E 3 analysis of sampled system

3.5 3.9
$\boxed{3.12}$ $\boxed{3.15}$
Q34

* keys points from L3
  * determine stability, reachability, observability (DTS)
  * design state observers
  * sampled-data control, based on state or output
  * design deadbeat controller

* Extra notes
  * finish up poles and zeros of E2.7. (don't forget the matlab commands)
  * why not the same approach for CTS ?
    why we need a new set of tools for Sampled system ?

E 3.5

① why?

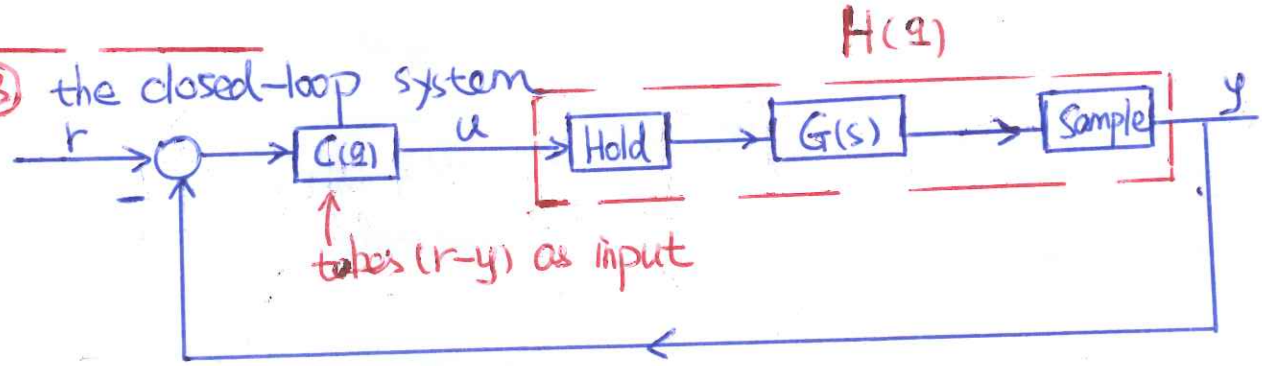
* analyze stability and asympototic convergence
  - stability → poles
  - asympototic convergence → final value theorem + output signal

*

② extra notes

* if it's possible, print the problems and bring to the classroom (they are too long to write down)

③ the closed-loop system



Step 1: derive the closed-loop transfer function

$$H(q) = \frac{1}{q(q-0.5)}$$

forward-shift operator: $q\,x(k) = x(k+1)$ (P_{10} of L_2)

simply, $H(z) = \dfrac{1}{z(z-0.5)}$ (one in time domain q, one in frequency domain z) $C(z) = K$

$$H_{cl}(z) = \frac{C(z)H(z)}{1 + C(z)H(z)}$$

negative feedback

($(r-y)HC = y$ prove)

$$\Rightarrow H_d(Z) = \frac{K}{Z^2-0.5Z+K}$$

step 2: stability

def: BIBO. ($\angle 3$. P4)

2nd: all poles are within the unit ~~cycle~~ cir cle

$\Rightarrow$ $\boxed{\text{roots}}$ of ~~$H_d(Z)$~~ $Z^2-0.5Z+K=0$ characteristic Polynomial

> Given a second order polynomial
> $$Z^2+a_1Z+a_2=0$$
> all roots are inside unit circle if
> $$a_2<1$$
> $$a_2>-1+a_1$$
> $$a_2>-1-a_1$$
> Jury's Stability Criterion

apply this

$K<1$

$K>-1.5$ since $K>0 \Rightarrow 0<K<1$

$K>-0.5$

step 3: steady state response (stationary value)

① in time-domain

unitary step $U(Z)=\frac{Z}{Z-1}$

$$Y(Z) = \frac{K}{Z^2-0.5Z+K}U(Z) = \frac{K}{Z^2-0.5Z+K}\cdot\frac{Z}{Z-1}$$

$\to$ Z-inverse transform $\to y(k) \to k\to\infty \to y(\infty)$ Stationary value

