Chapter 2: Basic Topology

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Notation.

(1) E° or $\mathrm{int}(E)$ is the interior of E.

(2) $\overline{E}$ is the closure of E.

(3) $\widetilde{E}$ is the complement of E.

(4) $B(p;r)$ or $B(p)$ is the set of all points q in a metric space (M,d) such that $d_M(p,q) < r$.

Exercise 2.1. *Prove that the empty set is a subset of every set.*

Proof. By Definitions 1.3,

(1) The set which contains no element will be called the **empty set**,

(2) If A and B are sets, and if every element of A is an element of B, we say that A is a **subset** of B,

every element of the empty set (there are none) belongs to every set. That is, the empty set is a subset of every set. $\square$

Exercise 2.2. *A complex number z is said to be algebraic if there are integers $a_0, ..., a_n$, not all zero, such that*

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic numbers is countable. (Hint: For every positive integer N there are only finitely many equations with

$$n + |a_0| + |a_1| + \cdots + |a_n| = N.$$

Might assume $a_0 \neq 0$.

For example, all rational numbers are algebraic since $p = \frac{\alpha}{\beta}$ (where $\alpha, \beta \in \mathbb{Z}$) is a root of $\beta z - \alpha = 0$.

Besides, $z = \sqrt{2}+\sqrt{3}$ is algebraic since $z^4-10z^2+1=0$. In fact, $z = \pm\sqrt{2}\pm\sqrt{3}$ are also algebraic since $z^4-10z^2+1 = (z-\sqrt{2}-\sqrt{3})(z+\sqrt{2}-\sqrt{3})(z-\sqrt{2}+\sqrt{3})(z+\sqrt{2}+\sqrt{3})$.

Lemma. *The set of all polynomials over $\mathbb{Z}$ is countable implies that the set of algebraic numbers is countable.*

Proof of Lemma. By definition, we write the set of algebraic numbers as

$$S = \bigcup_{f(x)\in\mathbb{Z}[x]} \{z \in \mathbb{C} : f(z) = 0\}.$$

Since each polynomial of degree n has at most n roots, $\{z \in \mathbb{C} : f(z) = 0\}$ is finite for each given $f(x) \in \mathbb{Z}[x]$. So S is a countable union (by assumption) of finite sets, and hence at most countable. S is infinite since every integer α is a root of $f(z) = z - \alpha$. So S is countable. $\square$

Thus, it suffices to show that *the set of all polynomials over $\mathbb{Z}$ is countable.*

Proof (Hint). For every positive integer N there are only finitely many equations with $n + |a_0| + |a_1| + \cdots + |a_n| = N$. Write

$$P_N = \{f(x) \in \mathbb{Z}[x] : n + |a_0| + |a_1| + \cdots + |a_n| = N\}$$

where $f(x) = a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n$ with $a_0 \neq 0$, and

$$P = \bigcup_{N=1}^{\infty} P_N.$$

P is the set of all polynomials over $\mathbb{Z}$.

Each P_N is finite for given N (since the equation $n+|a_0|+|a_1|+\cdots+|a_n| = N$ has finitely many solutions $(n, a_0, a_1, ..., a_n) \in \mathbb{Z}^{n+2}$). So P is a countable union of finite sets, and hence at most countable. P is infinite since $\mathbb{Z}$ is a subring of $\mathbb{Z}[x]$. So P is countable. $\square$

Proof (Theorem 2.13).

(1) *$\mathbb{Z}^N$ is countable for any integer $N > 0$.* Theorem 2.13.

(2) *The set of all polynomials over $\mathbb{Z}$ is countable.* Let

$$P_n = \{f \in \mathbb{Z}[x] : \deg f = n\},$$

and

$$P = \bigcup_{n=1}^{\infty} P_n = \mathbb{Z}[x].$$

Claim: P_n is countable. Define a 1-1 map $\varphi_n : P_n \to \mathbb{Z}^{n+1}$ by

$$\varphi_n(a_0 z^n + a_1 z^{n-1} + \cdots + a_n) = (a_0, a_1, ..., a_{n-1}, a_n).$$

By (1) and Theorem 2.8, P_n is countable. (P_n is infinite since $a_n \in \mathbb{Z}$.) Now P is a countable union of countable sets, and hence countable by Theorem 2.12.

$\square$

Proof (Unique factorization theorem).

(1) *The set of prime numbers is countable.* Write all primes in the ascending order as $p_1, p_2, ..., p_n, ...$ where $p_1 = 2, p_2 = 3, ..., p_{10001} = 104743, ...$ (See ProjectEuler 7: 10001st prime. Use sieve of Eratosthenes to get p_{10001}.)

(2) *The set of all polynomials over $\mathbb{Z}$ is countable.* Let

$$P_n = \{f \in \mathbb{Z}[x] : \deg f = n\},$$

and

$$P = \bigcup_{n=1}^{\infty} P_n = \mathbb{Z}[x].$$

Claim: P_n is countable. Define a map $\varphi_n : P_n \to \mathbb{Z}^+$ by

$$\varphi_n(a_0 z^n + a_1 z^{n-1} + \cdots + a_n) = p_1^{\psi(a_0)} p_2^{\psi(a_1)} \cdots p_{n+1}^{\psi(a_n)},$$

where ψ is a 1-1 correspondence from $\mathbb{Z}$ to $\mathbb{Z}^+$ (Example 2.5). By the unique factorization theorem, φ_n is 1-1. So P_n is countable by Theorem 2.8. (P_n is infinite since $a_n \in \mathbb{Z}$.) Now P is a countable union of countable sets, and hence countable by Theorem 2.12.

$\square$

Exercise 2.3. *Prove that there exist real numbers which are not algebraic.*

Proof (Exercise 2.2). If all real numbers were algebraic, then $\mathbb{R}$ is countable by Exercise 2.2, contrary to the fact that $\mathbb{R}$ is uncountable (Corollary to Theorem 2.43). $\square$

Proof (Liouville, 1844).

(1) **Lemma.** *If ξ is a real algebraic number of degree $n > 1$, then there is a constant $A > 0$ (depending on ξ) such that*

$$\left| \xi - \frac{h}{k} \right| \geq \frac{A}{k^n}$$

for all rational numbers $\frac{h}{k}$.

(a) If $\left|\xi - \frac{h}{k}\right| \geq 1$, pick $A = 1 > 0$.

(b) If $\left|\xi - \frac{h}{k}\right| < 1$, let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ be an irreducible polynomial of degree $n > 1$ over $\mathbb{Z}$ such that $f(\xi) = 0$. By the mean value theorem,

$$f(\xi) - f\left(\frac{h}{k}\right) = \left(\xi - \frac{h}{k}\right) f'(c)$$

for some $c \in \left(\xi - \frac{h}{k}, \xi + \frac{h}{k}\right) \subseteq (\xi - 1, \xi + 1)$. Notice that

(i) $f(\xi) = 0$ by definition.

(ii) $f\left(\frac{h}{k}\right) \neq 0$ since $\frac{h}{k}$ cannot be a root of $f(x)$. Otherwise f is of degree 1, contrary to the assumption of f.

(iii) $\left|f\left(\frac{h}{k}\right)\right| \geq \frac{1}{k^n}$ since

$$\begin{aligned} f\left(\frac{h}{k}\right) &= a_0 + a_1\left(\frac{h}{k}\right) + \cdots + a_n\left(\frac{h}{k}\right)^n \neq 0, \\ k^n f\left(\frac{h}{k}\right) &= a_0 k^n + hk^{n-1} a_1 + \cdots + h^n a_n \neq 0, \\ k^n \left|f\left(\frac{h}{k}\right)\right| &\geq 1. \end{aligned}$$

(iv) $|f'(c)| \leq \sup_{x \in [\xi-1, \xi+1]} |f'(x)|$ since $c \in [\xi - 1, \xi + 1]$ and $f'(x)$ is continuous or bounded on a compact set $[\xi - 1, \xi + 1]$.

By (i)-(iv),

$$\begin{aligned} \left|f(\xi) - f\left(\frac{h}{k}\right)\right| &= \left|\left(\xi - \frac{h}{k}\right) f'(c)\right|, \\ \frac{1}{k^n} \leq \left|f\left(\frac{h}{k}\right)\right| &= \left|\xi - \frac{h}{k}\right| |f'(c)| \leq \left|\xi - \frac{h}{k}\right| \cdot \sup_{x \in [\xi-1, \xi+1]} |f'(x)|. \end{aligned}$$

Pick $A = (1 + \sup_{x \in [\xi-1, \xi+1]} |f'(x)|)^{-1} > 0$.

By (a)(b), we arrange $A = \min(1, (1 + \sup_{x \in [\xi-1, \xi+1]} |f'(x)|)^{-1}) > 0$ to fit the inequality.

(2) $\xi = \sum_{n=0}^{\infty} 10^{-n!}$ *is transcendental.*

(a) Let $k_j = 10^{j!}$, $h_j = 10^{j!} \sum_{n=0}^{j} 10^{-n!}$. Then

$$\left|\xi - \frac{h_j}{k_j}\right| = \sum_{n=j+1}^{\infty} 10^{-n!} < \sum_{n=(j+1)!}^{\infty} 10^{-n} = \frac{A_j}{k_j^j}$$

where $A_j = \frac{10}{9} \cdot 10^{-j!}$.

(b) If ξ were a real algebraic number of degree $d > 1$, then by Lemma and (a),

$$\frac{A}{k_j^d} < \left|\xi - \frac{h_j}{k_j}\right| < \frac{A_j}{k_j^j} < \frac{A_j}{k_j^d}$$

for some $A > 0$ and $j \geq d$, or $0 < A < A_j$. Since j is arbitrary, $A_j \to 0$ as $j \to \infty$, contrary to $A > 0$.

(c) If ξ were a real algebraic number of degree $d = 1$, $\xi = \frac{h}{k}$ is a rational number. So

$$\left|\xi - \frac{h_j}{k_j}\right| = \left|\frac{h}{k} - \frac{h_j}{k_j}\right| = \left|\frac{hk_j - kh_j}{kk_j}\right| \geq \left|\frac{1}{kk_j}\right| = \frac{|k|^{-1}}{k_j}$$

for all j. (It is impossible that $hk_j - kh_j = 0$ or $\frac{h}{k} = \frac{h_i}{k_j}$ since $|\frac{h}{k} - \frac{h_j}{k_j}| = \sum_{n=j+1}^{\infty} 10^{-n!} > 0$ for all j.) Again by (a),

$$\frac{|k|^{-1}}{k_j} \leq \left|\xi - \frac{h_j}{k_j}\right| < \frac{A_j}{k_j^j} < \frac{A_j}{k_j},$$

or $0 < |k|^{-1} < A_j$. (Similar to (b).) Since j is arbitrary, $A_j \to 0$ as $j \to \infty$, contrary to $|k|^{-1} > 0$.

