Notes on the book:
Apostol, Modular Functions and Dirichlet Series in Number Theory, 2nd edition

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Chapter 1: Elliptic functions

Exercise 1.11.

If $k \geq 2$ and $\tau \in H$ prove that the Eisenstein series

$$G_{2k}(\tau) = \sum_{(m,n) \neq (0,0)} (m + n\tau)^{-2k}$$

has the Fourier expansion

$$G_{2k}(\tau) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) e^{2\pi i n \tau}.$$

Proof.

(1) Let $q = e^{2\pi i \tau}$. Similar to Lemma 1.3 on page 19, we have

$$(2k-1)! \sum_{m=-\infty}^{+\infty} \frac{1}{(\tau + m)^{2k}} = (2\pi i)^{2k} \sum_{r=1}^{\infty} r^{2k-1} q^r.$$

(2) Similar to Theorem 1.18, we have

$$G_{2k}(\tau) = \sum_{(m,n) \neq (0,0)} (m + n\tau)^{-2k}$$

$$= \sum_{\substack{m=-\infty \\ m \neq 0 (n=0)}}^{+\infty} m^{-2k} + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{+\infty} \left((m + n\tau)^{-2k} + (m - n\tau)^{-2k} \right)$$

$$= 2\zeta(2k) + 2 \sum_{n=1}^{\infty} \sum_{m=-\infty}^{+\infty} (m + n\tau)^{-2k}$$

$$= 2\zeta(2k) + 2 \sum_{n=1}^{\infty} \frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{r=1}^{\infty} r^{2k-1} q^{nr}$$

$$= 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \underbrace{\sum_{d|n} d^{2k-1}}_{=\sigma_{2k-1}(n)} q^n.$$

In the last double sum we collect together those terms for which nr is constant.

$\square$

Exercise 1.12.

Refer to Exercise 1.11. If $\tau \in H$ prove that

$$G_{2k}\left(-\frac{1}{\tau}\right) = \tau^{2k} G_{2k}(\tau)$$

and deduce that

$$G_{2k}\left(\frac{i}{2}\right) = (-4)^k G_{2k}(2i) \qquad\qquad \text{for all } k \geq 2,$$

$$G_{2k}(i) = 0 \qquad\qquad \text{if } k \text{ is odd,}$$

$$G_{2k}(e^{\frac{2\pi i}{3}}) = 0 \qquad\qquad \text{if } k \not\equiv 0 \pmod 3.$$

Proof.

(1)

$$G_{2k}\left(-\frac{1}{\tau}\right) = \sum_{(m,n)\neq(0,0)} \left(m - \frac{n}{\tau}\right)^{-2k}$$

$$= \tau^{2k} \sum_{(m,n)\neq(0,0)} (\tau m - n)^{-2k}$$

$$= \tau^{2k} G_{2k}(\tau).$$

(2) Let $\tau = 2i$. We have $G_{2k}\left(\frac{i}{2}\right) = (-4)^k G_{2k}(2i)$.

(3) Let $\tau = i$. We have $G_{2k}(i) = (-1)^k G_{2k}(i)$. Hence $G_{2k}(i) = 0$ if k is odd.

(4) Let $\tau = e^{\frac{\pi i}{3}}$. We have $G_{2k}(e^{\frac{2\pi i}{3}}) = e^{\frac{2k\pi i}{3}} G_{2k}(e^{\frac{\pi i}{3}})$. Since

$$e^{\frac{2\pi i}{3}} = -1 + e^{\frac{\pi i}{3}}$$

and each Eisenstein series is a periodic function of τ of period 1, we have $G_{2k}(e^{\frac{2\pi i}{3}}) = G_{2k}(e^{\frac{\pi i}{3}})$. So $G_{2k}(e^{\frac{2\pi i}{3}}) = e^{\frac{2k\pi i}{3}} G_{2k}(e^{\frac{2\pi i}{3}})$. Therefore $G_{2k}(e^{\frac{2\pi i}{3}}) = 0$ if $k \not\equiv 0 \pmod 3$.

