Chapter 7: Sequences and Series of Functions

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Exercise 7.1. *Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.*

Proof (Cauchy criterion). Let $\{f_n\}$ be a uniformly convergent sequence of bounded functions.

(1) Since f_n is bounded, there exists M_n such that $|f_n(x)| \leq M_n$.

(2) Since $\{f_n\}$ converges uniformly, given $1 > 0$ there exists an integer N such that
$$|f_n(x) - f_m(x)| \leq 1 \text{ whenever } n, m \geq N$$
(Theorem 7.8 (Cauchy criterion for uniformly convergence)). Especially,
$$|f_n(x)| \leq |f_n(x) - f_N(x)| + |f_N(x)| \leq 1 + M_N \text{ whenever } n \geq N.$$

(3) Thus, $\{f_n\}$ is uniformly bounded by $M = \max\{M_1, \dots, M_{N-1}, M_N + 1\}$.

□

Exercise 7.2. *If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n + g_n\}$ converge uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E.*

Proof. Let $\{f_n\} \to f$ uniformly and $\{g_n\} \to g$ uniformly.

(1) *Show that $\{f_n + g_n\}$ converges uniformly.* Given $\varepsilon > 0$. Since $\{f_n\} \to f$ uniformly and $\{g_n\} \to g$ uniformly, there exist two integers N_1 and N_2 such that
$$|f_n(x) - f(x)| \leq \frac{\varepsilon}{2} \text{ whenever } n \geq N_1, x \in E$$
$$|g_n(x) - g(x)| \leq \frac{\varepsilon}{2} \text{ whenever } n \geq N_2, x \in E.$$
Take $N = \max\{N_1, N_2\}$, we have
$$\begin{aligned} &|(f_n(x) + g_n(x)) - (f(x) + g(x))| \\ =&|(f_n(x) - f(x)) + (g_n(x) - g(x))| \\ \leq&|f_n(x) - f(x)| + |g_n(x) - g(x)| \\ \leq&\frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ =&\varepsilon \end{aligned}$$

whenever $n \geq N$, $x \in E$. Hence $\{f_n + g_n\}$ converges to $f + g$ uniformly on E.

(2) *Show that $\{f_n g_n\}$ converges uniformly if, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions.* Given $\varepsilon > 0$.

(a) By Exercise 7.1, both $\{f_n\}$ and $\{g_n\}$ are uniformly bounded. So there exist M_1 and M_2 such that

$$|f_n(x)| \leq M_1 \text{ and } |g_n(x)| \leq M_2$$

for all n and $x \in E$. Also, $|f(x)| \leq M_1 + 1$ and $|g(x)| \leq M_2 + 1$.

(b) Since $\{f_n\} \to f$ uniformly and $\{g_n\} \to g$ uniformly, there exist two integers N_1 and N_2 such that

$$\begin{aligned}
|f_n(x) - f(x)| &\leq \frac{\varepsilon}{2(M_2+1)} \text{ whenever } n \geq N_1, x \in E \\
|g_n(x) - g(x)| &\leq \frac{\varepsilon}{2(M_1+1)} \text{ whenever } n \geq N_2, x \in E.
\end{aligned}$$

(Note that each denominator of $\frac{\varepsilon}{2(M_j+1)}$ $(j = 1, 2)$ is well-defined and positive!) Take $N = \max\{N_1, N_2\}$, we have

$$\begin{aligned}
&|f_n(x)g_n(x) - f(x)g(x)| \\
=&|[f_n(x) - f(x)]g_n(x) + f(x)[g_n(x) - g(x)]| \\
\leq&|f_n(x) - f(x)||g_n(x)| + |f(x)||g_n(x) - g(x)| \\
\leq&\frac{\varepsilon}{2(M_2+1)} \cdot M_2 + (M_1 + 1) \cdot \frac{\varepsilon}{2(M_1+1)} \\
\leq&\varepsilon
\end{aligned}$$

whenever $n \geq N$, $x \in E$. Hence $\{f_n g_n\}$ converges to fg uniformly on E.

