E4 Computer realization of controllers | 3.11, 4.1, 4.3
4.5 Q3

* key points from L4
  - approximation of controller design (continuous - controller → discrete-time)
  - structure computer program to implement control algorithms
  - decide on reasonable sampling time

extra notes
  - 3.11 about constructing observers.

E3.11

*** why?**

design observers to estimate $x(k)$ from current and past y and u ?

- direct calculation
- full-state observer (dynamic observer)
- reduced-order observer P13-20, L3

*** extra notes**

* related to Dim of x (No. of unknown variables and linear equations)

(a) given DTS, $x(k+1) = \Phi x(k) + \Gamma u(k)$

$$y(k) = C\,x(k)$$

general formula, method P13-14, L3

step1. observability matrix. to $\boxed{\text{go back}}$ $(n-1)$ time steps

at time k

$$y(k) = C\,x(k) = C\,\Phi\,x(k-1) + C\,\Gamma\,u(k-1)$$

$$y(k-1) = C\,x(k-1)\cancel{= C\Phi x(k)}$$

to estimate $x(k-n+1)$

$$\begin{bmatrix} y(k-1) \\ y(k) \end{bmatrix} = \begin{bmatrix} C \\ C\Phi \end{bmatrix} x(k-1) + \begin{bmatrix} 0 \\ C\Gamma \end{bmatrix} u(k-1)$$

at time k, given $y(k-1), y(k), u(k-1)$,
if the obserability matrix $\begin{bmatrix} C \\ C\Phi \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0.22 & 1 \end{bmatrix}$ full rank (invertible)

$$\Rightarrow x(k-1) = \begin{bmatrix} C \\ C\Phi \end{bmatrix}^{-1} \begin{bmatrix} y(k-1) \\ y(k) \end{bmatrix} \cancel{*} - \begin{bmatrix} C \\ C\Phi \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ C\Gamma \end{bmatrix} u(k-1)$$

$$= \begin{bmatrix} -4.55 & 4.55 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y(k-1) \\ y(k) \end{bmatrix} - \begin{bmatrix} 0.1365 \\ 0 \end{bmatrix} u(k-1)$$

- it means $x(k-1)$ can be computed using $\begin{bmatrix} y(k-1) \\ y(k) \end{bmatrix}$ and $u(k-1)$

- step 2: $\boxed{\text{move forward}}$ in time to compute $x(k)$ from $x(k-n+1)$

$$x(k) = \Phi x(k-1) + \Gamma u(k-1)$$

$$= \Phi \left(\begin{bmatrix} C \\ C\Phi \end{bmatrix}^{-1} \begin{bmatrix} y(k-1) \\ y(k) \end{bmatrix} - \begin{bmatrix} C \\ C\Phi \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ C\Gamma \end{bmatrix} u(k-1) \right] + \Gamma u(k-1)$$

$$= \begin{bmatrix} -3.55 & 3.55 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y(k-1) \\ y(k) \end{bmatrix} + \begin{bmatrix} 0.114 \\ 0 \end{bmatrix} u(k-1)$$

- in general

- $x(k)$ is estimated as a linear combination of

$$x(k) = Q \underbrace{\begin{bmatrix} y(k-n+1) \\ \vdots \\ y(k) \end{bmatrix}}_{Y_k} + R \underbrace{\begin{bmatrix} u(k-n+1) \\ \vdots \\ u(k-1) \end{bmatrix}}_{U_{k-1}}$$

Do NOT need to go back to $x(0)$!

P_{14} of L_3: simple but sensitive to noise and disturbance

(b) the full state estimator (dynamic observer)

$\hat{x}(k+1|k)$: estimate of $x(k+1)$ based on All past measurements $y(0), y(1) \cdots y(k)$

conditional

dynamic observer

$$\hat{x}(k+1|k) = \underbrace{\Phi \hat{x}(k|k-1) + \Gamma u(k)}_{\text{exactly the original system}} + \underbrace{K}_{\text{estimation gain}} \underbrace{[y(k) - C\hat{x}(k|k-1)]}_{\text{based on measurement}}$$

