Problem 6.1.

(a) First, we need to find the closed-loop system (substituting $u = Kx$ into the state equation of the system $\dot{x} = Ax + Bu$):

$$\begin{cases} \dot{x} = x + u \\ u = -2x \end{cases} \xrightarrow{\text{closed-loop}} \dot{x} = x - 2x = -x.$$

In order to study the stability, one approach is apply Lyapunov Stability Theorem, which says that if there exists function $V : \mathbb{R}^n \to \mathbb{R}$ such that:

- $V(0) = 0$;
- $V(x) \geqslant 0$ for all $x \in \mathbb{R}^n$;
- $\dot{V}(x) < 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$;

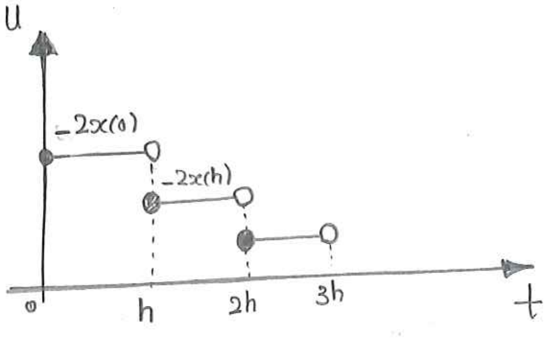
then the equilibrium point of the system at origin ($x = 0$) is globally asymptotically stable. We usually use quadratic function as a Lyapunov Candidate function. For example, in this problem, if we use $V(x) = x^2$ as a Lyapunov function, then

- $V(0) = 0^2 = 0$;
- $x^2 \geqslant 0, \ \forall x \in \mathbb{R}$;
- $\dot{V}(x) = 2x\dot{x} = 2x(-x) = -2x^2 < 0, \ \forall x \in \mathbb{R} \setminus \{0\}$.

It follows that the closed-loop system is globally asymptotically stable.

(b) Assume that $x(t)$ is sampled periodically and the controller is followed by a Zero Order Hold, i.e.,

$$u(t) = -2x(kh), \ t \in [kh, (k+1)h)$$



The closed-loop system is given by:

$$\dot{x}(t) = x(t) - 2x(kh), \quad t \in [kh, (k+1)h).$$

Thus, the solution of the system is

$$x(t) = x(kh) \cdot e^{t-kh} + \left(\int_0^{t-kh} e^{s} ds \right) (-2x(kh))$$

$$= x(kh) \cdot e^{t-kh} + \left(e^{t-kh} - 1 \right)(-2x(kh))$$

$$= \left(2 - e^{t-kh}\right) x(kh), \quad t \in [kh, (k+1)h). \qquad (1)$$

Note that we used the following equation to find the solution of the system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ x(t_0) = x_0 \end{cases} \longrightarrow x(t) = e^{A(t-t_0)} x_0 + \int_{t_0}^{t} e^{A(t-s)} Bu(s) ds.$$

From (1), when $t = (k+1)h$, we get

$$x((k+1)h) = \left(2 - e^{+h}\right) x(kh),$$

which implies that

$$x((k+1)h) = \left(2 - e^{h}\right)^2 x((k-1)h) \dots = \left(2 - e^{h}\right)^{k+1} x(0).$$

Thus, $\lim_{k \to \infty} x(kh) = 0$ if $|2 - e^h| < 1$. Therefore, if

$$-1 < 2 - e^h < 1$$

$$\longleftrightarrow \quad 1 < e^h < 3$$

$$\longleftrightarrow \quad 0 < h < \ln 3 \simeq 1.1$$

then, the closed-loop system is stable.

c Let $e(t) = x(t_k) - x(t)$, $t \in [t_k, t_{k+1})$. Since $u(t) = -2x(t_k)$, we have $u(t) = -2(e(t) + x(t))$ and the closed-loop system is given by

$$\dot{x}(t) = x(t) - 2(e(t) + x(t)) = -x(t) - 2e(t). \qquad (2)$$

In order to study the stability, we consider the candidate Lyapunov function $V(x) = x^2$. The time derivative of $V(x)$ along (2) is

$$\begin{aligned} \dot{V} = 2x\dot{x} = 2x(-x-2e) &= -2x^2 - 4xe \quad \Big) \; x^2 = |x|^2 \\ &= -2|x|^2 - 4xe \quad \Big) \; ab \leqslant |a| \cdot |b| \\ &\leqslant -2|x|^2 + 4|x||e| \\ &= -2|x|(|x| - 2|e|). \end{aligned}$$

Therefore, if $|e| < \frac{1}{2}|x|$ for all $t \geqslant 0$, then $\dot{V} < 0$ and, hence, the closed-loop system is stable.

Stability Condition: $|e(t)| < \frac{1}{2}|x(t)|$, $\forall t \geqslant 0$.

