Quantized Cooperative Control

June 13, 2010

MENG GUO

KTH Electrical Engineering

Master's Degree Project
Stockholm, Sweden 2011

XR-EE-RT 2011:013

December 2010 to June 2011

Quantized Cooperative Control
— Diploma Thesis —

MENG GUO[1]

Automatic Control Laboratory
School of Electrical Engineering, KTH Royal Institute of Technology, Sweden

Supervisor
Dr. D. V. Dimarogonas
KTH Stockholm

Examiner
Dr. D. V. Dimarogonas
KTH Stockholm

Stockholm, June 13, 2011

[1] mengg@kth.se

Abstract

In this thesis project, we consider the cooperative control of multi-agent systems under limited communication between the individual agents. In particular, quantized values of the relatives states between neighboring agents are used as the control parameters for each agent.

As an introductory part, the theoretical framework for the distributed consensus problem under perfect communication is reviewed with the focus on the system stability and convergence. We start from the common problem setup that single integrator agents with a static tree communication topology, where the stability constraints and convergence guaranty are derived for different quantization models: uniform, logarithmic and dynamic. Then the conclusions are extended to switching tree topologies, tree topologies with disconnected time intervals and finally general undirected graphs. The control performance like convergence rate and the area of convergence set are compared between systems with and without quantization effects, and also among the systems with different quantizers.

Furthermore, similar techniques are applied to other system dynamics with quantized control inputs. We investigate additional constraints on the stability of the corresponding discrete time system due to the presence of quantization effects. Explicit upper bounds on the sampling time that guarantee convergence are derived. As expected, the sampling frequency has to be increased accordingly under different quantization models. The multi-agent system composed of second-order agents under general undirected communication graphs is also taken into account with quite different analytical tools. Finally we switch to an alternative model that takes the relative quantized states as control parameters instead. Distinctive convergence properties are found between these two models and detailed comparisons are made. Throughout this report, all results obtained are supported by numerical simulations.

Acknowledgments

First and foremost, I offer my sincerest gratitude to my supervisor, Dr. Dimos V. Dimarogonas for giving me this opportunity as his Master Thesis student. From the first day I read the term "multi-agent system" to now I am fascinated by this challenging area, I have learned a lot and been enlightened so much by following his guidance, not only in the graduate course but also during our fruitful discussions. This thesis would not be possible without his help, support and great patience all the time.

Then I want to thank my parents for always being there. They have been giving me their unequivocal caring, dedication and love throughout, as always in my lifetime, for which my mere expression of thanks likewise does not suffice. Also I would like to express my gratitude to my grandparents and other family members, for all the time being so supportive to my study and life in Sweden.

Last, but by no means the least, great thanks to my friends and all the kind people I met in China and Sweden, for the greatly precious memory we shared together.

Meng Guo
May 21, 2011

Contents

Notation

E	Edge set in graph G
Δ	Degree matrix of graph G
δ_l	Gain of the logarithmic quantizer
δ_u	Gain of the uniform quantizer
$\lambda_2(M)$	Second largest eigenvalue of matrix M
$\lambda_{\max}(M)$	Maximal eigenvalue of matrix M
$\lambda_{\min}(M)$	Minimal eigenvalue of matrix M
$\mathbf{0}_N$	Column vector of zeros, with length N
$\mathbf{1}_N$	Column vector of ones, with length N
$\mathcal{N}_i$	Neighboring set of agent i
$\mathcal{V}$	Vertex set in graph G
μ	Zooming variable of dynamic quantizer
$\mathbf{I}$	$N \times N$ Identity matrix
A	Adjacency matrix of graph G
a_{ij}	The (i, j) element of adjacency matrix A
B	Incidence matrix of graph G
G	Underlying communication graph
L	Laplacian matrix of graph G
q_d	Dynamic quantizer
q_l	Logarithmic quantizer
q_u	Uniform quantizer
T_s, h	Sampling time

1 Introduction

The main purpose of this Master thesis project is to analyze the constraints on networked multi-agent systems due to the existence of quantization effects. This chapter serves as a brief introduction to the area of multi-agent systems and distributed control strategies, followed by a particular focus on the consensus problem and quantization effect. It is concluded with project goals and the outline of this report.

1.1 Multi-agent System and Cooperative Control

In recent years there has been significant research in the area of coordinated control of groups of autonomous vehicles. With the help of highly developed wireless communication technologies and embedded micro-processors, it becomes economically feasible that each vehicle, sensor or even nanostructure can be equipped with an individual controller and communication unit to cooperate and communicate with other participants in a dynamic and intelligent way. On the other hand, numerous bio-inspired principles and phenomena give rise to a deeper understanding of the role of interactions among group members for the collective performance of the whole group.

Agents in such networks are required to behave in accordance with each other in order to achieve certain global objectives, while they normally only have access to local information and limited sensing capabilities. This kind of distributed and networked multi-agent systems can be utilized to solve a large variety of problems in a quite efficient and robust fashion. There are some examples like formation flight[5], flocking[25], sensor networks[1], energy networks[2] and social networks. They all belong to the scope of cooperative control in the sense that the agents cooperate and share information with each other to achieve the desired global objectives, but without access to the global information of the whole system. The fundamental feature of networked systems that distinguishes them from traditional systems is the role of cooperation and communication, that leads to collective behaviors of the group.

Consequently, with increasing demand on the performance of multi-agent systems, the critical step for distributed control is to design appropriate control protocols for each agent so that the group can converge to the desired configuration, even in the presence of limited and unreliable information exchange and dynamically changing communication topologies.

1.2 Consensus Problem and Quantization

Agreement or *consensus problem* is one of the fundamental problems in the area of multi-agent coordination, where the group of agents are controlled to converge to a desired configuration, regarding a certain quantity of interest that depends on the state of all agents [21]. The corresponding *consensus protocol* is an interaction rule that specifies the control effort applied to each agent and the information exchange between neighboring agents.

Consensus and alignment problems have a long history in computer science [18] and have recently been studied in the context of cooperative control of multiple vehicle systems [19], [21], [22], [23]. Consensus protocols serve as the preliminary tool that has many potential applications in other multi-agent problems such as flocking, rendezvous, swarming, attitude alignment and distributed estimation. One thing to note here is that when we say the consensus problem, it doesn't necessarily mean that all agents have to converge to the same value within all dimensions and it is possible to reach an agreement in several dimensions of the state space. For instance, with respect to the multi-agent system of aerial vehicles, we can design control algorithms so that they remain at the same attitude but with different velocities, or at the same attitude with the same velocities.

Most of the popular literature mentioned above on multi-agent consensus problem assumes perfect communication with unlimited bandwidth and precision between communicating agents. However, in reality it has been recognized that restricted communication bandwidth and noisy information exchange will greatly impact the system stability and performance. The following example illustrates distinctive trajectories of a 50-agent system converging to a consensus point with and without quantized communication: average-consensus based on perfect communication in Figure 1.1 and non-smooth trajectories with large consensus errors due to uniform quantized (2.4) information in Figure 1.2.

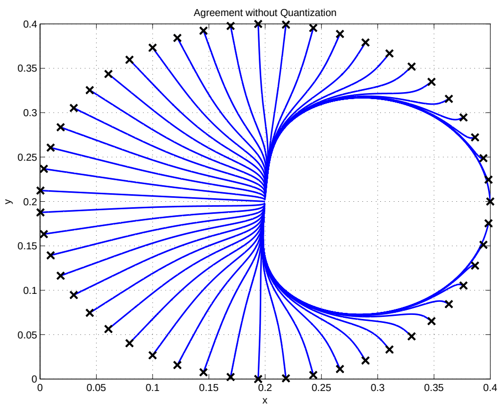


Figure 1.1: Consensus under perfect communication

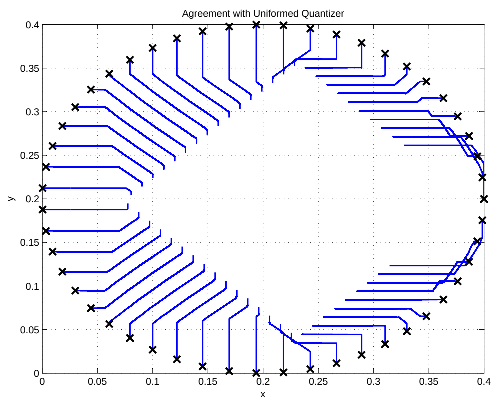


Figure 1.2: Consensus under quantized information

Moreover, in order to implement the control algorithms into digital platforms and save computation and communication cost, it is inevitable to use quantized value in some way or another. Due to the highly nonlinear properties of quantization functions, it is very hard to obtain explicit solutions of a system involving quantized elements and the traditional methods for analyzing linear systems are not feasible anymore. In this thesis, we mainly consider two types of quantized systems, two kinds of agent dynamics, and three quantization models.

1.3 Goals of this Project

The main goal of this project is to investigate the impact of quantized information on the stability and performance of the networked multi-agent systems. The associated convergence set and convergence rate should also be addressed. Furthermore, it is desired to provide detailed comparisons between the nominal system without quantization and the quantized system with different quantization models. All these discussions would be taken with respect to different communication topologies.

1.4 Outline

The remainder of this report is organized as follows: Chapter 2 presents necessary background knowledge about algebraic graph theory and preliminaries of the consensus problem, after which the models and numerical properties of three different quantizers are introduced. In Chapter 3, we treat the consensus problem of single integrator agents under static and switching tree communication topologies. Then we explore the results to general undirected topologies, where we also give detailed discussions about the convergence rate and convergence area under different quantizers. The stability and convergence of the discrete time multi-agent system with quantization are discussed in Chapter 4 to derive explicit constraints on the sampling time. The case of multi-agent system with second-order agents can be found in Chapter 5. Chapter 6 is devoted to extensions regarding different consensus protocols that use relative quantized states as control parameters. The thesis concludes with conclusions and future work in Chapter 7. Theoretical results in all parts are supported by computer simulations.

2 Background and Problem Formulation

This chapter includes the essential backgrounds and basic problem formulations for this thesis. The multi-agent system under consideration consists of limited number of single integrator agents that share quantized information under undirected communication topologies. Section 2.1 provides basic definitions and key matrices in algebraic graph theory that serve as instrumental tools for analyzing network related systems. In Section 2.2 detailed models of three different quantizers are given and also their numerical properties that are useful for later applications. We introduce the simplified models of first order and second order agents in Section 2.3, with corresponding widely used consensus protocols. At last existing important conclusions concerning the nominal consensus problem are briefly listed in Section 2.4.

2.1 Algebraic Graph Theory

As mentioned in the introductory chapter, concepts from *algebraic graph theory* [10] have been widely used in the modeling, controller design and performance evaluation of networked multi-agent systems. Graphs provide straightforward abstractions reflecting the information flow between agents in a network. In the following we will introduce basic elements and definitions in algebraic graph theory, with particular emphasis on the adjacency and Laplacian matrices associated with undirected graphs.

A finite and undirected graph G is normally defined as $G = \{V, E\}$, where V is the *vertex set* with N *vertices* $V = \{1, \dots, N\}$ and the *edge set* $E = \{(i, j) \in V \times V | i \in \mathcal{N}_j\}$, where $\mathcal{N}_j$ denotes agent j's *communication set* that includes all agents (neighbors) with which j can communicate. The communication graph is assumed *undirected* in the sense that if j is adjacent to i then i is adjacent to j, i.e., $i \in \mathcal{N}_j \Leftrightarrow j \in \mathcal{N}_i, i \neq j$. When the communication topology is called static, the communication sets $\mathcal{N}_i$ of each vertex remains static and graph G is time invariant. When the communication topology is time-varying, it means the sets $\mathcal{N}_i$ may change over time and G is time-varying by losing and/or adding edges, i.e., $G = G(t)$. A *path* of length r from vertex i to j is a sequence of $r + 1$ distinct vertices starting with i and ending with j such that consecutive vertices are adjacent. If there is a path between any two vertices, then G is called *connected.* A connected graph is called a *tree* if it contains no cycles. Two figures 2.1 and 3.1 below show a connected graph and a tree graph. In practical applications, these vertices are considered as real identities or *agents* such as vehicles, animals and robots.

Here we follow the same notation as [10]. The *adjacency matrix* $A = A(G) = \{a_{ij}\}$ is the $N \times N$ matrix given by $a_{ij} = 1$, if $(i, j) \in E$, and $a_{ij} = 0$, otherwise. Vertices i, j are called *adjacent* if $(i, j) \in E$ and this edge is denoted by e_{ij}. Clearly, $A(G)$ is symmetric for

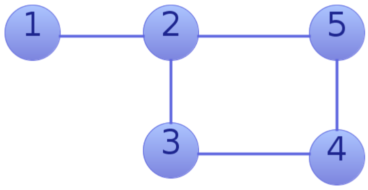


Figure 2.1: Connected Graph

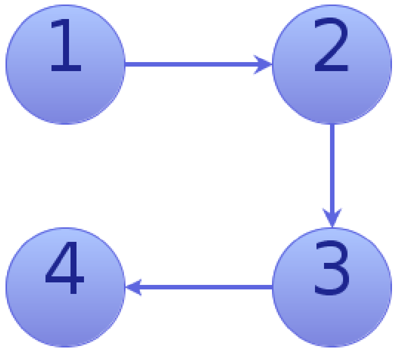


Figure 2.2: Tree Graph

undirected graph G as $a_{ij} = a_{ji}$. Let Δ be the $N \times N$ diagonal matrix of d_i's, where the *degree* of each vertex i is given by $d_i = \sum_j a_{ij}$. For unweighted graphs, d_i is equal to $|\mathcal{N}_i|$, the number of vertices that are adjacent to vertex i in G. G is called a *weighted* graph if its edges are assigned real positive weights instead of just one or zero. The *Laplacian* matrix of G is an asymmetric and positive semidefinite matrix defined as $L(G) = \Delta - A$. It is easily verified that the rows of the Laplacian matrix sum to zero, i.e., $L\mathbf{1}_N = \mathbf{0}_N$. Moreover, it is proved in [22] that L has a single zero eigenvalue with the corresponding eigenvector $\mathbf{1}_N$ if and only if G has a spanning tree. Back to the graph example in Figure 2.1, its corresponding adjacency, degree and Laplacian matrices are given as

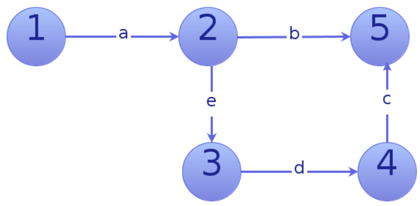
$$\Delta = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$L = \Delta - A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$

An arbitrarily *oriented graph* G is the assignment of a direction/orientation to each edge $e \in E$. The oriented graph is described by the *incidence matrix* $B = B(G) = \{b_{ij}\}$, which is the $\{0, \pm 1\}$ matrix with rows and columns indexed by the vertices and edges of G respectively, such that $b_{ij} = 1$ if the vertex i is the head of the edge e_{ij}, and $b_{ij} = -1$ if vertex i is the tail of the edge e_{ij}, and $b_{ij} = 0$ otherwise. This incidence matrix always has a column sum equal to zero because every edge has to have exactly one tail and one head.

The incidence matrix $B(G)$ of the same graph varies with different assignments of the edges orientation. But the Laplacian matrix which is given by $L = BB^T$ is a fact invariant to different incidence matrices [10]. The incidence matrix associated with the oriented graph in Figure 2.3 of the undirected graph in Figure 2.1 is shown above as an example.

If G contains cycles, the edges of each cycle have a direction, where each edge is directed towards its successor according to the cyclic order. A cycle C is represented by a vector v_C with $|E|$ elements. For each edge, the corresponding element of v_C is equal to 1 if the direction



$$B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Figure 2.3: Oriented Graph

of the edge with respect to C coincides with the orientation assigned to the graph for defining B, and -1, if the direction with respect to C is opposite to the orientation. The elements corresponding to edges not in C are zero. The cycle space of G is the subspace spanned by vectors representing cycles in G.

2.2 Quantization Function

Quantization effects have a special importance in industrial engineering since most systems today are hybrid and controlled via finite-bit digital platforms. During analog-to-digital conversions, the difference between actual analog value and quantized digital value is the quantization distortion. The error signal is normally considered as an additional random noise signal with stochastic behavior [24]. Through this project, we take into account three types of quantizer: uniform, logarithmic and dynamic. Detailed models of each kind are provided consecutively.

2.2.1 Uniform Quantizer

First of all, we introduce the uniform quantizer, which is the most commonly used quantizer in practice due to finite-bit computation in digital equipments. As in most relevant literature, the uniform quantizer $q_u : \mathbb{R} \to \mathbb{R}$ with the uniform gain δ_u is defined as

$$q_u(x) = \delta_u \, [\frac{x}{\delta_u}] \tag{2.1}$$

where $[\cdot]$ denotes the nearest integer operation and $[\frac{1}{2}] = 1$. Explicitly the relation is given as follows and illustrated in Figure 2.4

$$q_u(x) = \begin{cases} k\,\delta_u & x \in [(k-\frac{1}{2})\delta_u,\, (k+\frac{1}{2})\delta_u),\; k \geq 1,\; k \in \mathbb{Z} \\ 0 & x = 0 \\ k\,\delta_u & x \in ((k-\frac{1}{2})\delta_u,\, (k+\frac{1}{2})\delta_u],\; k \leq -1,\; k \in \mathbb{Z} \end{cases} \tag{2.2}$$

Regarding the numerical properties of the uniform quantizer (2.1), we will list several of them that are useful in the remaining parts.

Lemma 2.1. *With respect to the uniform quantizer defined in (2.1), the following relations hold:*

$$x \cdot q_u(x) \geq 0 \qquad |q_u(x) - x| \leq \frac{\delta_u}{2} \qquad q_u(-x) = -q_u(x)$$

$$\frac{1}{2}\, {q_u}^2(x) \leq x \cdot q_u(x) < \frac{3}{2}\, {q_u}^2(x) \qquad \frac{2}{3}\, x^2 < x \cdot q_u(x) \leq 2\, x^2 \quad \textit{if } q_u(x) \neq 0$$

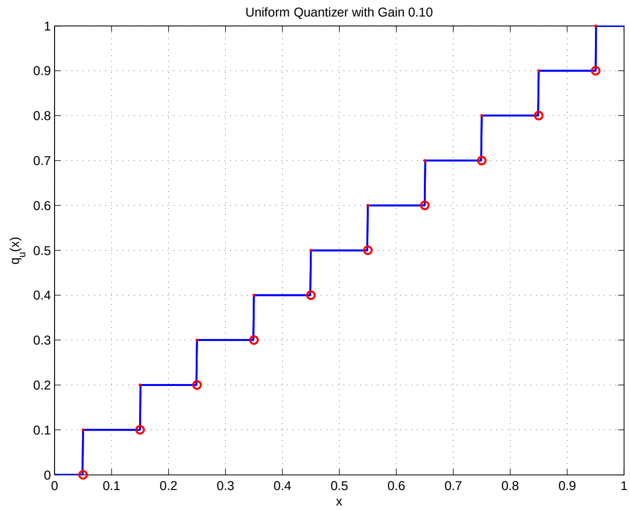


Figure 2.4: Uniform Quantizer with Gain 0.1

Proof. The first three properties can be easily verified by inserting the expression of $q_u(x)$ from (2.2). Here we focus on the last two with quadratic terms. When $q_u(x) \neq 0$ namely $k \neq 0$, we have

$$\frac{x}{q_u(x)} \in [\frac{|k|-\frac{1}{2}}{|k|}, \frac{|k|+\frac{1}{2}}{|k|}) \quad \Rightarrow \quad x \cdot q_u(x) \in [\frac{|k|-\frac{1}{2}}{|k|}, \frac{|k|+\frac{1}{2}}{|k|})\, q_u^2(x)$$

$$\Rightarrow \quad x \cdot q_u(x) \in [1 - \frac{1}{2|k|}, 1 + \frac{1}{2|k|})\, q_u^2(x) \quad \Rightarrow \quad \frac{1}{2}\, {q_u}^2(x) \leq x \cdot q_u(x) < \frac{3}{2}\, {q_u}^2(x)$$

where the condition $q_u(x) \neq 0$ can be omitted since $x \cdot q_u(x) = q_u^2(x) = 0$ when $q_u(x) = 0$. Using similar techniques in another way around when $q_u(x) \neq 0 \Rightarrow x \neq 0$, we can bound $\frac{q_u(x)}{x}$ in the way that

$$\frac{q_u(x)}{x} \in (\frac{|k|}{|k|+\frac{1}{2}}, \frac{|k|}{|k|-\frac{1}{2}}] \quad \Rightarrow \quad x \cdot q_u(x) \in (\frac{|k|}{|k|+\frac{1}{2}}, \frac{|k|}{|k|-\frac{1}{2}}]\, x^2$$

if $q_u(x) \neq 0$ namely $k \neq 0$, it holds that

$$\frac{2}{3}\, x^2 < x \cdot q_u(x) \leq 2\, x^2 \quad (q_u(x) \neq 0)$$

we can not omit the condition that $q_u(x) \neq 0$ because if $x \neq 0$ but $q_u(x) = 0$,i.e., $|x| < \frac{\delta_u}{2}$, above inequality is not true as $x \cdot q_u(x) = 0$ when $x^2 > 0$. Figure 2.5 verifies the results above, boundaries of $q_u(x) \cdot x$ by x^2 (the left) and $q_u^2(x)$ (the right).

