Notes on the book:
Apostol, Modular Functions and Dirichlet Series in Number Theory, 2nd edition

Meng-Gen Tsai
plover@gmail.com

July 27, 2021

Contents

Chapter 1: Elliptic functions **2**

Exercise 1.1. 2
Supplement 1.1.1. 3
Exercise 1.2. 3
Exercise 1.5. 4
Exercise 1.6. 5
Exercise 1.7. 6
Exercise 1.8. 7
Exercise 1.10. 8
Exercise 1.11. 9
Exercise 1.12. 10
Exercise 1.13. 10
Exercise 1.14. (Lambert series) 12
Exercise 1.15. 14

Chapter 1: Elliptic functions

Exercise 1.1.

Given two pairs of complex numbers (ω_1, ω_2) *and* (ω_1', ω_2') *with nonreal ratios* ω_1/ω_2 *and* ω_1'/ω_2'*. Prove that they generate the same set of periods if, and only if, there is a* 2×2 *matrix* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *with integer entries and determinant* ± 1 *such that*

$$\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix}.$$

Proof.

(1) ($\Longrightarrow$) Suppose (ω_1, ω_2) and (ω_1', ω_2') generate the same set of periods. In particular, there is a 2×2 matrix $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{M}_{2\times 2}(\mathbb{Z})$ (resp. $A' := \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathsf{M}_{2\times 2}(\mathbb{Z})$) such that

$$\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = A \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix}, \qquad \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} = A' \begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix}.$$

Hence it suffices to show $\det(A) = \pm 1$.

(2) Note that

$$\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = AA' \begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix}.$$

Hence

$$AA' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Take the determinant on the both sides to get

$$\det(A)\det(A') = 1.$$

Since $\det(\mathsf{M}_{2\times 2}(\mathbb{Z})) \subseteq \mathbb{Z}$, $\det(A) = \pm 1$.

(3) ($\Longleftarrow$) $\Omega(\omega_1', \omega_2') \subseteq \Omega(\omega_1, \omega_2)$ is obvious. Note that

$$\begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} = \underbrace{\frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}_{\in \mathsf{M}_{2\times 2}(\mathbb{Z})} \begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix}.$$

Thus $\Omega(\omega_1, \omega_2) \subseteq \Omega(\omega_1', \omega_2')$. Therefore $\Omega(\omega_1, \omega_2) = \Omega(\omega_1', \omega_2')$.

□

Supplement 1.1.1.

(Exercise I.1.1 in the textbook: *Jörgen Neukirch, Algebraic Number Theory.*) *$\alpha \in \mathbb{Z}[i]$ is a unit if and only if $N(\alpha) = 1$.*

Proof.

(1) ($\Longrightarrow$) Since α is a unit, there is $\beta \in \mathbb{Z}[i]$ such that $\alpha\beta = 1$. So $N(\alpha\beta) = N(1)$, or $N(\alpha)N(\beta) = 1$. Since the image of N is nonnegative integers, $N(\alpha) = 1$.

(2) ($\Longleftarrow$) $N(\alpha) = \alpha\overline{\alpha}$, or $1 = \alpha\overline{\alpha}$ since $N(\alpha) = 1$. That is, $\overline{\alpha} \in \mathbb{Z}[i]$ is the inverse of $\alpha \in \mathbb{Z}[i]$. (Or we solve the equation $N(\alpha) = a^2 + b^2 = 1$, and show that all four solutions (± 1 and $\pm i$) are units.)

(3) Conclusion: a unit $\alpha = a + bi$ of $\mathbb{Z}[i]$ is satisfying the equation $N(\alpha) = a^2 + b^2 = 1$ by (1)(2). That is, the only unit of $\mathbb{Z}[i]$ are ± 1 and $\pm i$.

$\square$

Exercise 1.2.

Let $S(0)$ denote the sum of the zeros of an elliptic function f in a period parallelogram, and let $S(\infty)$ denote the sum of the poles in the same parallelogram. Prove that $S(0) - S(\infty)$ is a period of f. (Hint: Integrate $z\frac{f'(z)}{f(z)}$.)

Proof.

