Chapter 15: Bernoulli Numbers

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Supplement. Equation (4) on page 231. *Prove that*

$$x \cot x = 1 - 2 \sum_{n=1}^{\infty} \frac{x^2}{n^2\pi^2 - x^2}.$$

Proof (Exercise 6.73 in the book Graham, Knuth and Patashnik, Concrete Mathematics, Second Edition).

(1) *Show that*

$$\cot x = \frac{1}{2^n} \sum_{k=0}^{2^n-1} \cot \frac{x + k\pi}{2^n}$$

for all integers $n \geq 1$. Notice that

$$\begin{aligned} \cot(x + \pi) &= \cot x, \\ \cot\left(x + \frac{\pi}{2}\right) &= -\tan x, \\ \cot x &= \frac{1}{2}\left(\cot \frac{x}{2} - \tan \frac{x}{2}\right). \end{aligned}$$

Use mathematical induction. The case $n = 1$ is the same as the note. Assume the case $n = m$ holds. For $n = m + 1$,

$$\begin{aligned} \sum_{k=0}^{2^{m+1}-1} \cot \frac{x + k\pi}{2^{m+1}} &= \sum_{k=0}^{2^m-1} \cot \frac{x + k\pi}{2^{m+1}} + \sum_{k=2^m}^{2^{m+1}-1} \cot \frac{x + k\pi}{2^{m+1}} \\ &= \sum_{k=0}^{2^m-1} \cot \frac{x + k\pi}{2^{m+1}} + \sum_{k=0}^{2^m-1} \cot \frac{x + (2^m + k)\pi}{2^{m+1}} \\ &= \sum_{k=0}^{2^m-1} \cot \frac{x + k\pi}{2^{m+1}} + \sum_{k=0}^{2^m-1} \cot\left(\frac{x + k\pi}{2^{m+1}} + \frac{\pi}{2}\right) \\ &= \sum_{k=0}^{2^m-1} \left(\cot \frac{x + k\pi}{2^{m+1}} - \tan \frac{x + k\pi}{2^{m+1}}\right) \\ &= \sum_{k=0}^{2^m-1} \left(\cot \frac{x + k\pi}{2^{m+1}} - \tan \frac{x + k\pi}{2^{m+1}}\right) \\ &= 2 \sum_{k=0}^{2^m-1} \cot \frac{x + k\pi}{2^m}. \end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{2^{m+1}}\sum_{k=0}^{2^{m+1}-1}\cot\frac{x+k\pi}{2^{m+1}} &= \frac{1}{2^{m+1}}\cdot 2\sum_{k=0}^{2^m-1}\cot\frac{x+k\pi}{2^m}\\ &= \frac{1}{2^m}\sum_{k=0}^{2^m-1}\cot\frac{x+k\pi}{2^m}\\ &= \cot x.\end{aligned}$$

(2) By rearranging the index of summation of the identity in (1), we have

$$x\cot x = \frac{x}{2^n}\cot\frac{x}{2^n} - \frac{x}{2^n}\tan\frac{x}{2^n} + \sum_{k=1}^{2^{n-1}-1}\frac{x}{2^n}\left(\cot\frac{x+k\pi}{2^n} + \cot\frac{x-k\pi}{2^n}\right)$$

for all integers $n \geq 1$.

(3) Notice that $\lim_{x\to 0} x\cot x = 1$. Let $n \to \infty$, the result is established.

□

Exercise 15.1. *Using the definition of the Bernoulli number show* $B_{10} = \frac{5}{66}$ *and* $B_{12} = -\frac{691}{2730}$.

Proof.

(1) It is known that $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, and $B_m = 0$ for odd $m > 1$.

(2) Recall the implicit recurrence relation,

$$\sum_{k=0}^{m}\binom{m+1}{k}B_k = [m=0],$$

where $[m=0]$ is the Iverson brackets which is equal to the Kronecker delta δ_{m0}.

(3) So

$$\begin{aligned}&0 = 1 + 9B_1 + 36B_2 + 84B_3 + 126B_4 + 126B_5 + 84B_6 + 36B_7 + 9B_8,\\ &0 = 1 + 9B_1 + 36B_2 + 126B_4 + 84B_6 + 9B_8,\\ &0 = 1 + 9\left(-\frac{1}{2}\right) + 36\left(\frac{1}{6}\right) + 126\left(-\frac{1}{30}\right) + 84\left(\frac{1}{42}\right) + 9B_8,\\ &0 = \frac{3}{10} + 9B_8,\end{aligned}$$

Thus $B_8 = -\frac{1}{30}$.