□

Exercise 2.4. *Is the set of all irrational real numbers countable?*

Proof (Reductio ad absurdum). If $\mathbb{R} - \mathbb{Q}$ were countable, then $\mathbb{R} = \mathbb{Q} \bigcup (\mathbb{R} - \mathbb{Q})$ is countable (Theorem 2.12), contrary to the fact that $\mathbb{R}$ is uncountable (Corollary to Theorem 2.43). □

Exercise 2.5. *Construct a bounded set of real numbers with exactly three limit points.*

Proof (Exercise 2.12). Let

$$K_p = \{p\} \bigcup \left\{ p + \frac{1}{n} : n \in \mathbb{Z}^+ \right\} \subseteq \mathbb{R}^1$$

be a compact set of $\mathbb{R}^1$ with exactly one limit point $p \in \mathbb{R}^1$ (Exercise 2.12). Then

$$K_{1989} \cup K_6 \cup K_4$$

is a compact set of $\mathbb{R}^1$ with exactly three limit points $1989, 6, 4 \in \mathbb{R}^1$. □

Exercise 2.6. *Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\overline{E}$ have the same limit points. (Recall that $\overline{E} = E \cup E'$.) Do E and E' always have the same limit points?*

Proof.

(1) *Show that E' is closed.*

(a) *Use Definition 2.18 (d).*

(i) It suffices to show every limit point of E' is a limit point of E. Given a limit point p of E', so that every open neighborhood U of p contains a point $q_0 \neq p$ such that $q_0 \in E'$.

(ii) Since q_0 is a limit point of E, there is an open neighborhood V of q_0 contains a point $q \neq q_0$ such that $q \in E$, where

$$V = U \cap B\left(q_0; \frac{1}{2} d_E(p, q_0)\right) \subseteq U$$

($B(x; r)$ is the open ball with center at x and radius r).

(iii) By the construction of V, for such open neighborhood U of p, there is $q \neq p$ and $q \in V \subseteq U$ and $q \in E$. That is, p is a limit point of E.

(b) *Use Definition 2.18 (e).*

(i) To show E' is closed or $X - E'$ is open, it suffices to show every point of $X - E'$ is an interior point of $X - E'$.

(ii) Given a point $p \in X - E'$, or p is not a limit point of E. There is an open neighborhood U of p contains no point $q \neq p$ such that $q \in E$.

(iii) To show U is an open neighborhood of p such that $U \subseteq X - E'$, it suffices to no point $q \neq p$ such that $q \in E'$. If there were a limit point q of E such that $q \neq p$ and $q \in U$, then

$$V = U \cap B\left(q; \frac{1}{2} d_E(p, q)\right) \subseteq U$$

is an open neighborhood of q contains no point of E, contrary to the assumption $q \in E'$. So $U \subseteq X - E'$ is an open neighborhood of $p \in X - E'$.

(2) *Show that $E' = \overline{E}'$.* It suffices to show $E' \supseteq \overline{E}'$. ($E' \subseteq \overline{E}'$ holds trivially since $E \subseteq \overline{E}$). Given a limit point p of $\overline{E} = E \cup E'$.

(a) p is a limit point of E. Nothing to do.

(b) p is a limit point of E'. Since p is a limit point of E' and E' is a closed set, $p \in E'$, or p is a limit point of E.

In any case, $E' \supseteq \overline{E}'$.

(3) *E and E' might not have the same limit points.* Let

$$E = \left\{ \frac{1}{n} : n \in \mathbb{Z}^+ \right\} \subseteq \mathbb{R}^1.$$

Then $E' = \{0\}$ and thus $(E')' = \varnothing$.

$\square$

Exercise 2.7. *Let $A_1, A_2, A_3, ...$ be subsets of a metric space.*

(a) *If $B_n = \bigcup_{i=1}^n A_i$, prove that $\overline{B_n} = \bigcup_{i=1}^n \overline{A_i}$, for $n = 1, 2, 3, ...$*

(b) *If $B = \bigcup_{i=1}^\infty A_i$, prove that $\overline{B} \supseteq \bigcup_{i=1}^\infty \overline{A_i}$.*

Show, by an example, that this inclusion can be proper.

Proof of (a).

(1) *Show that $\overline{B_n} \subseteq \bigcup_{i=1}^n \overline{A_i}$.* Since $A_i \subseteq \overline{A_i}$ for any i, we have

$$B_n = \bigcup_{i=1}^n A_i \subseteq \bigcup_{i=1}^n \overline{A_i}.$$

Since $\bigcup_{i=1}^n \overline{A_i}$ is a union of finitely many closed set $\overline{A_i}$, $\bigcup_{i=1}^n \overline{A_i}$ is closed (Theorem 2.24(d)). By Theorem 2.27(c), $\overline{B_n} \subseteq \bigcup_{i=1}^n \overline{A_i}$.

(2) *Show that $\overline{B_n} \supseteq \bigcup_{i=1}^n \overline{A_i}$.* Same argument in the proof of (b).

$\square$

Proof of (b). Since $\bigcup_{j=1}^\infty A_j \supseteq A_i$ for any i, by the monotonicity of closure, we have $\overline{\bigcup_{j=1}^\infty A_j} \supseteq \overline{A_i}$ for any i, or $\overline{B} \supseteq \bigcup_{i=1}^\infty \overline{A_i}$. $\square$

Proof of proper inclusion in (b). Let

$$A_n = \left(\frac{1}{n}, \infty \right) \subseteq \mathbb{R}^1$$

for any $n \in \mathbb{Z}^+$. Then

$$\bigcup_{n=1}^\infty A_n = (0, \infty) \Longrightarrow \overline{\bigcup_{n=1}^\infty A_n} = \overline{(0, \infty)} = [0, \infty),$$

$$\overline{A_n} = \left[\frac{1}{n}, \infty \right) \Longrightarrow \bigcup_{n=1}^\infty \overline{A_n} = \bigcup_{n=1}^\infty \left[\frac{1}{n}, \infty \right) = (0, \infty).$$

□

Exercise 2.8. *Is every point of every open set $E \subseteq \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.*

It is not true for all metric spaces X. The (discrete) metric in Exercise 2.10 implies no limit point exists in X.

Proof.

(1) *Show that for every open set $E \subseteq \mathbb{R}^k$, $E \subseteq E'$.* Given any point $\mathbf{p} \in E$, we shall show $\mathbf{p}$ is a limit point of E.

  (a) Since E is open, there is an open neighborhood $B(\mathbf{p}; r_0) \subseteq E$ for some $r_0 > 0$.

  (b) *In particular, given any $s \in \mathbb{R}$ such that $0 < s < r_0$, we can find*

$$\mathbf{q} \in B(\mathbf{p}; s) \subseteq B(\mathbf{p}; r_0) \subseteq E$$

  such that $\mathbf{q} \neq \mathbf{p}$. Explicitly, write

$$\mathbf{p} = (p_1, \dots, p_k)$$

  and choose

$$\mathbf{q} = \left(p_1 + \frac{s}{89}, p_2, \dots, p_k\right) \neq \mathbf{p}$$

  (since $s > 0$). Clearly, $\mathbf{q}$ is well-defined in $\mathbb{R}^k$ and $|\mathbf{q} - \mathbf{p}| = \frac{s}{89} < s$ or $\mathbf{q} \in B(\mathbf{p}; s)$.

  (c) Now given every open neighborhood $B(\mathbf{p}, r)$ of $\mathbf{p}$. We can choose $s \in \mathbb{R}$ such that $0 < s < \min\{r_0, r\} \leq r_0$. (might pick $s = \frac{1}{64}\min\{r_0, r\}$.) By (b), there exists $\mathbf{q} \neq \mathbf{p}$ such that

$$\mathbf{q} \in B(\mathbf{p}; s) \subseteq B(\mathbf{p}; r) \subseteq E.$$

(2) *Give an example of a closed set $E \subseteq \mathbb{R}^k$ such that $E \not\subseteq E'$.* Pick $E = \{\mathbf{0}\}$. So $E' = \varnothing$ and thus $E \not\subseteq E'$.

□

Exercise 2.9. *Let E° denote the set of all interior points of a set E. [See Definition 2.18(e); E° is called the interior of E.]*

(a) *Prove that E° is always open.*

(b) *Prove that E is open if and only if $E^\circ = E$.*

(c) *If G is contained in E and G is open, prove that G is contained in E°.*

(d) *Prove that the complement of E° is the closure of the complement of E.*

(e) *Do E and $\overline{E}$ always have the same interiors?*

(f) *Do E and E° always have the same closures?*

Similar to Theorem 2.27.

Proof of (a). It is equivalent to show that $E^\circ \subseteq (E^\circ)^\circ$.

(1) Given any point $x \in E^\circ$, there is $r > 0$ such that $B(x; r) \subseteq E$.

(2) It suffices to show that $B\left(x; \frac{2}{r}\right) \subseteq E^\circ$. Given any point $y \in B\left(x; \frac{2}{r}\right)$, we will show that there is an open neighborhood $B\left(y; \frac{2}{r}\right)$ of y such that $B\left(y; \frac{2}{r}\right) \subseteq E$.

(3) Given any point $z \in B\left(y; \frac{2}{r}\right)$, we have

$$d(z, x) \leq d(z, y) + d(y, x) < \frac{2}{r} + \frac{2}{r} = r,$$

or $z \in B(x; r) \subseteq E$. Therefore, $B\left(y; \frac{2}{r}\right) \subseteq E$, or $y \in E^\circ$, or $B\left(x; \frac{2}{r}\right) \subseteq E^\circ$, or $x \in (E^\circ)^\circ$, or $E^\circ \subseteq (E^\circ)^\circ$.

□

Proof of (b).

(1) ($\Longrightarrow$)(Definition 2.18) Since E is open, every point of E is an interior point of E. Hence $E \subseteq E^\circ$. Note that $E^\circ \subseteq E$ is trivial, and thus $E^\circ = E$.

(2) ($\Longleftarrow$)((a)) By (a), $E = E^\circ$ is always open.

(3) ($\Longleftarrow$)(Definition 2.18) Every point of E is an interior point of E since $E = E^\circ$. Hence E is open by Definition 2.18(f).

□

Proof of (c). $G \subseteq E$ implies $G^\circ \subseteq E^\circ$. $G = G^\circ$ since G is open ((b)). Hence $G = G^\circ \subseteq E^\circ$, that is, E° is the largest open set contained in E. (Similarly, $\overline{E}$ is the smallest closed set containing E.) □

Proof of (d). Show that $X - E^\circ = \overline{X - E}$ and $(X - E)^\circ = X - \overline{E}$.