(1) Similar to Theorem 1.8, the integral

$$\frac{1}{2\pi i}\int_C z\frac{f'(z)}{f(z)}$$

taken around the boundary C of a cell (no zeros or poles on its boundary) counts the difference between the sum of the zeros and the sum of the poles inside the cell, that is,

$$S(0) - S(\infty) = \frac{1}{2\pi i}\int_C z\frac{f'(z)}{f(z)}.$$

(The proof is similar to the proof of the argument principle.)

(2) Let C_1 be the path from 0 to ω_1, C_2 be the path from ω_1 to $\omega_1 + \omega_2$, C_3

be the path from $\omega_1 + \omega_2$ to ω_2, and C_4 be the path from ω_2 to 0. Hence

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{C_1} z\frac{f'(z)}{f(z)} + \frac{1}{2\pi i}\int_{C_3} z\frac{f'(z)}{f(z)} \\
=& \frac{1}{2\pi i}\int_{C_1} z\frac{f'(z)}{f(z)} + \frac{1}{2\pi i}\int_{-C_1} (z+\omega_2)\frac{f'(z+\omega_2)}{f(z+\omega_2)} \\
=& \frac{1}{2\pi i}\int_{C_1} z\frac{f'(z)}{f(z)} - \frac{1}{2\pi i}\int_{C_1} (z+\omega_2)\frac{f'(z)}{f(z)} \\
=& -\omega_2\frac{1}{2\pi i}\int_{C_1} \frac{f'(z)}{f(z)}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{C_2} z\frac{f'(z)}{f(z)} + \frac{1}{2\pi i}\int_{C_4} z\frac{f'(z)}{f(z)} \\
=& \frac{1}{2\pi i}\int_{-C_4} (z+\omega_1)\frac{f'(z+\omega_1)}{f(z+\omega_1)} + \frac{1}{2\pi i}\int_{C_4} z\frac{f'(z)}{f(z)} \\
=& -\frac{1}{2\pi i}\int_{C_4} (z+\omega_1)\frac{f'(z)}{f(z)} + \frac{1}{2\pi i}\int_{C_4} z\frac{f'(z)}{f(z)} \\
=& -\omega_1\frac{1}{2\pi i}\int_{C_4} \frac{f'(z)}{f(z)}
\end{aligned}$$

Therefore

$$S(0) - S(\infty) = -\omega_1\frac{1}{2\pi i}\int_{C_4}\frac{f'(z)}{f(z)} - \omega_2\frac{1}{2\pi i}\int_{C_1}\frac{f'(z)}{f(z)}.$$

So it suffices to show that $\frac{1}{2\pi i}\int_{C_1}\frac{f'(z)}{f(z)} \in \mathbb{Z}$. (Other cases are similar.)

(3) By choosing one branch of log, we have

$$\begin{aligned}
\frac{1}{2\pi i}\int_{C_1}\frac{f'(z)}{f(z)} &= \frac{1}{2\pi i}\log\frac{f(\omega_1)}{f(0)} \\
&= \frac{1}{2\pi i}\log(1) && (f(\omega_1) = f(0)) \\
&= \frac{1}{2\pi i}(2\pi i m) \text{ for some } m \in \mathbb{Z} \\
&= m \in \mathbb{Z}.
\end{aligned}$$

□

Exercise 1.5.

Prove that every elliptic function f can be expressed in the form

$$f(z) = R_1[\wp(z)] + \wp'(z)R_2[\wp(z)]$$

where R_1 and R_2 are rational functions and $\wp$ has the same set of periods as f.

Proof.

$$\begin{aligned} f(z) &= \underbrace{\frac{f(z)+f(-z)}{2}}_{\text{even}} + \wp'(z) \underbrace{\frac{f(z)-f(-z)}{2\wp'(z)}}_{\text{even}} \\ &= R_1[\wp(z)] + \wp'(z)R_2[\wp(z)] \text{ for some rational functions } R_1, R_2 \end{aligned}$$

(by Exercise 1.4). □

Exercise 1.6.

Let f and g be two elliptic functions with the same set of periods. Prove that there exists a polynomial $P(x,y)$, not identically zero, such that

$$P[f(z), g(z)] = C$$

where C is a constant (depending on f and g but not on z).

Proof.

