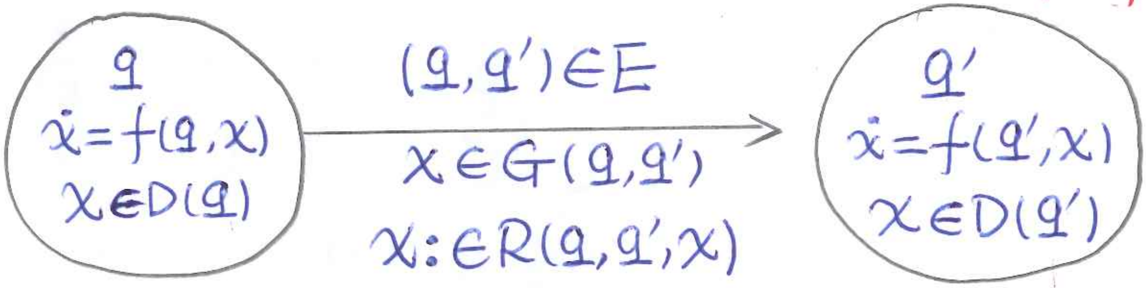
E10

10.2 ④	10.3 ②
10.6 ①	10.11 ③
M33	

* key points
  * mixed time and event-triggered dynamics lead to Hybrid dynamics
  * hybrid systems are a particular class of transition systems
  * hybrid automaton is a formal model of a hybrid system

① Formal definition

$$H = (Q, X, Init, f, D, E, G, R)$$ P9 of L10



- Q, discrete state space. X, continuous state space
- $Init \subseteq Q \times X$, initial states
- $f: Q \times X \to X$, vector fields
- $D: Q \longrightarrow 2^{X}$, domains
- $E \subset Q \times Q$, edges
- $G: E \longrightarrow 2^{X}$, guards
- $R: E \times X \to 2^{X}$, resets

② Solution of hybrid automaton P10 of L10

chi
$$\chi = (\tau, q, x)$$

time trajectory, time line on which the solution is defined

state trajectory, state evolution (both q and x)

③ Properties P20~29 of L10

liveness, determinism, Zenoness, stability, reachability

Ex 10.2

formulate/model the quantized control system as a hybrid automaton

① quantizer with 'k' bits $\Rightarrow$ level of quantizer 2^k

$= \{0, 1, \cdots 2^k - 1\} \cdot D$ (assume the sign not included) here

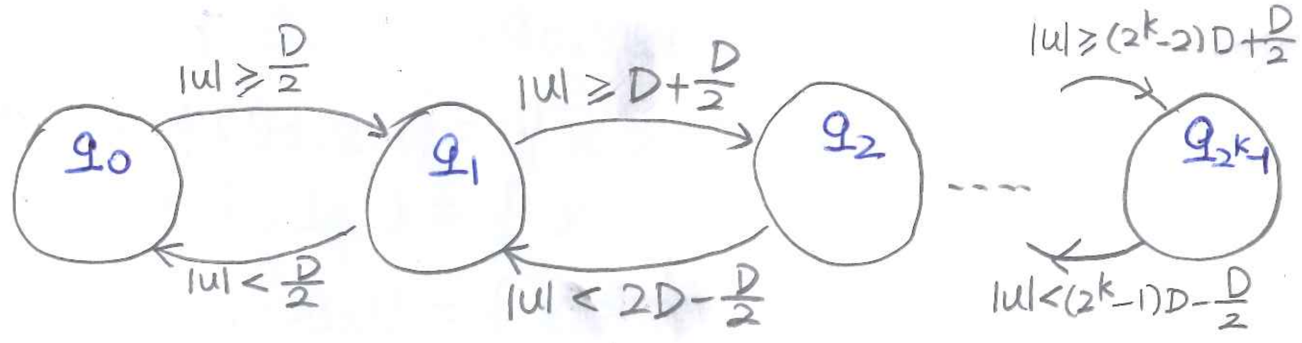
② when should discrete transitions take place?