□

Exercise 1.14. (Lambert series)

*A series of the form $\sum_{n=1}^{\infty} f(n)\frac{x^n}{1-x^n}$ is called a **Lambert series**. Assuming absolute convergence, prove that*

$$\sum_{n=1}^{\infty} f(n)\frac{x^n}{1 - x^n} = \sum_{n=1}^{\infty} F(n)x^n,$$

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where

$$F(n) = \sum_{d|n} f(d).$$

Apply this result to obtain the following formulas, valid for $|x| < 1$.

(a)
$$\sum_{n=1}^{\infty} \frac{\mu(n)x^n}{1 - x^n} = x.$$

(b)
$$\sum_{n=1}^{\infty} \frac{\varphi(n)x^n}{1 - x^n} = \frac{x}{(1 - x)^2}.$$

(c)
$$\sum_{n=1}^{\infty} \frac{n^\alpha x^n}{1 - x^n} = \sum_{n=1}^{\infty} \sigma_\alpha(n)x^n.$$

(d)
$$\sum_{n=1}^{\infty} \frac{\lambda(n)x^n}{1 - x^n} = \sum_{n=1}^{\infty} x^{n^2}.$$

(e) *Use the result in (c) to express $g_2(\tau)$ and $g_3(\tau)$ in terms of Lambert series in $x = e^{2\pi i \tau}$.*

Note. In (a), $\mu(n)$ is the Möbius function; In (b), $\varphi(n)$ is Euler's totient; and in (d), $\lambda(n)$ is Liouville's function.

Proof. Similar to the proof of Exercise 1.11.

$$\sum_{n=1}^{\infty} f(n) \frac{x^n}{1 - x^n} = \sum_{n=1}^{\infty} f(n) \sum_{r=1}^{\infty} x^{rn}$$

$$= \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} f(n) x^{rn}$$

$$= \sum_{n=1}^{\infty} \underbrace{\left(\sum_{d|n} f(d) \right)}_{=F(n)} x^n.$$

□

Proof of (a). Theorem 2.1 in the textbook: *T. M. Apostol, Introduction to Analytic Number Theory* shows that

$$F(n) := \sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

Hence

$$\sum_{n=1}^{\infty} \mu(n) \frac{x^n}{1 - x^n} = \sum_{n=1}^{\infty} F(n) x^n = x.$$

□

Proof of (b). Theorem 2.2 in the textbook: *T. M. Apostol, Introduction to Analytic Number Theory* shows that $F(n) := \sum_{d|n} \varphi(d) = n$. Hence

$$\sum_{n=1}^{\infty} \varphi(n) \frac{x^n}{1 - x^n} = \sum_{n=1}^{\infty} F(n) x^n = \sum_{n=1}^{\infty} n x^n = \frac{x}{(1 - x)^2}.$$

□

Proof of (c). Since

$$F(n) := \sum_{d|n} d^\alpha = \sigma_\alpha(n),$$

we have

$$\sum_{n=1}^{\infty} n^\alpha \frac{x^n}{1 - x^n} = \sum_{n=1}^{\infty} F(n) x^n = \sum_{n=1}^{\infty} \sigma_\alpha(n) x^n.$$

□

Proof of (d). Theorem 2.19 in the textbook: *T. M. Apostol, Introduction to Analytic Number Theory* shows that

$$F(n) := \sum_{d|n} \lambda(d) = \begin{cases} 1 & \text{if } n \text{ is a square}, \\ 0 & \text{otherwise}. \end{cases}$$

Hence

$$\sum_{n=1}^{\infty} \lambda(n) \frac{x^n}{1 - x^n} = \sum_{n=1}^{\infty} F(n) x^n = \sum_{n=1}^{\infty} x^{n^2}.$$

□

Proof of (e).