(1) By Exercise 1.5, we have

$$f(z) = R_1[\wp(z)] + \wp'(z)R_2[\wp(z)]$$

for some rational functions R_1, R_2 and $\wp$ has the same set of periods as f. By cleaning the denominators of R_1 and R_2, we might assume

$$S[\wp(z)]f(z) = R_1[\wp(z)] + \wp'(z)R_2[\wp(z)]$$

for some polynomials R_1, R_2, S.

(2) So

$$\begin{aligned} &\wp'(z)R_2[\wp(z)] = S[\wp(z)]f(z) - R_1[\wp(z)] \\ \Longrightarrow\ &\wp'(z)^2 R_2[\wp(z)]^2 = (S[\wp(z)]f(z) - R_1[\wp(z)])^2 \\ \Longrightarrow\ &(4\wp(z)^3 - 60G_4\wp(z) - 140G_6)R_2[\wp(z)]^2 \\ &\quad = (S[\wp(z)]f(z) - R_1[\wp(z)])^2. \qquad \text{(Theorem 1.12)} \\ \Longrightarrow\ &F(\wp(z), f(z)) = 0 \end{aligned}$$

for some polynomials $F(x,y) \in \mathbb{C}[x,y]$. Note that $F(x,y)$ is of degree 2 if we view $F \in (\mathbb{C}[x])[y]$.

(3) Similarly,

$$G(\wp(z), g(z)) = 0$$

for some polynomials $G(x,y) \in \mathbb{C}[x,y]$.

(4) Let $P = \mathrm{Res}_x(F, G)$ be the resultant of two polynomials F and G with respect t x to eliminate $\wp(z)$. Note that P is a nonzero polynomial (since F and G are nonzero) and $P[f(z), g(z)] = 0$. So P is our desired polynomial.

□

Exercise 1.7.

The discriminant of the polynomial $f(x) = 4(x - x_1)(x - x_2)(x - x_3)$ *is the product* $16\{(x_2 - x_1)(x_3 - x_2)(x_3 - x_1)\}^2$. *Prove that the discriminant of* $f(x) = 4x^3 - ax - b$ *is* $a^3 - 27b^2$.

Proof.

(1) Since

$$f'(x) = 4(x - x_2)(x - x_3) + 4(x - x_1)(x - x_3) + 4(x - x_1)(x - x_2),$$

we have

$$\begin{aligned} f'(x_1) &= 4(x_1 - x_2)(x_1 - x_3), \\ f'(x_2) &= 4(x_2 - x_1)(x_2 - x_3), \\ f'(x_3) &= 4(x_3 - x_1)(x_3 - x_2). \end{aligned}$$

Hence

$$f'(x_1)f'(x_2)f'(x_3) = -4\mathrm{disc}(f)$$

where $\mathrm{disc}(f)$ is the discriminant of $f(x)$.

(2) As $f(x) = 4x^3 - ax - b$, we have $f'(x) = 12x^2 - a$. So

$$f'(x_1)f'(x_2)f'(x_3) = (12x_1^2 - a)(12x_2^2 - a)(12x_3^2 - a).$$

Note that

$$\begin{aligned} x_1x_2x_3 &= \frac{b}{4}, \\ x_1x_2 + x_2x_3 + x_3x_1 &= -\frac{a}{4}, \\ x_1 + x_2 + x_3 &= 0, \end{aligned}$$

we have

$$\begin{aligned} x_1^2x_2^2x_3^2 &= \frac{b^2}{4^2}, \\ x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_1^2 &= (x_1x_2 + x_2x_3 + x_3x_1)^2 - 2x_1x_2x_3(x_1 + x_2 + x_3) \\ &= \frac{a^2}{4^2}, \\ x_1^2 + x_2^2 + x_3^2 &= (x_1 + x_2 + x_3)^2 - 2(x_1x_2 + x_2x_3 + x_3x_1) \\ &= \frac{a}{2}. \end{aligned}$$

(3) Hence

$$\begin{aligned}
f'(x_1)f'(x_2)f'(x_3) &= (12x_1^2 - a)(12x_2^2 - a)(12x_3^2 - a) \\
&= 12^3(x_1^2x_2^2x_3^2) - 12^2a(x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_1^2) \\
&\quad + 12a^2(x_1^2 + x_2^2 + x_3^2) - a^3 \\
&= 12^3 \cdot \frac{b^2}{4^2} - 12^2a \cdot \frac{a^2}{4^2} + 12a^2 \cdot \frac{a}{2} - a^3 \\
&= -4(a^3 - 27b^2).
\end{aligned}$$

Therefore

$$\operatorname{disc}(4x^3 - ax - b) = a^3 - 27b^2.$$

□

Exercise 1.8.