∎

2.2.2 Logarithmic Quantizer

Another important type of quantizers is the logarithmic quantizer [24]. $q_l : \mathbb{R} \to \mathbb{R}$,

$$q_l(x) = \text{sign}(x) \cdot e^{q_u(\ln(|x|))} \quad (x \neq 0) \tag{2.3}$$

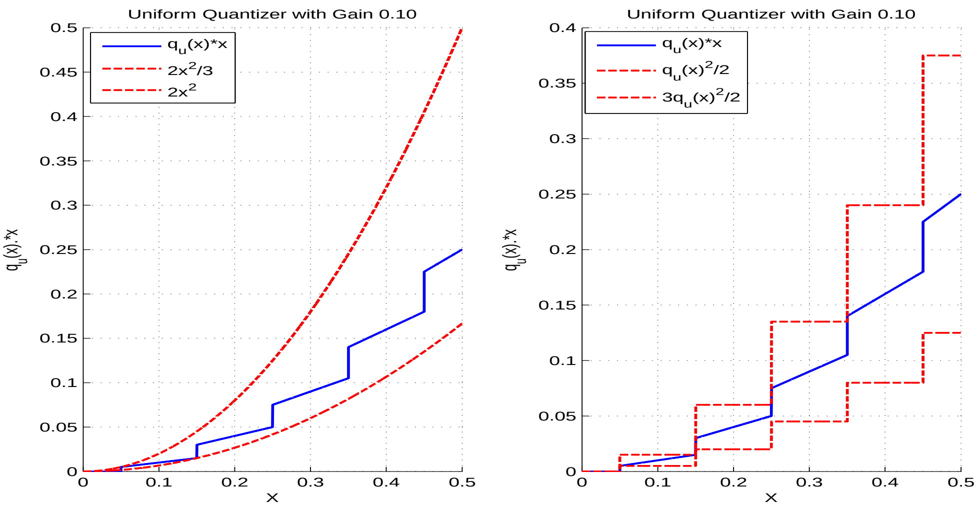


Figure 2.5: Boundaries on $q_u(x) \cdot x$

where q_u is the uniform quantizer with gain δ_u, as defined in (2.1). In particular, $q_l(0)$ is defined to be zero. We give the explicit definition of logarithmic quantizer as follows and also illustrated in Figure 2.6

$$q_l(x) = \begin{cases} e^{k\,\delta_u} & x \in [e^{(k-\frac{1}{2})\delta_u},\, e^{(k+\frac{1}{2})\delta_u}),\ k \in \mathbb{Z} \\ 0 & x = 0 \\ -e^{k\,\delta_u} & x \in (-e^{(k+\frac{1}{2})\delta_u},\, -e^{(k-\frac{1}{2})\delta_u}],\ k \in \mathbb{Z} \end{cases} \tag{2.4}$$

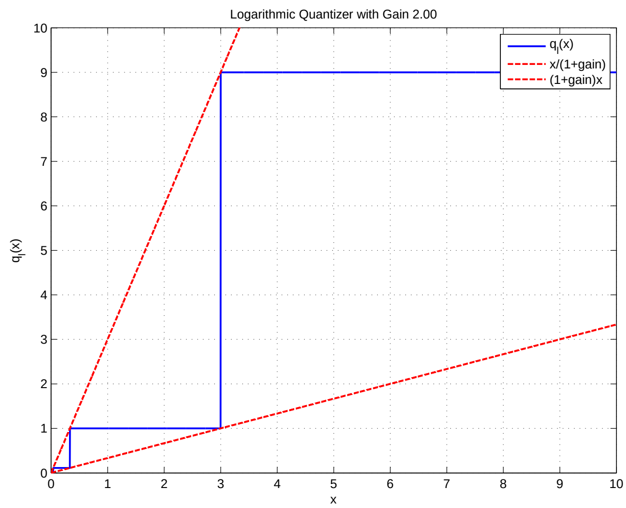


Figure 2.6: Logarithmic Quantizer with Gain 0.1

We may notice that the logarithmic quantizer is capable of adjusting the size of quantization step according to the input value, but it is more computationally expensive than the uniform one with the exponential and logarithmic functions. Some of its numerical properties are listed in the following lemma.

Lemma 2.2. *With respect to the logarithmic quantizer defined in (2.3), the following relations hold:*

$$x \cdot q_l(x) \geq 0 \qquad q_l(-x) = -q_l(x) \qquad |q_l(x) - x| \leq \delta_l |x| \quad \textit{where } \delta_l = e^{\frac{\delta_u}{2}} - 1$$

$$\frac{1}{1+\delta_l}\, {q_l}^2(x) \leq x \cdot q_l(x) < (1+\delta_l)\, {q_l}^2(x) \qquad \frac{1}{1+\delta_l}\, x^2 < x \cdot q_l(x) \leq (1+\delta_l)\, x^2$$

Proof. The symmetric property and passivity can be easily verified by checking the expression of $q_u(x)$ from (2.4). As the starting point, we need to compute the gain δ_l of a logarithmic quantizer. From the model (2.1), $q_u(x) \in [x - \frac{\delta_u}{2},\ x + \frac{\delta_u}{2}) \ \Rightarrow\ q_u(\ln(|x|)) \in (\ln(|x|) - \frac{\delta_u}{2},\ \ln(|x|) + \frac{\delta_u}{2})$. Since $q_l(x) = -q_l(-x)$ we now consider the case $x > 0$ only. $\ln(q_l(x)) \in (\ln(x) - \frac{\delta_u}{2},\ \ln(x) + \frac{\delta_u}{2})$. Denote $a = e^{\frac{\delta_u}{2}} \geq 1$, we have $\ln(q_l(x)) \in (\ln(\frac{x}{a}),\ \ln(a\,x))$, namely $q_l(x) \in (\frac{x}{a},\ a\,x)$ as the function $\ln(\cdot)$ is strictly increasing. We have $q_l(x) - x \in ((\frac{1}{a} - 1)x,\ (a-1)x)$ where $a > 1$ so $|a-1| > |\frac{1}{a} - 1|$. As a result

$$|q_l(x) - x| < |a-1|x = (a-1)x = (e^{\frac{\delta_u}{2}} - 1)x = \delta_l\, x$$

Given the desired logarithm gain δ_l, the model of a logarithmic quantizer is fixed and implementable. Regarding the last two relations, when $x \neq 0$ we have

$$\frac{q_l(x)}{x} \in (e^{-\frac{\delta_u}{2}},\, e^{\frac{\delta_u}{2}}] = (\frac{1}{1+\delta_l},\, 1+\delta_l] \quad \Rightarrow \quad x \cdot q_l(x) \in (\frac{1}{1+\delta_l},\, 1+\delta_l]\, x^2$$

The condition $x \neq 0$ is omitted in the sense that $q_l(x)\, x = x^2$ when $x = 0$. Same arguments can be applied with respect to $q_l(x)$, by replacing x^2 with $q_l^2(x)$ as $q_l(x) = 0$ if and only if $x = 0$. Figure 2.7 verifies the results above, boundaries of $q_l(x) \cdot x$ by x^2 (the left) and $q_l^2(x)$ (the right). ∎

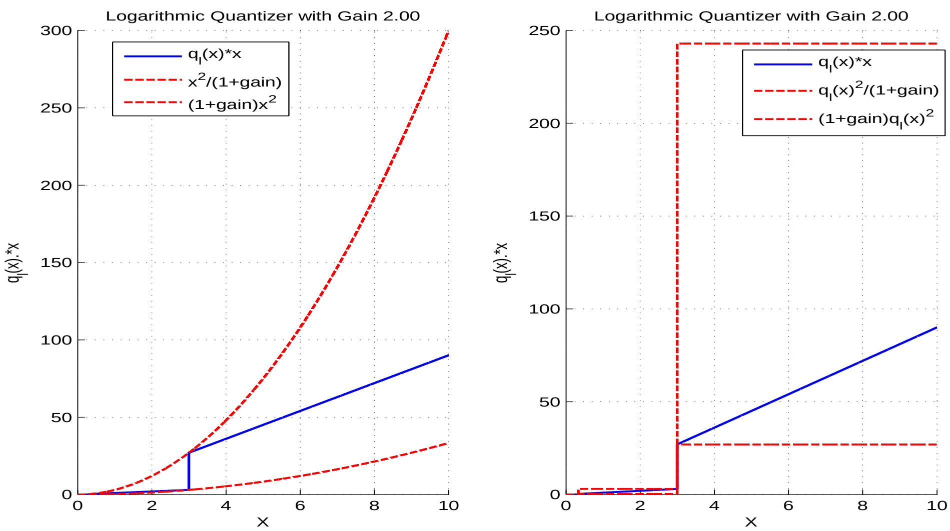


Figure 2.7: Boundaries on $q_l(x) \cdot x$

2.2.3 Dynamic Quantizer

The dynamic quantizer is a variation of uniform quantizer that adapts the value of uniform gain according to some predefined rules. It has the definition $q_d : \mathbb{R} \to \mathbb{R}$,

$$q_d(x) = \mu \, q_u(\frac{x}{\mu}) \quad (\mu > 0) \tag{2.5}$$

where μ is the time-varying gain or the *zooming variable*. The original description and mathematic model of dynamic quantizer can be found in [15]. One example is showed in Figure 2.8 where the zooming variable follows a second order polynomial. Similar to Lemma 2.1, the following relations hold

$$x \cdot q_d(x) \geq 0 \qquad\qquad |q_d(x) - x| \leq \mu \, \delta_u$$

The procedure of how to implement a dynamic quantizer given a quantized linear system and a candidate Lyapunov function is throughly discussed in [15] with detailed theoretical proofs.

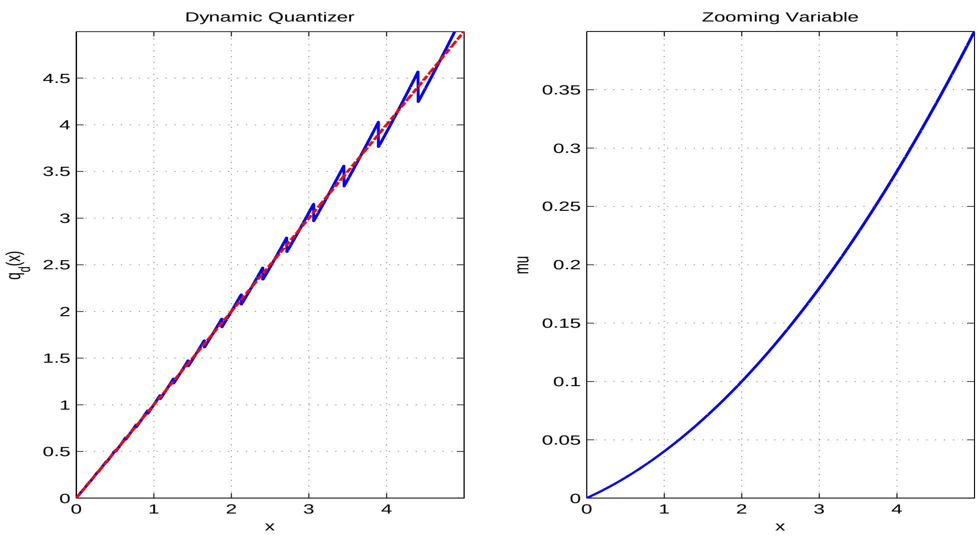


Figure 2.8: Dynamic Quantizer and the Zooming variable

2.3 System Model and Consensus Protocol

The vertex set of a graph G described in Section 2.1 actually represents a group of individual dynamic agents or *small system* in the whole multi-agent system. For instance, the networked system may be comprised of numerous members like vehicles, robots, aircrafts and so on. However within the scope of this thesis, we mainly consider two simplified agent models: single integrator and double integrator.

In the case of the *single integrator* agents, we have

$$\dot{z}_i = u_i, \qquad i \in \{1, \ldots, N\} \tag{2.6}$$

where $z_i = [x_i \; y_i]^T \in \mathbb{R}^2$ denotes the position and $u_i \in \mathbb{R}^2$ the control input of agent i. The design objective is to construct feedback controllers (consensus protocols) that lead the multi-agent system to an agreement, i.e., all agents converge to a consensus point.

The agreement control law for system (2.6) proposed in [9] is given by

$$u_i = -\sum_{j\in\mathcal{N}_i}(z_i - z_j) \tag{2.7}$$

where each agent only knows the state of agents that belong to its communication set $\mathcal{N}_i$ at each time instant. Thus the closed-loop equations of the nominal system (without quantization) satisfy

$$\dot{z}_i = -\sum_{j\in\mathcal{N}_i}(z_i - z_j,), \qquad i \in \{1,\ldots,N\}$$

or equivalently $\dot{z} = -(L\otimes I_2)z$ [9], where $\otimes$ denotes the Kronecker product. We may notice that actually $\dot{z} = -(L\otimes I_2)z$ can be separated into two sub-systems $\dot{x} = -Lx$ and $\dot{y} = -Ly$, that evolve independently with each other. Thus we can consider the stability and convergence in one dimension only and the others should obey the same results.

Since a broad class of vehicles requires a second-order dynamic model, a *double integrator* system is also considered

$$\dot{z}_i = v_i, \quad \dot{v}_i = u_i, \qquad i \in \{1,\ldots,N\} \tag{2.8}$$

where $z_i \in \mathbb{R}^2, v_i \in \mathbb{R}^2$ denotes the position and velocity, and $u_i \in \mathbb{R}^2$ is the acceleration input of agent i.

With respect to the second order system, the desired final configuration is same position and same velocity, i.e., all agents moving with common speed as one point. The agreement control protocol for the second-order system (2.8) proposed in [22] is given by

$$u_i = -\sum_{j\in\mathcal{N}_i}[(z_i - z_j) + \gamma(v_i - v_j)] \tag{2.9}$$

where $\gamma > 0$. The closed-loop system is given by $\dot{z} = v$ and $\dot{v} = -(L\otimes I_2)z - \gamma(L\otimes I_2)v$, which can also be separated independently in x and y directions.

Through this thesis, we only treat the system behavior in the x-coordinates but the same analysis that follows holds in the y-coordinates, and also other coordinates in higher dimensions. All equations can be extended to include higher dimensions by introducing Kronecker product between matrices. Thus the conclusions can be extended to agent models of arbitrary dimension.

Moreover, we denote by $\bar{x}$ the m-dimensional stack *edge vector* of relative differences (*head–tail*) of agent pairs that form an edge in G, where m is the number of edges. Then it can be shown that $\dot{\bar{x}} = B^T\dot{x} = -B^TLx = -B^TB\,\bar{x}$. Hence the first order nominal system is also given by $\dot{\bar{x}} = -B^TB\;\bar{x}$. Furthermore following relevant relations are easily verified: $Lx = B\bar{x}, \bar{x} = B^Tx$. Since $\bar{x} = 0 \Rightarrow B\bar{x} = 0 \Rightarrow Lx = 0$, then if G is connected, the requirement $Lx = 0$ guarantees that x has all its elements equal [10],[9], namely reaching an agreement. Using the same procedure, we can rewrite the second order nominal system into $\dot{\bar{x}} = \bar{v}$, $\dot{\bar{v}} = -B^TB\,\bar{x} - \gamma B^TB\,\bar{v}$, where $\bar{x} = B^Tx$ and $\bar{v} = B^Tv$.

However the additional constraint of quantized relative measurements is imposed in this thesis, which also has been justified in [6]. In particular, without any access to other agents' absolute states, each agent i is assumed to have only quantized measurements $q(x_i - x_j)$ of the relative position of all of its neighbors $j \in \mathcal{N}_i$, where $q(.)$ could be any kind of quantization functions described in Section 2.2. Consequently, the consensus protocols (2.7) and (2.9) with quantization become

$$u_i = -\sum_{j\in\mathcal{N}_i} q\left(x_i - x_j\right) \tag{2.10}$$

$$u_i = -\sum_{j\in\mathcal{N}_i} \left[\, q(x_i - x_j) + \gamma q(v_i - v_j)\right] \tag{2.11}$$

2.4 Solution without Quantization

Since the consensus problem has been a quite popular research area for a long time, many important results have been obtained. Though most of them are developed under the assumption of perfect communication, the following results from [21], [22] and [23] are listed here due to their importance in the thesis:

Theorem 2.3 (Olfati-Saber and Murray 2004). *Consider a network of integrators $\dot{x}_i = u_i$, where each node applies protocol (2.7). Assume G is a strongly connected digraph. Then, protocol (2.7) globally asymptotically solves the consensus problem.*

Proof. See the proof of Corollary 1 in [21] ■

Theorem 2.4 (Ren and Beard 2005). *Given a matrix $A = [a_{ij}] \in M_N(\mathbb{R})$, where $a_{ii} \leq 0$, $a_{ij} \geq 0\, \forall i \neq j$, and $\sum_{j=1}^{N} a_{ij} = 0$ for each j, then A has at least one zero eigenvalue and all of nonzero eigenvalues are in the open left half plane. Furthermore, A has exactly one zero eigenvalue if and only if the directed graph associated with A has a spanning tree.*

Proof. See the proof of Lemma 3.3 in [22] ■

Theorem 2.5 (Ren and Beard 2007). *Consensus protocol (2.9) for the second-order system (2.8) drives the agents to consensus if and only if the information exchange topology has a spanning tree.*

Proof. See the proof of Lemma 4.1 in [23] ■

3 Convergence under Undirected Topology

In this chapter, in order to find out the constraints of quantized communication on the system stability and convergence, we take the discussion in a step by step fashion. We start from the most familiar scenario of first order systems under a static tree, then to switching trees, and then time-varying trees with disconnected time intervals in Section 3.1. General undirected graphs with weighted edges are treated in Section 3.2 where we introduce the main framework for analyzing the convergence properties of the quantized consensus problems. At last the system stability with logarithmic quantizers under the directed graph is analyzed in Section 3.3.

3.1 Tree Topology

3.1.1 Static Tree Graph

In the presence of quantized relative distance measurements, the control law (2.10) imposes the closed-loop system

$$\dot{x}_i = -\sum_{j \in \mathcal{N}_i} q(x_i - x_j) \tag{3.1}$$

Assume that all agents have identical quantizers with the same parameter, i.e., the quantization function $q(\cdot)$ is the same for all agent i. Since $q\left(-a\right) = -q\left(a\right), \, a \in \mathbb{R}$ holds for all three types of quantizers (2.4), (2.6), (2.8), similar to the edge dynamics $\dot{\bar{x}} = -B^T B \bar{x}$ of the nominal system introduced in Section 2.3 we get

$$\dot{\bar{x}} = -B^T B \, q\left(\bar{x}\right) \tag{3.2}$$

where $q(\bar{x})$ is the m-dimensional stack vector of edge pairs $q\left(x_i - x_j\right)$ with $(i, j) \in E$, x_i being the head and x_j the tail. The Laplacian matrix L is always positive semidefinite [22]. By Lemma 1 in [6], it is true that the so-called *edge Laplacian* matrix $B^T B$ is always positive definite in the case of a tree graph as no cycle space exists.

For the sake of concision, we denote by $M = B^T B$ the edge Laplacian thus (3.2) becomes $\dot{\bar{x}} = -M \, q(\bar{x})$. When the underlying graph G is a tree, M remains to be positive definite and M^{-1} always exists, which is also positive definite [11]. Thus let $V = \bar{x}^T M^{-1} \bar{x}$ be a candidate Lyanunov function.

The level sets of V define compact sets with respect to the agents' relative distance. In this way, we can apply LaSalle's Invariance Principle [13]. Specifically, for any $(i, j) \in E$ we have

$$V \leq c \Rightarrow \lambda_{\min}(M^{-1}) |\bar{x}|^2 \leq c \Rightarrow |x_i - x_j| \leq |\bar{x}| \leq \sqrt{\frac{c}{\lambda_{\min}(M^{-1})}}$$

then for any $(i, j) \in V \times V$, we have

$$|x_i - x_j| = |x_i - x_k + x_k - \cdots - x_j| \leq (N-1)\sqrt{\frac{c}{\lambda_{\min}(M^{-1})}}$$

as the maximum length of the path connecting i and any j is $N-1$. Since $M = B^T B$ is symmetric, M^{-1} is also symmetric, i.e., $M^T = M$ and $(M^{-1})^T = M^{-1}$. The time derivative of V along the trajectories of the closed loop system (3.1) is given by

$$\begin{aligned}\dot{V} &= (\dot{\bar{x}})^T M^{-1}\bar{x} + \bar{x}^T M^{-1}(\dot{\bar{x}}) \\ &= -q(\bar{x})^T M^T M^{-1}\bar{x} - \bar{x}^T M^{-1} M q(\bar{x}) \\ &= -2\bar{x}^T q(\bar{x}) = -2\sum_{i=1}^{m} \bar{x}_i \, q(\bar{x}_i)\end{aligned}$$

From Lemma 2.1 and 2.2 we know that $a \cdot q(a) \geq 0$, $a \in \mathbb{R}$ and $q(0) = 0$ holds for all three kinds of quantizers mentioned in Section 2.2. Thus $\dot{V} \leq 0$ and the equality $\dot{V} = 0$ is satisfied only when $\bar{x}_i \, q(\bar{x}_i) = 0$ holds, which means that either $\bar{x}_i = 0$ or $q(\bar{x}_i) = 0$ for each edge $\bar{x}_i$, $i = 1, \ldots, m$. When $\bar{x}_i = 0$, it in turn implies $q(\bar{\mathbf{x}}_i) = 0$. As a result, $\dot{V} \leq 0$ and $\dot{V} = 0$ only when $q(\bar{x}) = 0$.

Theorem 3.1. *Assume that the system (2.6) evolves under the control law (2.10) and the static communication graph is a tree, then all agents are guaranteed to converge to*

$$\mathcal{I}(\boldsymbol{x}) = \{\boldsymbol{x} | \, q(x_i - x_j) = 0, \forall (i, j) \in E\}$$

Proof. We have proved that the Lyapunov function $V = \bar{x}^T M^{-1}\bar{x}$ is positive definite and compact with respect to the relative states. $\dot{V} \leq 0$ for all $t \geq 0$. Application of LaSalle's Invariance Principle ensures the convergence of the system to the largest invariant subset $\dot{V} = 0$. Hence all agents converge to the set that $\{\bar{\mathbf{x}} | \, q(\bar{x}_i) = 0, i = 1, \ldots, m\}$, equivalently $\mathcal{I}(\mathbf{x}) = \{\mathbf{x} | \, q(x_i - x_j) = 0, \forall (i, j) \in E\}$. ∎

Different results can be shown regarding three different models of quantizers, depending on when the condition $q(x_i - x_j) = 0$ holds.