- predication + correction style (
- kalman filter for LTS with gaussian noises.

reconstruction error

$$\tilde{x}(k+1|k) = x(k+1) - \hat{x}(k+1|k)$$

$$\Rightarrow \tilde{x}(k+1|k) = (\Phi - KC)\tilde{x}(k|k-1)$$

P16 of L3

Thus

$\tilde{x}(k+1|k) \to 0$ as $k \to \infty$, if $|\lambda_i(\Phi - KC)| < 1$

$$\text{set } K = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

$$\Rightarrow \Phi - KC = \begin{bmatrix} 0.78 & -k_1 \\ 0.22 & 1-k_2 \end{bmatrix}$$

$\Rightarrow$ characteristical polynominal $(\lambda - 0.78)(\lambda + k_2 - 1) + 0.22k_1 = 0$

$$\Rightarrow \lambda^2 + (k_2 - 1.78)\lambda + [0.22k_1 - 0.78(k_2 - 1)] = 0$$

E3.5 : Jury's stability criterion

$0.22k_1 - 0.78(k_2 - 1) < 1$ $\qquad k_1 + k_2 > 0$

$0.22k_1 - 0.78(k_2 - 1) > k_2 - 2.78$ $\qquad \Rightarrow k_1 - 3.54k_2 < 1$

$0.22k_1 - 0.78(k_2 - 1) > 0.78 - k_2$ $\qquad k_1 - 8.09k_2 > -16.18$

if set both poles to zero (very fast observer)

$$k_2 = 1.78 \quad k_1 = 2.76$$

(c) reduced-order observer P18-19, L3

* the dynamic observer has a unit delay from $y_{(k)}$ to $\hat{x}(x+1|k)$

- instead

$\hat{x}(k+1|k+1)$, the estimation of $x(k)$ based on All $y(0),\cdots y(k-1), y(k)$

=> reduced-order observer

$$\hat{x}(k+1|k+1) = \Phi\hat{x}(k|k) + \Gamma u(k) + k\left[y(k+1) - C\,\cancel{\hat{x}(k+1|k+1)}\right]$$

$$(\Phi\hat{x}(k|k) + \Gamma u(k))$$

- similarly, the reconstruction error

$$\tilde{x}(k\cancel{+1}|k\cancel{+1}) = -\hat{x}(k\cancel{+1}|k) + x(k)$$

$$\Rightarrow \quad \tilde{x}(k+1|k+1) = (I-kC)\Phi\tilde{x}(k|k)$$

- k should be chosen s.t. $|\lambda_i[(I-kC)\Phi]| < 1$

- $y(k+1) - C\hat{x}(k+1|k+1)$

$$= Cx(k+1) - C\hat{x}(k+1|k+1)$$

$$= C\tilde{x}(k+1|k+1)$$

$$= C(I-kC)\Phi\tilde{x}(k|k)$$

$$= (I - \underline{Ck})C\Phi\tilde{x}(k|k)$$

one output, then I_1

if $Ck = I_1$, $\Rightarrow y(k+1) = C\hat{x}(k+1|k+1)$ ↙ y is estimated with no error

Thus two conditions:

1° $Ck = I_1$ 1 $\Rightarrow k_2 = 1$

2° $|\lambda_i[(I-kC)\Phi]| < 1 \Rightarrow \begin{pmatrix} 0.78 - k_1\cdot 0.22 & -k_1 \\ 0 & 0 \end{pmatrix}$

poles at 0, chose $k_1 = \frac{0.78}{0.22} = 3.55$

E4.1

* why

- Pole placement for PI Controller
- discrete transfer function ⟶ state-space representation
- correspond to <u>design in discre-time</u> $P_{4,-5}$ of $L4$

continuous-time plant ⟶ sampled plant ⟶ design discrete-time controller

* extra notes

(a) the closed-loop system

$$\frac{H(z)}{1+H(z)H_c(z)} \quad \text{or} \quad \frac{H(z)H_c(z)}{1+H(z)H_c(z)}$$

poles: roots of numerator of $H(z)H_c(z)$

$$1+H(z)H_c(z)=\frac{\cancel{z-1}}{(z-0.5)(z-2)}\cdot\frac{(k+k_i)z-k}{\cancel{z-1}}+1$$

$$=\frac{(k+k_i)z-k+(z-0.5)(z-2)}{(z-0.5)(z-2)}$$

⟹ <u>characteristic polynomial</u> $z^2+(k+k_i-2.5)z+1-k=0$

- place both poles at origin

$$\begin{aligned} k+k_i-2.5&=0 \\ 1-k&=0 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} k&=1 \\ k_i&=1.5 \end{aligned}$$

