

Notes on the book: *Atiyah and Macdonald, Introduction to Commutative Algebra*

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Chapter 1: Rings and Ideals

Exercise 1.1.

Let x be a nilpotent element of A . Show that $1 + x$ is a unit of A . Deduce that the sum of a nilpotent element and a unit is a unit.

Proof.

- (1) Suppose $x^m = 0$ for some odd integer $m \geq 0$. Then

$$1 = 1 + x^m = (1 + x)(1 - x + x^2 - \cdots + (-1)^{m-1}x^{m-1}),$$

or $1 + x$ is a unit.

- (2) If u is any unit and x is any nilpotent, $u + x = u \cdot (1 + u^{-1}x)$ is a product of two units (using that $u^{-1}x$ is nilpotent and applying (1)) and hence a unit again.

□

Proof (Proposition 1.9).

- (1) *The nilradical is a subset of the Jacobson radical.*
- (a) The nilradical \mathfrak{N} of A is the intersection of all the prime ideals of A by Proposition 1.8.
 - (b) The Jacobson radical \mathfrak{J} of A is the intersection of all the maximal ideals of A by definition.
- (2) By Proposition 1.9, $x \in \mathfrak{J}$ if and only if $1 - xy$ is a unit in A for all $y \in A$. So $1 + x = 1 - (-x) \cdot 1$ is a unit in A since x is a nilpotent and \mathfrak{J} is an ideal.

□

Exercise 1.2.

Let A be a ring and let $A[x]$ be the ring of polynomials in an indeterminate x , with coefficients in A . Let $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$. Prove that

- (i) f is a unit in $A[x]$ if and only if a_0 is a unit in A and a_1, \dots, a_n are nilpotent. (Hint: If $b_0 + b_1x + \cdots + b_mx^m$ is the inverse of f , prove by induction on r that $a_n^{r+1}b_{m-r} = 0$. Hence show that a_n is nilpotent, and then use Exercise 1.1.)
- (ii) f is nilpotent if and only if a_0, a_1, \dots, a_n are nilpotent.

- (iii) f is a zero-divisor if and only if there exists $a \neq 0$ such that $af = 0$. (Hint: Choose a polynomial $g = b_0 + b_1x + \cdots + b_mx^m$ of least degree m such that $fg = 0$. Then $a_nb_m = 0$, hence $a_ng = 0$ (because a_ng annihilates f and has degree $< m$). Now show by induction that $a_{n-r}g = 0$ ($0 \leq r \leq n$).)
- (iv) f is said to be **primitive** if $(a_0, a_1, \dots, a_n) = (1)$. Prove that if $f, g \in A[x]$, then fg is primitive if and only if f and g are primitive.

Proof of (i).

- (1) (\Leftarrow) holds by Exercise 1.1.
- (2) (\Rightarrow) There exists the inverse g of f , say $g = b_0 + b_1x + \cdots + b_mx^m$ satisfying $1 = fg$. Clearly, $1 = a_0b_0$, or a_0 is a unit in A . Also,

$$\begin{aligned} 0 &= a_nb_m, \\ 0 &= a_nb_{m-1} + a_{n-1}b_m, \\ 0 &= a_nb_{m-2} + a_{n-1}b_{m-1} + a_{n-2}b_m, \\ &\dots \end{aligned}$$

A direct computing shows that

$$\begin{aligned} 0 &= a_n^1b_m, \\ 0 &= a_n(a_nb_{m-1} + a_{n-1}b_m) \\ &= a_n^2b_{m-1} + a_{n-1}a_nb_m \\ &= a_n^2b_{m-1}, \\ 0 &= a_n^2(a_nb_{m-2} + a_{n-1}b_{m-1} + a_{n-2}b_m) \\ &= a_n^3b_{m-2} + a_{n-1}a_n^2b_{m-1} + a_{n-2}a_n^2b_m \\ &= a_n^3b_{m-2}, \\ &\dots \end{aligned}$$

So we might have $a_n^{r+1}b_{m-r} = 0$ for $r = 0, 1, 2, \dots, m$.

- (3) Show that $a_n^{r+1}b_{m-r} = 0$ for $r = 0, 1, 2, \dots, m$ by induction on r .
- (a) As $r = 0$, $a_nb_m = 0$ by comparing the coefficient of $fg = 1$ at x^{n+m} .
- (b) For any $r > 0$, comparing the coefficient of $fg = 1$ at x^{n+m-r} ,

$$0 = a_nb_{m-r} + a_{n-1}b_{m-r+1} + \cdots + a_{n-r}b_m.$$

Multiplying by a_n^r on the both sides,

$$\begin{aligned} 0 &= a_n^{r+1}b_{m-r} + a_{n-1}a_n^rb_{m-r+1} + \cdots + a_{n-r}a_n^rb_m \\ &= a_n^{r+1}b_{m-r}. \end{aligned}$$

by the induction hypothesis.

- (4) a_n is a nilpotent. Putting $r = m$ in $a_n^{r+1}b_{m-r} = 0$ and get $a_n^{m+1}b_0 = 0$. Notice that b_0 is a unit, $a_n^{m+1} = 0$, or a_n is a nilpotent.
- (5) Consider $f - a_n x^n = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$, a polynomial $\in A[x]$ of degree $n-1$. Note that f is a unit and $a_n x^n$ is a nilpotent. By Exercise 1.1, $f - a_n x^n$ is a unit too. Applying the (2)(3)(4) again, a_{n-1} is a nilpotent as $n-1 > 0$, that is, applying descending induction on n then yields the desired property.

□

Proof of (ii).

- (1) (\Leftarrow) holds since the nilradical of any ring is an ideal.
- (2) (\Rightarrow) $f^N = 0$ for some $N > 0$. So $0 = f^N = a_0^N + \cdots + a_n^N x^{nN}$. Compare the coefficient in the lowest term to get $a_0^N = 0$, or a_0 is a nilpotent.
- (3) Note that $f - a_0 = a_1 x + \cdots + a_n x^n \in A[x]$ is nilpotent since f and a_0 are nilpotent. $f - a_0$ is a nilpotent too. Continue the same argument in (2), the result is established.

□

Proof of (iii).

- (1) (\Leftarrow) holds trivially.
- (2) (\Rightarrow) Pick a polynomial $g = b_0 + b_1 x + \cdots + b_m x^m$ of least degree m such that $fg = 0$. Especially, $a_n b_m = 0$.
- (3) Consider

$$\begin{aligned} a_n g &= a_n b_0 + \cdots + a_n b_{m-1} x^{m-1} + a_n b_m x^m \\ &= a_n b_0 + \cdots + a_n b_{m-1} x^{m-1} \end{aligned}$$

(since $a_n b_m = 0$). $a_n g$ is a polynomial over A of having degree strictly less than m . Notice that $f \cdot (a_n g) = a_n \cdot (fg) = 0$. By minimality of m , $a_n g = 0$.

- (4) Induction on the degree n of f .
- (a) As $n = 0$, $f = a_0$. There exists $b_m \neq 0$ such that $b_m f = b_m a_0 = 0$ by (2).
- (b) For any zero-divisor f of degree n , there is a polynomial $g = b_0 + b_1 x + \cdots + b_m x^m$ of least degree m such that $fg = 0$. By (2)(3),

$$\begin{aligned} (f - a_n x^n) \cdot g &= fg - a_n x^n g \\ &= 0 - 0 \\ &= 0. \end{aligned}$$

That is, $f - a_n x^n$ is a zero-divisor of degree $n - 1$. By the induction hypothesis, there exists $b_m \neq 0$ such that $b_m(f - a_n x^n) = 0$. So $b_m f = b_m(f - a_n x^n) + b_m a_n x^n = 0 + 0 = 0$.

(c) By (a)(b), (\implies) holds by mathematical induction.

□

Proof of (iv). Note that

- (1) $f \notin \mathfrak{m}[x]$ for any maximal ideal \mathfrak{m} of A if and only if f is primitive.
- (2) For any maximal ideal \mathfrak{m} of A , A/\mathfrak{m} is a field (or an integral domain).
- (3) $A[x]$ is an integral domain if A is an integral domain.
- (4) $A[x]/\mathfrak{m}[x] \cong (A/\mathfrak{m})[x]$ as a ring isomorphism.

Hence,

$$\begin{aligned}
 f, g : \text{primitive} &\iff f, g \notin \mathfrak{m}[x] \text{ for any maximal ideal } \mathfrak{m} \\
 &\iff f, g \neq 0 \text{ in } (A/\mathfrak{m})[x] \text{ for any maximal ideal } \mathfrak{m} \\
 &\iff fg \neq 0 \text{ in } (A/\mathfrak{m})[x] \text{ for any maximal ideal } \mathfrak{m} \\
 &\iff fg \notin \mathfrak{m}[x] \text{ for any maximal ideal } \mathfrak{m} \\
 &\iff fg : \text{primitive}.
 \end{aligned}$$

□

Exercise 1.3.

Generalize the results of Exercise 1.2 to a polynomial ring $A[x_1, \dots, x_r]$ in several indeterminates.

Generalization. Let

$$f = \sum_{(i)} a_{(i)} x^{(i)} \in A[x_1, \dots, x_r]$$

where $\sum_{(i)}$ is the summation over $(i) = (i_1, \dots, i_r)$ with $i_1 + \dots + i_r = n$. Then

- (i) f is a unit in $A[x_1, \dots, x_r]$ if and only if $a_{(0)}$ is a unit in A and all other $a_{(i)}$ are nilpotent.
- (ii) f is nilpotent if and only if all $a_{(i)}$ are nilpotent.
- (iii) f is a zero-divisor if and only if there exists $a \neq 0$ such that $af = 0$.
- (iv) If $f, g \in A[x_1, \dots, x_r]$, then fg is primitive if and only if f and g are primitive.

Proof. Use the mathematical induction to prove (i)(ii)(iii) and apply the same argument in Exercise 1.2 (iv) to prove (iv). \square

Exercise 1.4.

In the ring $A[x]$, the Jacobson radical is equal to the nilradical.

Proof.

- (1) The nilradical \mathfrak{N} is a subset of the Jacobson radical \mathfrak{J} . It suffices to show that $\mathfrak{J} \subseteq \mathfrak{N}$.

(2)

$$\begin{aligned}
 & f \in \mathfrak{J} \\
 \iff & 1 - fy \text{ is a unit in } A[x] \text{ for all } y \in A[x] && \text{(Proposition 1.9)} \\
 \implies & 1 - xf \text{ is a unit in } A[x] && (y = x) \\
 \implies & \text{All coefficients of } f \text{ are nilpotent} && \text{(Exercise 1.2 (i))} \\
 \implies & f \text{ is nilpotent} && \text{(Exercise 1.2 (ii))} \\
 \implies & f \in \mathfrak{N}.
 \end{aligned}$$

\square

Exercise 1.5.

Let A be a ring and let $A[[x]]$ be the ring of formal power series $f = \sum_{n=0}^{\infty} a_n x^n$ with coefficients in A . Show that

- (i) *f is a unit in $A[[x]]$ if and only if a_0 is a unit in A .*
- (ii) *If f is nilpotent, then a_n is nilpotent for all $n \geq 0$. Is converse true? (See Exercise 7.2.)*
- (iii) *f belongs to the Jacobson radical of $A[[x]]$ if and only if a_0 belongs to the Jacobson radical of A .*
- (iv) *The contraction of a maximal ideal \mathfrak{m} of $A[[x]]$ is a maximal ideal of A , and \mathfrak{m} is generated by \mathfrak{m}^c and x .*
- (v) *Every prime ideal of A is the contraction of a prime ideal of $A[[x]]$.*

Proof of (i).