Uniform Quantizer

Suppose identical uniform quantizers are equipped on all agents, from the definition (2.4) it is obvious that $|q_u(a)| \geq \delta_u$ when $|a| \geq \frac{\delta_u}{2}$ and $|q_u(a)| = 0$ only when $|a| < \frac{\delta_u}{2}$. So $\dot{V} = 0$ holds when $|\bar{x}_i| < \frac{\delta_u}{2}$, $i = 1, \cdots, m$, namely $|x_i - x_j| < \frac{\delta_u}{2}$, $\forall (i, j) \in E$, which means that the pair of agents in one communication edge should have absolute difference less than $\frac{\delta_u}{2}$.

By following the conclusion in Theorem 3.1, the closed-loop system with uniform quantizers is guaranteed to converge to the consensus set

$$\{\mathbf{x} | \, |x_i - x_j| < \frac{\delta_u}{2}, \forall (i, j) \in E\} \tag{3.3}$$

which further implies convergence to the set $\{\mathbf{x} | \, |\bar{x}| < \frac{\delta_u}{2}\sqrt{m}\}$, a ball with radius $\frac{\delta_u}{2}\sqrt{m}$, centered in the desired equilibrium point $\bar{x} = Lx = 0$. This point is equal to the average of the initial states by virtue of Lemma 3.7 which will be stated later. It can be easily verified that when $|\bar{x}_i| < \frac{\delta_u}{2}$ for all $i = 1, \cdots, m$, then $u_i = 0$ for all $i = 1, \cdots, N$, keeping all agents still inside the consensus set (3.3). This can be achieved in finite time independent of the communication topology as long as it remains a tree structure.

Logarithmic Quantizer

Assume that all agents have the same logarithmic quantizer with gain δ_l. From the definition of logarithmic quantizer (2.6), $q_l(\bar{x}_i) = 0$ only when $\bar{x}_i = 0$. As a result, the invariance set $\mathcal{I}(\mathbf{x}) = \{\mathbf{x}|\, q(x_i - x_j) = 0,\, \forall(i,j) \in E\} \Rightarrow \{\mathbf{x}|\, x_i = x_j,\, \forall(i,j) \in E\}$. Hence the closed system is guaranteed to converge to $\bar{x} = Lx = 0 \Rightarrow x \in \text{span}\{\mathbf{1}\}$, namely to the consensus point asymptotically. This result holds for arbitrary logarithmic gain satisfying

$$\delta_l = e^{\frac{\delta_u}{2}} - 1 > 0 \tag{3.4}$$

Compared with the results in Section 2 of [7], a smaller convergence set (3.3) is obtained with the uniform quantizer and the bounds for the logarithmic gain (3.4) to guarantee asymptotic convergence are much less conservative.

Dynamic Quantizer

If dynamic quantizers are used, we will utilize a centralized procedure to tune the zooming variable μ in order to achieve asymptotic convergence. In particular, it is centralized in the sense that all agents start from the same μ and share the same update rule at the same time instance T. But the control laws are still independent and distributed. If identical dynamic quantizers are used, based on the "zoom in, zoom out" strategy detailedly introduced in [15], we update the zooming variable μ at the same time instant for all agents.

With similar implementation of dynamic state quantizer as in [14], several key parameters should be mentioned here: the maximal range R of the uniform quantizer, the update interval (or dwell time) T and the zooming rate Ω. For our case, we have the following settings:

$$\begin{aligned}
R &> \frac{\sqrt{\lambda_{\max}(M)}}{\sqrt{\lambda_{\min}(M)}}\delta_u(1+\epsilon) \\
T &= [\frac{1}{\lambda_{\max}(M)}R^2 - \frac{1}{\lambda_{\min}(M)}\delta_u^2(1+\epsilon)^2]/[\delta_u^2(1+\epsilon)\epsilon] \\
\Omega &= \frac{\sqrt{\lambda_{\max}(M)}\delta_u(1+\epsilon)}{\sqrt{\lambda_{\min}(M)}R}
\end{aligned} \tag{3.5}$$

for any scalar $\epsilon > 0$.

An open-loop zooming-out stage is followed by a closed-loop zooming-in stage. Firstly with the initial value $\mu_0 = 1$ and for the open-loop system increase μ with a constant increment till $|\frac{x(t_0)}{\mu(t_0)}| \leq R\frac{\sqrt{\lambda_{\min}(M)}}{\sqrt{\lambda_{\max}(M)}}$ is satisfied at $t = t_0$. Then apply the control law $u_i = -\sum_{j\in\mathcal{N}_i} q_d\,(x_i - x_j)$ and let $\mu(t) = \mu(t_0)$ for $t \in [t_0,\ t_0 + T]$. After each constant time interval T, update the zooming variable by $\mu^+ = \Omega\,\mu^-$, i.e., $\mu(t) = \Omega^k\,\mu(t_0)$ for $t \in [t_0 + kT,\ t_0 + (k+1)T]$. Then it is guaranteed that $\bar{x}$ converges to zero exponentially for $t \geq t_0$. It is straightforward to prove the asymptotic stability of the closed-loop system with our Lyapunov function V, based on the proof of Lemma 1 in [14].

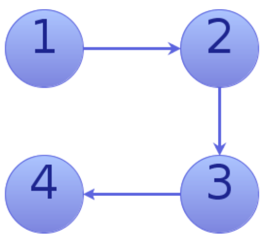
The discussions above can be summarized in the following theorem:

Theorem 3.2. *Assume that static communication graph G is a tree. Then the closed loop system (3.1) has the following convergence properties:*

- *In the case of uniform quantizers, the system converges to the consensus set (3.3) in finite time.*

- *In the case of logarithmic quantizers, the system asymptotically converges to the average consensus point for all logarithmic gains $\delta_l > 0$.*
- *In the case of dynamic quantizers, the system with settings (3.5) and the described algorithm, converges asymptotically to the average consensus point.*

Example 3.1. This example will be devoted to the consensus simulation of four first-order agents with uniform, logarithmic and dynamic quantizers. In particular, the underlying communication graph is a static line graph as in Figure 3.1 with the corresponding incidence matrix to the right:



$$B = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Figure 3.1: Line Graph

Starting from the initial position $(0.1, 0.175)$, $(0, 0)$, $(0.15, 0.05)$, $(0.2, 0.15)$, these agents obey the system dynamics (3.1) and three kind of quantizers are used separately to illustrate different results. First of all, uniform quantizers with uniform gain $\delta_u = 0.01$ are used. Figure 3.2 and 3.3 below illustrate the agents' trajectories and the final convergence error in both X and Y axis. Specifically, in Figure 3.3 we can see that though the agents can not converge to the agreement point but the difference between neighboring pairs $(1, 2)$, $(2, 3)$ and $(3, 4)$ are all smaller than $\frac{\delta_u}{2} = 0.005$, which approves our conclusions.

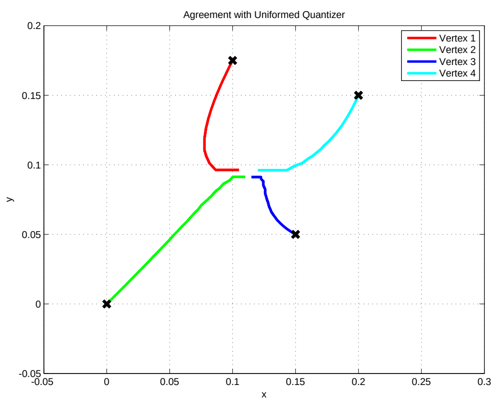


Figure 3.2: Convergence with uniform quantizers

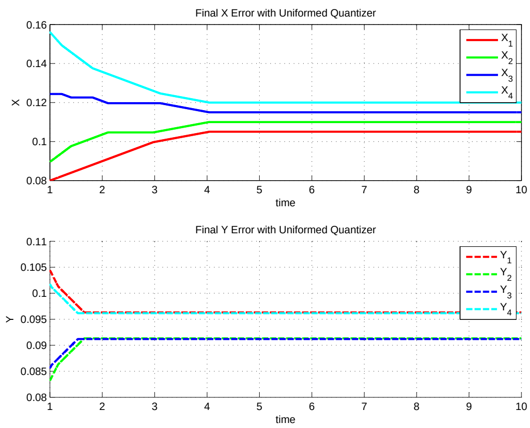


Figure 3.3: Final value in X, Y direction

Then logarithmic quantizers with gain $\delta_l = 1.5$ are used. Figure 3.4 below shows the trajectories in this case. An average consensus can be reached in an asymptotic way as in Figure 3.5 and all agents approaches the same agreement point.

At last, dynamic quantizers with gain $\delta_u = 0.01$ and the initial zooming variable $\mu = 1$ are utilized to demonstrate its implementation in the multi-agent system. Based on the settings

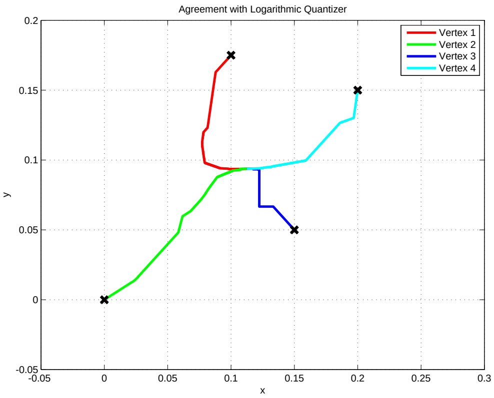


Figure 3.4: System with logarithmic quantizers

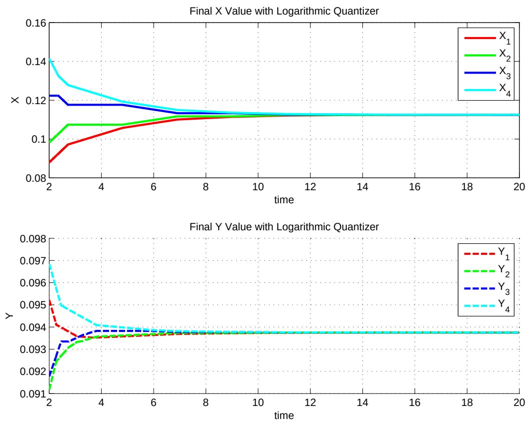


Figure 3.5: Final value in X, Y direction

in (3.5), other parameters are chosen as the maximal range $R = 0.0317$, updating interval $T = 118.9\,s$, zooming rate $\Omega = 0.7692$. Figure 3.6 below shows the system trajectories and Figure 3.7 illustrates the consecutive updating of zooming variable μ. In particular, asymptotic convergence to the average consensus can be seen in Figure 3.6.

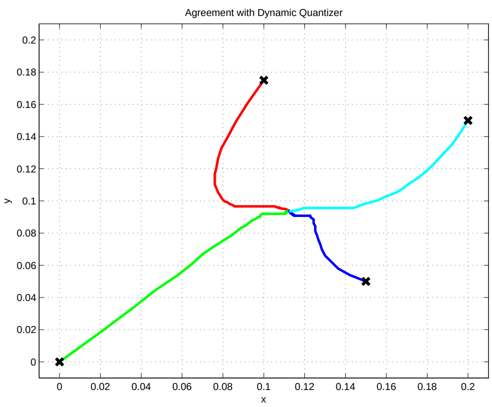


Figure 3.6: Convergence with dynamic quantizers

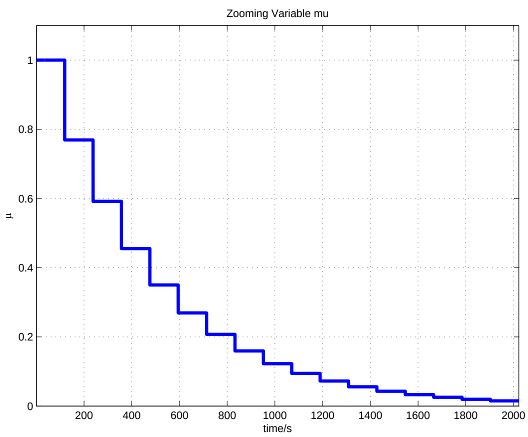


Figure 3.7: Updating of zooming variable mu

3.1.2 Time-varying Tree topology

In this section we treat the case when the communication topology is time-varying, allowing each agent to lose or create new communication links to other agents as the closed-loop system evolves. In particular, we consider that the underlying topology is switching among different tree topologies.

It is easy to notice that the dimension and elements of stack vector $\bar{x}$ change discontinuously whenever edges are added or deleted when the communication topology is time-varying. It seems that the old Lyapunov function is not feasible anymore. However in the this section, we will consider the same Lyapunov function $V = \bar{x}^T(B^TB)^{-1}\bar{x}$ and show that it can serve as a common Lyapunov function to prove convergence under switching tree topologies.

Inserting $\bar{x} = B^T x$ into V, we get

$$V = x^T B(B^T B)^{-1} B^T x \tag{3.6}$$

where V is now described by the state vector x rather than edge vector $\bar{x}$. Since the state vector x is continuous over all switching instances, the function V would be continuous under different communication topologies if the matrix $H = B(B^T B)^{-1} B^T$ remains the same when we have topology updates and thus, different incident matrices B.

First note that for undirected graphs with N vertices, it is always the case that any tree topology has exactly $N-1$ edges [10], which means that $m = N-1$ always holds. Thus, any incident matrix B of a tree topology has dimension $N \times (N-1)$. We assume its *singular value decomposition* to be $B = U\Sigma W^T$, where $U_{(N\times N)}$ is the left singular matrix, composed by the normalized eigenvectors of $BB^T_{(N\times N)}$, and $W_{((N-1)\times(N-1))}$ is the right singular matrix composed by the normalized eigenvectors of $B^T B_{((N-1)\times(N-1))}$, and the singular value matrix $\Sigma_{(N\times(N-1))}$ has the structure

$$\Sigma = \left[\begin{array}{cccc} \lambda_{N-1} & 0 & \cdots & 0 \\ 0 & \lambda_{N-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_1 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]_{N\times(N-1)}$$

where λ_i for $i = (N-1), \dots, 1$ are the singular values of B in descending order. We need to provide the following properties of matrix $B^T B$ first to obtain the main results later.

Lemma 3.3. *Assume B is the incidence matrix of any tree graph, then the matrix $B^T B$ has $N-1$ positive eigenvalues and the matrix $BB^T = L$ has N nonnegative eigenvalues, one of them equals to zero and the rest are the same as eigenvalues of $B^T B$.*

Proof. Using $B = U\Sigma W^T$ and denoting $T = \Sigma^T \Sigma$, we have

$$\begin{aligned} B^T B &= (U\Sigma W^T)^T U\Sigma W^T = W\Sigma^T U^T U\Sigma W^T \\ &= W\Sigma^T \Sigma W^T = WTW^T \end{aligned}$$

$$T = \Sigma^T \Sigma = \begin{bmatrix} \lambda_{N-1}^2 & 0 & \cdots & 0 \\ 0 & \lambda_{N-2}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_1^2 \end{bmatrix}_{(N-1)\times(N-1)}$$

where WTW^T is the singular value decomposition of $B^T B$. Moreover, after similar calculations for the matrix BB^T, and denoting $S = \Sigma\, \Sigma^T$, we get

$$\begin{aligned} BB^T &= U\Sigma W^T (U\Sigma W^T)^T = U\Sigma W^T W\Sigma^T U^T \\ &= U\Sigma\Sigma^T U^T = USU^T \end{aligned}$$

$$S = \Sigma\,\Sigma^T = \left[\begin{array}{cccc|c} \lambda_{N-1}^2 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_{N-2}^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_1^2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right]_{N\times N}$$

where USU^T is the singular value decomposition of BB^T, which is shown by matrix S to have N nonnegative eigenvalues. One of them is zero and the others are identical to the eigenvalues of B^TB by comparing S and T above.

Consider the zero eigenvalue of BB^T, since $BB^T = L$ [10] and $L\mathbf{1}_N = \mathbf{0}_N \Rightarrow BB^T\mathbf{1}_N = 0\cdot\mathbf{1}_N$, which means that $\mathbf{1}_N$ is the eigenvector associated with the eigenvalue zero, which is normalized to be $\frac{1}{\sqrt{N}}\mathbf{1}_N$. ∎

Lemma 3.4. *The matrices $H = B(B^TB)^{-1}B^T$ are identical for all possible incidence matrices B corresponding to an undirected tree graph.*

Proof. Again plugging $B = U\Sigma W^T$ into H we get

$$\begin{aligned} H = B(B^TB)^{-1}B^T &= U\Sigma W^T(W\Sigma^T\Sigma W^T)^{-1}W\Sigma^T U^T \\ &= U\Sigma W^T(WTW^T)^{-1}W\Sigma^T U^T = U\Sigma\,T^{-1}\Sigma^T U^T \end{aligned}$$

where

$$T^{-1} = \begin{bmatrix} \frac{1}{\lambda_{N-1}^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda_{N-2}^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\lambda_1^2} \end{bmatrix}$$

Denote by $G = \Sigma\,T^{-1}\Sigma^T$, which is calculated to be

$$G = \Sigma\,T^{-1}\Sigma^T = \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right]_{N\times N}$$

Consider the matrix $U_{N\times N} = [u_{N-1}\; u_{N-2} \dots u_0]$, composed of the normalized eigenvectors u_i of BB^T. Denote by $u_i(k)$ the k_{th} element of the column vector u_i and since U is the left singular orthogonal matrix and $UU^T = I_N$, we get

$$\sum_{k=0}^{N-1} u_k(i)\,u_k(i) = 1 \qquad \sum_{k=0}^{N-1} u_k(i)\,u_k(j) = 0\;(i \neq j)$$

for any $i,\ j \in \{1, \dots, N\}$. In Lemma 3.3 we have shown that the last eigenvector corresponding to the eigenvalue zero, is $u_0 = \frac{1}{\sqrt{N}}\mathbf{1}_N$.

Compute each entry of $H = UGU^T$ element-wise with $G_{N\times N}$ and $U_{N\times N}$ defined above:

$$H(i,j) = \sum_{k=1}^{N-1} u_k(i)\, u_k(j) = \sum_{k=0}^{N-1} u_k(i)\, u_k(j) - u_0(i)\, u_0(j)$$
$$= \begin{cases} 1 - \frac{1}{\sqrt{N}}\frac{1}{\sqrt{N}} = \frac{N-1}{N} & (i = j) \\ 0 - \frac{1}{\sqrt{N}}\frac{1}{\sqrt{N}} = -\frac{1}{N} & (i \neq j) \end{cases}$$

by which we can draw the conclusion that the matrix $H = B(B^TB)^{-1}B^T$ is the same, for any incidence matrix B corresponding to a tree graph.

$$H = \begin{bmatrix} \frac{N-1}{N} & -\frac{1}{N} & \cdots & -\frac{1}{N} \\ -\frac{1}{N} & \frac{N-1}{N} & \cdots & -\frac{1}{N} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{N} & -\frac{1}{N} & \cdots & \frac{N-1}{N} \end{bmatrix}_{N\times N}$$

This completes the proof. ■

Now that the time-varying communication graph remains a tree for continuous evolution, and under the same control protocol (2.10), consider the candidate Lyapunov function $V_g = x^T Hx$. Since $V_g = x^T B(B^TB)^{-1}B^T x = \bar{x}^T(B^TB)^{-1}\bar{x} \geq 0$, V_g is positive semidefinite and compact with respect to the agents' edge relative distance as stated in the static case.

The notation $\mathcal{T} = \{t_1, t_2, \cdots t_j\}$ is used for the set of switching instants. Through the time (especially at $\mathcal{T}$), the function V_g is continuous as the matrix H is constant from Lemma 3.4 and the state x is continuous. So if V_g is decreasing between any two switching instants, we can say that V_g is decreasing over all time. During any time interval $[t_j, t_{j+1}]$, assume the corresponding tree topology has the incidence matrix B_j. Using the same Lyapunov function as the static case $V_j = \bar{x}^T(B_j^TB_j)^{-1}\bar{x} = x^T Hx = V_g$, we have $\dot{V}_g = \dot{V}_j = -2\bar{x}\, q(\bar{x}) \leq 0$ and $\dot{V}_g = 0$ holds when $q(\bar{x}) = 0$. Applying the same analysis as Theorem 3.1 and 3.2, we can have the following theorem:

Theorem 3.5. *Assume that the system (2.6) evolves under the control law (2.10) and the finite switching communication graph $G(t)$ remains a tree. Then all agents are guaranteed to converge to*

$$\mathcal{I}(\boldsymbol{x}) = \{\boldsymbol{x}|\, q(x_i - x_j) = 0, \forall (i,j) \in E_n\}$$

where E_n denotes the edge set of the last topology.

With identical uniform quantizers and finite switching tree topology, all agents converge to

$$\mathcal{I}_n = \{\mathbf{x}||x_i - x_j| < \frac{\delta_u}{2}, (i,j) \in E_n\} \tag{3.7}$$

in finite time, where E_n denotes the edges set of the last topology. Whereas with same logarithmic quantizers and all possible switching tree topologies, asymptotic convergence to the average consensus is guaranteed for any logarithmic gain $\delta_l > 0$.

Theorem 3.6. *Assume that the finite switching communication graph $G = G(t)$ remains a tree for all continuous evolution intervals. Then system (3.1) has the following convergence properties:*

- *In the case of uniform quantizers, the system converges to the consensus set (3.7), which depends on the last tree topology in finite time.*
- *In the case of logarithmic quantizers, the system asymptotically converges to the average consensus point for all logarithmic gains $\delta_l > 0$.*

All results above are independent of the way that the topology switches as long as the tree structure is kept.