(b) state-space representation of controller $H_c(z)$ (not unique)

$$H_c(z) = \frac{2.5z-1}{z-1}$$

= make sure this coefficient for z^n is 1

$\boxed{\text{Canonical state-space model}}$

Given $H(z) = \dfrac{b_n z^n + b_{n-1} z^{n-1} \cdots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0} = \dfrac{y(k)}{u(k)}$

then

$$x_1(k+1) = x_2(k)$$
$$x_2(k+1) = x_3(k)$$
$$\vdots$$
$$x_{n-1}(k+1) = x_n(k)$$
$$x_n(k+1) = -a_0 x_1(k) - a_1 x_2(k) \cdots - a_{n-1} x_n(k) + u(k)$$

$$x(k+1) = \begin{bmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [b_0 \ b_1 \cdots b_{n-1}]\, x(k)$$

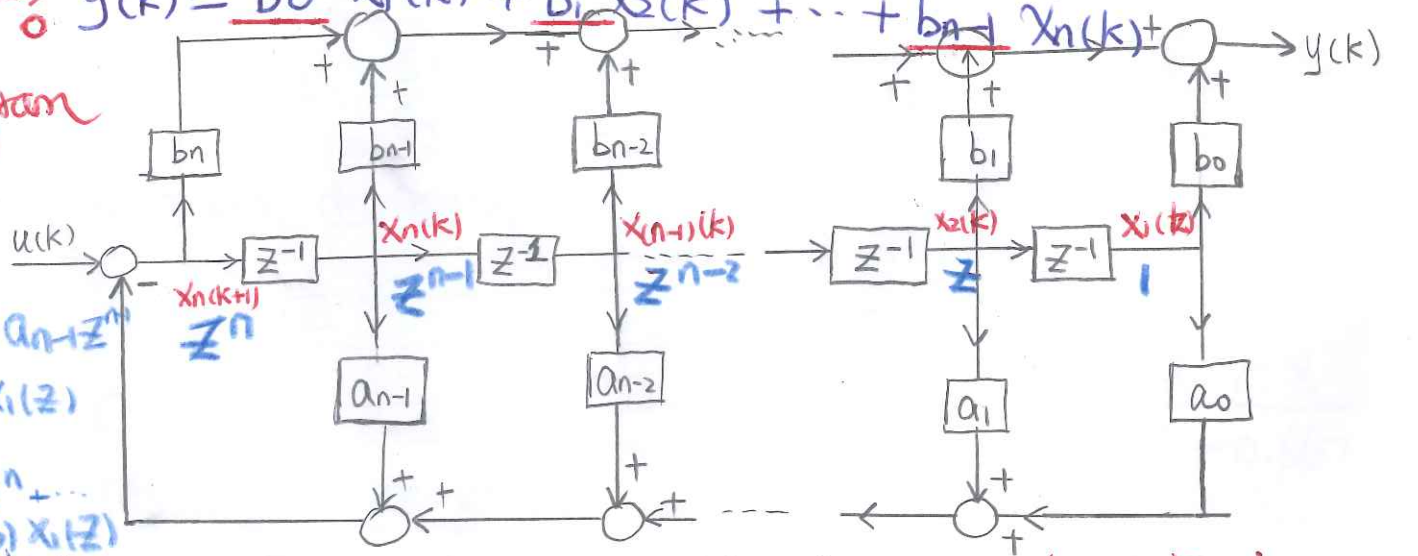
- $y(k) = (b_0 - b_n a_0) x_1(k) + (b_1 - b_n a_1) x_2(k) + \cdots + (b_{n-1} - b_n a_{n-1}) x_n(k) + \underline{b_n u(k)}$

