Chapter 2: Number Fields and Number Rings

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Exercise 2.28. *Let $f(x) = x^3+ax+b$, a and $b \in \mathbb{Z}$, and assume f is irreducible over $\mathbb{Q}$. Let α be a root of f.*

(a) *Show that $f'(\alpha) = -\frac{2a\alpha+3b}{\alpha}$.*

(b) *Show that $2a\alpha + 3b$ is a root of*
$$\left(\frac{x-3b}{2a}\right)^3 + a\left(\frac{x-3b}{2a}\right) + b.$$
Use this to find $N^{\mathbb{Q}[\alpha]}_{\mathbb{Q}}(2a\alpha + 3b)$.

(c) *Show that $disc(\alpha) = -(4a^3 + 27b^2)$.*

(d) *Suppose $\alpha^3 = \alpha+1$. Prove that $\{1, \alpha, \alpha^2\}$ is an integral basis for $\mathbb{A}\cap\mathbb{Q}[\alpha]$. (See Exercise 2.27(e).) Do the same if $\alpha^3 + \alpha = 1$.*

Proof of (a).

(1) *Show that $\alpha \neq 0$.* If α were 0, then $f(\alpha) = f(0) = b$. So $f(x) = x^3 + ax = x(x^2 + a)$ is reducible, contrary to the irreducibility of f.

(2) Since α be a root of f, $f(\alpha) = 0$, or $\alpha^3 + a\alpha + b = 0$, or $\alpha^3 = -a\alpha - b$.

(3)
$$\begin{aligned}
f'(x) = 3x^2 + a &\Longrightarrow f'(\alpha) = 3\alpha^2 + a \\
&\Longleftrightarrow \alpha f'(\alpha) = 3\alpha^3 + a && (\alpha \neq 0) \\
&\Longleftrightarrow \alpha f'(\alpha) = 3(-a\alpha - b) + a\alpha && (\alpha^3 = -a\alpha - b) \\
&\Longleftrightarrow \alpha f'(\alpha) = -2a\alpha - 3b.
\end{aligned}$$
So $f'(\alpha) = -\frac{2a\alpha+3b}{\alpha}$.

□

Proof of (b).

(1) Since $\alpha^3 + a\alpha + b = 0$,
$$\left(\frac{(2a\alpha+3b)-3b}{2a}\right)^3 + a\left(\frac{(2a\alpha+3b)-3b}{2a}\right) + b = 0.$$
That is, $2a\alpha + 3b$ is a root of $\left(\frac{x-3b}{2a}\right)^3 + a\left(\frac{x-3b}{2a}\right) + b$.

(2) $N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(2a\alpha+3b)$ is the product of three roots of $\left(\frac{x-3b}{2a}\right)^3 + a\left(\frac{x-3b}{2a}\right) + b$. Hence,

$$\begin{aligned}
N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(2a\alpha+3b) &= (2a)^3\left[\left(\frac{-3b}{2a}\right)^3 + a\cdot\frac{-3b}{2a} + b\right] \\
&= 8a^3\left[\frac{-27b^3}{8a^3} - \frac{b}{2}\right] \\
&= -27b^3 - 4a^3 b.
\end{aligned}$$

□

Proof of (c).

$$\begin{aligned}
\operatorname{disc}(\alpha) &= (-1)^{\frac{n(n-1)}{2}} N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(f'(\alpha)) && \text{(Theorem 2.8)} \\
&= -N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}\left(-\frac{2a\alpha+3b}{\alpha}\right) && (n = 3 \text{ and (a)}) \\
&= \frac{N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(2a\alpha+3b)}{N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(\alpha)} \\
&= \frac{-27b^3 - 4a^3 b}{b} && \text{((b))} \\
&= -27b^2 - 4a^3.
\end{aligned}$$

□

Proof of (d).

(1) $\alpha^3 = \alpha + 1$. $\alpha^3 - \alpha - 1 = 0$. So $\operatorname{disc}(\alpha) = -23$ (by (c)). Since $\operatorname{disc}(\alpha)$ is squarefree, $\{1, \alpha, \alpha^2\}$ forms an integral basis for $\mathbb{A} \cap \mathbb{Q}[\alpha]$ (Exercise 2.27(e)).

(2) $\alpha^3 + \alpha = 1$. $\alpha^3 + \alpha - 1 = 0$. So $\operatorname{disc}(\alpha) = -31$ (by (c)). The rest is similar to (1).

□