

Solutions to the book: *Fulton, Algebraic Curves*

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Contents

Chapter 1: Affine Algebraic Sets	4
1.1. Algebraic Preliminaries	4
Problem 1.1.*	4
Problem 1.5.*	5
Problem 1.6.*	5
1.2. Affine Space and Algebraic Sets	6
Problem 1.8.*	6
Problem 1.9.	7
Problem 1.11.	7
Problem 1.15.*	8
1.3. The Ideal of a Set of Points	9
Problem 1.18.*	9
Problem PLACEHOLDER	10
1.4. The Hilbert Basis Theorem	10
1.5. Irreducible Components of an Algebraic Set	10
1.6. Algebraic Subsets of the Plane	10
1.7. Hilbert's Nullstellensatz	10
1.8. Modules; Finiteness Conditions	10
1.9. Integral Elements	10
1.10. Field Extensions	10
Chapter 2: Affine Varieties	11
2.1. Coordinate Rings	11
Problem 2.1.*	11
Problem PLACEHOLDER	11
2.2. Polynomial Maps	12
2.3. Coordinate Changes	12
2.4. Rational Functions and Local Rings	12
2.5. Discrete Valuation Rings	12

2.6. Forms	12
2.7. Direct Products of Rings	12
2.8. Operations with Ideals	12
2.9. Ideals with a Finite Number of Zeros	12
2.10. Quotient Modules and Exact Sequences	12
2.11. Free Modules	12
Chapter 3: Local Properties of Plane Curves	13
3.1. Multiple Points and Tangent Lines	13
Problem PLACEHOLDER	13
3.2. Multiplicities and Local Rings	13
3.3. Intersection Numbers	13
Chapter 4: Projective Varieties	14
4.1. Projective Space	14
Problem PLACEHOLDER	14
4.2. Projective Algebraic Sets	14
4.3. Affine and Projective Varieties	14
4.4. Multiprojective Space	14
Chapter 5: Projective Plane Curves	15
5.1. Definitions	15
Problem PLACEHOLDER	15
5.2. Linear Systems of Curves	15
5.3. Bézout's Theorem	15
5.4. Multiple Points	15
5.5. Max Noether's Fundamental Theorem	15
5.6. Applications of Noether's Theorem	15
Chapter 6: Varieties, Morphisms, and Rational Maps	16
6.1. The Zariski Topology	16
6.2. Varieties	16
6.3. Morphisms of Varieties	16
6.4. Products and Graphs	16
6.5. Algebraic Function Fields and Dimension of Varieties	16
6.6. Rational Maps	16
Chapter 7: Resolution of Singularities	17
7.1. Rational Maps of Curves	17
Problem PLACEHOLDER	17
7.2. Blowing up a Point in \mathbf{A}^2	17
7.3. Blowing up a Point in \mathbf{P}^2	17
7.4. Quadratic Transformations	17
7.5. Nonsingular Models of Curves	17

Chapter 8: Riemann-Roch Theorem	18
8.1. Divisors	18
Problem PLACEHOLDER	18
8.1. The Vector Spaces $L(D)$	18
8.1. Riemann's Theorem	18
8.1. Derivations and Differentials	18
8.1. Canonical Divisors	18
8.6. Riemann-Roch Theorem	18

Chapter 1: Affine Algebraic Sets

1.1. Algebraic Preliminaries

Problem 1.1.*

Let R be a domain.

- (a) If f, g are forms of degree r, s respectively in $R[x_1, \dots, x_n]$, show that fg is a form of degree $r + s$.
- (b) Show that any factor of a form in $R[x_1, \dots, x_n]$ is also a form.

Proof of (a).

- (1) Write

$$f = \sum_{(i)} a_{(i)} x^{(i)},$$
$$g = \sum_{(j)} b_{(j)} x^{(j)},$$

where $\sum_{(i)}$ is the summation over $(i) = (i_1, \dots, i_n)$ with $i_1 + \dots + i_n = r$ and $\sum_{(j)}$ is the summation over $(j) = (j_1, \dots, j_n)$ with $j_1 + \dots + j_n = s$.

