

Chapter 5: Differentiation

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Exercise 5.1. Let f be defined for all real x , and suppose that

$$|f(x) - f(y)| \leq (x - y)^2$$

for all real x and y . Prove that f is a constant.

Proof.

(1) Write

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq |x - y|$$

if $x \neq y$.

(2) Given any $y \in \mathbb{R}$,

$$\left| \frac{f(x) - f(y)}{x - y} \right| \rightarrow 0 \quad \text{as } x \rightarrow y,$$

or $|f'(y)| = 0$.

(3) Or using ε - δ argument. Fix $y \in \mathbb{R}$. Given any $\varepsilon > 0$, there exists $\delta = \varepsilon > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} - 0 \right| \leq |x - y| < \delta = \varepsilon$$

whenever $|x - y| < \delta$. That is, $|f'(y)| = 0$.

(4) So $f'(y) = 0$ for any $y \in \mathbb{R}$. By Theorem 5.11 (b), f is a constant.

□

Exercise 5.2. Suppose $f'(x) > 0$ in (a, b) . Prove that f is strictly increasing in (a, b) , and let g be its inverse function. Prove that g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)} \quad (a < x < b).$$

Proof. Let $E = (a, b)$.

- (1) Theorem 5.10 implies that for any $a < p < q < b$ there exists $\xi \in (p, q)$ such that

$$f(p) - f(q) = (p - q)f'(\xi).$$

Since $\xi \in (p, q) \subseteq E$, by assumption $f'(\xi) > 0$. Hence $f(p) - f(q) = (p - q)f'(\xi) < 0$ (here $p - q < 0$), or

$$f(p) < f(q)$$

if $p < q$. Therefore, f is strictly increasing in (a, b) .

- (2) Show that f is one-to-one in E if f is strictly increasing in E . If $f(p) = f(q)$, then it cannot be $p > q$ or $p < q$ ((1)). So that $p = q$, or f is injective.
- (3) Show that g is well-defined. Theorem 5.2 and Theorem 4.17.
- (4) Show that $g'(f(x)) = \frac{1}{f'(x)}$. Given $y \in f(E)$, say $y = f(x)$ for some $x \in E$. Given any $s \in f(E)$ with $s \neq y$. Here $s = f(t)$ for some $t \in E$ and $t \neq x$.

$$\begin{aligned} \lim_{s \rightarrow y} \frac{g(s) - g(y)}{s - y} &= \lim_{f(t) \rightarrow f(x)} \frac{g(f(t)) - g(f(x))}{f(t) - f(x)} \\ &= \lim_{t \rightarrow x} \frac{t - x}{f(t) - f(x)} \\ &= \lim_{t \rightarrow x} \frac{1}{\frac{f(t) - f(x)}{t - x}} \\ &= \frac{1}{f'(x)}. \end{aligned} \quad (f' > 0)$$

Here $s \rightarrow y$ if and only if $t \rightarrow x$ since both f and g are continuous and one-to-one. Hence g is differentiable and $g'(f(x)) = \frac{1}{f'(x)}$.

□

Exercise 5.3. Suppose g is a real function on \mathbb{R}^1 , with bounded derivative (say $|g'| \leq M$). Fix $\varepsilon > 0$, and define $f(x) = x + \varepsilon g(x)$. Prove that f is one-to-one if ε is small enough. (A set of admissible values of ε can be determined which depends only on M .)

Proof.

- (1) Note that $f'(x) = 1 + \varepsilon g'(x)$ (Theorem 5.3). Since $|g'| \leq M$,

$$1 - \varepsilon M \leq f'(x) \leq 1 + \varepsilon M.$$

- (2) Pick

$$\varepsilon = \frac{1}{M + 1} > 0.$$

Thus,

$$f'(x) \geq \frac{1}{M+1} > 0.$$

By Exercise 5.2, $f(x)$ is strictly increasing in \mathbb{R} or one-to-one in \mathbb{R} .

□

Exercise 5.4. *If*

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$

where C_0, \dots, C_n are real constants, prove that the equation

$$C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n = 0$$

has at least one real root between 0 and 1.

Proof. Let

$$g(x) = C_0x + \frac{C_1}{2}x^2 + \cdots + \frac{C_{n-1}}{n}x^n + \frac{C_n}{n+1}x^{n+1} \in \mathbb{R}[x].$$

Then $g(0) = g(1) = 0$, and $g'(x) = C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n$. By the mean value theorem (Theorem 5.10), there exists a point $\xi \in (0, 1)$ at which

$$g(1) - g(0) = g'(\xi)(1 - 0),$$

or $g'(\xi) = 0$. That is, there exists a real root $x = \xi$ between 0 and 1 at which $C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n = 0$. □

Exercise 5.5. *Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \rightarrow 0$ as $x \rightarrow +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \rightarrow 0$ as $x \rightarrow +\infty$.*

Proof. Given any $x > 0$. Since f is differentiable for every $x > 0$, f is differentiable on $[x, x+1]$. By Theorem 5.2 and Theorem 5.10 (the mean value theorem), there is a point $\xi \in (x, x+1)$ at which

$$f(x+1) - f(x) = [(x+1) - x]f'(\xi)$$

or

$$g(x) = f'(\xi).$$

As $x \rightarrow +\infty$, $\xi \rightarrow +\infty$. Hence

$$\lim_{x \rightarrow +\infty} g(x) = \lim_{\xi \rightarrow +\infty} f'(\xi) = 0.$$

□

Exercise 5.6. Suppose

- (a) f is continuous for $x \geq 0$,
- (b) $f'(x)$ exists for $x > 0$,
- (c) $f(0) = 0$,
- (d) f' is monotonically increasing.

Put

$$g(x) = \frac{f(x)}{x} \quad (x > 0)$$

and prove that g is monotonically increasing.

Proof.

- (1) It suffices to show that $g'(x) \geq 0$ for $x > 0$ (Theorem 5.11(a)), that is, to show that

$$g'(x) = \frac{xf'(x) - f(x)}{x^2} \geq 0 \quad (x > 0),$$

or

$$xf'(x) - f(x) \geq 0 \quad (x > 0)$$

since $x^2 > 0$ for all nonzero x .

- (2) Given $x > 0$. By (a)(b), we apply the mean value theorem (Theorem 5.10) on f to get

$$f(x) - f(0) = (x - 0)f'(\xi)$$

for some $\xi \in (0, x)$. By (c),

$$f(x) = xf'(\xi).$$

By (d),

$$f(x) = xf'(\xi) \leq xf'(x).$$

Hence $xf'(x) - f(x) \geq 0$, or g is monotonically increasing.

□

Note. g is increasing strictly if f is increasing strictly.

Exercise 5.7. Suppose $f'(x)$, $g'(x)$ exist, $g'(x) \neq 0$, and $f(x) = g(x) = 0$. Prove that

$$\lim_{t \rightarrow x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}.$$

(This holds also for complex functions.)

Proof.

$$\begin{aligned}
 \frac{f'(t)}{g'(t)} &= \frac{\lim_{t \rightarrow x} \frac{f(t)-f(x)}{t-x}}{\lim_{t \rightarrow x} \frac{g(t)-g(x)}{t-x}} \\
 &= \lim_{t \rightarrow x} \frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}} && \text{(Both limits exist and } g' \neq 0) \\
 &= \lim_{t \rightarrow x} \frac{f(t)}{g(t)}. && (f(x) = g(x) = 0)
 \end{aligned}$$

This proof is also true for complex functions. \square

Exercise 5.8. Suppose $f'(x)$ is continuous on $[a, b]$ and $\varepsilon > 0$. Prove that there exists $\delta > 0$ such that

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \varepsilon$$

whenever $0 < |t - x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$. (This could be expressed by saying f is **uniformly differentiable** on $[a, b]$ if f' is continuous on $[a, b]$.) Does this hold for vector-valued functions too?

Proof.

- (1) Since $f'(x)$ is continuous on a compact set $[a, b]$, $f'(x)$ is uniformly continuous on $[a, b]$. So given any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f'(t) - f'(x)| < \varepsilon$$

whenever $0 < |t - x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$.

- (2) For such $t < x$ in (1), by the mean value theorem (Theorem 5.10), there exists a point $\xi \in (t, x)$ at which

$$f'(\xi) = \frac{f(t) - f(x)}{t - x}.$$

Note that ξ is also satisfying $0 < |t - \xi| < |t - x| < \delta$ and $a \leq \xi \leq b$. Hence by (1) we also have

$$|f'(\xi) - f'(x)| < \varepsilon,$$

or

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \varepsilon.$$

- (3) Suppose $\mathbf{f}'(x)$ is continuous on $[a, b]$ and $\varepsilon > 0$. Prove that there exists $\delta > 0$ such that

$$\left| \frac{\mathbf{f}(t) - \mathbf{f}(x)}{t - x} - \mathbf{f}'(x) \right| < \varepsilon$$

whenever $0 < |t - x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$.

- (a) Write

$$\mathbf{f}(x) = (f_1(x), \dots, f_k(x)) \in \mathbb{R}^k.$$

By Remarks 5.16, $\mathbf{f}(x)$ is differentiable at a point x if and only if each f_1, \dots, f_k is differentiable at x . So that

$$\mathbf{f}'(x) = (f'_1(x), \dots, f'_k(x)) \in \mathbb{R}^k.$$

By Theorem 4.10, $\mathbf{f}'(x)$ is continuous if and only if each f_1, \dots, f_k is continuous.

- (b) Similar to (1)(2), Since $f'_i(x)$ is continuous on a compact set $[a, b]$ where $1 \leq i \leq k$, $f'_i(x)$ is uniformly continuous on $[a, b]$. So given any $\varepsilon > 0$ there exists $\delta_i > 0$ such that

$$|f'_i(t) - f'_i(x)| < \frac{\varepsilon}{\sqrt{k}}$$

whenever $0 < |t - x| < \delta_i$, $a \leq x \leq b$, $a \leq t \leq b$. Take $\delta = \min_{1 \leq i \leq k} \delta_i > 0$.

- (c) For such $t < x$ in (1), by the mean value theorem (Theorem 5.10), there exists a point $\xi_i \in (t, x)$ at which

$$f'_i(\xi_i) = \frac{f_i(t) - f_i(x)}{t - x}.$$

Note that ξ_i is also satisfying $0 < |t - \xi_i| < |t - x| < \delta$ and $a \leq \xi_i \leq b$. Hence by (1) we also have

$$|f'_i(\xi_i) - f'_i(x)| < \frac{\varepsilon}{\sqrt{k}},$$

or

$$\left| \frac{f_i(t) - f_i(x)}{t - x} - f'_i(x) \right| < \frac{\varepsilon}{\sqrt{k}}.$$

- (d) Hence

$$\left| \frac{\mathbf{f}(t) - \mathbf{f}(x)}{t - x} - \mathbf{f}'(x) \right| = \left(\sum_{i=1}^k \left| \frac{f_i(t) - f_i(x)}{t - x} - f'_i(x) \right|^2 \right)^{\frac{1}{2}} < \varepsilon.$$

□

Exercise 5.9. Let f be a continuous real function on \mathbb{R}^1 , of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \rightarrow 3$ as $x \rightarrow 0$. Does it follow that $f'(0)$ exists?

