E.2 Models of Sampled Systems

2.1 2.4
~~2.5~~ 2.7 2.9
M33

① key points from L2

* derive Zero-order-hold sampling of continuous system in preparation for L3. analysis (stability and observability)
* from continuous statespace (or discrete) model to IO discrete model
* poles and zeros of continuous system and its sampled discrete model relation, transformation
* pulse-transfer function
* step response
* with or without delay

② Extra notes: (follow-up of last time)

* remind the time-delay property

wait til 2.9.

$$Z\{y(k-1)\} = z^{-1}Y(z) + y(-1)$$

$$Z\{y(k-2)\} = z^{-2}Y(z) + y(-1)z^{-1} + y(-2)$$

we didn't use this
we one-sided Z-transform for [redacted] $k \geq 0$

* nyquist frequency, better use cut-off frequency or bandwidth

more accurate, with larger n (slow convergence)

* use taylor-expansion to approximate e^{Ah}

approximate e^{Ah}	exp(A), element-wise
	expm(A) ✓

$$e^{Ah} = I + Ah + \frac{1}{2!}A^2h^2 + \frac{1}{3!}A^3h^3 \cdots + \frac{1}{n!}A^nh^n$$

we have $\begin{bmatrix} \cosh & \sinh \\ -\sinh & \cosh \end{bmatrix}$, for $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $h=0.5$, ~~or 2~~

$h = 0.5$

$M\_true = \begin{bmatrix} 0.87 & -0.47 \\ 0.47 & 0.87 \end{bmatrix}$, converges $n = 3$

$h=2$, $n=9$, $h=0.1$, $n=3$

$h=10$, N.A.

~~h=2~~

~~doesn't work~~!!

works if nilpotent
i.e. $M^q = 0$
for some q

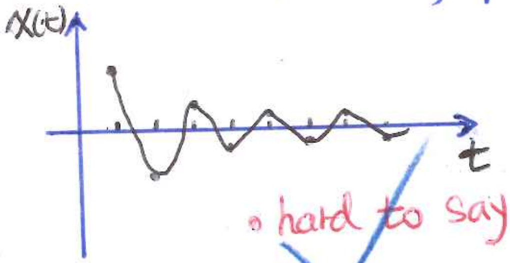
python.
linear algebra
linalg.expm

Exercise 2.1

① why do we want to solve this problem?

$$\boxed{PC} \xrightarrow{u_k} \boxed{ZoH} \xrightarrow{u} \boxed{sys} \longrightarrow y$$

* how to sample a continuous-time system
* more about asymptotic property, convergence, steady state



∘ hard to say about transit behaviors between the discrete time.

$\boxed{P_5 \text{ of } L_2}$

② extra notes

$$\dot{x} = Ax + Bu \qquad y = Cx + Du \quad \xrightarrow{\text{sampled}}$$

for $t \in [t_k, t_{k+1}]$

$$x(t) = e^{A(t-t_k)} x(t_k) + \int_{t_k}^{t} e^{A(t-\tau)} B u(\tau)\, d\tau$$

If $u(\tau)$ constant for $\tau \in [t_k, t]$

$$= e^{A(t-t_k)} x(t_k) + \int_{t_k}^{t} e^{A(t-\tau)} d\tau \cdot B u(t_k)$$

$$= e^{A(t-t_k)} x(t_k) + \int_{0}^{t-t_k} e^{As} ds \cdot B u(t_k)$$

time-varying system $\quad s = t-\tau \Rightarrow \tau = t - s$

dif $e^{A(t_{k+1}-t_k)}$ for dif. $|t_{k+1}-t_k|$

$$x_{t_{k+1}} = \Phi(t_{k+1}, t_k)\, x(t_k) + \Gamma(t_{k+1}, t_k)\, u(t_k)$$

$$y_{t_k} = C x_{t_k} + D u_{t_k}$$

$\boxed{\text{uniform, periodic sampling}}$

③ • given the sampling period as h

$$\dot{x} = -ax + bu \qquad y = cx \quad \Longrightarrow \quad x(t) = e^{-a(t-kh)} x(kh) + \left(\int_0^{t-kh} e^{-a\tau} d\tau\right) \cdot b\, u(kh)$$

why?