- (2) Hence,

$$fg = \sum_{(i)} \sum_{(j)} a_{(i)} b_{(j)} x^{(i)} x^{(j)}$$
$$= \sum_{(i), (j)} a_{(i)} b_{(j)} x^{(k)}$$

where $(k) = (i_1 + j_1, \dots, i_n + j_n)$ with $(i_1 + j_1) + \dots + (i_n + j_n) = r + s$. Each $x^{(k)}$ is the form of degree $r + s$ and $a_{(i)} b_{(j)} \in R$. Hence fg is a form of degree $r + s$.

□

Proof of (b).

- (1) Given any form $f \in R[x_1, \dots, x_n]$, and write $f = gh$. It suffices to show that g is a form as well. (So does h .)
- (2) Write

$$g = g_0 + \dots + g_r, \quad h = h_0 + \dots + h_s$$

where $g_r \neq 0$ and $h_s \neq 0$. So

$$f = gh = g_0h_0 + \cdots + g_rh_s.$$

Since R is a domain, $R[x_1, \dots, x_n]$ is a domain and thus $g_rh_s \neq 0$. The maximality of r and s implies that $\deg f = r + s$. Therefore, by the maximality of $r + s$, $f = g_rh_s$, or $g = g_r$, or g is a form.

□

Problem 1.5.*

Let k be any field. Show that there are an infinitely number of irreducible monic polynomials in $k[x]$. (Hint: Suppose f_1, \dots, f_n were all of them, and factor $f_1 \cdots f_n + 1$ into irreducible factors.)

Proof (Due to Euclid).

- (1) If f_1, \dots, f_n were all irreducible monic polynomials, then we consider

$$g = f_1 \cdots f_n + 1 \in k[x].$$

So there is an irreducible monic polynomial $f = f_i$ dividing g for some i since

$$\deg g = \deg f_1 + \cdots + \deg f_n \geq 1.$$

- (2) However, f would divide the difference

$$g - f_1 \cdots f_{i-1}f_{i+1} \cdots f_n = 1,$$

contrary to $\deg f_i \geq 1$.

□

Problem 1.6.*

Show that any algebraically closed field is infinite. (Hint: The irreducible monic polynomials are $x - a$, $a \in k$.)

Proof (Due to Euclid).

- (1) Let k be an algebraically closed field. If a_1, \dots, a_n were all elements in k , then we consider a monic polynomials

$$f(x) = (x - a_1) \cdots (x - a_n) + 1 \in k[x].$$

- (2) Since k is algebraically closed, there is an element $a \in k$ such that $f(a) = 0$. By assumption, $a = a_i$ for some $1 \leq i \leq n$, and thus $f(a) = f(a_i) = 1$, contrary to the fact that a field is a commutative ring where $0 \neq 1$ and all nonzero elements are invertible.

□

1.2. Affine Space and Algebraic Sets

Problem 1.8.*

Show that the algebraic subsets of $\mathbf{A}^1(k)$ are just the finite subsets, together with $\mathbf{A}^1(k)$ itself.

Proof.

- (1) Show that $k[x]$ is a PID if k is a field.
- (a) Let I be an ideal of $k[x]$.
 - (b) If $I = \{0\}$ then $I = (0)$ and I is principal.
 - (c) If $I \neq \{0\}$, then take f to be a polynomial of minimal degree in I . It suffices to show that $I = (f)$. Clearly, $(f) \subseteq I$ since I is an ideal. Conversely, for any $g \in I$,

$$g(x) = f(x)h(x) + r(x)$$

for some $h, r \in k[x]$ with $r = 0$ or $\deg r < \deg f$. Now as

$$r = g - fh \in I,$$

$r = 0$ (otherwise contrary to the minimality of f), we have $g = fh \in (f)$ for all $g \in I$.