(1) (Theorem 2.27 and (c))

$$\begin{aligned}
X - E^\circ &= X - \bigcup_{\text{Open } V \subseteq E} V \\
&= \bigcap_{\text{Open } V \subseteq E} (X - V) \\
&= \bigcap_{\text{Closed } W \supseteq X - E} W \\
&= \overline{X - E}. \\
X - \overline{E} &= X - \bigcap_{\text{Closed } W \supseteq E} W \\
&= \bigcup_{\text{Closed } W \supseteq E} (X - W) \\
&= \bigcup_{\text{Open } V \subseteq X - E} V \\
&= (X - E)^\circ.
\end{aligned}$$

(2) (Brute-force)

$$\begin{aligned}
x \in E^\circ &\Longleftrightarrow \exists r > 0 \text{ such that } B(x; r) \subseteq E \\
&\Longleftrightarrow \exists r > 0 \text{ such that } B(x; r) \cap (X - E) = \varnothing \\
&\Longleftrightarrow x \notin \overline{X - E} \\
&\Longleftrightarrow x \in X - \overline{X - E}. \\
x \in (X - E)^\circ &\Longleftrightarrow \exists r > 0 \text{ such that } B(x; r) \subseteq (X - E) \\
&\Longleftrightarrow \exists r > 0 \text{ such that } B(x; r) \cap E = \varnothing \\
&\Longleftrightarrow x \notin \overline{E} \\
&\Longleftrightarrow x \in X - \overline{E}.
\end{aligned}$$

Note that $X - E^\circ = \overline{X - E}$ is equivalent to $(X - E)^\circ = X - \overline{E}$ by mapping $E \mapsto X - E$. $\square$

Proof of (e). No.

(1) Let $X = \mathbb{R}^1$ equipped with the Euclidean metric, and $E = \mathbb{Q} \subseteq X$.

(2) $E^\circ = \varnothing$ since $\widetilde{\mathbb{Q}}$ is dense in $\mathbb{R}$.

(3) $(\overline{E})^\circ = (\mathbb{R}^1)^\circ = \mathbb{R}^1$ since $\mathbb{Q}$ is dense in $\mathbb{R}$ and $\mathbb{R}^1$ is open.

$\square$

Proof of (f). No.

(1) Let $X = \mathbb{R}^1$ equipped with the Euclidean metric, and $E = \mathbb{Q} \subseteq X$.

(2) $\overline{E} = \mathbb{R}^1$ since $\mathbb{Q}$ is dense in $\mathbb{R}$.

(3) $\overline{E^\circ} = \overline{\varnothing} = \varnothing$ since $\widetilde{\mathbb{Q}}$ is dense in $\mathbb{R}$.

□

Exercise 2.10. *Let X be an infinite set. For $p \in X$ and $q \in X$, define*

$$d(p,q) = \begin{cases} 1 & (\text{if } p \neq q) \\ 0 & (\text{if } p = q). \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

(The statement holds even if X is finite.) We called d the discrete metric, and the corresponding topology on X induces the discrete topology. Conversely, if X has the discrete topology, X is always metrizable by the discrete metric.

Proof.

(1) *$d(p,q)$ is a metric.*

  (a) *$d(p,q) > 0$ if $p \neq q$; $d(p,p) = 0$.* Trivial.
  (b) *$d(p,q) = d(q,p)$.* Trivial.
  (c) *$d(p,q) \leq d(p,r) + d(r,q)$ for any $r \in X$.* If $p = q$, nothing to do. If $p \neq q$, $r \neq p$ or $r \neq q$ for any $r \in X$. (Assume not true, $r = p$ and $r = q$ implies that $p = q$ which is a contradiction.) In any case $d(p,r) + d(r,q) \geq 1 = d(p,q)$.

(2) *Every subset is open.* Let E be any subset of X. Then every point $p \in E$ is an interior point of E. In fact, we can pick one open neighborhood $U = B\left(p; \frac{1}{2}\right)$ of p containing only one point $p \in E$ or $U = \{p\}$, and such open neighborhood U is a subset of E. So every subset of X is open.

(3) *Every subset is closed.* Since every subset is open, every subset is closed by Theorem 2.23.

  Supplement. Might use Definition 2.18 (d) to prove directly since there are no limit points in X if we consider one open neighborhood $U = B\left(p; \frac{1}{2}\right)$ of p. Therefore, every subset is closed. Again we apply Theorem 2.23 to get that every subset is open without using Definition 2.18 (f).

(4) *A subset is compact iff it is finite.*

(a) *Any finite subset is compact.* Say $E = \{p_1, p_2, ..., p_k\}$, and $\{G_\alpha\}$ be an open covering of E. From $\{G_\alpha\}$ we pick G_{α_1} containing p_1, G_{α_2} containing p_2, ..., and G_{α_k} containing p_k. This process can be done in the finitely many steps. Therefore,

$$\{G_{\alpha_1}, G_{\alpha_2}, ..., G_{\alpha_k}\}$$

is a finite subcovering of $\{G_\alpha\}$ covering E.

(b) *Any infinite subset is not compact.* Take a collection

$$\mathscr{G} = \{G_p = \{p\}\}$$

of open subsets where p runs all points in E. Clearly, $\{G_p\}$ is an open covering. Assume

$$\mathscr{G}' = \{G_{p_1}, G_{p_2}, ..., G_{p_k}\}$$

is any finite subcovering of $\mathscr{G}$. Since E is infinite, there exist a point $p \in E$ such that $p \neq p_1$, $p \neq p_2$, ..., $p \neq p_k$. Therefore, $\mathscr{G}'$ does not cover p, or $\mathscr{G}$ does not contains any finite subcovering $\mathscr{G}'$.

□

Notice that every subset is bounded. Therefore, every subset is closed and bounded, but only finite subset is compact, i.e., Heine-Borel theorem is not true in the infinite discrete topology.

Exercise 2.11. *For $x \in \mathbb{R}^1$ and $y \in \mathbb{R}^1$, define*

$$\begin{aligned}
d_1(x, y) &= (x - y)^2, \\
d_2(x, y) &= \sqrt{|x - y|}, \\
d_3(x, y) &= |x^2 - y^2|, \\
d_4(x, y) &= |x - 2y|, \\
d_5(x, y) &= \frac{|x - y|}{1 + |x - y|}.
\end{aligned}$$

Determine, for each of these, whether it is a metric or not.

Proof.

(1) $d = d_1$ *is not a metric.* (Reductio ad absurdum) If d were a metric, then

$$d(0, 2) > d(0, 1) + d(1, 2),$$

contrary to Definition 2.15(c) that $d(p, q) \leq d(p, r) + d(r, q)$.

(2) $d = d_2$ *is a metric. It suffices to show that* $d'(x,y) = \sqrt{d(x,y)}$ *is a metric if* $d(x,y)$ *is a metric.* For any $p, q, r \in \mathbb{R}^1$,

(a) $d'(p,q) = \sqrt{d(p,q)} > 0$ if $p \neq q$; $d'(p,p) = \sqrt{d(p,p)} = 0$.

(b) $d'(p,q) = \sqrt{d(p,q)} = \sqrt{d(q,p)} = d'(q,p)$.

(c)

$$\begin{aligned}
&\sqrt{d(p,r)+d(r,q)} \leq \sqrt{d(p,r)} + \sqrt{d(r,q)} \\
\Longleftrightarrow &(\sqrt{d(p,r)+d(r,q)})^2 \leq (\sqrt{d(p,r)} + \sqrt{d(r,q)})^2 \\
\Longleftrightarrow &d(p,r)+d(r,q) \leq d(p,r)+d(r,q) + 2\sqrt{d(p,r)}\sqrt{d(r,q)} \\
\Longleftrightarrow &0 \leq 2\sqrt{d(p,r)}\sqrt{d(r,q)}.
\end{aligned}$$

(d)

$$\begin{aligned}
d'(p,q) &= \sqrt{d(p,q)} \\
&\leq \sqrt{d(p,r)+d(r,q)} && \text{(Triangle inequality)} \\
&\leq \sqrt{d(p,r)} + \sqrt{d(r,q)} && \text{((c))} \\
&= d'(p,r) + d'(r,q).
\end{aligned}$$

By Definition 2.15, d' is a metric.

(3) $d = d_3$ *is not a metric.* (Reductio ad absurdum) If d were a metric, then

$$d(1,-1) = 0,$$

contrary to Definition 2.15(a): $d(p,q) > 0$ if $p \neq q$; $d(p,p) = 0$.

(4) $d = d_4$ *is not a metric.* (Reductio ad absurdum) If d were a metric, then

$$d(1,1) = 1,$$

contrary to Definition 2.15(a): $d(p,q) > 0$ if $p \neq q$; $d(p,p) = 0$.

(5) $d = d_5$ *is a metric. It suffices to show that* $d'(x,y) = \frac{d(x,y)}{1+d(x,y)}$ *is a metric if* $d(x,y)$ *is a metric.* For any $p, q, r \in \mathbb{R}^1$,

(a) $d'(p,q) = \frac{d(p,q)}{1+d(p,q)} > 0$ if $p \neq q$; $d'(p,p) = \frac{d(p,p)}{1+d(p,p)} = 0$.

(b) $d'(p,q) = \frac{d(p,q)}{1+d(p,q)} = \frac{d(q,p)}{1+d(q,p)} = d'(q,p)$.

(c) Write $x = d(p,q)$, $y = d(p,r)$ and $z = d(r,q)$. So $x, y, z \geq 0$ and

$$\begin{aligned}
&x \leq y + z \\
\Longleftrightarrow &x + x(y+z) \leq y + z + x(y+z) \\
\Longleftrightarrow &x(1+y+z) \leq (1+x)(y+z) \\
\Longleftrightarrow &\frac{x}{1+x} \leq \frac{y+z}{1+y+z}.
\end{aligned}$$

(d)

$$
\begin{aligned}
d'(p,q) &= \frac{d(p,q)}{1+d(p,q)} \\
&\leq \frac{d(p,r)+d(r,q)}{1+d(p,r)+d(r,q)} && ((c)) \\
&= \frac{d(p,r)}{1+d(p,r)+d(r,q)} + \frac{d(r,q)}{1+d(p,r)+d(r,q)} \\
&= \frac{d(p,r)}{1+d(p,r)} + \frac{d(r,q)}{1+d(r,q)} \\
&= d'(p,r) + d'(r,q).
\end{aligned}
$$

(e) Or we can show $d'(p,q) \leq d'(p,r) + d'(r,q)$ by

$$
\begin{aligned}
& \frac{x}{1+x} \leq \frac{y}{1+y} + \frac{z}{1+z} \\
\Longleftrightarrow & x(1+y)(1+z) \leq y(1+z)(1+x) + z(1+x)(1+y) \\
\Longleftrightarrow & x + xy + xz + xyz \\
& \quad \leq (y + xy + yz + xyz) + (z + xz + yz + xyz) \\
\Longleftrightarrow & x \leq y + z + 2yz + xyz \\
\Longleftarrow & x \leq y + z && (d \text{ is nonnegative})
\end{aligned}
$$

By Definition 2.15, d' is a metric.