② final value theorem

> Given $X(Z)=Z\{x(k)\}$, then
> $$x(k)\big|_{k\to\infty} = \lim_{Z\to1}(Z-1)X(Z)$$

apply this

$$y(k)\big|_{k\to\infty} = \frac{K}{Z^2-0.5Z+K}\cdot\frac{Z}{\cancel{Z-1}}\cdot\cancel{Z-1}\Big|_{Z\to1}$$

$$= \frac{K}{K+0.5} < 1$$ (Static error always exist. no mater K)

E 3.9

* study
  * state-feedback control
  * derive closed-loop system,
  * $\boxed{\text{deadbeat controller}}$ design
  * $\boxed{\text{pole placement}}$

*extra notes

1° demension $x(k): 2\times 1$, $u(k): 1\times 1$, $L: 1\times 2$

assume $L = [\,l_1 \quad l_2\,]$

2° $x(k+1) = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{\triangleq A = \Phi} x(k) + \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{\triangleq B = \Gamma} u(k)$

$y(k) = \underbrace{[\,c_1 \quad c_2\,]}_{\triangleq C} x(k)$,

$u(k) = \underbrace{[-l_1 \quad -l_2]}_{= L} x(k)$

$\Rightarrow x(k+1) = \underline{(A - BL)}\, x(k)$

$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} b_1 l_1 & b_1 l_2 \\ b_2 l_1 & b_2 l_2 \end{bmatrix} = \begin{bmatrix} a_{11} - b_1 l_1 & a_{12} - b_1 l_2 \\ a_{21} - b_2 l_1 & a_{22} - b_2 l_2 \end{bmatrix}$

3° characteristic equation

$$\det[zI_2-(A-BL)]=0$$

$$\Rightarrow [z-(a_{11}-b_1l_1)][z-(a_{22}-b_2l_2)]-(a_{12}-b_1l_2)(a_{21}-b_2l_1)=0$$

$$\Rightarrow z^2-[(a_{11}-b_1l_1)+(a_{22}-b_2l_2)]z+(a_{11}-b_1l_1)(a_{22}-b_2l_2)-(a_{12}-b_1l_2)(a_{21}-b_2l_1)=0$$

should be the same as

$$z^2+p_1z+p_2=0$$ — p_1, p_2 determined by poles desired

$$\Rightarrow -p_1 = a_{11}+a_{22}-(b_1 \overset{l_1}{} + b_2 \overset{l_2}{})$$

$$p_2=(a_{11}-b_1l_1)(a_{22}-b_2l_2)-(a_{12}-b_1l_2)(a_{21}-b_2l_1)$$

two unknown variable, two equation

$$\Rightarrow \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \frac{1}{a_{21}b_1^2-a_{12}b_2^2+b_1b_2(a_{22}-a_{11})} \cdot \begin{pmatrix} a_{21}b_1-a_{11}b_2 & -b_2 \\ -a_{12}b_2+a_{22}b_1 & b_1 \end{pmatrix} \cdot \begin{pmatrix} p_1-(a_{11}+a_{22}) \\ p_2-(a_{11}a_{22}-a_{12}a_{21}) \end{pmatrix}$$

norm (determinant) of controllability matrix (B AB)

4° deadbeat controller (P28 of L3)

$\boxed{\text{Put all poles at the origin}}$

then the origin can be reached in at most n steps

set $p_1=p_2=0$, we get $\begin{pmatrix} l_1 \\ l_2 \end{pmatrix}$ (proof see slide)

E 3.12

* why

* disturbance attenuation by feedback control

* obserability

* extra notes

Step 1. rewrite the system dynamics

$$x(k+1) = \underbrace{\begin{bmatrix} 0.5 & 1 \\ 0.5 & 0.7 \end{bmatrix}}_{=\Phi} x(k) + \underbrace{\begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}}_{=\Gamma} u(k) + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{=\Phi_{xw}} v(k)$$

$$y(k) = [1 \quad 0]\, x(k)$$

- since $v(k)$ is a constant, can be modeled by the system

$$w(k+1) = w(k)$$

$$v(k) = w(k)$$

- replace $v(k)$ by $w(k)$

Step 2. (a) if both state and v can be measured

$$u(k) = -L\,x(k) - L_w\,w(k)$$

$\Rightarrow$ closed-loop system

$$x(k+1) = \Phi\,x(k) + \Phi_{xw}\,w(k) - \Gamma L\,x(k) - \Gamma L_w\,w(k)$$

$$y(k) = C\,x(k)$$

$$\Rightarrow \quad X(z) = [zI_2 - (\Phi - \Gamma L)]^{-1} (\Phi_{xw} - \Gamma L_w) W(z)$$

$$Y(z) = C\, X(z)$$

$$\Rightarrow \quad Y(z) = C[zI_2 - (\Phi - \Gamma L)]^{-1} (\Phi_{xw} - \Gamma L_w) W(z)$$