Example 3.2. In this example, we will show how the same system as in Example 3.1 evolves under switching tree topologies. The following finite switching communication graph is used

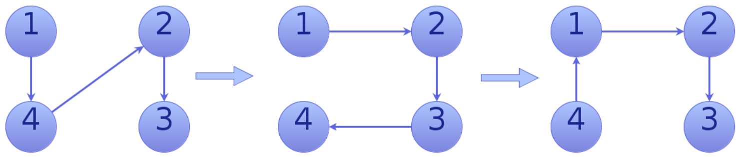


Figure 3.8: Switching Tree Topology

Same parameters and settings as in Example 3.1, the uniform gain δ_u is 0.01 and the logarithmic gain δ_l is 1.5. Red dots indicate the time instance when topology switches and black crosses are the starting point in Figure 3.9. We can see that average consensus can not be reached but the final difference between neighbors (1, 4), (1, 2), (2, 3) in Figure 3.10 is less than 0.005.

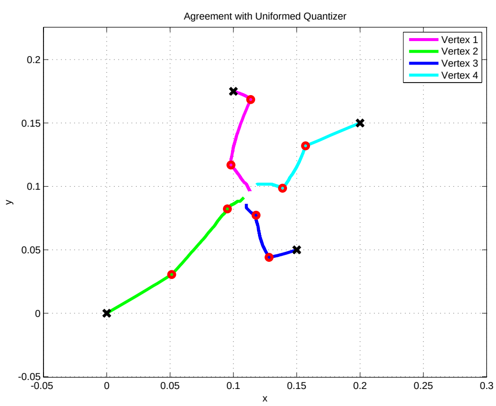


Figure 3.9: Convergence with uniform quantizers

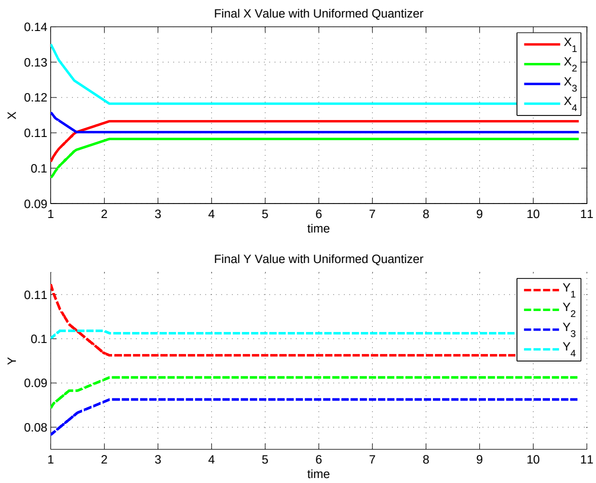


Figure 3.10: Final value in X, Y direction

When it comes to logarithmic quantizers, it is shown in Figure 3.11 that the average consensus can be reached in a asymptotic way and the consensus error between neighbors in Figure 3.12 becomes zero asymptotically.

3.1.3 Loss of Connectivity in Time-intervals

Above results from Theorem 3.2 and 3.6 are useful whenever the communication topology retains the tree structure at all switching instances. A more practical situation however

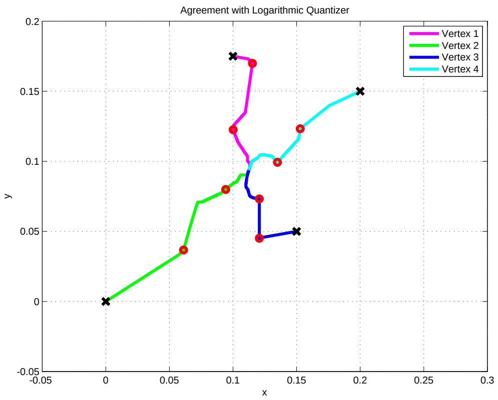


Figure 3.11: Convergence with Logarithmic quantizers

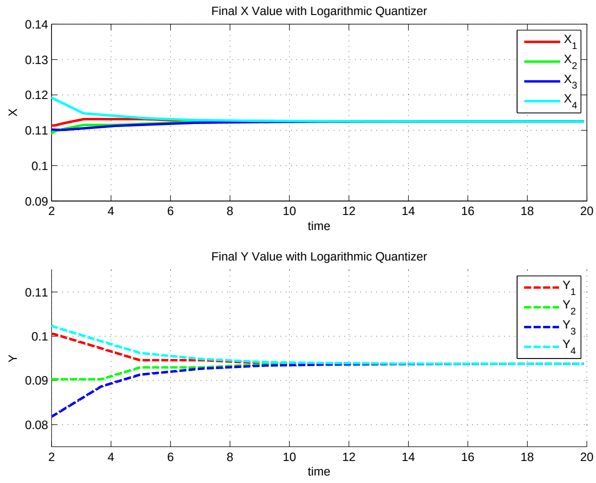


Figure 3.12: Final value in X, Y direction

occurs if we allow for the tree assumption to be lost during some time intervals. In particular, similar to the third section in [7], we assume that in between instants where the team switches to a different tree structure, there are time intervals where the connected tree assumption is not guaranteed to hold.

Lemma 3.7. *With control protocol (2.10) and system (3.1), the sum of agents' state keeps constant over all time under undirected topologies.*

Proof. By taking the derivative of the sum, we get

$$\sum_{i=1}^{N} \dot{x}_i = \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} q(x_i - x_j)$$

since $q(x_i - x_j) = -q(x_j - x_i)$ holds for all quantizers and the graph is undirected, we have

$$\sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} q(x_i - x_j) = 0 \Rightarrow \sum_{i=1}^{N} x_i = \sum_{i=1}^{N} x_i(0)$$

Denote by $\frac{1}{N}\sum_{i=1}^{N} x_i(0) = C$ the average of initial states, where C is an arbitrary constant, then $\frac{1}{N}\sum_{i=1}^{N} x_i(t) = C$ over all time. ■

Lemma 3.8. *Consider three consecutive switching instances t_1, t_2, t_3, with a tree topology G during $[t_1, t_2]$, and that one communication edge within G is disconnected during $[t_2, t_3]$. Then with an existing tree subgraph during $[t_2, t_3]$, the function $V_g(G) = x^T H x$ is strictly decreasing during $[t_1, t_3]$,*

- *with logarithmic quantizers.*
- *with uniform quantizers, as long as all its subgraphs have not converged to their different consensus sets.*

Proof. Using the same Lyapunov function $V_g = x^T Hx$ as before, we substitute constant matrix H and take into account the invariance of system center. V_g can be computed as

$$\begin{aligned} V_g(G) &= \frac{1}{N}\sum_{i=1}^{N} x_i[Nx_i - (x_1 + x_2 + \ldots + x_N)] \\ &= \frac{1}{N}\sum_{i=1}^{N} x_i(Nx_i - NC) = \sum_{i=1}^{N} x_i^2 - NC^2 \end{aligned} \tag{3.8}$$

When one communication edge in G is disconnected during $[t_2, t_3]$, G can be separated into two subgraphs G_1 and G_2, which are either both trees or a single agent and a tree. Denote the vertex set of the topology G_1 by V_1, and the vertex set of G_2 by V_2, then $V_1 \cup V_2 = V$ holds, which means that $N_1 = |V_1|$ and $N_2 = |V_2|$ should satisfy $N_1 + N_2 = N = |V|$.

For the same reason as in Lemma 3.7 and the above equation (3.8), the sums of agent states within G_1 and G_2 are also kept constant and denoted by C_1 and C_2. So we get

$$V_g(G_1) = \sum_{j\in V_1} x_i^2 - N_1C_1^2 \quad V_g(G_2) = \sum_{k\in V_2} x_k^2 - N_2C_2^2$$

where $N_1 + N_2 = N$ and $C_1 + C_2 = C$. Then we have

$$\begin{aligned} V_g(G) &= (\sum_{j\in G_1} x_i^2 + \sum_{k\in G_2} x_k^2) - NC^2 \\ &= V_g(G_1) + V_g(G_2) + K \end{aligned}$$

where the constant $K = N_1C_1^2 + N_2C_2^2 - NC^2$.

If G_1 is a tree topology, then it has been proven in the static topology case that $V_g(G_1)$ is strictly decreasing with the logarithmic quantizer, or with the uniform quantizer before it reaches its convergence set (3.3). If G_1 is a single agent, $V_g(G_1)$ remains constant without any information exchange. The same result holds for G_2. This implies that during the reset interval function $V_g(G_1) + V_g(G_2)$ should be strictly decreasing with logarithmic quantizers, or with uniform quantizers as long as G_1 and G_2 have not converged to and stay at their individual consensus sets. Consequently, $V_g(G)$ is strictly decreasing during $[t_1, t_3]$ with logarithmic quantizers, or with uniform quantizers as long as its subgraphs have not converged to their consensus sets. ∎

In the case of several communication edges being lost, we can apply Lemma 3.8 recursively to draw the conclusion that as long as there are still existing tree subgraphs, the global Lyapunov function V_g is ensured to be strictly decreasing in the presence of logarithmic quantizers because of the asymptotic convergence. While in the case of uniform quantizers the function V_g is strictly decreasing with existing tree subgraphs as long as all its subgraphs have not reached their different consensus sets.

Since V_g is a continuous function over all time, it is guaranteed that if the system is reset to have a connected tree topology from some point in time, the average consensus for the whole system is achieved with logarithmic quantizers while the set convergence (3.3) around the agreement point is guaranteed with uniform quantizers. Otherwise with unbounded interval time, all subgraphs within the system would asymptotically converge to (with the logarithmic quantizer) or stay at (with the uniform quantizer) different consensus sets (no change for a single agent and average consensus for a tree subgraph).

3.2 General Undirected Graphs

In contrast to the previous section where we concentrate on the convergence of the multi-agent system under tree topologies, we will extend the results to general undirected graphs that contain weighted edges and non-empty cycle space. Specifically, a more general consensus protocol for first order agents (2.6) is proposed in [22]

$$u_i = -\sum_{j=1}^{N} a_{ij}\, q(x_i - x_j)$$

where ($a_{ij} \geq 0$) are the *weights* or *reliability* associated to each edge $e \in E$. We have $a_{ij} = 0$ if there is no information exchange between agent i and j and the absolute value of a_{ij} depends on the weights or trust agent i puts on this link e_{ij}. We always assume that the graph G is undirected and it is required that $a_{ij} = a_{ji}$ always holds for $i, j = 1, \ldots, N$. The closed-loop system is given by

$$\dot{x}_i = -\sum_{j=1}^{N} a_{ij}\, q(x_i - x_j) \qquad (a_{ij} \geq 0) \tag{3.9}$$

Lemma 3.9. *Along the trajectories of system (3.9), the sum of agent states* $\sum_{i=1}^{N} x_i$ *is constant over all time for undirected topologies.*

Proof. By taking the time derivative of the sum of states, we get

$$\sum_{i=1}^{N} \dot{x}_i = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\, q(x_i - x_j)$$

Since $q(x_i - x_j) = -q(x_j - x_i)$ holds for all quantizers in Section 2.2 and $a_{ij} = a_{ji}$ for undirected topologies, we have

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\, q(x_i - x_j) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} (a_{ij} - a_{ji})\, q(x_i - x_j) = 0$$

$$\Rightarrow \quad \sum_{i=1}^{N} \dot{x}_i = 0 \quad \Rightarrow \quad \sum_{i=1}^{N} x_i(t) = \sum_{i=1}^{N} x_i(0)$$

Denote the initial average of the states by $\frac{1}{N}\sum_{i=1}^{N} x_i(0) = C$. Then $\frac{1}{N}\sum_{i=1}^{N} x_i(t) = C$, which remains constant over all time. So if an agreement is reached under undirected topologies, it is certain that this agreement point equals to the average consensus. ∎

3.2.1 Stability and Convergence

In order to investigate the stability and convergence of the closed-loop system under general undirected graphs, it is tempting to rely on the edge Laplacian matrix $B^T B$ as in Section 3.1. However due to the fact that cycles may exist in a general undirected graph, $B^T B$ is not positive definite anymore and its inverse $(B^T B)^{-1}$ does not exist. So the Lyapunov function

$V_g = x^T B(B^T B)^{-1} B^T x$ introduced in Lemma 3.8 becomes infeasible in this case. However the function V_g (3.8) gives us some hints about the structure of the global Lyapunov function we may use to consider general undirected graphs. Let the candidate Lyapunov function for system (3.9) to be

$$V = \frac{1}{N}\sum_{i=1}^{N}(x_i - \frac{1}{N}\sum_{i=1}^{N} x_i)^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - C)^2$$

which is positive semidefinite and $V = 0$ when all states equal to the initial average. V can be viewed as a quadratic disagreement function to the average consensus. The level sets of V define compact sets with respect to the agents' state. Specifically, for $i = 1, \ldots, N$ we have

$$V \le c \Rightarrow (x_i - C)^2 \le Nc \Rightarrow |x_i - C| \le \sqrt{Nc} \Rightarrow C + \sqrt{Nc} \le x_i \le C - \sqrt{Nc}$$

The time derivative of V along the trajectories of the closed-loop system is:

$$\begin{aligned}\dot{V} &= \frac{2}{N}(\sum_{i=1}^{N} x_i \cdot \dot{x_i} - C\sum_{i=1}^{N}\dot{x_i}) = -\frac{2}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} x_i \cdot a_{ij}\, q(x_i - x_j) \\ &= -\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}(a_{ij}x_i - a_{ji}x_j)\, q(x_i - x_j) \\ &= -\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}[(x_i - x_j)\, q(x_i - x_j)] \le 0\end{aligned}$$

because $(x_i - x_j)\, q(x_i - x_j) \ge 0$ for all three quantizers we consider in Section 2.2. It indicates that V is strictly decreasing until $a_{ij}(x_i - x_j)\, q(x_i - x_j) = 0$ for all $i, j = 1, 2 \ldots, N$. There are three cases when $a_{ij}(x_i - x_j)\, q(x_i - x_j) = 0$ is satisfied: 1. $a_{ij} = 0$. 2. $x_i - x_j = 0$. 3. $q(x_i - x_j) = 0$. Case 2 and 3 can be combined as $q(0) = 0$ holds for both uniform and logarithmic quantizers. So $\dot{V} = 0$ either when $a_{ij} = 0$ or $q(x_i - x_j) = 0$. For undirected graphs, we mentioned that $a_{ij} = a_{ji}$ and $a_{ij} = 0$ if agent i and agent j are not neighbors, i.e., $(i, j) \notin E$. Consequently, $\dot{V} = 0$ occurs only when $q(x_i - x_j) = 0$ for all pairs of agents $(i, j) \in E$.

Theorem 3.10. *Assume the underlying communication graph G is undirected and there is a finite switching sequence. Then the quantized closed-loop system (3.9) is guaranteed to converge to*

$$\mathcal{I}(x) = \{\boldsymbol{x} |\, q(x_i - x_j) = 0,\, \forall (i, j) \in E_n\} \tag{3.10}$$

where E_n denotes the edge set of the last topology.

Proof. The proposed Lyapunov function V is positive semidefinite and continuous over all switching instances and is also compact with respect to the states. $\dot{V} \le 0$ is satisfied for all time. Application of LaSalle's Invariance Principle for hybrid systems ensures the convergence of the system to the largest invariant subset $\dot{V} = 0$. Since the switching is finite, the last graph determines the last invariance set. Thus all agents converge to the set $\mathcal{I}(x) = \{\mathbf{x} |\, q(x_i - x_j) = 0,\, \forall (i, j) \in E_n\}$, where E_n is the edge set of the last topology. Of course, this conclusion is valid for the static case where we have an invariant graph. ■

Theorem 3.11. *With a finite switching and undirected communication topology, the closed-loop system (3.9) is guaranteed to converge to*

- $\{\boldsymbol{x} |\, |x_i - x_j| < \frac{\delta_u}{2}, \forall (i,j) \in E_n\}$ *in the case of uniform quantizers.*
- $\{\boldsymbol{x} |\, x_i = x_j, \forall (i,j) \in E_n\}$ *in the case of logarithmic quantizers.*

where E_n denotes the edge set of the last topology.

Proof. Applying the conclusions of Theorem 3.10 to different models of uniform and logarithmic quantizers, we have that $q(x_i - x_j) = 0$ indicates $|x_i - x_j| < \frac{\delta_u}{2}$ for uniform quantizers and $x_i - x_j = 0$ for logarithmic quantizers. Then it is straightforward to conclude the above results by determining the invariant set in Theorem 3.10. This theorem applies also to the case with static graphs. ∎

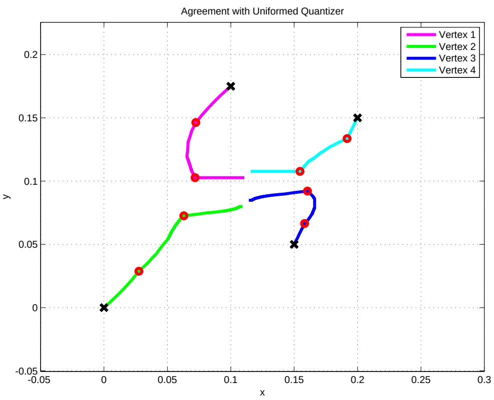


Figure 3.13: Undirected graph with uniform quantizers

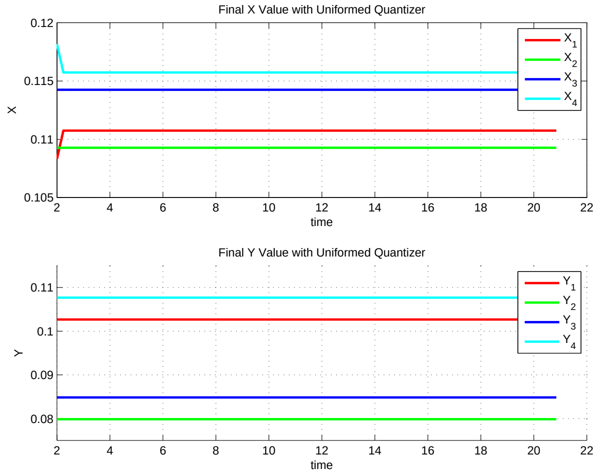


Figure 3.14: Final value in X, Y direction

Example 3.3. General undirected graphs with weighted edges are used in this simulation. We chose a time-varying topology that switches between several undirected graphs, connected and disconnected. This example illustrates how the system (3.9) evolves under the finite switching communication graph in Figure 3.15. These real numbers on communication links denote the weights a_{ij} we chose in the protocols.

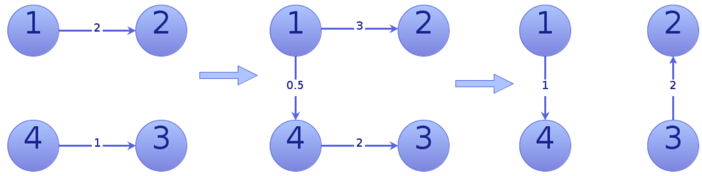


Figure 3.15: Time-varying undirected Topology

Figure 3.13 shows the agents' trajectories with uniform quantizers $\delta_u = 0.01$. Black crosses denote the initial position and red dots show the time instants when the communication topology switches. The steady state of each agent in X, Y direction is illustrated in Figure

3.14. Neighboring agents $(1,4)$ and $(2,3)$ all have difference less than $\frac{\delta_u}{2} = 0.005$ at the final configuration.

Logarithmic quantizers $\delta_l = 1.5$ are used under same environment settings. Certain advantages of logarithmic quantizers can be seen in Figure 3.16 and 3.17. Independent of the way the underlying graph switches, the steady state depends on the edge set of the last topology, which is $(1,4)$, $(2,3)$ in this example. So the differences between communicating neighbors $(1,4)$, $(2,3)$ become zero asymptotically.

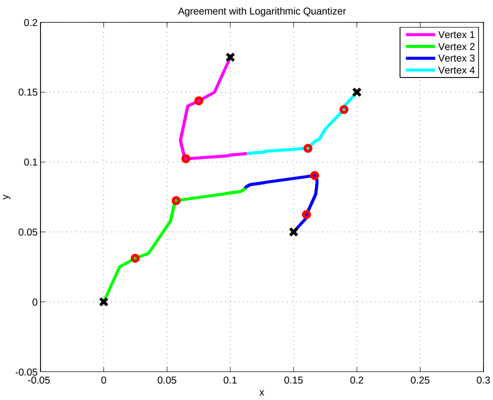


Figure 3.16: Undirected graph with logarithmic quantizers

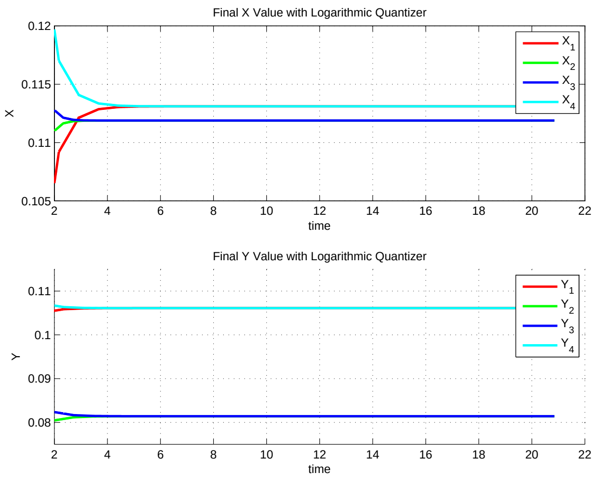


Figure 3.17: Final value in X, Y direction

However without requirements on the connectivity of the communication graph, the above theorem does not ensure the convergence to the average consensus as in Example 3.3. The following theorem considers the sufficient condition for the average consensus.

