that given any open neighborhood  $B(p; r)$  of  $p \in X - E$ , there exists  $q \in E$  such that  $q \in B(p; r)$ . Pick any  $n \in \mathbb{Z}^+$  such that  $\frac{1}{n} < r$  (Theorem 1.20(a)). By the construction of  $E_{\frac{1}{n}}$ , there is  $q \in E_{\frac{1}{n}}$  such that  $p \in B(q; \frac{1}{n})$ , or  $d_X(p, q) < \frac{1}{n} < r$ , or  $q \in B(p; r)$ .

□

### Supplement.

- (1) A topological space  $X$  is said to be limit point compact or weakly countably compact if every infinite subset of  $X$  has a limit point in  $X$ .
- (2) In a metric space, limit point compactness, compactness, and sequential compactness are all equivalent. For general topological spaces, however, these three notions of compactness are not equivalent.

- (3) A metric space  $X$  is totally bounded if and only if for every real number  $\delta > 0$ , there exists a finite collection of open balls in  $X$  of radius  $\delta$  whose union contains  $X$ .

**Exercise 2.25.** Prove that every compact metric space  $K$  has a countable base, and that  $K$  is therefore separable. (Hint: For every positive integer  $n$ , there are finitely many neighborhood of radius  $\frac{1}{n}$  whose union covers  $K$ .)

*Proof (Exercise 2.24(a)).*

- (1) Show that every compact metric space  $K$  is limit point compact. Given any subset  $E \subseteq K$ . It suffices to show that if  $E$  has no limit point, then  $E$  must be finite.
- (a) Since  $E$  has no limit point,  $E$  is closed.
  - (b) For any point  $p \in E$ . Since  $p$  is not a limit point, there is an open neighborhood  $B(p)$  such that  $B(p)$  contains no point other than  $p$ .
  - (c) Similar to the proof of Theorem 2.35, let

$$\mathcal{F} = \{B(p) : p \in E \text{ with } B(p) \cap E = \{p\}\} \cup \tilde{E}.$$

Hence  $\mathcal{F}$  is an open covering of  $K$ .

- (d) Since  $K$  is compact by assumption, there is a finite subcovering  $\mathcal{F}'$  of  $K$ . Since  $\tilde{E}$  does not intersect  $E$ , each  $B(p) \in \mathcal{F}'$  contains only one point of  $E$  and so  $E$  is finite.
- (2) Since  $K$  is limit point compact,  $K$  is separable (Theorem 2.24).

□

*Proof (Exercise 2.24(b)).*

- (1) Show that every compact metric space  $K$  is totally bounded. Given any real number  $\delta > 0$ , define an open covering  $\mathcal{F}$  of  $K$  by

$$\mathcal{F} = \{B(p; \delta) : p \in K\}.$$

Since  $K$  is compact, there exists a finite subcovering  $\mathcal{F}'$  of  $K$ .  $\mathcal{F}'$  is our desired finite collection of open balls in  $X$  of radius  $\delta$  whose union contains  $X$ .

- (2) Since  $K$  is totally bounded,  $K$  is separable (Theorem 2.24).

□

*Proof (Hint).*

- (1) Given any positive integer  $n > 0$ , define an open covering  $\mathcal{F}_n$  of  $K$  by

$$\mathcal{F}_n = \left\{ B\left(p; \frac{1}{n}\right) : p \in K \right\}.$$

Since  $K$  is compact, there exists a finite subcovering  $\mathcal{G}_n$  of  $K$ .

- (2) Show that every compact metric space  $K$  is second-countable.

- (a) Define

$$\mathcal{B} = \bigcup_{n \geq 1} \mathcal{G}_n$$

be a collection. Since  $\mathcal{B}$  is a countable union of finite set  $\mathcal{G}_n$ ,  $\mathcal{B}$  is countable. Hence it suffices to show that for every  $p \in K$  and every open set  $G \subseteq K$  such that  $p \in G$ , there is  $B \in \mathcal{B}$  such that  $x \in B \subseteq G$ .

- (b) Since  $G$  is open, there is an open neighborhood  $B(p; r)$  of  $p$  such that  $B(p; r) \subseteq G$ .  
(c) For such  $r > 0$ , there is  $n \in \mathbb{Z}^+$  with  $0 < \frac{1}{n} < \frac{r}{2}$  (Theorem 1.20(a)). So  $p$  is in some  $B(q; \frac{1}{n}) \in \mathcal{G}_n \subseteq \mathcal{B}$  since  $\mathcal{G}_n$  is a subcovering of  $K$ .  
(d) Show that  $B(q; \frac{1}{n}) \subseteq B(p; r) \subseteq G$ . For any  $z \in B(q; \frac{1}{n})$ ,

$$d_K(z, p) \leq d_K(z, q) + d_K(q, p) < \frac{1}{n} + \frac{1}{n} < \frac{r}{2} + \frac{r}{2} = r.$$

That is,  $z \in B(p; r)$ , or  $B(q; \frac{1}{n}) \subseteq B(p; r) \subseteq G$ .

By (a)(b)(c)(d),  $K$  is second-countable.

- (3) Show that every second-countable metric space is separable. Supplement (4) to Exercise 2.23.

□

**Exercise 2.26.** Let  $X$  be a metric space in which every infinite subsets has a limit point. Prove that  $X$  is compact.