$\square$

Exercise 2.12. *Let $K \subseteq \mathbb{R}^1$ consist of 0 and the numbers $\frac{1}{n}$, for $n = 1, 2, 3, \ldots$. Prove that K is compact directly from the definition (without using the Heine-Borel theorem).*

Proof. Let $\{G_\alpha\}$ be an open covering of K. There is an open set $G_0 \in \{G_\alpha\}$ containing 0. So there exists an open neighborhood $U = B(0; r)$ of 0 such that $U \subseteq G_0$. So U contains all points $q = \frac{1}{n}$ of K whenever $n > \frac{1}{r}$. To construct a finite subcovering of $\{G_\alpha\}$, we need to pick finitely many open sets from $\{G_\alpha\}$ to cover the remaining points $q = \frac{1}{n}$ where $n = 1, 2, ..., \left[\frac{1}{r}\right]$, say G_1 contains $q = \frac{1}{1}$, G_2 contains $q = \frac{1}{2}$, ..., $G_{\left[\frac{1}{r}\right]}$ contains $q = \frac{1}{\left[\frac{1}{r}\right]}$. (Might be duplicated.) Hence,

$$\left\{ G_0, G_1, G_2, ..., G_{\left[\frac{1}{r}\right]} \right\}$$

is a finite subcovering of $\{G_\alpha\}$ covering K. $\square$

Proof (Heine-Borel theorem).

(1) K is closed. In fact, the only limit point of K is 0, which is in K.

(a) $p = 0$ *is a limit point.* Given $r > 0$. There always exists $n \in \mathbb{Z}^+$ such that $r > \frac{1}{n}$. So any open neighborhood $B(0; r)$ of $p = 0$ contains at least one point $q = \frac{1}{n} \neq 0$ in K.

(b) $p < 0$ *is not a limit point.* Pick an open neighborhood $B(p; r)$ of p where $r = |p| > 0$. Then $B(p; r) \cap K = \varnothing$.

(c) $p > 0$ *is not a limit point.* There always exists $m \in \mathbb{Z}^+$ such that $p > \frac{1}{n}$ whenever $n \geq m$. Pick an open neighborhood $B(p; r)$ of p where $r = p - \frac{1}{m} > 0$. Then $B(p; r)$ does not have all points $q = \frac{1}{n} \in K$ whenever $n \geq m$. By Theorem 2.20, p cannot be a limit point of K.

(2) K is bounded. There is a real number $M = 2$ and a point $q = 0 \in \mathbb{R}^1$ such that $|p - q| = |p| < 2$ for all $p \in K$.

By Heine-Borel theorem, K is compact in $\mathbb{R}^1$. $\square$

Exercise 2.13. *Construct a compact set of real numbers whose limit points form a countable set.*

Proof (Exercise 2.12). Let $K(p; r) \subseteq \mathbb{R}^1$ be

$$K(p; r) = \left\{ p + \frac{r}{n} : n = 2, 3, \dots \right\} \bigcup \{p\}$$

and

$$K = \left(\bigcup_{i=0}^{\infty} K(2^{-i}; 2^{-i}) \right) \bigcup \{0\}.$$

(1) *The set of limit points of K is $K' = \{2^{-i} : i = 0, 1, 2, \dots\} \bigcup \{0\}$, which is (infinitely) countable.*

(a) The unique limit point of $K(2^{-i}; 2^{-i})$ is 2^{-i} for each $i = 0, 1, 2, \dots$ (Exercise 2.12).

(b) 0 is a limit point of K.

(c) No other limit points of K. Similar to the argument of the proof of Exercise 2.12.

(2) K is closed. All limit points are in K.

(3) K is bounded. There is a real number $M = 2$ and a point $q = 0 \in \mathbb{R}^1$ such that $|p - q| = |p| < 2$ for all $p \in K$.

By Heine-Borel theorem, K is compact in $\mathbb{R}^1$, and has infinitely countable limit points. $\square$

Exercise 2.14. *Give an example of an open cover of the segment* $(0, 1)$ *which has no finite subcover.*

Proof. In $\mathbb{R}^1$, take a collection

$$\mathscr{G} = \left\{ G_n = \left(\frac{1}{n}, 1 \right) \right\}$$

of open subsets where $n \in \mathbb{Z}^+$.

(1) *$\mathscr{G}$ is an open covering of $(0, 1) \subseteq \mathbb{R}^1$.* Actually, given $x \in (0, 1)$, there exists an positive integer n such that $x > \frac{1}{n}$. That is, $x \in \left(\frac{1}{n}, 1\right) = G_n$.

(2) *There is no finite subcovering of $\mathscr{G}$.* Assume

$$\mathscr{G}' = \{G_{n_1}, G_{n_2}, ..., G_{n_k}\}$$

is any finite subcovering of $\mathscr{G}$ where $n_1 < n_2 < ... < n_k$. Take $x \in \left(0, \frac{1}{n_k}\right) \neq \varnothing$, $x = \frac{1}{2n_k}$ for example. Then $x \notin G_{n_1}$, $x \notin G_{n_1}$, ..., $x \notin G_{n_k}$, which contradicts that $\mathscr{G}'$ is a finite subcovering of $\mathscr{G}$ covering $(0, 1)$.

□

Exercise 2.15. *Show that Theorem 2.36 and its Corollary become false (in $\mathbb{R}^1$, for example) if the word "compact" is replaced by "closed" or by "bounded."*

Recall:

(1) Theorem 2.36: *If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.*

(2) Corollary: *If $\{K_n\}$ is a sequence of nonempty compact sets such that K_n contains K_{n+1} $(n = 1, 2, 3, \ldots)$, then $\bigcap K_n$ is not empty.*

Proof. Let $X = \mathbb{R}^1$ with the usual Euclidean metric.

(1) For the closeness, let $K_n = [n, \infty) \subseteq X$.

(2) For the boundedness, let $K_n = \left(0, \frac{1}{n}\right) \subseteq X$.

In any case, $K_1 \supseteq K_2 \supseteq \cdots$ and $\bigcap K_n = \varnothing$. □

Exercise 2.16. *Regard $\mathbb{Q}$, the set of all rational numbers, as a metric space, with $d(p, q) = |p - q|$. Let E be the set of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Show that E is closed and bounded in Q, but that E is not compact. Is E open in $\mathbb{Q}$?*

Lemma. *Assume $S \subseteq T \subseteq M$. Then S is compact in (M,d) if, and only if, S is compact in the metric subspace (T,d).*

Proof of Lemma.

(1) ($\Longrightarrow$) Let $\mathscr{F}$ be an open covering of S in (T,d), say $S \subseteq \bigcup_{A\in\mathscr{F}} A$ where A is open in T. Then $A = B \cap T$ for some open set B in M (Theorem 3.33). Let $\mathscr{G}$ be the collection of B. Then

$$S \subseteq \bigcup_{A\in\mathscr{F}} A = \bigcup_{B\in\mathscr{G}} (B \cap T) \subseteq \bigcup_{B\in\mathscr{G}} B,$$

or $\mathscr{G}$ be an open covering of S in (M,d). Since S is compact in (M,d), $\mathscr{G}$ contains a finite subcovering, say

$$S \subseteq B_1 \cap \cdots \cap B_p.$$

So

$$S \cap T \subseteq (B_1 \cap T) \cap \cdots \cap (B_p \cap T),$$

or

$$S \subseteq A_1 \cap \cdots \cap A_p$$

(since $S \subseteq T$ or $S \cap T = S$). So there is a finite subcovering of $\mathscr{F}$ covering S, or S is compact in (T,d).

(2) ($\Longleftarrow$) Let $\mathscr{G}$ be an open covering of S in (M,d), say $S \subseteq \bigcup_{B\in\mathscr{G}} B$ where B is open in M. Then $A = B \cap T$ is open in T. Let $\mathscr{F}$ be the collection of A. Then

$$S \cap T \subseteq \bigcup_{B\in\mathscr{G}} (B \cap T) = \bigcup_{A\in\mathscr{F}} A,$$

or $\mathscr{F}$ be an open covering of $S \cap T = S$ in (T,d). Since S is compact in (T,d), $\mathscr{F}$ contains a finite subcovering, say

$$S \subseteq A_1 \cap \cdots \cap A_p.$$

Clearly, $S \subseteq B_1 \cap \cdots \cap B_p$ since $A = B \cap T \subseteq B$. So there is a finite subcovering of $\mathscr{G}$ covering S, or S is compact in (M,d).

□

Proof. Write $E_0 = (\sqrt{2}, \sqrt{3}) \bigcup (-\sqrt{3}, -\sqrt{2})$, and $E = E_0 \bigcap \mathbb{Q}$.

(1) E is a subset of $\mathbb{Q}$.

(2) *Show that E is bounded in $\mathbb{Q}$.* Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there is $p \in \mathbb{Q}$ such that $\sqrt{2} < p < \sqrt{3}$, or $p \in E$. Let $r = p + \sqrt{3} > 0$. Therefore, $E \subseteq B(p; r)$ for some $r > 0$ and $p \in E$, or E is bounded.

(3) *Show that E is closed in $\mathbb{Q}$.* It suffices to show its complement is open in $\mathbb{Q}$. Given any

$$p \in \widetilde{E} = ((-\infty, -\sqrt{3}] \cup [-\sqrt{2}, \sqrt{2}] \cup [\sqrt{3}, \infty)) \cap \mathbb{Q}.$$

$p \leq -\sqrt{3}$ or $-\sqrt{2} \leq p \leq \sqrt{2}$ or $p \geq \sqrt{3}$.

(a) $p \leq -\sqrt{3}$. $p \neq -\sqrt{3}$ since $p \in \mathbb{Q}$ and $-\sqrt{3}$ is irrational. So $p < -\sqrt{3}$ and thus there exists $q \in \mathbb{Q}$ such that $p < q < -\sqrt{3}$ since $\mathbb{Q}$ is dense in $\mathbb{R}$. Let $r = \max\{-\sqrt{3} - q, q - p\} > 0$. The ball $B(q; r)$ is contained in $\widetilde{E}$.

(b) $-\sqrt{2} \leq p \leq \sqrt{2}$. Similar to (a).

(c) $p \geq \sqrt{3}$. Similar to (a).

By (a)(b), $\widetilde{E}$ is open in $\mathbb{Q}$, or E is closed in $\mathbb{Q}$.

(4) *Show that E is not compact in $\mathbb{Q}$.* (Reductio ad absurdum) If E_0 were compact in the metric space $\mathbb{Q}$, E_0 is compact in the metric space $\mathbb{R}$ (Lemma), which is absurd.

(5) *Show that E is open.* Similar to (3).

□

Exercise 2.17. *Let E be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in $[0, 1]$? Is E compact? Is E perfect?*

Proof.

(1) *Show that E is uncountable.* Same as Theorem 2.14. Or show that E is perfect and then apply Theorem 2.43.

(2) *Show that E is not dense in* $[0, 1]$. Note that $E \subseteq \left[\frac{4}{9}, \frac{7}{9}\right]$. So

$$B\left(0; \frac{1}{64}\right) \bigcap E \subseteq B\left(0; \frac{1}{64}\right) \bigcap \left[\frac{4}{9}, \frac{7}{9}\right] = \varnothing$$

or 0 is not a limit point of E. Hence E is not dense in $[0, 1]$.