Proof.

- (1) Show that $f'(0) = 3$. It is equivalent to show that

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = 3.$$

Write $F(x) = f(x) - f(0)$ and $G(x) = x - 0$ on \mathbb{R}^1 . So that

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{F(x)}{G(x)} = 0.$$

- (2) Note that

$$\lim_{x \rightarrow 0} \frac{F'(x)}{G'(x)} = \lim_{x \rightarrow 0} \frac{f'(x)}{1} = 3.$$

- (3) Since f is continuous on \mathbb{R}^1 , F is continuous on \mathbb{R}^1 . Hence

$$\lim_{x \rightarrow 0} F(x) = F(\lim_{x \rightarrow 0} x) = F(0) = 0.$$

Also, G is continuous on \mathbb{R}^1 implies that

$$\lim_{x \rightarrow 0} G(x) = G(\lim_{x \rightarrow 0} x) = G(0) = 0.$$

- (4) Apply L'Hospital's rule (Theorem 5.13) to (2)(3), we have

$$\lim_{x \rightarrow 0} \frac{F(x)}{G(x)} = 3,$$

or $f'(0) = 3$.

□

Exercise 5.10. Suppose f and g are complex differentiable functions on $(0, 1)$, $f(x) \rightarrow 0$, $g(x) \rightarrow 0$, $f'(x) \rightarrow A$, $g'(x) \rightarrow B$ as $x \rightarrow 0$, where A and B are complex numbers, $B \neq 0$. Prove that

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \frac{A}{B}.$$

Compare with Example 5.18. (Hint:

$$\frac{f(x)}{g(x)} = \left(\frac{f(x)}{x} - A \right) \frac{x}{g(x)} + A \frac{x}{g(x)}.$$

Apply Theorem 5.13 to the real and imaginary parts of $\frac{f(x)}{x}$ and $\frac{g(x)}{x}$.)

Proof (Hint).

(1) Write

$$f(x) = f_1(x) + if_2(x)$$

for $x \in (0, 1)$, where both f_1 and f_2 are real functions. By Remarks 5.16, it is clear that

$$f'(x) = f'_1(x) + if'_2(x).$$

(2) Write

$$A = A_1 + iA_2$$

where both A_1 and A_2 are real numbers. Then as $x \rightarrow 0$, we have

(a) $f(x) \rightarrow 0$ if and only if $f_1(x) \rightarrow 0$ and $f_2(x) \rightarrow 0$.

(b) $f'(x) \rightarrow A$ if and only if $f'_1(x) \rightarrow A_1$ and $f'_2(x) \rightarrow A_2$.

Hence by L'Hospital's rule (Theorem 5.13),

$$\lim_{x \rightarrow 0} \frac{f_i(x)}{x} = \lim_{x \rightarrow 0} \frac{f'_i(x)}{1} = A_i$$

($i = 1, 2$) or

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{f(x)}{x} &= \lim_{x \rightarrow 0} \frac{f_1(x) + if_2(x)}{x} \\ &= \lim_{x \rightarrow 0} \frac{f_1(x)}{x} + i \lim_{x \rightarrow 0} \frac{f_2(x)}{x} \\ &= A_1 + iA_2 \\ &= A. \end{aligned}$$

Similarly,

$$\lim_{x \rightarrow 0} \frac{g(x)}{x} = B.$$

Note that $B \neq 0$, and thus

$$\lim_{x \rightarrow 0} \frac{x}{g(x)} = \frac{1}{B}.$$

(3) Hence

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow 0} \left[\left(\frac{f(x)}{x} - A \right) \frac{x}{g(x)} + A \frac{x}{g(x)} \right] \\ &= \lim_{x \rightarrow 0} \left(\frac{f(x)}{x} - A \right) \cdot \lim_{x \rightarrow 0} \frac{x}{g(x)} + \lim_{x \rightarrow 0} A \frac{x}{g(x)} \\ &= 0 \cdot \frac{1}{B} + \frac{A}{B} \\ &= \frac{A}{B}. \end{aligned}$$

- (4) *Compare with Example 5.18.* Define $f(x) = x$ and $g(x) = x + x^2 \exp(\frac{i}{x^2})$ as in Example 5.18. Note that $f(x) \rightarrow 0$, $g(x) \rightarrow 0$, $f'(x) \rightarrow 1$ and $g'(x) \rightarrow \infty$ as $x \rightarrow 0$. By Example 5.18

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = 1 \neq 0 = \frac{1}{\infty} = \lim_{x \rightarrow 0} \frac{A}{B}.$$

□

Exercise 5.11. Suppose f is defined in a neighborhood of x , and suppose $f''(x)$ exists. Show that

$$\lim_{h \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

Show by an example that the limit may exist even if $f''(x)$ does not. (Hint: Use Theorem 5.13.)

Proof (Theorem 5.13).

- (1) Write $F(h) = f(x+h) + f(x-h) - 2f(x)$ and $G(h) = h^2$. It is equivalent to show that

$$\lim_{h \rightarrow 0} \frac{F(h)}{G(h)} = f''(x).$$

We might apply Theorem 5.13 (L'Hospital rule) to get it.

- (2) Show that $\lim_{h \rightarrow 0} F(h) = 0$ and $\lim_{h \rightarrow 0} G(h) = 0$. It is clear that $\lim_{h \rightarrow 0} G(h) = \lim_{h \rightarrow 0} h^2 = 0$ since $x \mapsto x^2$ is continuous on \mathbb{R}^1 . Besides, since f is continuous at x (by applying Theorem 5.2 twice),

$$\lim_{h \rightarrow 0} F(h) = f(x) + f(x) - 2f(x) = 0.$$

- (3) Show that

$$\lim_{h \rightarrow 0} \frac{F'(h)}{G'(h)} = \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x-h)}{2h}$$

is well-defined. Since $f''(x)$ exists in a neighborhood $B(x; r)$ of x (where $r > 0$), $f'(x)$ exists and is continuous in $B(x; r)$ (Theorem 5.2). As $0 < |h| < \frac{r}{2}$,

$$x+h \in B\left(x+h; \frac{r}{2}\right) \subseteq B(x; r)$$

and

$$x-h \in B\left(x-h; \frac{r}{2}\right) \subseteq B(x; r).$$

So $f'(x+h)$ and $f'(x-h)$ exist in $B(x; r)$ as $0 < |h| < \frac{r}{2}$. Hence

$$\lim_{h \rightarrow 0} \frac{F'(h)}{G'(h)} = \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x-h)}{2h}$$

is well-defined (Theorem 5.3 and Theorem 5.5 (the chain rule)).

(4) Show that

$$\lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x-h)}{2h} = f''(x).$$

Since $f''(x)$ exists, by definition

$$\lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{h} = f''(x)$$

and

$$\lim_{h \rightarrow 0} \frac{f'(x-h) - f'(x)}{-h} = f''(x).$$

Sum up two expressions to get

$$2f''(x) = \lim_{h \rightarrow 0} \frac{f'(x-h) - f'(x-h)}{h}.$$

(5) By (2)(3)(4) and Theorem 5.13 (L'Hospital rule), the result is established.

(6) Given $f(x) = x|x|$ on \mathbb{R}^1 . Show that

$$\lim_{h \rightarrow 0} \frac{f(h) + f(-h) - 2f(0)}{h^2} = 0$$

but $f''(x)$ does not exist at $x = 0$. Clearly,

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(h) + f(-h) - 2f(0)}{h^2} &= \lim_{h \rightarrow 0} \frac{h|h| + (-h)|-h| - 2 \cdot 0}{h^2} \\ &= \lim_{h \rightarrow 0} \frac{h|h| - h|h| - 0}{h^2} \\ &= \lim_{h \rightarrow 0} 0 \\ &= 0. \end{aligned}$$

But $f''(x)$ does not exist by Exercise 5.12.

□

Exercise 5.12. If $f(x) = |x|^3$, compute $f'(x)$, $f''(x)$ for all real x , and show that $f^{(3)}(0)$ does not exist.

Proof.

(1) Write

$$f(x) = \begin{cases} x^3 & (x \geq 0), \\ -x^3 & (x \leq 0). \end{cases}$$

(2) Show that $f'(x) = 3x|x|$. It is trivial that

$$f'(x) = \begin{cases} 3x^2 & (x > 0), \\ -3x^2 & (x < 0). \end{cases}$$

Note that

$$\lim_{x \rightarrow 0} f'(x) = 0.$$

Apply the same argument in Exercise 5.9, we have

$$f'(0) = 0.$$

Hence f' exists and $f'(x) = 3x|x|$ for any $x \in \mathbb{R}$.

(3) Show that $f''(x) = 6|x|$. Similar to (2).

$$f''(x) = \begin{cases} 6x & (x > 0), \\ -6x & (x < 0). \end{cases}$$

Note that

$$\lim_{x \rightarrow 0} f''(x) = 0.$$

Apply the same argument in Exercise 5.9, we have

$$f''(0) = 0.$$

Hence f'' exists and $f''(x) = 6|x|$ for any $x \in \mathbb{R}$.

(4) Show that $f^{(3)}(0)$ does not exist.

$$f'''(x) = \begin{cases} 6 & (x > 0), \\ -6 & (x < 0). \end{cases}$$

There are some proofs for showing that $f^{(3)}(0)$ does not exist.

(a) Since

$$\lim_{t \rightarrow 0+} \frac{f''(t) - f''(0)}{t - 0} = \lim_{t \rightarrow 0+} \frac{6t}{t} = 6$$

and

$$\lim_{t \rightarrow 0-} \frac{f''(t) - f''(0)}{t - 0} = \lim_{t \rightarrow 0-} \frac{-6t}{t} = -6,$$

$f^{(3)}(0)$ does not exist.

(b) (Reductio ad absurdum) If f were differentiable on \mathbb{R}^1 , then

$$\lim_{t \rightarrow 0+} f'''(t) = 6$$

and

$$\lim_{t \rightarrow 0-} f'''(t) = -6,$$

or f''' has a simple discontinuity at $x = 0$, contrary to Corollary to Theorem 5.12.