- (1) (\implies) If $g = \sum_{n=0}^{\infty} b_n x^n$ is an inverse of f , then $fg = 1$ implies that $a_0 b_0 = 1$ so that a_0 is a unit in A .
- (2) (\impliedby) Our goal is to find $g = \sum_{n=0}^{\infty} b_n x^n$ such that the Cauchy product $fg = \sum_{n=0}^{\infty} c_n x^n$ is equal to $1 \in A[x]$. Here $c_n = \sum_{r=0}^n a_r b_{n-r}$. By the assumption we have that $c_0 = 1$ and $c_1 = c_2 = \dots = 0$. Hence

$$\begin{aligned} b_0 &= a_0^{-1} \\ b_1 &= -a_0^{-1} a_1 b_0 \\ &\dots \\ b_n &= a_0^{-1} \sum_{r=1}^n a_r b_{n-r} \\ &\dots \end{aligned}$$

by induction.

□

Proof of (ii).

- (1) The proof is the same as Exercise 1.2 (ii).
- (2) The converse is true if A is Noetherian (by Exercise 7.2).
- (3) The converse is not always true. Take

$$A = \mathbb{F}_2[t, t^{-2}, t^{-2^2}, \dots]/(t)$$

and

$$f(x) = \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} t^{-2^n} x^n \in A[x].$$

Note that A is not Noetherian and all a_n are nilpotent in A . To show f is not nilpotent in $A[x]$, it suffices to show that f^{2^r} is not equal to zero for all positive integers r .

- (4) Note that \mathbb{F}_2 is a field of characteristic 2. So

$$f^{2^r} = \sum_{n=1}^{\infty} a_n^{2^r} x^n = \sum_{n=1}^{\infty} t^{2^{r-n}} x^n = \sum_{n=r+1}^{\infty} t^{2^{r-n}} x^n \neq 0$$

for all r .

□

Proof of (iii).

$$\begin{aligned}
& f \text{ in the Jacobson radical of } A[[x]] \\
& \iff 1 - fg \in A[[x]] \text{ is unit for all } g = \sum_{n=0}^{\infty} b_n x^n \in A[[x]] \quad (\text{Proposition 1.9}) \\
& \iff 1 - a_0 b_0 \in A \text{ is unit for all } b_0 \in A \quad ((i)) \\
& \iff a_0 \text{ belongs to the Jacobson radical of } A. \quad (\text{Proposition 1.9})
\end{aligned}$$

□

Proof of (iv).

- (1) Note that $x = 0 + x$ belongs to the Jacobson radical of $A[[x]]$ since 0 obviously belongs to the Jacobson radical of A (by (iii)).
- (2) So $x \in \mathfrak{m}$ or $(x) \subseteq \mathfrak{m}$ for any maximal ideal in $A[[x]]$. So it is clear that $\mathfrak{m} = \mathfrak{m}^c + (x)$.
- (3) Moreover, \mathfrak{m}^c is a maximal ideal since $A/\mathfrak{m}^c \cong A[[x]]/\mathfrak{m}$ is a field.

□

Proof of (v).

- (1) Similar to (iv). Suppose \mathfrak{p} is a prime ideal of A . Let $\mathfrak{q} = \mathfrak{p} + (x)$ be an ideal of $A[[x]]$.
- (2) $\mathfrak{q}^c = \mathfrak{p}$ clearly. Besides, \mathfrak{q}^c is a prime ideal since

$$A[[x]]/\mathfrak{q}^c \cong A/\mathfrak{p}$$

is an integral domain.

□

Supplement 1.5.1.

(Exercise II.1.2 in the textbook: Jrgen Neukirch, *Algebraic Number Theory*.) A p -adic integer $a = a_0 + a_1 p + a_2 p^2 + \cdots$ is a unit in the ring \mathbb{Z}_p if and only if $a_0 \neq 0$.

Proof.

- (1) (\implies) If $b = b_0 + b_1 p + b_2 p^2 + \cdots$ is an inverse of a , then $ab = 1$ implies that $a_0 b_0 = 1$ so that a_0 is a unit in $\mathbb{Z}/p\mathbb{Z}$ or $a_0 \neq 0$.

(2) (\Leftarrow) Our goal is to find

$$b = b_0 + b_1p + b_2p^2 + \cdots \in \mathbb{Z}_p$$

such that the Cauchy product

$$ab = c_0 + c_1p + c_2p^2 + \cdots$$

is equal to $1 \in \mathbb{Z}_p$. Here $c_n = \sum_{\nu=0}^n a_\nu b_{n-\nu}$. By the assumption we have that $c_0 = 1$ and $c_1 = c_2 = \cdots = 0$. Hence

$$b_0 = a_0^{-1}$$

$$b_1 = -a_0^{-1}a_1b_0$$

\dots

$$b_n = a_0^{-1} \sum_{\nu=1}^n a_\nu b_{n-\nu}$$

\dots

by induction.

□

Exercise 1.6.

A ring A is such that every ideal not contained in the nilradical contains a nonzero idempotent (that is, an element e such that $e^2 = e \neq 0$). Prove that the nilradical and Jacobson radical of A are equal.

Proof.

(1) $\mathfrak{N} \subseteq \mathfrak{J}$ clearly.

(2) Since

$$\begin{aligned} a \notin \mathfrak{N} &\implies (a) \not\subseteq \mathfrak{N} \\ &\implies \text{there exists a nonzero idempotent } e \in (a) \\ &\implies e = ar \text{ for some } r \in A \\ &\implies 0 = e - e^2 = e(1 - e) = ar(1 - ar) \\ &\implies 1 - ar \text{ is a zero-divisor, not a unit} \\ &\implies a \notin \mathfrak{J}, \end{aligned} \tag{Proposition 1.9}$$

we have $\mathfrak{J} \subseteq \mathfrak{N}$.

□

Exercise 1.7.

Let A be a ring in which every element satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal in A is maximal.

Proof. It suffices to show that for any prime ideal \mathfrak{p} in A , A/\mathfrak{p} is a field.

- (1) Take any $0 \neq \bar{x} \in A/\mathfrak{p}$, which is represented by $x \in A - \mathfrak{p}$. By assumption there exists $n \geq 2$ such that $x^n = x$. So $\bar{x}^n = \bar{x}$ or $\bar{x}(\bar{x}^{n-1} - 1) = 0$.
- (2) Since \mathfrak{p} is prime, A/\mathfrak{p} is an integral domain. That is, $\bar{x} = 0$ (impossible) or $\bar{x}^{n-1} - 1 = 0$. Write $\bar{x} \cdot \bar{x}^{n-2} = 1$ in A/\mathfrak{p} . So \bar{x}^{n-2} is an inverse of $\bar{x} \neq 0$ in A/\mathfrak{p} , which implies that A/\mathfrak{p} is a field (since \bar{x} is arbitrary).
- (3) A/\mathfrak{p} is a field if and only if \mathfrak{p} is maximal.

□

Exercise 1.8.

Let A be a ring $\neq 0$. Show that the set of prime ideals of A has minimal elements with respect to inclusion.

Similar to Theorem 1.3.

Proof (Zorn's Lemma).

- (1) Let Σ be the set of all prime ideals of A .
- (2) Order Σ by \supseteq , that is, $\mathfrak{p} \leq \mathfrak{q}$ if $\mathfrak{p} \supseteq \mathfrak{q}$.
- (3) Σ is not empty, since every ring $A \neq 0$ has at least one maximal ideal (or prime ideal) (Theorem 1.3).
- (4) To apply Zorn's lemma we must show that every chain in Σ has a lower bound in Σ ; let then (\mathfrak{p}_α) be a chain of prime ideals in Σ , so that for each pair of indices α, β we have either $\mathfrak{p}_\alpha \subseteq \mathfrak{p}_\beta$ or $\mathfrak{p}_\beta \subseteq \mathfrak{p}_\alpha$. Let $\mathfrak{p} = \bigcap_\alpha \mathfrak{p}_\alpha$.
- (5) Show that \mathfrak{p} is a prime ideal. Clearly \mathfrak{p} is an ideal. Given any $xy \in \mathfrak{p}$ and $x \notin \mathfrak{p}$. So xy is in all prime ideals \mathfrak{p}_α . By assumption $x \notin \mathfrak{p}$, there is some β such that $x \notin \mathfrak{p}_\beta$, or $x \notin \mathfrak{p}_\alpha$ whenever $\alpha \geq \beta$. So $y \in \mathfrak{p}_\alpha$ whenever $\alpha \geq \beta$. Since $y \in \mathfrak{p}_\beta$, $y \in \mathfrak{p}_\gamma$ whenever $\beta \geq \gamma$. Therefore, $y \in \mathfrak{p}_\alpha$ for all α , or $y \in \mathfrak{p}$, or \mathfrak{p} is prime.

□

Exercise 1.9.

Let \mathfrak{a} be an ideal $\neq (1)$ in a ring A . Show that $\mathfrak{a} = r(\mathfrak{a}) \iff \mathfrak{a}$ is an intersection of prime ideals.

Proof.

- (1) (\implies) . By Proposition 1.14, $\mathfrak{a} = r(\mathfrak{a})$ is the intersection of the prime ideals which contain \mathfrak{a} .
- (2) (\impliedby) .

$$\begin{aligned}
 \mathfrak{a} &= \bigcap \{\mathfrak{p} \in \text{some subset of } \text{Spec}(A)\} \\
 &= \bigcap \{\mathfrak{p} \in \text{some subset of } \text{Spec}(A) : \mathfrak{p} \supseteq \mathfrak{a}\} \\
 &\supseteq \bigcap \{\mathfrak{p} \in \text{Spec}(A) : \mathfrak{p} \supseteq \mathfrak{a}\} \\
 &= r(\mathfrak{a}) \\
 &\supseteq \mathfrak{a}.
 \end{aligned}$$

□

Exercise 1.10.

Let A be a ring, \mathfrak{N} its nilradical. Show the following are equivalent:

- (i) A has exactly one prime ideal;
- (ii) every element of A is either a unit or nilpotent;
- (iii) A/\mathfrak{N} is a field.

Proof.

$$\begin{aligned}
 &A/\mathfrak{N} \text{ is a field} \\
 \implies &\mathfrak{N} \text{ is a maximal ideal} \\
 \implies &\mathfrak{p} = \mathfrak{N} \text{ for every prime ideal } \mathfrak{p} && (\text{Proposition 1.8}) \\
 \implies &A \text{ has exactly one prime ideal } \mathfrak{p} \\
 \implies &\mathfrak{p} = \mathfrak{N} \\
 \implies &A \text{ has exactly one maximal ideal } \mathfrak{p} \\
 \implies &\text{Given any } a \in A, a \text{ is a unit or } a \in \mathfrak{p} = \mathfrak{N}. && (\text{Corollary 1.5}) \\
 \implies &A/\mathfrak{N} \text{ is a field.}
 \end{aligned}$$

□

Exercise 1.11. (Boolean ring)

A ring A is **Boolean** if $x^2 = x$ for all $x \in A$. In a Boolean ring A , show that

- (i) $2x = 0$ for all $x \in A$;
- (ii) every prime ideal \mathfrak{p} is maximal, and A/\mathfrak{p} is a field with two elements;
- (iii) every finitely generated ideal in A is principal.

Proof of (i). Note that $2x = x + x = (x + x)^2 = (2x)^2 = 4x^2 = 4x$. So $2x = 0$. \square

Proof of (ii). Same as Exercise 1.7 with $n = 2$. \square

Proof of (iii).

- (1) By induction, it suffices to show that if $\mathfrak{a} = (x, y)$ is an ideal in A , then $\mathfrak{a} = (z)$ for some $z \in A$.
- (2) Take $z = x + y + xy$. $(z) \subseteq \mathfrak{a}$ obviously.
- (3) Conversely, note that

$$x = x^2 = x(z - y - xy) = xz - \overbrace{xy}^{=2xy=0} - \underbrace{x^2y}_{=xy} = xz \in (z).$$

Also $y \in (z)$ similarly. So $\mathfrak{a} \subseteq (z)$ and thus $\mathfrak{a} = (z)$ is principal.

\square

Exercise 1.12.

A local ring contains no idempotent $\neq 0, 1$.

Proof.

- (1) If e is an idempotent $\neq 0, 1$ in a local ring A with the maximal ideal \mathfrak{m} , then by definition $0 = e(1 - e)$ shows that both $e \neq 0$ and $1 - e \neq 0$ are not unit.
- (2) Thus $e \in \mathfrak{m}$ and $1 - e \in \mathfrak{m}$. So $1 = (1 - e) + e$ is a unit in \mathfrak{m} , which is absurd.