(3) *Show that E is compact. It is equivalent to show that E is closed and bounded (Theorem 2.41).* Let a decimal expansion of $x \in (0, 1)$ be $0.x_1x_2\cdots$.

(a) *Show that $\widetilde{E}$ is open.* Since $E \subseteq \left[\frac{4}{9}, \frac{7}{9}\right]$, it suffices to show that every point $x \in (0, 1) \cap \widetilde{E}$ is an interior point of $\widetilde{E}$. Say a decimal expansion of x containing at least one digit $x_n \neq 4, 7$. Note that

$$|x - y| \geq 10^{-n} > 0$$

for any $y = 0.y_1y_2\cdots \in E$. Hence there is an open neighborhood $B(x; 10^{-n})$ of x such that $B(x; 10^{-n}) \cap E = \varnothing$, or $B(x; 10^{-n}) \subseteq \widetilde{E}$, or x is an interior point of $\widetilde{E}$.

(b) *Show that E is closed.* Given any limit point $x \in \mathbb{R}^1$ of E, we want to show that $x \in E$. (Reductio ad absurdum) Similar to (a).

(c) *Show that E is bounded.* $E \subseteq B(0; 1)$.

(4) *Show that E is perfect.*

(a) E is closed (by (3)).

(b) *Show that every point of E is a limit point of E.* Given any $x \in E$. Given any open neighborhood $B(x; r)$ of x, there is a positive integer n such that

$$\frac{3}{10^n} < r.$$

For such n, pick $y = 0.x_1x_2\cdots x_{n-1}y_n\cdots x_{n+1}\cdots \in E$ where

$$y_n = \begin{cases} 4 & (x_n = 7), \\ 7 & (x_n = 4). \end{cases}$$

$y \neq x$, and $|y - x| = \frac{3}{10^n} < r$. So that there is $y \neq x$ such that $y \in B(x; r)$, or x is a limit point of E.

□

Exercise 2.18. *Is there a nonempty perfect set in $\mathbb{R}^1$ which contains no rational number?*

Yes.

Lemma. *$x \in \mathbb{Q}$ if and only if has repeating decimal expansion.*

Proof of Lemma.

(1) ($\Longleftarrow$) Given any repeating decimal

$$x = x_0.x_1x_2\cdots x_n\overline{x_{n+1}\cdots x_{n+m}}$$

where $x_0 \in \mathbb{Z}$ and $x_1, \ldots, x_{n+m} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Thus $x = p/q$ where

$$p = (10^m - 1)\sum_{i=0}^{n} 10^{n-i}x_i + \sum_{j=1}^{m} 10^{m-j}x_{n+j} \in \mathbb{Z}$$

and

$$q = 10^n(10^m - 1) \in \mathbb{Z}.$$

(2) ($\Longrightarrow$) (Euler's totient function) Given any $x = p/q$ where $p, q \in \mathbb{Z}$, $q > 0$.

(a) Write $q = 2^a 5^b q_1$ where a, b are nonnegative integers and $(q_1, 10) = 1$ (Unique factorization theorem).

(b) Let $n = \max\{a, b\}$. Then $2^{n-a}5^{n-b}q = 10^n q_1$.

(c) Since $(q_1, 10) = 1$, $10^m \equiv 1 \pmod{q_1}$ where $m = \varphi(q_1)$ is Euler's totient function of q_1. Hence $10^m - 1 = q_1 q_2$ for some $q_2 \in \mathbb{Z}$, or

$$2^{n-a}5^{n-b}q_2 q = 10^n(10^m - 1).$$

Here $2^{n-a}5^{n-b}q_2$, n, m are nonnegative integers.

(d) Now write

$$x = \frac{p}{q} = \frac{2^{n-a}5^{n-b}q_2 p}{10^n(10^m - 1)} = \frac{(10^m - 1)q_3 + r}{10^n(10^m - 1)} = \frac{q_3}{10^n} + \frac{r}{10^n(10^m - 1)}$$

where $q_3, r \in \mathbb{Z}$ with $0 \leq r < 10^m - 1$. Might assume $q_3 \geq 0$. (If $q_3 < 0$, apply the same argument to $-q_3$ and then add the minus symbol "$-$" in the front of a decimal expansion.) Hence

$$x = x_0.x_1x_2\cdots x_n\overline{x_{n+1}\cdots x_{n+m}}$$

where

$$\begin{aligned} x_0 &= \left\lfloor \frac{q_3}{10^n} \right\rfloor \\ x_i &= \text{last digit of } \left\lfloor \frac{q_3}{10^{n-i}} \right\rfloor & (1 \leq i \leq n) \\ x_{n+j} &= \text{last digit of } \left\lfloor \frac{r}{10^{m-j}} \right\rfloor & (1 \leq j \leq m) \end{aligned}$$

(3) ($\Longrightarrow$) (Pigeonhole principle) Given any $x = p/q$ where $p, q \in \mathbb{Z}$, $q > 0$.

(a) Might assume $p \geq 0$. (If $p < 0$, apply the same argument to $-p$ and then add the minus symbol "$-$" in the front of the decimal expansion.) Write

$$x = x_0.x_1x_2\cdots.$$

(b) Apply Euclidean algorithm to get

$$p = x_0 q + r_0 \quad \text{with} \quad 0 \leq r_0 < q.$$

x_0 is the integer part of $x = p/q$. Continue Euclidean algorithm to get x_1 by

$$10r_0 = x_1 q + r_1 \quad \text{with} \quad 0 \leq r_1 < q.$$

In general, for $n \geq 1$, x_n is given by

$$10r_{i-1} = x_i q + r_i \quad \text{with} \quad 0 \leq r_i < q.$$

(c) The pigeonhole principle shows that there must be two equal remainders, that is,
$$r_n = r_{n+m} \quad \text{with} \quad m > 0.$$
By induction, $r_{n+k} = r_{n+m+k}$ for any $k \geq 0$. Thus $x_{n+k} = x_{n+m+k}$ holds for any $k > 0$, that is, x has a decimal expansion
$$x = x_0.x_1x_2 \cdots x_n\overline{x_{n+1} \cdots x_{n+m}}.$$

□

Proof (Exercise 2.17). Let A be the set of all $y \in [0,1]$ whose decimal expansion contains only the digits 4 and 7. Though $A \cap \mathbb{Q} \neq \varnothing$ since $\frac{4}{9} \in A$, we can shift A by a number $\xi = \sum_{n=0}^{\infty} 10^{-n!}$ (Exercise 2.3), that is, we construct
$$E = \{y + \xi : y \in A\}$$
and show that E is our desired nonempty perfect set in $\mathbb{R} - \mathbb{Q}$.

(1) Any number $x \in E$ has decimal expansion $x = 0.x_1x_2 \cdots$ with $x_n \in \{5, 8\}$ if n is a factorial number; otherwise $x_n \in \{4, 7\}$.

(2) E is a perfect set (Exercise 2.17).

(3) $E \subseteq \mathbb{R} - \mathbb{Q}$. It suffices to show that each $x \in E$ has no repeating decimal expansions (Lemma). It is clear by the construction of $\xi = \sum_{n=0}^{\infty} 10^{-n!}$.

□

Proof (Exercise 2.3). Let E be a subset of Liouville numbers as
$$E = \left\{ \sum_{n=0}^{\infty} \frac{a_n}{10^{n!}} : a_n \in \{4, 7\} \right\}.$$
E is perfect. (The same argument of Exercise 2.17.) Besides, all numbers of E are transcendental. (Set $k_j = 10^{j!}$ and $h_j = 10^{j!} \sum_{n=0}^{j} \frac{a_n}{10^{n!}}$ and apply the same argument of Exercise 2.3.) □

Note. Or using Lemma to prove all numbers of E are irrational.

Proof (Theorem 3.32). Let
$$E = \left\{ \sum_{n=1989}^{\infty} \frac{a_n}{n!} : a_n \in \{6, 4\} \right\}.$$
E is perfect. (The same argument of Exercise 2.17.) Besides, all numbers of E are irrational (The same argument of Theorem 3.32.) □

Proof (Non constructive existence proof). By Cantor-Bendixson theorem (Exercise 2.28), *it suffices to find a uncountable closed set in* $\mathbb{R} - \mathbb{Q}$.

(1) Write $\mathbb{Q} = \{r_1, r_2, \ldots\}$ since $\mathbb{Q}$ is countable. Let

$$I_n = B\left(r_n; \frac{1}{2^{n+1}}\right) \supseteq \{r_n\}$$

and

$$A = \bigcup_{n=1}^{\infty} I_n \supseteq \mathbb{Q}.$$

Hence A is an open subset in $\mathbb{R}$.

(2) Let $E = \mathbb{R} - A$. By construction, E is closed and $E \cap \mathbb{Q} = \varnothing$.

(3) *Show that E is uncountable. It suffices to show that $m^*(E) > 0$.* In fact, the outer measure of U is

$$m^*(A) \leq \sum_{n=1}^{\infty} m^*(I_n) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Thus,

$$m^*(E) \geq m^*(\mathbb{R}) - m^*(A) = \infty - 1 = \infty.$$

Hence, the set of all condensation points of E is our desired nonempty perfect set in $\mathbb{R} - \mathbb{Q}$. $\square$

Note. In fact, we can replace $\mathbb{Q}$ by the set of all real algebraic numbers (Exercise 2.2).

Exercise 2.19.

(a) *If A and B are disjoint closed sets in some metric space X, prove that they are separated.*

(b) *Prove the same for disjoint open sets.*

(c) *Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p, q) < \delta$, define B similarly, with $>$ in place of $<$. Prove that A and B are separated.*

(d) *Prove that every connected metric space with at least two points is uncountable. Hint: Use (c).*

Proof of (a). Since

$$\begin{aligned}
A \cap \overline{B} &= A \cap B && (B \text{ is closed}) \\
&= \varnothing, && (A \text{ and } B \text{ are disjoint}) \\
\overline{A} \cap B &= A \cap B && (A \text{ is closed}) \\
&= \varnothing. && (A \text{ and } B \text{ are disjoint})
\end{aligned}$$

A and B are separated. $\square$

Proof of (b)(Theorem 2.27(c)). Note that $\widetilde{A}$ is a closed set containing B. Since $\overline{B}$ is the smallest closed set containing B, $\widetilde{A} \supseteq \overline{B}$ (Theorem 2.27(c)). Hence

$$A \cap \overline{B} \subseteq A \cap \widetilde{A} = \varnothing.$$

Similarly, $\overline{A} \cap B = \varnothing$. Hence A and B are separated. $\square$

Proof of (c). Since both

$$A = \{q \in X : d(p,q) < \delta\} \text{ and } B = \{q \in X : d(p,q) > \delta\}$$

are open in X, they are separated by (b). $\square$

Proof of (d). Let X be a connected metric space.

(1) Let $p, q \in X$ with $p \neq q$. Hence $d_X(p,q) = r > 0$ (Definition 2.15(a)).

(2) Given any $\delta \in (0, r)$. Define

$$A = \{x \in X : d(p,x) < \delta\} \text{ and } B = \{x \in X : d(p,x) > \delta\}.$$

$p \in A \neq \varnothing$ and $q \in B \neq \varnothing$.