- (2) Let Y be an algebraic subset of $\mathbf{A}^1(k)$, say $Y = V(I)$ for some ideal I of $k[x]$. Since $k[x]$ is a PID, $I = (f)$ for some $f \in k[x]$.
- (a) If $f = 0$, then $I = (0)$ and $Y = V(0) = \mathbf{A}^1(k)$.
 - (b) If $f \neq 0$, then $f(x) = 0$ has finitely many roots in k , say $a_1, \dots, a_m \in k$. Hence,

$$Y = V(I) = V(f) = \{f(a) = 0 : a \in k\} = \{a_1, \dots, a_m\}$$

is a finite subsets of $\mathbf{A}^1(k)$.

By (a)(b), the result is established.

□

Notes.

- (1) By the Hilbert basis theorem, $k[x]$ is Noetherian as k is Noetherian. Hence, for any algebraic subset $Y = V(I)$ of $\mathbf{A}^1(k)$, we can write $I = (f_1, \dots, f_m)$. Note that

$$Y = V(I) = V(f_1) \cap \dots \cap V(f_m).$$

Now apply the same argument to get the same conclusion.

- (2) Suppose $k = \bar{k}$. $\mathbf{A}^1(k)$ is irreducible, because its only proper closed subsets are finite, yet it is infinite (because k is algebraically closed, hence infinite).

Problem 1.9.

If k is a finite field, show that every subset of $\mathbf{A}^n(k)$ is algebraic.

Proof.

- (1) Every subset of $\mathbf{A}^n(k)$ is finite since $|\mathbf{A}^n(k)| = |k|^n$ is finite.
- (2) Note that $V(x_1 - a_1, \dots, x_n - a_n) = \{(a_1, \dots, a_n)\} \subseteq \mathbf{A}^n(k)$ (property (5) in this section) and any finite union of algebraic sets is algebraic (property (4) in this section). Thus, every subset of $\mathbf{A}^n(k)$ is algebraic (by (1)).

□

Problem 1.11.

Show that the following are algebraic sets:

- (a) $\{(t, t^2, t^3) \in \mathbf{A}^3(k) : t \in k\}$;
- (b) $\{(\cos(t), \sin(t)) \in \mathbf{A}^2(\mathbb{R}) : t \in \mathbb{R}\}$;
- (c) *the set of points in $\mathbf{A}^2(\mathbb{R})$ whose polar coordinates (r, θ) satisfy the equation $r = \sin(\theta)$.*

Proof of (a).

- (1) The twisted cubic curve

$$Y = \{(t, t^2, t^3) \in \mathbf{A}^3(k) : t \in k\} = V(x^2 - y) \cap V(x^3 - z)$$

is algebraic. We say that Y is given by the parametric representation $x = t, y = t^2, z = t^3$.

- (2) The generators for the ideal $I(Y)$ are $x^2 - y$ and $x^3 - z$.
- (3) Y is an affine variety of dimension 1.
- (4) The affine coordinate ring $A(Y)$ is isomorphic to a polynomial ring in one variable over k .

□

Proof of (b). The circle

$$\{(\cos(t), \sin(t)) \in \mathbf{A}^2(\mathbb{R}) : t \in \mathbb{R}\} = V(x^2 - y^2 - 1)$$

is algebraic. □

Proof of (c). The circle

$$\{(r, \theta) : r = \sin(\theta)\} = V(x^2 + y^2 - y)$$

is algebraic again. □

Problem 1.15.*

Let $V \subseteq \mathbf{A}^n(k)$, $W \subseteq \mathbf{A}^m(k)$ be algebraic sets. Show that

$$V \times W = \{(a_1, \dots, a_n, b_1, \dots, b_m) : (a_1, \dots, a_n) \in V, (b_1, \dots, b_m) \in W\}$$

is an algebraic set in $\mathbf{A}^{n+m}(k)$. It is called the **product** of V and W .