\square

Construction of an algebraic closure of a field (E. Artin)

Exercise 1.13.

Let K be a field and let Σ be the set of all irreducible monic polynomials f in one indeterminate with coefficients in K . Let A be the polynomial ring over K generated by indeterminates x_f , one for each $f \in \Sigma$. Let \mathfrak{a} be the ideal of A generated by the polynomials $f(x_f)$ for all $f \in \Sigma$. Show that $\mathfrak{a} \neq (1)$.

Let \mathfrak{m} be a maximal ideal of A containing \mathfrak{a} and let $K_1 = A/\mathfrak{m}$. Then K_1 is an extension field of K in which each $f \in \Sigma$ has a root. Repeat the construction with K_1 in place of K , obtaining a field K_2 , and so on. Let $L = \bigcup_{n=1}^{\infty} K_n$. Then L is a field in which each $f \in \Sigma$ splits completely into linear factors. Let \overline{K} be the set of all elements of L which are algebraic over K . Then \overline{K} is an algebraic closure of K .

Proof.

- (1) Show that $\mathfrak{a} \neq (1)$. (Reductio ad absurdum) If $\mathfrak{a} = (1)$, then we can write

$$1 = \sum_{i=1}^n g_i(x) f_i(x_{f_i}) \in A$$

where $x = (x_{f_1}, \dots, x_{f_n}, x_{g_1}, \dots, x_{g_r})$ is a tuple with finitely many indeterminates. It is possible since it is a finite sum.

- (2) Let L be an algebraic extension of K such that each f_i has a root $a_i \in L$ ($i = 1, \dots, n$).
- (3) Take $x = (a_1, \dots, a_n, 0, \dots, 0)$ in the equation $1 = \sum_{i=1}^n g_i(x) f_i(x_{f_i})$ to get

$$\begin{aligned} 1 &= \sum_{i=1}^n g_i(a_1, \dots, a_n, 0, \dots, 0) f_i(a_i) \\ &= \sum_{i=1}^n g_i(a_1, \dots, a_n, 0, \dots, 0) \cdot 0 \\ &= 0, \end{aligned}$$

which is absurd.

□

Exercise 1.14.

In a ring A , let Σ be the set of all ideals in which every element is a zero-divisor. Show that the set Σ has maximal elements and that every maximal element of Σ is a prime ideal. Hence the set of zero-divisors in A is a union of prime ideals.

Proof.

- (1) Suppose $1 \neq 0$.
- (2) Show that the set Σ has maximal elements. Order Σ by inclusion. Σ is not empty, since $0 \in \Sigma$. To apply Zorn's lemma we must show that every chain in Σ has an upper bound in Σ ; let then (\mathfrak{a}_α) be a chain of ideals in Σ , so that for each pair of indices α, β we have either $\mathfrak{a}_\alpha \subseteq \mathfrak{a}_\beta$ or $\mathfrak{a}_\beta \subseteq \mathfrak{a}_\alpha$.
- (3) Let $\mathfrak{a} = \bigcup_\alpha \mathfrak{a}_\alpha$. Then \mathfrak{a} is an ideal and every element of \mathfrak{a} is a zero-divisor. Hence $\mathfrak{a} \in \Sigma$, and \mathfrak{a} is an upper bound of the chain. Hence by Zorn's lemma, Σ has maximal elements.
- (4) Show that every maximal element of Σ is a prime ideal. Let \mathfrak{p} be a maximal element in Σ . Suppose $x, y \notin \mathfrak{p}$. Then there are non-zero-divisors in $\mathfrak{p} + (x)$ and $\mathfrak{p} + (y)$, and their product is an element of $\mathfrak{p} + (xy)$ that is again a non-zero-divisor. So $xy \notin \mathfrak{p}$.
- (5) Hence the set of zero-divisors in A is a union of prime ideals (by the construction in (2) and the result of (4)).

□

The prime spectrum of a ring**Lemma 1.15.1.**

For any $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$, $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$.

Proof.

- (1) If $\mathfrak{p} \supseteq \mathfrak{a}$. We are done.
- (2) If $\mathfrak{p} \not\supseteq \mathfrak{a}$, there exists $a \in \mathfrak{a} - \mathfrak{p}$. So for any $b \in \mathfrak{b}$, $b \in \mathfrak{p}$ since $ab \in \mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$ and \mathfrak{p} is a prime ideal, that is, $\mathfrak{p} \supseteq \mathfrak{b}$.

By (1)(2), $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$. □

Exercise 1.15.

Let A be a ring and let X be the set of all prime ideals of A . For each subset E of A , let $V(E)$ denote the set of all prime ideals of A which contain E . Prove that

- (i) if \mathfrak{a} is the ideal generated by E , then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$.
- (ii) $V(0) = X$, $V(1) = \emptyset$.
- (iii) if $(E_i)_{i \in I}$ is any family of subsets of A , then

$$V\left(\bigcup_{i \in I} E_i\right) = \bigcap_{i \in I} V(E_i).$$

- (iv) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ for any ideals $\mathfrak{a}, \mathfrak{b}$ of A .

The results show that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. The resulting topology is called the **Zariski topology**. The topological space X is called the **prime spectrum** of A , and is written $\text{Spec}(A)$.

Note that if $E_1 \subseteq E_2$, then $V(E_1) \supseteq V(E_2)$.

Proof of (i).

- (1) Show that $V(E) = V(\mathfrak{a})$.

- (a) Show that $V(E) \subseteq V(\mathfrak{a})$. Given any $\mathfrak{p} \in V(E)$, $\mathfrak{p} \supseteq E$. For any $a \in \mathfrak{a}$, since \mathfrak{a} is generated by E , we can write a as a finite sum $a = \sum \alpha\beta$ where $\alpha \in A$ and $\beta \in E$. Since $E \subseteq \mathfrak{p}$, all $\beta \in \mathfrak{p}$. Since \mathfrak{p} is an ideal, $a = \sum \alpha\beta \in \mathfrak{p}$. That is, $\mathfrak{p} \supseteq \mathfrak{a}$, or $\mathfrak{p} \in V(\mathfrak{a})$.
- (b) $V(E) \supseteq V(\mathfrak{a})$ since $\mathfrak{a} \supseteq E$.

- (2) Show that $V(\mathfrak{a}) = V(r(\mathfrak{a}))$.

- (a) Show that $V(\mathfrak{a}) \subseteq V(r(\mathfrak{a}))$. Given any $\mathfrak{p} \in V(\mathfrak{a})$,

$$\begin{aligned} \mathfrak{p} \in V(\mathfrak{a}) &\implies \mathfrak{p} \supseteq \mathfrak{a} \\ &\implies \mathfrak{p} \supseteq \text{the intersection of the prime ideals } \mathfrak{p} \supseteq \mathfrak{a} \\ &\implies \mathfrak{p} \supseteq r(\mathfrak{a}) \text{ (by Proposition 1.14)} \\ &\implies \mathfrak{p} \in V(r(\mathfrak{a})). \end{aligned}$$

- (b) $V(\mathfrak{a}) \supseteq V(r(\mathfrak{a}))$ since $r(\mathfrak{a}) \supseteq \mathfrak{a}$.

□

Proof of (ii).

- (1) $V(1) = \emptyset$ since no prime ideal contains 1 by definition.
- (2) $V(0) = X$ since 0 is in every ideal (especially in every prime ideal).

□

Proof of (iii).

$$\begin{aligned}
 \mathfrak{p} \in V\left(\bigcup_{i \in I} E_i\right) &\iff \mathfrak{p} \supseteq \bigcup_{i \in I} E_i \\
 &\iff \mathfrak{p} \supseteq E_i \text{ for all } i \in I \\
 &\iff \mathfrak{p} \in V(E_i) \text{ for all } i \in I \\
 &\iff \mathfrak{p} \in \bigcap_{i \in I} V(E_i).
 \end{aligned}$$

□

Proof of (iv).

- (1) *Show that $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$.*
 - (a) $V(\mathfrak{a} \cap \mathfrak{b}) \subseteq V(\mathfrak{a}\mathfrak{b})$ since $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b}$.
 - (b) *Show that $V(\mathfrak{a} \cap \mathfrak{b}) \supseteq V(\mathfrak{a}\mathfrak{b})$.* Given any $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$, $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$. By Lemma 15.1.1, $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$. Notice that $\mathfrak{a} \supseteq \mathfrak{a} \cap \mathfrak{b}$ and $\mathfrak{b} \supseteq \mathfrak{a} \cap \mathfrak{b}$. In any case, $\mathfrak{p} \supseteq \mathfrak{a} \cap \mathfrak{b}$, $\mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b})$.
- (2) *Show that $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.*
 - (a) *Show that $V(\mathfrak{a}\mathfrak{b}) \subseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$.* Given any $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$, $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$. By Lemma 15.1.1, $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$, $\mathfrak{p} \in V(\mathfrak{a})$ or $\mathfrak{p} \in V(\mathfrak{b})$, $\mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$.
 - (b) *Show that $V(\mathfrak{a}\mathfrak{b}) \supseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$.* Given any $\mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$, $\mathfrak{p} \in V(\mathfrak{a})$ or $\mathfrak{p} \in V(\mathfrak{b})$, $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$. Notice that $\mathfrak{a} \supseteq \mathfrak{a}\mathfrak{b}$ and $\mathfrak{b} \supseteq \mathfrak{a}\mathfrak{b}$. In any cases, $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$, or $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$.

□

Exercise 1.16.

Draw pictures of $\text{Spec}(\mathbb{Z})$, $\text{Spec}(\mathbb{R})$, $\text{Spec}(\mathbb{C}[x])$, $\text{Spec}(\mathbb{R}[x])$, $\text{Spec}(\mathbb{Z}[x])$.

Proof.

- (1) *Show that $\text{Spec}(\mathbb{Z}) = \{(0)\} \cup \{(p) : p \text{ is a rational prime}\}$.* Note that \mathbb{Z} is a PID. So all non-trivial prime ideals are of the form (π) where π are irreducible.

- (2) Show that $\text{Spec}(\mathbb{R}) = \{(0)\}$. Note that \mathbb{R} is a field.
- (3) Show that $\text{Spec}(\mathbb{C}[x]) = \{(0)\} \cup \{(x - z) : z \in \mathbb{C}\}$. Note that $\mathbb{C}[x]$ is a PID and \mathbb{C} is algebraically closed. Hence all non-trivial prime ideals are of the form $(x - z)$ where $z \in \mathbb{C}$.
- (4) Show that $\text{Spec}(\mathbb{R}[x])$ are
- (i) (0) .
 - (ii) $\{(x - r) : r \in \mathbb{R}\}$.
 - (iii) $\{(x - z)(x - \bar{z}) : z \in \mathbb{C}, \text{Im}(z) > 0\}$.

Here is the proof.

- (a) Note that $\mathbb{R}[x]$ is a PID and all non-trivial prime ideals are of the form (f) where f are irreducible. Might assume f is monic. By the fundamental theorem of algebra, f has a root $z \in \mathbb{C}$.
 - (b) The case $r := z \in \mathbb{R}$. $x - r$ is a factor of f . Hence $f = x - r$.
 - (c) The case $z \in \mathbb{C} \setminus \mathbb{R}$. Since the conjugate of f is also in $\mathbb{R}[x]$, \bar{z} is also a root of f . So $(x - z)(x - \bar{z}) \in \mathbb{R}[x]$ is an irreducible factor of f . Hence $f = (x - z)(x - \bar{z})$ by the irreducibility of f .
- (5) Show that $\text{Spec}(\mathbb{Z}[x])$ are
- (i) (0) .
 - (ii) (p) where p are rational primes.
 - (iii) (f) where $f \in \mathbb{Z}[x]$ are irreducible.
 - (iv) (p, f) where p are rational primes and $f \in \mathbb{Z}[x]$ are irreducible when viewed in $\mathbb{F}_p[x]$.