$$y(t) = c\,x(t), \quad t \in [kh, kh+h]$$

in

$$\Rightarrow x(kh+h) = e^{-ah} x(kh) + \left(\int_0^h e^{-a\tau} d\tau\right) b\, u(kh) \qquad -\frac{1}{a} e^{-a\tau}\Big|_0^h$$

$$= e^{-ah} x(kh) + \frac{b}{a}(1 - e^{-ah})\, u(kh) \qquad = -\frac{1}{a}(e^{-ah}) - \left(-\frac{1}{a}\right) = \frac{1}{a} \cdot (1 - e^{-ah})$$

$$y(kh) = c\,x(kh)$$

$\boxed{P_7 \text{ of } L_2}$

• Poles of the sampled system: e^{-ah} (p)

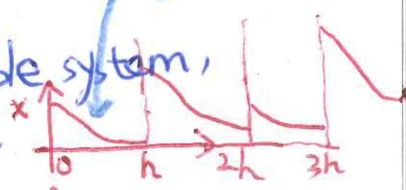
sampling frequency: $\frac{1}{h}$ (f)

h	f	p
$\to 0$	$\to \infty$	$\to 1$
$\to \infty$	$\to 0$	$\to 0$

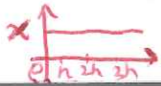
observation: for stable system,

traj of 0 input

long time / no time for u to effect: if $h \to \infty$,



no time for u to change: if $h \to 0$,



Exercise 2.4

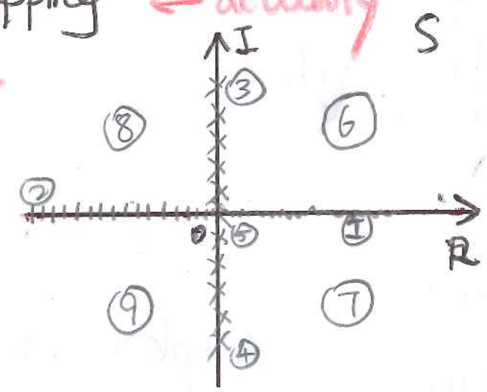
① Why do we want to solve this problem?

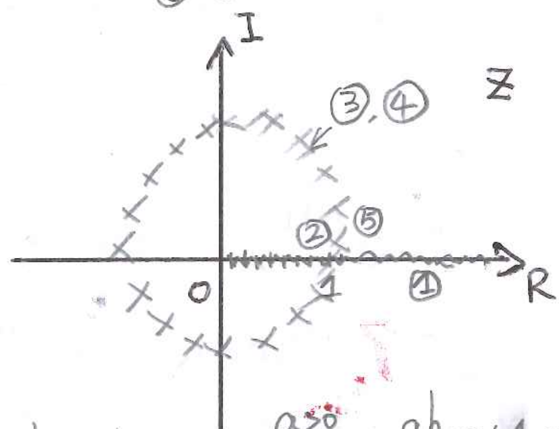
* from sampled continuous system
* pole mapping ← actually (looks like $C \to T$)
* ~~stability~~

$\boxed{P_{22} \text{ of } L_2}$

⑥, ⑦ outside unit circle

⑧ ⑨ inside unit circle





① real positive axis $\xrightarrow[b=0]{a>0}$ $e^{ah} \in (1, \infty)$

② real negative axis $\xrightarrow[b=0]{a<0}$ $e^{ah} \in (0, 1)$

③ negative Im axis $\xrightarrow[b>0]{a=0}$ unit circle

④ positive. same ↑ ⑤ origin $\xrightarrow[b=0]{a=0}$ 1

② extra notes

Location of poles of sampled system

P_{21} of L_2 $\boxed{A, \lambda_i \rightarrow e^{Ah}, e^{\lambda_i h}}$

since $\lambda_i \in$ s-domain, complex domain. $e^{\lambda_i h} \in$ Z-plane, complex domain

$\lambda = a + bj$

$\rightarrow e^{\lambda h} = e^{ah} \cdot e^{bhj}$

$= e^{ah}(\cos bh + \sin bh\, j)$

$= e^{ah}\cos bh + e^{ah}\sin bh \cdot j$

$\lambda = \rho \cdot e^{\theta j}$

$\cancel{\rightarrow e^{\lambda h} = e^{\rho h} \cdot e^{e^{\theta j}}}$

$\cancel{= e^{\rho h} e^{\cos\theta} [e^{\sin\theta j}]}$

③ (a) from Exercise 2.1,

$$\begin{aligned}\dot{x} &= -ax + bu \\ y &= cx\end{aligned} \implies \begin{aligned} x(kh+h) &= e^{-ah}x(kh) + \frac{b}{a}(1-e^{-ah})u(kh) \\ y(kh) &= c\,x(kh)\end{aligned}$$