(disappears if $b_n = 0$)

if $b_n = 0$

$\Rightarrow y(k) = b_0 x_1(k) + b_1 x_2(k) + \cdots + b_{n-1} x_n(k)$

Diagram



$U(z) = (z^n + a_{n-1}z^{n-1} \cdots + a_0) X_1(z)$

$Y(z) = (b_n z^n + \cdots + b_0) X_1(z)$

apply to our case:

$$x(k+1) = x(k) + u(k)$$
$$y(k) = 1.5x(k) + 2.5u(k)$$

$\left(\begin{array}{ll} b_1 = 2.5 & b_0 = -1 \\ a_0 = -1, & n = 1 \end{array} \right)$

E 4.3

* why?

- approximate continuous-time design into discrete time:
- approximate C(s) by a computer $C(s) \rightarrow C(z)$
- different approximation method

only approximation. NOT exact like e^{Ah}

P7-9, L4

extra notes

(a) Euler's method (forward different)

$$\boxed{\frac{dx}{dt} \approx \frac{x(t+h)-x(t)}{h} = \frac{q-1}{h}x(t)}$$

$$G(s) = 4 \cdot \frac{s+1}{s+2} \xrightarrow[h=0.25]{s=\frac{z-1}{h}} H(z) = 4 \cdot \frac{z-0.75}{z-0.5}$$

(b) Backward difference

$$\boxed{\frac{dx}{dt} \approx \frac{x(t)-x(t-h)}{h} = \frac{q-1}{qh}x(t)}$$

✓ common

$$G(s) = 4 \cdot \frac{s+1}{s+2} \xrightarrow[h=0.25]{s=\frac{z-1}{zh}} H(z) = 3.33\,\frac{z-0.8}{z-0.667}$$

(c) Tustin's approximation

$$\boxed{\frac{dx}{dt} \approx \frac{2}{h} \cdot \frac{q-1}{q+1}\, x(t)}$$

$$G(s) = 4\frac{s+1}{s+2} \xrightarrow[s=\frac{2}{h}\cdot\frac{z-1}{z+1}]{h=0.25} H(z) = 3.6\,\frac{z-0.778}{z-0.6}$$

(d) Tustin's approximation with pre-warping @ w_c

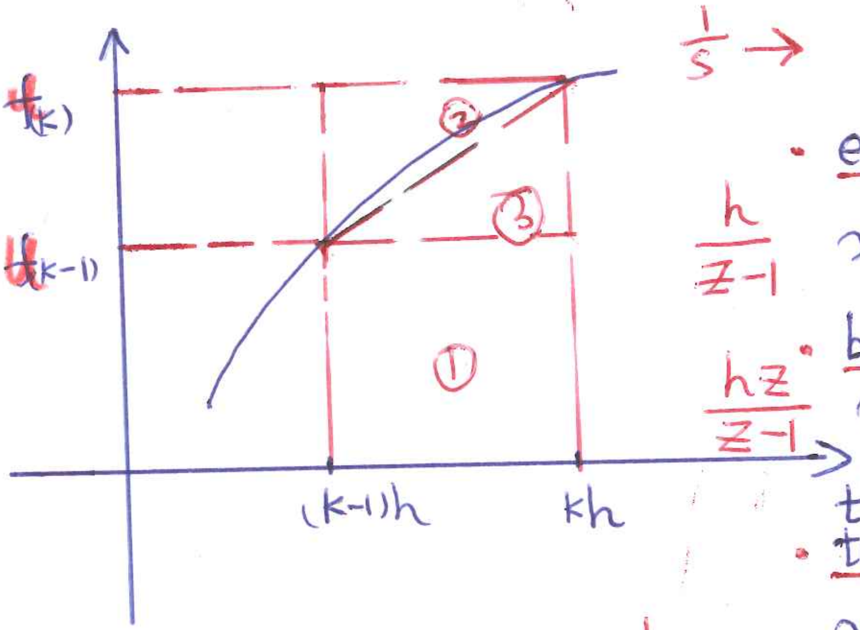
$$s = \frac{w}{\tan(w_c T_s/2)}\,\frac{z-1}{z+1}$$