Before giving a proof, it is worth taking a look at the book: *David Mumford, The red book of varieties and schemes*.

- (a) Let $\phi : \mathbb{Z} \rightarrow \mathbb{Z}[x]$ be the natural inclusion map. Hence $\phi^* : \text{Spec}(\mathbb{Z}[x]) \rightarrow \text{Spec}(\mathbb{Z})$ is continuous (Exercise 1.21). Suppose $\mathfrak{P} \in \text{Spec}(\mathbb{Z}[x])$, then $\phi^*(\mathfrak{P}) = (0)$ or (p) where p is a rational prime.
- (b) The case $\phi^*(\mathfrak{P}) = (0)$. A non-trivial prime ideal \mathfrak{P} must be generated by a set of nonconstant polynomials which, since \mathfrak{P} is prime, may be assumed to be irreducible in $\mathbb{Z}[x]$. Note that $\mathbb{Z}[x]$ is not a PID.
- (c) By Gauss' lemma, these polynomials are also irreducible in $\mathbb{Q}[x]$. Since $\mathbb{Q}[x]$ is a Euclidean domain, if there are at least two distinct irreducible polynomials f, g generating \mathfrak{P} , then $1 = af + bg$ for some $a, b \in \mathbb{Q}[x]$. Clearing all denominators to get that $n = \tilde{a}f + \tilde{b}g$ for some $\tilde{a}, \tilde{b} \in \mathbb{Z}[x]$ and some $n \in \mathbb{Z} \setminus \{0\}$, contrary to $\phi^*(\mathfrak{P}) = (0)$. Therefore, $\mathfrak{P} = (f)$ for one irreducible polynomial $f \in \mathbb{Z}[x]$.

(d) The case $\phi^*(\mathfrak{P}) = (p)$ where p is a rational prime. Note that

$$\begin{aligned}\mathbb{Z}[x]/\mathfrak{P} &\cong (\mathbb{Z}[x]/p\mathbb{Z}[x])/(\mathfrak{P}/p\mathbb{Z}[x]) \\ &\cong \underbrace{(\mathbb{Z}/p\mathbb{Z})}_{:=\mathbb{F}_p}[x]/(\mathfrak{P}/p\mathbb{Z}[x])\end{aligned}$$

is an integral domain (since \mathfrak{P} is prime). So $\mathfrak{P}/p\mathbb{Z}[x]$ is a prime ideal in $\mathbb{F}_p[x]$. Note that $\mathbb{F}_p[x]$ is a PID and all non-trivial prime ideals are of the form (f) where f are irreducible.

(e) As $\mathfrak{P}/p\mathbb{Z}[x] = (0)$, $\mathfrak{P} = p\mathbb{Z}[x] = (p) \in \mathbb{Z}[x]$.

(f) As $\mathfrak{P}/p\mathbb{Z}[x] = (f)$ where $f \in \mathbb{Z}[x]$ is irreducible when viewed in $\mathbb{F}_p[x]$, $\mathfrak{P} = (p, f)$.

□

Exercise 1.17.

For each $f \in A$, let X_f denote the complement of $V(f)$ in $X = \text{Spec}(A)$. The sets X_f are open. Show that they form a basis of open sets for the Zariski topology, and that

- (i) $X_f \cap X_g = X_{fg}$.
- (ii) $X_f = \emptyset \iff f$ is nilpotent.
- (iii) $X_f = X \iff f$ is a unit.
- (iv) $X_f = X_g \iff r((f)) = r((g))$.
- (v) X is quasi-compact (compact), that is, every open covering of X has a finite subcovering.
- (vi) More generally, each X_f is quasi-compact.
- (vii) An open subset of X is quasi-compact if and only if it is a finite union of sets X_f .

The sets X_f are called basic open sets of $X = \text{Spec}(A)$.

(Hint: To prove (v), remark that it is enough to consider a covering of X by basic open sets X_{f_i} ($i \in I$). Show that the f_i generate the unit ideal and hence that there is an equation of the form

$$1 = \sum_{i \in J} g_i f_i \quad (g_i \in A)$$

where J is some finite subset of I . Then the X_{f_i} ($i \in J$) cover X .)

Proof of basis. It is equivalent to Exercise 1.15 (iii). Given any open set O in X . Write $O = X - V(\mathfrak{a})$ for some ideal \mathfrak{a} of A . Since

$$V(\mathfrak{a}) = V\left(\bigcup_{f \in \mathfrak{a}} (f)\right) = \bigcap_{f \in \mathfrak{a}} V(f),$$

we have

$$O = X - V(\mathfrak{a}) = X - \bigcap_{f \in \mathfrak{a}} V(f) = \bigcup_{f \in \mathfrak{a}} (X - V(f)) = \bigcup_{f \in \mathfrak{a}} X_f,$$

or any open set is a union of basic open sets. \square

Proof of (i). $X_f \cap X_g = X_{fg} \iff V(f) \cup V(g) = V(fg)$ holds by Exercise 1.15 (iv). \square

Proof of (ii).

$$\begin{aligned} X_f = \emptyset &\iff V(f) = X \\ &\iff f \in \mathfrak{p} \text{ for all prime ideal } \mathfrak{p} \text{ of } A \\ &\iff f \in \mathfrak{N}, \text{ the nilradical of } A \text{ (Proposition 1.8)} \\ &\iff f \text{ is nilpotent (Proposition 1.7)} \end{aligned}$$

\square

Proof of (ii) (Using (iv)).

$$\begin{aligned} X_f = \emptyset &\iff X_f = X_0 && \text{(Exercise 15(ii))} \\ &\iff r(f) = r(0) && \text{((iv))} \\ &\iff f \in r(f) = r(0) \\ &\iff f^m = 0 \text{ for some } m > 0 \\ &\iff f \text{ is nilpotent} \end{aligned}$$

\square

Proof of (iii).

$$\begin{aligned} X_f = X &\iff V(f) = \emptyset \\ &\iff f \notin \mathfrak{p} \text{ for all prime ideal } \mathfrak{p} \text{ of } A \\ &\iff f \text{ is unit (Corollary 1.5)} \end{aligned}$$

\square

Proof of (iii)(Using (iv)).

$$\begin{aligned}
X_f = X &\iff X_f = X_1 && \text{(Exercise 15(ii))} \\
&\iff r(f) = r(1) && \text{((iv))} \\
&\iff f \in r(f) = r(1) \\
&\iff f^m = 1 \text{ for some } m > 0 \\
&\iff f \text{ is unit}
\end{aligned}$$

□

Proof of (iv).

(1) *Show that $X_f \subseteq X_g \iff r((f)) \subseteq r((g))$. Actually,*

$$\begin{aligned}
X_f \subseteq X_g &\implies V(f) \supseteq V(g) \\
&\implies \{\mathfrak{p} \in \text{Spec}(A) : \mathfrak{p} \supseteq (f)\} \supseteq \{\mathfrak{p} \in \text{Spec}(A) : \mathfrak{p} \supseteq (g)\} \\
&\implies \bigcap_{(f) \subseteq \mathfrak{p} \in \text{Spec}(A)} \mathfrak{p} \subseteq \bigcap_{(g) \subseteq \mathfrak{p} \in \text{Spec}(A)} \mathfrak{p} \\
&\stackrel{1.14}{\implies} r(f) \subseteq r(g) \\
&\implies V(r(f)) \supseteq V(r(g)) \\
&\implies V(f) \supseteq V(g) \\
&\implies X_f \subseteq X_g.
\end{aligned}$$

(2) By (1),

$$\begin{aligned}
X_f \subseteq X_g &\iff r((f)) \subseteq r((g)), \\
X_f \supseteq X_g &\iff r((f)) \supseteq r((g)).
\end{aligned}$$

Hence,

$$X_f = X_g \iff r((f)) = r((g)).$$

□

Proof of (v). Notice that it is enough to consider a covering of X by basic open sets X_{f_i} ($i \in I$).

(1) Since X is covered by $X_{f_i} (i \in I)$,

$$\begin{aligned}
X = \bigcup_{i \in I} X_{f_i} &\implies X - V(1) = \bigcup_{i \in I} (X - V(f_i)) \\
&\implies V(1) = \bigcap_{i \in I} V(f_i) \\
&\implies V(1) = V\left(\sum_{i \in I} f_i\right) \\
&\implies r(1) = r\left(\sum_{i \in I} f_i\right).
\end{aligned}$$

Hence, $1 \in r(1) = r\left(\sum_{i \in I} f_i\right)$ can be expressed as

$$1 = 1^m = \sum_{j \in J} g_j f_j$$

where J is a finite subset of I and $g_j \in A$. That is, $(1) = \sum_{j \in J} f_j$.

(2) Hence, $V(1) = V\left(\sum_{j \in J} f_j\right)$. Therefore, X is covered by finite subcovering $\{X_{f_j}\} (j \in J)$.

□

Proof of (v)(Using (vi)). Since $X = X_1$, X is quasi-compact by (vi). □

Proof of (vi). Notice that it is enough to consider a covering of X_f by basic open sets $X_{f_i} (i \in I)$.

(1) Since X_f is covered by $X_{f_i} (i \in I)$,

$$\begin{aligned}
X_f = \bigcup_{i \in I} X_{f_i} &\implies X - V(f) = \bigcup_{i \in I} (X - V(f_i)) \\
&\implies V(f) = \bigcap_{i \in I} V(f_i) \\
&\implies V(f) = V\left(\sum_{i \in I} f_i\right) \\
&\implies r(f) = r\left(\sum_{i \in I} f_i\right).
\end{aligned}$$

Hence, $f \in r(f) = r\left(\sum_{i \in I} f_i\right)$ can be expressed as

$$f^m = \sum_{j \in J} g_j f_j$$

where J is a finite subset of I and $g_j \in A$. That is, $f^m \in \sum_{j \in J} f_j$.

(2) Show that $V\left(\sum_{j \in J} f_j\right) = V(f)$.

(a) (\subseteq) For any prime ideal $\mathfrak{p} \supseteq \sum_{j \in J} f_j$, $f^m \in \mathfrak{p}$ or $f \in \mathfrak{p}$ (since \mathfrak{p} is prime). So $\mathfrak{p} \supseteq (f)$, or $V\left(\sum_{j \in J} f_j\right) \subseteq V(f)$.

(b) (\supseteq)

$$\sum_{j \in J} f_j \subseteq \sum_{i \in I} f_i \implies V\left(\sum_{j \in J} f_j\right) \supseteq V\left(\sum_{i \in I} f_i\right) = V(f).$$

(3) Therefore, X_f is covered by finite subcovering $\{X_{f_j}\}(j \in J)$.

□

Proof of (vi) (Using (v)). Exercise 3.21 (i) shows that X_f is the spectrum of A_f . By (v), X_f is quasi-compact. □

Proof of (vii).

(1) (\implies) Given an open subset O . Since X_f form a basis of open sets,

$$O = \bigcup_{f \in \mathfrak{a}} X_f \text{ for some ideal } \mathfrak{a} \text{ of } A$$

Especially, $\{X_f\}_{f \in \mathfrak{a}}$ is an open covering of O . Since O is quasi-compact, there exists a finite subcovering $\{X_f\}_{f \in J}$ of O , where J is a finite subset of \mathfrak{a} (as a set). That is, $O = \bigcup_{f \in J} X_f$ is a finite union of sets X_f .

(2) (\impliedby) Since X_f is quasi-compact, any finite union of quasi-compact sets is quasi-compact again.

□

Exercise 1.18.

For psychological reasons it is sometimes convenient to denote a prime ideal of A by a letter such as x or y when thinking of it as a point of $X = \text{Spec}(A)$. When thinking of x as a prime ideal of A , we denote it by \mathfrak{p}_x (logically, of course, it is the same thing). Show that

- (i) The set $\{x\}$ is closed (we say that x is a “closed point”) in $\text{Spec}(A)$ if and only if \mathfrak{p}_x is maximal;
- (ii) $\overline{\{x\}} = V(\mathfrak{p}_x)$;
- (iii) $y \in \overline{\{x\}}$ if and only if $\mathfrak{p}_x \subseteq \mathfrak{p}_y$;

- (iv) X is a T_0 -space (this means that if x, y are distinct points of X , then either there is a neighborhood of x which does not contain y , or else there is a neighborhood of y which does not contain x).