□

Proof (Cauchy criterion).

(1) *Show that $\{f_n + g_n\}$ converges uniformly.* Given $\varepsilon > 0$. Since $\{f_n\}$ and $\{g_n\}$ converge uniformly, there exist two integers N_1 and N_2 such that

$$\begin{aligned}
|f_n(x) - f_m(x)| &\leq \frac{\varepsilon}{2} \text{ whenever } n, m \geq N_1, x \in E \\
|g_n(x) - g_m(x)| &\leq \frac{\varepsilon}{2} \text{ whenever } n, m \geq N_2, x \in E.
\end{aligned}$$

Take $N = \max\{N_1, N_2\}$, we have

$$\begin{aligned}
&|(f_n(x) + g_n(x)) - (f_m(x) + g_m(x))| \\
=&|(f_n(x) - f_n(x)) + (g_n(x) - g_m(x))| \\
\leq&|f_n(x) - f_n(x)| + |g_n(x) - g_m(x)| \\
\leq&\frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
=&\varepsilon
\end{aligned}$$

whenever $n, m \geq N$, $x \in E$. Hence $\{f_n + g_n\}$ converges uniformly on E.

(2) *Show that $\{f_n g_n\}$ converges uniformly if, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions.* Given $\varepsilon > 0$.

(a) By Exercise 7.1, both $\{f_n\}$ and $\{g_n\}$ are uniformly bounded. So there exist M_1 and M_2 such that

$$|f_n(x)| \leq M_1 \text{ and } |g_n(x)| \leq M_2$$

for all n and $x \in E$. Also, $|f(x)| \leq M_1 + 1$ and $|g(x)| \leq M_2 + 1$.

(b) Since $\{f_n\} \to f$ uniformly and $\{g_n\} \to g$ uniformly, there exist two integers N_1 and N_2 such that

$$\begin{aligned}
|f_n(x) - f_m(x)| \leq \frac{\varepsilon}{2(M_2 + 1)} &\text{ whenever } n, m \geq N_1, x \in E \\
|g_n(x) - g_m(x)| \leq \frac{\varepsilon}{2(M_1 + 1)} &\text{ whenever } n, m \geq N_2, x \in E.
\end{aligned}$$

Take $N = \max\{N_1, N_2\}$, we have

$$\begin{aligned}
&|f_n(x)g_n(x) - f_m(x)g_m(x)| \\
=&|[f_n(x) - f_m(x)]g_n(x) + f_m(x)[g_n(x) - g_m(x)]| \\
\leq&|f_n(x) - f_m(x)||g_n(x)| + |f_m(x)||g_n(x) - g_m(x)| \\
\leq&\frac{\varepsilon}{2(M_2 + 1)} \cdot M_2 + M_1 \cdot \frac{\varepsilon}{2(M_1 + 1)} \\
\leq&\varepsilon
\end{aligned}$$

whenever $n \geq N$, $x \in E$. Hence $\{f_n g_n\}$ converges uniformly on E.

□

Exercise 7.3. *Construct sequences $\{f_n\}$, $\{g_n\}$ which converge uniformly on some set E, but such that $\{f_n g_n\}$ does not converge uniformly on E (of course, $\{f_n g_n\}$ must converge on E).*

We provides some examples here.

Proof ($f_n(x) = x + \frac{1}{n}$).

(1) Define $\{f_n(x)\}$ on $E = \mathbb{R}$ by $f_n(x) = x + \frac{1}{n}$ and $f(x) = x$. Clearly, $\{f_n(x)\}$ converges to $f(x)$ pointwise.

(2) *Show that $\{f_n\}$ converges uniformly.* Given $\varepsilon > 0$. There exists an integer $N \geq \frac{1}{\varepsilon}$ such that

$$|f_n(x) - f(x)| = \frac{1}{n} \leq \frac{1}{N} \leq \varepsilon$$

whenever $n \geq N$ and $x \in E$. Hence $\{f_n\} \to f$ uniformly.