Now, we find the solution of the closed-loop system. For $t \in [t_k, t_{k+1})$, we have

$$\begin{aligned} x(t) &= e^{t-t_k} x(t_k) + \left(\int_0^{t-t_k} e^s \, ds \right) u(t_k) \\ &= e^{t-t_k} x(t_k) + (e^{t-t_k} - 1) u(t_k) \\ &= e^{t-t_k} x(t_k) + (e^{t-t_k} - 1)(-2x(t_k)) \\ &= (2 - e^{t-t_k}) x(t_k). \end{aligned}$$

Thus, $e(t)$ for $t \in [t_k, t_{k+1})$ is given by

$$\begin{aligned} e(t) = x(t_k) - x(t) &= \frac{1}{2 - e^{t-t_k}} x(t) - x(t) \\ &= \frac{e^{t-t_k} - 1}{2 - e^{t-t_k}} x(t). \end{aligned}$$

To enforce $|e(t)| < \frac{1}{2}|x(t)|$, we obtain

$$\left| \frac{e^{t-t_k} - 1}{2 - e^{t-t_k}} \right| < \frac{1}{2}$$

$$\Longleftrightarrow e^{t-t_k} - 2 < 2e^{t-t_k} - 2 < 2 - e^{t-t_k}$$

$$\Longleftrightarrow \quad 0 < e^{t-t_k} < \frac{4}{3}$$

$$\Longleftrightarrow \quad t - t_k < \ln \frac{4}{3} \quad .$$

Therefore, $t_{k+1} - t_k < \ln \frac{4}{3}$.

(d) Event-based controller:

(1) Monitor: Keep monitoring $x(t)$ and check if

$$|e(t)| < \frac{1}{2}|x(t)|$$

holds, where $e(t) = x(t) - x(t_k)$. If $|e(t)| = \frac{1}{2}|x(t)|$, go to "Sample" step.

(2) Sample: Sample $x(t)$ to obtain $x(t_k)$. Save it and go to "Update" step.

(3) Update: Set $u(t) = -2x(t_k)$, go back to "Monitor" step.

(4) ZOH: Hold u.

Periodic Controller:

(1) Monitor: Keep monitoring t and check if $t = kh$ holds. If so, go to "Sample" step.

(2) Sample : sample $x(t)$ to obtain $x(kh)$, go to "Update" step.

(3) Update : Set $u(t) = -2x(kh)$, go back to "Monitor".

(4) ZOH : Hold u.

Problem 6.2 :

(a) First, we find the closed-loop system :

$$\begin{cases} \dot{x} = Ax + Bu \\ u = Kx \end{cases} \longrightarrow \dot{x} = (A + BK)x = \underbrace{\begin{bmatrix} -1 & 0 \\ -2 & -3 \end{bmatrix}}_{A_{cl}} x$$

Note that $\dot{x} = A_{cl}x$ is stable if and only if all eigenvalues of A_{cl} have negative real parts. Thus, we calculate the eigenvalues of A_{cl} :

$$\det|sI - A_{cl}| = 0 \longleftrightarrow \begin{vmatrix} s+1 & 0 \\ 2 & s+3 \end{vmatrix} = 0$$

$$\longleftrightarrow (s+1)(s+3) = 0$$

$$\longleftrightarrow \begin{cases} s = -1, \\ s = -3. \end{cases}$$

This shows that the closed-loop system is stable.

b) As $e(t) = x(t_k) - x(t)$ and $u(t) = Kx(t_k)$ for $t \in [t_k, t_{k+1})$, we have

$$\begin{aligned} \dot{x}(t) &= Ax(t) + BK\,x(t_k) \\ &= Ax(t) + BK(e(t) + x(t)) \\ &= (A + BK)x(t) + BK\,e(t) \end{aligned}$$

$$= \begin{bmatrix} -1 & 0 \\ -2 & -3 \end{bmatrix} x(t) + \begin{bmatrix} -3 & -1 \\ -3 & -1 \end{bmatrix} e(t). \qquad (3)$$

(c) In order to study the stability, we consider the candidate Lyapunov function $V(x) = x^T \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} x$. The time-derivative of $V(x)$ along (3) is given by

$$\dot{V} = 3x_1 \dot{x}_1 + x_2 \cdot \dot{x}_2$$

$$= -3x_1^2 - 2x_1x_2 - 3x_2^2 - 9x_1e_1 - 3x_1e_2 - 3x_2e_1 - x_2e_2$$

$$= -(x_1 + x_2)^2 - 2(x_1^2 + x_2^2) - 9x_1e_1 - 3x_1e_2 - 3x_2e_1 - x_2e_2$$

$$\leq -2(x_1^2 + x_2^2) + 9|x_1||e_1| + 3|x_1||e_2| + 3|x_2||e_1| + |x_2||e_2|$$

$$= -2\|x\|_2^2 + 16\|x\|_2\|e\|_2$$

$$= -2\|x\|_2 \left(\|x\|_2 - 8\|e\|_2 \right). \qquad (4)$$

Note that we used the fact that

$$\begin{cases} \|x\|_2^2 = |x_1|^2 + |x_2|^2 \\ |x_1|, |x_2| \leq \|x\|_2 \end{cases}$$

to get the third equality. From (4), it follows that if

$$\|e\| \leq \frac{1}{8}\|x\|,$$

then $\dot{V} < 0$ and, hence, the closed-loop system converges to the origin.