○ Hybrid automaton

$$H = (Q, X, Init, f, D, E, G, R)$$

- $Q = \{q_0, q_1 \cdots q_{2^k-1}\}$
- $X \in X_{pc}$ (include u, v) $\mathbb{R}^{n+2}$?
- $Init = \{(q_i, Init(X_{pc}))\}$
- $D(q_i) = \{ \stackrel{\not\geq}{} i \cdot D - \frac{D}{2} \leq |u| < i \cdot D + \frac{D}{2} \}$
- $E = \{(q_i, q_{i+1}), \forall i = 0, \cdots 2^k - 2\} \cup \{(q_{i+1}, q_i), \forall i = 0, \cdots 2^k - 2\}$
- $G(q_i, q_{i+1}) = \{ \stackrel{\not\geq}{} |u| \geq (i \cancel{+1}) \cdot D + \frac{D}{2} \}$
- $G(q_{i+1}, q_i) = \{ \stackrel{\not\geq}{} |u| < (i+1) \cdot D - \frac{D}{2} \}$
- $R(E, x) = \{x\}$

diagram



Ex 10.3

- nuclear reactor
  - both rods outside the reactor: temperature increases by

$$\dot{x} = 0.1x - 50$$

initial $T = 510$ degree, until it reaches 550 degree

  - rods can be placed only if it has not been there for at least 20 seconds (only one) (in other words left and place rod takes 20s)
  - after placing rods

rod 1: $\dot{x} = 0.1x - 56$

rod 2: $\dot{x} = 0.1x - 60$

Model the system as a hybrid system

Def: $H = (Q, X, Init, f, D, E, G, R)$

- discre states, $Q = \{q_0, q_1, q_2\}$ none rod, rod 1, rod 2 in the reactor
- continuous state, $X = R + C_1 + C_2$ (time since rod 1, 2 are lifted out of reactor) $\subset R^3$ or $R \times R^+ \times R^+$
- initial state $= \{(q_0, (510, 20, 20))\}$
- Vector field $f(q_0, z) = \begin{bmatrix} 0.1x - 50 \\ 1 \\ 1 \end{bmatrix}$ or $\dot{x} = 0.1x - 50$, $\dot{c}_1 = 1$, $\dot{c}_2 = 1$

$z \in R^3$, $z = (x, c_1, c_2)$

$f = (0.1x - 50, 1, 1)^T$ similar for $f(q_1, z)$, $f(q_2, z)$

- domain, $D(q_0) = \{z = (x, c_1, c_2) \in R^3 \mid x < 550\}$, $D(q_1) = \{510 < x\} = D(q_2)$
- edge, $E = \{(q_0, q_1), (q_0, q_2), (q_1, q_0), (q_2, q_0)\}$
- guards, $G((q_0, q_1)) = \{z = (x, c_1, c_2) \in R^3 \mid x \geq 550, \underline{c_1 \geq 20}\}$

$G((q_1, q_0)) = \{x \leq 510\}$

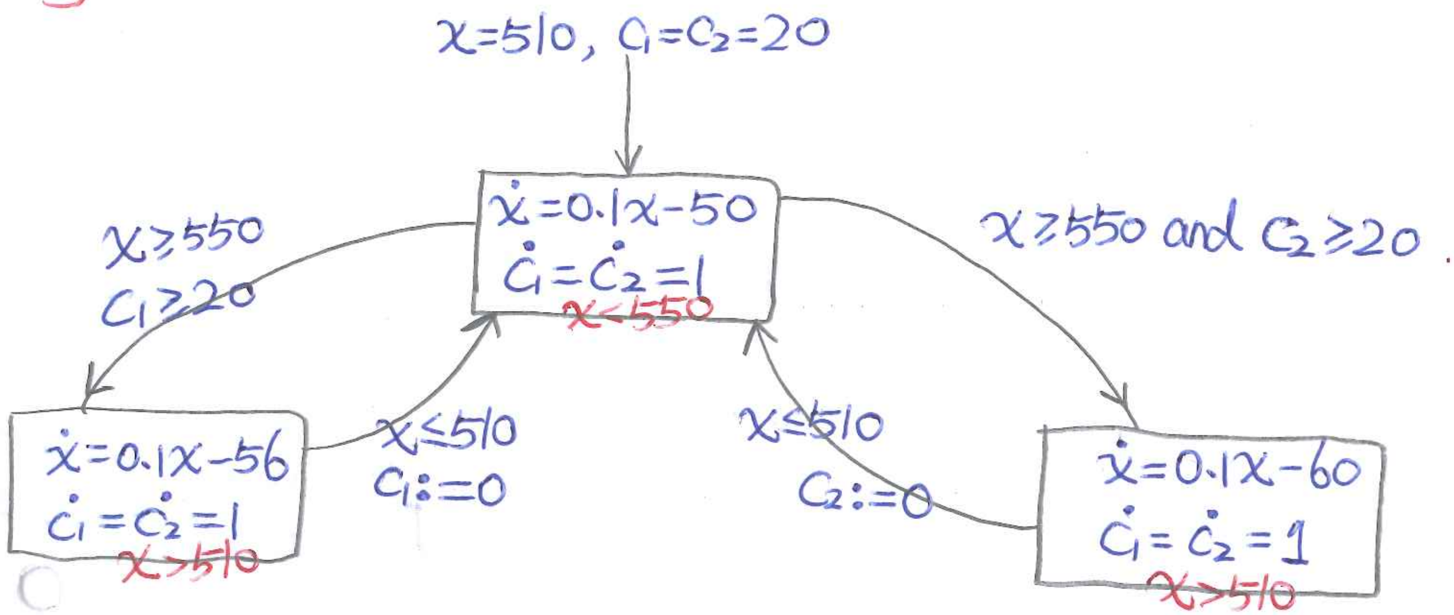
$G((q_0, q_2)) = \{x \geq 550, \underline{c_2 \geq 20}\}$