By replacing $y(kh)$ with $x(kh)$ in (a)

$$\begin{cases} x(kh) = 0.5\,x(kh-h) + 6\,u(kh-h) \\ y(kh) = x(kh)\end{cases}$$

$$\implies \begin{aligned} e^{-ah} &= 0.5 \\ \frac{b}{a}(1-e^{-ah}) &= 6 \\ c &= 1\end{aligned} \implies \begin{aligned} a &= -\frac{\ln 2}{h} \\ b &= -12 \cdot \frac{\ln 2}{h} \\ c &= 1\end{aligned}$$

negative real axis !!!

For 1st or 2nd system

(b) prove e^{Ah} always has positive ~~poles~~ eigenvalues ← to do!

eigenvalues: $\det(\lambda I - A) = 0 \implies (\lambda+0.5)(\lambda+0.3) = 0$

$\implies \lambda = -0.5, -0.3$

(c) eigenvalues: $\det(\lambda I + 0.5) = 0 \implies \lambda = -0.5$

impossible to find

ume that system matrix is real, then

1° real poles, $\lambda_i = a$

2° complex conjugate, $\lambda_1 = a+bj$, $\lambda_2 = a-bj$

$$\Big\Downarrow e^{\lambda_i h}$$

1° e^{ah}, always >0

2° $e^{ah}\cos bh + e^{ah}\sin bh \cdot j$

$e^{ah}\cos bh - e^{ah}\sin bh \cdot j$, complex conjugate

$$\Big\Downarrow \text{conclusion}$$

if negative real parts exists (more than one),

they must be the same (at least)

then imaginary part be conjugate.

ew words on zeros, ~~and~~ poles

consider DTS:
$$\left.\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= C x(k) + D u(k) \end{aligned}\right\} \text{from E2.1}$$

or E2.7
ext time!

how poles and zeros will affect the stability of the DTS

poles: 1° eigenvalues of Φ. 2° roots of denominator $H(z)$

3° $e^{\lambda_i h}$, $\lambda_i = eig(A)$

zeros: $(zI - \Phi) X(z) = \Gamma U(z)$

$Y(z) = C X(z) + D U(z)$

$$\Rightarrow \begin{bmatrix} 0 \\ Y(z) \end{bmatrix} = \begin{bmatrix} zI-\Phi & -\Gamma \\ C & D \end{bmatrix} \begin{bmatrix} X(z) \\ U(z) \end{bmatrix}$$

$$\left\{ z \in C \;\middle|\; \det\left(\begin{bmatrix} zI-\Phi & -\Gamma \\ C & D \end{bmatrix} \right) = 0 \right\}$$ why?

(P19 of L2)
~20

Exercise 2.9

① why do we want to solve this problem?

- * step response of a discrete-time system
- * NOT just a repetition of E 1.5
- * pulse-transfer function (def. in P_{11} of L2)

② extra notes

* what if the difference equation is given equivalently as II

$$y(k+1) - 1.5\,y(k) + 0.5\,y(k-1) = u(k)$$

and the same initial condition

* result we got

$$y(k) = 0.5 \cdot 0.5^k u_0(k) + [(2k-2) + 0.5^{k-1}]\, u_0(k-1) \quad (k \geq 0)$$

But set $k=-1$, $y(-1) = 0 \neq -1$

Reason: * all analysis are performed using one-sided transform, $k \geq 0$

* we are only interested in the future response of $y(k)$ after excitation of $u(k)$

③ Set $k = -1$

$$\Rightarrow y(1) - 1.5\,y(0) + 0.5\,y(-1) = u(0)$$

$$\Rightarrow y(1) = 1.25$$

Same as E 1.5, check! then we are done

Solution to II:

z-transform both sides

$$zY(z) - z\underbrace{y(0)}_{0.5} - 1.5Y(z) + 0.5[z^{-1}Y(z) + \underbrace{y(-1)}_{-1}] = U(z)$$

$$\Rightarrow Y(z) = \frac{z}{z^2 - 1.5z + 0.5} U(z) + \frac{0.5z^2 - 0.5z}{z^2 - 1.5z + 0.5}$$