Theorem 3.12. *Assume the time-varying communication graph G remains undirected and connected for all time. Then the multi-agent system (3.9) is guaranteed to converge to*

- *the convergence set $\{\boldsymbol{x} | |\boldsymbol{x} - \mathbf{c}| < \frac{\delta_u}{4}\sqrt{\frac{N(N^2-1)}{3}}\}$, a ball centered at the average agreement point, in the case of uniform quantizers.*
- *the average consensus in the case of logarithmic quantizers.*

Proof. Since now the undirected graph G is connected, then there exists a path between any two vertices. It is easy to notice that the maximum length of these paths is $N-1$. In the case of uniform quantizers, since

$$|\mathbf{x} - \mathbf{c}|^2 = \sum_{i=1}^{N} (x_i - C)^2 = \frac{1}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} (x_i - x_j)^2$$

which is determined by the square sum of relative difference of all agent pairs. We first consider two extreme situations:

Assume that there are exactly $N-1$ edges $\{e_1, e_2 \cdots, e_{N-1}\}$ in G, which corresponds to a tree. At steady state we have that $|x_i - x_j| < \frac{\delta_u}{2}, (i,j) \in E$. Denote $x_{\max} = \max\{x_1, x_1, \cdots, x_N\}$

and $x_{\min} = \min\{x_1, x_1, \cdots, x_N\}$. There is a path connecting $x_{\max}$ and $x_{\min}$, with maximum length $N-1$.

$$|x_i - x_j| \le |x_{\max} - x_{\min}| = |(x_{\max} - x_m) + (x_m - x_t) + \cdots + (x_t - x_{\min})| \le \frac{\delta_u}{2}\,(N-1)$$

for any $(i,j) \in V \times V$. But a tighter constraint can be computed by merging $|x_i - x_j|$ consecutively as

$$\begin{aligned}
\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}(x_i - x_j)^2 &\le (N-1)(\frac{\delta_u}{2})^2 + (N-2)(\frac{\delta_u}{2}\cdot 2)^2 + \cdots + (\frac{\delta_u}{2}\cdot(N-1))^2 \\
&= (\frac{\delta_u}{2})^2 \sum_{k=1}^{N-1}(N-k)\,k^2 \quad (\text{No. of Elements} \sum_{k=1}^{N-1} k = \frac{N^2-N}{2}) \\
&= (\frac{\delta_u}{2})^2\,\frac{N^2(N^2-1)}{12} \\
\Rightarrow |\mathbf{x} - \mathbf{c}| &= \sqrt{\frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}(x_i - x_j)^2} \le \frac{\delta_u}{4}\sqrt{\frac{N(N^2-1)}{3}}
\end{aligned} \tag{3.11}$$

Another extreme situation is that there are $N_e = |E| = \frac{N^2-N}{2}$ edges, which corresponds to a complete graph where any two of the agents are connected. Thus the steady state would be $|x_i - x_j| \le \frac{\delta_u}{2}$ for any $(i,j) \in V \times V$. We have

$$\begin{aligned}
\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}(x_i - x_j)^2 &\le \frac{N^2-N}{2}(\frac{\delta_u}{2})^2 \\
\Rightarrow |\mathbf{x} - \mathbf{c}| &= \sqrt{\frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}(x_i - x_j)^2} \le \frac{\delta_u}{2}\sqrt{\frac{N-1}{2}}
\end{aligned} \tag{3.12}$$

as a result the system will converge to the ball $|\mathbf{x} - \mathbf{c}| \le \frac{\delta_u}{2}\sqrt{\frac{N-1}{2}}$, if the graph G is always completed.

Between the above two extreme situations, we take into account the ordinary cases that the connected graph has edges number satisfying $N - 1 < N_e \le \frac{N^2-N}{2}$, by manipulating and merging $|x_i - x_j|$ consecutively,

$$\begin{aligned}
\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}(x_i - x_j)^2 \le{}& [(N_e)(\frac{\delta_u}{2})^2 + (N_e - 1)(\frac{\delta_u}{2}\cdot 2)^2 + \cdots + k(\frac{\delta_u}{2}\cdot(N_e + 1 - k))^2] \\
&+ (\frac{N^2-N}{2} - \frac{(N_e+k)(N_e-k+1)}{2})(\frac{\delta_u}{2}\cdot(N_e + 2 - k))^2 \\
={}& g(N_e, N, k, \delta_u) \\
\Rightarrow \quad |\mathbf{x} - \mathbf{c}| \le{}& \sqrt{\frac{1}{N} g(N_e, N, k, \delta_u)}
\end{aligned}$$

where k is the smallest integer bigger than the positive root of $(N_e + k)(N_e - k + 1) = N^2 - N$, namely $k = \lceil \frac{1+\sqrt{4[N_e^2+N_e-(N^2-N)]+1}}{2} \rceil$.

Since it is difficult to obtain explicit expression of the above function $g(N_e, N, k, \delta_u)$, we simulate the above algorithm instead of having an explicit expression. To illustrate the relation between convergence radius and the number of edges, the upper bounds are calculated for thirty agents and $\delta_u = 0.01$. As we expected, more edges would lead to a smaller convergence set around the average consensus.

With respect to the case of logarithmic quantizers, the closed-loop system is guaranteed to asymptotically converge to $\{\mathbf{x}| \, x_i = x_j, \forall (i,j) \in E_n\}$, as stated in Theorem 3.11. Now since the graph G remains connected, the agent pairs $(i,j) \in \mathcal{P}$ with $x_i = x_j$ can be jointed together by equality, leading to the agreement point $x_1 = x_2 = \cdots = x_N$. By virtue of Lemma 3.9, this point is the average consensus. ∎

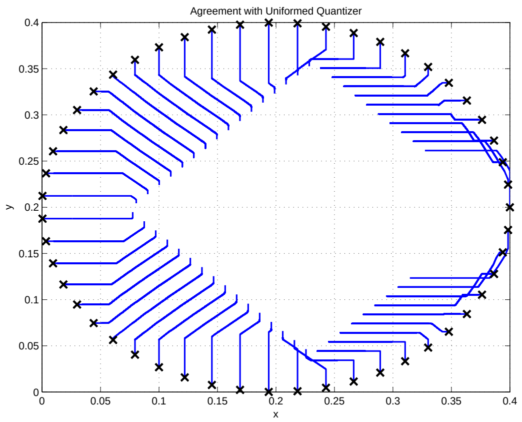


Figure 3.18: Convergence with line graph

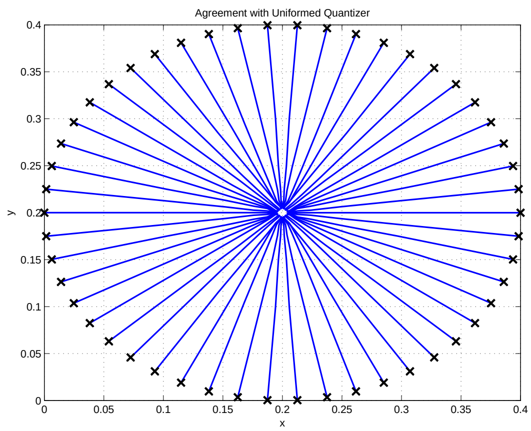


Figure 3.19: Convergence with complete graph

Example 3.4. In this example, we illustrate a larger scale multi-agent system with 50 agents. The same system (3.2) is simulated under a linear communication graph (3.11) and a complete graph (3.12), with the uniform quantizer $\delta_u = 0.02$ to compare the size of the final convergence area.

As expected, in Figure 3.18 the difference between all neighboring agents is less than $\frac{\delta_u}{2} = 0.01$ but the largest difference among these agents reaches more than 0.25 due to accumulation of errors. While in Figure 3.19 since a complete communication graph is used and any two of the members have communication, the corresponding steady configuration has consensus error less than 0.01 between any pair of agents.

From the above analyses, we may notice that the uniform gain greatly affects the system performance since smaller δ_u leads to more accurate consensus. On the other hand, more bits are needed to represent the same real number with smaller uniform gain and as a result more bandwidth and computational energy are consumed. Moreover, from Example 3.4 above we already see that the structure of underlying communication graph also has great impact on the consensus performance as in Figure 3.18 and 3.19, which is related to the connectivity $\lambda_2(L)$ of the graph as discussed in [21].

3.2.2 Convergence Rate

In the previous section, we mentioned that the connectivity of a graph can effect the size of convergence area under uniform quantizers. In this part, we will introduce the relation between the algebraic connectivity and the convergence rate to the average consensus, as in [21], [22]. The algebraic connectivity $\lambda_2(L)$ is the second largest eigenvalue or the first non-trivial eigenvalue of the Laplacian matrix. In the following propositions we consider the convergence rate in the presence of quantized information.

Uniform Quantizer

We have derived that $\dot{V} = -\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}[(x_i - x_j)\; q(x_i - x_j)]$, which is negative semidefinite. In order to obtain the convergence rate, we need to find out the relation between V and $\dot{V}$. From Lemma 2.1 we have $(x_i - x_j)\cdot q_u(x_i - x_j) > \frac{2}{3}(x_i - x_j)^2$ when $q_u(x_i - x_j) \neq 0$. The term $(x_i - x_j)\cdot q_u(x_i - x_j)$ in $\dot{V}$ can be lower bounded by $\frac{2}{3}(x_i - x_j)^2$ before any pair of agents satisfy $q_u(x_i - x_j) = 0$.

$$
\begin{aligned}
N\cdot\dot{V} &\leq -\frac{2}{3}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}[(x_i - x_j)^2] = -\frac{2}{3}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}[(x_i - C) - (x_j - C)]^2 \\
&= -\frac{4}{3}\;[\sum_{i=1}^{N}(x_i - C)^2\sum_{j=1}^{N} a_{ij} - \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(x_i - C)(x_j - C)] \\
&= -\frac{4}{3}\begin{bmatrix} x_1 - C \\ x_2 - C \\ \vdots \\ x_N - C \end{bmatrix}^T \begin{bmatrix} \sum_{j=1}^{N} a_{1j} & -a_{12} & \cdots & -a_{1N} \\ -a_{21} & \sum_{j=1}^{N} a_{2j} & \cdots & -a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{N1} & -a_{N2} & \cdots & \sum_{j=1}^{N} a_{Nj} \end{bmatrix} \begin{bmatrix} x_1 - C \\ x_2 - C \\ \vdots \\ x_N - C \end{bmatrix} \\
&= -\frac{4}{3}\;(\mathbf{x} - \mathbf{c})^T L\;(\mathbf{x} - \mathbf{c})
\end{aligned}
$$

Assume the graph G is static and the Laplacian matrix is time-invariant. By the Courant-Fisher Theorem and the fact that $L\;\mathbf{1}_N = \mathbf{0}_N$, we get:

$$
\lambda_2(L) = \min_{\mathbf{z}\neq 0 \quad \mathbf{1}^T\mathbf{z}=0} \frac{\mathbf{z}^T L\mathbf{z}}{\mathbf{z}^T\mathbf{z}} \quad \Rightarrow \quad \mathbf{z}^T L\mathbf{z} \geq \lambda_2(L)\;\mathbf{z}^T\mathbf{z}
$$

since $\mathbf{1}^T(\mathbf{x} - \mathbf{c}) = 0$, we can substitute $\mathbf{z}$ by $\mathbf{x} - \mathbf{c}$:

$$
N\dot{V} = -\frac{4}{3}\;(\mathbf{x} - \mathbf{c})^T L\;(\mathbf{x} - \mathbf{c}) \leq -\frac{4}{3}\lambda_2(L)\;(\mathbf{x} - \mathbf{c})^T(\mathbf{x} - \mathbf{c}) = -\frac{4}{3}\lambda_2(L)\;NV
$$

By applying the Comparison Lemma, we get the following estimates of the convergence rate for uniform quantizers:

$$
V(t) \leq e^{-\frac{4}{3}\lambda_2(L)t}V(0) \Rightarrow \quad |\mathbf{x}(t) - \mathbf{c}| \leq e^{-\frac{2}{3}\lambda_2(L)t}|\mathbf{x}(0) - \mathbf{c}|
$$

which only holds before one pair of the agents satisfies $|x_i - x_j| < \frac{\delta_u}{2}$. Assume the graph G is connected, the $\lambda_2(L)$ is strictly positive [22]. So the system is guaranteed to converge at certain rate before one pair of agents have absolute difference less than $\frac{\delta_u}{2}$.

Logarithmic Quantizer

Following exactly the same analysis as above, but considering now that $q_l(a) \cdot a > \frac{1}{1+\delta_l} a^2$, $\forall a \in \mathbb{R}$ from the Lemma 2.2, we have

$$V(t) \leq e^{-\frac{2}{1+\delta_l}\lambda_2(L)t} V(0) \Rightarrow \quad |\mathbf{x}(t) - \mathbf{c}| \leq e^{-\frac{1}{1+\delta_l}\lambda_2(L)t} |\mathbf{x}(0) - \mathbf{c}|$$

for $t \geq 0$. Clearly smaller logarithmic gain yields to a faster convergence of the closed-loop system. If the graph G is connected and $\lambda_2(L) > 0$ [22], the agents converges to the average consensus at certain rate for all time $t \geq 0$. We conclude the above discussion with the following theorem:

Theorem 3.13. *Assume the static communication graph G is undirected and connected. Then the closed system (3.9) converges to the invariant set (3.10):*

- *with a rate $\frac{2}{3}\lambda_2(L)$ before the time when one pair of agents satisfies $|x_i - x_j| < \frac{\delta_u}{2}$, in the case of uniform quantizers.*
- *with a rate $\frac{1}{1+\delta_l}\lambda_2(L)$ in the case of logarithmic quantizers.*

The above theorem also holds for time-varying undirected and connected graphs, with small modifications that $\lambda_2(L)$ is replaced by $\lambda_{\min} = \min\{\lambda_2(L(G(t)))\}$. In other words, the system at least has a convergence rate based on the minimal $\lambda_2(L)$ of all possible Laplacian matrices.

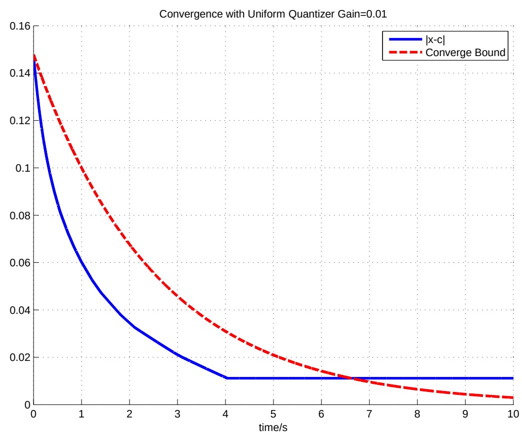


Figure 3.20: Convergence with Uniform Quantizer

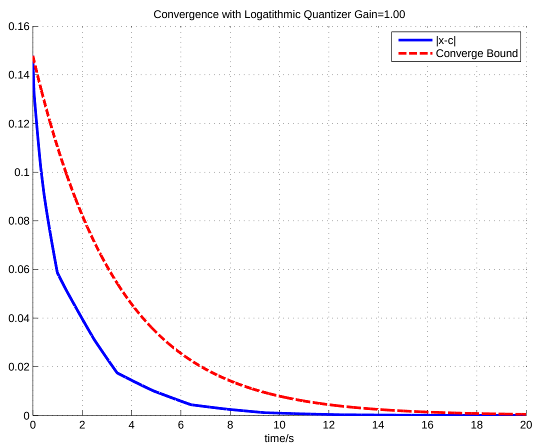


Figure 3.21: With Logarithmic Quantizer

Example 3.5. To illustrate the convergence rate under different quantization models, we simulate the process with a tree graph and same parameters as in Example 3.1. The trajectories of $|\mathbf{x} - \mathbf{c}|$ are plotted in blue line and the exponential reference curve with certain rate in red dashed line.

As stated in Theorem 3.13, the system with uniform quantizers has the convergence rate $\frac{2}{3}\lambda_2(L)$ until some of the agent pairs have difference smaller than $\frac{\delta_u}{2}$ as in Figure 3.20. On the other hand, with logarithmic quantizers as in Figure 3.21 a convergence rate $\frac{1}{1+\delta_l}\lambda_2(L)$ is preserved all time to ensure the asymptotic convergence to $|\mathbf{x} - \mathbf{c}| = 0$, namely the average consensus point.

3.3 Directed Topology

In this section, we consider the same system dynamics $\dot{x}_i = -\sum_{j=1}^{N} a_{ij}\, q(x_i - x_j)$ under directed communication graphs, which means $a_{ij} \neq a_{ji}$. The techniques we used in Section 3.2 are not feasible anymore and the sum of states is not maintained constant. Here we only tackle the system stability with the presence of logarithm quantizers.

Theorem 3.14. *The closed-loop system (3.9) with logarithmic quantizers is guaranteed to reach average consensus asymptotically, if the directed graph has a spanning tree.*

Proof. Denote $x_{\max} = \max\{x_1, x_2, \cdots, x_N\}$ and $x_{\min} = \min\{x_1, x_2, \cdots, x_N\}$. If multiple agents have the maximum value, a random one is chosen and the same way for minimum value. Propose the candidate Lyapunov function:

$$V = x_{\max} - x_{\min}$$

It is easily verified that $\dot{V} \leq 0$ and the level sets of V are compact with respect to the relative states. In particular, if $V \leq c$, then we have

$$|x_i - x_j| \leq |x_{\max} - x_{\min}| \leq c$$

for any $(i, j) = 1, \cdots, N$. Furthermore, since $x_i \leq x_{\max}$ and $x_i \geq x_{\min}$, we have

$$\dot{x}_{\max} = -\sum_{j=1}^{N} a_{ij}\, q_l(x_{\max} - x_j) \leq 0$$
$$\dot{x}_{\min} = -\sum_{j=1}^{N} a_{ij}\, q_l(x_{\min} - x_j) \geq 0$$

which means that the function V is non-increasing. We will focus on the case when $\dot{V} = 0$, namely $\dot{x}_{\max} = \dot{x}_{\min} = 0$.

Suppose k is the agent for which the maximum is achieved at time t. If many we choose anyone of those. When $\dot{x}_k = 0$, which is given as

$$\dot{x}_k = -\sum_{j=1}^{N} a_{kj}\, q_l(x_k - x_j) = 0$$

since $x_j \leq x_k = x_{\max}$, it means $x_j = x_k$, $\forall j \in N_k$. Namely, all agents that agent k communicates with should have the same value as k. Repeat the above arguments with respect to all agents $j \in N_k$ and all agents $i \in N_j$, etc. In particular, all agents i having a directed path from i to k belongs to $\mathcal{L}$. We denote $\mathcal{L} = \{i \,|\, x_i = x_k,\ i \in \mathcal{V}\}$, where $\mathcal{V}$ is the vertex set. Thus agents $\mathcal{V} \setminus \mathcal{L}$ should have a value less than x_k.

A similar discussion can be used for the minimum part $x_{\min}$ and suppose s is the agent that achieves minimum value at time t then $x_s = x_{\min}$. Moreover we denote $\mathcal{S} = \{i \,|\, x_i = x_s,\ i \in \mathcal{V}\}$. In particular, all agents i having a directed path from i to s belongs to $\mathcal{S}$. Thus agents $\mathcal{V} \setminus \mathcal{S}$ should have a value greater than x_s.

Hence the vertex $\mathcal{V}$ can be separated into three groups: $\mathcal{L}$, $\mathcal{S}$ and $\mathcal{V} \setminus (\mathcal{L} \cup \mathcal{S})$. Agents $i \in \mathcal{V} \setminus (\mathcal{L} \cup \mathcal{S})$ should satisfy $x_{\min} < x_i < x_{\max}$. If the directed graph always has a spanning tree, it holds that there is a directed path from the root of the spanning tree r to any other vertex $i \in \mathcal{V}$, which yields to $r \in \mathcal{L}$ and $r \in \mathcal{S}$. Hence we get $x_{\max} = x_{\min}$, namely $\mathcal{L} = \mathcal{S}$. Furthermore, it yields $\mathcal{V} \setminus (\mathcal{L} \cup \mathcal{S}) = \emptyset$ and $\mathcal{L} \cap \mathcal{S} = \mathcal{V}$, indicating all agents have the same value, i.e., the consensus is reached. ■

One more thing to note here is that the final consensus is not necessarily the average consensus or other predefined agreement point because the sum of states is not constant. The center of the system drifts with time, which depends on both the initial states and the logarithmic gain.

4 Sampled System

With the development of digital sensors and controllers, in many cases the control algorithms can only be implemented with sampled data at discrete sampling instants even though the agents have continuous dynamics. Since the sampled system requires less control effort and are less resource (energy and bandwidth) consuming, it is more suitable for real life applications. Therefore it becomes necessary to analysis the stability of sampled system and the constraints of quantization on it. It is easy to notice that the sampling period plays an important role in the stability analysis of sampled systems. If the sampling period goes to zero, the sampled system should have approximately the same characteristics as the continuous system. Main concern in our case is the upper bound of the sampling time or how large the interval could be to guarantee that the sampled system still has decent performance. Main results of the sampled consensus problem can be found in [26], where the perfect communication is assumed. In this chapter we consider the same system but with the presence of quantized information.

4.1 System Model

First of all, without quantization effects, the sampled version of system (3.9) is described in [26] as

$$x(kh + h) = (\mathbf{I} - h\,L)\,x(kh) \qquad k = 0, 1, \ldots$$

where $h > 0$ is the sampling period. By virtue of the Proposition 1 in [26], the sampled system above is stable and converges to an agreement if the communication graph G has a spanning tree and the sampling time h satisfies

$$0 < h \leq \min_{\lambda \in \{\lambda(L) - \{0\}\}} \frac{2\mathrm{Re}(\lambda)}{|\lambda|^2}$$

where $\{\lambda(L) - \{0\}\}$ denotes the set of eigenvalues of the Laplacian matrix L excluding the trivial zero eigenvalue. Since we only consider undirected topologies, the Laplacian matrix L is symmetric and all eigenvalues are real. i.e., $\mathrm{Re}(\lambda) = \lambda$ and $|\lambda| = \lambda$, thus $\frac{2\mathrm{Re}(\lambda)}{|\lambda|^2} = \frac{2}{\lambda}$. The necessary condition for the system stability then becomes

$$0 < h \leq \frac{2}{\lambda_{\max}(L)}$$

What happens if the relative states are quantized as the system (3.1)? In the following content, we will investigate the impact of quantized information on this upper bound of the sampling period.