$G((q_2, q_0)) = \{x \leq 510\}$

- resets, $R((q_0, q_1), z) = \underline{\{z\}} = R((q_0, q_2), z)$

$R((q_1, q_0), z) = \underline{\{c_1 = 0\}}$, $R((q_2, q_0), z) = \underline{\{c_2 = 0\}}$

diagram



Ex 10.6

Given the hybrid system H in digram

(a) describe it by $H=(Q, X, Init, f, D, E, G, R)$

(b) find $D(q_3)$ so that H is live

(c) plot the solution

(a)
- discrete states, $Q=\{q_1, q_2, q_3\}$
- Continuous state, $X=\mathbb{R}$
- init, $(q_1, 0)$
- $f(q_1,x)=2$, $f(q_2,x)=-1$, $f(q_3,x)=x+2$
- $D(q_1)=\{x\in\mathbb{R} \mid x<5\}$, $D(q_2)=\{x\in\mathbb{R} \mid x>3\}$, $D(q_3)$ to be defined
- $E=\{(q_1,q_2),(q_2,q_3)\}$
- $G((q_1,q_2))=\{x\in\mathbb{R} \mid x\geq 5\}$
  $G((q_2,q_3))=\{x\in\mathbb{R} \mid x\leq 3\}$
- $R((q_1,q_2),x)=\{x\}$
  $R((q_2,q_3),x)=-2$

(b) liveness   P21-22 of L10

(live: there exists at least one (infinite) solution from (q_0,x_0) for all $(q_0,x_0)\in Init$)

clearly, $x=-2$ after transition from q_2 to q_3

$\dot{x}=-2+2=0$ implies $x=-2$ always

H is live if $\{x=-2\} \in D(q_3)$

(c) plot the solution of H

$\chi = (\tau, q, x)$

P11~12 of L10

time trajectory

$\tau = \{[0, 2.5), [2.5, 4.5), [4.5, +\infty)\}$ finite!

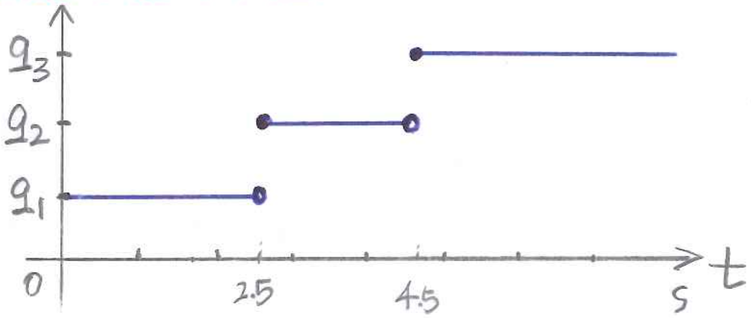
(a sequence of time intervals).