(3) If there were no $y_\delta \in X$ such that $d(p, y_\delta) = \delta$, we can write $X = A \cup B$ as a union of two nonempty separated sets ((c)), contrary to the connectedness of X.

(4) Collect these y as E. Since d is a function, there is a one-to-one map from $(0, r)$ to E defined by $\delta \mapsto y_\delta$ in (3). Since $(0, r)$ is uncountable, $X \supseteq E$ is uncountable.

$\square$

Exercise 2.20. *Are closures and interiors of connected sets always connected? (Look at subsets of $\mathbb{R}^2$.)*

Proof.

(1) *Interiors of connected sets are not always connected.* Let $X = \mathbb{R}^2$ with the usual Euclidean metric be a metric space. Take

$$E = B(89;1) \bigcup B(64;1) \bigcup \{(x,0) \in \mathbb{R}^2 : 64 \leq x \leq 89\}.$$

E is connected and

$$E^\circ = B(89;1) \bigcup B(64;1)$$

is disconnected.

(2) *Closures of connected sets are always connected. It suffices to show that E is disconnected if $\overline{E}$ is disconnected.*

   (a) Write $\overline{E} = A \cup B$ as a union of two nonempty separated sets. Here $A \neq \varnothing$, $B \neq \varnothing$, $A \cap \overline{B} = \varnothing$ and $\overline{A} \cap B = \varnothing$.

   (b) Write
$$E = (A \cap E) \bigcup (B \cap E)$$
   and we will show that E is disconnected.

   (c) *Show that $A \cap E$ and $B \cap E$ are separated.* In fact,
$$(A \cap E) \cap \overline{B \cap E} \subseteq A \cap \overline{B} = \varnothing,$$
$$\overline{A \cap E} \cap (B \cap E) \subseteq \overline{A} \cap B = \varnothing.$$

   (d) *Show that $A \cap E$ and $B \cap E$ are nonempty.* (Reductio ad absurdum) If $A \cap E = \varnothing$, then
$$E = (A \cap E) \bigcup (B \cap E) = B \cap E \Longrightarrow E \subseteq B.$$
   So
$$\begin{aligned} A &= (A \cup B) \bigcap A && (A \subseteq A \cup B) \\ &= \overline{E} \bigcap A \\ &\subseteq \overline{B} \bigcap A && (E \subseteq B) \\ &= \varnothing \end{aligned}$$
   which contradicts $A \neq \varnothing$ in (a). Therefore, $A \cap E \neq \varnothing$. Similarly, $B \cap E \neq \varnothing$.

Hence, E is disconnected if $\overline{E}$ is disconnected, or closures of connected sets are always connected.

□

Exercise 2.21. *Let A and B be separated subsets of some $\mathbb{R}^k$, suppose $\mathbf{a} \in A$, $\mathbf{b} \in B$, and define*
$$\mathbf{p}(t) = (1-t)\mathbf{a} + t\mathbf{b}$$
for $t \in \mathbb{R}^1$. Put $A_0 = \mathbf{p}^{-1}(A)$, $B_0 = \mathbf{p}^{-1}(B)$. [Thus $t \in A_0$ if and only if $\mathbf{p}(t) \in A$.]

(a) *Prove that A_0 and B_0 are separated subsets of $\mathbb{R}^1$.*

(b) *Prove that there exists $t_0 \in (0,1)$ such that $\mathbf{p}(t_0) \notin A \bigcup B$.*

(c) *Prove that every convex subset of $\mathbb{R}^k$ is connected.*

Proof of (a).

(1) Note that

(a) $\mathbf{a} \neq \mathbf{b}$ or $|\mathbf{a} - \mathbf{b}| > 0$ since $A \cap B = \varnothing$.

(b) $|\mathbf{p}(t) - \mathbf{p}(s)| = |t - s||\mathbf{a} - \mathbf{b}|$ by a direct calculation.

(c) $\mathbf{p}(t) = \mathbf{p}(s)$ if and only if $t = s$ by (a)(b).

(2) *Show that $A_0 \cap \overline{B_0} = \varnothing$.* (Reductio ad absurdum) If there were $t \in A_0 \cap \overline{B_0}$, then $t \in A_0$ and t is a limit point of B_0.

(a) $t \in A_0$ implies that $\mathbf{p}(t) \in A$.

(b) *Show that t is a limit point of $B_0 \Longrightarrow \mathbf{p}(t)$ is a limit point of B.* Given any $\varepsilon > 0$, there is $s \in B_0$ such that

$$|t - s| < \frac{\varepsilon}{|\mathbf{a} - \mathbf{b}|} \quad \text{with} \quad s \neq t$$

since t is a limit point of B_0. So by (1),

$$|\mathbf{p}(t) - \mathbf{p}(s)| = |t - s||\mathbf{a} - \mathbf{b}| < \varepsilon.$$

Here $\mathbf{p}(s) \in B$ and $\mathbf{p}(s) \neq \mathbf{p}(t)$. So $\mathbf{p}(t)$ is a limit point of B.

By (a)(b), $\mathbf{p}(t) \in A \cap \overline{B} = \varnothing$, contrary to the assumption that A and B are separated.

(3) *Show that $\overline{A_0} \cap B_0 = \varnothing$.* Similar to (2).

By (2)(3), A_0 and B_0 are separated. $\square$

Proof of (b). (Reductio ad absurdum) If $\mathbf{p}(t)$ were in $A \bigcup B$ for all $t \in (0,1)$, we will *show that $[0,1]$ is separated by $A_0 \cap [0,1]$ and $B_0 \cap [0,1]$ to get a contradiction.*

(1) $\mathbf{p}(t)$ were in $A \bigcup B$ for all $t \in [0,1]$ since $\mathbf{p}(0) = \mathbf{a} \in A \bigcup B$ and $\mathbf{p}(1) = \mathbf{b} \in A \bigcup B$. Therefore,

$$[0,1] \subseteq \mathbf{p}^{-1}(A \cup B) = \mathbf{p}^{-1}(A) \cup \mathbf{p}^{-1}(B) = A_0 \cup B_0.$$

(2) Let $A_1 = A_0 \cap [0,1]$ and $B_1 = B_0 \cap [0,1]$. So $[0,1] = A_1 \bigcup B_1$.

(3) *Show that $A_1 \neq \varnothing$ and $B_1 \neq \varnothing$.*

$$\begin{aligned} \mathbf{p}(0) \in A &\Longleftrightarrow 0 \in \mathbf{p}^{-1}(A) = A_0 \\ &\Longleftrightarrow 0 \in A_0 \text{ and } 0 \in [0,1] \\ &\Longleftrightarrow 0 \in A_0 \cap [0,1] = A_1. \end{aligned}$$

Similarly, $1 \in B_1$.

Note. That's why we consider $[0,1]$ instead of $(0,1)$.

(4) *Show that* $A_1 \cap \overline{B_1} = \varnothing$ *and* $\overline{A_1} \cap B_1 = \varnothing$. Since $A_1 \subseteq A_0$ and $B_1 \subseteq B_0$, $A_1 \cap \overline{B_1} \subseteq A_0 \cap \overline{B_0} = \varnothing$ or $A_1 \cap \overline{B_1} = \varnothing$. Similarly, $\overline{A_1} \cap B_1 = \varnothing$.

By (2)(3)(4), $[0,1]$ is separated, contrary to the connectedness of $[0,1]$ (Theorem 2.47). $\square$

Proof of (c).

(1) Let E be a convex subset of $\mathbb{R}^k$. Recall

$$\mathbf{p}(t) = (1-t)\mathbf{a} + t\mathbf{b} \in E$$

whenever $\mathbf{a}, \mathbf{b} \in E$ and $t \in (0,1)$.

(2) (Reductio ad absurdum) If E were separated by A and B, pick $\mathbf{a} \in A \subseteq E$ and $\mathbf{b} \in B \subseteq E$.

(3) By (b), there exists $t_0 \in (0,1)$ such that $\mathbf{p}(t_0) \not\in A \cup B = E$, contrary to the convexity of E.

$\square$

Exercise 2.22. *A metric space is called separable if it contains a countable dense subset. Show that* $\mathbb{R}^k$ *is separable. (Hint: Consider the set of points which have only rational coordinates.)*

Proof. Let E be the set of points which have only rational coordinates.

(1) *Show that E is countable.* $\mathbb{Q}$ is countable and thus $E = \mathbb{Q}^k$ is countable (Theorem 2.13).

(2) *Show that E is dense.* Given any $\mathbf{p} = (p_1, \dots, p_k) \in \mathbb{R}^k$. We want to show that $\mathbf{p}$ is a limit point of E.

   (a) Given any open neighborhood $B(\mathbf{p}; r)$ of $\mathbf{p}$, $r > 0$.

   (b) Since $\mathbb{Q}$ is dense in $\mathbb{R}$ (Theorem 1.20), every coordinate of $\mathbf{p}$ is a limit point of $\mathbb{Q}$. In particular, for every $i = 1, 2, \dots, k$, the open neighborhood $B\left(p_i, \frac{r}{\sqrt{k}}\right)$ of p_i contains a point $q_i \neq p_i$ and $q_i \in \mathbb{Q}$.

   (c) Collect all q_i in (b) and define $\mathbf{q} = (q_1, \dots, q_k) \in \mathbb{Q}^k = E$. By construction $\mathbf{q} \neq \mathbf{p}$ and

$$\begin{aligned}
|\mathbf{p} - \mathbf{q}| &= \sqrt{(p_1 - q_1)^2 + \cdots + (p_k - q_k)^2} \\
&< \sqrt{\left(\frac{r}{\sqrt{k}}\right)^2 + \cdots + \left(\frac{r}{\sqrt{k}}\right)^2} \\
&= \sqrt{k \cdot \frac{r^2}{k}} \\
&= r
\end{aligned}$$

   or $\mathbf{q} \in B(\mathbf{p}; r)$.

By (a)(b)(c), E is dense in $\mathbb{R}^k$.

By (1)(2), $\mathbb{R}^k$ is separable. $\square$

Exercise 2.23. *A collection $\{V_\alpha\}$ of open subsets of X is said to be a base for X if the following is true: For every $x \in X$ and every open set $G \subseteq X$ such that $x \in G$, we have $x \in V_\alpha \subseteq G$ for some α. In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$.*

Prove that every separable metric space has a countable base. (Hint: Take all neighborhoods with rational radius and center in some countable dense subset of X.)

Note. $\mathbb{R}^k$ has a countable base (Exercise 2.22).

Proof (Hint). Let X be a separable metric space, and E be a countable dense subset of X. Let $\mathscr{B}$ be a collection of all neighborhoods with rational radius and center in E.

(1) $\mathscr{B}$ is countable (Theorem 2.12).