The differential equation for $\wp$ shows that $\wp'(z) = 0$ if $z = \frac{\omega_1}{2}, \frac{\omega_2}{2}$ or $\frac{\omega_1+\omega_2}{2}$. Show that

$$\wp''\left(\frac{\omega_1}{2}\right) = 2(e_1 - e_2)(e_1 - e_3)$$

and obtain corresponding formulas for $\wp''\left(\frac{\omega_2}{2}\right)$ and $\wp''\left(\frac{\omega_1+\omega_2}{2}\right)$.

Proof.

(1) Differentiation of the equation

$$4\wp(z)^3 - g_2\wp(z) - g_3 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$$

in Theorem 1.14 to get

$$\begin{aligned}
12\wp(z)^2\wp'(z) - g_2\wp'(z) &= 4\wp'(z)(\wp(z) - e_2)(\wp(z) - e_3) \\
&\quad + 4\wp'(z)(\wp(z) - e_1)(\wp(z) - e_3) \\
&\quad + 4\wp'(z)(\wp(z) - e_1)(\wp(z) - e_2).
\end{aligned}$$

Since $\wp''(z) = 6\wp(z)^2 - \frac{g_2}{2}$, we have

$$\begin{aligned}
\wp''(z) &= 2(\wp(z) - e_2)(\wp(z) - e_3) \\
&\quad + 2(\wp(z) - e_1)(\wp(z) - e_3) \\
&\quad + 2(\wp(z) - e_1)(\wp(z) - e_2).
\end{aligned}$$

(2) Hence

$$\begin{aligned}
\wp''\left(\frac{\omega_1}{2}\right) &= 2(e_1 - e_2)(e_1 - e_3), \\
\wp''\left(\frac{\omega_2}{2}\right) &= 2(e_2 - e_1)(e_2 - e_3), \\
\wp''\left(\frac{\omega_1 + \omega_2}{2}\right) &= 2(e_3 - e_1)(e_3 - e_2).
\end{aligned}$$

□

Exercise 1.10.

Let ω_1 and ω_2 be complex numbers with nonreal ratio. Let $f(z)$ be an entire function and assume there are constants a and b such that

$$f(z+\omega_1) = af(z), \qquad f(z+\omega_2) = bf(z),$$

for all z. Prove that $f(z) = Ae^{Bz}$, where A and B are constant.

Proof.

(1) Might assume that $a \neq 0$ and $b \neq 0$ (otherwise $f = 0$ on $\mathbb{C}$).

(2) Define
$$g(z) := \frac{f(z)}{e^{Bz}}.$$
It suffices to show g is a constant. Note that $g(z)$ is entire (since f and $e^{Bz} \neq 0$ are entire). By Theorem 1.4, it suffices to show g is doubly periodic, that is, to show
$$g(z+\omega_1) = g(z) \text{ and } g(z+\omega_2) = g(z)$$
for suitable B.

(3) Note that
$$\begin{aligned} & g(z+\omega_1) = g(z) \text{ and } g(z+\omega_2) = g(z) \\ \Longleftrightarrow\ & \frac{a}{e^{B\omega_1}} \cdot g(z) = g(z) \text{ and } \frac{b}{e^{B\omega_2}} \cdot g(z) = g(z) \\ \Longleftrightarrow\ & e^{B\omega_1} = a \text{ and } e^{B\omega_2} = b \\ \Longleftrightarrow\ & \exists\, B \text{ such that } e^{B\omega_1} = a \text{ and } e^{B\omega_2} = b. \end{aligned}$$
Take B such that $e^{B(\omega_1-\omega_2)} = \frac{a}{b}$ (since $\frac{a}{b}$ is well-defined, $\omega_1 - \omega_2 \neq 0$, and $z \mapsto \exp(z)$ is a onto map from $\mathbb{C}$ to $\mathbb{C} \smallsetminus \{0\}$). Hence g is doubly periodic.

□

Exercise 1.11.

