

Solutions to Algebraic Curves

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Chapter 1: Affine Algebraic Sets

1.1. Algebraic Preliminaries

Problem 1.1.*

Let R be a domain.

- (a) If F, G are forms of degree r, s respectively in $R[X_1, \dots, X_n]$, show that FG is a form of degree $r + s$.
- (b) Show that any factor of a form in $R[X_1, \dots, X_n]$ is also a form.

Proof of (a).

- (1) Write

$$F = \sum_{(i)} a_{(i)} X^{(i)},$$
$$G = \sum_{(j)} b_{(j)} X^{(j)},$$

where $\sum_{(i)}$ is the summation over $(i) = (i_1, \dots, i_n)$ with $i_1 + \dots + i_n = r$ and $\sum_{(j)}$ is the summation over $(j) = (j_1, \dots, j_n)$ with $j_1 + \dots + j_n = s$.

- (2) Hence,

$$FG = \sum_{(i)} \sum_{(j)} a_{(i)} b_{(j)} X^{(i)} X^{(j)}$$
$$= \sum_{(i),(j)} a_{(i)} b_{(j)} X^{(k)}$$

where $(k) = (i_1 + j_1, \dots, i_n + j_n)$ with $(i_1 + j_1) + \dots + (i_n + j_n) = r + s$. Each $X^{(k)}$ is the form of degree $r + s$ and $a_{(i)} b_{(j)} \in R$. Hence FG is a form of degree $r + s$.

□

Proof of (b).

- (1) Given any form $F \in R[X_1, \dots, X_n]$, and write $F = GH$. It suffices to show that G (or H) is a form as well.
- (2) Write

$$G = G_0 + \dots + G_r,$$
$$H = H_0 + \dots + H_s$$

where $G_r \neq 0$ and $H_s \neq 0$. So

$$F = GH = G_0H_0 + \cdots + G_rH_s.$$

Since R is a domain, $R[X_1, \dots, X_n]$ is a domain and thus $G_rH_s \neq 0$. The maximality of r and s implies that $\deg(F) = r + s$. Therefore, by the maximality of $r + s$, $F = G_rH_s$, or $G = G_r$, or G is a form.

□

Problem 1.5.*

Let k be any field. Show that there are an infinitely number of irreducible monic polynomials in $k[X]$. (Hint: Suppose F_1, \dots, F_n were all of them, and factor $F_1 \cdots F_n + 1$ into irreducible factors.)

Proof (Due to Euclid).

- (1) If F_1, F_2, \dots, F_n were all irreducible monic polynomials, then we consider

$$G = F_1F_2 \cdots F_n + 1 \in k[X].$$

So there is an irreducible monic polynomial F dividing G since

$$\deg G = \deg F_1 + \deg F_2 + \cdots + \deg F_n \geq 1.$$

- (2) F can not be any of F_i for all i ; otherwise F would divide the difference $G - F_1F_2 \cdots F_n = 1$. That is, $F \neq F_i$ for all i , contrary to the assumption.

□

Problem 1.6.*

Show that any algebraically closed field is infinite. (Hint: The irreducible monic polynomials are $X - a$, $a \in k$.)

Proof (Due to Euclid).

- (1) Let k be an algebraically closed field. If a_1, \dots, a_n were all elements in k , then we consider a monic polynomials

$$F(X) = (X - a_1) \cdots (X - a_n) + 1 \in k[X].$$

- (2) Since k is algebraically closed, there is an element $a \in k$ such that $F(a) = 0$. By assumption, $a = a_i$ for some $1 \leq i \leq n$, and thus $F(a) = F(a_i) = 1$, contrary to the fact that a field is a commutative ring where $0 \neq 1$ and all nonzero elements are invertible.

□

1.2. Affine Space and Algebraic Sets