$$= \frac{z}{z^2 - 1.5z + 0.5} U(z) + \frac{0.5z}{z - 0.5}$$

Pulse transfer function

Same result! but easier to compute

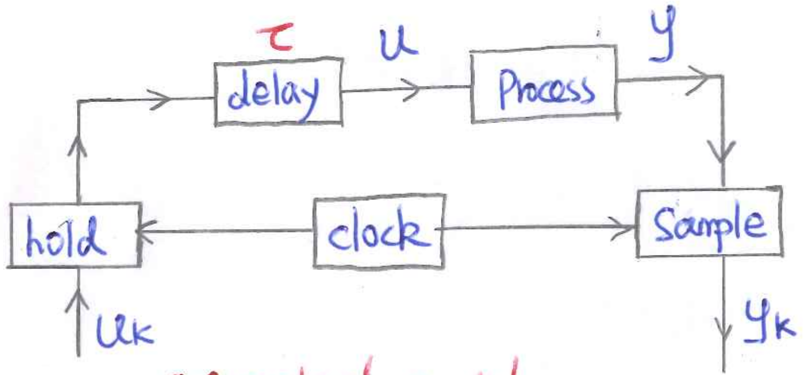
tips: few time delay or advance terms as possible

Exercise 2.7

① why do we want to solve this problem?

* how to derive statespace model from s-transfer function in CTS
* derive statespace model of sampled continuous system with time delay
* pulse transfer function and pulse response of the sampled system
* poles and zeros ✓

② extra notes



(conceptual model

③ $\boxed{\text{step 1:}}$ continuous transfer function → statespace model (not unique)

$$G(s) = \frac{1}{s+1} e^{-s\tau} = \frac{Y(s)}{U(s)}$$

- first. (w.l.g) set $y(t) = x(t)$
- second. $\frac{X(s)}{U(s)} = \frac{e^{-s\tau}}{s+1} \Rightarrow s X(s) + X(s) = e^{-s\tau} U(s)$

$$\xrightarrow{\mathcal{L}^{-1}} \dot{x}(t) + x(t) = u(t-\tau)$$

- third, state space

$$\dot{x}(t) = -x(t) + u(t-\tau)$$
$$y(t) = x(t)$$

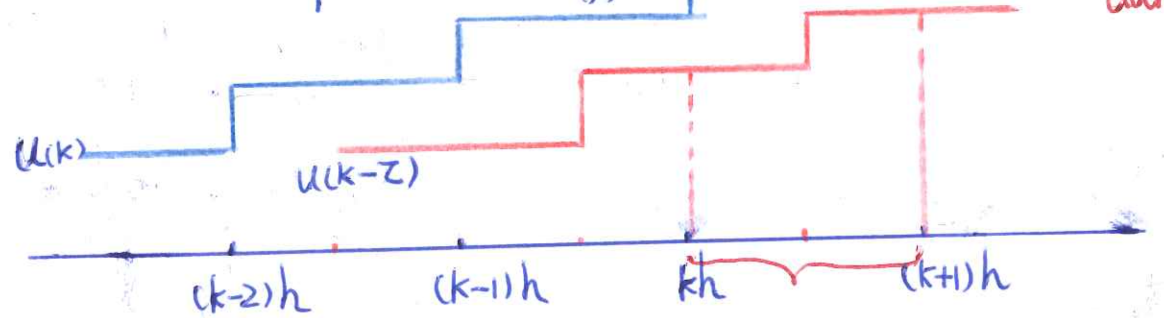
$\boxed{\text{step 2}}$: continuous statespace to discrete statespace (uniform sampling) with period h — sol. to (a)

① without delay, see E2.1

② with delay $0 < \tau < h$, see E2.6 and sol.

higher order → ③ with delay, $\tau > h$, here (if know how to solve ③ then similar for ②) in this case $\boxed{\tau = 1.5 \in (h, 2h)} = \frac{3}{2}h$ P17 of L2

what are the inputs that affect ~~the system states~~: during $[kh, (k+1)h]$



$u(k-\tau)$ is the delayed $u(k)$

Thus, for $t \in (kh, (k+1)h)$, $u(t) = \begin{cases} u((k-2)h), & t \in (kh, (k+\frac{1}{2})h) \\ u((k-1)h), & t \in ((k+\frac{1}{2})h, (k+1)h) \end{cases}$