(1) Let $q = x = e^{2\pi i \tau}$.

$$g_2(\tau) = \frac{4\pi^4}{3}\left\{1 + 240\sum_{k=1}^{\infty}\sigma_3(k)q^k\right\} \qquad \text{(Theorem 1.18)}$$

$$= \frac{4\pi^4}{3}\left\{1 + 240\sum_{k=1}^{\infty}\frac{k^3 q^k}{1 - q^k}\right\} \qquad ((c)).$$

(2) Similarly,

$$g_3(\tau) = \frac{8\pi^6}{27}\left\{1 - 504\sum_{k=1}^{\infty}\sigma_5(k)q^k\right\} \qquad \text{(Theorem 1.18)}$$

$$= \frac{8\pi^6}{27}\left\{1 - 504\sum_{k=1}^{\infty}\frac{k^5 q^k}{1 - q^k}\right\} \qquad ((c)).$$

□

Exercise 1.15.

Let

$$G(x) = \sum_{n=1}^{\infty}\frac{n^5 x^n}{1 - x^n},$$

and let

$$F(x) = \sum_{\substack{n=1 \\ (n\ odd)}}^{\infty}\frac{n^5 x^n}{1 + x^n}.$$

(a) *Prove that* $F(x) = G(x) - 34G(x^2) + 64(x^4).$

(b) *Prove that*

$$\sum_{\substack{n=1 \\ (n\ odd)}}^{\infty}\frac{n^5}{1 + e^{n\pi}} = \frac{31}{504}.$$

Proof of (a).

(1) Note that the identity

$$\sum_{n=1}^{\infty}\frac{x^n}{1 + x^n} = \sum_{n=1}^{\infty}\frac{x^n}{1 - x^n} - 2\sum_{n=1}^{\infty}\frac{x^{2n}}{1 - x^{2n}}.$$

(2) Let $H(x) = \sum_{n=1}^{\infty}\frac{n^5 x^n}{1 + x^n}$. Thus $H(x) = G(x) - 2G(x^2)$ by (1).

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(3) Note that

$$H(x) = \sum_{\substack{n=1 \\ (n\ \text{odd})}}^{\infty} \frac{n^5 x^n}{1 + x^n} + \sum_{\substack{n=1 \\ (n\ \text{even})}}^{\infty} \frac{n^5 x^n}{1 + x^n}$$

$$= F(x) + \sum_{n=1}^{\infty} \frac{(2n)^5 x^{2n}}{1 + x^{2n}}$$

$$= F(x) + 32 \sum_{n=1}^{\infty} \frac{n^5 x^{2n}}{1 + x^{2n}}$$

$$= F(x) + 32 H(x^2).$$

Hence by (2),

$$F(x) = H(x) - 32 H(x^2)$$
$$= [G(x) - 2G(x^2)] - 32[G(x^2) - 2G(x^4)]$$
$$= G(x) - 34 G(x^2) + 64 G(x^4).$$

□

Proof of (b).

(1) By Exercise 1.14(e), we have

$$g_3(\tau) = \frac{8\pi^6}{27} \left\{ 1 - 504 \sum_{k=1}^{\infty} \frac{k^5 q^k}{1 - q^k} \right\} = \frac{8\pi^6}{27}(1 - 504 G(q)).$$

where $q = x = e^{2\pi i \tau}$. Hence

$$g_3(\tau) - 34 g_3(2\tau) + 64 g_3(4\tau) = \frac{8\pi^6}{27}(31 - 504 F(q)).$$

(2) Take $\tau = \frac{i}{2}$. So

$$F(q) = F(e^{-\pi}) = \sum_{\substack{n=1 \\ (n\ \text{odd})}}^{\infty} \frac{n^5}{1 + e^{n\pi}}$$

$$g_3(\tau) = g_3\left(\frac{i}{2}\right) = (-4)^3 g_3(2i) = -64 g_3(2i) \qquad \text{(Exercise 1.12)}$$

$$g_3(2\tau) = g_3(i) = 0 \qquad\qquad\qquad\qquad\qquad \text{(Exercise 1.12)}$$

$$g_3(4\tau) = g_3(2i).$$

Hence

$$g_3(\tau) - 34 g_3(2\tau) + 64 g_3(4\tau) = -64 g_3(2i) - 0 + 64 g_3(2i) = 0$$

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and thus
$$\sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{n^5}{1 + e^{n\pi}} = \frac{31}{504}.$$

$\square$