□

Note. Given $k > 0$. We can construct one real function f on \mathbb{R}^1 , say

$$f(x) = \begin{cases} |x|^k & (k \text{ is odd}), \\ x|x|^{k-1} & (k > 0 \text{ is even}), \end{cases}$$

such that all $f^{(0)}(0) = \dots = f^{(k-1)}(0) = 0$ exist but $f^{(k)}(0)$ does not exist.

Exercise 5.13. Suppose a and c are real numbers, $c > 0$, and f is defined on $[-1, 1]$ by

$$f(x) = \begin{cases} x^a \sin(x^{-c}) & (\text{if } x \neq 0), \\ 0 & (\text{if } x = 0). \end{cases}$$

Prove the following statements:

- (a) f is continuous if and only if $a > 0$.
- (b) $f'(0)$ exists if and only if $a > 1$.
- (c) f' is bounded if and only if $a \geq 1 + c$.
- (d) f' is continuous if and only if $a > 1 + c$.
- (e) $f''(0)$ exists if and only if $a > 2 + c$.
- (f) f'' is bounded if and only if $a \geq 2 + 2c$.
- (g) f'' is continuous if and only if $a > 2 + 2c$.

Note that f is not well-defined as a real function if $x < 0$. Hence we modify the definition of f for the case $x < 0$:

$$f(x) = \begin{cases} |x|^a \sin(|x|^{-c}) & (\text{if } x \neq 0), \\ 0 & (\text{if } x = 0). \end{cases}$$

Proof of (a).

- (1) Since $|x|^a \sin(|x|^{-c})$ is continuous on $\mathbb{R}^1 - \{0\}$, f is continuous if and only if

$$\lim_{x \rightarrow 0} |x|^a \sin(|x|^{-c}) = 0.$$

- (2) Given $a > 0$. Show that

$$\lim_{x \rightarrow 0} |x|^a \sin(|x|^{-c}) = 0.$$

Since $|x|^a \rightarrow 0$ as $x \rightarrow 0$ and $|\sin(|x|^{-c})|$ is bounded by 1, the limit $\lim_{x \rightarrow 0} |x|^a \sin(|x|^{-c})$ exists and is equal to 0.

(3) Given $a = 0$. Show that

$$\lim_{x \rightarrow 0} |x|^a \sin(|x|^{-c}) = \lim_{x \rightarrow 0} \sin(|x|^{-c})$$

does not exist although $|x|^a \sin(|x|^{-c}) = \sin(|x|^{-c})$ is bounded on $[-1, 1] - \{0\}$.

(a) Take $x_n = (\frac{\pi}{2} + 2n\pi)^{-\frac{1}{c}} \neq 0$ for $n = 1, 2, 3, \dots$. The sequence $\{x_n\}$ converges to 0, and

$$\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} \sin(|x_n|^{-c}) = \lim_{n \rightarrow \infty} 1 = 1.$$

(b) Similarly, take $y_n = (2n\pi)^{-\frac{1}{c}} \neq 0$ for $n = 1, 2, 3, \dots$. The sequence $\{y_n\}$ converges to 0, and

$$\lim_{n \rightarrow \infty} f(y_n) = 0.$$

(c) By (a)(b), $\lim_{x \rightarrow 0} |x|^a \sin(|x|^{-c})$ does not exist (Theorem 4.2).

(d) Clearly, $|\sin(|x|^{-c})| \leq 1$ as $\sin(|x|^{-c})$ is well-defined.

(4) Given $a < 0$. Show that

$$\lim_{x \rightarrow 0} |x|^a \sin(|x|^{-c})$$

does not exist. Similar to (3), we take the same $\{x_n\}$ and $\{y_n\}$ as (3) to get the similar result:

$$\begin{aligned} \lim_{n \rightarrow \infty} f(x_n) &= \infty, \\ \lim_{n \rightarrow \infty} f(y_n) &= 0. \end{aligned}$$

By Theorem 4.2, $\lim_{x \rightarrow 0} |x|^a \sin(|x|^{-c})$ does not exist.

(5) By (2)(3)(4), f is continuous if and only if $a > 0$.

□

Proof of (b).

(1) By definition,

$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \operatorname{sgn}(x) |x|^{a-1} \sin(|x|^{-c}).$$

Here $\operatorname{sgn}(x)$ is the sign function defined by

$$\operatorname{sgn}(x) = \begin{cases} 1 & (x > 0), \\ 0 & (x = 0), \\ -1 & (x < 0). \end{cases}$$

- (2) Similar to (2)(3)(4) in the proof of (a), $f'(0) = 0$ exists if and only if $a - 1 > 0$.

□

Proof of (c).

- (1) Write $E = [-1, 1] - \{0\}$. f' is bounded if and only if $f'(0)$ exists and f' is bounded on E .
- (2) Given any $x \in E$,

$$\begin{aligned} f'(x) &= \operatorname{sgn}(x) (a|x|^{a-1} \sin(|x|^{-c}) + |x|^a \cos(|x|^{-c})(-c)|x|^{-c-1}) \\ &= \operatorname{sgn}(x)|x|^{a-c-1} (a|x|^c \sin(|x|^{-c}) - c \cos(|x|^{-c})). \end{aligned}$$

- (3) *Given $a - c - 1 \geq 0$. Show that f' is bounded on E .* Since $\operatorname{sgn}(x)$ is bounded by 1 on E , $|x|^{a-c-1}$ is bounded by 1 on E and $a|x|^c \sin(|x|^{-c}) - c \cos(|x|^{-c})$ is bounded by $|a| + |c|$ on E , f' is bounded on E .
- (4) *Given $a - c - 1 < 0$. Show that f' is unbounded on E .* Take $x_n = (2n\pi)^{-\frac{1}{c}} \neq 0$ for $n = 1, 2, 3, \dots$. The sequence $\{x_n\}$ converges to 0, and

$$\lim_{n \rightarrow \infty} f'(x_n) = \lim_{n \rightarrow \infty} -c(2n\pi)^{-\frac{a-c-1}{c}} = -\infty.$$

- (5) By (b), $f'(0)$ exists if and only if $a > 1$. By (3)(4), f' is bounded on E if and only if $a - c - 1 \geq 0$. Since $c > 0$, f' is bounded on $[-1, 1]$ if and only if $a - c - 1 \geq 0$.

□

Proof of (d). Similar to the proof of (a).

- (1) Write $E = [-1, 1] - \{0\}$. By (b)(c),

$$f'(x) = \begin{cases} 0 & \text{if } x = 0, \\ \operatorname{sgn}(x)|x|^{a-c-1} (a|x|^c \sin(|x|^{-c}) - c \cos(|x|^{-c})) & \text{if } x \in E. \end{cases}$$

Clearly, f' is continuous on E . Hence, f' is continuous if and only if $\lim_{x \rightarrow 0} f'(x) = f'(0) = 0$.

- (2) *Given $a - c - 1 > 0$. Show that $\lim_{x \rightarrow 0} f'(x) = 0$.* Since $|x|^{a-c-1} \rightarrow 0$ as $x \rightarrow 0$, $\operatorname{sgn}(x)$ is bounded by 1 on E , and $a|x|^c \sin(|x|^{-c}) - c \cos(|x|^{-c})$ is bounded by $|a| + |c|$ on E ,

$$\operatorname{sgn}(x)|x|^{a-c-1} (a|x|^c \sin(|x|^{-c}) - c \cos(|x|^{-c})) \rightarrow 0$$

as $x \rightarrow 0$. The result is established.

(3) Given $a - c - 1 = 0$. Show that $\lim_{x \rightarrow 0} f'(x)$ does not exist.

(a) Take $x_n = (\frac{\pi}{2} + 2n\pi)^{-\frac{1}{c}} \neq 0$ for $n = 1, 2, 3, \dots$. The sequence $\{x_n\}$ converges to 0, and

$$\begin{aligned}\lim_{n \rightarrow \infty} f'(x_n) &= \lim_{n \rightarrow \infty} \operatorname{sgn}(x_n) (a|x_n|^c \sin(|x_n|^{-c}) - c \cos(|x_n|^{-c})) \\ &= \lim_{n \rightarrow \infty} \frac{a}{\frac{\pi}{2} + 2n\pi} \\ &= 0.\end{aligned}$$

(b) Similarly, take $y_n = (2n\pi)^{-\frac{1}{c}} \neq 0$ for $n = 1, 2, 3, \dots$. The sequence $\{y_n\}$ converges to 0, and

$$\begin{aligned}\lim_{n \rightarrow \infty} f'(y_n) &= \lim_{n \rightarrow \infty} \operatorname{sgn}(y_n) (a|y_n|^c \sin(|y_n|^{-c}) - c \cos(|y_n|^{-c})) \\ &= \lim_{n \rightarrow \infty} -c \\ &= -c \neq 0.\end{aligned}$$

(c) By (a)(b), $\lim_{x \rightarrow 0} f'(x)$ does not exist (Theorem 4.2).

(4) Given $a - c - 1 < 0$. Show that $\lim_{x \rightarrow 0} f'(x)$ does not exist. It is the same as (4) in the proof of (c).

(5) By (2)(3)(4), f' is continuous if and only if $\lim_{x \rightarrow 0} f'(x) = 0$ if and only if $a - c - 1 > 0$.

□

Proof of (e). Similar to the proof of (b).

(1) Write $E = [-1, 1] - \{0\}$. By the proof of (d),

$$f'(x) = \begin{cases} 0 & \text{if } x = 0, \\ \operatorname{sgn}(x)|x|^{a-c-1} (a|x|^c \sin(|x|^{-c}) - c \cos(|x|^{-c})) & \text{if } x \in E. \end{cases}$$

By definition

$$\begin{aligned}f''(0) &= \lim_{x \rightarrow 0} \frac{f'(x) - f'(0)}{x - 0} \\ &= \lim_{x \rightarrow 0} |x|^{a-c-2} (a|x|^c \sin(|x|^{-c}) - c \cos(|x|^{-c})).\end{aligned}$$

(Here $\operatorname{sgn}(x)^2 = 1$ if $x \neq 0$.)

(2) Similar to (2)(3)(4) in the proof of (d), $f''(0) = 0$ exists if and only if $(a - c - 1) - 1 = a - c - 2 > 0$.

□

Proof of (f). Similar to the proof of (c).

(1) Write $E = [-1, 1] - \{0\}$. f'' is bounded if and only if $f''(0)$ exists and f'' is bounded on E .