(4) Again,

$$
\begin{aligned}
0 = &1 + 11B_1 + 55B_2 + 165B_3 + 330B_4 + 462B_5 + 462B_6 + \\
&330B_7 + 165B_8 + 55B_9 + 11B_{10}, \\
0 = &1 + 11B_1 + 55B_2 + 330B_4 + 462B_6 + 165B_8 + 11B_{10}, \\
0 = &1 + 11\left(-\frac{1}{2}\right) + 55\left(\frac{1}{6}\right) + 330\left(-\frac{1}{30}\right) + 462\left(\frac{1}{42}\right) + \\
&165\left(-\frac{1}{30}\right) + 11B_{10}, \\
0 = &-\frac{5}{6} + 11B_{10},
\end{aligned}
$$

Thus $B_{10} = \frac{5}{66}$.

(4) Finally,

$$
\begin{aligned}
0 = &1 + 13B_1 + 78B_2 + 286B_3 + 715B_4 + 1287B_5 + 1716B_6 + \\
&1716B_7 + 1287B_8 + 715B_9 + 286B_{10} + 78B_{11} + 13B_{12}, \\
0 = &1 + 13B_1 + 78B_2 + 715B_4 + 1716B_6 + 1287B_8 + 286B_{10} + 13B_{12}, \\
0 = &1 + 13\left(-\frac{1}{2}\right) + 78\left(\frac{1}{6}\right) + 715\left(-\frac{1}{30}\right) + 1716\left(\frac{1}{42}\right) + \\
&1287\left(-\frac{1}{30}\right) + 286\left(\frac{5}{66}\right) + 13B_{12}, \\
0 = &\frac{691}{210} + 13B_{12},
\end{aligned}
$$

Thus $B_{12} = -\frac{691}{2730}$.

□

Exercise 15.2. *If $a \in \mathbb{Z}$, show $a(a^m - 1)B_m \in \mathbb{Z}$ for all $m > 0$.*

Proof.

(1) *Trivial cases.* If $m = 1$, $a(a-1)B_1 = -\frac{1}{2}a(a-1) \in \mathbb{Z}$ for any $a \in \mathbb{Z}$. For odd $m > 1$, $B_m = 0$ or $a(a^m - 1)B_m = 0 \in \mathbb{Z}$ (Proposition 15.1.1).

(2) *Consider that $m > 1$ and even.* By Theorem 3,

$$
B_{2m} + \sum_{p-1 \mid 2m} \frac{1}{p} \in \mathbb{Z}
$$

where the sum is over all primes p such that $p-1 \mid 2m$. So it suffices to show

$$a(a^{2m}-1)\sum_{p-1|2m}\frac{1}{p}\in\mathbb{Z},$$

or

$$a(a^{2m}-1)\frac{1}{p}\in\mathbb{Z}$$

for any $a\in\mathbb{Z}$ and any prime p such that $p-1 \mid 2m$.

(3) *Consider all possible a.* If $p \mid a$, it is trivial. If $p \nmid a$, $a^{p-1}\equiv 1\ (p)$ by Fermat's Little Theorem, or $a^{2m}\equiv 1\ (p)$ by $p-1 \mid 2m$. In any cases, $a(a^{2m}-1)\frac{1}{p}\in\mathbb{Z}$.

□

Exercise 15.6. *For $m\geq 3$, show $|B_{2m+2}|>|B_{2m}|$. (Hint: Use Theorem 2.)*

Proof. By Theorem 2,

$$2\zeta(2m)=(-1)^{m+1}\frac{(2\pi)^{2m}}{(2m)!}B_{2m}.$$

Thus,

$$\frac{|B_{2m+2}|}{|B_{2m}|}=\frac{\zeta(2m+2)(2m+2)(2m+1)}{\zeta(2m)(2\pi)^2}>\frac{1\cdot 8\cdot 7}{\zeta(6)\cdot(2\pi)^2}=\frac{13230}{\pi^8}>1,$$

or $|B_{2m+2}|>|B_{2m}|$. □

Exercise 15.8. *Consider the power series expansion of* $\tan x$ *about the origin;*

$$\sum_{k=1}^{\infty}T_k\frac{x^{2k-1}}{(2k-1)!}.$$

Show

$$T_k=(-1)^{k-1}\frac{B_{2k}}{2k}(2^{2k}-1)2^{2k}.$$

Note that $T_k\in\mathbb{Z}$ for all k by Exercise 3.

Proof.