(2) *$\mathscr{B}$ is a base for X.* Similar to Exercise 2.9(a). Given any $p \in X$ and every open set $G \subseteq X$ such that $p \in G$. Since p is in an open set G, there exists an open neighborhood $B(p;r)$ of p such that $B(p;r) \subseteq G$.

(3) Let r_0 be rational such that $0 < r_0 < \frac{r}{2}$ (Theorem 1.20(b)). Since E is dense in X, there is $q \in E$ such that $d_X(p,q) < r_0$. For such $r_0 \in \mathbb{Q}$ we pick an open neighborhood $B(q;r_0)$ of q. Clearly, $B(q;r_0) \in \mathscr{B}$.

(4) $p \in B(q;r_0)$ since $d_X(p,q) < r_0$.

(5) *Show that $B(q;r_0) \subseteq B(p;r) \subseteq G$.* For any $z \in B(q;r_0)$, $d_X(z,p) \leq d_X(z,q) + d_X(q,p) < r_0 + r_0 < \frac{r}{2} + \frac{r}{2} = r$. That is, $z \in B(p;r)$.

By (3)(4)(5), (2) is established. By (1)(2), $\mathscr{B}$ is a countable base for X. $\square$

Supplement.

(1) In topology, a second-countable space, also called a completely separable space, is a topological space whose topology has a countable base.

(2) Every second-countable space is separable.

(3) The reverse implication of (2) does not hold in general. However, for metric spaces the properties of being second-countable and separable are equivalent.

(4) *Show that every second-countable metric space X is separable.*

(a) Let $\mathscr{B} = \{B_n : n \in \mathbb{Z}^+\}$ be a countable base of X.

(b) For every $B_n \in \mathscr{B}$, pick any point p_n of B_n and collect them as

$$E = \{p_n : p_n \in B_n \text{ for } n \in \mathbb{Z}^+\}.$$

(c) E is countable.

(d) *Show that E is dense.* Given any $x \in X$. For any open neighborhood $B(x)$ of x, $B(x)$ is a union of subcollection of $\mathscr{B}$. That is, there is always a point in E by the construction of E.

□

Exercise 2.24. *Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable.*

(Hint: Fix $\delta > 0$, and pick $x_1 \in X$. Having chosen $x_1, \ldots, x_j \in X$, choose x_{j+1}, if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \ldots, j$. Show that this process must stop after finite number of steps, and that X can therefore be covered by finite many neighborhoods of radius δ. Take $\delta = \frac{1}{n}$ $(n = 1, 2, 3, \ldots)$ and consider the centers of the corresponding neighborhoods.)

Note. The reverse implication does not hold (Exercise 2.10).

Proof (Hint).

(1) Fix $\delta > 0$, and pick $x_1 \in X$. *Show that every limit point compact metric space X is totally bounded.*

  (a) Having chosen $x_1, \ldots, x_j \in X$, choose x_{j+1}, if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \ldots, j$. Let E_δ be the set of these x_i.

  (b) *Show that this process must stop after finite number of steps, and that X can therefore be covered by finite many neighborhoods of radius δ.* (Reductio ad absurdum)

    (i) If not, E_δ is an infinite subset of X. By assumption there is a limit point of E_δ, say $p \in X$.

    (ii) In particular, an open neighborhood $B\left(p; \frac{\delta}{64}\right)$ of p contains a point $x_n \in E_\delta$ with $p \neq x_n$.

    (iii) The neighborhood $B\left(p; \frac{\delta}{64}\right)$ contains no other point $x_m \in E_\delta$ with $m \neq n$. If so,

$$d_X(x_n, x_m) \leq d_X(x_n, p) + d_X(p, x_m) < \frac{\delta}{64} + \frac{\delta}{64} < \delta,$$

    contrary to the construction of E_δ.

    (iv) Note that $p \notin E_\delta$ as a corollary to (iii).

(v) So another open neighborhood $B(p; r)$ of p with $r = d_X(p, x_n) > 0$ contains no points $x_m \in E_\delta$ with $p \neq x_m$, contrary to the assumption that p is a limit point of E_δ.

(2) *Show that every totally bounded metric space X is separable.* Take $\delta = \frac{1}{n}$ $(n = 1, 2, 3, \ldots)$ in (1), and union all $E_{\frac{1}{n}}$ as

$$E = \bigcup_{n=1}^{\infty} E_{\frac{1}{n}} \subseteq X.$$

Show that E is a countable dense subset of X.

(a) *Show that E is countable.* Since E is the countable union of finite set $E_{\frac{1}{n}}$, E is countable (Theorem 2.12).

(b) *Show that E is dense in X.* Given any $p \in X$. *It suffices to show that given any open neighborhood $B(p;r)$ of $p \in X - E$, there exists $q \in E$ such that $q \in B(p;r)$.* Pick any $n \in \mathbb{Z}^+$ such that $\frac{1}{n} < r$ (Theorem 1.20(a)). By the construction of $E_{\frac{1}{n}}$, there is $q \in E_{\frac{1}{n}}$ such that $p \in B\left(q; \frac{1}{n}\right)$, or $d_X(p, q) < \frac{1}{n} < r$, or $q \in B(p; r)$.

□

Supplement.

(1) A topological space X is said to be limit point compact or weakly countably compact if every infinite subset of X has a limit point in X.

(2) In a metric space, limit point compactness, compactness, and sequential compactness are all equivalent. For general topological spaces, however, these three notions of compactness are not equivalent.

(3) A metric space X is totally bounded if and only if for every real number $\delta > 0$, there exists a finite collection of open balls in X of radius δ whose union contains X.

Exercise 2.25. *Prove that every compact metric space K has a countable base, and that K is therefore separable. (Hint: For every positive integer n, there are finitely many neighborhood of radius $\frac{1}{n}$ whose union covers K.)*

Proof (Exercise 2.24(a)).

(1) *Show that every compact metric space K is limit point compact.* Given any subset $E \subseteq K$. It suffices to show that if E has no limit point, then E must be finite.

(a) Since E has no limit point, E is closed.

(b) For any point $p \in E$. Since p is not a limit point, there is an open neighborhood $B(p)$ such that $B(p)$ contains no point other than p.

(c) Similar to the proof of Theorem 2.35, let

$$\mathscr{F} = \{B(p) : p \in E \text{ with } B(p) \cap E = \{p\}\} \bigcup \widetilde{E}.$$

Hence $\mathscr{F}$ is an open covering of K.

(d) Since K is compact by assumption, there is an finitely subcovering $\mathscr{F}'$ of K. Since $\widetilde{E}$ does not intersect E, each $B(p) \in \mathscr{F}'$ contains only one point of E and so E is finite.

(2) Since K is limit point compact, K is separable (Theorem 2.24).

□

Proof (Exercise 2.24(b)).

(1) *Show that every compact metric space K is totally bounded.* Given any real number $\delta > 0$, define an open covering $\mathscr{F}$ of K by

$$\mathscr{F} = \{B(p; \delta) : p \in K\}.$$

Since K is compact, there exists a finite subcovering $\mathscr{F}'$ of K. $\mathscr{F}'$ is our desired finite collection of open balls in X of radius δ whose union contains X.

(2) Since K is totally bounded, K is separable (Theorem 2.24).

□

Proof (Hint).

(1) Given any positive integer $n > 0$, define an open covering $\mathscr{F}_n$ of K by

$$\mathscr{F}_n = \left\{ B\left(p; \frac{1}{n}\right) : p \in K \right\}.$$

Since K is compact, there exists a finite subcovering $\mathscr{G}_n$ of K.

(2) *Show that every compact metric space K is second-countable.*

(a) Define

$$\mathscr{B} = \bigcup_{n \geq 1} \mathscr{G}_n$$

be a collection. Since $\mathscr{B}$ is a countable union of finite set $\mathscr{G}_n$, $\mathscr{B}$ is countable. Hence it suffices to show that for every $p \in K$ and every open set $G \subseteq K$ such that $p \in G$, there is $B \in \mathscr{B}$ such that $x \in B \subseteq G$.

(b) Since G is open, there is an open neighborhood $B(p;r)$ of p such that $B(p;r) \subseteq G$.

(c) For such $r > 0$, there is $n \in \mathbb{Z}^+$ with $0 < \frac{1}{n} < \frac{r}{2}$ (Theorem 1.20(a)). So p is in some $B\left(q; \frac{1}{n}\right) \in \mathscr{G}_n \subseteq \mathscr{B}$ since $\mathscr{G}_n$ is a subcovering of K.

(d) *Show that $B\left(q; \frac{1}{n}\right) \subseteq B(p;r) \subseteq G$.* For any $z \in B(q; \frac{1}{n})$,

$$d_K(z,p) \leq d_K(z,q) + d_K(q,p) < \frac{1}{n} + \frac{1}{n} < \frac{r}{2} + \frac{r}{2} = r.$$

That is, $z \in B(p;r)$, or $B\left(q; \frac{1}{n}\right) \subseteq B(p;r) \subseteq G$.

By (a)(b)(c)(d), K is second-countable.

(3) *Show that every second-countable metric space is separable.* Supplement (4) to Exercise 2.23.

□

Exercise 2.26. *Let X be a metric space in which every infinite subsets has a limit point. Prove that X is compact.*

By Exercises 2.23 and 2.24, X has a countable base. It follows that every open cover of X has a countable subcovering $\{G_n\}$, $n = 1, 2, 3, \ldots$. If no finite subcollection of $\{G_n\}$ covers X, then the complement F_n of $G_1 \cup \cdots \cup G_n$ is nonempty for each n, but $\bigcap F_n$ is empty. If E is a set contains a point from each F_n, consider a limit point of E, and obtain a contradiction.

Note. In every metric space, we have

$$\begin{aligned}\{\text{compact}\} &\Longleftrightarrow \{\text{limit point compact}\}\\ &\Longrightarrow \{\text{totally bounded}\}\\ &\Longrightarrow \{\text{separable}\}\\ &\Longleftrightarrow \{\text{second-countable}\}\\ &\Longleftrightarrow \{\text{Lindelof}\}.\end{aligned}$$

Proof (Hint).

(1) Since X is limit point compact, X is separable (Exercise 2.24). Since X is separable, X is second-countable (Exercise 2.23).

(2) *Show that X is Lindelof if X is second-countable.* Let X be a second-countable metric space. Let $\mathscr{B} = \{B_n\}$ be a countable base of X. Given any open covering $\mathscr{F}$ of X.

(a) Iterate each $B_n \in \mathscr{B}$, pick one $G_n \in \mathscr{F}$ containing B_n, and collect them as
$$\mathscr{G} = \{G_n : G_n \supseteq B_n \text{ for } n \in \mathbb{Z}^+\}.$$
(G_n might be duplicated.)

(b) $\mathscr{G}$ is a countable subset of $\mathscr{F}$.

(c) $\mathscr{G}$ covers X since $\mathscr{B}$ is a countable base of X.

(3) Hence, given any open covering $\mathscr{F}$ of X, there is a countable subcovering $\mathscr{G} = \{G_n\}$ of X. (Reductio ad absurdum) If there were no finite subcovering of $\mathscr{G}$, then the complement F_n of $G_1 \cup \cdots \cup G_n$ is nonempty for each n, but $\cap F_n$ is empty.