If $k \geq 2$ and $\tau \in H$ prove that the Eisenstein series

$$G_{2k}(\tau) = \sum_{(m,n)\neq(0,0)} (m+n\tau)^{-2k}$$

has the Fourier expansion

$$G_{2k}(\tau) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)e^{2\pi i n\tau}.$$

Proof.

(1) Let $q = e^{2\pi i\tau}$. Similar to Lemma 1.3 on page 19, we have

$$(2k-1)!\sum_{m=-\infty}^{+\infty}\frac{1}{(\tau+m)^{2k}} = (2\pi i)^{2k}\sum_{r=1}^{\infty}r^{2k-1}q^r.$$

(2) Similar to Theorem 1.18, we have

$$\begin{aligned}
G_{2k}(\tau) &= \sum_{(m,n)\neq(0,0)}(m+n\tau)^{-2k} \\
&= \sum_{\substack{m=-\infty \\ m\neq 0 (n=0)}}^{+\infty} m^{-2k} + \sum_{n=1}^{\infty}\sum_{m=-\infty}^{+\infty}((m+n\tau)^{-2k}+(m-n\tau)^{-2k}) \\
&= 2\zeta(2k) + 2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{+\infty}(m+n\tau)^{-2k} \\
&= 2\zeta(2k) + 2\sum_{n=1}^{\infty}\frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{r=1}^{\infty}r^{2k-1}q^{nr} \\
&= 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\underbrace{\sum_{d|n}d^{2k-1}}_{=\sigma_{2k-1}(n)}q^n.
\end{aligned}$$

In the last double sum we collect together those terms for which nr is constant.

□

Exercise 1.12.

Refer to Exercise 1.11. If $\tau \in H$ *prove that*

$$G_{2k}\left(-\frac{1}{\tau}\right) = \tau^{2k}G_{2k}(\tau)$$

and deduce that

$$\begin{aligned}
G_{2k}\left(\frac{i}{2}\right) &= (-4)^k G_{2k}(2i) && \textit{for all } k \geq 2,\\
G_{2k}(i) &= 0 && \textit{if } k \textit{ is odd},\\
G_{2k}(e^{\frac{2\pi i}{3}}) &= 0 && \textit{if } k \not\equiv 0 \pmod 3.
\end{aligned}$$

Proof.

(1)

$$\begin{aligned}
G_{2k}\left(-\frac{1}{\tau}\right) &= \sum_{(m,n)\neq(0,0)} \left(m - \frac{n}{\tau}\right)^{-2k}\\
&= \tau^{2k} \sum_{(m,n)\neq(0,0)} (\tau m - n)^{-2k}\\
&= \tau^{2k} G_{2k}(\tau).
\end{aligned}$$

(2) Let $\tau = 2i$. We have $G_{2k}\left(\frac{i}{2}\right) = (-4)^k G_{2k}(2i)$.

(3) Let $\tau = i$. We have $G_{2k}(i) = (-1)^k G_{2k}(i)$. Hence $G_{2k}(i) = 0$ if k is odd.

(4) Let $\tau = e^{\frac{\pi i}{3}}$. We have $G_{2k}(e^{\frac{2\pi i}{3}}) = e^{\frac{2k\pi i}{3}} G_{2k}(e^{\frac{\pi i}{3}})$. Since

$$e^{\frac{2\pi i}{3}} = -1 + e^{\frac{\pi i}{3}}$$

and each Eisenstein series is a periodic function of τ of period 1, we have $G_{2k}(e^{\frac{2\pi i}{3}}) = G_{2k}(e^{\frac{\pi i}{3}})$. So $G_{2k}(e^{\frac{2\pi i}{3}}) = e^{\frac{2k\pi i}{3}} G_{2k}(e^{\frac{2\pi i}{3}})$. Therefore $G_{2k}(e^{\frac{2\pi i}{3}}) = 0$ if $k \not\equiv 0 \pmod 3$.

□

Exercise 1.13.