$[\tau_i, \tau_i')$

$\tau_i' = \tau_{i+1}$

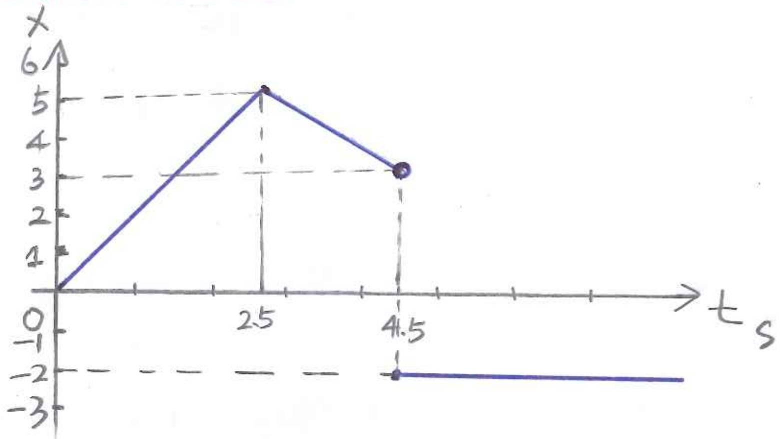
State trajectory

- discrete state



be clear also in HW 3

- Continous state



Ex 11.1

(a) model the three-ball system as a hybrid system

$H = (Q, X, \text{init}, f, D, E, G, R)$

- discrete states, $Q = \{q\}$ given
- Continuous state, $X = \mathbb{R}^3$; $x = (v_1, v_2, v_3)$

$\text{init} = (q, (1, 0, 0))$

$f(q, x) = (0, 0, 0)^T$

Domain, $D(q) = \{x \in \mathbb{R}^3 \mid v_1 \leq v_2 \leq v_3\}$

$E = \{(q, q)\}$

$G((q, q)) = \{x \in \mathbb{R}^3 \mid v_1 > v_2\} \cup \{x \in \mathbb{R}^3 \mid v_2 > v_3\}$

$$R((q,q),x) = \begin{cases} \{(\frac{v_1+v_2}{2}, \frac{v_1+v_2}{2}, v_3)\} & \text{if } v_1 > v_2 \\ \{(v_1, \frac{v_2+v_3}{2}, \frac{v_2+v_3}{2})\} & \text{if } v_2 > v_3 \end{cases}$$

(b) Def: Zeno if $\tau_\infty = \sum_{i=1}^{\infty} (\tau_i' - \tau_i) < \infty$ P25 of L10

Namely, an infinite number of discrete transitions over a finite time interval.

$(1,0,0) \to (\frac{1}{2}, \frac{1}{2}, 0) \to (\frac{1}{2}, \frac{1}{4}, \frac{1}{4}) \to (\frac{3}{8}, \frac{3}{8}, \frac{1}{4})$

...

which is an infinite sequence.

(c) at the k_{th} transition

$$v_1(\tau_k) - v_3(\tau_k) = \frac{1}{2^k}$$

$k \to \infty$, then $v_1(\tau_\infty) = v_2(\tau_\infty) = v_3(\tau_\infty)$

Since the for each transition, the sum remains the same

$\Rightarrow x_{\tau_\infty} = (1/3, 1/3, 1/3)$

Physical interpretation: after infinite number of hits, the three balls will have the same velocity (one third of the initial velocity)

momentum remains

energy decreased

Ex. 12 simulation and bisimulation

12.1 12.2
12.5
Q33

key points from Lecture

* simulation and bismulation
* quotient
* simulation relation verification, reachability

extra notes

transition system with initial and final states

Ex 12.1 Given T and T', find the bisimulation relation

Def. relation (P7 of L12)

→ relation R from A to B is a subset of A×B.

$\boxed{aRb \text{ if } (a,b)\in R \ ,\ a\in A, b\in B}$

Simulation relation (P8 of L12)

→ given T and T', a relation $\sim \in S\times S'$ is a simulation relation, if

- $\forall s\in S_0 \Rightarrow \exists s'\in S_0'$ that $s\sim s'$
- $s\sim s' \wedge s\in S_F \Rightarrow s'\in S_F'$
- $s\sim s' \wedge s \xrightarrow{\sigma} r \Rightarrow \exists r'\in S'$. s.t. $s' \xrightarrow{\sigma} r'$ and $r\sim r'$

Bisimulation relation (P9 of L12)

→ if $\sim$ is a simulation relation from T to T', and $\sim' = \{(s',s) \mid (s,s')\in\sim\}$ is a simulation from T' to T then $\sim$ is a bisimulation relation