(2) Given any $x \in E$,

$$f''(x) = |x|^{a-2c-2} \cdot [(a(a-1)|x|^{2c} - c^2) \sin(|x|^{-c}) - c(2a-c-1)|x|^c \cos(|x|^{-c})].$$

(3) Given $a - 2c - 2 \geq 0$. Show that f'' is bounded on E . Since $|x|^{a-2c-2}$ is bounded by 1 on E and

$$\begin{aligned} & |(a(a-1)|x|^{2c} - c^2) \sin(|x|^{-c}) - c(2a-c-1)|x|^c \cos(|x|^{-c})| \\ & \leq |a(a-1)| + |c^2| + |c(2a-c-1)| \end{aligned}$$

is bounded on E , f'' is bounded on E .

(4) Given $a - 2c - 2 < 0$. Show that f'' is unbounded on E . Take $x_n = (\frac{\pi}{2} + 2n\pi)^{-\frac{1}{c}} \neq 0$ for $n = 1, 2, 3, \dots$. The sequence $\{x_n\}$ converges to 0, and

$$\begin{aligned} & \lim_{n \rightarrow \infty} f''(x_n) \\ &= \lim_{n \rightarrow \infty} \underbrace{\left(a(a-1) \left(\frac{\pi}{2} + 2n\pi \right)^{-2} - c^2 \right)}_{\rightarrow -c^2 \neq 0} \underbrace{\left(\frac{\pi}{2} + 2n\pi \right)^{-\frac{a-2c-2}{c}}}_{\rightarrow \infty} \\ &= -\infty. \end{aligned}$$

(5) By (e), $f''(0)$ exists if and only if $a - c - 2 > 0$. By (3)(4), f'' is bounded on E if and only if $a - 2c - 2 \geq 0$. Since $c > 0$, f'' is bounded on $[-1, 1]$ if and only if $a - 2c - 2 \geq 0$.

□

Proof of (g). Similar to the proof of (a) or (d).

(1) Write $E = [-1, 1] - \{0\}$. By (e)(f),

$$f''(x) = \begin{cases} 0 & \text{if } x = 0, \\ |x|^{a-2c-2} [(a(a-1)|x|^{2c} - c^2) \sin(|x|^{-c}) - c(2a-c-1)|x|^c \cos(|x|^{-c})] & \text{if } x \in E. \end{cases}$$

Clearly, f'' is continuous on E . Hence, f'' is continuous if and only if $\lim_{x \rightarrow 0} f''(x) = f''(0) = 0$.

(2) Given $a - 2c - 2 > 0$. Show that $\lim_{x \rightarrow 0} f''(x) = 0$. Since $|x|^{a-2c-2} \rightarrow 0$ as $x \rightarrow 0$ and

$$(a(a-1)|x|^{2c} - c^2) \sin(|x|^{-c}) - c(2a-c-1)|x|^c \cos(|x|^{-c})$$

is bounded by $|a(a-1)| + |c^2| + |c(2a-c-1)|$ on E ,

$$|x|^{a-2c-2} \cdot [(a(a-1)|x|^{2c} - c^2) \sin(|x|^{-c}) - c(2a-c-1)|x|^c \cos(|x|^{-c})] \rightarrow 0$$

as $x \rightarrow 0$. The result is established.

(3) Given $a - 2c - 2 = 0$. Show that $\lim_{x \rightarrow 0} f''(x)$ does not exist.

(a) Take $x_n = \left(\frac{\pi}{2} + 2n\pi\right)^{-\frac{1}{c}} \neq 0$ for $n = 1, 2, 3, \dots$. The sequence $\{x_n\}$ converges to 0, and

$$\begin{aligned} & \lim_{n \rightarrow \infty} f''(x_n) \\ &= \lim_{n \rightarrow \infty} (a(a-1)|x_n|^{2c} - c^2) \sin(|x_n|^{-c}) - c(2a-c-1)|x_n|^c \cos(|x_n|^{-c}) \\ &= \lim_{n \rightarrow \infty} \frac{a(a-1)}{\left(\frac{\pi}{2} + 2n\pi\right)^2} - c^2 \\ &= -c^2 \end{aligned}$$

(b) Similarly, take $y_n = \left(\frac{3\pi}{2} + 2n\pi\right)^{-\frac{1}{c}} \neq 0$ for $n = 1, 2, 3, \dots$. The sequence $\{y_n\}$ converges to 0, and

$$\begin{aligned} & \lim_{n \rightarrow \infty} f''(y_n) \\ &= \lim_{n \rightarrow \infty} (a(a-1)|y_n|^{2c} - c^2) \sin(|y_n|^{-c}) - c(2a-c-1)|y_n|^c \cos(|y_n|^{-c}) \\ &= \lim_{n \rightarrow \infty} -\frac{a(a-1)}{\left(\frac{3\pi}{2} + 2n\pi\right)^2} + c^2 \\ &= c^2. \end{aligned}$$

(c) By (a)(b), $\lim_{x \rightarrow 0} f''(x)$ does not exist (Theorem 4.2).

(4) Given $a - 2c - 2 < 0$. Show that $\lim_{x \rightarrow 0} f''(x)$ does not exist. It is the same as (4) in the proof of (f).

(5) By (2)(3)(4), f'' is continuous if and only if $\lim_{x \rightarrow 0} f''(x) = 0$ if and only if $a - 2c - 2 > 0$.

□

Exercise 5.14. Let f be a differentiable real function defined in (a, b) . Prove that f is convex if and only if f' is monotonically increasing. Assume next $f''(x)$ exists for every $x \in (a, b)$, and prove that f is convex if and only if $f''(x) \geq 0$ for all $x \in (a, b)$.

Proof.

(1) Show that f' is monotonically increasing if f is convex.

(a) Since f is convex, by definition (Exercise 4.23)

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

whenever $a < x < b$, $a < y < b$, $0 < \lambda < 1$.

(b) As $x \neq y$, we have

$$\begin{aligned} f(y) - f(x) &\geq \frac{f(x + \lambda(y - x)) - f(x)}{\lambda} \\ &= \frac{f(x + \lambda(y - x)) - f(x)}{\lambda(y - x)} \cdot (y - x) \end{aligned}$$

and let $\lambda \rightarrow 0$ to get

$$f(y) - f(x) \geq f'(x)(y - x)$$

(since $f'(x)$ exists). Similarly, we have

$$f(x) - f(y) \geq f'(y)(x - y).$$

(c) Given any $y > x$, we have

$$f'(y)(y - x) \geq f(y) - f(x) \geq f'(x)(y - x).$$

Hence $f'(y) \geq f'(x)$ whenever $y > x$, or f' is monotonically increasing.

(2) Show that f is convex if f' is monotonically increasing. Given any $y > x$ and any $0 < \lambda < 1$.

(a) By Theorem 5.10 (the mean value theorem), there is a point $x < \xi < y$ such that

$$f(y) - f(x) = f'(\xi)(y - x).$$

Since f' is monotonically increasing,

$$f'(y)(y - x) \geq f(y) - f(x) \geq f'(x)(y - x).$$

(b) Write $z = \lambda x + (1 - \lambda)y$. Hence

$$\begin{aligned} f(y) - f(z) &\geq f'(z)(y - z), \\ f(z) - f(x) &\leq f'(z)(z - x), \end{aligned}$$

or

$$\begin{aligned} f(y) &\geq f(z) + f'(z)(y - z), \\ f(x) &\geq f(z) + f'(z)(x - z), \end{aligned}$$

or

$$\begin{aligned}\lambda f(x) + (1 - \lambda)f(y) &\geq \lambda[f(z) + f'(z)(x - z)] \\ &\quad + (1 - \lambda)[f(z) + f'(z)(y - z)] \\ &= f(z) \\ &= f(\lambda x + (1 - \lambda)y).\end{aligned}$$

Hence f is convex.

- (3) Show that $f''(x) \geq 0$ if f is convex and f'' exists. By (1), f' is monotonically increasing since f is convex. Given any $x \neq y$, we have

$$\frac{f'(y) - f'(x)}{y - x} \geq 0.$$

Let $y \rightarrow x$, we have $f''(x) \geq 0$ if f'' exists.

- (4) Show that f is convex if f'' exists and $f''(x) \geq 0$. By Theorem 5.11(a), f' is monotonically increasing. By (2), f is convex.

□

Exercise 5.15 (Landau-Kolmogorov inequality on the half-line). Suppose $a \in \mathbb{R}^1$, f is a twice-differentiable real function on (a, ∞) , and M_0, M_1, M_2 are the least upper bounds of $|f(x)|, |f'(x)|, |f''(x)|$, respectively, on (a, ∞) . Prove that

$$M_1^2 \leq 4M_0M_2.$$

(Hint: If $h > 0$, Taylor's theorem shows that

$$f'(x) = \frac{1}{2h}[f(x + 2h) - f(x)] - hf''(\xi)$$

for some $\xi \in (x, x + 2h)$. Hence

$$|f'(x)| \leq hM_2 + \frac{M_0}{h}.)$$

To show that $M_1^2 = 4M_0M_2$ can actually happen, take $a = -1$, define

$$f(x) = \begin{cases} 2x^2 - 1 & (-1 < x < 0), \\ \frac{x^2 - 1}{x^2 + 1} & (0 \leq x < \infty), \end{cases}$$

and show that $M_0 = 1, M_1 = 4, M_2 = 4$. Does $M_1^2 \leq 4M_0M_2$ hold for vector-valued functions too?

Note.

(1) Write

$$M_1 \leq 2M_0^{\frac{1}{2}}M_2^{\frac{1}{2}}.$$

2 is called the Landau-Kolmogorov constant, which is the best possible by the above example.

(2) In general, suppose $a \in \mathbb{R}^1$, f is a n th differentiable real function on (a, ∞) , and M_0, M_k, M_n are the least upper bounds of $|f(x)|, |f^{(k)}(x)|, |f^{(n)}(x)|$, respectively, on (a, ∞) where $1 \leq k < n$. Then

$$M_k \leq C(n, k)M_0^{1-\frac{k}{n}}M_n^{\frac{k}{n}}.$$

Proof.

(1) Consider some trivial cases.

- (a) If $M_0 = 0$, then $f(x) = 0$ on $(a, +\infty)$. So that $f'(x) = f''(x) = 0$ on $(a, +\infty)$, or $M_1 = M_2 = 0$. The inequality holds.
- (b) If $M_2 = 0$, then $f''(x) = 0$ on $(a, +\infty)$. So that $f'(x) = \alpha$ for some constant $\alpha \in \mathbb{R}^1$ (Theorem 5.11(b)), and $f(x) = \alpha x + \beta$ for some constant $\beta \in \mathbb{R}^1$ (by applying Theorem 5.11(b) to $x \mapsto f(x) - \alpha x$). Hence $M_1 = |\alpha|$ and

$$M_0 = \begin{cases} +\infty & (\alpha \neq 0), \\ |\beta| & (\alpha = 0). \end{cases}$$

In any case, the inequality holds.