Ramanujan's tau function $\tau(n)$ is defined by the Fourier expansion

$$\Delta(\tau) = (2\pi)^{12} \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n \tau},$$

derived in Theorem 1.19. Prove that

$$\tau(n) = 8000\{(\sigma_3 \circ \sigma_3) \circ \sigma_3\}(n) - 147(\sigma_5 \circ \sigma_5)(n),$$

where $f \circ g$ denotes the Cauchy product of two sequences,

$$(f \circ g)(n) = \sum_{k=0}^{n} f(k) g(n-k),$$

and $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$ *for* $n \geq 1$*, with* $\sigma_3(0) = \frac{1}{240}$*,* $\sigma_5(0) = -\frac{1}{504}$*. (Hint: Theorem 1.18.)*

Proof.

(1) Let $q = e^{2\pi i \tau}$. Write

$$g_2(\tau) = \frac{4\pi^4}{3}\left\{1 + 240\sum_{k=1}^{\infty}\sigma_3(k)q^k\right\} = \frac{4\pi^4}{3}\left\{240\sum_{k=0}^{\infty}\sigma_3(k)q^k\right\},$$
$$g_3(\tau) = \frac{8\pi^6}{27}\left\{1 - 504\sum_{k=1}^{\infty}\sigma_5(k)q^k\right\} = \frac{8\pi^6}{27}\left\{-504\sum_{k=0}^{\infty}\sigma_5(k)q^k\right\}$$

(Theorem 1.18).

(2) Similar to the proof of Theorem 1.19,

$$\begin{aligned}
\Delta(\tau) &= g_2(\tau)^3 - 27g_3(\tau)^2 \\
&= \frac{64\pi^{12}}{27}\left\{\left(240\sum_{k=0}^{\infty}\sigma_3(k)q^k\right)^3 - \left(-504\sum_{k=0}^{\infty}\sigma_5(k)q^k\right)^2\right\} \\
&= (2\pi)^{12}\left\{8000\left(\sum_{k=0}^{\infty}\sigma_3(k)q^k\right)^3 - 147\left(\sum_{k=0}^{\infty}\sigma_5(k)q^k\right)^2\right\} \\
&= (2\pi)^{12}\sum_{n=0}^{\infty}\left\{8000\{(\sigma_3 \circ \sigma_3) \circ \sigma_3\}(n) - 147(\sigma_5 \circ \sigma_5)(n)\right\} q^n \\
&= (2\pi)^{12}\sum_{n=1}^{\infty}\left\{8000\{(\sigma_3 \circ \sigma_3) \circ \sigma_3\}(n) - 147(\sigma_5 \circ \sigma_5)(n)\right\} q^n.
\end{aligned}$$

(Here $8000\{(\sigma_3 \circ \sigma_3) \circ \sigma_3\}(0) - 147(\sigma_5 \circ \sigma_5)(0) = 0$.)

(3) Therefore

$$\tau(n) = 8000\{(\sigma_3 \circ \sigma_3) \circ \sigma_3\}(n) - 147(\sigma_5 \circ \sigma_5)(n)$$

for $n \geq 1$.

□

Exercise 1.14. (Lambert series)

A series of the form $\sum_{n=1}^{\infty} f(n)\frac{x^n}{1-x^n}$ *is called a* ***Lambert series****. Assuming absolute convergence, prove that*

$$\sum_{n=1}^{\infty} f(n)\frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} F(n)x^n,$$

where

$$F(n) = \sum_{d|n} f(d).$$

Apply this result to obtain the following formulas, valid for $|x| < 1$.

(a)

$$\sum_{n=1}^{\infty} \frac{\mu(n)x^n}{1-x^n} = x.$$

(b)

$$\sum_{n=1}^{\infty} \frac{\varphi(n)x^n}{1-x^n} = \frac{x}{(1-x)^2}.$$

(c)

$$\sum_{n=1}^{\infty} \frac{n^{\alpha}x^n}{1-x^n} = \sum_{n=1}^{\infty} \sigma_{\alpha}(n)x^n.$$

(d)

$$\sum_{n=1}^{\infty} \frac{\lambda(n)x^n}{1-x^n} = \sum_{n=1}^{\infty} x^{n^2}.$$

(e) *Use the result in (c) to express $g_2(\tau)$ and $g_3(\tau)$ in terms of Lambert series in $x = e^{2\pi i\tau}$.*

Note. In (a), $\mu(n)$ is the Möbius function; In (b), $\varphi(n)$ is Euler's totient; and in (d), $\lambda(n)$ is Liouville's function.