Same as E 2.1

$$x((k+1)h) = e^{Ah} x(kh) + \int_{kh}^{(k+1)h} e^{A((k+1)h-s)} B u(s)\, ds.$$

$$= e^{Ah} x(kh) + \int_{kh}^{kh+\frac{1}{2}h} e^{A((k+1)h-s)} B\, u((k-2)h)\, ds$$

$$+ \int_{kh+\frac{1}{2}h}^{kh+h} e^{A((k+1)h-s)} B\, u((k-1)h)\, ds$$

$$\int_{kh}^{kh+\frac{1}{2}h} e^{A((k+1)h-s)}\, ds$$

replace $s = (k+1)h - \cancel{\delta}\rho$. (change of variable) integration by substitution

$$= \int_{\cancel{\delta}\rho=h}^{\cancel{\delta}=\frac{h}{2}} e^{A\rho} (-d\rho) = \int_{\rho=\frac{h}{2}}^{\rho=h} e^{A\rho} d\rho = \int_{\frac{h}{2}}^{h} e^{A\rho} d\rho$$

Thus $x((k+1)h) = e^{Ah} x(kh) + \left(\int_{\frac{h}{2}}^{h} e^{A\rho} d\rho\right) B\, u((k-2)h)$

$$+ \left(\int_0^{\frac{h}{2}} e^{A\rho} d\rho\right) B\, u((k-1)h)$$

replace A, B: $A = -1$, $B = 1$, $h = 1$

$$x(k+1) = e^{-1} x(k) + \left(\int_{\frac{1}{2}}^{1} e^{-\rho} d\rho\right) u(k-2)$$

$$+ \left(\int_0^{\frac{1}{2}} e^{-\rho} d\rho\right) u(k-1)$$

$$\int_a^b e^{-x} dx = -e^{-x}\Big|_a^b = e^{-x}\Big|_b^a$$

$$x(k+1) = e^{-1}x(k) + (e^{-1/2} - e^{-1})u(k-2) + (1-e^{-1/2})u(k-1) \quad (*)$$

$$y(k) = x(k)$$

In state-space format

$$\begin{bmatrix} x(k+1) \\ u(k-1) \\ u(k) \end{bmatrix} = \underbrace{\begin{bmatrix} e_a^{-1} & e_b^{-1/2} - e_a^{-1} & 1-e_b^{-1/2} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{=\Phi} \begin{bmatrix} x(k) \\ u(k-2) \\ u(k-1) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{=\Gamma} \underline{u(k)}$$

important

$$y(k) = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{=C} \begin{bmatrix} x(k) \\ u(k-2) \\ u(k-1) \end{bmatrix}$$

3rd order system

z^3

$\boxed{\text{Step 3:}}$ pulse-transfer function

$$H(z) = C(zI_3 - \Phi)^{-1}\Gamma$$

least favorite things to do by hand: invert

Computational intensive

- compute minors
- compute cofactor matrix
- determinant of 3x3
- product

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{e^{-1}-z} & \frac{e^{-1}-e^{-1/2}}{z(e^{-1}-z)} & \frac{e^{-1}-e^{-1/2}-z+e^{-1/2}z}{z^2(e^{-1}-z)} \\ 0 & \frac{1}{z} & \frac{1}{z^2} \\ 0 & 0 & \frac{1}{z} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Matlab
pzmap, pole, zero
impulse, ss2tf

numerator

$$= \frac{e^{-1}-e^{-1/2}-z+e^{-1/2}z}{z^2(e^{-1}-z)}$$

denominator

(Can be verified by (*) Z-transform of both side)

remove constant part

Pulse-response

$$h(k) = \mathcal{Z}^{-1}\left\{ \underbrace{z^{-2}}_{k-2} \left[\underbrace{\frac{e^{-1}-e^{-0.5}}{e^{-1}}}_{\delta[k]} + \frac{e^{-1.5}-e^{-0.5}}{e^{-1}} \cdot \underbrace{\frac{z}{e^{-1}-z}}_{(e^{-1})^k = e^{-k}} \right] \right\}$$

$\boxed{\text{Step 4:}}$ poles: eigenvalue of Φ: $0, 0, e^{-1}$

zeros: $\det\left(\begin{bmatrix} z-e^{-1} & e^{-1}-e^{-1/2} & e^{-1/2}-1 & 0 \\ 0 & z & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \right) = 0$, $z = \frac{e^{-1}-e^{-0.5}}{1-e^{-0.5}} = -0.6$

linear dependent