Proof of (i).

$$\{x\} = \overline{\{x\}} \stackrel{(ii)}{\iff} \{x\} = V(\mathfrak{p}_x) \iff \mathfrak{p}_x \text{ is maximal.}$$

□

Proof of (ii). Since $\overline{\{x\}}$ is the intersection of all closed sets containing x and Exercise 1.15 (iii), we have

$$\overline{\{x\}} = \bigcap_{\mathfrak{a} \subseteq \mathfrak{p}_x} V(\mathfrak{a}) = V\left(\sum_{\mathfrak{a} \subseteq \mathfrak{p}_x} \mathfrak{a}\right) = V(\mathfrak{p}_x).$$

□

Proof of (iii).

$$y \in \overline{\{x\}} \stackrel{(ii)}{\iff} y \in V(\mathfrak{p}_x) \iff \mathfrak{p}_y \supseteq \mathfrak{p}_x.$$

□

Proof of (iv).

- (1) Suppose x and y are two points in X such that $y \in \overline{\{x\}}$ and $x \in \overline{\{y\}}$. Note that $x = y$ implies that X is a T_0 -space. So it suffices to show that $x = y$.
- (2) By (iii), $\mathfrak{p}_y \supseteq \mathfrak{p}_x$ and $\mathfrak{p}_x \supseteq \mathfrak{p}_y$. So $\mathfrak{p}_x = \mathfrak{p}_y$ or $x = y$.

□

Exercise 1.19.

A topological space X is said to be irreducible if $X \neq \emptyset$ and if every pair of non-empty open sets in X intersect, or equivalently if every non-empty open set is dense in X . Show that $\text{Spec}(A)$ is irreducible if and only if the nilradical of A is a prime ideal.

Proof. Use the notations in Proposition 1.7 and Exercise 1.17.

$\text{Spec}(A)$ is irreducible

$$\begin{aligned} &\iff X_f \cap X_g \neq \emptyset \text{ for nonempty } X_f, X_g \in \text{Spec}(A) \\ &\iff X_{fg} \neq \emptyset \text{ for nonempty } X_f, X_g \in \text{Spec}(A) && \text{(Exercise 1.17 (i))} \\ &\iff fg \notin \mathfrak{N} \text{ for } f, g \notin \mathfrak{N} && \text{(Exercise 1.17 (ii))} \\ &\iff \mathfrak{N} \text{ is prime.} \end{aligned}$$

□

Exercise 1.20.

Let X be a topological space.

- (i) If Y is an irreducible subspace of X , then the closure \overline{Y} of Y in X is irreducible.
- (ii) Every irreducible subspace of X is contained in a maximal irreducible subspace.
- (iii) The maximal irreducible subspaces of X are closed and cover X . They are called the irreducible components of X . What are the irreducible components of a Hausdorff space?
- (iv) If A is a ring and $X = \text{Spec}(A)$, then the irreducible components of X are the closed sets $V(\mathfrak{p})$, where \mathfrak{p} is a minimal prime ideal of A (Exercise 1.8).

Proof of (i).

- (1) Y is irreducible if and only if Y cannot be represented as the union of two proper closed subspaces.

$$\begin{aligned}
 & \forall \text{ nonempty open sets } U_1 \text{ and } U_2, U_1 \cap U_2 \neq \emptyset \\
 \iff & \forall \text{ nonempty open sets } U_1 \text{ and } U_2, X - (U_1 \cap U_2) \neq X \\
 \iff & \forall \text{ nonempty open sets } U_1 \text{ and } U_2, (X - U_1) \cup (X - U_2) \neq X \\
 \iff & \forall \text{ proper closed sets } Y_1 \text{ and } Y_2, Y_1 \cup Y_2 \neq X \\
 \iff & \nexists \text{ proper closed sets } Y_1 \text{ and } Y_2, Y_1 \cup Y_2 = X.
 \end{aligned}$$

- (2) If \overline{Y} were reducible, there are two closed set Y_1 and Y_2 such that

$$\overline{Y} \subseteq Y_1 \cup Y_2, \quad \overline{Y} \not\subseteq Y_i (i = 1, 2).$$

- (a) $Y \subseteq \overline{Y} \subseteq Y_1 \cup Y_2$.
- (b) $Y \not\subseteq Y_i (i = 1, 2)$. If not, $Y \subseteq Y_i$ for some i . Take closure to get $\overline{Y} \subseteq \overline{Y_i} = Y_i$ (since Y_i is closed), contrary to the assumption.

By (a)(b), Y is reducible, which is absurd.

□

Proof of (ii).

- (1) This is a standard application of Zorn's lemma.

- (2) Suppose Y is an irreducible subspace of X . Let Σ be the set of all irreducible subspaces of X containing Y . Order Σ by inclusion. Σ is not empty, since $Y \in \Sigma$. To apply Zorn's lemma we must show that every chain in Σ has an upper bound in Σ ; let then (Y_α) be a chain in Σ . Let $Z = \bigcup_\alpha Y_\alpha$. $Z \supseteq Y$ clearly.
- (3) *Show that Z is irreducible.* Given two non-empty open sets U and V contained in $Z = \bigcup_\alpha Y_\alpha$. Then $U \cap Y_\alpha \neq \emptyset$ and $V \cap Y_\beta \neq \emptyset$ for some α, β . Since (Y_α) is a chain, we might have $V \cap Y_\alpha \supseteq V \cap Y_\beta \neq \emptyset$ if $\beta \leq \alpha$. (The case $\alpha \leq \beta$ is similar.) So $U \cap V \cap Z \supseteq U \cap V \cap Y_\alpha \neq \emptyset$ since Z contains an irreducible subspace Y_α in X .
- (4) Hence $Z \in \Sigma$, and Z is an upper bound of the chain (Y_α) . Hence by Zorn's lemma Σ has a maximal element.

□

Proof of (iii).

- (1) *Show that the maximal irreducible subspaces of X are closed.* Suppose Y is a maximal irreducible subspaces of X . So \overline{Y} of Y in X is irreducible (by part (i)). The maximality of Y implies that $Y = \overline{Y}$.
- (2) *Show that the maximal irreducible subspaces of X cover X .* Note that each element $P \in X$ forms an irreducible subset $\{P\}$ and thus $\{P\}$ is contained in one irreducible component (by (ii)).
- (3) One point subsets are the irreducible components of a Hausdorff space.

□

Proof of (iv).

- (1) Suppose Y is an irreducible components of X . *Show that $Y = V(\mathfrak{p})$ where \mathfrak{p} is a prime ideal.* Similar to the proof of Exercise 1.19.
- (2) *Show that \mathfrak{p} is a minimal prime ideal of A .* Suppose $\mathfrak{q} \subseteq \mathfrak{p}$. Then $V(\mathfrak{q}) \supseteq V(\mathfrak{p})$. By the maximality of $Y = V(\mathfrak{p})$, $V(\mathfrak{q}) = V(\mathfrak{p})$ or $r(\mathfrak{q}) = r(\mathfrak{p})$ or $\mathfrak{q} = \mathfrak{p}$. Hence \mathfrak{p} is a minimal prime ideal of A .

□

Exercise 1.21.

Let $\phi : A \rightarrow B$ be a ring homomorphism. Let $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$. If $\mathfrak{q} \in Y$, then $\phi^{-1}(\mathfrak{q})$ is a prime ideal of A , i.e., a point of X . Hence ϕ induces a mapping $\phi^* : Y \rightarrow X$. Show that

- (i) If $f \in A$ then $\phi^{*-1}(X_f) = Y_{\phi(f)}$, and hence that ϕ^* is continuous.
- (ii) If \mathfrak{a} is an ideal of A , then $\phi^{*-1}(V(\mathfrak{a})) = V(\mathfrak{a}^e)$.
- (iii) If \mathfrak{b} is an ideal of B , then $\overline{\phi^*(V(\mathfrak{b}))} = V(\mathfrak{b}^c)$.
- (iv) If ϕ is surjective, then ϕ^* is a homeomorphism of Y onto the closed subset $V(\ker(\phi))$ of X . (In particular, $\text{Spec}(A)$ and $\text{Spec}(A/\mathfrak{N})$ (where \mathfrak{N} is the nilradical of A) are naturally homeomorphic.)
- (v) If ϕ is injective, then $\phi^*(Y)$ is dense in X . More precisely, $\phi^*(Y)$ is dense in X if and only if $\ker(\phi) \subseteq \mathfrak{N}$.
- (vi) Let $\psi : B \rightarrow C$ be another ring homomorphism. Then $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.
- (vii) Let A be an integral domain with just one nonzero prime ideal \mathfrak{p} , and let K be the field of fractions of A . Let $B = (A/\mathfrak{p}) \times K$. Define $\phi : A \rightarrow B$ by $\phi(x) = (\bar{x}, x)$, where \bar{x} is the image of x in A/\mathfrak{p} . Show that ϕ^* is bijective but not a homeomorphism.

Proof of (i). Since

$$\begin{aligned}
& \mathfrak{q} \in Y_{\phi(f)} = Y - V(\phi(f)) \\
& \iff \mathfrak{q} \notin V(\phi(f)) = \{\text{all prime ideals in } B \text{ containing } \phi(f)\} \\
& \iff \phi(f) \notin \mathfrak{q} \\
& \iff f \notin \phi^{-1}(\mathfrak{q}) \\
& \iff \phi^{-1}(\mathfrak{q}) \notin V(f) = \{\text{all prime ideals in } A \text{ containing } f\} \\
& \iff \phi^*(\mathfrak{q}) = \phi^{-1}(\mathfrak{q}) \in X_f,
\end{aligned}$$

ϕ^* is continuous. \square

Proof of (ii).

- (1) Use the same notation of Proposition 1.17. Show that

$$\mathfrak{b}^c \supseteq \mathfrak{a} \iff \mathfrak{b} \supseteq \mathfrak{a}^e.$$

Suppose $\mathfrak{b}^c \supseteq \mathfrak{a}$, then $\mathfrak{b}^{ce} \supseteq \mathfrak{a}^e$. Proposition 1.17 (i) suggests that $\mathfrak{b} \supseteq \mathfrak{b}^{ce} \supseteq \mathfrak{a}^e$. The converse is similar.

- (2) So

$$\begin{aligned}
& \mathfrak{q} \in \phi^{*-1}(V(\mathfrak{a})) \\
& \iff \phi^*(\mathfrak{q}) \in V(\mathfrak{a}) = \{\text{all prime ideals containing } \mathfrak{a}\} \\
& \iff \phi^*(\mathfrak{q}) \supseteq \mathfrak{a} \\
& \iff \mathfrak{q}^c \supseteq \mathfrak{a} \\
& \iff \mathfrak{q} \supseteq \mathfrak{a}^e \\
& \iff \mathfrak{q} \in V(\mathfrak{a}^e) = \{\text{all prime ideals containing } \mathfrak{a}^e\}.
\end{aligned} \tag{1)$$

□

Proof of (iii).