(4) Let E bet a set contains a point from each F_n. E is infinite and thus E has a limit point, say p. $p \in G_n$ for some n since $\mathscr{G} = \{G_n\}$ is an open covering of X. Since G_n is open, there is an open neighborhood $B(p)$ of p such that $B(p) \subseteq G_n$. By the construction of F_n,
$$B(p) \cap F_m = \varnothing$$
whenever $m \geq n$, contrary to the assumption that p is a limit point of E.

Hence, X is compact if X is limit point compact. $\square$

Supplement.

(1) Lindelof space is a topological space in which every open covering has a countable subcovering.

(2) *Show that X is second-countable if X is Lindelof.* Same as the Proof (Hint) of Exercise 2.25 except changing the word "compact" to "Lindelof" and "finite" to "countable." $\square$

(3) In every metric space, we have
$$\{\text{compact}\} \Longleftrightarrow \{\text{limit point compact}\} \Longleftrightarrow \{\text{sequentially compact}\}.$$

Exercise 2.27. *Define a point p in a metric space X to be a condensation point of a set $E \subseteq X$ if every neighborhood of p contains uncountably many points of E.*

Suppose $E \subseteq \mathbb{R}^k$, E is uncountable, and let P be the set of all condensation points of E. Prove that P is perfect and that at most countably many points of E are not in P. In other words, show that $\widetilde{P} \cap E$ is at most countable.

(Hint: Let $\{V_n\}$ be a countable base of $\mathbb{R}^k$, let W be the union of those V_n for which $E \cap V_n$ is at most countable, and show that $P = \widetilde{W}$.)

Note. The statement is also true for separable metric space.

Proof.

(1) Let $\{V_n\}$ be a countable base of $\mathbb{R}^k$ (Exercise 2.22 and 2.23). Let W be the union of those V_n for which $E \cap V_n$ is at most countable.

(2) *Show that $P = \widetilde{W}$.*

(a) ($P \subseteq \widetilde{W}$) Given any $x \in P$.

$$\begin{aligned} x \in P \Longrightarrow & x \text{ is a condensation points of } E \\ \Longrightarrow & \forall\, V_n \ni x, \exists B(x) \subseteq V_n \text{ such that } E \cap B(x) \text{ is uncountable} \\ \Longrightarrow & E \cap V_n \text{ is uncountable} \\ \Longrightarrow & x \notin W. \end{aligned}$$

(b) ($P \supseteq \widetilde{W}$) Given any $x \in \widetilde{W}$. Let $P(V_n)$ be the proposition that $E \cap V_n$ is at most countable.

$$\begin{aligned} x \in \widetilde{W} \Longrightarrow & x \notin W = \bigcup_{P(V_n)} V_n \\ \Longrightarrow & x \notin V_n \text{ for which } E \cap V_n \text{ is at most countable} \\ \Longrightarrow & \forall\ B(x) \text{ of } x,\ x \in V_m \subseteq B(x) \text{ for some } V_m \qquad (\{V_n\}\text{: base of } X) \\ \Longrightarrow & E \cap V_m \text{ is uncountable} \\ \Longrightarrow & E \cap B(x) \supseteq E \cap V_m \text{ is uncountable} \\ \Longrightarrow & x \text{ is a condensation point of } E \\ \Longrightarrow & x \in P. \end{aligned}$$

(3) *Show that P is closed.* P is the complement of an open subset W.

(4) *Show that $P \subseteq P'$.* (Reductio ad absurdum)

(a) If there were an isolated point $x \in P$, then there exists an open neighborhood $B(x)$ of x such that $B(x) \cap P = \{x\}$.

(b) Since x is a condensation point of E, there are uncountably many points of E in $B(x)$, and such points y are not a condensation points of E except $y = x$.

(c) Given any point $y \in E \cap B(x)$ with $y \neq x$. Since y is not a condensation point, there exists a neighborhood $B(y)$ of y such that $B(y) \cap E$ is at most countable. Since $\{V_n\}$ is a base, for each $B(y)$ there exists $V_{n(y)}$ such that $y \in V_{n(y)} \subseteq B(y)$. Hence

$$V_{n(y)} \cap E \subseteq B(y) \cap E$$

is at most countable.

(d) Hence,

$$\begin{aligned} E \cap B(x) - \{x\} &\subseteq \bigcup_{y \in E \cap B(x) - \{x\}} V_{n(y)} \\ &= \bigcup_{n(y)} V_{n(y)} \end{aligned}$$

is a countable union of at most countable sets, which is countable. Hence $E \cap B(x) - \{x\}$ or $E \cap B(x)$ is countable, contrary to the assumption that $E \cap B(x)$ is uncountable.

(5) *Show that $E \cap \widetilde{P}$ is at most countable.*

$$E \cap \widetilde{P} = E \bigcap \left(\bigcup_{P(V_n)} V_n \right) = \bigcup_{P(V_n)} (E \cap V_n)$$

is at most countable.

□

Exercise 2.28. *Prove that every closed set in a separable metric space is the union of a (possible empty) perfect set and a set which is at most countable. (Corollary: Every countable closed set in $\mathbb{R}^k$ has isolated points.) (Hint: Use Exercise 2.27.)*

Proof (Exercise 2.27). Let E be a closed set in a separable metric space.

(1) E contains all limit points of E, especially contains all condensation points of E. So we can write

$$E = P \cup (E - P)$$

where P is the set of all condensation points of E.

(2) By Exercise 2.27, P is perfect and $E - P = E \cap \widetilde{P}$ is at most countable.

□

Cantor-Bendixson theorem.

(1) Closed sets of a Polish space X have the perfect set property in a particularly strong form: any closed subset of X may be written uniquely as the disjoint union of a perfect set and a countable set.

(2) A Polish space is a separable completely metrizable topological space; that is, a space homeomorphic to a complete metric space that has a countable dense subset.

Exercise 2.29. *Prove that every open set in $\mathbb{R}^1$ is the union of an at most countable collection of disjoint segments. (Hint: Use Exercise 2.22.)*

Proof. Let E be an open subset of $\mathbb{R}^1$.

(1) For each $x \in E$, let I_x denote the largest open interval containing x and contained in E. More precisely, since E is open, x is contained in some small (non-trivial) interval, and therefore if

$$a_x = \inf\{a < x : (a, x) \subseteq E\} \text{ and } b_x = \sup\{b > x : (x, b) \subseteq E\}$$

we must have $a_x < x < b_x$ (with possibly infinite values for a_x and b_x).

(2) Let $I_x = (a_x, b_x)$, then by construction we have $x \in I_x$ as well as $I_x \subseteq E$. Hence

$$E = \bigcup_{I_x \in \mathscr{F}} I_x,$$

where $\mathscr{F} = \{I_x\}_{x \in E}$.

(3) Suppose that two intervals I_x and I_y intersect. Then their union (which is also an open interval) is contained in E and contains x (and y). Since I_x is maximal, $I_x \cup I_y \subseteq I_x$, and similarly $I_x \cup I_y \subseteq I_y$. This can happen only if $I_x = I_y$.

(4) Therefore, any two distinct intervals in $\mathscr{F}$ must be disjoint. Hence $\mathscr{F}$ is countable since each open interval $I_x \in \mathscr{F}$ contains a rational number.

□

Exercise 2.30. *Imitate the proof of Theorem 2.43 to obtain the following result:*

> *If $\mathbb{R}^k = \bigcup_{n=1}^{\infty} F_n$, where each F_n is a closed subset of $\mathbb{R}^k$, then at least one F_n has a nonempty interior.*
>
> *Equivalent statement: If G_n is a dense open subset of $\mathbb{R}^k$, for $n = 1, 2, 3, \ldots$, then $\bigcap_{n=1}^{\infty} G_n$ is not empty (in fact, it is dense in $\mathbb{R}^k$).*

(This is a special case of Baire's theorem; see Exercise 3.22 for the general case.)

Baire category theorem. *If G_n is a dense open subset of $\mathbb{R}^k$, for $n = 1, 2, 3, \ldots$, then*

$$\bigcap_{n=1}^{\infty} G_n$$

is dense in $\mathbb{R}^k$.

Proof of Baire category theorem. Given any open set G_0 in $\mathbb{R}^k$, will show that

$$\bigcap_{n=0}^{\infty} G_n \neq \varnothing.$$

(1) Since G_1 is dense, $G_0 \cap G_1$ is nonempty. Take any one point $\mathbf{x}_1$ in the open set $G_0 \cap G_1$, there exists open neighborhood

$$V_1 = \{\mathbf{y} \in \mathbb{R}^k : |\mathbf{y} - \mathbf{x}_1| < r_1\}$$

of $\mathbf{x}_1$ such that

$$\overline{V_1} = \{\mathbf{y} \in \mathbb{R}^k : |\mathbf{y} - \mathbf{x}_1| \leq r_1\} \subseteq G_0 \cap G_1.$$

(2) Suppose V_n has been constructed, take any one point $\mathbf{x}_{n+1}$ in the open set $V_n \cap G_{n+1}$, there exists open neighborhood

$$V_{n+1} = \{\mathbf{y} \in \mathbb{R}^k : |\mathbf{y} - \mathbf{x}_{n+1}| < r_{n+1}\}$$

of $\mathbf{x}_1$ with r_{n+1} such that

$$\overline{V_{n+1}} = \{\mathbf{y} \in \mathbb{R}^k : |\mathbf{y} - \mathbf{x}_{n+1}| \leq r_{n+1}\} \subseteq V_n \cap G_{n+1}.$$

(3) Note that

   (a) each $\overline{V_n}$ is nonempty (containing $\mathbf{x}_n$) and compact.
   (b) $\overline{V_1} \supseteq \overline{V_2} \supseteq \cdots$ (since $\overline{V_{n+1}} \subseteq V_n \cap G_{n+1} \subseteq V_n \subseteq \overline{V_n}$).

   By Corollary to Theorem 2.36,

$$\bigcap_{n=1}^{\infty} \overline{V_n} \neq \varnothing.$$

(4) Pick $\mathbf{x} \in \bigcap_{n=1}^{\infty} \overline{V_n}$. Hence

$$\begin{aligned}
\mathbf{x} \in \bigcap_{n=1}^{\infty} \overline{V_n} &\Longleftrightarrow \mathbf{x} \in \overline{V_n} \text{ for all } n = 1, 2, 3, \ldots \\
&\Longrightarrow \mathbf{x} \in \overline{V_1} \subseteq G_0 \cap G_1 \text{ and } \mathbf{x} \in \overline{V_{n+1}} \subseteq V_n \cap G_{n+1} \subseteq G_{n+1} \\
&\Longrightarrow \mathbf{x} \in G_0 \cap G_1 \cap \cdots = \bigcap_{n=0}^{\infty} G_n \\
&\Longrightarrow \bigcap_{n=0}^{\infty} G_n \neq \varnothing.
\end{aligned}$$

□