4.2 Stability under Tree Topology

With the presence of quantized communication, we obtained the closed-loop system (3.1) earlier that

$$\dot{x}_i = u_i = -\sum_{j \in \mathcal{N}_i} q(x_i - x_j)$$

Now if we sample the control input u_i with sampling period h that $u_i(t) = u_i(kh)$ for $t \in [kh,\, kh + h)$, then the sampled system is given by

$$x_i(kh + h) - x_i(kh) = -\sum_{j \in \mathcal{N}_i} q(x_i(kh) - x_j(kh))h \tag{4.1}$$

which can be described in edge vector format as in the continuous case that

$$\bar{x}(kh + h) = \bar{x}(kh) - Mq(\bar{x}(kh))\, h$$

From now on, for simplicity we omit the time unit h so $x(k+1) = x(kh+h)$ and $x(k) = x(kh)$. Parallel to the continuous case, we propose the discrete time candidate Lyapunov function:

$$V(k) = \bar{x}^T(k)\, M^{-1}\bar{x}(k)$$

which is the sampled version of the continuous Lyapunov function we used. $V(k)$ is positive definite and compact with respect to the edge vector $\bar{x}$. Instead of taking the derivate of V, to analyze the stability of the discrete time system (4.1), we compute the difference $V(k+1) - V(k)$

$$\begin{aligned}
&V(k+1) - V(k) \\
&= \bar{x}^T(k+1)\, M^{-1}\bar{x}(k+1) - \bar{x}^T(k)\, M^{-1}\bar{x}(k) \\
&= [\, \bar{x}(k) - Mq(\bar{x}(k))\, h\,]^T\, M^{-1}[\, \bar{x}(k) - Mq(\bar{x}(k))\, h\,] - \bar{x}^T(k)\, M^{-1}\bar{x}(k) \\
&= -2\, h\, \bar{x}^T(k)\, q(\bar{x}(k)) + h^2[q(\bar{x}(k))]^T\, M\, q(\bar{x}(k))
\end{aligned}$$

Lemma 4.1. *Assume the underlying graph G is a static tree, the quantization function satisfies $a \cdot q(a) \geq \gamma\, [q(a)]^2, a \in \mathbb{R}$ and sampling time satisfies*

$$0 < h < \gamma \frac{2}{\lambda_{\max}(L)}$$

then the sampled system (4.1) is guaranteed to converge to the invariant set $\{\boldsymbol{x} |\, q(x_i - x_j) = 0, \forall (i,j) \in E\}$

Proof. Since $a \cdot q(a) \geq \gamma\, [q(a)]^2, a \in \mathbb{R}$, the first term of $V(k+1) - V(k)$ satisfies

$$\bar{x}^T(k)\, q(\bar{x}(k)) = \sum_{i=1}^{N-1} \bar{x}_i(k)\, q(\bar{x}_i(k)) \geq \sum_{i=1}^{N-1} \gamma\, [q(\bar{x}_i(k))]^2 = \gamma\, |q(\bar{x}(k))|^2$$

which can be inserted into $V(k+1) - V(k)$ to get

$$\begin{aligned}
V(k+1) - V(k) &\leq -2\gamma h |q(\bar{x}(k))|^2 + h^2[q(\bar{x}(k))]^T\, M\, q(\bar{x}(k)) \\
&\leq -2\gamma h |q(\bar{x}(k))|^2 + h^2\, \lambda_{\max}(M)\, |q(\bar{x}(k))|^2 \\
&= h\, |q(\bar{x}(k))|^2\, [\, h\, \lambda_{\max}(M) - 2\gamma]
\end{aligned}$$

where $\lambda_{\max}(M)$ is the maximal eigenvalue of M, which has $N-1$ positive eigenvalues (3.3). It is also the largest eigenvalue of the Laplacian matrix L as proved in Lemma 3.3: $\lambda_{\max}(M) = \lambda_{\max}(L)$. So if the sampling time h satisfying

$$0 < h \leq \gamma \frac{2}{\lambda_{\max}(L)}$$

then $V(k+1) - V(k) \leq 0$ and the equality holds when $q(\bar{x}(k)) = 0$. Application of LaSalle's invariance principle for discrete time systems ensures the system stability and convergence to the largest invariant set $\{\bar{\mathbf{x}} | \, q(\bar{x}) = 0\}$, namely $\{\mathbf{x} | \, q(x_i - x_j) = 0, \forall (i,j) \in E\}$. ∎

Theorem 4.2. *Assume the underlying communication graph is a static tree, then the sampled system (4.1) converges to*

- $\{\boldsymbol{x} | \, |x_i - x_j| < \frac{\delta_u}{2}, \forall (i,j) \in E\}$ *in the case of uniform quantizers, if the sampling time satisfies* $0 < h \leq \frac{1}{\lambda_{\max}(L)}$
- $\{\boldsymbol{x} | \, x_i = x_j, \forall (i,j) \in E\}$ *in the case of logarithmic quantizers, if the sampling time satisfies* $0 < h \leq \frac{1}{1+\delta_l} \frac{2}{\lambda_{\max}(L)}$

Proof. The proof is a straightforward implication of Lemma 4.1. From Lemma 2.1, the following relation $a \cdot q_u(a) \geq \frac{1}{2} [q_u(a)]^2$ holds, which means $\gamma = \frac{1}{2}$ for uniform quantizers. Thus if the sampling time h satisfies

$$0 < h \leq \frac{1}{\lambda_{\max}(L)}$$

Lemma 4.1 ensures the sampled system (4.1) with uniform quantizers is stable and converges to $\{\bar{\mathbf{x}} | \, q_u(\bar{\mathbf{x}}) = 0\}$, namely $\{\mathbf{x} | \, |x_i - x_j| < \frac{\delta_u}{2}, \forall (i,j) \in E\}$.

If identical logarithmic quantizers are used instead, we have $\gamma = \frac{1}{1+\delta_l}$ as $a \cdot q_l(a) \geq \frac{1}{1+\delta_l} [q_l(a)]^2$ from Lemma 2.2. By Lemma 4.1, if the sampling time h satisfies

$$0 < h \leq \frac{1}{1+\delta_l} \frac{2}{\lambda_{\max}(L)}$$

then the discrete time system with logarithmic quantizers is stable and converges to the set $\{\bar{\mathbf{x}} | \, q_l(\bar{\mathbf{x}}) = 0\}$, namely $\{\mathbf{x} | \, x_i = x_j, \forall (i,j) \in E\}$. ∎

Note that all above analyses hold also for switching tree topologies by replacing $\lambda_{\max}$ with $\max\{\lambda_{\max}(L(t))\}$, which gives stricter upper bound on the sampling time by taking into account the maximal eigenvalue of all possible Laplacian matrices. Moreover, if the graph G is connected, we have the average consensus set $\{\mathbf{x} | \, x_1 = x_2 = \cdots = x_N\}$ as the largest invariant set for logarithmic quantizers.

Example 4.1. The same setup that four two-dimensional agents and the tree graph as in Example 3.1 is simulated but the control input is updated at discrete time. In the uniform case, based on Theorem 4.2, we choose the sampling time being $0.3\,s < \frac{1}{\lambda_{\max}(L)} = 0.3939\,s$ and the whole system still converges to the invariance set that all neighbors have difference less than $\frac{\delta_u}{2} = 0.005$ as in Figure 4.1 and 4.2.

While in the logarithmic case we chose $\delta_l = 1.5$, based on Theorem 4.2, to ensure stability we chose the sampling time to be $0.2\,s < \frac{1}{1+\delta_l} \frac{2}{\lambda_{\max}(L)} = 0.2343\,s$. Even though the trajectories

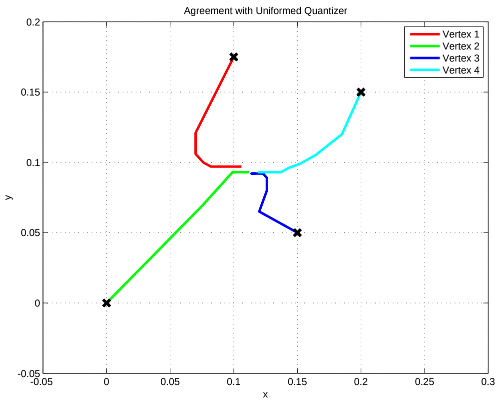


Figure 4.1: Sampled System with uniform quantizers

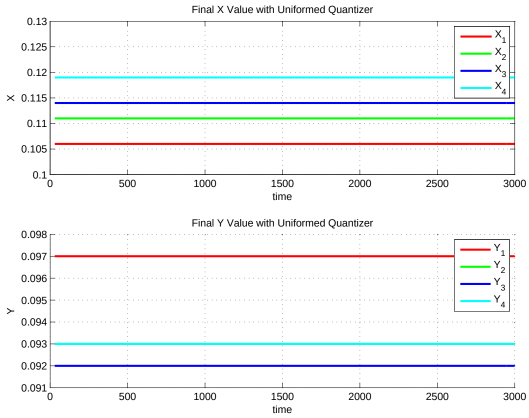


Figure 4.2: Final value in X, Y direction

in Figure 4.3 are not as smooth as before, these agents still converge to the average consensus asymptotically, which can be seen from Figure 4.4. A counter example is shown in Figure 4.5, where the sampling time equals to $0.3\,s > 0.2343\,s$, which violates the proposed constraints. As a result, the system experiences chattering and oscillation as in Figure 4.6, which causes the system instability.

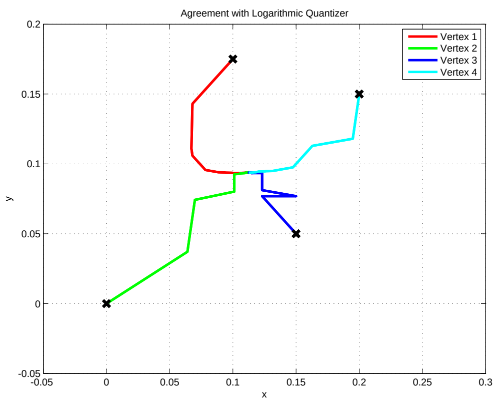


Figure 4.3: Sampled System with logarithmic quantizers

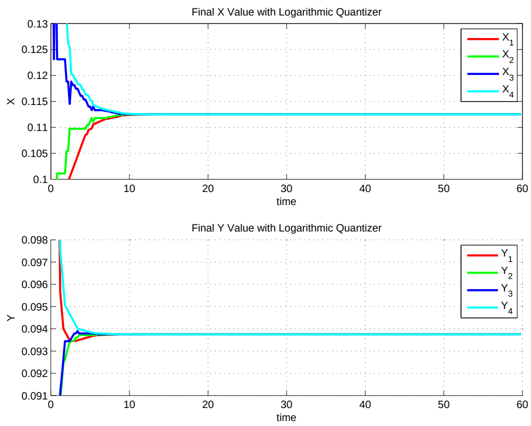


Figure 4.4: Final value in X, Y direction

4.3 Stability under General Undirected Graph

Instead of having the standard tree graph with normalized weights, we need to consider the same system (3.9) under general undirected graphs with weighted edges. The case of the continuous closed-loop system $\dot{x}_i = -\sum_{j=1}^{N} a_{ij}\, q(x_i - x_j)$, has been solved in section 3.2. If it is sampled with the sampling period $T_s > 0$ that $u_i(t) = u_i(kT_s) = \sum_{j=1}^{N} a_{ij}\, q(x_i(kT_s) -$

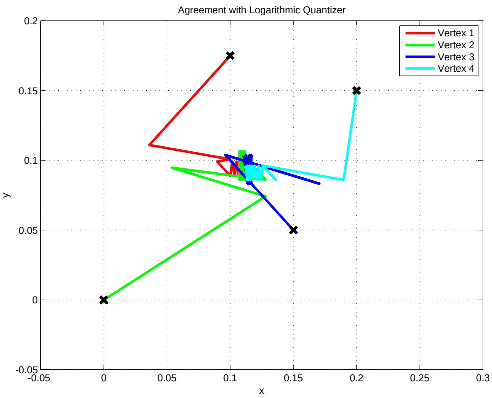


Figure 4.5: Sampled System with logarithmic quantizers

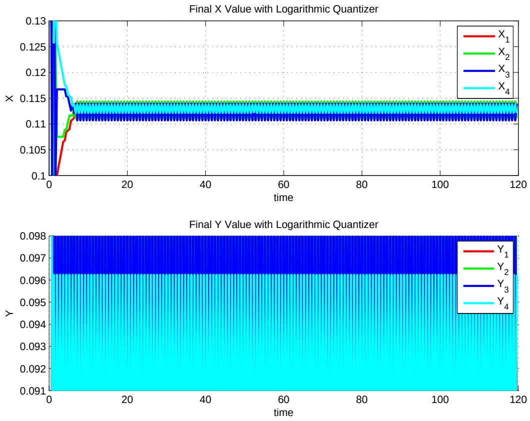


Figure 4.6: Final value in X, Y direction

$x_j(kT_s)), t \in [kT_s, kT_s + T_s)$, we have the sampled system in the form

$$x_i(k+1) = x_i(k) - \sum_{j=1}^{N} a_{ij}\, q(x_i(k) - x_j(k))\, T_s \tag{4.2}$$

We follow the same reasoning as in the continuous case that the sum of states $\sum_{i=1}^{N} x_i(k)$ remains constant and denote the invariant center $\frac{1}{N}\sum_{i=1}^{N} x_i(k) = \frac{1}{N}\sum_{i=1}^{N} x_i(0) = C$.

Let the second norm of the disagreement vector $V(k) = \sum_{i=1}^{N}(x_i(k) - C)^2$ be a candidate Lyapunov function, which is positive semidefinite and compact with respect to the agent states. It can be easily verified that $V(k) = \sum_{i=1}^{N} x_i^2(k) - NC^2$, hence the difference $V(k+1) - V(k)$ is computed as

$$\begin{aligned}
&V(k+1) - V(k) \\
&= \sum_{i=1}^{N} x_i^2(k+1) - \sum_{i=1}^{N} x_i^2(k) = \sum_{i=1}^{N} [x_i(k+1) + x_i(k)][x_i(k+1) - x_i(k)] \\
&= \sum_{i=1}^{N} [2x_i(k) - \sum_{j=1}^{N} a_{ij}\, q(x_i(k) - x_j(k))\, T_s][-\sum_{j=1}^{N} a_{ij}\, q(x_i(k) - x_j(k))\, T_s] \\
&= -2T_s \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\, x_i(k)\, q(x_i(k) - x_j(k)) \;+ T_s^2 \sum_{i=1}^{N}[\sum_{j=1}^{N} a_{ij}\, q(x_i(k) - x_j(k))]^2 \\
&= -T_s \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\, [x_i(k) - x_j(k)]\, q(x_i(k) - x_j(k)) \;+ T_s^2 \sum_{i=1}^{N}[\sum_{j=1}^{N} a_{ij}\, q(x_i(k) - x_j(k))]^2
\end{aligned}$$

Lemma 4.3. *Assume the communication graph G remains static and undirected. The quantization function satisfies $a \cdot q(a) \geq \gamma\, [q(a)]^2, \gamma > 0,\ \forall a \in \mathbb{R}$ and sampling time satisfies*

$$0 < h < \gamma \frac{1}{\max\limits_{i} \sum_{j=1}^{N} a_{ij}}$$

then the sampled system (4.7) is guaranteed to converge to the invariant set $\{\boldsymbol{x}|\, q(x_i - x_j) = 0, \forall (i,j) \in E\}$

Proof. In the first place, we need to recall the *Cauchy-Schwarz Inequality*:

$$\sum_{i=1}^{N} a_i\, b_i \leq (\sum_{i=1}^{N} a_i^2)^{\frac{1}{2}}\, (\sum_{i=1}^{N} b_i^2)^{\frac{1}{2}}$$

for any $a, b \in \mathbb{R}$. It can be applied to the second part of the $V(k+1) - V(k)$, which satisfies

$$\begin{aligned}
[\sum_{j=1}^{N} a_{ij}\, q(x_i(k) - x_j(k))]^2 &= [\sum_{j=1}^{N} \sqrt{(a_{ij})}\, \sqrt{(a_{ij})}\, q(x_i(k) - x_j(k))]^2 \\
&\leq \sum_{j=1}^{N} a_{ij}\, \sum_{j=1}^{N} a_{ij} [q(x_i(k) - x_j(k))]^2
\end{aligned}$$

Then since it is assumed that the quantization function satisfies $a \cdot q(a) \geq \gamma\, [q(a)]^2$, the first part of $V(k+1) - V(k)$ can be easily bounded with $\gamma q(x_i - x_j)^2$. Thus $V(k+1) - V(k)$ has the following bounds:

$$\begin{aligned}
&V(k+1) - V(k) \\
&\leq -\gamma\, T_s \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\, [q(x_i(k) - x_j(k))]^2 + T_s^2 \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\, \sum_{j=1}^{N} a_{ij} [q(x_i(k) - x_j(k))]^2
\end{aligned}$$

Denote by $z_i = \sum_{j=1}^{N} a_{ij}\, [q(x_i(k) - x_j(k))]^2 \geq 0$ and $d_i = \sum_{j=1}^{N} a_{ij} \geq 0$. Then

$$\begin{aligned}
&V(k+1) - V(k) \\
&\leq -\gamma\, T_s \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\, [q(x_i(k) - x_j(k))]^2 + T_s^2 \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\, \sum_{j=1}^{N} a_{ij} [q(x_i(k) - x_j(k))]^2 \\
&= -\gamma\, T_s \sum_{i=1}^{N} z_i + T_s^2 \sum_{i=1}^{N} d_i\, z_i \\
&\leq -\gamma\, T_s \sum_{i=1}^{N} z_i + T_s^2 \max_i d_i \sum_{i=1}^{N} z_i = T_s\, (T_s\, \max_i d_i - \gamma) \sum_{i=1}^{N} z_i
\end{aligned}$$

To ensure the stability and convergence of the system, it is required that $V(k+1) - V(k) \leq 0$, which in turn yields $(T_s \max_i d_i - \gamma) \leq 0$, thus

$$0 < T_s \leq \gamma\, \frac{1}{\max\limits_i d_i} = \gamma\, \frac{1}{\max\limits_i \sum_{j=1}^{N} a_{ij}}$$

Furthermore, by Geršgorin Disk Theorem and Theorem 2 in [21], all eigenvalues λ_i of the Laplacian matrix L should satisfy

$$|\lambda_i - \max_i d_i| \leq \max_i d_i$$

Then let $\lambda_{\max} = \max\limits_i\{\lambda_i\}$, which also should lie in the above region

$$|\lambda_{\max} - \max_i d_i| \leq \max_i d_i \Rightarrow \lambda_{\max} \leq 2 \max_i d_i \Rightarrow \max_i d_i \geq \frac{\lambda_{\max}}{2}$$

which implies that

$$0 < T_s \leq \gamma \frac{1}{\max\limits_i d_i} \leq \gamma \frac{2}{\lambda_{\max}(L)}$$

As expected, we have tighter constraints on the sampling time than the continuous counterparts if $0 < \gamma < 1$. Then $V(k+1) - V(k) \leq 0$ and the equality holds when $z_i = 0 \Rightarrow q(x_i - x_j) = 0$, $(i, j) \in E$. Application of the LaSalle's invariance principle for discrete time systems ensures the system stability and convergence to $\{\mathbf{x} |\, q(x_i - x_j) = 0,\ (i, j) \in E\}$. ■

Theorem 4.4. *Assume the static communication graph remains undirected. Then the sampled system (4.7) converges to*

- $\{\boldsymbol{x} |\, |x_i - x_j| < \frac{\delta_u}{2}, \forall (i, j) \in E\}$ *in the case of uniform quantizers, if the sampling time satisfies* $0 < T_s \leq \frac{1}{2} \frac{1}{\max\limits_i \sum_{j=1}^N a_{ij}}$
- $\{\boldsymbol{x} |\, x_i = x_j, \forall (i, j) \in E\}$ *in the case of logarithmic quantizers, if the sampling time satisfies* $0 < T_s \leq \frac{1}{1+\delta_l} \frac{1}{\max\limits_i \sum_{j=1}^N a_{ij}}$

Proof. Regarding uniform quantizers, we obtained in Lemma 2.1 that $a \cdot q_u(a) \geq \frac{1}{2} [q_u(a)]^2$, $\forall a \in \mathbb{R}$. Thus $\gamma = \frac{1}{2}$ and by virtue of Lemma 4.3, if the sampling time T_s satisfies

$$0 < T_s \leq \frac{1}{2} \frac{1}{\max\limits_i d_i} = \frac{1}{2} \frac{1}{\max\limits_i \sum_{j=1}^N a_{ij}}$$

all agents would converge to $\{\bar{\mathbf{x}} |\, q_u(\bar{\mathbf{x}}) = 0\}$, namely $\{\mathbf{x} |\, |x_i - x_j| < \frac{\delta_u}{2}, \forall (i, j) \in E\}$.

If logarithmic quantizers are used instead, we have $a \cdot q_l(a) \geq \frac{1}{1+\delta_l} [q_l(a)]^2$, $\forall a \in \mathbb{R}$ from Lemma 2.2. So $\gamma = \frac{1}{1+\delta_l}$ and by virtue of Lemma 4.3 we can obtain the corresponding bounds on sampling time T_s

$$0 < T_s \leq \frac{1}{1+\delta_l} \frac{1}{\max\limits_i d_i} = \frac{1}{1+\delta_l} \frac{1}{\max\limits_i \sum_{j=1}^N a_{ij}}$$

and final convergence to $\{\mathbf{x} |\, q_l(x_i - x_j) = 0, \forall (i, j) \in E\}$, namely $\{\mathbf{x} |\, x_i = x_j, \forall (i, j) \in E\}$. ■

Example 4.2. The static communication graph G in Figure 4.7 with weighted edges (numbers on the edges) is used in this example and the control law is updated in a discrete-time way. The sampling period is used as the simulation time step.