By Exercises 2.23 and 2.24,  $X$  has a countable base. It follows that every open cover of  $X$  has a countable subcovering  $\{G_n\}$ ,  $n = 1, 2, 3, \dots$ . If no finite subcollection of  $\{G_n\}$  covers  $X$ , then the complement  $F_n$  of  $G_1 \cup \dots \cup G_n$  is nonempty for each  $n$ , but  $\bigcap F_n$  is empty. If  $E$  is a set contains a point from each  $F_n$ , consider a limit point of  $E$ , and obtain a contradiction.



*Note.* In every metric space, we have

$$\begin{aligned}
\{\text{compact}\} &\iff \{\text{limit point compact}\} \\
&\implies \{\text{totally bounded}\} \\
&\implies \{\text{separable}\} \\
&\iff \{\text{second-countable}\} \\
&\iff \{\text{Lindelof}\}.
\end{aligned}$$

*Proof (Hint).*

- (1) Since  $X$  is limit point compact,  $X$  is separable (Exercise 2.24). Since  $X$  is separable,  $X$  is second-countable (Exercise 2.23).
- (2) *Show that  $X$  is Lindelof if  $X$  is second-countable.* Let  $X$  be a second-countable metric space. Let  $\mathcal{B} = \{B_n\}$  be a countable base of  $X$ . Given any open covering  $\mathcal{F}$  of  $X$ .
  - (a) Iterate each  $B_n \in \mathcal{B}$ , pick one  $G_n \in \mathcal{F}$  containing  $B_n$ , and collect them as
$$\mathcal{G} = \{G_n : G_n \supseteq B_n \text{ for } n \in \mathbb{Z}^+\}.$$

( $G_n$  might be duplicated.)
  - (b)  $\mathcal{G}$  is a countable subset of  $\mathcal{F}$ .
  - (c)  $\mathcal{G}$  covers  $X$  since  $\mathcal{B}$  is a countable base of  $X$ .
- (3) Hence, given any open covering  $\mathcal{F}$  of  $X$ , there is a countable subcovering  $\mathcal{G} = \{G_n\}$  of  $X$ . (Reductio ad absurdum) If there were no finite subcovering of  $\mathcal{G}$ , then the complement  $F_n$  of  $G_1 \cup \cdots \cup G_n$  is nonempty for each  $n$ , but  $\cap F_n$  is empty.
- (4) Let  $E$  be a set contains a point from each  $F_n$ .  $E$  is infinite and thus  $E$  has a limit point, say  $p$ .  $p \in G_n$  for some  $n$  since  $\mathcal{G} = \{G_n\}$  is an open covering of  $X$ . Since  $G_n$  is open, there is an open neighborhood  $B(p)$  of  $p$  such that  $B(p) \subseteq G_n$ . By the construction of  $F_n$ ,

$$B(p) \cap F_m = \emptyset$$

whenever  $m \geq n$ , contrary to the assumption that  $p$  is a limit point of  $E$ .

Hence,  $X$  is compact if  $X$  is limit point compact.  $\square$

### Supplement.

- (1) Lindelof space is a topological space in which every open covering has a countable subcovering.

- (2) Show that  $X$  is second-countable if  $X$  is Lindelof. Same as the Proof (Hint) of Exercise 2.25 except changing the word “compact” to “Lindelof” and “finite” to “countable.”  $\square$
- (3) In every metric space, we have

$$\{\text{compact}\} \iff \{\text{limit point compact}\} \iff \{\text{sequentially compact}\}.$$

**Exercise 2.27.** Define a point  $p$  in a metric space  $X$  to be a condensation point of a set  $E \subseteq X$  if every neighborhood of  $p$  contains uncountably many points of  $E$ .

Suppose  $E \subseteq \mathbb{R}^k$ ,  $E$  is uncountable, and let  $P$  be the set of all condensation points of  $E$ . Prove that  $P$  is perfect and that at most countably many points of  $E$  are not in  $P$ . In other words, show that  $\tilde{P} \cap E$  is at most countable.

(Hint: Let  $\{V_n\}$  be a countable base of  $\mathbb{R}^k$ , let  $W$  be the union of those  $V_n$  for which  $E \cap V_n$  is at most countable, and show that  $P = \tilde{W}$ .)

**Exercise 2.28.** Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is at most countable. (Corollary: Every countable closed set in  $\mathbb{R}^k$  has isolated points.) (Hint: Use Exercise 2.27.)

**Exercise 2.29.** Prove that every open set in  $\mathbb{R}^1$  is the union of an at most countable collection of disjoint segments. (Hint: Use Exercise 2.22.)

**Exercise 2.30.** Imitate the proof of Theorem 2.43 to obtain the following result:

If  $\mathbb{R}^k = \bigcup_{n=1}^{\infty} F_n$ , where each  $F_n$  is a closed subset of  $\mathbb{R}^k$ , then at least one  $F_n$  has a nonempty interior.

Equivalent statement: If  $G_n$  is a dense open subset of  $\mathbb{R}^k$ , for  $n = 1, 2, 3, \dots$ , then  $\bigcup_{n=1}^{\infty} G_n$  is not empty (in fact, it is dense in  $\mathbb{R}^k$ ).

(This is a special case of Baire’s theorem; see Exercise 3.22 for the general case.)