Proof. Similar to the proof of Exercise 1.11.

$$\begin{aligned}
\sum_{n=1}^{\infty} f(n)\frac{x^n}{1-x^n} &= \sum_{n=1}^{\infty} f(n) \sum_{r=1}^{\infty} x^{rn} \\
&= \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} f(n)x^{rn} \\
&= \sum_{n=1}^{\infty} \underbrace{\left(\sum_{d|n} f(d) \right)}_{=F(n)} x^n.
\end{aligned}$$

□

Proof of (a). Theorem 2.1 in the textbook: *T. M. Apostol, Introduction to Analytic Number Theory* shows that

$$F(n) := \sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

Hence

$$\sum_{n=1}^{\infty} \mu(n) \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} F(n) x^n = x.$$

□

Proof of (b). Theorem 2.2 in the textbook: *T. M. Apostol, Introduction to Analytic Number Theory* shows that $F(n) := \sum_{d|n} \varphi(d) = n$. Hence

$$\sum_{n=1}^{\infty} \varphi(n) \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} F(n) x^n = \sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}.$$

□

Proof of (c). Since

$$F(n) := \sum_{d|n} d^{\alpha} = \sigma_{\alpha}(n),$$

we have

$$\sum_{n=1}^{\infty} n^{\alpha} \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} F(n) x^n = \sum_{n=1}^{\infty} \sigma_{\alpha}(n) x^n.$$

□

Proof of (d). Theorem 2.19 in the textbook: *T. M. Apostol, Introduction to Analytic Number Theory* shows that

$$F(n) := \sum_{d|n} \lambda(d) = \begin{cases} 1 & \text{if } n \text{ is a square}, \\ 0 & \text{otherwise}. \end{cases}$$

Hence

$$\sum_{n=1}^{\infty} \lambda(n) \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} F(n) x^n = \sum_{n=1}^{\infty} x^{n^2}.$$

□

Proof of (e).

(1) Let $q = x = e^{2\pi i \tau}$.

$$\begin{aligned} g_2(\tau) &= \frac{4\pi^4}{3}\left\{1 + 240\sum_{k=1}^{\infty} \sigma_3(k) q^k\right\} && \text{(Theorem 1.18)} \\ &= \frac{4\pi^4}{3}\left\{1 + 240\sum_{k=1}^{\infty} \frac{k^3 q^k}{1-q^k}\right\} && \text{((c)).} \end{aligned}$$

(2) Similarly,

$$\begin{aligned} g_3(\tau) &= \frac{8\pi^6}{27}\left\{1 - 504\sum_{k=1}^{\infty} \sigma_5(k) q^k\right\} && \text{(Theorem 1.18)} \\ &= \frac{8\pi^6}{27}\left\{1 - 504\sum_{k=1}^{\infty} \frac{k^5 q^k}{1-q^k}\right\} && \text{((c)).} \end{aligned}$$

□

Exercise 1.15.

Let

$$G(x) = \sum_{n=1}^{\infty} \frac{n^5 x^n}{1-x^n},$$

and let

$$F(x) = \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^5 x^n}{1+x^n}.$$

(a) *Prove that* $F(x) = G(x) - 34G(x^2) + 64(x^4)$.

(b) *Prove that*

$$\sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^5}{1+e^{n\pi}} = \frac{31}{504}.$$

(c) *Use Theorem 12.17 in the textbook: T. M. Apostol, Introduction to Analytic Number Theory to prove the more general result*

$$\sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^{4k+1}}{1+e^{n\pi}} = \frac{2^{4k+1}-1}{8k+4} B_{4k+2}.$$

Proof of (a).

(1) Consider the general case. *Let*

$$G(x) = \sum_{n=1}^{\infty} \frac{n^{4k+1}x^n}{1-x^n},$$

and let

$$F(x) = \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^{4k+1}x^n}{1+x^n}.$$

Show that $F(x) = G(x) - (2^{4k+1}+2)G(x^2) + 2^{4k+2}G(x^4)$.

(2) The identity

$$\sum_{n=1}^{\infty} \frac{x^n}{1+x^n} = \sum_{n=1}^{\infty} \frac{x^n}{1-x^n} - 2\sum_{n=1}^{\infty} \frac{x^{2n}}{1-x^{2n}}$$

is always true. Hence $H(x) := \sum_{n=1}^{\infty} \frac{n^{4k+1}x^n}{1+x^n} = G(x) - 2G(x^2)$.