Final-value theorem

$$y(k)|_{k\to\infty} = \lim_{z\to 1} (z-1) Y(z) = C[I_2 - (\Phi - \Gamma L)]^{-1} (\Phi_{xw} - \Gamma L_w) W(z)(z-1)$$

$$= C[I_2 - (\Phi - \Gamma L)]^{-1} (\Phi_{xw} - \Gamma L_w) \lim_{z\to 1} W(z)(z-1)$$

$$= C[I_2 - (\Phi - \Gamma L)]^{-1} (\Phi_{xw} - \Gamma L_w)\, w(k)|_{k\to\infty}$$

To eliminate the influence of w, design L_w such that $C[I_2 - (\Phi - \Gamma L)]^{-1} (\Phi_{xw} - \Gamma L_w) = 0$ (given L. $\underbrace{[I_2 - (\Phi - \Gamma L)]^{-1}}_{2\times 2}$ $\underbrace{(\Phi_{xw} - \Gamma L_w)}_{2\times 1}$

(b) only state can be measured, not the disturbance

however since we know it's constant

$$\hat{w}(k) = [1 \quad 0] \left(x(k) - \Phi x(k-1) - \Gamma u(k-1) \right) \quad w(k-1)$$

The feedback control law

$$u(k) = -L x(k) - L_w \hat{w}(k)$$

where L and L_w are the same as (a)

(c) only the output can be measured.

estimator of $x(k), w(k)$ has to be constructed

observer

denote $\tilde{x}(k) = \underbrace{\hat{x}(k)}_{\text{estimation}} - x(k)$, $\tilde{w}(k) = \underbrace{\hat{w}(k)}_{\text{estimation}} - w(k)$

$$\underbrace{\begin{pmatrix} \hat{x}(k+1) \\ \hat{w}(k+1) \end{pmatrix}}_{\hat{S}_{k+1}} = \underbrace{\begin{pmatrix} \Phi & \Phi_{xw} \\ 0 & 1 \end{pmatrix}}_{\triangleq \Phi_n} \underbrace{\begin{pmatrix} \hat{x}(k) \\ \hat{w}(k) \end{pmatrix}}_{\hat{S}_k} + \underbrace{\begin{pmatrix} \Gamma \\ 0 \end{pmatrix}}_{\triangleq \Gamma_n} u(k) + \underbrace{\begin{pmatrix} K \\ K_w \end{pmatrix}}_{\triangleq K_n} \varepsilon(k)$$

$$\varepsilon(k) = y(k) - \underbrace{(C \quad 0)}_{\triangleq C_n} \begin{pmatrix} \hat{x}(k) \\ \hat{w}(k) \end{pmatrix}$$

dynamic observer

$\Rightarrow$ k, k_w need to determined such that $\tilde{x}(k), \tilde{\omega}(k) \to 0$

$$\begin{pmatrix} \tilde{x}(k+1) \\ \tilde{\omega}(k+1) \end{pmatrix} = \begin{pmatrix} \Phi - kC & \Phi_{xw} \\ -k_w C & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}(k) \\ \tilde{\omega}(k) \end{pmatrix}$$

how

$$\Phi_n - k_n C_n = \begin{bmatrix} kC & 0 \\ k_w C & 0 \end{bmatrix}$$

make sure all eigenvalues of < 1

$k = (k_1 \ \ k_2)$, k_w

$$\Rightarrow z^3 + (k_1 - 2.2)z^2 + (1.05 - 1.7k_1 + k_2 + k_w) z + 0.7k_1 + 0.15 - 0.7 k_w - k_2 = 0$$