Proof.

- (1) Write

$$\begin{aligned} V &= V(S_V) = \{a \in \mathbf{A}^n(k) : f(a) = 0 \forall f \in S_V\} \\ W &= V(S_W) = \{b \in \mathbf{A}^m(k) : g(b) = 0 \forall g \in S_W\}, \end{aligned}$$

where $S_V \subseteq k[x_1, \dots, x_n]$ and $S_W \subseteq k[y_1, \dots, y_m]$. It suffices to show that

$$V \times W = V(S),$$

where $S \subseteq k[x_1, \dots, x_n, y_1, \dots, y_m]$ is the union of S_V and S_W .

- (2) Here we can identify S_V with the subset of $k[x_1, \dots, x_n, y_1, \dots, y_m]$ by noting that

$$k[x_1, \dots, x_n] \hookrightarrow (k[y_1, \dots, y_m])[x_1, \dots, x_n] = k[x_1, \dots, x_n, y_1, \dots, y_m].$$

Here we regard k as a subring of $k[y_1, \dots, y_m]$. Similar treatment to S_W .

- (3) By construction, $V \times W \subseteq V(S)$. Conversely, given any $(a, b) \in V(S)$, we have $h(a, b) = 0$ for all $h \in S = S_V \cup S_W$ (by (2)). By construction, $f(a) = 0$ for all $f \in S_V$ since f only involve x_1, \dots, x_n . Hence, $a \in V$. Similarly, $b \in W$. Therefore, $(a, b) \in V \times W$.

□

1.3. The Ideal of a Set of Points

Problem 1.18.*

Let I be an ideal in a ring R . If $a^n \in I$, $b^m \in I$, show that $(a + b)^{n+m} \in I$. Show that $\text{Rad}(I)$ is an ideal, in fact a radical ideal. Show that any prime ideal is radical.

Proof.

- (1) Show that $(a + b)^{n+m} \in I$ if $a^n \in I$, $b^m \in I$. By the binomial theorem,

$$(a + b)^{n+m} = \sum_{i=0}^{n+m} a^i b^{n+m-i}.$$

For each term $a^i b^{n+m-i}$, either $i \geq n$ holds or $n + m - i \geq m$ holds, and thus $a^i b^{n+m-i} \in I$ (since $a^n \in I$, $b^m \in I$ and I is an ideal). Hence, the result is established.

- (2) Show that $\text{Rad}(I)$ is an ideal.

- (a) $0 \in \text{Rad}(I)$ since $0 = 0^1 \in I$ for any ideal in R .
- (b) $(a + b)^{n+m} \in I$ if $a^n \in I$, $b^m \in I$ by (1).
- (c) $(-a)^{2n} = (a^n)^2 \in I$ if $a^n \in I$ (since I is an ideal).
- (d) $(ra)^n = r^n a^n \in I$ if $a^n \in I$ and $r \in R$ (since I is an ideal and R is commutative).

- (3) Show that $\text{Rad}(\text{Rad}(I)) = \text{Rad}(I)$. It suffices to show $\text{Rad}(\text{Rad}(I)) \subseteq \text{Rad}(I)$. Given any $a \in \text{Rad}(\text{Rad}(I))$. By definition $a^n \in \text{Rad}(I)$ for some positive integer n . Again by definition $(a^n)^m = a^{nm} \in I$ for some positive integer m . As nm is a positive integer, $a \in \text{Rad}(I)$.

- (4) Show that every prime ideal \mathfrak{p} is radical. Given any $a \in \text{Rad}(\mathfrak{p})$, that is, $a^n \in \mathfrak{p}$ for some positive integer. Write $a^n = aa^{n-1}$ if $n > 1$. By the primality of \mathfrak{p} , $a \in \mathfrak{p}$ or $a^{n-1} \in \mathfrak{p}$. If $a \in \mathfrak{p}$, we are done. If $a^{n-1} \in \mathfrak{p}$, we continue this descending argument until the power of a is equal to 1. Hence \mathfrak{p} is radical.