ensures that the same frequency response at w_c

$$C(jw_c) = H(e^{jw_c T_s}) = \alpha\cdot\frac{z-1}{z+1}$$

$$\Rightarrow jw_c = \alpha\cdot\frac{e^{jw_c T_s}-1}{e^{jw_c T_s}+1} \Rightarrow \alpha = \frac{w_c}{\tan(w_c T_s/2)}$$

$$\Rightarrow G(s) = 4\cdot\frac{s+1}{s+2} \xrightarrow{s=\frac{1.6}{\tan(0.2)}\cdot\frac{z-1}{z+1}} H(z) = 3.596\cdot\frac{z-0.775}{z-0.6}$$



$\frac{1}{s} \rightarrow$

- euler's forward

$\frac{h}{z-1}$ $\quad x(k) \approx x(k-1) + h f(k-1)$

- backward

$\frac{hz}{z-1}$ $\quad x(k) \approx x(k-1) + h f(k)$

- tustin

$\frac{h}{2}\frac{z+1}{z-1}$ $\quad x(k) \approx x(k-1) + \frac{h}{2}[f(k) + f(k-1)]$

$$\frac{1-\cos x}{\sin x} = \frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \tan\frac{x}{2}$$

$$\frac{X(s)}{U(s)} = \frac{1}{s} \longrightarrow \Downarrow\ \frac{X(z)}{U(z)} =$$

trigonometric identities

$\sin(x\pm y) = \sin x\cos y \pm \cos x \sin y$, $\quad \cos x = 1 - 2\sin^2\frac{x}{2}$

$\cos(x\pm y) = \cos x \cos y \mp \sin x \sin y$, $\quad \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

E 4.5

* why?

- how to choose sampling frequency. P_{25-28}, L4

extra notes:

step response 10% ~ 90%

Rule of Thumb

① choose h such that there are 4~10 samples per rise time

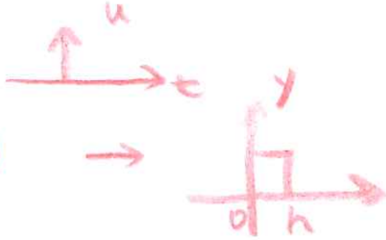
for second-order system

$G(s)=\frac{\omega_0^2}{s^2+2\zeta\omega_0 s+\omega_0^2}$ $10\omega_0 < \omega_c < 30\omega_0$ natural frequency

(much higher than shannon Sampling theorem)

② h that phase margin can be decreased by 5° ~ 15° by ZOH + Sample

- Laplace transform, transfer function of ZOH



$$ZOH(s) = \frac{1-e^{-sh}}{s} \quad (P_{12} \text{ of } L4)$$

P11

- Sampling contributes a gain $\frac{1}{h}$ (P_{11-12} of $L4$)

- thus ZOH + sampler gives

$$\frac{1-e^{-sh}}{sh}$$

$|G(j\omega_c)| = 1$

- frequency response at ω_c, cross-over frequency

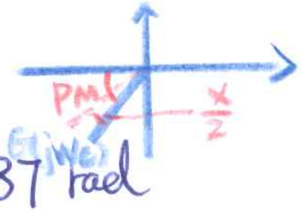
$$\frac{1-e^{-jh\omega_c}}{j\omega_c h} = \frac{\sin(\omega_c h)}{\omega_c h} - \frac{1-\cos\omega_c h}{\omega_c h} \cdot j$$

set $\omega_c h = x$

$$\frac{\sin x}{x} - \frac{1-\cos x}{x} j, \text{ angle } \arctan\left(-\frac{1-\cos x}{\sin x}\right)$$

$$= -\frac{x}{2}$$

$\tan\frac{x}{2}$



- $5° \to 0.087$ rad
- $15° \to 0.26$ rad $\longrightarrow 0.087 < \frac{x}{2} < 0.26$

$$\Rightarrow 0.17 < \omega_c h < 0.52$$

(in lecture, an anti-aliasing filter is added) different!