- (c) If $M_0 = +\infty$ and $M_2 \neq 0$, there is nothing to do.
 - (d) If $M_2 = +\infty$ and $M_0 \neq 0$, there is nothing to do.
- (2) By (1), we suppose that $0 < M_0 < +\infty$ and $0 < M_2 < +\infty$. Given $x \in (a, +\infty)$ and $h > 0$. By Taylor's theorem (Theorem 5.15):

$$f(x+2h) = f(x) + 2hf'(x) + 2h^2f''(\xi)$$

for some $\xi \in (x, x+2h) \subseteq (a, +\infty)$. Thus

$$\begin{aligned} 2h|f'(x)| &\leq |f(x+2h)| + |f(x)| + 2h^2|f''(\xi)| \\ &\leq 2M_0 + 2h^2M_2, \\ |f'(x)| &\leq \frac{M_0}{h} + hM_2 \end{aligned}$$

holds for all $h > 0$. In particular, take

$$h = \sqrt{\frac{M_0}{M_2}}$$

to get

$$|f'(x)| \leq 2\sqrt{M_0 M_2}.$$

Thus $2\sqrt{M_0 M_2}$ is an upper bound of $|f'(x)|$ for all $x \in (a, +\infty)$. Hence

$$M_1 \leq 2\sqrt{M_0 M_2}$$

or

$$M_1^2 \leq 4M_0 M_2.$$

(3) Define

$$f(x) = \begin{cases} 2x^2 - 1 & (-1 < x < 0), \\ \frac{x^2 - 1}{x^2 + 1} & (0 \leq x < \infty). \end{cases}$$

Show that $M_0 = 1$, $M_1 = 4$, $M_2 = 4$. Similar to Exercise 5.12,

$$f'(x) = \begin{cases} 4x & (-1 < x \leq 0), \\ \frac{4x}{(x^2 + 1)^2} & (0 \leq x < \infty). \end{cases}$$

(Here $\lim_{x \rightarrow 0+} f'(x) = 0$ and $\lim_{x \rightarrow 0-} f'(x) = 0$. So $f'(0) = 0$ by Exercise 5.9.) Also,

$$f''(x) = \begin{cases} 4 & (-1 < x \leq 0), \\ \frac{-12x^2 + 4}{(x^2 + 1)^3} & (0 \leq x < \infty). \end{cases}$$

(Here $\lim_{x \rightarrow 0+} f''(x) = 4$ and $\lim_{x \rightarrow 0-} f''(x) = 4$. So $f''(0) = 4$ by Exercise 5.9.) Hence, $M_0 = 1$, $M_1 = 4$, $M_2 = 4$.

(4) Given

$$\mathbf{f}(x) = (f_1(x), \dots, f_k(x))$$

be a twice-differentiable vector-valued function from (a, ∞) to \mathbb{R}^k . and M_0 , M_1 , M_2 are the least upper bounds of $|\mathbf{f}(x)|$, $|\mathbf{f}'(x)|$, $|\mathbf{f}''(x)|$, respectively, on (a, ∞) . Show that

$$M_1^2 \leq 4M_0 M_2.$$

Similar to (1), we suppose that $0 < M_0 < +\infty$ and $0 < M_2 < +\infty$. Given any $\mathbf{v} = (v_1, \dots, v_k) \in \mathbb{R}^k$, $\mathbf{v} \cdot \mathbf{f}$ is a twice-differentiable real function on (a, ∞) . Similar to (2), Given $x \in (a, +\infty)$ and $h > 0$. By Taylor's theorem (Theorem 5.15):

$$(\mathbf{v} \cdot \mathbf{f})(x + 2h) = (\mathbf{v} \cdot \mathbf{f})(x) + 2h(\mathbf{v} \cdot \mathbf{f})'(x) + 2h^2(\mathbf{v} \cdot \mathbf{f})''(\xi)$$

for some $\xi \in (x, x + 2h) \subseteq (a, +\infty)$. Thus by the Schwarz inequality (Theorem 1.37(d))

$$\begin{aligned} 2h|(\mathbf{v} \cdot \mathbf{f})'(x)| &\leq |(\mathbf{v} \cdot \mathbf{f})(x + 2h)| + |(\mathbf{v} \cdot \mathbf{f})(x)| + 2h^2|(\mathbf{v} \cdot \mathbf{f})''(\xi)| \\ &\leq |\mathbf{v}||\mathbf{f}(x + 2h)| + |\mathbf{v}||\mathbf{f}(x)| + 2h^2|\mathbf{v}||\mathbf{f}''(\xi)| \\ &\leq (2M_0 + 2h^2 M_2)|\mathbf{v}|, \end{aligned}$$

$$|(\mathbf{v} \cdot \mathbf{f})'(x)| \leq \left(\frac{M_0}{h} + hM_2 \right) |\mathbf{v}|$$

holds for any \mathbf{v} and $h > 0$. In particular, we take

$$\mathbf{v} = \mathbf{f}'(y)$$

and

$$h = \sqrt{\frac{M_0}{M_2}}$$

to get

$$|\mathbf{f}'(x) \cdot \mathbf{f}'(y)| \leq 2\sqrt{M_0 M_2} |\mathbf{f}'(y)| \leq 2M_1 \sqrt{M_0 M_2}.$$

Note that x and y are arbitrary (in $(a, +\infty)$). In particular, we take $x = y$ to get

$$|\mathbf{f}'(x)|^2 \leq 2M_1 \sqrt{M_0 M_2}.$$

Thus $2M_1 \sqrt{M_0 M_2}$ is an upper bound of $|\mathbf{f}'(x)|^2$ for all $x \in (a, +\infty)$. Hence

$$M_1^2 \leq 2M_1 \sqrt{M_0 M_2}$$

or

$$M_1^2 \leq 4M_0 M_2.$$

□

Supplement (Landau-Kolmogorov inequality on the real line). Suppose f is a twice-differentiable real function on $(-\infty, +\infty)$, and M_0, M_1, M_2 are the least upper bounds of $|f(x)|, |f'(x)|, |f''(x)|$, respectively, on $(-\infty, +\infty)$. Prove that

$$M_1^2 \leq 2M_0 M_2.$$

Proof.

- (1) Similar to (1) in Landau-Kolmogorov inequality on the half-line, we suppose that $0 < M_0 < +\infty$ and $0 < M_2 < +\infty$.
- (2) Similar to (2) in Landau-Kolmogorov inequality on the half-line. Given $x \in \mathbb{R}^1$ and $h > 0$. By Taylor's theorem (Theorem 5.15):

$$f(x+2h) = f(x) + 2hf'(x) + 2h^2 f''(\xi_1) \quad (\text{I})$$

$$f(x-2h) = f(x) - 2hf'(x) + 2h^2 f''(\xi_2) \quad (\text{II})$$

for some $\xi_1 \in (x, x+2h)$ and $\xi_2 \in (x, x-2h)$. So (I) subtracts (II):

$$f(x+2h) - f(x-2h) = 4hf'(x) + 2h^2 f''(\xi_1) - 2h^2 f''(\xi_2).$$

Thus

$$\begin{aligned} 4h|f'(x)| &\leq |f(x+2h)| + |f(x-2h)| + 2h^2|f''(\xi_1)| + 2h^2|f''(\xi_2)| \\ &\leq 2M_0 + 4h^2 M_2, \\ |f'(x)| &\leq \frac{M_0}{2h} + hM_2 \end{aligned}$$

holds for all $h > 0$. In particular, take

$$h = \sqrt{\frac{M_0}{2M_2}}$$

to get

$$|f'(x)| \leq \sqrt{2M_0M_2}.$$

Thus $\sqrt{2M_0M_2}$ is an upper bound of $|f'(x)|$ for all $x \in \mathbb{R}^1$. Hence

$$M_1 \leq \sqrt{2M_0M_2}$$

or

$$M_1^2 \leq 2M_0M_2.$$

□

Note.

- (1) Write

$$M_1 \leq \sqrt{2}M_0^{\frac{1}{2}}M_2^{\frac{1}{2}}.$$

$\sqrt{2}$ is called the Landau-Kolmogorov constant, which is the best possible.

- (2) In general, suppose f is a n th differentiable real function on \mathbb{R}^1 , and M_0, M_k, M_n are the least upper bounds of $|f(x)|, |f^{(k)}(x)|, |f^{(n)}(x)|$, respectively, on \mathbb{R}^1 where $1 \leq k < n$. Then

$$M_k \leq C(n, k)M_0^{1-\frac{k}{n}}M_n^{\frac{k}{n}}.$$

Exercise 5.16. Suppose f is twice-differentiable on $(0, \infty)$, f'' is bounded on $(0, \infty)$, and $f(x) \rightarrow 0$ as $x \rightarrow \infty$. Prove that $f'(x) \rightarrow 0$ as $x \rightarrow \infty$. (Hint: Let $a \rightarrow \infty$ in Exercise 5.15.)

Proof.

- (1) Write $|f''| \leq M$ for some real M since f'' is bounded on $(0, \infty)$.
 (2) Given any $a > 0$. As in Exercise 5.15, define M_0, M_1, M_2 are the least upper bounds of $|f(x)|, |f'(x)|, |f''(x)|$ on (a, ∞) . Note that $M_2 \leq M$ for any $a > 0$ (by (1)). So that

$$M_1^2 \leq 4M_0M_2 \leq 4MM_0$$

for any $a > 0$.

- (3) By assumption, $M_0 \rightarrow 0$ as $a \rightarrow \infty$. (So given any $\varepsilon > 0$, there exists a real A such that

$$0 \leq M_0 < \frac{\varepsilon}{4M+1}$$

whenever $a \geq A$. Hence

$$M_1^2 \leq 4MM_0 \leq 4M \cdot \frac{\varepsilon}{4M+1} < \varepsilon.$$

whenever $a \geq A$.) Therefore $M_1^2 \rightarrow 0$ as $a \rightarrow \infty$, or $f'(x) \rightarrow 0$ as $x \rightarrow \infty$.

□

Exercise 5.17. Suppose f is a real, three times differentiable function on $[-1, 1]$, such that

$$f(-1) = 0, \quad f(0) = 0, \quad f(1) = 1, \quad f'(0) = 0.$$

Prove that $f^{(3)}(x) \geq 3$ for some $x \in (-1, 1)$. Note that equality holds for $\frac{1}{2}(x^3 + x^2)$. (Hint: Use Theorem 5.15, with $\alpha = 0$ and $\beta = \pm 1$, to show that there exist $s \in (0, 1)$ and $t \in (-1, 0)$ such that

$$f^{(3)}(s) + f^{(3)}(t) = 6.)$$

We can drop the assumption that $f(0) = 0$ actually.

Proof (Hint).