- (1) Might assume that $\mathfrak{b} = r(\mathfrak{b})$ is radical by Exercise 1.15 (i).
- (2) Show that $\overline{\phi^*(V(\mathfrak{b}))} \supseteq V(\mathfrak{b}^c)$. Write $\overline{\phi^*(V(\mathfrak{b}))} = V(\mathfrak{a})$ for some radical ideal \mathfrak{a} in A since $\phi^*(V(\mathfrak{b}))$ is closed. So

$$\begin{aligned}
 V(\mathfrak{a}^e) &= \phi^{*-1}(V(\mathfrak{a})) = \phi^{*-1}(\overline{\phi^*(V(\mathfrak{b}))}) \supseteq V(\mathfrak{b}) & ((ii)) \\
 \implies r(\mathfrak{a}^e) &\subseteq r(\mathfrak{b}) \\
 \implies r(\mathfrak{a})^e &\subseteq r(\mathfrak{a}^e) \subseteq r(\mathfrak{b}) \\
 \implies \mathfrak{a}^e &\subseteq \mathfrak{b} \\
 \implies \mathfrak{a} &\subseteq \mathfrak{b}^c \\
 \implies V(\mathfrak{a}) &\supseteq V(\mathfrak{b}^c).
 \end{aligned}$$

- (3) Show that $\overline{\phi^*(V(\mathfrak{b}))} \subseteq V(\mathfrak{b}^c)$. It suffices to show that $\phi^*(V(\mathfrak{b})) \subseteq V(\mathfrak{b}^c)$ since $V(\mathfrak{b}^c)$ is closed. Suppose $\mathfrak{p} \in \phi^*(V(\mathfrak{b}))$. Then there is $\mathfrak{q} \in V(\mathfrak{b})$ such that

$$\mathfrak{p} = \phi^*(\mathfrak{q}) = \mathfrak{q}^c \supseteq \mathfrak{b}^c.$$

So $\mathfrak{p} \in V(\mathfrak{b}^c)$.

□

Proof of (iv). Note that $A/\ker \phi \cong B$ since ϕ is surjective. The correspondence theorem shows that $\phi^* : Y \rightarrow V(\ker \phi)$ is bijective. As the continuity of ϕ^* is given by (i), ϕ^* is a homeomorphism of Y onto $V(\ker(\phi)) \subseteq X$. □

Proof of (v).

- (1) It suffices to show that $\phi^*(Y)$ is dense in X if and only if $\ker(\phi) \subseteq \mathfrak{N}$.
- (2)

$$\begin{aligned}
 &\phi^*(Y) \text{ is dense in } X \\
 \iff &X = \overline{\phi^*(Y)} = \overline{\phi^*(V(0))} = V(0^c) = V(\ker \phi) \\
 \iff &\ker \phi \text{ is contained in every prime ideal of } A \\
 \iff &\ker \phi \subseteq \mathfrak{N}.
 \end{aligned}$$

□

Proof of (vi).

$$(\psi \circ \phi)^*(\mathfrak{p}) = (\psi \circ \phi)^{-1}(\mathfrak{p}) = \phi^{-1}(\psi^{-1}(\mathfrak{p})) = \phi^*(\psi^*(\mathfrak{p})) = (\phi^* \circ \psi^*)(\mathfrak{p})$$

for every prime ideal \mathfrak{p} in $\text{Spec}(C)$. □

Proof of (vii).

- (1) Show that ϕ^* is bijective. Note that

$$\begin{aligned} X &= \text{Spec}(A) = \{(0), \mathfrak{p}\} \\ Y &= \text{Spec}(B) = \{A/\mathfrak{p} \times (0), (0) \times K\} \end{aligned}$$

and thus

$$\begin{aligned} \phi^*(A/\mathfrak{p} \times (0)) &= (0) \\ \phi^*((0) \times K) &= (\mathfrak{p}). \end{aligned}$$

Hence ϕ^* is a bijection.

- (2) Show that ϕ^* is not a homeomorphism. Note that $\overline{\{(0)\}} = X$ (Exercise 1.18 (iii)) and Y is equipped with the discrete topology since each prime ideal of B is maximal (Exercise 1.18 (i)). So ϕ^* cannot be a homeomorphism.

□

Exercise 1.22.

Let $A = \prod_{i=1}^n A_i$ be a direct product of rings A_i . Show that $\text{Spec}(A)$ is the disjoint union of open (and closed) subspaces X_i , where X_i is canonically homeomorphic with $\text{Spec}(A_i)$.

Conversely, let A be any ring. Show that the following statements are equivalent:

- (i) $X = \text{Spec}(A)$ is disconnected.
- (ii) $A \cong A_1 \times A_2$ where neither of the rings A_1, A_2 is the zero ring.
- (iii) A contains an idempotent $\neq 0, 1$. In particular, the spectrum of a local ring is always connected (Exercise 1.12).

Proof.

- (1) Show that $\text{Spec}(A)$ is the union of closed subspaces X_i , where $X_i \cong \text{Spec}(A_i)$. Let $\phi_i : A \rightarrow A_i$ be the projection map. So

$$\ker \phi_i = A_1 \times \cdots \times A_{i-1} \times 0 \times A_{i+1} \times \cdots \times A_n.$$

So

$$\text{Spec}(A) = V(0) = V\left(\bigcap_{i=1}^n \ker \phi_i\right) = \bigcup_{i=1}^n V(\ker \phi_i)$$

where $X_i := V(\ker \phi_i) \cong \text{Spec}(A_i)$ (Exercise 1.21).

(2) Show that $V(\ker \phi_i)$ and $V(\ker \phi_j)$ are disjoint if $i \neq j$.

$$V(\ker \phi_i) \cap V(\ker \phi_j) = V(\ker \phi_i + \ker \phi_j) = V(A) = V(1) = \emptyset.$$

(3) Show that $V(\ker \phi_i)$ is open. $\text{Spec}(A) = \bigcup_{j=1}^n V(\ker \phi_j)$ and $V(\ker \phi_i) \cap V(\ker \phi_j) = \emptyset$ (if $i \neq j$) implies that $\text{Spec}(A) \setminus V(\ker \phi_i) = \bigcup_{j \neq i} V(\ker \phi_j)$ is closed. Thus $V(\ker \phi_i)$ is open.

(4) ((ii) \implies (i)) See (1)(2)(3).

(5) ((i) \implies (iii)) Write X as a disjoint union of two nonempty closed sets $V(\mathfrak{a}), V(\mathfrak{b})$ where $\mathfrak{a}, \mathfrak{b}$ are radical ideals in A (Exercise 1.15). Since

$$\begin{aligned} V(0) = X &= V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{ab}) \\ V(1) = \emptyset &= V(\mathfrak{a}) \cap V(\mathfrak{b}) = V(\mathfrak{a} + \mathfrak{b}), \end{aligned}$$

there exist $a \in \mathfrak{a}, b \in \mathfrak{b}$ such that $a + b = 1$ and $(ab)^n = 0$ for one positive integer n . So $ab = 0$ since \mathfrak{ab} is radical. (Note that $\mathfrak{a} + \mathfrak{b} = 1$ and Exercise 1.13 on page 9.) So

$$a^2 = a(1 - b) = a - ab = a$$

is an idempotent. Also $a \neq 0, 1$ since $V(\mathfrak{a}), V(\mathfrak{b})$ are proper subsets of X .

(6) ((iii) \implies (ii)) Take an idempotent $e \neq 0, 1$ in A . Two ideals (e) and $(1 - e)$ are proper and coprime. So $(e) \cap (1 - e) = (e)(1 - e) = (0)$ (Proposition 1.10 (i)). Proposition 1.10 (ii) and (iii) imply that the ring homomorphism

$$A \rightarrow A/(e) \times A/(1 - e)$$

is an isomorphism. Also $A/(e), A/(1 - e) \neq 0$ since $e \neq 0, 1$.

□

Exercise 1.23.

Let A be a Boolean ring (Exercise 1.11), and let $X = \text{Spec}(A)$.

- (i) For each $f \in A$, the set X_f (Exercise 1.17) is both open and closed in X .
- (ii) Let $f_1, \dots, f_n \in A$. Show that $X_{f_1} \cup \dots \cup X_{f_n} = X_f$ for some $f \in A$.
- (iii) The sets X_f are the only open subsets of X which are both open and closed.
- (iv) X is a compact Hausdorff space.

Proof of (i).

- (1) Show that X is the disjoint union of subspaces X_f and X_{1-f} . Note that every element in a Boolean ring is an idempotent. Hence

$$\begin{aligned} X_f \cap X_{1-f} &= X_{f(1-f)} = X_0 = \emptyset \\ X_f \cup X_{1-f} &= X \setminus (V(f) \cap V(1-f)) = X \setminus \underbrace{V(f + (1-f))}_{=V(1)=\emptyset} = X. \end{aligned}$$

- (2) Hence $X_f = X \setminus X_{1-f}$ is both open and closed.

□

Proof of (ii). Similar to (i),

$$\begin{aligned} X_{f_1} \cup \cdots \cup X_{f_n} &= X \setminus (V(f_1) \cap \cdots \cap V(f_n)) \\ &= X \setminus V(f_1, \dots, f_n) \\ &= X \setminus V(f) && \text{(Exercise 1.11 (iii))} \\ &= X_f \end{aligned}$$

for some $f \in A$. □

Proof of (iii).

- (1) Suppose Y is both open and closed in X .
- (2) Since Y is closed and X is quasi-compact (Exercise 1.17 (vi)), Y is quasi-compact.
- (3) Since Y is open, Y is a finite union of sets X_{f_i} for $i = 1, \dots, n$ (Exercise 1.17 (vii)). Hence $Y = X_f$ for some $f \in A$ (by (ii)).

□

Proof of (iv).

- (1) The compactness of X is followed by Exercise 1.17 (v).
- (2) Show that X is Hausdorff. Exercise 1.18 shows that X is a T_0 -space. This means that if x, y are distinct points of X , we might assume that there is a neighborhood U of x which does not contain y .
- (3) Write $U = X_f$ for some $f \in A$ (by Exercise 1.17 and (ii)). As $x \in X_f$, $y \in X \setminus X_f = X_{1-f}$ and $X_f \cap X_{1-f} = \emptyset$ by (i). Hence X is Hausdorff.

□

Exercise 1.26. (Maximal spectrum)

Let A be a ring. The subspace of $\text{Spec}(A)$ consisting of the maximal ideals of A , with the induced topology, is called the **maximal spectrum** of A and is denoted by $\text{Max}(A)$. For arbitrary commutative rings it does not have the nice functorial properties of $\text{Spec}(A)$ (see Exercise 1.21), because the inverse image of a maximal ideal under a ring homomorphism need not be maximal.

Let X be a compact Hausdorff space and let $C(X)$ denote the ring of all real-valued continuous functions on X (add and multiply functions by adding and multiplying their values). For each $x \in X$, let \mathfrak{m}_x be the set of all $f \in C(X)$ such that $f(x) = 0$. The ideal \mathfrak{m}_x is maximal, because it is the kernel of the (surjective) homomorphism $C(X) \rightarrow \mathbb{R}$ which takes f to $f(x)$. If \tilde{X} denotes $\text{Max}(C(X))$, we have therefore defined a mapping $\mu : X \rightarrow \tilde{X}$, namely $x \mapsto \mathfrak{m}_x$.

We shall show that μ is a homeomorphism of X onto \tilde{X} .

- (i) Let \mathfrak{m} be any maximal ideal of $C(X)$, and let $V = V(\mathfrak{m})$ be the set of common zeros of the functions in \mathfrak{m} : that is,

$$V = \{x \in X : f(x) = 0 \text{ for all } f \in \mathfrak{m}\}.$$

Suppose that V is empty. Then for each $x \in X$ there exists $f_x \in \mathfrak{m}$ such that $f_x(x) \neq 0$. Since f_x is continuous, there is an open neighborhood U_x of x in X on which f_x does not vanish. By compactness a finite number of the neighborhoods, say U_{x_1}, \dots, U_{x_n} , cover X . Let

$$f = f_{x_1}^2 + \dots + f_{x_n}^2.$$

Then f does not vanish at any point of X , hence is a unit in $C(X)$. But this contradicts $f \in \mathfrak{m}$, hence V is not empty. Let x be a point of V . Then $\mathfrak{m} \subseteq \mathfrak{m}_x$, hence $\mathfrak{m} = \mathfrak{m}_x$ because \mathfrak{m} is maximal. Hence μ is surjective.

- (ii) By Urysohn's lemma (this is the only non-trivial fact required in the argument) the continuous functions separate the points of X . Hence $x \neq y \implies \mathfrak{m}_x \neq \mathfrak{m}_y$, and therefore μ is injective.