(1) By the equation (6) on page 232,

$$x\cot x=1+\sum_{k=2}^{\infty}B_k\frac{(2ix)^k}{k!}.$$

Since $B_k = 0$ for odd $k > 1$,

$$x \cot x = 1 + \sum_{k=1}^{\infty} B_{2k} \frac{(2ix)^{2k}}{(2k)!} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} x^{2k},$$

or

$$\cot x = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} x^{2k-1}.$$

Combine the first term $\frac{1}{x}$ into the summation,

$$\cot x = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} x^{2k-1}.$$

(2) Note that $\tan x = \cot x - 2\cot(2x)$. By (1),

$$\begin{aligned}
\tan x &= \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} x^{2k-1} - 2 \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} (2x)^{2k-1} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k (1 - 2^{2k}) 2^{2k} B_{2k}}{(2k)!} x^{2k-1} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k (1 - 2^{2k}) 2^{2k} B_{2k}}{(2k)!} x^{2k-1}.
\end{aligned}$$

Write $T_k = (-1)^{k-1}(2^{2k} - 1)2^{2k}\frac{B_{2k}}{2k}$. Therefore, $\tan x = \sum_{k=1}^{\infty} T_k \frac{x^{2k-1}}{(2k-1)!}$.

By Exercise 3, $(2^{2k} - 1)2^{2k}\frac{B_{2k}}{2k} \in \mathbb{Z}$, or $T_k \in \mathbb{Z}$ for all k. $\square$

Exercise 15.13. *Show* $B_m(x+1) - B_m(x) = mx^{m-1}$.

Proof. Let $f(t, x) = \frac{te^{tx}}{e^t - 1}$.

(1)

$$f(t, x+1) - f(t, x) = \frac{te^{t(x+1)}}{e^t - 1} - \frac{te^{tx}}{e^t - 1} = te^{tx}.$$

Expand te^{tx} in a power series about the origin as follows

$$\begin{aligned}
te^{tx} &= t\sum_{m=0}^{\infty} x^m \frac{t^m}{m!} \\
&= \sum_{m=0}^{\infty} x^m \frac{t^{m+1}}{m!} \\
&= \sum_{m=1}^{\infty} x^{m-1} \frac{t^m}{(m-1)!} \\
&= \sum_{m=1}^{\infty} mx^{m-1} \frac{t^m}{m!} \\
&= \sum_{m=0}^{\infty} mx^{m-1} \frac{t^m}{m!}.
\end{aligned}$$

So,

$$f(t, x+1) - f(t, x) = \sum_{m=0}^{\infty} mx^{m-1} \frac{t^m}{m!}.$$

(2) By Exercise 15.12,

$$\begin{aligned}
f(t, x+1) - f(t, x) &= \sum_{m=0}^{\infty} B_m(x+1)\frac{t^m}{m!} - \sum_{m=0}^{\infty} B_m(x)\frac{t^m}{m!} \\
&= \sum_{m=0}^{\infty} (B_m(x+1) - B_m(x))\frac{t^m}{m!}.
\end{aligned}$$

By (1)(2), comparing coefficients of t^m yields

$$mx^{m-1} = B_m(x+1) - B_m(x).$$

□

Exercise 15.14. *Use Exercise 13 to give a new proof of Theorem 1:*

$$S_m(n) = \frac{1}{m+1}(B_{m+1}(n) - B_{m+1}).$$

Proof. By Exercise 13,

$$B_{m+1}(k) - B_{m+1}(k-1) = (m+1)(k-1)^m$$

for any k. So,

$$\begin{aligned}
\sum_{k=1}^{n}(B_{m+1}(k) - B_{m+1}(k-1)) &= \sum_{k=1}^{n}(m+1)(k-1)^m, \\
B_{m+1}(n) - B_{m+1}(0) &= (m+1)S_m(n).
\end{aligned}$$

Note that $B_{m+1}(0) = B_{m+1}$ for any m. So Theorem 1 is established by a new proof. $\square$

Exercise 15.16. *For $n \geq 1$, show $\frac{d}{dx}B_n(x) = nB_{n-1}(x)$.*

Proof. For $n \geq 1$,

$$\frac{d}{dx}B_n(x) = \sum_{k=0}^{n}(n-k)\binom{n}{k}B_k x^{n-k-1} = \sum_{k=0}^{n-1}(n-k)\binom{n}{k}B_k x^{n-k-1}.$$

Note that

$$(n-k)\binom{n}{k} = n\binom{n-1}{k}.$$

So

$$\begin{aligned}\frac{d}{dx}B_n(x) &= \sum_{k=0}^{n-1} n\binom{n-1}{k}B_k x^{n-k-1} \\ &= n\sum_{k=0}^{n-1}\binom{n-1}{k}B_k x^{n-k-1} \\ &= nB_{n-1}(x).\end{aligned}$$

$\square$