(3) *Show that $\{f_n^2\}$ does not converge uniformly.* Clearly, $\{f_n(x)^2\}$ converges to $f(x)^2$ pointwise. Hence

$$\sup_{x \in E} |f_n(x)^2 - f(x)^2| = \sup_{x \in E} \left| \frac{2x}{n} + \frac{1}{n^2} \right| \to \infty$$

as $n \to \infty$ (by considering $x = n^2 \in E$). Hence $\{f_n^2\}$ does not converge uniformly (Theorem 7.9).

$\square$

Proof ($f_n(x) = \frac{1}{x}$, $g_n(x) = \frac{1}{n}$).

(1) Let $E = (0, 1)$. Let $\{f_n(x)\}$ on E be $f_n(x) = \frac{1}{x}$ and $\{g_n(x)\}$ on E be $g_n(x) = \frac{1}{n}$. Clearly, $\{f_n(x)\}$ converges to $f(x) = \frac{1}{x}$ pointwise and $\{g_n(x)\}$ converges to $g(x) = 0$ pointwise.

(2) *Show that $\{f_n\}$ converges uniformly.* Given $\varepsilon > 0$. There exists an integer $N = 1$ such that

$$|f_n(x) - f(x)| = 0 \leq \varepsilon$$

whenever $n \geq N$ and $x \in E$. Hence $\{f_n\} \to f$ uniformly.

(3) *Show that $\{g_n\}$ converges uniformly.* Given $\varepsilon > 0$. There exists an integer $N \geq \frac{1}{\varepsilon}$ such that

$$|g_n(x) - g(x)| = \frac{1}{n} \leq \frac{1}{N} \leq \varepsilon$$

whenever $n \geq N$ and $x \in E$. Hence $\{g_n\} \to g$ uniformly.

(4) *Show that $\{f_n g_n\}$ does not converge uniformly.* Clearly, $\{f_n(x)g_n(x)\}$ converges to $f(x)g(x) = 0$ pointwise. Hence

$$\sup_{x \in E} |f_n(x)g_n(x) - 0| = \sup_{x \in E} \left| \frac{1}{nx} \right| \to \infty$$

as $n \to \infty$ (by considering $x = \frac{1}{n^2} \in E$). Hence $\{f_n g_n\}$ does not converge uniformly (Theorem 7.9).

$\square$

Proof (Exercise 9.2 in Tom M. Apostol, Mathematical Analysis, 2nd edition).

(1) Let $E = [\alpha, \beta] \subseteq \mathbb{R}$ be a bounded interval. Define two sequences $\{f_n\}$ and $\{g_n\}$ on E as follows:

$$f_n(x) = x\left(1 + \frac{1}{n}\right) \text{ if } x \in \mathbb{R},\ n = 1, 2, \cdots,$$

$$g_n(x) = \begin{cases} \frac{1}{n} & \text{if } x = 0 \text{ or if } x \text{ is irrational}, \\ b + \frac{1}{n} & \text{if } x \text{ is rational} \neq 0, \text{ say } x = \frac{a}{b},\ b > 0. \end{cases}$$

Here we assume that $\gcd(a, b) = 1$. Clearly, $f(x) = x$ and

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or if } x \text{ is irrational}, \\ b & \text{if } x \text{ is rational} \neq 0, \text{ say } x = \frac{a}{b},\ b > 0. \end{cases}$$

Let $M = \max\{|\alpha|, |\beta|\} \geq 0$.

(2) *Show that $\{f_n\}$ converges uniformly.* Given $\varepsilon > 0$. There exists an integer $N \geq \frac{M}{\varepsilon}$ such that

$$|f_n(x) - f(x)| = \frac{|x|}{n} \leq \frac{M}{N} \leq \varepsilon$$

whenever $n \geq N$ and $x \in E$. Hence $\{f_n\} \to f$ uniformly.