- (iii) Let $f \in C(X)$; let

$$U_f = \{x \in X : f(x) \neq 0\}$$

and let

$$\tilde{U}_f = \{\mathfrak{m} \in \tilde{X} : f \notin \mathfrak{m}\}.$$

Show that $\mu(U_f) = \tilde{U}_f$. The open sets U_f (resp. \tilde{U}_f) form a basis of the topology of X (resp. \tilde{X}) and therefore μ is a homeomorphism. Thus X can be reconstructed from the ring of functions $C(X)$.

Proof.

- (1) *Show that the inverse image of a maximal ideal under a ring homomorphism need not be maximal.* Let $\phi : \mathbb{Z}[x] \rightarrow \mathbb{R}[x]$ be a natural inclusion map. The ideal $\mathfrak{P} = (x)$ in $\mathbb{R}[x]$ is maximal. But $\phi^{-1}(\mathfrak{P}) = (x)$ in $\mathbb{Z}[x]$ is not maximal since $(x) \subsetneq (x, 2)$ in $\mathbb{Z}[x]$.

- (2) *Show that $\mu(U_f) = \tilde{U}_f$.*

$$\begin{aligned} x \in U_f &\iff x \in X \text{ such that } f(x) \neq 0 \\ &\iff x \in X \text{ such that } f \notin \mathfrak{m}_x \\ &\iff \mathfrak{m}_x \in \tilde{X} \text{ such that } f \notin \mathfrak{m}_x \\ &\iff \mu(x) = \mathfrak{m}_x \in \tilde{U}_f. \end{aligned}$$

- (3) *Show that U_f form a basis of the topology of X .* Let U be open in X . For any $x \in U$, it suffices to find $f \in C(X)$ such that $x \in U_f \subseteq U$. Note that one-point set $\{x\}$ is closed (since X is Hausdorff). By Urysohn's lemma, there is $f \in C(X)$ such that $f = 1$ on $\{x\}$ and $f = 0$ on $X \setminus U$.

- (4) *Show that \tilde{U}_f form a basis of the topology of \tilde{X} .* Let $\tilde{U} = \tilde{W} \cap \tilde{X}$ be open in \tilde{X} where \tilde{W} is open in $\text{Spec}(C(X))$ (w.r.t. the induced topology). For any $\mathfrak{m} \in \tilde{U} = \tilde{W} \cap \tilde{X} \subseteq \tilde{W}$, Exercise 1.17 shows that

$$\mathfrak{m} \in \text{Spec}(C(X))_f \subseteq \tilde{W}$$

for some $f \in C(X)$. So

$$\mathfrak{m} \in \underbrace{\text{Spec}(C(X))_f}_{=\tilde{U}_f} \cap \tilde{X} \subseteq \underbrace{\tilde{W} \cap \tilde{X}}_{=\tilde{U}}.$$

□

Affine algebraic varieties

Exercise 1.27. (Hilbert's Nullstellensatz)

Let k be an algebraically closed field and let

$$f_\alpha(t_1, \dots, t_n) = 0$$

be a set of polynomial equations in n variables with coefficients in k . The set X of all points $x = (x_1, \dots, x_n) \in k^n$ which satisfy these equations is an **affine algebraic variety**.

Consider the set of all polynomials $g \in k[t_1, \dots, t_n]$ with the property that $g(x) = 0$ for all $x \in X$. This set is an ideal $I(X)$ in the polynomial ring, and is called the **ideal of the variety** X . The quotient ring

$$P(X) = k[t_1, \dots, t_n]/I(X)$$

is the ring of polynomial functions on X , because two polynomials g, h define the same polynomial function on X if and only if $g - h$ vanishes at every point of X , that is, if and only if $g - h \in I(X)$.

Let ξ_i be the image of t_i in $P(X)$. The ξ_i ($1 \leq i \leq n$) are the **coordinate functions** on X : if $x \in X$, then $\xi_i(x)$ is the i th coordinate of x . $P(X)$ is generated as a k -algebra by the coordinate functions, and is called the **coordinate ring** (or affine algebra) of X .

As in Exercise 1.26, for each $x \in X$ let \mathfrak{m}_x be the ideal of all $f \in P(X)$ such that $f(x) = 0$; it is a maximal ideal of $P(X)$. Hence, if $\tilde{X} = \text{Max}(P(X))$, we have defined a mapping $\mu : X \rightarrow \tilde{X}$, namely $x \mapsto \mathfrak{m}_x$. It is easy to show that μ is injective: if $x \neq y$, we must have $x_i \neq y_i$ for some i ($1 \leq i \leq n$), and hence $\xi_i - x_i$ is in \mathfrak{m}_x but not in \mathfrak{m}_y , so that $\mathfrak{m}_x \neq \mathfrak{m}_y$. What is less obvious (but still true) is that μ is surjective. This is one form of Hilbert's Nullstellensatz (see Chapter 7).

Proof.

- (1) Show that μ is surjective. If \mathfrak{m} is a maximal ideal of $P(X)$, then $B := P(X)/\mathfrak{m}$ is a finitely generated k -algebra. Note that B is also a field, Corollary 5.24 implies that B is a finite algebraic extension of k .
- (2) In fact, $B \cong k$ since $k = \bar{k}$. Let x_i be the image of ξ_i in k for each i . So $\xi_i - x_i = 0 \in k \cong B$ or $\xi_i - x_i \in \mathfrak{m}$. So

$$\mathfrak{m} \subseteq (\xi_1 - x_1, \dots, \xi_n - x_n) = \mathfrak{m}_x.$$

Hence $\mathfrak{m} = \mathfrak{m}_x$ by the maximality of \mathfrak{m} .

□

Chapter 2: Modules

Exercise 2.1.

Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$ if m, n are coprime.

It suffices to show that

$$(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$$

where d is the greatest common divisor of m and n .

Outlines.

- (1) Define $\tilde{\varphi}$ by

$$\begin{array}{ccc} \tilde{\varphi}: & (\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z}) & \longrightarrow \mathbb{Z}/d\mathbb{Z} \\ & \Downarrow & \Downarrow \\ & (x + m\mathbb{Z}, y + n\mathbb{Z}) & \longmapsto xy + d\mathbb{Z}. \end{array}$$

$\tilde{\varphi}$ is well-defined and \mathbb{Z} -bilinear.

- (2) By the universal property, $\tilde{\varphi}$ factors through a \mathbb{Z} -bilinear map

$$\varphi: (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \rightarrow \mathbb{Z}/d\mathbb{Z}$$

(such that $\varphi(x \otimes y) = \tilde{\varphi}(x, y)$).

- (3) To show that φ is isomorphic, might find the inverse map $\psi: \mathbb{Z}/d\mathbb{Z} \rightarrow (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$ of φ . Define ψ by

$$\begin{array}{ccc} \psi: & \mathbb{Z}/d\mathbb{Z} & \longrightarrow (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \\ & \Downarrow & \Downarrow \\ & z + d\mathbb{Z} & \longmapsto (z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}). \end{array}$$

ψ is well-defined and \mathbb{Z} -linear.

- (4) $\psi \circ \varphi = \text{id}$.

- (5) $\varphi \circ \psi = \text{id}$.

Proof of (1).

- (a) $\tilde{\varphi}$ is well-defined. Say $x' = x + am$ for some $a \in \mathbb{Z}$ and $y' = y + bn$ for some $b \in \mathbb{Z}$. Then $x'y' - xy = yam + xbn + abmn \in \mathbb{Z}/d\mathbb{Z}$. That is, $\tilde{\varphi}$ is independent of coset representative.

(b) $\tilde{\varphi}$ is \mathbb{Z} -bilinear.

(i) For any $\lambda \in \mathbb{Z}$, $\tilde{\varphi}(\lambda x, y) = \tilde{\varphi}(x, \lambda y) = \lambda \tilde{\varphi}(x, y)$. In fact,

$$\begin{aligned}\tilde{\varphi}(\lambda(x + m\mathbb{Z}), y + n\mathbb{Z}) &= \tilde{\varphi}(\lambda x + m\mathbb{Z}, y + n\mathbb{Z}) = \lambda xy + d\mathbb{Z}, \\ \tilde{\varphi}(x + m\mathbb{Z}, \lambda(y + n\mathbb{Z})) &= \tilde{\varphi}(x + m\mathbb{Z}, \lambda y + n\mathbb{Z}) = \lambda xy + d\mathbb{Z}, \\ \tilde{\varphi}(x_1 + m\mathbb{Z}, y + n\mathbb{Z}) &= \lambda(xy + d\mathbb{Z}) = \lambda xy + d\mathbb{Z}.\end{aligned}$$

(ii) $\tilde{\varphi}(x_1 + x_2, y) = \tilde{\varphi}(x_1, y) + \tilde{\varphi}(x_2, y)$. In fact,

$$\begin{aligned}\tilde{\varphi}((x_1 + x_2) + m\mathbb{Z}, y + n\mathbb{Z}) &= (x_1 + x_2)y + d\mathbb{Z}, \\ \tilde{\varphi}(x_1 + m\mathbb{Z}, y + n\mathbb{Z}) + \tilde{\varphi}(x_2 + m\mathbb{Z}, y + n\mathbb{Z}) &= (x_1 y + d\mathbb{Z}) + (x_2 y + d\mathbb{Z}) \\ &= (x_1 + x_2)y + d\mathbb{Z}.\end{aligned}$$

(iii) $\tilde{\varphi}(x, y_1 + y_2) = \tilde{\varphi}(x, y_1) + \tilde{\varphi}(x, y_2)$. Similar to (ii).

□

Proof of (3).

(a) ψ is well-defined. Say $z' = z + cd$ for some $c \in \mathbb{Z}$. Note that $d = \alpha m + \beta n$ for some $\alpha, \beta \in \mathbb{Z}$. Thus

$$\begin{aligned}\psi(z' + d\mathbb{Z}) &= \psi(z + cd + d\mathbb{Z}) \\ &= \psi(z + c(\alpha m + \beta n) + d\mathbb{Z}) \\ &= (z + c(\alpha m + \beta n) + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\ &= (z + c\beta n + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\ &= (z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) + (c\beta n + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\ &= \psi(z + d\mathbb{Z}) + (1 + m\mathbb{Z}) \otimes (c\beta n + n\mathbb{Z}) \\ &= \psi(z + d\mathbb{Z}).\end{aligned}$$

(b) ψ is \mathbb{Z} -linear.

(i) For any $\lambda \in \mathbb{Z}$, $\psi(\lambda z) = \lambda \psi(z)$. In fact,

$$\begin{aligned}\psi(\lambda(z + d\mathbb{Z})) &= \psi(\lambda z + d\mathbb{Z}) = (\lambda z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}), \\ \lambda \psi(z + d\mathbb{Z}) &= \lambda((z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z})) = (\lambda z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}).\end{aligned}$$

(ii) $\psi(z_1 + z_2) = \psi(z_1) + \psi(z_2)$.

$$\begin{aligned}\psi((z_1 + z_2) + d\mathbb{Z}) &= (z_1 + z_2 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}), \\ \psi(z_1 + d\mathbb{Z}) + \psi(z_2 + d\mathbb{Z}) &= (z_1 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) + (z_2 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\ &= (z_1 + z_2 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}).\end{aligned}$$

□

Proof of (4). For any $(x + m\mathbb{Z}) \otimes (y + n\mathbb{Z}) \in (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$,

$$\begin{aligned}\psi(\varphi((x + m\mathbb{Z}) \otimes (y + n\mathbb{Z}))) &= \psi(xy + d\mathbb{Z}) \\ &= (xy + m\mathbb{Z}) \otimes (1 + n\mathbb{Z}) \\ &= (x + m\mathbb{Z}) \otimes (y + n\mathbb{Z}).\end{aligned}$$

□

Proof of (5). For any $z + d\mathbb{Z} \in \mathbb{Z}/d\mathbb{Z}$,

$$\begin{aligned}\varphi(\psi(z + d\mathbb{Z})) &= \varphi((z + m\mathbb{Z}) \otimes (1 + n\mathbb{Z})) \\ &= z + d\mathbb{Z}.\end{aligned}$$

□

Exercise 2.2.

