# Chapter 1: Affine Algebraic Sets

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**Problem 1.1.*** *Let $R$ be a domain.*

(a) *If $F$, $G$ are forms of degree $r$, $s$ respectively in $R[X_1, \ldots, X_n]$, show that $FG$ is a form of degree $r + s$.*

(b) *Show that any factor of a form in $R[X_1, \ldots, X_n]$ is also a form.*

*Proof of (a).*

(1) Write

$$F = \sum_{(i)} a_{(i)} X^{(i)},$$
$$G = \sum_{(j)} b_{(j)} X^{(j)},$$

where $\sum_{(i)}$ is the summation over $(i) = (i_1, \ldots, i_n)$ with $i_1 + \cdots + i_n = r$ and $\sum_{(j)}$ is the summation over $(j) = (j_1, \ldots, j_n)$ with $j_1 + \cdots + j_n = s$.

(2) Hence,

$$\begin{aligned} FG &= \sum_{(i)} \sum_{(j)} a_{(i)} b_{(j)} X^{(i)} X^{(j)} \\ &= \sum_{(i),(j)} a_{(i)} b_{(j)} X^{(k)} \end{aligned}$$

where $(k) = (i_1 + j_1, \ldots, i_n + j_n)$ with $(i_1 + j_1) + \cdots + (i_n + j_n) = r + s$. Each $X^{(k)}$ is the form of degree $r + s$ and $a_{(i)} b_{(j)} \in R$. Hence $FG$ is a form of degree $r + s$.

□

*Proof of (b).*

(1) Given any form $F \in R[X_1, \ldots, X_n]$, and write $F = GH$. *It suffices to show that $G$ (or $H$) is a form as well.*

(2) Write

$$G = G_0 + \cdots + G_r,$$
$$H = H_0 + \cdots + H_s$$

where $G_r \neq 0$ and $H_s \neq 0$. So

$$F = GH = G_0 H_0 + \cdots + G_r H_s.$$

Since $R$ is a domain, $R[X_1, \ldots, X_n]$ is a domain and thus $G_r H_s \neq 0$. The maximality of $r$ and $s$ implies that $\deg(F) = r + s$. Therefore, by the maximality of $r + s$, $F = G_r H_s$, or $G = G_r$, or $G$ is a form.

□

**Problem 1.5.*** *Let $k$ be any field. Show that there are an infinitely number of irreducible monic polynomials in $k[X]$. (Hint: Suppose $F_1, \ldots, F_n$ were all of them, and factor $F_1 \cdots F_n + 1$ into irreducible factors.)*

*Proof (Due to Euclid).*

(1) If $F_1, F_2, \ldots, F_n$ were all irreducible monic polynomials, then we consider

$$G = F_1 F_2 \cdots F_n + 1 \in k[X].$$

So there is an irreducible monic polynomial $F$ dividing $G$ since

$$\deg G = \deg F_1 + \deg F_2 + \cdots + \deg F_n \geq 1.$$

(2) $F$ can not be any of $F_i$ for all $i$; otherwise $F$ would divide the difference $G - F_1 F_2 \cdots F_n = 1$. That is, $F \neq F_i$ for all $i$, contrary to the assumption.

□

**Problem 1.6.*** *Show that any algebraically closed field is infinite. (Hint: The irreducible monic polynomials are $X - a$, $a \in k$.)*

*Proof (Due to Euclid).*

(1) Let $k$ be an algebraically closed field. If $a_1, \ldots, a_n$ were all elements in $k$, then we consider a monic polynomials

$$F(X) = (X - a_1) \cdots (X - a_n) + 1 \in k[X].$$

(2) Since $k$ is algebraically closed, there is an element $a \in k$ such that $F(a) = 0$. By assumption, $a = a_i$ for some $1 \leq i \leq n$, and thus $F(a) = F(a_i) = 1$, contrary to the fact that a field is a commutative ring where $0 \neq 1$ and all nonzero elements are invertible.

□