(3) *Show that $\{g_n\}$ converges uniformly.* Given $\varepsilon > 0$. There exists an integer $N \geq \frac{1}{\varepsilon}$ such that

$$|g_n(x) - g(x)| = \frac{1}{n} \leq \frac{1}{N} \leq \varepsilon$$

whenever $n \geq N$ and $x \in E$. Hence $\{g_n\} \to g$ uniformly.

(4) *Show that $\{f_n g_n\}$ does not converge uniformly.*

(a) Clearly, $\{f_n(x)g_n(x)\}$ converges to $f(x)g(x)$ pointwise where

$$f(x)g(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or if } x \text{ is irrational}, \\ a & \text{if } x = \frac{a}{b} \text{ is rational} \neq 0,\ b > 0. \end{cases}$$

(b) Note that

$$f_n(x)g_n(x) = \begin{cases} \frac{x}{n}\left(1 + \frac{1}{n}\right) & \text{if } x = 0 \text{ or if } x \text{ is irrational}, \\ \left(a + \frac{x}{n}\right)\left(1 + \frac{1}{n}\right) & \text{if } x = \frac{a}{b} \text{ is rational} \neq 0,\ b > 0. \end{cases}$$

Therefore,

$$f_n(x)g_n(x) - f(x)g(x) = \begin{cases} \frac{x}{n}\left(1 + \frac{1}{n}\right) & \text{if } x = 0 \text{ or if } x \text{ is irrational}, \\ \frac{x}{n}\left(1 + b + \frac{1}{n}\right) & \text{if } x = \frac{a}{b} \text{ is rational} \neq 0,\ b > 0. \end{cases}$$

(c) Hence

$$\begin{aligned}\sup_{x\in E}|f_n(x)g_n(x)-f(x)g(x)| &\geq \sup_{x\in E\cap\mathbb{Q}}|f_n(x)g_n(x)-f(x)g(x)| \\ &= \sup_{x\in E\cap\mathbb{Q}}|a|\left(\frac{1}{n}+\frac{1}{bn}+\frac{1}{bn^2}\right) \\ &\geq \sup_{x\in E\cap\mathbb{Q}}|a|\left(\frac{1}{n}\right) \\ &= \sup_{x\in E\cap\mathbb{Q}}\frac{|a|}{n}.\end{aligned}$$

(d) *Given any irrational number $\gamma \in E$, there exists a sequence*

$$\left\{r_m = \frac{a_m}{b_m}\right\}$$

of nonzero rational numbers in E such that $\lim r_m = \gamma$. Show that $\{a_m\}$ is unbounded. If it is true, we can find $x_n = r_{m_n} = \frac{a_{m_n}}{b_{m_n}}$ such that $|a_{m_n}| \geq n^2$ and

$$\sup_{x\in E}|f_n(x)g_n(x)-f(x)g(x)| \geq \sup_{x\in E\cap\mathbb{Q}}\frac{|a|}{n} \geq \frac{n^2}{n} = n \to \infty$$

as $n \to \infty$.

(e) (Reductio ad absurdum) If $\{a_m\}$ were bounded, then there exists a **constant** subsequence of $\{a_{m_k}\}$ such that $\lim a_{m_k} = a \in \mathbb{Z}$. Since $\lim_{m\to\infty} r_m = \gamma$, $\lim_{k\to\infty} r_{m_k} = \gamma$ or

$$\lim_{k\to\infty} b_{m_k} = \lim_{k\to\infty}\frac{a_{m_k}}{r_{m_k}} = \frac{a}{\gamma}$$

(it is well-defined since r_{m_k} and γ cannot be zero). Since all b_{m_k} are positive integers, the limit $\lim b_{m_k} = b$ is a positive integer too, or $b = \frac{a}{\gamma} \in \mathbb{Z}^+$, or $\gamma = \frac{a}{b} \in \mathbb{Z}$, which is absurd.

Therefore, $\{f_n g_n\}$ does not converge uniformly.

□