Let A be a ring, \mathfrak{a} an ideal, M an A -module. Show that $(A/\mathfrak{a}) \otimes_A M$ is isomorphic to $M/\mathfrak{a}M$. (Hint: Tensor the exact sequence $0 \rightarrow \mathfrak{a} \rightarrow A \rightarrow A/\mathfrak{a} \rightarrow 0$ with M .)

Proof (Hint). There is a natural exact sequence E :

$$E : 0 \rightarrow \mathfrak{a} \xrightarrow{i} A \xrightarrow{\pi} A/\mathfrak{a} \rightarrow 0$$

where i is the inclusion map (and π is the projection map). Tensor E with M :

$$E' : \mathfrak{a} \otimes_A M \xrightarrow{i \otimes 1} A \otimes_A M \xrightarrow{\pi \otimes 1} (A/\mathfrak{a}) \otimes_A M \rightarrow 0$$

is exact, or

$$(A/\mathfrak{a}) \otimes_A M \cong A \otimes_A M / \text{im}(i \otimes 1).$$

By Proposition 2.14, There is an unique isomorphism $A \otimes_A M \rightarrow M$ defined by $a \otimes x \mapsto ax$. This isomorphism sends $\text{im}(i \otimes 1)$ to $\mathfrak{a}M$. Therefore,

$$(A/\mathfrak{a}) \otimes_A M \cong M/\mathfrak{a}M.$$

□

Proof (Brute-force).

(1) Define $\tilde{\varphi}$ by

$$\begin{array}{ccc} \tilde{\varphi}: & A/\mathfrak{a} \times M & \longrightarrow M/\mathfrak{a}M \\ & \Downarrow & \Downarrow \\ & (a + \mathfrak{a}, x) & \longmapsto ax + \mathfrak{a}M. \end{array}$$

$\tilde{\varphi}$ is well-defined and A -bilinear.

- (2) By the universal property, $\tilde{\varphi}$ factors through a A -bilinear map

$$\varphi: A/\mathfrak{a} \otimes_A M \rightarrow M/\mathfrak{a}M$$

(such that $\varphi(a \otimes x) = \tilde{\varphi}(a, x)$).

- (3) To show that φ is isomorphic, might find the inverse map $\psi: M/\mathfrak{a}M \rightarrow A/\mathfrak{a} \otimes_A M$ of φ . Define ψ by

$$\begin{array}{ccc} \psi: & M/\mathfrak{a}M & \longrightarrow A/\mathfrak{a} \otimes_A M \\ & \Downarrow & \Downarrow \\ & x + \mathfrak{a}M & \longmapsto (1 + \mathfrak{a}) \otimes x. \end{array}$$

ψ is well-defined and A -linear.

- (4) $\psi \circ \varphi = \text{id}$.

- (5) $\varphi \circ \psi = \text{id}$.

□

Exercise 2.3.

Let A be a local ring, M and N finitely generated A -modules. Prove that if $M \otimes_A N = 0$, then $M = 0$ or $N = 0$. (Hint: Let \mathfrak{m} be the maximal ideal, $k = A/\mathfrak{m}$ the residue field. Let $M_k = k \otimes_A M \cong M/\mathfrak{m}M$ by Exercise 2.2. By Nakayama's lemma, $M_k = 0 \implies M = 0$. But $M \otimes_A N = 0 \implies (M \otimes_A N)_k = 0 \implies M_k \otimes_k N_k = 0 \implies M_k = 0$ or $N_k = 0$ since M_k, N_k are vector spaces over a field.)

The conclusion might be false if A is not local. For example, Exercise 2.1.

Proof (Hint). Let \mathfrak{m} be the maximal ideal, $k = A/\mathfrak{m}$ the residue field. Let $M_k = k \otimes_A M$.

- (1) (*Base extension*) Show that $(M \otimes_A N)_k = M_k \otimes_k N_k$. In fact, by Proposition 2.14

$$\begin{aligned}
 (M \otimes_A N)_k &= k \otimes_A (M \otimes_A N) \\
 &= (k \otimes_A M) \otimes_A N \\
 &= M_k \otimes_A N \\
 &= (M_k \otimes_k k) \otimes_A N \\
 &= M_k \otimes_k (k \otimes_A N) \\
 &= M_k \otimes_k N_k.
 \end{aligned}$$

(2)

$$\begin{aligned}
 M \otimes_A N = 0 &\implies (M \otimes_A N)_k = 0 \\
 &\implies M_k \otimes_k N_k = 0 && ((1)) \\
 &\implies M_k = 0 \text{ or } N_k = 0 && (M_k, N_k: \text{ vector spaces}) \\
 &\implies M/\mathfrak{m}M = 0 \text{ or } M/\mathfrak{m}M = 0 && (\text{Exercise 2.2}) \\
 &\implies M = 0 \text{ or } N = 0. && (\text{Nakayama's lemma})
 \end{aligned}$$

□

Exercise 2.4.

Let M_i ($i \in I$) be any family of A -modules, and let M be their direct sum. Prove that M is flat \Leftrightarrow each M_i is flat.

Proof. Given any A -module homomorphism $f : N' \rightarrow N$.

- (1) Similar to Proposition 2.14(iii), we have two isomorphisms

(a)

$$\varphi : \bigoplus_{i \in I} (N' \otimes M_i) \cong N' \otimes_A \bigoplus_{i \in I} M_i$$

defined by

$$\varphi((x \otimes m_i)_{i \in I}) = x \otimes (m_i)_{i \in I}$$

where $x \in N'$, $m_i \in M_i$ ($i \in I$).

(b)

$$\psi : N \otimes_A \bigoplus_{i \in I} M_i \cong \bigoplus_{i \in I} (N \otimes M_i)$$

defined by

$$\psi(y \otimes (m_i)_{i \in I}) = (y \otimes m_i)_{i \in I}$$

where $y \in N$, $m_i \in M_i$ ($i \in I$).

(2) $f : N' \rightarrow N$ induces an A -module homomorphism

$$f \otimes \text{id}_M : N' \otimes_A M \rightarrow N \otimes_A M.$$

(3) $\psi \circ f \otimes \text{id}_M \circ \varphi$ defines an A -module homomorphism

$$\psi \circ f \otimes \text{id}_M \circ \varphi : \bigoplus_{i \in I} (N' \otimes M_i) \rightarrow \bigoplus_{i \in I} (N \otimes M_i)$$

which sends $(x \otimes m_i)_{i \in I}$ to $(f(x) \otimes m_i)_{i \in I}$. That is,

$$\psi \circ f \otimes \text{id}_M \circ \varphi = \bigoplus_{i \in I} f \otimes \text{id}_{M_i}$$

(4) Show that M is flat if and only if each M_i is flat. Suppose f is injective.

$$\begin{aligned} & M_i \text{ is flat } \forall i \in I \\ \iff & f \otimes \text{id}_{M_i} \text{ is injective } \forall i \in I \\ \iff & \bigoplus_{i \in I} f \otimes \text{id}_{M_i} \text{ is injective} && \text{(Injectivity)} \\ \iff & \psi \circ f \otimes \text{id}_M \circ \varphi \text{ is injective} && ((3)) \\ \iff & f \otimes \text{id}_M \text{ is injective} && (\varphi, \psi \text{ are isomorphic}) \\ \iff & M \text{ is flat.} \end{aligned}$$

□

Exercise 2.5.

Let $A[x]$ be the ring of polynomials in one indeterminate over a ring A . Prove that $A[x]$ is a flat A -algebra. (Hint: Use Exercise 2.4.)

Proof (Hint).

(1) A is a flat A -module by Proposition 2.14(iv).

(2) As an A -module,

$$A[x] \cong \bigoplus_{n \in \mathbb{Z}^+} Ax^n \cong \bigoplus_{n \in \mathbb{Z}^+} A$$

(since $Ax^n \cong A$).

(3) By Exercise 2.4, $A[x] \cong \bigoplus_{n \in \mathbb{Z}^+} A$ is flat.

□

Exercise 2.8.

- (i) If M and N are flat A -modules, then so is $M \otimes_A N$.
- (ii) If B is a flat A -algebra and N is a flat B -module, then N is flat as A -module.

Proof of (i). Given any exact sequence of A -modules $0 \rightarrow N_1 \rightarrow N_2 \rightarrow N_3 \rightarrow 0$. Since M is flat,

$$0 \rightarrow N_1 \otimes_A M \rightarrow N_2 \otimes_A M \rightarrow N_3 \otimes_A M \rightarrow 0$$

is exact. Since N is flat,

$$0 \rightarrow (N_1 \otimes_A M) \otimes_A N \rightarrow (N_2 \otimes_A M) \otimes_A N \rightarrow (N_3 \otimes_A M) \otimes_A N \rightarrow 0$$

is exact. By Proposition 2.14 (ii),

$$0 \rightarrow N_1 \otimes_A (M \otimes_A N) \rightarrow N_2 \otimes_A (M \otimes_A N) \rightarrow N_3 \otimes_A (M \otimes_A N) \rightarrow 0$$

is exact, or $M \otimes_A N$ is flat. \square

Proof of (ii). Given any exact sequence of A -modules $0 \rightarrow N_1 \rightarrow N_2 \rightarrow N_3 \rightarrow 0$. Since B is a flat A -algebra (A -module),

$$0 \rightarrow N_1 \otimes_A B \rightarrow N_2 \otimes_A B \rightarrow N_3 \otimes_A B \rightarrow 0$$

is exact. Since N is a flat B -module,

$$0 \rightarrow (N_1 \otimes_A B) \otimes_B N \rightarrow (N_2 \otimes_A B) \otimes_B N \rightarrow (N_3 \otimes_A B) \otimes_B N \rightarrow 0$$

is exact. By “Exercise 2.15” on page 27,

$$0 \rightarrow N_1 \otimes_A (B \otimes_B N) \rightarrow N_2 \otimes_A (B \otimes_B N) \rightarrow N_3 \otimes_A (B \otimes_B N) \rightarrow 0$$

is exact. By Proposition 2.14 (iv),

$$0 \rightarrow N_1 \otimes_A N \rightarrow N_2 \otimes_A N \rightarrow N_3 \otimes_A N \rightarrow 0$$

is exact, or N is flat. \square

Exercise 2.9.

Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be an exact sequence of A -modules. If M' and M'' are finitely generated, then so is M .

Proof.

(1) Write

$$0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0.$$

Also write

$$\begin{aligned} x_1, \dots, x_n &\text{ as generators of } M', \\ z_1, \dots, z_m &\text{ as generators of } M'' \end{aligned}$$

(since M' and M'' are finitely generated).

(2) Since the map $g : M \rightarrow M''$ is surjective, there exists $y_j \in M$ such that $g(y_j) = z_j$ for $j = 1, \dots, m$.

(3) Show that M is generated by

$$f(x_1), \dots, f(x_n), y_1, \dots, y_m.$$

Given any $y \in M$.

$$\begin{aligned} y \in M &\implies g(y) \in M'' \\ &\implies g(y) = \sum_{j=1}^m s_j z_j \text{ where } s_j \in A \\ &\implies g(y) = \sum_{j=1}^m s_j g(y_j) \\ &\implies g(y) = g\left(\sum_{j=1}^m s_j y_j\right) \\ &\implies y - \sum_{j=1}^m s_j y_j \in \ker(g) = \operatorname{im}(f) \\ &\implies \exists x \in M' \text{ such that } f(x) = y - \sum_{j=1}^m s_j y_j \end{aligned}$$

Write $x = \sum_{i=1}^n r_i x_i$ where $r_i \in A$. So,

$$\begin{aligned} y \in M &\implies f\left(\sum_{i=1}^n r_i x_i\right) = y - \sum_{j=1}^m s_j y_j \\ &\implies \sum_{i=1}^n r_i f(x_i) = y - \sum_{j=1}^m s_j y_j \\ &\implies y = \sum_{i=1}^n r_i f(x_i) + \sum_{j=1}^m s_j y_j. \end{aligned}$$

Hence, every $y \in M$ is a linear combination of $f(x_1), \dots, f(x_n), y_1, \dots, y_m$, or M is finitely generated (by $f(x_1), \dots, f(x_n), y_1, \dots, y_m$).

□