(3) Note that

$$\begin{aligned} H(x) &= \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^{4k+1}x^n}{1+x^n} + \sum_{\substack{n=1 \\ (n \text{ even})}}^{\infty} \frac{n^{4k+1}x^n}{1+x^n} \\ &= F(x) + \sum_{n=1}^{\infty} \frac{(2n)^{4k+1}x^{2n}}{1+x^{2n}} \\ &= F(x) + 2^{4k+1}\sum_{n=1}^{\infty} \frac{n^{4k+1}x^{2n}}{1+x^{2n}} \\ &= F(x) + 2^{4k+1}H(x^2). \end{aligned}$$

Hence

$$\begin{aligned} F(x) &= H(x) - 2^{4k+1}H(x^2) \\ &= [G(x) - 2G(x^2)] - 2^{4k+1}[G(x^2) - 2G(x^4)] \\ &= G(x) - (2^{4k+1}+2)G(x^2) + 2^{4k+2}G(x^4). \end{aligned}$$

□

Proof of (b). Take $k = 1$ in part (c), we have

$$\sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^5}{1+e^{n\pi}} = \frac{31}{12} \cdot \frac{1}{42} = \frac{31}{504}.$$

□

Proof of (c).

(1) Let $q = e^{2\pi i\tau}$. So

$$\begin{aligned} G_{4k+2}(\tau) &= 2\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!}\sum_{n=1}^{\infty}\sigma_{4k+1}(n)q^n && \text{(Exercise 1.11)} \\ &= 2\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!}G(q) && \text{(Exercise 1.14(c))} \end{aligned}$$

Hence

$$\begin{aligned} &G_{4k+2}(\tau) - (2^{4k+1}+2)G_{4k+2}(2\tau) + 2^{4k+2}G_{4k+2}(4\tau) \\ =& \left[2\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!}G(q)\right] \\ &- (2^{4k+1}+2)\left[2\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!}G(q^2)\right] \\ &+ 2^{4k+2}\left[2\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!}G(q^4)\right] \\ =& (1 - (2^{4k+1}+2) + 2^{4k+2}) \cdot 2\zeta(4k+2) \\ &+ \frac{2(2\pi i)^{4k+2}}{(4k+1)!}[G(q) - (2^{4k+1}+2)G(q^2) + 2^{4k+2}G(q^4)] \\ =& (2^{4k+2}-2)\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!}F(q). \end{aligned}$$

(2) By taking $\tau = \frac{i}{2}$, we have

$$F(q) = F(e^{-\pi}) = \sum_{\substack{n=1 \\ (n\text{ odd})}}^{\infty} \frac{n^{4k+1}}{1+e^{n\pi}}$$

and

$$\begin{aligned} &G_{4k+2}(\tau) - (2^{4k+1}+2)G_{4k+2}(2\tau) + 2^{4k+2}G_{4k+2}(4\tau) \\ =& G_{4k+2}\left(\frac{i}{2}\right) - (2^{4k+1}+2)G_{4k+2}(i) + 2^{4k+2}G_{4k+2}(2i) \\ =& (-4)^{2k+1}G_{4k+2}(2i) - (2^{4k+1}+2)\cdot 0 + 2^{4k+2}G_{4k+2}(2i) \\ =& 0. \end{aligned}$$

(Exercise 1.12). Hence

$$0 = (2^{4k+2}-2)\zeta(4k+2) + \frac{2(2\pi i)^{4k+2}}{(4k+1)!}\sum_{\substack{n=1 \\ (n\text{ odd})}}^{\infty} \frac{n^{4k+1}}{1+e^{n\pi}}.$$

(3) Theorem 12.17 in the textbook: *T. M. Apostol, Introduction to Analytic Number Theory* shows that

$$\zeta(4k+2) = (-1)^{2k+1+1}\frac{(2\pi)^{4k+2}B_{4k+2}}{2(4k+2)!} = \frac{(2\pi)^{4k+2}B_{4k+2}}{2(4k+2)!}.$$

Hence

$$\sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^{4k+1}}{1+e^{n\pi}} = \frac{2^{4k+1}-1}{8k+4} B_{4k+2}.$$

$\square$