The whole system still converges to the invariance set that all neighbors $(1, 2)$, $(2, 3)$, $(1, 3)$, $(3, 4)$ have state difference less than $\frac{\delta_u}{2} = 0.005$ as in Figure 4.8 and 4.9 if we choose the sampling time $0.09\, s < \frac{1}{2} \frac{1}{\max\limits_i \sum_{j=1}^N a_{ij}} = 0.1\, s$ based on the Theorem 4.4.

If the logarithmic quantizers with gain $\delta_l = 1.5$ are used, to ensure the system stability we set the sampling time to be $0.05\, s < \frac{1}{1+\delta_l} \frac{1}{\max\limits_i \sum_{j=1}^N a_{ij}} = 0.08\, s$. As proved in Theorem 4.4 the closed-loop system asymptotically converges to the average consensus, which is visualized in Figure 4.10 and 4.11.

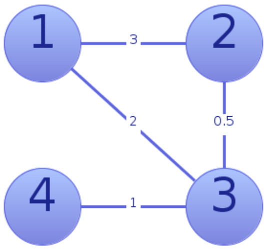


Figure 4.7: General Undirected Topology

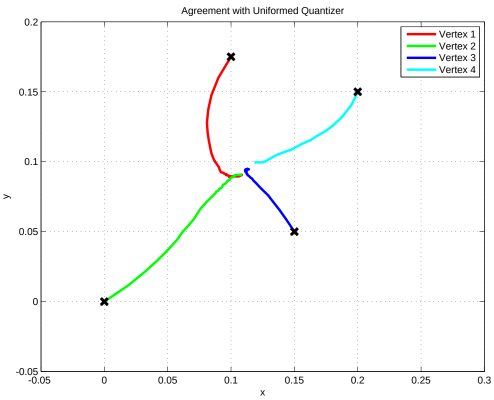


Figure 4.8: Sampled System with uniform quantizers

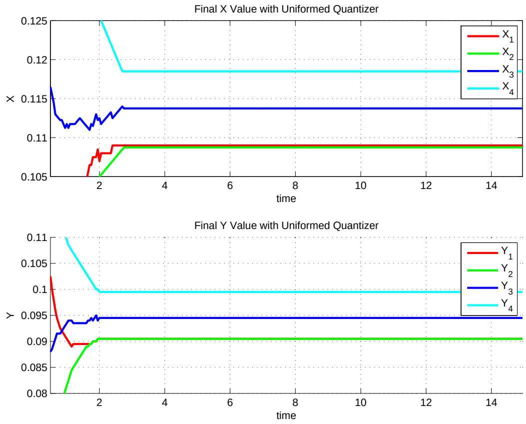


Figure 4.9: Final value in X, Y direction

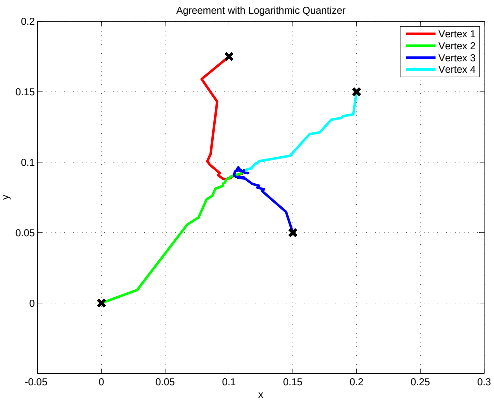


Figure 4.10: Sampled System with logarithmic quantizers

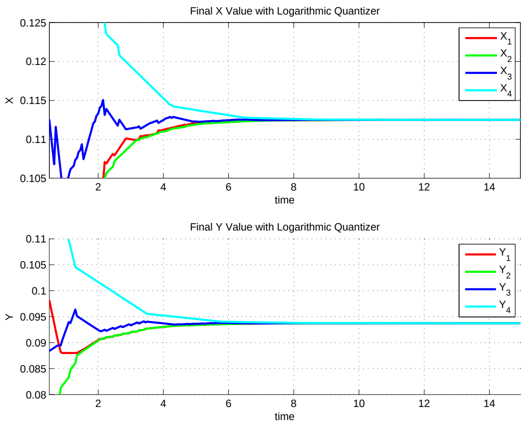


Figure 4.11: Final value in X, Y direction

5 Second Order System

In this chapter we mainly discuss the convergence of the second-order consensus problem with quantized communication. Instead of considering the tree topology first, we start directly from general undirected graphs and overcome the fact that the edge Laplacian matrix $B^T B$ is not positive definite anymore. In particular, a convergence set under uniform quantizers and upper bounds for the logarithmic gain are derived. Supporting lemmas are given first. Again we treat only the system behavior in the x-coordinates but all the analysis that follows holds in the y-coordinates, and also other coordinates in higher dimensions.

5.1 System Dynamics

Assume each agent has the distributed control law $u_i = -\sum_{j \in \mathcal{N}_i} [q(x_i - x_j) + \gamma q(v_i - v_j)]$, which has the same structure as (2.9) but with quantized relative states. The quantization function $q(\cdot)$ is identical for all agents. Then the closed-loop system is given by:

$$\begin{aligned} \dot{\bar{x}} &= \bar{v} \\ \dot{\bar{v}} &= -B^T B q(\bar{x}) - \gamma B^T B q(\bar{v}) \end{aligned} \tag{5.1}$$

Create the stack vector $y = [\bar{x}^T \quad \bar{v}^T]^T$ and denote $M = B^T B$. We can rewrite (5.1) into matrix form:

$$\begin{aligned} \dot{y} &= \begin{bmatrix} 0_{m\times m} & I_m \\ 0_{m\times m} & 0_{m\times m} \end{bmatrix} y + \begin{bmatrix} 0_{m\times m} \\ I_m \end{bmatrix} \bar{u} \\ \bar{u} &= \begin{bmatrix} -M & -\gamma M \end{bmatrix} q(y) \end{aligned} \tag{5.2}$$

As mentioned in the introductory chapter, the corresponding consensus objective for the second order system is to enforce a group of agents converge to one point and move with the same velocity.

5.2 Stability and Convergence

Before we state the main results about the stability and convergence of the system (5.2), we need the following lemmas.

Lemma 5.1. *If the undirected graph G is connected, then $B^T B$ and the Laplacian matrix $L = BB^T$ both have non-negative eigenvalues and moreover the same positive ones.*

Proof. Since G is connected and assuming that G has m edges, if $m = N-1$ then G is a tree and B^TB is positive definite. The result was given previously in Lemma 3.3. On the other hand, when $m \geq N$, there are exactly $m+1-N$ cycles in G, which gives exactly $m+1-N$ dimensional null space of B [10]. We assume that the singular value decomposition of the incidence matrix $B_{N\times m}$ is $B = U\Sigma W^T$, where $U_{(N\times N)}$ is the left singular matrix, composed by the normalized eigenvectors of $BB^T_{(N\times N)}$, and $W_{(m\times m)}$ is the right singular matrix composed by the normalized eigenvectors of $B^TB_{(m\times m)}$. Since B is overdetermined [11], its singular value matrix $\Sigma_{(N\times m)}$ (with exactly $m+1-N$ zero singular values) has the structure

$$\Sigma = \left[\begin{array}{cccc|cc} \lambda_{N-1} & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \lambda_{N-2} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & \lambda_1 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & \cdots & 0 \end{array}\right]_{N\times m}$$

where λ_i for $i = (N-1), \ldots, 1$ are the non-zero singular values of B in descending order. There are $m+1-N$ zero singular values, correspond to the cycle space of G.

Using $B = U\Sigma W^T$ and denoting $T = \Sigma^T\Sigma$, we have

$$\begin{aligned} B^TB &= (U\Sigma W^T)^T U\Sigma W^T = W\Sigma^T U^T U\Sigma W^T \\ &= W\Sigma^T\Sigma W^T = WTW^T \end{aligned}$$

$$T = \Sigma^T\Sigma = \left[\begin{array}{cccc|cc} \lambda_{N-1}^2 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \lambda_{N-2}^2 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & \lambda_1^2 & \cdots & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{array}\right]_{m\times m}$$

which implies that WTW^T is the singular value decomposition of B^TB. Moreover, using similar calculations for the matrix BB^T, and denoting $S = \Sigma\,\Sigma^T$, we get

$$\begin{aligned} L = BB^T &= U\Sigma W^T(U\Sigma W^T)^T = U\Sigma W^T W\Sigma^T U^T \\ &= U\Sigma\Sigma^T U^T = USU^T \end{aligned}$$

$$S = \Sigma\,\Sigma^T = \left[\begin{array}{cccc|c} \lambda_{N-1}^2 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_{N-2}^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_1^2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right]_{N\times N}$$

This implies that USU^T is the singular value decomposition of L, which has exactly $N-1$ positive eigenvalues, that are identical to corresponding positive eigenvalues of B^TB. Similar conclusions that L has non-negative eigenvalues and exactly one zero eigenvalue if the associated graph has a spanning tree, can be found in [22]. ■

Lemma 5.2. *If G is undirected and connected, then $\bar{x}^TB^TB\bar{x} \geq \lambda_2(L)|\bar{x}|^2$*

Proof. First of all, we need to clarify that $B\bar{x} = 0$ only if $\bar{x} = 0$. We have $B\bar{x} = 0 \Rightarrow BB^T x = 0 \Rightarrow Lx = 0 \Rightarrow x \in span\{\mathbf{1}\} \Rightarrow \bar{x} = 0$. Denote the $m+1-N$ column eigenvectors of $B^T B$ associated with the $m+1-N$ zero eigenvalues by $[c_1, c_2, \cdots, c_{m+1-N}]$, which can be obtained as signed edge vectors depending on the direction of cycles in G. By definition $Bc_i = 0$. Moreover, it can be proved that $c_i \perp \bar{x}$, $\forall i = 1, \cdots, m+1-N$:

$$c_i^T \bar{x} = c_i^T B^T x = (Bc_i)^T x = 0$$

By the Courant-Fischer Theorem [11]

$$\min_{\substack{c_i \perp \bar{x}, \bar{x} \neq 0 \\ i=1,\cdots,m+1-N}} \frac{\bar{x}^T B^T B \bar{x}}{\bar{x}^T \bar{x}} = \lambda_{m+2-N}(B^T B)$$

where $\lambda_{m+2-N}(B^T B) = \lambda_1^2 = \lambda_2(L)$. Thus $\bar{x}^T B^T B \bar{x} \geq \lambda_2(L)|\bar{x}|^2$, which completes the proof. ■

Propose the Lyapunov candidate function

$$V(y) = y^T \begin{bmatrix} \gamma M & \frac{1}{2} I_m \\ \frac{1}{2} I_m & \frac{\gamma}{2} I_m \end{bmatrix} y$$

Before we can use this Lyapunov function, we need to decide the condition on which $V(y) \geq 0$ is satisfied.

Lemma 5.3. *$V(y) \geq 0$ on condition that $\gamma > \sqrt{\frac{1}{2\lambda_2(L)}}$.*

Proof.

$$V(y) = \begin{bmatrix} \bar{x}^T & \bar{v}^T \end{bmatrix} \begin{bmatrix} \gamma M & \frac{1}{2} I_m \\ \frac{1}{2} I_m & \frac{\gamma}{2} I_m \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{v} \end{bmatrix}$$

Substituting $\bar{x} = B^T x$ and $\bar{v} = B^T v$, we have

$$\begin{aligned} V(y) &= \begin{bmatrix} x^T & v^T \end{bmatrix} \begin{bmatrix} \gamma BB^T BB^T & \frac{1}{2} BB^T \\ \frac{1}{2} BB^T & \frac{\gamma}{2} BB^T \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} \\ &= \begin{bmatrix} x^T & v^T \end{bmatrix} \begin{bmatrix} \gamma LL & \frac{1}{2} L \\ \frac{1}{2} L & \frac{\gamma}{2} L \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} x^T & v^T \end{bmatrix} P \begin{bmatrix} x \\ v \end{bmatrix} \end{aligned}$$

In order to find the eigenvalues of P, we should solve the equation that $\det(\lambda I_{2m} - P) = 0$, where $\det(\lambda I_{2m} - P)$ is the characteristic polynomial of P [23]. Since γLL and $\frac{1}{2}L$ commute and given the block matrix P, we have

$$\begin{aligned} \det(\lambda I_{2m} - P) &= \det(\begin{bmatrix} \lambda I_m - \gamma L^2 & -\frac{1}{2} L \\ -\frac{1}{2} L & \lambda I_m - \frac{\gamma}{2} L \end{bmatrix}) \\ &= \det(\lambda^2 I_m - \frac{\gamma}{2} \lambda L - \frac{\gamma}{2} \lambda L^2 + \frac{\gamma^2}{2} L^3 - \frac{1}{4} L^2) \\ &= \prod_{i=1}^{m} [\lambda^2 I_m - \frac{\gamma}{2} \lambda \theta_i - \frac{\gamma}{2} \lambda \theta_i^2 + \frac{\gamma^2}{2} \theta_i^3 - \frac{1}{4} \theta_i^2] \\ &= \prod_{i=1}^{m} [\lambda^2 - (\theta_i^2 + \theta_i)\gamma\, \lambda + \frac{\gamma^2}{2} \theta_i^3 - \frac{1}{4} \theta_i^2] = 0 \end{aligned}$$

where θ_i is the i_{th} eigenvalue of L. The zero eigenvalue $\theta_1 = 0$ gives two zero eigenvalue as the roots of this second polynomial. To guarantee that P is positive semidefinite, we require

$$\frac{\gamma^2}{2}\theta_i^3 - \frac{1}{4}\theta_i^2 > 0 \Rightarrow \gamma^2 > \frac{1}{2\theta_i}$$

which should hold for all positive eigenvalues θ_i of L, i.e., $\gamma^2 > \frac{1}{2\theta_i}$ for $i = 2, \cdots, N$. We can derive the sufficient condition that $\gamma > \sqrt{\frac{1}{2\lambda_2(L)}}$ as $\lambda_2(L) \leq \cdots \leq \lambda_N(L)$. ■

Then the time derivative of $V(y)$ along the trajectories of the closed-loop system is given by

$$\begin{aligned}
\dot{V}(y) &= (\dot{y})^T \begin{bmatrix} \gamma M & \frac{1}{2}I_m \\ \frac{1}{2}I_m & \frac{\gamma}{2}I_m \end{bmatrix} y + y^T \begin{bmatrix} \gamma M & \frac{1}{2}I_m \\ \frac{1}{2}I_m & \frac{\gamma}{2}I_m \end{bmatrix} \dot{y} \\
&= -y^T \begin{bmatrix} M & 0_{m\times m} \\ 0_{m\times m} & \gamma^2 M - I_m \end{bmatrix} y + y^T \begin{bmatrix} -M & -\gamma M \\ -\gamma M & -\gamma^2 M \end{bmatrix} (q(y) - y) \\
&= -y^T Q y + y^T W (q(y) - y)
\end{aligned}$$

Lemma 5.4. *If* $\gamma > \sqrt{\frac{1}{\lambda_2(L)}}$, *then* $y^T Q y \geq \min\{\lambda_2(L), \gamma^2\lambda_2(L) - 1\}|y|^2$.

Proof. By virtue of Lemma 5.2, we have $\bar{x}^T M \bar{x} \geq \lambda_2(L)|\bar{x}|^2$. Then the first part of $\dot{V}$ can be lower bounded by

$$\begin{aligned}
y^T Q y &= \begin{bmatrix} x^T & v^T \end{bmatrix} \begin{bmatrix} M & 0_{m\times m} \\ 0_{m\times m} & \gamma^2 M - I_m \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} \\
&= \bar{x}^T M \bar{x} + \gamma^2 \bar{v}^T M \bar{v} - \bar{v}^T \bar{v} \\
&\geq \lambda_2(L)|\bar{x}|^2 + (\gamma^2\lambda_2(L) - 1)|\bar{v}|^2
\end{aligned}$$

If $\gamma > \sqrt{\frac{1}{\lambda_2(L)}}$, we have $\gamma^2\lambda_2(L) - 1 > 0$ which ensures $y^T Q y$ to be positive semidefinite. Then the above equality implies

$$\begin{aligned}
y^T Q y &\geq \min\{\lambda_2(L), \gamma^2\lambda_2(L) - 1\}(|\bar{x}|^2 + |\bar{v}|^2) \\
&= \min\{\lambda_2(L), \gamma^2\lambda_2(L) - 1\}|y|^2
\end{aligned}$$

Set $\lambda_{\min}(Q) = \min\{\lambda_2(L), \gamma^2\lambda_2(L) - 1\}$, then $y^T Q y \geq \lambda_{\min}(Q)|y|^2$. ■

Lemma 5.5. $\|W\|_2 = (1 + \gamma^2)\lambda_{\max}(L)$

Proof. The Euclidean norm or second norm of a matrix A is the largest singular value of A or the square root of the largest eigenvalue of the positive-semidefinite matrix $A^T A$. In order to obtain the eigenvalue of matrix $W^T W$, we use similar approach as in Lemma 5.3 and calculate $\det(\lambda I_{2m} - W^T W)$. We have:

$$\begin{aligned}
\det(\lambda I_{2m} - W^T W) &= \det(\begin{bmatrix} \lambda I_m - (1+\gamma^2)M^2 & -\gamma(1+\gamma^2)M^2 \\ \gamma(1+\gamma^2)M^2 & \lambda I_m - (1+\gamma^2)\gamma^2 M^2 \end{bmatrix}) \\
&= \det(\lambda^2 I_m - \lambda M^2 (1+\gamma^2)^2) \\
&= \prod_{i=1}^{m} [\lambda(\lambda - \theta_i^2 (1+\gamma^2)^2)] = 0
\end{aligned}$$

which means W^TW has an eigenvalue at 0 with multiplicity m and another m non-zero eigenvalues corresponding to each θ_i of M. The maximal one should be $\lambda_{\max}(W^TW) = \lambda^2_{\max}(M)(1+\gamma^2)^2$, yielding $\|W\| = (1+\gamma^2)\lambda_{\max}(M) = (1+\gamma^2)\lambda_{\max}(L)$. ■

Finally with the condition $\gamma > \sqrt{\frac{1}{\lambda_2(L)}}$ and combining Lemma 5.4 and 5.5, we can bound the time derivative of $V(y)$ in such a way that

$$\begin{aligned} \dot{V}(y) &\le -\lambda_{\min}(Q)|y|^2 + |y|\|W\||q(y)-y| \\ &= -\lambda_{\min}(Q)|y|^2 + |y|(1+\gamma^2)\lambda_{\max}(L)\,|q(y)-y| \end{aligned} \tag{5.3}$$

- **Uniform Quantizer**

  In the case of uniform quantizers, we have $q = q_u$ and $|q_u(y) - y| \le \frac{\delta_u}{2}\sqrt{2m}$. Then (5.3) satisfies

  $$\dot{V}(y) \le -\lambda_{\min}(Q)|y|(|y| - \frac{(1+\gamma^2)\lambda_{\max}(L)}{\lambda_{\min}(Q)}\frac{\delta_u}{2}\sqrt{2m})$$

  Thus, all solutions of the closed-loop system (5.1) enter the ball

  $$\{y|\,|y| \le \frac{(1+\gamma^2)\lambda_{\max}(L)}{2\lambda_{\min}(Q)}\sqrt{2m}\delta_u\} \tag{5.4}$$

  where $\lambda_{\min}(Q) = \min\{\lambda_2(L), \gamma^2\lambda_2(L) - 1\}$. This region is centered at the point where $|y| = 0$, namely the consensus point $\bar{x} = \bar{v} = 0$.

- **Logarithmic Quantizer**

  With logarithmic quantizers, we have $|q_l(y) - y| \le \delta_l\,|y|$, which can be used to bound $\dot{V}(y)$ to get

  $$\begin{aligned} \dot{V}(y) &\le -\lambda_{\min}(Q)|y|^2 + |y|(1+\gamma^2)\lambda_{\max}(L)\,|q(y)-y| \\ &\le -|y|^2(\lambda_{\min}(Q) - (1+\gamma^2)\lambda_{\max}(L)\,\delta_l) \end{aligned}$$

  Thus the convergence to the agreement point $|y| = 0$, namely $\bar{x} = 0$ and $\bar{v} = 0$ is guaranteed if the logarithmic gain δ_l satisfies

  $$0 < \delta_l < \frac{\lambda_{\min}(Q)}{(1+\gamma^2)\lambda_{\max}(L)} \tag{5.5}$$

  where $\lambda_{\min}(Q) = \min\{\lambda_2(L), \gamma^2\lambda_2(L) - 1\}$, which means that the logarithmic gain should be smaller than an upper bound which depends on the communication topology.

Theorem 5.6. *Assume that the undirected graph G is static and connected and that if $\gamma > \sqrt{\frac{1}{\lambda_2(L)}}$. Then the closed-loop system (5.1) has the following convergence properties:*

- *In the case of uniform quantizers, the system converges to the consensus set (5.4), centered at the desired agreement point.*

- *In the case of logarithmic quantizers, the system asymptotically converges to the consensus point and moving with the same velocity, for all logarithmic gains δ_l satisfying the constraint (5.5).*

Example 5.1. In contrast to our normal setup, we simulate the second-order agents that moves only along x-coordinates in order to put emphasis on both the trajectories of velocity and position. The same tree communication topology is used as in Example 3.1. The weight γ is set to $1.65 > \sqrt{\frac{1}{\lambda_2(L)}} = 1.30$ and $\lambda_{\min}(Q) = \min\{\lambda_2(L), \gamma^2\lambda_2(L) - 1\} = 0.58$.

The whole system converges to $|y| = \sqrt{|\bar{x}^2 + \bar{v}^2|} \leq 0.3$ with uniform quantizers $\delta_u = 0.01$ as in Figure 5.1. The figures on the right are zoomed vision that is focused on the final steady configuration. In the second part, logarithmic quantizers with gain $\delta_l = 0.05 < \frac{\lambda_{\min}(Q)}{(1+\gamma^2)\lambda_N(L)} = 0.056$ are used to guarantee that the group of agents converges to the agreement point and moves with the same velocity as shown in Figure 5.2. The figures on the right indicate that the relative differences of position and velocity asymptotically converge to zero.