- (1) Use Taylor's theorem (Theorem 5.15), with $\alpha = 0$ and $\beta = \pm 1$,

$$f(1) = f(0) + f'(0) + \frac{f''(0)}{2} + \frac{f'''(s)}{6} \tag{I}$$

$$f(-1) = f(0) - f'(0) + \frac{f''(0)}{2} - \frac{f'''(t)}{6} \tag{II}$$

for some $s \in (0, 1)$ and $t \in (-1, 0)$.

- (2) (I) subtracts (II) implies that

$$f(1) - f(-1) = 2f'(0) + \frac{f'''(s)}{6} + \frac{f'''(t)}{6}.$$

By assumption, $f(-1) = 0$, $f(1) = 1$ and $f'(0) = 0$. Hence

$$f^{(3)}(s) + f^{(3)}(t) = 6$$

for some $s \in (0, 1)$ and $t \in (-1, 0)$. So either $f^{(3)}(s) \geq 3$ or $f^{(3)}(t) \geq 3$ for some $s, t \in (-1, 1)$.

□

Exercise 5.18. Suppose f is a real function on $[a, b]$, n is a positive integer, and $f^{(n-1)}$ exists for every $t \in [a, b]$. Let α , β , and P be as in Taylor's theorem (Theorem 5.15). Define

$$Q(t) = \frac{f(t) - f(\beta)}{t - \beta}$$

for $t \in [a, b]$, $t \neq \beta$, differentiate

$$f(t) - f(\beta) = (t - \beta)Q(t)$$

$n - 1$ times at $t = \alpha$, and derive the following version of Taylor's theorem:

$$f(\beta) = P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^n.$$

Proof. □

Exercise 5.19. Suppose f is defined in $(-1, 1)$ and $f'(0)$ exists. Suppose $-1 < \alpha_n < \beta_n < 1$, $\alpha_n \rightarrow 0$, and $\beta_n \rightarrow 0$ as $n \rightarrow \infty$. Define the difference quotients

$$D_n = \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}$$

Prove the following statements:

- (a) If $\alpha_n < 0 < \beta_n$, then $\lim D_n = f'(0)$.
- (b) If $0 < \alpha_n < \beta_n$ and $\left\{ \frac{\beta_n}{\beta_n - \alpha_n} \right\}$ is bounded, then $\lim D_n = f'(0)$.
- (c) If f' is continuous in $(-1, 1)$, then $\lim D_n = f'(0)$.

Give an example in which f is differentiable in $(-1, 1)$ (but f' is not continuous at 0) and in which α_n , β_n tend to 0 in such a way that $\lim D_n$ exists but is different from $f'(0)$.

Exercise 5.20. Formulate and prove an inequality which follows from Taylor's theorem and which remains valid for vector-valued function.

Proof.

- (a) Suppose \mathbf{f} is a function of $[a, b]$ into \mathbb{R}^m , n is a positive integer, $\mathbf{f}^{(n-1)}$ is continuous on $[a, b]$, $\mathbf{f}^{(n)}(t)$ exists for every $t \in (a, b)$. Let α , β be distinct points of $[a, b]$, and define

$$\mathbf{P}(t) = \sum_{k=0}^{n-1} \frac{\mathbf{f}^{(k)}(\alpha)}{k!} (t - \alpha)^k.$$

Then there exists a point x between α and β such that

$$|\mathbf{f}(\beta) - \mathbf{P}(\beta)| \leq (\beta - \alpha)^n \left| \frac{\mathbf{f}^{(n)}(x)}{n!} \right|.$$

For $n = 1$, this is just Theorem 5.19.

(b) Similar to the proof of Theorem 5.19. Put

$$\mathbf{z} = \mathbf{f}(\beta) - \mathbf{P}(\beta).$$

Define

$$\varphi(t) = \mathbf{z} \cdot \mathbf{f}(t) \quad (\alpha \leq t \leq \beta).$$

Then $\varphi(t)$ is a function of $[a, b]$ into \mathbb{R}^1 , and

$$\varphi^{(k)}(t) = \mathbf{z} \cdot \mathbf{f}^{(k)}(t)$$

where $0 \leq k \leq n$. Also, $\varphi^{(n-1)}$ is continuous on $[\alpha, \beta]$, and $\varphi^{(n)}(t)$ exists for every $t \in (\alpha, \beta)$.

(c) By Taylor's theorem (Theorem 5.15), there exists $x \in (\alpha, \beta)$ such that

$$\varphi(\beta) = Q(\beta) + \frac{\varphi^{(n)}(x)}{n!}(\beta - \alpha)^n$$

where

$$Q(t) = \sum_{k=0}^{n-1} \frac{\varphi^{(k)}(\alpha)}{k!} (t - \alpha)^k.$$

By (b), we have $Q(t) = \mathbf{z} \cdot \mathbf{P}(t)$ and thus

$$\mathbf{z} \cdot (\mathbf{f}(\beta) - \mathbf{P}(\beta)) = \mathbf{z} \cdot \frac{\mathbf{f}^{(n)}(x)}{n!} (\beta - \alpha)^n.$$

Note that $\mathbf{z} = \mathbf{f}(\beta) - \mathbf{P}(\beta)$ and Schwarz inequality (Theorem 1.37(d)). Hence

$$\begin{aligned} |\mathbf{f}(\beta) - \mathbf{P}(\beta)|^2 &= \left| (\mathbf{f}(\beta) - \mathbf{P}(\beta)) \cdot \frac{\mathbf{f}^{(n)}(x)}{n!} (\beta - \alpha)^n \right| \\ &\leq |\mathbf{f}(\beta) - \mathbf{P}(\beta)| \left| \frac{\mathbf{f}^{(n)}(x)}{n!} \right| (\beta - \alpha)^n \end{aligned}$$

or

$$|\mathbf{f}(\beta) - \mathbf{P}(\beta)| \leq \left| \frac{\mathbf{f}^{(n)}(x)}{n!} \right| (\beta - \alpha)^n$$

(whether $\mathbf{f}(\beta) - \mathbf{P}(\beta)$ is zero nor not).

□

Exercise 5.21. Let E be a closed subset of \mathbb{R}^1 . We saw in Exercise 4.22, that there is a real continuous function f on \mathbb{R}^1 whose zero set is E . Is it possible, for each closed set E , to find such an f which is differentiable on \mathbb{R}^1 , or one which is n times differentiable, or even one which has derivatives of all orders on \mathbb{R}^1 ?

Exercise 5.22. Suppose f is a real function on $(-\infty, +\infty)$. Call x a **fixed point** of f if $f(x) = x$.

- (a) If f is differentiable and $f'(t) \neq 1$ for every real t , prove that f has at most one fixed point.
- (b) Show that the function f defined by

$$f(t) = t + (1 + e^t)^{-1}$$

has no fixed point, although $0 < f'(t) < 1$ for all real t .

- (c) However, if there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t , prove that a fixed point x of f exists, and that $x = \lim x_n$, where x_1 is an arbitrary real number and

$$x_{n+1} = f(x_n)$$

for $n = 1, 2, 3, \dots$

- (d) Show that the process describe in (c) can be visualized by the zig-zag path

$$(x_1, x_2) \rightarrow (x_2, x_2) \rightarrow (x_2, x_3) \rightarrow (x_3, x_3) \rightarrow (x_3, x_4) \rightarrow \dots$$

Exercise 5.23. The function f defined by

$$f(x) = \frac{x^3 + 1}{3}$$

has three fixed points, say α, β, γ , where

$$2 < \alpha < 1, \quad 0 < \beta < 1, \quad 1 < \gamma < 2.$$

For arbitrarily chosen x_1 , define $\{x_n\}$ by setting $x_{n+1} = f(x_n)$.

- (a) If $x_1 < \alpha$, prove that $x_n \rightarrow -\infty$ as $n \rightarrow \infty$.
- (b) If $\alpha < x_1 < \gamma$, prove that $x_n \rightarrow \beta$ as $n \rightarrow \infty$.
- (c) If $\gamma < x_1$, prove that $x_n \rightarrow +\infty$ as $n \rightarrow \infty$.

Thus β can be located by this method, but α and γ cannot.

Exercise 5.24. The process described in part (c) of Exercise 5.22 can of course also be applied to functions that map $(0, \infty)$ to $(0, \infty)$. Fix some $\alpha > 1$, and put

$$f(x) = \frac{1}{2} \left(x + \frac{\alpha}{x} \right), \quad g(x) = \frac{\alpha + x}{1 + x}.$$

Both f and g have $\sqrt{\alpha}$ as their fixed point in $(0, \infty)$. Try to explain, on the basis of properties of f and g , why the convergence in Exercise 3.16, is so much more rapid than it is in Exercise 3.17. (Compare f' and g' , draw the zig-zags suggested in Exercise 5.22.)

Exercise 5.25. Suppose f is twice differentiable on $[a, b]$, $f(a) < 0$, $f(b) > 0$, $f'(x) \geq \delta > 0$, and $0 \leq f''(x) \leq M$ for all $x \in [a, b]$. Let ξ be the unique point in (a, b) at which $f(\xi) = 0$. Complete the details in the following outline of **Newton's method** for computing ξ .

- (a) Choose $x_1 \in (\xi, b)$, and define $\{x_n\}$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Interpret this geometrically, in terms of a tangent to the graph of f .

- (b) Prove that $x_{n+1} < x_n$ and that

$$\lim_{n \rightarrow \infty} x_n = \xi.$$

- (c) Use Taylor's theorem to show that

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

for some $t_n \in (\xi, x_n)$.

- (d) (Quadratic convergence) If $A = \frac{M}{2\delta}$, deduce that

$$0 \leq x_{n+1} - \xi \leq \frac{1}{A} [A(x_1 - \xi)]^{2^n}.$$

(Compare with Exercise 3.16 and 3.18.)

- (e) Show that Newton's method amounts to finding a fixed point of the function g defined by

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

How does $g'(x)$ behave for x near ξ ?

(f) Put $f(x) = x^{\frac{1}{3}}$ on $(-\infty, +\infty)$ and try Newton's method. What happens?

Proof of (a) (Wikipedia). The equation of the tangent line to the curve $y = f(x)$ at $x = x_n$ is

$$y = f'(x_n)(x - x_n) + f(x_n).$$

The x -intercept of this line (the value of x which makes $y = 0$) is taken as the next approximation, x_{n+1} , to the root, so that the equation of the tangent line is satisfied when $(x, y) = (x_{n+1}, 0)$:

$$0 = f'(x_n)(x - x_n) + f(x_n).$$

Solving for x_{n+1} gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

□

Proof of (b).

(1) Show that $x_n \geq \xi$ for all n . Induction on n .

(a) $n = 1$ is clearly true: $x_1 > \xi$ by assumption.