□

Problem PLACEHOLDER

PLACEHOLDER

Proof.

(1) PLACEHOLDER

1.4. The Hilbert Basis Theorem

1.5. Irreducible Components of an Algebraic Set

1.6. Algebraic Subsets of the Plane

1.7. Hilbert's Nullstellensatz

1.8. Modules; Finiteness Conditions

1.9. Integral Elements

1.10. Field Extensions

Chapter 2: Affine Varieties

2.1. Coordinate Rings

Problem 2.1.*

Show that the map which associates to each $f \in k[x_1, \dots, x_n]$ a polynomial function in $\mathcal{F}(V, k)$ is a ring homomorphism whose kernel is $I(V)$.

Proof.

- (1) Define a map $\alpha : k[x_1, \dots, x_n] \rightarrow \mathcal{F}(V, k)$ by $\alpha : f \mapsto f|_V$.
- (2) Clearly α is a ring homomorphism.
- (3) Show that $\ker(\alpha) = I(V)$. In fact, given any $f \in k[x_1, \dots, x_n]$, we have $\alpha(f) = 0$ if and only if $f(a) = 0$ for all $a \in V$ if and only if $f \in I(V)$. Hence, $\mathcal{F}(V, k) \cong k[x_1, \dots, x_n]/I(V) = \Gamma(V)$ as a ring isomorphism.

□

Problem PLACEHOLDER

PLACEHOLDER

Proof.

- (1) PLACEHOLDER

- 2.2. Polynomial Maps
- 2.3. Coordinate Changes
- 2.4. Rational Functions and Local Rings
- 2.5. Discrete Valuation Rings
- 2.6. Forms
- 2.7. Direct Products of Rings
- 2.8. Operations with Ideals
- 2.9. Ideals with a Finite Number of Zeros
- 2.10. Quotient Modules and Exact Sequences
- 2.11. Free Modules

Chapter 3: Local Properties of Plane Curves

3.1. Multiple Points and Tangent Lines

Problem PLACEHOLDER

PLACEHOLDER

Proof.

(1) PLACEHOLDER

3.2. Multiplicities and Local Rings

3.3. Intersection Numbers

Chapter 4: Projective Varieties

4.1. Projective Space

Problem PLACEHOLDER

PLACEHOLDER

Proof.

- (1) PLACEHOLDER

4.2. Projective Algebraic Sets

4.3. Affine and Projective Varieties

4.4. Multiprojective Space

Chapter 5: Projective Plane Curves

5.1. Definitions

Problem **PLACEHOLDER**

PLACEHOLDER

Proof.

- (1) **PLACEHOLDER**

5.2. Linear Systems of Curves

5.3. Bézout's Theorem

5.4. Multiple Points

5.5. Max Noether's Fundamental Theorem

5.6. Applications of Noether's Theorem

Chapter 6: Varieties, Morphisms, and Rational Maps

6.1. The Zariski Topology

6.2. Varieties

6.3. Morphisms of Varieties

6.4. Products and Graphs

6.5. Algebraic Function Fields and Dimension of Varieties

6.6. Rational Maps

Chapter 7: Resolution of Singularities

7.1. Rational Maps of Curves

Problem PLACEHOLDER

PLACEHOLDER

Proof.

(1) PLACEHOLDER

7.2. Blowing up a Point in A^2

7.3. Blowing up a Point in P^2

7.4. Quadratic Transformations

7.5. Nonsingular Models of Curves

Chapter 8: Riemann-Roch Theorem

8.1. Divisors

Problem **PLACEHOLDER**

PLACEHOLDER

Proof.

(1) **PLACEHOLDER**

8.2. The Vector Spaces $L(D)$

8.3. Riemann's Theorem

8.4. Derivations and Differentials

8.5. Canonical Divisors

8.6. Riemann-Roch Theorem