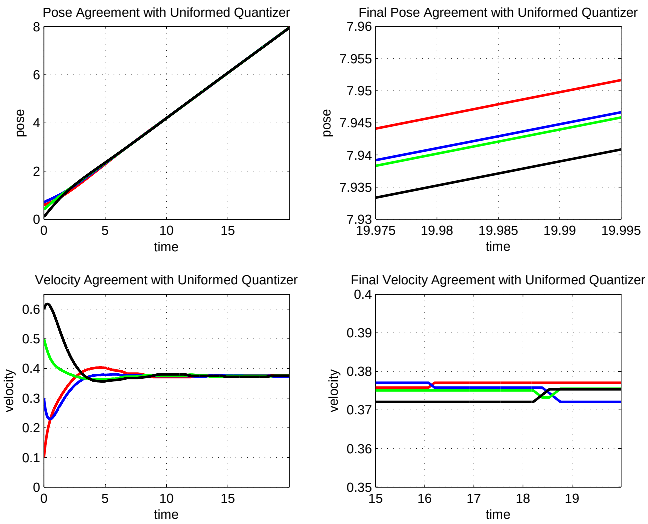


Figure 5.1: Second order system with uniform quantizers

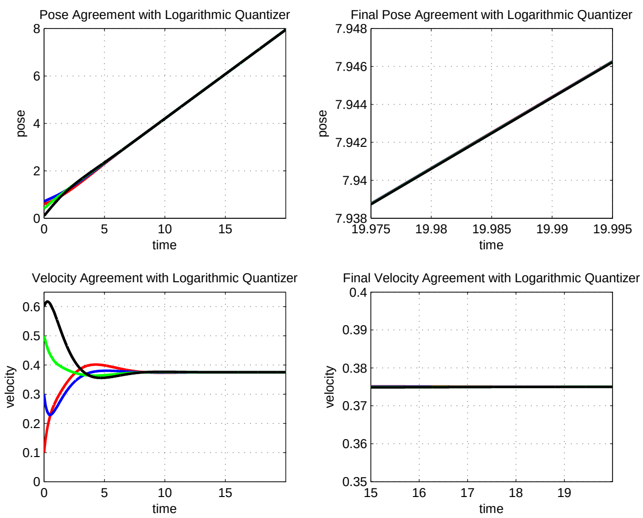


Figure 5.2: Second order system with logarithmic quantizers

6 Alternative Quantization Control Design

In this chapter we consider an alternative control approach that uses relative quantized states as control parameters, instead of using quantized relative measurements as in previous discussions. One practical explanation for this model would be that each member of the group transmits and receives quantized state value to and from its neighbors, then uses the relative differences as inputs to steer towards the consensus point.

6.1 Model Description

As in Section 3.2 we consider the first order agents under general weighted and undirected graphs, thus the following closed-loop system is given

$$\dot{x}_i = -\sum_{j=1}^{N} a_{ij}[q(x_i) - q(x_j)] \qquad (a_{ij} \geq 0) \tag{6.1}$$

where $q(x_i)$ and $q(x_j)$ are the quantized states of agent i and j. The weights $a_{ij} > 0$ if agent i communicates with j, otherwise $a_{ij} = 0$. Relative quantized states are used as control parameters. Through the following discussion, we make the assumption that all agents have identical quantizers and the underlying communication graph G is always undirected so that the information exchanges are always symmetric. The closed-loop system 6.1 can be described in closed form as $\dot{\mathbf{x}} = -L\, q(\mathbf{x})$, where $q(\mathbf{x})$ is the stack vector of quantized states.

6.2 Stability and Convergence

First of all, we can use the same reasoning as in Lemma 3.9 that the sum of states is invariant since $\sum_{i=1}^{N} \dot{x}_i = 0$. We denote the initial state by $\frac{1}{N}\sum_{i=1}^{N} x_i(0) = C$. Then $\frac{1}{N}\sum_{i=1}^{N} x_i(t) = C$, which holds for all time.

Theorem 6.1. *Assume that the static communication graph G is an undirected and the quantization function is non-decreasing, then the closed-loop system (6.1) is guaranteed to converge to*

$$\{\boldsymbol{x} |\, q(x_i) = q(x_j),\, \forall (i,j) \in E\}$$

Proof. Let the quadratic disagreement function

$$V = |\mathbf{x} - \mathbf{c}|^2 = \sum_{i=1}^{N} (x_i - C)^2$$

be the candidate Lyapunov function. As stated in Section 3.2.1, this function is positive definite and compact with respect to the states value. The time derivative of V along the trajectories of the closed-loop system is given by

$$\begin{aligned}\dot{V} &= \frac{2}{N}(\sum_{i=1}^{N} x_i \cdot \dot{x_i} - C\sum_{i=1}^{N}\dot{x_i}) = -\frac{2}{N}\sum_{i=1}^{N}\sum_{i=1}^{N} x_i \cdot a_{ij}\ (q(x_i) - q(x_j)) \\ &= -\sum_{i=1}^{N}\sum_{i=1}^{N}(a_{ij}\, x_i - a_{ji}\, x_j)\ [q(x_i) - q(x_j)] \\ &= -\sum_{i=1}^{N}\sum_{i=1}^{N} a_{ij}(x_i - x_j)\ [q(x_i) - q(x_j)]\end{aligned}$$

Both the uniform quantizer and logarithmic quantization models we investigate are non-decreasing, which means if $x_i \geq x_j$ then $q(x_i) \geq q(x_j)$. Hence $\dot{V} \leq 0$ and the equality holds when $q(x_i) = q(x_j)$ for $\forall(i,j) \in E$ as $x_i = x_j \rightarrow q(x_i) = q(x_j)$. By virtue of LaSalle's invariance principle for hybrid systems, the system (6.1) converges to the largest invariance set $\{\mathbf{x}|\, q(x_i) = q(x_j),\, \forall(i,j) \in E\}$. ■

If we assume the corresponding graph is connected, the final invariant set would be $\{\mathbf{x}|\, q(x_1) = q(x_2) = \cdots = q(x_N)\}$, which means that the system converges to the configuration that all agents have the same quantized states. This phenomenon can be explained in the intuitive way that since all agents use quantized levels of its neighbors as references, it in turn leads to the desired configuration that all agents end up at the same level. In the coming parts, we need to treat the uniform quantizer and logarithmic quantizer separately because their different characteristics have a great impact on the system performance.

6.2.1 Uniform Quantizer

In the case of uniform quantizers and a connected communication graph G, by Theorem 6.1, the final invariance set would be $\{\mathbf{x}|\, q_u(x_1) = q_u(x_2) = \cdots = q_u(x_N)\}$. From Lemma 2.1 it explicitly implies

$$x_i \in [-\frac{\delta_u}{2} + k\delta_u,\ \frac{\delta_u}{2} + k\delta_u), \forall\, x_i \in V \qquad k \in \mathbb{Z} \tag{6.2}$$

where the value of integer k can be determined by implying the condition that $\frac{1}{N}\sum_{i=1}^{N} x_i = C$

$$C \in [-\frac{\delta_u}{2} + k\delta_u,\ \frac{\delta_u}{2} + k\delta_u) \quad \Rightarrow \quad k \in (\frac{C}{\delta_u} - \frac{1}{2},\ \frac{C}{\delta_u} + \frac{1}{2}]$$

thus the value of k is unique as there is only one integer in the above region. It is obvious that $|x_i - x_j| < \delta_u$ for any $(i,j) \in V \times V$. Then the norm of disagreement vector $|\mathbf{x} - \mathbf{c}|$ can be bounded as

$$|\mathbf{x} - \mathbf{c}|^2 = \sum_{i=1}^{N}(x_i - C)^2 = \frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}(x_i - x_j)^2 < \frac{1}{2N}(N^2 - N)\delta_u^2 = \frac{N-1}{2}\delta_u^2$$

Thus the closed-loop system is guaranteed to converge to the set $\{\mathbf{x}|\, |\mathbf{x} - \mathbf{c}| < \delta_u\sqrt{\frac{N-1}{2}}\}$.

6.2.2 Logarithmic Quantizer

With logarithmic quantizers and a connected graph G, by Theorem 6.1, the final invariant set would be $\{\mathbf{x}|\, q_l(x_1) = q_l(x_2) = \cdots = q_l(x_N)\}$. From Lemma 2.2 it explicitly means that

$$|x_i| \in [e^{-\frac{\delta_u}{2}+k\delta_u},\, e^{\frac{\delta_u}{2}+k\delta_u}), \forall\, x_i \in V \qquad k \in \mathbb{Z} \tag{6.3}$$

where the value of integer k can be solved by implying the condition that $\frac{1}{N}\sum_{i=1}^{N} x_i = C$:

$$|C| \in [e^{-\frac{\delta_u}{2}+k\delta_u},\, e^{\frac{\delta_u}{2}+k\delta_u}) \quad \Rightarrow \quad k \in (\frac{\ln|C|}{\delta_u} - \frac{1}{2}, \frac{\ln|C|}{\delta_u} + \frac{1}{2}]$$

the value of k can be determined as there is only one integer satisfying the above bounds. From (6.3) we have

$$|x_i - x_j| < |e^{\frac{\delta_u}{2}+k\delta_u} - e^{-\frac{\delta_u}{2}+k\delta_u}|$$

for any pair of agents $(i,j) \in V \times V$. Then the closed-loop system converges to

$$\begin{aligned}|\mathbf{x} - \mathbf{c}| &< \sqrt{\frac{1}{2N}(N^2 - N)(e^{\frac{\delta_u}{2}+k\delta_u} - e^{-\frac{\delta_u}{2}+k\delta_u})^2} = \sqrt{\frac{N-1}{2}}\, e^{k\delta_u}|e^{\frac{\delta_u}{2}} - e^{-\frac{\delta_u}{2}}| \\ &\leq \sqrt{\frac{N-1}{2}}\, |C|(e^{\delta_u} - 1) = \sqrt{\frac{N-1}{2}}\, |C|(\delta_l^2 + 2\delta_l)\end{aligned}$$

So larger logarithmic gain leads to larger consensus errors. The above discussions can be summarized into the following theorem:

Theorem 6.2. *Assume that the static communication graph G is undirected and connected. Then the closed-loop system (6.1) has the following convergence properties:*

- *In the case of uniform quantizers, the system converges to $\{\boldsymbol{x}|\, |\boldsymbol{x} - \boldsymbol{c}| \leq \delta_u\sqrt{\frac{N-1}{2}}\}$*
- *In the case of logarithmic quantizers, the system converges to $\{\boldsymbol{x}|\, |\boldsymbol{x} - \boldsymbol{c}| \leq \sqrt{\frac{N-1}{2}}\, |C|(\delta_l^2 + 2\delta_l)\}$*

Now we will compare the performance of these two approaches: quantized relative states and relative quantized states. Since the nominal consensus protocol we consider here is the same as in (6.1) and (3.9), we can compare the size of invariance set directly. For uniform quantizers, we have from Theorem 3.11 that at steady state all neighbors have relative difference less than $\frac{\delta_u}{2}$ while from Theorem 6.1 all agents in the group have relative differences less than δ_u. From this point of view, in the case of uniform quantizers, the system using relative quantized states as control inputs gives better performance than the one using quantized relative states.

However, when it comes to the case of logarithmic quantizers, from Theorem 3.11 the average consensus is asymptotically reached by using quantized relative states if G is connected, while in Theorem 6.1 the invariant set is that all agents are at the same logarithmic quantization level. Especially when the logarithmic gain is very large, one quantization level (6.3) can be very board. If the system starts from the initial states that are far from the center, there is a high possibility that the whole system reaches the invariant set quite early, which is also far from the average consensus. For instance, when the logarithmic gain $\delta_l = 1.6$, real numbers in the region [12.18, 33.11) have the same quantized value 20.08. The worst case would be all agents initially have states in [12.18, 33.11) and the system won't move at all as illustrated in Figure 6.3, which is much worse than the result in Theorem 3.11 that an average consensus is guaranteed with a connected graph and logarithmic quantizers.

Example 6.1. The weighted and undirected communication topology as in Figure 4.7 is used in this example. The system (6.1) is evaluated with respect to convergence and stability. As stated in Theorem 6.1, all agents have relative difference less than $\delta_u = 0.01$ at the steady state, which is illustrated in Figure 6.1 and 6.2.

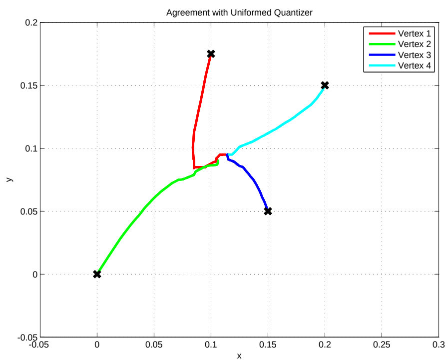


Figure 6.1: System 6.1 with uniform quantizers

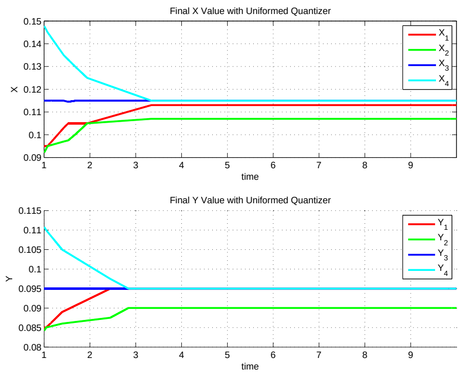


Figure 6.2: Final value in X, Y direction

If the logarithmic quantizer $\delta_l = 1.2$ is used, based on Theorem 6.1 the whole system converges to the configuration that all agents have the same logarithmically quantized value as shown in Figure 6.4. We can see that there is a large consensus error in Figure 6.3 and the system stops at the steady configuration quite early.

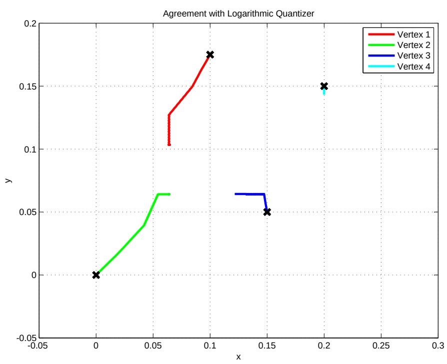


Figure 6.3: System 6.1 with logarithmic quantizers

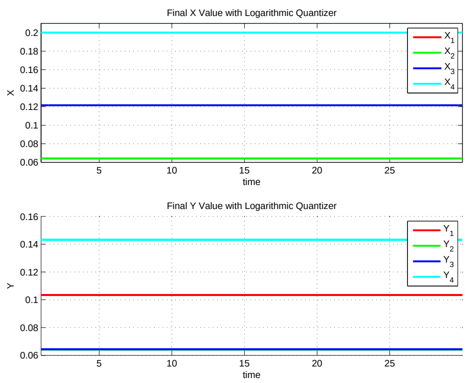


Figure 6.4: Final value in X, Y direction

7 Conclusion and Future Work

7.1 Conclusion

Stability of multi-agent systems under distributed control laws that are composed of quantized values of relative states between neighboring agents, were considered. We distinguished the system performance with uniform, logarithmic and dynamic quantizers as well as under static and time-varying communication topologies. It was proved that general undirected graphs guarantee stability and convergence in these cases. We also discussed the impact of quantization on the system convergence rate to the average consensus. Similar conclusions were shown to hold when the control inputs are applied in a discrete time fashion. Explicit upper bounds for the sampling time to ensure stability and convergence were derived. Second order dynamical agents were also taken into account where we obtained the convergence set for uniform quantizers and the upper bounds of the logarithmic gain. On the other hand, we also considered an alternative quantized protocol that uses relative quantized states as control parameters. Different convergence properties were investigated by comparisons between these two different approaches.

7.2 Future Work

At the beginning of Chapter 3, we clarified that through this thesis we mainly consider undirected graphs and always assume that all agents are homogeneous with identical quantizers. Future research may include extending the results to more general cases: directed graphs and heterogeneous agents. From simulation results, all conclusions should hold in directed counterparts but a formal mathematical proof is still lacking. There are mainly two obstacles in the way. First of all, the sum of the states is not kept constant anymore, which means the consensus point drifts with time and average consensus is not guaranteed. Moreover, if we approximate the quantization function $q(x)$ by a first order linearization $c(t)x$ at every time instance, consequently we get rid of the quantization function but the Laplacian matrix is weighted and time-varying now, which is quite similar to the case of infinite switching nominal system discussed in [9]. It is proved that a global quadratic Lyapunov function may not exist especially when the number of agents becomes large and topology switching is too frequent. Due to the above reasons, the mechanisms we used for undirected graphs in this thesis may not be feasible to these setups. Other non-quadratic Lyapunov functions like $x_{\max} - x_{\min}$ proposed in Section 3.3 could be one of the future research directions.

Another promising extension would be to take into account general non-smooth nonlinear protocols, using similar mechanisms as in Section 3.2 and 6.2. Also sampled systems with non-smooth protocols can be tackled with similar techniques as in Chapter 4. With respect to the second order system, even though the results we obtained are good enough to be used as a guidance on how to choose the quantizers, simulation experiments reveal that there are less conservative constrains regarding the system stability and convergence.

Furthermore, since the multi-agent cooperative control is a quite broad field, we could extend the discussion about the impact of quantization to other precision critical applications like event-triggered control where the control signal is updated based on state errors, or formation control that relies on the relative distance.

References

[1] B. D. O. Anderson , Iman Shames, Guoqiang Mao, and Baris Fidan, Formal Theory of Noisy Sensor Network Localization, *Siam j. Discrete Math*, vol. 24, no. 2, pp. 684-698.

[2] P. Bergamo, D. Maniezzo, A. Giovanardi, G. Mazzini, M. Zorzi, Distributed Power Control for Power-aware Energy-efficient Routing in Ad Hoc Networks, *EW2002*, pp. 237-243, Italy, Feb. 2002

[3] R. Carli, F. Fagnani, and S. Zampieri. On the state agreement with quantized information. *17th Intern. Symp. Networks and Systems*, pages 1500-1508, 2006

[4] F. Ceragioli, C. D. Persis, P. Frasca. Quantized average consensus: Discontinuities and hysteresis. *8th IFAC Symposium on Nonlinear Control Systems*, 2010

[5] D. V. Dimarogonas and K. J. Kyriakopoulos, A connection between formation infeasibility and velocity alignment in kinematic multi-agent systems, *Automatica*, Vol. 44, No. 10, pp. 2648-2654, October 2008.

[6] D. V. Dimarogonas and K. J. Kyriakopoulos. On the rendezvous problem for multiple nonholonomic agents. *IEEE Transaction on Automatic Control*, 52(5):916-922, 2007

[7] D. V. Dimarogonas and K. H. Johansson. Quantized Agreement under Time-varying Communication Topology. *2008 American Control Conference*, 4376-4381, 2008

[8] F. Fagnani, K. H. Johansson, A. Speranzon, and S. Zampieri. On multi-vehicle rendezvous under quantized communication. *16th Intern. Symp. Networks and Systems*, 2004

[9] J. A. Fax and R. M. Murray. Graph laplacian and stabilization of vehicle formations. *15th IFAC World Congress*, 2002

[10] C. Godsil and G. Royle. Algebraic Graph Theory, *Springer Graduate Texts in Mathematics*, 207, 2001

[11] R. A. Horn, C. R. Johnson. Matrix Analysis. *Cambridge University Press*, 1990

[12] A. Kashyap, T. Basar, and R. Srikant. Quantized consensus. *Automatica*, 43(7):1192-1203, 2007

[13] H. K. Khalil, Nonlinear Systems, *Prentice Hall*, 2000

[14] D. Liberzon. Hybrid feedback stabilization of systems with quantized signals, *Automatica*, 39(9):1543-1554, 2003.

[15] D. Liberzon. *Switching in Systems and Control.* Birkhäuser, Boston, 2003

[16] K. H. Johansson, A. Speranzon, and S. Zampieri. on quantization and communication topologies in multi-vehicle rendezous. *16th IFAC World Congress*, 2005.

[17] J. Lygeros, K. H. Johansson, S. N. Simic, J. Zhang, and S. S. Sastry. Dynamical Properties of Hybrid Automata. *Transaction on Automatic Control*, 48(1), 2003

[18] N. A. Lynch, Distributed algorithms, *San Francisco, CA: Morgan Kaufmann*, 1997

[19] L. Moreau, Stability of multiagent systems with time-dependent communication links, *IEEE Trans. Autom. Control*, vol. 50, no. 2, pp. 169-182, Feb. 2005.

[20] A. Nedic, A. Olshevsky, A. Ozdaglar, and J.N. Tsitsiklis. Distributed Subgradient Methods and Quantization Effects. *Proc. of CDC*, 2008.

[21] R. Olfati-Saber and R.M. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(9):1520-1533, 2004

[22] W. Ren and R. W. Beard. Consensus seeking in multiagent systems under dynamically changing interaction topologies, *IEEE Transactions on Automatic Control*, Vol. 50, No. 5, May 2005, pp. 655-661

[23] W. Ren, Ella Atkins. Distributed multi-vehicle coordinated control via local information exchange, *Int. J. Robust Nonlinear Control*, 2007;17:1002-1033

[24] A. Speranzon. Consensus and Communication in Multi-Robot Control Systems. *PhD Thesis*, KTH, 2006

[25] H. G. Tanner, Ali Jadbabaie and George J. Pappas, Stable Flocking of Mobile Agents, Part I: Fixed Topology, *IEEE Conference on Decision and Control*, 2003, pp. 2010-2015.

[26] G. Xie, H. Liu, L. Wang, Y. Jia, Consensus in networked multi-agent systems via sampled control: Fixed topology case, *American Control Conference*, 3902, 2009