(b) Assume the induction hypothesis that for the single case $n = k$ holds. By the mean value theorem (Theorem 5.10), there is a point $\xi_k \in (\xi, x_k)$

$$f(x_k) - f(\xi) = f'(\xi_k)(x_k - \xi),$$

or

$$f(x_k) = f'(\xi_k)(x_k - \xi)$$

(since $f(\xi) = 0$). Since $f'' \geq 0$, f' is monotonically increasing (Theorem 5.11(a)). Hence $f'(\xi_k) \leq f'(x_k)$ and thus

$$f(x_k) = f'(\xi_k)(x_k - \xi) \leq f'(x_k)(x_k - \xi).$$

Since $f'(x_k) > 0$ by assumption,

$$\xi \leq x_k - \frac{f(x_k)}{f'(x_k)} = x_{k+1}.$$

(c) Since both the base case in (a) and the inductive step in (b) have been proved as true, by mathematical induction $x_n \geq \xi$ for all n .

(2) Show that $x_{n+1} < x_n$ for all n .

(a) Since $f' > 0$, $f'(x_n) > 0$ for all n .

(b) Since $f' > 0$, f is strictly increasing (Theorem 5.10). Hence $f(x_n) > f(\xi) = 0$ for all n (by (1)).

(c) By (a)(b), $\frac{f(x_n)}{f'(x_n)} > 0$ or

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} < x_n.$$

(3) By Theorem 3.14, $\{x_n\}$ converges to some real number $\zeta \geq \xi$. Note that f and f' are continuous by the existence of f'' (Theorem 5.2), we have

$$\lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} x_n - \frac{f(\lim_{n \rightarrow \infty} x_n)}{f'(\lim_{n \rightarrow \infty} x_n)}$$

provided $f' \neq 0$ (Theorem 4.9 and Theorem 4.4). Hence

$$\zeta = \zeta - \frac{f(\zeta)}{f'(\zeta)}$$

or $f(\zeta) = 0$. By the uniqueness of ξ , $\zeta = \xi$ or $\lim x_n = \xi$ as desired.

□

Proof of (c). By Taylor's theorem (Theorem 5.15),

$$f(\xi) = f(x_n) + f'(x_n)(\xi - x_n) + \frac{f''(t_n)}{2}(\xi - x_n)^2$$

for some $t_n \in (\xi, x_n)$. Note that $f(\xi) = 0$, $f'(x_n) \neq 0$ and $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, we have the desired result. □

Proof of (d). Clearly, $0 \leq x_{n+1} - \xi$ for all n (by (b)). Besides, by (c)

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

Note that $f'' \leq M$ and $f' \geq \delta > 0$ by assumption, and thus

$$x_{n+1} - \xi \leq \frac{M}{2\delta}(x_n - \xi)^2 = A(x_n - \xi)^2.$$

By induction,

$$x_{n+1} - \xi \leq \frac{1}{A}(A(x_1 - \xi))^{2^n}.$$

□

Note. Compare with Exercise 3.16 and Exercise 3.18. Might assume that $p > 1$.

(1) Fix a positive number α . Let $f(x) = x^p - \alpha$ on $E = (a, b)$ where $a = \frac{1}{2}\alpha^{\frac{1}{p}}$ and

$$b = \begin{cases} 2\alpha^{\frac{1}{p}} & (p = 2), \\ \left(\frac{2(p-1)}{p}\right)^{\frac{1}{p-2}} \alpha^{\frac{1}{p}} & (p > 2). \end{cases}$$

$E = (a, b)$ is well-defined since $a < b$. Besides, $\xi = \alpha^{\frac{1}{p}} \in E = (a, b)$.

(2) By construction,

$$f(a) < 0 \text{ and } f(b) > 0.$$

By $f'(x) = px^{p-1}$ and $f''(x) = p(p-1)x^{p-2}$,

$$f'(x) \geq pa^{p-1} > 0,$$

$$0 \leq f''(x) \leq p(p-1)b^{p-2}.$$

on E . Write

$$\delta = pa^{p-1} = \frac{p}{2^{p-1}} \alpha^{\frac{p-1}{p}},$$

$$M = p(p-1)b^{p-2} = 2(p-1)^2 \alpha^{\frac{p-2}{p}}.$$

(3) Hence the Newton's method works for $f(x) = x^p - \alpha$. That is, as we define $\{x_n\}$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{p-1}{p}x_n + \frac{\alpha}{p}x_n^{-p+1},$$

we have $\lim x_n = \xi = \alpha^{\frac{1}{p}}$. And

$$0 \leq x_{n+1} - \xi \leq \frac{1}{A}(A(x_1 - \xi))^{2^n}.$$

Here

$$A = \frac{M}{2\delta} = \frac{2^{p-1}(p-1)^2}{p\alpha^{\frac{1}{p}}}.$$

(4) Note that

$$\beta = \frac{p\alpha^{\frac{1}{p}}}{(p-1)^2} \neq \frac{p\alpha^{\frac{1}{p}}}{2^{p-1}(p-1)^2} = \frac{1}{A}.$$

where β is defined in the proof of Exercise 3.18. Note that $f'(x_n) \geq f'(\xi)$ (since f' is monotonically increasing and all $x_n \geq \xi$), and thus A can be chosen by a better estimation:

$$A = \frac{M}{2f'(\xi)} = \frac{(p-1)^2}{p\alpha^{\frac{1}{p}}} = \frac{1}{\beta}.$$

Now it is exactly the same as Exercise 3.16 and Exercise 3.18.

Proof of (e).

(1) Define $g(x) = x - \frac{f(x)}{f'(x)}$ on $[a, b]$. $g(\xi) = \xi$ if and only if $f(\xi) = 0$.

(2) By the construction of g , g is differentiable and

$$g'(x) = 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2}.$$

(3) Hence

$$|g'(x)| \leq \left| \frac{f(x)f''(x)}{f'(x)^2} \right| = \frac{|f(x)||f''(x)|}{|f'(x)|^2} \leq \frac{M}{\delta^2} |f(x)|.$$

As $x \rightarrow \xi$, $|f(x)| \rightarrow 0$. Therefore, $|g'(x)| \rightarrow 0$ or $g'(x) \rightarrow 0$ as $x \rightarrow \xi$.

□

Proof of (f).

(1) It is clearly that $f(x) = 0$ if and only if $x = 0$. Write $\xi = 0$.

(2) Note that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = -2x_n,$$

or

$$x_n = (-2)^{n-1}x_1$$

for any $x_1 \in (\xi, \infty)$ where $n = 1, 2, 3, \dots$. Hence, the sequence $\{x_n\}$ does not converge for any choice of $x_1 \in (\xi, \infty)$. In this case we cannot find ξ satisfying $f(\xi) = 0$ by Newton's method.

(3) In fact,

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} \rightarrow 0 \text{ as } x \rightarrow \pm\infty.$$

Hence such $\delta > 0$ satisfying $f'(x) \geq \delta > 0$ does not exist.

□

Exercise 5.26. Suppose f is differentiable on $[a, b]$, $f(a) = 0$, and there is a real number A such that $|f'(x)| \leq A|f(x)|$ on $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$. (Hint: Fix $x_0 \in [a, b]$, let

$$M_0 = \sup |f(x)|, \quad M_1 = \sup |f'(x)|$$

for $a \leq x \leq x_0$. For any such x ,

$$|f(x)| \leq M_1(x_0 - a) \leq A(x_0 - a)M_0.$$

Hence $M_0 = 0$ if $A(x_0 - a) < 1$. That is, $f = 0$ on $[a, x_0]$. Proceed.)

Proof (Hint).

- (1) If $A = 0$, then $f'(x) = 0$ or $f(x)$ is constant on $[a, b]$ (Theorem 5.11(b)). Since $f(a) = 0$, $f(x) = 0$ on $[a, b]$.

(2) Suppose that $A > 0$. Fix $x_0 \in [a, b]$, let

$$M_0 = \sup |f(x)|, \quad M_1 = \sup |f'(x)|$$

for $a \leq x \leq x_0$. Since $|f'(x)| \leq A|f(x)|$ on $[a, b]$,

$$|f'(x)| \leq A|f(x)| \leq AM_0.$$

Since AM_0 is an upper bound for $|f'(x)|$,

$$M_1 \leq AM_0.$$

(3) Given any $x \in [a, x_0]$. Since f is differentiable on $[a, x_0] \subseteq [a, b]$, by the mean value theorem (Theorem 5.10), there is $\xi \in (a, x)$ such that

$$f(x) - f(a) = f'(\xi)(x - a).$$

Note that $f(a) = 0$ by assumption. So that

$$\begin{aligned} |f(x)| &= |f'(\xi)|(x - a) \\ &\leq M_1(x - a) && \text{(Definition of } M_1) \\ &\leq AM_0(x - a) && ((2)) \\ &\leq AM_0(x_0 - a). && (x \in [a, x_0]) \end{aligned}$$

Since $AM_0(x_0 - a)$ is an upper bound for $|f(x)|$,

$$M_0 \leq AM_0(x_0 - a).$$

Take

$$x_0 = \min \left\{ \frac{1}{2A} + a, b \right\}$$

so that $M_0 \leq AM_0(x_0 - a) \leq \frac{M_0}{2}$. $M_0 = 0$ or $f(x) = 0$ on $[a, x_0]$.

(4) Take a partition

$$P = \{a = x_{-1}, x_0, \dots, x_n = b\}$$

of $[a, b]$ such that each subinterval $[x_{i-1}, x_i]$ satisfying $\Delta x_i = x_i - x_{i-1} < \frac{1}{2A}$. By (3), $f(x) = 0$ on $[x_{-1}, x_0]$. Apply the same argument in (3), $f(x) = 0$ on $[x_0, x_1]$. Continue this process, $f(x) = 0$ on each subinterval and thus on the whole interval $[a, b]$.

□

Note. It holds for vector-valued functions too:

Suppose \mathbf{f} is a vector-valued differentiable function on $[a, b]$, $\mathbf{f}(a) = 0$, and there is a real number A such that $|\mathbf{f}'(x)| \leq A|\mathbf{f}(x)|$ on $[a, b]$. Prove that $\mathbf{f}(x) = 0$ for all $x \in [a, b]$.

The proof is similar except using Theorem 5.19 ($|\mathbf{f}(b) - \mathbf{f}(a)| \leq (b - a)|\mathbf{f}'(x)|$) in addition.