Set $\lambda_{1,2,3} = 0$

$\Rightarrow$ $k_1 = 2.2$, $k_2 = -0.64$, $k_w = 3.33$

then $u(k) = -L\hat{x}(k) - L_w \hat{\omega}(k)$, L, L_w are the same as (a)

$$S_{k+1} = \Phi_n S_k + \Gamma_n u_k, \quad y_k = C_n S_k \quad \cdots ①$$

$$\hat{S}_{k+1} = \Phi_n \hat{S}_k + \Gamma_n u_k + k_n [y_k - C_n \hat{S}_k] \quad \cdots ②$$

the estimation error

$$\check{S}_{k+1} = \hat{S}_{k+1} - S_{k+1}$$

$$= \Phi(\hat{S}_k - S_k) + k_n [C_n S_k - C_n \hat{S}_k]$$

$$= (\Phi - k_n C_n)(\hat{S}_k - S_k)$$

E 3.15

* why

* derive sampled system

* controller design by pole placement

* stability analysis by Lyapunov arguments

* extra notes

(a) Sampled system (see E2.1)

$$\dot{x} = Ax + Bu, \quad y = Cx \xrightarrow[h]{\text{uniform}} x(kh+h) = \underbrace{e^{Ah}}_{\Phi} x(kh) + \underbrace{\int_0^h e^{As}ds \cdot B}_{\tau} \cdot u(kh)$$

$$y(kh+h) = Cx(kh)$$

Thus $e^{Ah} = \begin{pmatrix} e^{-1} & 0 \\ 0 & e^{-2} \end{pmatrix} = \Phi$

$$\int_0^h e^{As}ds = \begin{pmatrix} \int_0^1 e^{-s}ds & 0 \\ 0 & \int_0^1 e^{-2s}ds \end{pmatrix} \Rightarrow \begin{pmatrix} 1-e^{-1} \\ \frac{1-e^{-2}}{2} \end{pmatrix} = T$$

(b) assume the Controller has the form

$$u(k) = [\,l_1 \quad l_2\,]\; x(k)$$

$$\Rightarrow x(k+1) = (\Phi - \Gamma L)\, x(k) \qquad \text{closed-loop}$$

where

$$\Phi - \Gamma L = \begin{pmatrix} e^{-1} - (1-e^{-1}) l_1 & -(1-e^{-1}) l_2 \\ -\frac{1-e^{-2}}{2} l_1 & e^{-2} - \frac{1-e^{-2}}{2} l_2 \end{pmatrix}$$

we want to put poles at 0.1, 0.2 asymptotically stable → back to E 3.9

since

$$\mathrm{trace}(M) = \sum_{i=1}^{n} \lambda_i, \quad \det(M) = \prod_{i=1}^{n} \lambda_i$$

$$0.1 + 0.2 = e^{-1} - (1-e^{-1}) l_1 + e^{-2} - \frac{1-e^{-2}}{2} l_2$$

$$0.02 = [e^{-1} - (1-e^{-1}) l_1][e^{-2} - \frac{1-e^{-2}}{2} l_2] - (1-e^{-1}) l_2 \cdot \frac{1-e^{-2}}{2} l_1$$

$$\Rightarrow l_1 \approx 0.3, \quad l_2 \approx 0.02$$

(c) $$\Phi_c = \Phi - \Gamma L = \begin{bmatrix} 0.17 & -0.014 \\ -0.13 & 0.12 \end{bmatrix}$$

Lyapunov stability analysis: $\boxed{\text{P5 of L3,}}$ linear system

find a Lyapunov function $V: \mathbb{R}^2 \to \mathbb{R}$ for $x(k+1)h = \Phi_c x(kh)$

$$V(x) = \underline{x^T P x} \geq 0$$

$\boxed{\text{if and only if}}$

and there exists Q, s.t (Q positive semidefinite)

$$\underline{\Phi_c^T P \Phi_c - P = -Q} \leq 0$$

here we set $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\Phi_c^T P \Phi_c - P = \begin{bmatrix} 0.046 & -0.018 \\ -0.018 & 0.015 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

actually asympotitally stable!

which is negative semidefinite as eigenvalues are -0.94, -0.99

proof: $\nabla V = x_{k+1}^T P x_{k+1} - x_k^T P x_k$

$$= x_k^T \Phi_c P \Phi_c x_k^T - x_k^T P x_k = x_k^T (\Phi_c^T P \Phi_c - P) x_k \leq 0$$