Exercise 5.27. Let ϕ be a real function defined on a rectangle R in the plane, given by $a \leq x \leq b$, $\alpha \leq y \leq \beta$. A **solution** of the initial-value problem

$$y' = \phi(x, y), \quad y(a) = c \quad (\alpha \leq c \leq \beta)$$

is, by definition, a differentiable function f on $[a, b]$ such that $f(a) = c$, $\alpha \leq f(x) \leq \beta$, and

$$f'(x) = \phi(x, f(x)) \quad (a \leq x \leq b)$$

Prove that such a problem has at most one solution if there is a constant A such that

$$|\phi(x, y_2) - \phi(x, y_1)| \leq A|y_2 - y_1|$$

whenever $(x, y_1) \in R$ and $(x, y_2) \in R$. (Hint: Apply Exercise 26 to the difference of two solutions.) Note that this uniqueness theorem does not hold for the initial-value problem

$$y' = y^{\frac{1}{2}}, \quad y(0) = 0,$$

which has two solutions: $f(x) = 0$ and $f(x) = \frac{x^2}{4}$. Find all other solutions.

Proof (Hint).

- (1) Suppose f_1 and f_2 are two solutions of that problem. Define $f = f_1 - f_2$. f is differentiable on $[a, b]$, $f(a) = f_1(a) - f_2(a) = c - c = 0$. And

$$\begin{aligned} |f'(x)| &= |f_1'(x) - f_2'(x)| \\ &= |\phi(x, f_1(x)) - \phi(x, f_2(x))| \\ &\leq A|f_1(x) - f_2(x)| \end{aligned}$$

on $[a, b]$. By Exercise 5.26, $f(x) = 0$ on $[a, b]$, or $f_1(x) = f_2(x)$ on $[a, b]$.

- (2) *The initial-value problem*

$$y' = y^{\frac{1}{2}}, \quad y(0) = 0,$$

which has two solutions: $f(x) = 0$ and $f(x) = \frac{x^2}{4}$. Find all other solutions.

Note. It does not exist a real A such that $|\phi(x, y_2) - \phi(x, y_1)| \leq A|y_2 - y_1|$ in this initial-value problem.

- (a) Clearly, $f(x) = 0$ and $f(x) = \frac{x^2}{4}$ are two solutions for the initial-value problem.

- (b) Suppose $f(x) \neq 0$ on $[0, \infty)$. Since $f'(x) = f(x)^{\frac{1}{2}}$, $f(x) \geq 0$. Since $f(x)$ is continuous (Theorem 5.2), the set

$$E = \{x \in [0, \infty) : f(x) > 0\}$$

is open in \mathbb{R}^1 (Theorem 4.8). By Exercise 2.29 we write E as the union of an at most countable collection of disjoint segments, say

$$E = \bigcup_{(a_i, b_i) \subseteq [0, \infty)} (a_i, b_i)$$

where all (a_i, b_i) segments are disjoint. Note that E is nonempty.

- (c) For any segment (a_i, b_i) , define $g(x) = f(x)^{\frac{1}{2}}$ on (a_i, b_i) . (Clearly, $g(a_i) = f(a_i) = 0$ by the definition of E .) Thus

$$g'(x) = \frac{1}{2}f(x)^{-\frac{1}{2}}f'(x) = \frac{1}{2}.$$

Hence

$$g(x) = \frac{1}{2}x + c$$

for some constant $c \in \mathbb{R}^1$. So

$$f(x) = g(x)^2 = \left(\frac{1}{2}x + c\right)^2.$$

$f(a_i) = 0$ implies that $c = -\frac{a_i}{2}$. Hence

$$f(x) = \frac{1}{4}(x - a_i)^2$$

on (a_i, b_i) .

- (d) By (c), if $b_i < \infty$ is defined as a real number, then $f(b_i) = 0$ by definition of E . Note that

$$\lim_{x \rightarrow b_i^-} f(x) = \frac{1}{4}(b_i - a_i)^2 > 0,$$

which is absurd. Hence $b_i = \infty$ and thus E is of the form

$$E = (a, \infty) \quad (a \geq 0).$$

Therefore,

$$f(x) = \begin{cases} 0 & (0 \leq x \leq a), \\ \frac{1}{4}(x - a)^2 & (x > a \geq 0). \end{cases}$$

□

Exercise 5.28. Formulate and prove an analogous uniqueness theorem for systems of differential equations of the form

$$y'_j = \phi_j(x, y_1, \dots, y_k), \quad y_j(a) = c_j \quad (j = 1, \dots, k)$$

Note that this can be rewritten in the form

$$\mathbf{y}' = \boldsymbol{\phi}(x, \mathbf{y}), \quad \mathbf{y}(a) = \mathbf{c}$$

where $\mathbf{y} = (y_1, \dots, y_k)$ ranges over a k -cell, $\boldsymbol{\phi}$ is the mapping of a $(k+1)$ -cell into the Euclidean k -space whose components are the function ϕ_1, \dots, ϕ_k , and \mathbf{c} is the vector (c_1, \dots, c_k) . Use Exercise 5.26, for vector-valued functions.

Proof.

- (1) A **solution** of the initial-value problem

$$\mathbf{y}' = \boldsymbol{\phi}(x, \mathbf{y}), \quad \mathbf{y}(a) = \mathbf{c}$$

is, by definition, a differentiable function \mathbf{f} on $[a, b]$ such that $\mathbf{f}(a) = \mathbf{c}$, and

$$\mathbf{f}'(x) = \boldsymbol{\phi}(x, \mathbf{f}(x)) \quad (a \leq x \leq b).$$

Then this problem has at most one solution if there is a constant A such that

$$|\boldsymbol{\phi}(x, \mathbf{y}_2) - \boldsymbol{\phi}(x, \mathbf{y}_1)| \leq A|\mathbf{y}_2 - \mathbf{y}_1|$$

whenever $(x, \mathbf{y}_1) \in R$ and $(x, \mathbf{y}_2) \in R$ where R is a $(k+1)$ -cell defined by

$$R = [a, b] \times [\alpha_1, \beta_1] \times \dots \times [\alpha_k, \beta_k].$$

- (2) Similar to Exercise 5.27, Suppose \mathbf{f}_1 and \mathbf{f}_2 are two solutions of that problem. Define $\mathbf{f} = \mathbf{f}_1 - \mathbf{f}_2$. \mathbf{f} is differentiable on $[a, b]$, $\mathbf{f}(a) = \mathbf{f}_1(a) - \mathbf{f}_2(a) = \mathbf{c} - \mathbf{c} = 0$. And

$$\begin{aligned} |\mathbf{f}'(x)| &= |\mathbf{f}'_1(x) - \mathbf{f}'_2(x)| \\ &= |\boldsymbol{\phi}(x, \mathbf{f}_1(x)) - \boldsymbol{\phi}(x, \mathbf{f}_2(x))| \\ &\leq A|\mathbf{f}_1(x) - \mathbf{f}_2(x)| \end{aligned}$$

on $[a, b]$. By Note in Exercise 5.26, $\mathbf{f}(x) = 0$ on $[a, b]$, or $\mathbf{f}_1(x) = \mathbf{f}_2(x)$ on $[a, b]$.

□

Exercise 5.29. Specialize Exercise 5.28 by considering the system

$$\begin{aligned} y'_j &= y_{j+1} \quad (j = 1, \dots, k-1), \\ y'_k &= f(x) - \sum_{j=1}^k g_j(x)y_j \end{aligned}$$

where f, g_1, \dots, g_k are continuous real functions on $[a, b]$, and derive a uniqueness theorem for solutions of the equation

$$y^{(k)} + g_k(x)y^{(k-1)} + \dots + g_2(x)y' + g_1(x)y = f(x),$$

subject to initial conditions

$$y(a) = c_1, \quad y'(a) = c_1, \quad \dots, \quad y^{(k-1)}(a) = c_k.$$

Proof.

(1) Write

$$\begin{aligned} \mathbf{y} &= (y_1, \dots, y_k) \\ &= (y, y', y'', \dots, y^{(k-1)}), \\ \phi(x, \mathbf{y}) &= \left(y_2, y_3, \dots, y_{k-1}, f(x) - \sum_{j=1}^k g_j(x)y_j \right) \\ &= \left(y', y'', \dots, y^{(k-1)}, f(x) - \sum_{j=1}^k g_j(x)y^{(j-1)} \right), \\ \mathbf{c} &= (c_1, \dots, c_k). \end{aligned}$$

So that

$$\mathbf{y}' = \phi(x, \mathbf{y}), \quad \mathbf{y}(a) = \mathbf{c}$$

where \mathbf{y} ranges over a k -cell R .

(2) To show that the problem has at most one solution, by Exercise 5.28 it suffices to show that there is a constant A such that

$$|\phi(x, \mathbf{y}) - \phi(x, \mathbf{z})| \leq A|\mathbf{y} - \mathbf{z}|$$

whenever $(x, \mathbf{y}) \in R$ and $(x, \mathbf{z}) \in R$.

(3) Since all g_j ($1 \leq j \leq k$) are real continuous functions on a compact set $[a, b]$, all g_j are bounded (Theorem 4.15), say $|g_j| \leq M$ on $[a, b]$ for some $M_j \in \mathbb{R}^1$ ($1 \leq j \leq k$).

(4) Write $\mathbf{y} = (y_1, \dots, y_k)$ and $\mathbf{z} = (z_1, \dots, z_k)$. So

$$\begin{aligned}
& |\phi(x, \mathbf{y}) - \phi(x, \mathbf{z})|^2 \\
&= \left| \left(y_2 - z_2, y_3 - z_3, \dots, y_{k-1} - z_{k-1}, -\sum_{j=1}^k g_j(x)(y_j - z_j) \right) \right|^2 \\
&= \sum_{j=2}^{k-1} (y_j - z_j)^2 + \left(-\sum_{j=1}^k g_j(x)(y_j - z_j) \right)^2 \\
&\leq \sum_{j=2}^{k-1} (y_j - z_j)^2 + \sum_{j=1}^k g_j(x)^2 \sum_{j=1}^k (y_j - z_j)^2 && \text{(Theorem 1.35)} \\
&\leq \sum_{j=2}^{k-1} (y_j - z_j)^2 + \sum_{j=1}^k M_j^2 \sum_{j=1}^k (y_j - z_j)^2 && ((3)) \\
&\leq \sum_{j=1}^k (y_j - z_j)^2 + \sum_{j=1}^k M_j^2 \sum_{j=1}^k (y_j - z_j)^2 && (x^2 \geq 0 \forall x \in \mathbb{R}^1) \\
&\leq \left(1 + \sum_{j=1}^k M_j^2 \right) |\mathbf{y} - \mathbf{z}|^2.
\end{aligned}$$

Hence $|\phi(x, \mathbf{y}) - \phi(x, \mathbf{z})| \leq A|\mathbf{y} - \mathbf{z}|$ for some $A = \left(1 + \sum_{j=1}^k M_j^2 \right)^{\frac{1}{2}}$.

□